Propositional Logic

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Resolution Rule of Inference

Given two clauses, infer a new clause:
from clauses $\{p\} \vee C_1$ and $\{\neg p\} \vee C_2$
infer clause $C_1 \vee C_2$. This new clause is called the **resolvent** of input clauses with respect to p.
Example: from clauses $[w, p, q]$ and $[w, s, \neg p]$ infer $[w, q, s]$ as resolvent wrt p.
Special case: $[p]$ and $[\neg p]$ resolve to $[\,]$
A derivation of a clause c from a set S of clauses is a sequence $c_1, c_2, \cdots, c_n$ of clauses, where the last clause $c_n = c$, and for each c_i , either

1. $c_i \in S$ or
2. c_i is a resolvent of two earlier clauses in the derivation

Write: $S \rightarrow c$ if there is a derivation of c from S.

Properties

Theorem

Resolvent is entailed by input clauses:

$$\{(p \vee \alpha, \neg p \vee \beta)\} \models \alpha \vee \beta$$

Proof: See the lecture notes

Notice: if $S \models T$, denote $A = \{M|M \textit{ satisfies } S\}$ and $B = \{M|M \textit{ satisfies } T\}$, then $A \subseteq B$.

- if $S \rightarrow [\,]$, let $A = \{M|M \textit{ satisfies } S\}$, $B = \{M|M \textit{ satisfies } [\,]\}$. Then $B = \emptyset$ and $A \subseteq B$, so $A = \emptyset$. So, S has got no model, i.e. S is inconsistent.

Proof by Resolution Refutation

Theorem:

If $KB \cup \{[\neg g]\} \rightarrow [\,]$, then $KB \models g$

Proof:(by refutation)

Given $KB \cup \{[\neg g]\} \rightarrow [\,]$, $KB \cup \{[\neg g]\}$ has got no model. (1)

Assume that $KB \not\models g$, i.e. $\exists M.(I_M(KB) = 1 \wedge I_M(g) = 0)$

Then $I_M(\neg g) = 0$, i.e. $I_M(KB \cup \{[\neg g]\}) = 1$.

Thus, M is a model of $KB \cup \{[\neg g]\}$, which is inconsistent with (1).

Proved.

Exercise 1

Heads, I win; tails, you lose. Express these statements (plus other statements you might need) in propositional calculus, and then use resolution to prove that I win.

Solution 1 - Proof by refutation

Statements:

$Head \supset IWin$

$Tail \supset Ylose$

$Head \vee Tail$

$YLose \supset IWin$

The clausal form:

1. $\neg Head \vee Iwin$
2. $\neg Tail \vee YLose$
3. $Head \vee Tail$
4. $\neg YLose \vee IWin$

Resolution proof:

5. $Head \vee YLose$	resolving 2 with 3
6. $YLose \vee IWin$	resolving 5 with 1
7. $IWin$	resolving 6 with 4
8. $\neg IWin$	negation of goal
9. [ ]	resolving 7 with 8

Exercise 2 - A Lady or a Tiger?

In this puzzle a prisoner is faced with a decision where he must open one of two doors. Behind each door is either a lady or a tiger. There might be two tigers, two ladies or one of each. If the prisoner opens a door and finds a lady he will marry her and if he opens a door and finds a tiger he will be eaten alive. Of course, the prisoner would prefer to be married than eaten alive. Each of the doors has a sign bearing a statement that may be either true or false.

The statement on door I says, "**In this room is a lady, and in the other room is a tiger.**" The statement on door II says, "**In one of these rooms is a lady, and in one of these rooms is a tiger.**" The prisoner is informed that one of the two statements is true and one is false. Which door should he open?

Solution 2 - A Lady or a Tiger?

Since if the statement on door I is true, then the statement on door II is also true, which contradicts to the information that one of the two statements is true and one is false, it can be concluded that the statement on door I is false.

More interesting variations of the "Lady or Tiger" problem can be found at click here or go to http://clasen.blogspot.hk/2012_02_01_archive.html

Solution 2 - Proof by Propositional Logic

KB:

$S1 \equiv L1 \wedge T2$ Rule 1

$S2 \equiv (L1 \vee L2) \wedge (T1 \vee T2) \equiv (L1 \wedge T2) \vee (L2 \wedge T1)$ Rule 2

$(S1 \wedge \neg S2) \vee (\neg S1 \wedge S2)$ Rule 3

Method 1:

Rule 2 gives Rule 4: $\neg((L1 \wedge T2) \vee (L2 \wedge T1)) \vee S2$,

Simplifying R4 gives R5: $\neg(L1 \wedge T2) \vee S2$

R1 gives R6: $\neg S1 \vee (L1 \wedge T2)$

Resolving R5 with R6 gives R7: $S2 \vee \neg S1$ or equivalently, $\neg(S1 \wedge \neg S2)$

Resolving R7 with R3 gives: $\neg S1 \wedge S2$

Method 2:

Simplifying (Rule 1 and Rule 2) gives:

$(\neg S1 \vee (L1 \wedge T2)) \wedge (\neg S2 \vee (L1 \wedge T2) \vee (L2 \wedge T1))$

$= \neg S1 \neg S2 \vee \neg S1(L1T2 \vee L2T1) \vee (L1T2 \wedge \neg S2) \vee (L1T2(L1T2 \vee L2T1))$

$= (\neg S1 + L1T2) \wedge (L1T2 + L2T1)$

Hence the second term implies that S2 is true.