Serial #

HKUST – Department of Computer Science and Engineering

COMP 2711: Discrete Math Tools for CS – Spring 2013

Midterm Examination

Date: Tuesday, 19 March 2013   Time: 19:00–21:30

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: ________

Instructions

- This is a closed book exam. It consists of 16 pages and 8 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- **Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of** 16, **and** $5^{\underline{3}}$ **instead of** 60**. Calculators may be used for the exam.**
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	Total
Score									

Student ID: ________________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ____________________________

Student's Signature:      ______________________
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Student ID: ______________

Problem 1: (10 points) Twelve Easter eggs are numbered 1, 2, ..., 12. Eggs 1 to 4 are colored red, eggs 5 to 8 are colored green, and eggs 9 to 12 are colored purple. A child is allowed to choose 5 of the eggs.

(a) In how many different ways can the child choose 5 eggs?

(b) What is the answer to (a) if the child must choose at least one egg from each color?

(c) What is the answer to (a) if the number of red eggs chosen must be more than the total number of green and purple eggs?

Explain how your answers are derived.

Answer: (a) There are totally 12 eggs. The child chooses 5 of them. The number of different ways he can do that is: $\binom{12}{5}$.

(b) The wanted number equals the total number of ways minus the number of ways to choose the eggs from only two of the three colors.

The number of ways to choose 5 eggs from among the red and green eggs is $\binom{8}{5}$. The numbers for the red-purple and green-purple combination are the same. The three subsets of choice plans are disjoint. So, the answer to the question is:

$$\binom{12}{5} - 3\binom{8}{5}.$$

(c) Here are are possible choice plans that satisfy the requirement:

3 red, 2 green or purple

4 red, 1 green or purple

So, the answer is:

$$\binom{4}{3}\binom{8}{2} + \binom{4}{4}\binom{8}{1}.$$

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Problem 2: (12 points) Consider $S_3 = \{1, 2, 3\}$ and $S_n = \{1, 2, \ldots, n\}$, where $n > 3$.

(a) How many different functions f are there from S_3 to S_n?

(b) How many different functions f are there from S_3 to S_n such that $f(1) \neq f(2)$?

(c) How many different functions f are there from S_3 to S_n such that $f(1) \neq f(2)$ and $f(2) \neq f(3)$?

(d) How many different functions f are there from S_3 to S_n such that $f(1) \neq f(2)$, $f(2) \neq f(3)$, and $f(1) < f(3)$?

Explain how your answers are derived.

Answer: (a) To define f are from S_3 to S_n, we need to specify $f(1)$, $f(2)$, and $f(3)$. Each of those values can be any of the n elements in S_n. So, the answer is: n^3.

(b) We determine $f(1)$, $f(2)$ and $f(3)$ in order. There are n possibilities for $f(1)$, $n-1$ possibilities for $f(2)$, and n possibilities for $f(3)$. So, the answer is: $n^2(n-1)$.

(c) We determine $f(2)$, $f(1)$ and $f(3)$ in order. There are n possibilities for $f(2)$, $n-1$ possibilities for $f(1)$, and $n-1$ possibilities for $f(3)$. So, the answer is: $n(n-1)^2$.

(d) We determine $f(2)$, $f(1)$ and $f(3)$ in order. There are n possibilities for $f(2)$. After $f(2)$ is determined, there are $n-1$ numbers remaining. We need to choose two of them as values for $f(1)$ and $f(3)$. There are $\binom{n-1}{2}$ different ways to choose the two numbers. So, the answer is: $n\binom{n-1}{2}$.

Problem 3: (10 points) Let n be a positive integer. Give a combinatorial proof of the following equation:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Note: An algebraic proof will not be accepted as a correct solution.

Answer:

- Left side: select two items from $2n$ item.
- Right side: split $2n$ items into left half and right half each contain n items. We have $\binom{n}{2}$ ways to select two items both are in the left half, $\binom{n}{2}$ ways to select two items both are in the right half, and n^2 ways to select two items such that one in left half and the other in right half. That's all possible cases, so the number of ways to select two items from $2n$ items is $2\binom{n}{2} + n^2$.

Both left and right hand side are counting the number of ways to select two items from $2n$ items, so they are the same.

Problem 4: (16 points) There are 8 boys and 8 girls to pair up in a dance party.

(a) In how many different ways can the boys be paired up with the girls?

(b) Assuming the pair-up is done at random, what is the probability that a boy called John is paired up with a girl called Mary? In other words, what is the probability $P(E)$ of the event E that "John is paired up with Mary"?

(c) Given that John is already paired up with Mary, what is the probability that another student called Jack is paired up with another girl called Jill? Answer this question by computing the conditional probability $P(F|E)$, where F is the event that "Jack is paired up with Jill".

(d) What is the answer for (a) if there are 16 boys and 8 girls. Note that in this case not all boys can have a dance partner.

Explain how your answers are derived. For (b) and (c), clearly specify the sample space and describe the events as subsets of the sample space.

[WORKSPACE FOR THIS PROBLEM INCLUDES THIS AND THE NEXT PAGES]

Answer: (a) Let us number of the boys from 1 to 8. Boy number 1 can be paired up with any of the 8 girls; boy number 2 can be paired up any of the remaining 7 girls; and so on. So the answer is: $8!$.

(b) Let us number of girls from 1 to 8. The sample space is:

$$S = \{g_1 g_2 \dots g_8 | g_i \in \{1, 2, \dots, 8\} \text{ and } g_i \neq g_j \text{ for different } i \text{ and } j\},$$

where g_i stands for the girl who is paired up with boy number i. As explained in part (a), $|S| = 8!$.

Suppose John is boy number 1 and Marry is girl number 1. Then the event E is as follows:

$$E = \{g_1 g_2 \dots g_8 | g_i \in \{1, 2, \dots, 8\}, g_i \neq g_j \text{ for different } i \text{ and } j, \text{ and } g_1 = 1\}.$$

The first position of each sequence in E is fixed. It is easy to see that $|E| = 7!$. So,

$$P(E) = \frac{|E|}{|S|} = \frac{7!}{8!} = \frac{1}{8}.$$

[WORKSPACE FOR PROBLEM 4]

(c) Suppose Jack is boy number 2 and Jill is girl number 2. Then the event F is as follows:

$F = \{g_1 g_2 \ldots g_8 | g_i \in \{1, 2, \ldots, 8\}, g_i \neq g_j$ for different i and j, and $g_2 = 2\}$.

The intersection $E \cap F$ is as follows:

$E \cap F = \{g_1 g_2 \ldots g_8 | g_i \in \{1, 2, \ldots, 8\}, g_i \neq g_j$ for different i and $j, g_1 = 1$, and $g_2 = 2\}$.

The first two positions of each sequence in the intersection are fixed. So $|E \cap F| = 6!$. Hence

$$P(E \cap F) = \frac{|E \cap F|}{|S|} = \frac{6!}{8!} = \frac{1}{7 \times 8}.$$

Consequently, we have

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{1/(7 \times 8)}{1/8} = \frac{1}{7}.$$

(d) There are totally 16 boys. Only 8 of them will have dance partners. The number of ways to pick 8 boys from among 16 is $\binom{16}{8}$. Let b_1, b_2, $\ldots$, b_8 be the 8 boys picked. Boy b_1 can be paired up with any of the 8 girls; Boy b_2 can be paired up any of the remaining 7 girls; and so on. So the answer is: $\binom{16}{8} 8!$.

There is another way to solve the problem. Number the girls from 1 to 8. Girl number 1 can be paired up with any of the 16 boys; girl number 2 can then be paired up with with any of the remaining 15 boys; and so on. So the answer is $16 \times 15 \times \ldots \times 9 = 16^{\underline{8}}$, which is the same as $\binom{16}{8} 8!$.

Problem 5: (13 points) Consider a party with n ($n \geq 7$) people. Assume that each person's birthday has equal probability ($\frac{1}{7}$) of falling on each of the seven days of the week. For each i ($1 \leq i \leq 7$), let E_i be the event that nobody's birthday falls on the i-th day of the week. For example, E_1 and E_7 mean that nobody's birthdays are on Monday and Sunday respectively.

(a) What is $P(E_i)$?

(b) For any $1 \leq i_1 < i_2 < \ldots < i_k \leq 7$ ($k \leq 7$), what is $P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$?

(c) What is the probability that, for each day of the week, there is at least one person whose birthday falls on that day. This means that there is at least one person whose birthday is Monday; there is at least one other person whose birthday is on Tuesday; and so on.

Explain how your answers are derived.

Answer: (a) For each j ($1 \leq j \leq n$), let F_j be the event that the j-th person's birthday is not on the i-th day of the week. Then $P(F_j) = 6/7$ and the different events are mutually independent. So, we have

$$P(E_i) = P(\cap_{j=1}^{n} F_j) = \prod_{j=1}^{n} P(F_j) = (\frac{6}{7})^n.$$

(b) Similar derivations lead to the following:

$$P(E_{i_1} \cap E_{i_1} \cap \ldots \cap E_{i_k}) = (\frac{7-k}{7})^n$$

(c) The complement of the event in the question is: There is at least one day of the week on which nobody's birthday falls. It is $\cup_{i=1}^{7} E_i$.

By the inclusion-exclusion principle, we have:

$$\begin{aligned} P(\cup_{i=1}^{7} E_i) &= \sum_{k=1}^{7} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq 7} P(E_{i_1} \cap E_{i_1} \cap \ldots \cap E_{i_k}) \\ &= \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} (\frac{7-k}{7})^n \end{aligned}$$

So, the answer to the question is:

$$1 - P(\cup_{i=1}^{7} E_i) = 1 - \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} (\frac{7-k}{7})^n.$$

Problem 6: (13 points) In a mini lottery, 10 balls are numbered from 1 to 10, and three balls are randomly drawn to determine the winning numbers. A lottery ticket consists of three numbers between 1 and 10. If those three numbers match the three numbers drawn, the ticket holder wins.

Suppose the lottery is held every day and Tom buys one ticket every day.

(a) What is the probability that Tom wins on a particular day? Answer this question using a number.

(b) What is the expected number of days until Tom wins the lottery for the first time?

(c) Suppose the answer to (b) is k. Will Tom win at least once **for sure** in the first k days? If yes, explain why. If not, calculate the probability that Tom wins at least once during those k days.

Explain how your answers are derived. For (a) and (c), round off the number to the fourth digit after the decimal point.

[WORKSPACE FOR THIS PROBLEM INCLUDES THIS AND THE NEXT PAGES]

Answer: (a) The sample is:

$$S = \{\text{3-element subsets of } \{1, 2, \ldots, 10\}\}.$$

The size of the sample space is $|S| = \binom{10}{3} = 120$. Hence the probability that any particular ticket wins is $\frac{1}{120} = 0.0083$.

(b) Whether Tom wins over the days can be viewed as an independent Bernoulli process with success probability $1/120$. The wanted number is the expected number of trials until the first success. So, the answer is $\frac{1}{1/120} = 120$.

(c) No, it is not for sure the Tom will win at least once for sure in 120. For any integer $0 \leq m \leq 120$, the probability that he wins exactly m times in 120 days is the Binomial distribution:

$$\binom{120}{m} 0.0083^m 0.9917^{(120-m)}.$$

In particular, the probability that he never wins (i.e., win 0 times) is

$$\binom{120}{0} 0.0083^0 0.9917^{120} \approx 0.3678.$$

So the probability that he wins at least once is

$$1 - 0.3678 = 0.6322.$$

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[WORKSPACE FOR PROBLEM 6]

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Problem 7: (14 points) A gift for Tom is randomly hidden at one of n locations. Let X be the number of locations that Tom checks until he finds the gift. For example, $X = 1$ if Tom finds the gift after checking just one location and $X = n$ if he finds the gift after checking n locations.

(a) What is the distribution function of X? What are the expectation $E(X)$ and the variance $V(X)$ of X? Show your calculations.

(b) Now suppose that, each time after Tom checks a location and does not find the gift there, the gift is taken away from its current location and randomly replaced at any of the n locations. What is the expected number of checks Tom needs to do until he finds the gift? What is the variance? Explain your answer.

Note that the gift might be replaced at one of the locations that Tom has checked before. Thus a location might be checked more than once, and the number of checks might be different from the number of locations checked.

(c) Intuitively, in which case the variance is larger, (a) or (b)? Give your reasons.

[WORKSPACE FOR THIS PROBLEM INCLUDES THIS AND THE NEXT PAGES]

Answer: (a) The possible values of X are $\{1, 2, \ldots, n\}$. The distribution function is given by:

$$P(X = i) = \frac{1}{n} \quad \forall i \in \{1, 2, \ldots, n\}.$$

The expectation $E(X)$ is:

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} iP(X = i) = \sum_{i=1}^{n} i\frac{1}{n} \\ &= \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}. \end{aligned}$$

The variance $V(x)$ is:

$$\begin{aligned} V(X) &= \sum_{i=1}^{n}(i - E(X))^2 P(X = i) = \sum_{i=1}^{n}(i - \frac{n+1}{2})^2\frac{1}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n}(i - \frac{n+1}{2})^2. \end{aligned}$$

[WORKSPACE FOR PROBLEM 7]

(b) In this case, the gift-finding process is an independent Bernoulli trial process with success probability $p = \frac{1}{n}$. So, the expected number of checks before Tom finds the gift is

$$\frac{1}{p} = n.$$

The variance is:

$$\sum_{i=1}^{\infty}(i - \frac{1}{p})^2 p(1-p)^{i-1} = \frac{1-p}{p^2}$$

The answer with $\sum$ notation is accepted.

(c) The variance is larger in (b). In (a), the number of possible locations decreases with time, while it does not in (b).

Problem 8: (12 points) In how many different ways can we write 100 as the sum of three non-negative integers? In other words, how many lists $x_1x_2x_3$ of three non-negative integers x_1, x_2 and x_3 are there such that $x_1 + x_2 + x_3 = 100$?

You might need to use the Bijection principle for this question. Clearly state the two sets involved, and the function between the two sets.

Answer: The collection that we need to count is:

$$S = \{x_1x_2x_3 | \text{the } x_i \geq 0\text{'s are integers such that } x_1 + x_2 + x_3 = 100\ \}.$$

This set is difficult to count directly. To get another set that is easy to count, consider arranging 100 balls on one line and dividing the line of balls into three segments by inserting two separators between the balls. Some of the segments might be empty. Let T be the set of all possible ways to place the two separators.

To define a function from T to S, consider one particular way to place the two separators. We can get three non-negative integers as follows: Let x_1 be the number of balls one the left side of the first separator, x_2 be the number of balls between the two separators, and x_3 be the number of balls on the right side of the second separator. It is clear that $x_1 + x_2 + x_3 = 100$. It is also obvious that the function is a bijection.

To count T, consider placing the 100 balls and the two separators on one line. There are totally 102 positions. The two separators can be placed at any two of those positions. Therefore, the size of T is:

$$\binom{102}{2}.$$

Because there is a bijection between S and T, the size of S is the same.