Hong Kong University of Science and Technology

COMP170: Discrete Mathematical Tools for Computer Science

Spring 2009

Final Exam

21 May 2009, 12:30–3:00pm, Rm 3007

Instructions

1. This is a closed-book exam consisting of 11 questions.
2. Please write your name, student number and email address on the cover page of the answer booklet.
3. Please sign the honor code statement on the second page of the answer booklet.
4. All answers *must* be put on the answer booklet. Only the answer booklet needs to be handed in at the end of the exam.
5. In the answer booklet each question starts on a new page. This is for clarity and is not meant to imply that your answer needs to fill up all the space provided.
6. Unless otherwise specified, you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
7. Your answers may be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam.
8. Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.
9. Please put your student ID card on the desk so the TA can check it.
10. All mobile phones must be turned off completely during the exam or else you will be disqualified.

You may use the following identities in the exam without having to provide a proof:

1. $$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

2. For any real number $r \neq 1$,
$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}.$$

3. For any real number $r \neq 1$,
$$\sum_{i=1}^{n} ir^i = \frac{nr^{n+2} - (n+1)r^{n+1} + r}{(1-r)^2}.$$

4. For any real number $0 < r < 1$,
$$\sum_{i=1}^{\infty} i^2 r^i = \frac{r(r+1)}{(1-r)^3}.$$

5. Inclusion-exclusion theorem:
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$$

6. $f(n) = O(g(n))$ if there exist some integer $n_0 > 0$ and some real number $c > 0$ such that $\forall n \geq n_0, f(n) \leq c \cdot g(n)$.

Question 1: [10 points]

For each of the following parts, is the statement **true** or **false**?
You need to prove your answer, or else you will receive no mark even if the answer (**true** or **false**) is correct.

(a) Let n and k be nonnegative integers such that $0 \leq k \leq n$. The identity $n^{\underline{k}} = \binom{n}{k} k!$ always holds.

(b) Let A and B be two finite sets and f be an injective function from A to B. Then the size of B must be larger than that of A, i.e., $|B| > |A|$.

(c) Because 1571 and 3534 are relatively prime, i.e., $\gcd(1571, 3534) = 1$, we can set $p = 1571$ and $q = 3534$, $n = pq$, $T = (p-1)(q-1)$ and so on to generate a public key and a secret key for using the RSA cryptosystem.

(d) Let U be a universe consisting of all students,
CS(x) be a predicate for 'x is a Computer Science student', and
APlus(x) be a predicate for 'x gets an A+ grade'.
The statement 'there exists a Computer Science student who gets an A+ grade' can be expressed as the following quantified logical statement:

$$\exists x \in U \Big(\mathrm{CS}(x) \Rightarrow \mathrm{APlus}(x)\Big).$$

(e) Let A and B be two events in some sample space S, and
$\overline{A}$ be the complement of A in S.
The conditional probability $P(A|B)$ can be expressed as:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}.$$

Question 2: [7 points]

(a) A farmer has x cows in his farm. He only knows that $100 \leq x \leq 200$ but he does not know the exact value of x.

When the farmer forms groups of 20 cows each, there are 13 cows left. However, when he forms groups of 9 cows each, there is no cow left.

How many cows does he have (i.e., what is the value of x)?

(b) Suppose $500 \leq x \leq 600$ instead of $100 \leq x \leq 200$, what is the value of x?

Question 3: [6 points]

In a tennis tournament there are $2n$ participants. In the first round of the tournament each participant plays just once, so there are n games, each involving a pair of players.

(a) How many different ways are there to arrange the pairing for the first round? Explain your answer.

(b) Suppose the answer to part (a) is $P(n)$. Express $P(n)$ as a recurrence relation in one or more of the earlier terms $P(i), 1 \leq i < n$.

Question 4: [7 points]

Consider the following finite series:

$$S(n) = \sum_{k=1}^{n} k \cdot k!$$

where n is any positive integer.

(a) By considering the values of $S(n)$ for some smallest values of n:

$$\begin{aligned}
S(1) &= 1 \\
S(2) &= 5 \\
S(3) &= 23 \\
S(4) &= 119 \\
S(5) &= 719,
\end{aligned}$$

guess a closed form for $S(n)$ without involving the summation ($\sum$) notation.

(b) Prove the correctness of your guess for a closed form in part (a).

Question 5: [15 points]

Consider the following recurrence:

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(n/m) + bn^d & \text{if } n > 1 \end{cases}$$

where m is a positive integer, a, b, c, d are positive real numbers with $a \neq m^d$, and n is a nonnegative integer power of m.

(a) By iterating the recurrence, find a closed formula for $T(n)$.
In the formula, n should only appear in the form n^k where k is some real number, but it should not appear in an exponent such as in the form $k^{f(n)}$ where k is a real number and $f(n)$ is a function of n.

(b) Prove the correctness of the closed formula obtained in part (a) by induction.

(c) Using the result obtained in part (a), find a closed formula for the following recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/3) + n^2 & \text{if } n > 1 \end{cases}$$

Question 6: [10 points]

Prove the correctness of the following statement by induction:

$$\text{If } T(n) \leq 9T(n/3) + n, \text{ then } T(n) = O(n^2).$$

You may assume that n is an integer power of 3.

Question 7: [7 points]

A jury of 12 members in a court has to decide whether or not a defendant is guilty. An absolute majority is needed to decide that the defendant is guilty, i.e., at least seven members of the jury have to say YES.

Suppose we know that four members of the jury will say YES and three will say NO. Of the rest, four will each toss a fair coin to decide and the remaining one will just go along with the majority.

(a) Of the four tossers, at least how many are needed to say YES for the defendant to be found guilty? Explain your answer.

(b) What is the probability that the defendant is found guilty?

Question 8: [10 points]

Suppose there are 100 distinct events, denoted as $E_i, 1 \leq i \leq 100$, and their union is the entire sample space S. The events have the following properties:

(i) Each event contains 1000 outcomes.

(ii) The intersection of any two distinct events contains 200 outcomes.

(iii) The intersection of any three distinct events contains 6 outcomes.

(iv) The intersection of any four or more distinct events is empty.

We define a probability distribution on S such that each outcome in S is associated with the same probability.

(a) Are the 100 events pairwise disjoint? Explain your answer.

(b) Express the probability $P(E_i)$ in terms of E_i and S.

(c) Using the inclusion-exclusion principle for probability, compute the probability $P(E_i)$ for each event E_i.

Question 9: [8 points]

In an exam with 100 **true-or-false** questions, suppose the probability that the professor assigns **true** to be the correct answer for each question is 1/3 and that for a **false** answer is 2/3.

Two students have no idea how to answer all the questions. Student 1 decides to answer the questions by flipping a fair coin; student 2 simply writes down **false** for all the questions and then sleeps through the rest of the exam period.

Let S_1 and S_2 denote the exam scores earned by students 1 and 2, respectively. Compute the expected values $E(S_1), E(S_2)$ and variances $V(S_1), V(S_2)$.

Question 10: [12 points]

We have a hash table of size $n \geq 2$. Items are hashed into the hash table one at a time. For the kth item, where $k \in \{1, 2, \ldots\}$ is an item index, the probability of hashing it into any of the n locations is always the same (i.e., equal to $1/n$).

We are interested in n specific events, each of which corresponds to hashing an item into a currently empty location of the hash table. Let $k_1, k_2, \ldots, k_n$, where $k_1 < k_2 < \cdots < k_n$, be the n item indices corresponding to the n events. We define the following random variables:

$$X_1 = k_1$$
$$X_i = k_i - k_{i-1}, \ 1 < i \leq n$$
$$X = \sum_{i=1}^{n} X_i.$$

(a) Compute the expected values $E(X_i), 1 \leq i \leq n$.

(b) Compute the expected value $E(X)$, which is the expected number of items needed to fill all n locations of the hash table with at least one item in each location.

(c) If Y is any random variable, express the variance $V(Y)$ in terms of the expected values $E(Y)$ and $E(Y^2)$. Show your derivation.

(d) Compute the variances $V(X_i), 1 \leq i \leq n$.

(e) Compute the variance $V(X)$. You may express it using the summation ($\sum$) notation.

Question 11: [8 points]

There are $n \geq 1$ points randomly placed on the circumference of a circle. What is the probability that all n points lie along a semicircular arc?

For example, the 3 points in the left figure below lie along a semicircular arc but those in the right figure do not.

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