Multi-Market Cournot Equilibria with Heterogeneous Resource-Constrained Firms

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Abstract

We study Cournot competition among firms in a multi-market framework where each of the firms face different budget / capacity constraints. We assume independent linear inverse demand functions for each market, and completely characterize the resulting unique equilibrium. Specifically, we introduce the notions of *augmented* and *cutoff* budgets for firms and markets, respectively. We show, for example, that firm i operates in market j if and only if firm i's augmented budget is greater than market j's cutoff budget. We also study the properties of the equilibrium as a function of the number of firms N while keeping the aggregate budget fixed. In a numerical study, we show that increasing N increases the total output across all markets although this monotonicity can fail to hold at the individual market level. Similarly, we show that that while the firms' cumulative payoff decreases in N, the consumer surplus and social surplus increase in N.

Keywords: Cournot competition, non-cooperative games, heterogeneous products, heterogeneous firms, capacity constraints.

1 Introduction

In this note we study a model of Cournot competition in a multi-firm multi-market environment in which each of the firms face different budget or capacity constraints. (We will use budget and capacity interchangeably throughout.) In this setting, we completely characterize the resulting unique Cournot market equilibrium and investigate the effects that these capacity constraints have on the equilibrium outcome, including total output across markets, firms' payoffs and social welfare.

From a modeling standpoint, we adopt one of the most commonly used framework in the Cournot literature with independent linear inverse demand functions for each market and linear and symmetric cost functions (see Section 2 for further discussion of our modeling assumptions). At the same time we consider a fundamental – and curiously unexplored – extension of this standard model by explicitly incorporating firm-level budget constraints that limit the production decisions of the firms and couple the equilibrium outcome across markets. This novel feature of our model not only brings our analysis closer to reality but also produces new insights on the nature of the resulting Cournot equilibrium. For example, in the presence of budget constraints it is no longer true that every firm operates in every market or that there is a positive output on every market in equilibrium.

To formalize these ideas we introduce the notions of *augmented* and *cutoff* budgets for firms and markets, respectively (see Definition 3.1), which capture in a parsimonious way the intricate interactions among firms and markets. (To the best of our knowledge these are new concepts that have not been discussed previously in the literature). We show, for example, that in equilibrium firm i operates in market j if and only if firm i's augmented budget is greater than market j's cutoff budget. We also derive monotonicity results on the equilibrium market outputs with respect to both firm size (as measured by budget or capacity) and market size. We also study the properties of the equilibrium as a function of the number of firms N while keeping the aggregate budget fixed. In a numerical study, we show that increasing N increases the total output across all markets although this monotonicity can fail to hold at the individual market level. In other words, increasing competition can lead to some markets shutting down in equilibrium. Similarly, we show that that while the firms' cumulative payoff decreases in N, the consumer surplus and social surplus increase in N.

We conclude this introduction by reviewing some related work. While the literature on oligopolistic competition is vast, two main approaches have emerged: Cournot / quantity competition (Cournot, 1838), and Bertrand / price competition (Bertrand, 1883). In the former, firms compete by choosing what quantities to produce and prices are then determined so as to clear the market. In the latter, firms compete by setting prices after which consumers then decide what quantities to buy from each firm. The two approaches often result in very different outcomes and so the appropriate form of competition will depend on the particular industry and its characteristics. For example, Bertrand competition is more suitable for industries where quantities are easily adjusted in response to realized demand, e.g. digital goods such as software, video games or music. In contrast, Cournot competition is more suitable for industries where production (or production capacities) must be planned in advance, e.g. semiconductor manufacturing, electricity markets. There is a considerable body of work comparing the Bertrand and Cournot paradigms including, for example, Levitan and Shubik (1972) who characterize Cournot and Bertrand equilibria in a duopoly market with symmetric capacity-constrained firms and linear demand. Kreps and Scheinkman (1983) investigate the relationship between Cournot and Bertrand equilibrium in the context of a two-stage duopoly game in which firms decide on their level of production capacity in the first stage. Then in the second stage they engage in price competition but are constrained by their individual level of capacities from the first stage. They show that the Cournot outputs, i.e. the production quantities in a single-stage Cournot game, emerge as an equilibrium in this two-stage game; see also Moreno and Ubeda, 2006 for more recent work on this problem.

As mentioned earlier, the literature on oligopolistic competition is vast and we cannot do justice to it here. Instead we refer the reader to the textbooks Okuguchi and Szidarovszky (1999) and Vives (2001). They provide detailed treatments covering such questions as existence, uniqueness, and stability of equilibria, as well as properties of the equilibria and how they relate to market structures, etc.

Some work from the operations literature is particularly relevant, however, and concerns settings in which firms compete across multiple products and markets[1]. For example, the firms in Kluberg and Perakis (2012) produce multiple differentiated products and face asymmetric production constraints. Their model is more general in that the linear cost functions vary with firm and they allow more general affine inverse demand functions that allows for cross-market dependence in such a way that the products are gross substitutes for each other. However, most of their analysis assumes a single product per firm and they don't explicitly characterize the equilibrium. In contrast, in our model each of the firms can produce each product but the product markets are independent in the sense that a change in the quantity produced for one product has no impact on the clearing price for other products. Nonetheless the product markets are all coupled via the capacity constraints and we explicitly characterize the unique equilibrium. Motivated by problems in communications networks and airports, Perakis and Sun (2014) consider Cournot competition in service industries where the firms compete for users who are sensitive to both prices and congestion. They consider congestion with and without spillover costs and they quantify the efficiency of an unregulated oligopoly w.r.t the optimal social welfare. Other papers that also model competition across multiple markets include Allon and Federgruen (2009), Perakis and Roels (2007) and Federgruen and Hu (2015) but the form of competition in these papers is not Cournot and the firms are not budget / capacity constrained.

More recently, there has been work on Cournot competition across markets with a network structure. For example, Bimpikis et al. (2019) use a bipartite graph to model which subset of markets (of a homogeneous good) a firm can supply to. They characterize the unique Cournot equilibrium under a linear inverse demand function and relate it to supply paths in the underlying network structure. Related work includes Abolhassani et al. (2014) who study a more general version of Cournot competition in networked markets and Cai et al. (2019) who consider a similar problem but focus on the role of a market-maker in determining the resulting Cournot equilibrium and whether or not whether or not there is a unique equilibrium. Motivated by the operations of online platforms, Lin et al. (2017) contrast the market efficiency of open access versus discriminatory access platforms using a networked Cournot competition model. An important difference between our work and this existing literature on networked Cournot competition is that we explicitly impose

[1]We note that there exists an extensive operations management literature devoted to the study of Cournot equilibrium in a single market under various operational characteristics on the firms production function. Some representative examples include Deo and Corbett (2009), Downward et al. (2010) Jansen and Özaltin (2017), and references therein.

capacity constraints on the quantities that each firm can supply across markets.

The remainder of this note is organized as follows. In Section 2 we formulate the Cournot equilibrium problem. Then in Section 3 we introduce the notions of augmented and cutoff budgets before using them to fully characterize the equilibrium. In Section 4 we use our results from Section 3 to study the sensitivity of the equilibrium (aggregate output, consumer surplus, social welfare etc.) to the total number of firms whilst keeping the aggregate budget fixed. We conclude in Section 5 where we also outline some directions for future research.

2 Problem Formulation

We consider a setting where N firms engage in Cournot competition in M different markets. We assume that firm $i \in [N]$ is budget constrained and can spend no more than $B_i \geq 0$ dollars in total. (For a positive integer k, we let $[k] := \{1, 2, \ldots, k\}$). Firm i competes in market $j \in [M]$ by allocating an amount $x_{ij} \geq 0$ of its budget subject to the budget constraint $\sum_{j=1}^{M} x_{ij} \leq B_i$.

The profits that firm i makes on market j depend on its budget allocation x_{ij} a well as on the cumulative budget spent by all firms in the market. Specifically, we assume that firm i's profit in market j is equal to $r_j(X_j)\, x_{ij}$, where $X_j := \sum_{i=1}^{N} x_{ij}$ is the total budget allocated to market j by all N firms and the function $r_j(\cdot)$ models market's j return per unit of investment. Given a budget allocation $\{x_{ij}\}_{j\in[M]}$, firm i collects a net profit $\sum_{j=1}^{M} r_j(X_j)\, x_{ij}$. We will assume a linear demand model so that $r_j(x) = R_j - x/\beta_j$ for two positive parameters R_j and β_j for all $j \in [M]$. Without loss of generality, we rank the markets so that $0 \leq R_1 \leq R_2 \leq \cdots \leq R_M$ and we will refer to the pair (R_j, β_j) as the j^{th} market.

For future reference, we will denote by $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ the collection of markets, and by $\mathbf{B} = (B_1, \ldots, B_N)$ the vector of budgets. Again without loss of generality, we assume the elements in $\mathbf{B}$ have been ordered so that

$$B_1 \geq B_2 \geq \cdots \geq B_N > 0. \tag{1}$$

Let $X_{ij-} := X_j - x_{ij}$ be the total budget allocated to market j by all firms except firm i and define $\mathbf{X}_{i-} := (X_{i1-} \ \ldots \ X_{iM-})$. For a given value $\mathbf{X}_{i-}$, firm's i best-response budget-allocation strategy $\{x^*_{ij}(\mathbf{X}_{i-})\}_{j\in[M]}$ solves the optimization problem:

$$\Pi_i(\mathbf{X}_{i-}) := \max_{x_{ij}\geq 0} \sum_{j=1}^{M} r_j(X_{ij-} + x_{ij})\, x_{ij} \qquad \text{subject to} \qquad \sum_{j=1}^{M} x_{ij} \leq B_i. \tag{2}$$

Definition 2.1 (Cournot Equilibrium) *Consider a collection of markets $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ and a set of N firms with budgets $\mathbf{B} = (B_1, \ldots, B_N)$. A* Cournot equilibrium *is a set of budget allocations $\{x^*_{ij}\}_{j\in[M]}$ for $i \in [N]$ chosen by the firms so that:*

(i) *$\{x^*_{ij}\}_{j\in[M]}$ solves the optimization problem (2) for $i \in [N]$.*

(ii) *They satisfy the fixed-point condition:*

$$X_{ij-} = \sum_{k\neq i} x^*_{kj}(\mathbf{X}_{k-}) \qquad \textit{for all } i \in [N] \quad \textit{and} \quad j \in [M]. \tag{3}$$

Before proceeding to analyze the Cournot game and solving for its equilibrium, we describe several applications that can be modelled using this framework. In each case the players are assumed to be in Cournot competition.

1. Production Scheduling: There are N manufacturers who can produce M different products. The total production capacity of the i^{th} manufacturer is B_i and it allocates x_{ij} units of this capacity to the production of product j. Other settings are also possible. For example, we could consider the settings where B_i represents a *time* or *space* budget.

2. Airline Revenue Management: There are N airlines and a total of M routes under consideration. The i^{th} airline has a total of B_i passenger seats to allocate to the M routes. Then x_{ij} represents the number of seats allocated by the i^{th} airline to the j^{th} route (i.e., partition booking limits). Market competition in the airline industry has been studied in Brander and Zhang (1990), Kim and Singal (1993), Hu et al. (2012), Alves and Forte (2015) and references therein.

3. Intertemporal Competition with Exhaustible Resources: There are N firms each endowed with a finite amount of an exhaustible resource (e.g., crude oil or minerals). Firms compete by deciding the amount of resource (i.e., budget) that they want to put on the market over time. In other words, the notion of a market in our formulation corresponds to a particular period in time. With this interpretation, our model can be used to study dynamic Cournot games with limited production resources (e.g., Maskin and Tirole, 1987, Ludkovski and Sircar, 2012).

4. Financial Hedging in Supply Chains: There are N retailers who purchase a *single* product from a producer at time $t = 0$. There are M possible states of nature at time $t = 1$, each of which occurs with probability p_j, for $j = 1, \ldots, M$. The producer allows the ordering quantity to be state dependent with q_{ij} denoting the quantity purchased by retailer i in state j. The producer charges v_j per unit ordered in state j. A so-called *complete financial market* assumption allows the retailer to allocate (via, for example, a dynamic financial trading strategy) the budget B across the M states. This means the i^{th} retailer only has to satisfy the budget constraint *in expectation*, i.e. she must satisfy $\sum_{j=1}^{M} p_j v_j q_{ij} \leq B_i$ which becomes $\sum_{j=1}^{M} x_{ij} \leq B_i$ upon defining $x_{ij} := p_j v_j q_{ij}$. A similar adjustment can be made to cast each retailer's objective, i.e. their net expected revenue, in the form of the objective in (2). Further details are provided in Caldentey and Haugh (2021) who only consider the symmetric case where the budgets B_i are identical. They also allow for the possibility of the retailers using costly debt in addition to dynamic trading.

Discussion of Model Assumptions: One aspect of our model that deserves further discussion is our choice of linear inverse demand functions which are quite common in the oligopoly literature. See, for example, Levitan and Shubik (1972), Singh and Vives (1984), Szidarovszky and Okuguchi (1988), Bernstein and Federgruen (2004), Yao et al. (2008), Farahat and Perakis (2011), Kluberg and Perakis (2012) and Federgruen and Hu (2015) as well as the textbook by Vives (2001). Establishing existence and uniqueness of a Cournot equilibrium using these demand functions is often quite straightforward using standard techniques. Indeed establishing existence and uniqueness for more general concave inverse demand functions can often be tackled using the concave games framework of Rosen (1965).

Our choice of linear inverse demand functions is also restricted by the assumption of independent markets so that a change in the quantity produced for one market has no impact on the clearing price in other markets. A second restriction of our model is that we assume all of the firms have access to all of the markets. This contrasts, for example, with the network Cournot literature discussed in Section 1 where a bipartite graph (firms on one side and markets on the other) is used to model which firms can access which markets. A third restriction of our model relates to production costs. We can easily incorporate linear production costs that vary by market as long as they are identical across firms. (Such costs can be absorbed into the definition of the R_j's and so we don't need to model them explicitly here.) We cannot handle non-linear production costs, however, nor costs that vary by firm. Nonetheless, imposing these restrictions allow us to fully characterize the unique Cournot equilibrium despite the presence of capacity constraints. Indeed, to the best of our knowledge, we are the first to produce such a characterization in multi-market oligopoly games with capacity constraints. Moreover, our characterization independently establishes existence and uniqueness by construction.

Finally we note that we could also formulate our model via decision variables z_{ij} that represent the *quantity* of product j produced by firm i. In this case the budget constraint becomes $\sum_{j=1}^{M} w_j z_{ij} \leq B_i$ where w_j is the production cost (common to all firms) for producing one unit of product j. However, we can then easily reformulate the problem in terms of the dollar amounts $x_{ij} = w_j z_{ij}$ to obtain the problem formulation above. (In this case a w_j^2 term will be absorbed into the β_j term.) The advantage of working with dollar quantities x_{ij} and absorbing linear costs into the R_j terms is that this results in simple definitions of augmented and cutoff budgets in Section 3 which in turn allow for a much easier interpretation of our equilibrium characterization.

3 The Cournot Equilibrium

In this section we provide a complete derivation of the Cournot equilibrium. One of the main difficulties in finding a Cournot equilibrium in a multi-market setting when firms are budget constrained is that it is generally not the case that every firm operates in every market in equilibrium. For example, a low budget firm might be better off not competing in a market with a small market size to avoid an inefficient allocation of its budget.

A key step in the derivation is that we can provide a relatively simple characterization of the markets in which a firm operates by introducing the concepts of '*augmented*' and '*cutoff*' budgets.

Definition 3.1 (Augmented and Cutoff Budgets)

(i) *For a vector of firms' budgets* $\mathbf{B} = (B_1, \ldots, B_N)$, *the* augmented budget *associated to firm* i *is given by*

$$\mathcal{B}_i := i\, B_i + \sum_{k=i}^{N} B_k, \qquad \text{for all } i \in [N]. \tag{4}$$

(ii) *For a set of linear markets* $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$*, the* cutoff budget *of market* j *is given by*

$$\mathbb{B}_j := \sum_{k=j}^{M} \beta_k \,(R_k - R_j), \qquad \text{for all } j \in [M]. \tag{5}$$

We also define $\mathbb{B}_0 := \sum_{k=1}^{M} \beta_k \, R_k$.

Recall that the firms have been ordered so that $B_1 \geq B_2 \geq \ldots \geq B_N$. It follows that the sequence $\{\mathcal{B}_i\}$ is also non-increasing in i. It is also worth noting that $\mathcal{B}_i$ does not depend on $\{B_1, \ldots, B_{i-1}\}$, i.e., on the $(i-1)$ highest budgets. Similarly, since the markets have been ordered so that $0 \leq R_1 \leq \cdots \leq R_M$, it follows that the $\{\mathbb{B}_j\}$'s are non-increasing in j and the value of $\mathbb{B}_j$ is independent of the characteristics of markets $\{1, 2, \ldots, j-1\}$.

As we shall see, the significance of the augmented and cutoff budgets is that in equilibrium, firm i operates in market j if and only if $\mathcal{B}_i > \mathbb{B}_j$. Before we can formally state our equilibrium result, one further definition is needed.

Definition 3.2 *For a given collection of linear markets* $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$*, we define the function* $H(x)$ *according to*

$$H(x) := \inf \Big\{ z \geq 0 \ \ \text{such that} \ \ \sum_{j=1}^{M} \beta_j \,(R_j - z)^+ \leq x \Big\}, \quad \text{for any } x \geq 0.$$

We note that $H(\cdot)$ is a continuous, non-increasing and piece-wise linear function that satisfies $H(\mathbb{B}_j) = R_j$ for all $j \in [M]$. We are now ready to state our main result.

Theorem 1 (Cournot Equilibrium)

Given a collection of linear markets $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ *and a set of firms with budgets* $\mathbf{B} = (B_1, \ldots, B_N)$ *satisfying the conditions in (1), the Cournot equilibrium* x^*_{ij} *satisfy*

$$x^*_{ij} = \beta_j \left[\frac{R_j}{1 + n^*_j} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+, \tag{6}$$

where the $\{\mathcal{B}_i\}$*'s are the firms' augmented budgets defined in equation (4) and*

$$n^*_j := \max \big\{ i \in [N] \ \text{such that} \ \ \mathcal{B}_i > \mathbb{B}_j \big\} \tag{7}$$

is the number of firms operating in market j *where the* $\{\mathbb{B}_j\}$*'s are the markets' cutoff budgets defined in (5).*

It is interesting to note that (6) implies that firm i's allocations x^*_{ij} depend on the vector of budgets $\mathbf{B} = (B_1, \ldots, B_N)$ only through the values of $(\mathcal{B}_i, \mathcal{B}_{i+1}, \ldots, \mathcal{B}_N)$, which according to (4) are all independent of the value of the highest $i-1$ budgets $(B_1, B_2, \ldots, B_{i-1})$. At the same time, $(\mathcal{B}_i, \mathcal{B}_{i+1}, \ldots, \mathcal{B}_N)$ do depend on i, i.e. on the *number* of firms that have a budget greater than or equal to firm i's budget. In other words, the allocation decisions of a firm are unaffected by the size (but not the number) of larger firms.

An interesting special case is the symmetric case where we have $B_i = B$ for all i. In this case $\mathcal{B}_i = (N+1)B$ and the expression for x^*_{ij} in (6) also simplifies considerably. In particular, since $H(\mathcal{B}_k) = H((N+1)B)$ is independent of k, both summations in (6) can be expressed as telescoping sums and considerable simplification occurs. This leads to the following corollary.

Corollary 1 (Symmetric Firms) *Suppose the N firms are identical with $B_i = B$ for all $i \in [N]$. Then in equilibrium*

$$x^*_{ij} \;=\; \frac{\beta_j (R_j - H((N+1)B))^+}{1+N}. \tag{8}$$

The following proposition builds on Theorem 1. It (i) establishes monotonicity properties of the equilibrium ordering quantities and (ii) highlights the central role played by the augmented and cutoff budgets in the equilibrium.

Proposition 1 *Given a collection of linear markets $\boldsymbol{\mathcal{M}} := \{(R_j, \beta_j) \colon j \in [M]\}$ and a set of firms with budgets $\mathbf{B} = (B_1, \ldots, B_N)$ ordered as in (1), the Cournot equilibrium satisfies:*

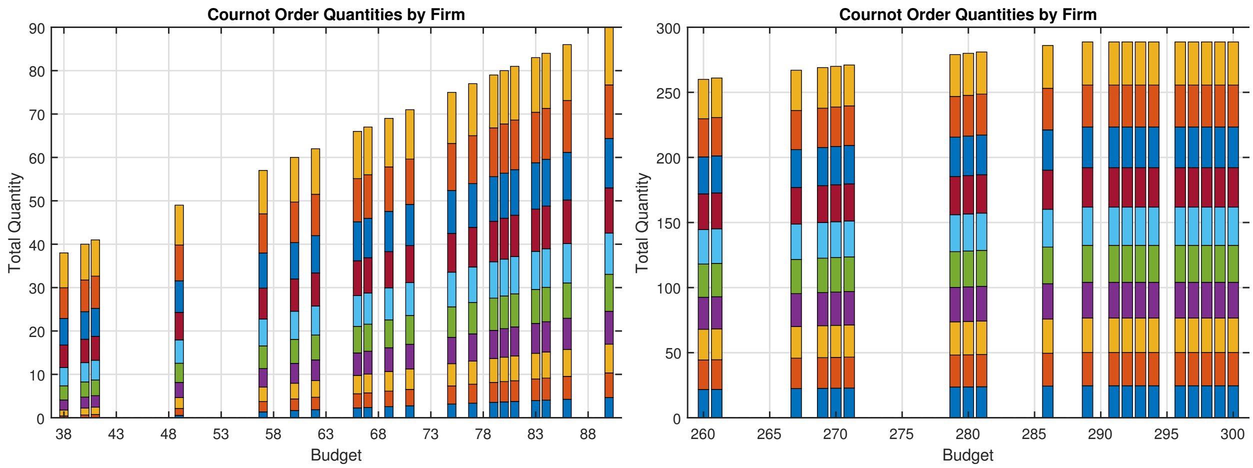
(i) $x^*_{ij} > 0$, *i.e. firm i operates in market j, if and only if $\mathcal{B}_i > \mathbb{B}_j$. In particular, $x^*_{iM} > 0$ for all $i \in [N]$ since $\mathbb{B}_M = 0$ (see equation 5). In addition, x^*_{ij} is non-increasing in i and x^*_{ij}/β_j is non-decreasing in j.*

(ii) $X^*_j = \sum_{i=1}^{N} x^*_{ij} > 0$, *i.e. market j is active, if and only if $\mathcal{B}_1 > \mathbb{B}_j$. Furthermore,*

$$X^*_j = \beta_j \left[\frac{n^*_j R_j}{1+n^*_j} - \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} \right].$$

(iii) $\sum_{j=1}^{M} x^*_{ij} = B_i$, *i.e. firm i's budget constraint is binding, if and only if $\mathcal{B}_i \le \mathbb{B}_0$.*

Figure 1 depicts the Cournot equilibrium ordering quantities for $M = 10$ markets and $N = 20$ firms. In panel (a) all budgets are binding whereas in panel (b) only the smallest ten budgets are binding. Different bars correspond to different firms and different colors correspond to different markets. Moreover, we assumed the β_j's were constant w.r.t j and so the monotonicity w.r.t both i and j from part (i) of Proposition 1 is evident. In particular, (i) we see firms increase their allocations to each market as their budgets increase and (ii) for a fixed budget, i.e. firm, we see the allocation to market j increases in j. (For a fixed budget, the j^{th} segment in the bar corresponds to the j^{th} market.)



(a) All budgets binding. (b) Smallest ten budgets binding.

Figure 1: Cournot equilibrium ordering quantities for $M = 10$ markets and $N = 20$ firms. In Figure 1(a) all budgets are binding whereas in Figure 1(b) only the first ten firms have their budgets binding. Other details are provided in the main text.

4 Sensitivity Analysis on the Number of Firms

We now investigate how the outcome of the Cournot equilibrium in Theorem 1 changes as a function of the total number of firms N. To this end, let us denote by $B_i(N)$ and $X_j^*(N)$ the budget of firm $i \in [N]$ and the equilibrium output in market $j \in [M]$, respectively, when there are N competing firms. In order to have a meaningful comparison of how $X_j^*(N)$ varies with N, we assume that the cumulative budget $B_{\mathrm{C}} := \sum_{i=1}^{N} B_i(N)$ is constant and therefore independent of N.

So as to capture different budget distributions across the N firms, we will assume the budget of firm i (when there are N firms in the market) is equal to

$$B_i^{\mathrm{N}} = \left[f\left(\frac{i-1}{N}\right) - f\left(\frac{i}{N}\right) \right] B_{\mathrm{C}}, \qquad i \in [N], \tag{9}$$

for some given non-decreasing and convex function $f : [0,1] \to [0,1]$ with $f(0) = 1$ and $f(1) = 0$. The monotonicity and convexity of f guarantee that $B_1^{\mathrm{N}} \geq B_2^{\mathrm{N}} \geq \cdots \geq B_N^{\mathrm{N}}$ as assumed in (1).

Combining equations (9) and (4) we see that the augmented budget of firm i equals

$$\mathcal{B}_i^{\mathrm{N}} = \left[(i+1)\, f\left(\frac{i-1}{N}\right) - i\, f\left(\frac{i}{N}\right) \right] B_{\mathrm{C}}.$$

As a concrete example, consider the exponential family of functions defined as

$$\mathcal{F} := \left\{ f_\alpha(x) = (e^{-\alpha x} - e^{-\alpha})/(1 - e^{-\alpha}) \colon \alpha \geq 0 \right\}.$$

The value of α controls the degree of heterogeneity in the distribution of B_{C} across the firms. On one extreme we have $\lim_{\alpha \downarrow 0} B_i^\alpha(N) = B_{\mathrm{C}}/N$ so that the cumulative budget is uniformly distributed across firms as α approaches 0. On the other extreme we have $\lim_{\alpha \to \infty} B_i^\alpha = B_{\mathrm{C}}\, \mathbb{1}(i = 1)$ so that the cumulative budget is allocated entirely to firm 1 in the limit as α goes to infinity.

The six panels in Figure 2 depict aggregate Cournot equilibrium outputs as a function of the number of firms (N) when there are $M = 10$ markets for each value of $\alpha \in \{0, 5, 10, \infty\}$. For the three panels on the top row we set $B_{\mathrm{C}} = 0.9\,\mathbb{B}_0$ while for the three panels in the bottom row we set $B_{\mathrm{C}} = 1.1\,\mathbb{B}_0$. (Recall from part (iii) of Proposition 1 that firm i's budget is binding if and only if $\mathcal{B}_i \leq \mathbb{B}_0$. This means the "average" firm has a binding budget constraint in the top row of panels and a non-binding budget constraint in the bottom row of panels.) The two panels on the left display the total output across all M markets, i.e. $\sum_{j=1}^{M} X_j^*$. The middle and right panels depict the values of X_1^* and X_M^*, respectively.

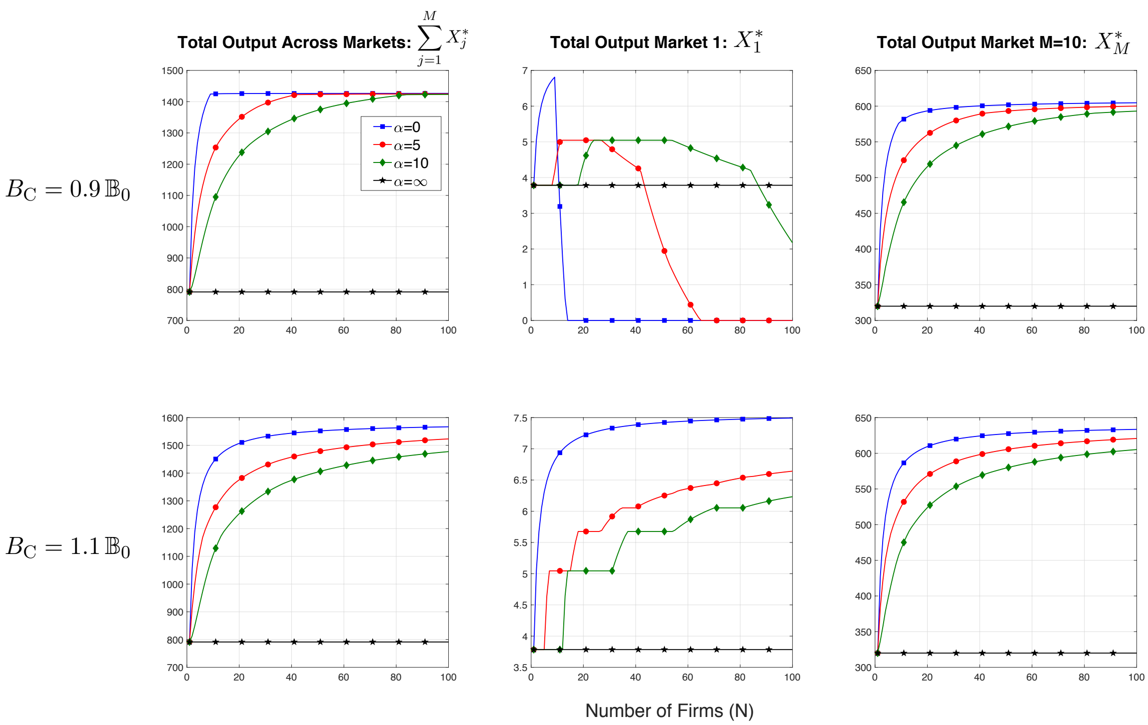


Figure 2: Aggregate Cournot equilibrium outputs for the case in which $M = 10$ for four distributions of budgets corresponding to $\alpha \in \{0, 5, 10, \infty\}$ as function of the number of firms (N).

We see that increasing the value of N, i.e. increasing competition among firms, increases the total output across all markets. (When $\alpha = \infty$ there is effectively only 1 firm as explained above so in this case the total output is independent of N.) The effect of competition on the output of a particular market can be positive or negative, however. For example, when $B_{\mathrm{C}} = 0.9\,\mathbb{B}_0$, the total output X_1^* in market 1 can decrease and even become zero as N increases. Similarly, the more uniformly distributed the total budget B_{C} is among the N firms (corresponding to smaller values of α), the larger the cumulative output $\sum_{j=1}^{M} X_j^*$. However, this aggregate monotonicity does not hold across each of the individual markets. This is demonstrated by the totals for X_1^* in the middle panel of Figure 2 when $B_{\mathrm{C}} = 0.9\,\mathbb{B}_0$. The results in the figure also suggest that – except in the extreme case when $\alpha = \infty$ – the equilibrium outputs in every market converge to the same limit as $N \to \infty$ independently of the value of α. We formalize this observation in the following proposition under the additional condition $\lim_{N\to\infty} B_1(N) = 0$, which is satisfied for the exponential family above for every $\alpha < \infty$. In other words, we consider an asymptotic regime with a very large number of small firms that collectively have a fixed cumulative budget B_{C}.

Proposition 2 *Consider the asymptotic regime in which the number of firms N goes to infinity and $\lim_{N\to\infty} B_1(N) = 0$. Then*

$$\lim_{N\to\infty} X_j^*(N) = \beta_j \left(R_j - H(B_{\mathrm{C}})\right)^+ \qquad \textit{for all } j \in [M].$$

It follows that $\lim_{N\to\infty} X_j^*(N) = 0$ for all $j < k^* := \min\{j \geq 1 \colon \mathbb{B}_j < B_{\mathrm{C}}\}$.

It is worth noting that the limiting output quantities in the previous proposition are such that the margin $r_j(X_j^*)$ is constant and equal to $H(B_{\mathrm{C}})$ for all $j \geq k^*$. Intuitively, in the limit as $N \to \infty$, each firm becomes infinitesimally small and their individual strategies have no impact on a market's return.

We next look at how the value of N and the distribution of budgets across the firms impact the firms' payoffs and consumers' surplus. To this end, we denote by

$$\Pi_{\mathrm{C}} := \sum_{i=1}^{N} \Pi_i = \sum_{j=1}^{M} \left(R_j - \frac{X_j^*}{\beta}\right) X_j^*$$

the firms' cumulative payoff across all markets. Similarly, we define

$$\mathcal{S} := \sum_{j=1}^{M} \frac{(X_j^*)^2}{2\,\beta_j}$$

to be the consumers' total surplus across all markets[2]. Finally, we define the social surplus across all markets to be $\mathcal{W} := \Pi_{\mathrm{C}} + \mathcal{S}$.

Figure 3 depicts the values of Π_{C} (left panel), $\mathcal{S}$ (middle panel) and $\mathcal{W}$ (right panel) as a function of N for four distributions of the total budget B_{C} across firms using (9) and the exponential family with $\alpha \in \{0, 5, 10, \infty\}$. The dashed-line on each panel corresponds to the solution that a social planner would obtain by solving the aggregate social surplus maximization problem:

$$\max_{X_j \geq 0} \sum_{j=1}^{M} \left(R_j - \frac{X_j}{\beta_j}\right) X_j + \sum_{j=1}^{M} \frac{(X_j)^2}{2\,\beta_j} \qquad \text{subject to} \quad \sum_{j=1}^{M} X_j \leq B_{\mathrm{C}}.$$

As we can see from the figure, and except for the limiting case with $\alpha = \infty$, the firms' total payoff decreases in N while the consumers' surplus increases. In aggregate, the net effect is that the social surplus increases with N. Also, these effects are more pronounced for small value of α, i.e., as the cumulative budget B_{C} is more uniformly distributed across firms.

Remark 1 We note that Theorem 1 and Proposition 1 can be used to do some quick and basic analysis of mergers, a topic of interest to many researchers e.g. Bimpikis et al. (2019). For example, suppose two firms merge and the augmented budget of the merged firm is smaller than the cutoff budget of the

[2] Recall that the consumers' surplus in a market is the area under the demand curve and above a horizontal line at the equilibrium price.

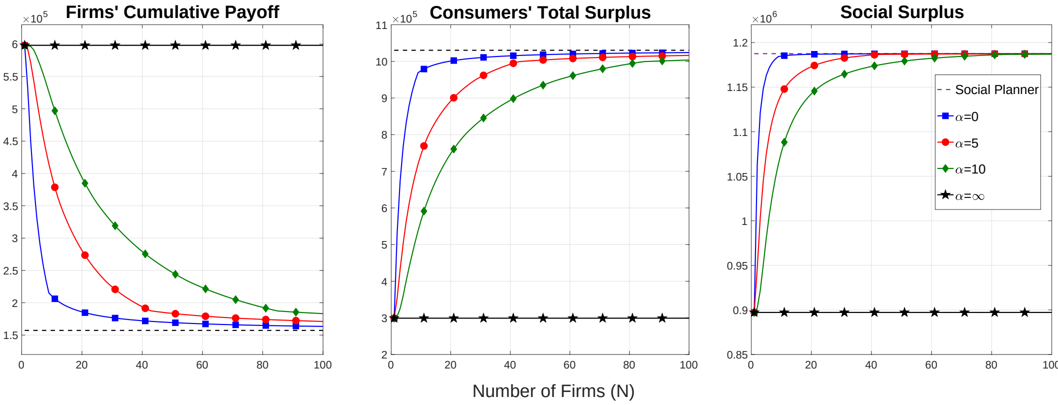


Figure 3: Firms' cumulative payoff (Π_{C}), consumers' total surplus ($\mathcal{S}$) and social surplus ($\mathcal{W}$) as a function of N for four distributions of the total budget B_{C} across firms using (9) and the exponential family with $\alpha \in \{0, 5, 10, \infty\}$. The dashed-line on each panel corresponds to the social planner solution. Each plot corresponds to the average over 100 simulations in which the demand function $r_j(X_j) = R_j - X_j/\beta_j$ on each market $j \in [M]$ was randomly generated with $R_j \sim \mathrm{Uniform}[0, 1000]$ and $\beta_j \sim -\log(\mathrm{Uniform}[0, 1])$. The total budget B_{C} on each simulation was set at $B_{\mathrm{C}} = 0.9\,\mathbb{B}_0$.

j^{th} market. Then it follows from (7) that n_j^* does not change post-merger. It then follows from (6) that the equilibrium ordering quantities in the j^{th} market of all firms which are bigger than the merged firm are unchanged post-merger. $\square$

5 Conclusions

In this note we have considered a Cournot competition model where a number of firms compete on quantities across a number of independent product markets. We assume independent linear inverse-demand functions and that each of the firms are budget constrained. Together these assumptions allow us to define an explicit ordering of the firms and markets, and define the notions of augmented and cutoff budgets. We then characterize the unique Cournot equilibrium in terms of these augmented and cutoff budgets.

There are several potential directions for further research. One natural extension is to extend our model to incorporate a network (or compatibility) structure that restricts the set of markets in which a firm can participate. Another interesting direction would be to consider more general (concave) inverse-demand functions for the different markets and still provide a (possibly partial) characterization of the equilibrium. Some preliminary analysis that we have done in this direction suggests that one of the key properties that we use in our proof of Theorem 1 –namely, the fact that we can order the firms and markets to define augmented and cutoff budgets– would not necessarily hold under a more general demand model. (It should be possible to confirm existence and uniqueness using the results of Rosen, 1965.) Another potential direction is the question of mergers as discussed in Remark 1 of Section 4. Finally, we also observed in Section 4 that the consumer surplus and social surplus increased in the number of firms N whilst keeping the total aggregate budget fixed. While we witnessed this behavior numerically for a broad range of budget distributions (as characterized by the parameter α), it would be instructive to establish some of these results more rigorously.

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A Appendix: Proofs

Proof of Theorem 1: We start by characterizing the best response that firm i uses as a function of the strategies of the other firms. Taking $\mathbf{X}_{i-}$ as fixed, it is straightforward to obtain that the optimal solution to (2) satisfies

$$x_{ij} = \frac{(\beta_j\, R_j - \beta_j\, \lambda_i - X_{ij-})^+}{2}, \tag{A-1}$$

where $\lambda_i \geq 0$ is the Lagrange multiplier corresponding to the i^{th} firm's budget constraint. In particular, $\lambda_i \geq 0$ is the smallest real such that $\sum_{j=1}^{M} x_{ij} \leq B_i$. Given the ordering of the budgets B_i, it follows that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ when they are chosen optimally. Equation (A-1) and the ordering of the Lagrange multipliers then implies that for each market j, there is a number $n_j \in \{0, 1, \ldots, N\}$ such that $x_{ij} = 0$ for all $i > n_j$. In other words, n_j is the number of firms that operate in market j. We therefore obtain the following system of equations

$$x_{ij} \;\; = \;\; \beta_j\, R_j - \beta_j\, \lambda_i - X_j, \qquad \text{for} \quad i = 1, \ldots, n_j \tag{A-2}$$

where $X_j = \sum_{i=1}^{n_j} x_{ij}$ is the total budget allocation in market j. For each $j = 1, \ldots, M$, this is a system with n_j linear equations in n_j unknowns which we can easily solve once the n_j's are known. Summing the x_{ij}'s from $i = 1$ to n_j we obtain

$$X_j = \frac{\beta_j}{n_j + 1} \left[n_j\, R_j - \sum_{i=1}^{n_j} \lambda_i \right]. \tag{A-3}$$

Substituting this value of X_j into (A-2), and using the fact that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$, we see the optimal ordering quantities x_{ij} for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ satisfy

$$x_{ij} = \frac{\beta_j}{n_j + 1} \left[R_j - \lambda_i\, (n_j + 1) + \sum_{k=1}^{n_j} \lambda_k \right]^+. \tag{A-4}$$

To complete the characterization of the Cournot equilibrium we must compute the values of the Lagrange multipliers $\{\lambda_i,\ i = 1, \ldots, N\}$ as well as the n_j's. For reasons that will soon become apparent, it will be convenient to replace the Lagrange multipliers by an equivalent set of unknowns $\{\alpha_i,\ i = 1, \ldots, N\}$ that we define below.

Suppose $x_{ij} = 0$ for some market j. Then (A-1) implies $\beta_j\, R_j - \beta_j\, \lambda_i - X_j \leq 0$ which, after substituting for X_j using (A-3), implies that

$$R_j \leq \lambda_i\, (1 + n_j) - \sum_{k=1}^{n_j} \lambda_k. \tag{A-5}$$

It follows that n_j depends on j only through the value of R_j, that is, $n_j = n_j(R_j)$. If we replace R_j and n_j in (A-5) with a generic $R \in [0, \infty)$ and $n(R)$, respectively, we can define α_i to be that value of R where the i^{th} retailer moves from ordering zero to ordering a positive quantity. It therefore satisfies

$$\alpha_i = \lambda_i\,(1 + n(R)) - \sum_{k=1}^{n(R)} \lambda_k \tag{A-6}$$

and we note the α_i's are non-decreasing in i. Abusing notation slightly, we see[3] that $n(\alpha_i) = i - 1$ and so (A-6) implies

$$\alpha_i = i\,\lambda_i - \sum_{k=1}^{i-1} \lambda_k \quad \text{ for } \;i = 1, \ldots, N. \tag{A-7}$$

Using (A-7) recursively, one can show that

$$\lambda_i = \frac{\alpha_i}{i} + \sum_{k=1}^{i-1} \frac{\alpha_k}{k\,(k+1)}. \tag{A-8}$$

Substituting this expression in (A-4), it follows that for all $i = 1, \ldots, N$

$$\begin{aligned}
x_{ij} &= \beta_j \left[\frac{R_j}{1+n_j} - \lambda_i + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j}\right]^+ = \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i} + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} - \sum_{k=1}^{i-1} \frac{\alpha_k}{k\,(k+1)}\right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i+1} + \sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} - \sum_{k=1}^{i} \frac{\alpha_k}{k\,(k+1)}\right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j} - \frac{\alpha_i}{i+1} + \sum_{k=1}^{n_j} \frac{\alpha_k}{k(k+1)} - \sum_{k=1}^{i} \frac{\alpha_k}{k\,(k+1)}\right]^+ \qquad \text{(A-9)}
\end{aligned}$$

where (A-9) follows from the identity

$$\sum_{k=1}^{n_j} \frac{\lambda_k}{1+n_j} = \frac{1}{1+n_j} \sum_{k=1}^{n_j} \left(\frac{\alpha_k}{k} + \sum_{l=1}^{k-1} \frac{\alpha_l}{l(l+1)}\right) = \sum_{k=1}^{n_j} \frac{\alpha_k}{k(k+1)}$$

which in turns follows from (A-8) and reversing the order of summation. It should be clear from the discussion above that

$$n_j = \max\left\{i \in \{0\} \cup [N] \text{ such that } \alpha_i \leq R_j\right\} \tag{A-10}$$

where $\alpha_0 := 0$ and we therefore only need to derive the values of the α_i's to complete the proof of the theorem.

From (A-8) and the ordering of the firms' budgets, it follows that if firm i's budget constraint is not binding then firm k's budget constraint is also not binding, for all $k = 1, \ldots, i$. As a result, if $\sum_{j=1}^{M} x_{ij} < B_i$ then $\lambda_k = 0$ for all $k = 1, \ldots, i$ and (A-8) implies that $\alpha_k = 0$ for all $k = 1, \ldots, i$. Hence, if $\alpha_i > 0$ the budget constraint is binding and $\sum_{j=1}^{M} x_{ij} = B_i$ as required.

To complete the proof, we need to show that α_i is equal to $H(\mathcal{B}_i)$ for all $i \in [N]$. To this end, we

[3] We are assuming that the N budgets are distinct so that $B_{k-1} > B_k$. This then implies $x_i(\alpha_k) > 0$ for all $i \leq k-1$. The case where some budgets coincide is straightforward to handle. We also emphasize that the α_i's need not be in the range $[\min_j R_j, \max_j R_j]$.

make use of Lemma 1 stated immediately after this proof. Setting V_j set to 1 in the lemma implies

$$\sum_{j=1}^{M} x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} x_{lj} = \sum_{j\,:\,\alpha_i<R_j} \frac{\beta_j}{i+1} \left(R_j - \alpha_i\right) \tag{A-11}$$

for $i = 1, \ldots, N$. But the budget constraints for the N firms also imply

$$\sum_{j=1}^{M} x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} x_{lj} \le B_i + \frac{1}{i+1} \sum_{l=i+1}^{N} B_l \tag{A-12}$$

which, when combined with (A-11), leads to

$$\sum_{j\,:\,\alpha_i<R_j} \beta_j \left(R_j - \alpha_i\right) \le (i+1)\, B_i + \sum_{l=i+1}^{N} B_l = \mathcal{B}_i \tag{A-13}$$

for $i = 1, \ldots, N$. We can use (A-13) sequentially to determine the α_i's. Beginning at $i = N$, we see that the N^{th} firm's budget constraint is equivalent to

$$\sum_{j\,:\,\alpha_N<R_j} \beta_j \left(R_j - \alpha_N\right) \le \mathcal{B}_N. \tag{A-14}$$

The optimality condition on λ_i implies that it is the smallest non-negative real that satisfies the i^{th} budget constraint. Since the optimal λ_i's are non-decreasing in i, we see from (A-12) that α_i is therefore the smallest real greater than or equal to 0 satisfying the i^{th} budget constraint. Therefore, beginning with $i = N$ we can check if $\alpha_N = 0$ satisfies (A-14) and if it does, then we know the N^{th} budget constraint is not binding. If $\alpha_N = 0$ does not satisfy (A-14) then we set α_N equal to that value (greater than one) that makes (A-14) an equality. In particular, we obtain that the optimal value of α_N is $H(\mathcal{B}_N)$, as desired.

Note that if $\alpha_N = 0$ then none of the budget constraints are binding. In particular, this implies $\alpha_i = 0$ and $\lambda_i = 0$ for all $i = 1, \ldots N$. Moreover, $\alpha_i = H(\mathcal{B}_i)$ must be satisfied for all i since it is true for $i = N$ and since the B_i's are decreasing. Suppose now that the budget constraint is binding for firms $i + 1, \ldots, N$ and consider the i^{th} firm. Then the i^{th} firm's budget constraint is equivalent[4] to (A-13) and we can again use precisely the same argument as before to argue that $\alpha_i = H(\mathcal{B}_i)$ holds. □

We now state and prove the lemma that we used for proving Theorem 1.

Lemma 1 *Consider a collection of linear markets* $\boldsymbol{\mathcal{M}} := \{R_j - x/\beta_j \colon j \in [M]\}$ *and the budget allocations* $\{x_{ij}\}$ *in (A-9). Then, for any vector* $(V_1, \ldots, V_M)$ *we have*

$$(i+1) \sum_{j=1}^{M} V_j\, x_{ij} + \sum_{k=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{kj} = \sum_{j\,:\,\alpha_i<R_j} V_j\, \beta_j \left(R_j - \alpha_i\right). \tag{A-15}$$

[4]Equivalence follows because the second terms on either side of the inequality sign in (A-12) are equal by assumption.

Proof of Lemma 1: We first recall that the α_i's are non-decreasing in i. We also note that $n(\alpha_j) = k-1$ for all α_j satisfying $\alpha_{k-1} \leq \alpha_j < \alpha_k$ where j indexes products and k indexes retailers. Setting $\alpha_{N+1} := \infty$ and noting that $x_{ij} = 0$ if $\alpha_i \geq \alpha_j$, we can combine these results and (A-4) to write

$$x_{ij} = \sum_{k=i}^{N} \frac{\beta_j}{k+1} \left[R_j - (k+1)\,\lambda_i + \sum_{j=1}^{k} \lambda_j \right] \mathbb{1}\Big(\alpha_j \in [\alpha_k, \alpha_{k+1})\Big). \tag{A-16}$$

Letting $\Omega_k := \{j : \ \alpha_k \leq \alpha_j < \alpha_{k+1}\}$, we see that (A-16) implies

$$\begin{aligned}
\sum_{l=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{lj} &= \sum_{l=i+1}^{N} \sum_{k=l}^{N} \sum_{j\in\Omega_k} \frac{V_j\,\beta_j}{k+1} \left[R_j - (k+1)\,\lambda_l + \sum_{s=1}^{k} \lambda_s \right] \\
&= \sum_{k=i+1}^{N} \sum_{l=i+1}^{k} \sum_{j\in\Omega_k} \frac{V_j\,\beta_j}{k+1} \left[R_j - (k+1)\,\lambda_l + \sum_{s=1}^{k} \lambda_s \right] \\
&= \sum_{k=i+1}^{N} \sum_{j\in\Omega_k} \frac{V_j\,\beta_j}{k+1} \left[(k-i)\,R_j - (k+1) \sum_{l=i+1}^{k} \lambda_l + (k-i) \sum_{s=1}^{k} \lambda_s \right].
\end{aligned} \tag{A-17}$$

(A-16) also implies

$$\sum_{j=1}^{M} V_j\, x_{ij} = \sum_{k=i}^{N} \sum_{j\in\Omega_k} \frac{V_j\,\beta_j}{k+1} \Big[R_j - (k+1)\,\lambda_i + \sum_{l=1}^{k} \lambda_l \Big]. \tag{A-18}$$

If we use the convention $\sum_{l=i+1}^{i} \lambda_l = 0$, then the first sum on the right-hand-side of (A-17) can run from $k = i$ to N instead of $k = i+1$ to N. We can then add $1/(i+1)$ times (A-17) with (A-18) to obtain

$$\sum_{j=1}^{M} V_j\, x_{ij} + \frac{1}{i+1} \sum_{l=i+1}^{N} \sum_{j=1}^{M} V_j\, x_{lj} = \sum_{k=i}^{N} \sum_{j\in\Omega_k} \frac{V_j\,\beta_j}{k+1}\, Z_{ikj} \tag{A-19}$$

where

$$Z_{ikj} := \left[R_j - (k+1)\,\lambda_i + \sum_{l=1}^{k} \lambda_l + \frac{(k-i)\,R_j - (k+1)\, \sum_{l=i+1}^{k} \lambda_l + (k-i)\, \sum_{s=1}^{k} \lambda_s}{i+1} \right].$$

Some straightforward algebra together with (A-7) can be used to show

$$Z_{ikj} = \frac{k+1}{i+1}\,(R_j - \alpha_i)$$

and so by the definition of Ω_k we can substitute for Z_{ikj} in (A-19) and obtain (A-15). $\square$

Proof of Proposition 1:

(i) Let us prove that $x^*_{ij} > 0$ if and only if $\mathcal{B}_i > \mathbb{B}_j$. First suppose that $\mathcal{B}_i > \mathbb{B}_j$. It follows from (7) that $n_j^* \geq i$ and (6) implies that

$$x_{ij}^* = \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=i+1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \geq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=i+1}^{n_j^*} \frac{H(\mathcal{B}_i)}{k(k+1)} \right]^+$$
$$= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + H(\mathcal{B}_i) \left(\frac{1}{i+1} - \frac{1}{n_j^*+1} \right) \right]^+$$
$$= \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{n_j^*+1} \right]^+ > 0,$$

where the weak inequality uses the facts that the sequence $\{\mathcal{B}_i\}$ is non-increasing in i and the function $H(\cdot)$ is non-increasing and the strict inequality follows from the fact that $R_j = H(\mathbb{B}_j)$ and the assumption $\mathcal{B}_i > \mathbb{B}_j$.

Let us suppose now that $\mathcal{B}_i \leq \mathbb{B}_j$. In this case (7) implies that $n_j^* \leq i-1$ and (6) leads to

$$x_{ij}^* = \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} - \sum_{k=n_j^*+1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \leq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{n_j^*+1})}{i+1} - \sum_{k=n_j^*+1}^{i} \frac{H(\mathcal{B}_{n_j^*+1})}{k(k+1)} \right]^+$$
$$= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{n_j^*+1})}{i+1} - H(\mathcal{B}_{n_j^*+1}) \left(\frac{1}{n_j^*+1} - \frac{1}{i+1} \right) \right]^+$$
$$= \beta_j \left[\frac{R_j - H(\mathcal{B}_{n_j^*+1})}{n_j^*+1} \right]^+ = 0,$$

where the last equality follows $n_j^* + 1 \leq i$ and so $H(\mathcal{B}_{n_j^*+1}) \geq H(\mathcal{B}_i) \geq H(\mathbb{B}_j) = R_j$.

To prove that x_{ij}^* is non-increasing in i note that

$$\begin{aligned}
x_{ij}^* &= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)} \right]^+ \\
&\geq \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)} \right]^+ \\
&= \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ = x_{i+1j}^*.
\end{aligned}$$

Finally, the fact that x_{ij}^*/β_j is non-decreasing in j follows trivially from (6) and the fact that n_j^* is non-decreasing in j.

(ii) From part (i) we have that $x_{ij}^* > 0$ if and only if $i \le n_j^*$. It follows that

$$
\begin{aligned}
X_j^* &= \sum_{i=1}^{N} x_{ij}^* = \sum_{i=1}^{n_j^*} \beta_j \left[\frac{R_j}{1+n_j^*} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)} \right]^+ \\
&= \beta_j \frac{n_j^* R_j}{1+n_j^*} - \beta_j \sum_{i=1}^{n_j^*} \frac{H(\mathcal{B}_i)}{i+1} + \beta_j \sum_{i=1}^{n_j^*-1} \sum_{k=i+1}^{n_j^*} \frac{H(\mathcal{B}_k)}{k(k+1)} \\
&= \beta_j \frac{n_j^* R_j}{1+n_j^*} - \beta_j \sum_{i=1}^{n_j^*} \frac{H(\mathcal{B}_i)}{i+1} + \beta_j \sum_{i=1}^{n_j^*} \frac{(i-1)\, H(\mathcal{B}_i)}{i(i+1)} = \beta_j \left[\frac{n_j^* R_j}{1+n_j^*} - \sum_{i=1}^{n_j^*} \frac{H(\mathcal{B}_i)}{i(i+1)} \right].
\end{aligned}
$$

(iii) Let us prove that $\sum_{j=1}^{M} x_{ij}^* = B_i$ if and only if $\mathcal{B}_i \le \mathbb{B}_0$ using backward induction over i. To this end, we find convenient to define $m_i^* := \min\{j \in [M] \colon \mathcal{B}_i > \mathbb{B}_j\}$ for $i \in [N]$, so that $x_{ij}^* > 0$ if and only if $j \ge m_i^*$. Also, using the definition of $H(\cdot)$ in Definition 3.2 we have that

$$
H(\mathcal{B}_i) = \frac{\sum_{k=m_i^*}^{M} \beta_k R_k - \mathcal{B}_i}{\sum_{k=m_i^*}^{M} \beta_k} \quad \text{for all } i \in [N] \text{ such that } \mathcal{B}_i \le \mathbb{B}_0. \tag{A-20}
$$

•) Suppose $i = N$. We consider two cases: (a) $\mathcal{B}_N \le \mathbb{B}_0$ and (b) $\mathcal{B}_N > \mathbb{B}_0$.

(a) Suppose $\mathcal{B}_N \le \mathbb{B}_0$. In this case, (7) implies that $n_j^* = N$ for all $j \ge m_N^*$ and so (6) implies

$$
\begin{aligned}
\sum_{j=1}^{M} x_{Nj}^* &= \sum_{j=m_N^*}^{M} x_{Nj}^* = \sum_{j=m_N^*}^{M} \beta_j \left[\frac{R_j}{1+N} - \frac{H(\mathcal{B}_N)}{1+N} \right] \\
&= \frac{1}{1+N} \sum_{j=m_N^*}^{M} \beta_j R_j - \frac{H(\mathcal{B}_N)}{1+N} \sum_{j=m_N^*}^{M} \beta_j = \frac{\mathcal{B}_N}{1+N} = B_N,
\end{aligned}
$$

where the second-to-last equality uses (A-20) and the last equality follows from the fact that $\mathcal{B}_N = (1+N)\, B_N$ (see equation (4)).

(b) Suppose $\mathcal{B}_N > \mathbb{B}_0$. In this case, $m_N^* = 1$, $n_j^* = N$ for all $j \in [M]$ and $H(\mathcal{B}_N) = 0$. It follows that

$$
\sum_{j=1}^{M} x_{Nj}^* = \sum_{j=1}^{M} \beta_j \left[\frac{R_j}{1+N} - \frac{H(\mathcal{B}_N)}{1+N} \right] = \frac{1}{1+N} \sum_{j=1}^{M} \beta_j R_j = \frac{\mathbb{B}_0}{1+N} < \frac{\mathcal{B}_N}{1+N} = B_N.
$$

We conclude that the result holds for $i = N$.

•) Suppose that the result holds for $i+1$, namely, $\sum_{j=1}^{M} x_{i+1j}^* = B_{i+1}$ if and only if $\mathcal{B}_{i+1} \le \mathbb{B}_0$.

•) Let us prove the result for i. We consider again the two cases: (a) $\mathcal{B}_i \le \mathbb{B}_0$ and (b) $\mathcal{B}_i > \mathbb{B}_0$.

(a) Suppose $\mathcal{B}_i \le \mathbb{B}_0$, then $\mathcal{B}_{i+1} \le \mathbb{B}_0$ and by the induction hypothesis $\sum_{j=1}^{M} x_{i+1j}^* = B_{i+1}$. It follows that

$$\sum_{j=1}^{M} x^*_{ij} = \sum_{j=1}^{M} x^*_{i+1j} + \sum_{j=1}^{M}(x^*_{ij} - x^*_{i+1j}) = B_{i+1} + \sum_{j=1}^{M}(x^*_{ij} - x^*_{i+1j}).$$

Now, for all $j \in [M]$ such that $j < m^*_i$, we have that $x^*_{ij} = x^*_{i+1,j} = 0$. On the other hand, for $j \in [M]$ such that $m^*_i \le j < m^*_{i+1}$, we have $x^*_{i+1j} = 0 < x^*_{ij}$ and $n^*_j = i$. It follows from (6) that

$$x^*_{ij} = \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_i)}{i+1}\right] = \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{i+1}\right].$$

Finally, for $j \ge m^*_{i+1}$, we have $0 < x^*_{i+1j} \le x^*_{ij}$ and from (6) we get that

$$\begin{aligned}
x^*_{ij} &= \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_i)}{i+1} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i} \frac{H(\mathcal{B}_k)}{k(k+1)}\right] \\
&= \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} - \frac{H(\mathcal{B}_i)}{i+1} + \frac{H(\mathcal{B}_{i+1})}{i+2} + \frac{H(\mathcal{B}_{i+1})}{(i+1)(i+2)}\right] \\
&= \beta_j \left[\frac{R_j}{1+n^*_j} - \frac{H(\mathcal{B}_{i+1})}{i+2} + \sum_{k=1}^{n^*_j} \frac{H(\mathcal{B}_k)}{k(k+1)} - \sum_{k=1}^{i+1} \frac{H(\mathcal{B}_k)}{k(k+1)} + \frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1}\right] \\
&= x^*_{i+1j} + \beta_j \left[\frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1}\right].
\end{aligned}$$

As a result

$$\begin{aligned}
\sum_{j=1}^{M} x^*_{ij} &= B_{i+1} + \sum_{j=1}^{M}(x^*_{ij} - x^*_{i+1j}) \\
&= B_{i+1} + \sum_{j=m^*_i}^{m^*_{i+1}-1} \beta_j \left[\frac{R_j - H(\mathcal{B}_i)}{i+1}\right] + \sum_{j=m^*_{i+1}}^{N} \beta_j \left[\frac{H(\mathcal{B}_{i+1}) - H(\mathcal{B}_i)}{i+1}\right] \\
&= B_{i+1} + \frac{1}{i+1}\Big[\sum_{j=m^*_i}^{m^*_{i+1}-1} \beta_j\, R_j + H(\mathcal{B}_{i+1}) \sum_{j=m^*_{i+1}}^{N} \beta_j - H(\mathcal{B}_i) \sum_{j=m^*_i}^{N} \beta_j\Big] \\
&= B_{i+1} + \frac{1}{i+1}\left[\mathcal{B}_i - \mathcal{B}_{i+1}\right] = B_i,
\end{aligned}$$

where the second-to-last equality uses (A-20) and the last equality follows from (4).

(b) Suppose $\mathcal{B}_i > \mathbb{B}_0$. In this case, $H(\mathcal{B}_i) = 0$ and $m^*_i = 1$. Using a similar argument as in part (a) one can show that

$$\sum_{j=1}^{M} x^*_{ij} = \sum_{j=m^*_{i+1}}^{M} x^*_{i+1j} + \frac{1}{i+1}\Big[\sum_{j=1}^{m^*_{i+1}-1} \beta_j\, R_j + H(\mathcal{B}_{i+1}) \sum_{j=m^*_{i+1}}^{N} \beta_j - \sum_{j=1}^{N} \beta_j\Big]. \tag{A-21}$$

Suppose that $\mathcal{B}_{i+1} \le \mathbb{B}_0$, then (A-21) together with (A-20) and the induction hypothesis imply that

$$\begin{aligned}\sum_{j=1}^{M} x_{ij}^* &= B_{i+1} + \frac{1}{i+1}\Big[\sum_{j=1}^{m_{i+1}^*-1} \beta_j\, R_j + \sum_{j=m_{i+1}^*}^{N} \beta_j\, R_j - \mathcal{B}_{i+1} - \sum_{j=1}^{N} \beta_j\Big] \\ &= B_{i+1} + \frac{1}{i+1}\Big[\sum_{j=1}^{N} \beta_j\, R_j - \mathcal{B}_{i+1}\Big] = B_{i+1} + \frac{1}{i+1}\Big[\mathbb{B}_0 - \mathcal{B}_{i+1}\Big] \\ &< B_{i+1} + \frac{1}{i+1}\Big[\mathcal{B}_i - \mathcal{B}_{i+1}\Big] = B_i.\end{aligned}$$

Suppose now that $\mathcal{B}_{i+1} > \mathbb{B}_0$ then $m_{i+1}^* = 1$. It follows from (A-21) and the induction hypothesis that

$$\sum_{j=1}^{M} x_{ij}^* = \sum_{j=1}^{M} x_{i+1j}^* < B_{i+1}.$$

But since $B_i \geq B_{i+1}$, we conclude that $\sum_{j=1}^{M} x_{ij}^* < B_i$. $\square$

Proof of Proposition 2: Recall from Corollary 1 that for fixed N

$$X_j^*(N) = \beta_j \left[\frac{n_j^*(N)\, R_j}{1 + n_j^*(N)} - \sum_{k=1}^{n_j^*(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)}\right],$$

where $n_j^*(N) = \max\big\{i \in [N] \text{ such that } \mathcal{B}_i(N) > \mathbb{B}_j\big\}$ and $\mathcal{B}_i(N) = i\, B_i(N) + \sum_{k=i}^{N} B_k(N)$. Note that $\mathcal{B}_1(N) = B_1(N) + \sum_{i=1}^{N} B_i(N) = B_1(N) + B_{\mathrm{C}}$ and so $\lim_{N\to\infty} \mathcal{B}_1(N) = B_{\mathrm{C}}$. Thus, we have that $\lim_{N\to\infty} X_j^*(N) = 0$ for all $j \in [M]$ such that $\mathbb{B}_j \geq B_{\mathrm{C}}$, i.e., for all $j < k^* = \min\{j \geq 1 \colon \mathbb{B}_j < B_{\mathrm{C}}\}$.

Suppose $j \geq k^*$ and let $\epsilon > 0$ be such that $B_{\mathrm{C}} - \epsilon > \mathbb{B}_{k*}$. We define

$$n_\epsilon(N) := \max\big\{n \in [N] \text{ such that } \mathcal{B}_i(N) \geq B_{\mathrm{C}} - \epsilon \text{ for all } i \leq n\big\}.$$

Since $\mathcal{B}_1(N) = B_1(N) + B_{\mathrm{C}}$ we have that $n_\epsilon(N) \geq 1$ for all N.

We next show that $\lim_{N\to\infty} n_\epsilon(N) = \infty$. We prove this claim by contradiction. Suppose, otherwise, that there exists an increasing integer-valued sequence $\{N_k\}$ and an integer $\bar{n}_\epsilon$ such that $\lim_{k\to\infty} N_k = \infty$ and $\lim_{k\to\infty} n_\epsilon(N_k) = \bar{n}_\epsilon$. It follows that there exists an integer $\bar{k}_\epsilon$ such that $n_\epsilon(N_k) = \bar{n}_\epsilon$ for all $k \geq \bar{k}_\epsilon$ and so $\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) < B_{\mathrm{C}} - \epsilon$ for all $k \geq \bar{k}_\epsilon$. But, for k sufficiently large

$$\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) = (\bar{n}_\epsilon + 1)\, B_{\bar{n}_\epsilon+1}(N_k) + \sum_{i=\bar{n}_\epsilon+1}^{N_k} B_i(N_k) = (\bar{n}_\epsilon + 1)\, B_{\bar{n}_\epsilon+1}(N_k) - \sum_{i=1}^{\bar{n}_\epsilon} B_i(N_k) + \sum_{i=1}^{N_k} B_i(N_k).$$

Taking limit as $k \to \infty$ and noticing that $\bar{n}_\epsilon$ is finite (independent of k) and $B_i(N_k) \downarrow 0$ as N_k goes to infinity, we get

$$\lim_{k\to\infty} \mathcal{B}_{\bar{n}_\epsilon+1}(N_k) = \lim_{k\to\infty} \sum_{i=1}^{N_k} B_i(N_k) = B_{\mathrm{C}} > B_{\mathrm{C}} - \epsilon.$$

But this contradicts $\mathcal{B}_{\bar{n}_\epsilon+1}(N_k) < B_{\mathrm{C}} - \epsilon$ for all $k \geq \bar{k}_\epsilon$. We conclude that $\lim_{N\to\infty} n_\epsilon(N) = \infty$. As a direct consequence we get that $\lim_{N\to\infty} n_j^*(N) = \infty$ for all $j \geq k^*$ since $n_j^*(N) \geq n_\epsilon(N)$. Thus, for $j \geq k^*$

$$X_j^*(N) = \beta_j \left[\frac{n_j^*(N)\, R_j}{1 + n_j^*(N)} - \sum_{k=1}^{n_\epsilon(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} - \sum_{k=n_\epsilon(N)+1}^{n_j^*(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} \right].$$

From this expression and the facts that (i) $H(\mathcal{B}_k(N)) \leq H(0) = a_M < \infty$ for all k and N and (ii) $n_\epsilon(N) \to \infty$ and $n_j^*(N) \to \infty$ as $N \to \infty$, we get that

$$\lim_{N\to\infty} X_j^*(N) = w_j \left[R_j - \lim_{N\to\infty} \sum_{k=1}^{n_\epsilon(N)} \frac{H(\mathcal{B}_k(N))}{k(k+1)} \right].$$

But, $B_{\mathrm{C}} - \epsilon \leq \mathcal{B}_k(N) \leq \mathcal{B}_1(N) = B_1(N) + B_{\mathrm{C}}$ for all $k \leq n_\epsilon(N)$. It follows that

$$\beta_j \big(R_j - H(B_{\mathrm{C}} - \epsilon)\big) \leq \lim_{N\to\infty} X_j^*(N) \leq \beta_j \big(R_j - H(B_{\mathrm{C}})\big).$$

From the continuity of H, letting $\epsilon \downarrow 0$ we conclude that

$$\lim_{N\to\infty} X_j^*(N) \leq \beta_j \big(R_j - H(B_{\mathrm{C}})\big) \quad \text{for all } j \geq k^*. \;\; \square$$