

# Feedback Vertex Set with Bounded Cycle Length: Approximation, Tractability and Beyond the Worst-Case Analysis

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# Outline

- 1 Problem Overview
- 2 Approximation Schemes
- 3 Fixed-Parameter Tractability of FVS-BCL
- 4 Certified Algorithms
- 5 Conclusion

- Dominating Set - BWCA is studied
- Feedback Vertex Set (FVS) - Approximation is studied

# Approximation Algorithms for Feedback Vertex Set (FVS)

- $\min\{2\Delta^2, 4 \log n\}$  where  $\Delta$  is the max. degree in  $G$ .  
(Bar-Yehuda et al., 1994)  
Primal-dual algorithm on undirected graphs with general vertex weights.
- **2-Approximation (Bafna et al., 1995)**  
Local ratio technique with improved efficiency.
- **2-Approximation (Becker and Geiger, 1996)**  
Greedy-like approximation algorithm.
- **2-Approximation (Chudak et al., 1998)**  
A primal-dual algorithm.
- **Hardness: APX-Complete (Dinur and Safra, 2005)**  
NP-hard to approximate within a factor better than 1.36 via reduction from Vertex Cover.

# Approximation Algorithms for Feedback Vertex Set (FVS) in planar graphs

- **PTAS for FVS (Kleinberg and Kumar, 2001)**
- **PTAS for FVS (Le and Zheng, 2020)**  
Using a local search heuristic
- **EPTAS for unweighted FVS (Demaine and Hajiaghayi)**  
Using bidimensionality
- **PTAS for weighted FVS (Cohen-Addad et al., 2016)**  
Reduction from weighted feedback vertex set to vertex-weighted connected dominating set
- **EPTAS for weighted FVS (Open question.)**

# Problems Studied (Emphasis on Focus Areas)

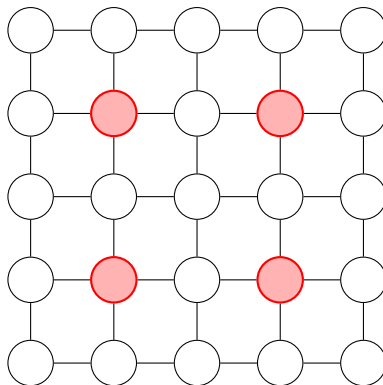
- Feedback Vertex Set with Four Cycle Length (FVS-4CL)
- Feedback Vertex Set with Bounded Cycle Length (FVS-BCL)
- $r$ -Dominating Set ( $r$ -DS)

**FVS-4CL** is the primary focus of this work.

We aim to generalize results to **FVS-BCL**.

(Our hope in the beginning was generalizing our results to **FVS**).

# FVS-4CL on a 5x5 grid



## Definition (Feedback Vertex Set with Four Cycle Length (**FVS-4CL**))

**Input:** A graph  $G$  and an integer  $k$ .

**Question:** Does there exist a set  $X$  of at most  $k$  vertices of  $G$  such that  $G - X$  does not contain any  $C_4$ ?

- **Inapproximability of Feedback Vertex Set for Bounded Length Cycles (Guruswami and Lee, 2014)**

For any integer constant  $\rho \geq 3$  and  $\epsilon > 0$ , it is hard to find a  $(\rho - 1 - \epsilon)$ -approximate solution to the problem of intersecting every cycle of length at most  $\rho$ .



We obtain:

- $(1 + 2\epsilon)$ -approximation algorithm for the FVS-4CL problem with a running time of  $2^{\mathcal{O}(\text{tw}^2)} \cdot n^{\mathcal{O}(1)}$ .
- $\left(1 + \frac{\lfloor \rho/2 \rfloor}{\epsilon}\right)$ -approximation for the FVS-BCL problem with a running time of  $f(\text{tw}, \rho) \cdot n^{\mathcal{O}(1)}$  for some computable function  $f$ .

# Baker's Technique for Unweighted FVS-4CL

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**Algorithm** Baker's technique for the unweighted FVS-4CL

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**Require:** Planar graph  $G = (V, E)$ , parameter  $\ell \leftarrow \frac{1}{\epsilon}$

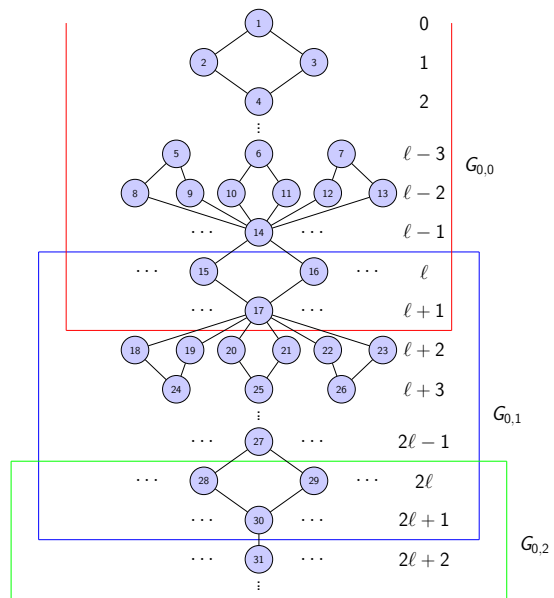
**Ensure:**  $(1 + \epsilon)$ -approximation for FVS-4CL

- 1: Perform BFS from some arbitrary vertex  $r$
- 2:  $S \leftarrow \emptyset$   
    *i: shift; j: slice*
- 3: **for** each  $i = 0$  to  $\ell - 1$  **do**
- 4:   Let  $G_{i,j}$  be the subgraph induced on vertices at levels  $j \cdot \ell + i$  through  $(j + 1) \cdot \ell + i + 1$  for all  $j \geq 0$ .
- 5:   Let  $S_{i,j}$  be the minimum unweighted FVS-4CL of  $G_{i,j}$  using Algorithm 22 as a subroutine (weights are all ones).
- 6:   Let  $S_i = \bigcup_j S_{i,j}$
- 7:    $S \leftarrow S \cup \{S_i\}$
- 8: **end for**
- 9: **return**  $S_{i^*}$  from  $S$  with minimum cardinality

# BFS Layered Tree Structure

**Idea:** Break graph into layers via BFS.

- Nodes grouped by distance from root.
- Overlap grouped by distance from root mod  $\ell$
- Subgraphs  $G_{0,0}, G_{0,1}, G_{0,2}, \dots$



# Approximation Bound via Subgraph Solutions

- Let  $S_{i,j}$  be the minimum unweighted feedback vertex set (FVS-4CL) for subgraph  $G_{i,j}$ , computed using Algorithm 22.
- Let  $F \equiv \text{OPT}$  be the optimum solution for the full graph  $G$ , and define  $F_{i,j} := F \cap V(G_{i,j})$ , the restriction of  $F$  to subgraph  $G_{i,j}$ .
- We obtain the following bound on the total solution size:

$$\underbrace{|S_{i^*}| \leq \sum_j |S_{i^*,j}|}_{\text{by Lemma 8 and Result 1}} \leq \underbrace{\sum_j |F_{i^*,j}|}_{\text{by Lemma 9}} \leq \underbrace{\left(1 + \frac{2}{\ell}\right) \cdot |F|}_{\text{by Lemma 10}} = (1 + 2\epsilon) \cdot |F|$$

# Weighted Bounds for FVS-BCL Problems

## Weighted FVS-4CL:

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq \left(1 + \frac{2}{\ell}\right) \cdot w(F) = (1 + 2\epsilon) \cdot w(F)$$

## Weighted FVS-BCL (break cycles of length $\rho$ ):

$$w(S_{i^*}) \leq \sum_j w(S_{i^*,j}) \leq \sum_j w(F_{i^*,j}) \leq \left(1 + \frac{\lfloor \rho/2 \rfloor}{\ell}\right) \cdot w(F)$$

# Fixed-Parameter Tractability of FVS-BCL

# Fixed-Parameter Tractability of FVS-BCL

- Monadic Second Order Logic for FVS-BCL
- Dynamic Programming Algorithm for FVS-4CL using Nice Tree Decompositions

- An extension of First-Order Logic
- Object variables: *vertices*:  $v_1, v_2, \dots$  and *edges*:  $e_1, e_2, \dots$
- Set variables: sets of vertices  $V_1, V_2, \dots$  and sets of edges  $E_1, E_2, \dots$
- Binary relation  $\in: \{\text{object variable}\} \times \{\text{set variable}\} \rightarrow \{0, 1\}$ .  
Therefore  $v \in V$  iff  $v$  is an element of the corresponding set  $V$ .
- The  $Adj(e, v_i, v_j)$  relation. It detects whether edge  $e$  is an edge from vertex  $v_i$  to vertex  $v_j$  where  $v_i \neq v_j$ .
- Quantification over set variables:  $\forall V_i, \forall E_i$  and  $\exists V_i, \exists E_i$ .



# MSOL Formulation: Unweighted FVS-4CL

$$\min_{F \subseteq V} |F| :$$

$$\forall v : \forall u : \forall w : \forall z :$$

$$v \in (V \setminus F) \wedge u \in (V \setminus F) \wedge w \in (V \setminus F) \wedge z \in (V \setminus F) \quad (1)$$

$$\wedge v \neq w \wedge u \neq z$$

$$\wedge \neg((v, u) \in E \wedge (u, w) \in E \wedge (w, z) \in E \wedge (v, z) \in E)$$

$$\min_{F \subseteq V} w(F) :$$

$$\forall v : \forall u : \forall w : \forall z :$$

$$v \in (V \setminus F) \wedge u \in (V \setminus F) \wedge w \in (V \setminus F) \wedge z \in (V \setminus F) \quad (2)$$

$$\wedge v \neq w \wedge u \neq z$$

$$\wedge \neg((v, u) \in E \wedge (u, w) \in E \wedge (w, z) \in E \wedge (v, z) \in E)$$

# MSOL Formulation: Weighted FVS- $\rho$ CL

$$\min_{F \subseteq V} w(F) :$$

$$\forall v_1 : \forall v_2 : \dots \forall v_\rho :$$

$$v_1 \in (V \setminus F) \wedge v_2 \in (V \setminus F) \cdots \wedge v_\rho \in (V \setminus F)$$

$$\wedge (v_1 \neq v_2 \wedge v_1 \neq v_3 \wedge \cdots \wedge v_1 \neq v_\rho)$$

$$\wedge (v_2 \neq v_1 \wedge v_2 \neq v_3 \wedge \cdots \wedge v_2 \neq v_\rho)$$

...

$$\wedge (v_\rho \neq v_1 \wedge v_\rho \neq v_2 \wedge \cdots \wedge v_\rho \neq v_{\rho-1})$$

$$\wedge \neg((v_1, v_2) \in E \wedge (v_2, v_3) \in E \wedge \cdots \wedge (v_{\rho-1}, v_\rho) \in E \wedge (v_\rho, v_1) \in E)$$

(3)

# Courcelle's Theorem and MSOL Solvability of FVS- $\rho$ CL

Courcelle's Theorem states that any graph property definable in MSOL can be decided in linear time on graphs of bounded treewidth.

**Theorem 7.11, p. 183**

**Parameterized Algorithms, Cygan et al., 2015**

## Corollary (MSOL Solvability of FVS- $\rho$ CL on Bounded-Treewidth Graphs)

*Let  $\rho \geq 3$  be a constant and  $G = (V, E)$  be a graph of treewidth at most  $tw$ , with vertex-weight function  $w : V \rightarrow \mathbb{N}$ . Then the minimum-weight set breaking all  $\rho$ -cycles (FVS- $\rho$ CL) can be computed in time:*

$$f(\rho, tw) \cdot n$$

*for some computable function  $f$  depending only on  $\rho$  and  $tw$ .*

# Dynamic Programming Algorithm for FVS-4CL over Nice Tree Decompositions

# Algorithm Design for FVS-4CL

- ① **Solution:** For the FVS-4CL problem, a solution for graph  $G$  is a set  $F$  such that  $G - F$  contains no 4-cycles.
- ② **Partial Solution:** For subgraph  $G_i = (V_i, E_i)$ , a partial solution  $F_i$  is a subset  $F_i \subseteq V_i$ , a restriction of a full solution.
- ③ **Extension of Partial Solution:** A solution  $F$  extends  $F_i$  if  $F \cap V_i = F_i$ .
- ④ **Characteristic of a Partial Solution:** For  $X_i$ , vertices are partitioned as:
  - $I \subseteq X_i$ : vertices in the partial solution.
  - $\mathcal{F} = \{(v, u) \in X_i \times X_i : \exists w \in V_i \setminus X_i, (v, w), (u, w) \in E_i, w \notin F_i\}$
$$ch(G_i, F_i) = (I, \mathcal{F})$$

The valuation table  $c[i, I, \mathcal{F}] \in \mathbb{N} \cup \{\infty\}$  gives the min. weight of  $F_i$ :

$$c[i, I, \mathcal{F}] = \min\{w(W) : W \text{ is a FVS-4CL of } G_i \wedge W \cap X_i = I\}$$

- ⑤ **Full Set of Characteristics:** For node  $X_i$ , valuations exist for all

$$I \in \{0, 1\}^{|X_i|} \quad \text{and} \quad \mathcal{F} \in \{0, 1\}^{|X_i \times X_i|}$$

There are at most  $2^{(tw+1)} \cdot 2^{\binom{tw+1}{2}} = 2^{\mathcal{O}(tw^2)}$  entries.

# DP Transitions over Tree Decomposition Nodes

**Leaf Node:**  $X_i = \emptyset$

$$c[i, \emptyset, \emptyset] = 0$$

**Introduce Node:** Let  $X_i = X_j \cup \{v\}$

$$c[i, I \cup \{v\}, \mathcal{F}] = w(v) + c[j, I, \mathcal{F}]$$

$$c[i, I, \mathcal{F}] = \begin{cases} \infty & \text{if } \exists u, w \notin I : (u, w) \in \mathcal{F}, (v, u), (v, w) \in E_i \\ c[j, I, \mathcal{F}] & \text{otherwise} \end{cases}$$

**Forget Node:** Let  $X_i = X_j \setminus \{v\}$

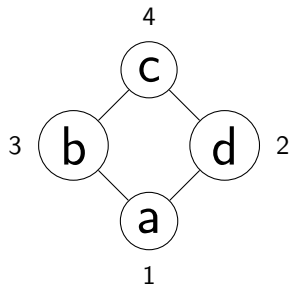
$$c[i, I, \mathcal{F}] = \min \left( c[j, I \cup \{v\}, \mathcal{F}], c[j, I, \mathcal{F} \cup \{(u, w) : (v, u), (v, w) \in E_i\}] \right)$$

**Join Node:** Let  $X_i = X_{j_1} = X_{j_2}$

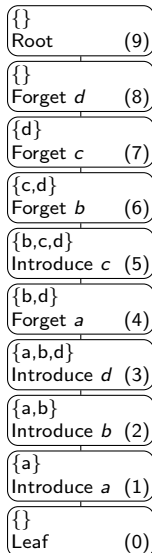
$$c[i, I, \mathcal{F}] = \min_{\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}} (c[j_1, I, \mathcal{F}_1] + c[j_2, I, \mathcal{F}_2] - w(I))$$

Infeasible if  $\exists u, w \notin I : (u, w) \in \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow c[i, I, \mathcal{F}] = \infty$

# Rhombus Graph and Tree Decomposition



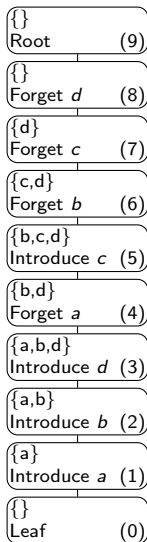
Rhombus Graph  $G$



**Nice Tree Decomposition**  
of the graph  $G$  with a width of 2.



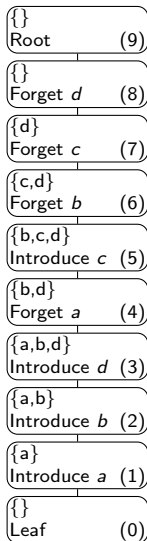
# Rhombus Graph and Tree Decomposition



$i$	$I$	$\mathcal{F}$	Value
0	$\emptyset$	$\emptyset$	0
1	$\{a\}$	$\emptyset$	1
1	$\emptyset$	$\emptyset$	0
2	$\{a, b\}$	$\emptyset$	4
2	$\{a\}$	$\emptyset$	1
2	$\{b\}$	$\emptyset$	3
2	$\emptyset$	$\emptyset$	0
3	$\{a, b, d\}$	$\emptyset$	6
3	$\{a, b\}$	$\emptyset$	4
3	$\{a, d\}$	$\emptyset$	3
3	$\{b, d\}$	$\emptyset$	5
3	$\{a\}$	$\emptyset$	1
3	$\{b\}$	$\emptyset$	3
3	$\{d\}$	$\emptyset$	2
3	$\emptyset$	$\emptyset$	0
4	$\{b, d\}$	$\emptyset$	6
4	$\{b, d\}$	$\{(b, d)\}$	5
4	$\{b\}$	$\emptyset$	4
4	$\{b\}$	$\{(b, d)\}$	3

$i$	$I$	$\mathcal{F}$	Value
4	$\{d\}$	$\emptyset$	3
4	$\{d\}$	$\{(b, d)\}$	2
4	$\emptyset$	$\emptyset$	1
4	$\emptyset$	$\{(b, d)\}$	0
5	$\{b, c, d\}$	$\emptyset$	10
5	$\{b, c, d\}$	$\{(b, d)\}$	9
5	$\{b, d\}$	$\emptyset$	6
5	$\{b, d\}$	$\{(b, d)\}$	5
5	$\{b, c\}$	$\emptyset$	8
5	$\{b, c\}$	$\{(b, d)\}$	7
5	$\{b\}$	$\emptyset$	4
5	$\{b\}$	$\{(b, d)\}$	3
5	$\{c, d\}$	$\emptyset$	7
5	$\{c, d\}$	$\{(b, d)\}$	6
5	$\{d\}$	$\emptyset$	3
5	$\{d\}$	$\{(b, d)\}$	2
5	$\{c\}$	$\emptyset$	5
5	$\{c\}$	$\{(b, d)\}$	4
5	$\emptyset$	$\emptyset$	1
5	$\emptyset$	$\{(b, d)\}$	$\infty$

# Rhombus Graph and Tree Decomposition



$i$	$I$	$\mathcal{F}$	Value
6	$\{c, d\}$	$\emptyset$	10
6	$\{c, d\}$	$\emptyset$	9
6	$\{d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	8
6	$\{c\}$	$\emptyset$	7
6	$\emptyset$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	3
6	$\{c, d\}$	$\emptyset$	7
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	3
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	5
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
6	$\emptyset$	$\emptyset$	$\infty$

$i$	$I$	$\mathcal{F}$	Value
6	$\{c, d\}$	$\emptyset$	6
6	$\{d\}$	$\emptyset$	2
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	1
7	$\{d\}$	$\emptyset$	2
7	$\emptyset$	$\emptyset$	1
8	$\emptyset$	$\emptyset$	1
9	$\emptyset$	$\emptyset$	1

# Beyond the Worst-Case Analysis

# Beyond the Worst-Case Analysis

- Perturbation resilience
- $m$ -Stitching and  $\Pi$ - $m$ -Stitching
- Certified Algorithms

# Definitions: $\gamma$ -Perturbation and $\gamma$ -Stability

## Definition ( $\gamma$ -Perturbation for Vertex-Optimization Problems)

Let  $(G, w)$  be a weighted graph. For any  $\gamma \in \mathbb{R}_{\geq 0}$ , a  $\gamma$ -**perturbation** of the weight function  $w : V \rightarrow \mathbb{N}$  is a function  $w' : V \rightarrow \mathbb{R}$  such that:

$$w(v) \leq w'(v) \leq \gamma \cdot w(v) \quad \forall v \in V.$$

**the number may be different for each parameter!**

## Definition ( $\gamma$ -Stability)

Let  $\Pi$  be a vertex-minimization problem. For any  $\gamma \in \mathbb{R}_{\geq 0}$ , a weighted graph  $(G, w)$  is called a  $\gamma$ -**stable instance** of  $\Pi$  if it admits a unique optimal solution  $S$  that remains optimal under all  $\gamma$ -perturbations of the weight function  $w$ .

# Definition: Certified Algorithm

## Definition (Certified Algorithm)

A  $\gamma$ -**certified solution** to an instance  $(G, w)$  of a weighted vertex-optimization problem  $\Pi$  is a pair  $(S, w')$ , where:

- $w'$  is a  $\gamma$ -**perturbation** of the original weight function  $w$ , and
- $S$  is an **optimal solution** for the instance  $(G, w')$ .

A  $\gamma$ -**certified algorithm** for  $\Pi$  maps each instance  $(G, w)$  to a  $\gamma$ -certified solution.

A  $\gamma$ -certified algorithm:

- **Exactly solves all  $\gamma$ -perturbation-resilient instances.**
- **Always returns a  $\gamma$ -approximate solution.**
- **Gives a  $\gamma$ -approximate solution** for the compliment problem.

# Definition: $m$ -stitching

## Definition ( $m$ -stitching)

Assume  $m \geq 0$  is an integer,  $J$  is an induced subgraph of  $G$ , and  $S_1, S_2 \subseteq V(G)$ . Then we define the  $m$ -stitch of  $S_2$  onto  $S_1$  along  $J$  as the set:

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^m[J]).$$



# Illustration of 2-stitching

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^2[J]).$$

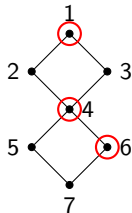


Figure: Vertex set  $S_1$

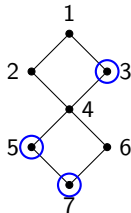


Figure: Vertex set  $S_2$

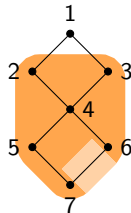


Figure:  $J$  and  $N_G^2[J]$

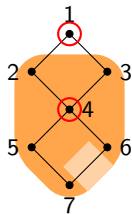


Figure:  $S_1 \setminus J$

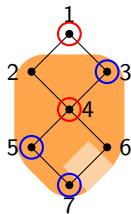


Figure:  $S_2 \cap N_G^2[J]$

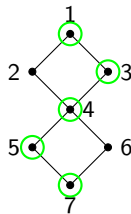


Figure: Final set  $S_3$

# Meta-Theorem for Minor-Closed Graph Classes

## Theorem (Meta-Theorem for Minor-Closed Graph Classes)

Let  $\mathcal{G}$  be a minor-closed graph class whose local treewidth is bounded by  $g(r) = \lambda \cdot r$ , for fixed  $\lambda \in \mathbb{R}$  and  $r \in \mathbb{N}$ .

Let  $\Pi$  be a vertex-minimization problem such that:

- 1  $\Pi$  is guessable.
- 2  $\Pi$  is  $m$ -stitchable.
- 3 There exists an algorithm  $A_\Pi$  that solves  $\Pi$ - $m$ -stitching in time  $f(t) \cdot |V(G)|^{\mathcal{O}(1)}$ , where  $t = \text{tw}(G[N_G^m(J)])$  and  $f$  is computable.

Then, for each  $\epsilon > 0$ , there exists a  $(1 + \epsilon)$ -certified algorithm for  $\Pi$  running in time  $f(\lambda \cdot m/\epsilon) \cdot |V(G)|^{\mathcal{O}(1)}$  on any input  $(G, w : V(G) \rightarrow \mathbb{N})$ , with  $G \in \mathcal{G}$  and polynomially-bounded weights.

(Bumpus et al., 2024)

# $(1 + \epsilon)$ -Certified Algorithm for FVS-4CL

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**Algorithm**  $(1 + \epsilon)$ -Certified algorithm for FVS-4CL

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**Require:** Vertex-weighted planar graph  $(G, w : V(G) \rightarrow \mathbb{N})$ ,  $\epsilon > 0$

**Ensure:** A vertex set  $S^* \subseteq V(G)$  and a  $(1 + \epsilon)$ -perturbation  $w'$  of  $w$  such that  $S^*$  is optimal for FVS-4CL on  $(G, w')$

- 1:  $\kappa \leftarrow \lceil \frac{2m}{\epsilon} \rceil + 2m$ , where  $m \leftarrow 2$
- 2: Let  $S^*$  be a feasible solution (FVS-4CL is guessable)
- 3: Perform BFS from an arbitrary vertex  $r$
- 4: **while** there exists a subgraph  $J_{\kappa-2m}$  of width  $\kappa - 2m$  such that  $w_A((G, w), S^*, J_{\kappa-2m}) < w(S^*)$  **do**
- 5:    $S^* \leftarrow A((G, w), S^*, J_{\kappa-2m})$
- 6: **end while**
- 7: Define  $w' : V(G) \rightarrow \mathbb{R}^+$  by:

$$w'(x) = \begin{cases} w(x) & \text{if } x \in S^* \\ (1 + \epsilon)w(x) & \text{otherwise} \end{cases}$$

# Summary of Contributions

- Establish **EPTAS** for **FVS-BCL** in planar graphs.
- Tractability:
  - Solved the **FVS-4CL** problem using **dynamic programming over nice tree decompositions**.
  - Solved the **FVS-BCL** problem using a **MSOL** formulation.
- Designed  $(1 + \epsilon)$ -certified algorithms for **FVS-4CL** and **FVS-BCL**

- Extend the DP algorithm over nice tree decompositions to FVS-BCL
- Improve the current algorithm that has a running time of  $2^{\mathcal{O}(tw^2)} \cdot n^{\mathcal{O}(1)}$ 
  - Cut & Count technique, which obtained a  $3^{\mathcal{O}(tw)} \cdot n^{\mathcal{O}(1)}$  randomized algorithm (**Cygan et al., 2011**)
  - Deterministic  $2^{\mathcal{O}(tw)} \cdot n^{\mathcal{O}(1)}$  rank-based approach (**Bodlaender et al., 2015**)
- Improve the certified algorithm for FVS-BCL.

# Thank you!

Questions?

# Appendix

- A.1 Approximation Schemes
- A.2 Tractability
- A.3 Certified Algorithms

# Appendix Approximation Schemes



# Bounds on the Optimum Solution (Unweighted Case)

## Lemma (Bound on the Optimum Solution in Subgraphs)

Let  $S_{i,j}$  be the minimum unweighted feedback vertex set (FVS-4CL) for subgraph  $G_{i,j}$ , computed using Algorithm 22, and let  $F \equiv \text{OPT}$  be the optimum solution for the full graph  $G$ .

Define  $F_{i,j} := F \cap V(G_{i,j})$ , i.e., the restriction of the global solution  $F$  to the subgraph  $G_{i,j}$ .

Then:

$$|S_{i,j}| \leq |F_{i,j}|$$

## Result (Optimum solution bound for the whole graph)

Let  $S_i$  be the union of optimal solutions defined on Line 6 of Algorithm 1 for some shift  $i$ . Let  $F \equiv \text{OPT}$  be the optimum solution in  $G$  and let  $F_{i,j} = G_{i,j} \cap F$ . For those sets it holds that:

$$|S_i| \leq \sum_j |S_{i,j}| \leq \sum_j |F_{i,j}|$$

# Bounds on the Optimum Solution (Unweighted Case)

## Lemma (Bound for the vertices on the boundaries (Unweighted Case))

Let  $F_i = F \cap \{\text{vertices at levels } i \bmod \ell\}$ . The sets  $F_i$  are disjoint and  $\bigcup_i F_i = F$ . We claim that:

$$\exists q \in \{0, 1, \dots, \ell - 1\} : |F_q| + |F_{q+1}| \leq \frac{2}{\ell} \cdot |F|.$$

## Lemma (Bound for the cardinality of the intersection sets)

Let  $F \equiv \text{OPT}$  be the optimal solution and  $F_{i,j} = F \cap G_{i,j}$  be the intersection sets with the subgraphs. Then we have:

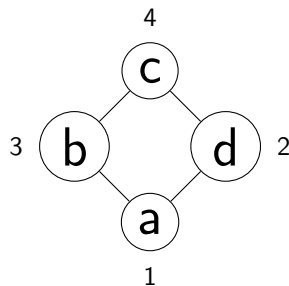
$$\sum_j |F_{i^*,j}| \leq |F| + \frac{2}{\ell} \cdot |F|$$

for some specific integer  $i^*$ .

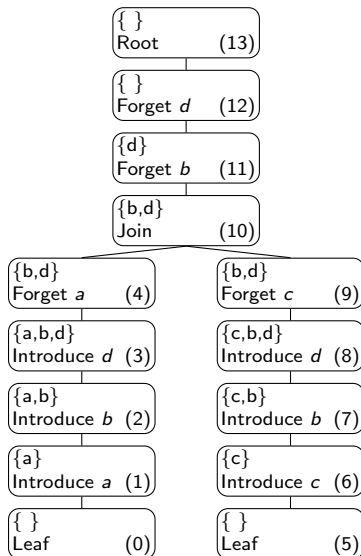
$$|S_i^*| \leq \sum_j |S_{i^*,j}| \leq \sum_j |F_{i^*,j}| \leq \left(1 + \frac{2}{\ell}\right) \cdot |F| = (1 + 2\epsilon) \cdot |F|$$

# Appendix Tractability

# Rhombus Graph and second Nice Tree Decomposition



**Rhombus Graph  $G$**



# Rhombus graph and second Nice Tree Decomposition

$i$	$I$	$\mathcal{F}$	Value
0	$\emptyset$	$\emptyset$	0
1	$\{a\}$	$\emptyset$	1
1	$\emptyset$	$\emptyset$	0
2	$\{a, b\}$	$\emptyset$	4
2	$\{a\}$	$\emptyset$	1
2	$\{b\}$	$\emptyset$	3
2	$\emptyset$	$\emptyset$	0
3	$\{a, b, d\}$	$\emptyset$	6
3	$\{a, b\}$	$\emptyset$	4
3	$\{a, d\}$	$\emptyset$	3
3	$\{b, d\}$	$\emptyset$	5
3	$\{a\}$	$\emptyset$	1
3	$\{b\}$	$\emptyset$	3
3	$\{d\}$	$\emptyset$	2
3	$\emptyset$	$\emptyset$	0
4	$\{b, d\}$	$\emptyset$	6
4	$\{b, d\}$	$\{b, d\}$	5
4	$\{b\}$	$\emptyset$	4
4	$\{b\}$	$\{b, d\}$	3
4	$\{d\}$	$\emptyset$	3
4	$\{d\}$	$\{b, d\}$	2
4	$\emptyset$	$\emptyset$	1
4	$\emptyset$	$\{b, d\}$	0

$i$	$I$	$\mathcal{F}$	Value
5	$\emptyset$	$\emptyset$	0
6	$\{c\}$	$\emptyset$	4
6	$\emptyset$	$\emptyset$	0
7	$\{c, b\}$	$\emptyset$	7
7	$\{c\}$	$\emptyset$	4
7	$\{b\}$	$\emptyset$	3
7	$\emptyset$	$\emptyset$	0
8	$\{c, b, d\}$	$\emptyset$	9
8	$\{c, b\}$	$\emptyset$	7
8	$\{c, d\}$	$\emptyset$	6
8	$\{b, d\}$	$\emptyset$	5
8	$\{c\}$	$\emptyset$	4
8	$\{b\}$	$\emptyset$	3
8	$\{d\}$	$\emptyset$	2
8	$\emptyset$	$\emptyset$	0
9	$\{b, d\}$	$\emptyset$	9
9	$\{b, d\}$	$\{b, d\}$	5
9	$\{b\}$	$\emptyset$	7
9	$\{b\}$	$\{b, d\}$	3
9	$\{d\}$	$\emptyset$	6
9	$\{d\}$	$\{b, d\}$	2
9	$\emptyset$	$\emptyset$	4
9	$\emptyset$	$\{b, d\}$	0

$i$	$I$	$\mathcal{F}$	Value
10	$\{b, d\}$	$\emptyset$	10
10	$\{b, d\}$	$\{b, d\}$	5
10	$\{b\}$	$\emptyset$	8
10	$\{b\}$	$\{b, d\}$	3
10	$\{d\}$	$\emptyset$	7
10	$\{d\}$	$\{b, d\}$	2
10	$\emptyset$	$\emptyset$	5
10	$\emptyset$	$\{b, d\}$	1
11	$\{d\}$	$\emptyset$	2
11	$\emptyset$	$\emptyset$	1
12	$\emptyset$	$\emptyset$	1
13	$\emptyset$	$\emptyset$	1

# Appendix Certified Algorithms

# Meta-Theorem for Minor-Closed Graph Classes

## Theorem (Meta-Theorem for Minor-Closed Graph Classes)

Let  $\mathcal{G}$  be a minor-closed graph class whose local treewidth is bounded by  $g(r) = \lambda \cdot r$ , for fixed  $\lambda \in \mathbb{R}$  and  $r \in \mathbb{N}$ .

Let  $\Pi$  be a vertex-minimization problem such that:

- 1  $\Pi$  is guessable.
- 2  $\Pi$  is  $m$ -stitchable.
- 3 There exists an algorithm  $A_\Pi$  that solves  $\Pi$ - $m$ -stitching in time  $f(t) \cdot |V(G)|^{\mathcal{O}(1)}$ , where  $t = \text{tw}(G[N_G^m(J)])$  and  $f$  is computable.

Then, for each  $\epsilon > 0$ , there exists a  $(1 + \epsilon)$ -certified algorithm for  $\Pi$  running in time  $f(\lambda \cdot m/\epsilon) \cdot |V(G)|^{\mathcal{O}(1)}$  on any input  $(G, w : V(G) \rightarrow \mathbb{N})$ , with  $G \in \mathcal{G}$  and polynomially-bounded weights.

## Definition (Guessable)

A problem  $\Pi$  is *guessable* if there is an algorithm that outputs a feasible solution with no requirement for optimality in polynomial time.

### In the case of FVS-BCL:

This set is  $F \leftarrow V$  for a graph  $G = (V, E)$ .



## Lemma: FVS-4CL is 2-stitchable

### Lemma (FVS-4CL is 2-stitchable)

*Let  $(G, w)$  be any instance of the FVS-4CL problem, let  $J \subseteq V(G)$  be a subset of the vertices in the graph, and let  $S_1$  and  $S_2$  be any two feasible solutions to the problem in  $G$ . Then the set*

$$S_3 := (S_1 \setminus J) \cup (S_2 \cap N_G^2[J])$$

*is a feasible solution to the problem.*

# Stitch-FVS-4CL Algorithm

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## Algorithm Stitch-FVS-4CL

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**Require:** Vertex-weighted planar graph  $(G, w : V(G) \rightarrow \mathbb{N})$ , a feasible solution  $S_1$  on  $G$ , and a vertex set  $J \subseteq V(G)$

**Ensure:** Feasible solution  $S'$  on  $G$ , such that for all feasible solutions  $S^*$ ,

$$w(S') \leq w(S^* \oplus_{G,J}^2 S_1)$$

- 1:  $H \leftarrow G[N_G^2[J] \setminus (S_1 \setminus J)]$
  - 2: Let  $S_2$  be the output of algorithm  $A$  on input  $(H, w)$
  - 3: **if**  $w(S_2 \oplus_{G,J}^2 S_1) < w(S_1)$  **then**
  - 4:     **return**  $S_2 \oplus_{G,J}^2 S_1$
  - 5: **else**
  - 6:     **return**  $S_1$
  - 7: **end if**
-

# Case-distinction proof

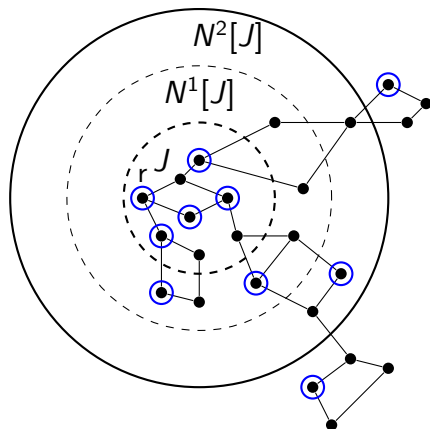


Figure: Sets  $J \subseteq N_G^1[J] \subseteq N_G^2[J]$ ;  
Solution  $S_1$  in blue

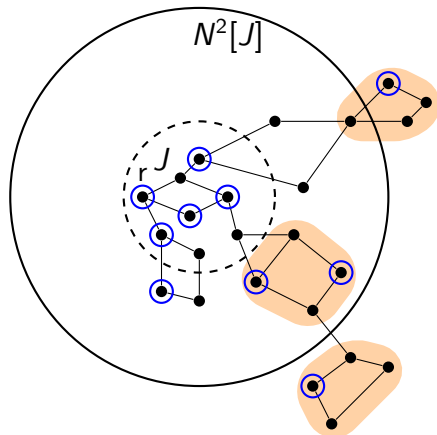


Figure:  $\forall v \in \{v_1, v_2, v_3, v_4\}$ , such that  
 $v \in S_1$ , it holds that  $v \notin J$

# Case-distinction proof

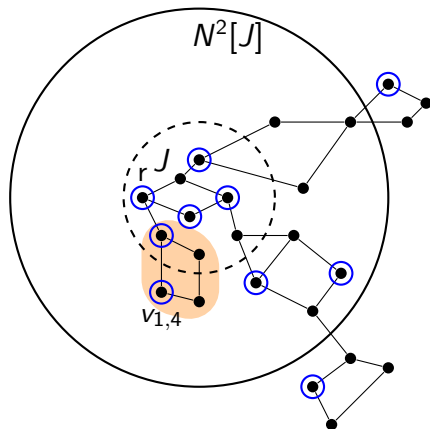


Figure:  $\exists v \in \{v_1, v_2, v_3, v_4\}$ , such that  $v \in S_1: v \in J$

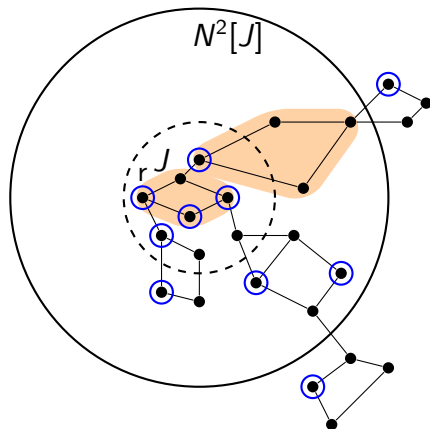


Figure:  $\forall v \in \{v_1, v_2, v_3, v_4\}$ , such that  $v \in S_1: v \in J$

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