# Polymorphic sessions and sequential composition of types

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PLACES 123

RECAP ON • CFSTs are <u>not</u> restricted to tail recursion CFSTs

- RECAP ON CFSTs
- · CFSTs are not restricted to tail recursion
- · 5 Sequential composition operator

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#### RECAP ON CFSTs

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#### RECAP ON CFSTs

- · CFSTs are not restricted to tail recursion
- 5 Sequential composition operator
- · Skip Corresponding neutral element
- · CFSTs are not characterized by regular languages
- Type equivalence : decidable (via translation to simple grammars)

· Promoted CFSTs to the higher-order setting

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 !(?int)

```
[PLACES '22] Higher - order Context - free Session types in System F
```

• Promoted CFSTs to the higher-order setting !(?int)

```
Input Tree = 0 { Node : Input Tree ; ! (?int) ; Input Tree ,

Leaf : skip }
```

incorporated using equations

 Added polymorphism, using De Brugin indices to refer to polymorphic variables

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Y 0; ! int

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$$\forall S_0$$
; int  $\forall T_0 \rightarrow I$  int bound by the first enclosing  $\forall$ 

 Added polymorphism, using De Brugin indices to refer to polymorphic variables

· But .. we only allowed functional polymorphism !

- We pronded: syntactic (rule based) and semantic (bisimulation based) definitions of equivalence;
  - the bisimilarity of simple grammars.

#### [PLACES'22] -> [PLACES'23]

GOAL. Full polymorphism

 $\forall \dots \ T \rightarrow \text{of Kind } S$ 

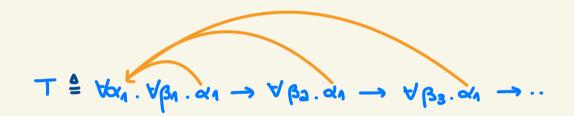
#### GOAL. Full polymorphism

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NEW DESIGN CHOICES

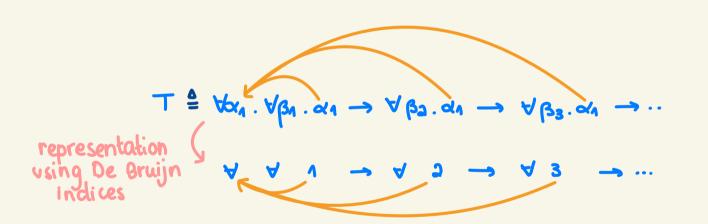
equation 
$$X = O \rightarrow V^{S}X$$
expansion  $X = O \rightarrow V^{S}X$ 

$$T \triangleq \forall \alpha_1 . \forall \beta_1 . \alpha_1 \rightarrow \forall \beta_2 . \alpha_1 \rightarrow \forall \beta_3 . \alpha_1 \rightarrow \cdots$$



$$T \stackrel{\triangle}{=} \forall \alpha_1 . \forall \beta_1 . \alpha_1 \rightarrow \forall \beta_2 . \alpha_1 \rightarrow \forall \beta_3 . \alpha_1 \rightarrow \cdots$$

$$Y \stackrel{:}{=} \forall \beta \ \alpha \rightarrow Y$$



$$T \triangleq \forall \alpha_1 . \forall \beta_1 . \alpha_1 \rightarrow \forall \beta_2 . \alpha_1 \rightarrow \forall \beta_3 . \alpha_1 \rightarrow \cdots$$

$$\forall \forall \forall 1 \rightarrow \forall 2 \rightarrow \forall 3 \rightarrow \cdots$$

$$\forall \text{NO FINITE REPRESENTATION}$$

$$U \stackrel{\triangle}{=} \forall \alpha_1 \cdot \forall \beta_1 \cdot \alpha_1 \rightarrow \forall \alpha_2 \cdot \beta_1 \rightarrow \forall \beta_2 \cdot \alpha_2 \rightarrow \forall \alpha_3 \cdot \beta_2 \rightarrow \cdots$$

$$\bigcup \stackrel{\bullet}{=} \forall \alpha_1 . \forall \beta_1 . \alpha_1 \rightarrow \forall \alpha_2 . \beta_1 \rightarrow \forall \beta_2 . \alpha_2 \rightarrow \forall \alpha_3 . \beta_2 \rightarrow \cdots$$

$$\downarrow \quad \forall \alpha . \times$$

$$\times \stackrel{\dot{=}}{=} \forall \beta . \alpha \rightarrow \forall \alpha . \beta \rightarrow \times \quad \text{(*)}$$

$$U \stackrel{\triangle}{=} \forall \alpha_1 . \forall \beta_1 . \alpha_1 \rightarrow \forall \alpha_2 . \beta_1 \rightarrow \forall \beta_2 . \alpha_2 \rightarrow \forall \alpha_3 . \beta_2 \rightarrow \cdots$$

$$\forall \forall \alpha . \times \\
\times \stackrel{\triangle}{=} \forall \beta . \alpha \rightarrow \forall \alpha . \beta \rightarrow \times \text{ (4)}$$

$$\Rightarrow \text{CANNOT BE WRITTEN USING CONVENTIONAL } \mu\text{-TYPES}$$

WHAT WE
DID WITH
WHAT WE
LEARNED

· We kept using equations

# WHAT WE DID WITH WHAT WE LEARNED

- · We kept using equations
- .. We dropped the use of De Bruijn indices and worked with variable names

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- We kept using equations
- .. We dropped the use of De Bruijn indices and worked with variable names
- .. We still do not rename variables when expanding equations we do not adopt the variable convention

WHAT WE

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· We kept using equations

.. We dropped the use of De Bruijn indices and worked with variable names

.. We still do not rename variables when expanding equations — we do not adopt the variable convention

$$X \doteq \forall \beta. \alpha \rightarrow \forall \alpha. \beta \rightarrow X \rightarrow \text{renaming} \qquad X \doteq \forall \beta. \alpha \rightarrow \forall \alpha'. \beta \rightarrow X$$

```
BACK TO OUR GOAL
```

## Full polymorphism

```
HeteroTree = & { Leaf · skip,

Node: HeteroTree; (Vx:T.?x);

HeteroTree
}

kinds
```

TYPE EQUIVALENCE WITH DE BRUIJN INDICES

 $A_{k}^{\perp} \perp = A_{k}^{\perp} \cap$   $\perp = \cap$ 

TYPE EQUIVALENCE WITH DE BRUIJN INDICES

 $\frac{A_{\mathbf{k}} \perp \sim A_{\mathbf{k}} \cap}{\perp \sim \cap}$ 

TYPE EQUIVALENCE WITH VARIABLE NAMES

T ~ U [α/β]
∀α: K. Τ~ ∀β: K. U

## TYPE EQUIVALENCE WITH DE BRUIJN INDICES

 $\frac{A_{\mathbf{k}} \perp \sim A_{\mathbf{k}} \cap A_{\mathbf{k}}}{\bot \sim A_{\mathbf{k}}}$ 

TYPE EQUIVALENCE WITH VARIABLE NAMES

T~U[«/β] «
Va:K.T~Vβ:K.U

but

we do not want to make these substitutions on the fly

CANONICAL RENAMING

$$N(\alpha) = \alpha$$

$$N(\alpha) = \alpha$$

$$N(\forall \alpha : K.T) = \forall \alpha_i : K.N(T[\alpha_i/\alpha])$$
fresh vanable

 $\alpha_i = \text{first variable not free in } \forall \alpha : K. T$ 

$$N(\alpha) = \alpha$$

$$N(\forall \alpha : K.T) = \forall \alpha_i : K. N(T[\alpha_i/\alpha]),$$
where  $\alpha_i = first(\forall \alpha : K.T)$ 

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$$eg.$$

$$N(\forall \alpha: \tau. \ \forall \beta \cdot s. \beta) = \forall x \cdot \tau. \ N(\forall \beta: s. \beta)$$

$$= \forall x: \tau. \ N(\forall \beta: s. \beta)$$

$$= \forall x \cdot \tau. \ \forall x: s. \beta$$
bound here

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where  $\alpha_i = first(\forall \alpha : K.T)$ 

$$eg.$$

$$N(\forall \alpha: \tau. \ \forall \beta \cdot s. \beta) = \forall x \cdot \tau. \ N(\forall \beta: s. \beta [ \forall \lambda ) ]$$

$$= \forall x : \tau. \ N(\forall \beta: s. \beta)$$

$$= \forall x \cdot \tau. \ \forall x : s. x$$
bound here

The canonical renaming uses the least amount of variable names possible.

$$N(\alpha) = \alpha$$

$$N(\forall \alpha : K.T) = \forall \alpha_i : K. N(T[\alpha_i/\alpha]),$$
where  $\alpha_i = first(\forall \alpha : K.T)$ 

$$N(\forall \alpha: K. T; U) = \forall \alpha: K.(N(T[\alpha:/\alpha]); N(U))$$
where  $\alpha: = first(\forall \alpha: K. T; U)$ 

$$N(\alpha) = \alpha$$

$$N(\forall \alpha : K.T) = \forall \alpha_i : K.N(T[\alpha_i/\alpha]),$$
where  $\alpha_i = first(\forall \alpha : K.T)$ 

$$N(\forall \alpha: K. \top; U) = \forall \alpha: K. (N(\top [\alpha: /\alpha]); N(U))$$
where  $\alpha: = first(\forall \alpha: K. \top; U)$ 

TYPE EQUIVALENCE

e.g., N((∀a: s.d); d)

e.g., N((Va:s.a); d) = V b:s. (N(a[r/a]); N(a))

e.g., N((Va: s.a); d) = V &: s. (N(a[ [ /a]); N(a)) = V V. s. (N(V); N(a))

$$N(\forall \alpha: K.T; U) = \forall \alpha_i: K.(N(T[\alpha i/\alpha]); N(U))$$
where  $\alpha_i = first(\forall \alpha: K.T; U)$ 

```
e.g.,

N((\forall \alpha: s. \alpha); \alpha) = \forall \forall: s. \left( n(\alpha)); \left( \alpha) \right) = \forall \forall: s. \left( n(\alpha)) = \forall \forall: s. \left( \forall : \alpha \right) = \forall: s. \left( \forall : \alpha \right)
```

e.g., N(Va: s.a; VB:T 1B)

e.g., N(∀α:s.α; ∀β:Τ 'β) = ∀ δ:s. (N(α[Υ/α]); N(∀β:T.!β))

e.g., N(∀α:s.α; ∀β:Τ 'β) = ∀ ४:s. (N(α[Υ/α]); N(∀β:Τ.!β)) = ∀ γ. s.(γ; ∀γ:Τ. N(!β[४/β]))

e.g., N(∀α: s.α; ∀β: Τ 'β) = ∀ δ: s. (N(α[Υ/α]); N(∀β: Τ.!β)) = ∀ δ. s. (∀; ∀ δ: Τ. N(!β)) = ∀ δ: s. (δ; ∀ Γ: Τ. N(!δ)) CANONICAL N(V RENAMING

$$N(\forall \alpha: K.T; U) = \forall \alpha_i: K.(N(T[\alpha_i/\alpha]); N(U))$$
  
where  $\alpha_i = first(\forall \alpha: K.T; U)$ 

e.g., N(∀α: s.α; ∀β: Τ 'β) = ∀δ: s. (N(α[Υ/α]); N(∀β: Τ. !β)) = ∀δ: s. (δ; ∀ζ: Τ. N(!૪)) = ∀δ: s. (ζ; ∀ζ: Τ. !૪)

## MERCI!