Theoretical Statistical Physics (MKTP1)
Version: 27.3.2021

1 Introduction to probability theory

Bayes' theorem

$$p(B|A) = \frac{p(A|B)\cdot p(B)}{p(A)} = \frac{p(A|B)\cdot p(B)}{\sum_{B'} p(A|B)\cdot p(B')}$$

Expactation and covariance

$$\langle f\rangle = \sum_i f(i)p_i \text{ or } \langle f\rangle = \int f(x)p(x)dx$$

$$\mu = \langle i\rangle = \sum_i ip_i \text{ or } \mu = \langle x\rangle = \int xp(x)dx$$

$$\sigma^2 = \langle i^2\rangle - \langle i\rangle^2$$

$$\sigma_{ij}^2 = \langle ij\rangle - \langle i\rangle^2$$

Binomial distribution

$$\frac{N!}{(N-i)!i!} = \binom{N}{i} \text{ binomial coefficient}$$

$$p_i = \binom{N}{i}\cdot p^i q^{N-i} \text{ distribution}$$

$$\mu = \langle i\rangle = N\cdot p$$

$$\langle i^2\rangle = p\cdot N + p^2\cdot N\cdot(N-1)$$

$$\sigma^2 = N\cdot p\cdot q$$

$$\sum_{i=0}^N p_i = \sum_{i=0}^N \binom{N}{i}\cdot p^i q^{N-i} = (p+q)^N = 1$$

Gauss distribution

$$p(x) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{1}{2}}}\cdot e^{-\frac{x-\mu}{2\sigma^2}}, \quad \langle x^2\rangle = \sigma^2$$

Poisson distribution

$$p(k;\mu) = \frac{\mu^k}{k!}e^{-\mu}, \quad E[k]=\mu,\ V[k]=\mu$$

Information entropy

$$S = -\sum_i p_i\ln(p_i)$$

2 The microcanonical ensemble

$E \approx$ const , $V =$ const , $N =$ const.

The fundamental postulate

$$\Omega(E) = \sum_{n:E-\delta E\le E_n\le E} 1$$

$$\Omega(E;\delta E) = \frac{1}{h^{3N}N!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E} d\vec q d\vec p$$

$$S = -k_B\sum_{i=1}^{\Omega} p_i\ln(p_i) = k_B\ln(\Omega)$$

n_0 different particles

$$\Omega = \frac{1}{h^{3N}\prod_{j=0}^{n_0}N_j!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E} d\vec q d\vec p$$

Equilibrium conditions

Entropy S must be maximal

Thermal contact

$$\left.\frac{\partial S(E,V,N)}{\partial E}\right|_{V,N} = \frac{1}{T(E,V,N)}$$

Contact with volume excahnge

$$\left.\frac{\partial S(E,V,N)}{\partial V}\right|_{E,N} = \frac{p(E,V,N)}{T(E,V,N)}$$

Contact with exchange of particle number

$$\left.\frac{\partial S(E,V,N)}{\partial N}\right|_{E,V} = -\frac{\mu(E,V,N)}{T(E,V,N)}$$

Equations of state

$$dE = TdS - pdV + \mu dN$$

Specific heat

$$c_v = \frac{dE}{dT}$$

solution concept

- Set up Hamiltonian
- Calculate phasevolume Ω
- Calculate entropy S
- determine T,p,μ
- Calculate $U = \langle E\rangle$
- thermodynamic potentials:
  $F(T,V,N) = U - TS$
  $\hat H(S,p,N) = U + pV$
  $G(T,p,N) = U + pV - TS$

Ideal Gas

$$\mathcal{H} = \sum_{i=1}^{3N}\frac{p_i^2}{2m} + V(q_1,\dots,q_{3N})$$

microcanonical partition sum for an ideal gas

$$\Omega(E) = \frac{V^N\pi^{3N/2}(2mE)^{3N/2}}{h^{3N}N!\left(\frac{3N}{2}\right)!}$$

$$S = k_BN\left\{\ln\left[\left(\frac{V}{N}\right)\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right] + \frac{5}{2}\right\}$$

Equations of state fo ideal gas

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{3}{2}\frac{Nk_B}{E} \to U = \frac{3}{2}Nk_BT$$

$$p = T\left(\frac{\partial S}{\partial V}\right)_{E,N} = TNk_B\frac{1}{V} \to pV = Nk_BT$$

$$\mu = k_BT\ln\left(\frac{N\lambda^3}{V}\right) \text{ chemical potentail}$$

$$\lambda = \frac{h}{\sqrt{2\pi mk_BT}} \quad \text{Thermische de Broglie}$$

Einstein model for specific heat of a solid

$E = \hbar\omega\left(\frac{N}{2}+\mathcal{Q}\right) \to \mathcal{Q} = \left(\frac{E}{\hbar\omega}-\frac{N}{2}\right)$

$$\Omega(E,N) = \frac{(Q+N)!}{Q!N!}$$

$$S = k_B\left[Q\ln\left(\frac{Q+N}{Q}\right) + N\ln\left(\frac{Q+N}{N}\right)\right]$$

$$= k_BN\left[(e+\frac{1}{2})\ln(e+\frac{1}{2}) - (e-\frac{1}{2})\ln(e-\frac{1}{2})\right]$$

$$e = E/E_0\,;E_0 = N\hbar\omega\,;\beta = \hbar\omega/k_BT$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} \Rightarrow E = N\hbar\omega\left(\frac{1}{2}+\frac{1}{e^\beta-1}\right)$$

Entropic elasticity of polymers

$N_+ - N_- = \frac{L}{a} = m \to N_+ = \frac{1}{2}(N+m)$

$\Omega = \frac{N!}{N_+!N_-!} = \frac{N!}{\left(\frac{1}{2}(N+m)\right)!\left(\frac{1}{2}(N-m)\right)!}$

if both directions are possible $x2$

$$S = -k_B\left(N_+\ln\left(\frac{N_+}{N}\right) + N_-\ln\left(\frac{N_-}{N}\right)\right)$$

Statistical deviation from average

Two ideal gases in thermal conact $T_1 = T_2$

$$S_i = \frac{3}{2}k_BN_i\ln(E_i) + \text{independent of} E_i$$

$$S = S_1 + S_2$$

$$dS = 0 \ \to\ \frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2}$$

$$\to \overline{E}_1 = \frac{N_1}{N}E$$

consider small deviation:

$$E_1 = \overline{E}_1 + \Delta E, \quad E_2 = \overline{E}_2 - \Delta E$$

$$S(\overline{E}_1+\Delta E) \approx \frac{3}{2}k_B\left[N_1\ln\overline{E}_1 + N_2\ln\overline{E}_2 - \frac{N_1}{2}\left(\frac{\Delta E}{\overline{E}_1}\right)^2 - \frac{N_2}{2}\left(\frac{\Delta E}{\overline{E}_2}\right)^2\right]$$

$$\to \Omega = \overline{\Omega}e^{\left[-\frac{3}{4}\left(\frac{\Delta E}{E}\right)^2N^2\left(\frac{1}{N_1}+\frac{1}{N_2}\right)\right]}$$

3 The canonical ensemble

$T =$ const , $V =$ const , $N =$ const.

Boltzmann distribution

$$p_i = \frac{1}{Z}e^{-\beta E_i} \quad \text{Boltzmann distribution}$$

$$Z = \sum_i e^{-\beta E_i} \quad \text{partition sum}$$

For classical Hamiltonian systems:

$$p(\vec q,\vec p) = \frac{1}{ZN!h^{3N}}e^{-\beta\mathcal{H}(\vec q,\vec p)}$$

$$Z_N(T,V) = \frac{1}{N!h^{3N}}\iint d\vec q d\vec p e^{-\beta\mathcal{H}(\vec q,\vec p)}$$

For common Hamiltonian:

$$Z_N(T,V) = \frac{1}{\lambda^{3N}N!}\int_V d\vec q e^{-\beta\hat V(\vec q)}$$

Free energy

$$F(T,V,N) = -k_BT\ln Z_N(T,V)$$

$$\langle E\rangle = U = -\partial_\beta\ln Z_N$$

total differential:

$$dF = dE + d(TS) = -SdT - pdV + \mu N$$

equations of state

$$S = -\frac{\partial F}{\partial T}, \quad p = -\frac{\partial F}{\partial V}, \quad \mu = \frac{\partial F}{\partial N}$$

Non-interacting systems

ϵ_{ij} is the j^{th} state of the i^{th} element

$$Z = \sum_{j_1}\sum_{j_2}\dots\sum_{j_N} e^{-\beta\sum_{i=1}^N\epsilon_{ij_i}}$$

$$= \left(\sum_{j_1}e^{-\beta\epsilon_{1j_1}}\right)\dots\left(\sum_{j_N}e^{-\beta\epsilon_{Nj_1N}}\right)$$

$$= z_1\cdot\dots\cdot z_N = \prod_{i=1}^N z_i$$

$$\to F = -k_BT\sum_{i=1}^N\ln(z_i) = -k_BT\ln(Z)$$

$$Z = z^N, \qquad F = -k_BTNln(z)$$

Equipartition theorem

f are the degrees of freedom.
harmonic Hamiltonian with $f = 2$

$$\mathcal{H} = Aq^2 + Bp^2$$

$$z \propto \int dqdpe^{-\beta\mathcal{H}}$$

$$= \left(\frac{\pi}{A\beta}\right)^{\frac{1}{2}}\cdot\left(\frac{\pi}{B\beta}\right)^{\frac{1}{2}}$$

$$\propto \left(T^{\frac{1}{2}}\right)^f$$

For sufficiently high temperture (classical limit), each quadratic term in the Hamiltonian contributes a factor $T^{\frac{1}{2}}$ to the partition sum (**'equipartition theorem'**)

$$F = -k_BT\ln(z) = -\frac{f}{2}k_BTln(T)$$

$$S = -\frac{\partial F}{\partial T} = \frac{f}{2}k_B(\ln(T)+1)$$

$$U = -\partial_\beta\ln(z) = \frac{f}{2}k_BT$$

$$c_v = \frac{dU}{dT} = \frac{f}{2}k_B$$

Molecular gases

N molecules; x different mode types:
$Z = Z_{trans}\cdot Z_{vib}\cdot Z_{rot}\cdot Z_{elec}\cdot Z_{nuc}$
$Z_x = z_x^N$

Vibrational modes

often described by the Morse potential:
$V(r) = E_0\left(1-e^{-\alpha(r-r_0)}\right)^2$

An exact solution of the Schrödinger equation gives:

$E_n = \hbar\omega_0\left(n+\frac{1}{2}\right) - \frac{\hbar^2\omega_0^2}{eE_0}\left(n+\frac{1}{2}\right)^2$

$\omega_0 = \frac{\alpha}{2\pi}\sqrt{\frac{2E_0}{\mu}}, \quad \mu = \frac{m}{2}$

For $\hbar\omega_0 \ll E_0$ we can use the harmonic approximation:

$z_{vib} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega_0}}$

$T_{vib} \approx \frac{\hbar\omega_0}{k_B} \approx 6.140K$ for H_2

Rotational modes

standart approximation is the one of a rigid rotator. The moment of inertia is given as:

$I = \mu r_0^2 \quad T_{rot} = \frac{\hbar^2}{Ik_B}$

$\to E_l = \frac{\hbar^2}{2I}l(l+1)$

Nuclear contributions: ortho- and parahydrogen

$$S = 1\,,z_{ortho} = \sum_{l=1,3,5,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$

$$S = 0\,,z_{para} = \sum_{l=0,2,4,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$

Specific heat of a solid

Debye model

$$\to \omega(k) = \left(\frac{4\kappa}{m}\right)^{\frac{1}{2}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}$$

Debye frequency:

$$\omega_D = c_s\left(\frac{6\pi^2N}{V}\right)^{\frac{1}{3}}$$

$$c_s = \left.\frac{d\omega}{dk}\right|_{k=0} = \sqrt{\frac{\kappa}{m}}a$$

density of states in ω-space:

$$D(\omega) = 3\frac{\omega^2}{\omega_D^3} \quad \text{for } \omega \le \omega_D$$

count modes in frequency-space:

$$\sum_{modes}(\dots) = 3\sum_k(\dots) = 3N\int_0^{\omega_D}d\omega D(\omega)(\dots)$$

partition sum:

$$z(\omega) = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$\to Z = \prod_{modes} z(\omega)$$

$$\to E = -\partial_\beta \ln(Z) = \sum_{modes} \hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}+\frac{1}{2}\right)$$

$$= E_0 + 3N\int_0^{\omega_D} d\omega \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{3\omega^2}{\omega_D^3}$$

$$c_v(T) = \frac{\partial E}{\partial T} = \frac{3\hbar^2 N}{k_B T^2}\int_0^{\omega_D} d\omega \frac{3\omega^2}{\omega_D^3}\frac{e^{\beta\hbar\omega}\omega^2}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

$$u = \beta\hbar\omega$$

$$c_v(T) = \frac{9Nk_B}{u_m^3}\int_0^{u_m}\frac{e^u u^4}{(e^u-1)^2}du$$

the limit for $\hbar\omega_D \ll k_B T$:

$$c_v(T) = 3Nk_B$$

the limit for $k_B T \ll \hbar\omega_D$: ($T_D = \frac{\hbar\omega_D}{k_B}$)

$$c_v(T) = \frac{12\pi^4}{5}Nk_B\left(\frac{T}{T_D}\right)^3$$

Black body radiation

$$E = \frac{4\sigma}{c}VT^4, \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

$$c_v = \frac{16\sigma}{c}VT^3$$

$$J = \frac{P}{A} = \sigma T^4 \quad \text{Stefan- Boltzmann law}$$

Plank's law for black body radiation

$$u(\omega) := \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/(k_BT)}-1}$$

The Plank distribution has a maximum at:

$\hbar\omega_{max} = 2.82 k_B T$ Wien's displacement law

4 The grandcanonical ensemble

$T, \mu = const.$

$$p_N(q,p) = \frac{1}{\Xi_\mu(T,V)}e^{-\beta(H_N(q,p)-\mu N)}$$

$$\Xi_\mu(T,V) = \sum_{N=0}^\infty \frac{1}{h^{3N}N!}\iint d^{3N}q\, d^{3N}p\, e^{-\beta(H_N-\mu N)}$$

$$\to \Xi_z = \sum_{N=0}^\infty z^N Z_N(T,V)$$

$z = e^{\beta\mu} \to$ Fugacity

Mean phase space observable

$$\langle F\rangle = \frac{1}{\Xi_\mu(T,V)}\sum_{N=0}^\infty\frac{1}{h^{3N}N!}\iint d^{3N}q\, d^{3N}p \ldots$$
$$\ldots e^{-\beta(H_N-\mu N)}F_N(q,p)$$

mean particle number:

$$\langle N\rangle = \frac{1}{\beta}\left(\frac{\partial}{\partial\mu}\ln\left(\Xi_\mu(T,V)\right)\right)_{T,V}$$
$$= z\left(\frac{\partial}{\partial z}\ln\left(\Xi_z(T,V)\right)\right)_{T,V}$$

pressure:

$$p = -\left\langle\frac{\partial H}{\partial V}\right\rangle = \frac{1}{\beta}\left(\frac{\partial}{\partial V}\ln\left(\Xi_\mu(T,V)\right)\right)$$

energy U:

$$U = \langle H\rangle = -\left(\frac{\partial}{\partial\beta}\ln\left(\Xi_\mu(T,V)\right)\right)_{\mu,V} + \mu\langle N\rangle$$
$$= -\left(\frac{\partial}{\partial\beta}\ln\left(\Xi_z(T,V)\right)\right)_{z,V}$$

Grandcanonical potential

grandcanonical potential:

$$\Psi(T,V,\mu) = -k_B T\ln\left(\Xi_\mu(T,V)\right)$$

p is maximal, if Ψ is minimal.
Total differential:

$$d\Psi = -SdT - pdV - \langle N\rangle d\mu$$

Equations of state:

$$S = -\frac{\partial\Psi}{\partial T}, p = -\frac{\partial\Psi}{\partial V}, N = -\frac{\partial\Psi}{\partial\mu}$$

Fluctuations

$$\sigma_N^2 = \langle N^2\rangle - \langle N\rangle^2 = \frac{1}{\beta^2}\left(\partial_\mu^2\ln(\Xi_\mu)\right)$$

$$\frac{\sigma_N}{\langle N\rangle} \propto \frac{1}{\sqrt{N}}$$

Ideal gas

$$Z_N(T,V) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N, \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$

$$\Xi = \sum_{N=0}^\infty Z_N(T,V)z^N$$
$$= \sum_{N=0}^\infty \frac{1}{N!}\left(e^{\beta\mu}\frac{V}{\lambda^3}\right)^N$$
$$= e^{z\frac{V}{\lambda^3}} \quad \text{fugacity: } z := e^{\beta\mu}$$

$$\langle N\rangle = \frac{1}{\beta}\partial_\mu\ln(Z_G) = \frac{V}{\lambda^3}d^{\beta\mu}$$

$$\mu = k_B T\ln\left(\frac{N\lambda^3}{V}\right)$$

Molecular adsorption onto a surface

$$Z_G = z_G^N; z_G = 1 + e^{-\beta(\epsilon-\mu)}$$
$$\langle n\rangle = \frac{1}{e^{-\beta(\mu-\epsilon)}+1} \quad \text{per site}$$
$$\langle\epsilon\rangle = \epsilon\langle n\rangle$$

5 Quantum fluids

Fermion vs. bosons

Particles with half-integer (integer) spin are called fermions (bosons). Their total wave function (space and spin) must be antisymmetric (symmetric) under the exchange of any pair of identical particles.

Canonical ensemble

two particles that are distributed over two states with energies 0 and ϵ

$$Z_F = e^{-\beta\epsilon} \quad \text{Fermi-Dirac}$$
$$Z_B = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \quad \text{Bose-Einstein}$$
$$Z_M = \frac{1+2e^{-\beta\epsilon}+e^{-2\beta\epsilon}}{2} \quad \text{Maxwell-Boltzmann}$$

Grand canonical ensemble

Fermions:

$$z_F = 1 + e^{-\beta(\epsilon-\mu)}$$

average occupation number n_F:

$$n_F = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \quad \text{Fermi function}$$

For $T \to 0$, the fermi function approaches a step function:

$$n_F = \Theta(\mu-\epsilon)$$

Bosons:

$$z_B = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}$$

average occupation number n_B:

$$n_B = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

- Fermions tend to fill up energy states one after the other
- Bosons tend to condense all into the same low energy state

The ideal Fermi fluid

density of states:

$$D(\epsilon) = \frac{V}{2\pi N}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\epsilon}$$

Fermi energy

$$N = \sum_{\vec{k},m_s} n_{\vec{k},m_s} = N\int_0^\infty d\epsilon D(\epsilon)n_F(\epsilon)$$

Limit $T \to 0$. $\mu(T=0)$ is called Fermi energy:

$$\epsilon_F = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2\rho^{\frac{2}{3}}}{2m}$$

specific heat

$$\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \text{ for } T \ll \frac{\epsilon_F}{k_B}$$

$$c_V = \frac{\partial E}{\partial T}\bigg|_V = N\frac{\pi^2}{3}k_B^2 D(\epsilon_F)T$$

$$c_V = N\frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}k_B$$

Fermi pressure

$$p \overset{T\to 0}{\to} \frac{2}{5}\frac{N}{V}\epsilon_F = \frac{(2\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{mv^{\frac{5}{3}}}$$

The ideal Bose fluid

$\epsilon = \frac{\hbar^2 k^2}{2m}$ and conserved particle number N.

$$N = \frac{N}{\lambda^3}g_{\frac{3}{2}}(z)$$

$$z = e^{\beta\mu}, \quad \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$

$$T_c = \frac{2\pi}{\left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}\frac{\hbar^2\rho^{\frac{2}{3}}}{k_B m}$$

$$E = \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{\frac{5}{2}}(z) = \frac{3}{2}k_B T N_e\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}$$

$$c_V = \frac{15}{4}k_B N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \text{ (for } T \le T_c)$$

$$c_V = \frac{15}{4}k_B N\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} - \frac{9}{4}k_B N\frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} \quad (T > T_c)$$

Classical limit

$\mu \to -\infty$ the two grandcanonical distr. become the Maxwell-Boltzmann distr.

$$n_{F/B} = \frac{1}{e^{\beta(\epsilon-\mu)}\pm 1} \to e^{\beta\mu}e^{-\beta\epsilon}$$
$$N = g\frac{V}{\lambda^3}e^{\beta\mu}$$
$$E = \frac{3}{2}k_B T N$$

6 Phase transitions

Ising model

Hamiltonian

$$\mathcal{H} = -\sum_{i,j}J_{ij}S_iS_j - \mu B_0\sum_i S_i$$

special cases:

Ferromagnetic systems:
$\mathcal{H} = -J\sum_{\langle i,j\rangle}\vec{J}_i\vec{J}_j - \mu\vec{B}\sum_i\vec{J}_i$
lattice gases:
$\mathcal{H} = -\sum_{\langle i,j\rangle}J_{ij}S_iS_j$

1. Dimensional

Only Next Neighbor and $B_0 = 0$
$J_{i,i+1} \to J_i, \quad \mathcal{H} = -\sum_{i=1}^{N-1}J_iS_iS_{i+1}, \quad j_i = \beta J_i$

$$Z_N = \sum_{S_1}\ldots\sum_{S_N}\exp\left(\sum_{i=1}^{N-1}j_iS_iS_{i+1}\right)$$
$$= 2^N\prod_{i=1}^{N-1}\cosh(\beta J_i)$$

Spin correlation function:

$$\langle S_iS_{i+1}\rangle = \tanh(\beta J)$$

spontanious magnetisation:

$$M_S(T) = \mu\langle S\rangle$$
$$M_S^2(T) = \mu^2\lim_{j\to\infty}\langle S_iS_{i+1}\rangle$$

No phase transition for $T > 0$. But for $T = 0$
$M_S(T=0) = \mu$

Transfer matrix

$$j = \beta J, \quad b = \beta\mu B_0, \quad S_i = \pm 1$$

$$T_{i,i+1} = e^{jS_iS_{i+1}+\frac{1}{2}b(S_i+S_{i+1})}$$
$$\to e^{-\beta\mathcal{H}} = T_{1,2}\cdot T_{2,3}\ldots T_{N,1}$$
$$T = \begin{pmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1)\end{pmatrix}$$
$$Z_N = \lambda_1^N + \lambda_2^N = E_+^N + E_-^N$$

for $N \gg 1 \to E_+ \gg E_-$

Renormalization of the Ising chain

$$K' = \frac{1}{2}\ln(\cosh(2K))$$

Renormalization of the 2d Ising model

$$\overline{K}' = K' + K_1 = \frac{3}{8}\ln(\cosh(4K))$$

The 2d Ising model

$$\beta\mathcal{H} = -K\sum_{r,c} S_{r,c}S_{r+1,c} - K\sum_{r,c} S_{r.c}S_{r,c+1}$$

$$1 = \sinh(2K_c)$$

$$K_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.4407$$

$$T_c = 2J/\ln\left(1+\sqrt{2}\right) \approx 2.269J/k_B$$

Perturbation theory

$$F \le F_u = F_0 + \langle\mathcal{H}_1\rangle_0 \,\text{Bogoliubov inequality}$$

Mean field theory for the Ising model

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_i S_j$$

$$\mathcal{H}_0 = -B\sum_i S_i$$

$$F_0 = -Nk_BT\ln\left(e^{\beta B} + e^{-\beta B}\right)$$

$$= -Nk_BT\ln(2\cosh(\beta B))$$

$$F \le F_0 + \langle\mathcal{H} - \mathcal{H}_0\rangle_0$$

$$= -Nk_BT\ln(2\cosh(\beta B)) - N\frac{z}{2}\langle S\rangle_0^2$$

$$+ N\langle S\rangle_0 = F_u$$

$$\rightarrow z = 2\cdot\text{dimension}$$

$$B = Jz\langle S\rangle_0 = Jz\tanh(\beta B)$$

$$K_c = \frac{1}{z} \rightarrow T_c = \frac{zJ}{k_B}$$

---

7 Classical fluids

Virial expansion

$$F = Nk_BT\left[\ln(\rho\lambda^3) - 1 + B_2\rho\right]$$

$$p = \rho k_BT\left[1 + B_2\rho\right]$$

Second virial coefficient

$$B_2(T) = -2\pi\int r^2\,dr\left(e^{-\beta U(r)} - 1\right)$$

8 Others

Stirling's formula

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n)$$

de Broglie relation

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m}$$