**COMP 2711 Discrete Mathematical Tools for CS**
**Spring 2016 – Written Assignment # 4**
**Distributed: March 9, 2016 – Due: March 16, 2016**

**Solutions**

Your solutions should contain (i) your name, (ii) your student ID #, (ii) your email address, (iv) your lecture section and (v) your tutorial section. Your work should be submitted before 4PM of the due date in the collection bin outside Room 4210 (Lift 21). Make sure you submit into the box labeled with the correct lecture section.

For all questions you should provide a short explanation as to how you derived the solution. That is, if the solution is 20, you shouldn't just write down 20. You need to explain why it's 20.

**Problem 1:** Consider the sets $S_4 = \{a, b, c, d\}$ and $S_5 = \{1, 2, 3, 4, 5\}$?
(a) How many functions are there from the set $S_4$ to $S_5$?
(b) How many *one-to-one* functions are there from the set $S_4$ to $S_5$?
(c) How many *onto* functions are there from the set $S_4$ to $S_5$?
(d) How many *bijections* are there from the set $S_4$ to $S_5$?
(e) How many functions are there from the set $S_5$ to $S_4$?
(f) How many *one-to-one* functions are there from the set $S_5$ to $S_4$?
(g) How many *onto* functions are there from the set $S_5$ to $S_4$?
(h) How many *bijections* are there from the set $S_5$ to $S_4$?
(i) How many functions are there from the set $S_4$ to $S_4$?
(j) How many *one-to-one* functions are there from the set $S_4$ to $S_4$?
(k) How many *onto* functions are there from the set $S_4$ to $S_4$?
(l) How many *permutations* are there from the set $S_4$ to $S_4$?

Solution: (a) From problem 1, $5^4 = 625$.

(b) There is a bijection between the set of one-to-one functions from $S_4$ to $S_5$ and the set of 4-element permutations of $S_5$. So the answer is the number of 4-element permutations of $S_5$, which is $5^{\underline{4}} = 5 \times 4 \times 3 \times 2 = 120$.

(c) None. If $S_4 = \{a, b, c, d\}$ then $\{f(a), f(b), f(c), f(d)\}$ is a set of size at most 4 but $S_5$ has size 5.

(d) None. Since a bijection is both one-to-one and onto and there is no onto function from $S_4$ to $S_5$.

(e) From problem 1, $4^5 = 1024$.

(f) None. If $f$ is one-to-one then the set $\{f(1), f(2), f(3), f(4), f(5)\}$ contains 5 distinct elements but $S_4$ contains only 4 elements.

(g) Since $|S_5| = 5$ and $|S_4| = 4$, for $f$ to be onto, there will be two elements $x$ and $y$ such that $f(x) = f(y)$. There are $\binom{5}{2}$ possible ways of choosing this $\{x, y\}$ pair. Once we have chosen the unique pair such that $f(x) = f(y)$ there are 4! ways of assigning values in $S_4$ to the $f(x_i)$. So the answer is

$$\binom{5}{2} \cdot 4! = 240.$$

(h) None. Since there is no one-to-one function from $S_5$ to $S_4$.

(i) From problem 1, $4^4$.

(j), (k) and (l): The answer to *all* of these problems is $4! = 24$. It is possible to solve them each separately (which you had to do). After the 2nd tutorial, though, you will know that a function from a set to another set of the same size is one-to-one if and only if it is onto. If the two sets are the same, this says that the function is one-to-one if and only if it is onto if and only if it is a permutation. So, all three problems have the same answer which is the number of permutations of 4 items, i.e., $4! = 24$.

**Problem 2:** A base ten number is a string of five digits, where the digits are from the set $\{0, 1, \ldots, 9\}$ but the first digit cannot be 0 (so 52375 is a valid number but 02323 and 2323 are not).
(a) How many five-digit base ten numbers are there?
(b) How many five-digit numbers have no two consecutive digits equal?
(c) How many have at least one pair of consecutive digits equal?

SOLUTION: (a) First, note that "a five-digit base ten number" means a string of five digits, where the first digit is not 0 and each of the following digits is in the set $\{0, 1, \ldots, 9\}$. By the product rule, the number of these is $9 \times 10^4$, or 90000.

(b) If no two consecutive digits can be equal, then there are nine choices for the first digit, nine for the second (any digit other than the first), nine for the third (any digit other than the second), and so on. By the product principle, the total number is $9^5$.

(c) By the sum principle, the total number of five-digit numbers equals the number of five-digit numbers that have no two consecutive digits equal plus the number of five-digit numbers that have at least one pair of consecutive digits equal. Thus, letting $x$ denote the number of the latter, we have $9 \times 10^4 = 9^5 + x$; so, $x = 9 \times 10^4 - 9^5 = 30951$.

**Problem 3:** Suppose you are organizing a panel discussion on allowing karoke on campus. Participants will sit behind a long table in the order in which you list them. You must choose 4 administrators from a group of 10 and 5 students from a group of 15.

(a) If the administrators must sit together in a group and the students must sit together in a group, in how many ways can you choose and list the 9 people?

(b) If you must alternate students and administrators, in how many ways can you choose and list them?

SOLUTION: (a) Assume the seats are numbered $1, 2, \ldots, 9$. Once the participants are chosen you need to decide how to place them in the seats. If the administrators must sit together in a group and the students must sit together in another group, there are two cases: (i) the administrators are in seats $1, 2, 3, 4$ and the students in seats $5, 6, 7, 8, 9$, or (ii) the administrators are in seats $6, 7, 8, 9$ and the students in seats $1, 2, 3, 4, 5$.

For the first case, the number of ways of choosing 4 administrators and placing them in seats $1, 2, 3, 4$ is $10^{\underline{4}}$ and the number of ways of choosing 5 students and placing them in seats $5, 6, 7, 8, 9$ is $15^{\underline{5}}$. So the number of ways of forming the panel is $10^{\underline{4}} \times 15^{\underline{5}}$. The second case is similar. Since the two sets corresponding to the two cases are disjoint, we apply the sum principle to get the total number as $10^{\underline{4}} \times 15^{\underline{5}} \times 2 = 3{,}632{,}428{,}800$.

(b) There is only one way to alternate the students and the administrators, which is to place the students in seats $1, 3, 5, 7, 9$ and the administrators in seats $2, 4, 6, 8$. Similar to each of the two cases in (a), the number of ways of choosing the panel is $10^{\underline{4}} \times 15^{\underline{5}} = 1{,}816{,}214{,}400$.

**Problem 4:** In class we stated that
*each row of Pascal's triangle first increases and then decreases.*
In this question you will prove this statement.

(a) Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ prove that

$$\text{if } 0 < k \leq n/2 \text{ then } \binom{n}{k-1} < \binom{n}{k}.$$

(b) Using part (a) and the fact that $\binom{n}{k} = \binom{n}{n-k}$ prove that
*each row of Pascal's triangle first increases and then decreases.*

SOLUTION: (a)

$$\begin{aligned}
\binom{n}{k} \div \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} \cdot \frac{(k-1)!(n-k+1)!}{n!} \\
&= \frac{n-k+1}{k} \\
&= \frac{n}{k} - 1 + \frac{1}{k}
\end{aligned}$$

Since $0 < k \le n/2$, we have $n/k \ge 2$ and $1/k > 0$ and hence $n/k - 1 + 1/k > 1$. So, $\binom{n}{k} > \binom{n}{k-1}$.

(b) Each row of Pascal's triangle lists $\binom{n}{k}$ for $k = 0, 1, \ldots, n$.

When $0 < k < n/2$, we proved in part (a) that $\binom{n}{k-1} < \binom{n}{k}$. So the first half of the row is increasing.

When $n/2 \le k < n$, we have $0 < n - k \le n/2$. By the result of part (a),

$$\binom{n}{k+1} = \binom{n}{n-k-1} < \binom{n}{n-k} = \binom{n}{k}.$$

So the second half of the row is decreasing.

In summary, each row of Pascal's triangle first increases and then decreases.

**Problem 5:** Give two proofs that

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

Your first proof should be purely algebraic, i.e., just plug in the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and show that the left side equals the right side. Your second proof should be combinatorial, i.e., it should show that the left and right sides are just two different ways to count the same thing.

SOLUTION: Proof 1:

$$\binom{n}{k}\binom{k}{j} = \frac{n!k!}{k!(n-k)!j!(k-j)!} = \frac{n!}{(n-k)!j!(k-j)!}.$$

Similarly, $\binom{n}{j}\binom{n-j}{k-j}$ equals the same expression.

Proof 2: The number of ways to choose $k$ items from $n$ items and then choose $j$ items from the chosen $k$ items is $\binom{n}{k}\binom{k}{j}$. We can also carry out this process in the following way: First choose $j$ items from $n$ items, and then choose $k-j$ more items from the remaining $n-j$ items. The number of ways to do this is $\binom{n}{j}\binom{n-j}{k-j}$.

**Problem 6:** (a) If you have thirteen distinct chairs to paint, in how many ways can you paint seven of them orange and six of them red?

(b) Now, how many ways can you paint four of them green, three of them blue, and six of them red?

SOLUTION: (a) $13!/(7!\,6!) = 1{,}716$. This is the number of ways to label seven of the chairs with the label orange, and six of the chairs with the label red.

(b) $13!/(4!\,3!\,6!) = 60{,}060$. This is the number of ways to label four of the chairs with the label green, three of the chairs with the label blue, and six of the chairs with the label red.

**Problem 7:** Show that for any $n+1$ distinct integers, we can pick two of them such that their difference is divisible by $n$. (Hint: Use the Pigeonhole Principle on the remainders of the integers when divided by $n$).

SOLUTION: If we divide each of the $n+1$ distinct integers by $n$, we get $n+1$ remainders. On the other hand, there are only $n$ possible values for the remainders are $0, 1, \ldots, n-1$. By the Pigeonhole Principle, two of the remainders must be the same.

Let $a$ and $b$ be the two integers with the same remainder. Then, the remainder of $a-b$ when divided by $n$ is 0. This means that $a-b$ is divisible by $n$. The statement is proved.

**Problem 8:** **(Challenge)** Suppose there are 20 balls and 3 boxes. The three boxes are labeled with 'A', 'B' and 'C' respectively. In how many different ways can we distribute the balls into the boxes under each of the following conditions?

(a) The twenty balls are labeled with integers $1, 2, \ldots, 20$ respectively so that each ball is distinct and is different from other balls.

(b) All the balls are identical and hence are indistinguishable from each other.

SOLUTION: (a) Each ball can be distributed into each of 3 boxes. Hence the answer is:

$$3^{20}.$$

(b) The challenge with this problem is that the balls are indistinguishable from each other. We reason as follows:

Suppose the 20 balls are all white and imagine that we have two other red balls. We arrange the 22 balls on one line and distribute the 20 white balls into the boxes as follows: All the white balls to the left of the first red ball go to Box A; all the white balls between the two red ball go to Box B; and all the white balls to the right the the second red ball go the Box C.

We see that each way to place the two red balls corresponds to one distinct way to distribute the 20 white balls. The number of different ways to place the two red balls is:

$$\binom{22}{2}.$$

This is the answer to the question.