Tutorial 10: Induction

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Question 1

Use induction to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all integers $n \geq 1$

Solution:
Denote the statement to be proven by $p(n)$.

- Base case:
  For $n = 1$, we have $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$. So $p(1)$ is true.
- Inductive hypothesis:
  Suppose $p(n-1)$ is true for some $n \geq 2$, i.e.

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n = \frac{(n-1)n(n+1)}{3}.$$

- Inductive step:
  Adding $n(n+1)$ to both sides of $p(n-1)$, gives

$$
\begin{aligned}
& 1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n + n(n+1) \\
= \quad & \frac{(n-1)n(n+1)}{3} + n(n+1) \\
= \quad & \frac{(n-1)n(n+1) + 3n(n+1)}{3} \\
= \quad & \frac{n(n+1)((n-1)+3)}{3} = \frac{n(n+1)(n+2)}{3}
\end{aligned}
$$

  which shows that $p(n)$ is true. Thus $p(n-1) \rightarrow p(n)$.
- Inductive conclusion:
  By the principle of mathematical induction, we can conclude that

$$
1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}
$$

  is true for all integers $n \geq 1$.

Question 2

Let $0 \leq j \leq n$, prove that

$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}.$$

Solution:

- Base case:
  For $n = 0$, the equation says $\binom{0}{0} = \binom{1}{1}$, so $p(0)$ is true.
- Inductive hypothesis:
  Suppose for $n = k - 1 \geq 0$, the equation is true.

- Inductive step:
  For the inductive hypothesis (second equality) and the Pascal relationship (third equality), we have

$$\sum_{i=j}^{k}\binom{i}{j}=\sum_{i=j}^{k-1}\binom{i}{j}+\binom{k}{j}=\binom{k}{j+1}+\binom{k}{j}=\binom{k+1}{j+1}.$$

  Thus the equation is true for $n=k$, i.e. $p(n-1)\rightarrow p(n)$.
- Inductive conclusion:
  From the principle of mathematical induction, the equation is true for all integers $n\geq 0$.

Question 3

Prove by induction that the number of subsets of an n-element set is 2^n for all $n \geq 0$.

Solution:
Denote the statement to be proven by $p(n)$.

- Base case:
  For $n = 0$, the set has no elements, so it is the empty set. The only subset of the empty set is the empty set. Since $2^0 = 1$, $p(0)$ is true.
- Inductive hypothesis:
  Assume $p(n-1)$ is true for some $n \geq 1$, i.e., the number of subsets of an $(n-1)$-element set is 2^{n-1}.

Question 3

- Inductive step:

  For any set S of size $n \geq 1$, identify a single element $x \in S$.

  The subsets of S can be partitioned into (i) those subsets that do not contain x and (ii) those subsets that do contain x.

  The number of subsets not containing x is the number of subsets of $S - \{x\}$, which, by the inductive hypotheses, is 2^{n-1}.

  The number of subsets containing x must be the same, because by removing x from each we get a subset not containing x.

  Thus, the total number of subsets is $2^{n-1} + 2^{n-1} = 2^n$.

- Inductive conclusion:

  By the principle of mathematical induction, the number of subsets of an n-element set is 2^n for all $n \geq 0$.

Question 3

Comment: In class we prove something very similar, when using a *recurrence relation method* to solve this problem.

First we used the same type of idea and induction to prove that $S(n)$, the number of subsets of an n item set, satisfies

$$S(0) = 1; \; S(n) = 2S(n-1) \text{ for } n > 1$$

Then we used induction AGAIN to prove that such an $S(n)$ satisfies $S(n) = 2^n$

So, in class, we actually used induction twice.
Here, we combined the two steps into one.