COMP3711: Design and Analysis of Algorithms

Tutorial 3 (Optional)

HKUST

Question 1

Professor Diogenes has n supposedly identical VLSI chips that in principle are capable of testing each other. The professor's test jig accommodates two chips at a time. When the jig is loaded, each chip tests the other and reports whether it is good or bad. A good chip always reports accurately whether the other chip is good or bad, but the answer of a bad chip cannot be trusted. Thus, the four possible outcomes of a test are as follows:

Chip A says	Chip B says	Conclusion
B is good	A is good	both are good, or both are bad
B is good	A is bad	at least one is bad
B is bad	A is good	at least one is bad
B is bad	A is bad	at least one is bad

Question 1

(a) Show that if more than $n/2$ chips are bad, the professor cannot necessarily determine which chips are good using any strategy based on this kind of pairwise test. Assume that the bad chips can conspire to fool the professor.

(b) Consider the problem of finding a single good chip from among n chips, assuming that more than $n/2$ of the chips are good. Show that $\lceil n/2 \rceil$ pairwise tests are sufficient to reduce the problem to one of nearly half the size.

(c) Show that the good chips can be identified with $\Theta(n)$ pairwise tests, assuming that more than $n/2$ of the chips are good. Give and solve the recurrence that describes the number of tests.

Solution 1(a)

Note that this is possible to have certain bad chips always work as a good chip during the test. Let $B > n/2$ denote the number of bad chips and $G < n/2$ denote the number of good chips. This is possible to have G bad chips always work as the good chips during the testing, such that any testing results are confused by these G bad chips. Thus, there is no way to recognize whether a chip is really a good chip from the pairwise testing results.

More precisely, there can be G bad chips that pretend to be good chips, which we call "fake good chips". Each of these fake good chips will do the following. When it tests a good chip, it will say "bad"; when it tests another fake good chip, it will say "good"; when it tests one of the remaining $B - G$ bad chips, it will say "bad". Note that if all the fake good chips follow this strategy, all the testing results would be the same if the G fake good chip and the G true good chips switched their roles.

Solution 1(b)

Assume that more than $n/2$ of the chips are good.

1. Color all the chips as black.
2. For $1 \leq j \leq \lfloor n/2 \rfloor$, test the chips in pair of $((2j-1)$-th chip, $(2j)$-th chip). For each test, if the result is both are good chips, color one of the two chips as white.
3. Let m denote the number of white chips.
4. If n is odd and m is even, color the n-th chip as white.
5. The reduced problem contains only the white chips. The number of white chips is at most $\lceil n/2 \rceil$.

Solution 1(b)

Correctness:

Assume n is even. Suppose there are G good chips and B bad chips. We know that $G > B$. Suppose x good chips paired with each other, and $G - x$ of them are paired with bad chips. Note that the number of bad chips paired with each other is $B - (G - x) = B - G + x$. Thus, the number of white good chips is $x/2$, and the number of white bad chips is at most $(B - G + x)/2 < x/2$. So we still have more good chips than bad ones among the white chips, and the problem has been reduced to the at most $n/2$ white chips.

Solution 1(b)

Assume n is odd. Suppose there are G good chips and B bad chips among the first $n-1$ chips. Similar to the previous prove, we have the number of white good chips is $x/2$ and the number of white bad chips is $(B-G+x)/2$. The number of white chips among the first $n-1$ chips is $m = x-(G-B)/2$. If m is odd, we already have more good chips than bad ones among the white chips (without the n-th chip). If m is even and the n-th chip is bad, we have $G-B = 4k > 0$ for some positive integer k which implies $x/2-(B-G+x)/2 \geq 2$. Thus, we still have more good chips than bad ones among the white chips (with the n-th chip). If m is even and the n-th chip is good, we have $G-B = 4k \geq 0$ for some non-negative integer k. This is obvious that we have more good chips than bad chips among the white chips (with the n-th chip).

Solution 1(c)

Recursively apply the method in part (b) until the problem size is at most 2. Return any one of the chips from the at most 2 chips which must be good. The total number of pairwise tests for finding one good chip is

$$T(n) = T(\lceil n/2 \rceil) + \lceil n/2 \rceil \quad \text{for } n > 2$$
$$T(n) = 1 \quad \text{Otherwise}$$

Solve this recurrence we have $T(n) = \Theta(n)$.
Finally, use this good chip to test all the remaining $n-1$ chips for identifying all the good chips. Therefore, the total number of pairwise tests is $\Theta(n) + n - 1 = \Theta(n)$.