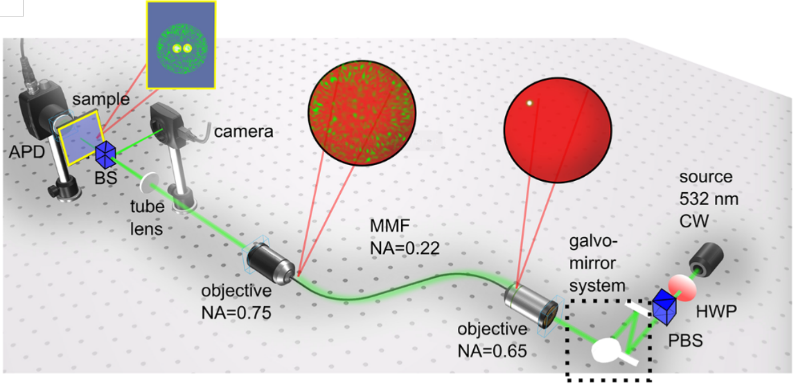
Sparsity-inducing bases for compressive sensing

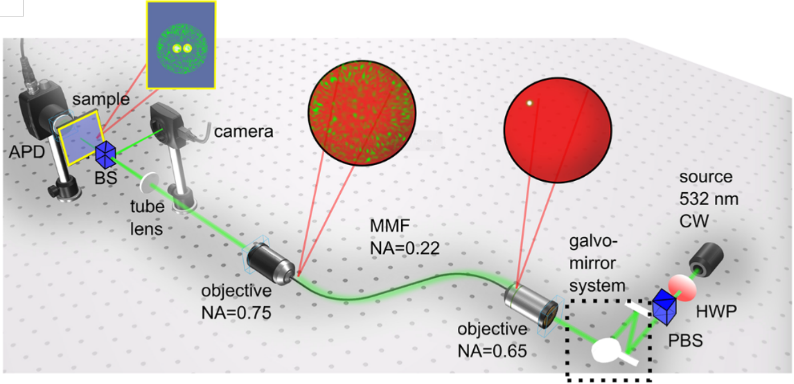
Nathan van den Berg Martijn Brouwer Robin van der Laag
Marcelo Geurts Galdámez Sean Tobin

Supervised by:
Ksenia Abrashitova (ARCNL), Tristan van Leeuwen (UU),
Rob Bisseling (UU), Ivan Kryven (UU)

21 April 2023

Experimental set-up





$$y = \sum_{j=1}^{n^2} a_j \cdot x_j = \langle a, x \rangle \qquad y_i = \langle a_i, x \rangle$$

Mathematical formulation

$$Ax = \begin{bmatrix} \dots & a_1 & \dots \\ & \vdots & \\ \dots & a_m & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n^2} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = y, \text{ where } A \in \mathbb{R}^{m \times n^2}.$$

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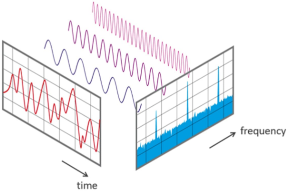
$$\underset{x \in \mathbb{R}^{n^2}}{\arg\min} \|Ax - y\|_2^2 + \lambda \|x\|_1 \qquad x = \mathrm{GPSR}(A, y, \lambda)$$

$$\underset{x \in \mathbb{R}^{n^2}}{\arg\min} \|(AB^{-1})(Bx) - y\|_2^2 + \lambda \|Bx\|_1 \qquad Bx = \mathrm{GPSR}(AB^{-1}, y, \lambda)$$

Goal: GPSR performs better when x is sparse, so we try to find B such that Bx is sparse.

Discrete Cosine Transform



$$x_k^c = \sum_{n=0}^{M-1} x_n \cos\left(\frac{\pi}{M}\left(n+\frac{1}{2}\right)k\right)$$

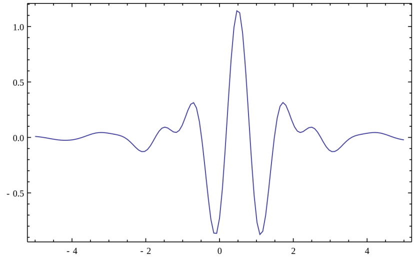
$$x^c = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \cos\left(\frac{\pi}{2M}\right) & \cos\left(\frac{3\pi}{2M}\right) & \dots & \cos\left(\frac{(M-\frac{1}{2})\pi}{M}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(\frac{(M-1)\pi}{2M}\right) & \cos\left(\frac{(M-1)3\pi}{2M}\right) & \dots & \cos\left(\frac{(M-1)(M-\frac{1}{2})\pi}{M}\right) \end{bmatrix} x$$

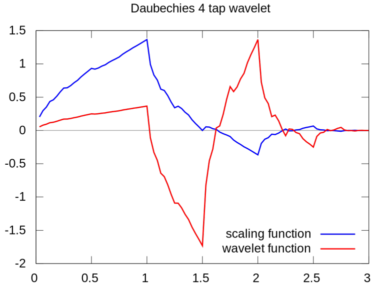
Discrete Cosine Transform

$$X^c = C(CX^T)^T$$

$$\begin{array}{cc} \text{Row 1} & \\ \text{Row 2} & \\ \text{Row 3} & \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{array}{c} \begin{array}{ccc} \text{Column 1} & \text{Column 1} & \text{Column 1} \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

DAUB4 Wavelet Transform





$$
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}
\xrightarrow{W_{\text{coef}}(8)}
\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \\ x'_5 \\ x'_6 \\ x'_7 \end{bmatrix}
\xrightarrow{P(8)}
\left[\begin{array}{c} x'_0 \\ x'_2 \\ x'_4 \\ x'_6 \\ \hline x'_1 \\ x'_3 \\ x'_5 \\ x'_7 \end{array}\right]
\xrightarrow{W_{\text{coef}}(4)}
\left[\begin{array}{c} x''_0 \\ x''_2 \\ x''_4 \\ x''_6 \\ \hline x'_1 \\ x'_3 \\ x'_5 \\ x'_7 \end{array}\right]
\xrightarrow{P(4)}
\left[\begin{array}{c} x''_0 \\ x''_4 \\ \hline x''_2 \\ x''_6 \\ \hline x'_1 \\ x'_3 \\ x'_5 \\ x'_7 \end{array}\right]
\xrightarrow{W_{\text{coef}}(2)}
\left[\begin{array}{c} x'''_0 \\ x'''_4 \\ \hline x''_2 \\ x''_6 \\ \hline x'_1 \\ x'_3 \\ x'_5 \\ x'_7 \end{array}\right]
$$

We want to find an $\hat{x} \in \mathbb{R}^{N^2}$, with $\hat{x} = Bx$ that is a solution to

$$\min_{\hat{x} \in \mathbb{R}^{n^2}} \|AB^{-1}\hat{x} - y\|_2^2 + \lambda \|\hat{x}\|.$$

Traditional Transforms

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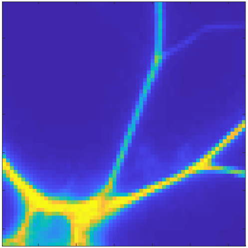
$$\min_{\hat{x} \in \mathbb{R}^{n^2}} \|AB^{-1}\hat{x} - y\|_2^2 + \lambda \|\hat{x}\|.$$

Using four different transforms B^{-1}:

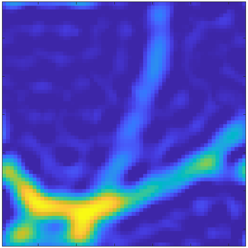
1. $B^{-1} = I$, no transformation,
2. $B^{-1} = C^{-1} \otimes C^{-1}$, with C the discrete cosine transformation matrix,
3. $B^{-1} = W^{-1} \otimes W^{-1}$, with W the Daubechies-4 wavelet transformation matrix,
4. $B^{-1} = (C \otimes C)(W^{-1} \otimes W^{-1})$, with C and W the same as above.

Recovered Images

Sample

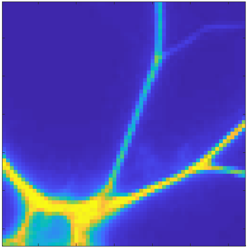


Baseline

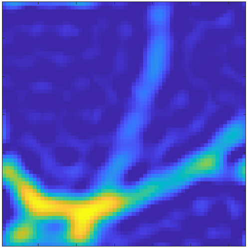


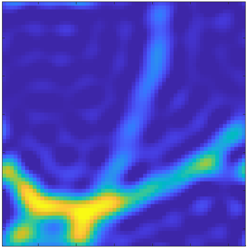
Recovered Images

Sample

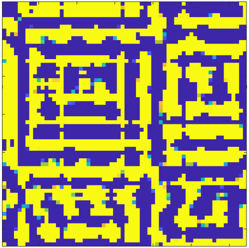


Baseline

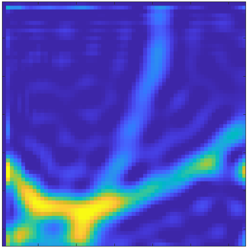




DCT

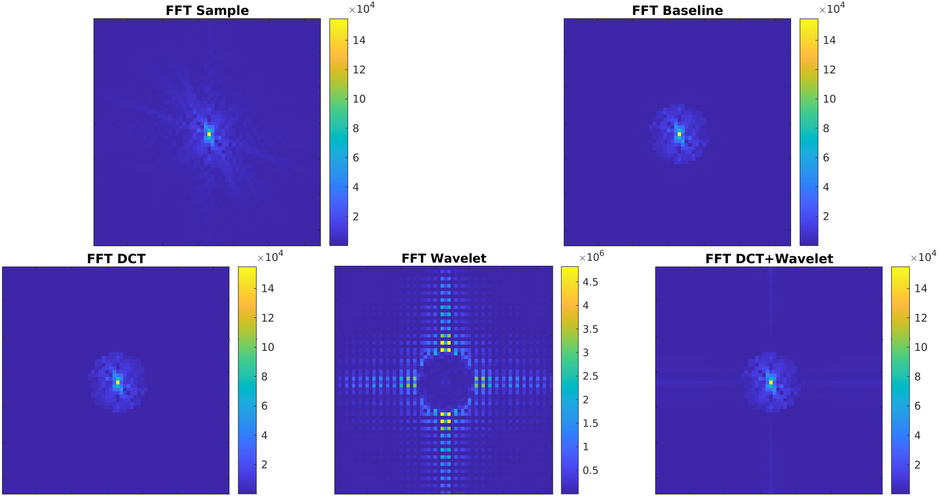


Wavelet



DCT + Wavelet

Fourier Transformed Images



Goal: find a basis transformation B such that the sparsity of Bx increases and with which the reconstruction quality using GPSR.

Machine learning approach

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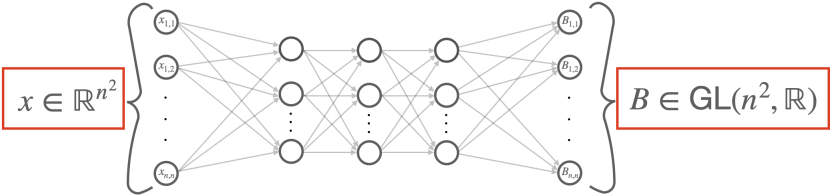
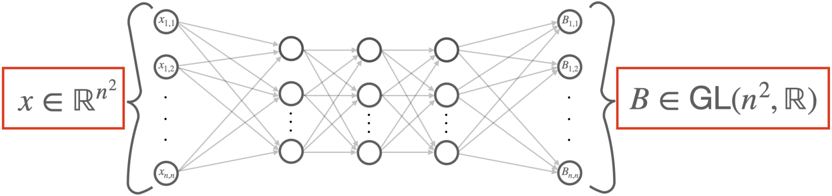
Machine learning approach: train a neural network with sparsity-inducing loss function that preserves orthogonality with flattened training samples x as input.

The network



$x \in \mathbb{R}^{n^2}$
$x_{1,1}$
$x_{1,2}$
$x_{n,n}$
$B_{1,1}$
$B_{1,2}$
$B_{n,n}$
$B \in \mathrm{GL}(n^2, \mathbb{R})$

The network



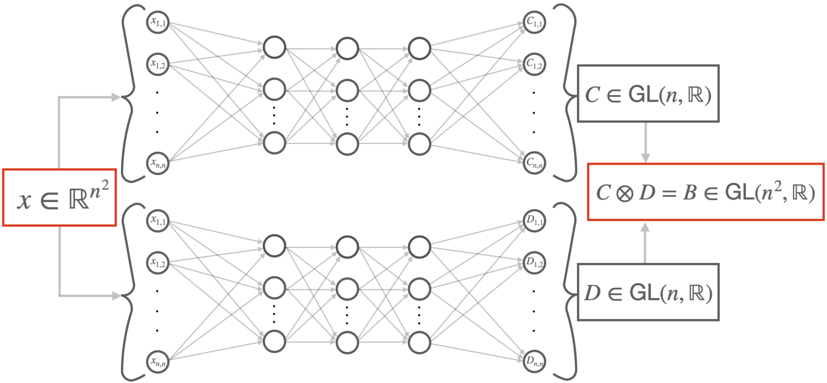
1. $loss_{sparse}[Bx] = \sum_i \sqrt{(Bx)_i^2 + \epsilon} \approx \|Bx\|_1, \quad \epsilon > 0$ (small).
2. $loss_{orth.}[B] = \|BB^\top - I_n\|_2^2 = \mathrm{MSE}(BB^\top - I_n)$.

Dimension reduction

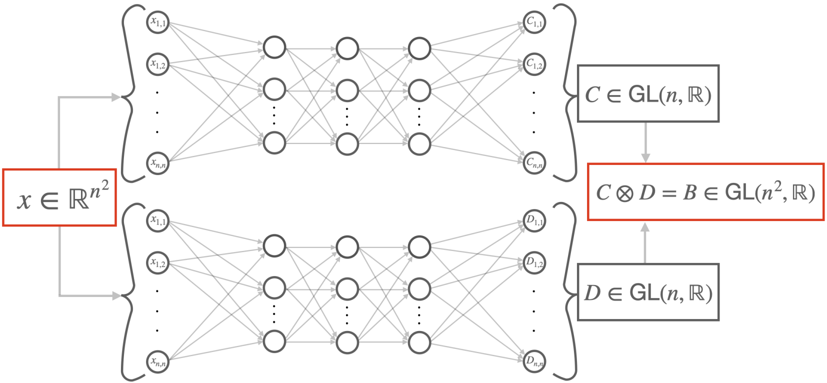
Let $C = [c_{ij}]_{i,j} \in \mathbb{R}^{n \times n}$ and $D = [d_{ij}]_{i,j} \in \mathbb{R}^{n \times n}$.
The Kronecker product of C and D is defined as,

$$B := C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \dots & c_{1n}D \\ c_{21}D & c_{22}D & \dots & c_{2n}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}D & c_{n2}D & \dots & c_{nn}D \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}.$$

Two linked neural networks



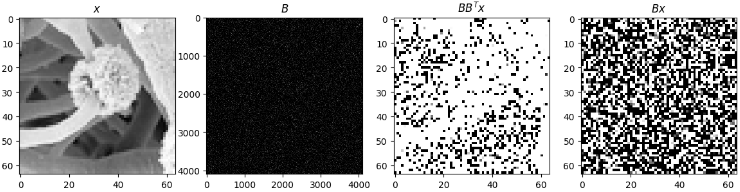
Two linked neural networks



$$loss(x; C, D) = loss_{sparse}[(C \otimes D)x] + \alpha\left(loss_{orth.}[C] + loss_{orth.}[D]\right)$$

$$= \sum_i \sqrt{[(C \otimes D)x]_i^2 + \epsilon} + \alpha\left(\|CC^\top - I_n\|_2^2 + \|DD^\top - I_n\|_2^2\right).$$

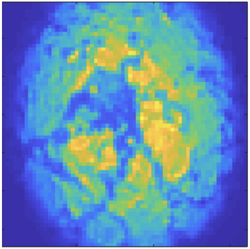
Training

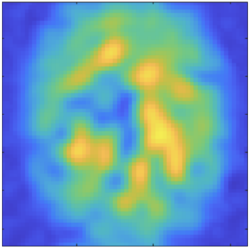


Iteration: 100
Sparsity x: 1
Sparsity Bx: 1993

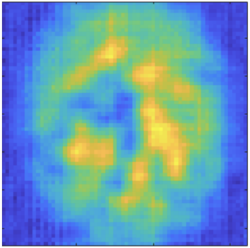
Results

Sample

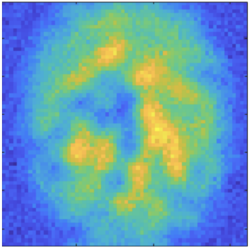




Identity B



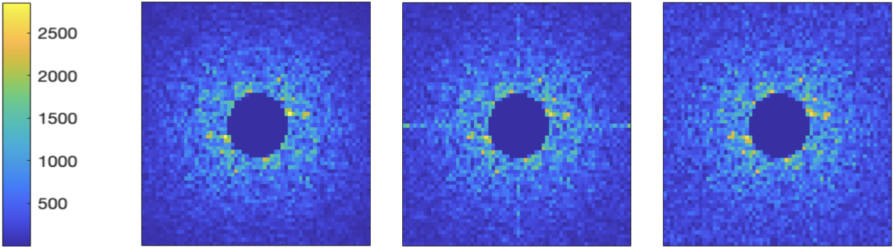
Kronecker B



Regular B

Quantifying errors

	Identity B	Kronecker B	Regular B
Correlation	0.93939	0.93393	0.9267
S.similarity	0.52892	0.47649	0.44159
FFT error	1362159.6975	1462938.2558	1640305.0802



Conclusions

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Questions?

2*Astrocyte	GPSR Output		Recovered Image	
	corr2	ssim	corr2	ssim
Baseline	NA	NA	0.91749	0.25872
DCT	0.94775	0.00716	0.92562	0.27034
DAUB4	-0.03108	-8.2600-05	-0.02966	-1.3560-05
DAUB4 + DCT	0.94489	0.23078	0.90678	0.23078

Table: 2D Correlation and the Structural similarity of the recovered sample/image using the different transformations on the 64×64 Astrocyte sample.

BONUS SLIDE2 (CIFAR10)