Morphological Image Processing for Binary Images

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Morphological Image Processing

1. It is a tool for extracting image components that are useful in the representation and description of region shape, such as boundaries, skeletons etc.
2. We use the language in the Set theory to represent morphological operations.
3. Link for more details: http://en.wikipedia.org/wiki/Mathematical_morphology

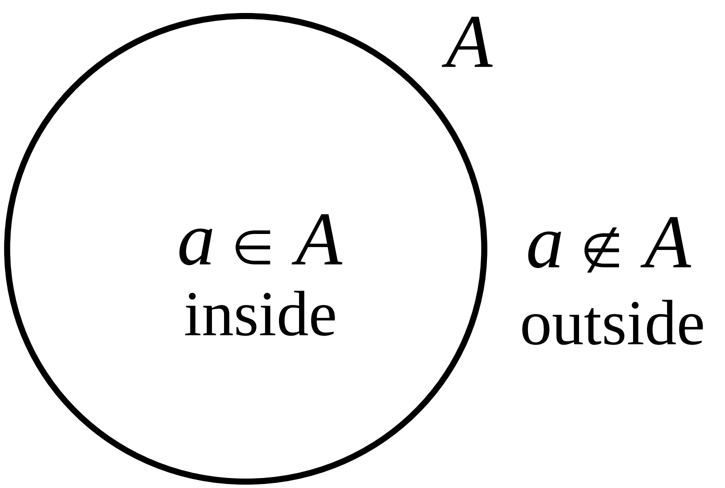
Basic Concepts from Set Theory

1. Let *A* be a set in Z^2 (2D integer space)
2. If $a = (a_1, a_2)$ is an element of *A*, where a_1 and a_2 are coordinates of pixels in 2D, then we write

$$a \in A$$

3. If a is not an element of A, then we write

$$a \notin A$$



Basic Concepts from Set Theory

4. If A is a set with no element, then we call it *null or empty set*.

$$A = \varnothing$$

5. A set is specified by the contents of two braces: { },
   a. e.g., $$C = \left\{ w \mid w = -d,\ \text{for } d \in D \right\}$$
   b. C represents a set of elements.
   c. w represents the elements.
   d. All elements inside C are formed by multiplying the element coordinates, d, by –1.

Basic Concepts from Set Theory

6. *A* is a subset of *B*: every element of a set *A* is also an element of another set *B*. (every element of *A* is in *B*)

$$A \subseteq B$$

7. Union of two sets *A* and *B*: the set of all elements belonging to either *A* or *B*. (All elements are either in *A* or *B*.)

$$C = A \cup B$$

Basic Concepts from Set Theory

8. Intersection of two sets A and B: the set of all elements belonging to both A and B. (All elements are in A and B.)

$$D = A \cap B$$

9. Two sets A and B are disjoint or mutually exclusive if their intersection is an empty set.

$$A \cap B = \varnothing$$

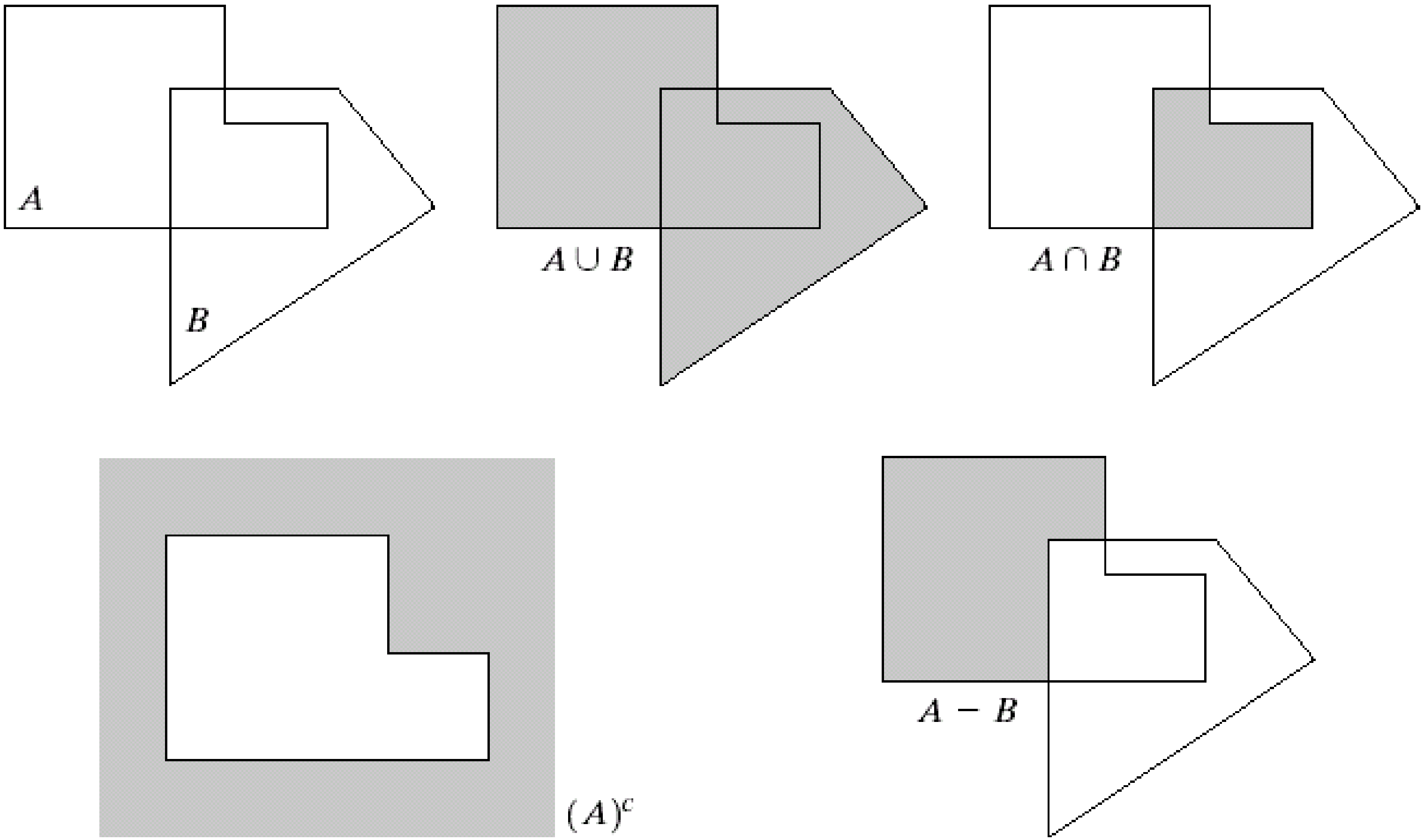
Basic Concepts from Set Theory

10. Complement of a set A: the set of elements not contained in A.

$$A^c = \{w \mid w \notin A\}$$

11. Difference of two sets A and B: the set of all elements belonging to A and not contained in B.

$$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$$



a b c
d e

FIGURE 9.1
(a) Two sets A and B. (b) The union of A and B. (c) The intersection of A and B. (d) The complement of A. (e) The difference between A and B.

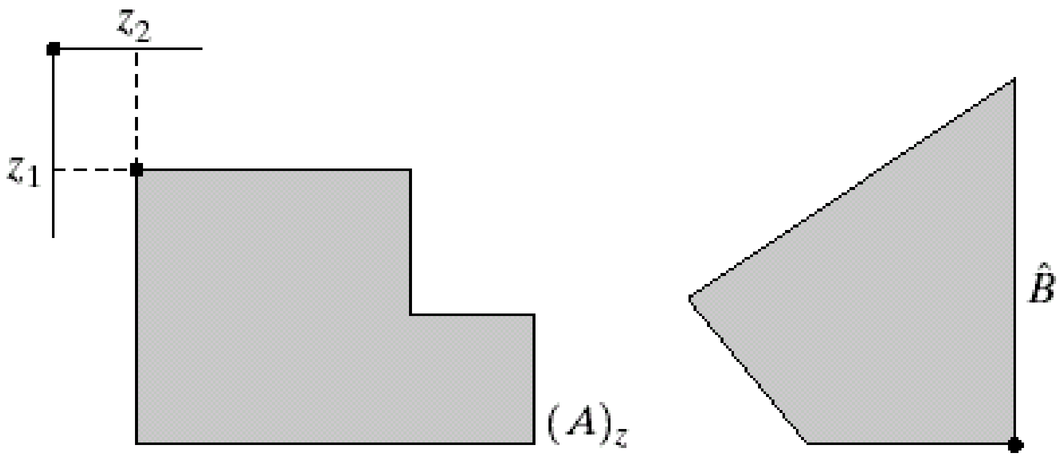
Basic Concepts from Set Theory

12. Reflection of set B:

$$\hat{B} = \{ w \mid w = -b, \text{ for } b \in B \}$$

13. Translation of set A:

$$(A)_z = \{ c \mid c = a + z, \text{ for } a \in A \}$$



a b

FIGURE 9.2
(a) Translation of A by z.
(b) Reflection of B. The sets A and B are from Fig. 9.1.

Logic Operations Involving Binary Images

1. All images are binary images.
2. In the images, black indicates a binary 1 and white indicates a 0.
3. Operations are limited to binary variables.

TABLE 9.1
The three basic logical operations.

p	q	p AND q (also $p \cdot q$)	p OR q (also $p + q$)	NOT (p) (also $\bar{p}$)
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

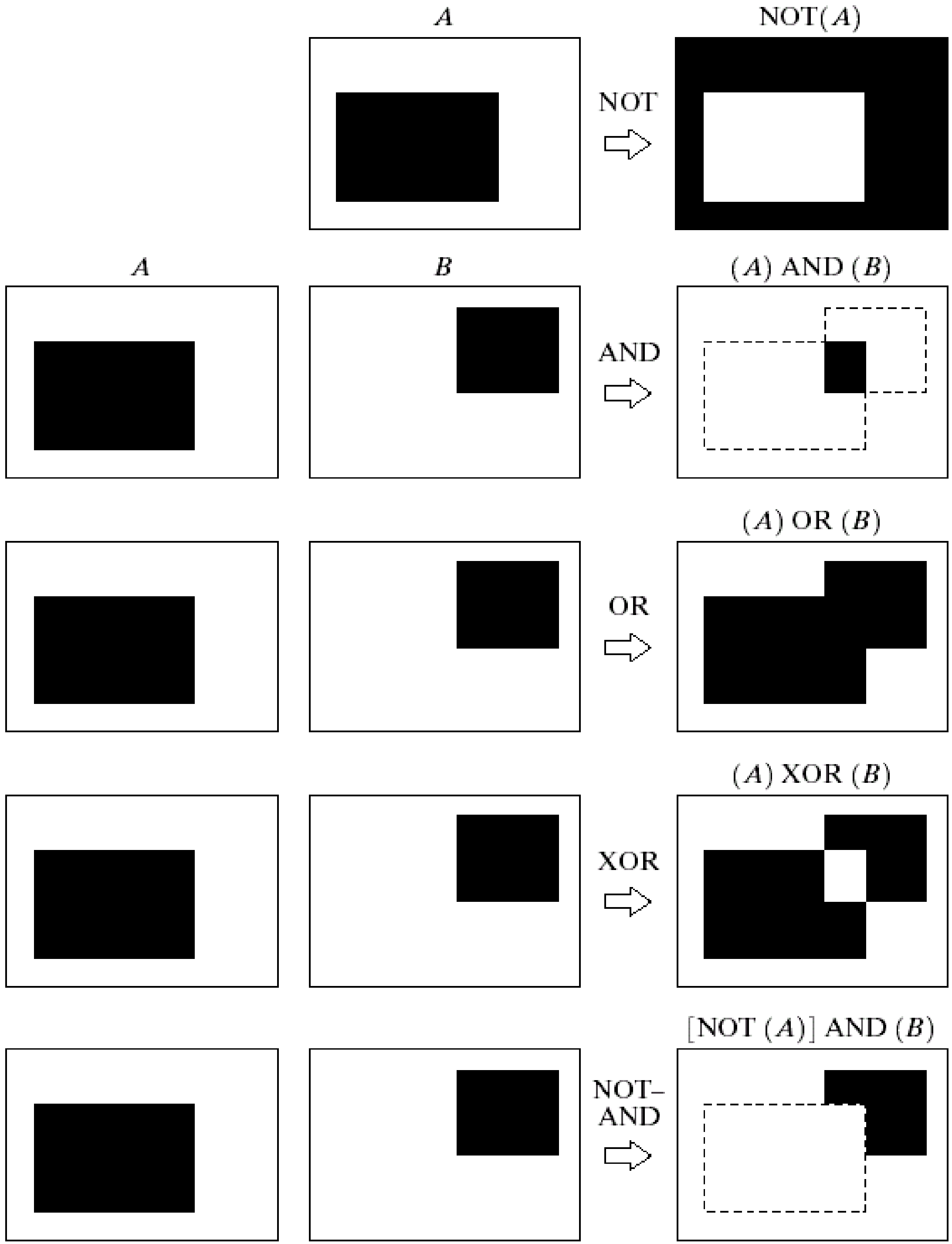


FIGURE 9.3 Some logic operations between binary images. Black represents binary 1s and white binary 0s in this example.

Black = 1
White = 0

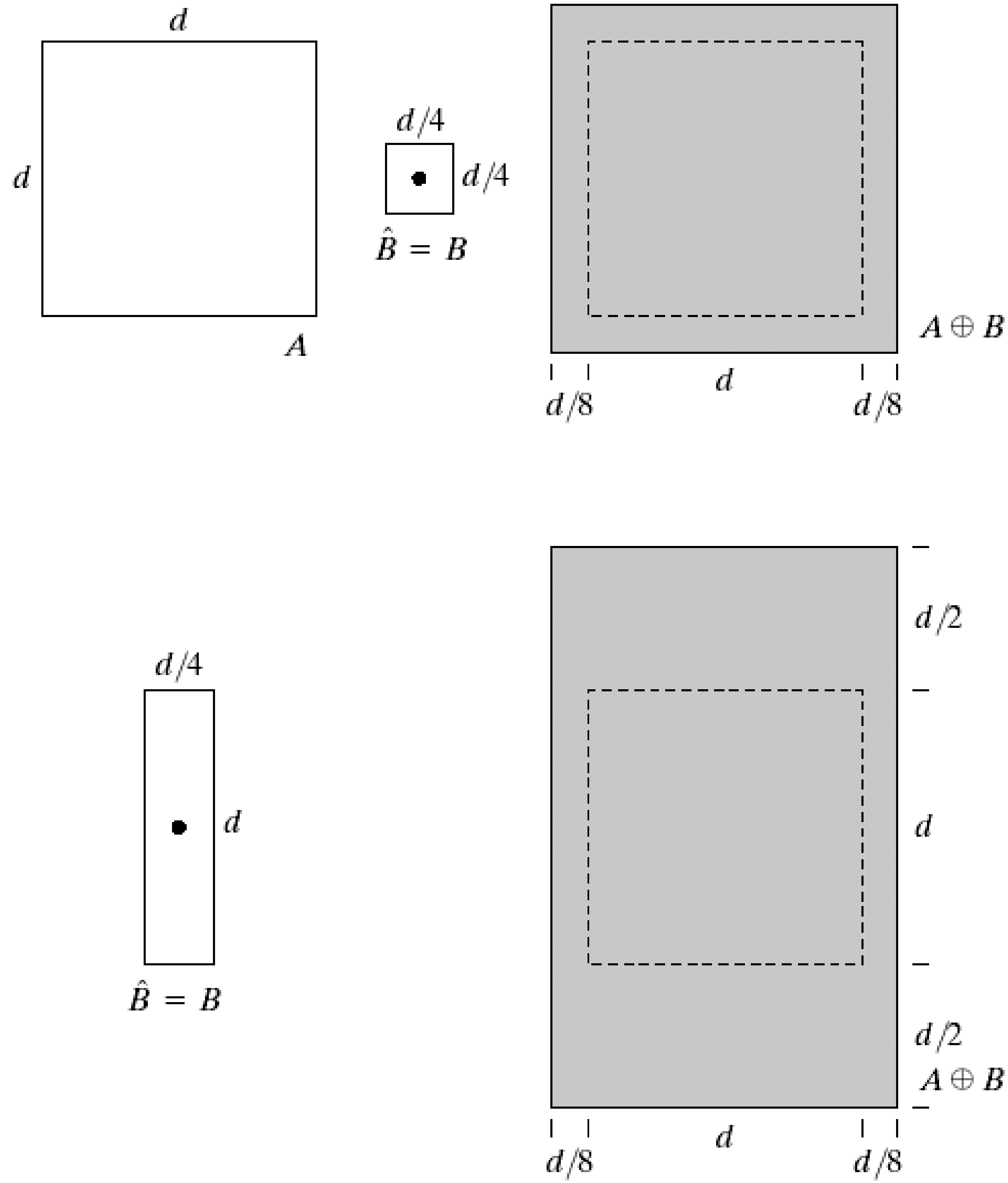
Dilation

1. Let A and B be sets in Z^2.
2. The dilation of A by B is defined as

$$A \oplus B = \left\{ z \mid \left(\hat{B} \right)_z \cap A \neq \varnothing \right\}$$

3. The set of all displacements, z, such that $\hat{B}$ and A overlap by at least one element.
4. Another definition.

$$A \oplus B = \left\{ z \mid \left[\left(\hat{B} \right)_z \cap A \right] \subseteq A \right\}$$



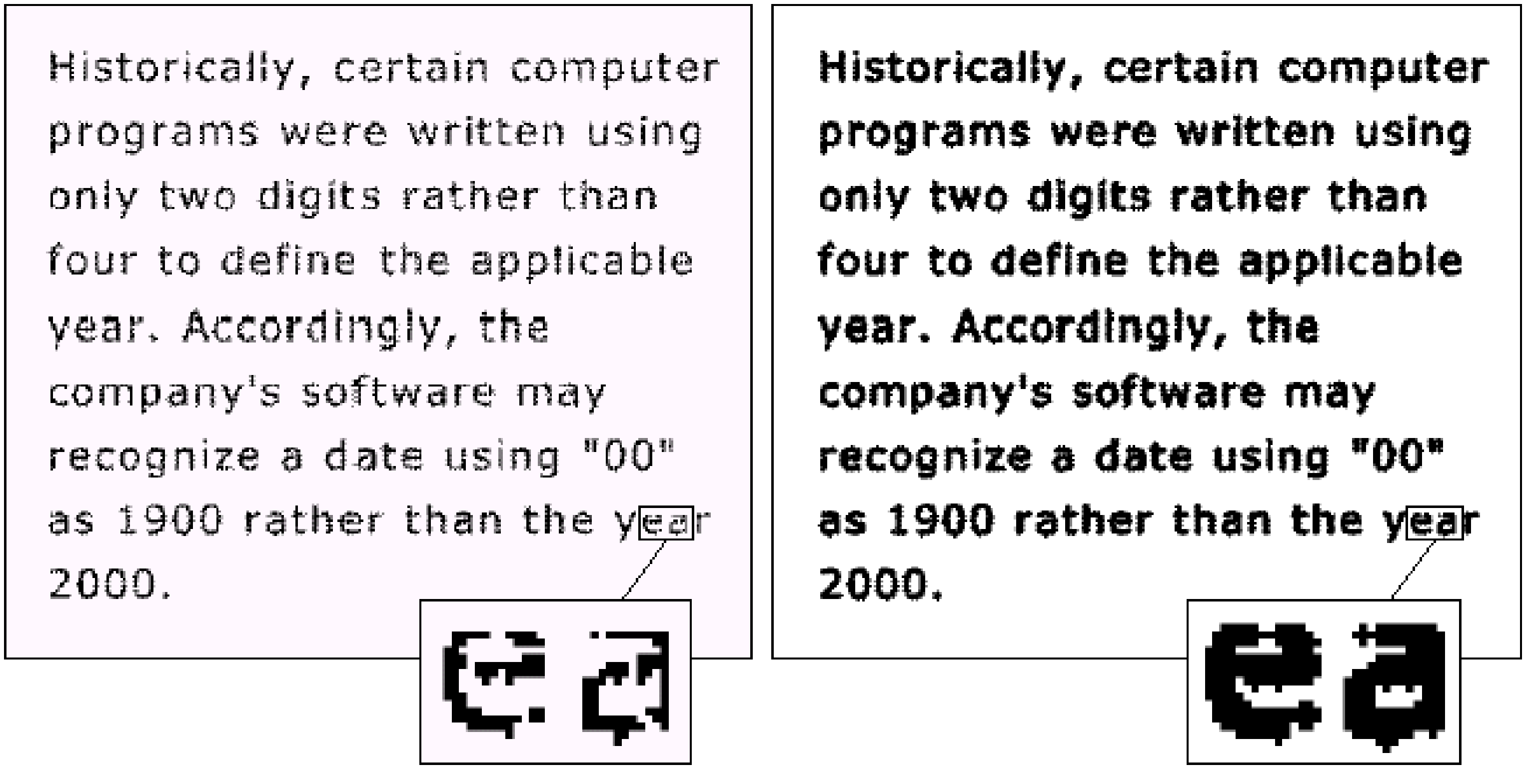
a b c
d e

FIGURE 9.4
(a) Set A.
(b) Square structuring element (dot is the center).
(c) Dilation of A by B, shown shaded.
(d) Elongated structuring element.
(e) Dilation of A using this element.

1. Dashed lines (in Figs. c and e) show the original set for reference.
2. Solid lines (in Figs c and e) show the limit beyond which any further displacements of the origin of $\hat{B}$ by z would cause the intersection of $\hat{B}$ and A to be empty.
3. z is in $A \oplus B$ when A and $\hat{B}$ overlap by at least one element.

Dilation: application

1. Bridging gaps in broken characters



$$B = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

a c
b

FIGURE 9.5
(a) Sample text of poor resolution with broken characters (magnified view).
(b) Structuring element.
(c) Dilation of (a) by (b). Broken segments were joined.

Erosion

1. Let A and B be sets in Z^2.
2. The erosion of A by B is defined as

$$A \Theta B = \left\{ z \mid \left(B \right)_z \subseteq A \right\}$$

3. The set of all displacements, z, such that B, translated by z, is contained in A.
4. Dilation and erosion are duals of each other.

$$\left(A \Theta B \right)^c = A^c \oplus \hat{B}$$

5. Dilation expands objects (represented by '1') in an image and erosion shrinks objects.

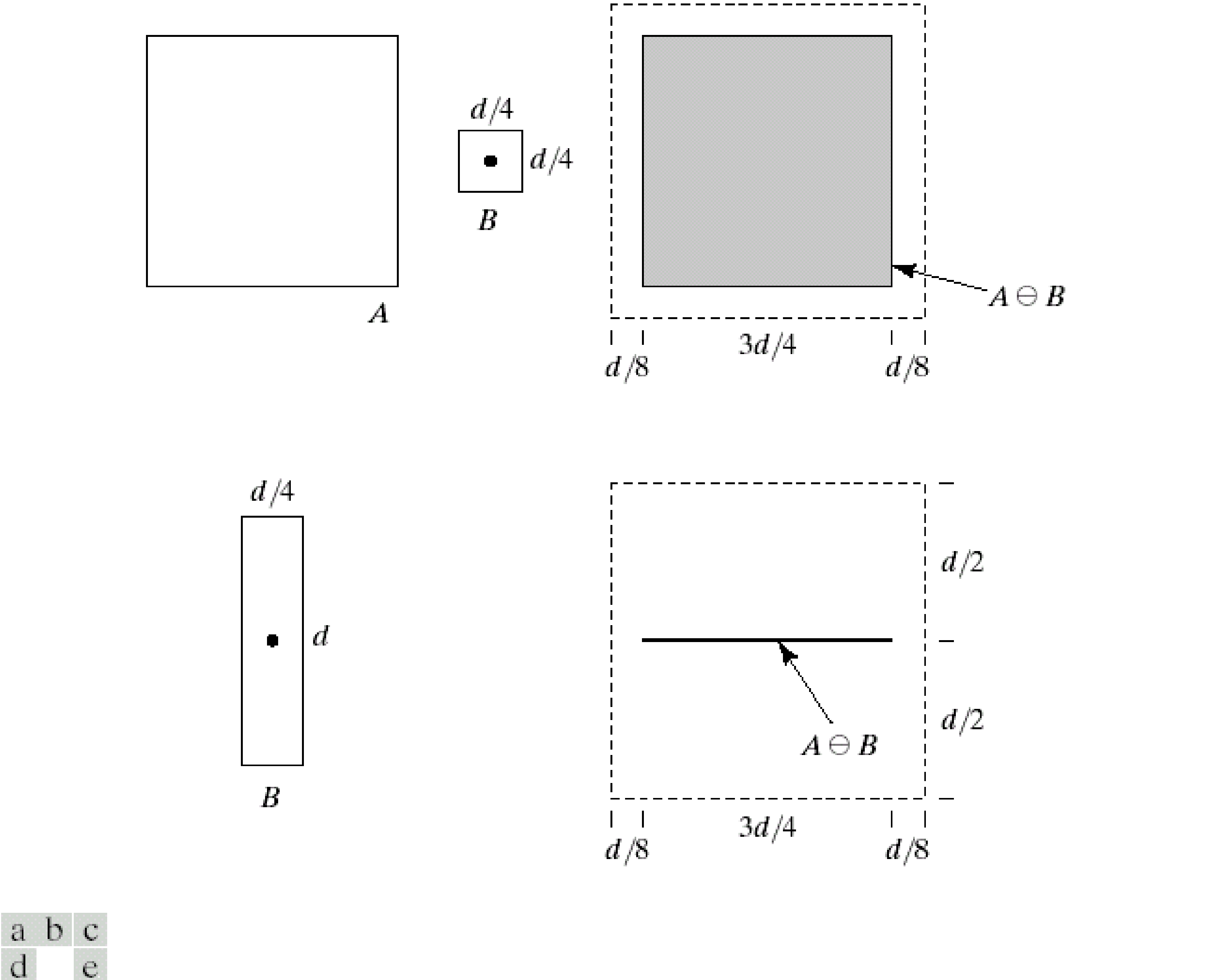
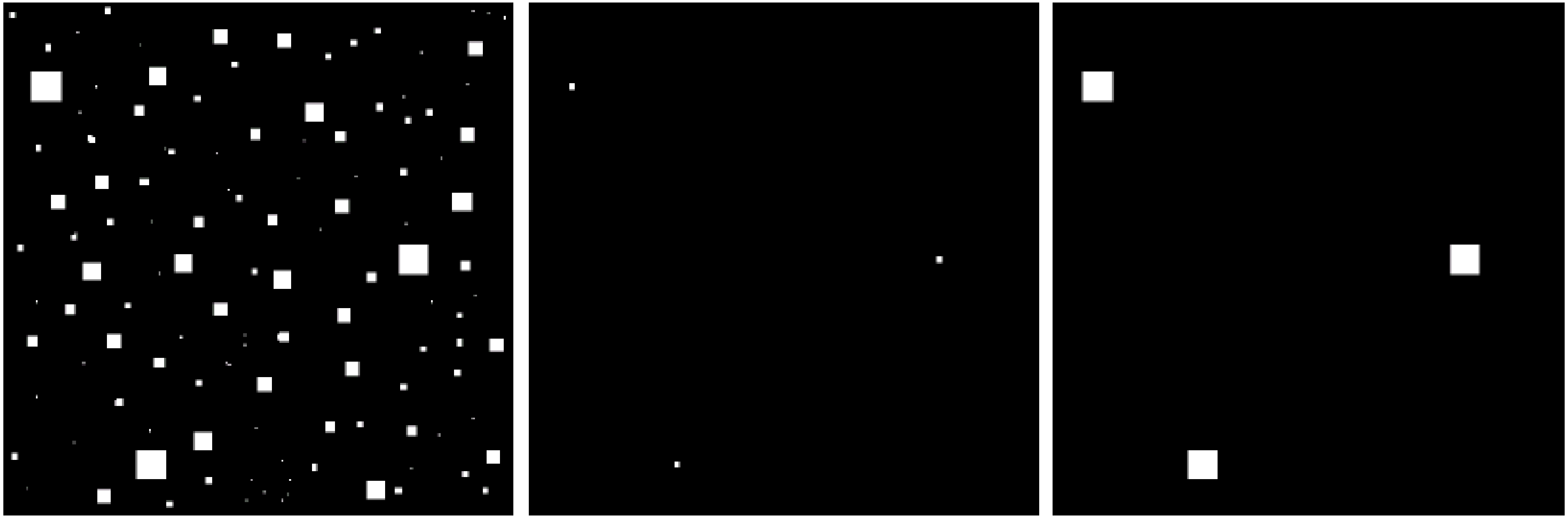


FIGURE 9.6 (a) Set A. (b) Square structuring element. (c) Erosion of A by B, shown shaded. (d) Elongated structuring element. (e) Erosion of A using this element.

1. Dashed lines (in Figs. c and e) show the original set for reference.
2. Solid lines/shared region (in Figs c and e) show the limit beyond which any further displacements of the origin of *B* by *z* would cause the erosion set not contained in *A*.
3. z is in $A \ominus B$ when *B* is completely contained by the set *A*.

Erosion: application

1. Eliminating irrelevant details from a binary image.
2. In the images, black indicates a binary 0 and white indicates a 1.
3. All elements in the structuring element have the same binary values as the objects of interest.



a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

Morphological Image Processing

1. We will cover
   a. Opening and Closing
   b. Boundary Extraction
   c. Region Filling
   d. Connected Components
   e. Skeletons

Opening

1. Recall: Dilation expands an object and erosion shrinks it.
2. In general, opening smoothes the object contour by removing corners and regions that cannot be reached by *B*.

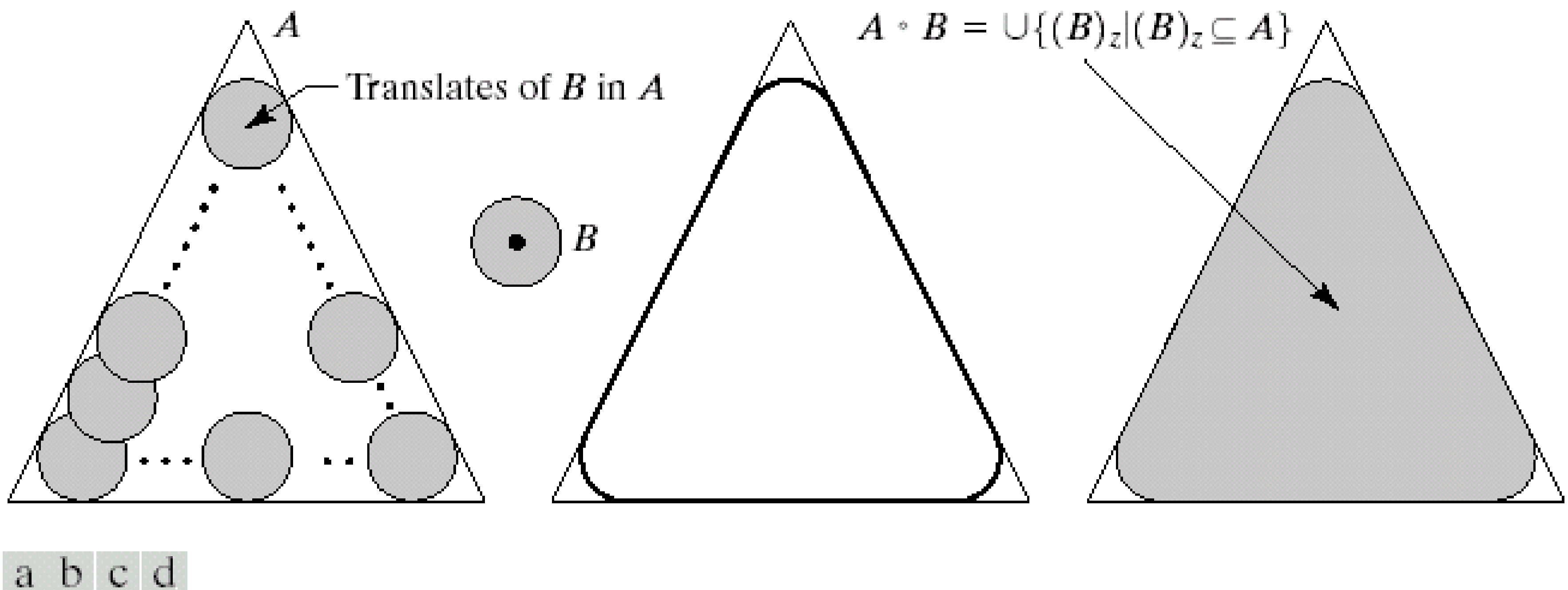


FIGURE 9.8 (a) Structuring element B "rolling" along the inner boundary of A (the dot indicates the origin of B). (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded).

Opening

3. Definition

$$A \circ B = \left(A \Theta B \right) \oplus B$$

4. It represents erosion of *A* by *B*, followed by dilation of the result by *B*.
5. It means that it takes the union of all translates of *B* that fit into *A*. (Imagine you roll the ball inside the object.)
6. Another definition

$$A \circ B = \cup \left\{ \left(B \right)_z \mid \left(B \right)_z \subseteq A \right\}$$

Opening

7. Subimage property

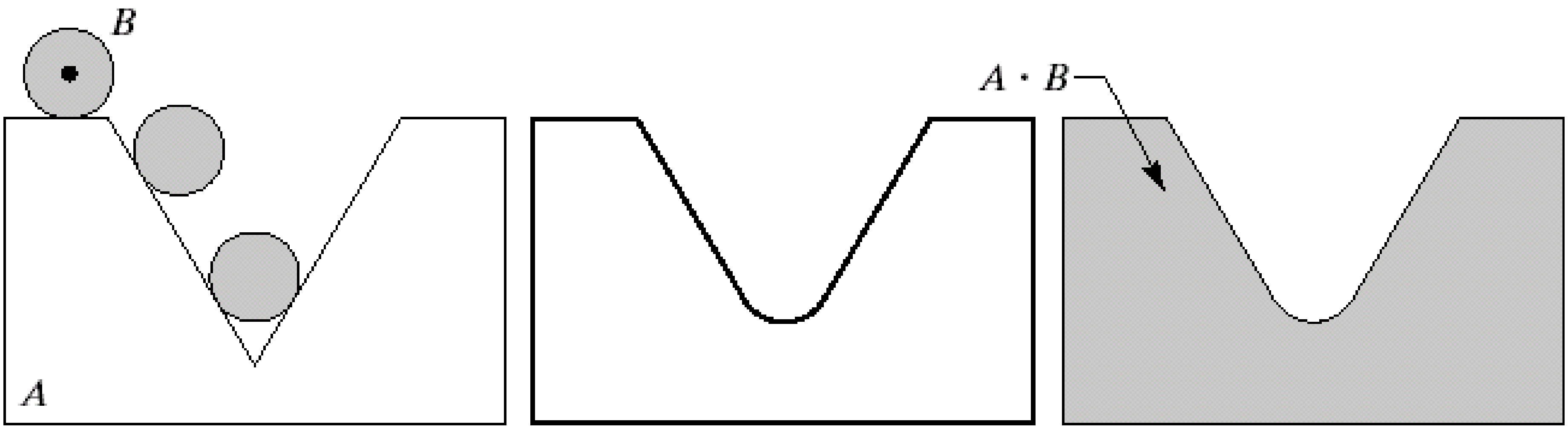
$$A \circ B \text{ is a subset (subimage) of } A$$

8. Convergence property

$$\left(A \circ B\right) \circ B = A \circ B$$

Closing

1. In general, closing can also smooth the object contour/boundary by filling the region that cannot be reached by *B*.



a b c

FIGURE 9.9 (a) Structuring element *B* "rolling" on the outer boundary of set *A*. (b) Heavy line is the outer boundary of the closing. (c) Complete closing (shaded).

Closing

2. Definition

$$A \bullet B = \left(A \oplus B\right) \Theta B$$

3. It represents dilation of *A* by *B*, followed by erosion of the result by *B*.
4. Imagine that you roll the ball outside the object instead of rolling the ball inside the object.
5. Opening and closing are dual

$$\left(A \bullet B\right)^c = A^c \circ \hat{B}$$

Closing

6. Subimage property

$$A \text{ is a subset (subimage) of } A \bullet B$$

7. Convergence property

$$\left(A \bullet B\right) \bullet B = A \bullet B$$

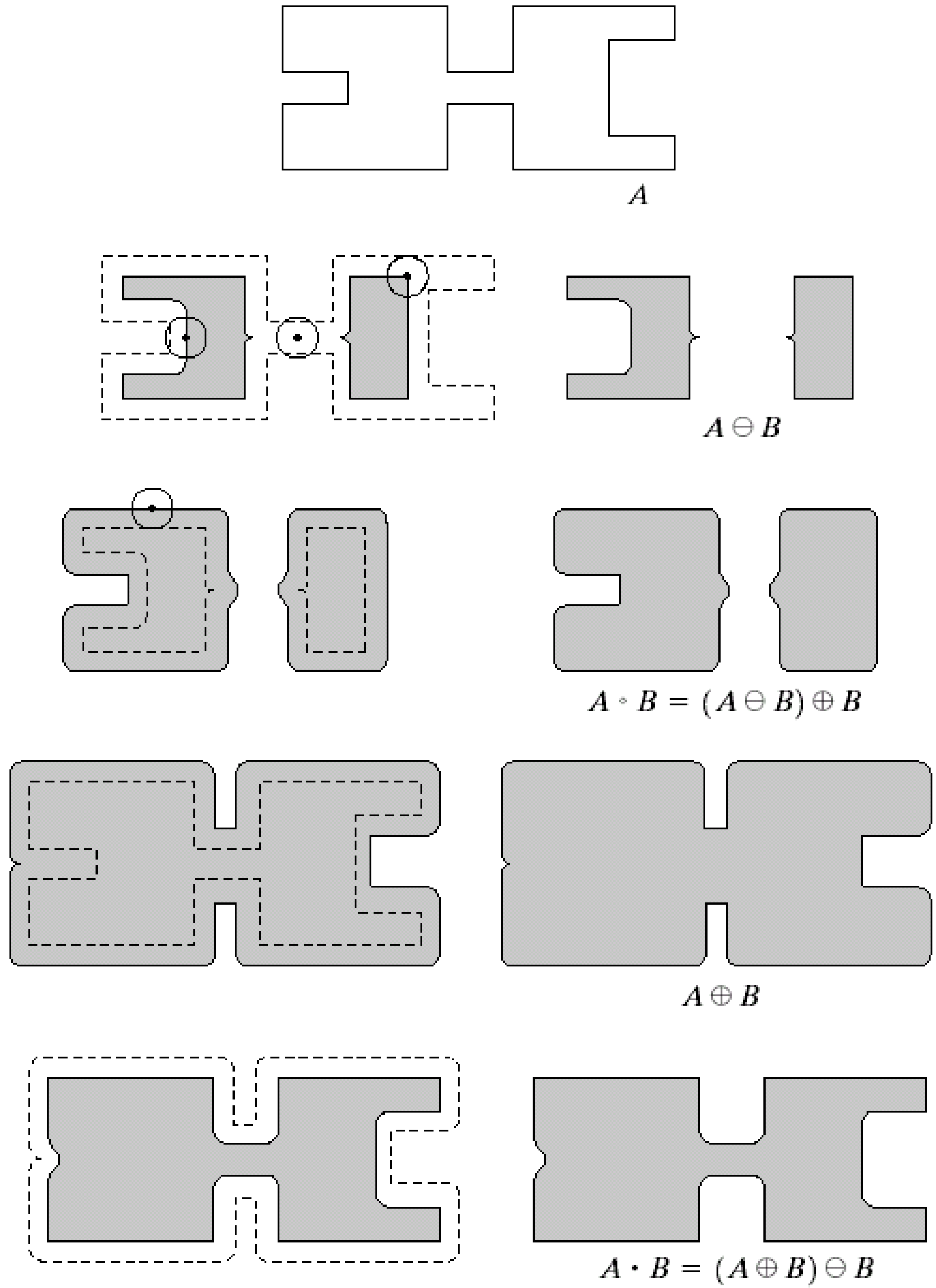
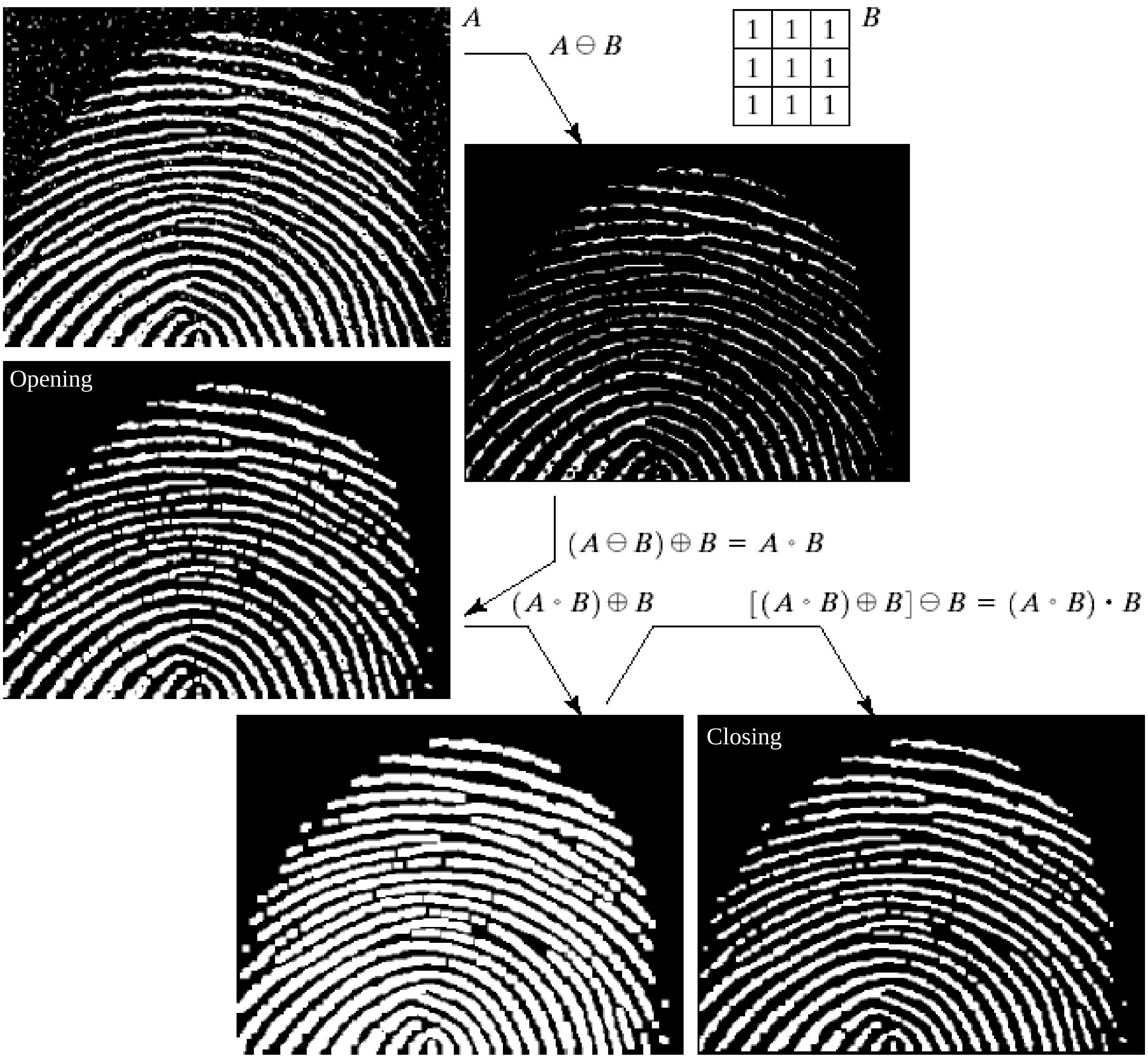


FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The dark dot is the center of the structuring element.



a b
d c
e f

FIGURE 9.11
(a) Noisy image.
(c) Eroded image.
(d) Opening of A.
(d) Dilation of the opening.
(e) Closing of the opening. (Original image for this example courtesy of the National Institute of Standards and Technology.)

Boundary Extraction

1. We define

$$\beta\left(A\right) \text{ denotes the boundary of a set } A$$

2. The boundary is extracted by using

$$\beta\left(A\right) = A - \left(A \Theta B\right)$$

3. It means the boundary is the difference between the object and the ‘eroded’ object.

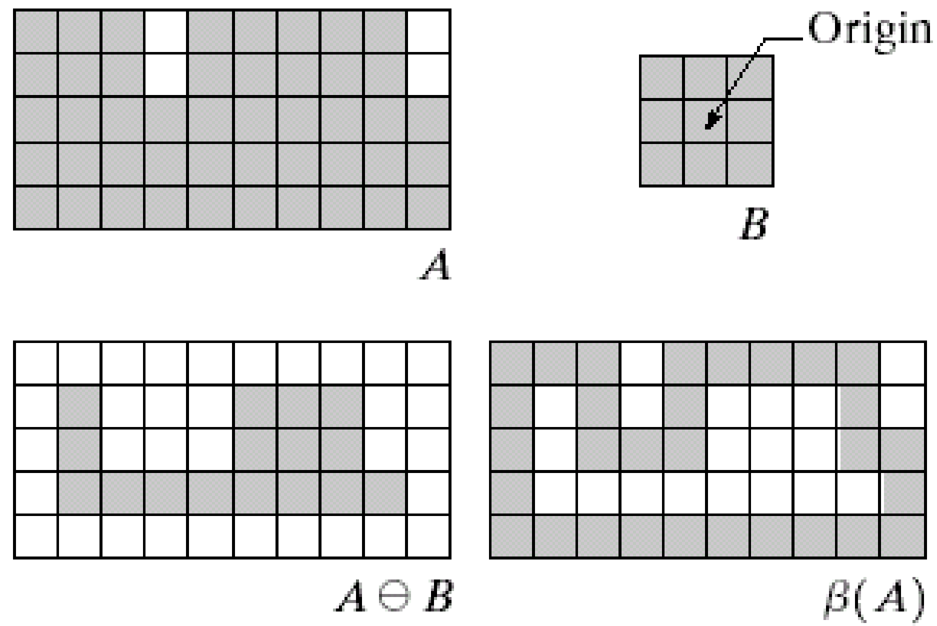


FIGURE 9.13 (a) Set A. (b) Structuring element B. (c) A eroded by B. (d) Boundary, given by the set difference between A and its erosion.

$$\beta(A) = A - (A \Theta B)$$

Shaded region = 1, white region = 0

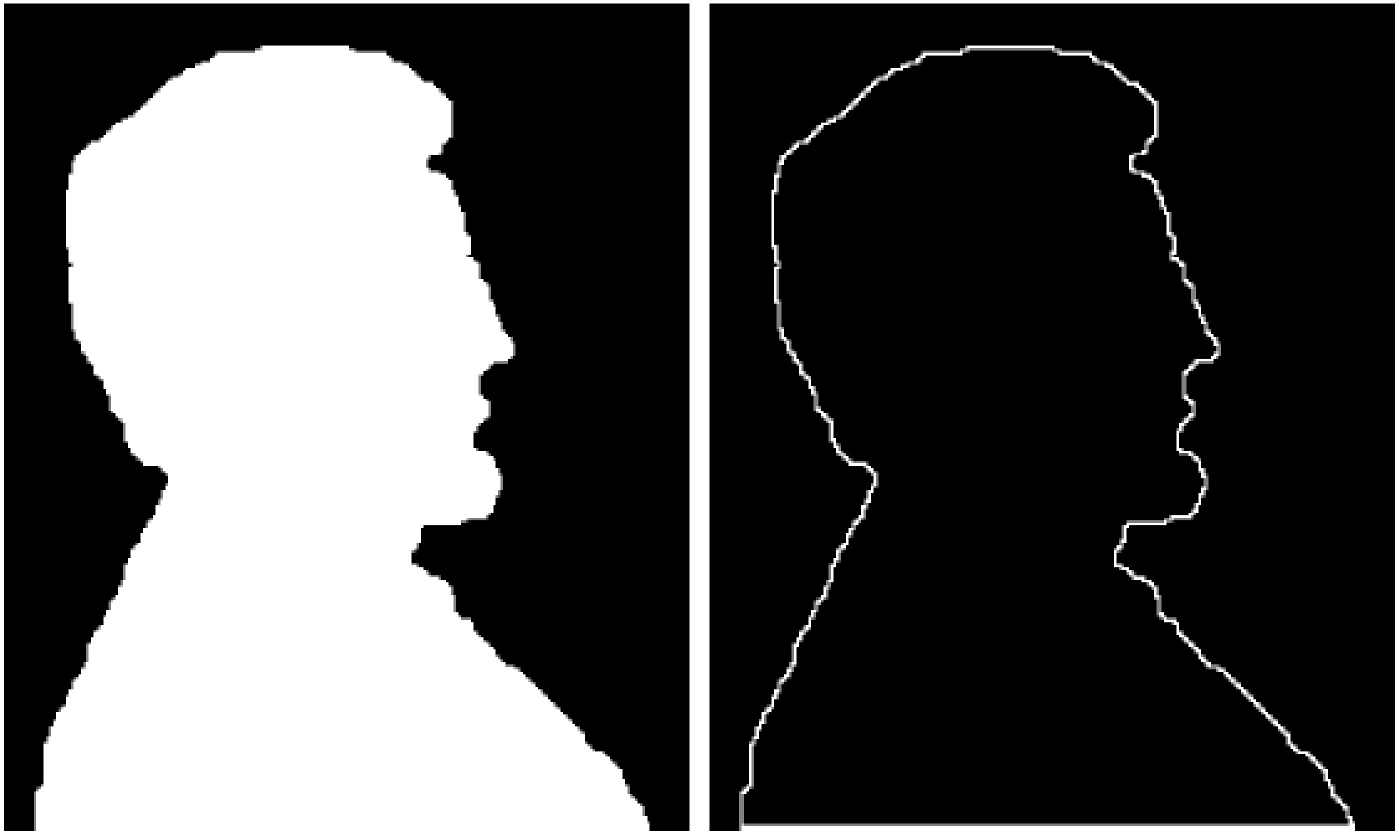


FIGURE 9.14 (a) A simple binary image, with 1's represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

Region Filling

1. Beginning with a point p inside the region boundary, the objective is to fill the bounded region with 1's.
2. The algorithm is defined as follows.

$$X_k = \left(X_{k-1} \oplus B \right) \cap A^c$$

3. $k = 1, 2, 3, \ldots$
4. $X_0 = p$
5. B is the structuring element.
6. Intersection with A^c limits the result to inside the region of interest.
7. The algorithm stops when $X_k = X_{k-1}$. This means the algorithm stops when there is no change in the region size.

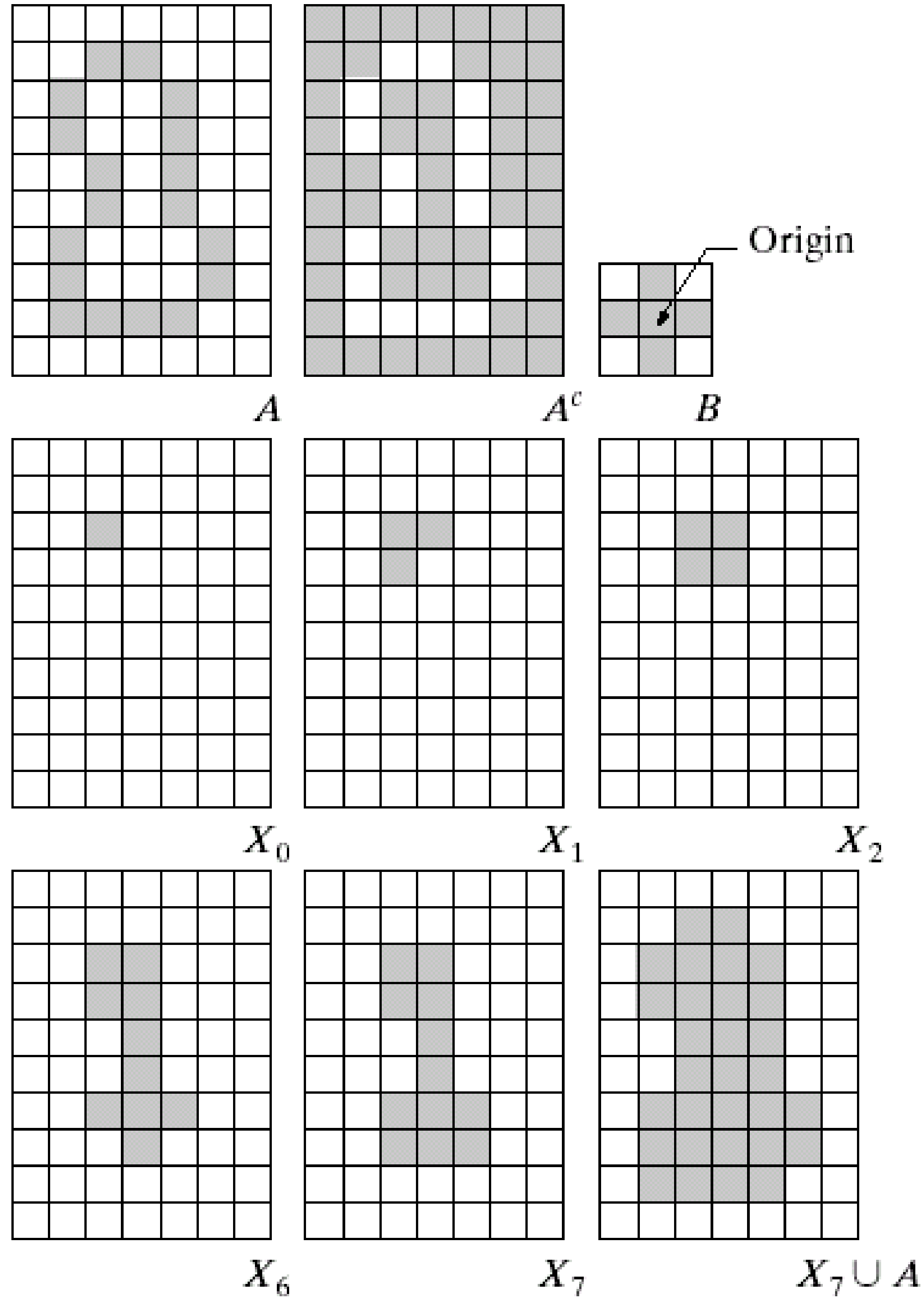
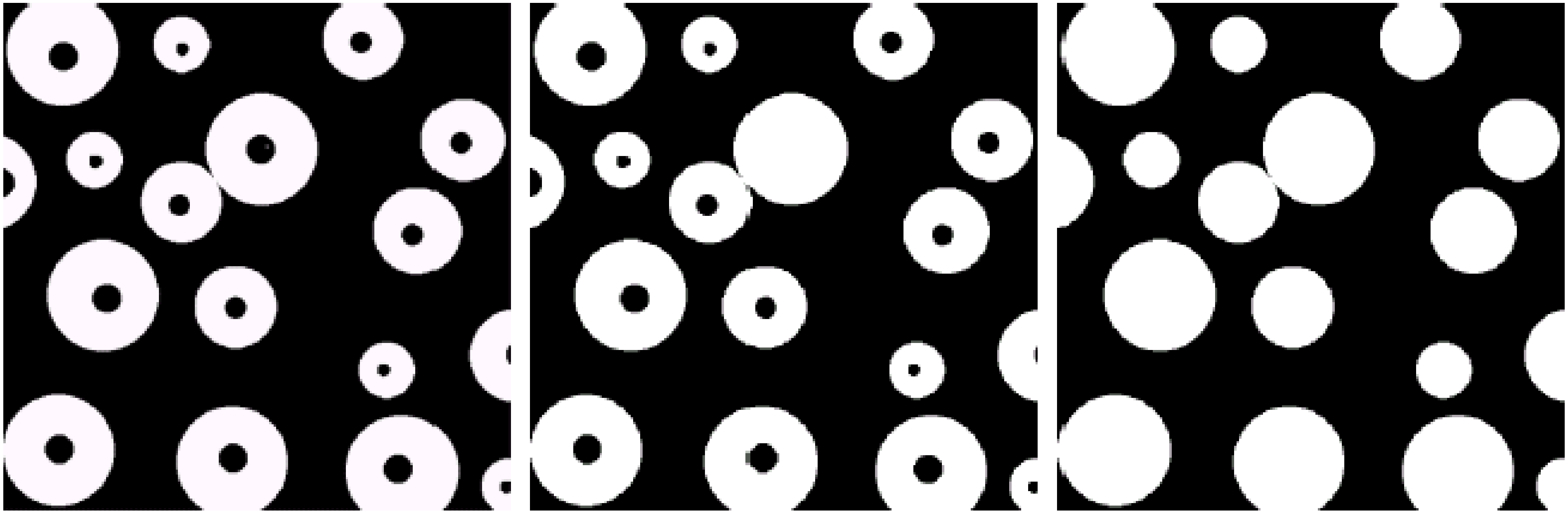


FIGURE 9.15 Region filling. (a) Set A. (b) Complement of A. (c) Structuring element B. (d) Initial point inside the boundary. (e)–(h) Various steps of Eq. (9.5-2). (i) Final result [union of (a) and (h)].

Shaded region = 1
White region = 0



a b c

FIGURE 9.16 (a) Binary image (the white dot inside one of the regions is the starting point for the region-filling algorithm). (b) Result of filling that region (c) Result of filling all regions.

Connected Components

1. Similar to region filling, the procedure is defined as follows.

$$X_k = \left(X_{k-1} \oplus B \right) \cap A$$

2. Beginning with a point *p* inside the object, the objective is to fill out the entire object with 1's.
3. The intersection with *A* at each iteration eliminates dilations centred on elements labelled 0.

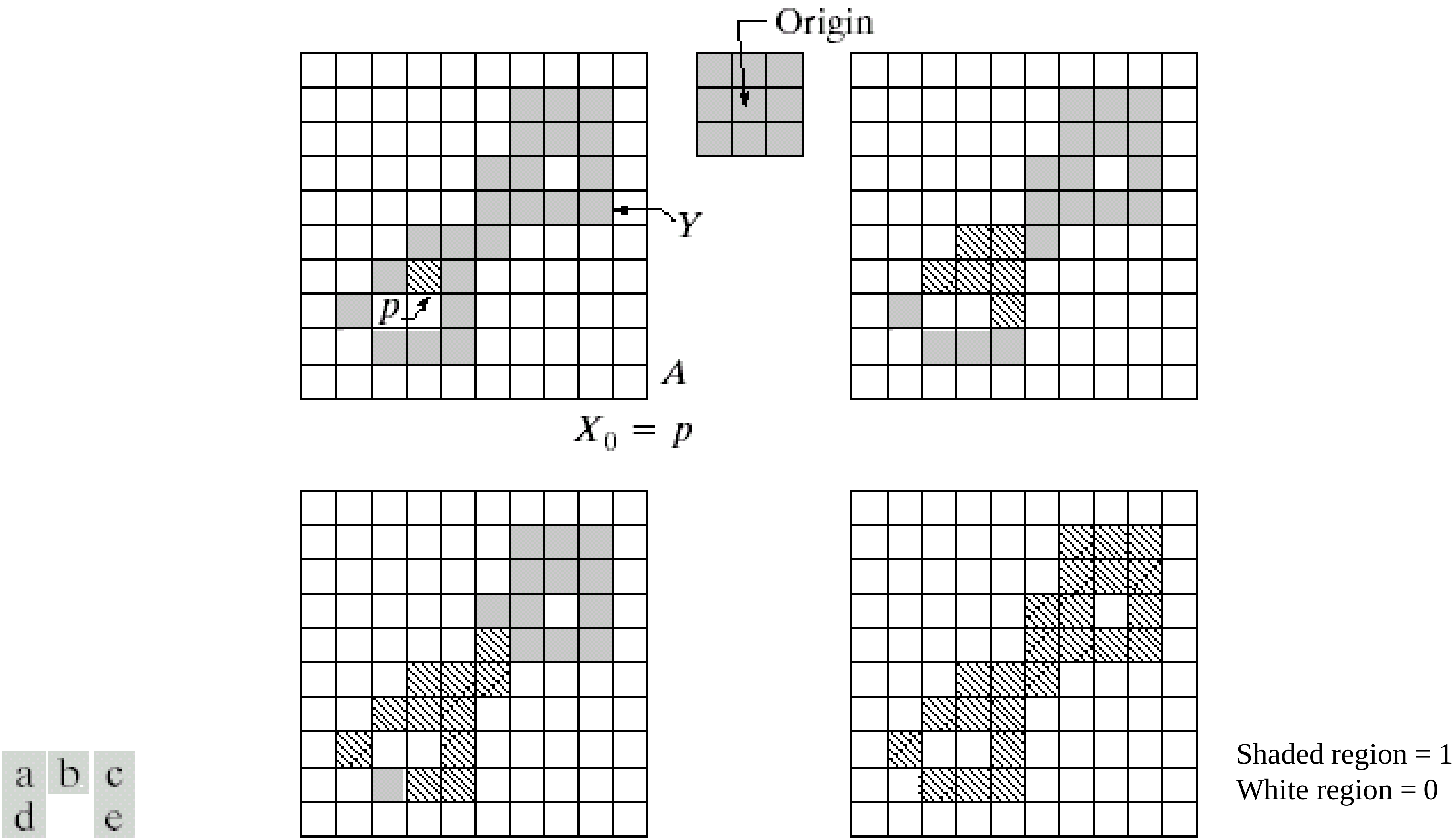
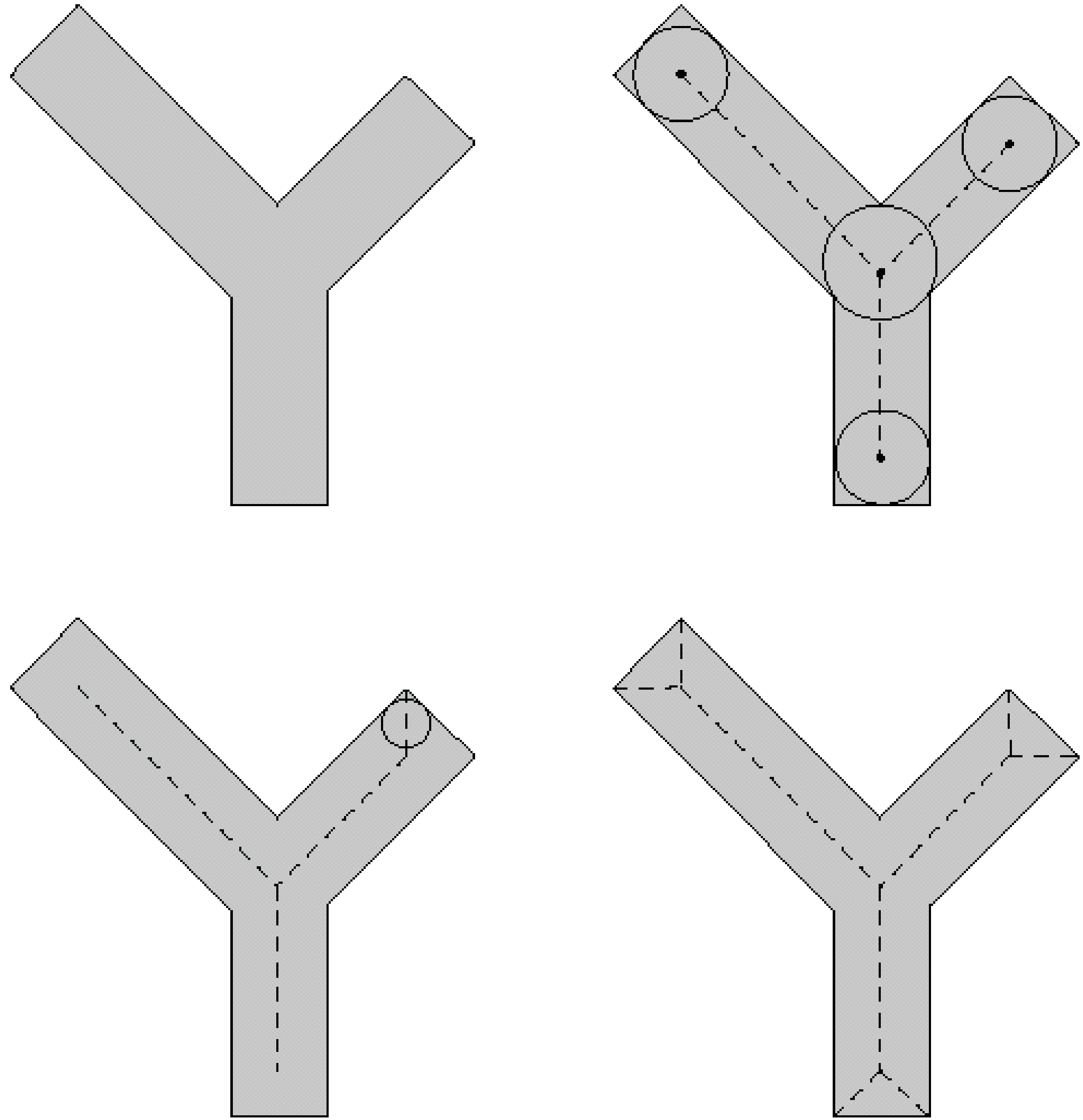


FIGURE 9.17 (a) Set A showing initial point p (all shaded points are valued 1, but are shown different from p to indicate that they have not yet been found by the algorithm). (b) Structuring element. (c) Result of first iterative step. (d) Result of second step. (e) Final result.

Skeletons

1. Definition of a skeleton, $S(A)$, of a set A is as follows
   a. If
      i. z is a point of $S(A)$ and
      ii. $(D)_z$ is the largest disk centred at z and contained in A,
   b. then
      i. one cannot find a larger disk (not necessarily centred at z) containing $(D)_z$ and included in A, and
      ii. the disk $(D)_z$ is called *maximum disk*.
      iii. the disk $(D)_z$ touches the boundary of A at two or more different places.



a b
c d

FIGURE 9.23
(a) Set A.
(b) Various positions of maximum disks with centers on the skeleton of A.
(c) Another maximum disk on a different segment of the skeleton of A.
(d) Complete skeleton.

2. The algorithm is given by $S(A)$ is the union of skeleton subsets $S_k(A)$

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

3. with the skeleton subset defined by

$$S_k(A) = (A \Theta kB) - (A \Theta kB) \circ B$$

4. where B is a structuring element, and

$$(A \Theta kB) = (\ldots(A \Theta B)\Theta B)\Theta \ldots)\Theta B$$

k successive erosions of *A by B*

5. *K* is the last iterative step before *A* erodes to an empty set.

$$K = \max\{k \mid (A \Theta kB) \neq \varnothing\}$$

Reconstruction of skeletons

1. The original set A can be reconstructed by using the equation

$$A = \bigcup_{k=0}^{K} \left(S_k\left(A\right) \oplus kB \right)$$

2. where

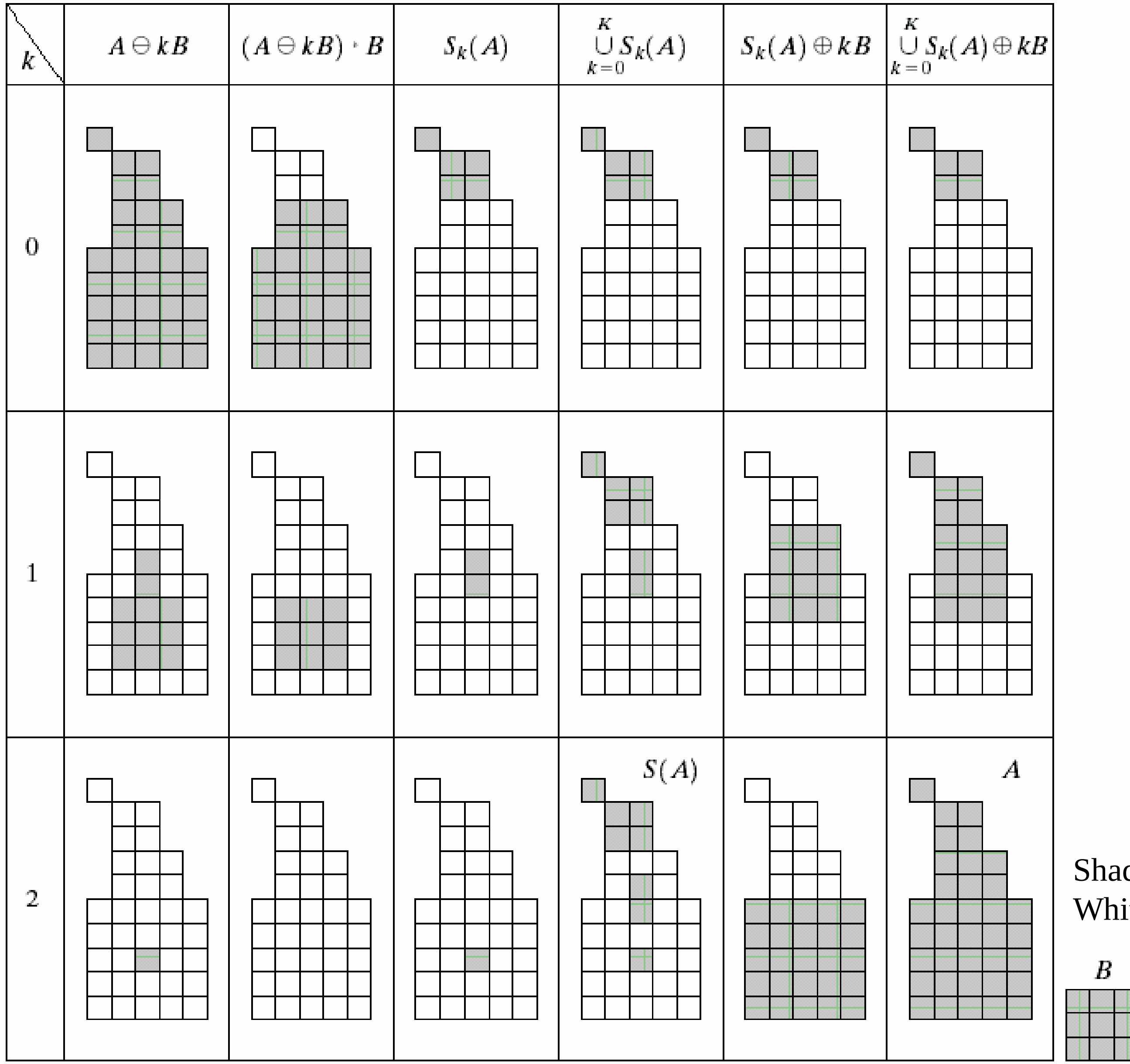
$$(S_k(A) \oplus kB) = \underbrace{\left(\left(\left((S_k(A) \oplus B) \oplus B\right) \oplus B\right) \oplus \cdots\right) \oplus B}_{k\ successive\ dilations}$$

k successive dilations of $S_k(A)$

k = 0

k = 1

k = 2



Shaded region = 1
White region = 0

FIGURE 9.24 Implementation of Eqs. (9.5-11) through (9.5-15). The original set is at the top left, and its morphological skeleton is at the bottom of the fourth column. The reconstructed set is at the bottom of the sixth column.

Morphological Image Processing for Gray-Scale Images

Morphological Image Processing

1. Extensions to gray-scale images.
   a. Dilation
   b. Erosion
   c. Opening
   d. Closing
2. Some applications.

Notations

1. Let $f(x,y)$ be the input image.
2. Let $b(x,y)$ be the structuring element.
3. (x,y) = position coordinates.
4. $f(x,y)$ and $b(x,y)$ are functions that assign a gray-level value to each position located at (x,y).

Dilation

1. 2D gray-scale dilation of f by b is defined as

$$(f \oplus b)(s,t) = \max\left\{ f(s-x, t-y) + b(x,y) \mid (s-x),(t-y) \in D_f; (x,y) \in D_b \right\}$$

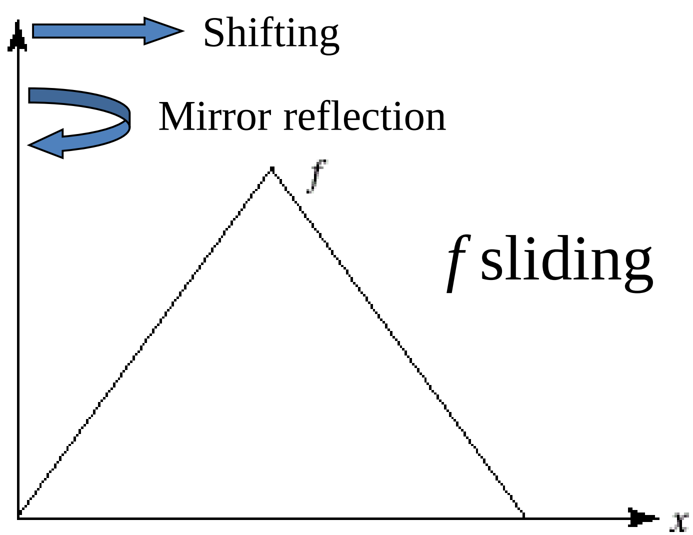
2. D_f and D_b are the domains of $f(s\text{-}x, t\text{-}y)$ and $b(x, y)$.
3. If the function b is zero, the gray-level dilation operation is the same as finding the local maximum.
4. $(s\text{-}x)$ and $(t\text{-}y)$ have to be in the domain of $f(s\text{-}x, t\text{-}y)$.
5. x and y have to be in the domain of $b(x, y)$.
6. Points 3 and 4 are similar to the conditions in the binary definition of dilation, which make sure that *the two sets have to overlap by at least one element*.

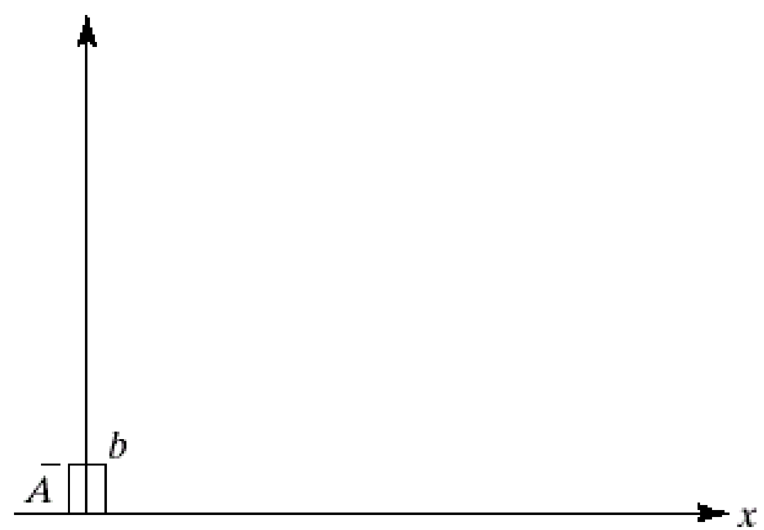
Dilation

7. 1D gray-scale dilation of f by b is defined as

$$(f \oplus b)(s) = \max\left\{ f(s-x) + b(x) \mid (s-x) \in D_f;\, x \in D_b \right\}$$

8. $f(-x)$ is a mirror reflection of $f(x)$.
9. $f(s-x)$ is equivalent to shifting to the right (left) when the value of s is positive (negative).





Dilation

10. As *f* sliding by *b* is no different with *b* sliding by *f*, the actual mechanics of gray-scale dilation are easier to visualize if *b* is the function that slides past *f*.

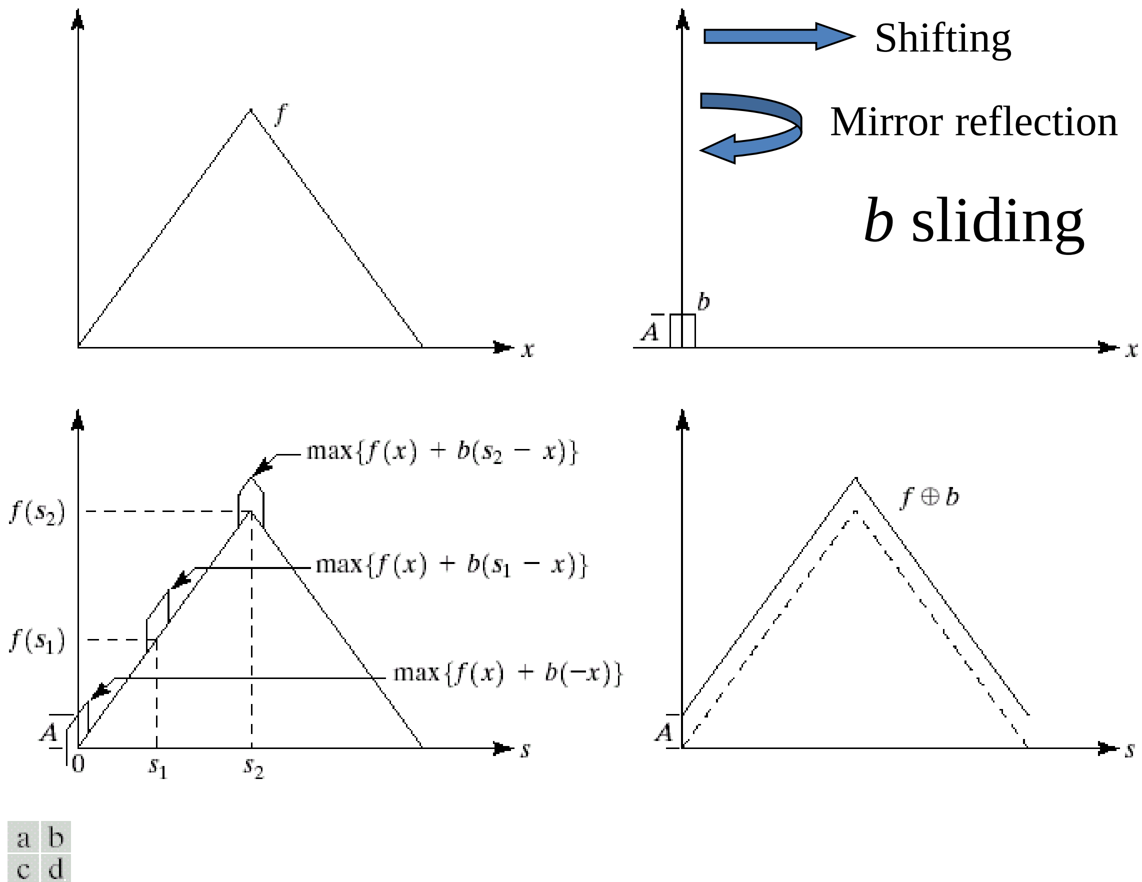
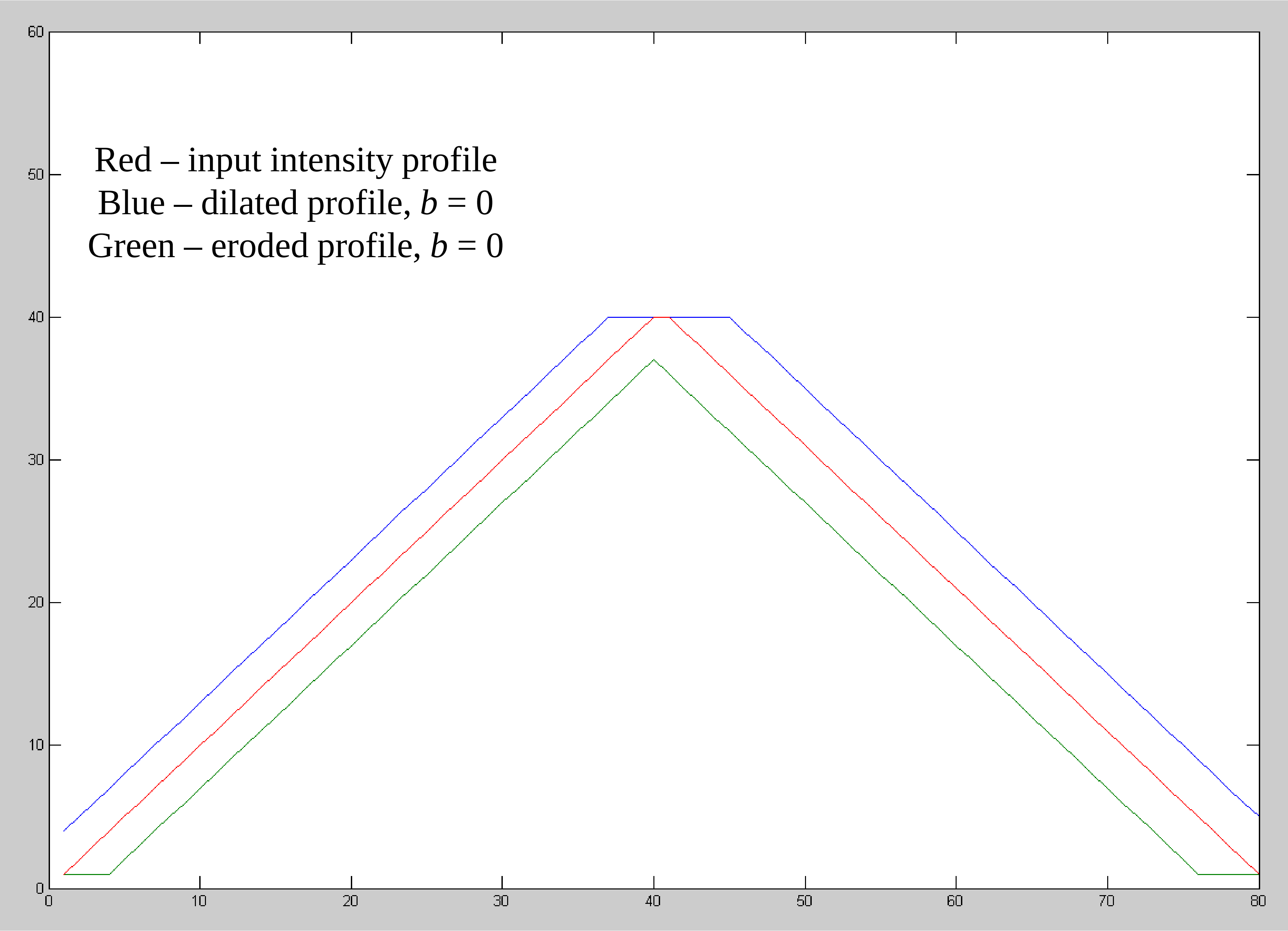
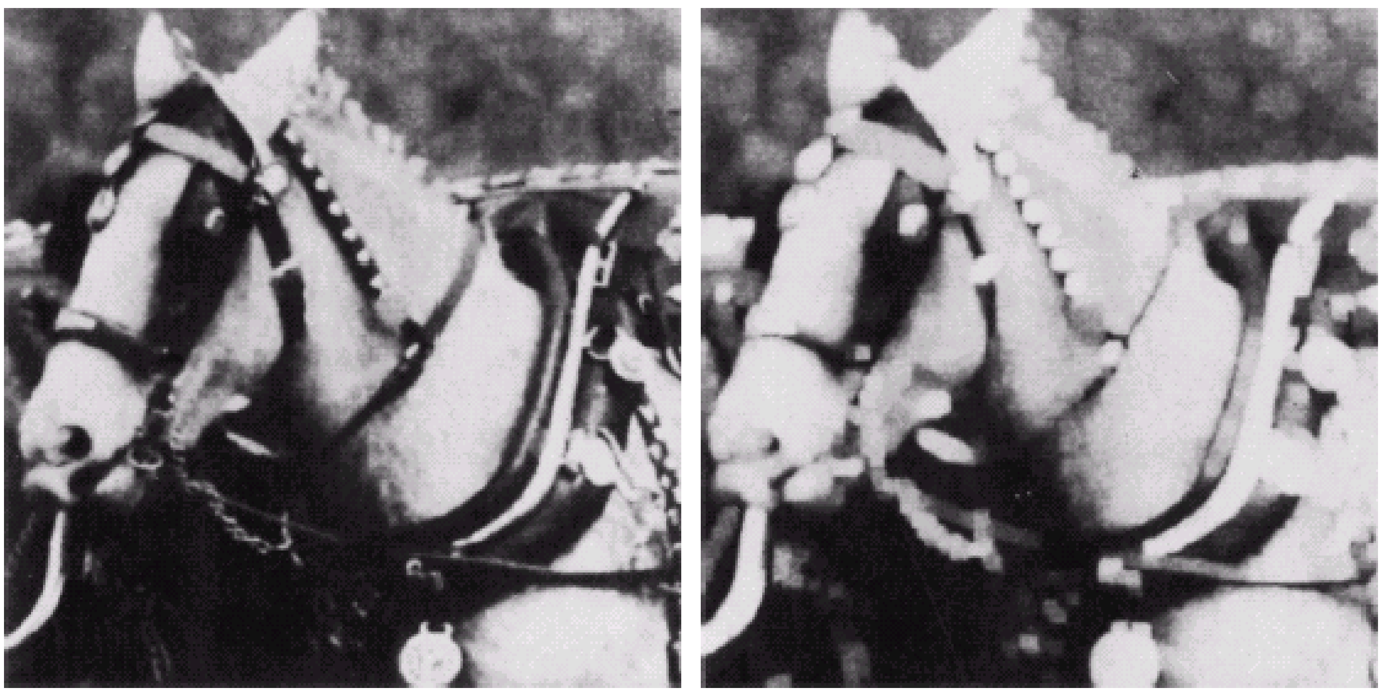


FIGURE 9.27 (a) A simple function. (b) Structuring element of height A. (c) Result of dilation for various positions of sliding b past f. (d) Complete result of dilation (shown solid).


Red – input intensity profile
Blue – dilated profile, $b = 0$
Green – eroded profile, $b = 0$
0
10
20
30
40
50
60
0
10
20
30
40
50
60
70
80

Dilation

11. The general effect of performance:
    a. If all the values of the structuring element are positive, then the output image tends to be brighter than the input.
    b. Dark small details either are reduced or eliminated.



a b
c

FIGURE 9.29
(a) Original image. (b) Result of dilation. (c) Result of erosion. (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

Erosion

1. 2D gray-scale erosion of f by b is defined as

$$(f \Theta b)(s,t) = \min\left\{ f(s+x, t+y) - b(x,y) \mid (s+x),(t+y) \in D_f; (x,y) \in D_b \right\}$$

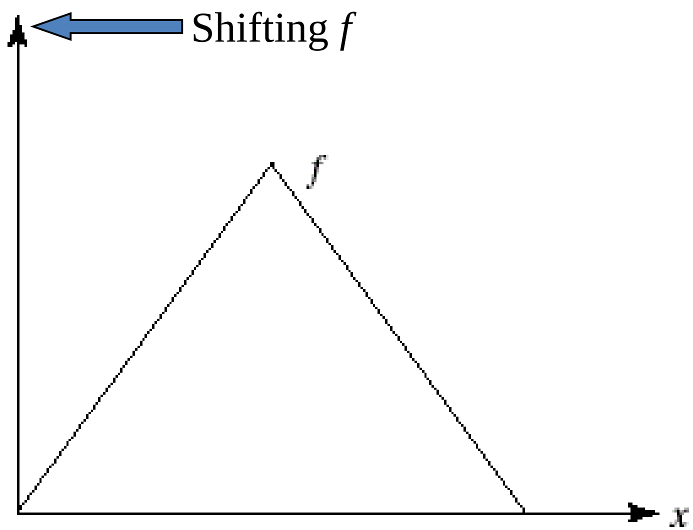
2. D_f and D_b are the domains of $f(s+x, t+y)$ and $b(x, y)$.
3. If the function b is zero, the gray-level erosion operation is the same as finding the local minimum.
4. $(s+x)$ and $(t+y)$ have to be in the domain of $f(s+x, t+y)$.
5. x and y have to be in the domain of $b(x, y)$.
6. Points 3 and 4 are similar to the conditions in the binary definition of erosion, which make sure that *the structuring element has to be completely contained by the set being eroded*.

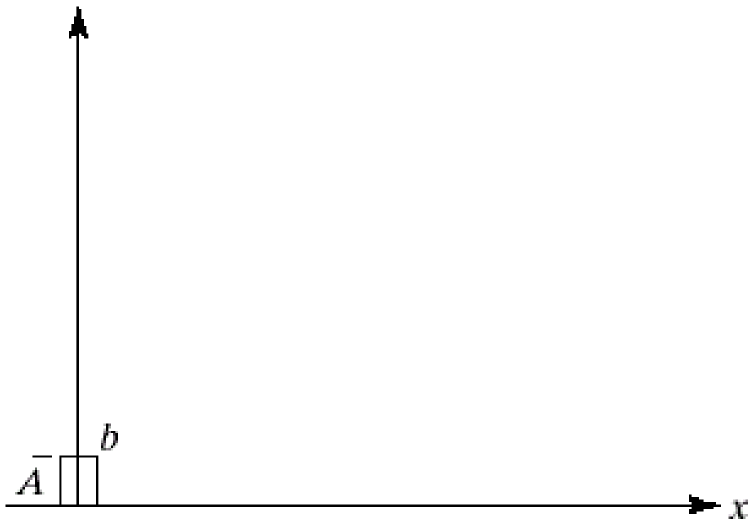
Erosion

7. 1D gray-scale erosion of f by b is defined as

$$(f \Theta b)(s) = \min\left\{ f(s+x) - b(x) \mid (s+x) \in D_f ; x \in D_b \right\}$$

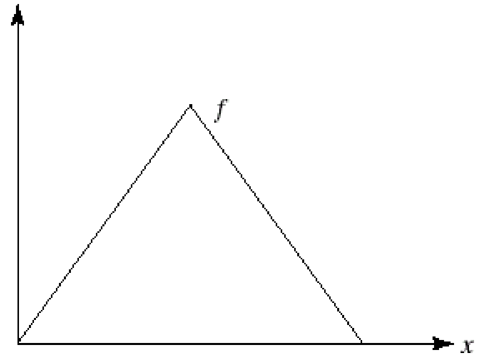
8. *f(s+x)* is equivalent to shifting to the left (right) when the value of *s* is positive (negative).





Erosion

9. As *f* sliding by *b* is no different with *b* sliding by *f*, the actual mechanics of gray-scale dilation are easier to visualize if *b* is the function that slides past *f*.



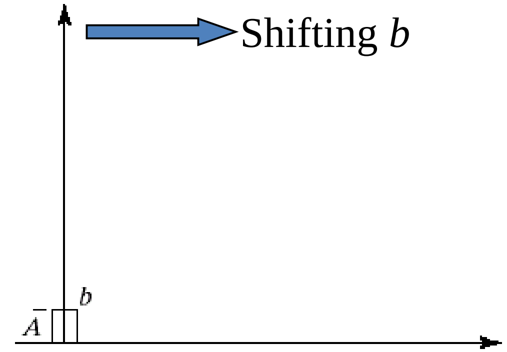
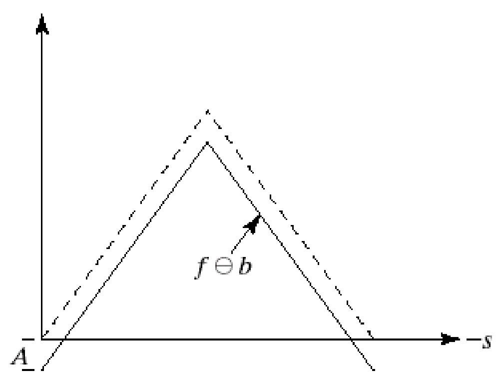


FIGURE 9.28 Erosion of the function shown in Fig. 9.27(a) by the structuring element shown in Fig. 9.27(b).



10. The general effect of performance:

   a. If all the values of the structuring element are positive, then the output image tends to be darker than the input.
   b. The effect of bright small details in the input image that are smaller in area than the structuring element is reduced.

Original image

Results of erosion

Erosion

11. Gray-scale dilation and erosion are duals with respect to function complementation and reflection.

$$\left(f \ominus b\right)^c (s,t) = \left(f^c \oplus \hat{b}\right)(s,t)$$

where

$$f^c = -f\left(x,y\right)$$

$$\hat{b} = b\left(-x,-y\right)$$

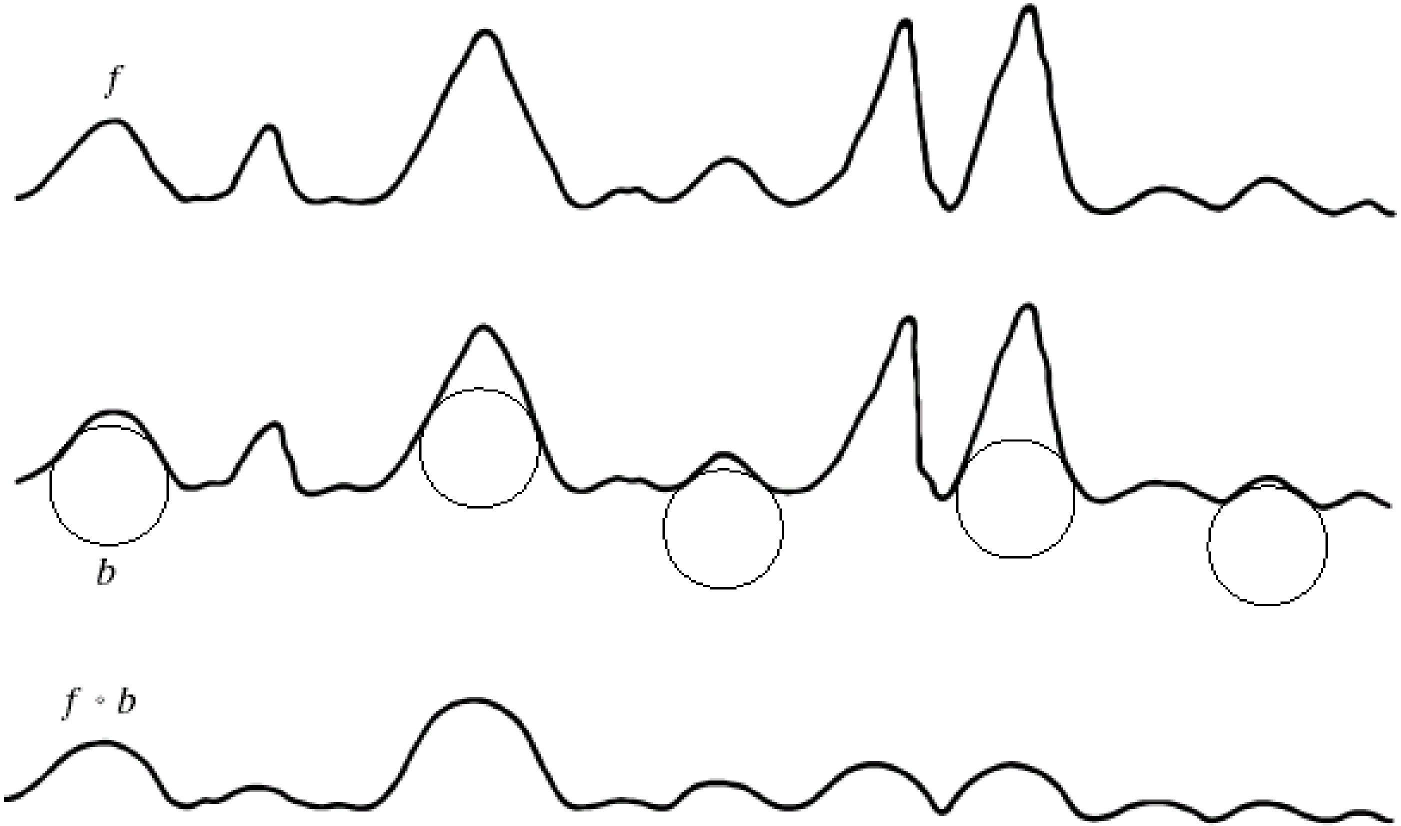
Opening

1. The opening of image f by structuring element *b* is given by

$$f \circ b = \left(f \Theta b \right) \oplus b$$

2. As in the binary case, opening is simply the erosion of f by *b*, followed by a dilation of the result by *b*.
3. Suppose that we open f by a spherical structuring element *b*, (a 'rolling ball').
4. It is equivalent to pushing the ball against the underside of the surface, while at the same time rolling it so that the entire underside of the surface is traversed.
5. $f \circ b$ = surface of the highest points reached by any part of the sphere as it slides over the entire under-surface of f.

Opening



a
b
c
d
e

FIGURE 9.30
(a) A gray-scale scan line.
(b) Positions of rolling ball for opening.
(c) Result of opening.
(d) Positions of rolling ball for closing. (e) Result of closing.

- All the peaks that were narrow with respect to the diameter of the ball were reduced in amplitude and sharpness.

- Opening operations usually are applied to remove small light details, while leaving the overall gray levels and larger bright features relatively undisturbed.

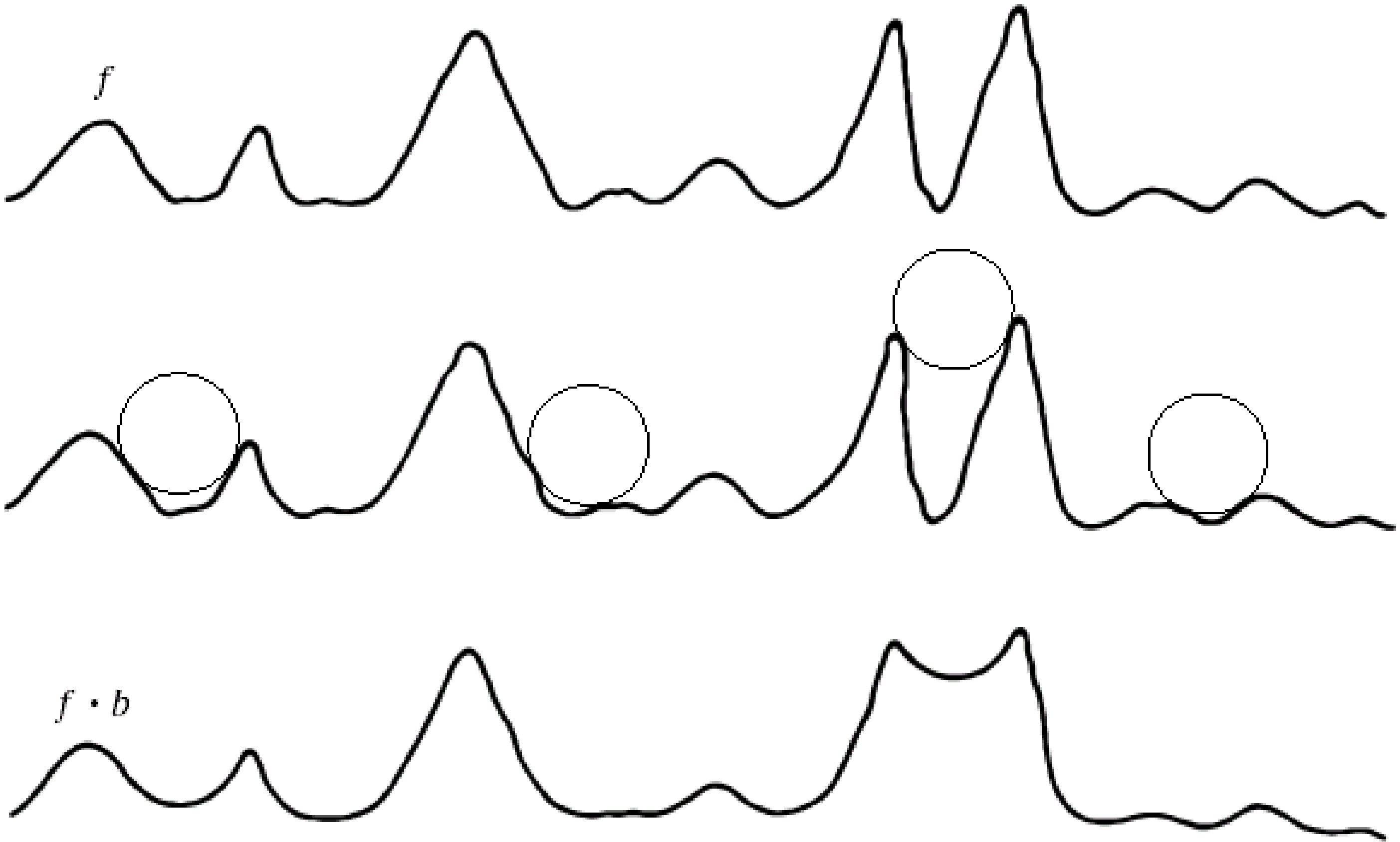
Closing

1. The closing of image f by structuring element b is given by

$$f \bullet b = \left(f \oplus b \right) \Theta b$$

2. As in the binary case, closing is simply the dilation of f by b, followed by a erosion of the result by b.
3. Suppose that we close f by a spherical structuring element b, (a ‘rolling ball’).
4. It is equivalent to rolling the ball on the top of the surface so that the entire surface is traversed.
5. $f \bullet b$ = surface of the lowest points reached by any part of the sphere as it slides over the entire surface of f.

Closing



- Peaks essentially are left in their original form.
- Closing is generally used to remove dark details from an image, while leaving bright features relatively undisturbed.

a b

FIGURE 9.31 (a) Opening and (b) closing of Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

Duality

1. The opening and closing for gray-scale images are duals with respect to complementation and reflection.

$$\left(f \bullet b\right)^{c}(s,t) = \left(f^{c} \circ \hat{b}\right)(s,t)$$

where

$$f^{c} = -f\left(x,y\right)$$

$$-\left(f \bullet b\right) = \left(-f \circ \hat{b}\right)$$

Applications: Morphological smoothing

1. Morphological opening followed by a closing.
2. The net result is to remove or attenuate both bright and dark artifacts or noise.

FIGURE 9.32 Morphological smoothing of the image in Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

Applications: Morphological gradient

1. Morphological gradient is defined as

$$g = (f \oplus b) - (f \Theta b)$$

2. It highlights sharp gray-level transitions in the input image.

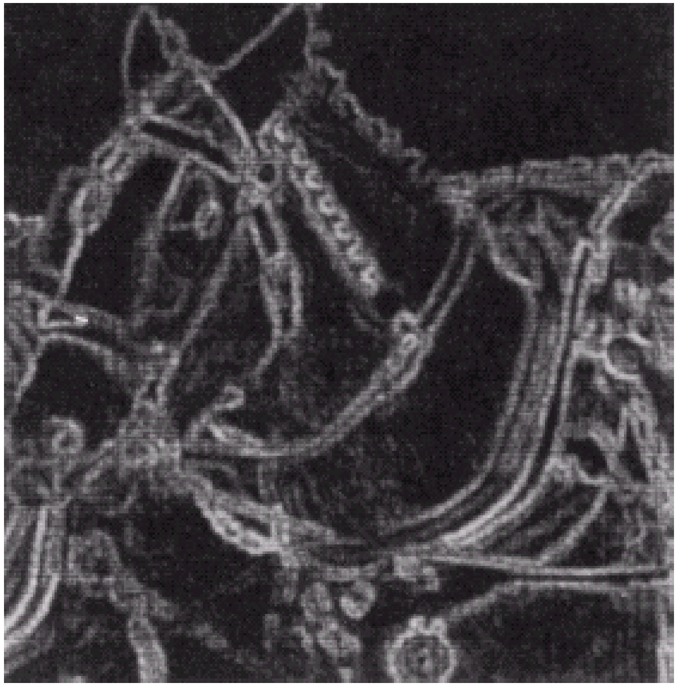


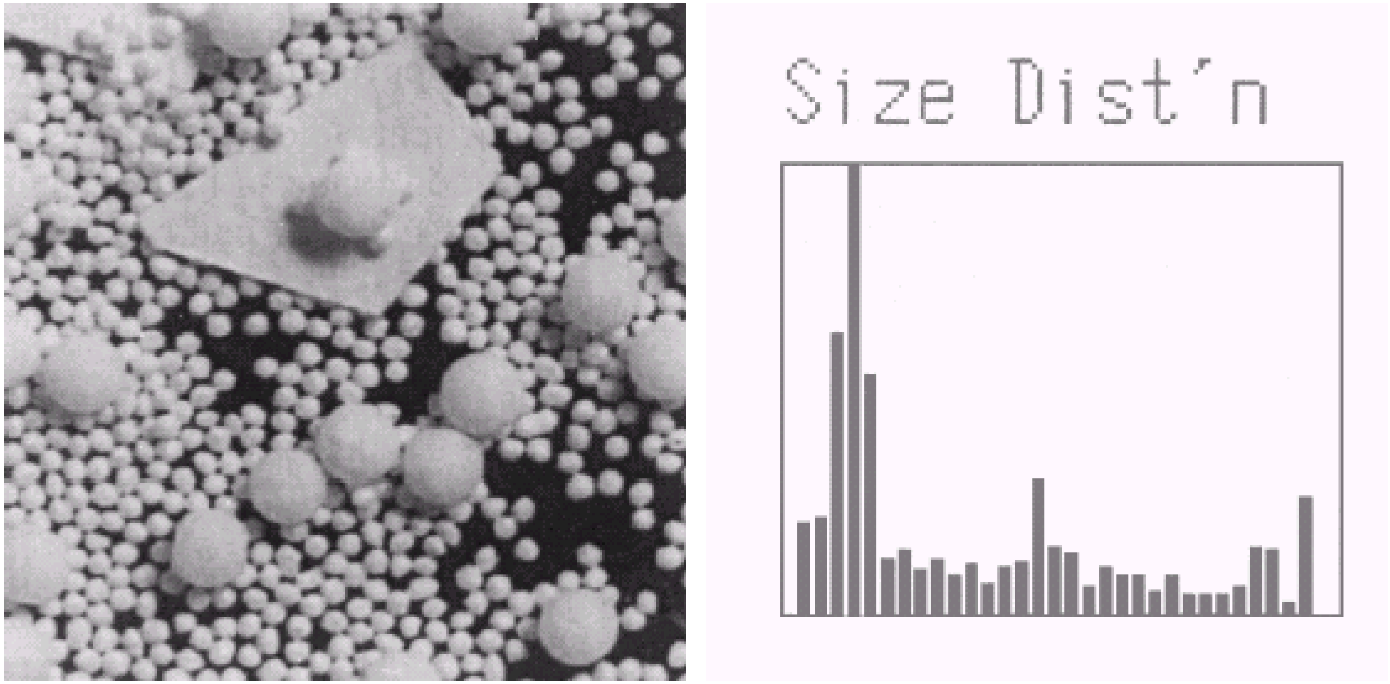
FIGURE 9.33 Morphological gradient of the image in Fig. 9.29(a). (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)

Applications: Granulometry

1. It deals with determining the size distribution of particles in an image.
2. Opening operations with structuring elements of increasing size are performed on the original image.
3. The difference between the original image and its opening is computed after each pass when a different structuring element is completed.
4. At the end of the process, these differences are normalized and then used to construct a histogram of particle-size distribution.
5. It is very useful for describing regions with a per-dominant particle-like character.

http://en.wikipedia.org/wiki/Granulometry_(morphology)

Applications: Granulometry



a b

FIGURE 9.36 (a) Original image consisting of overlapping particles; (b) size distribution. (Courtesy of Mr. A. Morris, Leica Cambridge, Ltd.)