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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – Spring 2010

Final Examination

Date: Monday May 31, 2010 Time: 16:30- 19:00

Sketch Solution Key

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: ________

Instructions

- This is a closed book examination. It consists of 28 pages and 9 questions.
- Please write your name, student ID, email, lecture and tutorial sections on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page (and sometimes has an extra page for you to do work on). This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Only use notation given in class. Do not use notation that you have learnt outside of this class that is nonstandard.
- Calculators may be used for the examination.

Questions	1	2	3	4	5	6	7	8	9	Total
Points	10	10	14	12	9	13	12	12	8	100
Score										

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As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

> I declare that the answers submitted for this examination are my own work.
>
> I understand that sanctions will be imposed, if I am found to have violated the University regulations governing academic integrity.
>
> Student's Name: ______________________
>
> Student's Signature: ______________________

Definitions and Formulas: This page contains some definitions used in this exam and a list of formulas (theorems) that you may use in the exam (without having to provide a proof). Note that you might not need all of these formulas on this exam.

Definitions:

1. $N = \{0, 1, 2, 3, \ldots\}$, the set of non-negative integers.
2. $Z^+ = \{1, 2, 3, \ldots\}$, the set of positive integers.
3. $Z_n = \{0, 1, 2, 3, \ldots, n-1\}$ for $n \in Z^+$.
4. R is the set of *real numbers*.
5. R^+ is the set of positive *real numbers*.
6. A *coin* is *fair* if it has probability $\frac{1}{2}$ of showing a head and probability $\frac{1}{2}$ of showing a tail.

Formulas:

1. $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$
2. If $0 < i < n$ then $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$.
3. $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.
4. $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
5. $\sum_{i=1}^{n-1} i = n(n-1)/2$
6. $\sum_{i=1}^{n-1} i^2 = \frac{2n^3-3n^2+n}{6}$
7. If $r \neq 1$ then $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.
8. If $r \neq 1$ then $\sum_{i=0}^{n} ir^i = \frac{nr^{n+2}-(n+1)r^{n+1}+r}{(1-r)^2}$.
9. The inclusion-exclusion theorem:
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$
10. If X is a random variable, then $E(X)$ denotes the *Expectation of* X and $V(X) = E\left((X - E(X))^2\right)$ denotes the *Variance of* X.
11. $f(n) = O(g(n))$ if there exist some $N > 0$ and positive constant c such that $\forall n > N$, $f(n) \le c \cdot g(n)$.
12. $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

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Problem 1: [10 pts]
For each of the following parts either give a solution x to the part (it is not necessary to show your work) or prove that no such solution exists.

(a) Consider the following set of modular equations:

$$\begin{aligned} x \bmod 26 &= 5 \\ x \bmod 51 &= 1 \end{aligned}$$

Does there exist an integer solution x, $0 \le x \le 1000$, satisfying both these equations?

(b) Does there exist an $x \in Z_{51}$ that solves

$$26 \cdot_{51} x = 4?$$

(c) Does there exist an $x \in Z_{51}$ that solves

$$27 \cdot_{51} x = 4?$$

Solution:

To start, note that $gcd(26, 51) = 1$ *and*

$$1 = 26 \cdot 2 + 51 \cdot (-1). \qquad (1)$$

(a) There is no solution.
Since $gcd(26, 51) = 1$ *the Chinese Remainder Theorem tells us there exists a unique solution to the equation in* $Z_{26 \cdot 51} = Z_{1326}$.
Solving using the method taught in class gives that this unique solution is $y = 1123$. *Since this is* > 1000 *there is no solution* x *to the equations satisfying* $0 \le x \le 1000$.

(b) Note that, from equation (1), the multiplicative inverse of 26 *in* Z_{51} *is* 2. *So, there exists a unique solution to the equation and it is*

$$x = 2 \cdot_{51} 4 = 8.$$

(c) There is no solution.
Suppose that such a solution x *existed. Then*

$$27 \cdot x = 51q + 4$$

for some integer q. *Then*

$$4 = 27x - 51q = 3(9x - 17q)$$

which is impossible since the right hand side of the equation is divisble by 3 *and the left hand side isn't.*

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Problem 2: [10 pts]

You are given an infinite supply of 3-cent and 7-cent postage stamps.

Note that some integer postage values can be formed using these stamps and some cannot. For example, it is impossible to form 8 cents of postage value using these stamps but 27 cents can be formed, e.g., by using three 7-cent stamps and two 3-cent stamps.

Which integer postage values can and which cannot be formed using just 3-cent and 7-cent postage stamps? You must prove the correctness of your answer.

Hint: Use induction

Solution: The answer is that all positive integer values except *for 1,2,4,5,8,11 are possible.*

To start, we say that a number is good *if it is possible to form it using the stamps and* bad *if it can't be.*

It is easy to see that 1,2,4,5,8,11 are bad.

It is also easy to see that
$3,\ 6 = 2\cdot 3,\ 7,\ 9 = 3\cdot 3,\ 10 = 3+7,\ 12 = 4\cdot 3,\ 13 = 2\cdot 3+7,\ 14 = 2\cdot 7$
are all good.

We now prove by induction that all numbers ≥ 12 *are good.*

Our induction hypothesis $p(n)$ *will be that all numbers* x, $12 \leq x \leq n$ *are good.*

We have already seen that the base case $p(14)$ *is true.*

Now assume that $p(n)$ *is true for some* $n \geq 14$. *Since* $12 \leq n-2 \leq n$, *we have that* $n-2$ *is good, i.e.,* $n-2 = 3a+7b$ *for some* $a, b \in N$. *Then*

$$n+1 = n-2+3 = 3(a+1)+7b$$

so $n+1$ *is good, so* $p(n+1)$ *is true.*

Then $p(n)$ *is true for all* $n \geq 14$, *so all* $n \geq 12$ *are good.*

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Problem 3: [14 pts]
A new restaurant at the university is offering a special deal.

There are 3 tables, A, B, C at the entry to the restaurant.
Each table contains 5 different types of food:
i.e., table A contains dishes A_1, A_2, A_3, A_4, A_5;
table B contains dishes B_1, B_2, B_3, B_4, B_5;
table C contains dishes C_1, C_2, C_3, C_4, C_5;

Answer the following questions. You only have to write the solution. It is not necessary to show your derivation.

(a) For this problem you should assume that all 15 dishes are different.

How many different ways are there to choose 6 dishes in total so that all the chosen dishes are different?

(b) For this problem you should again assume that all 15 dishes are different but you are now allowed to choose only two dishes from each table.

How many different ways are there to choose 2 dishes from each table (so that you have 6 dishes in total) and all the dishes are different?

(c) For this problem you should assume that all the tables share their first dish and all the other dishes are different, i.e., $A_1 = B_1 = C_1$, and there are 13 different dishes in total.

Now how many different ways are there to choose 2 dishes from each table so that all the dishes you have chosen are different?

(d) Now assume instead that two of the dishes on table A and B are the same, e.g., that $A_1 = B_1$, $A_2 = B_2$ and that all other dishes are different, i.e., that there are 13 different dishes in total.

Now how many different ways are there to choose 2 dishes from each table so that all the dishes you have chosen are different?

Solution:

(a)

$$\binom{15}{6} = 5005.$$

(b) You can choose 2 items from each table so the answer is

$$\binom{5}{2} \cdot \binom{5}{2} \cdot \binom{5}{2} = 10 \cdot 10 \cdot 10 = 1000.$$

(c) *Split the problem into two cases. In Case 1,* A_1 **is** *chosen from one of the tables; in Case 2,* A_1 **is not** *chosen.*

In Case 1, there are 3 possible tables from which A_1 *can be chosen; one* **other** *item is chosen from that table and* **two items each** *from the other 2 tables. So, the total number of Case 1 possibilities is*

$$3\binom{4}{1}\binom{4}{2}\binom{4}{2} = 3 \cdot 4 \cdot 6 \cdot 6 = 432.$$

In Case 2, since A_1 *is not chosen from any of the tables, two items are chosen from the* **four** *remaining items on each of the 3 tables. The number of possibilities is then*

$$\binom{4}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} = 6^3 = 216.$$

So, the final answer is

$$432 + 216 = 648.$$

Some students wrote the answer as

$$3\binom{4}{1}\binom{4}{2}\binom{4}{2} + \binom{4}{2}\binom{4}{2}\binom{4}{2}.$$

(d) *There are 4 different cases that need to be analyzed separately.*

(I) *Neither* A_1 *nor* A_2 *is chosen:*
This leaves three items to choose from at tables 1 and 2.
The total number of such cases is

$$\binom{3}{2} \cdot \binom{3}{2} \cdot \binom{5}{2} = 3 \cdot 3 \cdot 10 = 90.$$

(II) A_1 *is chosen and* A_2 *is not chosen:*
If A_1 *is chosen from table A then the total number of possibilities is*

$$\binom{3}{1}\binom{3}{2}\binom{5}{2}$$

while if A_1 *is chosen from table B the total number of possibilities is*

$$\binom{3}{2}\binom{3}{1}\binom{5}{2}.$$

So, the total number of case (II) possibilities is

$$2\binom{3}{1}\binom{3}{2}\binom{5}{2} = 2 \cdot 3 \cdot 3 \cdot 10 = 180.$$

(III) A_1 is not chosen and A_2 is chosen:
the analysis is the same as Case (II) giving

$$2\binom{3}{1}\binom{3}{2}\binom{5}{2} = 180.$$

(IV) Both A_1 and A_2 are chosen:
This splits into 4 different subcases.

(i) A_1 and A_2 are chosen from table A.
The number of possibilities is

$$\binom{3}{2}\binom{5}{2} = 3 \cdot 10 = 30.$$

(ii) A_1 and A_2 are chosen from table B.
The number of possibilities is again

$$\binom{3}{2}\binom{5}{2} = 3 \cdot 10 = 30.$$

(iii) A_1 is chosen from table A and A_2 from table B. The number of possibilities is

$$\binom{3}{1}\binom{3}{1}\binom{5}{2} = 3^2 \cdot 10 = 90.$$

(iv) A_1 is chosen from table B and A_2 from table A. The number of possibilities is again

$$\binom{3}{1}\binom{3}{1}\binom{5}{2} = 3^2 \cdot 10 = 90.$$

So, the total number of Case (IV) possibilities is

$$30 + 30 + 9090 = 240.$$

Adding all four cases together gives

$$90 + 180 + 180 + 380 = 690$$

as the final answer. Some students wrote this as follows (or something similar):

$$\binom{3}{2}\binom{3}{2}\binom{5}{2} + 2 \cdot 2\binom{3}{1}\binom{3}{2}\binom{5}{2} + 2\binom{3}{2}\binom{5}{2} + 2\binom{3}{1}\binom{3}{1}\binom{5}{2}.$$

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Problem 4: [12 pts]
Consider the following game using one fair coin.

It begins by flipping the coin once.
If a head is seen on the first flip, the coin is flipped two more times.
If a tail is seen on the first flip, the coin is flipped three more times.

(a) Describe the sample space corresponding to this game by listing all possible outcomes and the probability weight associated with each possible outcome.

(b) Let X be the number of heads seen when playing the game.
What are the possible values that X can achieve?

(c) For each possible value x of X, describe the event "$X = x$".
Recall that an event is a *subset of the sample space.* To answer this problem you need to give the subset of the sample space corresponding to each value of x.

(d) For all possible values x, give the probability that "$X = x$".

(e) What is $E(X)$?

(f) What is $V(X)$?

Note: For parts (a) and (d), probabilities must be given exactly, either as fractions or in decimal form. For parts (e) and (f), answers can be in either fractional or decimal form. If in fractional form, the answers must be exact. If in decimal form, the answers only need to be correct up to four significant digits.

Solution

(a)

HHH	$\frac{1}{8}$
HHT	$\frac{1}{8}$
HTH	$\frac{1}{8}$
HTT	$\frac{1}{8}$
$THHH$	$\frac{1}{16}$
$THHT$	$\frac{1}{16}$
$THTH$	$\frac{1}{16}$
$THTT$	$\frac{1}{16}$
$TTHH$	$\frac{1}{16}$
$TTHT$	$\frac{1}{16}$
$TTTH$	$\frac{1}{16}$
$TTTT$	$\frac{1}{16}$

(b) $0, 1, 2, 3$

(c)

$X=0$	$\{TTTT\}$
$X=1$	$\{HTT, THTT, TTHT, TTTH\}$
$X=2$	$\{HHT, HTH, THHT, THTH, TTHH\}$
$X=3$	$\{HHH, THHH\}$

(d)

$X=0$	$\frac{1}{16}$
$X=1$	$\frac{1}{8}+\frac{3}{16}=\frac{5}{16}$
$X=2$	$\frac{2}{8}+\frac{3}{16}=\frac{7}{16}$
$X=3$	$\frac{1}{8}+\frac{1}{16}=\frac{3}{16}$

(e)

$$E(X)=0\cdot\frac{1}{16}+1\cdot\frac{5}{16}+2\cdot\frac{7}{16}+3\cdot\frac{3}{16}=\frac{28}{16}=1\frac{3}{4}=\frac{7}{4}$$

(f)

$$\begin{aligned}
V(X) &= E\left(\left(X-\frac{7}{4}\right)^2\right)\\
&= \left(\frac{7}{4}\right)^2\cdot\frac{1}{16}+\left(\frac{3}{4}\right)^2\cdot\frac{5}{16}+\left(\frac{1}{4}\right)^2\cdot\frac{7}{16}+\left(\frac{5}{4}\right)^2\cdot\frac{3}{16}\\
&= \frac{49+45+7+75}{16^2}\\
&= \frac{176}{256}\\
&= \frac{11}{16}.
\end{aligned}$$

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Problem 5: [9 pts]

Flip a fair coin n times and consider the following events:

E is the event that *at least* one of the n flips is a **Head**.

F is the event that *all* of the n flips are **Head**s.

G is the event that *the first flip* is a **Head**.

Recall that $\Pr(\mathbf{E})$ denotes *"The probability of event* **E** *occurring"* while $\Pr(\mathbf{E} \mid \mathbf{G})$ denotes *"The probability of event* **E** *occurring conditioned on event* **G** *occurring".*

(a) What is $\Pr(\mathbf{E})$?

(b) What is $\Pr(\mathbf{F} \mid \mathbf{G})$?

(c) What is $\Pr(\mathbf{F} \mid \mathbf{E})$?

Explain how you derived your answers.

Solution:

(a) The probability that all tosses are **Tail***s is* 2^{-n}. *E is the complement of this so*

$$\Pr(E) = 1 - 2^{-n}.$$

(b) Note that $F \cap G = F$, $\Pr(F) = 2^{-n}$, *and* $\Pr(G) = 1/2$ *so*

$$\Pr(F|G) = \frac{\Pr(F \cap G)}{\Pr(G)} = \frac{\Pr(F)}{\Pr(G)} = \frac{2^{-n}}{1/2} = 2^{-(n-1)}.$$

(c) Note that $F \cap E = F$,

$$\Pr(F|E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{\Pr(F)}{\Pr(E)} = \frac{2^{-n}}{1 - 2^{-n}} = \frac{1}{2^n - 1}.$$

Problem 6: [13 pts]

In this problem, you must explain how you derived your answers.

Tony goes to a discussion party where there are n other people.
While there, he wants to talk with both Mary and John.

A computer randomly decides with whom Tony will talk. Each assigned conversation lasts for exactly 5 minutes, after which the computer selects another partner for Tony at random, and so on.

The computer has two settings. In the first setting, the computer selects Tony's next conversation partner at random *with replacement*. That is, at each time, each partner has probability $\frac{1}{n}$ of being chosen. Answer the following three questions in this setting.

In what follows, when we say *"The amount of time (in minutes) until Tony finishes talking with Mary for the first time"* the meaning is *all the minutes of conversation Tony has with other partners before he first talks with Mary and then the time he talks with Mary.*

(a) What is the expected amount of time (in minutes) until Tony finishes talking with Mary for the first time?

(b) What is the expected amount of time (in minutes) until Tony finishes talking with Mary for the first time and then with John?

That is, you must count the expected (in minutes) until Tony finishes talking with Mary for the first time, then the time until he talks with John, then the time he talks with John.

(c) What is the expected amount of time (in minutes) until Tony finishes talking with both Mary and John. In this question the order in which Tony talks with Mary and John does not matter. That is, Tony could first talk with Mary and then later with John, or first with John and then with Mary.

In its second setting, the computer selects Tony's next conversation partner *without replacment*, i.e., it never repeats a choice. That is, the computer chooses Tony's next partner uniformly from among the people with whom he hasn't spoken; if there are m people with whom he hasn't yet spoken, the computer will choose each one with probability $\frac{1}{m}$.
Answer the following question in this setting.

(d) What is the expected amount of time (in minutes) until Tony finishes talking with Mary for the first time?

Solution:

(a) *Let X be the number of people Tony talks with until he talks with Mary (including Mary in the count). X is the time until first success of independent Bernoulli trials, each with probability $p = \frac{1}{n}$ of success. So*

$$E(X) = \frac{1}{p} = n.$$

Let Z be the number of minutes until Tony finishes talking with Mary. then $Z = 5X$ so

$$E(Z) = 5E(X) = 5n.$$

(b) *Let X be as in part (a) and set Y be the number of people Tony talks with starting after he talks with Mary for the first time until he talks with John (including John in the count). Y is also the time until first success of independent Bernoulli trials, each with probability $p = \frac{1}{n}$ of success. So*

$$E(Y) = \frac{1}{p} = n.$$

Let Z be the number of minutes until Tony finishes talking with Mary and then with John. then $Z = 5(X + Y)$ so

$$E(Z) = 5E(X + Y) = 5E(X) + 5E(Y) = 10n.$$

(c) *Now let W be the number of people Tony talks with until Tony talks with either Mary or John (including John or Mary in the count). W is the time until first success of independent Bernoulli trials, each with probability $p = \frac{2}{n}$ of success. So*

$$E(W) = \frac{1}{p} = \frac{n}{2}.$$

Now let V be the number of people Tony talks with from that first success until he talks with the other of John or Mary (including them in teh count). V is the time until first success of independent Bernoulli trials, each with probability $p = \frac{1}{n}$ of success. So

$$E(W) = \frac{1}{p} = n.$$

Let Z be the number of minutes until Tony finishes talking with Mary and John (order not mattering). Then $Z = 5(W + V)$ so

$$E(Z) = 5E(W + V) = 5E(W) + 5E(V) = \frac{15}{2}n.$$

(d) *Let X be the number of people Tony talks with until he talks with Mary (including Mary in the count). It is not hard to see that, for* $1 \leq i \leq n$,

$$\Pr(X = i) = \frac{1}{n}.$$

So,

$$E(X) = \sum_{i=1}^{n} i \Pr(X = i) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Let Z be the number of minutes until Tony finishes talking with Mary. then $Z = 5X$ so

$$E(Z) = 5E(X) = \frac{5}{2}(n+1).$$

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Problem 7: [12 pts]

For this problem, you may assume that n is a non-negative power of 4. Recall that if $f(n)$ and $g(n)$ are functions, to prove that $f(n) = O(g(n))$, you must prove that there exist some $n_0 \geq 0$ and $c > 0$ such that

$$\forall n > n_0, \quad f(n) \leq cg(n).$$

(a) Suppose function $T(n)$ satisfies $T(1) = 5$ and, for $n > 1$

$$T(n) \leq 3T\left(\frac{n}{4}\right) + 5n.$$

Prove by induction that $T(n) = O(n)$.

(b) Suppose function $T(n)$ satisfies $T(1) = 5$ and, for $n > 1$

$$T(n) \leq 16T\left(\frac{n}{4}\right) + 5n.$$

Prove by induction that $T(n) = O(n^2)$.

Solution

(a) Let $P(n)$ be "$T(n) \leq cn$".

When $n = 1$, we want to prove that $T(1) \leq c \cdot 1$. In order to prove this, we must set the value of c such that

$$c \geq T(1) = 5$$

Thus, $n_o \geq 0$.

Now, suppose that the statement $P(n)$ is correct for $n = 1, 4, 4^2, 4^3, \ldots$.

Assume that $P(\frac{n}{4})$ is true. That is,

$$T(\frac{n}{4}) \leq c \cdot \frac{n}{4}$$

Consider $P(n)$.

$$\begin{aligned} T(n) &\leq 3T(\frac{n}{4}) + 5n \\ &\leq 3 \cdot (c \cdot \frac{n}{4}) + 5n \\ &= \frac{3cn}{4} + 5n \\ &= cn - \frac{cn}{4} + 5n \end{aligned}$$

In order to prove that $T(n) \leq cn$, we must set the value of c such that

$$-\frac{cn}{4} + 5n \leq 0$$

That is, c must satisfy the following inequality.

$$c \geq 20$$

We should set some values for c and n_o such that the above induction is correct. We can set $c = \max(5, 20) = 20$ and $n_o = 0$.

(b) It turns out that, similar to the problems seen in class, it is difficult to use induction to prove

$$\forall n > n_o, T(n) \leq cn^2$$

Instead, following the same method used in class, we will prove

$$\forall n > n_o, T(n) \leq c_1 n^2 - c_2 n$$

for some $n_o \geq 0$ and $c_1, c_2 > 0$. This will imply

$$\forall n > n_o, T(n) \leq c_1 n^2$$

Let $P(n)$ be "$T(n) \leq c_1 n^2 - c_2 n$".
When $n = 1$, we want to prove that $T(1) \leq c_1 \cdot 1^2 - c_2 \cdot 1$. In order to prove this, we must set the values of c_1 and c_2 such that

$$c_1 - c_2 \geq T(1) = 5$$

Thus, $n_o \geq 0$.
Now, suppose that the statement $P(n)$ is correct for $n = 1, 4, 4^2, 4^3, \ldots$.
Assume that $P(\frac{n}{4})$ is true. That is,

$$T(\frac{n}{4}) \leq c_1(\frac{n}{4})^2 - c_2\frac{n}{4}$$

Consider $P(n)$.

$$\begin{aligned} T(n) &\leq 16T(\frac{n}{4}) + 5n \\ &\leq 16 \cdot (c_1(\frac{n}{4})^2 - c_2\frac{n}{4}) + 5n \\ &= c_1 n^2 - 4c_2 n + 5n \\ &= c_1 n^2 - c_2 n - 3c_2 n + 5n \end{aligned}$$

In order to prove that $T(n) \leq c_1 n^2 - c_2 n$, we must set the value of c_2 such that

$$-3c_2 n + 5n \leq 0$$

That is, c_2 must satisfy the following inequality.

$$c_2 \geq \frac{5}{3}$$

We should set some values for c_1, c_2 and n_o such that the above induction is correct. We can set $c_2 = \frac{5}{3}$, $c_1 = c_2 + 5 = \frac{5}{3} + 5 = \frac{20}{3}$ and $n_o = 0$.

Problem 8: [12 pts]
Six married couples (i.e., 12 people) sit down at random in a row of 12 seats. That is, each one of the 12! different ways of seating the people is equally likely to occur.

We say that a couple *sits together* if the husband and wife in that couple sit next to each other.

In the following, you may express your answers using the summation ($\sum$) sign, binomial coefficients $\left(\binom{n}{m}\right)$, factorials ($n!$) and exponentials ($c^k$). Actual numerical solutions are not necessary.

It is also not necessary for you to show how you derived your answers.

(a) Consider two specific couples. The first couple c_1 is Peter and Mary and the second couple c_2 is John and Helen. Since Peter and John are friends, they want to sit next to each other. What is the probability that each of these two couples (i.e., c_1 and c_2) sits together, and Peter and John sit next to each other?

(b) What is the probability that every couple sits together?

(c) What is the probability that no couple sits together?

Solution

(a) *There are totally 12! ways to seat the 6 couples.*

Since each of these two couples sits together, and Peter and John sit next to each other, we treat the two couples as one single unit involving the four persons in these two couples. We can randomly permute the 9 *units in* 9! *different ways. For each permutation, there are 2 ways to seat the two couples (i.e., (Mary, Peter, John, Helen) and (Helen, John, Peter, Mary)). Therefore, the probability that each of these two couples sits together, and Peter and John sit next to each other is*

$$\frac{9! \cdot 2}{12!} = 0.001515$$

(b) *There are totally 12! ways to seat the 6 couples.*

If a couple sits together, we treat it as one single unit. There are 6! ways of permuting 6 single units. For each permutation, there are 2^6 *ways to seat the* 6 *bound couples. Therefore, the probability that every couple sits together is*

$$\frac{6! \cdot 2^6}{12!} = 0.0000962$$

(c) *There are totally 12! ways to seat the 6 couples.*

If a couple sits together, we treat it as one single unit. Thus, for the k specified couples to sit together, we can randomly permute the $(12-k)$ *units in* $(12-k)!$ *different ways. For each permutation, there are* 2^k *ways to seat the k bound couples. Therefore, the probability that* $k(1 \leq k \leq 6)$ *specified couples end up sitting together (regardless of whether the other* $6-k$ *couples sit together or not) is*

$$\frac{(12-k)! \cdot 2^k}{12!}$$

Let E_i *denote the event that the i-th couple sits together. The probability that at least one couple sits together can be computed using the inclusion-exclusion principle as*

$$P(\cup_{i=1}^{6} E_i) = \sum_{k=1}^{6} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k \\ 1 \leq i_1 < i_2 < \ldots < i_k \leq 6}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

Thus, we have

$$P(\cup_{i=1}^{6} E_i) = \sum_{k=1}^{6} (-1)^{k+1} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!}$$

Thus, the probability that no couple sits together is

$$\begin{aligned} 1 - P(\cup_{i=1}^{6} E_i) &= 1 - \sum_{k=1}^{6} (-1)^{k+1} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!} \\ &= \sum_{k=0}^{6} (-1)^{k} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!} \end{aligned}$$

Problem 9: [8 pts]

Each of (a), (b), (c) and (d) below contains a pair of statements, (i) and (ii).

For each pair, say whether (i) is equivalent to (ii), i.e., that, for all $p(x)$ and $q(x)$ (i) is true if and only if (ii) is true.

If they are equivalent, all you have to do is say that they are equivalent. If they are not equivalent, give a counterexample. A counter example should involve a specification of $p(x)$ and $q(x)$ and an explanation as to why the resulting statement is false.

(a) (i) $\Big(\forall x \in R \; (p(x)) \Big) \vee \Big(\forall x \in R \; (q(x)) \Big)$

(ii) $\forall x \in R \; \Big(p(x) \vee q(x) \Big)$

(b) (i) $\Big(\forall x \in R \; (p(x)) \Big) \wedge \Big(\forall x \in R \; (q(x)) \Big)$

(ii) $\forall x \in R \; \Big(p(x) \wedge q(x) \Big)$

(c) (i) $\Big(\forall x \in R \; (p(x)) \Big) \wedge \Big(\exists y \in R \; (q(y)) \Big)$

(ii) $\forall x \in R \; \Big(\exists y \in R \; (p(x) \wedge q(y)) \Big)$

(d) (i) $\Big(\forall x \in R \; (p(x)) \Big) \vee (\exists y \in R \; (q(y)))$

(ii) $\forall x \in R \; \Big(\exists y \in R \; (p(x) \vee q(y)) \Big)$

Solution

(a) Not equivalent. Let $p(x)$ be "$x \geq 0$" and $q(x)$ be "$x < 0$". (i) is false but (ii) is true.

(b) Equivalent.

(c) Equivalent.

(d) Equivalent.