COMP 2711 Discrete Mathematical Tools for CS
Written Assignment # 3
Distributed: 26 September 2014 – Due: 03 Octorber 2014
Solution Key

At the top of your solution, please write your (i) name, (ii) student ID #, (iii) email address and (iv) tutorial section.

<u>Some Notes:</u>

- Please write clearly and briefly. For all questions you should also provide a short explanation as to *how* you derived the solution. That is, if the solution is 20, you shouldn't just write down 20. You need to explain *why* it's 20.
- Please follow the guidelines on doing your own work and avoiding plagiarism given on the class home page. Don't forget to *acknowledge* individuals who assisted you, or sources where you found solutions.
- Some of these problems are taken (some modified) from the textbook.
- Please make a *copy* of your assignment before submitting it. If we can't find your paper in the submission pile, we will ask you to resubmit the copy.
- Your solutions should be submitted before 5PM of the due date, in the collection bin near Room 4210A (Lift 21).

Problem 1: The eight kings and queens are removed from a deck of cards, and then two of these cards are selected (from the eight). What is the probability that the king or queen of spades is among the cards selected?

Solution:
Let K_{spades} be the event that the king of spades is selected and Q_{spades} be the event that the queen of spades is selected. We want to find $P(K_{spades} \cup Q_{spades})$.
The total number of ways of selecting two cards from eight cards is $\binom{8}{2}$. We are using the space of two-element subsets as our model.
The king of spades can be selected in 1×7 ways and it is the same for the queen of spades, i.e.,

$$P(K_{spades}) = P(Q_{spades}) = \frac{1 \times 7}{\binom{8}{2}} = \frac{1}{4}$$

The probability that both of them are selected is:

$$P(K_{spades} \cap Q_{spades}) = \frac{1}{\binom{8}{2}} = \frac{1}{28}.$$

$$\begin{aligned} P(K_{\text{spades}} \cup Q_{\text{spades}}) &= P(K_{\text{spades}}) + P(Q_{\text{spades}}) \\ &\quad - P(K_{\text{spades}} \cap Q_{\text{spades}}) \\ &= \frac{1}{4} + \frac{1}{4} - \frac{1}{28} = \frac{13}{28}. \end{aligned}$$

Problem 2: Calculate

$$\sum_{\substack{i_1,i_2,i_3: \\ 1\le i_1<i_2<i_3\le 5}} i_1 \cdot i_2 \cdot i_3$$

Solution:
We consider the triples that satisfy the condition.

i_1	i_2	i_3	$i_1 \cdot i_2 \cdot i_3$	i_1	i_2	i_3	$i_1 \cdot i_2 \cdot i_3$
1	2	3	6	1	4	5	20
1	2	4	8	2	3	4	24
1	2	5	10	2	3	5	30
1	3	4	12	2	4	5	40
1	3	5	15	3	4	5	60

The result is: 225

Problem 3 In this problem, a *black card* is a spade or a club.
Remove one card from an ordinary deck of cards. What is the probability that it is an ace, a diamond, or black? Use the inclusion-exclusion formula to solve this problem.

Solution:
Let $E_1 = E_{ace}$ be the event that it is ace.
Let $E_2 = E_{diamond}$ be the event that it is diamond.
Let $E_3 = E_{black}$ be the event that it is black.

$$\begin{aligned} P(E_1) &= P(E_{ace}) &= 1/13 \\ P(E_2) &= P(E_{diamond}) &= 1/4 \\ P(E_3) &= P(E_{black}) &= 1/2 \end{aligned}$$

$$P(E_1 \cap E_2) = 1/52$$
$$P(E_1 \cap E_3) = 1/26$$

$$P(E_2 \cap E_3) = P(E_1 \cap E_2 \cap E_3) = 0$$

Now use the inclusion-exclusion formula,

$$\begin{aligned}
P(E_1 \cup E_2 \cup E_3) &= \sum_{i=1}^{3} P(E_i) - \sum_{1\le i<j\le 3} P(E_i \cap E_j) + P(E_1 \cap E_2 \cap E_3) \\
&= \left(\frac{1}{13} + \frac{1}{4} + \frac{1}{2}\right) - \left(\frac{1}{52} + \frac{1}{26} + 0\right) + 0 \\
&= 10/13
\end{aligned}$$

Problem 4: In this exercise you will solve the following problem:
If you roll eight dice, what is the probability that each of the numbers 1 through 6 appears on top at least once?

For $1 \le i \le 6$, let E_i be the event that number i doesn't show up on any of the dice.

(a) Write a formula for $P(E_i)$.

(b) Let $k \le 6$ and $1 \le i_1 < i_2 < \cdots < i_k \le 6$.
Write a formula for $P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k})$.

(c) Now use the inclusion-exclusion formula to write a formula for $P(E_1 \cup E_2 \cup \cdots \cup E_6)$.
This is the probability that *some* number doesn't appear when you roll eight die.
Your formula should use the summation sign, powers and binomial coeficients.

(d) Using the solution to (c), write down the probability that each of the numbers 1 through 6 appears on top at least once (a solution in the form of a sum is fine; it is not necessary to actually calculate the value of the sum).

Solution:

(a) $P(E_i) = (5/6)^8$, $1 \le i \le 6$.

(b) We can see that $P(E_1) = (5/6)^8$, $P(E_1 \cap E_2) = (4/6)^8$, and, in general,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \left(\frac{6-k}{6}\right)^8.$$

(c) According to the inclusion-exclusion formula,

$$P(E_1 \cup E_2 \cup \cdots \cup E_6) = p = \sum_{k=1}^{6} (-1)^{k+1} \binom{6}{k} \left(\frac{6-k}{6}\right)^8$$

(d) Thus, the final result for the original problem is:

$$1-p = 1-\sum_{k=1}^{6}(-1)^{k+1}\binom{6}{k}\left(\frac{6-k}{6}\right)^8 = \sum_{k=0}^{6}(-1)^k\binom{6}{k}\left(\frac{6-k}{6}\right)^8 = \frac{665}{5832}$$

Problem 5: In this exercise you will solve the following problem:
If you are hashing n keys into a hash table with k locations (buckets), what is the probability that every location gets at least one key? This probability can be expressed as a formula using the summation ($\sum$) symbol.

Hint: To solve this problem let E_i be the event that bucket E_i is empty. Then $E_1 \cup E_2 \cup \cdots \cup E_k$ is the event that at least one bucket is empty.

Let X be the event that every bucket gets at least one key. *Then X is the complement of $E_1 \cup E_2 \cup \cdots \cup E_k$ and the problem is asking you to find*

$$P(X) = 1 - P(E_1 \cup E_2 \cup \cdots \cup E_k).$$

You can now use the inclusion-exclusion formula to find $P(E_1 \cup E_2 \cup \cdots \cup E_k)$.

Solution:

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{j=1}^{k}(-1)^{j+1} \sum_{\substack{i_1, i_2, \dots i_j: \\ 1 \le i_1 < i_2 < \cdots < i_j \le k}} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_j}).$$

We can see that

$$P(E_1) = \frac{(k-1)^n}{k^n}, \quad P(E_1 \cap E_2) = \frac{(k-2)^n}{k^n}$$

and, in general,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_j}) = \frac{(k-j)^n}{k^n}.$$

There are $\binom{k}{j}$ subsets of size j so

$$P(X) = 1 - \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{(k-j)^n}{k^n} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(k-j)^n}{k^n}$$

Note that the last term in the sum, when $j = k$, is equal to zero, so an equally valid answer would be

$$P(X) = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\frac{(k-j)^n}{k^n}$$

Problem 6: Six married couples (i.e., 12 people) sit down at random in a row of 12 seats. That is, each one of the 12! different ways of seating the people is equally likely to occur.

We say that a couple *sits together* if the husband and wife in that couple sit next to each other.

In the following, you may express your answers using the summation ($\sum$) sign, binomial coefficients $\left(\binom{n}{m}\right)$, factorials ($n!$) and exponentials ($c^k$). Actual numerical solutions are not necessary.

(a) Consider two specific couples. The first couple c_1 is Peter and Mary and the second couple c_2 is John and Helen. Since Peter and John are friends, they want to sit next to each other. What is the probability that each of these two couples (i.e., c_1 and c_2) sits together, and Peter and John sit next to each other?

(b) What is the probability that every couple sits together?

(c) What is the probability that no couple sits together?

Solution

(a) There are totally 12! ways to seat the 6 couples.

Since each of these two couples sits together, and Peter and John sit next to each other, we treat the two couples as one single unit involving the four persons in these two couples. We can randomly permute the 9 *units in* 9! *different ways. For each permutation, there are* 2 *ways to seat the two couples (i.e., (Mary, Peter, John, Helen) and (Helen, John, Peter, Mary)). Therefore, the probability that each of these two couples sits together, and Peter and John sit next to each other is*

$$\frac{9! \cdot 2}{12!} = 0.001515$$

(b) There are totally 12! ways to seat the 6 couples.

If a couple sits together, we treat it as one single unit. There are 6! ways of permuting 6 single units. For each permutation, there are 2^6 *ways to seat the* 6 *bound couples. Therefore, the probability that every couple sits together is*

$$\frac{6! \cdot 2^6}{12!} = 0.0000962$$

(c) There are totally 12! ways to seat the 6 couples.

If a couple sits together, we treat it as one single unit. Thus, for the k *specified couples to sit together, we can randomly permute the* $(12-k)$

units in $(12-k)!$ *different ways. For each permutation, there are* 2^k *ways to seat the* k *bound couples. Therefore, the probability that* $k(1 \le k \le 6)$ *specified couples end up sitting together (regardless of whether the other* $6-k$ *couples sit together or not) is*

$$\frac{(12-k)! \cdot 2^k}{12!}$$

Let E_i *denote the event that the* i*-th couple sits together. The probability that at least one couple sits together can be computed using the inclusion-exclusion principle as*

$$P(\cup_{i=1}^{6} E_i) = \sum_{k=1}^{6} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k \\ 1 \le i_1 < i_2 < \ldots < i_k \le 6}} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k})$$

Thus, we have

$$P(\cup_{i=1}^{6} E_i) = \sum_{k=1}^{6} (-1)^{k+1} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!}$$

Thus, the probability that no couple sits together is

$$\begin{aligned} 1 - P(\cup_{i=1}^{6} E_i) &= 1 - \sum_{k=1}^{6} (-1)^{k+1} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!} \\ &= \sum_{k=0}^{6} (-1)^{k} \binom{6}{k} \frac{(12-k)! \cdot 2^k}{12!} \end{aligned}$$