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HKUST – Department of Computer Science
COMP170: Discrete Math Tools for CS – FALL 2005

Midterm Examination 1 – *Solution-Key*

Date: Tue, October 04, 2005 Time: 19:00–20:30 Venues: LTD, LTC, LTB

Name: ______________________ Student ID: ________________
Email: ______________________ Lecture and Tutorial: ________

Instructions

- This is a closed book exam. It consists of 15 pages and 9 questions.
- Please write your name, student ID, Email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	9	Total
Points	7	7	7	10	15	14	14	12	14	100
Score										

Student ID: ____________

As part of HKUST's introduction of an honor code, the HKUST senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ______________________

Student's Signature:  _____________________
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Problem 1: [7 pts] For the multiple choice problems below, **circle** the correct answer. For this question, no work needs to be shown. In what follows S_n is a set containing n items

(a) Assume $n > m$. The number of *one-to-one* functions from S_n to S_m is

(i) 0 (ii) n^m (iii) m^n (iv) $n^{\underline{m}}$ (v) $\binom{n}{m}$

(b) Assume $m > n$. The number of *one-to-one* functions from S_n to S_m is

(i) 0 (ii) n^m (iii) m^n (iv) $n^{\underline{m}}$ (v) $\binom{n}{m}$

(c) Assume $m > n$. The number of *onto* functions from S_n to S_m is

(i) 0 (ii) n^m (iii) m^n (iv) $n^{\underline{m}}$ (v) $\binom{m}{n}$

(d) Assume $m \leq n$. You are given n distinct chairs. How many different ways are there to color m chairs red and $n - m$ chairs blue.

(i) 2^n (ii) n^m (iii) m^n (iv) $n^{\underline{m}}$ (v) $\binom{n}{m}$

(e) You are given n distinct chairs. Each chair is colored either red or blue. How many different ways are there to color the chairs?

(i) 2^n (ii) 3^n (iii) $n!$ (iv) n^2 (v) $\binom{2n}{n}$

(f) How many elements in Z_{31} have a multiplicative inverse?

(i) 1 (ii) 30 (iii) 31 (iv) 15 (v) 16

(g) How many elements in Z_{32} have a multiplicative inverse?

(i) 1 (ii) 31 (iii) 32 (iv) 16 (v) 17

a) (i)
b) see below
c) (i)
d) (v)
e) (i)
f) (ii)
g) (iv)

The correct answer to (b) is $m^{\underline{n}}$. Due to a typographical error this answer was not included as one of the choices so, as announced during the exam, this question was marked as correct for everyone.

Student ID: ____________

Problem 2: [7 pts] Suppose that a computer password is eight (8) characters long where each character is an upper case letter from $\{A, B.C, \ldots, Z\}$. Suppose further that in a legal password, no letter can appear immediately after itself, e.g., `ABCABCXY` is a legal password but `ABBCDEFG` is not a legal password. How many legal passwords are there?

This is a list with 8 elements. There are 26 possibilities for the first element and 25 possibilities for the remaining 7 elements (since they can not be the same as the letter before them). The product principle then gives

$$26 * 25^7$$

legal passwords.

Student ID: ____________

Problem 3: [7 pts] You have twelve distinct doors to paint.

How many ways can you paint them so that 3 of them are red, 4 of them are blue and 5 of them are green?

First choose 3 *out of the* 12 *as red and then choose* 4 *out of the remaining* 9 *as blue. The* 5 *doors still unpainted will be green. This gives*

$$\binom{12}{3}\binom{9}{4} = \frac{12!}{3!\ 9!}\ \frac{9!}{4!\ 5!} = \frac{12!}{3!\ 4!\ 5!}.$$

Alternatively, this can be solved using trinomial coefficients:

$$\binom{12}{3\ 4\ 5} = \frac{12!}{3!\ 4!\ 5!}$$

Student ID: ____________

Problem 4: [10 pts] Let $S_5 = \{a, b, c, d, e\}$ and $S_6 = \{1, 2, 3, 4, 5, 6\}$.
How many different *onto* functions are there from S_6 to S_5?

Notice that an onto function f from S_6 to S_5 has the following property: there is exactly one pair $i, j \in S_6$ such that $f(i) = f(j)$. This can be thought of as partitioning S_6 into exactly 5 non-empty subsets. One of the subsets has two items; the others have exactly one item.

Once this pair i, j is known, we can built exactly $5! = 120$ different onto functions that have $f(i) = f(j)$ (by assigning a unique letter in S_5 to each subset in the partition).

So, the number of onto functions is the number of ways of choosing $i, j \in S_6$ times $5!$ which is

$$\binom{6}{2} 5! = 15 * 120 = 1800$$

Student ID: ____________

Problem 5: [15 pts] Suppose that we have 20 teachers and 50 students and want to form a student-teacher committee containing exactly 8 people, 3 of whom are teachers and 5 of whom are students.

a) How many different possible committees can we form?

b) Now suppose that one of the students is a PG student who is also a teacher. This student is allowed to serve on the committee as a teacher or as a student (if he serves as a teacher then he doesn't count as a student; if he serves as a student, then he doesn't count as a teacher). How many different possible committees can we form now?

c) Now suppose that we change (b) so that we have **two** PG students who are also teachers. With the same conditions as part (b), how many different possible committees can we form now?

(a) By the product principle this is just

$$\binom{20}{3}\binom{50}{5}$$

(b) The major difficulty in solving this problem was "double-counting", i.e., in whch the problem was split up into non-*disjoint cases.*

Here is one possible corect solution (there were many other ways to derive this). We split the problem up into up into two cases.

Case 1) The PG student is on the committee as a student.
In this case we still have to choose 3 *teachers out of* 20 *but now only have to choose 4 students out of* 49 *(since the 5th student member is the PG student). The number of such committees is*

$$\binom{20}{3}\binom{49}{4}$$

Case 2) The PG student is NOT on the committe as a student. In this case we choose 3 *teachers out of* 21 *teachers (since the PG student is one of our choices) and have to choose 5 students out of* 49 *The number of such committees is*

$$\binom{21}{3}\binom{49}{5}$$

Notice that every possible committee is counted in one of the two cases and no committee is counted in both cases. So, the final answer is Case 1 + Case 2 equals

$$\binom{20}{3}\binom{49}{4}+\binom{21}{3}\binom{49}{5}$$

(c) Again, the major difficulty students had in solving this problem was double-counting.

What follows is one possible correct solution.

We split the problem up into four cases

Case 1) PG student 1 is on the commitee as a student but PG student 2 is not
Case 2) PG student 2 is on the commitee as a student but PG student 1 is not
Case 3) Both PG students 1 and 2 are on the committee as students
Case 4) Neither of PG students 1 and 2 are on the committee as students

Using the same reasoning as in part (b) the number of committees for Case 1 is

$$\binom{21}{3}\binom{48}{4}$$

since 3 *teachers have to be chosen out of the 20 teachers and PG student 2, and 4 students need to be chosen out of the 48 remaining students*

Switching PG students 1 and 2 The same calculation applies to Case 2 to give

$$\binom{21}{3}\binom{48}{4}$$

For case 3 we choose the 3 teachers out of the 20 teachers but only need to choose 3 more students out of the 48 remaining students, so the answer is

$$\binom{20}{3}\binom{48}{3}$$

Finally, for case 4, we choose the 3 teachers from the 20 teachers + 2 Pg students, and choose the 5 students from the non-PG students to get

$$\binom{22}{3}\binom{48}{5}$$

Notice that every possible committee is counted in one of the four cases and no committee is counted in more than one case. So, the final answer is the sum of the four cases which is:

$$2\binom{21}{3}\binom{48}{4}+\binom{20}{3}\binom{48}{3}+\binom{22}{3}\binom{48}{5}$$

Student ID: ____________

Problem 6: [14 pts] Give a combinatorial proof of the identity

$$\binom{n}{3}\binom{n-3}{2} = \binom{n}{2}\binom{n-2}{3}.$$

Note: An algebraic proof of this identity will *not* be accepted as a solution.

Consider that we have to construct two committees out of a group of n people.
Committee one contains 3 people
Committe two contains 2 people.
A person can serve on at most one committee.

How many ways are there to choose these two committees?

Approach 1: First choose one of the $\binom{n}{3}$ possible committees of 3 people. Then choose the second committee out of the remaining $n-3$ people. There are $\binom{n-3}{2}$ possible ways of choosing the second committte. By the product principle, the total number of ways of choosing the two committes is then

$$\binom{n}{3}\binom{n-3}{2}$$

Approach 2: First choose one of the $\binom{n}{2}$ possible committees of 2 people. Then choose the first committee out of the remaining $n-2$ people. There are $\binom{n-2}{3}$ possible ways of choosing the first committte. By the product principle, the total number of ways of choosing the two committes is then

$$\binom{n}{2}\binom{n-2}{3}$$

Both approaches are counting the same thing *so the two equations are equal.*

Problem 7: [14 pts] Assume that $2 \leq k \leq n-2$. Consider the two identities

$$\binom{n}{k} = \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \tag{1}$$

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} \tag{2}$$

One of these identities is always true. One of these is false. Identify which one is true and which one is false.

For the one that is true, prove that it is always true (either an algebaic or combinatorial proof will be accepted).

For the one that is false, give an example of an $\mathbf{n}, \mathbf{k}$ pair for which the identity is false.

(1) is false and (2) is true.

To see that (1) is false you can plug in any *values for n and k.*

To see that (2) is true, recall that Pascal's relationship tells us that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Apply Pascal's relationship again to the two terms on the right to get:

$$\begin{aligned}\binom{n-1}{k} &= \binom{n-2}{k-1} + \binom{n-2}{k} \\ \binom{n-1}{k-1} &= \binom{n-2}{k-2} + \binom{n-2}{k-1}\end{aligned}$$

Plugging these two equalities back into the original Pascal's relationship gives

$$\begin{aligned}\binom{n}{k} &= \left(\binom{n-2}{k-1} + \binom{n-2}{k}\right) + \left(\binom{n-2}{k-2} + \binom{n-2}{k-1}\right) \\ &= \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}\end{aligned}$$

Proving (2).

Problem 8: [12 pts] Consider the equation

$$10 \cdot_{25} x = 3$$

Does this equation have a solution x in Z_{25}?

If yes, give a value for x that solves the equation.

If no, prove that such an x does not exist.

No.

Suppose that such an x existed. Then there would be some q such that

$$10 \cdot x = 25 \cdot q + 3$$

Or

$$3 = 10 \cdot x - 25 \cdot q = 5(2 \cdot x - 5 \cdot q).$$

Since 5 *divides the right hand side this implies that* 5 *divides* 3 *as well.*

Since 5 *does not divide* 3 *we have reached a contradiction, so no such x exists.*

Student ID: ____________

Problem 9: [14 pts]
(a) Does 65 have a multiplicative inverse in Z_{201}?

That is, is there an x in Z_{201} such that $65 \cdot_{201} x = 1$?

(b) Does 66 have a multiplicative inverse in Z_{201}?

In both (a) and (b,) if such an x exists, you should give it.

If such an x does not exist, prove that it does not exist.

Note that when giving the multiplicative inverse x it is *not necessary* to show how you derived it. It is to your benefit to do so, though. If you show your work and get a wrong answer for the inverse we can give you some partial credit. If you don't show your work, partial credit is not possible.

(a) 65 *does have an inverse.*
Using the extended GCD algorithm (or just by guessing) you can find that

$$(-34) * 65 + (11)201 = 1$$

so the multiplicative inverse of 65 mod 201 *is*

$$-34 \bmod 201 = 167.$$

(b) Such an x *does not exist because* $gcd(66, 201) = 3$ *and an inverse exists if and only if* $gcd(66, 201) = 1$.

In order to get full credit for this you needed to explain how you know that $gcd(66, 201) \neq 1$. *You could have run the GCD algorithm to find that* $gcd(66, 201) = 3$ *or just simply have noted that* $66 = 3 \cdot 22$ *and* $201 = 3 \cdot 67$.

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