Recurrence for a Probability Problem

Problem: A line of 100 people are waiting to enter an airplane. They each hold a ticket for one of the 100 seats on the flight. (For convenience, assume that the nth passenger in line has a ticket for seat number n, e.g., the first person has a ticket for seat number 1, etc.).

Unfortunately, the first person in line is crazy, will ignore the seat number on his ticket, and will pick a random seat out of the 100 to sit in. All of the other passengers are quite normal, though, and will go to their proper seat unless someone is already sitting in it. If someone is already sitting in their seat, they will choose an unoccupied seat at random and sit in it.

What is the probability that the last (100^{th}) person will sit in his own (the 100^{th}) seat?

This problem was taken from *Make Magazine*, v10, p122, "Aha! Puzzle This" by Michael H. Pryor.

Solution: Consider the problem with $N \geq 2$ passengers and N seats. Let $f(N)$ be the probability that the N-th passenger sits on the N-th seat.

It is obvious that $f(2) = \frac{1}{2}$.

We claim that, for $N > 2$

$$f(N) = \frac{1}{N} + \frac{1}{N} f(N-1) + \frac{1}{N} f(N-2) + \ldots + \frac{1}{N} f(2). \tag{1}$$

Here are the reasons: Let m be the seat that the crazy passenger takes.

- If $m = 1$, the N-th passenger for sure will get seat N. The probability of this case is $\frac{1}{N}$.
- If $2 <= m <= N-1$, every passenger before m sits in their own seat. When it comes time for passenger m to sit down, the situation is exactly the same as if we have $N-(m-1)$ people and $N-(m-1)$ seats, and the first passenger is the crazy passenger. In such a case, the probability for the last passenger to take seat N is $f(N-(m-1))$. The probability of each such case is $\frac{1}{N}$.
- If $m = N$, the N-th passenger has no chance of getting seat N.

Multiplying both sides of Equation 1 by N, we get

$$Nf(N) = 1 + f(N-1) + f(N-2) + \ldots + f(2),\ \forall N > 2. \tag{2}$$

Applying the Equation (2) to the case of $N-1$, we have

$$(N-1)f(N-1) = 1 + f(N-2) + f(N-3) + \ldots + f(2),\ \forall N \geq 3 \tag{3}$$

Together, Equations 2 and 3 imply that

$$f(N) = f(N-1)\ \forall N \geq 3$$

Therefore,

$$f(N) = f(N-1) = \ldots = f(2) = \frac{1}{2}.$$