Foundations of Financial Engineering

The Merton Structural Model

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Structural Models

Structural approach to credit modeling began with Merton (1974).

It is based on the fundamental accounting equation:

$$\text{Assets} = \text{Debt} + \text{Equity} \tag{1}$$

- states that asset-value of a firm equals value of firm's debt + equity
- we are assuming no taxes.

(1) follows because all profits generated by a firm's assets will ultimately accrue to the debt- and equity-holders.

Structural Models

The capital structure of a firm is such that debt-holders are senior to equity holders

- implies that in event of bankruptcy debt-holders must be paid off in full before equity-holders can receive anything.

This insight allowed Merton to write time T equity value, E_T, as a call option on firm value, V_T, with strike equal to face value of debt, D_T.

Merton's model therefore implies

$$E_T = \max\left(0,\ V_T - D_T\right) \tag{2}$$

with default occurring if $V_T < D_T$.

Merton's Structural Model

Note that (2) implicitly assumes that the firm is wound up at time T and that default can only occur at that time

- not very realistic assumptions
- they have been relaxed in many directions since Merton's original work
- nonetheless, can gain many insights from working with (2).

Can take V_t to be value of a traded asset (why?) so that risk-neutral pricing applies.

If firm does not pay dividends then could assume that $V_t \sim \text{GBM}(r, \sigma)$

- so $E_t =$ Black-Scholes price of a call option with maturity T, strike D_T and underlying security value V_t.

Can compute other quantities such as the (risk-neutral) probability of default etc.

e.g. A Merton Lattice Model

Consider following example:

$V_0 = 1,000$, $T = 7$ years, $\mu = 15\%$, $\sigma = 25\%$ and $r = 5\%$.
$\#$ of time periods $= 7$.
Face value of debt is 800 and coupon on the debt is zero.

First task is to construct lattice model for V_t. Can do this following our usual approach to lattice construction:

- $\nu = (\mu - \sigma^2/2)$
- $\ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$
- $d = 1/u$
- risk-neutral probability of an up-move is $q = (e^{r\Delta t} - d)/(u - d)$.

A Merton Lattice Model

Firm Price Lattice

```
                                                                 6940.6
                                                         5262.6  3990.2
                                                 3990.2  3025.5  2294.0
                                         3025.5  2294.0  1739.4  1318.9
                                 2294.0  1739.4  1318.9  1000.0   758.2*
                         1739.4  1318.9  1000.0   758.2   574.9   435.9*
                 1318.9  1000.0   758.2   574.9   435.9   330.5   250.6*
         1000.0   758.2   574.9   435.9   330.5   250.6   190.0   144.1*

         t = 0   t = 1   t = 2   t = 3   t = 4   t = 5   t = 6   t = 7
```

Note default only possible at time T.

Merton's Model

Now ready to price the equity and debt, i.e. corporate bonds, of the firm.

We price the equity first by simply viewing it as a regular call option on V_T with strike $K = 800$ and using risk-neutral backward evaluation.

The bond or debt price can then be computed similarly or by simply observing that it must equal the difference between the firm-value and equity value at each time and state.

Equity Lattice

							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1127.8	687.1	358.4	128.3	0.0	0.0
	758.6	435.7	207.1	63.8	0.0	0.0	0.0
499.7	269.9	117.4	31.7	0.0	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Debt Lattice

							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		611.6	631.7	641.6	630.0	574.9	435.9*
	560.3	564.3	551.1	511.1	435.9	330.5	250.6*
500.3	488.3	457.5	404.2	330.5	250.6	190.0	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Merton's Model

We see the initial values of the equity and debt are 499.7 and 500.3, respectively.

The yield-to-maturity, y, of the bond satisfies $500.3 = e^{-yT} \times 800$ which implies $y = 6.7\%$.

The credit spread is then given by $c = y - r = 1.7\%$ or 170 basis points.

Merton's Model

Can easily compute the true or risk-neutral probability of default by constructing an appropriate lattice.

Also easy to handle coupons: if debt pays a coupon of C per period, then we write $E_T = \max(0,\ V_T - D_T - C)$.

And in any earlier period we have

$$E_t = \max\left(0,\ \left[qE^u + (1-q)E^d\right]/R - C\right)$$

where $R = e^{r\Delta t}$.

Foundations of Financial Engineering

The Black-Cox Structural Model

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

The Black-Cox Model

Black-Cox model generalizes Merton model by allowing default to also occur before time T.

We now assume default occurs first time firm value falls below face value of debt.

In that case we can compute the value of the equity by placing 0 in those cells where default occurs

- and updating other cells using usual backwards evaluation approach.

Debt value at a given cell in the lattice given by difference between the firm and equity values in that cell.

We obtain the following equity and debt lattices:

Equity Lattice

							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1115.8	660.7	300.1	0.0	0.0	0.0
	703.9	328.5	0.0	0.0	0.0	0.0	0.0
350.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Debt Lattice

							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		623.6	658.2	699.9	758.2*	574.9*	435.9*
	614.9	671.5	758.2*	574.9*	435.9*	330.5*	250.6*
650.0	758.2*	574.9*	435.9*	330.5*	250.6*	190.0*	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

The Black-Cox Model

We see the debt-holders have benefitted from this new default regime: their value increased from 500.3 to 650.

Of course this increase has come at the expense of the equity holders whose value has fallen from 499.7 to 350.

In this case the credit spread on the bond is -200 basis points!

- unreasonable value of credit spread is evidence against the realism of default assumption made here.

Negative credit spreads generally not found in practice but have occurred here because the debt holders essentially own a down-and-in call option on the value of the firm with zero strike and barrier equal to the face value of the debt.

While it is true that a firm can default at any time, the barrier would generally be much lower than the face value of the long-term debt of 800.

Note that we could easily use a different and time-dependent default barrier to obtain a more realistic value of the credit spread.

Foundations of Financial Engineering

Ratings Models

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Ratings Models

An alternative approach to modeling default events is via ratings transitions.

Ratings are intended to signify the credit-worthiness of financial instruments with safer securities having higher ratings

- e.g. a security rated `AAA` is believed to be very secure and have almost no default risk.

These ratings are updated periodically as the prospects of the firms (or governments!) change.

Can model these periodic updates via a Markov chain and use historical ratings transitions to estimate the transition matrix, $\mathbf{P}$.

Ratings Models

		End State							
		AAA	AA	A	BBB	BB	B	CCC	Default
	AAA	90.81	8.33	0.68	0.06	0.12	-	-	-
	AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	-
Initial	A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
State	BBB	0.02	0.33	5.95	85.93	5.30	1.17	1.12	0.18
	BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
	B	-	0.11	0.24	0.43	6.48	83.47	4.07	5.20
	CCC	0.22	-	0.22	1.30	2.38	11.24	64.85	19.79

– A sample transition matrix for Standard & Poors ratings.

$\mathbf{P}_{i,j} =$ probability a firm with current rating i will be rated j one year or one quarter from now.

Final row of matrix omitted since a firm in default assumed to stay in default.

Ratings Models

Ratings transition models were popularized by CreditMetrics and J.P. Morgan in the late 1990's.

Their approach to credit risk was to assume the credit rating of a company was well-modeled by a Markov chain with transition matrix $\mathbf{P}$.

Then easy to compute the probability of default (or indeed losses / gains due to a deterioration / improvement in credit quality) over any period of time.

e.g. easy to see that $\mathbf{P}^k = k$-period transition matrix.

Ratings Models

More generally, could use a database of ratings transitions to estimate a continuous-time Markov model so that ratings transitions could occur at any time instant – this of course is more realistic.

To compute risk measures such as Value-at-Risk (VaR) for credit portfolios, it is necessary to model the joint ratings transition of many companies

- can be achieved using copula methods
- Monte-Carlo methods can then be used to estimate quantities of interest.

Foundations of Financial Engineering

Credit-Default Swaps

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Credit-Default Swaps

Credit default swaps (CDS's) are a very important class of derivative instrument

- developed in late 1990's
- and now ubiquitous in the financial markets.

A CDS allow investors (or speculators) to hedge (or take on) the default risk of a firm or government.

A CDS is structured like an insurance policy between two parties:

> Party A agrees to pay party B a fixed amount every period (typically every quarter), in return for protection against the default of a third party, C.

These payments constitute the premium leg and size of these payments are proportional to the notional amount, N.

When a default occurs, party B must pay party A the difference between N and the market value of the reference bond (with notional N) issued by party C.

Pricing a CDS in General

Also possible to have physical settlement: upon default the protection buyer delivers the reference bond to the protection seller and receives the face value of the bond in return

- this payment constitutes the default leg of the CDS.

Credit-Default Swaps

The three parties (A, B and C) are referred to as the protection buyer, the protection seller and the reference entity, respectively.

The CDS has a maturity date, T, and all payments cease at $\min(\tau, T)$ where τ is the default time of the reference entity, party C

- not true for a CDS written on an index containing multiple reference entities
- in the index case there can be many default-leg payments, one for each reference entity
  - notional of the CDS, however, is reduced appropriately after each default.

We price a CDS by equating the risk-neutral value of the premium and default legs and from this determining the fair annual spread that the protection buyer must pay to the protection seller.

Pricing a CDS: The Premium Leg

The Premium Leg:

Suppose the premium leg of the CDS has n payment times, $t_1, \ldots, t_n = T$

- we assume that default can only occur at one of these times.

Then fair value of the premium leg at time $t = 0$ is given by

$$P_0 = s_0^T N \sum_{i=1}^{n} Z_0^{t_i} \alpha_i \, P(\tau > t_i) \tag{3}$$

where $Z_0^{t_i}$ is the discount factor, N is the notional and $\alpha_i = t_i - t_{i-1}$.

Annualized CDS spread is then s_0^T

- use superscript 'T' since CDS spread is maturity-dependent in practice.

Pricing a CDS: The Default Leg

The Default Leg:

The fair value at time $t = 0$ of the default leg satisfies

$$D_0 = (1 - R)N \sum_{i=1}^{n} Z_0^{t_i} P(\tau = t_i) \tag{4}$$

where R is the so-called recovery rate.

$R = \%$ of face value of the reference bond that is recovered by a bond-holder upon default of the reference entity.

Common to assume that R is a fixed constant, e.g. 40%

- but in practice it is stochastic.

Obtaining the Fair CDS Spread

The spread, s_0^T, is obtained by equating P_0 with D_0 so that the CDS has zero value for both parties at the beginning of the contract.

This implies

$$s_0^T = \frac{(1-R)\sum_{i=1}^{n} Z_0^{t_i}\, P(\tau = t_i)}{\sum_{i=1}^{n} Z_0^{t_i} \alpha_i\, P(\tau > t_i)}. \tag{5}$$

Can (and often should) make simple adjustments to (5) to allow for default possibility in (t_{i-1}, t_i).

A plot of s_0^T against T then shows the term-structure of CDS spreads.

Foundations of Financial Engineering

Structural Models for Pricing a CDS

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Structural Models for Pricing a CDS

Recall that CDS spread calculated as

$$s_0^T = \frac{(1-R)\sum_{i=1}^n Z_0^{t_i}\, P(\tau = t_i)}{\sum_{i=1}^n Z_0^{t_i}\alpha_i\, P(\tau > t_i)}. \tag{6}$$

To compute a specific value for s_0^T, must have a (risk-neutral) default model in order to compute the various probabilities in (6).

Can certainly compute s_0^T using structural models as described earlier.

Structural Models for Pricing a CDS

Recall the debt value lattice from Black-Cox model:

```
Debt Lattice
                                                                  800.0
                                                        761.0     800.0
                                              723.9     761.0     800.0
                                    688.6     723.9     761.0     800.0
                          653.2     684.7     715.3     742.0     758.2*
                623.6     658.2     699.9     758.2*    574.9*    435.9*
      614.9     671.5     758.2*    574.9*    435.9*    330.5*    250.6*
650.0 758.2*    574.9*    435.9*    330.5*    250.6*    190.0*    144.1*

t = 0 t = 1     t = 2     t = 3     t = 4     t = 5     t = 6     t = 7
```

Can use lattice to build separate lattices for premium and default leg cash-flows

- trial and error could be used to find the fair premium, s_0^T.

Alternatively could use lattices to compute the required default probabilities

- can then use (6) to calculate s_0^T.

Structural Models for Pricing a CDS

One of the main weaknesses of the structural approach to pricing CDS's is that the default event is said to be predictable.

> **Example:** Consider the Black-Cox model where default is impossible in next time period unless firm value is very close to the default boundary.
>
> And if firm value is far from the default boundary then default in the next several periods will also be impossible.
>
> In contrast, if we let the length of a time period, Δt, go to zero and the firm value process is extremely close to the default boundary then default will happen for sure very soon.

This is what we mean when we say default is predictable.

Structural Models for Pricing a CDS

In the real-world, however, default is not predictable and the default of firms can come as a complete surprise.

This is true for firms such as Enron and Parmalat where default was caused by the discovery of huge accounting frauds.

Even when the market sees that a firm, e.g. Lehman Brothers, is in financial difficulty so that the CDS spreads of that firm have widened, the actual default event itself is still a surprise

- and unpredictable (in a mathematical sense).

Intensity models are commonly used to circumvent this problem

- default events in these models are unpredictable
- they do not model the economic value of the firm and hence are often termed reduced-form models.

(Inhomogeneous) Poisson process intensity models are examples of deterministic intensity models.

Cox process models are examples of stochastic intensity models.

Foundations of Financial Engineering

Using (Inhomogeneous) Poisson Processes to Model Default

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Using Poisson Processes to Model Default

CDS spread now calculated as

$$s_0^T = \frac{(1-R)\sum_{i=1}^{n} Z_0^{t_i}\, P(t_{i-1} < \tau \leq t_i)}{\sum_{i=1}^{n} Z_0^{t_i} \alpha_i\, P(\tau > t_i)} \tag{7}$$

since we assume default event can take place any time in $(t_{i-1}, t_i]$ for any i.

Suppose the arrival of default follows a Poisson process with parameter λ.

Then $P(\tau \leq t) = 1 - e^{-\lambda t}$ and so we can easily compute s_0^T using (7).

Using Poisson Processes to Model Default

Can also compute time $t = 0$ value, V_0, of a zero-coupon bond with face value F and maturity T.

If the recovery-rate upon default is a known value, R, that is paid at the time of default, τ, then we have

$$\begin{aligned} V_0 &= \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT} F 1_{\{\tau > T\}} + \int_0^T e^{-rt} RF\, 1_{\{\tau = t\}}\right] \\ &= e^{-(r+\lambda)\,T} F \ + \ RF \int_0^T \lambda e^{-(r+\lambda)\,t}\, dt \\ &= e^{-(r+\lambda)\,T} F \ + \ RF \frac{\lambda}{r+\lambda}\left(1 - e^{-(r+\lambda)\,T}\right). \end{aligned} \tag{8}$$

Using an Inhomogeneous Poisson Process

A clear weakness with the Poisson model is that there is just a single parameter, λ, that we are free to choose.

In practice, however, there are typically liquid CDS spreads s_0^T, for several values of T

- typical values are $T = 1, 3, 5, 7$ and 10 years.

Unless term structure of credit spreads is constant then not possible to choose λ so that model values of s_0^T coincide with corresponding market values for all values of T.

Can resolve this problem using an inhomogeneous Poisson process

- arrival rate, λ_t, is now a deterministic function of time, t.

For such a process, probability of zero arrivals in interval $(t, t + \Delta t)$ is then approximately $e^{-\lambda_t \Delta t}$.

Using an Inhomogeneous Poisson Process

Using this we obtain the survival probability

$$P(\tau > t) = e^{-\int_0^t \lambda_s \, ds}. \tag{9}$$

Can now use (9) to compute fair CDS spread according to our expression

$$s_0^T = \frac{(1-R)\sum_{i=1}^n Z_0^{t_i} \, P(t_{i-1} < \tau \leq t_i)}{\sum_{i=1}^n Z_0^{t_i} \alpha_i \, P(\tau > t_i)}. \tag{10}$$

Can also calibrate the inhomogeneous Poisson process model to the CDS spreads observed in the market.

In particular, can assume λ_t is piecewise constant on the intervals $(0, T_1]$, $(T_1, T_2], \ldots, (T_{n-1}, T_n]$ where $T_1 < T_2 < \cdots < T_n$ are CDS market maturities.

Can then use (10) to first calibrate λ_{0, T_1} and to then calibrate each of $\lambda_{T_1, T_2}, \ldots, \lambda_{T_{n-1}, T_n}$ in turn.

Foundations of Financial Engineering

Stochastic Intensity Models

Martin B. Haugh

Department of Industrial Engineering and Operations Research
Columbia University

Stochastic Intensity Models

While the inhomogeneous Poisson model can be calibrated to the CDS spreads seen in the market, it has a glaring weakness:

credit spreads at all future times are known today at time $t = 0$

- follows because the intensity, λ_t, is deterministic.

In practice, however, credit spreads are stochastic

- so we would like to consider models where the default intensity is stochastic.

Will consider one important and tractable class of stochastic intensity models: the doubly-stochastic or Cox process

- a Cox process is in fact an entire class of (point) processes.

Stochastic Intensity Models

Before defining such a process let N_t be the number of "arrivals" up to time t.

When N_t follows a Poisson process we know N_t has a Poisson distribution with parameter λt

- or parameter $\int_0^t \lambda_u \, du$ in the case of an inhomogeneous Poisson process.

A Cox process generalizes this by allowing the intensity λ_u to be stochastic.

In a Cox process, however, conditional on knowing $\{\lambda_u\}_{0 \leq u \leq T}$, the $\#$ of arrivals in interval $[s, t]$ where $0 \leq s < t \leq T$ has an inhomogeneous Poisson distribution with parameter $\int_s^t \lambda_u \, du$

- such a process remains very tractable.

Cox Processes

Example: If τ is the first arrival of a Cox process then we can compute the survival probability as follows:

$$
\begin{aligned}
P(\tau > t) &= \mathsf{E}\left[1_{\{\tau>t\}}\right] \\
&= \mathsf{E}\left[\mathsf{E}\left[1_{\{\tau>t\}} \mid \{\lambda_u\}_{0\leq u\leq t}\right]\right] \\
&= \mathsf{E}\left[e^{-\int_0^t \lambda_u \, du}\right] \qquad (11)
\end{aligned}
$$

where (11) follows from the fact that the point process is an inhomogenous Poisson process on $[0, t]$ conditional on $\{\lambda_u\}_{0\leq u\leq t}$.

Pricing a Bond with a Cox Process Default Model

Recall our earlier expression for the price of a zero-coupon bond with face value F and maturity T:

$$V_0 \quad = \quad e^{-(r+\lambda)\,T} F \;+\; RF \frac{\lambda}{r+\lambda}\left(1 - e^{-(r+\lambda)\,T}\right). \tag{12}$$

– assumes recovery-rate is R is known and paid at time of default, τ.

Assuming a (risk-neutral) Cox process, can generalize (12) to obtain

$$\begin{aligned} V_0 \quad &= \quad \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT} F 1_{\{\tau > T\}} + \int_0^T e^{-rt} RF\, 1_{\{\tau = t\}}\right] \\ &= \quad \mathsf{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T (r+\lambda_u)\,du} F\right] \;+\; RF\, \mathsf{E}_0^{\mathbb{Q}}\left[\int_0^T \lambda_t e^{-\int_0^t (r+\lambda_s)\,ds}\, dt\right] \\ &= \quad \mathsf{E}_0^{\mathbb{Q}}\left[e^{-\int_0^T (r+\lambda_u)\,du} F\right] \;+\; RF \int_0^T \mathsf{E}_0^{\mathbb{Q}}\left[\lambda_t e^{-\int_0^t (r+\lambda_s)\,ds}\right] dt. \end{aligned} \tag{13}$$

Pricing a Bond with a Cox Process Default Model

A Cox process is only fully specified once we specify dynamics for λ_t.

We can consider a lattice model for λ_t but we also note that there are continuous-time models

- e.g. the CIR model, for which (13) can be computed analytically.

Parameters of these models would then be chosen so that the model CDS spreads, s_0^T, match market CDS spreads for different maturities, T

- this is the process of calibration.