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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – Spring 2010

Midterm Examination 1

Date: Thursday, March 11, 2010 Time: 19:00–20:30 Venues: LT A,B,C

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: __________

Instructions

- This is a closed book exam. It consists of 17 pages and 7 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- **Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.**
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the $_nP_k$ and $_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	Total
Points	15	20	10	13	14	16	12	100
Score								

Student ID: ______________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ____________________

Student's Signature: ____________________
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Student ID: ______________

Problem 1: [15 pts] A by-election is being held to fill 5 seats in a council that have recently become vacant. A total of 9 candidates have registered to run in the election. The candidates can be divided into 3 groups based on their political affiliations with: 4 in Group A, 3 in Group B, and 2 in Group C.

(a) In how many ways can a voter choose 5 candidates?

(b) In how many ways can a voter choose 5 candidates such that at least 3 candidates from Group A are chosen?

(c) In how many ways can a voter choose 5 candidates such that at least 1 candidate from each group is chosen?

ANS:

(a) $\binom{9}{5} = 126.$

(b) We partition this into two cases: choosing either 3 or 4, from group A and, respectively, 2 or 1 candidates from the combiniation of groups B and C. This is

$$\binom{4}{3}\binom{5}{2} + \binom{4}{4}\binom{5}{1} = 4 \cdot 10 + 1 \cdot 5 = 45.$$

(c) In this case we need to partition into all the possible scenarios of choosing from (A, B, C)*, which are* $(1,2,2), (1,3,1), (2,1,2), (2,2,1), (3,1,1)$*. This gives*

$$\binom{4}{1}\binom{3}{2}\binom{2}{2}+\binom{4}{1}\binom{3}{3}\binom{2}{1}+\binom{4}{2}\binom{3}{1}\binom{2}{2}+\binom{4}{2}\binom{3}{2}\binom{2}{1}+\binom{4}{3}\binom{3}{1}\binom{2}{1}$$

or

$$4 \cdot 3 \cdot 1 + 4 \cdot 1 \cdot 2 + 6 \cdot 3 \cdot 1 + 6 \cdot 3 \cdot 2 + 4 \cdot 3 \cdot 2 = 98$$

Student ID: ____________

Problem 2: [20 pts] Recall that $S_n = \{1, 2, 3, \ldots, n\}$.

(a) How many onto functions f are there from S_5 to S_4?

(b) How many onto functions f are there from S_5 to S_4 such that $f(1) \leq f(2) \leq f(3) \leq f(4) \leq f(5)$?

(c) How many one-to-one functions f are there from S_4 to S_5?

(d) How many one-to-one functions f are there from S_4 to S_5 such that $f(1) \leq f(2) \leq f(3) \leq f(4)$?

(e) How many one-to-one functions f are there from S_4 to S_5 such that $f(x) \geq x$ for each $x \in S_4$?

ANS:

(a)

$$\binom{5}{2} 4! = 240$$

(b) Since f is onto, $f(1), f(2), f(3), f(4), f(5)$, must take on exactly *4 different values. This means that exactly 1 of the $\leq$ in*

$$f(1) \leq f(2) \leq f(3) \leq f(4) \leq f(5)$$

becomes an $=$, while the remaining $3 \leq$ signs become $<$.
Once the location of the $=$ is chosen, the values of the $f(i)$ are uniquely forced. So, there are exactly 4 possibilities (the number of ways of choosing the location of the $=$ sign).

(c) Choose the 4 items in S_5 onto which the items in S_4 are mapped. Once those 4 items are chosen, there are $4!$ ways of mapping them. So, the answer is

$$\binom{5}{4} 4! = 5! = 120.$$

(d) Since f is one-to-one there is exactly one item in S_5 onto which f does not map something in S_4. Once that item is chosen, the values of $f(1), f(2), f(3), f(4)$ are uniquely forced . So, the answer is 5.

(e) $f(4)$ can be 4 or 5.
$f(3)$ can be chosen to be either of the two values in $\{3, 4, 5\} - \{f(4)\}$.
$f(2)$ can be chosen to be either of the two values in
$\{2, 3, 4, 5\} - \{f(3), f(4)\}$.
$f(1)$ can be chosen to be either of the two values in
$\{1, 2, 3, 4, 5\} - \{f(2), f(3), f(4)\}$.
So, by the product principle, the answer is: $2^4 = 16$

Student ID: ____________

Problem 3: [10 pts] Give a combinatorial proof of the identity, for all $n \geq 13$.

$$\binom{n}{4}\binom{n-4}{6}\binom{n-10}{3} = \binom{n}{6}\binom{n-6}{3}\binom{n-9}{4}$$

Note: An algebraic proof of this identity will not be accepted as a solution.

ANS: *Consider the problem of "how to color n items so that 4 are red, 6 are green, 3 are blue and the remaining $n-13$ are yellow.*

The left hand side of the inequality obviously counts this. Notice that the problem is the same if we change the order in which we ask the question. Consider the problem of how to color n items so that 6 are green, 3 are blue, 4 are red and the remaining $n-13$ are yellow. This is what the right hand side of the equation is counting, so the two sides are the same.

Problem 4: [13 pts] There will be n students receiving prizes at an award ceremony tomorrow. The students will be seated on a platform in a row of n seats facing the audience.

(a) How many different ways are there to seat the n students in the n seats?

(b) Let $n \geq 3$.
Label three specific students as student **A**, student **B** and student **C**. Suppose that we require that

1. Student **A** sits next to student **B** and
2. Student **A** sits next to student **C**.

How many ways are there to seat the n students now? Explain how you derived your answer.

(c) Let $n \geq 4$.
Forget about the requirements of part (b). Now suppose that one of the other students is student **D**. The new requirements are that

1. Student **A** sits next to student **B** and
2. Student **C** sits next to student **D**.

How many ways are there to seat the n students now? Explain how you derived your answer.

(d) Let $n \geq 4$.
In addition to the requirements of (c) we now want that **A** must sit next to either **C** or **D** That is, the full requirements are that

1. Student **A** sits next to student **B** and
2. Student **C** sits next to student **D** and
3. Student A sits next to either student **C** or student **D**.

How many ways are there to seat the n students now? Explain how you derived your answer.

Ans:

(a) $n!$

(b) The three students must sit together either as BAC or as CAB occurring. Suppose that CAB occurs. We can consider the 3 students as one unit that must be seated together along with the $n-3$ other students that must be seated. This gives $(n-2)!$ possible solutions. Similarly, there are $(n-2)!$ solutions when BAC occurs. The final solution is thus,

$$2(n-2)!$$

(c) *We must have either AB or BA*

Similarly, we must have CD or DC.

Suppose that we have AB and CD. Each of these two sets is seated along with $n-4$ other students. There are $(n-2)!$ such ways of doing this.

Each of the other 3 ways of seating $\{A, B\}$ and $\{C, D\}$ also lead to $(n-2)!$ ways. In total, this gives

$$4(n-2)!$$

(d) *The four students must sit together in one of the four configurations $DCAB$, $BACD$, $CDAB$, $BADC$.*

Suppose that $DCAB$ occurs. There are $(n-3)!$ ways of placing the $DCAB$ down with the $n-4$ other students.

A similar analysis holds for the three other configutations. So, the final answer is

$$4(n-3)!$$

Problem 5: [14 pts] Let m and n be two positive integers. Suppose you are told that

$$\begin{aligned} m &= qn + r & \qquad 0 \le r < n \\ m &= q'n + r' & \qquad 0 \le r' < n \end{aligned}$$

where q, q', r, r' are integers.

Prove that $q' = q$ and $r = r'$.

Your proof must be from scratch. That is, it should start from the two given equations and may not use any of the Theorems and Lemmas taught in class. In particular, it may not use Euclid's Division Theorem.

ANS: *The statement implies that*

$$n(q - q') = r' - r,$$

which in turn implies

$$n|(q - q')| = |r' - r|.$$

But, since $0 \le r, r' < n$ *we have that*

$$|r - r'| < n.$$

So,

$$n|(q - q')| < n.$$

This implies that $|q - q'| = 0$, *so* $q = q'$.

This implies $|r' - r| = n|(q - q')| = 0$, *so* $r = r'$.

Student ID: ____________

Problem 6: [16 pts]

(a) Does there exist an x in Z_{154} that solves

$$21 \cdot_{154} x = 5?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove that such an x does not exist.

(b) Does there exist an x in Z_{85} that solves

$$33 \cdot_{85} x = 1?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove that such an x does not exist.

ANS:

(a) No. First note that $21 = 3 * 7$ *and* $154 = 22 * 7$.
If there was such an x *then* $21x = 154q + 5$ *for some* q.
Then $5 = 21x - 154q = 7(3x - 22q)$
Since 7 *does not divide* 5, *this is impossible.*

Note that to solve this problem it would not have been enough to say that "gcd$(21, 154) = 7 \neq 1$ *so* 21 *does not have an inverse in* Z_{154}*."*

Some problems in the form

$$21 \cdot_{154} x = b$$

actually do *have solutions* $x \in Z_{154}$. *For example*

$$21 \cdot_{154} x = 42$$

has the solution $x = 2$.

(b) Running the extended GCD algorithm gives $1 = 85 \cdot 7 + 33 \cdot (-18)$.
So, the muliplicative inverse of 33 *in* Z_{85} *is*

$$(-18) \bmod 85 = 67.$$

As a reality check, note that $67 * 33 = 2211 = 26 * 85 + 1$.

Problem 7: [12 pts] Let a, b, c be integers. For each of the following four statements either (i) prove that the statement is always correct or (ii) give a counterexample.

A counterexample for (1) or (2) would be a value a for which the statement is false. A counterexample for (3) or (4) would be a triple a, b, c for which the statement is false.

While proving that a statement is correct you can either prove it from scratch or reference any theorem or lemma given in class. If you use a theorem or lemma from class you should state the theorem or lemma explicitly, i.e., state its assumptions and conclusions.

(1) For all integers a

$$(a \bmod 35) \bmod 5 \quad = \quad a \bmod 5$$

(2) For all integers a

$$(a \bmod 36) \bmod 5 \quad = \quad a \bmod 5$$

(3)

$$\Big(\big((a \cdot b) \bmod 5\big) \cdot c\Big) \bmod 7 \quad = \quad \Big(a \cdot \big((b \cdot c) \bmod 7\big)\Big) \bmod 5$$

(4)

$$\Big(\big((10 \cdot a \cdot b) \bmod 5\big) \cdot c\Big) \bmod 7 \quad = \quad \Big(10 \cdot a \cdot \big((b \cdot c) \bmod 7\big)\Big) \bmod 5$$

Ans:

(1) True
From Euclid's division theorem we know that a can be uniquely written as

$$a = 35q_1 + r_1, \quad 0 \le r_1 < 35$$

and

$$a = 5q_2 + r_2, \quad 0 \le r_2 < 5$$

where $r_1 = a \bmod 35$ and $r_2 = a \bmod 5$.
For these values we have

$$35q_1 + r_1 = 5q_2 + r_2$$

so

$$r_1 = 5(q_2 - 7q_1) + r_2.$$

But, again from Euclid's division theorem, this implies

$$r_2 = r_1 \bmod 5$$

which is what we wanted to prove.

(2) *False*
Let $a = 41$.
Then

$$(a \bmod 36) \bmod 5 = 0 \neq 1 = a \bmod 5.$$

(3) *False*
Let $a = 1,\ b = 10,\ c = 1$. *Then*

$$\Big(\big((a \cdot b) \bmod 5\big) \cdot c\Big) \bmod 7 = 0$$

while

$$\Big(a \cdot \big((b \cdot c) \bmod 7\big)\Big) \bmod 5 = 3.$$

(4) *True (both sides always equal 0)*