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LECTURE 18: LINK ANALYSIS: PAGERANK AND HITS

COMP4641: Social Information Network Analysis and Engineering
Wednesday May 6th 2015

How to Organize the Web?

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- **How to organize the Web?**
- **First try:** Human curated **Web directories**
  - Yahoo, DMOZ, LookSmart
- **Second try: Web Search**
  - **Information Retrieval** investigates: Find relevant docs in a small and trusted set
    - Newspaper articles, Patents, etc.
  - **But:** Web is **huge**, full of untrusted documents, random things, web spam, etc.


Web Search: 2 Challenges

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2 challenges of web search:

- **(1) Web contains many sources of information Who to "trust"?**
  - **Trick:** Trustworthy pages may point to each other!
- **(2) What is the "best" answer to query "newspaper"?**
  - No single right answer
  - **Trick:** Pages that actually know about newspapers might all be pointing to many newspapers

Ranking Nodes on the Graph

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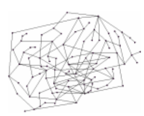
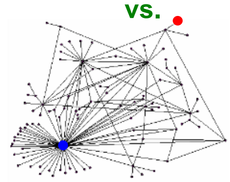
- **All web pages are not equally "important"**
  www.joe-schmoe.com vs. www.mit.edu
- **We already know:** There is large diversity in the web-graph node connectivity. **Let's rank the pages by the link structure!**



vs.



Link Analysis Algorithms

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- **We will cover the following Link Analysis approaches** to compute importances of nodes in a graph:
  - Hubs and Authorities (HITS)
  - Page Rank

Sidenote: Various notions of **node centrality: Node u**
- **Degree centrality** = degree of u
- **Betweenness centrality** = #shortest paths passing through u
- **Closeness centrality** = avg. length of shortest paths from u to all other nodes of the network
- **Eigenvector centrality** = like PageRank

Link Analysis

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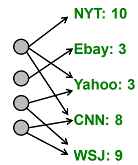
- **Goal** (back to the newspaper example):
  - Don't just find newspapers. Find "experts" – pages that link in a coordinated way to good newspapers
- **Idea: Links as votes**
  - **Page is more important if it has more links**
    - In-coming links? Out-going links?
- **Hubs and Authorities**
  Each page has 2 scores:
  - **Quality as an expert (hub):**
    - Total sum of votes of pages it pointed to
  - **Quality as an content (authority):**
    - Total sum of votes of experts
  - Principle of repeated improvement



Hubs and Authorities

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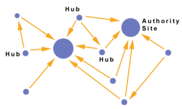
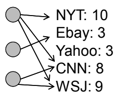
Interesting pages fall into two classes:

1. **Authorities** are pages containing useful information
   - Newspaper home pages
   - Course home pages
   - Home pages of auto manufacturers



2. **Hubs** are pages that link to authorities
   - List of newspapers
   - Course bulletin
   - List of US auto manufacturers



Counting in-links: Authority

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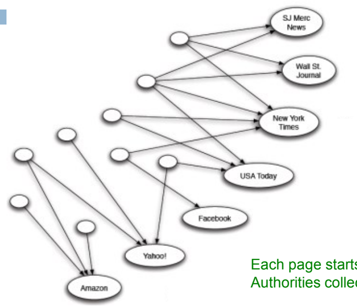

Each page starts with hub score 1
Authorities collect their votes

(Note this is idealized example. In reality graph is not bipartite and each page has both the hub and authority score)

Expert Quality: Hub

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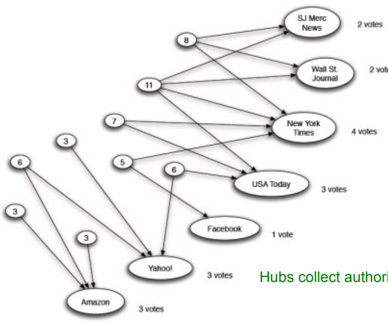

Hubs collect authority scores

(Note this is idealized example. In reality graph is not bipartite and each page has both the hub and authority score)

Reweighting

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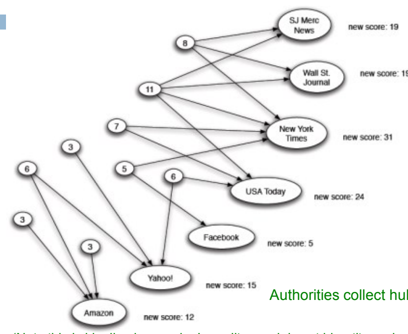

Authorities collect hub scores

(Note this is idealized example. In reality graph is not bipartite and each page has both the hub and authority score)

Mutually Recursive Definition

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- **A good hub links to many good authorities**
- **A good authority is linked from many good hubs**
- **Model using two scores for each node:**
  - **Hub** score and **Authority** score
  - Represented as vectors h and a

Hubs and Authorities

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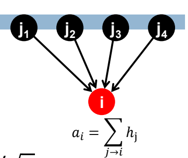
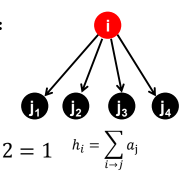
- **Each page i has 2 scores:**
  - Authority score: a_i
  - Hub score: h_i



HITS algorithm:

- Initialize: $a_j(0) = 1/\sqrt{n},\ h_i(0) = 1/\sqrt{n}$
- Then keep iterating until convergence:
  - $\forall i$: Authority: $a_i(t+1) = \sum_{j \to i} h_j(t)$
  - $\forall i$: Hub: $h_i(t+1) = \sum_{i \to j} a_j(t)$
  - $\forall i$: Normalize: $\sum_i (a_i\ (t+1))2 = 1, \sum_j (h_j\ (t+1))2 = 1$



Hubs and Authorities

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- **HITS converges to a single stable point**
- **Notation:**
  - Vector $a = (a_1 \dots, a_n)$, $h = (h_1 \dots, h_n)$
  - Adjacency matrix $\boldsymbol{A}$ (n x n): $A_{ij} = 1$ if $i \to j$
- **Then** $h_i = \sum_{i \to j} a_j$
  can be rewriten as $h_i = \sum_j A_{ij} \cdot a_j$
- **So:** $\boldsymbol{h} = \boldsymbol{A} \cdot \boldsymbol{a}$
- **And likewise:** $\boldsymbol{a} = \boldsymbol{A^T} \cdot \boldsymbol{h}$

Hubs and Authorities

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- **HITS algorithm in vector notation:**
  - Set: $a_i = h_i = \frac{1}{\sqrt{n}}$

  Repeat until convergence:
  - $h = A \cdot a$
  - $a = A^T \cdot h$
  - Normalize a and h
- **Then:** $\underbrace{\boldsymbol{a} = \boldsymbol{A^T} \cdot (\underbrace{\boldsymbol{A} \cdot \boldsymbol{a}}_{\text{new } h})}_{\text{new } a}$
- **Thus, in $2k$ steps:**
  $a = (A^T \cdot A)^k \cdot a$
  $h = (A \cdot A^T)^k \cdot h$

Convergence criterion:

$$\sum_i \left(h_i^{(t)} - h_i^{(t-1)}\right)^2 < \varepsilon$$

$$\sum_i \left(a_i^{(t)} - a_i^{(t-1)}\right)^2 < \varepsilon$$

a is updated (in 2 steps):
$a = A^T(A\,a) = (A^T A)\,a$

h is updated (in 2 steps):
$h = A\,(A^T h) = (A\,A^T)\,h$

Repeated matrix powering

Eigenvalues & Eigenvectors

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- **Definition:**
  - Let $R \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x}$
    for some scalar λ, vector $\boldsymbol{x}$, matrix R
  - Then $\boldsymbol{x}$ **is an eigenvector**, and λ **is its eigenvalue**
- **Fact:**
  - If R is symmetric ($R_{ij} = R_{ji}$)
    (in our case $R = A^T \cdot A$ and $R = A \cdot A^T$ are symmetric)
  - Then R has n orthogonal unit eigenvectors $x_1 \dots x_n$ that form a basis (coordinate system) with eigenvalues $\lambda_1 \dots \lambda_n$ ($|\lambda_i| \geq |\lambda_{i+1}|$)
- **Authority** a is eigenvector of $R = ATA$ associated with largest eigenvalue λ_1
- Similarly: **hub** h is eigenvector of $R = A\,AT$ with the largest eigenvalue

PAGERANK

Links as Votes

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- **Still the same idea: Links as votes**
  - **Page is more important if it has more links**
    - In-coming links? Out-going links?
- **Think of in-links as votes:**
  - www.stanford.edu has 23,400 in-links
  - www.joe-schmoe.com has 1 in-link
- **Are all in-links are equal?**
  - Links from important pages count more
  - Recursive question!

PageRank: The "Flow" Model

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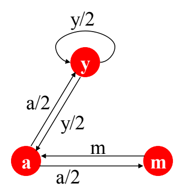
- **A "vote" from an important page is worth more**
- **A page is important if it is pointed to by other important pages**
- **Define a "rank" r_j for node j**

$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_i}$$

d_i … out-degree of node i



"Flow" equations:
$r_y = r_y/2 + r_a/2$
$r_a = r_y/2 + r_m$
$r_m = r_a/2$

PageRank: Matrix Formulation

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- **Stochastic adjacency matrix M**
  - Let page j has d_j out-links
  - If $j \rightarrow i$, then $M_{ij} = \frac{1}{d_j}$ else $M_{ij} = 0$
    - **M is a column stochastic matrix**
      - Columns sum to 1
- **Rank vector r:** vector with an entry per page
  - r_i is the importance score of page i
  - $\sum_i r_i = 1$
- **The flow equations can be written**

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_i}}$$

Random Walk Interpretation

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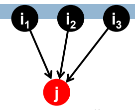

$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_{out}(i)}}$$

- **Imagine a random web surfer:**
  - At any time t, surfer is on some page i
  - At time $t+1$, the surfer follows an out-link from i uniformly at random
  - Ends up on some page j linked from i
  - Process repeats indefinitely
- **Let:**
  - $\boldsymbol{p(t)}$ … vector whose i^{th} coordinate is the prob. that the surfer is at page i at time t
  - So, $\boldsymbol{p(t)}$ is a probability distribution over pages

The Stationary Distribution

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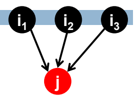

- **Where is the surfer at time $t+1$?**
  - Follows a link uniformly at random

$$p(t+1) = M \cdot p(t)$$

$$p(t+1) = \mathrm{M} \cdot p(t)$$

- Suppose the random walk reaches a state $p(t +$

PageRank: How to solve?

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Given a web graph with n nodes, where the nodes are pages and edges are hyperlinks

- Assign each node an initial page rank
- Repeat until convergence ($\Sigma_i \, |\mathbf{r}_i^{(t+1)} - \mathbf{r}_i^{(t)}| < \varepsilon$)
  - Calculate the page rank of each node

$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

d_i …. out-degree of node i

PageRank: How to solve?

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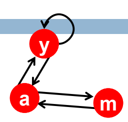

	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2 + r_m$
$r_m = r_a/2$

- **Power Iteration:**
  - Set $r_j = 1$
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
    - And iterate
- **Example:**

r_y		1/3	1/3	5/12	9/24		6/15
r_a	=	1/3	3/6	1/3	11/24	…	6/15
r_m		1/3	1/6	3/12	1/6		3/15

Iteration 0, 1, 2, …

PageRank: Three Questions

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$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}} \quad \text{or equivalently} \quad r = Mr$$

- **Does this converge?**
- **Does it converge to what we want?**
- **Are results reasonable?**

Does this converge?

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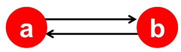

$$r_j^{(t+1)} = \sum_{i\to j} \frac{r_i^{(t)}}{d_i}$$

- **Example:**

r_a	=	1	0	1	0
r_b		0	1	0	1

Iteration 0, 1, 2, …

Does it converge to what we want?

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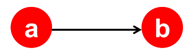

$$r_j^{(t+1)} = \sum_{i\to j} \frac{r_i^{(t)}}{d_i}$$

- **Example:**

r_a	=	1	0	0	0
r_b		0	1	0	0

Iteration 0, 1, 2, …

RageRank: Problems

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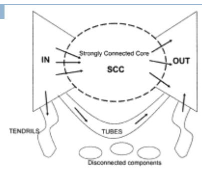
2 problems:

- **(1)** Some pages are **dead ends** (have no out-links)
  - Such pages cause importance to "leak out"



- **(2) Spider traps** (all out-links are within the group)
  - Eventually spider traps absorb all importance

Problem: Spider Traps

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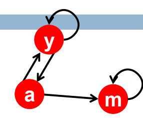
- **Power Iteration:**
  - Set $r_j = 1$
  - $r_j = \sum_{i\to j} \frac{r_i}{d_i}$
    - And iterate



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	1

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2$
$r_m = r_a/2 + r_m$

- **Example:**

r_y		1/3	2/6	3/12	5/24		0
r_a	=	1/3	1/6	2/12	3/24	…	0
r_m		1/3	3/6	7/12	16/24		1

Iteration 0, 1, 2, …

Solution: Random Teleports

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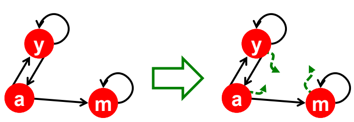
- **The Google solution for spider traps: At each time step, the random surfer has two options**
  - With prob. β, follow a link at random
  - With prob. 1-β, jump to some page uniformly at random
  - Common values for β are in the range 0.8 to 0.9
- **Surfer will teleport out of spider trap within a few time steps**



Problem: Dead Ends

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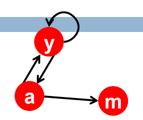
- **Power Iteration:**
  - Set $r_j = 1$
  - $r_j = \sum_{i\to j} \frac{r_i}{d_i}$
    - And iterate



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2$
$r_m = r_a/2$

- **Example:**

r_y		1/3	2/6	3/12	5/24		0
r_a	=	1/3	1/6	2/12	3/24	…	0
r_m		1/3	1/6	1/12	2/24		0

Iteration 0, 1, 2, …

Solution: Always Teleport

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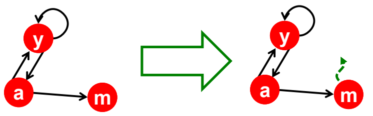
- **Teleports:** Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

	y	a	m
y	½	½	⅓
a	½	0	⅓
m	0	½	⅓

Solution: Random Jumps

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- **Google's solution: At each step, random surfer has two options:**
  - With probability β, follow a link at random
  - With probability 1-β, jump to some random page
- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1-\beta)\frac{1}{n}$$

d_i … out-degree of node i

The above formulation assumes that M has no dead ends. We can either preprocess matrix M **(bad!)** or explicitly follow random teleport links with probability 1.0 from dead-ends. See P. Berkhin, *A Survey on PageRank Computing*, Internet Mathematics, 2005.

PageRank & Eigenvectors

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- **PageRank as a principal eigenvector**

  $r = M \cdot r$ or equivalently $r_j = \sum_i \frac{r_i}{d_i}$

- **But we really want:**

  $r_j = \beta \sum_i \frac{r_i}{d_i} + (1-\beta)\frac{1}{n}$

- **Let's define:**

  $M'_{ij} = \beta M_{ij} + (1-\beta)\frac{1}{n}$

- **Now we get what we want:**

  $\boldsymbol{r} = \boldsymbol{M'} \cdot \boldsymbol{r}$

- **What is $1-\beta$?**
  - In practice 0.15 (5 links and jump)

d_i … out-degree of node i

Note: M is a sparse matrix but M' is dense (all entries $\neq 0$). In practice we never "materialize" M but rather we use the "sum" formulation

PageRank: The Complete Algorithm

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See P. Berkhin, *A Survey on PageRank Computing*, Internet Mathematics, 2005.

- **Input: A and β**
  - Adjacency matrix A of a directed graph with spider traps and dead ends
  - Parameter β
- **Output: PageRank vector r**
  - Set: $r_j^{(0)} = 1/n$
  - **Repeat until:** $\sum_j \left| r_j^{(t)} - r_j^{(t-1)} \right| < \varepsilon$
    - $\forall j$: $r'^{(t)}_j = \sum_{i \to j} \beta \frac{r_i^{(t-1)}}{d_i}$, if in-deg. of j is 0 then $r'^{(t)}_j = 0$
    - Now re-insert the leaked PageRank:
      $\forall j$: $r_j^{(t)} = r'^{(t)}_j + (1-S)/n$

Where: $S = \sum_j r'^{(t)}_j$

Example

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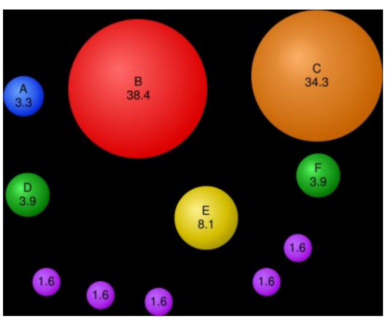

PageRank and HITS

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- **PageRank and HITS are two solutions to the same problem:**
  - **What is the value of an in-link from *u* to *v*?**
  - In the PageRank model, the value of the link depends on the links into *u*
  - In the HITS model, it depends on the value of the other links out of *u*
- **The destinies of PageRank and HITS post-1998 were very different**