Basics: Sets and Functions

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Contents

Set

Definition

A **set** is an unordered collection of objects.

Remark

This is only an intuitive definition which is not part of a formal theory of sets.

Element

Definition

The objects in a set are called the **elements** or **members** of the set. A set is said to **contain** its elements. We write $a \in A$ to denote that a is an element of the set A and $a \notin A$ to denote that a is not an element of A.

Example 1

The set of all odd positive integers less than 100 can be denoted by $\{1, 3, 5, \ldots, 99\}$. We can also use the **set builder** notation to express the set as

$$\{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 100\}.$$

Empty Set and Singleton Set

Definition

The **empty set** or **null set**, denoted by $\emptyset$ or $\{\}$, is a special set containing no elements.

Definition

A set with one element is called a **singleton set**.

Example 2

Note that $\emptyset \neq \{\emptyset\}$. The latter is a singleton set.

Data Type in Computer Science

Remark

The concept of a **data type** in computer science is built upon the concept of a set. In particular, a data type is the name of a set, together with the operations that can be performed on objects from that set. For example, `boolean` is the name of the set $\{0, 1\}$ together with operations on one or more elements of this set, such as `AND`, `OR`, and `NOT`.

Set Equality

Definition

Two sets A and B are **equal**, denoted by $A = B$, if and only if they have the same elements, i.e., for every x, $x \in A$ if and only if $x \in B$.

Example 3

$\{1, 2, 3\} = \{3, 1, 2\} = \{1, 2, 2, 2, 3, 3\}$.

Subset

Definition

A set A is said to be a **subset** of a set B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B.

Theorem 1.1

For every set S, (a) $\emptyset \subseteq S$ and (b) $S \subseteq S$.

Remark

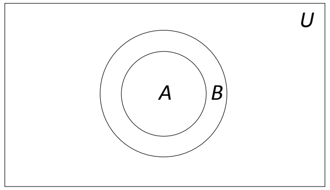
Every nonempty set is guaranteed to have at least two subsets, the empty set and the set itself.

Proper Subset

Definition

A set A is said to be a **proper subset** of a set B, denoted by $A \subset B$, if and only if $A \subseteq B$ but $A \neq B$.

We can use a **Venn diagram** to illustrate, among other things, the subset relationship.



Set Equality

Theorem 1.2

Let A and B be two sets. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Remark

This theorem provides a useful way to show that two sets are equal.

Cardinality

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a **finite set** and that n is the **cardinality** of S, denoted by $|S|$.

Definition

A set is said to be **infinite** if it is not finite.

Power Set

Definition

Given a set S, the **power set** of S, denoted by $P(S)$, is the set of all subsets of S.

Remark

If a set has n elements where n is a nonnegative integer, then its power set has 2^n elements.

Examples

Example 4

The power set of the set $\{a, b, c\}$ is
$P(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Example 5

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Ordered Tuple

Definition

The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element. An ordered 2-tuple is more commonly called an **ordered pair**.

Definition

Two ordered n-tuples $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_n)$ are **equal** if and only if $m = n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

Cartesian Product

Definition

Let A and B be two sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$, i.e.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Relation

Definition

Let A and B be two sets. A subset R of $A \times B$ is called a **relation** from the set A to the set B.

Example 6

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. The Cartesian product is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

and $R = \{(1, a), (1, c), (2, a), (2, b)\} \subset A \times B$ is a relation from A to B.

Cartesian Product

Definition

The **Cartesian product** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$, i.e.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Union

Definition

Let A and B be two sets. The **union** of A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

Definition

Let $A_1, A_2, \ldots, A_n$ be n sets. The **union** of the collection of n sets, denoted by $\bigcup_{i=1}^{n} A_i$, is the set that contains those elements that are members of at least one set in the collection.

Intersection

Definition

Let A and B be two sets. The **intersection** of A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

Definition

Let $A_1, A_2, \ldots, A_n$ be n sets. The **intersection** of the collection of n sets, denoted by $\bigcap_{i=1}^{n} A_i$, is the set that contains those elements that are members of all the sets in the collection.

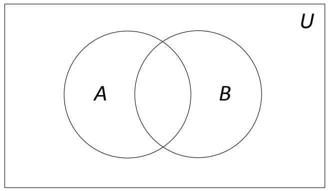
Definition

Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Union, Intersection, and Cardinality

Theorem 1.3

Let A and B be two finite sets. The cardinality of their union $|A \cup B| = |A| + |B| - |A \cap B|$.



Difference and Complement

Definition

Let A and B be two sets. The **difference** of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. It is also called the **complement of** B **with respect to** A.

Example 7

$\{1,3,5\} - \{1,2,3\} = \{5\}$.

Definition

Let U be the universal set. The **complement** of a set A, denoted by $\overline{A}$, is the complement of A with respect to U. In other words, it is $U - A$.

Set Identities

Set identities	
Identity	Name
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Set Identities (cont'd)

Set identities	
Identity	Name
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Function

Definition

Let A and B be nonempty sets. A **function** from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Remark

Functions are sometimes also called **mappings** or **transformations**.

Definition

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b. The **range** of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f **maps** A to B.

Examples

Example 8

Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set $\{00, 01, 10, 11\}$.

Example 9

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, we take the codomain of f to be the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Injective Function

Definition

A function f is said to be **injective** (or **one-to-one**) if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. An injective function is also called an **injection**.

Example 10

Is the function $f(x) = x + 1$ from the set of real numbers to the set of real numbers injective?

Example 11

Is the function $f(x) = x^2$ from the set of integers to the set of integers injective?

Increasing and Decreasing

Definition

A function f whose domain and codomain are subsets of the set of real numbers is called **increasing** if $f(x) \leq f(y)$, and **strictly increasing** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called **decreasing** if $f(x) \geq f(y)$, and **strictly decreasing** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

Surjective Function

Definition

A function f from the set A to the set B is said to be **surjective** (or **onto**) if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A surjective function is also called a **surjection**.

Example 12

Is the function $f(x) = x + 1$ from the set of integers to the set of integers surjective?

Example 13

Is the function $f(x) = x^2$ from the set of integers to the set of integers surjective?

Bijection

Definition

A function f is a **bijection** (or **one-to-one correspondence**) if it is both one-to-one and onto.

Theorem 2.1

Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.

Definition

A set that is either finite or has the same cardinality as the set of positive integers is called **countable**. A set that is not countable is called **uncountable**.

Inverse Function

Definition

Let f be a one-to-one correspondence from the set A to the set B. The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Definition

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence because the inverse of such a function does not exist.

Examples

Example 14

Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Example 15

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$. Is f invertible?

Composition

Definition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition** of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

Example 16

Let f and g be functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?