COMP2611: Computer Organization

Data Representation

1. Binary numbers and 2's Complement

Numbers

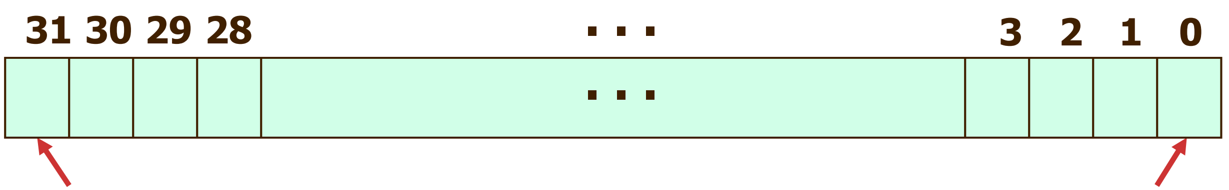
- Bits: are the basis for binary number representation in digital computers

- What you will learn here:
  - How to convert between binary and decimal and vice-versa?
  - How to represent negative numbers
  - How to represent fractions and real numbers
  - What is a representable range of numbers in a computer?
  - How to handle numbers that go beyond the representable range

- What you will not learn here (will be covered in Computer arithmetic):
  - Arithmetic operations: How to add, subtract, multiply, divide binary numbers
  - How to build the hardware that takes care of arithmetic operations

Base, Representation and Value

- Numbers can be represented in any base
  - Human: decimal (base 10, has 10 digits 0,1,…,9);
  - Computer: binary (base 2, has 2 digits, 0,1)
- Positional Notation: value of the `i`th digit `d` is $d \times Base^i$
  - $1001_2 = (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)_{10} = 9_{10}$
- Bits are grouped and numbered 0, 1, 2, 3 … from right to the left:
  - Byte: a group of 8 bits
  - Word: a group of 32 or 64 bit



most significant bit (MSb) least significant bit (LSb)

- Value of the 32-bit binary numbers =

$$(b_{31} \times 2^{31}) + (b_{30} \times 2^{30}) + \ \ldots + (b_1 \times 2^1) + (b_0 \times 2^0)$$

Hexadecimal Representation

- Hexadecimal (base 16) numbers are commonly used
- To avoid reading and writing long binary numbers

Conversion to hexadecimal

- Since base 16 is a power of 2, we can simply convert by replacing each group of four bits by a single hexadecimal digit, and vice versa

Example of hexadecimal-to-binary conversion:

0_{hex} - 9_{hex} for 0000_2 - 1001_2

a_{hex} - f_{hex} for 1010_2 – 1111_2

i.e. $0000\ 1010\ 0000\ 0101\ 0000\ 1100\ 0000\ 0110_2$

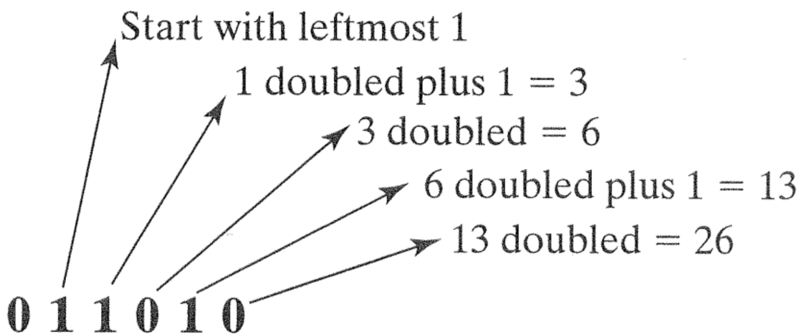
= 0 a 0 5 0 c 0 6_{hex}

= 0x0a050c06 # 0x to indicate it is a hexadecimal

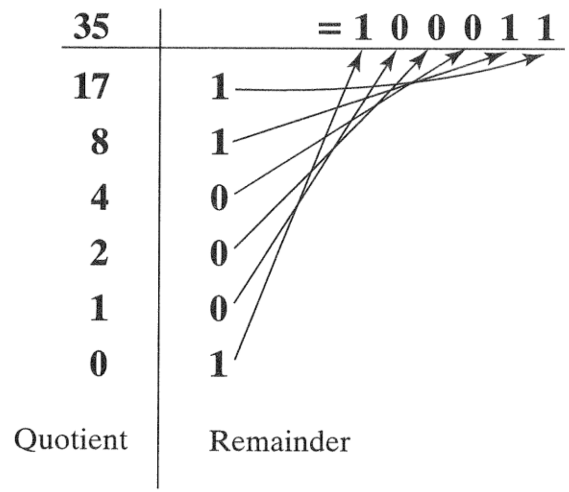
= 168102918_{10}

Simple recipes: Converting Binary <-> Decimal

- Binary to decimal:
  - Double and add method:
    - For each position i starting at the leftmost 1,
      - Double the sum, add bit i to the sum
  - Ex: 011010



- Decimal to binary:
  - Successive divisions by 2:
    - Inverse of double and add
    - In each step the remainder of the division is the next bit of the sequence



- Remark:
  - useful powers of 2 to memorize

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
1	2	4	8	16	32	64	128	256	512	1024

- All computers use 2's complement representation for signed numbers
- The most significant bit is called the sign bit:
  - When it is 0 the number is non-negative
  - When it is 1 the number is negative
  - The positive half uses the same representation as before
  - The negative half uses the conversion from the positive value illustrated below:
  - Ex: What is the representation of -6 in 2’s complement on 4 bits?

```
i)   Start from the representation of +6
```

$0110_2 = 6_{10}$

```
ii) Invert bits to get 1’s complement
```

$1001_2 = -7_{10}$

```
iii) Add 1 to get 2’s complement
```

$1010_2 = -6_{10}$

Signed numbers? 2's Complement

- Ex: What is the representation of -6 in 2's complement on 8 bits?

i) R epresentation of +6	$0000\ 0110_2 = 6_{10}$
ii) Invert:	$1111\ 1001_2 = -7_{10}$
iii) Add 1	$1111\ 1010_2 = -6_{10}$

- Ex: What is the representation of -6 in 2's complement on 32 bits?

i) Start from the representation of +6

$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0110_2 = 6_{10}$

ii) Invert bits to get 1's complement

$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1001_2 = -7_{10}$

iii) Add 1 to get 2's complement

$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1010_2 = -6_{10}$

Signed numbers? 2's Complement

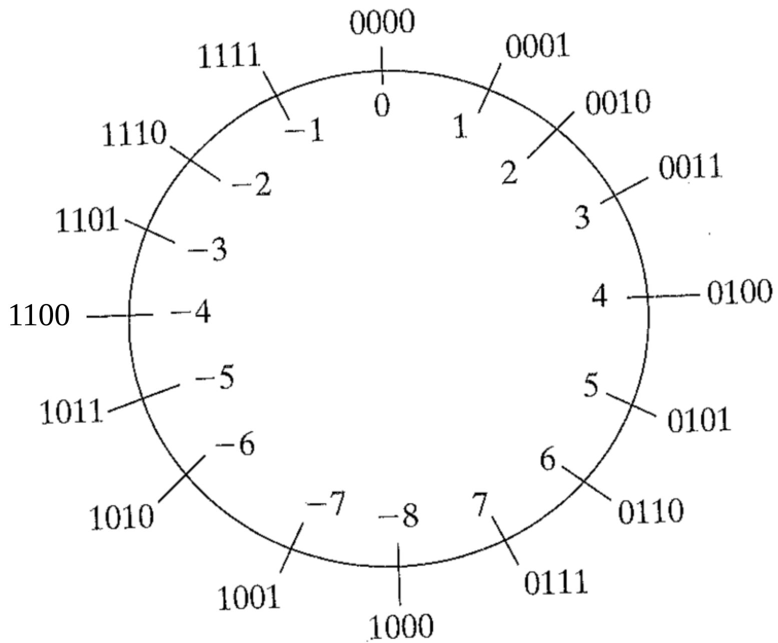
- Why invert and add 1 works?
- In One's Complement We have: if $x = 0$ then $\bar{x} = 1$

$$x + \bar{x} = 1111...111_2$$

- In 2's complement $111...111_2$ = -1, therefore

$$x + \bar{x} = 1111...111_2 = -1$$

$$\bar{x} + 1 = -x$$



Signed numbers? 2's Complement

- Largest integer represented by a 32 bit word:

  0111 1111 1111 1111 1111 1111 1111 1111_2 = $(2^{31} - 1)_{10}$ = $2{,}147{,}483{,}647_{10}$

- Smallest integer represented by a 32 bit word:

  1000 0000 0000 0000 0000 0000 0000 0000_2 = $-2^{31}{}_{10}$ = $-2{,}147{,}483{,}648_{10}$

- Examples:

  What is the largest/smallest integer represented by 8 bits (16 bits)

  What is the 16 bit representation of -255

  What number does the 32 bit 2's complement binary sequence 0xFFFFFFF1 represent

Examples

- Largest number

  8 bits -> $0111\ 1111_2 = $ 0x7F = 127 = 128 – 1 = $\mathbf{2^7 - 1}$

  16 bits -> $0111\ 1111\ 1111\ 1111_2 = $ 0x7FFF = 32767 = 32768 – 1 = $\mathbf{2^{15}} - 1$

- Smallest number

  8 bits -> $1000\ 0000_2$

  Invert and add 1: $0111\ 1111_2 + 1 = 1000\ 0000_2 = 2^7 = 128$

  => **- 128**

  16 bits -> $1000\ 0000\ 0000\ 0000_2$

  Invert – add 1: $0111\ 1111\ 1111\ 1111_2 + 1 = $ 0x8000 $= 2^{15} = 32768$

  => **- 32768**

Examples

- What is the 16 bit representation of -256

  256_{10} = 0000 0001 0000 0000$_2$

  Invert and add 1 : **-256_{10} = 1111 1111 0000 0000$_2$**

- What number does the 32 bit 2's complement binary sequence 0xFFFFFFF1 represent

  0xFFFFFFF1 (1111 1111 1111 1111 1111 1111 1111 0001)

  We can see this is a negative number because **the leftmost bit of the 32 bits is 1**

  Invert -> 0x0000000E (0000 0000 0000 0000 0000 0000 0000 1110)

  add 1 -> 0x0000000F = 15_{10}

  Therefore **0xFFFFFFF1 = -15_{10}**

- **Signed numbers**

  negative or **non-negative** integers, e.g. `int` in C/C++

- **Unsigned numbers**

  non-negative integers, e.g. `unsigned int` in C/C++

- **Ranges for signed and unsigned numbers**

  32 bit words signed:

  - from

    0111 1111 1111 1111 1111 1111 1111 1111_2 = $(2^{31} - 1)_{10}$ = $2{,}147{,}483{,}647_{10}$

  - to

    1000 0000 0000 0000 0000 0000 0000 0000_2 = $-2^{31}{}_{10}$ = $-2{,}147{,}483{,}648_{10}$

  32 bit words unsigned:

  - from

    0000 0000 0000 0000 0000 0000 0000 0000_2 = 0_{10}

  - to

    1111 1111 1111 1111 1111 1111 1111 1111_2 = $(2^{32} - 1)_{10}$ = $4{,}294{,}967{,}295_{10}$

Sign Extension

Why sign extension?

- Consider using a cast in C/C++ on a 32 bit machine

```
      int i;        /* signed integer represented on 32 bits */
      char a;       /* Character represented on 8 bits */

      i = (int) a;
```

- Assume we can do the same on the hardware and an instructions loads a 16 bit number into a 32 bit register (hardware variable)

  Bits 0~15 of **the register** will contain the **16bit value**

  What should be put in the remaining 16 bits (16~31) of the register?

Sign Extension

- Bits 0~15 of **the register** will contain the **16bit value**
- What should be put in the remaining 16 bits (16~31) of the register?

31	30	29	28	...	15	...	3	2	1	0
?	?	?	?	...	1	1 ...	1	1	1	0

- Depends on the sign of the 16 bit number

  If sign, of the 16 bits is 0 then fill with 0

  If sign is 1 then fill with 1

- For example:

  2 (16 bits -> 32 bits):

  0000 0000 0000 0010 -> 0000 0000 0000 0000 0000 0000 0000 0010

  -2 (16 bits -> 32 bits):

  1111 1111 1111 1110 -> 1111 1111 1111 1111 1111 1111 1111 1110

2. Floating Point Numbers

Why Floating Point?

- In addition to signed and unsigned integers, we also need to represent
  - **Numbers with fractions** (called real numbers in mathematics)
    - e.g. 3.1416
  - **Very small numbers**
    - e.g., 0.00000000001
  - **Very large numbers**
    - e.g., 1.23456×10^{10} (a number a 32-bit integer can't represent)

- In decimal representation, we have **decimal point**
- In binary representation, we call it **binary point**

- Such numbers are called **floating point** in computer arithmetic
- Because the binary point is not fixed in the representation

Scientific Notation & Normalized Scientific Notation

- **Scientific notation**

  A single digit to the left of the decimal point

  e.g. 1.23×10^{-3}, 0.5×10^{5}

- **Normalized scientific notation** (pay attention to this one)

  Scientific notation with no leading 0's

  e.g. 1.23×10^{-3}, 5.0×10^{4}

- Binary numbers can also be represented in scientific notation
- All normalized binary numbers always start with a 1

$$1.xxxxx..xx_{two} \times 2^{yy..yy}$$

Single-Precision Floating-Point Representation

- Single-precision uses 32 bits
- Sign-and-magnitude representation:

31 30 29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0

s	exponent	significand
1 bit	8 bits	23 bits

- Interpretation
  - S = sign; F = significand; E = exponent
  - Value represented = $(-1)^{S} \times F \times 2^{E}$
  - Roughly gives 7 decimal digits in precision
  - Exponent scale of about 10^{-38} to 10^{+38}

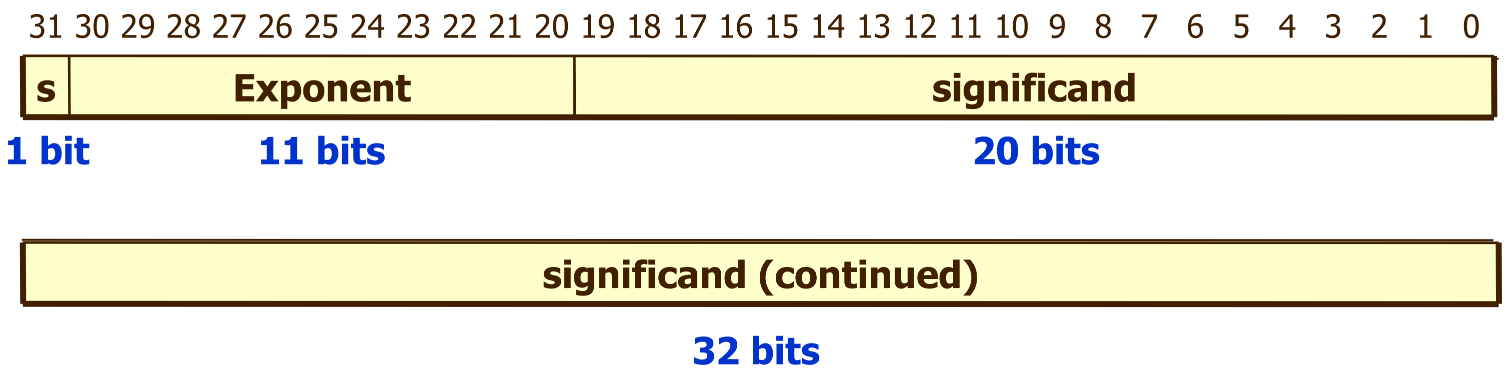
- Compromise between sizes of exponent and significand fields:
  - Increase size of **exponent** ⇒ increase representable range
  - Increase size of **significand** ⇒ increase accuracy

Overflow and Underflow

- **Overflow** (floating-point)
  - A positive exponent becomes too large to fit in the exponent field
- **Underflow** (floating-point)
  - A negative exponent becomes too large to fit in the exponent field
- To alleviate overflow/underflow problems in the single-precision
  - Use **double-precision representation**
  - It extends the exponent field from 8 bits to 11 bits

Double-Precision Floating-Point Representation

- Double-precision floating-point uses 64 bits
  - In 32 bit architectures like MIPS, each double-precision number requires two MIPS words
  - 11 bits for exponent, 52 bits for significand



- Provides precision of about 16 decimals
- Exponent scale from 10^{-308} to 10^{+308}
- For more precision IEEE 754 defines Quad precision with 128 bits, comprised of 1 sign bit, 15 exponent bits and 112 significand bits

IEEE 754 Floating-Point Standard

- Most computers use this standard for both single and double precision

- Why use a standard floating-point representation?
- Simplify porting floating-point programs across different computers

- To pack even more bits into the significand

  This standard makes the leading 1 bit (in 1.xx … xxx) implicit

  Interpretation: $(-1)^{s} \times (1 + 0.\text{significand}) \times 2^{E}$

  Effective number of bits used for representing the significand:

  - 24 (i.e., 23 + 1) – for single precision
  - 53 (i.e., 52 + 1) – for double precision

  Special case:

  - Since 0 has no leading 1, it is given the reserved exponent value 0 so that the hardware does not attach a leading 1 to it

- **Computation of significand**:

$$\text{Significand} = s_1 \times 2^{-1} + s_2 \times 2^{-2} + s_3 \times 2^{-3} + \ldots$$

The significand bits are denoted as $\boxed{s_1, s_2, s_3, \ldots,}$ from left to right

- **To allow <u>quick</u> comparisons in hardware implementation**:

The sign is in the most significant bit

The exponent is placed before the significand

(Comparisons mean "less than", "greater than", "equal to zero")

- **How to represent a NEGATIVE exponent?**

Biased exponent: a bias is implicitly added to the exponent

$$(-1)^{s} \times (1 + 0.\text{significand}) \times 2^{(E-\text{bias})}$$

bias = 127 for single precision, bias = 1023 for double precision

The most negative exponent = 0_2, the most positive = $11\ldots11_2$

Example

- Give the binary representation of -0.75_{10} in single & double precisions

- **Answer**

  $-0.75_{10} = -0.11_2$

  0.75 * 2 = **1.50**, S1 = **1**

  0.50 * 2 = **1.00**, S2 = **1**; stop because the fraction is 0.00

  Scientific notation: $-0.11_2 \times 2^0$

  Normalized scientific notation: $-1.1_2 \times 2^{-1}$

  Sign = 1 (negative), exponent = -1

  Single precision:

  S = 1, E = 01111110, significand = 100...00 (23 bits)

  = -1+127, (127 is the bias)

  Double precision:

  S = 1, E = 01111111110, significand = 100...00 (52 bits)

  = -1+1023, (1023 is the bias)

- What decimal number is represented by this word (single precision)?

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
1	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

- **Answer**:

$$(-1)^s \times (1 + Significand) \times 2^{(E-Bias)}$$

$$= (-1)^1 \times (1 + 0.25) \times 2^{(129-127)}$$

$$= -1 \times 1.25 \times 2^2$$

$$= -1.25 \times 4$$

$$= -5.0$$

IEEE Standard for floating point Arithmetic

Single precision:

Significand \ Exponent	0	**1 - 254**	255
0	0	$(-1)^S \times (1.F) \times (2)^{E-127}$	$(-1)^S \times (\infty)$
≠ 0	$(-1)^S \times (0.F) \times (2)^{-126}$		non-numbers e.g. 0/0 , $\sqrt{-1}$

Double precision:

Significand \ Exponent	0	**1 - 2046**	2047
0	0	$(-1)^S \times (1.F) \times (2)^{E-1023}$	$(-1)^S \times (\infty)$
≠ 0	$(-1)^S \times (0.F) \times (2)^{-1022}$		non-numbers e.g. 0/0 , $\sqrt{-1}$

Examples

0 00000000 00000000000000000000000 = 0

1 00000000 00000000000000000000000 = -0

0 11111111 00000000000000000000000 = + infinity

1 11111111 00000000000000000000000 = - infinity

0 11111111 01001100010001000001000 = **NaN (Not a Number)**

1 11111111 01001100010001000001000 = NaN

0 10000000 00000000000000000000000 = $+(1.0_2)\times(2)^{128-127}=2$

0 10000001 10100000000000000000000 = $+(1.101_2)\times(2)^{129-127}=6.5$

1 10000001 10100000000000000000000 = $-(1.101_2)\times(2)^{129-127}=-6.5$

0 00000001 00000000000000000000000 = $+(1.0_2)\times(2)^{1-127}=(2)^{-126}$

0 00000000 10000000000000000000000 = $+(0.1_2)\times(2)^{-126}=(2)^{-127}$

0 00000000 00000000000000000000001 = $+(2)^{-23}\times(2)^{-126}=(2)^{-149}$

❑ How to represent Characters

Characters are unsigned bytes e.g., in C++ `Char`

Usually follow the ASCII standard

Uses 8 bits unsigned to represent a character

b_7 b_6 b_5 →					0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
Bits b_4 ↓	b_3 ↓	b_2 ↓	b_1 ↓	Column → Row ↓	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	'	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FC	,	<	L	\	l	|
1	1	0	1	13	CR	GS	-	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

- **What does the following 32 bit pattern represent: 0x32363131**
  - If it were a 2's complement integer
    - `the MSb is 0 therefore this is a positive number`
    - `evaluation left as an exercise`
  - An unsigned number
    - `Same value as above`
  - A sequence of ASCII encoded bytes: `2611`
    - `Checking the ascii table gives:`
    - `0x32 = code for character '2'`
    - `0x36 = code for character '6'`
    - `0x32 = code for character '1'`
    - `0x32 = code for character '1'`
  - A 32 bit IEEE 754 floating point number
    - `s= 0, E = 01100100, S = 01101100011000100110001`
    - `This is a normalized number so E is biased.`

- Consider building a floating point number system like the IEEE754 standard on 8 bit only, with 3 bits being reserved for the exponent.

  What is the value of the bias?

  3

  What is the representation of 0?

  0 000 0000

  What is the representation of -4?

  -4 = - 1.0 x 2+2

  S=1, F= 0 and the biased exponent must be

  E – 3 = 2 or E = +5

  So - 4 = 1 101 0000

  What is the next value representable after – 4?

  1 101 0001 = - 4.125 so we can see that 4 bits for the significand is not accurate enough

  What does the byte 11111011 represent? **- NAN**

  What is the representation of –Inf? **1 111 0000**

Key Concepts to Remember

- **2's complement representation** for signed numbers
- **Floating-point numbers**
  - Representation follows closely the **scientific notation**
  - Almost all computers, including MIPS, follow **IEEE 754 standard**
- In MIPS,
  - **Single-precision** floating-point representation takes 32 bits
  - **Double-precision** floating-point representation takes 64 bits
  - In most hardware FP **co-processor** and separate **FP registers**
- **Overflow** (**underflow**) in floating-point representation occurs
  - When the exponent is too large (small) to be represented