Combinatorics

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Contents

Introduction

Examples of when counting is needed in computer science:

- Determining whether there are enough Internet protocol (IP) addresses to meet the demand
- Determining the time or space complexity of algorithms
- Computing the probabilities of events

Two basic counting principles:

- **Product rule** (or **sequential rule**)
- **Sum rule** (or **disjunctive rule**)

Product Rule

Rule (Product rule)

Suppose a procedure can be broken down into $m > 1$ tasks, $T_1, T_2, \ldots, T_m$, and carried out by performing the tasks in sequence. If each task T_i can be done in n_i ways regardless of how the previous tasks were done, then there are $n_1 n_2 \cdots n_m$ ways to carry out the procedure.

Examples

Example 1

Ten chairs are numbered from 1 to 10. How many ways are there to assign different chairs to Joseph and Mary?

Example 2

Let U and V be two finite sets consisting of $|U| = m > 0$ and $|V| = n > 0$ elements, respectively. How many different functions $f : U \to V$ can be defined?

Examples

Example 3

Let U and V be two finite sets consisting of $|U| = m > 0$ and $|V| = n > 0$ elements, respectively. How many different one-to-one (or injective) functions $f : U \to V$ can be defined?

Example 4

Let S be a finite set of cardinality $|S| = n > 0$. How many different subsets of S are there?

Example

Example 5

Consider the following algorithm for matrix multiplication:

```
for i = 1 to m
  for j = 1 to n
    s = 0
    for k = 1 to p
      s = s + A[i,k] × B[k,j]
    C[i,j] = s
  end for
end for
```

How many multiplications are executed?

Sum Rule

Rule (Sum rule – version 1)

Suppose a procedure can be carried out in one of $m > 1$ methods, $T_1, T_2, \ldots, T_m$. If each method T_i can be done in n_i ways and no two methods share at least one way (i.e., the methods are disjoint), then there are totally $\sum_{i=1}^{m} n_i$ different ways to carry out the procedure.

Rule (Sum rule – version 2)

Suppose a procedure can be broken down into $m > 1$ tasks, $T_1, T_2, \ldots, T_m$, and carried out by performing the tasks in sequence. If each task T_i can be carried out in n_i steps and no two tasks have any step in common (i.e., the tasks are disjoint), then the procedure can be carried out in $\sum_{i=1}^{m} n_i$ steps.

Sum Rule

Rule (Sum rule – version 3)

Let $S_1, S_2, \ldots, S_m$ be $m > 1$ disjoint finite sets. The number of elements in the union of these sets is equal to the sum of the numbers of elements in the sets, i.e.,

$$|S_1 \cup S_2 \cup \cdots \cup S_m| = |S_1| + |S_2| + \cdots + |S_m| \qquad \text{or} \qquad \left|\bigcup_{i=1}^{m} S_i\right| = \sum_{i=1}^{m} |S_i|.$$

Note that the sum rule cannot be applied if the tasks (or sets) are not disjoint. In that case, we need to apply the **inclusion-exclusion principle** which will be studied later.

Example

Example 6

There are 15 boys and 18 girls in a class. How many ways are there to choose a representative for the class?

Example

Example 7

Consider the following selection sort algorithm:

```
for i = 1 to n − 1
  for j = i + 1 to n
    if (A[i] > A[j])
      swap A[i] and A[j]
    end if
  end for
end for
```

How many logical comparisons of the `if` statement are executed?

Using Both Product and Sum Rules

Example 8

In a programming language, a variable name is a string of one or two alphanumeric characters which are not case-sensitive, i.e., an alphanumeric character is either one of the 26 English letters or one of the 10 digits. Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in the language?

Example 9

In a computer system, each user has a password of six to eight characters long where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Tree Diagrams

Some counting problems can be solved more easily using tree diagrams.

Example 10

How many bit strings of length four do not have two consecutive 1s?

Example 11

A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

Pigeonhole Principle

Pigeonhole Principle

Principle (Pigeonhole principle or Dirichlet drawer principle)

If $k+1$ or more objects (pigeons) are placed into k boxes (pigeonholes) where k is a positive integer, then there is at least one box (pigeonhole) containing two or more of the objects (pigeons).

Corollary 2.1 (Corollary of pigeonhole principle)

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

The challenge is to choose the objects (pigeons) and boxes (pigeonholes) correctly.

Examples

Example 12

Among any group of 367 people, there must be at least two with the same birthday.

Example 13

In any group of 27 English words, there must be at least two that begin with the same letter.

Example 14

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Examples

Example 15

Twenty-five points are marked inside the area bounded by a rectangle that is 6 cm long and 4 cm wide. Show that there are at least two points that are at most $\sqrt{2}$ cm apart.

Example 16

Let $a_1, a_2, \ldots, a_{1997}$ represent an arbitrary arrangement of the numbers $1, 2, \ldots, 1997$. Is the product $(a_1 - 1)(a_2 - 2)\cdots(a_{1997} - 1997)$ an odd or even number? Justify your answer.

Generalized Pigeonhole Principle

Principle (Generalized pigeonhole principle)

If n objects are placed into k boxes, then there is at least one box containing at least $\lceil n/k \rceil$ objects.

Proof of generalized pigeonhole principle.

We prove it by contradiction. Suppose none of the boxes contains more than $\lceil n/k \rceil - 1$ objects. Then, the total number of objects is at most $k(\lceil n/k \rceil - 1)$. We note that $\lceil n/k \rceil < n/k + 1$. This implies that $k(\lceil n/k \rceil - 1) < k((n/k + 1) - 1) = n$, which contradicts the fact that there are a total of n objects. □

Examples

Example 17

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 18

What is the minimum number of students required in a class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Examples

Example 19

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? How many cards must be selected to guarantee that at least three hearts are selected?

Example 20

Assume that phone numbers are of the form *NXX-NXX-XXXX*, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit phone numbers?

Permutations

Example 21

In how many ways can we select three students from a group of five students to stand in line for a photo?

Example 22

In how many ways can we arrange all five of the students above in a line for a photo?

Permutation

Definition

A **permutation** of a set of distinct elements is an ordered arrangement of the elements of the set.

Proposition 3.1

A permutation of a set is a **bijection** from the set to itself.

Example 23

Let $S = \{1, 2, 3\}$. The ordered arrangement $(3, 1, 2)$ is a permutation of S.

k-Permutation

Definition

A k-**permutation** (or k-**element permutation**) of a set of n distinct elements is an ordered arrangement of $k \leq n$ elements of the set.

Example 24

Let $S = \{1, 2, 3\}$. The ordered arrangement $(3, 2)$ is a 2-permutation of S.

Number of k-Permutations

Theorem 3.1

Let n and k be integers with $0 \leq k \leq n$. The number of k-permutations of a set with n distinct elements, denoted by $P(n, k)$, is equal to

$$P(n, k) = n(n-1)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i) = \frac{n!}{(n-k)!}.$$

Proof

Proof.

We first consider the case with $1 \leq k \leq n$. The first element of the permutation can be chosen in n ways. After the first element has been chosen, there are $n-1$ ways to choose the second element because there are $n-1$ elements left in the set after using the element picked for the first position. Similarly, there are $n-2$ ways to choose the third element, and so on, until there are exactly $n-(k-1) = n-k+1$ ways to choose the kth element. By the product rule, there are $n(n-1)\cdots(n-k+1) = n!/(n-k)!$ k-permutations of the set.

For $k = 0$, $P(n, 0) = 1$ because there is exactly one way to order zero elements, i.e., there is exactly one list with no elements in it, namely the empty list. Because $n!/(n-0)! = n!/n! = 1$, we see that the formula $P(n, k) = n!/(n-k)!$ also holds when $k = 0$. □

n-Permutation

Definition

An n-permutation of a set of n distinct objects is simply called a permutation of the set.

Examples

Example 25

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Example 26

Suppose a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wants. How many possible orders can the saleswoman use when visiting these cities?

Example 27

How many permutations of the letters $ABCDEFGH$ contain the string ABC?

k-Combinations

Example 28

How many different committees of three students can be formed from a group of four students?

Definition

A k-**combination** of a set of n distinct elements is an unordered selection of $k \leq n$ elements from the set.

Example 29

Let $S = \{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S.

Number of k-Combinations

Theorem 3.2

Let n and k be integers with $0 \le k \le n$. The number of k-combinations of a set with n distinct elements, denoted by $C(n,k)$ or $\binom{n}{k}$, is equal to

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Proof

Proof.

The k-permutations of the set can be obtained by forming the $C(n,k)$ k-combinations of the set, and then ordering the elements in each k-combination, which can be done in $P(k,k)$ ways. Consequently,

$$P(n,k) = C(n,k) \cdot P(k,k).$$

This implies that

$$C(n,k) = \frac{P(n,k)}{P(k,k)} = \frac{n!/(n-k)!}{k!/(k-k)!} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

□

Example

Example 30

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Corollary

Corollary 3.3

Let n and k be integers with $0 \le k \le n$. Then $C(n,k) = C(n,n-k)$.

Proof.

From Theorem 3.2, it follows that

$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

$$C(n,n-k) = \frac{n!}{(n-k)!\,[n-(n-k)]!} = \frac{n!}{(n-k)!\,k!}.$$

Hence, $C(n,k) = C(n,n-k)$.

□

Combinatorial Proof and Bijection Principle

Alternative (combinatorial) proof of corollary.

Suppose S is a set with n distinct elements. Every subset A of S with $k \leq n$ elements corresponds to the subset $S - A$ of S with $n - k$ elements. Consequently, $C(n, k) = C(n, n - k)$. □

Definition

A **combinatorial proof** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.

Principle (Bijection principle)

Two sets U and V have the same size, i.e., $|U| = |V|$, if and only if there is a one-to-one function $f : U \to V$ from U onto V.

Examples

Example 31

How many bit strings of length n contain exactly k 1s?

Example 32

Suppose there are 40 faculty members in the computer science department and 30 in the mathematics department. How many ways are there to select a committee to develop a discrete mathematics course at a university if the committee is to consist of four faculty members from the computer science department and three from the mathematics department?

Example 33

Twelve people sit down at a round table. We consider two seating charts equivalent if each person has the same person to the right in both seating charts. How many different seating charts are there?

Binomial Theorem

Definition

The number of k-combinations of a set with n elements, denoted by $C(n, k)$ or $\binom{n}{k}$, is also called a **binomial coefficient** because it occurs as a coefficient in the expansion of the power of a binomial expression such as $(x + y)^n$.

Theorem 4.1 (Binomial theorem)

Let x and y be variables and n be a nonnegative integer. Then

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.\end{aligned}$$

Proof

Proof of binomial theorem.

We give a combinatorial proof of the theorem here. The terms in the product when it is expanded are of the form $x^{n-k} y^k$ for $k = 0, 1, \ldots, n$. To count the number of terms of the form $x^{n-k} y^k$, note that to obtain such a term it is necessary to choose k ys from the n sums so that the other $n - k$ terms in the product are xs. Therefore, the coefficient of $x^{n-k} y^k$ is $\binom{n}{k}$. This proves the theorem. □

Examples

Example 34

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Example 35

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Examples

Example 36

What is the constant term in the expansion of $\left(x+\frac{1}{x^3}\right)^{12}$?

Example 37

What is the coefficient of x^{13} in the expansion of $(1+x^4+x^5)^{10}$?

Corollary

Corollary 4.2

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

Proof.

Using the binomial theorem with $x=y=1$, we can get

$$2^n=(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k=\sum_{k=0}^{n}\binom{n}{k}.$$

□

Alternative Proof

Alternative (combinatorial) proof of corollary.

A set with n elements has a total of 2^n different subsets. Each subset has zero elements, one element, two elements, ..., or n elements in it. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, ..., and $\binom{n}{n}$ subsets with n elements. Therefore,

$$\sum_{k=0}^{n}\binom{n}{k}$$

counts the total number of subsets of a set with n elements. This shows that

$$\sum_{k=0}^{n}\binom{n}{k}=2^n.$$

□

Corollary

Corollary 4.3

Let n be a positive integer. Then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

Proof.

Using the binomial theorem with $x = 1$ and $y = -1$, we can get

$$0 = (1 + (-1))^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}(-1)^k = \sum_{k=0}^{n}\binom{n}{k}(-1)^k.$$

This proves the corollary. □

Remark

Remark

The corollary implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

Corollary

Corollary 4.4

Let n be a positive integer. Then

$$\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n.$$

Proof.

Using the binomial theorem with $x = 1$ and $y = 2$, we can get

$$3^n = (1 + 2)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}2^k = \sum_{k=0}^{n}\binom{n}{k}2^k.$$

This proves the corollary. □

Pascal's Identity

The binomial coefficients satisfy many different identities. One of the most important identities is discussed below.

Theorem 4.5 (Pascal's identity)

Let n and k be integers with $0 < k < n$. Then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof

Combinatorial proof of Pascal's identity.

Suppose S is a set containing n elements. Let a be an element of S and $T = S - \{a\}$. Note that there are $\binom{n}{k}$ subsets of S containing k elements. However, a subset of S with k elements either contains a together with $k-1$ elements of T, or contains k elements of T and does not contain a. Because there are $\binom{n-1}{k-1}$ subsets of $k-1$ elements of T, there are $\binom{n-1}{k-1}$ subsets of k elements of S that contain a. Also, there are $\binom{n-1}{k}$ subsets of k elements of S that do not contain a because there are $\binom{n-1}{k}$ subsets of k elements of T. Consequently,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

□

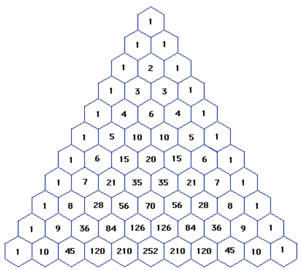
Pascal's Triangle

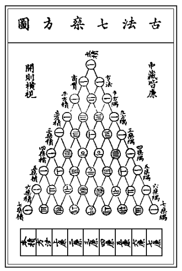
Corollary 4.6

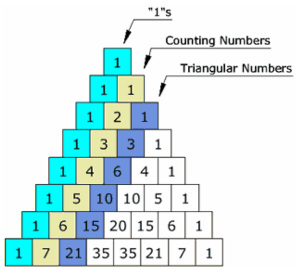
Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to give a geometric arrangement of the binomial coefficients in a triangle, called **Pascal's triangle**, as follows:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\cdots
\end{array}$$

Pascal's Triangle







Trinomial Theorem

Theorem 4.7 (Trinomial theorem)

Let x, y, z be variables and n be a nonnegative integer. Then

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} x^{k_1} y^{k_2} z^{k_3},$$

where

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!\, k_2!\, k_3!}$$

is a **trinomial coefficient**.

Proof

Proof of trinomial theorem.

The terms in the product when it is expanded are of the form $x^{k_1}y^{k_2}z^{k_3}$ where $k_1 + k_2 + k_3 = n$ and $0 \le k_1, k_2, k_3 \le n$. There are $\binom{n}{k_1}$ ways to choose the k_1 xs from the n sums and $\binom{n-k_1}{k_2}$ ways to choose the k_2 ys from the remaining $n - k_1$ sums. After that, the remaining $k_3 = n - k_1 - k_2$ sums contribute the k_3 zs. By the product rule,

$$\begin{aligned}\binom{n}{k_1, k_2, k_3} &= \binom{n}{k_1}\binom{n-k_1}{k_2} \\ &= \frac{n!}{k_1!\,(n-k_1)!}\cdot\frac{(n-k_1)!}{k_2!\,(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!\,k_2!\,(n-k_1-k_2)!} = \frac{n!}{k_1!\,k_2!\,k_3!}.\end{aligned}$$

□

Vandermonde's Identity

Theorem 4.8 (Vandermonde's identity)

Let m, n, and r be integers with $0 \le r \le m$ and $0 \le r \le n$. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}.$$

Proof

Combinatorial proof of Vandermonde's identity.

Suppose that there are m elements in one set and n elements in a second set. Then the total number of ways to choose r elements from the union of the two sets is $\binom{m+n}{r}$. Another way to choose r elements from the union is to choose $r - k$ elements from the first set and then k elements from the second set, where k is an integer with $0 \le k \le r$. By the product rule, this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Consequently,

$$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}.$$

□

Corollary

Corollary 4.9

Let n be a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

Proof.

Using Vandermonde's identity with $m = n = r$, we can get

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

□

Another Identity

Theorem 4.10

Let n and r be integers with $0 \le r \le n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof

Combinatorial proof.

The left-hand side counts the bit strings of length $n+1$ containing $r+1$ 1s. We consider an alternative way of counting the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ 1s. This final 1 must occur at location $r+1$, $r+2$, ..., or $n+1$. Furthermore, if the last 1 is the kth bit then there must be r 1s among the first $k-1$ locations. Consequently, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \le k \le n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

bit strings of length $n+1$ containing exactly $r+1$ 1s. Because both sides of the identity count the same objects, they must be equal. This completes the proof. □

Permutations with Indistinguishable Objects

Example 38

How many different character strings can be made by reordering the letters of the word *SUCCESS*?

Theorem 5.1

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Distributing Objects into Boxes

Many counting problems can be solved by enumerating the ways objects can be placed into boxes where the order these objects are placed into the boxes does not matter.

The objects can be either *distinguishable* or *indistinguishable*. Similarly, the boxes can also be either *distinguishable* or *indistinguishable*.

Closed formulae exist only when the boxes are distinguishable.

For simplicity, we will only consider the case with distinguishable objects and distinguishable boxes in this course.

Distinguishable Objects and Distinguishable Boxes

Example 39

How many ways are there to distribute hands of five cards to each of four players from the standard deck of 52 cards?

Remark

The solution above is equal to the number of permutations of 52 objects, with five indistinguishable objects of each of four different types and 32 of a fifth type.

Distinguishable Objects and Distinguishable Boxes

Theorem 5.2

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, is equal to

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Examples

Example 40

In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Example 41

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Example

Example 42

There are 1807 first year undergraduate students in a university. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Inclusion-Exclusion Principle

Theorem 6.1 (Inclusion-exclusion principle)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof

Proof.

We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation. Suppose a is a member of exactly r of the sets $A_1, A_2, \ldots, A_n$ where $1 \le r \le n$. This element is counted $C(r,1)$ times by $\sum |A_i|$. It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$. In general, it is counted $C(r,m)$ times by the summation involving m of the sets A_i. Thus, this element is counted exactly

$$C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1}C(r,r)$$

times by the expression on the right-hand side of this equation. As a corollary of the binomial theorem, we have shown before that

$$C(r,0) - C(r,1) + C(r,2) - \cdots + (-1)^r C(r,r) = 0.$$

Proof (cont'd)

Proof (cont'd).

Hence,

$$C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1}C(r,r) = C(r,0) = 1.$$

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the inclusion-exclusion principle. □

Example

Example 43

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Furthermore, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Alternative Form of Inclusion-Exclusion Principle

Remark (Alternative form of inclusion-exclusion principle)

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_i}, P_{i_2}, \ldots, P_{i_k}$ will be denoted by $N(P_{i_i} P_{i_2} \cdots P_{i_k})$. Writing these quantities in terms of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = N(P_{i_i} P_{i_2} \cdots P_{i_k}).$$

If the number of elements with none of the properties $P_1, P_2, \ldots, P_n$ is denoted by $N(P'_1 P'_2 \cdots P'_n)$ and the number of elements in the set is denoted by N, it follows that

$$N(P'_1 P'_2 \cdots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Alternative Form of Inclusion-Exclusion Principle (cont'd)

Remark (cont'd)

From the inclusion-exclusion principle, we see that

$$\begin{aligned} N(P'_1 P'_2 \cdots P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) \\ - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots \\ + (-1)^n N(P_1 P_2 \cdots P_n). \end{aligned}$$

Number of Prime Numbers

Example 44

Find the number of prime numbers not exceeding 100.

Remark

The method above is related to the **sieve of Eratosthenes**, which is an ancient algorithm for finding all prime numbers up to a specified integer.

Number of Surjective Functions

Example 45

How many surjective (or onto) functions are there from a set with six elements to a set with three elements?

Theorem 6.2

Let m and n be positive integers with $m \ge n$. Then, there are

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1} \cdot 1^m$$

surjective functions from a set with m elements to a set with n elements.

Example 46

How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?

Hatcheck Problem

Example 47 (Hatcheck problem or derangement problem)

A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

Remark

The answer is the number of ways that the hats can be arranged so that there is no hat in its original position divided by $n!$ which is the number of permutations of n hats. In this topic, we first find the number of permutations of n objects that leave no objects in their original position.

Derangement

Definition

A **derangement** is a permutation of objects that leaves no object in its original position.

Example 48

The permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345 because this permutation leaves 4 unchanged.

Number of Derangements

Theorem 6.3

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof

Proof.

Let a permutation have property P_i if element i remains unchanged in its position. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$, so

$$D_n = N(P_1' P_2' \cdots P_n').$$

Using the inclusion-exclusion principle, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \cdots P_n),$$

where N is the number of permutations of n elements.

Proof (cont'd)

Proof (cont'd).

We note that

$$\begin{aligned} N &= n! \\ N(P_i) &= (n-1)! \\ N(P_iP_j) &= (n-2)! \\ &\vdots \\ N(P_{i_1}P_{i_2}\cdots P_{i_m}) &= (n-m)! \\ &\vdots \end{aligned}$$

Proof (cont'd)

Proof (cont'd).

Because there are $C(n,m)$ ways to choose m elements from n, it follows that

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n}(n-n)! \\ &= n! - \frac{n!}{1!\,(n-1)!}(n-1)! + \frac{n!}{2!\,(n-2)!}(n-2)! - \cdots + (-1)^n\frac{n!}{n!\,0!}0! \\ &= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right]. \end{aligned}$$

□