HKUST – Department of Computer Science and Engineering
COMP 2711: Discrete Math Tools for CS – Fall 2015

Serial #

Midterm Examination

Date: Monday, 19 October 2015 Time: 19:00–21:30

Name: ____________________	Student ID: ________________
Email: ____________________	Lecture Section: ____________

Instructions

- This is a closed book exam. One cheat sheet written on double-sided A4 paper is allowed.
- The exam consists of 15 pages and 8 questions.
- Please write your name, student ID, email, lecture section on this page. For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. **You may use the back of the pages for your answers or rough work.** The last three pages are scrap paper and may also be used for rough work.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- **Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of** 16**, and** $5^{\underline{3}}$ **instead of** 60**. Calculators may be used for the exam.**
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	Total
Score									

Student ID: ______________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ______________________

Student's Signature:   ____________________
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Student ID: ______________

Problem 1: [14 pts] (Logic, Multiple Choice)
For each of the following parts, circle the correct answer or answers. There is no need to explain how your answers are derived.

(a) Let p and q be two propositions. Which of the following propositions is/are **equivalent** to $p \leftrightarrow q$?

1. $(p \to q) \wedge (q \to p)$
2. $(\neg p \to \neg q) \wedge (\neg q \to \neg p)$
3. $(p \wedge q) \wedge (\neg p \wedge \neg q)$
4. $(p \wedge q) \vee (\neg p \wedge \neg q)$
5. None of above.

(b) Let p and q be two propositions. Which of the following propositions is/are **tautology/tautologies**?

1. $p \to (p \to q)$
2. $\neg p \to (p \to q)$
3. $q \to (p \to q)$
4. $\neg q \to (p \to q)$
5. None of above.

(c) Let U be a domain (a.k.a. universe), and $P(x)$ and $Q(x)$ be two predicates. Which of the following statements is/are **equivalent** to $\forall x \in U(P(x) \vee Q(x))$?

1. $\exists x \in U(\neg P(x) \wedge \neg Q(x))$
2. $\forall y \in U(\neg P(y) \to Q(y))$
3. $\neg(\forall x \in U(\neg P(x)) \vee \forall x \in U(\neg Q(x)))$
4. $\neg(\exists x \in U(\neg P(x)) \wedge \exists x \in U(\neg Q(x)))$
5. None of above.

(d) Let U_1 and U_2 be two domains, and let $P(x)$ and $R(x, y)$ be two predicates. Which of the following statements logically **implies/imply** $\exists x \in U_2 \forall y \in U_1(P(x) \to R(x, y))$?

1. $\forall y \in U_1 \exists x \in U_2(P(x) \to R(x, y))$
2. $\exists x \in U_2 \forall y \in U_1(P(x))$
3. $\exists x \in U_2 \forall y \in U_1(R(x, y))$
4. $\forall x \in U_2 \forall y \in U_1(P(x) \to R(x, y))$
5. None of above.

Solution:

(a) 1, 2, 4

(b) 2, 3

(c) 2

(d) 3,4
Grading: 3, 3, 4, 4. Deduct 2 marks for each wrong circle or missing circle, capped by the total marks for the question.

Problem 2: [12 pts] (Propositional Logic, Inference)
In this question, p, q, r and s are logic propositions. In each part, a set of premises and a conclusion are given. You are asked to derive the conclusion from the premises using the rules of inference that you have learned in class. Show all steps of your derivations, and state clearly the assumption or the inference rule used in each step.

(a) Premises:
1. $\neg p$
2. $\neg r$
3. $\neg(p \vee q) \rightarrow r$

Conclusion: q

(b) Premises:
1. $\neg q$
2. $p \rightarrow q$
3. $\neg p \rightarrow (r \wedge s)$

Conclusion: r

(c) Premises:
1. $s \rightarrow r$
2. $p \wedge q$
3. $p \rightarrow \neg(q \wedge r)$

Conclusion: $\neg s$

[MORE WORKSPACE FOR PROBLEM 2 ON NEXT PAGE]

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[ADDITIONAL WORKSPACE FOR PROBLEM 2]

Solution: (a)

1.$\neg(p \vee q) \rightarrow r$	Premise
2.$\neg r$	Premise
3.$(p \vee q)$	Modus tollens (1,2)
4.$\neg p$	Premise
5.q	Disjunctive syllogism (3,4)

(b)

1.$p \rightarrow q$	Premise
2.$\neg q$	Premise
3.$\neg p$	Modus tollens (1,2)
4.$\neg p \rightarrow (r \wedge s)$	Premise
5.$(r \wedge s)$	Modus Ponens (3,4)
6.r	Simplification (5)

(c)

1.$p \wedge q$	Premise
2.p	Simplification
3.q	Simplification
4.$p \rightarrow \neg(q \wedge r)$	Premise
5.$\neg(q \wedge r)$	Modus Ponens (3,4)
6.$\neg q \vee \neg r$	DeMorgan's law, 5
7.$\neg r$	Disjunctive syllogism (3,6)
8.$s \rightarrow r$	Premise
9.$\neg s$	Modus tollens (7,8)

Grading: 4, 4, 4

Student ID: ____________

Problem 3: [12 pts] (Predicate Logic)
Consider the following three statements:

(a) A rich and famous person is a successful person.

(b) A rich person is successful if she/he is famous.

(c) A famous person is successful if she/he is rich.

Convert the statements into logic statements using the domain U (set of all persons), and predicates $rich(x)$ (person x is rich), $famous(x)$ (person x is famous), $successful(x)$ (person x is successful).

Are the three statements logically equivalent? If not, explain why. If yes, give a proof. Show all steps of your proof, and state clearly the assumption or the inference rule used in each step.

[MORE WORKSPACE FOR PROBLEM 3 ON NEXT PAGE]

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[ADDITIONAL WORKSPACE FOR PROBLEM 3]

Solution: Here are the logic statements:

(a) $\forall x \in U((rich(x) \wedge famous(x)) \rightarrow successful(x))$.

(b) $\forall x \in U(rich(x) \rightarrow (famous(x) \rightarrow successful(x)))$.

(c) $\forall x \in U(famous(x) \rightarrow (rich(x) \rightarrow successful(x)))$.

The three statements are equivalent because they are all equivalent to the same fourth statement, as shown below:

$$
\begin{array}{lll}
 & \forall x \in U((rich(x) \wedge famous(x)) \rightarrow successful(x)) & \\
\equiv & \forall x \in U(\neg(rich(x) \wedge famous(x)) \vee successful(x)) & p \rightarrow q \equiv \neg p \vee q \\
\equiv & \forall x \in U(\neg rich(x) \vee \neg famous(x) \vee successful(x)) & \text{DeMorgan's \& Associative Law}
\end{array}
$$

$$
\begin{array}{lll}
 & \forall x \in U(rich(x) \rightarrow (famous(x) \rightarrow successful(x))) & \\
\equiv & \forall x \in U(rich(x) \rightarrow (\neg famous(x) \vee successful(x))) & p \rightarrow q \equiv \neg p \vee q \\
\equiv & \forall x \in U(\neg rich(x) \vee (\neg famous(x) \vee successful(x))) & p \rightarrow q \equiv \neg p \vee q \\
\equiv & \forall x \in U(\neg rich(x) \vee \neg famous(x) \vee successful(x)) & \text{Associative Law}
\end{array}
$$

$$
\begin{array}{lll}
 & \forall x \in U(famous(x) \rightarrow (rich(x) \rightarrow successful(x))) & \\
\equiv & \forall x \in U(famous(x) \rightarrow (\neg rich(x) \vee successful(x))) & p \rightarrow q \equiv \neg p \vee q \\
\equiv & \forall x \in U(\neg famous(x) \vee (\neg rich(x) \vee successful(x))) & p \rightarrow q \equiv \neg p \vee q \\
\equiv & \forall x \in U(\neg rich(x) \vee \neg famous(x) \vee successful(x)) & \text{Associative \& Commutative Law}
\end{array}
$$

Grading: 2 for each correct statement, 2 for each proof.

Student ID: ____________

Problem 4: [12 pts] (Proofs)
A real number is rational if and only if it can be written as $\frac{a}{b}$, where a and b are integers. Let x and y be two real numbers such that $xy \neq 0$.

(a) Show, by **direct proof**, that if x and y are rational, then the product xy is also rational.

(b) Show, by **contraposition**, that if x is irrational, then $x^{\frac{1}{5}}$ is irrational.

(c) Show, by **contradiction**, that if x is irrational and xy is rational, then y is irrational.

Solution: (a) **Since x and y are rational**, they can be written as $x = \frac{a_1}{b_1}$ and $y = \frac{a_2}{b_2}$, where a_1, b_1, a_1 and b_2 are integers. Then we have $xy=\frac{a_1 a_2}{b_1 b_2}$. Since integers are closed under multiplication operations, $a_1 a_2$ and $b_1 b_2$ are both integers. Therefore xy is rational.

(b) **Assume $x^{\frac{1}{5}}$ were rational.** By part (a), $x = x^{\frac{1}{5}} x^{\frac{1}{5}} x^{\frac{1}{5}} x^{\frac{1}{5}} x^{\frac{1}{5}}$ must also be rational, which contradicts the premise of the statement. Therefore if x is irrational, then $x^{\frac{1}{5}}$ is irrational.

(c) **Assume x is irrational and xy is rational, and y were rational.** Let $y = \frac{a_2}{b_2}$. Then $\frac{1}{y} = \frac{b_2}{a_2}$, and hence would be rational. By part (1), $xy\frac{1}{y} = x$ would also be rational, which contradicts the assumption that x is irrational. Therefore, if x is irrational and xy is rational, then y is irrational.

Grading: 4, 4, 4

Student ID: ______________

Problem 5: [10 pts] (Combinatorial Proof)

Give a combinatorial proof for the following equation for integers $n \geq 1$ and $n \geq r$.

$$\binom{n}{r} = \sum_{i=r}^{n} \binom{i-1}{r-1}$$

Note: An algebraic proof of this statement will not be accepted as a solution.

Solution:

- Left side: number of ways to select r objects from n objects.
- Right side: the final object selected can be the rth, $(r+1)$th, ..., nth object. If the final object selected is the ith object, where $i = r, r+1, ..., n$, there are $\binom{i-1}{r-1}$ ways to select the first $r-1$ objects.

Both left and right hand side count the number of ways to select r objects from n objects, so they are equal.

Student ID: ____________

Problem 6: [14 pts] (Counting Lists)
In a computer system, a password is a string of 8 characters, where each character is either a digit or a lowercase letter from the English alphabet. How many possible passwords are there under each of the following conditions?

(a) The first character must be a letter.

(b) The first character must be a letter, and there must be at least one digit.

(c) The first character must be a letter, and there must be at least two digits.

(d) The first character must be a letter, and there must be exactly two digits, which cannot be next to each other.

Briefly explain how your answers are derived.

Solution:

(a) 26×36^7. There are 26 possibilities for the first character and 36 possibilities for each of the other characters.

(b) $26 \times 36^7 - 26^8$. There are 26^8 passwords without digits.

(c) $26 \times 36^7 - 26^8 - 7 \times 10 \times 26^7$. The are 26^8 passwords without digits. The number of passwords with exactly 1 digit is: $7 \times 10 \times 26^7$, because there are 7 possible positions for the digit, the digit has 10 possibilities, and the other 7 characters must be letters.

(d) Let us create a password in two steps. (1) Choose 6 letters, and arrange them on a line such that there is a space between each pair and there is a space at the end:

$$L - L - L - L - L - L -$$

(2) Choose two digits, place them in two of the spaces, and remove the remaining spaces. There are 26^6 possibilities for step 1, and $10^2 \times \binom{6}{2}$ possibilities for step 2. Hence the total number of passwords is $26^6 \times 10^2 \times \binom{6}{2}$.

Grading: 2, 3, 4, 5

Student ID: ____________

Problem 7: [11 pts] (Permutations and Pigeonhole Principle)
You are going to distribute kn cash coupons to kn members in your company such that each member will get one cash coupon, where $k, n \geq 1$. The values of the kn cash coupons are distinct. Suppose there are n ranks of seniority in the company, and there are exactly k members at each rank. If a more senior member receives a cash coupon worth less than the cash coupon given to a more junior member, you will receive a complaint.

(a) How many ways are there to distribute the kn cash coupons if you don't care about receiving complaints?

(b) How many ways are there to distribute the kn cash coupons such that you will not receive any complaints?

(c) Suppose you have an unlimited number of cash coupons, instead of kn cash coupons. Each member may receive 0 or more cash coupons, and you distribute the cash coupons randomly. For each rank r, let t_r be the total number of coupons members at the rank receive altogether. What is the minimum number of cash coupons you will need to distribute so that there is at least one rank r for which $t_r \geq 2k$?

Briefly explain how your answers are derived.

Solution: (a) $(kn)!$, the number of permutation of the kn cash coupons.

(b) $(k!)^n$. The value of each cash coupon given to a person at the lowest rank should be less than the value of any cash coupon given to a person in the second lowest rank, and so on. Therefore, once sorted, the k coupons distributed to each rank are fixed. Within each rank, there are $k!$ ways to distribute the k cash coupons to the k members. By the product principle, the answer is $(k!)^n$.

(c) $(2k-1)n+1$. This is the same as finding the minimum number x of pigeons such that at least one of the n pigeonholes contain at least $2k$ objects. By the pigeonhole principle, find the minimum x such that $\lceil x/n \rceil = 2k$. It is easy to see that this equality holds for $x \in \{(2k-1)n+1, \ldots, 2kn\}$. Thus, the minimum value of $x = (2k-1)n+1$.

Grading: 3, 4, 4.

Student ID: ____________

Problem 8: [15 pts] (Counting Combinations)
A shopping bag contains 12 pair of gloves. Each pair consists of two different gloves, one for the left hand and another for the right hand, and the two gloves are in the same color. Moreover, different pairs are in different colors. You blindly take 6 gloves (**not** 6 pairs of gloves) from the bag. How many possible ways are there if

(a) the number of colors of these 6 gloves is 3?

(b) the number of colors of these 6 gloves is 4?

(c) these 6 gloves have different colors?

Briefly explain how your answers are derived.

Solution: (a) *The number of colors of these 6 gloves is 3* means you have picked 3 pairs of gloves out of 12. So there are $\binom{12}{3}$ possible cases.

(b) *The number of colors of these 6 gloves is 4* means firstly you pick 2 pairs of gloves out of 12 so there are $\binom{12}{2}$ possibilities. Then the remaining 2 gloves are from 2 different pairs out of the remaining 10 pairs. You choose only one glove from each pair. So there are $\binom{10}{2}\cdot 2\cdot 2$ possibilities. Overall there are $\binom{12}{2}\binom{10}{2}\cdot 2\cdot 2$ cases.

(c) *These 6 gloves have different colors* means these gloves are from different pairs. Firstly you pick 6 pairs of gloves out of 12 pairs. There are $\binom{12}{6}$ possibilities. Then you choose only one glove from each pair. So there are $\binom{12}{6}\cdot 2^6$ cases in total.

Grading: 5, 5, 5.