Theoretical Statistical Physics (MKTP1)
Version: 22.3.2021

# 1 Introduction to probability theory

**Bayes' theorem**

$$p(B|A) = \frac{p(A|B)\cdot p(B)}{p(A)} = \frac{p(A|B)\cdot p(B)}{\sum_{B'} p(A|B)\cdot p(B')}$$

**Expactation and covariance**

$$\langle f\rangle = \sum_i f(i)p_i \text{ or } \langle f\rangle = \int f(x)p(x)dx$$

$$\mu = \langle i\rangle = \sum_i ip_i \text{ or } \mu = \langle x\rangle = \int xp(x)dx$$

$$\sigma^2 = \langle i^2\rangle - \langle i\rangle^2$$
$$\sigma_{ij}^2 = \langle ij\rangle - \langle i\rangle^2$$

**Binomial distribution**

$$\frac{N!}{(N-i)!i!} = \binom{n}{i} \text{ binomial coefficient}$$

$$p_i = \binom{N}{i}\cdot p^i q^{N-i} \text{ distribution}$$

$$\mu = \langle i\rangle = N\cdot p$$
$$\langle i^2\rangle = p\cdot N + p^2\cdot N\cdot(N-1)$$
$$\sigma^2 = N\cdot p\cdot q$$
$$\sum_{i=0}^N p_i = \sum_{i=0}^N \binom{N}{i}\cdot p^i q^{N-i} = (p+q)^N = 1$$

**Gauss distribution**

$$p(x) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{1}{2}}}\cdot e^{-\frac{x-\mu}{2\sigma^2}}, \quad \langle x^2\rangle = \sigma^2$$

**Poisson distribution**

$$p(k;\mu) = \frac{\mu^k}{k!}e^{-\mu}, \quad E[k]=\mu,\ V[k]=\mu$$

**Information entropy**

$$S = -\sum_i p_i\ln(p_i)$$

# 2 The microcanonical ensemble

$E \approx$ const , $V =$ const , $N =$ const.

**The fundamental postulate**

$$\Omega(E) = \sum_{n:E-\delta E\le E_n\le E} 1$$

$$\Omega(E;\delta E) = \frac{1}{h^{3N}N!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E} d\vec q d\vec p$$

$$S = -k_B\sum_{i=1}^{\Omega} p_i\ln(p_i) = k_B\ln(\Omega)$$

**$n_0$ different particles**

$$\Omega = \frac{1}{h^{3N}\prod_{j=0}^{n_0}N_j!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E} d\vec q d\vec p$$

**Equilibrium conditions**

Entropy $S$ must be maximal

**Thermal contact**

$$\left.\frac{\partial S(E,V,N)}{\partial E}\right|_{V,N} = \frac{1}{T(E,V,N)}$$

**Contact with volume excahnge**

$$\left.\frac{\partial S(E,V,N)}{\partial V}\right|_{E,N} = \frac{p(E,V,N)}{T(E,V,N)}$$

**Contact with exchange of particle number**

$$\left.\frac{\partial S(E,V,N)}{\partial N}\right|_{E,V} = -\frac{\mu(E,V,N)}{T(E,V,N)}$$

**Equations of state**

$$dE = TdS - pdV + \mu dN$$

**solution concept**

- Set up Hamiltonian
- Calculate phasevolume $\Omega$
- Calculate entropy $S$
- determine $T, p, \mu$
- Calculate $U = \langle E\rangle$
- thermodynamic potentials:
  $F(T,V,N) = U - TS$
  $\hat H(S,p,N) = U + pV$
  $G(T,p,N) = U + pV - TS$

**Ideal Gas**

$$\mathcal{H} = \sum_{i=1}^{3N}\frac{p_i^2}{2m} + V(q_1,\dots,q_{3N})$$

**microcanonical partition sum for an ideal gas**

$$\Omega(E) = \frac{V^N\pi^{3N/2}(2mE)^{3N/2}}{h^{3N}N!\left(\frac{3N}{2}\right)!}$$

$$S = k_BN\left\{\ln\left[\left(\frac{V}{N}\right)\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right] + \frac{5}{2}\right\}$$

**Equations of state fo ideal gas**

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{3}{2}\frac{Nk_B}{E} \to U = \frac{3}{2}Nk_BT$$

$$p = T\left(\frac{\partial S}{\partial V}\right)_{E,N} = TNk_B\frac{1}{V} \to pV = Nk_BT$$

$$\mu = k_BT\ln\left(\frac{N\lambda^3}{V}\right) \text{ chemical potentail}$$

$$\lambda = \frac{h}{\sqrt{2\pi mk_BT}} \quad \text{Thermische de Broglie}$$

**Einstein model for specific heat of a solid**

$$E = \hbar\omega\left(\frac{N}{2} + Q\right)$$

$$\Omega(E,N) = \frac{(Q+N)!}{Q!N!}$$

$$S = k_B\ln(\Omega)$$

$$= k_B\left[Q\ln\left(\frac{Q+N}{Q}\right) + N\ln\left(\frac{Q+N}{N}\right)\right]$$

$$= k_BN\left[(e+\frac{1}{2})\ln(e+\frac{1}{2}) - (e-\frac{1}{2})\ln(e-\frac{1}{2})\right]$$

$$e = E/E_0\ ; E_0 = N\hbar\omega$$

$$\to E = N\hbar\omega\left(\frac{1}{2} + \frac{1}{e^\beta - 1}\right)$$

**Entropic elasticity of polymers**

$$N_+ - N_- = \frac{L}{a}$$

$$N_+ = \frac{1}{2}\left(N + \frac{L}{a}\right)$$

$$\Omega = \frac{N!}{N_+!N_-!}$$

$$S = -k_B\left(N_+\ln\left(\frac{N_+}{N}\right) + N_-\ln\left(\frac{N_-}{N}\right)\right)$$

**Statistical deviation from average**

Two ideal gases in thermal conact $T_1 = T_2$

$$S_i = \frac{3}{2}k_BN_i\ln(E_i) + \text{independent of} E_i$$

$$S = S_1 + S_2$$

$$dS = 0 \to \frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2}$$

$$\to \overline{E}_1 = \frac{N_1}{N}E$$

consider small deviation:

$$E_1 = \overline{E}_1 + \Delta E, \quad E_2 = \overline{E}_2 - \Delta E$$

$$S(\overline{E}_1 + \Delta E) \approx \frac{3}{2}k_B\Big[N_1\ln\overline{E}_1 + N_2\ln\overline{E}_2$$
$$- \frac{N_1}{2}\left(\frac{\Delta E}{\overline{E}_1}\right)^2 - \frac{N_2}{2}\left(\frac{\Delta E}{\overline{E}_2}\right)^2\Big]$$

$$\to \Omega = \overline{\Omega}e^{\left[-\frac{3}{4}\left(\frac{\Delta E}{E}\right)^2 N^2\left(\frac{1}{N_1} + \frac{1}{N_2}\right)\right]}$$

# 3 The canonical ensemble

$T =$ const , $V =$ const , $N =$ const.

**Boltzmann distribution**

$$p_i = \frac{1}{Z}e^{-\beta E_i} \quad \text{Boltzmann distribution}$$

$$Z = \sum_i e^{-\beta E_i} \quad \text{partition sum}$$

**For classical Hamiltonian systems:**

$$p(\vec q,\vec p) = \frac{1}{ZN!h^{3N}}e^{-\beta\mathcal{H}(\vec q,\vec p)}$$

$$Z_N(T,V) = \frac{1}{N!h^{3N}}\iint d\vec q d\vec p e^{-\beta\mathcal{H}(\vec q,\vec p)}$$

For common Hamiltonian:

$$Z_N(T,V) = \frac{1}{\lambda^{3N}N!}\int_V d\vec q e^{-\beta\hat V(\vec q)}$$

**Free energy**

$$F(T,V,N) = -k_BT\ln Z_N(T,V)$$

$$\langle E\rangle = U = -\partial_\beta\ln Z_N$$

total differential:

$$dF = dE + d(TS) = -SdT - pdV + \mu N$$

**equations of state**

$$S = -\frac{\partial F}{\partial T}, \quad p = -\frac{\partial F}{\partial V}, \quad \mu = \frac{\partial F}{\partial N}$$

**Non-interacting systems**

$\epsilon_{ij}$ is the $j^{th}$ state of the $i^{th}$ element

$$Z = \sum_{j_1}\sum_{j_2}\dots\sum_{j_N} e^{-\beta\sum_{i=1}^N\epsilon_{ij_i}}$$

$$= \left(\sum_{j_1} e^{-\beta\epsilon_{1j_1}}\right)\dots\left(\sum_{j_N} e^{-\beta\epsilon_{Nj_1N}}\right)$$

$$= z_1\cdot\dots\cdot z_N = \prod_{i=1}^N z_i$$

$$\to F = -k_BT\sum_{i=1}^N\ln(z_i) = -k_BT\ln(Z)$$

$$Z = z^N, \qquad F = -k_BTNln(z)$$

**TODO: ADD EXAMPLES**

**Equipartition theorem**

f are the degrees of freedom.
harmonic Hamiltonian with $f = 2$

$$\mathcal{H} = Aq^2 + Bp^2$$

$$z \propto \int dqdpe^{-\beta\mathcal{H}}$$

$$= \left(\frac{\pi}{A\beta}\right)^{\frac{1}{2}}\cdot\left(\frac{\pi}{B\beta}\right)^{\frac{1}{2}}$$

$$\propto \left(T^{\frac{1}{2}}\right)^f$$

For sufficiently high temperture (classical limit), each quadratic term in the Hamiltonian contributes a factor $T^{\frac{1}{2}}$ to the partition sum ('equipartition theorem')

$$F = -k_BT\ln(z) = -\frac{f}{2}k_BTln(T)$$

$$S = -\frac{\partial F}{\partial T} = \frac{f}{2}k_B(\ln(T) + 1)$$

$$U = -\partial_\beta\ln(z) = \frac{f}{2}k_BT$$

$$c_v = \frac{dU}{dT} = \frac{f}{2}k_B$$

**Molecular gases**

$$Z = Z_{trans}\cdot Z_{vib}\cdot Z_{rot}\cdot Z_{elec}\cdot Z_{nuc}$$
$$Z_x = z_x^N$$

**Vibrational modes**

often described by the Morse potential:

$$V(r) = E_0\left(1 - e^{-\alpha(r-r_0)}\right)^2$$

An exact solution of the Schrödinger equation gives:

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) - \frac{\hbar^2\omega_0^2}{eE_0}\left(n + \frac{1}{2}\right)^2$$

$$\omega_0 = \frac{\alpha}{2\pi}\sqrt{\frac{2E_0}{\mu}}, \quad \mu = \frac{m}{2}$$

For $\hbar\omega_0 \ll E_0$ we can use the harmonic approximation:

$$z_{vib} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega_0}}$$

$$T_{vib} \approx \frac{\hbar\omega_0}{k_B} \approx 6.140K \text{ for } H_2$$

**Rotational modes**

standart approximation is the one of a rigid rotator. The moment of inertia is given as:

$$I = \mu r_0^2 \quad T_{rot} = \frac{\hbar^2}{Ik_B}$$

$$\to E_l = \frac{\hbar^2}{2I}l(l+1)$$

**Nuclear contributions: ortho- and parahydrogen**

$$z_{ortho} = \sum_{l=1,3,5,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$

$$z_{para} = \sum_{l=0,2,4,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$

Theoretical Statistical Physics (MKTP1)
Version: 22.3.2021

**Specific heat of a solid**

**Debye model**

$$\rightarrow \omega(k) = \left(\frac{4\kappa}{m}\right)^{\frac{1}{2}} \left|\sin\left(\frac{ka}{2}\right)\right|$$
$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}$$

Debye frequency:

$$\omega_D = c_s \left(\frac{6\pi^2 N}{V}\right)^{\frac{1}{3}}$$
$$c_s = \left.\frac{d\omega}{dk}\right|_{k=0} = \sqrt{\frac{\kappa}{m}}a$$

density of states in $\omega$−space:

$$D(\omega) = 3\frac{\omega^2}{\omega_D^3} \quad \text{for} \omega \le \omega_D$$

count modes in frequency-space:

$$\sum_{modes}(\dots) = 3\sum_k(\dots) = 3N\int_0^{\omega_D} d\omega D(\omega)(\dots)$$

partition sum:

$$z(\omega) = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$\rightarrow Z = \prod_{modes} z(\omega)$$

$$\rightarrow E = -\partial_\beta \ln(Z) = \sum_{modes}\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}+\frac{1}{2}\right)$$
$$= E_0 + 3N\int_0^{\omega_D} d\omega \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{3\omega^2}{\omega_D^3}$$

$$c_v(T) = \frac{\partial E}{\partial T}$$
$$= \frac{3\hbar^2 N}{k_B T^2}\int_0^{\omega_D} d\omega \frac{3\omega^2}{\omega_D^3}\frac{e^{\beta\hbar\omega}\omega^2}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

$$u = \beta\hbar\omega$$

$$c_v(T) = \frac{9Nk_B}{u_m^3}\int_0^{u_m}\frac{e^u u^4}{(e^u-1)^2}du$$

the limit for $\hbar\omega_D \ll k_B T$:

$$c_v(T) = 3Nk_B$$

the limit for $k_B T \ll \hbar\omega_D$: ($T_D = \frac{\hbar\omega_D}{k_B}$)

$$c_v(T) = \frac{12\pi^4}{5}Nk_B\left(\frac{T}{T_D}\right)^3$$

**Black body radiation**

$$E = \frac{4\sigma}{c}VT^4, \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$
$$c_v = \frac{16\sigma}{c}VT^3$$

$$J = \frac{P}{A} = \sigma T^4 \quad \text{Stefan- Boltzmann law}$$

Plank's law for black body radiation

$$u(\omega) := \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/(k_B T)}-1}$$

The Plank distribution has a maximum at:

$$\hbar\omega_{max} = 2.82 k_B T \quad \text{Wien's displacement law}$$

## 4 The grandcanonical ensemble

**Probability distribution**

$T$ and $\mu$ are fixed.

$$p_i = \frac{1}{Z_G}e^{-\beta(E_i-\mu N_i)} \text{ prob. distribution}$$
$$Z_G = \sum_i e^{-\beta(E_i-\mu N_i)} \text{ partition sum}$$
$$\Psi = -k_B T\ln(Z_G) \text{ thermodynamic potential}$$

**Grandcanonical potential**

The probability to have a macroscopic value $(E, N)$ is:

$$p(E,N) = \frac{1}{Z_G}\Omega(E,N)e^{-\beta(E-\mu N)}$$
$$= \frac{1}{Z_G}e^{-\beta(E-TS-\mu N)} = \frac{1}{Z_G}e^{-\beta\Psi(T,V,\mu)}$$

grandcanonical potential:

$$\Psi(T,V,\mu) := E - TS - \mu N$$

p is maximal, if $\Psi$ is minimal.
Total differential:

$$d\Psi = d(E - TS - \mu N)$$
$$= TdS - pdV + \mu dN - d(TS + \mu N)$$
$$= -SdT - pdV - Nd\mu$$

Equations of state:

$$S = -\frac{\partial\Psi}{\partial T}, p = -\frac{\partial\Psi}{\partial V}, N = -\frac{\partial\Psi}{\partial\mu}$$

**Fluctuations**

$$\langle N\rangle = \sum_i p_i N_i = \frac{1}{\beta}\partial_\mu \ln(Z_G)$$
$$\sigma_N^2 = \langle N^2\rangle - \langle N\rangle^2 = \frac{1}{\beta^2}\partial_\mu^2\ln(Z_G)$$
$$\frac{\sigma_N}{\langle N\rangle} \propto \frac{1}{N^{\frac{1}{2}}}$$

**Ideal gas**

$$Z(T,V,N) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N, \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$
$$Z_G = \sum_{N=0}^\infty Z(T,V,N)e^{\beta\mu N}$$
$$= \sum_{N=0}^\infty \frac{1}{N!}\left(e^{\beta\mu}\frac{V}{\lambda^3}\right)^N$$
$$= e^{z\frac{V}{\lambda^3}} \quad \text{fugacity: } z := e^{\beta\mu}$$
$$\langle N\rangle = \frac{1}{\beta}\partial_\mu\ln(Z_G) = \frac{V}{\lambda^3}e^{\beta\mu}$$
$$\mu = k_B T\ln\left(\frac{N\lambda^3}{V}\right)$$

**Molecular adsorption onto a surface**

$$Z_G = z_G^N; z_G = 1 + e^{-\beta(\epsilon-\mu)}$$
$$\langle n\rangle = \frac{1}{e^{-\beta(\mu-\epsilon)}+1} \text{ per site}$$
$$\langle\epsilon\rangle = \epsilon\langle n\rangle$$

## 5 Quantum fluids

**Fermion vs. bosons**

Particles with half-integer (integer) spin are called fermions (bosons). Their total wave function (space and spin) must be antisymmetric (symmetric) under the exchange of any pair of identical particles.

**Canonical ensemble**

two particles that are distributed over two states with energies 0 and $\epsilon$

$$Z_F = e^{-\beta\epsilon} \quad \text{Fermi-Dirac}$$
$$Z_B = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \quad \text{Bose-Einstein}$$
$$Z_M = \frac{1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}}{2} \quad \text{Maxwell-Boltzmann}$$

**Grand canonical ensemble**

Fermions:

$$z_F = 1 + e^{-\beta(\epsilon-\mu)}$$

average occupation number $n_F$:

$$n_F = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \quad \text{Fermi function}$$

For $T \to 0$, the fermi function approaches a step function:

$$n_F = \Theta(\mu - \epsilon)$$

Bosons:

$$z_B = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}}$$

average occupation number $n_B$:

$$n_B = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

- Fermions tend to fill up energy states one after the other
- Bosons tend to condense all into the same low energy state

**The ideal Fermi fluid**

density of states:

$$D(\epsilon) = \frac{V}{2\pi N}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\epsilon}$$

**Fermi energy**

$$N = \sum_{\vec{k},m_s} n_{\vec{k},m_s} = N\int_0^\infty d\epsilon D(\epsilon)n_F(\epsilon)$$

Limit $T \to 0$. $\mu(T = 0)$ is called Fermi energy:

$$\epsilon_F = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2\rho^{\frac{2}{3}}}{2m}$$

**specific heat**

$$\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \text{ for } T \ll \frac{\epsilon_F}{k_B}$$
$$c_V = \left.\frac{\partial E}{\partial T}\right|_V = N\frac{\pi^2}{3}k_B^2 D(\epsilon_F)T$$
$$c_V = N\frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}k_B$$

**Fermi pressure**

$$p \overset{T\to 0}{\rightarrow} \frac{2}{5}\frac{N}{V}\epsilon_F = \frac{(2\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{mv^{\frac{5}{3}}}$$

**The ideal Bose fluid**

$\epsilon = \frac{\hbar^2 k^2}{2m}$ and conserved particle number N.

$$N = \frac{N}{\lambda^3}g_{\frac{3}{2}}(z)$$
$$z = e^{\beta\mu}, \quad \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$
$$T_c = \frac{2\pi}{\left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{3}{2}}}\frac{\hbar^2\rho^{\frac{2}{3}}}{k_B m}$$
$$E = \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{\frac{5}{2}}(z) = \frac{3}{2}k_B T N_e\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}$$
$$c_V = \frac{15}{4}k_B N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \text{ ( for } T \le T_c)$$
$$c_V = \frac{15}{4}k_B N\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} - \frac{9}{4}k_B N\frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} \ (T > T_c)$$

**Classical limit**

$\mu \to -\infty$ the two grandcanonical distr. become the Maxwell-Boltzmann distr.

$$n_{F/B} = \frac{1}{e^{\beta(\epsilon-\mu)}\pm 1} \rightarrow e^{\beta\mu}e^{-\beta\epsilon}$$
$$N = g\frac{V}{\lambda^3}e^{\beta\mu}$$
$$E = \frac{3}{2}k_B T N$$

## 6 Phase transitions

**Ising model**

$$\mathcal{H} = -J\sum_{\langle ij\rangle}S_i S_j - B\mu\sum_i S_i$$
$$\beta\mathcal{H} = -K\sum_{\langle ij\rangle}S_i S_j - H\sum_i S_i$$
$$K = \beta J, \quad H = \beta B\mu$$
$$Z_N(K,H) = \sum_{S_1=\pm 1}\dots\sum_{S_N=\pm 1}e^{-\beta\mathcal{H}} = \sum_{\{S_i\}}e^{-\beta\mathcal{H}}$$

**examples:**

Ferromagnetic systems:

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}\vec{J}_i\vec{J}_j - \mu B\sum_i\vec{J}_i$$

lattice gases:

$$\mathcal{H} = -\sum_{\langle i,j\rangle}J_{ij}S_i S_j$$

**magnetisation**

$$M(K,H) = \left\langle\mu\sum_{i=1}^N S_i\right\rangle$$

**The 1D Ising model**

$$Z_N \overset{N\gg 1}{\approx} (2\cosh(K))^N$$
$$F = -k_B T N\ln\left(2\cosh\left(\frac{J}{k_B T}\right)\right)$$
$$\langle S_i S_{i+j}\rangle = (\tanh(K))^j = \left(e^{\ln(\tanh(K))}\right)^j = e^{-j/\zeta}$$
$$\zeta = -(\ln(\tanh(K)))^{-1} \text{ correlation length}$$

**Transfer matrix**

$$T_{i,i+1} = e^{KS_iS_{i+1}+\frac{1}{2}H(S_i+S_{i+1})}$$
$$\rightarrow e^{-\beta\mathcal{H}} = T_{1,2}\cdot T_{2,3}\dots T_{N,1}$$
$$T = \begin{pmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1)\end{pmatrix}$$
$$Z_N = \lambda_1^N + \lambda_2^N$$

**Renormalization of the Ising chain**

$$K' = \frac{1}{2}\ln(\cosh(2K))$$

**Renormalization of the 2d Ising model**

$$\overline{K}' = K' + K_1 = \frac{3}{8}\ln(\cosh(4K))$$

**The 2d Ising model**

$$\beta\mathcal{H} = -K\sum_{r,c} S_{r,c}S_{r+1,c} - K\sum_{r,c} S_{r.c}S_{r,c+1}$$
$$1 = \sinh(2K_c)$$
$$K_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.4407$$
$$T_c = 2J/\ln\left(1+\sqrt{2}\right) \approx 2.269J/k_B$$

**Perturbation theory**

$$F \le F_u = F_0 + \langle\mathcal{H}_1\rangle_0 \,\text{Bogoliubov inequality}$$

**Mean field theory for the Ising model**

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_i S_j$$
$$\mathcal{H}_0 = -B\sum_i S_i$$
$$\begin{aligned} F_0 &= -Nk_BT\ln\left(e^{\beta B} + e^{-\beta B}\right) \\ &= -Nk_BT\ln(2\cosh(\beta B)) \\ F &\le F_0 + \langle\mathcal{H}-\mathcal{H}_0\rangle_0 \\ &= -Nk_BT\ln(2\cosh(\beta B)) - N\frac{z}{2}\langle S\rangle_0^2 \\ &+ N\langle S\rangle_0 = F_u \\ \rightarrow z &= 2\cdot\text{dimension} \\ B &= Jz\langle S\rangle_0 = Jz\tanh(\beta B) \end{aligned}$$

$$K_c = \frac{1}{z} \rightarrow T_c = \frac{zJ}{k_B}$$

---

## 7 Classical fluids

**Virial expansion**

$$F = Nk_BT\left[\ln(\rho\lambda^3) - 1 + B_2\rho\right]$$
$$p = \rho k_BT\left[1 + B_2\rho\right]$$

**Second virial coefficient**

$$B_2(T) = -2\pi\int r^2 dr\left(e^{-\beta U(r)} - 1\right)$$

## 8 Others

**Stirling's formula**

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n)$$

**de Broglie relation**

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m}$$