Tutorial 1: Propositional Logics

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Logic in Programs

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(1) if ((i+j ≤ p+q) && (i ≤ p) &&
    ((j > q) || (List1[i] ≤ List2[j])))
(2)     List3[k] = List1[i]
(3)     i = i+1
(4) else
(5)     List3[k] = List2[j]
(6)     j = j+1
(7) k = k+1
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(1) if (((i+j ≤ p+q) && (i ≤ p) &&
    (j > q)) || ((i+j ≤ p+q) && (i ≤
    p) && (List1[i] ≤ List2[j])))
(2)     List3[k] = List1[i]
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(5)     List3[k] = List2[j]
(6)     j = j+1
(7) k = k+1
```

Consider the above two pieces of code. They are taken from two different versions of *Mergesort*. Do they do the same thing? (Note that && = "and", || = "or".)

Observation: The two pieces of code are the same except for line 1.

(1) ((i+j $\leq$ p+q) && (i $\leq$ p) && ((j $>$ q) || (List1[i] $\leq$ List2[j])))

(1') (((i+j $\leq$ p+q) && (i $\leq$ p) && (j $>$ q)) || ((i+j $\leq$ p+q) && (i $\leq$ p) && (List1[i] $\leq$ List2[j])))

Are they equivalent?

Let's rewrite using

$s \sim$ (i+j $\leq$ p+q) $t \sim$ (i $\leq$ p) $u \sim$ (j $>$ q) $v \sim$ (List[i] $\leq$ List2[j])

(1) s and t and (u or v)

(1') (s and t and u) or (s and t and v)

Now set $w \sim$ (s and t)

(1) w and (u or v) $\xleftrightarrow{\text{equal?}}$ (1') (w and u) or (w and v)

Logic in Programs

We just transformed our code into compound propositions and next want to develop a theory of how to determine whether two such propositions are equal (equivalent).

Question 1

Are $s \oplus t$ and $(s \wedge \neg t) \vee (\neg s \wedge t)$ logically equivalent? If yes, prove it by truth table. Otherwise, give a counterexample to show that they are not logically equivalent.

Solution:

s	t	$s \oplus t$	$s \wedge \neg t$	$\neg s \wedge t$	$(s \wedge \neg t) \vee (\neg s \wedge t)$
T	T	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	F	F	F

Question 2

Are $w \wedge (s \oplus t)$ and $(w \vee s) \oplus (w \vee t)$ logically equivalent? If yes, prove it by truth table. Otherwise, give a counterexample to show that they are not logically equivalent.

Solution:
No, they are not logically equivalent. Set $w = F$, $s = F$ and $t = T$. $w \wedge (s \oplus t) = F$ but $(w \vee s) \oplus (w \vee t) = T$.

Question 3

For each of the following pairs of propositions, either prove they are logically equivalent, or give a counterexample.

(a) $p \to q$ and $q \to p$

(b) $p \to q$ and $\neg q \to \neg p$

(c) $(p \to q) \to q$ and $p \to q$

(d) $(p \to p) \to q$ and q

(e) $q \to (p \wedge \neg p)$ and $\neg q$

(f) $q \to (p \wedge \neg p)$ and $\neg p$

Solution:

(a) Not equivalent. Set $p = T, q = F$, then $p \to q = F$ but $q \to p = T$.

(b) Equivalent. $p \to q \equiv \neg p \vee q \equiv (\neg\neg q) \vee (\neg p) \equiv \neg q \to \neg p$.
Or truth table.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(c) Not equivalent. Set $p = F, q = F$, then $(p \to q) \to q = F$ but $p \to q = T$

(d) Equivalent. $(p \to p) \to q \equiv T \to q \equiv q$. Or truth table.

p	q	$p \to p$	$(p \to p) \to q$	q
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	F	F

(e) Equivalent. $q \to (p \wedge \neg p) \equiv q \to F \equiv \neg q$. Or truth table.

p	q	$p \wedge \neg p$	$q \to (p \wedge \neg p)$	$\neg q$
T	T	F	F	F
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

(f) Not equivalent. Set $p = T, q = F$, then $q \to (p \wedge \neg p) = T$ but $\neg p = F$

Question 4

For each of the following pair of propositions, either prove that the two propositions are logically equivalent without using truth table, or give a counterexample to show that the propositions are not logically equivalent.

(a) (i) $p \to (q \to r)$
    (ii) $(p \wedge q) \to r$

(b) (i) $(\neg(p \to \neg q)) \to (\neg(p \wedge r) \vee s)$
    (ii) $(\neg r \to p) \to ((p \wedge q) \to s)$

Question 4

Solution:

(a) Consider

$$\begin{aligned} p \to (q \to r) &= \neg p \lor (q \to r) \\ &= \neg p \lor (\neg q \lor r) \\ &= \neg p \lor \neg q \lor r \end{aligned}$$

Consider

$$\begin{aligned} (p \land q) \to r &= \neg (p \land q) \lor r \\ &= (\neg p \lor \neg q) \lor r \\ &= \neg p \lor \neg q \lor r \end{aligned}$$

Thus, $p \to (q \to r)$ is equivalent to $(p \land q) \to r$.

(b) Set $p = T, q = T, r = F, s = F$,
$(\neg(p \to \neg q)) \to (\neg(p \land r) \lor s)$ becomes true and
$(\neg r \to p) \to ((p \land q) \to s)$ becomes false. Thus they are not equivalent.