

Instructions

Student ID: _____

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name: _____

Student's Signature: _____

Definitions and Formulas: This page contains some definitions used in this exam and a list of formulas (theorems) that you may use in the exam (without having to provide a proof). Note that you might not need all of these formulas on this exam.

Definitions

1. $N = \{0, 1, 2, 3, \dots\}$, the set of non-negative integers.
2. $Z^+ = \{1, 2, 3, \dots\}$, the set of positive integers.
3. R is the set of *real numbers*.
4. R^+ is the set of positive *real numbers*.

Formulas:

1. $\binom{n}{i} = \frac{n!}{i!(n-i)!}$
2. If $0 < i < n$ then $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$.
3. $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.
4. $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
5. $\neg \forall x \in U (p(x))$ is equivalent to $\exists x \in U (\neg p(x))$
6. $\sum_{i=1}^{n-1} i = n(n-1)/2$.
7. $\sum_{i=1}^{n-1} i^2 = \frac{2n^3 - 3n^2 + n}{6}$.
8. If $r \neq 1$ then $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$
9. If $r \neq 1$ then $\sum_{i=0}^n i r^i = \frac{nr^{n+2} - (n+1)r^{n+1} + r}{(1-r)^2}$
10. If $r \neq 1$ then $\sum_{i=0}^n i^2 r^i = \frac{r+r^2 - (n+1)^2 r^{n+1} + (2n^2+2n-1)r^{n+2} - n^2 r^{n+3}}{(1-r)^3}$

Problem 1: (10 points) Bob is constructing an RSA key-pair. He first chooses $p = 17$, $q = 11$ and then chooses $e = 7$. His public key is then the pair (n, e) where $n = pq = 187$.

(a) Bob then calculates his private key d .

What is the value of d ?

Show how you derived this.

(b) Alice wants to send Bob the message $M = 75$.

She encrypts the message using the public key pair $(n, e) = (187, 7)$.

What is the value of the encrypted message that Alice sends Bob?

Solution: (a) $T = (p - 1)(q - 1) = 16 \cdot 10 = 160$. Use the extended GCD:

$$\begin{aligned} 160 &= 22 \cdot 7 + 1 \cdot 6 \\ 7 &= 1 \cdot 6 + 1 \cdot 1 \end{aligned}$$

We want to know, what are x and y in the equation $7x + 160y = 1$? x is the inverse of $7 \bmod 160$, so x is our d .

Iterating “backwards” (we could also use the formal algorithm given in class) yields

$$\begin{aligned} 1 &= 1 \cdot 7 - 1 \cdot 6 \\ &= 1 \cdot 7 - 1 \cdot (1 \cdot 160 - 22 \cdot 7) \\ &= 1 \cdot 7 - 1 \cdot 160 + 22 \cdot 7 \\ &= 23 \cdot 7 - 1 \cdot 160 \end{aligned}$$

So, y is -1 , and $x = d = e^{-1} = 23$.

(b) Encrypted message $C = M^e \bmod n = 75^7 \bmod 187 = 114$. (Bob decrypts by doing $M = C^d \bmod n = 114^{23} \bmod 187 = 75$, but this is not asked in the question.)

Problem 2: (10 pts)(a) Efficiently calculate the value of $3^{134} \bmod 100$.

What is the value of the answer?

Describe how you got the answer and show the results of *all* of the multiplications that you performed while deriving the answer.Note: It is *not* necessary to justify each step of your derivation.

(b) How many multiplications did you use?

Solution: (a) *Use Repeated Squaring* $3^{134} \bmod 100 = 3^{128} \cdot 3^4 \cdot 3^2 \bmod 100$.

$$I_1 = (a \cdot a) \bmod 100 = 3^2 \bmod 100 = 9 \bmod 100 = 9$$

$$I_2 = (I_1 \cdot I_1) \bmod 100 = 9^2 \bmod 100 = 81 \bmod 100 = 81$$

$$I_3 = (I_2 \cdot I_2) \bmod 100 = 81^2 \bmod 100 = 6561 \bmod 100 = 61$$

$$I_4 = (I_3 \cdot I_3) \bmod 100 = 61^2 \bmod 100 = 3721 \bmod 100 = 21$$

$$I_5 = (I_4 \cdot I_4) \bmod 100 = 21^2 \bmod 100 = 441 \bmod 100 = 41$$

$$I_6 = (I_5 \cdot I_5) \bmod 100 = 41^2 \bmod 100 = 1681 \bmod 100 = 81$$

$$I_7 = (I_6 \cdot I_6) \bmod 100 = 81^2 \bmod 100 = 6561 \bmod 100 = 61$$

$$3^{134} \bmod 100 = (I_7(I_2 \cdot I_1) \bmod 100) \bmod 100 = (61(81 \cdot 9) \bmod 100) \bmod 100 = (61 \cdot 29) \bmod 100 = 69.$$

(b) 9 multiplications. 7 to calculate the I_i and 2 more to actually calculate the solution from the I_i .

Problem 3: (10 pts) How many solutions x with $0 \leq x < 264$ are there that satisfy

$$\begin{aligned}x \bmod 11 &= 5, \\x \bmod 24 &= 7?\end{aligned}$$

Give all of the solutions.

Solution: *Since 11 and 24 are relatively prime the Chinese Remainder Theorem tells us that there is exactly one solution.*

Calculation shows that $24 \cdot_{11} 6 = 1$ and $11 \cdot_{24} 11 = 1$. Setting $y = 5 \cdot 6 \cdot 24 + 7 \cdot 11 \cdot 11 = 1567$ gives

$$\begin{aligned}y \bmod 11 &= 5, \\y \bmod 24 &= 7.\end{aligned}$$

Setting $x = y \bmod 264 = 247$ gives the solution.

Problem 4: (6 pts) For each of the three statements below, state whether they are True or False and prove your answer.

For problem (c), $D(x, y)$ will denote that “ x divides y ”, e.g., $D(3, 9)$ is True while $D(3, 10)$ is False. Also, xy will denote the normal product $x \cdot y$.

(a) $\forall x \in R \left(\forall y \in R (x^2 + y^2 \geq 2xy) \right)$

(b) $\forall x \in R \left(\exists y \in R^+ ((y - 3) \geq (x - 2)) \right)$

(c) $\forall x \in Z^+ \left(\forall y \in Z^+ (\forall z \in Z^+ ([D(x, z) \wedge D(y, z)] \Rightarrow D(xy, z))) \right)$

Solution: (a) True. $x^2 + y^2 \geq 2xy \Leftrightarrow x^2 - 2xy + y^2 \geq 0 \Leftrightarrow (x - y)^2 \geq 0$.
Since $(x - y)^2 \geq 0$, the statement is true for all x and y .

(b) True. Choose $y = \max\{x + 1, 1\}$.

(c) False. Counter example: if $x = 3, y = 6, z = 12$, then $xy = 18$.

Problem 5: (11 pts) In each part below, you are asked whether one statement is equivalent to another. In each case, if yes, prove your answer. If no, give a counterexample. In (a)-(d) a counterexample would be some universe U , statement $p(x)$ and value $x \in U$; in (e), a counterexample would be a truth setting of the variables.

If you are using formulas and/or theorems proven in class, state explicitly what the formulas or theorems you are using are.

(a) Is $\neg(\forall x \in R^+ (p(x)))$ equivalent to $\exists x \in R ((x > 0) \wedge \neg p(x))$?

(b) Is $\neg(\exists x \in R^+ (p(x)))$ equivalent to $\forall x \in R (\neg(x > 0) \vee \neg p(x))$?

(c) Is $(\exists x \in U (p(x))) \vee (\exists y \in U (q(y)))$
equivalent to $\exists z \in U (p(z) \vee q(z))$?

(d) Is $(\exists x \in U (p(x))) \wedge (\exists y \in U (q(y)))$
equivalent to $\exists z \in U (p(z) \wedge q(z))$?

(e) Is $(p \wedge \neg q) \Rightarrow ((r \vee s) \wedge \neg t)$
equivalent to $((\neg r \vee t) \wedge (\neg s \vee t)) \Rightarrow (\neg p \vee q)$?

Solution: (a) Yes. From Lecture 8 Theorem 3.2 we know we can rewrite $\neg(\forall x \in R^+ (p(x)))$ as $\neg(\forall x \in R ((x > 0) \Rightarrow p(x)))$.

We then apply Theorem 3.3 (formula 5 on page 3 of this exam) to get the equivalent statement $(\exists x \in R \neg((x > 0) \Rightarrow p(x)))$.

$\neg s \vee t$ is equivalent to $s \Rightarrow t$, so we get: $(\exists x \in R \neg(\neg(x > 0) \vee p(x)))$.

Applying DeMorgan's Law yields $(\exists x \in R ((x > 0) \wedge \neg p(x)))$.

(b) Yes. We can rewrite $\neg(\exists x \in R^+ (p(x)))$ as $\neg(\exists x \in R ((x > 0) \wedge p(x)))$ (Theorem 3.2 in Lecture 8).

This is equivalent (Theorem 3.3) to $(\forall x \in R \neg((x > 0) \wedge p(x)))$.

Applying DeMorgan's Law yields $(\forall x \in R (\neg(x > 0) \vee \neg p(x)))$.

(c) Yes. If there exists an $x \in U$ that makes $p(x)$ true, then that same x also makes $p(x) \vee q(x)$ true. The same holds for a $y \in U$ and $q(y)$ respectively.

On the other hand if we find a $z \in U$ that makes $p(z) \vee q(z)$ true, this will make either $p(z)$ or $q(z)$ true and therefore the first statement.

So, we have just shown that the first statement is true if and only if the second statement is true. They are therefore equivalent.

(d) No. $U = Z$. $p(x) = x$ is even. $q(x) = x$ is odd. There exist even and odd integers in Z but none of them are both even and odd.

(e) Yes. Use Lecture 9, Principle 3.6: The statements $u \Rightarrow v$ and $\neg v \Rightarrow \neg u$ are equivalent. Note that $(p \wedge \neg q)$ is u , so $\neg u$ is $(\neg p \vee q)$, and v is $((r \vee s) \wedge \neg t)$, so $\neg v$ is $((\neg r \vee t) \wedge (\neg s \vee t))$, because

$$\begin{aligned} & \neg((r \vee s) \wedge \neg t) \\ &= \neg(r \vee s) \vee t \quad (\text{DeMorgan's Law}) \\ &= (\neg r \wedge \neg s) \vee t \quad (\text{DeMorgan's Law}) \\ &= (\neg r \vee t) \wedge (\neg s \vee t) \quad (\text{Distributive Law}). \end{aligned}$$

Problem 6: (8 pts)Find *all* values $n \in \mathbb{N}$ such that $3^n > n^2 - 10n + 50$.

Prove the correctness of your answer.

Solution: *We try the first few values to find:**when $n = 1$, $3 < 1 - 10 + 50 = 41$;**when $n = 2$, $9 < 4 - 20 + 50 = 34$;**when $n = 3$, $27 < 9 - 30 + 50 = 29$;**when $n = 4$, $81 > 16 - 40 + 50 = 26$;**when $n = 5$, $243 > 25 - 50 + 50 = 25$;**when $n = 6$, $729 > 36 - 60 + 50 = 26$;**...**The statement is false when $n \leq 3$ and it seems that it is correct for $n \geq 4$.**We prove this by induction. We will start our induction at $n = 4$.**Basis: We already saw that it is true for $n = 4$.**Hypothesis:**Assume that for $k \geq 4$, when $n = k$, $3^k > k^2 - 10k + 50$.**Step :**Consider when $n = k + 1$, what we need to prove is $3^{k+1} - ((k+1)^2 - 10(k+1) + 50) > 0$.*

$$\begin{aligned}
& 3^{k+1} - ((k+1)^2 - 10(k+1) + 50) \\
& > 3(k^2 - 10k + 50) - (k^2 - 8k + 41) \\
& = 2k^2 - 22k + 109 \\
& = 2(k - 5.5)^2 + 48.5 \\
& > 0
\end{aligned}$$

*So by the principle of mathematical induction, we see the inequality holds for all integers $n \geq 4$.*We could also derive the induction step from $k - 1$ to k ($k \geq 5$) as follows.Assume $3^{k-1} > ((k-1)^2 - 10(k-1) + 50)$.

$$\begin{aligned}
3^k = 3 \cdot 3^{k-1} & > 3((k-1)^2 - 10(k-1) + 50) \\
& = k^2 - 10k + 50 + (2k^2 - 26k + 133) \\
& = k^2 - 10k + 50 + 2(k - 6.5)^2 + 48.5 \quad (2(k - 6.5)^2 + 48.5 > 0) \\
& > k^2 - 10k + 50
\end{aligned}$$

Problem 7: (12 pts)

Prove by induction on n that $\forall n \in \mathbb{Z}^+, \sum_{i=0}^n \binom{n}{i} (-1)^i = 0$.

Solution: *Base Case* $n = 1$: $\sum_{i=0}^1 \binom{1}{i} (-1)^i = \binom{1}{0} \cdot 1 + \binom{1}{1} \cdot (-1) = 0$.

Inductive Hypothesis: Assume $n > 1$ and that formula holds for $n - 1$, i.e., $\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i = 0$.

Then

$$\begin{aligned}
 \sum_{i=0}^n \binom{n}{i} (-1)^i &= 1 + (-1)^n + \sum_{i=1}^{n-1} \binom{n}{i} (-1)^i \\
 &= 1 + (-1)^n + \sum_{i=1}^{n-1} \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) (-1)^i \\
 &= (-1)^0 + \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i \\
 &\quad + (-1)^{((n-1)+1)} + \sum_{i=0}^{n-2} \binom{n-1}{i} (-1)^{i+1} \\
 &= \binom{n-1}{0} (-1)^0 + \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^i \\
 &\quad + \binom{n-1}{n-1} (-1)^{(n-1)+1} + \sum_{i=0}^{n-2} \binom{n-1}{i} (-1)^{i+1} \\
 &= \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i + \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{i+1} \\
 &= 0 - \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \\
 &= 0 - 0 = 0.
 \end{aligned}$$

where the last two '0's come from the inductive hypothesis.

We have just proven the inductive step.

So by the principle of mathematical induction, we see the formula holds for all $n \in \mathbb{Z}^+$.

Problem 8: (9 pts)(a) Consider the recurrence below defined on $n \geq 2$.

$$T(n) = \begin{cases} 5 & \text{if } n = 2 \\ 4T(n-1) + 3 & \text{if } n > 2 \end{cases}$$

Give a closed-form, exact solution to the recurrence.

You only have to give the solution. You do not need to show how you derived it.

(b) Now, prove the correctness of your solution by induction.

Solution: (a) *Using Theorem 4.14, or by iterating the recurrence or by using recursion trees we get $T(n) = 6 \cdot 4^{n-2} - 1$.*(b) *Base Case $n = 2$: $T(2) = 6 \cdot 4^{2-2} - 1 = 5$.**Inductive Hypothesis: Assume $n > 2$ and that formula holds for $n - 1$, i.e., $T(n - 1) = 6 \cdot 4^{n-3} - 1$.**Then*

$$\begin{aligned} T(n) &= 4T(n-1) + 3 \\ &= 4(6 \cdot 4^{n-3} - 1) + 3 \\ &= 6 \cdot 4^{n-2} - 4 + 3 \\ &= 6 \cdot 4^{n-2} - 1 \end{aligned}$$

*We have just proven the inductive step.**So by the principle of mathematical induction, we see the formula holds for all integers $n > 1$.*

Problem 9: (12 pts)(a) Consider the recurrence below defined on $n \geq 0$.

$$T(n) = \begin{cases} 5 & \text{if } n = 0 \\ 3T(n-1) + n^2 3^n & \text{if } n > 0 \end{cases}$$

Give a closed-form, exact solution to the recurrence.

You only have to give the solution. You do not need to show how you derived it.

(b) Now, prove the correctness of your solution by induction.

Solution: (a) *Using Theorem 4.14, or by iterating the recurrence or by using recursion trees, we get*

$$T(n) = 3^n(5 + (2n^3 + 3n^2 + n)/6) = \frac{3^n}{6} (2n^3 + 3n^2 + n + 30).$$

(b) *Base Case $n = 0$:*

$$T(0) = \frac{3^0}{6} (30) = 5.$$

Inductive Hypothesis: Assume $n > 0$ and that formula holds for $n - 1$, i.e.,

$$T(n-1) = \frac{3^{n-1}}{6} (2(n-1)^3 + 3(n-1)^2 + (n-1) + 30).$$

Then

$$\begin{aligned} T(n) &= 3T(n-1) + n^2 3^n \\ &= 3 \frac{3^{n-1}}{6} (2(n-1)^3 + 3(n-1)^2 + (n-1) + 30) + n^2 3^n \\ &= \frac{3^n}{6} (2(n-1)^3 + 3(n-1)^2 + (n-1) + 30 + 6n^2) \\ &= \frac{3^n}{6} (2(n-1)^3 + 3(n-1)^2 + (n-1) + 30 + 6n^2) \\ &= \frac{3^n}{6} ((2n^3 - 6n^2 + 6n - 2) + (3n^2 - 6n + 3) + (n-1) + 30 + 6n^2) \\ &= \frac{3^n}{6} (2n^3 + 3n^2 + n + 30) \end{aligned}$$

*We have just proven the inductive step.**So by the principle of mathematical induction, we see that the formula holds for all integers $n \geq 0$.*

Problem 10: (12 pts)(a) Consider the recurrence below defined on $n \geq 1$?

$$T(n) = \begin{cases} 2 & \text{if } n = 1 \\ 2T\left(\frac{n}{4}\right) + 3n & \text{if } n > 1 \end{cases}$$

Give a closed-form, exact solution to the recurrence.

Your solution should assume that n is always a power of 4. You only have to give the solution. You do not need to show how you derived it.

(b) Now, prove the correctness of your solution by induction.

(a) Use recursion tree or iterate the recurrence:

$$\begin{aligned}
T(n) &= 2T(n/4) + 3n \\
&= 2(2T(n/16) + 3n/4) + 3n \\
&= 2^2T(n/4^2) + 3n/2^1 + 3n \\
&= 2^2(2T(n/4^3) + 3n/4^2) + 3n/2^1 + 3n \\
&= 2^3T(n/4^3) + 3n/2^2 + 3n/2^1 + 3n \\
&\vdots \\
&= 2^{\log_4 n} T(n/4^{\log_4 n}) + 3n/2^{\log_4(n)-1} + \dots + 3n/2^2 + 3n/2^1 + 3n \\
&= 2^{\log_4 n} T(1) + 3n \sum_{i=0}^{\log_4(n)-1} (1/2)^i \\
&= 2^{\log_4 n} \cdot 1 + 3n \frac{1 - (1/2)^{\log_4 n}}{1 - (1/2)} \\
&= 2^{1+\log_4 n} + 3n \cdot 2(1 - (1/2)^{\log_4 n}) \\
&= 2^{1+\log_4 n} + 6n(1 - (1/2)^{\log_4 n})
\end{aligned}$$

Now, noting that $2^{\log_4 n} = \sqrt{n}$ this gives

$$T(n) = 2\sqrt{n} + 6n \left(1 - \frac{1}{\sqrt{n}}\right) = 6n - 4\sqrt{n}.$$

(b) Base Case $n = 1$: $T(1) = 6 - 4\sqrt{1} = 2$.Inductive Hypothesis: If $m = 4^j$ with $1 \leq j < i$, then $T(m) = 6m - 4\sqrt{m}$.Now suppose $n = 2^i$.By the i.h., $T(n/4) = 6\frac{n}{4} - 4\sqrt{\frac{n}{4}} = 3\frac{n}{2} - 2\sqrt{n}$, so

$$\begin{aligned}
T(n) &= 2T(n/4) + 3n \\
&= 2\left(3\frac{n}{2} - 2\sqrt{n}\right) + 3n \\
&= 6n - 4\sqrt{n}
\end{aligned}$$

and we have just proven the inductive step. So by the principle of mathematical induction, we see the formula holds for all integers $n \geq 1$.

Alternatively, we can use induction on the longer form of the solution.

Base Case $n = 1$: $T(1) = 2^{1+\log_4 1} + 6 \cdot 1(1 - (1/2)^{\log_4 1}) = 2 + 0 = 2$.

Inductive Hypothesis: If $m = 4^j$ with $1 \leq j < i$, then $T(m) = 2^{1+\log_4 m} + 6m(1 - (1/2)^{\log_4 m})$.

Now suppose $n = 2^i$.

By the i.h., $T(n/4) = 2^{1+\log_4 n/4} + 6n/4(1 - (1/2)^{\log_4 n/4})$, so

$$\begin{aligned}
 T(n) &= 2T(n/4) + 3n \\
 &= 2(2^{1+\log_4 n/4} + 6n/4(1 - (1/2)^{\log_4 n/4})) + 3n \\
 &= 2^{1+\log_4 n} + 3n - 3n(1/2)^{\log_4 n/4} + 3n \\
 &= 2^{1+\log_4 n} + 6n - 6n(1/2)(1/2)^{\log_4(n)-1} \\
 &= 2^{1+\log_4 n} + 6n - 6n(1/2)^{\log_4 n} \\
 &= 2^{1+\log_4 n} + 6n(1 - (1/2)^{\log_4 n})
 \end{aligned}$$

We have just proven the inductive step.

So by the principle of mathematical induction, we see the formula holds for all integers $n \geq 1$.