Introduction to non-linear filtering

Introduction to non-linear filtering

References:

(1) P. Perona and J. Malik, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 12, No. 7, pp. 629-639, 1990.

(2) Joachim Weickert, *A review of nonlinear diffusion filtering*, Scale-Space Theory in Computer Vision, Lecture Notes in Computer Science, Vol. 1252, Springer, Berlin, pp. 3-28, 1997.

(3) G. Gerig, *Nonlinear Anisotropic Filtering of MRI Data*, IEEE Transactions on Medical Imaging, Vol. 11, No. 2, pp. 221-232, 1992.

Roadmap

1. Filtering based on Gaussian low-pass filter

2. Concepts of diffusion

3. Linear diffusion filtering

4. Non-linear isotropic diffusion filtering

5. Non-linear anisotropic diffusion filtering
   (Edge enhancing anisotropic diffusion filtering)

Filtering based on Gaussian low-pass filter

1. Low-pass Gaussian filtering of a 2D image

$$u\left(\vec{x},t\right)=\begin{cases} I\left(\vec{x}\right) & (t=0) \\ \left(G_{\sqrt{2t}} * I\right)\left(\vec{x}\right) & (t>0) \end{cases}$$

where $\vec{x}=(x,y)=$ Position vector

$I\left(\vec{x}\right)=$ Initial image

$u\left(\vec{x},t\right)=$ Filtered image

$*=$ Convolution operator

$$G_\sigma\left(\vec{x}\right)=\frac{1}{2\pi\sigma^2}\cdot\exp\left(\frac{-\left|\vec{x}\right|^2}{2\sigma^2}\right)$$

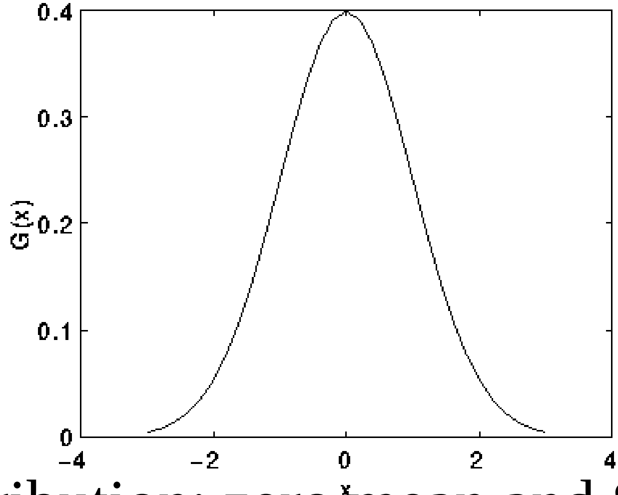
Filtering based on Gaussian low-pass filter

2. Gaussian filtering – it filters or can smooth an initial image I by convolving the image with a Gaussian filter.

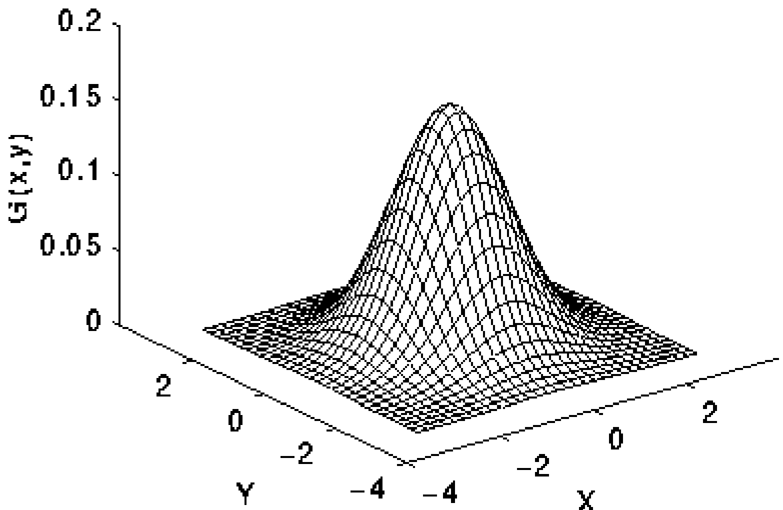
3. Convolution represents the weighted sum of local intensity values, in which the weights are determined by a Gaussian distribution with higher values in the center and lower values in the tails.

Filtering based on Gaussian low-pass filter

4. 1D Gaussian distribution: zero mean and SD = 1

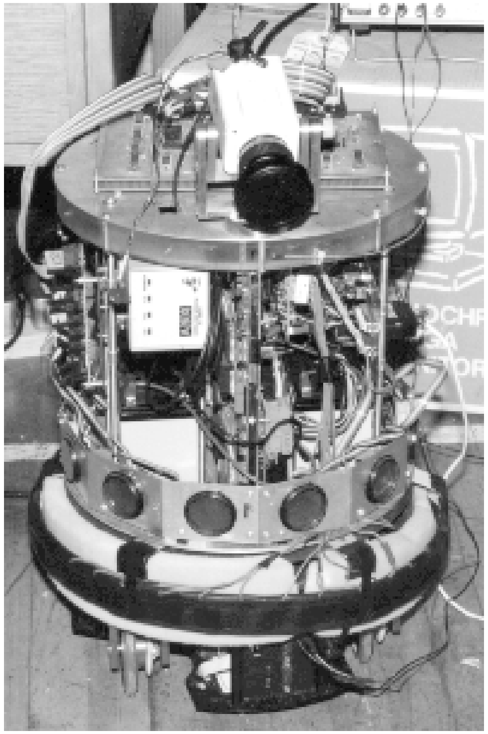


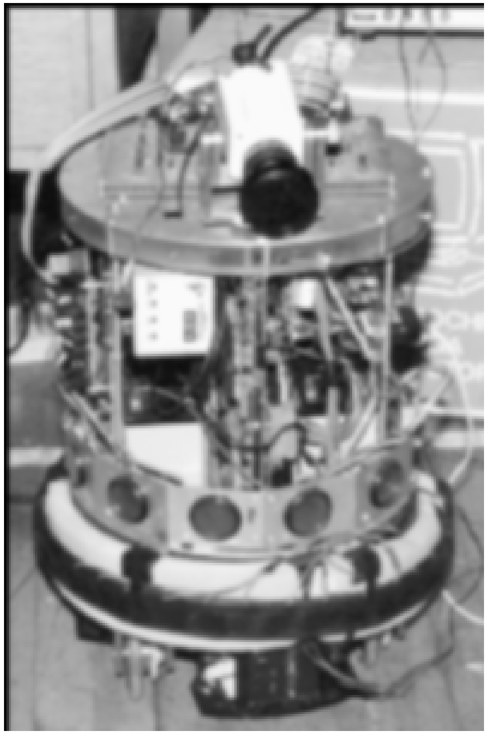
5. 2D Gaussian distribution: zero mean and SD = 1



http://www.dai.ed.ac.uk/HIPR2/gsmooth.htm

Filtering based on Gaussian low-pass filter



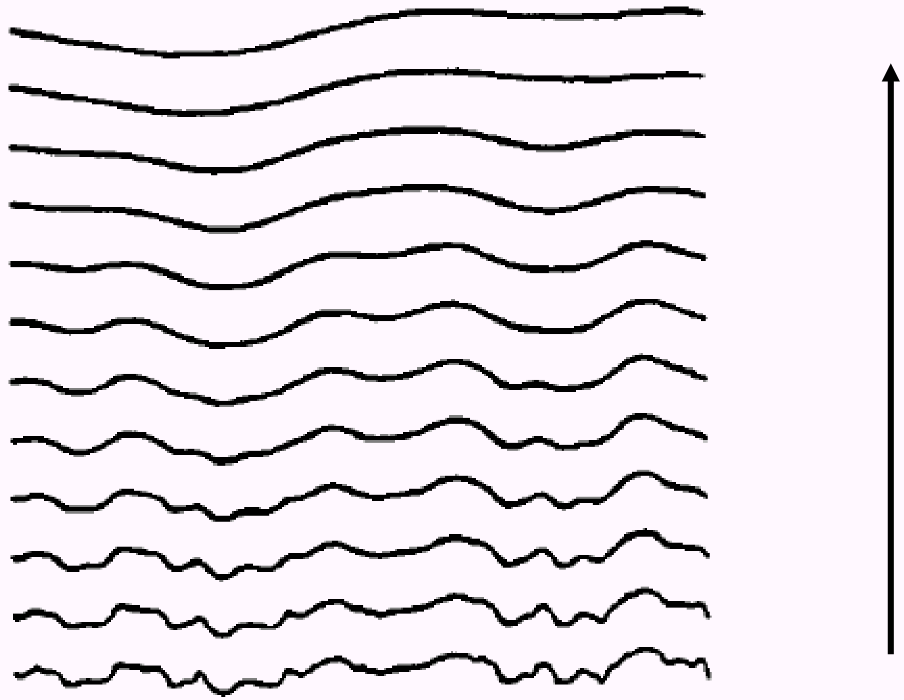


$\sigma = 1$ (5X5 kernel size)

$\sigma = 2$ (9X9 kernel size)

http://www.dai.ed.ac.uk/HIPR2/gsmooth.htm

Filtering based on Gaussian low-pass filter



The value of t (variance) is increasing.

Fig. 1. A family of 1-D signals $I(x, t)$ obtained by convolving the original one (bottom) with Gaussian kernels whose variance increases from bottom to top (adapted from Witkin [21]).

A family of smoother images is formed by convolving the original image with a Gaussian kernel (filter) of increasing variance.

Perona and Malik

Concepts of diffusion

1. The first law of diffusion (Fick's law) – mass flux (diffusion) is proportional to the concentration gradient (change in concentration)

$$\vec{J} = -D \cdot \nabla u$$

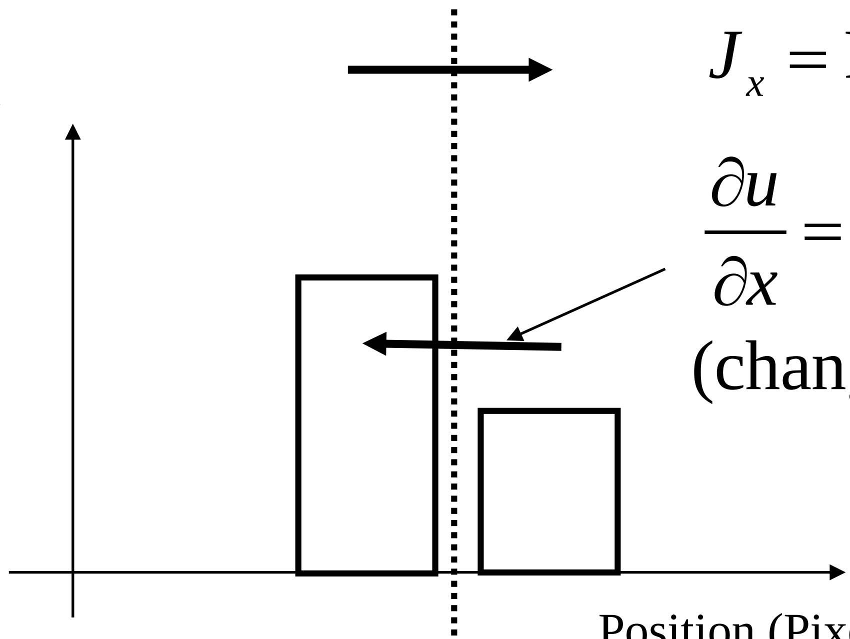
$$\left(J_x, J_y\right) = -D \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$$J_x = \text{Flux} = -\frac{\partial \text{u}}{\partial \text{x}}$$

$$\frac{\partial u}{\partial x} = \text{ gradient}$$

(change in concentration)

Concentration, *u*
(Intensity)

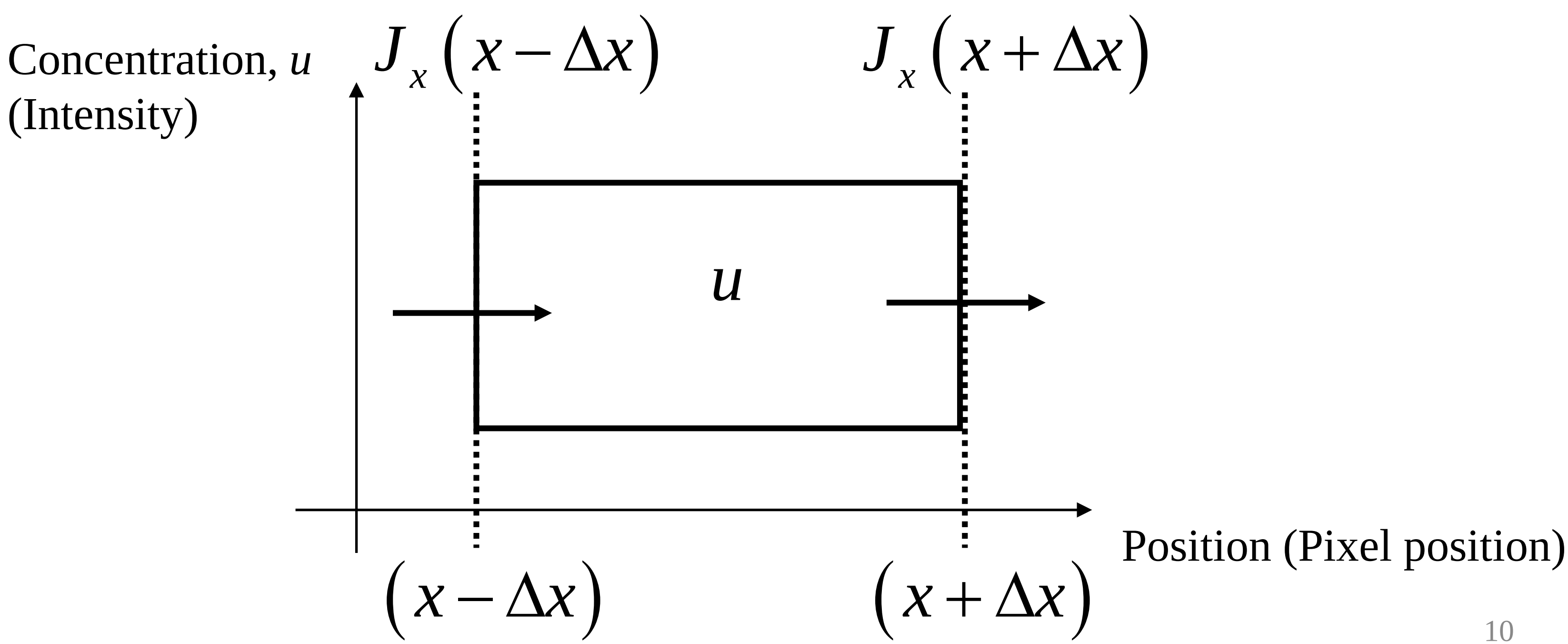


Position (Pixel position)

Concepts of diffusion

2. The second law of diffusion (Fick's law) – the rate of accumulation of concentration within a volume is proportional to the change of local concentration gradient (continuity equation).

For example, for 1D $$\frac{\partial u}{\partial t} = -\frac{\partial J_x}{\partial x}$$



Concepts of diffusion

3. Equation of diffusion (Heat equation)

E.g., for 1D $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$ where $D =$ diffusivity

or Conductance

or Diffusion constant

E.g., for 2D $\dfrac{\partial u}{\partial t} = \mathrm{div}\left(D \cdot \nabla u\right)$

where $D =$ diffusivity tensor

$$\mathrm{div}\left(\vec{x}\right) = \nabla \cdot \vec{x}$$

Concepts of diffusion

4. Linear diffusion filtering: $D = I$, it means heat (intensity) diffuses isotropically.

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\nabla u\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Initial condition

$$u\left(\vec{x}, t = 0\right) = I\left(\vec{x}\right)$$

Remark: The low-pass Gaussian filtering can be viewed as the solution of the heat equation.

Linear diffusion filtering

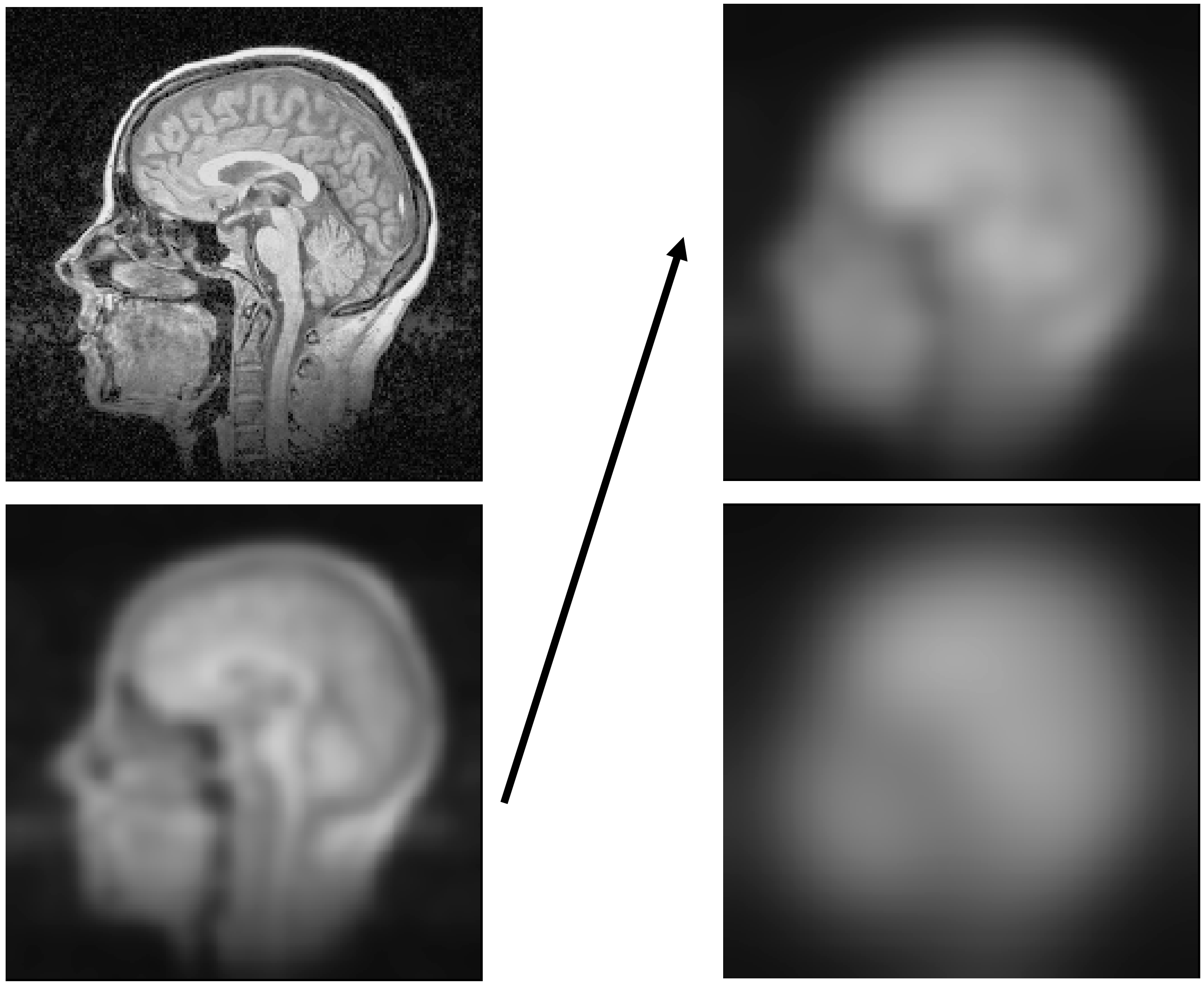
1. Assume that $D = I$.
2. Heat equation (2D)

$$\frac{\partial u}{\partial t} = \Delta u$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Initial condition $u\left(\vec{x}, t = 0\right) = I\left(\vec{x}\right)$

Linear diffusion filtering



Linear diffusion examples: t=0, 12.5, 50, 200.

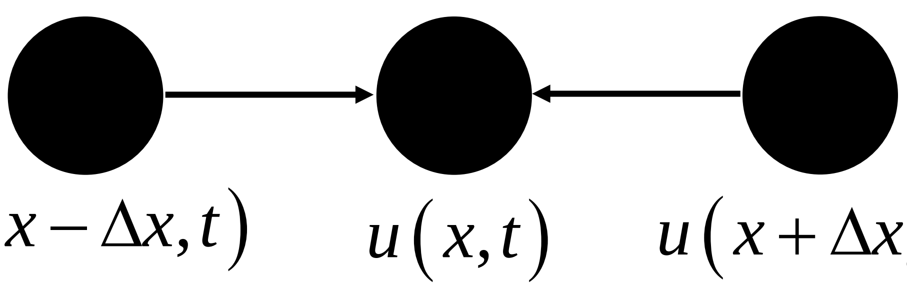
Linear diffusion filtering

3. Implementation. For example, in 1D

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition $\quad u\left(x, t=0\right) = I\left(x\right)$

$$u\left(x, t+\Delta t\right) = u\left(x, t\right) + \frac{\Delta t}{\left(\Delta x\right)^2}\left(u\left(x+\Delta x, t\right) - 2u\left(x, t\right) + u\left(x-\Delta x, t\right)\right)$$



Linear diffusion filtering

4. Gaussian smoothing / linear diffusion filtering does not only reduce noise, but also blurs important features such as edges and thus makes them harder to identify (see Fig. 1 on next page).

5. Gaussian smoothing / linear diffusion filtering dislocates edges when moving from finer to coarser scales (see Fig. 2 on next page).

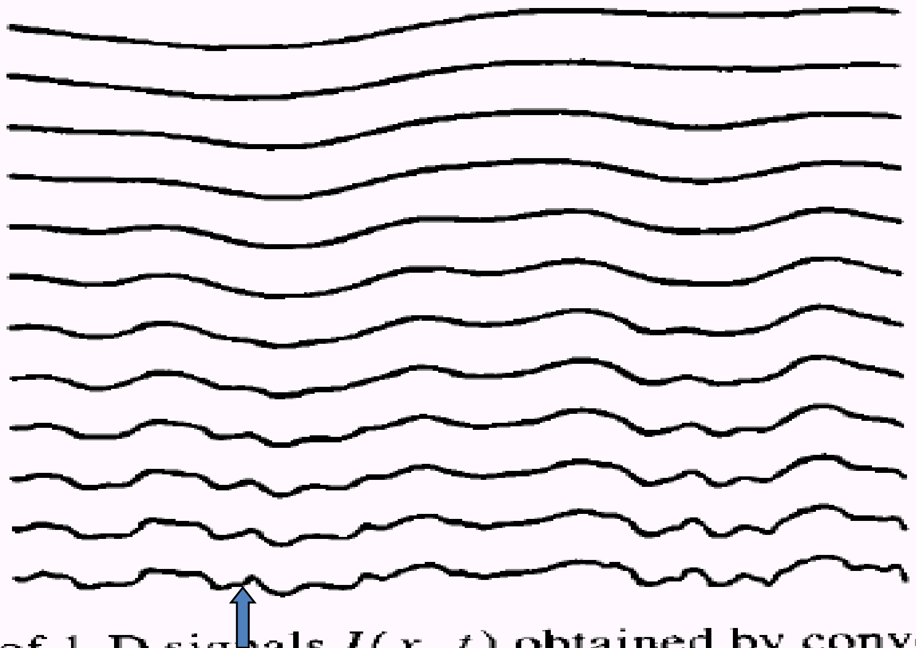


Fig. 1. A family of 1-D signals $I(x, t)$ obtained by convolving the original one (bottom) with Gaussian kernels whose variance increases from bottom to top (adapted from Witkin [21]).

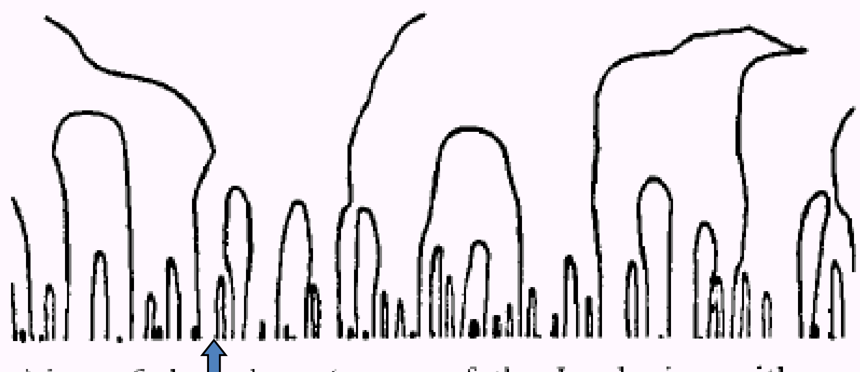


Fig. 2. Position of the edges (zeros of the Laplacian with respect to x) through the linear scale space of Fig. 1 (adapted from Witkin [21]).

Non-linear isotropic diffusion filtering

1. We want to encourage smoothing within a region in preference to smoothing across the boundaries.
2. This could be achieved by setting the diffusivity to be 0 on the boundary and to be non-zero in the interior of the region.

E.g., for 2D $$\frac{\partial u}{\partial t} = \mathrm{div}\left(D\ \nabla u\right)$$

where $$D = g\left(\left|\nabla u\right|\right)$$

$$g\left(\left|\nabla u\right|\right) = \exp\left(-\left(\frac{\left|\nabla u\right|}{K}\right)^2\right)$$ or $$g\left(\left|\nabla u\right|\right) = \frac{1}{1+\left(\frac{\left|\nabla u\right|}{K}\right)^2}$$

Non-linear isotropic diffusion filtering

3. Implementation. For example, in 2D

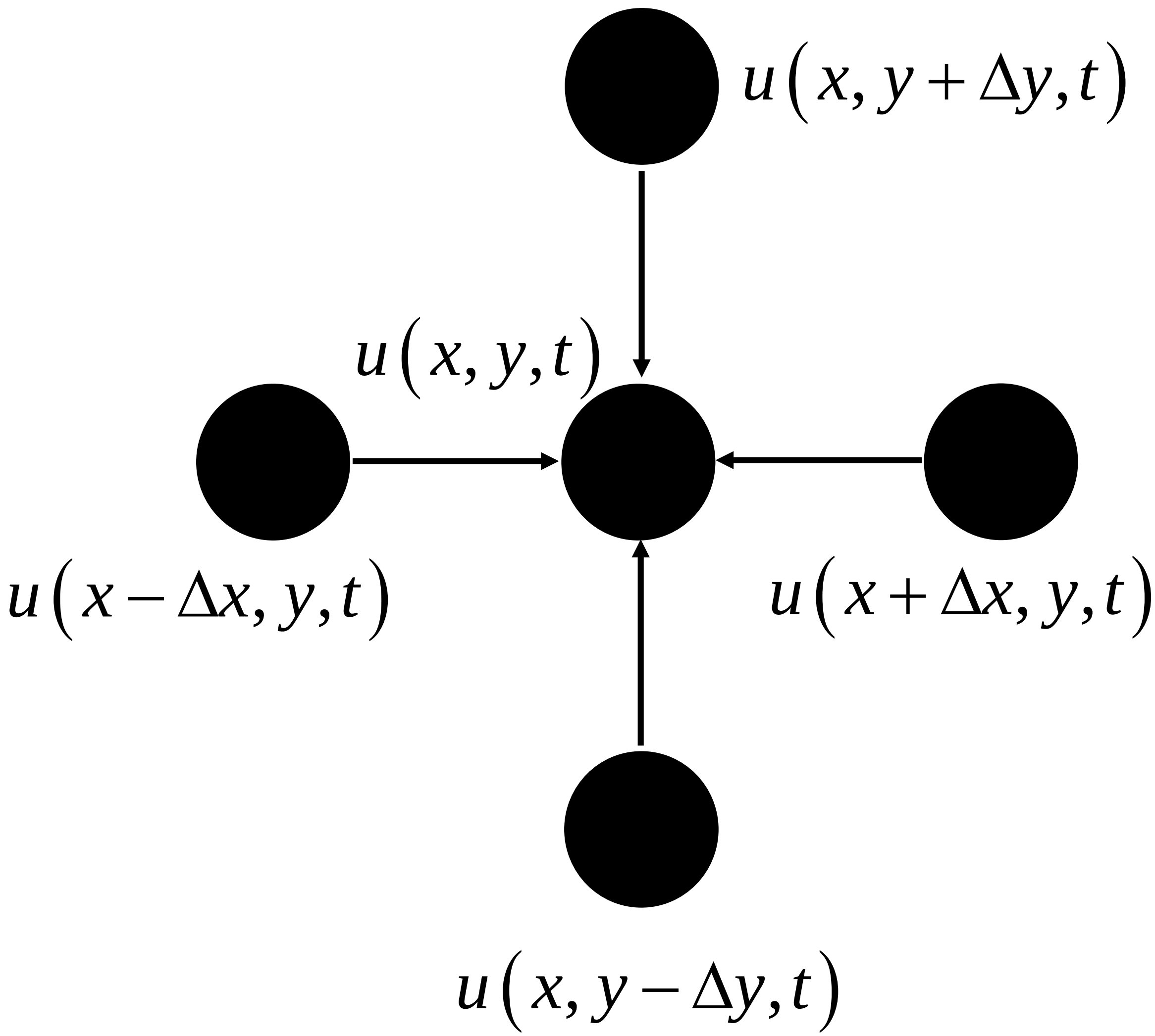
$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g \nabla u \right)$$

Initial condition $\quad u\left(\vec{x}, t=0\right) = I\left(\vec{x}\right)$

$$u\left(x, y, t+\Delta t\right) = u\left(x, y, t\right)$$

$$+\Delta t\left[\left(\Phi_N - \Phi_S\right) + \left(\Phi_E - \Phi_W\right)\right]$$

Non-linear isotropic diffusion filtering



Non-linear isotropic diffusion filtering

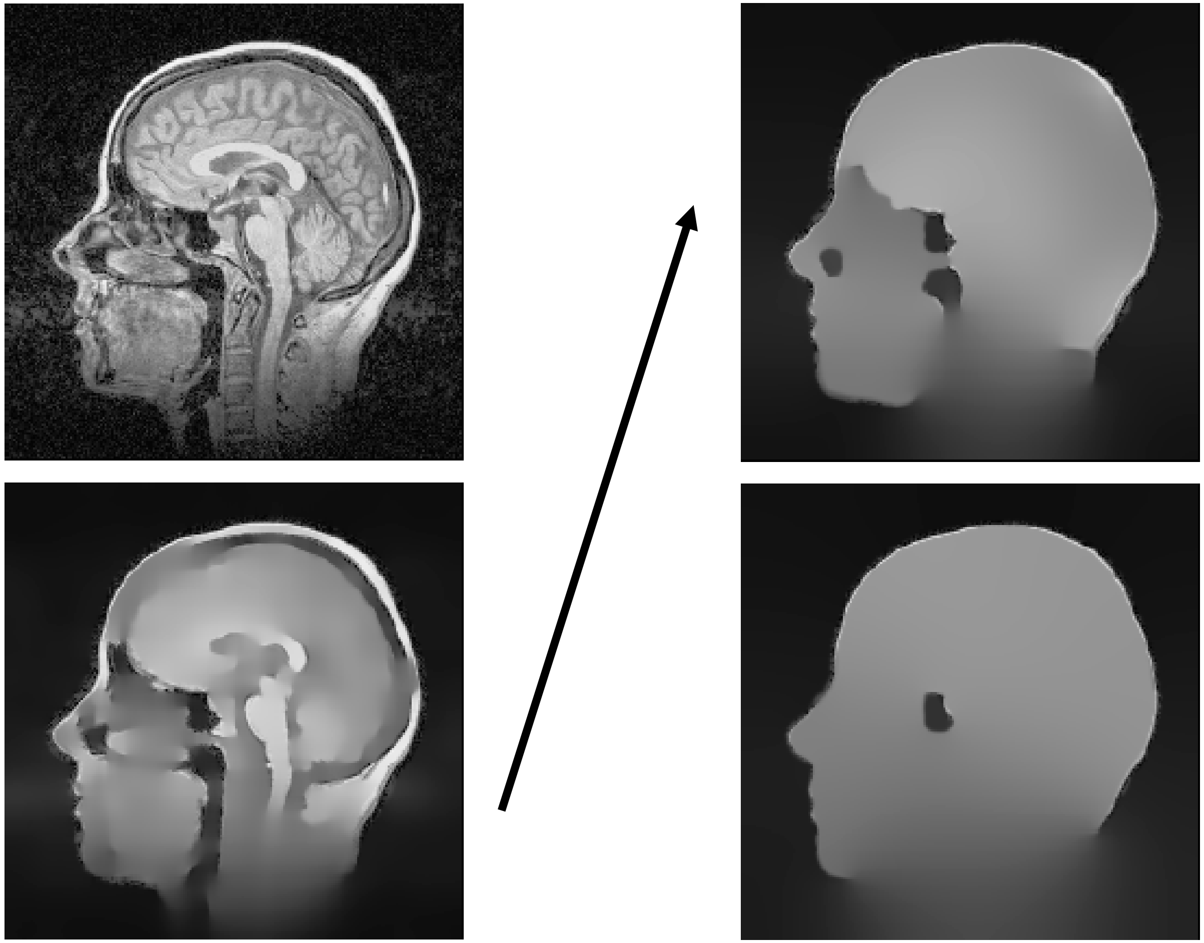
$$\Phi_N = g\left(\left|\nabla u\left(x, y+\frac{\Delta y}{2}, t\right)\right|\right)\left(\frac{u\left(x, y+\Delta y, t\right)-u\left(x, y, t\right)}{\Delta y^2}\right)$$

$$\Phi_S = g\left(\left|\nabla u\left(x, y-\frac{\Delta y}{2}, t\right)\right|\right)\left(\frac{u\left(x, y, t\right)-u\left(x, y-\Delta y, t\right)}{\Delta y^2}\right)$$

$$\Phi_E = g\left(\left|\nabla u\left(x+\frac{\Delta x}{2}, y, t\right)\right|\right)\left(\frac{u\left(x+\Delta x, y, t\right)-u\left(x, y, t\right)}{\Delta x^2}\right)$$

$$\Phi_W = g\left(\left|\nabla u\left(x-\frac{\Delta x}{2}, y, t\right)\right|\right)\left(\frac{u\left(x, y, t\right)-u\left(x-\Delta x, y, t\right)}{\Delta x^2}\right)$$

Non-linear isotropic diffusion filtering



Nonlinear isotropic diffusion examples: K=3,
t=0, 40, 400, 1500.

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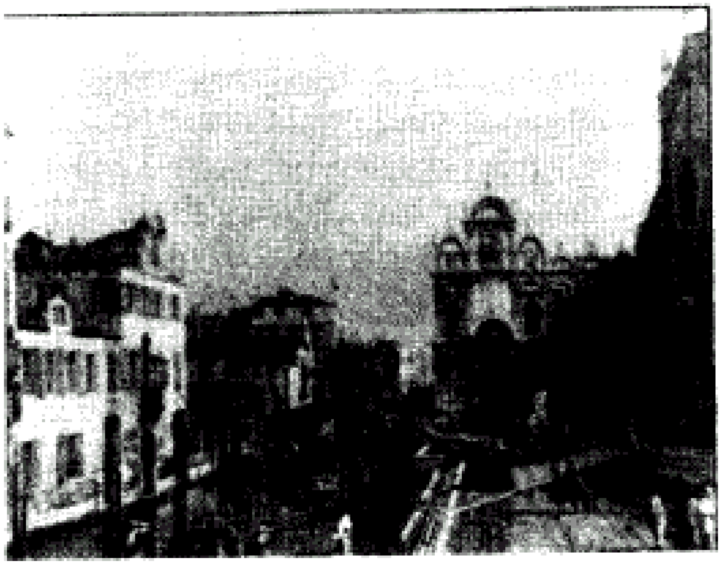
Linear isotropic diffusion

Nonlinear isotropic diffusion

$$\frac{\partial u}{\partial t} = \Delta u$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g \cdot \nabla u\right)$$

Perona and Malik



(a)

(b)

(a)

(b)

Fig. 10. Edges detected using (a) anisotropic diffusion and (b) Gaussian smoothing (Canny detector).

Perona and Malik

Non-linear anisotropic diffusion filtering

1. For isotropic diffusion, the diffusion direction is always parallel to the gradient vector ∇u.
2. Let $\vec{v} \perp \nabla u$ be a flux perpendicular to the direction of gradient ∇u. For example,

If $$\vec{v} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x} \right)$$ Tangent vector

then $$\frac{\partial u}{\partial t} = \mathrm{div}\left(\vec{v}\right) = 0$$ No smoothing/filtering along edge

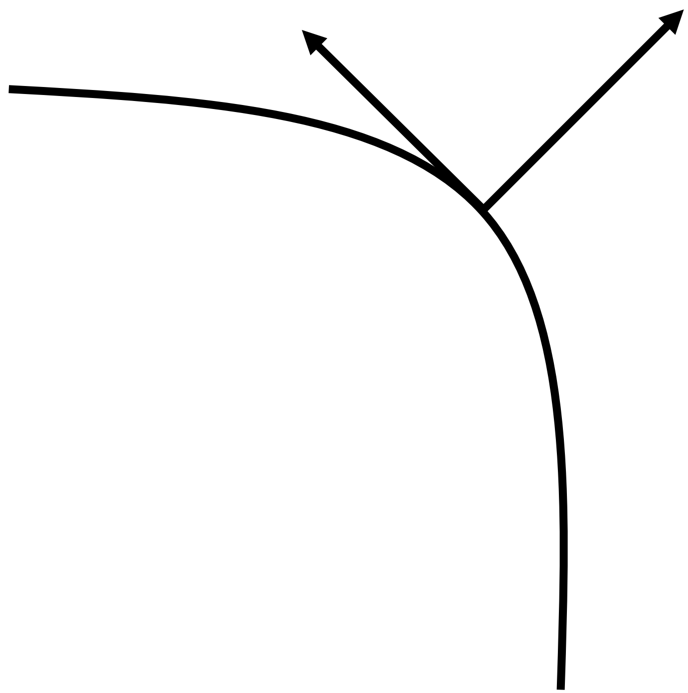
3. Diffusion favours the gradient direction.
   a. Interior region: linear filter
   b. Boundary region: no diffusion if gradient function $g\left(\left|\nabla u\right|\right)$ is used. Problem: no smoothing along edge.

Non-linear anisotropic diffusion filtering

Zero flux and hence
no diffusion
(along the negative
tangent vector)

Maximum flux for
diffusion
(along the negative
gradient vector)

$\vec{v} \perp \nabla u$ $\quad\quad$ ∇u



Edge-enhancing anisotropic diffusion

4. Smoothing along edge can be achieved by diffusing along the negative gradient and negative tangent vectors of a Gaussian intensity smoothed image u_σ .
5. Definitions:

$$u_\sigma = G_\sigma * u$$

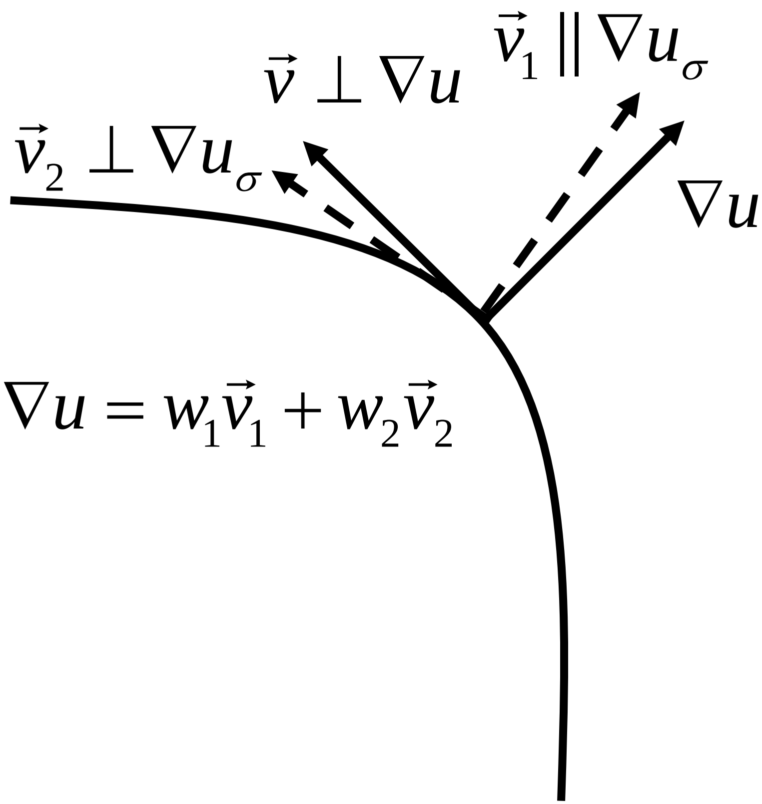
$$G_\sigma\left(x, y\right) = \frac{1}{2\pi\sigma^2} e^{\frac{-x^2-y^2}{2\sigma^2}}$$

Gradient vector of the smoothed image $\vec{v}_1 \parallel \nabla u_\sigma$

Tangent vector of the smoothed image $\vec{v}_2 \perp \nabla u_\sigma$

Edge-enhancing anisotropic diffusion

6. In general, ∇u_σ will not parallel to ∇u .



$$\nabla u = w_1 \vec{v}_1 + w_2 \vec{v}_2$$

$$\lambda_1 = g\left(\left|\nabla u_\sigma\right|\right)$$

$$\lambda_2 = 1$$

7. If we assign a relatively large weight to $\vec{v}_2$ and a very small weight to $\vec{v}_1$, then smoothing along the edge is achieved.

Edge-enhancing anisotropic diffusion

8. Changes have to be made in the heat equation

Original $$\frac{\partial u}{\partial t} = \operatorname{div}\left(D \cdot \nabla u\right)$$ Heat equation

$$\Big\downarrow$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\overleftrightarrow{D} \cdot \nabla u\right)$$

where $\overleftrightarrow{D}$ represents a matrix with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, and with eigenvalues λ_1 and λ_2 .

$$\overleftrightarrow{D} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$$

$$\vec{v}_1 \parallel \nabla u_\sigma \;, \vec{v}_2 \perp \nabla u_\sigma \;, \lambda_1 = g\left(\left|\nabla u_\sigma\right|\right) \text{ and } \lambda_2 = 1$$

Notes on Matrix, eigenvectors and eigenvalues

http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

$$\overleftrightarrow{A}\cdot\vec{v} = \lambda\vec{v}$$

$$\left(\overleftrightarrow{A} - \lambda\overleftrightarrow{I}\right)\cdot\vec{v} = \vec{0}$$

Let the eigenvectors be $\vec{v}_1$ and $\vec{v}_2$.

Let the eigenvalues be λ_1 and λ_2 .

$$\overleftrightarrow{A} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}^{-1}$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\overleftrightarrow{D} \cdot \nabla u\right)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\overleftrightarrow{D} \cdot \left(w_1 \vec{v}_1 + w_2 \vec{v}_2\right)\right)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(w_1 \overleftrightarrow{D} \cdot \vec{v}_1 + w_2 \overleftrightarrow{D} \cdot \vec{v}_2\right)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(w_1 \lambda_1 \vec{v}_1 + w_2 \lambda_2 \vec{v}_2\right)$$

$$\Downarrow$$

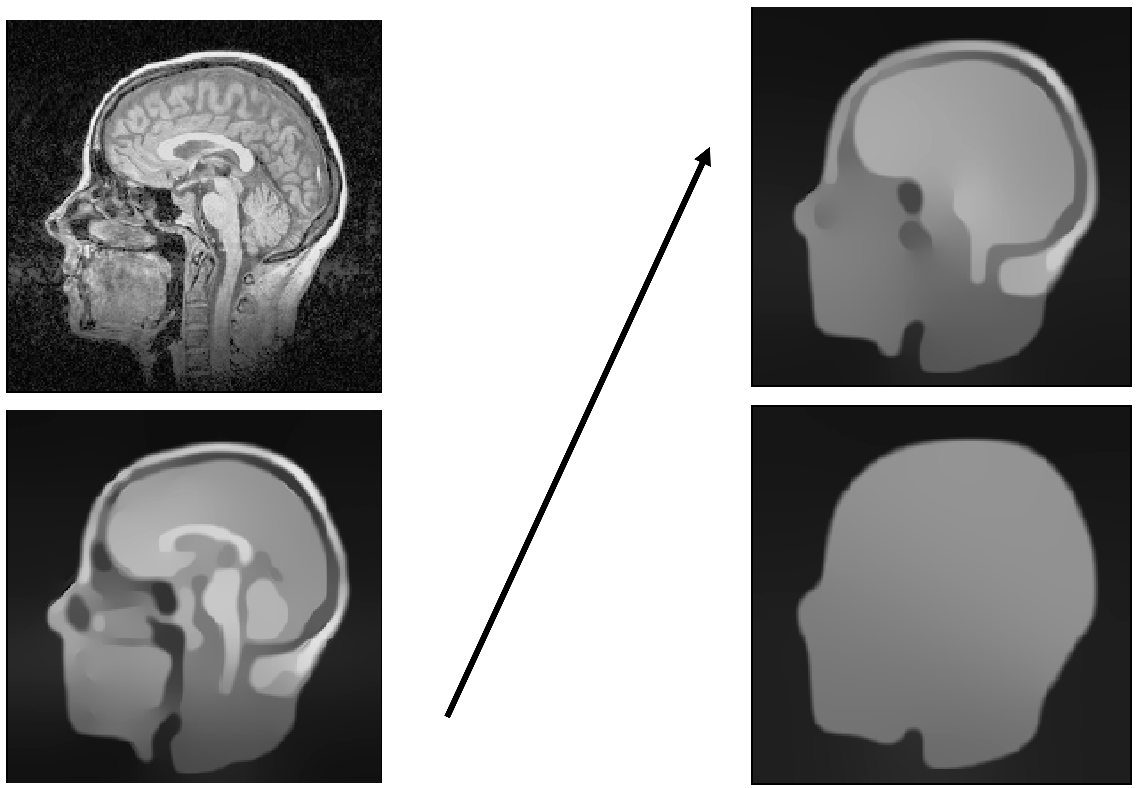
At the boundary, $\lambda_1 \approx 0$

Therefore, $\dfrac{\partial u}{\partial t} = w_2 \lambda_2 \mathrm{div}\left(\vec{v}_2\right)$

This represents diffusion along edge.

Edge-enhancing anisotropic diffusion

9. The new model behaves anisotropic.
10. When $\sigma \to 0$, the new model switches back to the isotropic diffusion method.



Nonlinear anisotropic diffusion examples: K=3, t=0, 250, 875, 3000.

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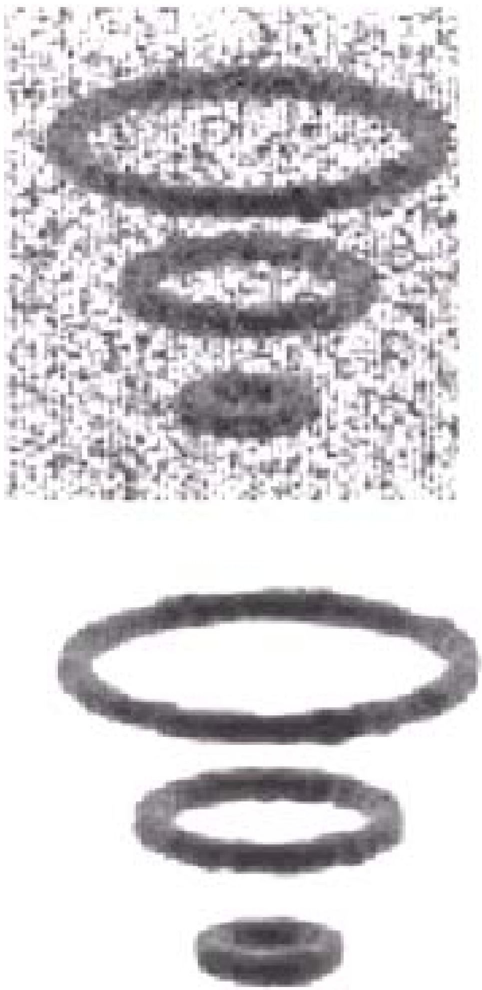
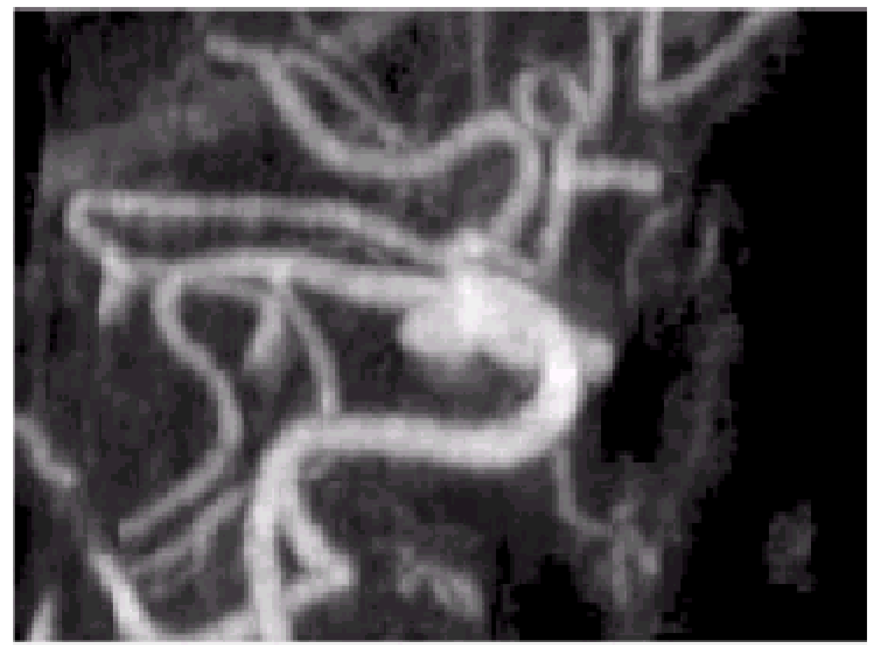
Edge-enhancing anisotropic diffusion



Enhancement of tubular structures
(Karl Krissian, et al.)



Enhancement of vessels
(Karl Krissian, et al.)

Coherence-enhancing anisotropic diffusion

Fig. 4. Coherence-enhancing anisotropic diffusion of a fingerprint image. (a) LEFT: Original image, $\Omega = (0, 256)^2$. (b) RIGHT: Filtered, $\sigma = 0.5$, $\rho = 4$, $t = 20$. From [74].

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Coherence-enhancing anisotropic diffusion

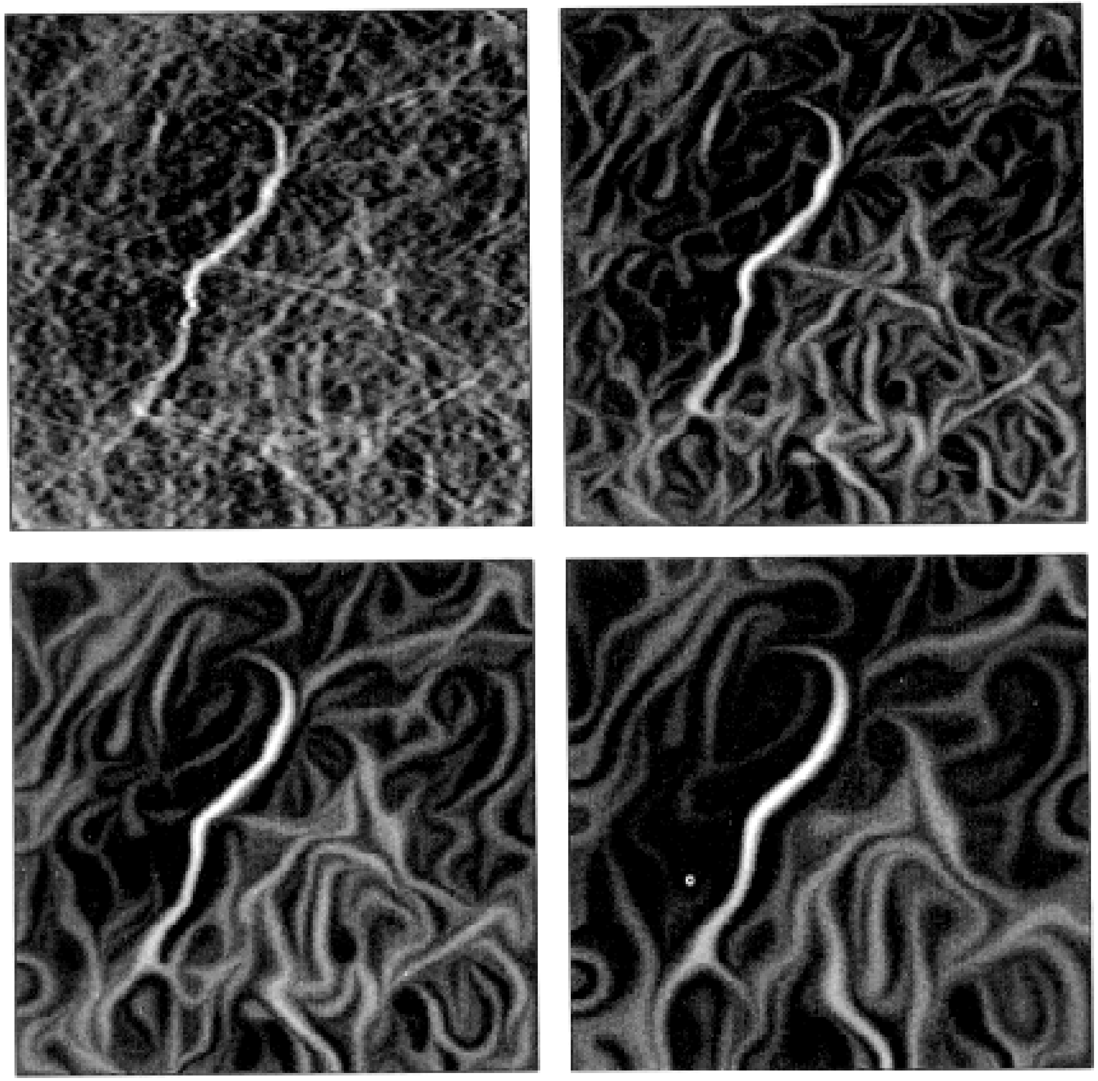

Figure 5.12: Scale-space behaviour of coherence-enhancing diffusion ($\sigma = 0.5$, $\rho = 2$). (a) TOP LEFT: Original fabric image, $\Omega = (0, 257)^2$. (b) TOP RIGHT: $t = 20$. (c) BOTTOM LEFT: $t = 120$. (d) BOTTOM RIGHT: $t = 640$.

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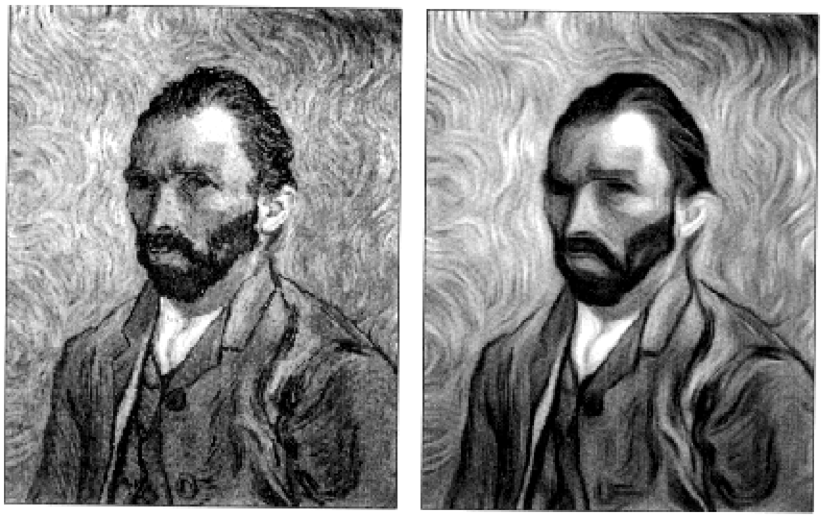

Figure 5.14: Image restoration using coherence-enhancing anisotropic diffusion. (a) LEFT: "Selfportrait" by van Gogh (Saint-Rémy, 1889, Paris, Museé d'Orsay). $\Omega = (0, 215) \times (0, 275)$. (b) RIGHT: Filtered, $\sigma = 0.5$, $\rho = 4$, $t = 6$.

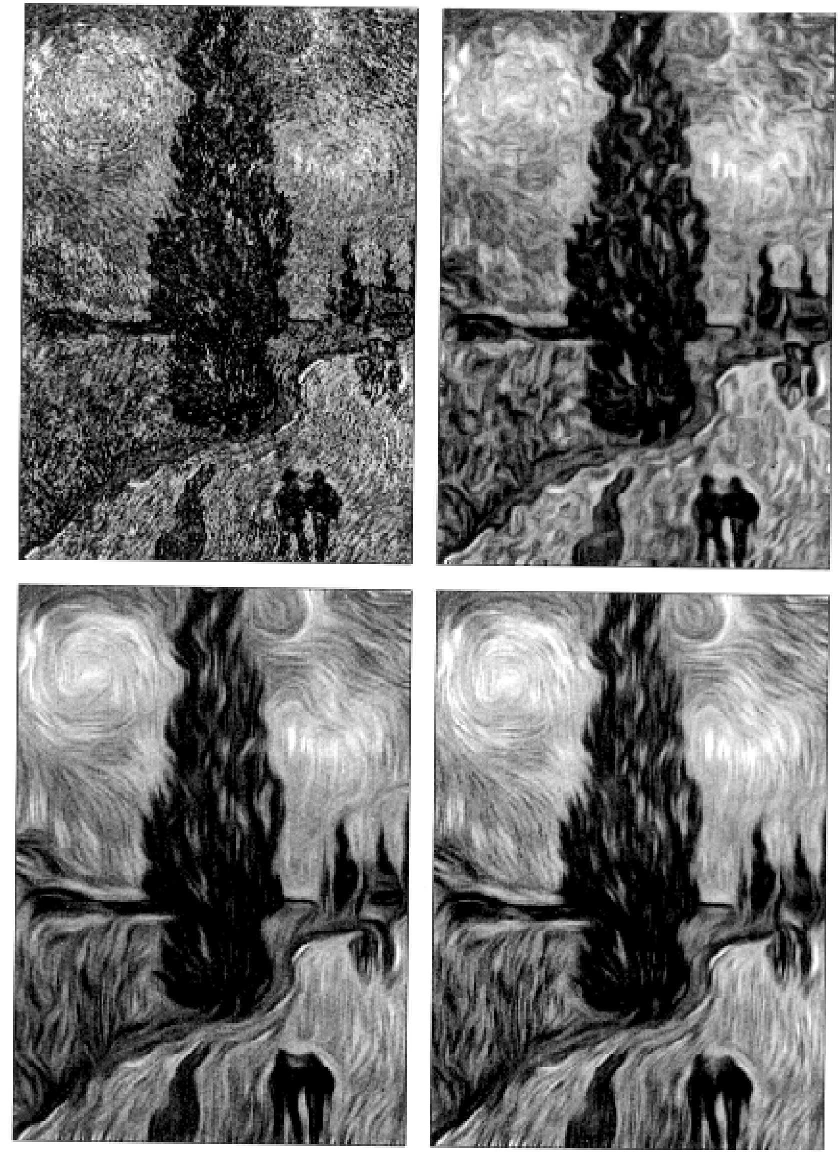

Figure 5.15: Impact of the integration scale on coherence-enhancing anisotropic diffusion ($\sigma = 0.5$, $t = 8$). (a) TOP LEFT: "Road with Cypress and Star" by van Gogh (Auvers-sur-Oise, 1890; Otterlo, Rijksmuseum Kröller-Müller), $\Omega = (0, 203) \times (0, 290)$. (b) TOP RIGHT: Filtered with $\rho = 1$. (c) BOTTOM LEFT: $\rho = 4$. (d) BOTTOM RIGHT: $\rho = 6$.

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