Certified Algorithms for Combinatorial Optimization Problems

Martin Sokolov

Utrecht University

June 19, 2025

Outline

Problems Studied (Emphasis on Focus Areas)

- Dominating Set
- Feedback Vertex Set (FVS)
- Feedback Vertex Set with Four Cycle Length (FVS-4CL)
- Feedback Vertex Set with Bounded Cycle Length (FVS-BCL)
- r-Dominating Set (r-DS)

FVS-4CL is the primary focus of this work.
We aim to generalize results to **FVS-BCL**.
(Our hope in the beginning was generalizing our results to **FVS**).

Approximation Algorithms for Feedback Vertex Set (FVS)

- $\min\{2\Delta^2,\ 4\log n\}$ **where Δ is the max. degree in G. (Bar-Yehuda et al., 1994)**
  Primal-dual algorithm on undirected graphs with general vertex weights.
- **2-Approximation (Bafna et al., 1995)**
  Local ratio technique with improved efficiency.
- **2-Approximation (Becker and Geiger, 1996)**
  Greedy-like approximation algorithm.
- **2-Approximation (Chudak et al., 1998)**
  A primal-dual algorithm.
- **Hardness: APX-Complete (Dinur and Safra, 2005)**
  NP-hard to approximate within a factor better than 1.36 via reduction from Vertex Cover.

Approximation Algorithms for Feedback Vertex Set (FVS) in planar graphs

- **PTAS for FVS (Kleinberg and Kumar, 2001)**
- **PTAS for FVS (Le and Zheng, 2020)**
  Using a local search heuristic
- **EPTAS for unweighted FVS (Demaine and Hajiaghayi)**
  Using bidimensionality
- **PTAS for weighted FVS (Cohen-Addad et al., 2016)**
  Reduction from weighted feedback vertex set to vertex-weighted connected dominating set
- **EPTAS for weighted FVS (Open question.)**

FVS with Bounded Cycle Length (FVS-BCL)

- **Inapproximability of Feedback Vertex Set for Bounded Length Cycles (Guruswami and Lee, 2014)**
  For any integer constant $\rho \geq 3$ and $\epsilon > 0$, it is hard to find a $(\rho - 1 - \epsilon)$-approximate solution to the problem of intersecting every cycle of length at most ρ.

EPTAS via Baker's Technique

We obtain:

- $(1+2\epsilon)$-approximation algorithm for the FVS-4CL problem with a running time of $2^{\mathcal{O}(\mathrm{tw}^2)} \cdot n^{\mathcal{O}(1)}$.
- $\left(1+\frac{\lfloor \rho/2 \rfloor}{\epsilon}\right)$-approximation for the FVS-BCL problem with a running time of $f\left(\mathrm{tw}, \rho\right) \cdot n^{\mathcal{O}(1)}$ for some computable function f.

Baker's Technique for Unweighted FVS-4CL

Algorithm Baker's technique for the unweighted FVS-4CL

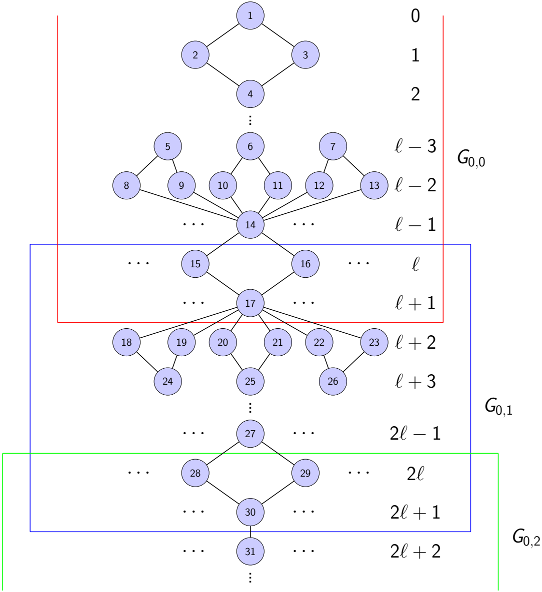
Require: Planar graph $G = (V, E)$, parameter $\ell \leftarrow \frac{1}{\epsilon}$
Ensure: $(1 + \epsilon)$-approximation for $\mathrm{FVS\text{-}4CL}$

```
Perform BFS from some arbitrary vertex r
S ← ∅
i: shift; j: slice
for each i = 0 to ℓ − 1 do
    Let G_{i,j} be the subgraph induced on vertices at levels j·ℓ + i through
    (j + 1)·ℓ + i + 1 for all j ≥ 0.
    Let S_{i,j} be the minimum unweighted FVS-4CL of G_{i,j} using Algo-
    rithm ?? as a subroutine (weights are all ones).
    Let S_i = ⋃_j S_{i,j}
    S ← S ∪ {S_i}
end for
return S_{i*} from S with minimum cardinality
```

BFS Layered Tree Structure

Idea: Break graph into layers via BFS.

- Nodes grouped by distance from root.
- Overlap grouped by distance from root mod ℓ
- Subgraphs $G_{0,0}, G_{0,1}, G_{0,2}, \ldots$



Bounds on the Optimum Solution (Unweighted Case)

Lemma (Bound on the Optimum Solution in Subgraphs)

Let $S_{i,j}$ *be the minimum unweighted feedback vertex set (FVS-4CL) for subgraph* $G_{i,j}$*, computed using Algorithm 1, and let* $F \equiv OPT$ *be the optimum solution for the full graph* G*.*
Define $F_{i,j} := F \cap V(G_{i,j})$*, i.e., the restriction of the global solution* F *to the subgraph* $G_{i,j}$*.*
Then:

$$|S_{i,j}| \leq |F_{i,j}|$$

Result (Optimum solution bound for the whole graph)

Let S_i *be the union of optimal solutions defined on Line 6 of Algorithm 1 for some shift* i*. Let* $F \equiv OPT$ *be the optimum solution in* G *and let* $F_{i,j} = G_{i,j} \cap F$*. For those sets it holds that:*

$$|S_i| \leq \sum_j |S_{i,j}| \leq \sum_j |F_{i,j}|$$

Bounds on the Optimum Solution (Unweighted Case)

Lemma (Bound for the vertices on the boundaries (Unweighted Case))

Let $F_i = F \cap \{$*vertices at levels* $i \bmod \ell\}$*. The sets* F_i *are disjoint and* $\bigcup_i F_i = F$*.*
We claim that:

$$\exists q \in \{0, 1, \ldots, \ell - 1\} : |F_q| + |F_{q+1}| \leq \frac{2}{\ell} \cdot |F|.$$

Lemma (Bound for the cardinality of the intersection sets)

Let $F \equiv OPT$ *be the optimal solution and* $F_{i,j} = F \cap G_{i,j}$ *be the intersection sets with the subgraphs. Then we have:*

$$\sum_j |F_{i^*,j}| \leq |F| + \frac{2}{\ell} \cdot |F|$$

for some specific integer i^**.*

$$|S_{i^*}| \leq \sum_j |S_{i^*,j}| \leq \sum_j |F_{i^*,j}| \leq |F| + \frac{2}{\ell} \cdot |F| = (1 + \frac{2}{\ell}) \cdot |F|$$

Weighted Bounds for FVS-BCL Problems

Weighted FVS-4CL:

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq w(F) + \frac{2}{\ell} \cdot w(F) = \left(1 + \frac{2}{\ell}\right) \cdot w(F)$$

Weighted FVS-BCL (break cycles of length ρ):

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq \left(1 + \frac{\lfloor \rho/2 \rfloor}{\ell}\right) \cdot w(F)$$

Fixed-Parameter Tractability of FVS-BCL

Fixed-Parameter Tractability of FVS-BCL

- Monadic Second Order Logic for FVS-BCL
- Dynamic Programming Algorithm for FVS-BCL using Nice Tree Decompositions

MSOL Formulation

- An extension of First-Order Logic
- Object variables: *vertices*: $v_1, v_2, \ldots$ and *edges*: $e_1, e_2, \ldots$.
- Set variables: sets of vertices $V_1, V_2, \ldots$ and sets of edges $E_1, E_2, \ldots$.
- Binary relation $\in$: $\{\text{object variable}\} \times \{\text{set variable}\} \rightarrow \{0, 1\}$.
  Therefore $v \in V$ *iff* v is an element of the corresponding set V.
- The $Adj(e, v_i, v_j)$ relation. It detects whether edge e is an edge from vertex v_i to vertex v_j where $v_i \neq v_j$.
- Quantification over set variables: $\forall V_i, \forall E_i$ and $\exists V_i, \exists E_i$.

MSOL Formulation: Unweighted FVS-4CL

$$
\begin{aligned}
&\min_{F \subseteq V} |F| : \\
&\quad \forall v : \forall u : \forall w : \forall z : \\
&\quad v \in (V \backslash F) \wedge u \in (V \backslash F) \wedge w \in (V \backslash F) \wedge z \in (V \backslash F) \\
&\quad \wedge\, v \neq w \wedge u \neq z \\
&\quad \wedge\, \neg((v,u) \in E \wedge (u,w) \in E \wedge (w,z) \in E \wedge (v,z) \in E)
\end{aligned} \tag{1}
$$

MSOL Formulation: Weighted FVS-4CL

$$\min_{F \subseteq V} w(F) : \\ \forall v : \forall u : \forall w : \forall z : \\ v \in (V \setminus F) \land u \in (V \setminus F) \land w \in (V \setminus F) \land z \in (V \setminus F) \\ \land\, v \neq w \land u \neq z \\ \land\, \neg((v, u) \in E \land (u, w) \in E \land (w, z) \in E \land (v, z) \in E) \tag{2}$$

MSOL Formulation: Weighted FVS-ρCL

$$
\begin{aligned}
\min_{F \subseteq V} \; & w(F) : \\
& \forall v_1 : \forall v_2 : \ldots \forall v_\rho : \\
& v_1 \in (V \backslash F) \land v_2 \in (V \backslash F) \cdots \land v_\rho \in (V \backslash F) \\
& \land (v_1 \neq v_2 \land v_1 \neq v_3 \land \cdots \land v_1 \neq v_\rho) \\
& \land (v_2 \neq v_1 \land v_2 \neq v_3 \land \cdots \land v_2 \neq v_\rho) \\
& \ldots \\
& \land (v_\rho \neq v_1 \land v_\rho \neq v_2 \land \cdots \land v_\rho \neq v_{\rho-1}) \\
& \land \neg((v_1, v_2) \in E \land (v_2, v_3) \in E \land \cdots \land (v_{\rho-1}, v_\rho) \in E \land (v_\rho, v_1) \in E)
\end{aligned} \tag{3}
$$

Courcelle's Theorem and MSOL Solvability of FVS-ρCL

Theorem (Courcelle's theorem)

Assume that ϕ is a MSOL formula and G is an n-vertex graph, with an evaluation of all free variables of ϕ. Suppose a tree decomposition of G of width t is given. Then there exists an algorithm that verifies whether ϕ is satisfied in G in time:

$$f(\|\phi\|, t) \cdot n^{\mathcal{O}(1)}$$

for some computable function f.

Corollary (MSOL Solvability of FVS-ρCL on Bounded-Treewidth Graphs)

Let $\rho \geq 3$ be a constant and $G = (V, E)$ be a graph of treewidth at most tw, *with vertex-weight function $w : V \to \mathbb{N}$. Then the minimum-weight set breaking all ρ-cycles (FVS-ρCL) can be computed in time:*

$$f(\rho, \mathrm{tw}) \cdot n^{\mathcal{O}(1)}$$

for some computable function f depending only on ρ and tw.

Algorithm Design for FVS-4CL

1. **Solution:** For the FVS-4CL problem, a solution for graph G is a set F such that $G - F$ contains no 4-cycles.
2. **Partial Solution:** For subgraph $G_i = (V_i, E_i)$, a partial solution F_i is a subset $F_i \subseteq V_i$, a restriction of a full solution.
3. **Extension of Partial Solution:** A solution F extends F_i if $F \cap V_i = F_i$.
4. **Characteristic of a Partial Solution:** For X_i, vertices are partitioned as:
   - $I \subseteq X_i$: vertices in the partial solution.
   - $\mathcal{F} \subseteq X_i \times X_i$: pairs (v, u) with a common neighbour $w \in V_i \setminus X_i$ not in F_i.

   $$ch(G_i, F_i) = \big(I, \{(v, u) \in X_i \times X_i : \exists w \in V_i \setminus X_i, (v, w), (u, w) \in E_i, w \notin F_i\}\big)$$

   The valuation table $c[i, I, \mathcal{F}] \in \mathbb{N} \cup \{\infty\}$ gives the minimum weight of partial solution F_i:

   $$c[i, I, \mathcal{F}] = \min\{w(W) : W \text{ is a FVS-4CL of } G_i \wedge W \cap X_i = I\}$$
5. **Full Set of Characteristics:** For node X_i, valuations exist for all

   $$I \in \{0,1\}^{|X_i|} \quad \text{and} \quad \mathcal{F} \in \{0,1\}^{|X_i \times X_i|}$$

   There are at most $2^{(\mathrm{tw}+1)} \cdot 2^{\binom{\mathrm{tw}+1}{2}} = 2^{\mathcal{O}(\mathrm{tw}^2)}$ entries.

DP Transitions over Tree Decomposition Nodes

Leaf Node: $X_i = \emptyset$

$$c[i, \emptyset, \emptyset] = 0$$

Introduce Node: Let $X_i = X_j \cup \{v\}$

$$c[i, I \cup \{v\}, \mathcal{F}] = w(v) + c[j, I, \mathcal{F}]$$

$$c[i, I, \mathcal{F}] = \begin{cases} \infty & \text{if } \exists u, w \notin I : (u, w) \in \mathcal{F},\ (v, u), (v, w) \in E_i \\ c[j, I, \mathcal{F}] & \text{otherwise} \end{cases}$$

Forget Node: Let $X_i = X_j \setminus \{v\}$

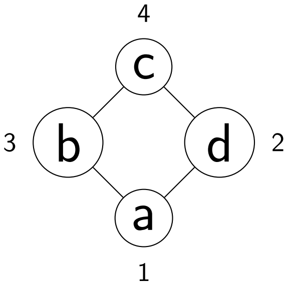
$$c[i, I, \mathcal{F}] = \min \begin{pmatrix} c[j, I \cup \{v\}, \mathcal{F}], \\ c[j, I, \mathcal{F} \cup \{(u, w) : (v, u), (v, w) \in E_i\}] \end{pmatrix}$$

Join Node: Let $X_i = X_{j_1} = X_{j_2}$

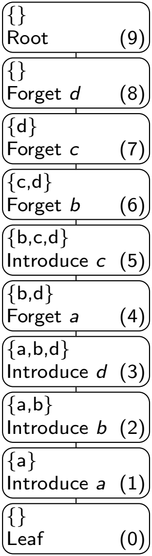
$$c[i, I, \mathcal{F}] = \min_{\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}} (c[j_1, I, \mathcal{F}_1] + c[j_2, I, \mathcal{F}_2] - w(I))$$

Infeasible if $\exists u, w \notin I : (u, w) \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow c[i, I, \mathcal{F}] = \infty$

Rhombus Graph and Tree Decomposition

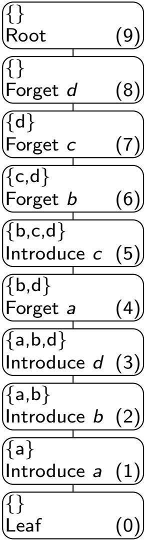


Rhombus Graph G



Nice Tree Decomposition
of the graph G with a width of 2.

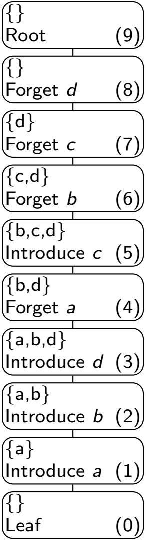
Rhombus Graph and Tree Decomposition



i	I	$\mathcal{F}$	**Value**
0	$\emptyset$	$\emptyset$	0
1	a	$\emptyset$	1
1	$\emptyset$	$\emptyset$	0
2	$\{a, b\}$	$\emptyset$	4
2	$\{a\}$	$\emptyset$	1
2	$\{b\}$	$\emptyset$	3
2	$\emptyset$	$\emptyset$	0
3	$\{a, b, d\}$	$\emptyset$	6
3	$\{a, b\}$	$\emptyset$	4
3	$\{a, d\}$	$\emptyset$	3
3	$\{b, d\}$	$\emptyset$	5
3	$\{a\}$	$\emptyset$	1
3	$\{b\}$	$\emptyset$	3
3	$\{d\}$	$\emptyset$	2
3	$\emptyset$	$\emptyset$	0
4	$\{b, d\}$	$\emptyset$	6
4	$\{b, d\}$	$\{b, d\}$	5
4	$\{b\}$	$\emptyset$	4
4	$\{b\}$	$\{b, d\}$	3

i	I	$\mathcal{F}$	**Value**
4	$\{d\}$	$\emptyset$	3
4	$\{d\}$	$\{b, d\}$	2
4	$\emptyset$	$\emptyset$	1
4	$\emptyset$	$\{b, d\}$	0
5	$\{b, c, d\}$	$\emptyset$	10
5	$\{b, c, d\}$	$\{b, d\}$	9
5	$\{b, d\}$	$\emptyset$	6
5	$\{b, d\}$	$\{b, d\}$	5
5	$\{b, c\}$	$\emptyset$	8
5	$\{b, c\}$	$\{b, d\}$	7
5	$\{b\}$	$\emptyset$	4
5	$\{b\}$	$\{b, d\}$	3
5	$\{c, d\}$	$\emptyset$	7
5	$\{c, d\}$	$\{b, d\}$	6
5	$\{d\}$	$\emptyset$	3
5	$\{d\}$	$\{b, d\}$	2
5	$\{c\}$	$\emptyset$	5
5	$\{c\}$	$\{b, d\}$	4
5	$\emptyset$	$\emptyset$	1
5	$\emptyset$	$\{b, d\}$	∞

Rhombus Graph and Tree Decomposition



i	I	$\mathcal{F}$	**Value**
6	$\{c, d\}$	$\emptyset$	10
6	$\{c, d\}$	$\emptyset$	9
6	$\{d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	8
6	$\{c\}$	$\emptyset$	7
6	$\emptyset$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	3
6	$\{c, d\}$	$\emptyset$	7
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	3
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
6	$\emptyset$	$\emptyset$	∞

i	I	$\mathcal{F}$	**Value**
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
7	$\{d\}$	$\emptyset$	2
7	$\emptyset$	$\emptyset$	1
8	$\emptyset$	$\emptyset$	1
9	$\emptyset$	$\emptyset$	1

Certified Algorithms

- Perturbation resilience
- m-Stitching and Π-m-Stitching
- Certified Algorithms

Definition: m-stitching

Definition (m-stitching)

Assume $m \geq 0$ is an integer, J is an induced subgraph of G, and $S_1, S_2 \subseteq V(G)$. Then we define the *m-stitch* of S_2 onto S_1 along J as the set:

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^m[J]).$$

Illustration of 2-stitching

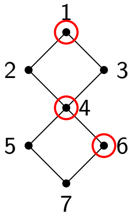


Figure: Vertex set S_1

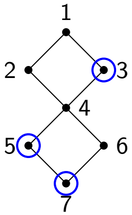


Figure: Vertex set S_2

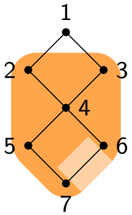


Figure: J and $N_G^2[J]$

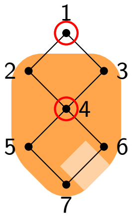


Figure: Remove J from S_1

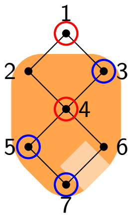


Figure: Add $S_2 \cap N_G^2[J]$

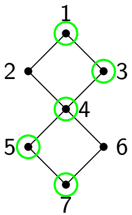


Figure: Final set S_3

Meta-Theorem for Minor-Closed Graph Classes

Definition

Let $\mathcal{G}$ be a minor-closed graph class whose local treewidth is bounded by $g(r) = \lambda \cdot r$, for fixed $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$.
Let Π be a vertex-minimization problem such that:

1. Π is guessable.
2. Π is m-stitchable.
3. There exists an algorithm A_Π that solves Π-m-stitching in time $f(t) \cdot |V(G)|^{\mathcal{O}(1)}$, where $t = \mathrm{tw}(G[N_G^m(J)])$ and f is computable.

Then, for each $\epsilon > 0$, there exists a $(1+\epsilon)$-certified algorithm for Π running in time $f(\lambda \cdot m/\epsilon) \cdot |V(G)|^{\mathcal{O}(1)}$ on any input $(G, w : V(G) \to \mathbb{N})$, with $G \in \mathcal{G}$ and polynomially-bounded weights.

Guessable Problems

Definition (Guessable)

A problem Π is *guessable* if there is an algorithm that outputs a feasible solution with no requirement for optimality in polynomial time.

In the case of FVS-BCL:
This set is $F \leftarrow V$ for a graph $G = (V, E)$.

Lemma: FVS-BCL is 2-stitchable

Lemma (FVS-BCL is 2-stitchable)

Let (G, w) *be any instance of the FVS-4CL problem, let* $J \subseteq V(G)$ *be a subset of the vertices in the graph, and let* S_1 *and* S_2 *be any two feasible solutions to the problem in* G*. Then the set*

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^2[J])$$

is a feasible solution to the problem.

Stitch-FVS-4CL Algorithm

Algorithm Stitch-FVS-4CL

Require: Vertex-weighted planar graph $(G, w : V(G) \to \mathbb{N})$, a feasible solution S_1 on G, and a vertex set $J \subseteq V(G)$

Ensure: Feasible solution S' on G, such that for all feasible solutions S^*, $w(S') \leq w(S^* \oplus^2_{G,J} S_1)$

$H \leftarrow G[N^2_G[J] \setminus (S_1 \setminus J)]$
Let S_2 be the output of algorithm A on input (H, w)
if $w(S_2 \oplus^2_{G,J} S_1) < w(S_1)$ **then**
    return $S_2 \oplus^2_{G,J} S_1$
else
    return S_1
end if

Case-distinction proof

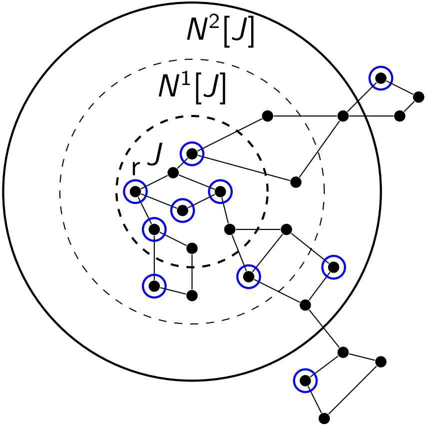


Figure: Sets $J \subseteq N^1_G[J] \subseteq N^2_G[J]$; Solution S_1 in blue

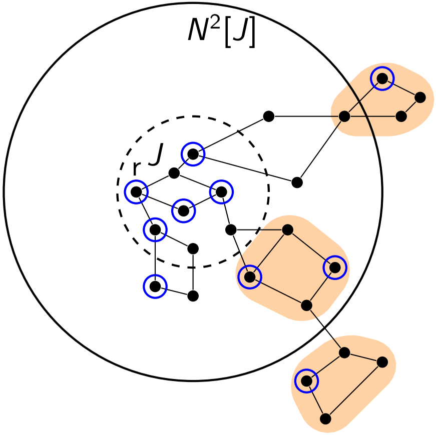


Figure: $\forall v \in \{v_1, v_2, v_3, v_4\}$, such that $v \in S_1$, it holds that $v \notin J$

Case-distinction proof

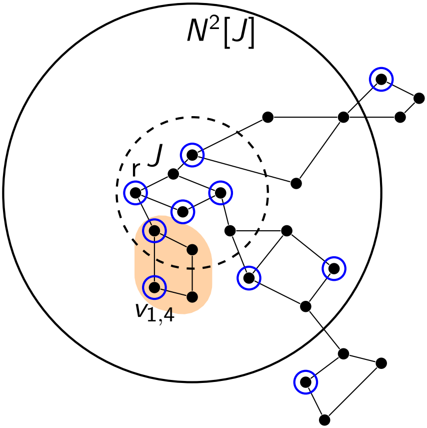


Figure: $\exists v \in \{v_1, v_2, v_3, v_4\}$, such that $v \in S_1$: $v \in J$

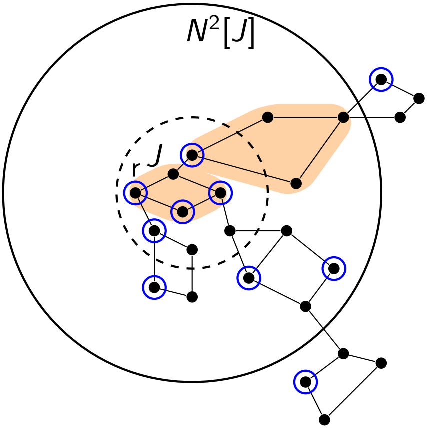


Figure: $\forall v \in \{v_1, v_2, v_3, v_4\}$, such that $v \in S_1$: $v \in J$

Meta-Theorem for Minor-Closed Graph Classes

Definition

Let $\mathcal{G}$ be a minor-closed graph class whose local treewidth is bounded by $g(r) = \lambda \cdot r$, for fixed $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$.
Let Π be a vertex-minimization problem such that:

1. Π is guessable.
2. Π is m-stitchable.
3. There exists an algorithm A_Π that solves Π-m-stitching in time $f(t) \cdot |V(G)|^{\mathcal{O}(1)}$, where $t = \mathrm{tw}(G[N_G^m(J)])$ and f is computable.

Then, for each $\epsilon > 0$, there exists a $(1 + \epsilon)$-certified algorithm for Π running in time $f(\lambda \cdot m/\epsilon) \cdot |V(G)|^{\mathcal{O}(1)}$ on any input $(G, w : V(G) \to \mathbb{N})$, with $G \in \mathcal{G}$ and polynomially-bounded weights.

$(1+\epsilon)$-Certified Algorithm for FVS-4CL

Algorithm $(1+\epsilon)$-Certified algorithm for FVS-4CL

Require: Vertex-weighted planar graph $(G, w : V(G) \to \mathbb{N})$, $\epsilon > 0$

Ensure: A vertex set $S^* \subseteq V(G)$ and a $(1+\epsilon)$-perturbation w' of w such that S^* is optimal for FVS-4CL on (G, w')

1: $\kappa \leftarrow \lceil \frac{2m}{\epsilon} \rceil + 2m$, where $m \leftarrow 2$

2: Let S^* be a feasible solution (FVS-4CL is guessable)

3: Perform BFS from an arbitrary vertex r

4: **while** there exists a subgraph $J_{\kappa-2m}$ of width $\kappa - 2m$ such that $w_A((G,w), S^*, J_{\kappa-2m}) < w(S^*)$ **do**

5: $\quad S^* \leftarrow A((G,w), S^*, J_{\kappa-2m})$

6: **end while**

7: Define $w' : V(G) \to \mathbb{R}^+$ by:

$$w'(x) = \begin{cases} w(x) & \text{if } x \in S^* \\ (1+\epsilon)w(x) & \text{otherwise} \end{cases}$$

Summary of Contributions

-

Future Work

-

Thank you!

Questions?