COMP 2711H Discrete Mathematical Tools for Computer Science
2014 Fall Semester
Homework 3
Handed out: Oct 27
Due: Nov 5

Problem 1. What is the probability that a set of 13 cards randomly selected from a standard set of 52 playing cards contains no pairs?

Problem 2. Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than mn is not divisible by either m or n?

Problem 3. Timothy can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will late. The probability that he is late when he rides his bicycle is only 5%. Timothy arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

(a) Suppose the boss assumes that there is a 1/3 chance that Timothy takes each of the three ways he can get to work. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem under this assumption?

(b) Suppose the boss knows that Timothy drives 30% of the time, takes the bus only 10% of the time,, and takes his bicycle 60% of the time. What estimate for the probability that Timothy drove his car does the boss obtain from Bayes' theorem using this information?

Problem 4. Show that the sum of the probabilities of a random variable with geometric distribution with parameter p, where $0 < p \leq 1$, equals 1.

Problem 5. A *run* is a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence $S, S, S, F, S, S, F, F, S$, where S represents success and F represents failure, there are three runs consisting of three successes, two successes, and one success, respectively. Let R denote the random variable on the set of sequences of n independent Bernoulli trials that counts the number of runs in this sequence. Find $E(R)$.

Problem 6. Let $X(s)$ be a random variable, where $X(s)$ is a nonnegative integer for all $s \in S$, and let A_k be the event that $X(s) \geq k$. Show that $E(X) = \sum_{k=1}^{\infty} p(A_k)$.

Problem 7. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

(a) What is the expected value of X_n?

(b) What is the variance of X_n?

Problem 8. Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.

Problem 9. Consider the following game. A person flips a coin repeatedly until a head comes up. This person receives a payment of 2^n dollars if the first head comes up at the n-th flip.

(a) Let X be a random variable equal to the amount of money the person wins. Show that the expected value of X does not exist (that is, it is infinite). Show that a *rational gambler*, that is, someone willing to pay to play the game as long as the price to play is not more than the expected payoff, should be willing to wager any amount of money to play this game. (This is knows as the *St. Petersburg paradox.* Why do you suppose it is called a paradox?)

(b) Suppse that the person receives 2^n dollars if the first head comes up on the n-th flip where $n < 8$ and $2^8 = 256$ dollars if the first head comes up on or after the eighth flip. What is the expected value of the amount of money the person wins? How much should a rational gambler be willing to pay to play this game?

Problem 10. Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.

(a) Find the probability that a particular ball lands in a specified bin.

(b) What is the expected number of balls that land in a particular bin?

(c) What is the expected number of balls tossed until a particular bin contains a ball?

(d) What is the expected number of balls tossed until all bins contain a ball?

Problem 11. Let X be a nonnegative random variable with mean μ. Show that for any positive real a,

$$p(X \geq a) \leq \mu/a.$$

This inequality is called *Markov's inequality.*

Problem 12. Let X be a random variable with mean μ and standard deviation σ. Show that for any positive real α,

$$p(|X - \mu| \geq \alpha\sigma) \leq 1/\alpha^2.$$

This inequality is called *Chebyshev's inequality.*

Problem 13. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $5\sqrt{n}$.

Problem 14. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed n times deviates from the mean by more than $\sqrt{n}$.

Problem 15. Recall the hatcheck problem studied in class. Use Chebyshev's inequality to show that the probability that more than 10 people get the correct hat back when a hatcheck person returns hats at random does not exceed $1/100$ no matter how many people check their hats.

Problem 16. You are given a biased coin (that is, the probability of getting "head" is not equal to $1/2$). Describe how you can use this biased coin to simulate a fair coin. In order to simulate one toss of a fair coin, you are allowed to toss the biased coin more than once. Further, you may assume that the probability of getting head for the biased coin remains the same for each coin toss.