Hong Kong University of Science and Technology

# COMP170: Discrete Mathematical Tools for Computer Science

*Spring 2009*

## Midterm Exam 1

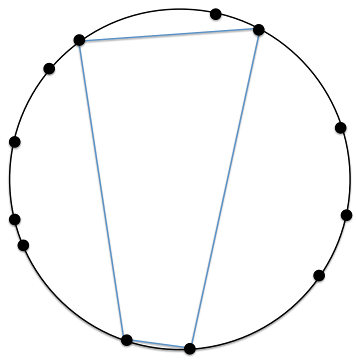
10 March 2009, 3:00–4:20pm, LT-E

## Instructions

1. This is a closed-book exam consisting of 6 questions.
2. Please write your name, student number and email address on the cover page of the answer booklet.
3. Please sign the honor code statement on the second page of the answer booklet.
4. All answers *must* be put on the answer booklet. Only the answer booklet needs to be handed in at the end of the exam.
5. In the answer booklet each question starts on a new page. This is for clarity and is not meant to imply that your answer needs to fill up all the space provided.
6. Unless otherwise specified, you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
7. Your answers may be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
8. Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.
9. Please put your student ID card on the desk so the TA can check it.
10. All mobile phones must be turned off completely during the exam or else you will be disqualified.

**Question 1:** [12 points]

Twelve points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some or all of the 12 points as vertices?
(The figure below shows one such polygon formed by four of the 12 points.)



**Question 2:** [14 points]

Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_{k-2}\}$ be two nonempty sets where $k > 3$ is a positive integer.

We define surjective functions from $A$ to $B$. How many such functions can be defined?

**Question 3:** [16 points]

Consider the following identity:

$$\binom{n}{r\ s\ t} = \binom{n-1}{(r-1)\ s\ t} + \binom{n-1}{r\ (s-1)\ t} + \binom{n-1}{r\ s\ (t-1)},$$

where $n, r, s, t$ are positive integers such that $r + s + t = n$.

(a) Prove the identity above by means of a combinatorial proof.

(b) Prove the identity above by means of an algebraic proof.

**Question 4:** [20 points]

We want to form 3-letter words using the letters in LITTLEST. For the purpose of this problem, a word is considered any ordered list of letters.

(a) How many distinct 3-letter words can be formed with no repetition of letters allowed?
(E.g., LET is legal but SEE is not.)

(b) How many distinct 3-letter words can be formed with unlimited repetition of letters allowed?
(E.g., both LET and SEE are legal.)

(c) How many distinct 3-letter words can be formed if repeats are allowed but no letter can be used more often than it appears in LITTLEST?
(E.g., TTL is legal but LEE is not.)

**Question 5:** [18 points]

We define a sequence of nonnegative integers called Fibonacci sequence as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

In other words, the Fibonacci numbers in the sequence are $0, 1, 1, 2, 3, 5, 8, \ldots$, with each number being the sum of the previous two.

(a) What is $\gcd(F_0, F_1)$?

(b) For $n \geq 2$, prove that $\gcd(F_n, F_{n-1}) = \gcd(F_{n-1}, F_{n-2})$.

(c) What is $\gcd(F_{99}, F_{100})$? Justify your result.

**Question 6:** [20 points]

(a) Using the extended GCD algorithm, find $\gcd(69, 259)$. Also find integers $x, y$ such that $69x + 259y = 1$. Show all steps in running the algorithm to obtain the results.

(b) Consider the following modular equation:

$$69 \cdot_{259} x = 3.$$

Does it have any solution $x \in Z_{259}$? If yes, show how to obtain the solution(s). If not, explain why not.

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