COMP 2711 Discrete Mathematical Tools for CS
2014 Fall Semester – Assignment # 7
Distributed: 13 November 2014 – Due: 5pm, 20 November 2014
Solutions

At the top of your solution, please write your (i) name, (ii) student ID #, (iii) email address and (iv) tutorial section.

Problem 1: Use a truth table to prove the DeMorgan's law that states $\neg(p \wedge q) = \neg p \vee \neg q$.

SOLUTION: Use a double truth table.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The columns under $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are identical, so the two statements are equivalent.

Problem 2: Which of the following statements (in which Z^+ stands for the positive integers and Z stands for all integers) is true and which is false? Don't forget to explain why.
a) $\forall z \in Z^+ \, (z^2 + 5z + 10 > 17)$
b) $\forall z \in Z \, (z^2 - z \geq 0)$
c) $\exists z \in Z^+ \, (z - z^2 > 0)$
d) $\exists z \in Z \, (z^2 - z = 6)$

SOLUTION: a) False, because $1^2 + 5 \cdot 1 + 10 = 16$.
b) There are many different ways of proving this. One way is to consider the following cases:

1. $z < 0$: true, because $z^2 - z \geq 0 \Leftrightarrow z^2 \geq z$. The LHS is positive, the RHS is negative.
2. $z = 0$: true by substitution ($0^2 - 0 \geq 0$).
3. $z > 0$: true, because $z^2 - z \geq 0 \Leftrightarrow z^2 \geq z$, so we get $z \geq 1$ which holds in this case.

c) False. To prove that it is false we want to prove its negation

$$\neg \exists z \in Z^+ \, (z^2 - z > 0). \tag{1}$$

Equivalently, we want to prove

$$\forall z \in Z^+ \, (z^2 - z \not> 0) \tag{2}$$

We can re-write it as follows.

$$\forall z \in Z^+ \,(z - z^2 \leq 0). \tag{3}$$

But, multiplying the inequality in (**??**) by -1 gives us $(z^2 - z \geq 0)$, which was already proved in part (b) (for an even bigger universe).

d) True, because $(-2)^2 - (-2) = 6$.

Problem 3: Show that the statements $s \Rightarrow t$ and $\neg s \vee t$ are equivalent using a truth table.

SOLUTION: Give the truth table for $s \Rightarrow t$ and $\neg s \vee t$, and compare them. They are the same.

s	t	$s \Rightarrow t$	$\neg s \vee t$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Problem 4: (Distributive "Laws")
(a) Is $w \wedge (u \oplus v)$ equivalent to $(w \wedge u) \oplus (w \wedge v)$?
(b) Is $w \vee (u \oplus v)$ equivalent to $(w \vee u) \oplus (w \vee v)$?

SOLUTION: (a) $\wedge$ distributes over $\oplus$. Compare the truth tables of $w \wedge (u \oplus v)$ and $(w \wedge u) \oplus (w \wedge v)$.

w	u	v	$w \wedge u$	$w \wedge v$	$(w \wedge u) \oplus (w \wedge v)$	$(u \oplus v)$	$w \wedge (u \oplus v)$
T	T	T	T	T	F	F	F
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	F	F	F
F	T	F	F	F	F	T	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

(b) $\vee$ doesn't distribute over $\oplus$. Try $w = T, u = T$, and $v = T$, where T stands for a statement that is always true, to get the value of $w \vee (u \oplus v)$ and $(w \vee u) \oplus (w \vee v)$.
$T \vee (T \oplus T) = T \neq F = (T \vee T) \oplus (T \vee T)$.

Problem 5: (a) Construct a contrapositive proof that for all real numbers x, if $x^2 - 2x \neq 3$ then $x \neq 3$.
(b) Construct a proof by contradiction that for all real numbers x, if $x^2 - 2x \neq 3$ then $x \neq 3$.

SOLUTION: Note that these are *possible* solutions, not the *only* solutions.

Even though both parts are proving the same thing, they are asking you to use different proof techniques. Essentially, they are both asking for a proof of "$p(x) \Rightarrow q(x)$" where
$p(x) =$ "$x^2 - 2x \neq 3$".
$q(x) =$ "$x \neq 3$".

A correct solution to (a) must prove this by proving the *contrapositive* "$\neg q(x) \Rightarrow \neg p(x)$". A correct solution to (b) proves "$p(x) \Rightarrow q(x)$" by assuming $p(x)$ and assuming $\neg q(x)$ and showing that this implies the correctness of both $r(x)$ and $\neg r(x)$ for some statement $r(x)$ (a contradiction).

(a)
"$\neg q(x) \Rightarrow \neg p(x)$" is the **contrapositive** of "$p(x) \Rightarrow q(x)$".
"$p(x) \Rightarrow q(x)$" is actually equivalent to "$\neg q(x) \Rightarrow \neg p(x)$".

Suppose that $x = 3$ ($\neg q(x)$).
Then $x^2 - 2x = 9 - 6 = 3$ ($\neg p(x)$).
Thus, if $x = 3$, then $x^2 - 2x = 3$ ($\neg q(x) \Rightarrow \neg p(x)$).
By contrapositive inference, if $x^2 - 2x \neq 3$, then $x \neq 3$ ($p(x) \Rightarrow q(x)$).

(b)
If by assuming $p(x)$ and $\neg q(x)$,
we can derive both $r(x)$ and $\neg r(x)$ for some statement $r(x)$,
we may conclude $p(x) \Rightarrow q(x)$.

Suppose that $x^2 - 2x \neq 3$ ($p(x)$).
Suppose as well that $x = 3$ ($\neg q(x)$).
Then $x^2 - 2x = 9 - 6 = 3$ ($\neg p(x)$),
which is a contradiction ($r(x) = p(x)$).
Therefore, if $x^2 - 2x \neq 3$, then $x \neq 3$ ($p(x) \Rightarrow q(x)$).