COMP 2711H Discrete Mathematical Tools for Computer Science
2014 Fall Semester
Homework 5
Handed out: Nov 19
Due: Nov 26

Problem 1. Let $P(n)$ be the following statement:

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for the positive integer n.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step, identifying where you use the inductive hypothesis.

(f) Explain why these steps show that this formula is true whenever n is a positive integer.

Problem 2. Find the formula for the sum of cubes given in Problem 1 using the method of differences.

Problem 3.

(a) Find a formula for

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n, or using any other method.

(b) Prove the formula you found or conjectured in part (a), using the method of induction.

Problem 4. Use mathematical induction to prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.

Problem 5. Use mathematical induction to prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

Problem 6. Use mathematical induction to prove that n lines separate the plane into $(n^2+n+2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

Problem 7. Use mathematical induction to prove that a two-dimensional $2^n \times 2^n$ checkerboard with one 1×1 square missing can be completely covered by 2×2 squares with one 1×1 square missing.

Problem 8. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for $n \geq 18$. You should give two different proofs, using weak and strong induction, respectively.

Problem 9. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n-1)/2$.

Problem 10. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \to P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

Problem 11. Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$.

Recursive step: If $(a,b) \in S$, then $(a,b+1) \in S$, $(a+1,b+1) \in S$, and $(a+2,b+1) \in S$.

(a) List the elements of S produced by the first four applications of the recursive definition.

(b) Use induction to prove that $a \leq 2b$ whenever $(a,b) \in S$.

Problem 12. The *reversal* of a string is the string consisting of the symbols of the string in reverse order. The reversal of the string w is denoted by w^R.

(a) Give a recursive definition of the reversal of a string. (*Hint:* First define the reversal of the empty string. Then write a string w of length $n+1$ as xy, where x is a string of length n, and express the reversal of w appropriately.)

(b) Use induction to prove that $(w_1w_2)^R = w_2^Rw_1^R$.

Problem 13. Recursively define the set of bit strings that have more zeros than ones.

Problem 14. Use induction to show that $\ell(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal nodes of T.

The set of leaves and the set of internal nodes of a full binary tree can be defined recursively as follows.

Basis step: The root r is a leaf of the full binary tree with exactly one node r. This tree has no internal nodes.

Recursive step: The set of leaves of the tree is the union of the sets of leaves of T_1 and of T_2. The internal nodes of T are the root r of T and the union of the set of internal nodes of T_1 and the set of internal nodes of T_2.

Problem 15. In this problem, you need to determine the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \dots, a_k$, where $a_1 = 1, a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \dots, k-1$.

(a) Find the answer to this problem by first writing a recurrence relation for the number of such sequences, and then solving it (using, say, the iterative approach).

(b) Find another way of solving this problem, which does not involve writing a recurrence relation.

Problem 16. A *ternary* string is a string that contains only 0's, 1's, and 2's.

(a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0's. How many ternary strings of length six contain two consecutive 0's?

(b) Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same. How many ternary strings of length six do not contain consecutive symbols that are the same?

Problem 17. Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes.

Problem 18. In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we canot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk. Write a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restricton, and solve it.

Problem 19. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmission, the other two signals require 2 microseconds each for transmission, and a signal in a message is followed immediately by the next signal? You need to give a closed-form solution.

Problem 20. Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample.

(a) $\log_2 f(n)$ is $O(\log_2 g(n))$.

(b) $2^{f(n)}$ is $O(2^{g(n)})$.

(c) $(f(n))^2$ is $O((g(n))^2)$.

Problem 21. Arrange the following running times in order of increasing asymptotic complexity:

$$n^{2.5}, \quad \sqrt{2n}, \quad n+10, \quad 10^n, \quad 100^n, \quad n^2 \log n$$

Note that you must write function $f(n)$ *before* function $g(n)$ if $f(n) = O(g(n))$. *Just give the answer; no explanation is needed.*

Problem 22. Consider the following recurrence relation for the running time $T(n)$ of a divide and conquer algorithm:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 3\,T(n/2) + n \qquad \text{if } n > 1. \end{aligned}$$

Assume that n is a power of 2. Answer the following questions regarding the recursion tree for this recurrence.

(a) Recall that there is a subproblem associated with each node of the recursion tree. How many subproblems are there at level i of the recursion tree? (Recall that the *root* is assumed to be at level 0.)

(b) What is the size of each subproblem at level i of the recursion tree?

(c) How much work is needed for the *combine* part for each subproblem at level i (*note: you must ignore the work done during the recursive calls*)?

(d) What is the work done summed over all the subproblems at level i (*again, you must ignore the work done during the recursive calls*)?

(e) How many levels are there in the recursion tree?

(f) Give a good asymptotic upper bound on the total work done summed over all the subproblems in the recursion tree. (*In other words, you need to give a good upper bound on $T(n)$. You should try to express your answer in the form of n raised to a suitable power.*)

Problem 23. Consider again the recurrence relation given in Problem 1. Establish an asymptotic upper bound for $T(n)$, using the *method of mathematical induction*. Make your bound as tight as possible. You may assume that n is a power of 2.

Problem 24. Give asymptotic upper bounds for $T(n)$. Make your bounds as tight as possible. In each case, you may assume that $T(1) = 1$ and n is a power of 2. *Just give the answers; no explanation is needed.*

(a) $T(n) = 2\,T(n/2) + n, \qquad \text{if } n > 1.$

(b) $T(n) = 3\,T(n/2) + n, \qquad \text{if } n > 1.$

(c) $T(n) = T(n-1) + n^3, \qquad \text{if } n > 1.$

(d) $T(n) = T(n/2) + 1, \qquad \text{if } n > 1.$

(e) $T(n) = 2\,T(n-1) + 1, \qquad \text{if } n > 1.$