Tutorial 7: Probability II

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Consider a game where one repeatedly throws a fair die until the total number of dots in all the throws reaches or exceeds 3. This means that if one gets 3,4,5, or 6 at the first throw, one stops right away. If one gets 1 or 2 at the first throw, however, one must continue.

(a) Describe the sample space for the game by listing all the possible outcomes. Here are two examples: The outcome "12" means that one gets 1 at the first throw and 2 at the second throw, and then the game is over. The outcome "113" means that one gets 1 at the first throw, 1 at the second throw and 3 at the third throw, and then the game is over.

Sol: All possible outcomes are:
{"3", "4", "5", "6", "12", "13", "14", "15", "16", "21", "22", "23", "24", "25", "26", "111", "112", "113", "114", "115", "116"}
i.e. There are 21 possible outcomes.

(b) Give the probability weight for each possible outcome.

Sol: For {"3", "4", "5", "6"}, the probability of each of these outcome is $1/6$.
For {"12", "13", "14", "15", "16", "21", "22", "23", "24", "25", "26"}, the probability of each of these outcome is $(1/6)^2$.
For {"111", "112", "113", "114", "115", "116"}, the probability of each of these outcome is $(1/6)^3$.

(c) Let X be the number of times one gets an odd number of dots during the game. What are the possible values of X? For the two outcomes "12" and "113", the values of X are 1 and 3 respectively.

Sol: The possible values are 0,1,2,3.

(d) For each possible value x of X, describe the event "$X = x$". Recall that an event is a *subset of the sample space*. To answer this question you need to give the subset of the sample space corresponding to each value of x.

Sol: The event of $X = 0$ includes the following subsets: {"22", "24", "26", "4", "6"}
The event of $X = 1$ includes the following subsets: {"3", "5", "12", "14", "16", "21", "23", "25" }
The event of $X = 2$ includes the following subsets: {"112", "114", "116", "13", "15" }
The event of $X = 3$ includes the following subsets: {"111", "113", "115" }

(e) For all possible values x, give the probability that "$X = x$".

Sol:

$$\begin{aligned} P(X=0) &= 2 \cdot \frac{1}{6} + 3 \cdot \left(\frac{1}{6}\right)^2 = \frac{5}{12} \\ P(X=1) &= 2 \cdot \frac{1}{6} + 6 \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{2} \\ P(X=2) &= 2 \cdot \left(\frac{1}{6}\right)^2 + 3 \cdot \left(\frac{1}{6}\right)^3 = \frac{5}{72} \\ P(X=3) &= 3 \cdot \left(\frac{1}{6}\right)^3 = \frac{1}{72} \end{aligned}$$

(f) What is $E(X)$?

Sol: $E(X) = 0 \cdot \frac{5}{12} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{72} + 3 \cdot \frac{1}{72} = \frac{49}{72}$

(g) What is $V(X)$?

Sol:

$$\begin{aligned}
V(X) &= E((X-E(X))^2) \\
&= (0-\frac{49}{72})^2\cdot\frac{5}{12}+(1-\frac{49}{72})^2\cdot\frac{1}{2}+(2-\frac{49}{72})^2\cdot\frac{5}{72}+ \\
&\quad (3-\frac{49}{72})^2\cdot\frac{1}{72} \\
&= \frac{2401}{5184}\cdot\frac{5}{12}+\frac{529}{5184}\cdot\frac{1}{2}+\frac{9025}{5184}\cdot\frac{5}{72}+\frac{27889}{5184}\cdot\frac{1}{72} \\
&= \frac{2401}{5184}\cdot\frac{30}{72}+\frac{529}{5184}\cdot\frac{36}{72}+\frac{9025}{5184}\cdot\frac{5}{72}+\frac{27889}{5184}\cdot\frac{1}{72} \\
&= \frac{2401\cdot 30+529\cdot 36+9025\cdot 5+27889}{5184\cdot 72} \\
&= \frac{164088}{373248}=\frac{2279}{5184}
\end{aligned}$$

Question 2

Two coins are placed in a bag. One of them is a fair coin, while the other is a special coin with tails on both sides. One of the coins is picked from the bag at random and this coin is tossed 2 times. Let $A_i (i = 1, 2)$ be the event that the coin comes up tail on the i-th toss.

(a) Calculate $P(A_1)$ and $P(A_2)$.

(b) Calculate $P(A_2|A_1)$.

(c) Calculate $P(A_2 \cup A_1)$.

(d) Are A_1 and A_2 independent? Why?

Question 2

Solution:

(a) Let K be the event that the coin picked is the fair coin, and $\bar{K}$ be the event that the coin picked is the magic coin. Then,

$$P(A_1) = P(K)P(A_1|K) + P(\bar{K})P(A_1|\bar{K}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$$

Similarly, $P(A_2) = \frac{3}{4}$

(b)

$$\begin{aligned} P(A_2|A_1) &= \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1}{\frac{3}{4}} \\ &= \frac{\frac{1}{8} + \frac{1}{2}}{\frac{3}{4}} = \frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{6} \end{aligned}$$

Question 2

(c)

$$\begin{aligned} P(A_2 \cup A_1) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= \frac{3}{4} + \frac{3}{4} - \frac{5}{8} \\ &= \frac{12}{8} - \frac{5}{8} \\ &= \frac{7}{8} \end{aligned}$$

(d) No, A_1 and A_2 are not independent because $P(A_2|A_1) \neq P(A_2)$.

Question 3

Let n be a positive integer and X be a random variable that is uniformly distributed over the set $Z_n = \{0, 1, \ldots, n-1\}$. This means that $P(X = i) = \frac{1}{n}$ for any $i \in Z_n$.
Further let a and b be two other positive integers. Consider the following event $E : aX + b = 0 \bmod n$.

(a) What is $P(E)$ when $a = 2, b = 7$ and $n = 17$?

(b) What is $P(E)$ when $a = 3, b = 12$ and $n = 33$?

Explain your answers.

Question 3

Solution:

(a) $E : 2X + 7 = 0 \bmod 17$
The value of X is in the range $[0, 16]$
So $2X + 7$ is in the range $[7, 39]$
$2X + 7 = 17k$ for some integer k only when $X = 5$
Therefore $P(E) = \frac{1}{17}$.

(b) $E : 3X + 12 = 0 \bmod 33$
The value of X is in the range $[0, 32]$
So $3X + 12$ is in the range $[12, 108]$
$3X + 12 = 33k$ for some integer k only when $X = 7$ or 18 or 29
Therefore $P(E) = 3 \cdot \frac{1}{33} = \frac{1}{11}$

Question 4

Time is continuous. In this question, suppose the time line is discretized into intervals, with the length of each interval being 1 second.

A web server has probability p $(0 < p < 1)$ of getting 1 visit in each time interval and $1 - p$ probability of getting 0 visit.

(a) How long is the server expected to wait until it gets 100 visits?

(b) How many visits is the server expected to get during 1 hour?

Solution:

(a) Let X_i be the number of seconds the server waits until it gets the i-th visit after $(i-1)$ visits.
This is a geometric random variable, so $E(X_i) = \frac{1}{p}$. The number of seconds for which the server is expected to wait until it gets the 100-th visit is then

$$\sum_{i=1}^{100} E(X_i) = 100 \cdot \frac{1}{p} = \frac{100}{p}$$

(b) Let Y_i be the number of visits that the server gets at the i-th second. It can be 0 or 1, and $E(Y_i) = p$.
There are 3600 seconds in one hour. So, the expected number of visits during an hour is:

$$E(\sum_{i=1}^{3600} Y_i) = \sum_{i=1}^{3600} E(Y_i) = \sum_{i=1}^{3600} p = 3600p$$