Tutorial 2: Predicate Logics

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Question 1

Consider the statement:

"For all prime number p, p is odd or p is 2"

(a) Express the above statement using predicate and universal quantifiers.

(b) Express the negation of the statement in (a) using an existential quantifier.

Solution:

(a) Let $r(p)$ stand for "p is a prime", $s(p)$ stand for "p is odd". The statement is: $\forall p \in Z^{+}(r(p) \rightarrow (s(p) \vee p = 2))$.

(b) $\exists p \in Z^{+}(r(p) \wedge \neg s(p) \wedge p \neq 2))$.

Question 2

For each of the following parts, select the answer (from (i) to (iv)) that are logically equivalent to it.

(a) $\exists x(P(x) \vee Q(x))$

- (i) $\forall x(P(x) \wedge Q(x))$
- (ii) $\neg(\forall x \neg P(x)) \vee \neg(\forall x \neg Q(x))$
- (iii) $\neg(\forall x \neg P(x)) \wedge \neg(\forall x \neg Q(x))$
- (iv) $\neg(\forall x \neg P(x)) \vee (\forall x \neg Q(x))$

(b) $\exists x P(x) \rightarrow \forall y Q(y)$

- (i) $\exists x \neg P(x) \vee \forall y Q(y)$
- (ii) $\exists x \neg P(x) \wedge \forall y Q(y)$
- (iii) $\exists x \neg P(x) \rightarrow \forall y Q(y)$
- (iv) $\forall x \neg P(x) \vee \forall y Q(y)$

(c) $\neg \forall x \exists y(P(y) \wedge Q(x, y))$

- (i) $\forall x \forall y(P(y) \rightarrow \neg Q(x, y))$
- (ii) $\exists x \forall y(P(y) \rightarrow \neg Q(x, y))$
- (iii) $\forall x \forall y(P(y) \rightarrow Q(x, y))$
- (iv) $\forall x \exists y(P(y) \rightarrow Q(x, y))$

Question 2

Solution:

(a) (ii)

(b) (iv)

(c) (ii)

Question 3

Consider the following quantification about elements in some universe U:

$$\neg\forall x \in U\ (\exists y \in U\ (P(x,y) \wedge Q(x,y))) \tag{1}$$

Let $R(x,y) = \neg P(x,y)$ and $S(x,y) = \neg Q(x,y)$
Express the quantification in Equation (1) in terms of $R(x,y)$ and $S(x,y)$. The negation sign ($\neg$) should *not* appear in the answer. Show how you derived your new quantification.

Solution:
Using the facts:

* $\neg \forall x \in U \ (p(x))$ is equivalent to $\exists x \in U \ (\neg p(x))$
* $\neg \exists x \in U \ (p(x))$ is equivalent to $\forall x \in U \ (\neg p(x))$
* DeMorgan's laws

we get

$$\begin{aligned}
& \neg \forall x \in U \ (\exists y \in U \ (P(x,y) \wedge Q(x,y))) \\
= \ & \exists x \in U \ (\neg \exists y \in U \ (P(x,y) \wedge Q(x,y))) \\
= \ & \exists \in U \ (\forall y \in U \ \neg (P(x,y) \wedge Q(x,y))) \\
= \ & \exists \in U \ (\forall y \in U \ (\neg P(x,y) \vee \neg Q(x,y))) \\
= \ & \exists x \in U \ (\forall y \in U \ (R(x,y) \vee S(x,y)))
\end{aligned}$$

Question 4

Is $(\exists x \in U\ (p(x))) \land (\exists y \in U\ (q(y)))$ logically equivalent to $\exists z \in U\ (p(z) \land q(z))$?

Solution:
No. Consider the following counterexample.
Set $U = Z^+$, $p(x) =$ "x is even" and $q(x) =$ "x is odd".

Obviously, $(\exists x \in U\ (p(x)))$ is true because there exists some even integers in Z^+ and $(\exists y \in U\ (q(y)))$ is true because there exists some odd integers in Z^+. So, $(\exists x \in U\ (p(x))) \land (\exists y \in U\ (q(y)))$ is true.

However, $\exists z \in U\ (p(z) \land q(z))$ is false because there is no integer $z \in Z^+$ that is both even and odd.

Therefore, they are not logically equivalent.