COMP 2711 Discrete Mathematical Tools for Computer Science
2016 Spring Semester – Assignment # 2
Distributed: 19 February 2016 – Due: 4pm, 26 February 2016

Your solutions should contain (i) your name, (ii) your student ID #, (ii) your email address, (iv) your lecture section and (v) your tutorial section. Your work should be submitted before 4PM of the due date in the collection bin outside Room 4210 (Lift 21).

Problem 1. Let $P(x, y)$ be a propositional function. Show that the implication $\exists x \forall y P(x, y) \to \forall y \exists x P(x, y)$ is a tautology.

Solution:

We have to prove the following:
"When $\exists x \forall y P(x, y)$ is true, $\forall y \exists x P(x, y)$ must also be true."
This is sufficient to show that $\exists x \forall y P(x, y) \to \forall y \exists x P(x, y)$ is a tautology, because $p \to q$ is false only when p is true and q is false.

Assume that $\exists x \forall y P(x, y)$ is true. This means there is at least one particular element x_1 such that $P(x_1, y)$ is true for any value of y. Next we show that the statement $\forall y \exists x P(x, y)$ is also true. For every element y, we do have at least one element, which is x_1, such that $P(x_1, y)$ is true. Therefore, we conclude that $\forall y \exists x P(x, y)$ must also be true.

Problem 2. In Problem 5 of Homework Assignment 1, does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion? Justify your answer briefly.

Solution: Yes. Here is the proof.

1.	$Q(a)$	Hypothesis, an arbitrary a
2.	$\forall x(Q(x) \to P(x))$	Premise
3.	$Q(a) \to P(a)$	2, Universal instantiation
4.	$P(a)$	1,3 Modus ponens
5.	$\forall x(P(x) \to \neg S(x))$	Premise
6.	$P(a) \to \neg S(a)$	5, Universal instantiation
7.	$\neg S(a)$	4,6 Modus ponens
8.	$\forall x(R(x) \to S(x))$	Premise
9.	$R(a) \to S(a)$	9, Universal instantiation
10.	$\neg R(a)$	7,9 Modus ponens
11.	$Q(a) \to \neg R(a)$	1, 10, implication
12.	$\forall x(Q(x) \to \neg R(x))$	11, Universal generalization

Problem 3: (a) Construct a contrapositive proof that for all real numbers x, if $x^2 - 2x \neq 3$ then $x \neq 3$.
(b) Construct a proof by contradiction that for all real numbers x, if $x^2 - 2x \neq 3$ then $x \neq 3$.

SOLUTION: We are asked to prove "$p(x) \Rightarrow q(x)$" in two different ways, where $p(x) =$ "$x^2 - 2x \neq 3$", $q(x) =$ "$x \neq 3$".

(a)
In the contrapositive proof, we prove the contraposition "$\neg q(x) \Rightarrow \neg p(x)$" of "$p(x) \Rightarrow q(x)$". We do this by assuming $\neg q(x)$ and deriving $\neg p(x)$.

Suppose that $x = 3$ ($\neg q(x)$).
Then $x^2 - 2x = 9 - 6 = 3$ ($\neg p(x)$).
Thus, if $x = 3$, then $x^2 - 2x = 3$ ($\neg q(x) \Rightarrow \neg p(x)$).
By contrapositive inference, if $x^2 - 2x \neq 3$, then $x \neq 3$ ($p(x) \Rightarrow q(x)$).

(b)
In the proof by contradiction, we assume $p(x)$ and $\neg q(x)$ and try to derive a contradiction.

Suppose that $x^2 - 2x \neq 3$ ($p(x)$).
Suppose as well that $x = 3$ ($\neg q(x)$).
Then $x^2 - 2x = 9 - 6 = 3$ ($\neg p(x)$),
which is a contradiction ($r(x) = p(x)$).
Therefore, if $x^2 - 2x \neq 3$, then $x \neq 3$ ($p(x) \Rightarrow q(x)$).

Problem 4. Prove that $\sqrt{20}$ is irrational.

Solution:
We will prove it by contradiction.
Assume that $\sqrt{20}$ is rational, i.e., there exist two integers a, b with $gcd(a, b) = 1$ and $\sqrt{20} = \frac{a}{b}$.

Then $20 = \frac{a^2}{b^2}$ and thus $20b^2 = a^2$. The prime factorization of 20 is $2 \cdot 2 \cdot 5$. This implies that 2 and 5 are divisors of a^2 and thus they are also divisors of a, which means a is a multiple of 10, i.e., $a = 10k$ for some integer k.

From $20b^2 = a^2$, we get $20b^2 = 100k^2$ and thus $b^2 = 5k^2$. This implies that b^2 is a multiple of 5 and thus b is a multiple of 5. Therefore, $gcd(a, b) > 1$.

This contradicts with our assumption that $\sqrt{20}$ is rational (i.e., a and b are the two integers such that $\sqrt{20} = \frac{a}{b}$ with $gcd(a, b) = 1$). Thus, $\sqrt{20}$ is irrational.

Problem 5. Is the following reasoning for finding the solutions of the equation $\sqrt{x+3} = 3 - x$ correct?

(1) $\sqrt{x+3} = 3 - x$: Given
(2) $x + 3 = 9 - 6x + x^2$: Square both sides
(3) $x^2 - 7x + 6 = 0$: Re-arrange the terms.
(4) $(x - 1)(x - 6) = 0$: Factor the right hand side.
(5) So, $x = 1$ or $x = 6$.

Solution: The reasoning is incorrect. For a fixed value of x, we have

$$(2) \leftrightarrow (3) \leftrightarrow (4) \leftrightarrow (5).$$

So, both $x = 1$ and $x = 6$ are solutions for (2). However, we don't have

$$(1) \leftrightarrow (2).$$

Instead, we only have

$$(1) \rightarrow (2).$$

Therefore, we cannot conclude that a value of x that satisfies (2) also satisfies (1). As a matter of fact, although $x = 6$ satisfies (2), it does not satisfy (1).

Problem 6. (Challenge Problem) Three friends Tom, Jerry and Spike play a game with a host. The host has 3 black hats and 2 white hats on his hand initially. He turns off the light, puts one hat on each player, and hides the other two hats. Then he turns on the light and asks each player to guess what color hat he is wearing.

The three players stand on one line and face one direction, so that Tom can see the hats of both Jerry and Spike, Jerry can see only the hat of Spike, and Spike cannot see anyone's hat.

Tom speaks first and says "I don't know". Then Jerry says "I don't know either". After that, Spike says "I know".

What color hat is Spike wearing? How does he know?

Solution: Because there are 2 white hats and 3 black hats, the following rule holds:

Tom sees 2 white hats $\rightarrow$ Tom knows the color of his hat.

However, Tom does not know. By contraposition, Jerry and Spike can conclude that Tom does not see 2 white hats. This means that at least one of Jerry or Spike is wearing a black hat. Because of this fact, the following rule holds:

Jerry sees 1 white hat $\rightarrow$ Jerry knows the color of his hat.

However, Jerry does not know. By contraposition, Spike can conclude that Jerry does not see 1 white hat. By negation, Spike knows that Jerry must see a black hat, and hence Spike must be wearing a black hat.