COMP 3721: Theory of Computation
Spring 2012 Final Exam

1. Print your name and student ID at the top of every page (in case the staple falls out!).
2. This is an open-book, open-notes, open-brain exam.
3. Time limit: 180 minutes.
4. You should answer all the questions on the exam. At least you should read all the questions.
5. When asked to describe a Turing machine, you can use either pseudocode or plain language, just like how you would describe an algorithm. You may assume the most convenient variant of the TM, unless you are explicitly told otherwise.
6. You can write on the back of the paper if you run out of space. Please let us know if you need more scratch paper.
7. Now, take a deep breath.

1. (18 pts) Assuming $\mathsf{P} \neq \mathsf{NP}$ and $\mathsf{NP} \neq \mathsf{coNP}$, put the following languages into the correct places in the diagram below.

   (a) $\{\phi : \phi$ is a Boolean formula that is satisfiable$\}$.

   (b) $\{\phi : \phi$ is a Boolean formula that is not satisfiable$\}$.

   (c) $H_1 = \{$"M" : Turing machine M halts on input string "M"$\}$.

   (d) $\overline{H_1}$.

   (e) $\{a^n b^m a^n b^m : n, m \geq 0\}$.

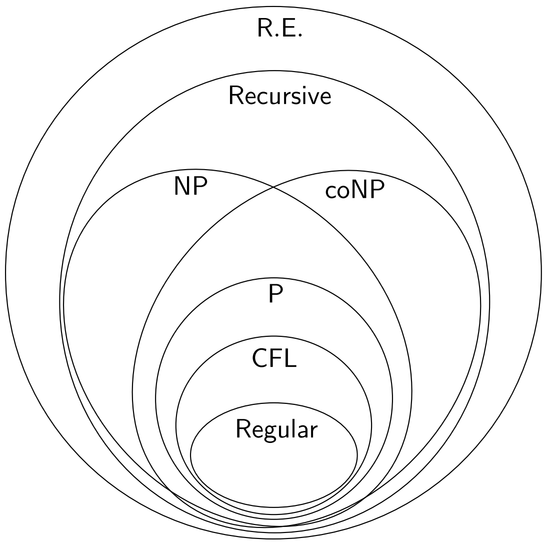
   (f) $\{a^{n+1} b^n : n \geq 0\}$.

   In the following languages, $D(x)$ denotes the set of all prefixes of the decimal representation of x. For example, $D(123) = \{1, 12, 123\}$, $D(5/3) = \{1, 1., 1.6, 1.66, 1.666, \dots\}$, $D(\pi) = \{3, 3., 3.1, 3.14, 3.141, \dots\}$.

   (g) $D(2^{32})$.

   (h) $D(1/7)$.

   (i) $D(\sqrt{2})$.



2. (4 pts) Continuing with the question above, is there a real number x such that $D(x)$ is outside R.E.? Give a brief justification.

3. (8 pts) For a nonnegative integer n, let $B(n)$ be the reverse binary presentation of n, for example:

   (a) $B(0) = 0$;

   (b) $B(3) = 11$;

   (c) $B(10) = 0101$.

   Now design a one-tape Turing Machine, which reads $B(n)$ on the tape, outputs $B(n+1)$ on the tape and then halts. You may either use the graphical notation of Turing machines or give the definition in the form of $M = (K, \Sigma, \delta, s, H)$. Suppose initially the tape is in the format of $\triangleright\underline{\sqcup}B(n)\sqcup$.

4. (10 pts) **Multiple Choice**

   For each of the following statements, indicate whether it is (a) true, (b) false, or (c) unknown based on our current scientific knowledge. You do not need to justify your answers.

   (a) No NP-complete problem can be solved in polynomial time.

   (b) $\mathsf{NP} = \mathsf{P} \cup \mathsf{NPC}$.

   (c) All problems in NP can be solved in exponential time, i.e., in time $O(2^{p(n)})$ where $p(n)$ is a polynomial of n.

   (d) If we can prove that SAT needs at least $\Omega(2^n)$ time to solve, then no NP-complete problem can be solved in polynomial time.

   (e) If we can prove that SAT needs at least $\Omega(2^n)$ time to solve, then any NP-complete problem needs at least $\Omega(2^n)$ time to solve.

5. (10 pts) Prove that the following problem is undecidable. Given two Turing machines M_1 and M_2, is $L(M_1) \subseteq L(M_2)$?

6. (20 pts) Let $\mathcal{N}$ be the set of all natural numbers. Recall that a Turing machine can also be used to compute a function $f : \mathcal{N} \rightarrow \mathcal{N}$. More precisely, assuming the alphabet $\Sigma = \{\triangleright, \sqcup, 0, 1\}$, we say that a Turing machine M *computes* f if for any $n \in \mathcal{N}$, when M is given input n (in binary form), the output of M is $f(n)$ (in binary form).

   (a) (5 pts) Using a counting argument, show that there exists a function $f : \mathcal{N} \rightarrow \mathcal{N}$ that is not computable, i.e., there is no Turing machine that computes f.

(b) (15 pts) We now define a concrete function that is not computable. Let $f(n)$ be the maximum number of steps a Turing machine with n states (besides the halting state) can run before it halts, when started on an empty tape. For example, $f(1) = 3$, with the following Turing machine achieving this. Note that the Turing machine must halt (so it cannot keep changing 0 to 1 and 1 back to 0, or moving to the right indefinitely).

δ	$\triangleright$	$\sqcup$	0	1
q_0	$(q_0, \rightarrow)$	$(q_0, 0)$	$(q_0, 1)$	$(h, 1)$

This function is clearly well defined. Show that it is not computable.

7. (15 pts) Suppose someday someone shows that SAT is in coNP. What will be the consequences?

   (a) (10 pts) Show that we will have $\mathsf{NP} \subseteq \mathsf{coNP}$.

(b) (5 pts) Show that we will also have $\mathsf{coNP} \subseteq \mathsf{NP}$, thus $\mathsf{NP} = \mathsf{coNP}$. [You may prove this part assuming (a) is true, even if you cannot prove (a).]

8. (15 pts) The decision version of Integer Programming is the following problem: Given n unknowns $x_1, \ldots, x_n$ and m inequalities

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2, \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m,
\end{aligned}$$

where the a_{ij}'s and b_j's are integers (can be either positive, negative, or zero), we want to decide whether there is a solution in which the x_{ij}'s are either 0 or 1 and satisfy all inequalities. Show that this problem is NP-complete. [Hint: Reduce from 3SAT.]