COMP 2711 Discrete Mathematical Tools for CS
2012 Spring Semester – Solution to Written Assignment # 8
Distributed: 20 Nov 2013 – Due: 27 Nov 2013

At the top of your solution, please write your (i) name, (ii) student ID #, (iii) email address and (iv) tutorial section.

Problem 1: Prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all integers $n \geq 1$.

SOLUTION: Denote the statement to prove by $p(n)$.

Base case: For $n = 1$, we have $1^3 = 1^2 2^2/4$. So $p(1)$ is true.

Inductive hypothesis: Suppose $p(n-1)$ is true for some $n \geq 2$, i.e.

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 = \frac{(n-1)^2 n^2}{4}.$$

Inductive step: Adding n^3 to both sides of $p(n-1)$, we have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 &= \frac{(n-1)^2 n^2}{4} + n^3 \\ &= \frac{(n-1)^2 n^2 + 4n^3}{4} \\ &= \frac{n^2((n-1)^2 + 4n)}{4} \\ &= \frac{n^2(n+1)^2}{4}, \end{aligned}$$

which shows that $p(n)$ is true, and so $p(n-1) \Rightarrow p(n)$.

Inductive conclusion: By the principle of mathematical induction, we can conclude that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

is true for all $n \geq 1$.

(For clarity, we have labelled all steps of the inductive proof in **bold***).*

Problem 2: Prove that every integer greater than 7 is a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.
(Hint: first prove the three base cases *of $n = 8, 9, 10$ and then prove the inductive step assuming that $n > 10$.)*

SOLUTION: Denote the statement by $p(n)$.

Base cases: Since

$$\begin{aligned} 8 &= 1 \cdot 3 + 1 \cdot 5 \\ 9 &= 3 \cdot 3 + 0 \cdot 5 \\ 10 &= 0 \cdot 3 + 2 \cdot 5, \end{aligned}$$

$p(8)$, $p(9)$ and $p(10)$ are true.

Inductive hypothesis: Suppose $p(n-1)$, $p(n-2)$ and $p(n-3)$ are true for some $n \geq 11$.

Inductive step: Since $p(n-3)$ is true, we can express $n-3$ as a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5, i.e., $n-3 = 3a + 5b$ for some nonnegative integers a and b. Hence $n = 3(a+1) + 5b$, implying that $p(n)$ is true, and so $p(n-3) \Rightarrow p(n)$.

Inductive conclusion: By the strong principle of mathematical induction, we can conclude that $p(n)$ is true for all integers greater than 7.

Note: It would *not* be correct to just use the base case $n = 8$ and proceed from there.

Problem 3: Consider the recurrence $M(n) = 2M(n-1)+2$, with base case of $M(1) = 1$.
(a) State the solution to this recurrence
(you may use Theorem 4.1 in the book).
(b) Use induction to prove that this solution is correct.

SOLUTION: (a)

Because our base case is $n = 1$, we shift the index by setting $T(n) = M(n+1)$.

T starts from 0.
$a = 2$, $r = 2$, $b = 1$:

$$M(n+1) = T(n) = r^n b + a\frac{1-r^n}{1-r} = 2^n \cdot 1 + 2\frac{1-2^n}{1-2} = 2^n + 2 \cdot (2^n - 1) = \frac{3}{2}2^{n+1} - 2$$

so, from Theorem 4.1,

$$M(n) = \frac{3}{2}2^n - 2.$$

We could also derive the same formula from scratch by Iterating the recurrence:

$$
\begin{aligned}
M(n) &= 2^1 M(n-1) + 2 \\
&= 2^1(2M(n-2)+2)+2 \\
&= 2^2 M(n-2) + 2^2 + 2 \\
&= 2^3 M(n-3) + 2^3 + 2^2 + 2 \\
&\vdots \\
&= 2^i M(n-i) + \sum_{j=1}^{i} 2^j.
\end{aligned}
$$

Because $M(1)$ is the base case taking $i = n-1$ gives

$$
\begin{aligned}
&= 2^{n-1} M(1) + \sum_{j=1}^{n-1} 2^j \\
&= 2^{n-1} M(1) + 2\sum_{j=0}^{n-2} 2^j \\
&= 2^{n-1} M(1) + 2\frac{2^{n-1}-1}{2-1} \\
&= 2^{n-1} M(1) + 2^n - 2.
\end{aligned}
$$

Substituting $M(1) = 1$,

$$
\begin{aligned}
&= 2^{n-1} + 2^n - 2 \\
&= \frac{3}{2} \cdot 2^n - 2.
\end{aligned}
$$

(b) We use mathematical induction to verify the correctness of our derivation.

1. Base case: $n = 1$. $M(1) = (3/2)2^1 - 2 = 1$. Thus, the base case is true.
2. Suppose inductively that

   $$M(n-1) = \frac{3}{2} \cdot 2^{n-1} - 2.$$

   Now, for $M(n)$, we have

   $$M(n) = 2M(n-1) + 2$$

$$
\begin{aligned}
&= 2 \cdot \frac{3}{2} \cdot 2^{n-1} - 2^2 + 2 \\
&= \frac{3}{2} \cdot 2^n - 2.
\end{aligned}
$$

3. From Steps 1 and 2 and the principle of mathematical induction, the statement is true for all n.

Problem 4: Consider a function $T(n)$ defined on integers n that are powers of 2. Suppose

$$T(1) = 1, \quad T(n) = 3T(n/2) + n^2.$$

Iterate the recurrence or use a recursion tree to find a closed-form expression for $T(n)$. Simplify the closed-form expression using the big Θ notation.

Answer: Iterating the recurrence, we get:

$$
\begin{aligned}
T(n) &= T(2^j) \\
&= 3T(2^{j-1}) + 2^{2j} \\
&= 3(3T(2^{j-2}) + 2^{2(j-1)}) + 2^{2j} \\
&= 3^2T(2^{j-2}) + \frac{3}{4}2^{2j} + 2^{2j} \\
&= 3^2(3T(2^{j-3}) + 2^{2(j-2)}) + \frac{3}{4}2^{2j} + 2^{2j} \\
&= 3^3T(2^{j-3}) + (\frac{3}{4})^2 2^{2j} + \frac{3}{4}2^{2j} + 2^{2j} \\
&\vdots \\
&= 3^jT(1) + (\frac{3}{4})^{j-1}2^{2j} + \ldots + \frac{3}{4}2^{2j} + 2^{2j} \\
&= 3^j + 2^{2j}\frac{1-(3/4)^j}{1-3/4} \\
&= 3^j + 4 \cdot 2^{2j} - 4 \cdot 3^j = 4 \cdot 2^{2j} - 3 \cdot 3^j \\
&= 4n^2 - 3n^{log_2 3} \\
&= \Theta(n^2).
\end{aligned}
$$

Problem 5: Challenge Problem: Leaving Dot-town

Every person living in Dot-town has a red or blue dot on his forehead but doesn't know the color of his own dot. Every day the people gather in the town square to talk with each other. If anyone ever figures out the color of his own dot he must leave town before the next gathering. People never leave Dot-town unless they figure out their own dot color. One day, a stranger comes to town and casually mentions that at least one person in town has a blue dot on their forehead.

1. Prove that, eventually, every person must leave town.
2. How long does it take before everyone has left town?

Before solving the problem we prove the following lemma

Lemma: Let n, j, k be positive integers. Suppose that there are n people with blue dots on their heads. Suppose further that at the gathering on day j, everyone knows the fact that there are at least k people with blue dots.

Then

- If $n = k$,
  * before the gathering on day j+1 all the people with blue dots leave town,
  * before the gathering on on day j+2 all people with red dots leave town.
- If $n > k$, at the gathering on day j+1 everyone remains in town and everyone knows the fact that there are at least k+1 people with blue dots.

Proof of Lemma:

(a) If $n = k$,
then at the gathering on day j everyone with a blue dot only sees $k-1$ people with blue dots. Therefore they KNOW that their dot must be blue. Therefore, they all leave town before the next day's gathering.

On the other hand, at the day j gathering, the people with red dots see k people with blue dots, so they have no idea what color their own dot is. But, at the day $j+1$ gathering, they saw that all the blue dot people had left. This implies that their dots must be red, so they leave by before the day $j+2$ gathering

(b) If $n > k$,
At the day j gathering everyone sees more than $k-1$ dots. Therefore, no one has enough information to know their own dot and does not leave before the next gathering. Since everyone remains at the next gathering, they all know that case (a) did not happen. This means that they all know that they are in case (b) so $n > k$ and therefore everyone knows the fact that there are at least $k+1$ people with blue dots

QED

Proof for Challenge Problem: Let n be the number of people with blue dots. Using the lemma, we can now easily prove by induction that

- For $j < n$, at the gathering on day j everyone remains in town and everyone knows the fact that there are at least k people with blue dots
- Everyone with a blue dot leaves after the gathering on day n
- Everyone with a red dot leaves after the gathering on day $n + 1$