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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – Spring 2011

Midterm Examination 1

Date: Friday, March 11, 2011   Time: 19:00–21:00

Name: ____________________ Student ID: ____________________

Email: ____________________ Lecture and Tutorial: __________

Instructions

- This is a closed book exam. It consists of ?? pages and 7 questions.
- Please write your name, student ID, email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- **Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.**
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	Total
Points								100
Score								

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As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

I declare that the answers submitted for this examination are my own work. I understand that sanctions will be imposed, if I am found to have violated the University regulations governing academic integrity. Student's Name: ______________________ Student's Signature: ______________________

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Problem 1: [pts] Consider the set $S = \{1, 2, 3, \ldots, 10, 11\}$ of 11 integers. In this question, you are asked to count the subsets of S with certain properties.

(a) How many subsets contain exactly two even integers and one odd integer?

(b) How many subsets contain equal numbers of odd integers and even integers?

(c) How many subsets contain at least two even integers and at most three odd integers.

Solution 1: [pts]

(a) $\binom{5}{2}\binom{6}{1} = 60$

(b) $\sum_{i=0}^{5}\binom{5}{i}\binom{6}{i} = 462$

(c) $\sum_{i=2}^{5}\sum_{j=0}^{3}\binom{5}{i}\binom{6}{j} = 1092$

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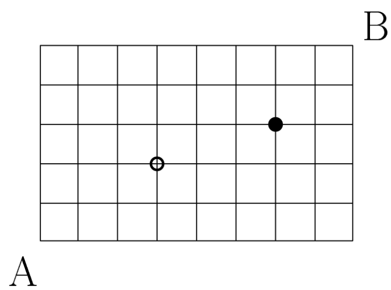
Problem 2: [pts] Imagine that the following 5×8 grid is part of the street map of a city. You need to walk from the lower left corner marked A to the upper right corner marked B. There are 8 blocks from left to right, and 5 blocks from bottom to top. So, you can reach point B from point A by walking 13 blocks.



(a) How many different 13-block paths are there in total?

(b) Suppose that you must pass through the two points marked $\bullet$ and $\circ$. How many different 13-block paths that you can take?

(c) Suppose that you want to avoid passing through the point marked $\bullet$. How many different 13-block paths that you can take?

Solution 2: [pts]

(a) $\binom{13}{5} = \binom{13}{8} = 1287$

(b) $\binom{5}{2}\binom{4}{1}\binom{4}{2} = 240$

(c) $\binom{13}{5} - \binom{9}{3}\binom{4}{2} = 783$

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Problem 3: [pts] Consider $S_3 = \{1,2,3\}$ and $S_n = \{1,2,3,\dots,n\}$ with $n \geq 3$.

(a) How many functions f are there from S_3 to S_n?

(b) How many one-to-one functions f are there from S_3 to S_n?

(c) How many onto functions f are there from S_3 to S_n?

(d) How many functions f are there from S_3 to S_n such that $f(1) \leq f(2) \leq f(3)$?

Solution 3: [pts]

(a) n^3

(b) $\binom{n}{3}3!$ or $n(n-1)(n-2)$ or $n^{\underline{3}}$

(c) For $n = 3$, the answer is 3!. For $n > 3$, the answer is 0

(d) $\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$

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Problem 4: [16 pts] Give a combinatorial proof of the identity, for all $n \geq 13$.

$$\binom{n}{10}\binom{n-10}{3}\binom{10}{6} = \binom{n}{13}\binom{13}{7}\binom{7}{4}$$

Note: An algebraic proof of this identity will not be accepted as a solution.

Solution 4: [16 pts] Suppose there is n white chairs in classroom and the order of the chairs are fixed.

- Left side actually equals $\binom{n}{10}\binom{n-10}{3}\binom{10}{6}\binom{4}{4}$
  Thus, left side is the
  # of ways to choose 10 chairs be painted as red or green later from the n chairs
  times
  # of ways to choose 3 chairs be painted as blue from the remaining $n - 10$ chairs
  times
  # of ways to choose 6 chairs be painted as red from the previous selected 10 chairs
  times
  # of ways to choose 4 chairs be painted as green from the remaining $10 - 6$ chairs

- Right side actually equals $\binom{n}{13}\binom{13}{7}\binom{6}{6}\binom{7}{4}\binom{3}{3}$
  Thus, right side is the
  # of ways to choose 13 chairs from the n chairs
  times
  # of ways to choose 7 chairs be painted as green or blue later from the selected 13 chairs
  times
  # of ways to choose 6 chairs be painted as red from the remaining $13 - 7$ chairs
  times
  # of ways to choose 4 chairs be painted as green from the previous selected 7 chairs
  times
  # of ways to choose 3 chairs be painted as blue from the remaining $7 - 4$ chairs

Thus, both left and right side are counting the # of ways to select 3 chairs be blue, 6 chairs be red and 4 chairs be green from the n white chairs, so they are equal.

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Problem 5: [10 pts]

Let m and n be two positive integers. Are the following equations always true? Give a proof for each of the true equations and give a counterexample for each of the false ones. Your proofs should be based Euclid's division theorem. Other Lemmas/Theorems/Corollaries from the course should not be used.

$$m \bmod n + (-m) \bmod n = 0.$$

$$m \bmod n + (-m) \bmod n = n$$

$$(m \bmod n + (-m) \bmod n) \bmod n = 0$$

$$m \bmod n = (n - m) \bmod n.$$

$$m \bmod n = (m - n) \bmod n.$$

Solution 5: [10 pts]

(a) False. Counter example: $m = 1$, $n = 4$, we have
$1 \bmod 4 + (-1) \bmod 4 = 1 + 3 = 4 \neq 0$.

(b) False. Counter example: $m = 8$, $n = 4$, we have
$8 \bmod 4 + (-8) \bmod 4 = 0 + 0 = 0 \neq 4$.

(c) True.

$$\begin{aligned} & (m \bmod n + (-m) \bmod n) \bmod n \\ = \;& (m - q_1 n - m - q_2 n) \bmod n \\ = \;& (-q_1 n - q_2 n) \bmod n \\ = \;& (-q_1 - q_2) n \bmod n \\ = \;& 0 \end{aligned}$$

Since any integer are multiple of n mod n must equal to 0.

(d) False. Counter example: $m = 1$, $n = 4$, we have
$1 \bmod 4 = 1 \neq 3 = (4 - 1) \bmod 4$.

(e) True. Let $r = (m - n) \bmod n$, where $0 \leq r < n$.

$$\begin{aligned} r &= (m - n) \bmod n \\ &= (m - n) - qn \\ &= m - (1 + q)n \\ &= m - q_2 n \\ \Rightarrow m &= q_2 n + r \end{aligned}$$

$m = q_2 n + r$ with $0 \leq r < n$ implies $r = m \bmod n = (m - n) \bmod n$.

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Problem 6: [pts]

(a) Write the $\cdot_6$ multiplication table for Z_6. Which elements of Z_6 have inverses?

(b) Prove that 91 does not have inverse in Z_{119}

Solution 6: [pts]

(a)

Z_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

1 and 5 has multiplicative inverses.

(b) This is equivalent to show that there is no such x that satisfies $(91 \cdot x) \bmod 119 = 1$.

$$\begin{aligned} 1 &= (91 \cdot x) \bmod 119 \\ \Rightarrow 1 &= 91 \cdot x - 119q \\ \Rightarrow 1 &= 7(13 \cdot x - 17q) \end{aligned}$$

Since 7 does not divide 1, this is impossible.

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Problem 7: [14 pts] A Caesar cipher with shift k letters (to the left or to the right) has been executed on some original plaintext message. The resulting ciphertext is `RFC MJB FMPQC ILMUQ RFC UYW`. What is k and what was the original message?

Solution 7: [14 pts] k is 2 to the left or 24 to the right. The original message is "`The old horse knows the way`".