Foundations of Financial Engineering

The Cash Account and Pricing Zero-Coupon Bonds

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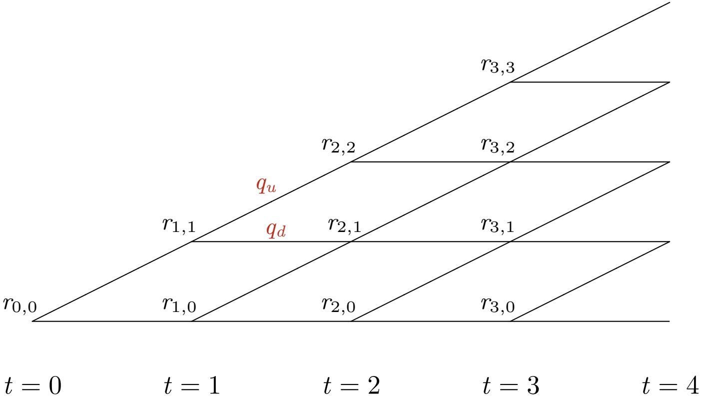
Binomial Models for the Short-Rate



The Cash-Account

Recall the cash-account is a particular security that in each period earns interest at the short-rate

- we use B_t to denote its value at time t and assume that $B_0 = 1$.

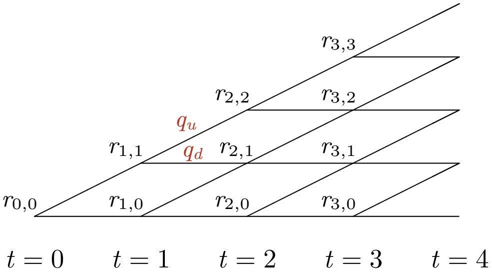
The cash-account is not risk-free since B_{t+s} is not known at time t for any $s > 1$

- it is locally risk-free since B_{t+1} is known at time t.

Note that B_t satisfies $B_t \;=\; (1 + r_{0,0})(1 + r_1) \ldots (1 + r_{t-1})$

- so that $B_t / B_{t+1} = 1/(1 + r_t)$.

Binomial Models for the Short-Rate



Martingale pricing for a "non-coupon" paying security then takes the form:

$$\begin{aligned} Z_{t,j} &= \mathsf{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_{t+1}} Z_{t+1}\right] \\ &= \frac{1}{1+r_{t,j}}\left[q_u \times Z_{t+1,j+1} \; + \; q_d \times Z_{t+1,j}\right] \end{aligned} \tag{1}$$

– $q_u > 0$ and $q_d > 0$ are the risk-neutral probabilities of an up- and down-move, respectively, of the short-rate.

Risk-Neutral Pricing with the Cash-Account

Risk-neutral pricing for a "coupon" paying security takes the form:

$$\begin{aligned} Z_{t,j} &= \frac{1}{1+r_{t,j}} \left[q_u \left(Z_{t+1,j+1} + C_{t+1,j+1} \right) \; + \; q_d \left(Z_{t+1,j} + C_{t+1,j} \right) \right] \\ &= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1} + C_{t+1}}{1+r_{t,j}} \right]. \end{aligned} \tag{2}$$

Can rewrite (2) as

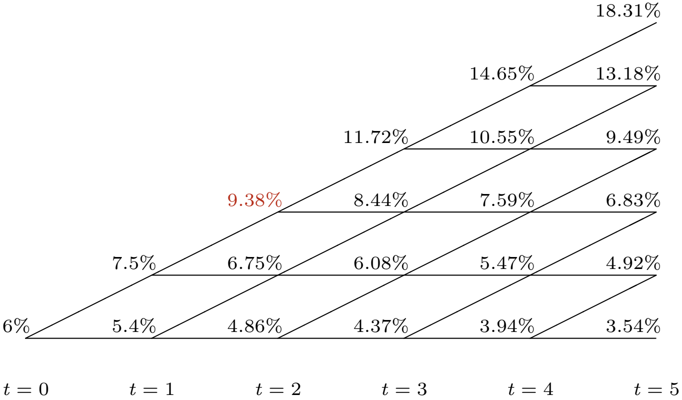
$$\frac{Z_t}{B_t} \; = \; \mathsf{E}_t^{\mathbb{Q}} \left[\frac{C_{t+1}}{B_{t+1}} \; + \; \frac{Z_{t+1}}{B_{t+1}} \right]. \tag{3}$$

More generally, can iterate (3) to obtain

$$\frac{Z_t}{B_t} \; = \; \mathsf{E}_t^{\mathbb{Q}} \left[\sum_{j=t+1}^{t+s} \frac{C_j}{B_j} \; + \; \frac{Z_{t+s}}{B_{t+s}} \right] \tag{4}$$

– just martingale pricing.

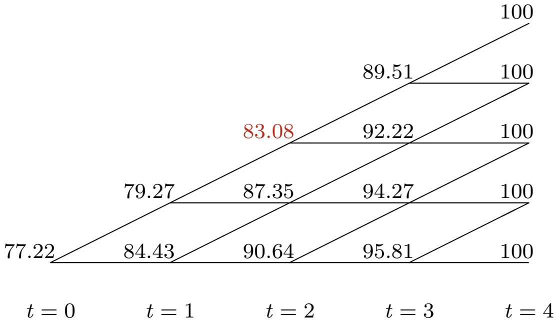
A Short-Rate Lattice Example



The short-rate, r, grows by a factor of $u = 1.25$ or $d = .9$ in each period

– not very realistic but more than sufficient for our purposes.

Pricing a ZCB that Matures at Time t=4

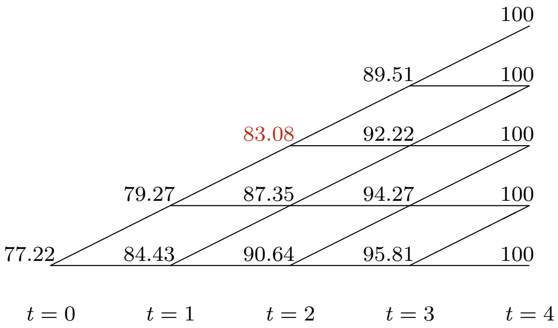


$$\text{e.g.}\quad 83.08 = \frac{1}{1 + .0938}\left[\frac{1}{2} \times 89.51 + \frac{1}{2} \times 92.22\right].$$

Can compute the term-structure by pricing ZCB's of every maturity and then backing out the spot-rates for those maturities

- so $s_4 = 6.68\%$ assuming per-period compounding, i.e., $77.22(1 + s_4)^4 = 100$.

Pricing a ZCB that Matures at Time t=4



Therefore can compute compute Z_0^1, Z_0^2, Z_0^3 and Z_0^4

– and then compute s_1, s_2, s_3 and s_4 to obtain the term-structure of interest rates at time $t = 0$.

At $t = 1$ we will compute new ZCB prices and obtain a new term-structure

– model for the short-rate, r_t, therefore defines a model for the term-structure!

Foundations of Financial Engineering

Fixed Income Derivatives: Options on Bonds

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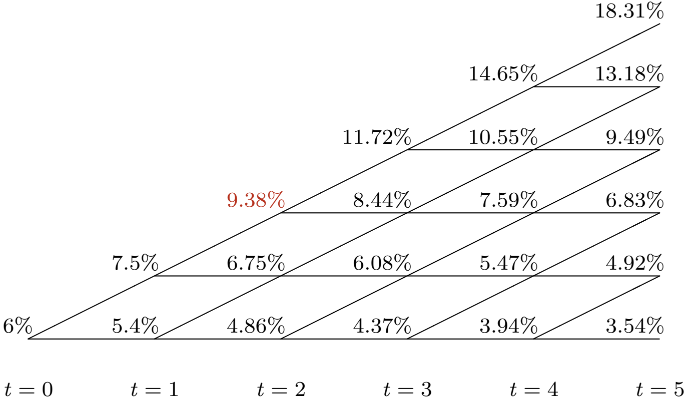
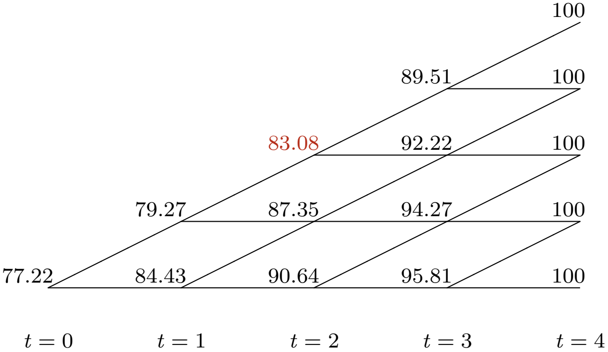
Our Short-Rate Lattice Example

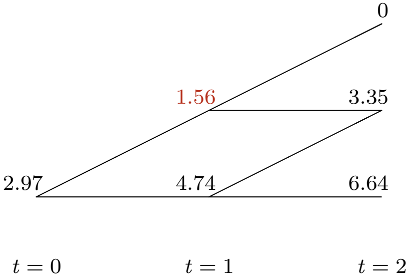


Pricing a ZCB that Matures at Time t=4



$$\text{e.g.} \quad 83.08 \; = \; \frac{1}{1 + .0938} \left[\frac{1}{2} \times 89.51 \; + \; \frac{1}{2} \times 92.22 \right].$$

Pricing a European Call Option on the ZCB



Strike $= \$84$

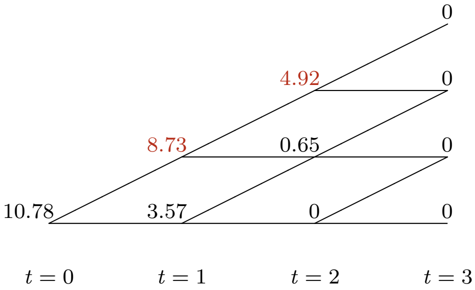
Option Expiration at $t = 2$

Option Payoff $= \max\left(0,\ Z^4_{2,\cdot} - 84\right)$

Underlying ZCB Matures at $t = 4$

$$\text{e.g.}\quad 1.56 \;=\; \frac{1}{1 + .075}\left[\frac{1}{2} \times 0 \;+\; \frac{1}{2} \times 3.35\right].$$

Pricing an American Put Option on a ZCB



Strike $= \$88$
Expiration at $t = 3$
Payoff at $t = 3$ is $\max(0,\, 88 - Z^4_{3,\cdot})$
Underlying ZCB Matures at $t = 4$

$$\text{e.g.}\quad 4.92 \;=\; \max\left\{88 - 83.08\,,\; \frac{1}{1+.0938}\left[\frac{1}{2}\times 0 \;+\; \frac{1}{2}\times 0\right]\right\}.$$

$$\text{e.g.}\quad 8.73 \;=\; \max\left\{88 - 79.27\,,\; \frac{1}{1+.075}\left[\frac{1}{2}\times 4.92 \;+\; \frac{1}{2}\times 0.65\right]\right\}.$$

Turns out it's optimal to early-exercise everywhere

– not a very realistic example.

Foundations of Financial Engineering

Fixed Income Derivatives: Bond Forwards and Futures

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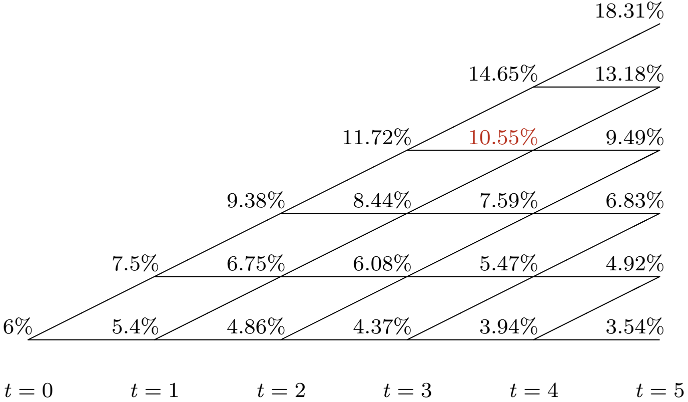
Our Short-Rate Lattice Example



Pricing a Forward on a Coupon-Bearing Bond

Want to compute forward price of a 2-year 10% coupon-bearing bond for delivery at $t = 4$. (Delivery takes place just after a coupon has been paid.)

Let $G_0 =$ forward price at $t = 0$. Recall that

$$0 \quad = \quad \mathsf{E}_0^{\mathbb{Q}}\left[\frac{Z_4^6 - G_0}{B_4}\right].$$

so we obtain

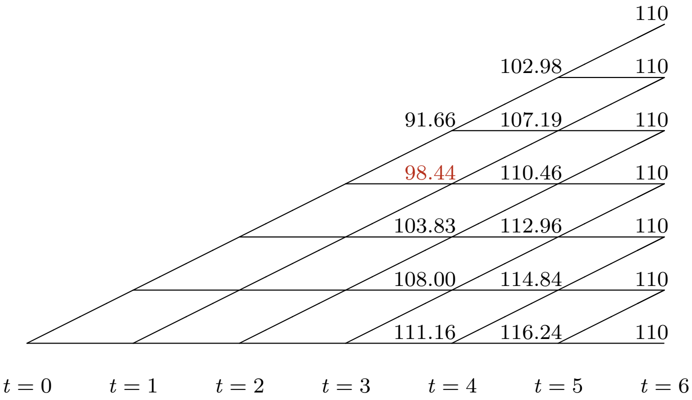
$$G_0 \;=\; \frac{\mathsf{E}_0^{\mathbb{Q}}\left[Z_4^6/B_4\right]}{\mathsf{E}_0^{\mathbb{Q}}\left[1/B_4\right]}. \tag{5}$$

where $Z_4^6 =$ ex-coupon bond price at $t = 4$

- easily computed via backwards evaluation.

Recall $\mathsf{E}_0^{\mathbb{Q}}\left[1/B_4\right]$ is time $t = 0$ price of a ZCB maturing at $t = 4$ but with a face value $\$1$ – have already calculated this to be $.7722$.

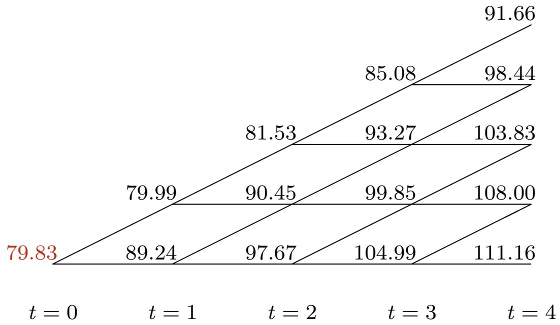
Pricing a Forward on a Coupon-Bearing Bond



First find ex-coupon price, Z_4^6, of the bond at time $t = 4$:

$$\text{e.g.} \quad 98.44 \; = \; \frac{1}{1 + .1055} \left[\frac{1}{2} \times 107.19 \; + \; \frac{1}{2} \times 110.46 \right].$$

Pricing a Forward on a Coupon-Bearing Bond



Now work backwards in lattice to compute $\mathsf{E}_0^{\mathbb{Q}}\left[Z_4^6/B_4\right] = 79.83$.

Can now use (5) to obtain

$$G_0 \;=\; \frac{79.83}{0.7722} \;=\; 103.38.$$

Pricing a Futures Contract on a Coupon-Bearing Bond

Let $F_k =$ time k price of a futures contract that expires after n periods.

Let $S_k =$ time k price of the security underlying the futures contract.

Then $F_n = S_n$ and we saw earlier that F_{n-1} must satisfy

$$\frac{0}{B_{n-1}} \quad = \quad \mathsf{E}_{n-1}^{\mathbb{Q}}\left[\frac{F_n - F_{n-1}}{B_n}\right]$$

so that $F_{n-1} = \mathsf{E}_{n-1}^{\mathbb{Q}}[F_n]$. More generally we have

$$F_k = \mathsf{E}_k^{\mathbb{Q}}[F_{k+1}] \quad \text{for } 0 \leq k < n$$

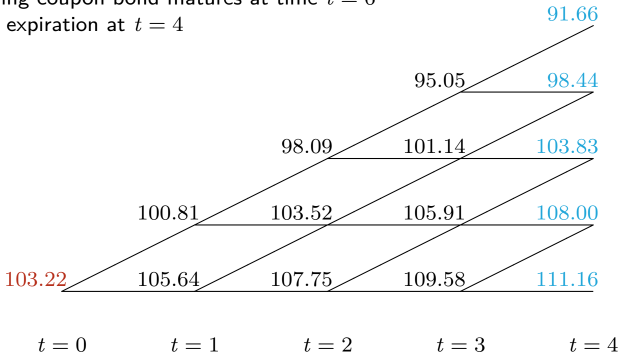
and the law of iterated expectations yields

$$F_0 \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[F_n\right] \;=\; \mathsf{E}_0^{\mathbb{Q}}\left[S_n\right]$$

– holds regardless of whether or not underlying security pays coupons etc.

A Futures Contract on a Coupon-Bearing Bond

Futures contract written on same coupon bond as earlier forward contract
Underlying coupon bond matures at time $t = 6$
Futures expiration at $t = 4$



Note that the forward price, 103.38, and futures price, 103.22, are close

– but not equal!

Foundations of Financial Engineering

Fixed Income Derivatives: Caplets and Floorlets

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Pricing a Caplet

A caplet is similar to a European call option on the interest rate, r_t.

- Usually settled in arrears but they may also be settled in advance.
- If maturity is τ and strike is c, then payoff of a caplet (settled in arrears) at time τ is

$$(r_{\tau-1} - c)^+$$

  – so the caplet is a call option on the short rate prevailing at time $\tau - 1$, settled at time τ.

A floorlet is the same as a caplet except the payoff is $(c - r_{\tau-1})^+$.

A cap consists of a sequence of caplets all of which have the same strike.

A floor consists of a sequence of floorlets all of which have the same strike.

Our Short-Rate Lattice

					18.31%
				14.65%	13.18%
			11.72%	10.55%	9.49%
		9.38%	8.44%	7.59%	6.83%
	7.5%	6.75%	6.08%	5.47%	4.92%
6%	5.4%	4.86%	4.37%	3.94%	3.54%
$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$

Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					.138
				.103	.099
			.080	.076	.068
		.064	.059	.053	.045
	.052	.047	.041	.035	.028
.042	.038	.032	.026	.021	.015

Note that it is easier to record the time $t = 6$ cash flows at their time 5 predecessor nodes, and then discount them appropriately:

– so $(r_5 - c)^+$ at $t = 6$ is worth $(r_5 - c)^+/(1 + r_5)$ at $t = 5$.

Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$

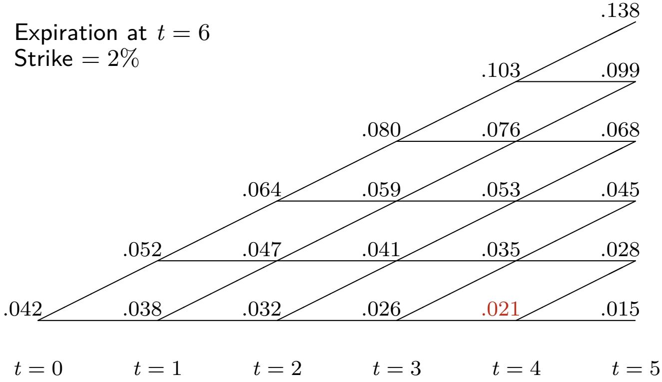
$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					.138
				.103	.099
			.080	.076	.068
		.064	.059	.053	.045
	.052	.047	.041	.035	.028
.042	.038	.032	.026	.021	.015

A sample calculation:

$$0.015 \quad = \quad \frac{\max(0,\ .0354 - .02)}{1 + .0354}$$

Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$



Now work backwards in the lattice to find the price at $t = 0$.

A sample calculation:

$$0.021 \quad = \quad \frac{1}{1.0394}\left[\frac{1}{2} \times 0.028 \; + \; \frac{1}{2} \times 0.015\right]$$

Foundations of Financial Engineering

Fixed Income Derivatives: Swaps and Swaptions

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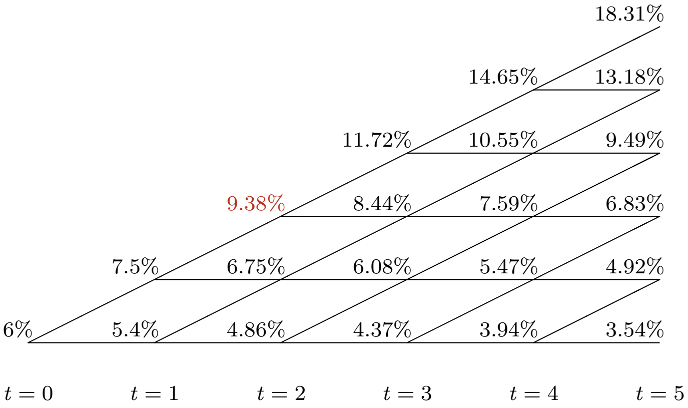
Our Short-Rate Lattice



Want to price an interest-rate swap with fixed rate of 5% that expires at $t = 6$

- first payment at $t = 1$ and final payment at $t = 6$
- payment of $\pm(r_{i,j} - K)$ made at time $t = i + 1$ if in state j at time i.

Pricing Swaps

Expiration, i.e. last payment, at $t = 6$
First payment, at $t = 1$
Strike $=$ fixed rate $= 5\%$

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					$.1125$
				$.1648$	$.0723$
			$.1793$	$.1014$	$.0410$
		$.1686$	$.1021$	$.0512$	$.0172$
	$.1403$	$.0829$	$.0400$	$.0122$	$-.0008$
$.0990$	$.0496$	$.0137$	$-.0085$	$-.0174$	$-.0141$

Note that it is easier to record the time t cash flows at their time $t-1$ predecessor nodes, and then discount them appropriately:

– so $(r_{5,5} - K)$ at $t = 6$ is worth $\pm(r_{5,5} - K)/(1 + r_{5,5}) = .0723$ at $t = 5$.

Pricing Swaps

Expiration, i.e. last payment, at $t = 6$
First payment, at $t = 1$
Strike = fixed rate = 5%

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1686	.1021	.0512	.0172
	.1403	.0829	.0400	.0122	$-.0008$
.0990	.0496	.0137	$-.0085$	$-.0174$	$-.0141$

A sample calculation:

$$0.1686 = \frac{1}{1.0938}\left[(.0938 - .05) + \frac{1}{2} \times 0.1793 + \frac{1}{2} \times 0.1021\right]$$

Pricing Swaptions

A swaption is an option on a swap.

Consider a swaption on the swap of the previous slide:

- Will assume that the option strike is 0%
  - not to be confused with the strike, i.e. fixed rate, of underlying swap.
- The swaption expiration is at $t = 3$.

Swaption value at expiration is therefore $\max(0,\ S_3)$ where $S_3 \equiv$ value of underlying swap contract at $t = 3$.

Value at dates $0 \leq t < 3$ computed in usual manner by working backwards in the lattice

- but underlying cash-flows of swap are not included at those times.

Pricing Swaptions

Fixed rate in Swap $= 5\%$
Underlying Swap Expiration at $t = 6$
Option Expiration at $t = 3$

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1286	.1021	.0512	.0172
	.0908	.0665	.0400	.0122	−.0008
.0620	.0406	.0191	**0**	−.0174	−.0141

Swaption price is computed by determining payoff at maturity, i.e $t = 3$ and then working backwards in the lattice.

A sample calculation:

$$.0908 \;=\; \frac{1}{1+.075}\left[\frac{1}{2}\times .1286 \;+\; \frac{1}{2}\times .0665\right]$$

Foundations of Financial Engineering

The Forward Equations

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The Forward Equations

$P^e_{i,j}$ denotes the time 0 price of a security that pays $\$1$ at time i, state j and 0 at every other time and state.

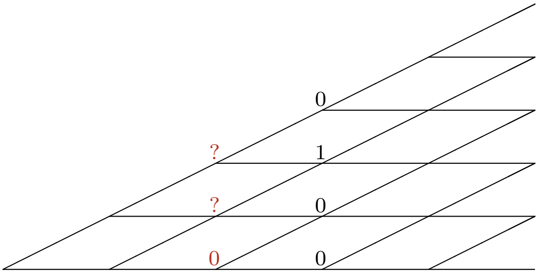
- call such a security an elementary security
- and $P^e_{i,j}$ is its state price.

Can see elementary security prices satisfy the forward equations:

$$
\begin{aligned}
P^e_{k+1,s} &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})}, \quad 0 < s < k+1 \qquad (6) \\
P^e_{k+1,0} &= \frac{1}{2}\frac{P^e_{k,0}}{(1+r_{k,0})} \\
P^e_{k+1,k+1} &= \frac{1}{2}\frac{P^e_{k,k}}{(1+r_{k,k})}.
\end{aligned}
$$

with $P^e_{0,0} = 1$.

Deriving the Forward Equations



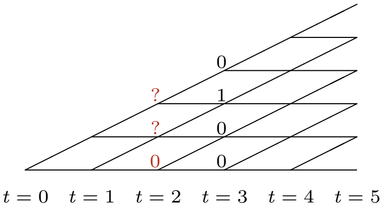
$t = 0$ $t = 1$ $t = 2$ $t = 3$ $t = 4$ $t = 5$

Consider the security that pays $\$1$ only at $t = 3$ and only in state 2

- value of this security is $P^e_{3,2}$ by definition.

Can also work backwards in lattice to price it:

Deriving the Forward Equations



Its value at node $N_{2,2}$ is

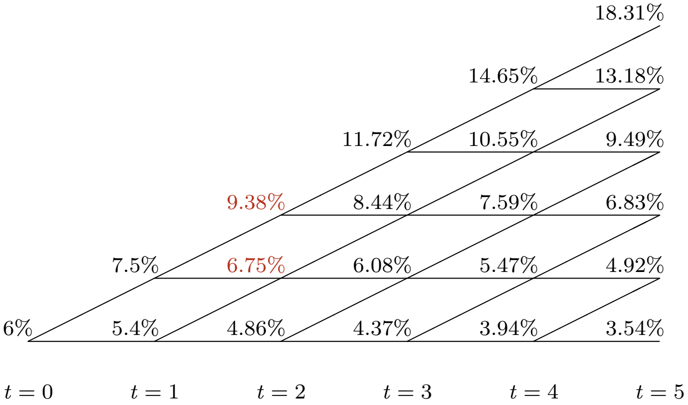
$$\frac{1}{1+r_{2,2}}\left[\frac{1}{2}\times 0 \ + \ \frac{1}{2}\times 1\right] \ = \ \frac{1}{2(1+r_{2,2})}$$

and its value at node $N_{2,1}$ is

$$\frac{1}{1+r_{2,1}}\left[\frac{1}{2}\times 1 \ + \ \frac{1}{2}\times 0\right] \ = \ \frac{1}{2(1+r_{2,1})}.$$

Therefore $P^e_{3,2} = \frac{1}{2(1+r_{2,2})} \times P^e_{2,2} \ + \ \frac{1}{2(1+r_{2,1})} \times P^e_{2,1} \ + \ 0 \times P^e_{2,0}$.

Our Short-Rate Lattice



Now compute the forward prices by iterating the equations forward starting with $P^e_{0,0} = 1$.

… and the Corresponding Elementary Prices

Key: Value at node $N_{i,j}$ is $P^e_{i,j}$

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
					0.0196
				0.0449	0.1041
			0.1003	0.1868	0.2193
		.2194	0.3079	0.2901	0.2293
	.4717	.4432	0.3143	0.1992	0.1190
1	.4717	.2238	0.1067	0.0511	0.0246

Sample calculation:

$$
\begin{aligned}
.3079 &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})} \\
&= \frac{.4432}{2(1+.0675)} + \frac{.2194}{2(1+.0938)}
\end{aligned}
$$

Derivative Prices Via Elementary Prices

Given the elementary prices the calculation of some security prices becomes very straightforward:

Example: Can calculate Z_0^4 as

$$\begin{aligned} Z_0^4 &= 100 \times (.0449 + .1868 + .2901 + .1992 + .0511) \\ &= 77.22 \end{aligned}$$

– as we saw earlier.

Derivative Prices Via Elementary Prices

Consider a forward-start swap that begins at $t = 1$ and ends at $t = 3$:

- Notional principal is $\$1$ million
- Fixed rate in the swap is 7%
- Payments at $t = i$ for $i = 2, 3$ are based (as usual) on fixed rate minus floating rate that prevailed at $t = i - 1$.

The "forward" feature of the swap is that it begins at $t = 1$

– first payment is then at $t = 2$ since payments are made in arrears.

Question: What is the value, V_0, of the forward swap today at $t = 0$?

Derivative Prices Via Elementary Prices

Question: What is the value, V_0, of the forward swap today at $t = 0$?

Solution: Can use the forward equations to obtain

$$
\begin{aligned}
V_0 &= \frac{(.07 - .0938)}{1.0938} \times .2194 + \frac{(.07 - .0675)}{1.0675} \times .4432 + \frac{(.07 - .0486)}{1.0486} \times .2238 \\
&\quad + \frac{(.07 - .075)}{1.075} \times .4717 + \frac{(.07 - .054)}{1.054} \times .4717 \\
&= \$5,800.
\end{aligned}
$$

Foundations of Financial Engineering

Model Calibration

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Model Calibration

Until now have focused on pricing derivative securities

- so have taken the model and its parameters as given.

But a model is no good unless (at the very least!) the model prices agree with the market prices of liquid securities such as caps, floors, swaptions etc

- so need to calibrate the model to market prices.

There are too many free parameters, e.g. $r_{i,j}$, $q_{i,j}$ for all i, j

- so we fix some parameters, e.g. $q_{i,j} = q = 1 - q = 1/2$ for all i, j
- and assume some parametric form for the $r_{i,j}$'s.

For example: the Ho-Lee model assumes $r_{i,j} \; = \; a_i + b_i\, j$

- has only $2n$ parameters: a_i and b_i for $i = 0, \ldots, n-1$
- standard deviation of 1-period rate is $b_i/2$
- not a very realistic model
- but an influential model in the term structure literature.

Will focus instead on the Black-Derman-Toy (BDT) model.

The Black-Derman-Toy Model

The BDT model assumes that the interest rate at node $N_{i,j}$ is given by

$$r_{i,j} \;=\; a_i e^{b_i j}$$

where $\log(a_i)$ is a drift parameter for $\log(r)$ and b_i is a volatility parameter for $\log(r)$.

Need to calibrate the model to observed term-structure in the market

- and, ideally, other security prices.

Can do this by choosing the a_i's and b_i's to match market prices

- for relatively few periods, can do this in *Excel* using the *Solver* Add-In
- would use more efficient algorithms and software in general.

Calibrating BDT to the Observed Term-Structure

Let $(s_1, \ldots, s_n)$ be the term-structure of interest rates observed in the market

- assume (without loss of generality) spot rates are compounded per period

Will use an n-period binomial lattice and will assume for now (!) that $b_i = b$ is known for all i.

Calibrating BDT to the Observed Term-Structure

Then have

$$
\begin{aligned}
\frac{1}{(1+s_i)^i} &= \sum_{j=0}^{i} P^e_{i,j} \\
&= \frac{P^e_{i-1,0}}{2(1+a_{i-1})} + \sum_{j=1}^{i-1} \left(\frac{P^e_{i-1,j}}{2(1+a_{i-1}\,e^{bj})} + \frac{P^e_{i-1,j-1}}{2(1+a_{i-1}\,e^{b(j-1)})} \right) \\
&\quad + \frac{P^e_{i-1,i-1}}{2(1+a_{i-1}\,e^{b(i-1)}).}
\end{aligned}
$$

Can now solve iteratively for the a_i's.

Foundations of Financial Engineering

An Application: Pricing a Payer Swaption in a BDT Model

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Pricing a Payer Swaption in a BDT Model

Consider a $2-8$ payer swaption in the BDT model with fixed rate $= 11.65\%$. This is:

- An option to enter an 8-year swap in 2 years time
- Settled in arrears so payments would take place in years 3 through 10
- Each payment based on prevailing short-rate of previous year
- "Payer" feature of option means that if option is exercised, the exerciser "pays fixed and receives floating"
- Owner of a receiver swaption would "receive fixed and pay floating".

We therefore use a 10-period lattice with 1 period corresponding to 1 year.

Will also assume $b_i = b = .005$ for all i

- Is this a significant assumption?

The a_i's are calibrated to match the market term structure

– see *Excel* spreadsheet for calibration details.

Pricing the Swaption in the Calibrated BDT Model

Will assume a notional principal of $\$1$.

Let S_2 denote the value of swap at $t = 2$:

- Can compute S_2 by discounting cash-flows back from $t = 10$ to $t = 2$
- Recall it's easier to record time t cash flows at their predecessor nodes, discounting appropriately
  - hence there are no payments recorded at $t = 10$ in the swaption lattice.
- Option is then exercised if and only if $S_2 > 0$ so value at $t = 2$ is $\max(0, S_2)$.

Then discount S_2 backwards to $t = 0$ to find swaption price at $t = 0$.

Pricing the Swaption in the Calibrated BDT Model

When we calibrate to ZCB's in market place we find swaption price of $\$13,339$ when $b = .005$.

When we take $b = .01$ we find a swaption price of $\$19,497$

- but must remember to recalibrate model when we change b.

So see a significant difference in swaption prices

- even though both models, i.e. the model with $b = .005$ and the model with $b = .01$, were calibrated to the same ZCB prices.

This is not surprising: swaption prices clearly depend on model volatility

- which is controlled by b
- and not influenced at all by calibrating to ZCB prices.

Has very important implications for the calibration process in general

- want our calibration securities to be "close" to the securities we want to price with the calibrated model.

Foundations of Financial Engineering

Fixed Income Derivatives Pricing in Practice

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Fixed Income Derivatives Pricing in Practice

In practice more complex models than binomial models are used to price fixed income derivatives today.

But the pricing philosophy is the same:

1. Specify a model under the $\mathbb{Q}(\boldsymbol{\theta})$-dynamics
   – $\boldsymbol{\theta}$ is a vector of parameters, e.g. the a_i's and b_i's in Ho-Lee and BDT
2. Price all securities using

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}(\boldsymbol{\theta})}\left[\sum_{j=t+1}^{t+s} \frac{C_j}{B_j} + \frac{Z_{t+s}}{B_{t+s}}\right] \tag{7}$$

3. Now choose $\boldsymbol{\theta}$ so that market prices of appropriate liquid securities agree with model prices of those securities
   – this is the model calibration procedure.

Model Calibration in Practice

Calibration problem typically requires minimizing a sum of squares:

$$\min_{\boldsymbol{\theta}} \sum_i \omega_i \left(P_i(\text{model}) - P_i(\text{market})\right)^2 \;+\; \lambda \,||\boldsymbol{\theta} - \boldsymbol{\theta}_{\text{prev}}||^2 \tag{8}$$

where:

- $P_i(\text{model})$ is the model price of the i^{th} calibration security
- $P_i(\text{market})$ is the market price of the i^{th} calibration security
- ω_i is a positive weight reflecting the importance of the i^{th} security or the confidence we have in its market price.
- $\boldsymbol{\theta}_{\text{prev}} \equiv$ previously calibrated model parameters
- λ is a parameter reflecting relative importance of remaining close to previous calibration.

Once model has been calibrated, i.e. (8) has been solved to our "satisfaction", can use the model to hedge and price more exotic or illiquid securities.

Model Calibration in Practice

But (8) is often difficult to solve

– a non-convex optimization problem with many local minima.

As market conditions change, need to re-calibrate frequently

– often many times a day.

If the model was "right" would only need to calibrate once!

Markets are too complex and there is (unfortunately) no right model

– or even a model that is close to "being right".

Derivatives pricing in practice then is little more than using observable market prices to interpolate or extrapolate to price non-observable security prices

– but risk-neutral pricing at the model level implies that we can extrapolate / interpolate in an arbitrage-free manner.

True probabilities and market risk aversion enter the derivatives prices via the calibration process.