Certified Algorithms for Combinatorial Optimization Problems

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Outline

Feedback Vertex Set (FVS)

FVS with Bounded Cycle Length (FVS-BCL)

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EPTAS via Baker's Technique

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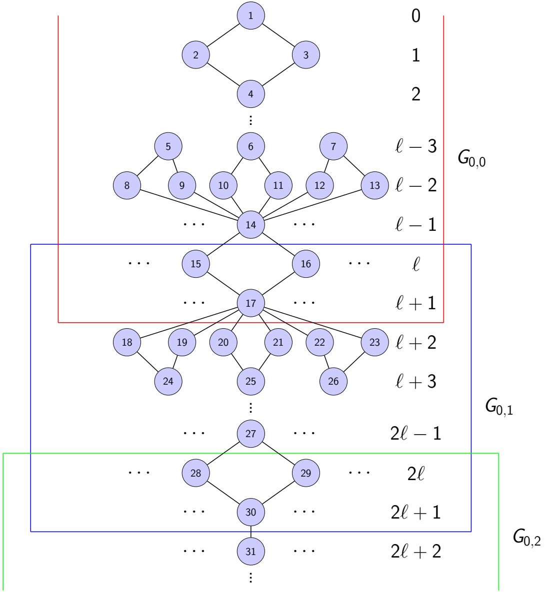
BFS Layered Tree Structure

Idea: Break graph into layers via BFS.

- Nodes grouped by distance from root.
- Overlap grouped by distance from root mod ℓ
- Subgraphs $G_{0,0}, G_{0,1}, G_{0,2}, \ldots$



Baker's Technique for Unweighted FVS-4CL

Algorithm Baker's technique for the unweighted FVS-4CL

Require: Planar graph $G = (V, E)$, parameter $\ell \leftarrow \frac{1}{\epsilon}$
Ensure: $(1+\epsilon)$-approximation for $\mathrm{FVS\text{-}4CL}$

Perform BFS from some arbitrary vertex r
$S \leftarrow \emptyset$
i: *shift*; j: *slice*
for each $i = 0$ to $\ell - 1$ **do**
  Let $G_{i,j}$ be the subgraph induced on vertices at levels $j \cdot \ell + i$ through $(j+1) \cdot \ell + i + 1$ for all $j \geq 0$.
  Let $S_{i,j}$ be the minimum unweighted $\mathrm{FVS\text{-}4CL}$ of $G_{i,j}$ using Algorithm **??** as a subroutine (weights are all ones).
  Let $S_i = \bigcup_j S_{i,j}$
  $S \leftarrow S \cup \{S_i\}$
end for
return S_{i^*} from S with minimum cardinality

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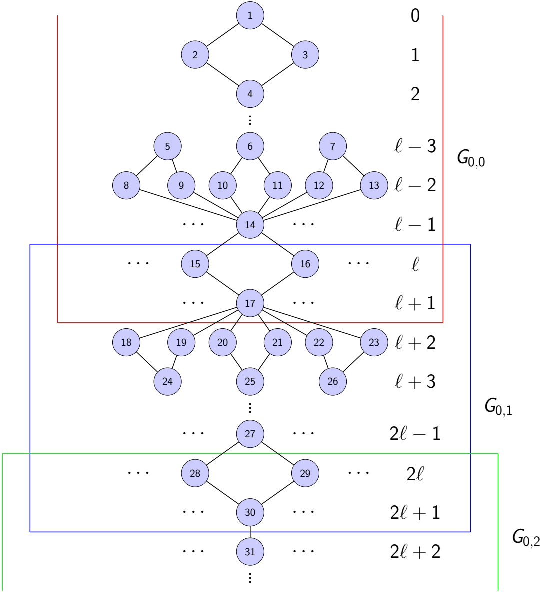

Bounds on the Optimum Solution (Unweighted Case)

Lemma (Bound on the Optimum Solution in Subgraphs)

Let $S_{i,j}$ be the minimum unweighted feedback vertex set (FVS-4CL) for subgraph $G_{i,j}$, computed using Algorithm 1, and let $F \equiv OPT$ be the optimum solution for the full graph G.
Define $F_{i,j} := F \cap V(G_{i,j})$, i.e., the restriction of the global solution F to the subgraph $G_{i,j}$.
Then:

$$|S_{i,j}| \leq |F_{i,j}|$$

Result (Optimum solution bound for the whole graph)

Let S_i be the union of optimal solutions defined on Line 6 of Algorithm 1 for some shift i. Let $F \equiv OPT$ be the optimum solution in G and let $F_{i,j} = G_{i,j} \cap F$. For those sets it holds that:

$$|S_i| \leq \sum_j |S_{i,j}| \leq \sum_j |F_{i,j}|$$

Bounds on the Optimum Solution (Unweighted Case)

Lemma (Bound for the vertices on the boundaries (Unweighted Case))

Let $F_i = F \cap \{\text{vertices at levels } i \bmod \ell\}$. *The sets* F_i *are disjoint and* $\bigcup_i F_i = F$.
We claim that:

$$\exists q \in \{0, 1, \ldots, \ell - 1\} : |F_q| + |F_{q+1}| \leq \frac{2}{\ell} \cdot |F|.$$

Lemma (Bound for the cardinality of the intersection sets)

Let $F \equiv OPT$ *be the optimal solution and* $F_{i,j} = F \cap G_{i,j}$ *be the intersection sets with the subgraphs. Then we have:*

$$\sum_j |F_{i^*,j}| \leq |F| + \frac{2}{\ell} \cdot |F|$$

for some specific integer i^*.

$$|S_{i^*}| \leq \sum_j |S_{i^*,j}| \leq \sum_j |F_{i^*,j}| \leq |F| + \frac{2}{\ell} \cdot |F| = (1 + \frac{2}{\ell}) \cdot |F|$$

Weighted Bounds for FVS-BCL Problems

Weighted FVS-4CL:

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq w(F) + \frac{2}{\ell} \cdot w(F) = \left(1 + \frac{2}{\ell}\right) \cdot w(F)$$

Weighted FVS-BCL (break cycles of length ρ):

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq \left(1 + \frac{\lfloor \rho/2 \rfloor}{\ell}\right) \cdot w(F)$$

FPT via Dynamic Programming on Tree Decompositions

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Algorithm Design for FVS-4CL

1. **Solution:** For the FVS-4CL problem, a solution for graph G is a set F such that $G - F$ contains no 4-cycles.
2. **Partial Solution:** For subgraph $G_i = (V_i, E_i)$, a partial solution F_i is a subset $F_i \subseteq V_i$, a restriction of a full solution.
3. **Extension of Partial Solution:** A solution F extends F_i if $F \cap V_i = F_i$.
4. **Characteristic of a Partial Solution:** For X_i, vertices are partitioned as:
   - $I \subseteq X_i$: vertices in the partial solution.
   - $\mathcal{F} \subseteq X_i \times X_i$: pairs (v, u) with a common neighbour $w \in V_i \setminus X_i$ not in F_i.

   $$ch(G_i, F_i) = \big(I, \{(v, u) \in X_i \times X_i : \exists w \in V_i \setminus X_i, (v, w), (u, w) \in E_i, w \notin F_i\}\big)$$

   The valuation table $c[i, I, \mathcal{F}] \in \mathbb{N} \cup \{\infty\}$ gives the minimum weight of partial solution F_i:

   $$c[i, I, \mathcal{F}] = \min\{w(W) : W \text{ is a FVS-4CL of } G_i \wedge W \cap X_i = I\}$$

5. **Full Set of Characteristics:** For node X_i, valuations exist for all

   $$I \in \{0,1\}^{|X_i|} \quad \text{and} \quad \mathcal{F} \in \{0,1\}^{|X_i \times X_i|}$$

   There are at most $2^{(\mathrm{tw}+1)} \cdot 2^{\binom{\mathrm{tw}+1}{2}} = 2^{\mathcal{O}(\mathrm{tw}^2)}$ entries.

DP Transitions over Tree Decomposition Nodes

Leaf Node: $X_i = \emptyset$

$$c[i, \emptyset, \emptyset] = 0$$

Introduce Node: Let $X_i = X_j \cup \{v\}$

$$c[i, I \cup \{v\}, \mathcal{F}] = w(v) + c[j, I, \mathcal{F}]$$

$$c[i, I, \mathcal{F}] = \begin{cases} \infty & \text{if } \exists u, w \notin I : (u, w) \in \mathcal{F},\ (v, u), (v, w) \in E_i \\ c[j, I, \mathcal{F}] & \text{otherwise} \end{cases}$$

Forget Node: Let $X_i = X_j \setminus \{v\}$

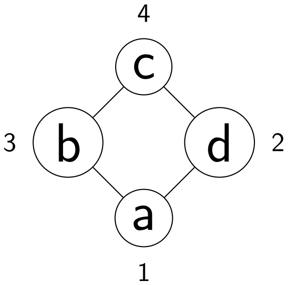
$$c[i, I, \mathcal{F}] = \min \begin{pmatrix} c[j, I \cup \{v\}, \mathcal{F}], \\ c[j, I, \mathcal{F} \cup \{(u, w) : (v, u), (v, w) \in E_i\}] \end{pmatrix}$$

Join Node: Let $X_i = X_{j_1} = X_{j_2}$

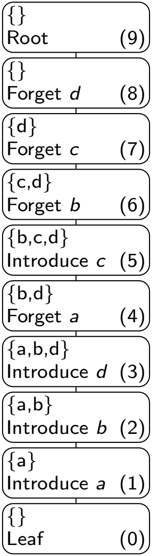
$$c[i, I, \mathcal{F}] = \min_{\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}} (c[j_1, I, \mathcal{F}_1] + c[j_2, I, \mathcal{F}_2] - w(I))$$

$$\text{Infeasible if } \exists u, w \notin I : (u, w) \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow c[i, I, \mathcal{F}] = \infty$$

Rhombus Graph and Tree Decomposition

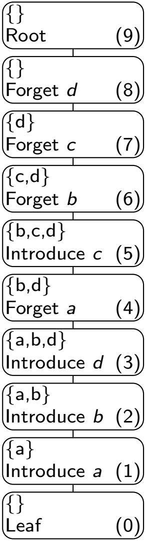


Rhombus Graph G



Nice Tree Decomposition
of the graph G with a width of 2.

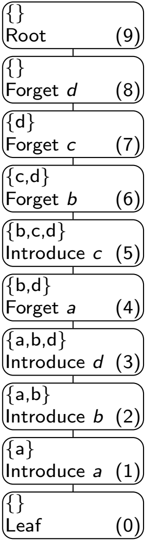
Rhombus Graph and Tree Decomposition



i	I	$\mathcal{F}$	**Value**
0	$\emptyset$	$\emptyset$	0
1	a	$\emptyset$	1
1	$\emptyset$	$\emptyset$	0
2	$\{a, b\}$	$\emptyset$	4
2	$\{a\}$	$\emptyset$	1
2	$\{b\}$	$\emptyset$	3
2	$\emptyset$	$\emptyset$	0
3	$\{a, b, d\}$	$\emptyset$	6
3	$\{a, b\}$	$\emptyset$	4
3	$\{a, d\}$	$\emptyset$	3
3	$\{b, d\}$	$\emptyset$	5
3	$\{a\}$	$\emptyset$	1
3	$\{b\}$	$\emptyset$	3
3	$\{d\}$	$\emptyset$	2
3	$\emptyset$	$\emptyset$	0
4	$\{b, d\}$	$\emptyset$	6
4	$\{b, d\}$	$\{b, d\}$	5
4	$\{b\}$	$\emptyset$	4
4	$\{b\}$	$\{b, d\}$	3

i	I	$\mathcal{F}$	**Value**
4	$\{d\}$	$\emptyset$	3
4	$\{d\}$	$\{b, d\}$	2
4	$\emptyset$	$\emptyset$	1
4	$\emptyset$	$\{b, d\}$	0
5	$\{b, c, d\}$	$\emptyset$	10
5	$\{b, c, d\}$	$\{b, d\}$	9
5	$\{b, d\}$	$\emptyset$	6
5	$\{b, d\}$	$\{b, d\}$	5
5	$\{b, c\}$	$\emptyset$	8
5	$\{b, c\}$	$\{b, d\}$	7
5	$\{b\}$	$\emptyset$	4
5	$\{b\}$	$\{b, d\}$	3
5	$\{c, d\}$	$\emptyset$	7
5	$\{c, d\}$	$\{b, d\}$	6
5	$\{d\}$	$\emptyset$	3
5	$\{d\}$	$\{b, d\}$	2
5	$\{c\}$	$\emptyset$	5
5	$\{c\}$	$\{b, d\}$	4
5	$\emptyset$	$\emptyset$	1
5	$\emptyset$	$\{b, d\}$	∞

Rhombus Graph and Tree Decomposition



i	I	$\mathcal{F}$	**Value**
6	$\{c, d\}$	$\emptyset$	10
6	$\{c, d\}$	$\emptyset$	9
6	$\{d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	8
6	$\{c\}$	$\emptyset$	7
6	$\emptyset$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	3
6	$\{c, d\}$	$\emptyset$	7
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	3
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
6	$\emptyset$	$\emptyset$	∞

i	I	$\mathcal{F}$	**Value**
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
7	$\{d\}$	$\emptyset$	2
7	$\emptyset$	$\emptyset$	1
8	$\emptyset$	$\emptyset$	1
9	$\emptyset$	$\emptyset$	1

MSOL Formulation

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Certified Algorithms

- Perturbation resilience
- m-Stitching and Π-m-Stitching
- Certified Algorithms

Definition: *m*-stitching

Definition (*m*-stitching)

Assume $m \geq 0$ is an integer, J is an induced subgraph of G, and $S_1, S_2 \subseteq V(G)$. Then we define the *m-stitch* of S_2 onto S_1 along J as the set:

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^m[J]).$$

Illustration of 2-stitching

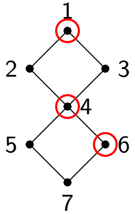


Figure: Vertex set S_1

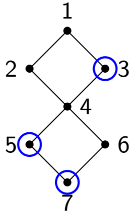


Figure: Vertex set S_2

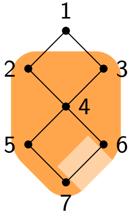


Figure: J and $N_G^2[J]$

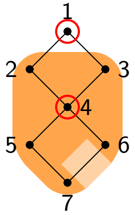


Figure: Remove J from S_1

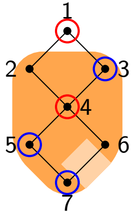


Figure: Add $S_2 \cap N_G^2[J]$

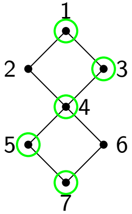


Figure: Final set S_3

Meta-Theorem for Minor-Closed Graph Classes

Definition

Let $\mathcal{G}$ be a minor-closed graph class whose local treewidth is bounded by $g(r) = \lambda \cdot r$, for fixed $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$.
Let Π be a vertex-minimization problem such that:

1. Π is guessable.
2. Π is m-stitchable.
3. There exists an algorithm A_Π that solves Π-m-stitching in time $f(t) \cdot |V(G)|^{\mathcal{O}(1)}$, where $t = \text{tw}(G[N_G^m(J)])$ and f is computable.

Then, for each $\epsilon > 0$, there exists a $(1+\epsilon)$-certified algorithm for Π running in time $f(\lambda \cdot m/\epsilon) \cdot |V(G)|^{\mathcal{O}(1)}$ on any input $(G, w : V(G) \to \mathbb{N})$, with $G \in \mathcal{G}$ and polynomially-bounded weights.

Guessable Problems

Definition (Guessable)

A problem Π is *guessable* if there is an algorithm that outputs a feasible solution with no requirement for optimality in polynomial time.

In the case of FVS-BCL:
This set is $F \leftarrow V$ for a graph $G = (V, E)$.

Lemma: FVS-BCL is 2-stitchable

Lemma (FVS-BCL is 2-stitchable)

Let (G, w) be any instance of the FVS-4CL problem, let $J \subseteq V(G)$ be a subset of the vertices in the graph, and let S_1 and S_2 be any two feasible solutions to the problem in G. Then the set

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^2[J])$$

is a feasible solution to the problem.

Stitch-FVS-4CL Algorithm

Algorithm Stitch-FVS-4CL

Require: Vertex-weighted planar graph $(G, w : V(G) \to \mathbb{N})$, a feasible solution S_1 on G, and a vertex set $J \subseteq V(G)$

Ensure: Feasible solution S' on G, such that for all feasible solutions S^*, $w(S') \leq w(S^* \oplus^2_{G,J} S_1)$

1: $H \leftarrow G[N^2_G[J] \setminus (S_1 \setminus J)]$
2: Let S_2 be the output of algorithm A on input (H, w)
3: **if** $w(S_2 \oplus^2_{G,J} S_1) < w(S_1)$ **then**
4:     **return** $S_2 \oplus^2_{G,J} S_1$
5: **else**
6:     **return** S_1
7: **end if**

Case-distinction proof

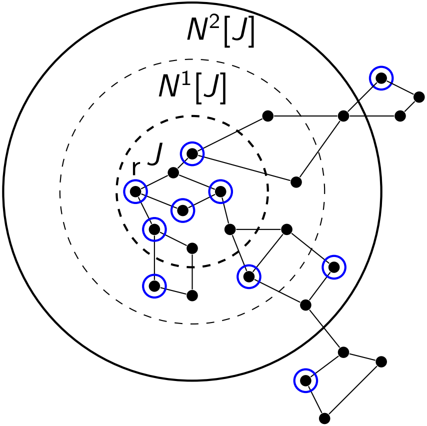


Figure: Sets $J \subseteq N^1_G[J] \subseteq N^2_G[J]$; Solution S_1 in blue

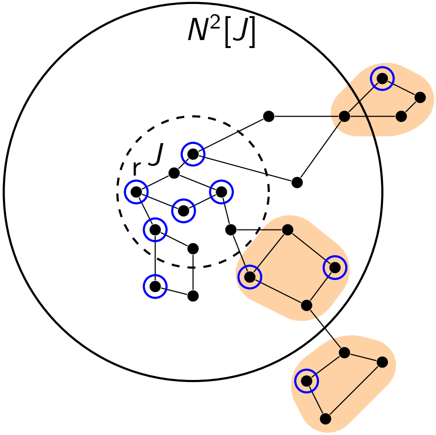


Figure: $\forall v \in \{v_1, v_2, v_3, v_4\}$, such that $v \in S_1$, it holds that $v \notin J$

Case-distinction proof

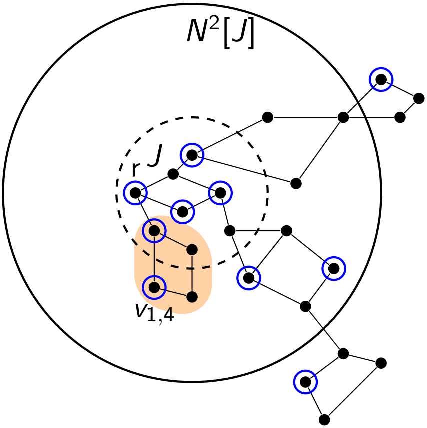
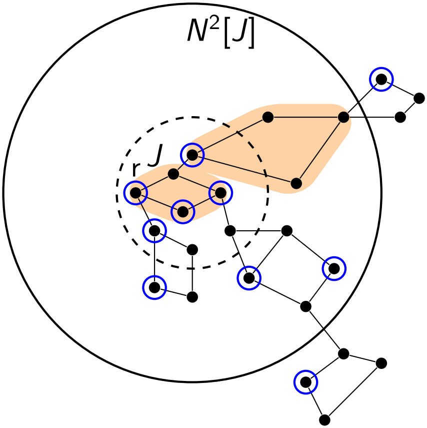


Figure: $\exists v \in \{v_1, v_2, v_3, v_4\}$, such that $v \in S_1$: $v \in J$



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Summary of Contributions

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Future Work

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Thank you!

Questions?