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HKUST – Department of Computer Science and Engineering
COMP170: Discrete Math Tools for CS – FALL 2006

Midterm Examination 1 – Sketch Solution Key

Date: Th, October 05, 2006 Time: 19:00–20:30 Venues: LTB, LTC, LTD

Name: ______________________	Student ID: __________________
Email: ______________________	Lecture and Tutorial: __________

Instructions

- This is a closed book exam. It consists of 17 pages and 8 questions.
- Please write your name, student ID, Email, lecture section and tutorial on this page.
- For each subsequent page, please write your student ID at the top of the page in the space provided.
- Please sign the honor code statement on page 2.
- Answer all the questions within the space provided on the examination paper. You may use the back of the pages for your rough work. The last three pages are scrap paper and may also be used for rough work. Each question is on a separate page. This is for clarity and is not meant to imply that each question requires a full page answer. Many can be answered using only a few lines.
- Unless otherwise specified you *must* always explain how you derived your answer. A number without an explanation will be considered an incorrect answer.
- Solutions can be written in terms of binomial coefficients and falling factorials. For example, $\binom{5}{3}+\binom{4}{2}$ may be written instead of 16, and $5^{\underline{3}}$ instead of 60. Calculators may be used for the exam (but are not necessary).
- Please *do not* use the ${}_nP_k$ and ${}_nC_k$ notation. Use $n^{\underline{k}}$ and $\binom{n}{k}$ instead.

Questions	1	2	3	4	5	6	7	8	Total
Points	10	20	10	15	10	10	11	14	100
Score									

Student ID: ______________

As part of HKUST's introduction of an honor code, the HKUST Senate has recommended that all students be asked to sign a brief declaration printed on examination answer books that their answers are their own work, and that they are aware of the regulations relating to academic integrity. Following this, please read and sign the declaration below.

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I declare that the answers submitted for
this examination are my own work.

I understand that sanctions will be
imposed, if I am found to have violated the
University regulations governing academic
integrity.

Student's Name:      __________________________

Student's Signature:    _____________________
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Student ID: ______________

Problem 1: [10 pts] Suppose that we have a student hall with 9 rooms labelled $A, B, \ldots, I$. How many ways are there to assign 10 students to the 9 hall rooms so that every student is assigned and no hall room is empty?

An *assignment* is a mapping which states, for each student, which room he/she will live in. Here are three *different* example assignments.

Assignment	Student									
	1	2	3	4	5	6	7	8	9	10
	Assigned to Room									
(1)	A	B	C	D	E	F	G	H	I	I
(2)	A	A	B	C	D	E	F	G	H	I
(3)	A	A	C	B	D	E	F	G	H	I

This is exactly the number of onto functions from S_{10} to S_9 (S_i, being a set with i elements).

As seen in the homeworks, we can solve this by first noticing that exactly one pair of students get assigned to the same room, while all the others get assigned an individual room. There are $\binom{10}{2}$ ways of choosing that pair. Once the pair is chosen consider the students as comprising 9 groups (8 groups of size 1 and one group of size 2). There are $9!$ ways of assigning the 9 groups to the rooms. So the answer is

$$\begin{aligned} \binom{10}{2} 9! &= \frac{9}{2} \cdot 10! \\ &= = 16,329,600 \end{aligned}$$

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Problem 2: [20 pts] Suppose you have 5 standard six-sided dice; one red, one green, one yellow, one black, one orange. Each die has the numbers 1-6 on its sides.

Roll all of the dice. An "outcome" of a roll is a number for each die. An outcome contains a
(i) *two-of-a-kind* if there are at least two dice showing the same number;
(ii) *three-of-a-kind* if there are at least three dice showing the same number;
(iii) *four-of-a-kind* if there are at least four dice showing the same number;
(iv) *five-of-a-kind* if all five dice show the same number.

The table below illustrates a few possible outcomes.

	Dice outcome					two of a kind	three of a kind	four of a kind	five of a kind
	R	**G**	**Y**	**B**	**O**				
(1)	1	1	1	1	1	$\surd$	$\surd$	$\surd$	$\surd$
(2)	5	1	5	5	5	$\surd$	$\surd$	$\surd$	X
(3)	1	3	1	1	3	$\surd$	$\surd$	X	X
(4)	1	3	3	1	1	$\surd$	$\surd$	X	X
(5)	1	2	3	4	5	X	X	X	X
(6)	1	2	6	2	2	$\surd$	$\surd$	X	X

(a) How many possible outcomes are there?

(b) How many possible outcomes contain a *five-of-a-kind*?

(c) How many possible outcomes contain a *four-of-a-kind* but no *five-of-a-kind*?

(d) How many possible outcomes contain a *three-of-a-kind* but no *four-of-a-kind*?

(e) How many possible outcomes contain a *three-of-a-kind* AND a *two-of-a-kind* that have different dice values but no *five-of-a-kind*?

As an example, outcomes (3) and (4) above would satisfy this, but none of the others would.

(a) $6^5 = 7776$

(b) 6

(c) *Each such outcome is uniquely determined by (i) the value of the four-of-a-kind, (ii) the color of the die that is not part of the four-of-a-kind and (iii) the value of that die. So the answer is*

$$6 \cdot 5 \cdot 5 = 150.$$

(d) *Each such outcome is uniquely determined by (i) the value of the three-of-a-kind, (ii) the colors of the dice that are not part of the three-of-a-kind and (iii) the values of those dice. So the answer is*

$$6 \cdot \binom{5}{2} \cdot 5^2 = 1500.$$

(e) *Each such outcome is uniquely determined by (i) the value of the three-of-a-kind, (ii) the colors of the dice that are not part of the three-of-a-kind and (iii) the values of those dice. So the answer is*

$$6 \cdot \binom{5}{2} \cdot 5 = 300.$$

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Problem 3: [10 pts] Is the following formula correct for all $n \geq 1$?

$$\sum_{i=0}^{n} 3^i \binom{n}{i} = 2^n \sum_{i=0}^{n} \binom{n}{i}.$$

If yes, prove it. If no, give a value of n for which the formula is incorrect.

The proof is that the both equal 4^n. *This can be derived (using the binomial theorem) by writing*

$$4^n = (3+1)^n = \sum_{i=0}^{n} 3^i 1^{n-i} \binom{n}{i}$$

and

$$4^n = 2^n 2^n = 2^n \sum_{i=0}^{n} \binom{n}{i}$$

where the equality $2^n = \sum_{i=0}^{n} \binom{n}{i}$ *was derived in class.*

Alternatively, the second equality can be derived as

$$4^n = (2+2)^n = \sum_{i=0}^{n} 2^i 2^{n-i} \binom{n}{i} = 2^n \sum_{i=0}^{n} \binom{n}{i}$$

Problem 4: [15 pts] A *permutation* of the letters A,B,C,D,E,F,G,H is a list (string) of the eight letters in some order.

A permutation *contains* the substring ABC if the letters appear in the given order. For example, permutations DEFABCGH and HGABCFED contain the substring ABC but ADBECFGH doesn't.

(a) How many *permutations* of the letters A,B,C,D,E,F,G,H contain the substring ABC?

(b) How many *permutations* of the letters A,B,C,D,E,F,G,H contain A before B and B before C?
For example, all of DEFABCGH, HGAFBCED and ADBECFGH contain the letters in the given order but CBADEFGH does not.

(a) *To solve this problem consider substring* 'ABC' *as an indivisible item and each of the other individual 5 letters as a separate indivisible item. Then the answer is the number of ways of writing a permutation of 6 items (the five letters and* 'ABC'*) which is*

$$6! = 720.$$

(b) *Such a permutation is uniquely defined by (i) choosing the locations of the three letters A,B,C (once the locations are chosen we know that 'A' must go in the first, 'B' in the 2nd and 'C', in the 3rd) and (ii) the locations of the five remaining letters in the 5 remaining spaces. This is therefore*

$$\binom{8}{3} \cdot 5! = \frac{8!}{3!} = 6720.$$

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Problem 5: [10 pts] An *anagram* is a distinct ordering of the letters. For example, the word "eat" has six anagrams "eat", "eta", "ate", "aet", "tea" and "tae", while the word "too" has only three anagrams "too", "oto" and "oot".

(a) How many anagrams does the word "mammal" have?

(b) How many anagrams does the word "mississippi" have?

(a) *This is the number of ways to label 6 items so that 3 are labeled 'm', 2 are labelled 'a' and 1 is labelled 'l' which is*

$$\binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = \frac{6!}{3! \cdot 2! \cdot 1!} = 60$$

(b) *This is the number of ways to label 11 items so that 1 is labeled 'm', 4 are labelled 'i', 4 are labelled 's' and 2 are labelled 'p' which is*

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2} = \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34650$$

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Problem 6: [10 pts] Give a combinatorial proof of the identity, for all $n \geq 9$,

$$\binom{n}{2}\binom{n-2}{3}\binom{n-5}{4} = \binom{n}{4}\binom{n-4}{3}\binom{n-7}{2}.$$

Note: An algebraic proof of this identity will *not* be accepted as a solution.

"Consider the problem of how to color n items so that 2 are red, 3 are green, 4 are blue and the remaining $n-9$ are yellow."

The left hand side of the inequality obviously counts this.

Notice that the problem is the same if we change the order in which we ask the question. "Consider the problem of how to color n items so that 4 are blue, 3 are green,2 are red and the remaining $n-9$ are yellow."

This is what the right hand side of the equation is counting, so the two sides are the same.

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Problem 7: [11 pts] Consider the following statement

$$gcd(j,k) = gcd(k-j, j).$$

Is this statement always true for k, j with $k \geq j \geq 0$?

Either *prove* that it is true for all k, j with $k \geq j \geq 0$, or give values for k, j with $k \geq j \geq 0$ such that $gcd(j,k) \neq gcd(k-j, j)$.

We will show that d is a common divisor of j and k if and only if d is a common divisor of $k-j$ and j. This shows that the set of common divisors of j and k is exactly the same as the set of common divisors of $k-j$ and j. This immediately implies that the greatest *common divisor of j and k is the same as the* greatest *common divisor of $k-j$ and j.*

To prove the if *direction suppose that d* **is a common divisor of j and k**. *Then, for some a and b*

$$j = a \cdot d \quad and \quad k = b \cdot d$$

so

$$k - j = (b-a)d$$

and d is a divisor of $k-j$. Thus d **is a common divisor of $k-j$ and j.**

To prove the only if *direction suppose that d* **is a common divisor of $k-j$ and j**. *Then, for some c and a*

$$j = a \cdot d \quad and \quad k - j = c \cdot d$$

so

$$k = (k-j) + j = (c+a)d$$

and d is a divisor of k. Thus d **is a common divisor of k and j.**

Problem 8: [14 pts]

(a) Consider the equation

$$3 \cdot_{100} x = 13$$

Does this equation have a solution x in Z_{100}?
If yes, give a value for x that solves the equation.
If no, prove that such an x does not exist.

(b) Consider the equation

$$15 \cdot_{100} x = 13$$

Does this equation have a solution x in Z_{100}?
If yes, give a value for x that solves the equation.
If no, prove that such an x does not exist.

Note that for both parts (a) and (b), if there does exist such an x then you only have to write the value *of x; it is* not necessary *to show how you derived it. It is to your benefit to do so, though. If you show your work and get a wrong answer for x we can give you some partial credit. If you don't show your work, partial credit is not possible.*

(a) *Yes.*
To start, first notice that $3 \cdot (-33) + 1 \cdot 100 = 1$. (This can either be derived using the extended GCD algorithm or just by observation.) This tells us that the multiplicative inverse of 3 in Z_{100} is $-33 \bmod 100 = 67$. So, the answer is

$$x = 67 \cdot_{100} 13 = 871 \bmod 100 = 71.$$

As a reality check, note that

$$3 \cdot_{100} 71 = 231 \bmod 100 = 13$$

so $x = 71$ really does solve our problem.

Comment: *Some students answered that $x = -29$ or $x = 171$. These answers were not awarded full points because $-29 \notin Z_{100}$ and $171 \notin Z_{100}$.*

(b) *No.*
This can be proven by contradiction. Suppose there was some solution x. Then $15 \cdot_{100} x = 13$ so there is some q such that $15 \cdot x = q \cdot 100 + 13$ or

$$15 \cdot x - q \cdot 100 = 13.$$

But the left hand side of this equation is divisble by 5 *while the right hand side is not, leading to a contradiction.*

Comment: *Some students answered that there is no solution because* $gcd(15, 100) = 5 \neq 1$. *The fact that* $gcd(15, 100) \neq 1$ *does imply that* $15 \cdot_{100} x = 1$ *does not have a solution; it does not imply that* $15 \cdot_{100} x = b$ *has no solution for all* $b \in Z_{100}$. *In fact, e.g.,* $15 \cdot_{100} x = 5$ *does have a solution. So this was not a correct answer.*