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HKUST – Department of Computer Science
COMP170: Discrete Math Tools for CS – FALL 2005

Final Examination

Date: Th Dec 15, 2005   Time: 16:30–19:30   Venue: Sports Hall

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: ________

Note: Some of the questions had typos in them that were pointed out during the exam. These were corrected in this paper

Instructions

- This is a closed book exam. It consists of 25 pages and 12 questions.
- Please write your name, student ID, Email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page (and sometimes has an extra page for you to do work on). This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Only use notation given in class. Do not use notation that you have learnt outside of this class that is nonstandard.
- Calculators may be used for the exam.

Questions	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points	5	9	8	8	9	9	9	10	8	9	8	8	100
Score													

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As part of HKUST's introduction of an honor code, the HKUST senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      ___________________________

Student's Signature:    _______________________
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Definitions and Formulas: This page contains some definitions used in this exam and a list of formulas (theorems) that you may use in the exam (without having to provide a proof). Note that you might not need all of these formulas on this exam.

Definitions

1. $N = \{0, 1, 2, 3, \ldots\}$, the set of non-negative integers.
2. $Z^+ = \{1, 2, 3, \ldots\}$, the set of positive integers.
3. R is the set of *real numbers*.
4. R^+ is the set of positive *real numbers*.

Formulas:

1. $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$
2. If $0 < i < n$ then $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$.
3. $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.
4. $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
5. $\sum_{i=1}^{n-1} i = n(n-1)/2$.
6. $\sum_{i=1}^{n-1} i^2 = \frac{2n^3-3n^2+n}{6}$.
7. If $r \neq 1$ then $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$
8. If $r \neq 1$ then $\sum_{i=0}^{n} ir^i = \frac{nr^{n+2}-(n+1)r^{n+1}+r}{(1-r)^2}$
9. The inclusion-exclusion theorem:
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$
10. If X is a random variable, then $E(X)$ denotes the *Expectation of* X and $V(X) = E\left((X - E(X))^2\right)$ denotes the *Variance of* X.

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Problem 1: (5 pts) For the multiple choice problems below, **circle** the correct answer. For this question, no work needs to be shown. **In solution key, items are set in bold font.**

(a) Suppose the logical statement $p \Rightarrow q$ is TRUE.
Which of the following is then always also TRUE?

(i) $p \vee \neg q$ (ii) $\neg p \Rightarrow \neg q$ **(iii)** $\neg q \Rightarrow \neg p$ (iv) $p \wedge \neg q$

(v) None of the above

b) Consider the following three statements
A) $\forall x \in R\ (\ x^2 \geq 2x)$
B) $\exists y \in N\ \Big(\ \forall x \in N\ (\ (x > y) \Rightarrow 2^x > 10 \cdot x^2)\Big)$
C) $\forall x \in R^+\ (\ x^3 \geq 3x^2 - 3x + 1\ )$.
Which of the three is **True**?

(i) only A **(ii) only B** (iii) only C (iv) only B and C

(iv) None of them

(c) How many elements in Z_{30} have a multiplicative inverse?

(i) 1 (ii) 7 **(iii) 8** (iv) 15 (v) 29

(d) Suppose you have a $\frac{1}{3}$-biased coin. That is, it has probability $\frac{1}{3}$ of coming up heads. You now flip it 10 times. What is the probability that the first five flips are Heads and the last five flips are Tails?

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) $\binom{10}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^5$

(iv) The same as the probability that the first 3 flips are Heads and the last 7 flips are Tails.

(v) None of the above

(e) There are 3 coins in a closed box: one \$1 coin, one \$5 coin and one \$10 coin. You now take the coins out of the box one at a time without replacement. X_1 is the value of the first coin you took out, X_2 the value of the second and X_3 the value of the third. Which statement below is correct?

(i) $V(X_1) < V(X_1 + X_2) < V(X_1 + X_2 + X_3)$

(ii) $V(X_1 + X_2 + X_3) < V(X_1) < V(X_1 + X_2)$

(iii) $V(X_1 + X_2 + X_3) < V(X_1) = V(X_1 + X_2)$

(iv) $V(X_1) = V(X_1 + X_2) < V(X_1 + X_2 + X_3)$

(v) None of the above

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Problem 2: (9 pts) For this problem, it is not necessary to show your work.
The problem is to calculate the number of occurrences of various hands of cards.

Recall that a card in a standard deck has two attributes:
i) 13 possible *values* or *kinds* in the order **A,2,3,4,5,6,7,8,9,10,J,Q,K**
ii) 4 suits: Diamond, Spade, Heart, Clubs
All $52 = 13 \cdot 4$ possible cards exist in the standard deck.

In this problem a *hand of 5 cards* means a subset of size 5 out of the 52 cards. So, for example, there is exactly one hand that contains all four **A**(ce)s and the **K**(ing) of Clubs.

Note the following definitions

A *pair* is two cards of the same kind but of different suits.
Three of a kind is three cards of the same kind but of different suits.
Four of a kind is four cards of the same kind but of different suits.

A *straight* is five cards increasing in value, not necessarily of the same suit,
e.g., **9** of diamonds, **10** of hearts, **J** of diamonds, **Q** of hearts and **K** of clubs.
Note that, in our definition, **A,2,3,4,5** *is a straight,*
but **10,J,Q,K,A** is not *a straight.*

A *full house* is five cards containing a *three of a kind* and another *pair* of a different value.
For example, a hand containing three **5**s and two **7**s is a full house.

a) How many hands of 5 cards contain a *full house*

b) How many hands of 5 cards contain a *three of a kind* but do not contain *four of a kind* and do not contain a *full house*?

c) How many hands of 5 cards contain a *straight*?

a)

$$13\binom{4}{3} \cdot 12\binom{4}{2} = 3,744$$

b)

$$13\binom{4}{3} \cdot \frac{48 \cdot 44}{2} = 54,912$$

c)

$$9 \cdot 4^5 = 9,216$$

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Problem 3: (8 pts)

(a) Consider the recurrence below defined on $n \geq 1$.

$$T(n) = \begin{cases} 5 & \text{if } n = 1 \\ 4T\left(\frac{n}{4}\right) + 3n & \text{if } n > 1 \end{cases}$$

Give a closed-form, exact solution to the recurrence.
Your solution should assume that n is always a power of 4. You only have to give the solution. You do not need to show how you derived it.

(b) Now, prove the correctness of your solution by induction.

(a) Iterating the recurrence gives us

$$T(n) = 3n \log_4 n + 5n$$

Alternatively, if $n = 4^i$, this can be written as

$$T(4^i) = 3 \cdot 4^i \cdot i + 5 \cdot 4^i.$$

(b) We prove this by induction on i.

If $i = 0$, the equation is obviously true. Suppose then, that the formula is true for 2^{i-1}, i.e.,

$$T(4^{i-1}) = 3 \cdot 4^{i-1} \cdot (i-1) + 5 \cdot 4^{i-1}.$$

Then, plugging back into the defining formula gives

$$\begin{aligned} T(4^i) &= 4T\left(4^{i-1}\right) + 34^i \\ &= 4\left(3 \cdot 4^{i-1} \cdot (i-1) + 5 \cdot 4^{i-1}\right) + 3 \cdot 4^i \\ &= 3 \cdot 4^i - 3 \cdot 4^i + 5 \cdot 4^i + 3 \cdot 4^i \\ &= 3 \cdot 4^i + 5 \cdot 4^i \end{aligned}$$

and we are done.

Problem 4: (8 pts)
Does 34 have a multiplicative inverse in Z_{55}?
That is, does there exist x such that $34 \cdot x = 1 \bmod 55$?

If the answer is no, prove it.
If the answer is yes, give the value of x (In this case, it is not necessary to show your work).

It does have an inverse.

Using the extended GCD algorithm we find that

$$13 \cdot 55 - 21 \cdot 34 = 1$$

so $x = -21 \bmod 55 = 34$.

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Problem 5: (9 pts) In this problem you must calculate probabilities. For each part, you must show *how* you derived your answer.

Assume that there is a $\frac{1}{3}$-biased coin, i.e., a coin such that $P(\textbf{Heads}) = \frac{1}{3}$. The coin is now tossed 4 times.

a) What is the probability that more **Heads** were seen than **Tails**?

b) Now suppose that you are told that, in the 4 coin tosses, *at least* one toss showed a **Head** and *at least* one toss showed a **Tail**. Conditioned on this information, what is the probability that more **Heads** were seen than **Tails**?

c) Further suppose that you are told that the *first* toss was a **Head** and the *second* toss was a **Tail**. Conditioned on this information, what is the probability that more **Heads** were seen than **Tails**?

a)Let A be the event that there are more Heads than tails. Then

$$\begin{aligned} P(A) &= P(H=3 \ \wedge \ T=1) + P(H=4 \ \wedge \ T=0) \\ &= \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 \\ &= 4\cdot\frac{2}{3^4} + \frac{1}{3^4} \\ &= \frac{9}{81} = \frac{1}{9} \end{aligned}$$

b) Let B be the event that at least one coin showed a head and one coin showed a tail.

$$\begin{aligned} P(B) &= P(H=1 \ \wedge \ T=3) + P(H=2 \ \wedge \ T=2) + P(H=3 \ \wedge \ T=1) \\ &= 1 - P(H=4 \ \wedge \ T=0) + P(H=0 \ \wedge \ T=4) \\ &= 1 - \frac{1}{81} - \frac{16}{81} = \frac{64}{81} \end{aligned}$$

Note that $B \cap A$ is the event that there are 3 heads and one tail so $P(B \cap A) = 4\cdot\frac{2}{3^4} = \frac{8}{81}$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{8}{81}}{\frac{64}{81}} = \frac{1}{8}$$

c) Let C be the event that the first *toss was a* **Head** *and the* second *toss was a* **Tail**. *Then $P(C) = \frac{2}{9}$. $A \cap C$ is the event that the second toss is a tail and all of the other 3 tosses are heads so $P(A \cap C) = \frac{2}{81}$*

$$\begin{aligned} P(A|C) &= \frac{P(A\cap C)}{P(C)} \\ &= \frac{\frac{2}{81}}{\frac{2}{9}} = \frac{1}{9} \end{aligned}$$

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Problem 6: (9 pts) In this problem you are given **two**, fair, six-sided dice. Each die has the numbers 1 through 6 printed on its sides.

You now roll the two dice again and again, each time counting the sum of the two numbers showing on the tops of the two dice. First you roll the dice until you see an *even sum*. Then you roll the dice until you see a sum divisible by 3. They you roll the dice until you see a sum divisible by 4.

The following two examples illustrate the process (the boldfaced numbers are the relevant ones divisible by 2, 3 and 4).

$$3\ 5\ 7\ 9\ \mathbf{2}\ 4\ 5\ 8\ 10\ \mathbf{12}\ 3\ 6\ 11\ 1\ \mathbf{8} \quad \Rightarrow \quad \text{Total number of rolls is 15}$$
$$9\ \mathbf{4}\ 5\ \mathbf{6}\ 3\ \mathbf{4} \quad \Rightarrow \quad \text{Total number of rolls is 6}$$

What is the expected total number of times you rolled the dice?
You must state which formulas and/or theorems you used to derive your answer.

Let X_1 be the time needed to roll an even number.
Let X_2 be the time needed to roll a number divisble by 3 starting *right after we see the first even number. Let X_3 be the time needed to roll a number divisble by 4* starting *right after we see the first number divisible by three seen after the firsteven number.*

We want to calculate $E(X_1 + X_2 + X_3)$. By linearity of expectation this is $E(X_1) + E(X_2) + E(X_3)$.

Recall that the expectation of a geometric random with parameter p is $\frac{1}{p}$.

Calculation shows that X_1 is geometric with $p = 1/2$, X_2 is geometric with $p = 1/3$ and X_3 is geometric with $p = 1/4$. Then

$$E(X_1) + E(X_2) + E(X_3) = \frac{1}{1/2} + \frac{1}{1/3} + \frac{1}{1/4} = 2 + 3 + 4 = 9.$$

Problem 7: (9 pts) For this problem, it is not necessary to show your work.

Suppose that you are randomly hashing n keys into a k element array. That is, each key is randomly assigned with equal probability to each of the k buckets. (Recall that a *bucket* is a location in the array.)

a) What is the probability (as a function of k and n) that *exactly* 2 keys get hashed to the first bucket? That is, after the hashing, the first bucket will contain *exactly* two keys.

b) What is the *expected number of buckets* (as a function of k and n) that will contain exactly two keys?

a) Let Y be the number of keys hashed to a particular bucket. Then Y has a binomial distribution with success probability $\frac{1}{k}$. So the answer is

$$P(Y) = \binom{n}{2}\left(\frac{1}{k}\right)^2\left(1-\frac{1}{k}\right)^{n-2}$$

Let X_i be the indicator random variable for the event that bucket i has exactly two keys. That is, X_i equals one if bucket i contains exactly two keys and is zero otherwise. Then, from part (a)

$$E(X_i) = P(\textit{bucket } i \textit{ contains exactly two keys}) = \binom{n}{2}\left(\frac{1}{k}\right)^2\left(1-\frac{1}{k}\right)^{n-2}$$

Let Z be the number of buckets containing exactly two keys. Then $Z = \sum_{i=1}^{k} X_i$ so, by linearity of expectation

$$E(Z) = \sum_{i=1}^{k} E(X_i) = k\binom{n}{2}\left(\frac{1}{k}\right)^2\left(1-\frac{1}{k}\right)^{n-2} = \binom{n}{2}\frac{1}{k}\left(1-\frac{1}{k}\right)^{n-2}$$

Problem 8: (10 pts) For this problem, you should explain how you derived your answers.

Suppose that you are randomly hashing 20 keys into a 10 element array. What is the probability that every bucket contains at least one key? For this problem, it is not necessary to write down the probability as a number. You can write it as the sum of terms.

Let A be the event that every bucket contains at least one key. Let $B = \overline{A}$ so $P(A) = 1 - P(B)$. We will write B as the union of simple events and then use the inclusion exclusion formula,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

So, let E_i be the event that bucket i does not *contain any key. Then*

$$B = E_1 \cup E_2 \cup \cdots \cup E_{10}.$$

Note that, for fixed $Ei_1, E_{i_2}, \ldots E_{i_k}$, the event $E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}$ is the event that no keys get hashed to any of the corresponding k buckets. Thus, all keys must be hashed to the other $10 - k$ buckets so

$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \left(\frac{10-k}{10}\right)^{20} = \left(1 - \frac{k}{10}\right)^{20}$$

Since there are $\binom{10}{k}$ ways of choosing k buckets we plug into the inclusion-exclusion formula to get

$$\begin{aligned} P(B) &= P\left(\bigcup_{i=1}^{10} E_i\right) \\ &= \sum_{k=1}^{10} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) \\ &= \sum_{k=1}^{10} (-1)^{k+1} \binom{10}{k} \left(1 - \frac{k}{10}\right)^{20} \end{aligned}$$

Note that, when $k = 10$, the corresponding term in the summation is 0, so we could also have written the sum as $\sum_{i=1}^{9}$, instead of $\sum_{i=1}^{10}$.

And, as discussed, $P(A) = 1 - P(B)$.

Problem 9: (8 pts) In this problem you are calculating expectations and variances of different random variables. For each part you must show your work (if you use theorems taught in class, state which theorems you are using) and then give the final results as a number (either a fraction or in decimal notation; your final answer can not be given as a sum).

Assume you have a fair, six-sided die which has the numbers 1 through 6 printed on its sides. You now throw the die 10 times. Let X_i be the value shown on the ith throw.

a) What is $E(X_1)$? What is $V(X_1)$?

b) Let $X = \sum_{i=1}^{10} X_i$, i.e, the sum of all of the throws.
What is $E(X)$? What is $V(X)$?

c) Let $c > 0$ be a constant.
What is $E(cX_1)$? What is $V(cX_1)$?

d) Let $Y = \sum_{i=1}^{10} i \cdot X_i$.
For example, if the 10 throws are 1, 3, 2, 4, 2, 5, 6, 3, 5, 5 then
$Y = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 4 + 5 \cdot 2 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot 3 + 9 \cdot 5 + 10 \cdot 5$.
What is $E(Y)$? What is $V(Y)$?

a) Calculation gives $E(X_1) = \sum_{i=1}^{6} \frac{1}{6} i = \frac{7}{2}$ *Further calculation gives*

$$V(X_1) = \frac{1}{6}\left[\left(-\frac{5}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2\right] = \frac{2}{24}\left[5^2 + 3^2 + 1^2\right] = \frac{35}{12}$$

b) By linearity of expectation

$$E(X) = \sum_{i=1}^{10} E(X_i) = 10 \cdot E(X_1) = 35$$

Since the X_i *are all independent of each other*

$$V(X) = \sum_{i=1}^{10} V(X_i) = 10 \cdot V(X_1) = \frac{350}{12}$$

c) $E(cX_1) = cE(X_1) = \frac{7c}{2}$. $V(cX_1) = c^2 V(X_1) = \frac{35c^2}{12}$

d) By linearity of expectation

$$E(Y) = \sum_{i=1}^{10} E(iX_i) = \sum_{i=1}^{10} iE(X_i) = \frac{7}{2}\sum_{i=1}^{10} i = \frac{7}{2} \cdot 55 = \frac{385}{2}$$

Since the X_i *are all independent of each other, the* iX_i *are also all independent of each other and*

$$V(Y) = \sum_{i=1}^{10} V(iX_i) = \sum_{i=1}^{10} i^2 V(X_i) = \frac{35}{12}\sum_{i=1}^{10} i^2 = \frac{35}{12} \cdot 385 = \frac{13475}{12}$$

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Problem 10: (9 pts)

For this problem you may assume that n is a nonnegative power of 4. Suppose function $T(n)$ satisfies $T(1) = 10$ and, for $n > 1$,

$$T(n) \leq 3T\left(\frac{n}{4}\right) + 2n$$

Prove that $T(n) = O(n)$. That is, you should prove that there exists some $n_0 \geq 0$ and $k \geq 0$ such that

$$\forall n > n_0, \quad T(n) \leq kn.$$

Hint: Prove by induction.

We let $n_0 = 0$. In order for the statment to be correct we must need

$$10 = T(1) \leq k \cdot 1 = k$$

so

$$10 \leq k \tag{1}$$

That will be our induction basis. Now suppose that the statment is correct for all $n = 4^j$, $j = 0, 1, 2, \ldots, i-1$. When $n = 4^i$ we us ethe induction hypothesis to get

$$\begin{aligned} T(n) &\leq 3T\left(\frac{n}{4}\right) + 2n \\ &\leq 3\left(k\frac{n}{4}\right) + 2n \\ &= \left(\frac{3}{4}k + 2\right) n \end{aligned}$$

We therefore will have $T(n) \leq kn$ if

$$\left(\frac{3}{4}k + 2\right) \leq k$$

or

$$8 \leq k \tag{2}$$

Thus, from (1) and (2), as long as $k \geq \max\{8, 10\} = 10$, the induction works.

Problem 11: (8 pts) Flip a fair coin 4 times. For each flip write down whether the outcome is **H**(eads) or **T**(ails). Define the following three events:

a) $A =$ No two consecutive flips were **H**.
b) $B =$ The number of **H**s were even, i.e., 0,2,4.
c) $C =$ The first flip and last flip are both **T**.

e,g,, **HTTH** is in $A \cap B$ but not in C.
TTHH is in B but not in $A \cup C$.
TTHT is in $A \cap C$ but not in B.

Now answer the following two questions. In both cases you must prove your answer.

a) Is B independent of A?
b) Is C independent of A?

Note that event A contains exactly the 8 outcomes

TTTT, HTTT, THTT, TTHT, TTTH, HTHT, THTH, HTTH

so $P(A) = \frac{8}{16} = \frac{1}{2}$

Event B contains $\binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8$ *events so* $P(B) = \frac{8}{16} = \frac{1}{2}$

Event $A \cap B$ contains exactly the four outcomes

TTTT, HTHT, THTH, HTTH

so $P(A \cap B) = \frac{4}{16} = \frac{1}{4}$

Event C contains exactly the 4 outcomes

THHT, THTT, TTHT, TTTT

so $P(C) = \frac{4}{16} = \frac{1}{4}$

Event $A \cap C$ contains exactly the three outcomes

TTTT, THTT, TTHT

so $P(A \cap C) = \frac{3}{16}$

We now use the theorem learnt in class that states that events D and E are independent iff $P(E)P(D) = P(D \cap E)$.

(a) $P(A)P(B) = \frac{1}{4} = P(A \cap B)$ *so A and B are independent.*

(b) $P(A)P(C) = \frac{1}{4} \neq \frac{3}{16} = P(A \cap C)$ *so A and C are not independent.*

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Problem 12: (8 pts) For this problem, you must explain how you derived your answer.

A student is taking a 100 multiple-choice question exam. Each question lists five possible answers.

Suppose the student knows 80% of the material which means that she has an 80% chance (i.e., a 0.8 probability) of knowing the answer to each individual question (and knowing the answer to a question is independent of knowing the answer to any other question).

Furthermore, suppose that when she knows the answer, she chooses the correct choice and that when she doesn't know the answer, she chooses a choice at random. Let X be the number of correct answers she chooses. Then

a) What is $E(X)$?

b) What is $V(X)$?

Let K_i be the event that she knows the correct answer to question i and C_i the event that she gets the answer correct. Then

$$P(C_i) = P(C_i|K_i)P(K_i) + P(C_i|\overline{K_i})P(\overline{K_i})$$

where $\overline{K_i}$ is the event she doesn't know the answer we are given that

$$\begin{aligned} P(C_i|K_i) &= 1 \\ P(K_i) &= 0.8 \\ P(C_i|\overline{K_i}) &= 0.2 \\ P(\overline{K_i}) &= 0.2 \end{aligned}$$

Putting it all together gives $P(C_i) = 0.84$.

The number of questions she gets correct is therefore the number of successes in an independent trial proces with probability of success $p = 0.84$. Therefore by the theorems we learnt in class

(a) $E(X) = 100 \cdot p = 84$.

(b) $V(X) = 100p(1-p) = 84 \cdot (.16) = 13.44$.