Lecture 1: Sets, Relations, and Functions (a Review)

Sets

- A *set* is a collection of objects.
  Example: $L = \{a, b, c, d\}$ is a set of four *elements*.
- An element of a set can also be a set.
- Ignore repetitions of elements in a set; ignore the order of elements.
- $a \in L$ denotes a is *in* L, and $z \notin L$ denotes z is not in L.
- A set containing only one element is called a *singleton*: $\{a\} \neq a$.
- The *empty set*, denoted by $\emptyset$, contains no elements.
- A set is *finite* if it has a finite number of elements. Otherwise, the set is *infinite*.
- A is a *subset* of B, denoted $A \subseteq B$, if every element in A is an element in B.
- Two sets A and B are *equal* ($A = B$) if $A \subseteq B$ and $B \subseteq A$.

- $A \subset B$ means A is a *proper subset* of B (i.e., $A \subseteq B$ and $A \neq B$). By this definition, $\emptyset$ is a proper subset of any non-empty set.

- We can define a set by listing all its elements: $L = \{a, b, c, d\}$ or $\mathcal{N} = \{0, 1, 2, \ldots\}$ or by specifying the conditions the elements should have: $O = \{x : x \in \mathcal{N}, x \text{ is not divisible by } 2\}$.

Set operations. Let A and B be sets.

- *Intersection*: $A \cap B = \{x : x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$, we say that A and B are *disjoint*.
- *Union*: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- *Difference*: $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Power sets and partition.

- The set of all subsets of a set A, denoted by 2^A, is called the *power set* of A. For example, $2^{\{c,d\}} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$.
- What is the size of 2^A for an arbitrary A? 2^A is "much larger than" A (we will define the meaning precisely later).
- A *partition* of A is any set of non-empty subsets of A, denoted by $A_1, A_2, \cdots$, such that (a) $A_1 \cup A_2 \cup \cdots = A$, and (b) $A_i \cap A_j = \emptyset$ for all $i \neq j$ (i.e. mutually disjoint).

Cartesian product of A and B, denoted by $A \times B$, is the set of all possible *ordered pairs* (a, b) with $a \in A$ and $b \in B$.

$$\{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

$A \times A = A^2$.

A **binary relation** defined on A and B is a subset of $A \times B$.
Similarly generated to n-**ary relations**.

A **function** from A to B, $f : A \to B$, is a special type of relation $R \subseteq A \times B$ such that for each element $a \in A$, there is exactly one ordered pair in R with the first component a.
A is called the *domain* of f and B is called the *range*.
Example:

> Let $f = R = \{(1, 3), (2, 4)\} \subseteq A \times B$. Then, $f(1) = 3$ and $f(2) = 4$.

$f(a)$ is called the *image* of a (under f).
Similarly generated to functions with n arguments.

Bijection.

- $f : A \rightarrow B$ is *one-to-one* (or *injective*) if for any two distinct $a, a' \in A$, $f(a) \neq f(a')$.
- $f : A \rightarrow B$ is *onto* (or *surjective*) if for any $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
- f is a *bijection* if it is both one-to-one and onto.
- Example:

  Let $A = \{1, 2\}$ and $B = \{3, 4\}$.
  $f_1 = \{(1, 3), (2, 4)\}$ is a bijection,
  while $f_2 = \{(1, 4), (2, 4)\}$ is neither one-one nor onto.

Reflexivity, symmetry, and transitivity. Let $R \subseteq A \times A$ be a relation.

- R is *reflexive*: $(a, a) \in R$ for each $a \in A$.
- R is *symmetric*: if $(a, b) \in R$, then $(b, a) \in R$
- R is *transitive*: if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
- R is an *equivalence relation* if it is reflexive, symmetric and transitive.

$R \subseteq A \times A$ can be represented as a directed graph where an arrow is drawn from a to b if and only if $(a, b) \in R$. Example:

> Let $A = \{1, 2, 3, 4\}$. Draw the directed graph to represent the equivalence relation $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$:

The "clusters" of an equivalence relation are called its equivalence classes.

> $[1] = \{1, 2, 3\}$, $[4] = \{4\}$.