Lecture 12: Closure Properties of CFLs

Theorem 1 *CFLs are closed under*

1. *union,*
2. *concatenation,*
3. *Kleene Star.*

Proof

Let $G_1 = (V_1, \Sigma_1, R_1, S_1)$, and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ be the two CFG generating the CFLs L_1 and L_2.
Rename the nonterminals, if necessary, so that $V_1 - \Sigma_1$ and $V_2 - \Sigma_2$ are disjoint sets.

1. Let $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2,$
   $R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}, S)$.

   Prove that $L(G) = L(G_1) \cup L(G_2)$:
   $w \in L(G)$ iff $S \Rightarrow_G^* w$.
   iff $S \Rightarrow_G \; S_1 \Rightarrow_G^* \; w$ or $S \Rightarrow_G \; S_2 \Rightarrow_G^* \; w$.
   iff $S_1 \Rightarrow_G^* \; w$ or $S_2 \Rightarrow_G^* \; w$.
   iff $S_1 \Rightarrow_{G_1}^* \; w$ or $S_2 \Rightarrow_{G_2}^* \; w$, since $S \notin V_1 \cup V_2$ and $(V_1 - \Sigma_1) \cap (V_2 - \Sigma_2) = \emptyset$.
   iff $w \in L(G_1)$ or $w \in L(G_2)$.
   iff $w \in L(G_1) \cup L(G_2)$.

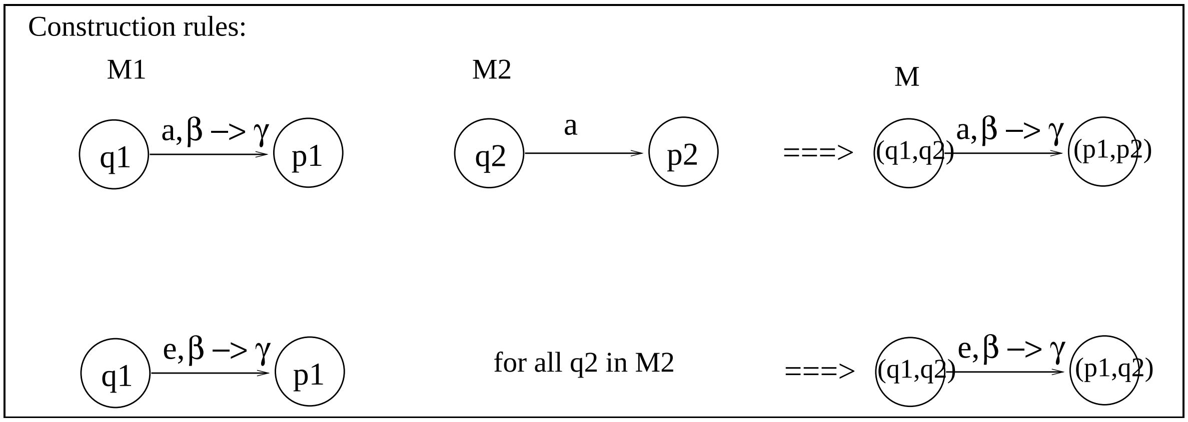
2. Let $G = (V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2,$
   $R_1 \cup R_2 \cup \{S \to S_1 S_2\}, S)$.
   Prove that $L(G) = L(G_1)L(G_2)$.
3. Let $G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \to e, S \to SS_1\}, S)$.
   Prove that $L(G) = L(G_1)^*$.

Note: CFLs are not closed under intersection and complementation. (the next lecture).

Theorem 2 *The intersection of a CFL and a regular language is a CFL.*

Proof:

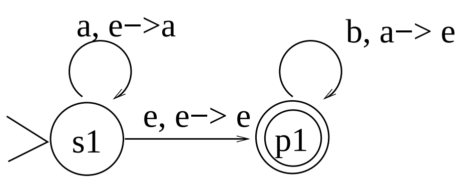
- Let L be a CFL L and R a regular language.
- There exists a PA $M_1 = (K_1, \Sigma_1, \Gamma_1, \Delta_1, s_1, F_1)$ that accepts L, and a DFA $M_2 = (K_2, \Sigma_2, \delta, s_2, F_2)$ that accepts R.
- We need to show that there exists a PA $M = (K, \Sigma, \Gamma, \Delta, s, F)$ that accepts $L \cap R$.
- Idea: Construct a PA M that operates in the same way as M_1 except that it also keeps track of the state transitions of M_2 while reading the same input symbol.
- The *cross product construction*:
  - $K = K_1 \times K_2$, $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Gamma = \Gamma_1$,
    $s = (s_1, s_2)$, $F = F_1 \times F_2$.
  - For each transition $((q_1, a, \beta), (p_1, \gamma)) \in \Delta_1$,
    and each state $q_2 \in K_2$,
    add the transition $(((q_1, q_2), a, \beta), ((p_1, \delta(q_2, a)), \gamma))$ to Δ.
  - For each transition $((q_1, e, \beta), (p_1, \gamma)) \in \Delta_1$,
    and each state $q_2 \in K_2$,
    add the transition $(((q_1, q_2), e, \beta), ((p_1, q_2), \gamma))$ to Δ.



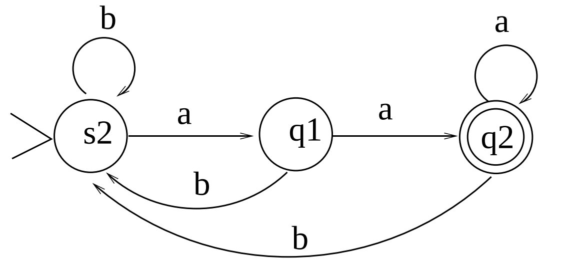
Note that these general construction rules may generate states that are unreachable. As usual, we can construct incrementally by starting from the start state and adding only reachable states.

- Finally, we need to prove that $L(M) = L(M_1) \cap L(M_2)$. That is, prove that $(s, w, e) \vdash_M (f, e, e)$ for some $f \in F$ if and only if $(s, w, e) \vdash_{M_1} (r, e, e)$ for some $r \in F_1$ and $(s, w) \vdash_{M_2} (t, e)$ for some $t \in F_2$.

Example:

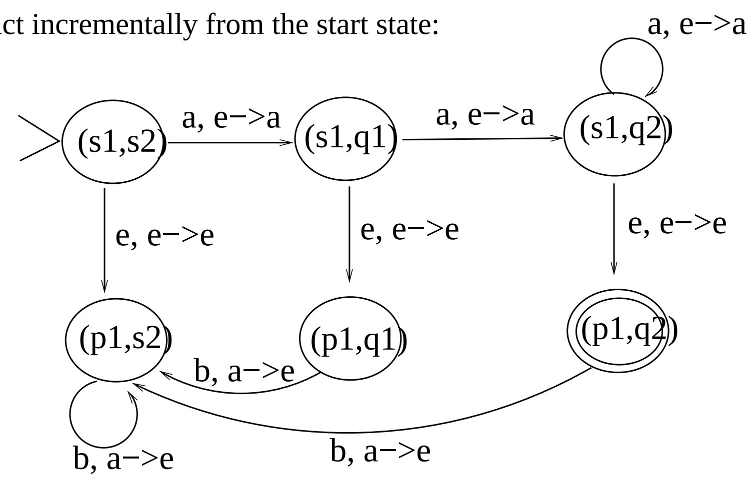


L1 = { $a^i b^i$: i>= 0}



L2 = { w in {a,b}* | w ends with aa}

Construct incrementally from the start state:



L = L1 intersect L2

Example:
$L =$ the set of all strings w in $\{a, b\}^*$ such that $|w|_a = |w|_b$ and w contains neither of the substrings $abaa$ or $babb$.

Is L context-free?

Answer:

- Let $L_1 = \{w \in \{a, b\}^* : |w|_a = |w|_b\}$.
  Let $R = \{w \in \{a, b\}^* : w$ contains neither of the substrings $abaa$ or $babb\}$.
- Then $L = L_1 \cap R$.
- Since complement of R, $\overline{R} = L((a \cup b)^*(abaa \cup babb)(a \cup b)^*)$, thus R is regular.
- Since L_1 is a CFL and R is regular, L is a CFL.