COMP 2711 Discrete Mathematical Tools for Computer Science
2016 Spring Semester – Assignment # 1
Distributed: 12 February 2016 – Due: 4pm, 19 February 2016

Solutions

Your solutions should contain (i) your name, (ii) your student ID #, (ii) your email address, (iv) your lecture section and (v) your tutorial section.
Some Notes:

- Please write clearly and briefly. For all questions you should also provide a short explanation as to *how* you derived the solution. That is, if the solution is 20, you shouldn't just write down 20. You need to explain *why* it's 20.
- Please follow the guidelines on doing your own work and avoiding plagiarism given on the class home page. Don't forget to *acknowledge* individuals who assisted you, or sources where you found solutions.
- Please make a *copy* of your assignment before submitting it. If we can't find your paper in the submission pile, we will ask you to resubmit the copy.
- Your solutions should be submitted before 4PM of the due date in the collection bin outside Room 4210 (Lift 21). Make sure you submit into the box labeled with the correct lecture section.

Problem 1. Let p, q, and r be the following propositions "you get an A on the final exam", "you do every exercise in this book", and "you get an A in this class", respectively. Write the following propositions using p, q, and r and logical connectives.

(a) You get an A in this class, but you do not do every exercise in this book.

(b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

(c) To get an A in this class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

(e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

(f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution:

(a) $r \wedge \neg q$

(b) $p \wedge q \wedge r$

(c) $r \rightarrow p$

(d) $p \wedge \neg q \wedge r$

(e) $(p \wedge q) \rightarrow r$

(f) $r \leftrightarrow (q \vee p)$

Problem 2. Determine whether the following compound propositions are tautologies. If yes, present two proofs of this fact (one, using truth tables, and the other without truth tables). If it is not a tautology, provide an assignment of truth values that makes the proposition false.

(a) $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$

(b) $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Solution:

(a) This compound proposition is not a tautology.
$(\neg p \wedge (p \rightarrow q)) \rightarrow \ q \equiv (\neg p \wedge (\neg p \vee q)) \rightarrow \neg q \equiv ((\neg p \wedge \neg p) \vee (\neg p \wedge q)) \rightarrow \neg q \equiv (\neg p \vee (\neg p \wedge q)) \rightarrow \neg q \equiv \neg p \rightarrow \neg q \equiv q \rightarrow p$ which is a contingency.
We can set $p = F, q = T$, such that $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is F.

(b) This compound proposition is a tautology.
First proof (by truth table):

p	q	$p \rightarrow q$	$\neg q \wedge (p \rightarrow q)$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$
T	T	T	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Second proof (by logic equivalence):
$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p \equiv (\neg q \wedge (\neg p \vee q)) \rightarrow \neg p \equiv (\neg q \wedge \neg p) \rightarrow \neg p \equiv \neg(q \vee p) \rightarrow \neg p$
$\equiv (q \vee p) \vee \neg p \equiv T$

Problem 3: Prove the following logical equivalences. You are allowed to use only the equivalences "$p \rightarrow q \equiv \neg p \vee q$" and the equivalences on slides 31-32 of the lecture note set L01. State the law you are using in each step.

(a) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(b) $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

(c) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

(d) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Solution: (a)

$(p \to q) \wedge (p \to r)$
$\equiv (\neg p \vee q) \wedge (\neg p \vee r)$ (given $p \to q \equiv \neg p \vee q$)
$\equiv \neg p \vee (q \wedge r)$ (distributive laws, from RHS to LHS)
$\equiv p \to (q \wedge r)$ (given $p \to q \equiv \neg p \vee q$)

(b)

$(p \to q) \vee (p \to r)$
$\equiv (\neg p \vee q) \vee (\neg p \vee r)$ (given $p \to q \equiv \neg p \vee q$)
$\equiv \neg p \vee ((q \vee (\neg p \vee r))$ (associative laws, considers "$(\neg p \vee r)$" a single proposition)
$\equiv \neg p \vee ((q \vee \neg p) \vee r)$ (associative law)
$\equiv (\neg p \vee (q \vee \neg p)) \vee r$ (associative law)
$\equiv (\neg p \vee (\neg p \vee q)) \vee r$ (commutative law)
$\equiv (\neg p \vee \neg p) \vee (q \vee r)$ (associative law)
$\equiv \neg p \vee (q \vee r)$ (idempotent law)
$\equiv p \to (q \vee r)$ (given $p \to q \equiv \neg p \vee q$)

(c)

$(p \to r) \wedge (q \to r)$
$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ (given $p \to q \equiv \neg p \vee q$)
$\equiv (r \vee \neg p) \wedge (r \vee \neg q)$ (commutative law)
$\equiv r \vee (\neg p \wedge \neg q)$
$\equiv (\neg p \wedge \neg q) \vee r$ (commutative law)
$\equiv \neg(p \vee q) \vee r$ (De Morgan's law)
$\equiv (p \vee q) \to r$ (given $p \to q \equiv \neg p \vee q$)

(d)

$(p \to r) \vee (q \to r)$
$\equiv (\neg p \vee r) \vee (\neg q \vee r)$ (given $p \to q \equiv \neg p \vee q$)
$\equiv (\neg p \vee \neg q) \vee r \vee r$ (associative law, commutative law)
$\equiv (\neg p \vee \neg q) \vee r$ (idempotent law)
$\equiv \neg(p \wedge q) \vee r$ (De Morgan's law)
$\equiv (p \wedge q) \to r$ (given $p \to q \equiv \neg p \vee q$)

Problem 4: Which of the following statements (in which Z^+ stands for the positive integers and Z stands for all integers) is true and which is false? Don't forget to explain why.
a) $\forall z \in Z^+ (z^2 + 5z + 10 > 17)$
b) $\forall z \in Z (z^2 - z \geq 0)$
c) $\exists z \in Z^+ (z - z^2 > 0)$
d) $\exists z \in Z (z^2 - z = 6)$

SOLUTION: a) False, because $1^2 + 5 \cdot 1 + 10 = 16$.

b) There are many different ways of proving this. One way is to consider the following cases:

1. $z < 0$: true, because $z^2 - z \geq 0 \Leftrightarrow z^2 \geq z$. The LHS is positive, the RHS is negative.
2. $z = 0$: true by substitution $(0^2 - 0 \geq 0)$.
3. $z > 0$: true, because $z^2 - z \geq 0 \Leftrightarrow z^2 \geq z$, so we get $z \geq 1$ which holds in this case.

c) False. To prove that it is false we want to prove its negation

$$\neg \exists z \in Z^+ \, (z - z^2 - > 0). \tag{1}$$

Equivalently, we want to prove

$$\forall z \in Z^+ \, (z - z^2 \not> 0) \tag{2}$$

We can re-write it as follows.

$$\forall z \in Z^+ \, (z - z^2 \leq 0). \tag{3}$$

But, multiplying the inequality in (3) by -1 gives us $(z^2 - z \geq 0)$, which was already proved in part (b) (for an even bigger universe).

d) True, because $(-2)^2 - (-2) = 6$.

Problem 5. Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a duck", "x is one of my poultry", "x is an officer", and "x is willing to waltz", respectively. Express each of the following statements using quantifiers; logical connectives; and $P(x), Q(x), R(x)$, and $S(x)$.

(a) No ducks are willing to waltz.
(b) No officers ever decline to waltz.
(c) All my poultry are ducks.
(d) My poultry are not officers.

Solution:

(a) $\forall x(P(x) \to \neg S(x))$
(b) $\forall x(R(x) \to S(x))$
(c) $\forall x(Q(x) \to P(x)$
(d) $\forall x(Q(x) \to \neg R(x))$