Hong Kong University of Science and Technology

COMP170: Discrete Mathematical Tools for Computer Science

Spring 2009

Midterm Exam 1

10 March 2009, 3:00–4:20pm, LT-E

Solutions

Question 1: Any subset of the 12 points with three or more members can be made into exactly one convex polygon. Thus, this problem is equivalent to counting the number of such subsets.

Since the number of k-element subsets is $\binom{12}{k}$, the total number of subsets is equal to

$$\begin{aligned}\sum_{k=3}^{12}\binom{12}{k} &= \sum_{k=0}^{12}\binom{12}{k} - \sum_{k=0}^{2}\binom{12}{k} \\ &= 2^{12} - \sum_{k=0}^{2}\binom{12}{k} \\ &= 4096 - 1 - 12 - 66 \\ &= 4017.\end{aligned}$$

Question 2: A surjective function from A to B has to be either one of the following two cases:

(i) It maps three elements in A to the same element in B and the remaining $k-3$ elements in A to the remaining $k-3$ elements in B.

(ii) It maps two elements in A to the same element in B, another two elements in A to a different element in B, and the remaining $k-4$ elements in A to the remaining $k-4$ elements in B.

For case (i), the number of ways of choosing the triple is $\binom{k}{3}$. Once the triple is chosen, there are $(k-2)!$ ways of mapping the $(k-2)$ groups in A to B.

For case (ii), the number of ways of choosing two pairs is $\frac{1}{2}\binom{k}{2}\binom{k-2}{2}$. Once the two pairs are chosen, there are $(k-2)!$ ways of mapping the $(k-2)$ groups in A to B.

By the sum principle, the total number of surjective functions is

$$\binom{k}{3}(k-2)! + \frac{1}{2}\binom{k}{2}\binom{k-2}{2}(k-2)! = \left[\binom{k}{3} + \frac{1}{2}\binom{k}{2}\binom{k-2}{2}\right](k-2)!$$

Question 3: (a) Let us consider the problem of putting n students into three classrooms so that the first, second and third classrooms contain r, s and t students, respectively. The number of ways of doing this is equal to the following trinomial coefficient:

$$\binom{n}{r\ s\ t}.$$

We now use a different way of counting, say by assuming that one specific student wears a hat. This hat-wearing student may be put in the first, second or third classroom. If the student is put in the first classroom, the number of ways of putting the other students is equal to

$$\binom{n-1}{(r-1)\ s\ t}.$$

Similarly, if the student is put in the second or third classroom, the number of ways of putting the other students is equal to

$$\binom{n-1}{r\ (s-1)\ t}$$

or

$$\binom{n-1}{r\ s\ (t-1)},$$

respectively. Since these three cases form three mutually disjoint sets that together cover all possibilities, we can apply the sum principle to show that

$$\binom{n}{r\ s\ t} = \binom{n-1}{(r-1)\ s\ t} + \binom{n-1}{r\ (s-1)\ t} + \binom{n-1}{r\ s\ (t-1)}.$$

(b) The right-hand side (RHS) of the identity can be expanded as:

$$\begin{aligned}
\text{RHS} &= \binom{n-1}{(r-1)\ s\ t} + \binom{n-1}{r\ (s-1)\ t} + \binom{n-1}{r\ s\ (t-1)} \\
&= \frac{(n-1)!}{(r-1)!\,s!\,t!} + \frac{(n-1)!}{r!\,(s-1)!\,t!} + \frac{(n-1)!}{r!\,s!\,(t-1)!} \\
&= \frac{r(n-1)! + s(n-1)! + t(n-1)!}{r!\,s!\,t!} \\
&= \frac{n!}{r!\,s!\,t!} \\
&= \binom{n}{r\ s\ t}
\end{aligned}$$

which is equal to the left-hand side (LHS).

Question 4: (a) Since there are 5 distinct letters in LITTLEST and no repetition of letters is allowed, by the product principle, there are $5 \times 4 \times 3 = 60$ distinct 3-letter words.

(b) Since there are 5 distinct letters in LITTLEST and unlimited repetition of letters is allowed, by the product principle, there are $5^3 = 125$ distinct 3-letter words.

(c) In LITTLEST, there are 3 occurrences of the letter T, 2 occurrences of L, and 1 each for I, E and S.

Each 3-letter word can be in either one of the following three cases:

(i) All three letters are distinct:
We have seen in part (a) that this can be done in 60 ways.

(ii) One letter appears twice:
We choose one of L and T to repeat, choose one of the remaining 4 different letters and choose where that letter is to go, giving $2 \times 4 \times 3 = 24$ ways.

(iii) One letter appears three times:
The only letter that can be chosen is T.

By the sum principle, we add the three cases to give the answer as $60 + 24 + 1 = 85$.

Question 5: (a) Since every integer divides 0, $\gcd(F_0, F_1) = \gcd(0, 1) = 1$.

(b) Let a positive integer d be a common divisor of F_n and F_{n-1}, i.e., there exist nonnegative integers a, b such that $F_n = da$ and $F_{n-1} = db$. Since $F_{n-2} = F_n - F_{n-1} = da - db = d(a-b)$, d is also a common divisor of F_{n-1} and F_{n-2}.

Similarly, let a positive integer e be a common divisor of F_{n-1} and F_{n-2}, i.e., there exist nonnegative integers f, g such that $F_{n-1} = ef$ and $F_{n-2} = eg$. Since $F_n = F_{n-1} + F_{n-2} = ef + eg = e(f+g)$, e is also a common divisor of F_n and F_{n-1}.

So F_n and F_{n-1} have the same set of common divisors as that for F_{n-1} and F_{n-2}, implying that the largest numbers from the two sets are the same, i.e., $\gcd(F_n, F_{n-1}) = \gcd(F_{n-1}, F_{n-2})$.

(c) From part (b), we can conclude that the greatest common divisor of any two consecutive Fibonacci numbers is the same. By combining parts (a) and (b), we have $\gcd(F_{99}, F_{100}) = \gcd(F_0, F_1) = 1$.

Question 6: (a) Let $j = 69$ and $k = 259$. The following table shows the execution of the extended GCD algorithm.

i	$k[i]$	$=$	$j[i]q[i]$	$+$	$r[i]$	$k[i]$	$j[i]$	$r[i]$	$q[i]$	$y[i]$	$x[i]$
0	259	$=$	$69 \cdot 3$	$+$	52	259	69	52	3	4	-15
1	69	$=$	$52 \cdot 1$	$+$	17	69	52	17	1	-3	4
2	52	$=$	$17 \cdot 3$	$+$	1	52	17	1	3	1	-3
3	17	$=$	$1 \cdot 17$	$+$	0	17	1	0	17	0	1

So we get

$$\begin{aligned} \gcd(69, 259) &= 1 = 69 \cdot (-15) + 259 \cdot 4 = -1035 + 1036 \\ x &= -15 \\ y &= 4. \end{aligned}$$

(b) Since $\gcd(69, 259) = 1$, the integer 69 has a unique multiplicative inverse in Z_{259} and it can be calculated as

$$(-15) \bmod 259 = 244.$$

The unique solution of the equation is thus equal to

$$244 \cdot_{259} 3 = 214.$$