Tutorial 6: Probability I

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Question 1

What is the probability that a hand of 5 cards chosen from an ordinary deck of 52 cards, will consist of cards of the same suit? The first question is

What is the sample space
(and associated probability weights)?

There are actually two different logical answers:

(a) The 5-element set model, in which each of the $\binom{52}{5}$ different hands of 5 cards is equally likely.

(b) The 5-element permutation model, in which each of the $52^{\underline{5}}$ different ordered hands of 5 cards is equally likely.

Solution (Using the 5-element set model):
There are $\binom{52}{5}$ different possible 5-sets.
Among these possibilities, there are

$$4(13 \cdot 12 \cdot 11 \cdot 10 \cdot 9)/5!$$

ways of choosing one suit. Thus, the probability is

$$\frac{(52 \cdot 12 \cdot 11 \cdot 10 \cdot 9)/5!}{(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)/5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{51 \cdot 50 \cdot 49 \cdot 48} = \frac{33}{16660}$$

which is approximately 0.00198.

Question 1

Solution (Using the 5-element permutation model):
There are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ different possible 5-element permutations.
Among these possibilities, we have $52 \cdot 12 \cdot 11 \cdot 10 \cdot 9$ ways of choosing one suit.
Thus, the probability is

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{51 \cdot 50 \cdot 49 \cdot 48} = \frac{33}{16660}$$

which is the same as the answer with sets as hands.

Question 2

A five-card hands chosen from a standard deck of playing cards consist of five cards in a row is called a *straight*. E.g.,
$\diamondsuit$ 9, $\clubsuit$ 10, $\clubsuit$ J, $\heartsuit$ Q, $\spadesuit$ K.

What is the probability that a hand of 5 cards chosen from an ordinary deck of 52 cards is a straight?

Explore whether you get the same answer by using 5-element sets as your model of hands, or 5-element permutations as your model of hands.

To simplify the problem, we assume the rules forbid an ace-low straight (A,2,3,4,5).

Question 2

Solution (5-element set model):
There are $\binom{52}{5}$ sets of 5 cards that can be dealt as hands. Among them, there are $9 \cdot 4^5$ possibilities of choosing a straight.
(There are $9 \cdot 4$ possibilities for the highest card;
For each additional card, there are four possibilities.)

Thus, the probability is

$$\frac{9 \cdot 4^5}{\binom{52}{5}} = \frac{192}{54145}$$

or approximately 0.003546

Solution (5-element permutation model):
There are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$ possiblle hands that can be dealt.

Among them, $5! \cdot 9 \cdot 4^5$ hands are a straight.
(There are $9 \cdot 4$ possibilities for the highest card in a straight.
For each additional card, there are four possibilities.)

Thus, the probability is

$$\frac{5! \cdot 9 \cdot 4^5}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$

which turns out to be the same as the answer with sets as hands.

When we say *throw m balls into n boxes* we mean that each of the n^m possible outcomes is uniformly likely.

(a) Throw 6 balls into 5 boxes.
What is the probability that every box contains at least one ball?

(b) Throw 150 balls into 100 boxes.
What is the probability that every box contains at least one ball? You can write the solution to this problem as a summation of numbers.

Solution (a):
The number of ways to throw 6 balls into 5 boxes so that every box contains at least one ball is exactly the same as the number of onto functions from
$\{1,2,3,4,5,6\}$ to $\{1,2,3,4,5\}$
which is $\binom{6}{2}5!$

There are 5^6 different ways of throwing the 6 balls into 5 boxes, so the probability is.

$$\frac{\binom{6}{2}5!}{5^6}.$$

Solution (b):
Let A be the event that every box contains at least 1 ball.
Let B be the event that at least 1 box is empty.
Since B is the complement of A, $P(A) = 1 - P(B)$.

Now, for $i = 1, \ldots, 100$, let E_i be the event that box i is empty.
Since $B = \bigcup_{i=1}^{100} E_i$, from the inclusion-exclusion formula,

$$P\left(\bigcup_{i=1}^{100} E_i\right) = \sum_{k=1}^{100} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le 100}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$$

Question 3

Let E_i be the event that box i is empty

Note that, for all $i_1, i_2, \ldots, i_k$:

$$\begin{aligned} P(E_{i_1}) &= \left(\frac{100-1}{100}\right)^{150} = \left(1-\frac{1}{100}\right)^{150} \\ P(E_{i_1} \cap E_{i_2}) &= \left(\frac{100-2}{100}\right)^{150} = \left(1-\frac{2}{100}\right)^{150} \end{aligned}$$

and, in general,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \left(\frac{100-k}{100}\right)^{150} = \left(1-\frac{k}{100}\right)^{150}$$

There are $\binom{100}{k}$ ways of choosing such k-tuples, so

$$\sum_{\substack{i_1, i_2, \ldots, i_k \\ 1 \le i_1 < i_2 < \cdots < i_k \le 100}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \binom{100}{k} \left(1-\frac{k}{100}\right)^{150}$$

Therefore,

$$\begin{aligned} P(B) &= \sum_{k=1}^{100}(-1)^{k+1} \sum_{\substack{i_1,i_2,\dots,i_k:\\ 1\le i_1<i_2<\dots<i_k\le n}} P(E_{i_1}\cap E_{i_2}\cap\cdots\cap E_{i_k}) \\ &= \sum_{k=1}^{100}(-1)^{k+1}\binom{100}{k}\left(1-\frac{k}{100}\right)^{150} \end{aligned}$$

and $P(A) = 1 - P(B)$

Question 4

Five married couples (i.e., 10 people) sit down at random in a row of 10 seats. That is, each one of the 10! different ways of seating the people is equally likely to occur.
We say that a couple *sits together* if the husband and wife in that couple sit next to each other.

(a) What is the probability that $k (1 \leq k \leq 5)$ specified couples end up sitting together (regardless of whether the other $5 - k$ couples sit together or not)?

(b) What is the probability that no couple sits together? You may use the summation ($\sum$) sign and $\binom{n}{m}$ to express your answer.

Solution (a):
There are totally 10! ways to seat the 5 couples.

If a couple sits together, we treat it as one single unit.
If there are k specified couples sitting together, we can think of the problem as *randomly permuting the (10-k)* units *in (10-k)! different ways.*
For each permutation, $\exists 2^k$ ways to set the k bound couples.
Therefore, the probability is

$$\frac{(10-k)!2^k}{10!}$$

Question 4

Solution (b):

Let E_i denote the event that the ith couple sits together.
The probability that at least one couple sits together can be computed using the inclusion-exclusion principle as

$$P\left(\bigcup_{i=1}^{5} E_i\right) = \sum_{k=1}^{5}(-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq 5}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$$

From part (a)

$$P\left(\bigcup_{i=1}^{5} E_i\right) = \sum_{k=1}^{5}(-1)^{k+1} \binom{5}{k} \frac{(10-k)!2^k}{10!}$$

Thus, the probability that no couple sits together is

$$\begin{aligned} 1 - P\left(\bigcup_{i=1}^{5} E_i\right) &= 1 - \sum_{k=1}^{5}(-1)^{k+1}\binom{5}{k}\frac{(10-k)!2^k}{10!} \\ &= \sum_{k=0}^{5}(-1)^{k}\binom{5}{k}\frac{(10-k)!2^k}{10!} \end{aligned}$$

Question 5

Suppose that boys and girls are equally likely to be born.

(a) In a family consisting of a mother, father, and two children of different ages, what is the probability that the family has two girls, given that one of the children is a girl?

(b) What is the probability that the children are both girls, given that the older child is a girl?

Solution (a):
Let G_2 be event that the family has two girls. $P(G_2) = 1/4$.

Let G_{1+} be event that the family has at least one girl. Then

$$P(G_{1+}) = 1 - P(\text{the family has two boys}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

We have $P(G_{1+} \cap G_2) = 1/4$. Therefore,

$$P(G_2|G_{1+}) = \frac{P(G_{1+} \cap G_2)}{P(G_{1+})} = \frac{1/4}{3/4} = \frac{1}{3}$$

Question 5

It might be easier to visualize this if you write out *all* of the outcomes as GG, GB, BG, BB, where, e.g., GB means that the youngest child is a girl and the oldest child is a boy. Now

$$\begin{aligned} G_{1+} &= \{GG, GB, BG\}, \\ G_2 \cap G_{1+} &= \{GG\} \end{aligned}$$

Conditioning on G_{1+} means that all of the three outcomes containing at least one girl are equally likely, so

$$P(G_2|G_{1+}) = \frac{1}{3}.$$

Question 5

Solution (b):
Let G_2 be event that the family has two girls. $P(G_2) = 1/4$.

Let G_{old} be event that the older child is a girl. $P(G_{old}) = 1/2$.

Since $P(G_{old} \cap G_2) = P(G_2) = 1/4$.
This gives,

$$P(G_2|G_{old}) = \frac{1/4}{1/2} = \frac{1}{2}$$

Again, it might be easier to visualize this by writing out:

$$\begin{aligned} G_{1+} &= \{BG, GG\}, \\ G_2 \cap G_{1+} &= \{GG\} \end{aligned}$$

Conditioning on G_{old} means that each of the two events $\{BG, GG\}$ are equally likely, in which case $P(G_2|G_{old})$, which is the probability of GG given G_{old}, is $\frac{1}{2}$.

Question 5

We have just shown that if a family has two children then

Probability both children are girls, given that one child is a girl $= \frac{1}{3}$

Probability both children are girls, given that the oldest child is a girl $= \frac{1}{2}$

It is instructive to understand the difference between these two problems.

Suppose that a family has 3 children of different ages,

(a) What is the probability that all three children are girls, given that at least 1 is a girl?

(b) What is the probability that all three children are girls, given that the youngest child is a girl?

Question 5 (more)

Solution (a):
Let G_3 be event that the family has three girls. $P(G_3) = 1/8$.
Let G_{1+} be event that family has at least one girl. Then

$$P(G_{1+}) = 1 - P(\text{the family has three boys}) = 1 - \frac{1}{8} = \frac{7}{8}.$$

We have $P(G_{1+} \cap G_3) = 1/8$. Therefore,

$$P(G_3|G_{1+}) = \frac{P(G_{1+} \cap G_3)}{P(G_{1+})} = \frac{1/8}{7/8} = \frac{1}{7}$$

Question 5 (more)

Solution (b):
Let G_3 be event that the family has three girls. $P(G_3) = 1/8$.
Let $G_{youngest}$ be event that the youngest child is a girl.
$P(G_{youngest}) = 1/2$.

We have $P(G_{youngest} \cap G_3) = 1/8$.

This gives $P(G_3 | G_{youngest}) = \frac{1/8}{1/2} = \frac{1}{4}$.

Question 6

If a student knows 75% of the material in a course, and if a 100-question multiple-choice test with five choices per question covers the material in a balanced way. What is the student's probability of getting a right answer to a question, given that the student guesses at the answer to each question whose answer he does not know?

Question 6

Solution:
Let K be the event that the student knows the correct answer and C be the event that the student gets the correct answer.

Then, we know the conditional probabilities $P(C|K) = 1$ and $P(C|\overline{K}) = \frac{1}{5}$.

Then $C = (C \cap K) \cup (C \cap \overline{K})$. Since the events $(C \cap K)$ and $(C \cap \overline{K})$ are disjoint. This gives

$$\begin{aligned} P(C) &= P(C \cap K) + P(C \cap \overline{K}) = P(C|K)P(K) + P(C|\overline{K})P(\overline{K}) \\ &= (1)(0.75) + \frac{1}{5}(0.25) = 0.8. \end{aligned}$$