Lecture 7: Properties of regular languages

Theorem 1 *The set of regular languages are closed under*

1. *Concatenation (L_1 and L_2 regular, then so is L_1L_2),*
2. *Union (L_1 and L_2 regular, then so is $L_1 \cup L_2$),*
3. *Kleene star (L regular, then so is L^*),*
4. *Complementation (L regular, then so is $\overline{L} = \Sigma^* - L$), and*
5. *Intersection (L_1 and L_2 regular, then so is $L_1 \cap L_2$).*

Proof:

1.-3. We have seen two ways of proving these statements in earlier lecture notes. The first way is to show that there exists a regular expression that represents the resulting language, the second way is to show that there exists an (N)FA that accepts the resulting language.

4. Since L is regular, L is accepted by some DFA M. Let M' be the same as M except that:

   A state is a final state in M' iff it is not a final state in M.

   $\overline{L}$ is accepted by the DFA M'. Hence, $\overline{L}$ is regular.

5. $$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}.$$

   Since the set of all regular languages is closed under union and complementation, thus it is also closed under intersection.

Summary:

There are 3 main ways to prove that a language is regular

- show that the language can be expressed as a regular expression
- show that the language can be recognized by an NFA or DFA
- use closure properties of regular languages

Example:

$\Sigma = \{0, 1, \ldots, 9\}$

Prove that the following language is regular.

$L =$ the set of decimal representations of natural numbers divisible by 2 or 3.
e.g. $0, 2, 3, 984330, 2378127, \ldots$

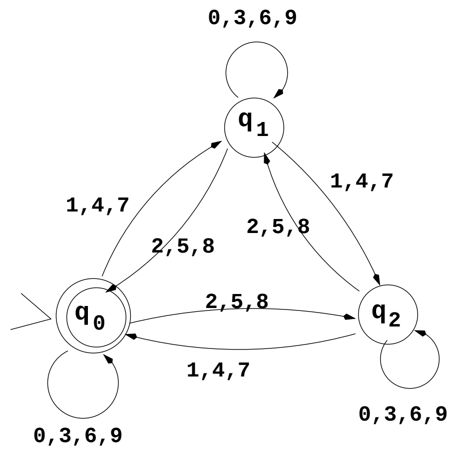
Proof:

- The set of decimal representation of natural numbers is
$$L_1 = \{0\} \cup \{1, 2, \ldots, 9\}\Sigma^*$$
which is regular.
- The set of decimal representation of natural numbers divisible by 2 can be expressed as
$$L_2 = L_1 \cap \Sigma^*\{0, 2, 4, 6, 8\}$$
which is regular

- The set of decimal representation of natural numbers divisible by 3 is

$$L_3 = L_1 \cap L_4$$

  where L_4 is the set of strings whose digits sum to a number which is divisible by 3 (but may begin with 0). We claim that the following FA accepts L_4.



  q_0: sum of digits mod 3 = 0.

  q_1: sum of digits mod 3 = 1.

  q_2: sum of digits mod 3 = 2.

  Since L_1 and L_4 are regular and the set of regular languages is closed under intersection, hence L_3 is regular.

- $L = L_2 \cup L_3$ and hence is regular