HKUST – Department of Computer Science and Engineering

COMP 2711: Discrete Math Tools for CS – Fall 2014

Serial #

Midterm Examination

Date: Thursday, 16 October 2014 Time: 19:00–21:30

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: ________

Instructions

- This is a closed book exam. It consists of 15 pages and 8 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- **Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of** 16**, and** $5^{\underline{3}}$ **instead of** 60**. Calculators may be used for the exam.**
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.
- The inclusion-exclusion theorem:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

Questions	1	2	3	4	5	6	7	8	Total
Score									

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As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name: ______________________________

Student's Signature: _________________________
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Problem 1: [14 pts] A by-election is being held to fill 5 seats in a council that have recently become vacant. A total of 9 candidates have registered to run in the election. The candidates are divided into 3 groups based on their political affiliations with: 4 in Group A, 3 in Group B, and 2 in Group C.

(a) In how many ways can a voter choose 5 candidates?

(b) In how many ways can a voter choose 5 candidates such that at least 3 candidates from Group A are chosen?

(c) In how many ways can a voter choose 5 candidates such that at least 1 candidate from each group is chosen?

Explain how your answers are derived.

ANS:

(a) $\binom{9}{5} = 126$.

(b) We partition this into two cases: choosing either 3 or 4, from group A and, respectively, 2 or 1 candidates from the combination of groups B and C. This is

$$\binom{4}{3}\binom{5}{2} + \binom{4}{4}\binom{5}{1} = 4 \cdot 10 + 1 \cdot 5 = 45.$$

(c) In this case we need to partition into all the possible scenarios of choosing from (A, B, C)*, which are* $(1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1)$*. This gives*

$$\binom{4}{1}\binom{3}{2}\binom{2}{2}+\binom{4}{1}\binom{3}{3}\binom{2}{1}+\binom{4}{2}\binom{3}{1}\binom{2}{2}+\binom{4}{2}\binom{3}{2}\binom{2}{1}+\binom{4}{3}\binom{3}{1}\binom{2}{1}$$

or

$$4 \cdot 3 \cdot 1 + 4 \cdot 1 \cdot 2 + 6 \cdot 3 \cdot 1 + 6 \cdot 3 \cdot 2 + 4 \cdot 3 \cdot 2 = 98$$

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Problem 2: [14 pts] There are n families joining a party. Each family consists of 3 people, a mother, a father and a child. In the party, there are $3n$ seats placed in a line.

(a) Suppose everyone is allowed to sit anywhere. How many different seating arrangements are there?

(b) Suppose the three people in each family must sit together. How many different seating arrangement are there?

(c) Suppose each child must sit between his/her parents. How many different seating arrangements are there?

(d) Suppose each child must sit next to his/her mother. How many different seating arrangements are there?

Explain how your answers are derived.

Solution:

(a) $(3n)!$

(b) $n!(3!)^n$. Consider a family as a unit. There are $n!$ ways to arrange the n units. In each unit there are 3! different seating orders.

(c) $n!(2!)^n$. Consider a family as a unit. There are $n!$ ways to arrange the n units. In each unit there are 2! different seating orders with the child sitting between his/her parents.

(d) $(2n)!2^n$. Consider the child and his/her mother as a unit. There are $2n$ units to arrange, and the number of ways is $(2n)!$. For each mother and child pair, there are 2 different seating orders.

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Problem 3: [11 pts]

Give a combinatorial proof of the identity, for all $n \geq 13$.

$$\binom{n}{4}\binom{n-4}{6}\binom{n-10}{3} = \binom{n}{6}\binom{n-6}{3}\binom{n-9}{4}$$

Note: An algebraic proof of this identity will not be accepted as a solution.

ANS: *Consider the problem of "how to color n items so that 4 are red, 6 are green, 3 are blue and the remaining $n-13$ are yellow.*

The left hand side of the inequality obviously counts this. Notice that the problem is the same if we change the order in which we ask the question. Consider the problem of how to color n items so that 6 are green, 3 are blue, 4 are red and the remaining $n-13$ are yellow. This is what the right hand side of the equation is counting, so the two sides are the same.

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Problem 4: [11 pts] An anagram is a distinct ordering of the letters. For example, the word "eat" has six anagrams "eat", "eta", "ate", "aet","tea" and "tae", while the word "too" has only three anagrams "too", "oto" and "oot".

(a) How many anagrams does the word "java" have?

(b) How many anagrams does the word "lollapalooza" have?

Explain your answers.

Answer: (a) This is the same as the number of ways to label 4 positions according to the following rules: Label 2 positions with 'a', 1 positions with 'j', and 1 position with 'v'. So, the answer is:

$$\frac{4!}{2!1!1!} = 12.$$

(b) This is the same as the number of ways to label 12 positions according to the following rules: Label 3 positions with 'a', 4 positions with 'l', 3 positions with 'o', 1 position with 'p', and 1 position with 'z'. So the answer is:

$$\frac{12!}{3!4!1!3!1!} =?$$

Problem 5: [10 pts]

Consider rolling two fair dice. Let X and Y be the two results. Further let A be the event that $X+Y$ is even, and B be the event that $X \times Y$ is even.

(a) What is $P(A)$?

(b) What is $P(B)$?

(c) What is $P(A \cap B)$?

(d) What is $P(A|B)$?

(e) Is A independent of B?

Explain how your answers are derived.

[MORE SPACE FOR THIS PROBLEM ON THE NEXT PAGE]

Answer: (a) Let E be the event that X is even and F be the event that Y is even. And let $\bar{E}$ and $\bar{F}$ be the complements of E and F respectively. It is obvious that $A = (E \cap F) \cup (\bar{E} \cap \bar{F})$. So,

$$\begin{aligned} P(A) &= P(E \cap F) + P(\bar{E} \cap \bar{F}) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

(b) It is obvious that $B = E \cup F$.

$$\begin{aligned} P(B) &= P(E \cup F) \\ &= P(E) + P(F) - P(E \cap F) \\ &= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

(c) Note that $A \cap B = E \cap F$. So,

$$P(A \cap B) = P(E \cap F) = \frac{1}{4}.$$

(d) By the definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

(d) A is not independent of B because $P(A|B) = \frac{1}{3} \neq P(A) = \frac{1}{2}$.

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Problem 5: [15 pts] There are 20 students in a kindergarten class. Each day, the teacher gives out 1 blue sticker to one of the students at random.

(a) What is the probability that a particular student gets at least one blue sticker after 30 days?

(b) What is the probability that every student gets at least one sticker after 30 days?

Explain how your answers are derived.

[MORE SPACE FOR THIS PROBLEM ON THE NEXT PAGE]

Answer: (a) Let us call the student Tom. Let E_i be the event that Tom gets a blue sticker on the i-th day, and F_i be the event that Tom does not get a blue sticker on the i-th day. The question asks for $P(E_1 \cup E_2 \cup \ldots \cup E_{30})$. It is quite easy to see that

$$\begin{aligned} P(E_1 \cup E_2 \cup \ldots \cup E_{30}) &= 1 - P(F_1 \cap F_2 \cap \ldots \cap F_{30}) \\ &= 1 - P(F_1)P(F_2)\ldots P(F_{30}) \\ &= 1 - (\frac{19}{20})^{30}. \end{aligned}$$

(b) Let G_j be the probability that the j-th student does not get any stickers after 30 days. The probability that every student gets at least one sticker after 30 days is:

$$1 - P(G_1 \cup G_2 \cup \ldots \cup G_{20}).$$

Using the inclusion-exclusion principle, we get

$$P(G_1 \cup G_2 \cup \ldots \cup G_{20}) = \sum_{k=1}^{20} (-1)^{k+1} \sum_{\substack{j_1, j_2, \ldots, j_k \\ 1 \le j_1 < j_2 < \ldots < j_k \le 20}} P(G_{j_1} \cap G_{j_2} \cap \ldots \cap G_{j_k})$$

$P(G_{j_1} \cap G_{j_2} \cap \ldots \cap G_{j_k})$ is the probability that the k specific students j_1, j_2, $\ldots$, j_k never get stickers in 30 days. It is:

$$P(G_{j_1} \cap G_{j_2} \cap \ldots \cap G_{j_k}) = (\frac{20-k}{20})^{30}.$$

So,

$$P(G_1 \cup G_2 \cup \ldots \cup G_{20}) = \sum_{k=1}^{20} (-1)^{k+1} \binom{20}{k} (\frac{20-k}{20})^{30}.$$

Conseqently, the answer to the question is:

$$1 - \sum_{k=1}^{20} (-1)^{k+1} \binom{20}{k} (\frac{20-k}{20})^{30}$$

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Problem 7: [13 pts]
In a group of n ($n \geq 2$) people, what is the expected number of people who share a birthday with at least one other person in the group? What is the expected number when $n = 100$?

Explain how your answer is derived. For the purpose of this question, assume that there are 365 days in a year.

Answer: Define a random variable:

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th person shares birthday with at least another person} \\ 0 & \text{otherwise} \end{cases}$$

If none of the other $n-1$ people share a birthday with the i-th person, their birthdays must fall on the other 364 days. So, we have

$$P(X_i = 1) = 1 - (\frac{364}{365})^{n-1}.$$

Hence,

$$E(X_i) = 1 - (\frac{364}{365})^{n-1}$$

The question asks for $E(X_1 + X_2 + \ldots + X_n)$. By linearity of expectation, we get

$$\begin{aligned} E(X_1 + X_2 + \ldots + X_n) &= E(X_1) + E(X_2) + \ldots + E(X_n) \\ &= n(1 - (\frac{364}{365})^{n-1}). \end{aligned}$$

When $n = 100$, the value is 23.8. (Is this a bit surprising to you?)

Problem 8: [12 pts] There are 4 aces in a standard deck of 52 cards. David draws a card at random. He keeps the card if it is an ace, but puts the card back to the deck if it is not an ace. The deck is then reshuffled. The process continues until David gets all the 4 aces.

What is the expected number of draws until David gets all the 4 aces? Explain how your answer is derived.

Answer: At the beginning, the probability of getting an ace at each draw is $\frac{4}{52} = \frac{1}{13}$. So, the expected number of draws to get the first ace is $1/\frac{1}{13} = 13$.

After the first ace is drawn, the probability of getting a second ace is $\frac{3}{51}$. So, the expected number of draws to get the second ace, after the first one, is $\frac{51}{3} = 17$.

Similarly, the expected number of draws to get the third ace, after the second one, is $\frac{50}{2} = 25$, and the expected number of draws to get the fourth ace, after the third one, is 49.

Putting together, we conclude that the expected number of draws until getting all the 4 aces is $13 + 17 + 25 + 49 = 104$.