COMP 2711 Discrete Mathematical Tools for CS
Written Assignment # 6
Distributed: 06 April 2016 – Due: 4:00pm, 13 April 2016
Solution Keys

Your solutions should contain (i) your name, (ii) your student ID #, (ii) your email address, (iv) your lecture section and (v) your tutorial section. Your work should be submitted to the collection bin outside Room 4210 (Lift 21).

Problem 1: If a student knows 75% of the material in a course, and if a 100-question multiple-choice test with five choices per question covers the material in a balanced way, what is the student's probability of getting a right answer to a question, given that the student guesses at the answer to each question whose answer he does not know?

Answer:

Let
K be the event that the student knows the correct answer
C be the event that the student gets the correct answer.

Then, we know the conditional probabilities (why?)

$$P(C \mid K) = 1 \quad \text{and} \quad P(C \mid \overline{K}) = \frac{1}{5}.$$

Then $C = (C \cap K) \cup (C \cap \overline{K})$. Since the events $(C \cap K)$ and $(C \cap \overline{K})$ are disjoint (why?) this gives

$$\begin{aligned} P(C) &= P(C \cap K) + P(C \cap \overline{K}) \\ &= P(C \mid K) \cdot P(K) + P(C \mid \overline{K}) \cdot P(\overline{K}) \\ &= (1) \cdot (0.75) + \frac{1}{5} \cdot (0.25) \\ &= .8 \end{aligned}$$

Problem 2: Suppose a student who knows 60% of the material covered in a chapter of a textbook is going to take a five-question objective (each answer is either right or wrong, not multiple choice or true-false) quiz. Let X be the random variable that gives the number of questions the student answers correctly for each quiz in the sample space of all quizzes the instructor could construct.
(a) What is the expected value of the random variable $X - 3$?
(b) What is the expected value of $(X - 3)^2$?
(c) What is the variance of X?

Answer:

X has a binomial distribution with $n = 5$ and $p = .6$ so $E(X) = np = 3$.

(a) $E(X - 3) = E(x) - 3 = 0$.

(b)

$$E((x-3)^2) = (-3)^2 \cdot .4^5 + (-2)^2 \cdot 5 \cdot .6 \cdot .4^4 + (-1)^2 \cdot 10 \cdot .6^2 \cdot .4^3 + (1)^2 \cdot 5 \cdot .6^4 \cdot .4^2 + 2^2 \cdot (.6)^5 = 1.2$$

(c) $Var(X) = E((x - 3)^2) = 1.2$

Alternatively, we can set X_i to be the indicator random variable as to whether question i is answered correctly ($X_i = 1$) or not ($X_i = 0$). Then $Var(X_i) = (.6) \cdot (.4) = .24$. Since $X = \sum_{i=1}^{5} X_i$ and the X_i are all independent,

$$Var(X) = \sum_{i=1}^{5} Var(X_i) = 5Var(X_1) = 1.2.$$

Problem 3: Show that if X and Y are independent and b and c are constant, then $X - b$ and $Y - c$ are independent.

Answer: *Let $X' = X - b$ and $Y' = Y - c$. Then,*

$$\begin{aligned} P((X' = x) \wedge (Y' = y)) &= P((X = x + b) \wedge (Y = y + c)) \\ &= P(X = x + b) \cdot P(Y = y + c) \\ &= P(X' = x) \cdot P(Y' = y) \end{aligned}$$

where the second equality comes from the independence of X and Y.

Problem 4: (a) Roll a fair die and let X be the number of dots showing on top. What are $E(X)$ and $Var(X)$?
(b)What are $E(2X)$ and $Var(2X)$?
(c) Now roll another die and let Y be the number of dots showing. What are $E(X + Y)$ and $Var(X + Y)$?

Answer: (a) $E(X) = 3.5$

$$\begin{aligned} Var(X) &= \frac{1}{6}[(1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 + (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2] \\ &= \frac{35}{12} \end{aligned}$$

(b) By Linearity of Expectation

$$E(2x) = 2E(X) = 7.$$

By the result of the previous question

$$Var(2X) = 4Var(X) = 4 \cdot \frac{35}{12} = \frac{35}{3}.$$

(c) By Linearity of Expectation

$$E(X+Y) = E(X) + E(Y) = 3.5 + 3.5 = 7.$$

Since X and Y are independent

$$Var(X+Y) = Var(X) + Var(Y) = 2 \cdot \frac{35}{12} = \frac{35}{6}.$$

Problem 5: Flip four fair coins. let X be the number of heads showing. Now flip four $\frac{1}{3}$-biased coins (that is, they have $P(H) = \frac{1}{3}$) and let Y be the number of heads showing.
(a) What is $E(X+Y)$?
(b) What is $Var(X+Y)$?

Answer: X is the number of successes in $n = 4$ independent trials with $p = \frac{1}{2}$. Therefore, by the theorems derived in class

$$E(X) = np = 2 \quad \text{and} \quad Var(X) = np(1-p) = n \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

Similarly Y is the number of successes in $n = 4$ independent trials with $p = \frac{1}{3}$. Therefore, by the theorems derived in class

$$E(Y) = np = \frac{4}{3} \quad \text{and} \quad Var(Y) = np(1-p) = n \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{8}{9}$$

(a) By Linearity of Expectation

$$E(X+Y) = E(X) + E(Y) = 2 + \frac{4}{3} = \frac{10}{3}.$$

Since X and Y are independent

$$Var(X+Y) = Var(X) + Var(Y) = 1 + \frac{8}{9} = \frac{17}{9}.$$

Problem 6: A standard *deck* contains 52 cards, 4 each of **2,3,4,5,6,7,8,9,10,J,Q,K,A**. Now start the following process. Pick a random card from the deck, show it, and then return it to the deck. Continue repeating this process, stopping when each type of card, **2,3,4,5,6,7,8,9,10,J,Q,K,A**, has been seen at least once. What is the expected number of cards that you will have drawn?

Answer:

To simplify our presentation let's rename the cards as **1,2,3,4,5,6,7,8,9,10,11,12,13**.

Let Y_i be the number of picks made before seeing i *different* numbers. $Y_1 = 1$ (since the first pick always gives us a number we have never seen before) and Y_{13} is the answer we want. For $i > 1$, consider $X_i = Y_i - Y_{i-1}$. This is the number of picks needed (starting from the first time we have seen $i-1$ numbers) to see the i^{th} number. Define $X_1 = 1$. Note that $Y_{13} = \sum_{i=1}^{13} X_i$.

The important observation is that, just as in the previous problem, when picking the cards, seeing any of the previously seen $i-1$ numbers is a failure, while seeing any of the previously unseen $(13-(i-1))$ ones is a success. Since there are 52 cards in total and 4 cards of each number, X_i is a geometric random variable with $p = \frac{4*(13-(i-1))}{52} = \frac{13-(i-1)}{13}$. $E(X_i) = \frac{13}{13-(i-1)}$. Similar calculations as before give

$$\begin{aligned}
E(Y_{13}) = E(\sum_{i=1}^{13} X_i) &= \sum_{i=1}^{13} E(X_i) \\
&= E(X_1) + \sum_{i=2}^{13} \frac{13}{13-(i-1)} \\
&= \sum_{i=1}^{13} \frac{13}{13-(i-1)} \\
&= 13 \sum_{j=1}^{13} \frac{1}{j} = 41.34.
\end{aligned}$$

Problem 7: (**Challenge**) There are $n \geq 1$ points randomly placed on the circumference of a circle. What is the probability that all n points lie along a semicircular arc?

For example, the 3 points in the left figure below lie along a semicircular arc but those in the right figure do not.

- **Answer:**

  Let P_1, P_2, $\ldots$, P_n denote the n points.

  If all n points lie along a semicircular arc, then there must exist a point, say P_i, such that the semicircular arc starting at P_i and going clockwise around the circle contains no other point P_j, $j \neq i$. Let E_i denote such an event. The probability of E_i is

  $$P(E_i) = \frac{1}{2^{n-1}}$$

  because each of the $n-1$ points other than P_i can only lie on half of the circumference.

  We note that if there exists a point P_i that satisfies the event E_i, then there does not exist a different point P_j that satisfies the corresponding event E_j . This implies that the n events are disjoint. Hence, the desired probability is

  $$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) = \frac{n}{2^{n-1}}.$$