Tutorial 4: Combinatorial proofs and Functions

Department of Computer Science and Engineering
Hong Kong University of Science and Technology

Consider the identity:

$$\binom{n}{2}\binom{n-2}{4} = \binom{n}{4}\binom{n-4}{2}$$

An example,

$$\begin{aligned}\binom{10}{2}\binom{8}{4} &= 45 \cdot 70 \\ &= 3150 \\ &= 210 \cdot 15 \\ &= \binom{10}{4}\binom{6}{2}\end{aligned}$$

In the next slide we will see that is easy to prove this algebraically using the formal algebraic definition of $\binom{n}{k}$. But, what does this proof mean?

We will then see a combinatorial interpretation of the equation. This interpretation will provide a second, combinatorial proof.

An algebraic proof

$$
\begin{aligned}
\binom{n}{2}\binom{n-2}{4} &= \frac{n!}{2!(n-2)!}\,\frac{(n-2)!}{4!(n-6)!} \\
&= \frac{n!}{2!4!(n-6)!} \\
&= \frac{n!}{4!(n-4)!}\,\frac{(n-4)!}{2!(n-6)!} \\
&= \binom{n}{4}\binom{n-4}{2}
\end{aligned}
$$

Combinatorial proof

A *combinatorial* identity is proven by counting some carefully chosen object in two different ways to obtain two different expressions of the same statement.

An example

Consider the problem of choosing 2 co-chairmen and a 4-person executive advisory board from members of a 10-person club.

- There are $\binom{10}{2}$ ways to choose the 2 co-chairmen and $\binom{8}{4}$ ways of choosing the board from the remaining 10-2=8 people. This gives $\binom{10}{2}\binom{8}{4}$.

- Alternatively, we can first choose 4-person board and then the 2 co-chairmen from the remaining 10-4=6 people. This gives $\binom{10}{4}\binom{6}{2}$.

- We have counted the same thing two ways.
  Thus, $\binom{10}{2}\binom{8}{4} = \binom{10}{4}\binom{6}{2}$.

General expression

Consider original general statement

$$\binom{n}{2}\binom{n-2}{4} = \binom{n}{4}\binom{n-4}{2}$$

- Left side is the
  \# ways to choose 2 co-chairmen from n people
  times
  \# ways to choose 4-person board from remaining $n-2$ people
- Right side is the
  \# ways to choose 4-person board from n people
  times
  \# ways to choose 2 co-chairmen from remaining $n-4$ people
- Both side count # ways to choose 2 co-chairmen and 4-person board from n people so they are equal!

An even more general expression

For arbitrary a, b (before $a = 2, b = 4$)

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{b}\binom{n-b}{a}$$

Examples:

$$\begin{aligned}
\binom{n}{5}\binom{n-5}{3} &= \binom{n}{3}\binom{n-3}{5} \\
\binom{n}{1}\binom{n-1}{10} &= \binom{n}{10}\binom{n-10}{1} \\
\binom{n}{5}\binom{n-5}{1} &= \binom{n}{1}\binom{n-1}{5}
\end{aligned}$$

An even more general expression

For arbitrary a, b (before $a = 2, b = 4$)

$$\binom{n}{a}\binom{n-a}{b} = \binom{n}{b}\binom{n-b}{a}$$

- Left side is the
  \# ways to choose 'a' co-chairmen from n people
  times
  \# ways to choose 'b'-person board from remaining $n - a$ people
- Right side is the
  \# ways to choose 'b'-person board from n people
  times
  \# ways to choose 'a' co-chairmen from remaining $n - b$ people
- Both side count # ways to choose 'a' co-chairmen and 'b' person board from n people so they are equal!

One to One, Onto functions

Recall

$f : X \Rightarrow Y$ is one-to-one if,

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2)$$

$f : X \Rightarrow Y$ is onto if,

$$\text{for every } y \in Y \text{ there is at least one } x \in X \text{ such that } f(x) = y$$

There is usually no relationship between whether a function is one-to-one or onto.

A generic f can be one-to-one and not onto, onto and not one-to-one, both or neither.

One special exception to this rule is if $|X| = |Y|$.

In this case, f is one-to-one if and only if f is onto.

One-to-one and onto

Explain why a function from an n-element set to an n-element set is one-to-one if and only if it is onto.

"$\Rightarrow$" If f is one-to-one, it takes exactly n distinct values. Since the range has only n values f must be onto. Thus, a one-to-one function from an n-element set to an n-element set is onto.

"$\Leftarrow$" If f is onto, then f takes n distinct values because it maps onto a set of size n. But in this case, we may conclude that because there are only n values of x, all the values of $f(x)$ are different. Therefore, f must be one-to-one.

Inverse function

The function g is called an inverse to the function f if

1. the domain of g is the range of f,
2. $g(f(x)) = x$ for every x in the domain of f, and
3. $f(g(y)) = y$ for each y in the range of f.

Example: If the function

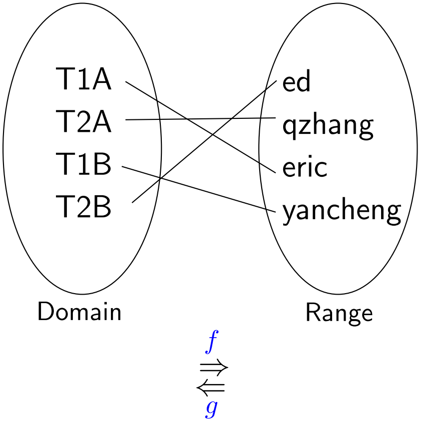
$$f : x \to 3x + 2,\ x \in \mathbb{R}$$

is given, then its inverse function is

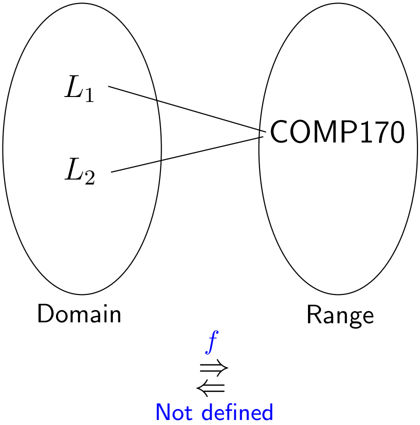
$$g : x \to \frac{x-2}{3},\ x \in \mathbb{R}$$

Another example

Invertible function



Non-invertible function



Inverse function and bijection

Proposition: a function $f : X \Rightarrow Y$ is a bijection if and only if f has an inverse function.

"$\Rightarrow$" If f is a bijection, we may define $g(y)$ to be the unique x such that $f(x) = y$. This defines a function because different values of y are related to different values of x. But then $f(g(y))$ is the result of applying f to the unique x with $f(x) = y$, so $f(g(y)) = y$. Also, $g(f(x))$ is the unique x that is related to $f(x)$, so $g(f(x)) = x$.

"$\Leftarrow$" If f has an inverse function, then f is onto.
The definition of one-to-one is that
if $f(x) = f(y)$ then $x = y$.
Now suppose that f has an inverse function g. If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$, which gives us $x = y$.
Therefore, if f has an inverse function, then f is one-to-one.

Inverse functions are unique

Proposition: a function that has an inverse function has only one inverse function.

Suppose g and h both satisfy the definition of being inverses to f. Suppose y is in the range of f and $g(y) = x$. Then,

$$f(g(y)) = f(x) \text{ and } h(f(g(y))) = h(f(x)) = x.$$

Because $f(g(y)) = y$, we have $h(y) = x$ as well.
Thus, for any y in the range of f, $h(y) = g(y)$, which means that g and h are equal. Thus, f has only one inverse function.