PROBLEM SET 8
MTH 5500 STOCHASTIC CALCULUS

- These problems cover the material of Chapter 10 (and some of 9).
- The problems to hand in are the ones with TO HAND IN. Needs to be handed in by Monday **May 15** by email before 10am.
- The assignment has to be done in **teams** of 4.
- The written part has to be handed in a pdf file. The numerical project needs to be handed in as a jupyter notebook .ipynb. The file names should be

  `team#_HW8_MTH5500_spring2023.pdf` `team#_HW8_MTH5500_spring2023.ipynb`.

  I like when you submit a `.zip` file for both!

(1) **Option portfolio.** We consider the following portfolio V of options for an underlying asset: 1 long put with $K = 100$, 2 short puts $K = 150$, 1 long put with $K = 250$ All options have the same expiration T.
  (a) Find the payoff V_T of the option at expiration T as a function of S_T.
  (b) Draw the graph of the payoff.
  (c) This strategy is called a *skip strike butterfly with Puts*. Give it a catchier name.
  (d) Draw the typical graph of the price of a put as a function of the strike price. Conclude from this that the value of the portfolio at time 0 is positive.

(2) **Greeks of Black-Scholes**. Verify Equation 10.28 for the Greeks of a European call in the Black-Scholes model.

(3) **European puts in Black-Scholes model.** Use put-call parity of Example 10.10 to answer the following questions in the Black-Scholes model:
  (a) Prove that a European put $(P_t, t \leq T)$ with strike price K has the price
  $$P_t = -S_t N(-d_+) + Ke^{-r(T-t)} N(-d_-),$$
  where $d_\pm$ are evaluated at $x = S_t$.
  (b) Derive formulas for the Greeks of the put based on Equation 10.28.
  (c) Compute $\frac{\partial P_t}{\partial K}$ and $\frac{\partial^2 P_t}{\partial K^2}$. Conclude that P_t is a convex increasing function of K.

(4) TO HAND IN **Delta hedging.** Consider a portfolio $V^{(1)}$ which is short 100 calls with maturity $T = 60/365$ and $K = 100$. The price of the underlying asset is $S_0 = 100$, the risk-free rate is $r = 0.05$ and the volatility is $\sigma = 0.1$.
  (a) If $V_0^{(1)} = 0$, determine how much money is to be put in risk-free assets to construct $V^{(1)}$.
  (b) Find a delta-neutral portfolio $V^{(2)}$ with 100 long calls and $V^{(2)}(0) = 0$.

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(c) Compare the values of the portfolio $V^{(1)}$ and $V^{(2)}$ after one day when $S_{1/365} =$ 100, 99 and 101.
(d) Plot the graph of $V^{(1)}_{1/365}$ and $V^{(2)}_{1/365}$ as a function of $S_{1/365}$ for the interval $[98, 102]$.

(5) **Cash-or-nothing option.** We consider a *cash-or-nothing* call option with strike price K and value at expiry T

$$O_T = \begin{cases} 1 & S_T > K, \\ 0 & S_T \leq K. \end{cases}$$

(a) Use the pricing formula 10.37 to show that in the Black-Scholes model.

$$O_t = e^{-r(T-t)} N(d_-), \ \ t \in [0, T].$$

(b) Consider a *cash-or-nothing* put with strike price K and value at expiry T

$$U_T = \begin{cases} 0 & S_T > K, \\ 1 & S_T \leq K. \end{cases}$$

Find a relation between the put and the call that holds at T for any outcome (a sort of put-call parity). Use this to find the price of the put at any time.
(c) Verify the previous price of the put is the same as the one given by the pricing formula.
(d) Use a cash-or-nothing call and an asset-or-nothing call (Example 10.18) to replicate a European call.

(6) TO HAND IN **Easy pricing in Black-Scholes.** Consider the Black-Scholes model under its risk-neutral probability $\widetilde{\mathrm{P}}$

$$\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_t \mathrm{d}\widetilde{B}_t \qquad S_0 > 0 \qquad D_t = e^{-rt},$$

where $(\widetilde{B}_t, t \geq 0)$ is a standard $\widetilde{\mathrm{P}}$-Brownian motion. Using risk-neutral pricing, we would like to price the option $(O_t, t \leq T)$ with payoff at expiration T

$$O_T = \log S_T \ .$$

(a) Show that

$$O_0 = e^{-rT}(\log S_0 + (r - \sigma^2/2)T).$$

(b) Show that

$$O_t = e^{-r(T-t)}(\log S_t + (r - \sigma^2/2)(T - t)).$$

(c) Suppose $r = 0$. Is O_t smaller or greater than $\log S_t$. Is this consistent with what you expect directly from the pricing formula?
Hint: Jensen's inequality

(7) **Bachelier at $r = 0$.** Consider the Bachelier model with SDE in the risk-neutral probability for $(S_t, t \leq T)$ given by

$$\mathrm{d}S_t = \sigma \mathrm{d}\widetilde{B}_t.$$

Show that the price of a European call in this model is

$$C_t = (S_t - K)N(b) + \sigma\sqrt{T-t}N'(b), \quad b = \frac{S_t - K}{\sigma\sqrt{T-t}}.$$

(8) TO HAND IN **Lookback option.** We consider the following Bachelier model for the price of an asset:

$$\begin{aligned} \mathrm{d}S_t &= 2\mathrm{d}t + 2\mathrm{d}B_t, & S_0 &= 0,\\ D_t &= 1, & r &= 0. \end{aligned}$$

Using risk-neutral valuation, we are interested in pricing lookback options with expiration $T = 1$.

(a) What is the risk-neutral probability $\widetilde{\mathrm{P}}$ for this model on $[0,1]$? What is the distribution of $(S_t, t \in [0,1])$ under $\widetilde{\mathrm{P}}$?

(b) Using the risk-neutral probability, find O_0 for the option with value at expiration given by

$$O_1 = \max_{t \leq 1} S_t.$$

(c) Using a similar method, show that the price at time 0 of an *asset-or nothing lookback call* with value at expiration and strike price $K > 0$

$$C_1 = \begin{cases} \max_{t\leq 1} S_t & \text{if } \max_{t \leq 1} S_t > K,\\ 0 & \text{if } \max_{t\leq 1} S_t \leq K, \end{cases}$$

is given by

$$C_0 = \frac{4}{\sqrt{2\pi}} e^{-K^2/8}.$$

(d) Find an adequate put-call parity relation. Use the two previous questions to price a lookback put at time 0 with value at expiration

$$P_1 = \begin{cases} 0 & \text{if } \max_{t\leq 1} S_t > K,\\ \max_{t\leq 1} S_t & \text{if } \max_{t\leq 1} S_t \leq K. \end{cases}$$

Numerical Projects ALL TO HAND IN.

(1) **Black-Scholes calculator.** Define a function in Python using `def` that takes as inputs the parameters T, K, S_t, t, r, σ and returns the prices of a European call and a European put in the Black-Scholes model as in Equation 10.19.
To evaluate the CDF of a standard normal random variable you can import the command `norm` from `scipy.stats` in Python.

(2) **Option strategies.** Consider the option strategies in Example 10.8 including the European call option. Consider the parameters $\sigma = 0.1$, $T = 1$, and $K = 480$ as in Example 10.16.

(a) Draw the graph of the payoff of of the six options as a function of S_T, the price of the underlying asset at expiration for $S_T = [400, 550]$. Use $a = 20$.
(b) Use the Black-Scholes formula of a call to generate the value of each of the strategies at time 0 at every 0.1, from $S_0 = 400$ to $S_0 = 600$. Plot it for $r = 0$ and $r = 0.05$. Notice the difference.

(3) **Monte-Carlo exotic option pricing.** Consider the Black-Scholes model with parameters

$$\sigma = 0.1 \quad S_0 = 500 \quad r = 0.05.$$

Use Monte-Carlo pricing to price the following options with expiration $T = 1$ using the average over 1000 paths with a discretization of 0.001.
(a) $O_1 = \max_{t\leq 1} S_t$,
(b) $O_1 = \exp\left(\int_0^1 \log S_t \, \mathrm{d}t\right)$.

(4) **Sampling bias à la Cameron-Martin.** We consider a Brownian motion with drift $\theta = 1$:

$$\widetilde{B}_t = B_t + t \ .$$

(a) Generate a sample $\mathcal{S}$ of 100,000 paths for $(\widetilde{B}_t, t \in [0, 1])$ using a 0.01 discretization.
(b) Using the function `numpy.random.choice`, sample 1000 paths in $\mathcal{S}$ not uniformly but proportionally to their weight:

$$M(\widetilde{B}) = e^{-\widetilde{B}_1 + 1/2} \ .$$

Again you will need to normalize the weights $M(\widetilde{B})$ so that the sum over the 100,000 paths is 1. Let's call this new sample $\widetilde{\mathcal{S}}$.
(c) Draw the histogram of $\widetilde{B}_{1/2}$ on the sample $\widetilde{\mathcal{S}}$. It should look like a Gaussian PDF with mean 0 and variance 1/2.
(d) Plot the first 10 paths from $\mathcal{S}$. Plot the first 10 paths from $\widetilde{\mathcal{S}}$.