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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – FALL 2007

Midterm Examination 1 – Solution Key

Date: Tue, October 09, 2007   Time: 19:00–20:30   Venues: LTA, LTB

Name: ____________	Student ID: ____________
Email: ____________	Lecture and Tutorial: ________

Instructions

- This is a closed book exam. It consists of 18 pages and 8 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	Total
Points	14	13	10	13	12	11	15	12	100
Score									

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As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ___________________________

Student's Signature:     _______________________
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Problem 1: [14 pts]

There will be n guests of honor at an award ceremony tomorrow. The guests will be seated on a platform in a row of n seats facing the audience.

(a) How many different ways are there to seat the n guests in the n seats?

(b) Let $n > 2$.
Suppose now that two of the guests had a fight and will not agree to sit next to each other. How many ways are there to seat the n guests so that the two who fought do not sit next to each other? Explain how you derived your answer.

(c) Let $n > 10$.
Now suppose that the guests include five married couples, and for each married couple, we want the husband and wife to sit next to each other. How many ways are there to seat the guests so that this happens? Explain how you derived your answer.
[Note: this question is independent of (b); that is, we no longer assume that there are two guests who fought each other.]

ANS:

(a) $n!$

(b) We first count the number of ways to sit them *next to each other*. We can do this by thinking of the two people as one group and all of the other individuals as groups and noticing that there are $(n-1)!$ different ways of seating the groups. Since there are 2 different ways of seating the two people in the same group, there are $2(n-1)!$ ways to seat them so that they're next to each other.

Then there are

$$n! - 2(n-1)! = (n-2)(n-1)!$$

ways of seating them so that they're *not* next to each other.

(c) As in the first part of (b), we treat each of the married pairs as a group. There are then $(n-5)!$ ways of placing the $n-5$ groups. Since each group has 2 different ways of being ordered, the total number is

$$2^5(n-5)! = 32(n-5)!.$$

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Problem 2: [13 pts]

Recall that $Z_n = \{0, 1, 2, \ldots, n-1\}$.

(a) How many 5-element subsets of Z_{10} contain at least one of the members of Z_3? Explain how you derived your answer.

(b) How many 5-element subsets of Z_{10} contain two odd and three even numbers? Explain how you derived your answer.

(c) Let n be any positive odd integer. Show that the number of even-sized subsets of Z_n is equal to the number of odd-sized subsets of Z_n.

ANS:

(a) There are $\binom{7}{5}$ sets that do not contain *any* element of Z_3. So, the number that contain at least one element of Z_3 is

$$\binom{10}{5} - \binom{7}{5}.$$

(b) By the product priciple,

$$\binom{5}{2} \cdot \binom{5}{3}.$$

(c) Let n be odd. Recall that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Let $O_n = \{1, 3, 5, \ldots, n\}$ and $E_n = \{0, 2, 4, \ldots, n-1\}$ be, respectively, the sets of odd and even non-negative integers $\leq n$. Now, if k is odd, then

$$f(k) = n - k$$

is actually a bijection between O_n and E_n. So

$$\sum_{k \in O_n} \binom{n}{k} = \sum_{k \in O_n} \binom{n}{n-k} = \sum_{k \in E_n} \binom{n}{k}$$

which is what we wanted to prove.

An alternative approach to (c) is to note that

$$2^n = (1+1)^n = \sum_{0 \le i \le n} \binom{n}{i} \quad \text{and} \quad 0 = (1-1)^n = \sum_{0 \le i \le n} (-1)^i \binom{n}{i}$$

Summing the two gives

$$\begin{aligned} 2^n &= \sum_{0 \le i \le n} \binom{n}{i} + \sum_{0 \le i \le n} (-1)^i \binom{n}{i} \\ &= \sum_{0 \le i \le n} \left(1 + (-1)^i\right) \binom{n}{i} \\ &= 2 \sum_{i \in E_n} \binom{n}{i} \end{aligned}$$

So

$$\sum_{i \in E_n} \binom{n}{i} = 2^{n-1}.$$

Then

$$\begin{aligned} \sum_{i \in O_n} \binom{n}{i} &= \sum_{0 \le i \le n} \binom{n}{i} - \sum_{i \in E_n} \binom{n}{i} \\ &= 2^n - 2^{n-1} \\ &= 2^{n-1} \\ &= \sum_{i \in E_n} \binom{n}{i}. \end{aligned}$$

Notice that this second proof *never* used the fact that n was odd. The statement is actually true for all n, not just n odd.

Problem 3: [10 pts]

Our office door has a lock whose keypad contains only the 8 digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. For technical reasons, legal key-codes should satisfy the following three requirements:

(i) They are 4 digits long with all of the digits being different.

(ii) They must end with an even digit.

(iii) They cannot have their first digit equal to 0.

For example, 1350 and 5432 are legal key-codes, while 1150, 1357 and 0536 are not legal key-codes.

How many legal key-codes are there?
Explain how you derived your answer.

ANS: We split the set of legal key-codes into two sets; those that end with a 0 and those that don't.

Those that end with a 0 have 7 possible first digits, 6 possible second ones and 5 possible third ones. So the total number is $7 \cdot 6 \cdot 5 = 210$.

Those that do not end with a 0 have 3 possible last digits, 6 possible first ones, 6 possible second ones and 5 possible third ones. So the total number is $3 \cdot 6 \cdot 6 \cdot 5 = 540$.

Adding the two gives $210 + 540 = 750$.

Problem 4: [13 pts]

(a) How many surjections (onto functions) are there from S_7 to S_6? Explain how you derived your answer.

(b) How many surjections (onto functions) are there from S_8 to S_6? Explain how you derived your answer.

ANS: A surjection from S_7 to S_6 must have the property that for some $a \in S_6$, there are two values $x, y \in S_7$ such that $f(x) = f(y) = a$, while for all other $a' \neq a$, there is only one x such that $f(x) = a'$.

We can therefore partition the surjections into sets classified by which two values x, y get grouped together. There are exactly $\binom{7}{2} = 21$ such sets. Note that once we have chosen the two items that get paired together, the items in S_7 are now divided into 6 different groups. There are exactly $6! = 720$ different ways to map these groups into S_6, so the total answer is

$$\binom{7}{2} 6! = 15{,}120.$$

ANS:

(a) Following the reasoning above we note that a surjection either maps three items from S_8 into the same item in S_6, mapping the remaining 5 items from S_8 into the remaining 5 items in S_6; or, it maps a pair of items in S_8 into the same item in S_6, another pair of items in S_7 into the same item in S_6 and the remaining 4 items in S_8 into the remaining 4 items in S_6.

The number of ways of choosing the triple is $\binom{8}{3}$. Once the triple is chosen, there are $6!$ ways of mapping the 6 groups into S_6.

The number of ways of choosing two pairs is $\frac{1}{2}\binom{8}{2}\binom{6}{2}$. Once the two pairs are chosen, there are $6!$ ways of mapping the 6 groups into S_6. The final solution is then

$$\binom{8}{3} 6! + \frac{1}{2}\binom{8}{2}\binom{6}{2} 6! = 266 \cdot 6! = 191{,}520$$

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Problem 5: [12 pts]

Consider a Cartesian coordinate system.

(a) Suppose we start at $(0,0)$ and at each step move either one unit to the right or one unit down. Our final destination is $(20,-10)$. How many different paths are there from $(0,0)$ to $(20,-10)$? Explain how you derived your answer.

(b) Suppose we start at $(0,0)$ and at each step move either one unit to the right or one unit down. We walk for 10 steps this way without any specific predetermined destination. How many different paths of 10 steps are there? Explain how you derived your answer.

(c) Suppose we start at $(0,0)$ and at each step move one unit either up, down, left or right. We walk for k steps this way without any specific predetermined destination. How many different paths of k steps are there? Explain how you derived your answer.

ANS:

(a) Every path consists of 30 steps, of which 20 are "right steps" and 10 are "down steps". The answer is then

$$\binom{30}{10} = \binom{30}{20}.$$

(b) There are ten steps, each of which has two choices. The answer is then

$$2^{10}.$$

(c) There are k steps, each of which has four choices. The answer is then

$$4^k.$$

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Problem 6: [11 pts]

We want to divide 12 students into three groups.

(a) If the three groups are of sizes 3, 4 and 5, how many ways are there to do the grouping? Explain how you derived your answer.

(b) If the three groups are all of the same size, how many ways are there to do the grouping? Explain how you derived your answer.

Ans:

(a) Since the three groups are of different sizes, this is equivalent to the number of ways of labeleling 12 objects with three different colors. The answer is thus equal to the trinomial coefficient:

$$\binom{12}{3\ 4\ 5} = \frac{12!}{3!\,4!\,5!}.$$

(b) Since the three groups are of the same size and hence indistinguishable, the answer is

$$\frac{1}{3!}\binom{12}{4\ 4\ 4} = \frac{12!}{3!\,4!\,4!\,4!}.$$

Problem 7: [15 pts]

Consider the following modular equation

$$a \cdot_n \bar{x} = b,$$

where n is a positive integer and $a, b, \bar{x} \in Z_n$. Given n, a, b, the equation has a solution if we can find $\bar{x} \in Z_n$ that satisfies the equation.

(a) One way to check whether a has a multiplicative inverse in Z_n is to try to find integers x, y satisfying some equation that involves x, y, a, n. What is the equation?

(b) Suppose a has a multiplicative inverse in Z_n. Consider the equation that you wrote down in part (a). Are the integers x, y that satisfy that equation unique?

If the answer is yes, prove this.

If the answer is no, give a counterexample. That is, find (i) a positive integer n, (ii) a in Z_n such that a has a multiplicative inverse in Z_n, and (iii) two different pairs x, y that satisfy the equation in part (a).

(c) Find a solution $\bar{x}$ in Z_{12} for the following equation:

$$5 \cdot_{12} \bar{x} = 8.$$

Show how you found the solution.

Is the solution unique? Explain how you know the answer to this question.

ANS:

(a) $ax + ny = 1$.

(b) Not unique. For example, let $n = 12$ and $a = 5$. Then the two pairs $x = -7, y = 3$ and $x = -19, y = 8$ both satisfy the equation

$$ax + ny = 5x + 12y = 1.$$

(c) We proved in class that if a has a multiplicative inverse a' in Z_n, then the *unique* solution to the equation is $\bar{x} = a' \cdot_n b$.

If $ax + ny = 1$ then $a' = x \bmod n$.

From the answer of part (b),

$$a' = -7 \bmod 12 = -19 \bmod 12 = 5.$$

So the unique solution is

$$\bar{x} = 5 \cdot_{12} 8 = 4.$$

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Problem 8: [12 pts]

(a) Does there exist an x in Z_{99} that solves

$$123 \cdot_{99} x = 5?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove the fact.

(b) Does there exist an x in Z_{100} that solves

$$123 \cdot_{100} x = 5?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove the fact.

ANS: No.
Suppose there was a solution x. Then there is some integer q such that

$$123\,x = 99\,q + 5$$

so

$$123\,x - 99\,q = 5.$$

But the left hand side of this equation is divisible by 3 and the right hand side is not. Contradiction. So such an x does not exist.

(b) Yes. $x = 35$.
To derive this solution you first use the extended GCD algorithm to find that

$$123 \cdot (-13) + 100 \cdot 16 = 1.$$

So

$$87 = (-13) \bmod 100$$

is the multiplicative inverse of 123 in Z_{100}. Therefore the unique solution to the problem is

$$87 \cdot_{100} 5 = (87 \cdot 5) \bmod 100 = 435 \bmod 100 = 35.$$