Lecture 4: Discrete Random Variables and the Binomial Lattice Model

Introduction to the Binomial Lattice Model

- **Conceptual Foundation:**
  - Define the binomial lattice model as a discrete-time model for asset price dynamics.
  - Recap Bernoulli and binomial random variables and connect them to price movements: up and down steps driven by Bernoulli trials.
- **Mathematical Framework:**
  - Asset price S_t at time t evolves as:

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p, \\ dS_t & \text{with probability } 1 - p, \end{cases}$$

    where $u > 1$, $d < 1$, and p is the probability of an upward movement.

Calibrating Parameters for Small Time Steps

- **Role of Calibration:**
  - Calibrate u, d, and p to ensure consistency with the expected return and volatility for small time intervals (Δt).
- **Approximation for Small Δt:**

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p = \frac{e^{r\Delta t} - d}{u - d},$$

  where σ is volatility and r is the risk-free rate.
- Highlight the intuitive link between the discrete-time model and continuous models (to be expanded in Lecture 5).

Implementing the Model in R

- **Programming Concepts:**
  - Start by implementing a simple binomial tree in R:
    - Generate the tree structure (price paths) iteratively or recursively.
    - Compute probabilities along paths to calculate option values or expected prices.
  - Use this example to introduce:
    - **Control and Flow Structures:**
      - `if` statements for branching logic.
      - Loops (`for`, `while`, `repeat`) for iterative computations.
    - **Modularization:**
      - Break the program into smaller functions:
        1. A function to compute u, d, and p.
        2. A function to generate the binomial tree.
        3. A function to calculate prices or option values.
      - Combine functions into a cohesive program.
    - **Lists in R:**
      - Use lists to store and organize data such as the levels of the binomial tree, probabilities, and computed prices.

- **Application and Visualization:**
  - Simulate an asset price evolution over N time steps.
  - Visualize the lattice structure using `plot` or `ggplot2` (optional).

- **Discussion Points:**
  - Emphasize the flexibility of R for modeling and simulation.
  - Highlight the importance of organizing and reusing code through modularization.

Transition to Lecture 5: Continuous Models and Wiener Processes

- **Bridge to Continuous Random Variables:**
  - **Limit of the Binomial Model:**
    - As $\Delta t \to 0$, the binomial model converges to the continuous-time geometric Brownian motion.
  - Introduce continuous random variables and the normal distribution as the limiting distribution for the binomial model.

- **Preview of Continuous Models:**

- **Random Walks and Wiener Processes:**
  * Define the random walk and its relationship to the Wiener process.
  * Discuss stock log returns modeled as $\mathcal{N}(\mu\Delta t, \sigma^2\Delta t)$.
- **Applications:**
  * Highlight how these concepts generalize asset price modeling for Lecture 5.