Homework 6 — Due 4/21

Portfolio Optimization Via Metropolis

The accompanying file `V.txt` contains the sample covariance matrix $\mathbf{V}$ for the month-to-month change in value ΔV per 1 dollar investment for each of 50 stocks from the S&P 100 over a five year time period. Use this data to solve the problems below. Your starter code for this project, `Metropolis.cpp`, reads in the covariance matrix.

Use the Metropolis algorithm to solve the problems below. These problems involve finding a minimum variance portfolio subject to certain constraints. (For problems 2 and 3, there are standard quadratic programming algorithms to solve the problem. But use Metropolis, it works well.) To implement Metropolis, you must: (1) specify the state space S; (2) specify the neighbors of each state; (3) specify the energy $E(\mathbf{x})$ of each state $\mathbf{x} \in S$; and (4) tinker with the temperature parameter T until the algorithm produces good results. In all that follows, a **portfolio**, call it $\mathbf{x}$, is a 50×1 column vector whose components sum to \$100. The monthly return (in percent) variance of such a portfolio is $\mathrm{Var}\,(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x}$.

1. Suppose $\mathbf{x} \neq \mathbf{y}$ are two distinct portfolios — they are permitted to have both long and short positions. Then $\mathbf{p} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is another portfolio for any real number λ. Show that $\mathrm{Var}\,(\mathbf{p})$ is quadratic in λ — of the form $a\lambda^2 + b\lambda + c$ for some real numbers a, b, and c with $a > 0$. You may use that $\mathbf{V}$ is PD to establish that $a > 0$. Calculate a in terms of $\mathbf{x}$, $\mathbf{y}$, and $\mathrm{Var}\,(\cdot)$. Use this to argue that $\mathrm{Var}\,(\mathbf{x})$ cannot have multiple local minima as $\mathbf{x}$ ranges over all portfolios. What does that say about the best temperature parameter for problems 2 and 3?

2 (No constraints). Find the minimum variance portfolio when both long and short positions are allowed for individual stocks. Find the allocation to each individual stock to the nearest 0.01 dollar (1 penny). Hint: with the right definition of "neighbor" each state will have 50×49 neighboring states. Compare your portfolio to the actual minimum variance portfolio, given by $100 \cdot \frac{1}{c} \cdot \mathbf{V}^{-1}\mathbf{e}$, and its variance, where $\mathbf{e}_{50\times 1} = (1, 1, \ldots, 1)^T$ and $c = \mathbf{e}^T\mathbf{V}^{-1}\mathbf{e}$. (Should be close.) Note: for this problem the state space is actually infinite, but that does not pose a problem.

3 (No short positions). Find the minimum variance portfolio when short positions are not allowed for individual stocks. Find the allocation to each individual stock to the nearest 0.01 dollar. Hint: if $\mathbf{x}$ is a portfolio with a short position let $\mathrm{Var}\,(\mathbf{x}) = 1000$, then no such portfolio will be selected as optimal.

4 ('Simple' portfolios only). Let us call a portfolio of these 50 stocks 'simple' if all the stocks present in the portfolio have equal weight (e.g., 20 stocks with weight \$5.00 each). This is non-standard terminology; don't use it in an interview! Use Metropolis (I know of no other way) to find the simple portfolio with at least 10 stocks that has the least return variance. (There are approximately 1.1×10^{15} simple portfolios with 10 or more stocks.) Hint: with the right definition of "neighbor" (which is different from problems 2 and 3) each state will have 50 neighboring states.

5 (Non-ground stable state). A state $\mathbf{x}$ is *stable* if $E(\mathbf{y}) > E(\mathbf{x})$ for all neighbors $\mathbf{y}$ of $\mathbf{x}$. In the context of problem 4, exhibit a stable state $\mathbf{x}$ whose energy $E(\mathbf{x})$ is not minimal, i.e., with $E(\mathbf{x}) > E_0$. To do this, run your algorithm with zero-temperature dynamics and see where it stops improving. Verify stability by computing the energy of its 50 neighbors. (You may have to try a few different seeds of the RNG to get this to work.)