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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – FALL 2008

Midterm Examination 1 – Solution Sketch

Date: Thursday, October 09, 2008 Time: 19:00–20:30 Venues: LT B,C,D

Name: ____________ Student ID: ____________
Email: ____________ Lecture and Tutorial: ____________

Instructions

- This is a closed book exam. It consists of 17 pages and 7 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- **Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.**
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3} + \binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	Total
Points	15	11	16	13	14	17	14	100
Score								

Student ID: ______________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ______________________

Student's Signature: ______________________
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Problem 1: [15 pts]

8 men and 8 women are invited to a party at which they are seated at a long rectangular table. The table is designed so that 8 people can sit on its top side and 8 people on its bottom side (no one sits at the other two sides). Each seat on the top is across from exactly one seat on the bottom.

The 8 men are seated on the top side of the table and the 8 women on the bottom side.

(a) How many ways are there to seat the 8 men and 8 women?

(b) Now suppose that one man and one woman have quarreled with each other and will not sit across from each other. How many ways are there to seat all of the men and women now?

(c) Now suppose that if a man and woman are married they sit exactly across from each other.
The party guests include exactly 3 married couples (husband-wife pairs). How many ways are there to seat everyone now?

Note, in part (c), only the husbands and wives are constrained to be across from each other. There are no other constraints. In particular, the quarrel constraint from (b) no longer applies.

Examples:

M_1	M_6	M_7	M_2	M_5	M_8	M_3	M_4
1	2	3	4	5	6	7	8
W_2	W_5	W_6	W_1	W_8	W_7	W_3	W_4

(i)

M_1	M_6	M_7	M_2	M_5	M_4	M_3	M_8
1	2	3	4	5	6	7	8
W_1	W_5	W_6	W_2	W_8	W_7	W_3	W_4

(ii)

In the diagrams, M_i is man i; W_i is woman i. Both (i) and (ii) are legal seating arrangements.

Suppose in part (b) that it is M_1 and W_1 that have quarelled. Then, (i) is a valid arrangement for (b) while (ii) is not.

Suppose in part (c) that the three couples are (M_1, W_1), (M_2, W_2) and (M_3, W_3). Then (ii) is a valid seating arrangement for (c) but (i) is not.

ANS:

(a) *There are* $8!$ *ways to seat the men and* $8!$ *ways of seating the women. By the product principle, the answer is*

$$(8!)^2 = 1,625,702,400$$

(b) *Suppose that* M_1 *and* W_1 *have quarelled. There are* $8!$ *ways of seating the men. There are then* 7 *possible locations to seat* W_1 *and, after that,* $7!$ *ways of seating the remaining* 7 *women. By the product principle, the answer is*

$$8! \cdot 7 \cdot 7! = 1,422,489,600.$$

(c) *There are, again,* $8!$ *ways of seatng the men. After the men are seated, the* 3 *wives must sit across from their husbands. The remaining* 5 *women have* $5!$ *seating arrangements. So, by the productprinciple, the answer is*

$$8! \cdot 5! = 4,838,400.$$

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Problem 2: [11 pts] Give a combinatorial proof of the identity, for all $n \geq 11$.

$$\binom{n}{3}\binom{n-3}{5}\binom{n-8}{3} = \binom{n}{3}\binom{n-3}{3}\binom{n-6}{5}$$

Note: An algebraic proof of this identity will not be accepted as a solution.

ANS: *Consider the problem of "how to color n items so that* 3 *are red,* 5 *are green,* 3 *are blue and the remaining* $n-11$ *are yellow.*

The left hand side of the inequality obviously counts this. Notice that the problem is the same if we change the order in which we ask the question. Consider the problem of how to color n items so that 3 *are red,* 3 *are blue,* 5 *are green and the remaining* $n-11$ *are yellow. This is what the right hand side of the equation is counting, so the two sides are the same.*

Problem 3: [16 pts]

You are arranging a special meal for your friends and go to a restaurant that has the following deal. They have two types of food:

Starters 10 types of starters are listed. Call this the A list

Main Courses 15 types of main courses are listed. Call this the B list

To create your own menu you are allowed to choose 3 items from the A list as starters and 2 items from the B list as main courses.

Note that, after choosing the foods, the dishes all put down on the table at the same time. So, a menu is just a properly chosen set of 5 foods with no order.

(a) How many different menus can you create?

(b) Now suppose that the restaurant is flexible and lets you choose starters as main courses.
That is, you can choose both, one or none of your main courses from the A list.
If you choose an item from the A list as a main course, then you can not also choose it as a starter.
Now how many different menus are there?

Examples. Suppose that the A list contains items $A_1, A_2, \ldots, A_{10}$ and the B list $B_1, B_2, \ldots, B_{15}$.

Then $\{A_1, A_3, A_5, B_4, B_6\}$ is a legal menu for both questions (a) and (b) while $\{A_1, A_3, A_5, A_7, B_6\}$ is a legal menu for question (b) but not for question (a).

Note that, in part (b), a menu is just a set of foods so a legal menu is only counted *once*. As an example, the set $\{A_1, A_3, A_5, A_7, B_6\}$ was a legal menu for (b). When listing this menu, you do *not* specify which A item in the set is chosen as a main course.

ANS:

(a)

$$\binom{10}{3}\binom{15}{2}.$$

(b) *Let i be the number of items from List A chosen as main courses. To solve this problem you must evaluate the three cases $i = 0, 1, 2$ separately and then add them up. For fixed value i, we choose $i + 3$ items from A and $2 - i$ items from B. So, the answer is*

$$\sum_{i=0}^{2}\binom{10}{3+i}\binom{15}{2-i} = \binom{10}{3}\binom{15}{2} + \binom{10}{4}\binom{15}{1} + \binom{10}{5}$$

Problem 4: [13 pts] Let $n > 2$ be an integer and $a \in Z_n$. For each of the two following statements either
(i) prove that the statement is correct for all such a and n or
(ii) give a counterexample. A counterexample would be a pair a, n for which the statement is false.

While proving that a statement is correct you can either prove it from scratch or reference any theorem or lemma given in class. If you use a theorem or lemma from class you should state the theorem or lemma explicitly, i.e., state its assumptions and conclusions.

(a) If the equation $a \cdot_n x = 1$ has a solution in Z_n, then the equation $a \cdot_n x = 2$ has a solution in Z_n,

(b) If the equation $a \cdot_n x = 2$ has a solution in Z_n, then the equation $a \cdot_n x = 1$ has a solution in Z_n,

ANS:

(a) True. In class we proved that if $a \cdot_n x = 1$ has a solution $x = a'$ in Z_n then $a \cdot_n x = b$ has a solution $x = a' \cdot_n b$. Letting $b = 2$ proves the statement.

(b) False. For a counterexample, Consider n $= 4$ and $x = 2$. Then $a \cdot_n x = 2$ has solution $x = 1$ but $a \cdot_n x = 1$ has no solution.

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Problem 5: [14 pts]

(a) Does there exist an x in Z_{79} that solves

$$53 \cdot_{79} x = 1?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove the fact.

(b) Does there exist an x in Z_{147} that solves

$$12 \cdot_{147} x = 7?$$

If yes, give the value of x (it is not necessary to show your work).
If no, prove the fact.

ANS:

(a) Yes. $x = 3$ *solves the equation.*
One way to find this would be to use the Extended GCD algorithm to calculate

$$1 = 3 \cdot 53 - 2 \cdot 79.$$

(b) No. If $12 \cdot_{147} x = 7$ *then there is some integer* q *such that*

$$12x = 147q + 7$$

or

$$3(4x - 49) = 7.$$

Since the left side of this equation is divisible by 3 *and the right side isn't, this is impossible.*

Problem 6: [17 pts] Recall that $S_n = \{1, 2, 3, \ldots, n\}$.

(a) Let $n \geq 2$. How many onto functions are there from S_n to S_2?

(b) How many onto functions are there from S_6 to S_4?

Ans:

(a) One way to define such an onto function is to let $X \subset S_n$ be all of the $x \in S_n$ such that $f(x) = 1$ while if $x \notin X$ then $f(x) = 2$. The only constraint on X is that $X \neq \emptyset$ and $X \neq S_n$. Since the total number of possible subsets of S_n is 2^n, the answer is

$$2^n - 2.$$

(b) Recall that, for $i \in S_4$, $f^{-1}(i)$ is the set of all items $x \in S_6$ such that $f(x) = i$. Examining

$$\Big(f^{-1}(1),\ f^{-1}(2), f^{-1}(3), f^{-1}(4)\Big)$$

a partition of S_6 that fully specifies f. So, we want to count the number of such partitions.

There are two possiblities.

- *One of the $f^{-1}(i)$ contains 3 item and the other 3 contain one item. There are $\binom{6}{3}$ ways of choosing the 3 items and once they are chosen there are $4!$ ways of ordering the sets.*
- *Two of the $f^{-1}(i)$ contains 2 items and the other 2 contain one item. There are $\frac{1}{2}\binom{6}{2}\binom{4}{2}$ ways to choose these sets and then $4!$ ways of ordering them.*

The total answer is therefore

$$\binom{6}{3}4! + \frac{1}{2}\binom{6}{2}\binom{4}{2}4! = 1560.$$

Problem 7: [14 pts] Consider the following statement:

$$gcd(k, k-j) = gcd(k, k+j).$$

Is this statement always true for k, j with $k > j > 0$?

Either prove that it is true for all k, j with $k > j > 0$, or give values for k, j with $k > j > 0$ such that $gcd(k, k-j) \neq gcd(k, k+j)$.

ANS:

The proof is very similar to that of Lemma 2.13 in the class notes.

(a) We first show that if d is a common divisor of k and $k-j$ then it is a divisor of $k+j$. This will imply that all common divisors of k and $k-j$ are common divisors of k and $k+j$.

(b) We will then show that if d is a common divisor of k and $k+j$ then it is a divisor of $k-j$. This will imply that all common divisors of k and $k+j$ are common divisors of k and $k-j$.

The combination of (a) and (b) implies that the set of common divisors of k and $k+j$ is exactly the same as the set of common divisors of k and $k-j$. Since the sets of common divisors are the same, the greatest common divisor of k and $k+j$ is the same as the greatest common divosr of k and $k-j$.

To prove (a) suppose that d is a common divisor of k and $k-j$. Then there are q_1 and q_2 such that $k = q_1 d$ and $k - j = q_2 d$. Then

$$k + j = k - [(k-j) - k] = (2q_1 - q_2)d$$

so d is a divisor of $k+j$.

Similarly, to prove (b), suppose that d is a common divisor of k and $k+j$. Then there are q_1 and q_2 such that $k = q_1 d$ and $k + j = q_2 d$. Then

$$k - j = k - [(k+j) - k] = (2q_1 - q_2)d$$

so d is a divisor of $k-j$.

There are actually many different proofs of the statement. For one alternative proof recall that we proved in class (Lemma 2.13)
If j, k, q, and r are nonnegative integers such that $k = jq + r$, then $\gcd(j,k) = gcd(r,j)$.

Then

1. $k + j = k \cdot 1 + j$ *so* $\gcd(k + j, k) = \gcd(k, j)$.
2. $k = (k - j) \cdot 1 + j$ *so* $\gcd(k, k - j) = \gcd(k - j, j)$.
3. $k = j \cdot 1 + (k - j)$ *so* $\gcd(k, j) = \gcd(j, k - j)$.

Combining these gives

$$\gcd(k + j, k) = \gcd(k, j) = \gcd(j, k - j) = \gcd(k, k - j).$$

Note: This was not the "intended" solution but came, slightly modified, from one of the test books.