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HKUST – Department of Computer Science and Engineering

COMP170: Discrete Math Tools for CS – FALL 2008

Final Examination – Sketch Solution Key

Date: Thursday Dec 11, 2008 Time: 12:30–3:00pm

Name: ______________	Student ID: ______________
Email: ______________	Lecture and Tutorial: __________

Instructions

- This is a closed book examination. It consists of 26 pages and 10 questions.
- Please write your name, student ID, email, lecture and tutorial sections on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page (and sometimes has an extra page for you to do work on). This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Only use notation given in class. Do not use notation that you have learnt outside of this class that is nonstandard.
- Calculators may be used for the examination.

Questions	1	2	3	4	5	6	7	8	9	10	Total
Points	8	9	9	14	16	8	8	8	8	12	100
Score											

Student ID: ________________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:       ______________________

Student's Signature:  ______________________
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Student ID: ______________

Definitions and Formulas: This page contains some definitions used in this exam and a list of formulas (theorems) that you may use in the exam (without having to provide a proof). Note that you might not need all of these formulas on this exam.

Definitions:

1. $N = \{0, 1, 2, 3, \ldots\}$, the set of non-negative integers.
2. $Z^+ = \{1, 2, 3, \ldots\}$, the set of positive integers.
3. R is the set of *real numbers*.
4. R^+ is the set of positive *real numbers*.

Formulas:

1. $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$
2. If $0 < i < n$ then $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$.
3. $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.
4. $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
5. $\sum_{i=1}^{n-1} i = n(n-1)/2$
6. $\sum_{i=1}^{n-1} i^2 = \frac{2n^3-3n^2+n}{6}$
7. If $r \neq 1$ then $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.
8. If $r \neq 1$ then $\sum_{i=0}^{n} ir^i = \frac{nr^{n+2}-(n+1)r^{n+1}+r}{(1-r)^2}$.
9. The inclusion-exclusion theorem:
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \leq i_1 < i_2 < \ldots < i_k \leq n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$
10. If X is a random variable, then $E(X)$ denotes the *Expectation of* X and $V(X) = E\left((X - E(X))^2\right)$ denotes the *Variance of* X.
11. $f(n) = O(g(n))$ if there exist some $N > 0$ and positive constant c such that $\forall n > N$, $f(n) \leq c \cdot g(n)$.
12. $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

Problem 1: [8 pts]
Consider the following two sets of modular equations:

(a)

$$\begin{aligned} x \bmod 21 &= 12 \\ x \bmod 25 &= 11 \end{aligned}$$

(b)

$$\begin{aligned} x \bmod 21 &= 12 \\ x \bmod 30 &= 11 \end{aligned}$$

For each of the two sets of equations answer the following question:

> Does there exist a unique solution for $x \in Z_{mn}$, where m and n are the divisors of the two modular equations?

Note: in (a), $(m, n) = (21, 25)$; in (b), $(m, n) = (21, 30)$.

For each set, explain why your answer is correct. Furthermore, if your answer is that there is a unique solution, give the solution.

Solution: (i) Since $m = 21$ and $b = 25$ are relatively prime, the Chinese remainder theorem guarantees that there is a unique solution.
To find x we first need to find $\bar{n} \in Z_m$ and $\bar{m} \in Z_n$ such that

$$n \cdot \bar{n} \bmod m = 1 \quad \text{and} \quad m \cdot \bar{m} \bmod n = 1.$$

By using the extended GCD algorithm we find that

$$6 \cdot 21 + (-5) \cdot 25 = 1.$$

Thus $\bar{m} = 6$ and $\bar{n} = (-5) \bmod 21 = 16$
Now let

$$y = 12 \cdot \bar{m} \cdot m + 11 \cdot \bar{n} \cdot n = 9186$$

and

$$x = y \bmod (mn) = 6186 \bmod 525 = 411.$$

Then, 122 is the unique solution.

(i) There is no solution.
The proof is by contradiction. Suppose that there is a solution x in Z_{mn}.
Consider the first equation: $x \bmod 21 = 12$. Since both 21 and 12 are divisible by 3, we must have that x is divisible by 3.

But, since 30 is also divisible by 3, this implies that x mod 30 is also divisible by 3. This contradicts the fact that 11 is not divisible by 3.

Note: The fact that $gcd(21, 30) \neq 1$ and therefore the Chinese Remainder Theorem can not be applied is not a valid solution to the problem. When the CRT can be applied, it tells us that there is a unique solution to the equalities. But, the fact that the CRT can not be applied, tells us nothing about the existence of solutions (or lack of them).

Problem 2: [9 pts]

There is a type of cereal that contains a toy in each box. There are 10 types of toys, $T_1, T_2, \ldots, T_{10}$ and every box has probability $\frac{1}{10}$ of containing each possible toy, independently of every other box.

Your little brother buys a box of cereal each week for 20 weeks and keeps all of the toys that he finds.

For each of the following three problems write a formula for the solution (it is not necesary to write an actual number).
For part (c) you may write your solution as a summation formula, e.g., using a Σ.
For parts (a) and (b) your formula may *not* use a summation.

(a) What is the probability that your brother has at least one copy of toy T_1 after 20 weeks?

(b) What is the expected number of *different* toys that he has collected after 20 weeks?
For example, if he has collected 11 copies of T_1, 8 copies of T_4 and 1 copy of T_7, he has collected 3 different toys.

(c) What is the probability that after 20 weeks your brother has collected *all* of the 10 different toys?

Solution: (a) By independence, the probability that he has *no* copy of toy T_1 is

$$\left(1 - \frac{1}{10}\right)^{20}.$$

So, the probability that he has at least one copy is the complement

$$1 - \left(1 - \frac{1}{10}\right)^{20}.$$

(b) Let X_i be the indicator random value that is 1 if he has at least 1 copy of toy T_i and 0 otherwise. Then, similar to the answer to the previous question,

$$E(X_i) = 1 - \left(1 - \frac{1}{10}\right)^{20}.$$

Now let X be the number of different toys that he has collected. By definition,

$$X = \sum_{i=1}^{10} X_i$$

so, by linearity of expectation,

$$E(X) = \sum_{i=1}^{10} E(X_i) = 10 - 10\left(1 - \frac{1}{10}\right)^{20}.$$

(c) Let E_i be the event that he has no copy of toy T_i. Then $A = \bigcup_{i=1}^{10} E_i$ is the event that there is at least one toy that he has no copy of. A^C, the complement of A, is the event that he has a copy of every toy, which is what we want.

We will now show how to use the inclusion-exclusion formula to calculate $P(A)$;the solution will then be $1 - P(A)$.

Recall the Inclusion-Exclusion formula:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{i_1,i_2,\ldots,i_k: \\ 1\le i_1<i_2<\ldots<i_k\le n}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$ is the probability that all of the toys he finds are one of the $10 - k$ unspecified ones, so

$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \left(\frac{10-k}{10}\right)^{20} = \left(1 - \frac{k}{10}\right)^{20}.$$

Thus, using the fact that there are $\binom{10}{k}$ ways of choosing an increasing k-tuple from 10 items, gives

$$\begin{aligned} P(A) = P\left(\bigcup_{i=1}^{10} E_i\right) &= \sum_{k=1}^{10}(-1)^{k+1} \sum_{\substack{i_1,i_2,\ldots,i_k: \\ 1\le i_1<i_2<\ldots<i_k\le n}} \left(1 - \frac{k}{10}\right)^{20} \\ &= \sum_{k=1}^{10}(-1)^{k+1}\binom{10}{k}\left(1 - \frac{k}{10}\right)^{20}. \end{aligned}$$

The final result is then $1 - P(A)$. This can also be written variously as

$$\begin{aligned} 1 - P(A) &= 1 - \sum_{k=1}^{10}(-1)^{k+1}\binom{10}{k}\left(1 - \frac{k}{10}\right)^{20} \\ &= \sum_{k=0}^{10}(-1)^{k}\binom{10}{k}\left(1 - \frac{k}{10}\right)^{20} \\ &= \sum_{k=0}^{9}(-1)^{k}\binom{10}{k}\left(1 - \frac{k}{10}\right)^{20}. \end{aligned}$$

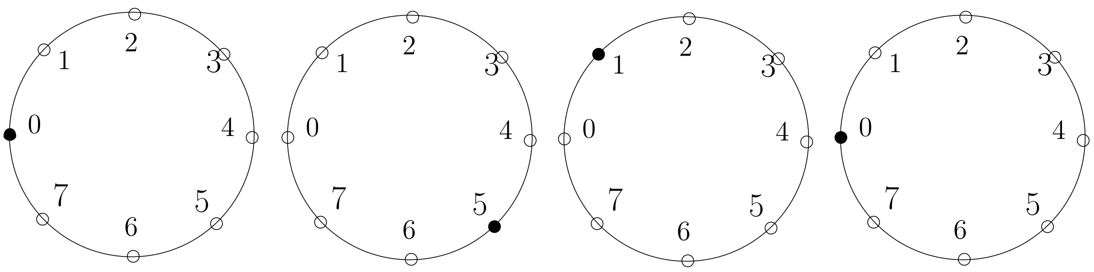
Problem 3: [9 pts]

You have an 8 sided die with the numbers 1-8 painted on its different sides. It is a fair die, so every time you roll it, each of the 8 numbers is equally likely to occur.

Now consider the following game. You have a circle with 8 spots on it, numbered $0, 1, 2, 3, 4, 5, 6, 7$. A game piece is placed on location 0 and you start rolling the 8-sided die. Each time you roll the die, you walk clockwise around the circle the number of spaces you have rolled.

Mathematically, if you are at location X and roll a Y, then you end up at location $(X + Y) \bmod 8$.

As an example, consider the diagrams below that occur one after another: (A) is the starting position. (B) occurs after rolling a 5; (C) after rolling a 4; and (D) after rolling a 7.



(A) (B) (C) (D)

For each of the questions below explain how you derived your answer.

(a) If you start at location 0, what is the expected number of rolls until the *first* time that you end up at location 0 again?

(b) If you start at location 0, what is the expected number of rolls until the *third* time that you end up at location 0 again?

(c) If you start at location 0, what is the expected number of rolls until you have landed on each of the other 7 locations on the circle $(1, 2, 3, 4, 5, 6, 7)$ **at least once** (for each location)?

Solution: (a) Let X be the number of rolls needed until returning to 0. Note that, at every roll, the probability of returning to 0 is $p = \frac{1}{8}$. This, X is just the time until first success when flipping a p-biased coin. As proven in class

$$E(X) = \frac{1}{p} = \frac{1}{1/8} = 8.$$

(b) Let X_1 be the number of rolls until the first return to 0; X_2 the number of rolls until the second return to 0; X_3 the number of rolls until the third return to 0. We are interested in $X = X_1 + X_2 + X_3$.

From linearity of expectation, $E(X) = E(X_1) + E(X_2) + E(X_3)$.

But, for each i, X_i is exactly the same as x in part (a) (formally, we say that each X_i has the same distribution as the X in part (a)). So, for each i, $E(X_i) = E(X) = 8$ and the answer is $E(X) = 3 \cdot 8 = 24$.

(c) This is very similar to the coupon collector problem we had in the notes.

Start rolling the die. Let $Y_0 = 0$ and, for $0 < i \le 7$, let Y_i be the roll on which we see the i'th newest item and Z_i the value of that item. That is,

* Y_i is the roll on which we see the first non-zero item and Z_1 is the item we see.
* Y_2 is the roll on which we see the first item that is not in $\{0, Y_1\}$ and Z_2 is the item seen on that roll.
* Y_3 is the roll on which we see the first item that is not in $\{0, Y_1, Y_2\}$ and Z_3 is the item seen on that roll.
* And so on ...

We are interested in Y_7, the roll on which we see the 7'th unseen item, ie., when we've first seen everyone.

Now, for $0 < i \le 7$, set $X_i = Y_i - Y_{i-1}$.

X_i is the number of rolls needed, starting from Y_{i-1}, to see the first item *not in* $\{0, Y_1, Y_2, \ldots, Y_{i-1}\}$. Since on each roll we have probability $i/8$ not seeing an item in $\{0, Y_1, Y_2, \ldots, Y_{i-1}\}$, X_i is the time until first success when flipping a coin with $p = i/8$. Thus, $E(X_i) = 1/p = 8/i$.

We can now use linearity of expectation to find

$$\begin{aligned}
E(7) &= E\left(\sum_{i=1}^{7}(Y_i - Y_{i-1})\right) \\
&= E\left(\sum_{i=1}^{7} X_i\right) \\
&= \sum_{i=1}^{7} E(X_i) = \sum_{i=1}^{7} 8/i = 8\sum_{i=1}^{7} 1/i
\end{aligned}$$

Problem 4: [14 pts]
For this problem you may assume that n is a nonnegative power of 5.
Recall that if $f(n)$ and $g(n)$ are functions, to prove that $f(n) = O(g(n))$ you must prove that there exist some $n_0 \geq 0$ and $c > 0$ such that

$$\forall n > n_0,\ f(n) \leq cg(n).$$

(a) Suppose function $T(n)$ satisfies $T(1) = 1$ and, for $n > 1$,

$$T(n) \leq 5T\left(\frac{n}{5}\right) + 5n.$$

Prove that $T(n) = O(n \log_5 n)$.

(b) Suppose function $T(n)$ satisfies $T(1) = 1$ and, for $n > 1$

$$T(n) \leq 25T\left(\frac{n}{5}\right) + 5n.$$

Prove that $T(n) = O(n^2)$.

Hint: Prove by induction.

Solution: (a) We show that $\exists k_0, c$, such that $\forall k > k_0$,

$$T(5^k) \leq c \cdot 5^k \cdot \log_5 5^k = c \cdot k5^k \qquad (*)$$

Base case: Let $k_0 = 0$. The base case is $k = 1$. For that case, we need

$$T(5) \leq c \cdot 5 \log_5 5 = 5c.$$

Since $T(5) \leq 5T(1) + 5 \cdot 5 = 30$, we can satisfy $(*)$ for $k = 1$ by setting $c \geq 30/5 = 6$.
Induction hypothesis: Assume (*) is true for the case of k.
Induction step: For the case of $k + 1$,

$$\begin{aligned} T(5^{k+1}) &\leq 5T(5^k) + 5 \cdot 5^{k+1} \\ &\leq 5ck5^k + 5 \cdot 5^{k+1} \\ &= 5ck5^k + 25 \cdot 5^k. \end{aligned}$$

To satisfy (*), we need $5ck5^k + 25 \cdot 5^k \leq c(k+1)5^{k+1} = 5c(k+1)5^k$, i.e., $ck + 5 \leq c(k+1)$. This will be true for all $c \geq 5$.

So, (*) is true for all $k > k_0$ for $k_0 = 0$ and $c = 6$. Hence $T(n) = O(n \log n)$.

(b) Attempting to prove $T(n) \leq cn^2$ directly by induction doesn't work, similar to a case we saw in the "advanced induction" notes in class. So, using the technique we saw in that same class, we will instead attempt to prove

$$T(n) \leq c_1 n^2 - c_2 n$$

for some c_1, c_2. This will immediately imply $T(n) \leq c_1 n^2$ and we will be done.

More specifically, restricting to the form $n = 5^k$ (and using the fact that then $n^2 = (5^k)^2 = 5^{2k}$) we will show that that $\exists k_0, c_1, c_2$, such that $\forall k > k_0$,

$$T(5^k) \leq c_1 5^{2k} - c_2 5^k \qquad (**)$$

Base case: Let $k_0 = 0$. The base case is $k = 1$. For that case, we need

$$T(5) \leq c_1 5^2 - c_2 5 = 25c_1 - 5c_2$$

Since $T(5) \leq 25T(1) + 5 \cdot 5 = 50$, we can satisfy $(*)$ for $k = 1$ by setting

$$c_1 \geq c_2/25 + 2.$$

Induction hypothesis: Assume (**) is true for the case of k.

Induction step: For the case of $k + 1$, we have

$$\begin{aligned} T(5^{k+1}) &\leq 25T(5^k) + 5 \cdot 5^{k+1} \\ &\leq 25(c_1 5^{2k} - c_2 5^k) + 5 \cdot 5^{k+1}. \end{aligned}$$

To satisfy (*), we want to show that

$$25(c_1 5^{2k} - c_2 5^k) + 5 \cdot 5^{k+1} \leq c_1 5^{2(k+1)} - c_2 5^{k+1},$$

i.e.,

$$-25c_2 5^k + 25 \cdot 5^k \leq -5c_2 5^k$$

or

$$-5c_2 + 5 \leq -c_2.$$

We can achieve this by setting $c \geq 5/4$.

So, (**) is true for all $k > k_0$ for $k_0 = 1$ and, e.g., $c_2 = 25$, $c_1 = 25/25 + 2 = 3$. Hence $T(n) = O(n^2)$.

Problem 5: [16 pts] A bag contains 80 fair coins and 20 magic coins.
Flipping a fair coin results in a head with probability 0.5, whereas flipping a magic coin results in a head with probability 1.0.

(a) Consider randomly drawing one coin from the bag and flipping it. What is the probability of seeing a head?

(b) Consider drawing *two coins* from the bag with replacement and flipping *each* of them *once*.
(*With replacement* means that the second coin is drawn *after* the first is returned to the bag.)
Let

$$X_1 = \begin{cases} 1 & \text{if first flip a head} \\ 0 & \text{if first flip a tail} \end{cases}, \quad X_2 = \begin{cases} 1 & \text{if second flip a head} \\ 0 & \text{if second flip a tail} \end{cases}.$$

Are X_1 and X_2 independent?
If the answer is yes, just write "yes". If the answer is no, prove it.

(c) Consider the same situation as in (b), i.e., that two coins are chosen with replacement and flipped.
Let X denote the number of heads you see. Note that X can be $0, 1, 2$.
What is $E(X)$? What is $V(X)$?

(d) Now consider drawing *one coin* from the bag and flipping it *twice*.
Let

$$Y_1 = \begin{cases} 1 & \text{if first flip a head} \\ 0 & \text{if first flip a tail} \end{cases}, \quad Y_2 = \begin{cases} 1 & \text{if second flip a head} \\ 0 & \text{if second flip a tail} \end{cases}.$$

Are Y_1 and Y_2 independent?
If the answer is yes, just write "yes". If the answer is no, prove it.

(e). Consider the same situation as in (d), i.e, one coin is drawn and flipped twice.
Let Y denote the number of heads you see. Note that Y can be $0, 1, 2$.
What is $\Pr(Y = 2)$? What is $E(Y)$? What is $V(Y)$?

Solution: (a). Let H = "Getting a head"; K="Drawing a fair coin"; $\bar{K}$ = "Drawing a magic coin".

$$\begin{aligned} P(H) &= P(H \cap K) + P(H \cap \bar{K}) \\ &= P(H|K)P(K) + P(H|\bar{K})P(\bar{K}) \\ &= 0.5 \times 0.8 + 1.0 \times 0.2 = 0.6 \end{aligned}$$

(b). Yes.

(c).

$$X = X_1 + X_2$$

$$E(X) = E(X_1) + E(X_2) = P(X_1 = H) + P(X_2 = H) = 1.2$$
$$V(X) = V(X_1) + V(X_2) = 0.24 + 0.24 = 0.28.$$

(d) The two variables are not independent. For example,

$$\begin{aligned}
P(H_1 \cap H_2) &= P(H_1 \cap H_2 \cap K) + P(H_1 \cap H_2 \cap \bar{K}) \\
&= P(H_1 \cap H_2|K)P(K) + P(H_1 \cap H_2|\bar{K})P(\bar{K}) \\
&= 0.25 \times 0.8 + 1.0 \times 0.2 \\
&= 0.4 \\
&\neq 0.6 \cdot 0.6 = P(H_1)P(H_2).
\end{aligned}$$

(e) Similar to (d), we have

$$\begin{aligned}
P(\bar{H}_1 \cap H_2) &= P(\bar{H}_1 \cap H_2|K)P(K) + P(\bar{H}_1 \cap H_2|\bar{K})P(\bar{K}) \\
&= 0.25 \times 0.8 + 0.0 \times 0.2 = 0.2;
\end{aligned}$$

$$\begin{aligned}
P(H_1 \cap \bar{H}_2) &= P(H_1 \cap \bar{H}_2|K)P(K) + P(H_1 \cap \bar{H}_2|\bar{K})P(\bar{K}) \\
&= 0.25 \times 0.8 + 0.0 \times 0.2 = 0.2;
\end{aligned}$$

$$\begin{aligned}
P(\bar{H}_1 \cap \bar{H}_2) &= P(\bar{H}_1 \cap \bar{H}_2|K)P(K) + P(\bar{H}_1 \cap \bar{H}_2|\bar{K})P(\bar{K}) \\
&= 0.25 \times 0.8 + 0.0 \times 0.2 = 0.2.
\end{aligned}$$

So, $P(Y = 0) = 0.2$, $P(Y = 1) = 0.4$, $P(Y = 2) = 0.4$. Consequently,

$$E(Y) = 1 \cdot 0.4 + 2 \cdot 0.4 = 1.2$$

And,

$$\begin{aligned}
V(Y) &= (0 - E(Y))^2 \cdot P(Y = 0) + (1 - E(Y))^2 \cdot P(Y = 1) + (2 - E(Y))^2 \cdot P(Y = 2) \\
&= (0 - 1.2)^2 \cdot 0.2 + (1 - 1.2)^2 \cdot 0.4 + (2 - 1.2)^2 \cdot 0.4 \\
&= 0.288 + 0.016 + 0.256 = 0.56
\end{aligned}$$

Problem 6: [8 pts]

Each of (a), (b) (c) and (d) below contains a pair of statements, (i) and (ii).

For each pair, say whether (i) is equivalent to (ii), i.e., that, for all statements $p(x)$, (i) is true if and only if (ii) is true

If they are equivalent, all you have to do is say that they are equivalent. If they are not equivalent, give a counterexample. A counter example should involve a specification of $p(x)$ and an explanation as to why the resulting statement is false.

(a). (i) $\forall x \in R^+ \ (p(x))$ (ii) $\forall x \in R \ \big((x > 0) \wedge p(x)\big)$

(b). (i) $\forall x \in R^+ \ (p(x))$ (ii) $\forall x \in R \ \big((x > 0) \Rightarrow p(x)\big)$

(c). (i) $\exists x \in R^+ \ (p(x))$ (ii) $\exists x \in R \ \big((x > 0) \wedge p(x)\big)$

(d). (i) $\exists x \in R^+ \ (p(x))$ (ii) $\exists x \in R \ \big((x > 0) \Rightarrow p(x)\big)$

Solution:

(a) Not equivalent. The second statement is false regardless what $p(x)$ is, because $x > 0$ is not true for all $x \in R$. The first statement can be true. For example, it is true if $p(x)$ is $(x + 1)^2 > x^2$.

(b) Equivalent.

(c) Equivalent.

(d) Not equivalent. Let $p(x)$ be $x^2 + 1 = 0$. Then $p(x)$ is false for $x = 1 \in R^+$. So the first statement is false.

On the other hand, the second statement is true for $x = -1$ because then $(x > 0)$ is false so $\big((x > 0) \Rightarrow p(x)\big)$ is true.

Problem 7: [8 pts]
Consider a competition in which $n = 2^k$ teams participate in an *elimination tournament*. An elimination tournament is split into rounds. In each round the teams pair off and play a game. Only the winners of each game proceed on to the next round.

For example, in the first round, $n/2$ games are played. The winners of those games proceed onto the next round, where they pair off and play $n/4$ games. The winners of those games proceed onto the next round, where they pair off and play $n/8$ games, etc. The competition stops when there is only one team remaining.

Let $T(n)$ be the number of *rounds* in the tournament. Then, $T(1) = 0$.

Answer the following three problems. For all of them, you should assume that n is a power of 2.

(a) Express $T(n)$ as a recurrence relation.

(b) Find a closed-form expression for $T(n)$ by solving the recurrence.
It is not necessary to show your derivation.

(c) Prove by induction that the closed form found in (b) is the solution to the recurrence relation derived in (a).

Solution: (a)

$$T(n) = T(n/2) + 1$$

(b)

$$T(n) = \log_2 n$$

(c) Base case: $T(1) = 0 = \log_2 1$.
Induction hypothesis: Assume the equation is true for $n = 2^k$.
Induction step: Consider $n = 2^{k+1}$.

$$\begin{aligned} T(n) &= T(2^{k+1})) \\ &= T(2^k) + 1 \\ &= \log_2 2^k + \log_2 2 \\ &= \log_2 2^{k+1} = \log_2 n. \end{aligned}$$

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Problem 8: [8 pts] Prove by induction that

$$\forall n \in Z^+, \qquad \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

As examples of the statement,

$$\begin{aligned} \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} &= \frac{2}{5}, \\ \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} &= \frac{3}{7}. \end{aligned}$$

Solution:

- Base case: When $n = 1$,

$$\frac{1}{1\cdot 3} = \frac{1}{3} = \frac{1}{2\cdot 1+1}.$$

- Induction Hypothesis: Assume that the equation is true for the case of $n = k$, i.e.

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}.$$

- Induction Step: For the case of $n = k+1$, we have

$$\begin{aligned} & \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ = & \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\ = & \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\ = & \frac{2k2+3k+1}{(2k+1)(2k+3)} \\ = & \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\ = & \frac{(k+1}{2(k+1)+1}. \end{aligned}$$

So, the equation is true for all n.

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Problem 9: [8 pts]
For this problem you may assume that n is a power of 3.

(a) Define

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 2T\left(\frac{n}{3}\right) + 5n & \text{if } n > 1 \end{cases}$$

Find a closed formula for $T(n)$ as a function of n (your formula may not contain a summation).

(b) Define

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 4T\left(\frac{n}{3}\right) + 2n & \text{if } n > 1 \end{cases}$$

Find a closed formula for $T(n)$ as a function of n (your formula may not contain a summation).

For both part (a) and part (b), it is not necessary to show how you derived your answer.

Solution: (a) Iterating the recurrence gives

$$\begin{aligned}
T(n) &= 5n + 2T\left(\frac{n}{3}\right) \\
&= 5n + 2\left[5\frac{n}{3} + 2T\left(\frac{n}{3}\right)\right] \\
&= 5n\left(1 + \frac{2}{3}\right) + 2^2 T\left(\frac{n}{3^2}\right) \\
&= 5n\left(1 + \frac{2}{3}\right) + 2^2\left[5\frac{n}{3^2} + 2T\left(\frac{n}{3^3}\right)\right] \\
&= 5n\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2\right) + 2^3 T\left(\frac{n}{3^3}\right) \\
&\vdots \\
&= 5n\left(1 + \frac{2}{3} + \ldots + \left(\frac{2}{3}\right)^{i-1}\right) + 2^i T\left(\frac{n}{3^i}\right) \\
&\vdots \\
&= 5n\left(1 + \frac{2}{3} + \ldots + \left(\frac{2}{3}\right)^{\log_3 n - 1}\right) + 2^{\log_3 n} T\left(\frac{n}{3^{\log_3 n}}\right) \\
&= 5n\frac{1 - \left(\frac{2}{3}\right)^{\log_3 n}}{1 - \frac{2}{3}} + 2^{\log_3 n} T(1)
\end{aligned}$$

Recalling that $T(1) = 1$ gives a closed formula for $T(n)$. This can be simplified further by noting that

$$2^{\log_3 n} = 2^{\log_3 2 \cdot \log_2 n} = \left(2^{\log_2 n}\right)^{\log_3 2} = n^{\log_3 2}$$

so

$$T(n) = 15n\left(1 - \frac{n^{\log_3 2}}{n}\right) - n^{\log_3 2} = 15n - 14n^{\log_3 2}.$$

(b) Iterating the recurrence gives

$$\begin{aligned}
T(n) &= 2n + 4T\left(\frac{n}{3}\right) \\
&= 2n + 4\left[2\frac{n}{3} + 4T\left(\frac{n}{3}\right)\right] \\
&= 2n\left(1 + \frac{4}{3}\right) + 4^2T\left(\frac{n}{3^2}\right) \\
&= 2n\left(1 + \frac{4}{3}\right) + 4^2\left[2\frac{n}{3^2} + 2T\left(\frac{n}{3^3}\right)\right] \\
&= 2n\left(1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2\right) + 4^3T\left(\frac{n}{3^3}\right) \\
&\vdots \\
&= 2n\left(1 + \frac{4}{3} + \ldots + \left(\frac{4}{3}\right)^{i-1}\right) + 4^iT\left(\frac{n}{3^i}\right) \\
&\vdots \\
&= 2n\left(1 + \frac{4}{3} + \ldots + \left(\frac{4}{3}\right)^{\log_3 n - 1}\right) + 4^{\log_3 n}T\left(\frac{n}{3^{\log_3 n}}\right) \\
&= 2n\frac{\left(\frac{4}{3}\right)^{\log_3 n} - 1}{\frac{4}{3} - 1} + 4^{\log_3 n}T(1)
\end{aligned}$$

As before, we can simplify by noting that $4^{\log_3 n} = n^{\log_3 4}$ to write

$$T(n) = 6n\left(\frac{n^{\log_3 4}}{n} - 1\right) + n^{\log_3 4} = 7n^{\log_3 4} - 6n.$$

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Problem 10: [12 pts]

For this problem, assume that there are 8 red balls labelled $1, 2, 3, 4, 5, 6, 7, 8$ and 8 blue balls labelled $1, 2, 3, 4, 5, 6, 7, 8$.

For each of the problems below, you will be asked how many different ways there are of choosing 2 red balls and 2 blue balls that satisfy certain conditions. In what follows, R_1, R_2 will always denote the smaller and larger numbers appearing, respectively, on the two red balls and B_1, B_2 will denote the smaller and larger numbers appearing, respectively, on the two blue balls.

Notes: "Choosing two balls" means choosing without replacement, so $R_1 < R_2$ and $B_1 < B_2$. Also, when counting, note that in some cases it may be possible for red balls to show the same values as blue balls.

For each problem explain how you derived your answer. Your solution does not have to be a number. Also, it may be written as the summation of terms.

(a) How many different ways are there of choosing 2 red balls and 2 blue balls so that both of the numbers on the chosen red balls are less than both of the numbers on the chosen blue balls?
That is, $R_2 < B_1$.

(b) How many different ways are there of choosing 2 red balls and 2 blue balls so that the smallest numbers and largest numbers on the four balls are both on blue balls?
That is, both $B_1 = \min(R_1, B_1)$ and $B_2 = \max(R_2, B_2)$.

(b) How many different ways are there of choosing 2 red balls and 2 blue balls so that the sum of the blue balls equals the sum of the red balls?
That is, $B_1 + B_2 = R_1 + R_2$.

Solution: Note that there are *many* ways to solve these three problems. In what follows, we provide one way for each problem.

(a) Since the red balls are smaller than the blue balls we have

$$R_1 < R_2 < B_1 < B_2$$

so the four chosen balls have four distinct numbers Knowing those four numbers fixes which are red (the two smallest) and blue (the two largest). Furthermore, given any four numbers, there is a unique way of choosing red and blue balls with those numbers satisfying the conditions.

So the answer is simply

$$\binom{8}{4} = 70.$$

(b) The conditions can be written as

$$B_1 \leq R_1 < R_2 \leq B_2 \tag{1}$$

Note: many students misunderstood the question and thought that the conditions were $B_1 < R_1 < R_2 < B_2$. In this case the answer would be exactly the same as for part (a).

We can split equation (1) into 3 different possibilities depending upon how many distinct numbers R_1, R_2, B_1, B_2 encompass.

1. 2 distinct numbers.
   In this case $B_1 = R_1$ and $B_2 = R_2$. There are $\binom{8}{2}$ ways of choosing these numbers
2. 3 distinct numbers. There are $\binom{8}{3}$ ways of choosing these numbers. In this case B_1 must be the smallest and B_2 the largest. Once these are fixed, either $R_1 = B_1$ and R_2 is the middle number or $R_2 = B_2$ and R_1 is the middle number. There are thus $2\binom{8}{3}$ total such combinations.
3. 4 distinct numbers. There are $\binom{8}{4}$ ways of choosing these numbers and, once they are chosen, the specifications of the blue and red balls are fixed.

So, the total answer is

$$\binom{8}{2} + 2\binom{8}{3} + \binom{8}{4}.$$

This can be simplified further by noting that it is equal to

$$\begin{aligned}\binom{8}{2} + \binom{8}{3} + \binom{8}{3} + \binom{8}{4} &= \binom{9}{3} + \binom{9}{4} \\ &= \binom{10}{4} = 210\end{aligned}$$

(c) Let $f(i)$ be the number of ways of writing the number i as the sum of two red (blue) dice. Then the number of ways that the two dice both sum to i is $(f(i))^2$. Summing over all i gives $\sum_{i=3}^{17}(f(i))^2$.
Working out the values we find that

$$\begin{aligned}1 &= f(3) = f(4) = f(14) = f(15), \\ 2 &= f(5) = f(6) = f(12) = f(13), \\ 3 &= f(7) = f(8) = f(10) = f(11)\end{aligned}$$

and

$$4 = f(9).$$

Combining, give the final answer as

$$4(1^2 + 2^2 + 3^2) + 4^2 = 72.$$