Theoretical Statistical Physics (MKTP1)
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1 Introduction to probability theory

Bayes' theorem

$$p(B|A)=\frac{p(A|B)\cdot p(B)}{p(A)}=\frac{p(A|B)\cdot p(B)}{\sum_{B'}p(A|B)\cdot p(B')}$$

Expactation and covariance

$$\langle f\rangle=\sum_i f(i)p_i \text{ or } \langle f\rangle=\int f(x)p(x)dx$$

$$\mu=\langle i\rangle=\sum_i ip_i \text{ or } \mu=\langle x\rangle=\int xp(x)dx$$

$$\sigma^2=\langle i^2\rangle-\langle i\rangle^2$$
$$\sigma_{ij}^2=\langle ij\rangle-\langle i\rangle^2$$

Binomial distribution

$$\frac{N!}{(N-i)!i!}=\binom{n}{i}\text{ binomial coefficient}$$
$$p_i=\binom{N}{i}\cdot p^iq^{N-i}\text{ distribution}$$
$$\mu=\langle i\rangle=N\cdot p$$
$$\langle i^2\rangle=p\cdot N+p^2\cdot N\cdot(N-1)$$
$$\sigma^2=N\cdot p\cdot q$$
$$\sum_{i=0}^N p_i=\sum_{i=0}^N\binom{N}{i}\cdot p^iq^{N-i}=(p+q)^N=1$$

Gauss distribution

$$p(x)=\frac{1}{(2\pi\sigma^2)^{\frac12}}\cdot e^{-\frac{x-\mu}{2\sigma^2}},\quad\langle x^2\rangle=\sigma^2$$

Poisson distribution

$$p(k;\mu)=\frac{\mu^k}{k!}e^{-\mu},\quad E[k]=\mu,\ V[k]=\mu$$

Information entropy

$$S=-\sum_i p_i\ln(p_i)$$

2 The microcanonical ensemble

$E\approx$ const , $V=$ const , $N=$ const.

The fundamental postulate

$$\Omega(E)=\sum_{n:E-\delta E\le E_n\le E}1$$
$$\Omega(E;\delta E)=\frac{1}{h^{3N}N!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E}d\vec q d\vec p$$
$$S=-k_B\sum_{i=1}^{\Omega}p_i\ln(p_i)=k_B\ln(\Omega)$$

n_0 different particles

$$\Omega=\frac{1}{h^{3N}\prod_{j=0}^{n_0}N_j!}\iint_{E-\delta E\le\mathcal H(\vec q,\vec p)\le E}d\vec q d\vec p$$

Equilibrium conditions
Entropy S must be maximal

Thermal contact

$$\frac{\partial S(E,V,N)}{\partial E}\Big|_{V,N}=\frac{1}{T(E,V,N)}$$

Contact with volume excahnge

$$\frac{\partial S(E,V,N)}{\partial V}\Big|_{E,N}=\frac{p(E,V,N)}{T(E,V,N)}$$

Contact with exchange of particle number

$$\frac{\partial S(E,V,N)}{\partial N}\Big|_{E,V}=-\frac{\mu(E,V,N)}{T(E,V,N)}$$

Equations of state

$$dE=TdS-pdV+\mu dN$$

Specific heat

$$c_v=\frac{dE}{dT}$$

solution concept

- Set up Hamiltonian
- Calculate phasevolume Ω
- Calculate entropy S
- determine T,p,μ
- Calculate $U=\langle E\rangle$
- thermodynamic potentials:
  $F(T,V,N)=U-TS$
  $\hat H(S,p,N)=U+pV$
  $G(T,p,N)=U+pV-TS$

Ideal Gas

$$\mathcal H=\sum_{i=1}^{3N}\frac{p_i^2}{2m}+V(q_1,\dots,q_{3N})$$

microcanonical partition sum for an ideal gas

$$\Omega(E)=\frac{V^N\pi^{3N/2}(2mE)^{3N/2}}{h^{3N}N!\left(\frac{3N}{2}\right)!}$$
$$S=k_BN\left\{\ln\left[\left(\frac VN\right)\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right]+\frac52\right\}$$

Equations of state fo ideal gas

$$\frac1T=\left(\frac{\partial S}{\partial E}\right)_{N,V}=\frac32\frac{Nk_B}{E}\to U=\frac32Nk_BT$$
$$p=T\left(\frac{\partial S}{\partial V}\right)_{E,N}=TNk_B\frac1V\to pV=Nk_BT$$
$$\mu=k_BT\ln\left(\frac{N\lambda^3}{V}\right)\text{ chemical potentail}$$
$$\lambda=\frac{h}{\sqrt{2\pi mk_BT}}\quad\text{Thermische de Broglie}$$

Einstein model for specific heat of a solid

$E=\hbar\omega\left(\frac N2+\mathcal Q\right)\to\mathcal Q=\left(\frac{E}{\hbar\omega}-\frac N2\right)$

$$\Omega(E,N)=\frac{(\mathcal Q+N)!}{\mathcal Q!N!}$$
$$S=k_B\left[\mathcal Q\ln\left(\frac{\mathcal Q+N}{\mathcal Q}\right)+N\ln\left(\frac{\mathcal Q+N}{N}\right)\right]$$
$$=k_BN\left[(e+\frac12)\ln(e+\frac12)-(e-\frac12)\ln(e-\frac12)\right]$$
$$e=E/E_0\,;E_0=N\hbar\omega$$
$$\frac1T=\frac{\partial S}{\partial E}\Rightarrow E=N\hbar\omega\left(\frac12+\frac{1}{e^\beta-1}\right)$$

Entropic elasticity of polymers

$N_+-N_-=\frac La=m\to N_+=\frac12(N+m)$
$\Omega=\frac{N!}{N_+!N_-!}=\frac{N!}{(\frac12(N+m))!(\frac12(N-m))!}$
if both directions are possible $x2$

$$S=-k_B\left(N_+\ln\left(\frac{N_+}{N}\right)+N_-\ln\left(\frac{N_-}{N}\right)\right)$$

Statistical deviation from average
Two ideal gases in thermal conact $T_1=T_2$

$$S_i=\frac32k_BN_i\ln(E_i)+\text{independent of}E_i$$
$$S=S_1+S_2$$
$$dS=0\to\frac{\partial S_1}{\partial E_1}=\frac{\partial S_2}{\partial E_2}$$
$$\to\overline E_1=\frac{N_1}{N}E$$

consider small deviation:

$$E_1=\overline E_1+\Delta E,\quad E_2=\overline E_2-\Delta E$$
$$S(\overline E_1+\Delta E)\approx\frac32k_B\Big[N_1\ln\overline E_1+N_2\ln\overline E_2-\frac{N_1}{2}\left(\frac{\Delta E}{\overline E_1}\right)^2-\frac{N_2}{2}\left(\frac{\Delta E}{\overline E_2}\right)^2\Big]$$
$$\to\Omega=\overline\Omega e^{\left[-\frac34\left(\frac{\Delta E}{E}\right)^2N^2\left(\frac1{N_1}+\frac1{N_2}\right)\right]}$$

3 The canonical ensemble

$T=$ const , $V=$ const , $N=$ const.

Boltzmann distribution

$$p_i=\frac1Ze^{-\beta E_i}\quad\text{Boltzmann distribution}$$
$$Z=\sum_ie^{-\beta E_i}\quad\text{partition sum}$$

For classical Hamiltonian systems:

$$p(\vec q,\vec p)=\frac{1}{ZN!h^{3N}}e^{-\beta\mathcal H(\vec q,\vec p)}$$
$$Z_N(T,V)=\frac{1}{N!h^{3N}}\iint d\vec qd\vec pe^{-\beta\mathcal H(\vec q,\vec p)}$$

For common Hamiltonian:

$$Z_N(T,V)=\frac{1}{\lambda^{3N}N!}\int_Vd\vec qe^{-\beta\hat V(\vec q)}$$

Free energy

$$F(T,V,N)=-k_BT\ln Z_N(T,V)$$
$$\langle E\rangle=U=-\partial_\beta\ln Z_N$$

total differential:

$$dF=dE+d(TS)=-SdT-pdV+\mu N$$

equations of state

$$S=-\frac{\partial F}{\partial T},\quad p=-\frac{\partial F}{\partial V},\quad\mu=\frac{\partial F}{\partial N}$$

Non-interacting systems
ϵ_{ij} is the j^{th} state of the i^{th} element

$$Z=\sum_{j_1}\sum_{j_2}\dots\sum_{j_N}e^{-\beta\sum_{i=1}^N\epsilon_{ij_i}}$$
$$=\left(\sum_{j_1}e^{-\beta\epsilon_{1j_1}}\right)\dots\left(\sum_{j_N}e^{-\beta\epsilon_{Nj_1N}}\right)$$
$$=z_1\cdot\dots\cdot z_N=\prod_{i=1}^Nz_i$$
$$\to F=-k_BT\sum_{i=1}^N\ln(z_i)=-k_BT\ln(Z)$$
$$Z=z^N,\quad F=-k_BTNln(z)$$

TODO: ADD EXAMPLES

Equipartition theorem
f are the degrees of freedom.
harmonic Hamiltonian with $f=2$

$$\mathcal H=Aq^2+Bp^2$$
$$z\propto\int dqdpe^{-\beta\mathcal H}$$
$$=\left(\frac{\pi}{A\beta}\right)^{\frac12}\cdot\left(\frac{\pi}{B\beta}\right)^{\frac12}$$
$$\propto\left(T^{\frac12}\right)^f$$

For sufficiently high temperture (classical limit), each quadratic term in the Hamiltonian contributes a factor $T^{\frac12}$ to the partition sum ('equipartition theorem')

$$F=-k_BT\ln(z)=-\frac f2k_BTln(T)$$
$$S=-\frac{\partial F}{\partial T}=\frac f2k_B(\ln(T)+1)$$
$$U=-\partial_\beta\ln(z)=\frac f2k_BT$$
$$c_v=\frac{dU}{dT}=\frac f2k_B$$

Molecular gases

$$Z=Z_{trans}\cdot Z_{vib}\cdot Z_{rot}\cdot Z_{elec}\cdot Z_{nuc}$$
$$Z_x=z_x^N$$

Vibrational modes

often described by the Morse potential:

$$V(r)=E_0\left(1-e^{-\alpha(r-r_0)}\right)^2$$

An exact solution of the Schrödinger equation gives:

$$E_n=\hbar\omega_0\left(n+\frac12\right)-\frac{\hbar^2\omega_0^2}{eE_0}\left(n+\frac12\right)^2$$
$$\omega_0=\frac{\alpha}{2\pi}\sqrt{\frac{2E_0}{\mu}},\quad\mu=\frac m2$$

For $\hbar\omega_0\ll E_0$ we can use the harmonic approximation:

$$z_{vib}=\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega_0}}$$
$$T_{vib}\approx\frac{\hbar\omega_0}{k_B}\approx6.140K\ \text{ for }H_2$$

Rotational modes

standart approximation is the one of a rigid rotator. The moment of inertia is given as:

$$I=\mu r_0^2\quad T_{rot}=\frac{\hbar^2}{Ik_B}$$
$$\to E_l=\frac{\hbar^2}{2I}l(l+1)$$

Nuclear contributions: ortho- and parahydrogen

$$z_{ortho}=\sum_{l=1,3,5,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$
$$z_{para}=\sum_{l=0,2,4,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$$

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Specific heat of a solid

Debye model

$$\to \omega(k) = \left(\frac{4\kappa}{m}\right)^{\frac{1}{2}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

$$\omega = \frac{2\pi}{T},\quad k = \frac{2\pi}{\lambda}$$

Debye frequency:

$$\omega_D = c_s\left(\frac{6\pi^2 N}{V}\right)^{\frac{1}{3}}$$

$$c_s = \left.\frac{d\omega}{dk}\right|_{k=0} = \sqrt{\frac{\kappa}{m}}a$$

density of states in ω–space:

$$D(\omega) = 3\frac{\omega^2}{\omega_D^3}\quad \text{for} \omega \le \omega_D$$

count modes in frequency-space:

$$\sum_{modes}(\dots) = 3\sum_{k}(\dots) = 3N\int_0^{\omega_D} d\omega D(\omega)(\dots)$$

partition sum:

$$z(\omega) = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$\to Z = \prod_{modes} z(\omega)$$

$$\to E = -\partial_\beta \ln(Z) = \sum_{modes}\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}+\frac{1}{2}\right)$$

$$= E_0 + 3N\int_0^{\omega_D} d\omega\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{3\omega^2}{\omega_D^3}$$

$$c_v(T) = \frac{\partial E}{\partial T} = \frac{3\hbar^2 N}{k_B T^2}\int_0^{\omega_D} d\omega\frac{3\omega^2}{\omega_D^3}\frac{e^{\beta\hbar\omega}\omega^2}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

$$u = \beta\hbar\omega$$

$$c_v(T) = \frac{9Nk_B}{u_m^3}\int_0^{u_m}\frac{e^u u^4}{(e^u-1)^2}du$$

the limit for $\hbar\omega_D \ll k_B T$:

$$c_v(T) = 3Nk_B$$

the limit for $k_B T \ll \hbar\omega_D$: $(T_D = \frac{\hbar\omega_D}{k_B})$

$$c_v(T) = \frac{12\pi^4}{5}Nk_B\left(\frac{T}{T_D}\right)^3$$

Black body radiation

$$E = \frac{4\sigma}{c}VT^4,\quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

$$c_v = \frac{16\sigma}{c}VT^3$$

$$J = \frac{P}{A} = \sigma T^4 \quad \text{Stefan- Boltzmann law}$$

Plank's law for black body radiation

$$u(\omega) := \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/(k_B T)}-1}$$

The Plank distribution has a maximum at:

$$\hbar\omega_{max} = 2.82k_B T \quad \text{Wien's displacement law}$$

4 The grandcanonical ensemble

Probability distribution

T and μ are fixed.

$$p_i = \frac{1}{Z_G}e^{-\beta(E_i-\mu N_i)} \text{ prob. distribution}$$

$$Z_G = \sum_i e^{-\beta(E_i-\mu N_i)} \text{ partition sum}$$

$$\Psi = -k_B T\ln(Z_G) \text{ grandcanonical pot.}$$

Grandcanonical potential

The probability to have a macroscopic value (E,N) is:

$$p(E,N) = \frac{1}{Z_G}\Omega(E,N)e^{-\beta(E-\mu N)}$$
$$= \frac{1}{Z_G}e^{-\beta(E-TS-\mu N)} = \frac{1}{Z_G}e^{-\beta\Psi(T,V,\mu)}$$

grandcanonical potential:

$$\Psi(T,V,\mu) := E - TS - \mu N$$

p is maximal, if Ψ is minimal.
Total differential:

$$d\Psi = d(E-TS-\mu N)$$
$$= TdS - pdV + \mu dN - d(TS+\mu N)$$
$$= -SdT - pdV - Nd\mu$$

Equations of state:

$$S = -\frac{\partial\Psi}{\partial T}, p = -\frac{\partial\Psi}{\partial V}, N = -\frac{\partial\Psi}{\partial\mu}$$

Fluctuations

$$\langle N\rangle = \sum_i p_i N_i = \frac{1}{\beta}\partial_\mu\ln(Z_G)$$

$$\sigma_N^2 = \langle N^2\rangle - \langle N\rangle^2 = \frac{1}{\beta^2}\partial_\mu^2\ln(Z_G)$$

$$\frac{\sigma_N}{\langle N\rangle} \propto \frac{1}{N^{\frac{1}{2}}}$$

Ideal gas

$$Z(T,V,N) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N,\ \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$

$$Z_G = \sum_{N=0}^{\infty} Z(T,V,N)e^{\beta\mu N}$$
$$= \sum_{N=0}^{\infty}\frac{1}{N!}\left(e^{\beta\mu}\frac{V}{\lambda^3}\right)^N$$
$$= e^{z\frac{V}{\lambda^3}} \quad \text{fugacity: } z := e^{\beta\mu}$$

$$\langle N\rangle = \frac{1}{\beta}\partial_\mu\ln(Z_G) = \frac{V}{\lambda^3}e^{\beta\mu}$$

$$\mu = k_B T\ln\left(\frac{N\lambda^3}{V}\right)$$

Molecular adsorption onto a surface

$$Z_G = z_G^N; z_G = 1 + e^{-\beta(\epsilon-\mu)}$$

$$\langle n\rangle = \frac{1}{e^{-\beta(\mu-\epsilon)}+1} \text{ per site}$$

$$\langle\epsilon\rangle = \epsilon\langle n\rangle$$

5 Quantum fluids

Fermion vs. bosons

Particles with half-integer (integer) spin are called fermions (bosons). Their total wave function (space and spin) must be antisymmetric (symmetric) under the exchange of any pair of identical particles.

Canonical ensemble

two particles that are distributed over two states with energies 0 and ϵ

$$Z_F = e^{-\beta\epsilon} \text{ Fermi-Dirac}$$

$$Z_B = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \text{ Bose-Einstein}$$

$$Z_M = \frac{1+2e^{-\beta\epsilon}+e^{-2\beta\epsilon}}{2} \text{ Maxwell-Boltzmann}$$

Grand canonical ensemble

Fermions:

$$z_F = 1 + e^{-\beta(\epsilon-\mu)}$$

average occupation number n_F:

$$n_F = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \text{ Fermi function}$$

For $T \to 0$, the fermi function approaches a step function:

$$n_F = \Theta(\mu-\epsilon)$$

Bosons:

$$z_B = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}$$

average occupation number n_B:

$$n_B = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

- Fermions tend to fill up energy states one after the other
- Bosons tend to condense all into the same low energy state

The ideal Fermi fluid

density of states:

$$D(\epsilon) = \frac{V}{2\pi N}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\epsilon}$$

Fermi energy

$$N = \sum_{\vec{k},m_s} n_{\vec{k},m_s} = N\int_0^\infty d\epsilon D(\epsilon)n_F(\epsilon)$$

Limit $T \to 0$. $\mu(T=0)$ is called Fermi energy:

$$\epsilon_F = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2\rho^{\frac{2}{3}}}{2m}$$

specific heat

$$\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right] \text{ for } T \ll \frac{\epsilon_F}{k_B}$$

$$c_V = \left.\frac{\partial E}{\partial T}\right|_V = N\frac{\pi^2}{3}k_B^2 D(\epsilon_F)T$$

$$c_V = N\frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}k_B$$

Fermi pressure

$$p \overset{T\to 0}{\to} \frac{2}{5}\frac{N}{V}\epsilon_F = \frac{(2\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{mv^{\frac{5}{3}}}$$

The ideal Bose fluid

$\epsilon = \frac{\hbar^2 k^2}{2m}$ and conserved particle number N.

$$N = \frac{N}{\lambda^3}g_{\frac{3}{2}}(z)$$

$$z = e^{\beta\mu},\ \lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$$

$$T_c = \frac{2\pi}{\left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}}\frac{\hbar^2\rho^{\frac{2}{3}}}{k_B m}$$

$$E = \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{\frac{5}{2}}(z) = \frac{3}{2}k_B T N_e\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}$$

$$c_V = \frac{15}{4}k_B N\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \text{ (for } T \le T_c)$$

$$c_V = \frac{15}{4}k_B N\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} - \frac{9}{4}k_B N\frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} \ (T > T_c)$$

Classical limit

$\mu \to -\infty$ the two grandcanonical distr. become the Maxwell-Boltzmann distr.

$$n_{F/B} = \frac{1}{e^{\beta(\epsilon-\mu)}\pm 1} \to e^{\beta\mu}e^{-\beta\epsilon}$$

$$N = g\frac{V}{\lambda^3}e^{\beta\mu}$$

$$E = \frac{3}{2}k_B T N$$

6 Phase transitions

Ising model

$$\mathcal{H} = -J\sum_{\langle ij\rangle}S_i S_j - B\mu\sum_i S_i$$

$$\beta\mathcal{H} = -K\sum_{\langle ij\rangle}S_i S_j - H\sum_i S_i$$

$$K = \beta J,\quad H = \beta B\mu$$

$$Z_N(K,H) = \sum_{S_1=\pm 1}\dots\sum_{S_N=\pm 1}e^{-\beta\mathcal{H}} = \sum_{\{S_i\}}e^{-\beta\mathcal{H}}$$

examples:

Ferromagnetic systems:

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}\vec{J}_i\vec{J}_j - \mu B\sum_i\vec{J}_i$$

lattice gases:

$$\mathcal{H} = -\sum_{\langle i,j\rangle}J_{ij}S_i S_j$$

magnetisation

$$M(K,H) = \left\langle\mu\sum_{i=1}^N S_i\right\rangle$$

The 1D Ising model

$$Z_N \overset{N\gg 1}{\approx} (2\cosh(K))^N$$

$$F = -k_B T N\ln\left(2\cosh\left(\frac{J}{k_B T}\right)\right)$$

$$\langle S_i S_{i+j}\rangle = (\tanh(K))^j = \left(e^{\ln(\tanh(K))}\right)^j = e^{-j/\zeta}$$

$$\zeta = -(\ln(\tanh(K)))^{-1} \text{ correlation length}$$

Transfer matrix

$$T_{i,i+1} = e^{KS_iS_{i+1}+\frac{1}{2}H(S_i+S_{i+1})}$$

$$\to e^{-\beta\mathcal{H}} = T_{1,2}\cdot T_{2,3}\dots T_{N,1}$$

$$T = \begin{pmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1)\end{pmatrix}$$

$$Z_N = \lambda_1^N + \lambda_2^N$$

Renormalization of the Ising chain

$$K' = \frac{1}{2}\ln(\cosh(2K))$$

Renormalization of the 2d Ising model

$$\overline{K}' = K' + K_1 = \frac{3}{8}\ln(\cosh(4K))$$

The 2d Ising model

$$\beta\mathcal{H} = -K\sum_{r,c} S_{r,c}S_{r+1,c} - K\sum_{r,c} S_{r.c}S_{r,c+1}$$
$$1 = \sinh(2K_c)$$
$$K_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.4407$$
$$T_c = 2J/\ln\left(1+\sqrt{2}\right) \approx 2.269J/k_B$$

Perturbation theory

$$F \le F_u = F_0 + \langle\mathcal{H}_1\rangle_0 \text{ Bogoliubov inequality}$$

Mean field theory for the Ising model

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_iS_j$$
$$\mathcal{H}_0 = -B\sum_i S_i$$
$$F_0 = -Nk_BT\ln\left(e^{\beta B}+e^{-\beta B}\right)$$
$$= -Nk_BT\ln(2\cosh(\beta B))$$
$$F \le F_0 + \langle\mathcal{H}-\mathcal{H}_0\rangle_0$$
$$= -Nk_BT\ln(2\cosh(\beta B)) - N\frac{z}{2}\langle S\rangle_0^2$$
$$+ N\langle S\rangle_0 = F_u$$
$$\rightarrow z = 2\cdot\text{dimension}$$
$$B = Jz\langle S\rangle_0 = Jz\tanh(\beta B)$$

$$K_c = \frac{1}{z} \rightarrow T_c = \frac{zJ}{k_B}$$

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7 Classical fluids

Virial expansion

$$F = Nk_BT\left[\ln(\rho\lambda^3) - 1 + B_2\rho\right]$$
$$p = \rho k_BT\left[1 + B_2\rho\right]$$

Second virial coefficient

$$B_2(T) = -2\pi\int r^2dr\left(e^{-\beta U(r)}-1\right)$$

8 Others

Stirling's formula

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n)$$

de Broglie relation

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m}$$