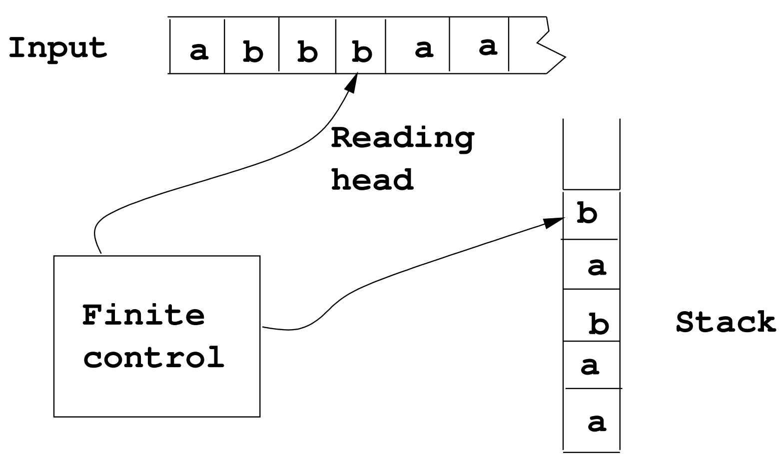
Lecture 10: Pushdown Automata

Every regular language is a CFL.
But some CFLs are nonregular.
Since FA $\equiv$ Regular languages, some CFLs cannot be recognized by any FA.
Examples of CFLs that are nonregular:

$$\{a^n b^n : n \geq 0\}$$
$$\{ww^R : w \in \{a,b\}^*\}$$

We need to consider a more powerful computation model to recognize CFL – Pushdown Automata (PA). It is an automaton equipped with a stack.



Formal definition of PA

A pushdown automata is defined as a 6-tuple $M = (K, \Sigma, \Gamma, \Delta, s, F)$ where

- K is a finite, nonempty set of states,
- Σ is an input alphabet,
- Γ is a stack alphabet,
- $s \in K$ is the initial state,
- $F \subseteq K$ is a set of final state,
- Δ is a transition relation, a finite subset of $(K \times (\Sigma \cup \{e\}) \times \Gamma^*) \times (K \times \Gamma^*)$.

Note that Δ is a relation, not a function, thus PAs are nondeterministic. Unlike FAs, deterministic pushdown automata are *not* equivalent in power with nondeterministic pushdown automata. Specifically, nondeterministic PA can recognize certain languages that are not recognizable by any deterministic pushdown automata (e.g., $\{ww^R : w \in \{a, b\}^*$, but we will not prove this fact).

Formal definition of PA

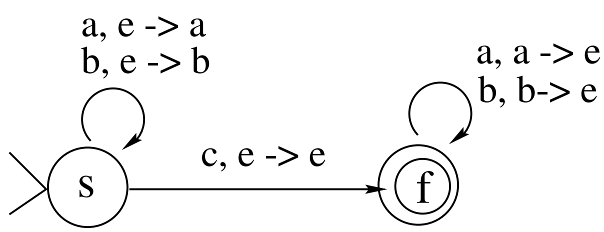
A **transition** $((p, a, \beta), (q, \gamma)) \in \Delta$ – intuitively, it means when in state p, read a, replace β on stack by γ (equivalently, pop $beta$, then push γ) and change state to q.
If γ or β is a string instead of a single character, then the *left-most* symbol in γ or β is the *topmost* in the stack.

E.g.

- $((p, a, b), (q, cb))$ — in state p, read a, pop b, push b, then push c (leftmost symbol is topmost in stack), and change state to q.
- $((p, b, e), (q, a))$ — in state p, read b, push a, change state to q.
- $((p, b, a), (q, e))$ — in state p, read b, pop a, change state to q.
- $((p, e, e), (q, e))$ — in state p, nondeterministically change state to q without reading anything or doing anything to stack.

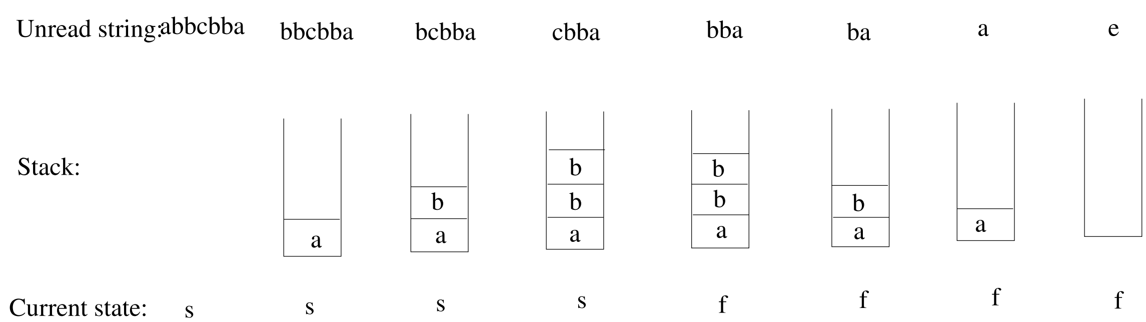
Example:

$$L = \{wcw^R : w \in \{a, b\}^*\}$$



$\sigma, \beta \rightarrow \gamma$ means the symbol read is σ, pop β, push γ.

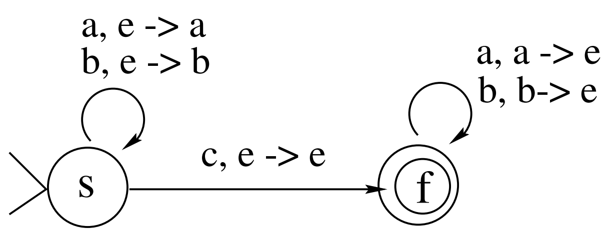
Consider *abbcbba*:



- A string is said to be accepted by M if, when the input is *completely read*, M is *at a final state* and *the stack is empty.* (see formal definition of string acceptance in slide 6).

Example

$$L = \{wcw^R : w \in \{a,b\}^*\}$$



Representing the PA as a 6-tuple:

A PA accepting L is $M = (K, \Sigma, \Gamma, \Delta, s, F)$ where
$K = \{s, f\}$
$\Sigma = \{a, b, c\}$
$\Gamma = \{a, b\}$
$F = \{f\}$
$\Delta = \{$
$((s, a, e), (s, a)),$
$((s, b, e), (s, b)),$
$((s, c, e), (f, e)),$
$((f, a, a), (f, e)),$
$((f, b, b), (f, e))\}$

Computation

- A **configuration**, (p, w, β) is a member of $K \times \Sigma^* \times \Gamma^*$
  - p is the current state
  - w is the remaining input to be read
  - β is the current content of stack from top to bottom (leftmost is topmost).

- Example:
  $(s, abcba, e) \vdash_M (s, bcba, a) \vdash_M (s, cba, ba) \vdash_M (f, ba, ba) \vdash_M (f, a, a) \vdash_M (f, e, e)$

- $\vdash_M$ means '**yields** in one step'

- $\vdash_M^*$ means 'yields in 0, 1 or more step'

- M **accepts** w if

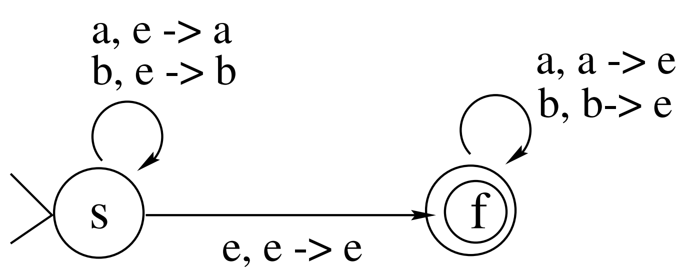
$$(s, w, e) \vdash_M^* (q, e, e) \text{ for some } q \in F$$

  - finish reading the input string
  - at a final state
  - the stack is empty

- The language accepted by M, $L(M)$, is the set of all strings accepted by M.

More example

$L = \{ww^R : w \in \{a, b\}^*\}$



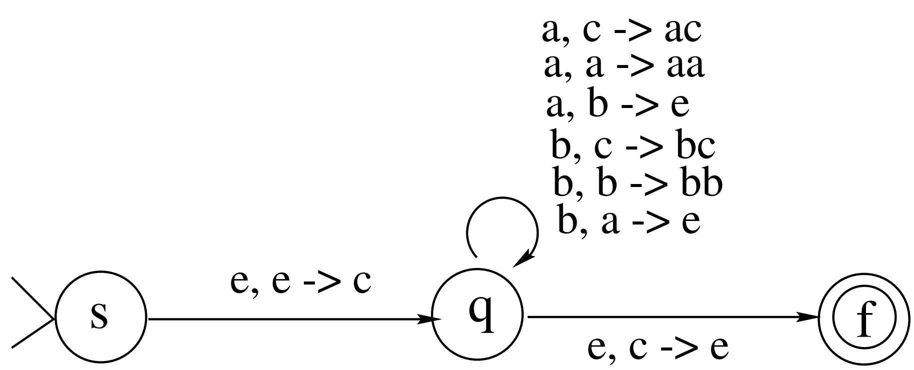
A PA that accepts L is $M = (K, \Sigma, \Gamma, \Delta, s, F)$ where
$K = \{s, f\}, \Sigma = \Gamma = \{a, b\}$, $F = \{f\}$
$\Delta = \{$
$((s, a, e), (s, a)),$
$((s, b, e), (s, b)),$
$((s, e, e), (f, e)),$
$((f, a, a), (f, e)),$
$((f, b, b), (f, e))\}$

$(s, abba, e) \vdash_M (s, bba, a) \vdash_M (s, ba, ba) \vdash_M (f, ba, ba) \vdash_M$
$(f, a, a) \vdash_M (f, e, e)$ – accept
$(s, abba, e) \vdash_M (f, abba, e)$ – fail
$(s, abba, e) \vdash_M (s, bba, a) \vdash_M (f, bba, a)$ – fail

Try to verify that $aabba \notin L$, $abbaa \notin L$.

More example

$$L = \{w \in a, b^* : w \text{ has the same number of a's and b's}\}$$



- $K = \{s, q, f\}$, $\Sigma = \{a, b\}$, $\Gamma = \{a, b, c\}$, $F = \{f\}$.
- $\Delta = \{$ $((s, e, e), (q, c))$,
  $((q, a, c), (q, ac))$,
  $((q, a, a), (q, aa))$,
  $((q, a, b), (q, e))$,
  $((q, b, c), (q, bc))$,
  $((q, b, b), (q, bb))$,
  $((q, b, a), (q, e))$,
  $((q, e, c), (f, e))\}$

$(s, abbbaa, e) \vdash_M \ . \ . \ .$