Operational measures for squeezing

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joined work with ***Daniel Lercher*** *and* ***Michael M. Wolf***

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What about the weird affiliation?

TNG TECHNOLOGY CONSULTING

- Worked with Michael M. Wolf as a graduate student at TUM.
- Graduated recently and left academia.
- Work for TNG Technology Consulting:
    - Munich based German speaking IT development and consulting firm
    - $>$ 50% PhD in Physics, Mathematics and Computer Science
    - special needs software consulting/development for various areas from telecommunications to autonomous driving
    - even a few quantum information people

We pave the way to investigate squeezing as a resource

If one specifies an error tolerance no larger than some error $\varepsilon > 0$ and allows for using n instances of a given resource, what communication rates are achievable?

In this talk:

- "New" resource theory with the usual questions: Squeezing of formation, distillation of squeezing, etc.
- Interesting due to connections with entanglement theory, experimental difficulties, maybe even on its own.
- Providing new tools to study the question in continuous variable quantum information.

Modeling the electromagnetic field in phase space

- The electromagnetic field can be modeled as non-interacting harmonic oscillators (second quantisation).
- Harmonic oscillator description: frequency ω_k and a set of position and momentum operators Q, P.
- Usually finitely many k are enough (e.g. in a cavity) $\Rightarrow$ $R = (Q_1, P_1, \ldots, Q_n, P_n)$.
- Photons are bosons $\Rightarrow$ R fulfil the CCR:

$$[R_k, R_l] = i\sigma_{kl}\mathbb{1}, \qquad \sigma = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Symplectic transformations $Sp(2n)$ leave the CCR invariant (corresponds to unitary transformations on the state).

For Gaussian states, the covariance matrix is your friend

- Q, P must be unbounded $\Rightarrow$ use bounded representations $W_\xi = \exp(-i\xi\sigma R)$.
- Define the *characteristic function* $\chi(\xi) = \mathrm{tr}(W_\xi \rho)$.
- The characteristic function of a state can often be described by its *moments*. Gaussian states are described by their first and second moments only:

$$\begin{aligned} d_k &:= \mathrm{tr}(\rho R_k) \\ \gamma_{kl} &:= \mathrm{tr}(\rho\{R_k - d_k \mathbb{1}, R_l - d_l \mathbb{1}\}_+). \end{aligned}$$

- Operations on the state (such as time evolution) correspond to operations on the moments (such as symplectic transformations).
- Heisenberg's uncertainty relation: $\gamma \geq i\sigma$.
- A squeezed state has an eigenvalue λ of γ with $\lambda < 1$.

If you want entanglement, you need squeezing

Theorem

Given a quantum state ρ with covariance matrix γ, for an arbitrary two-mode subsystem of a quantum state we have

$$E_N \propto \max\{0, -\log_2(\lambda_1 \lambda_2)\},$$

where λ_1, λ_2 are the smallest eigenvalues of γ and E_N is the logarithmic negativity (an entanglement measure). *[M.M. Wolf, J. Eisert, M. Plenio. PRL, 90, 2003]*

Theorem (No super-activation without squeezing)

Let T_1, T_2 be passive Gaussian quantum channels. If each channel either has a symmetric extension or satisfies the PPT property, then $Q(T_1 \otimes T_2) = 0$. *[D. Lercher, G. Giedke, M.M. Wolf. New J. Phys. 15, 2013]*

Squeezing measures as entanglement witnesses

- Variances (in position and momentum) can be measured very well in the lab.
- "Spin squeezing" measures have been used as entanglement measures for years.
- Similar "squeezing measures" have been proposed recently [M. Gessner, et al. *Quantum*, 2017-07-10]:

$$\xi^2(\gamma) = \min_{g \in \mathbb{R}^{2n}, \|g\|_2 = 1} (g^T \sigma^T \gamma_{\prod(\rho)} \sigma g)(g^T \gamma_\rho g)$$

where $\prod(\rho) = \prod_{i=1}^N \rho_i$ with the reduced density matrices ρ_i. The separability criterion reads:

$$\xi^2(\gamma_{\mathrm{sep}}) \geq 1$$

- Different goal: find entanglement, not study squeezing as is

Current squeezing measures work well for one-mode states

Currently: $G_{\text{squeeze}} = \lambda_{\mathrm{min}}(\gamma)$. [B. Kraus et al. *PRA* 67:0402314, 2003]
Problems:

- Consider multimode states

$$\begin{pmatrix} s & & & \\ & \frac{1}{s} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} s & & & \\ & \frac{1}{s} & & \\ & & s & \\ & & & \frac{1}{s} \end{pmatrix}, \quad \begin{pmatrix} s^2 & & & \\ & \frac{1}{s^2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

  The first and second should not have the same squeezing.
- Operational measures?

Squeezing is a resource

- First noted by Braunstein: Squeezing remains invariant under linear optics [S.L. Braunstein. *PRA*, **71**, 2005]
- Resource states: one-mode squeezed states: $\mathrm{diag}(s, s^{-1})$
- free operations:
  1. Linear optics (symplectic orthogonal matrices S acting via $\gamma \mapsto S^T \gamma S$),
  2. Free ancillary states ($\gamma \to \gamma \oplus \gamma_{\text{anc}}$),
  3. Add classical noise ($\gamma_{\text{noise}} \geq 0$ acting via $\gamma \mapsto \gamma + \gamma_{\text{noise}}$),
  4. Weyl rotations (no change in covariance matrix),
  5. Convex combinations ($\lambda\gamma + (1-\lambda)\tilde{\gamma}$),
  6. Homodyne detection:

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \quad \Rightarrow \quad \gamma \mapsto \lim_{d \to \infty} A - C(B + \mathrm{diag}(d, 1/d))^{-1} C^T.$$

Defining measures of squeezing which also work for multiple modes

Cost: A one-mode squeezer $S = \mathrm{diag}(s, s^{-1})$ has costs $\log(s)$.

Idea: For a symplectic matrix S minimise one-mode squeezing for decompositions $S = S_1 \ldots, S_m$ with S_i passive ($K(n)$) or a one-mode squeezer ($Z(n)$).
Then we have the measure:

$$F(S) = \inf\left\{\sum_{i=1}^{m} \log s_1^{\downarrow}(S_i) | S = S_1 \ldots S_m, S_i \in K(n) \cup Z(n)\right\}.$$

Minimal squeezing for symplectic matrices is given by the Euler decomposition

Proposition (M.I., D. Lercher, M.M. Wolf (2016))

For any symplectic matrix S we have

$$F(S) = \sum_{i=1}^{n} \log(s_i^{\downarrow}(S)).$$

Equality is achieved by the Euler decomposition $S = K_1 \operatorname{diag}(s_1, s_1^{-1}, \ldots, s_n, s_n^{-1}) K_2$ with passive K_1, K_2.

Proof

- Euler decomposition achieves the bound (clear).
- Otherwise, prove $F(S_1 \cdot S_2) \leq F(S_1) + F(S_2)$

$$F(S_1 \cdot S_2) = \log\left(\prod_{i=1}^{n} s_i^{\downarrow}(S_1 \cdot S_2)\right) \leq \log\left(\prod_{i=1}^{n} s_i^{\downarrow}(S_1) \prod_{i=1}^{n} s_i^{\downarrow}(S_2)\right)$$

(special case of a theorem by Gel'fand and Naimark [Bhatia. *Matrix Analysis* 1996]).

This already is an operational measure for pure states

- If $\mathrm{diag}(s, s^{-1})$ are resource states, the optimal way to prepare a pure state with covariance matrix γ is given by the Euler decomposition.
- Preparation costs can be read from the Euler decomposition.
- For general states: Take a pure state and add noise.

Idea: Suggestion for a measure for general states:

$$G(\gamma) := \inf\{F(S) | \gamma \geq S^T S, S \in Sp(2n)\}$$

Minimal squeezing for states is given by a simple optimisation problem

Minimise costs over all sequences

$$\gamma_0 \to \gamma_1 \to \ldots \to \gamma_N = \gamma$$

with γ_0 resource states and each operation being an allowed operation.

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Theorem (M.I., D. Lercher, M.M. Wolf (2016))

Given a quantum state ρ with covariance matrix γ, the minimal amount of one-mode squeezing needed for its creation is given by

$$G(\gamma) = \min\{F(S) | \gamma \geq S^T S, S \in Sp(2n)\}$$

Proof I: Bring everything to a normal form

- Idea 1: Prove that you can interchange operations to get a "normal form"
- Normal form: If $\mathcal{M}$ denotes a measurement, any set of allowed operations can be brought to the form.

$$\gamma = \mathcal{M}(S\gamma_0 \oplus \gamma_1 \oplus \ldots \oplus \gamma_m S^T + \gamma_{\text{noise}})$$

  with a collection of resource states γ_i and noise γ_{noise}.
- Measurements can always be done last [M. Nielsen, I. Chuang. *Quantum Computation and Quantum Information* 2000].
- Noise stays noise since conjugation preserves the sign of eigenvalues.

Proof II: Prove that measurements do not squeeze

- Idea 2: Prove that measurements don't squeeze.
- Recall that for a measurement with $\mathcal{M}(\tilde{\gamma}) = \gamma$:

$$\tilde{\gamma} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C}^T & \tilde{B} \end{pmatrix}.$$

- Then the measurement is given by:

$$\mathcal{M}(\tilde{\gamma}) = \tilde{A} - \tilde{C}(\tilde{B} + \mathrm{diag}(d, 1/d))^{-1} \tilde{C}^T.$$

- And finally Weyl + Cauchy interlacing theorem:

$$\prod_{j=1}^{m} s_j^{\downarrow}(\mathcal{M}(\tilde{\gamma})) \leq \prod_{j=1}^{m} s_j^{\downarrow}(\tilde{A}) \leq \prod_{j=1}^{m} s_j^{\downarrow}(\tilde{\gamma}) \leq \prod_{j=1}^{n} s_j^{\downarrow}(\tilde{\gamma}).$$

Detour: Cayley transformation

Regular Cayley transform: Transform upper complex half plane to unit disk.
Matrix Cayley transform:

- Transformation of positive half-plane $Z > 0$ to unit disc $||H|| < 1$ [D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, 1998].
- Transformation of skew-Hermition matrix to unitary matrices.
- Transformation of symplectic matrices into Hamiltonian matrices [V. Mehrmann *Lin. Alg. App. 241-243, 1996*].
- Symplectic Cayley transform transforms symplectic matrices into symmetric ones [M. deGosson, *Symplectic Geometry and Quantum Mechanics*, 2006].

Detour: Cayley transformation

$$\mathcal{C}:\{H\in\mathbb{R}^{n\times n}|\operatorname{spec}(H)\cap\{+1\}=\emptyset\}\mapsto\mathbb{R}^{n\times n},\quad H\mapsto\frac{\mathbb{1}+H}{\mathbb{1}-H}.$$
$$\mathcal{C}^{-1}:\{S\in\mathbb{R}^{n\times n}|\operatorname{spec}(H)\cap\{-1\}=\emptyset\}\mapsto\mathbb{R}^{n\times n},\quad S\mapsto\frac{S-\mathbb{1}}{S+\mathbb{1}}.$$

- $\mathcal{C}$ and $\mathcal{C}^{-1}$ are diffeomorphisms onto their respective images.
- $\mathcal{C}$ is operator monotone and operator convex on matrices A with $\operatorname{spec}(A)\subset(-1,1)$.
- $\mathcal{C}^{-1}$ is operator monotone and operator concave on matrices A with $\operatorname{spec}(A)\subset(-1,\infty)$.
- $\mathcal{C}:\mathbb{R}\to\mathbb{R}$ is $\log$-convex on $[0,1)$.
- $\mathcal{C}(\mathcal{H})=Sp(2n)\cap\{\gamma\geq i\sigma\}$, where

$$\mathcal{H}:=\left\{H=\begin{pmatrix}A & B\\ B & -A\end{pmatrix}\middle|A\in\mathbb{R}^{2n\times 2n}A^T=A,B^T=B,-\mathbb{1}<H<\mathbb{1}\right\}.$$

We can rewrite G using the Cayley transform

$$
\begin{aligned}
G(\gamma) &= \inf\{F(S) | \gamma \geq S^T S, S \in Sp(2n)\} \\
&= \inf\{F(\gamma_0^{1/2}) | \gamma \geq \gamma_0 \geq iJ\} \\
&= \inf\left\{ \frac{1}{2} \sum_{i=1}^{n} \log\left(\frac{1 + s_i(A + iB)}{1 - s_i(A + iB)} \right) \,\middle|\, \mathcal{C}^{-1}(\gamma) \geq H, H \in \mathcal{H} \right\}
\end{aligned}
$$

with (again)

$$
\mathcal{H} := \left\{ H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \,\middle|\, A \in \mathbb{R}^{2n \times 2n} A^T = A, B^T = B, -\mathbb{1} < H < \mathbb{1} \right\}.
$$

Proof of convexity I: We need to prove joint convexity

Define:

$$f(A,B)=\frac{1}{2}\sum_{i=1}^{n}\log\left(\frac{1+s_i(A+iB)}{1-s_i(A+iB)}\right)$$

We want to prove joint convexity for symmetric matrices:

$$f(tA+(1-t)A',tB+(1-t)B')\leq tf(A,B)+(1-t)f(A',B')\quad\forall\, t\in[0,1]$$

Then the convexity/concavity of the Cayley transform finishes the proof.

Proof of convexity II: This is done using Thompson and Lidskii inequalities

Use Thompson inequality and Lidskii theorem:

$$
\begin{aligned}
&s_j(t(A+iB)+(1-t)(A'+iB')) \\
&\quad = \lambda_j(|t(A+iB)+(1-t)(A'+iB')|) \\
&\quad \leq \lambda_j(U|t(A+iB)|U^* + V|(1-t)(A'+iB')|V^*) \\
&\quad \leq \lambda_j(U|t(A+iB)|U^*) + \sum_\pi p_\pi \lambda_{\pi(j)}(V|(1-t)(A'+iB')|V^*) \\
&\quad = t\lambda_j(|A+iB|) + (1-t)\sum_\pi p_\pi \lambda_{\pi(j)}(|A'+iB'|)
\end{aligned}
$$

with $p_\pi \geq 0$ and $\sum_\pi p_\pi = 1$.

Proof of convexity III: Finally, we use convexity of the Cayley transform

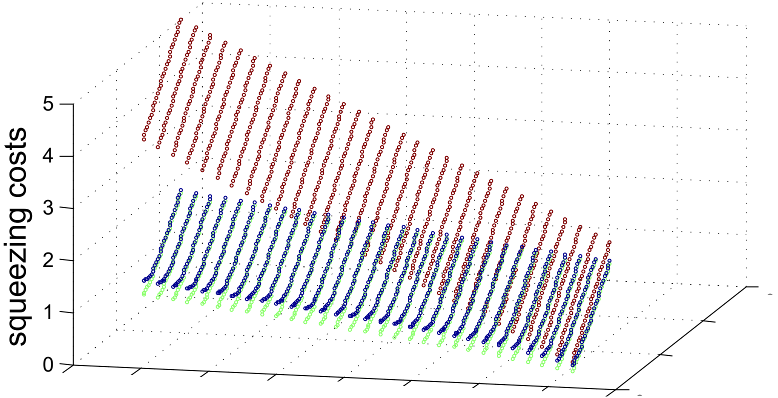
Then we have:

$$\sum_{j=1}^{n} \log \mathcal{C}[s_j(t(A+iB)+(1-t)(A'+iB'))]$$
$$\leq \sum_{i=1}^{n} t \log \mathcal{C}[\lambda_i(|A+iB|)] + (1-t)\sum_{\pi} p_\pi \left(\sum_{i=1}^{n} \log \mathcal{C}[\lambda_{\pi(i)}(|A'+iB'|)]\right).$$

The measure improves preparation procedures

We provide numerical calculations for examples (two-parameter family of states, code can be found at `https://github.com/Martin-Idel/operationalsqueezing`):

[L. Mišta, N. Korolkova. *PRA* 77:050302, 2008]



G is superadditive and probably subadditive

We have:

$$\frac{1}{2}(G(\gamma_A) + G(\gamma_B)) \leq G(\gamma_A \oplus \gamma_B) \leq G(\gamma_A) + G(\gamma_B).$$

Conjecture: G is subadditive

- Supported by numerical data.
- True at least if γ_A is pure.

We have bounds for G, but the upper bound is bad

Best bounds:

$$-\frac{1}{2}\sum_{\lambda_i^{\downarrow}(\gamma)<1} \log(\lambda_i^{\downarrow}(\gamma)) \leq G(\gamma) \leq F(S)$$

where S is the symplectic matrix in Williamson's theorem $\gamma = S^T DS$.

- lower bound achieved, if the eigenvectors to eigenvalues < 1 can be extended to an orthonormal symplectic basis.
- upper bound can be arbitrarily bad: Thermal state with $\gamma = n\mathbb{1}$ and $S = \mathrm{diag}(\sqrt{N-1}, 1/\sqrt{N-1}, \ldots) \in Sp(2n)$.

Detour: Set-valued analysis

- Work with functions with *sets* and not just points as values.
- Define continuity, norms, etc.

Definition

Let $X, Y \subseteq \mathbb{R}^{n\times m}$ and $f : X \to 2^Y$ be a set-valued function. It is *upper semicontinuous* (*upper hemicontinuous*) at $x_0 \in X$ if:
for all open neighbourhoods Q of $f(x_0)$ there exists an open neighbourhood W of x_0 such that $W \subseteq \{x \in X | f(x) \subset Q\}$.

Likewise, we call it *lower semicontinuous* (often called *lower hemicontinuous*) at a point x_0 if for any open set V intersecting $f(x_0)$, we can find a neighbourhood U of x_0 such that $f(x) \cap V \neq \emptyset$ for all $x \in U$.

Detour: Set-valued analysis

Just for fun:

Theorem

Let S be a non-empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $f : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in S$. Then f has a fixed point.

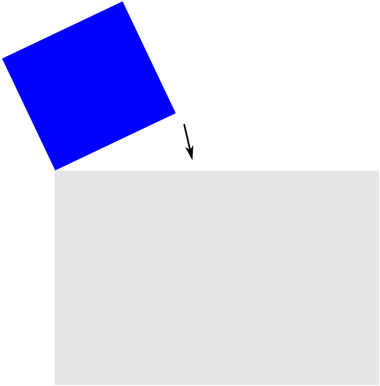
Proved in 1941 by Shizuo Kakutani and used in the one-page paper "Equilibrium points in N-person games" by John F. Nash.

G is probably continuous

Recall:
$G(\gamma) = \inf\{F(\tilde{\gamma}^{1/2}) | \gamma \geq \tilde{\gamma} \geq i\sigma\}$
For continuity, this means we have the intersection of two convex, non-empty and set-valued functions:

- $f(A) : A \geq i\sigma$.
- $g(B) : B \leq \gamma$ (this one varies continuously).

Heuristic: This should be continuous.



We should be able to prove continuity using set-valued analysis

Current state: G is lower semicontinuous on the set of covariance matrices and continuous on its interior.
Conjecture: G is continuous, since any compact intersection of set-valued functions consisting of matrix cones is continuous.

- Any intersection of non-empty compact sets with non-empty interior is continuous.
- Non-polynomial matrix cones make it somewhat difficult.

Potential other applications

Would directly prove continuity of all functions consisting of optimisation of continuous functions over convex cones.

Example: Gaussian entanglement of formation: e.g. [M.M. Wolf., *PRA* ***69***, 2004]

$$E_{\mathrm{form}}(\gamma_{AB}) = \min\{H(\gamma_p) | \gamma_{AB} \geq S^T S, S \in Sp(2n)\}$$

where γ_p is the reduced state of $S^T S$ and H is entanglement entropy.

On the road to more realistic measures

- $\log(s)$ could be interpreted as “interaction time” of the squeezing Hamiltonian.
- In experiments: linear difficulty until cutoff at about 15 *dB* squeezing

  e.g. [H. Vahlbruch et al., *PRL* ***117***, 2016].
- Maybe exponential difficulty without explicit cutoff.

Problematic parts:

- Convexity of G.
- Submultiplicativity of F (irrelevant for resource theory).

Can we do better using Lie algebras?

Idea: Maybe simple products of the form $S = S_1 \cdots S_n$ are not optimal. How about general paths on the symplectic group?

Proposition (M.I., D. Lercher, M.M. Wolf (2016))

General paths on $Sp(2n)$ cannot decrease squeezing costs.

We define the measure as follows:

$$\tilde{F}(S) := \inf \left\{ \int_0^1 \|\vec{c}_\alpha^{\,a}(t)\|_1 \,\mathrm{d}t \,\middle|\, \alpha \in C^r(S), \right.$$
$$\left. \dot{\alpha}(t) = (\vec{c}_\alpha^{\,p}(t) g^p(\alpha(t)), \vec{c}_\alpha^{\,a}(t) g^a(\alpha(t)))^T \right\}$$

with $c_\alpha \in L^\infty([0,1], \mathfrak{sp}(2n))$.

Main ideas: Use propagator and simple approximation arguments

- Approximate via step functions.
- Show that the error on the path can be upper bounded by the error of the approximation, in particular:

$$\|U_A(0,t) - U_{A(\varepsilon)}(0,t)\|_1 \leq \mathrm{const} \int_0^t \|c_\alpha(s) - c_{\alpha(\varepsilon)}\|_1 \,\mathrm{d}s.$$

Can we distill squeezing and use it in channels?

Question: What is the "Maximum output squeezing". How to calculate it? Can the normal form for channels with squeezed environment help? [M.I., R. König. *Quant. Inf. Comp.*, 2017].

Partial Answer: Distillation of squeezing is prohibited for Gaussian states [B. Kraus, et al. *PRA* ***67***, 2003], hence not for the usual measure. For our measure, there are clear upper bounds ⇒ not so interesting.

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Question: What about non-gaussian states?

Distillation is possible via linear optics and being studied [R. Filip. *PRA* ***88***, 2013].

Problem: You cannot work with the covariance matrices only.

I still have many open questions

- Fermionic quantum systems?
- Squeezing is related to the spectrum of the covariance matrix, while entanglement is related to the symplectic spectrum of submatrices. Can we have more explicit direct bounds?
- State interconvertibility is more complicated. Can we have even "better" measures?
- Can we have trade-off functions between squeezing and (e.g.) superactivation?