Tutorial 5: Counting

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Question 1

Consider a tennis club has $2n$ members.
We want to pair up the members (by twos) to play single matches.

(a) In how many ways can we pair up all the members of the club?

(b) Suppose that in addition to specifying who pairs up with whom, we also determine who serves first for each pairing. How many ways are there now to specify our pairs?

Question 1

Solution (a):

Sort the members in some order.

Take the first member and pair him with any of the other $2n-1$ members.

Take the next remaining member (in the given order) and pair him with any of the remaining $2n-3$ members.

Continue this way, at the ith step, choosing the next remaining member (in the order), then pairing him with any of the remaining $2n-(2i+1)$ members.

Note that every possible way of pairing the $2n$ members is generated exactly once by this procedure so it counts everything.

By the product principle, the number of ways is

$$\prod_{i=0}^{n-1}(2n-2i-1) = 1\cdot 3\cdot 5 \ldots (2n-3)\cdot(2n-1) = \frac{(2n)!}{n!2^n}.$$

Question 1

Solution (b):
For each pair that we have counted in part (a), there are two choices for who serves first. We can therefore multiply previous answer (as to how many pairings there are) by 2^n to get

$$2^n \frac{(2n)!}{n!2^n} = \frac{(2n)!}{n!}$$

Question 1

Alternative Solution:

We first consider the second part which specifies who plays first for each pairing.

The n players who play first in each pair form a set F of size n; the remaining n players (who play second) form another set S, also of size n.

Each pairing-up with designation of first player can be fully described by

(i) a partition F, S of the $2n$ into two sets of size n
(ii) a bijection from F to S.

If we are given a partition F, S and a bijection from F to S, then we can construct a unique pairing-up.

By bijection principle, the number of pairing-ups is the number of ways we can write a partition F, S and a bijection from F to S.

(i) $\#$ ways to partition $2n$ into F, S is $\binom{2n}{n}$.

(ii) Given part. F, S $\#$ of bijections from F to S is $n!$.

By product principle, total $\#$ of pairing-ups is

$$\binom{2n}{n} n! = \frac{(2n)!}{n!}$$

For the first part, note that each unordered pairing-up corresponds to 2^n ordered pairing-ups. So, total $\#$ of unordered pairing ups is $\frac{(2n)!}{2^n n!}$

Yet another alternative solution.

First, from the $2n$ members we have, choose 2 of them and put them on team 1.

Then, from the remaining $2n - 2$ members choose 2 of them on team 2.

Continue this way, at the i-th step, choose 2 of the remaining $2n - 2(i - 1)$ members and put them on team i.

Question 1

How many ways can we perform this process?

By the product principle, the number of ways is

$$
\begin{aligned}
& \binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{4}{2}\binom{2}{2} \\
= \quad & \frac{(2n)!}{2(2n-2)!}\frac{(2n-2)!}{2(2n-4)!}\frac{(2n-4)!}{2(2n-6)!}\cdots\frac{4!}{2(4-2)!}\frac{2!}{2(2-2)!} \\
= \quad & \frac{(2n)!}{2^n}
\end{aligned}
$$

Is this the answer to the original question?

No. We have a double counting problem!

Let's consider an example of the same problem with $n = 2$

- The tennis club has 4 people and we want to pair them up by twos).

Question 1

Suppose the members are a, b, c, d

Consider 2 possible solutions

- Solution 1: (a, b) (c, d)
- Solution 2: (c, d) (a, b)

They are the same kind of pairing but they are counted twice in the process (double counting), although in solution 1 (a, b) are paired up first while in solution 2 (c, d) are paired up first.

Question 1

How about $n = 3$?

Suppose $n = 3$ and we have members a, b, c, d, e, f

How may "double counting" do we have for the same kind of pairing (a, b) (c, d) (e, f)?

- Solution 1: (a, b) (c, d) (e, f)
- Solution 2: (a, b) (e, f) (c, d)
- Solution 3: (c, d) (a, b) (e, f)
- Solution 4: (c, d) (e, f) (a, b)
- Solution 5: (e, f) (a, b) (c, d)
- Solution 6: (e, f) (c, d) (a, b)

Answer: 6 or 3!.

3! is the way of ordering the 3 pairs.

Question 1

Generally, for $2n$ members, the counting process we introduced would produce $n!$ double counting.

- $n!$ is the way of ordering the n pairs

Our counting process counts

$$\frac{(2n)!}{2^n}$$

After eliminating the double counting, we have the solution to the tennis club problem

$$\frac{(2n)!}{n!2^n}$$

Question 2

A base ten number is a string of five digits, where the digits are from the set $\{0, 1, \ldots, 9\}$ but the first digit cannot be 0 (so 52375 is a valid number but 02323 and 2323 are not valid number).

(a) How many five-digit base ten numbers are there?

(b) How many five-digit base ten numbers with no two consecutive digits equal are there?

(c) How many five-digit base ten numbers with at least one pair of consecutive digits equal are there?

Question 2

Solution (a):
A "five-digit base ten number" means a string of five digits, where the first digit is not 0. Therefore, the first digit has 9 choices and the other digits have 10 choices. By the product principle, there are

$$9 \cdot 10^4 = 90000$$

such numbers.

Question 2

Solution (b):
If there are no two consecutive digits equal, then there are 9 choices for the first digit,
9 choices for the second digit (any digit other than the first one)
9 choices for the third digit (any digit other than the second one), and so on.
By the product principle, the total number is

$$9^5.$$

Solution (c):
By the sum principle, the total number of five-digit base ten numbers is the # of five-digit numbers that have no two consecutive digits equal plus the # of five-digit base ten numbers that have at least one pair of consecutive digits equal.
Let x denote the # of latter. Then

$$9 \cdot 10^4 = 9^5 + x$$

so,

$$x = 9 \cdot 10^4 - 9^5 = 30951.$$

Question 3

A basketball team has 12 players.
However, only 5 players play on the court at any given time during a game.

(a) How many ways can the coach choose the 5 players?

(b) More realistically, the 5 players playing in the game normally consist of

2 guards, 2 forwards, and 1 center.

If there are 5 guards, 4 forwards, and 3 centers on the team ,in how many ways can the coach choose the players on the court?

(c) What if one of the centers is equally skilled at playing forward and can be put in either position?

Question 3

Solution:

(a): This is problem of choosing a five-item subset from twelve items, so the answer is

$$\binom{12}{5}$$

(b): There are $\binom{5}{2}$ ways to choose the guards, $\binom{4}{2}$ ways to choose the forwards and $\binom{3}{1}$ ways to choose the center. By the product principle, the answer is

$$\binom{5}{2}\binom{4}{2}\binom{3}{1} = 180.$$

Solution:

(c) There are at least two ways of solving this. In the first, you partition based on whether the versatile player is playing center or not; if he is not, then he is available to play forward. This gives

$$\binom{5}{2}\binom{4}{2}\binom{1}{1} + \binom{5}{2}\binom{5}{2}\binom{2}{1} = 260$$

In the second, you partition based on whether the versatile player is playing forward or not; if he is not, then he is available to play center. This also gives

$$\binom{5}{2}\binom{4}{1}\binom{2}{1} + \binom{5}{2}\binom{4}{2}\binom{3}{1} = 260$$

Question 4

In a Cartesian coordinate system, how many paths are there from the origin to the point with integer coordinates (m, n) if the paths are built up of exactly $m + n$ horizontal and vertical line segments, each of length 1?

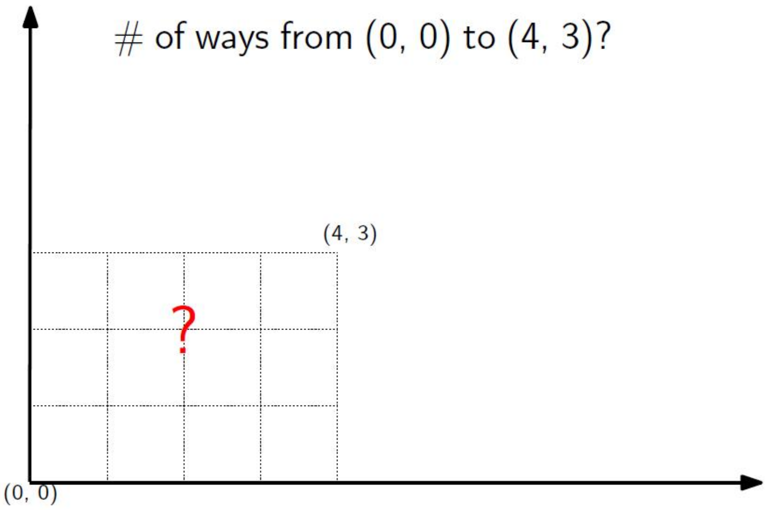
You should assume that all of the horizontal edges go from left to right and all the vertical edges from bottom to top. You should also assume that $m, n > 0$.

That is, you start at point $(0, 0)$ and each edge either goes

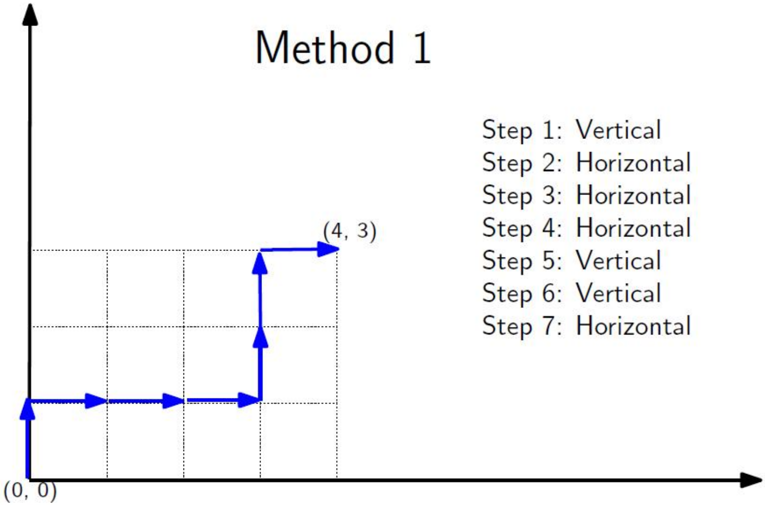
(i) vertically, from (i, j) to $(i, j + 1)$ or

(ii) horizontally, from (i, j) to $(i + 1, j)$.

Cartesian Coordinate Path Problem (Example)



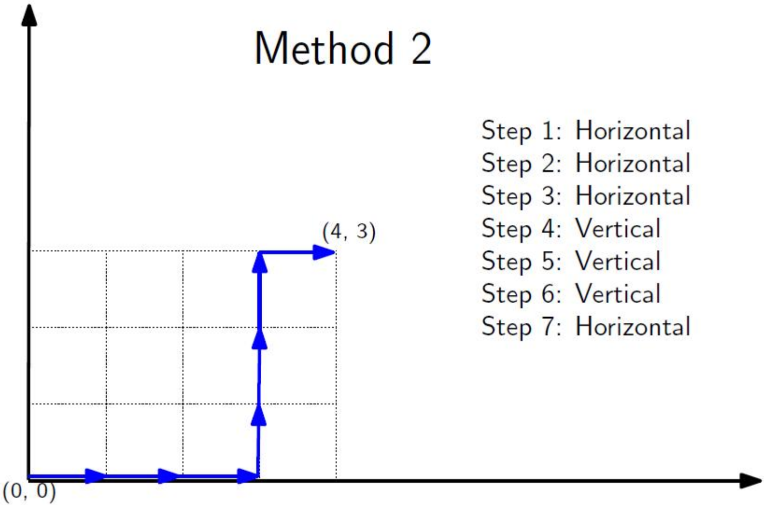
Cartesian Coordinate Path Problem (Example)



Step 1: Vertical
Step 2: Horizontal
Step 3: Horizontal
Step 4: Horizontal
Step 5: Vertical
Step 6: Vertical
Step 7: Horizontal

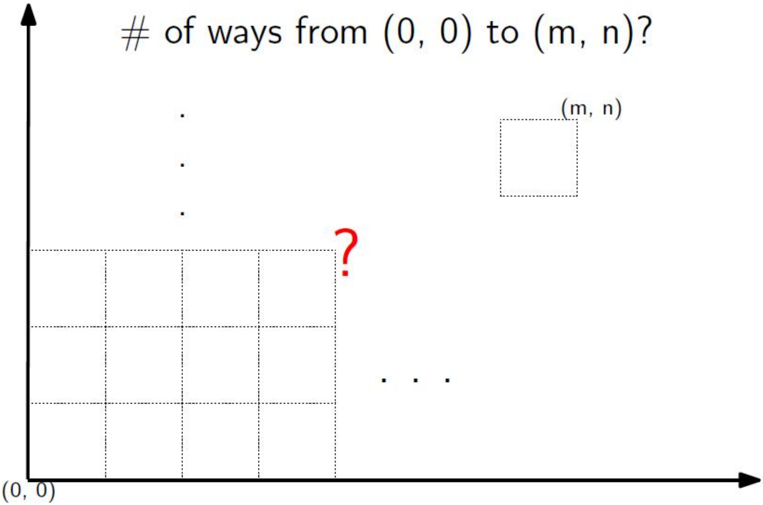
Cartesian Coordinate Path Problem (Example)

Method 2



Step 1: Horizontal
Step 2: Horizontal
Step 3: Horizontal
Step 4: Vertical
Step 5: Vertical
Step 6: Vertical
Step 7: Horizontal

Cartesian Coordinate Path Problem (Example)



Question 4

From any line segment on our path, we have two choices: horizontal (right) and vertical (up). Clearly, we need n vertical line segments and m horizontal line segments to reach (m, n) in exactly $m + n$ steps.

By deciding on our choice of n vertical lines, we have also decided the choice of m horizontal lines, because we must choose a horizontal line for each step that is not a vertical line. We could also start by choosing m horizontal lines, which automatically decides the choice of n vertical lines.

Thus, the number of ways to choose the path is the number of ways to choose the n places for vertical lines from $m + n$ places or the m places for horizontal lines from $m + n$ places.

Thus, the answer is $\binom{m+n}{m}$ or $\binom{m+n}{n}$.