An Introduction to Probability

Conditional Probability

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Conditional Probability

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- **Conditional probability** is a basic tool of probability theory.
- Particularly relevant in **Finance** for analyzing dependencies and risk.
- Often obscured by complex terminology despite simple ideas.

Motivation: A Striking Scenario

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- Imagine evaluating the safety of a **bond portfolio**:
  - Bonds are highly rated and diversified.
  - During a global recession:
    - Defaults occur, unraveling the portfolio.
    - Losses mount unexpectedly.
- Key lesson: **Underestimating event connections** leads to catastrophic risks.
- **Conditional probability** enables us to model dependencies effectively.

Why Conditional Probability Matters

Why Conditional Probability Matters

- Mastering conditional probability is crucial for:
  - Pricing financial instruments.
  - Assessing credit risk.
  - Making informed investment decisions.
- Neglecting it can lead to systemic failures:
  - Example: The **2007-2008 financial crisis**.

Case Study: The Financial Crisis (2007-2008)

Case Study: The Financial Crisis (2007-2008)

- Revealed the dangers of assuming independence between events.
- Highlighted failures in **structured finance**:
  - Underestimation of dependencies.
  - Misjudgment of risk profiles.
- Reference: For a comprehensive discussion, see [Tooze 2018].

Understanding Structured Finance

- **Bonds**:
  - Financial instruments with fixed payments and default risks.
  - Ratings by agencies like Moody's and Standard & Poor's:
    - High grade (AAA, AA) to speculative (BB, B) and default danger (CCC, C).

Rating Category	Moody's	Standard & Poor's
High grade	Aaa	AAA
	Aa	AA
Medium grade	A	A
	Baa	BBB
Speculative grade	Ba	BB
	B	B
Default danger	Caa	CCC
	Ca	CC
	C	C/D

Pooling and Tranching: An Innovation

Pooling and Tranching: An Innovation

- **Structured finance**:
  - Pools risky assets.
  - Divides cash flows into **tranches** with distinct risk profiles.
  - Enables creation of investment-grade securities from speculative-grade assets.
- Example products:
  - **Mortgage-backed securities (MBS)**.
  - Variants using similar financial engineering concepts.

Simplified Example

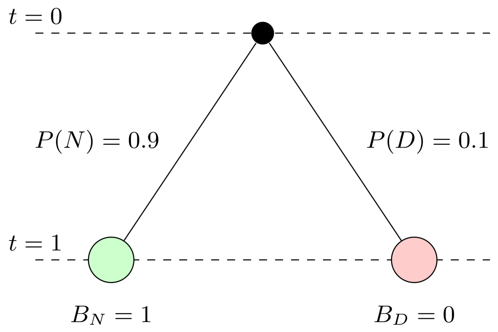
Simplified Example

- Based on Karl Schmedder's course on probability.
- Illustrates **structured finance** and its relation to probability.
- Develops an intuitive understanding of pooling, tranching, and conditional dependencies.

A Simple Event Tree for One Bond

A Simple Event Tree for One Bond

- Consider a single bond paying €1 at maturity in the future.
  - Default probability: 10% ($P(D) = 0.1$).
  - Non-default probability: 90% ($P(N) = 0.9$).
- Payoff structure:
  - No Default (N): €1.
  - Default (D): €0.
- Graphically represented as an event tree:



Understanding the Event Tree

Understanding the Event Tree

- Nodes represent states of the bond:
  - $t = 0$: Initial state.
  - $t = 1$: Outcomes (N or D).
- Probabilities:
  - $P(N) = 0.9$.
  - $P(D) = 0.1$.
- Analogy:
  - Coin toss with unequal probabilities:
    - Heads: 90% (No Default).
    - Tails: 10% (Default).
- Probabilistic interpretation:
  - Random experiment with outcomes N and D.

Combining Two Bonds: Independence Assumption

Combining Two Bonds: Independence Assumption

- Portfolio of two bonds.
- **Independence Assumption**:
  - Defaults occur independently.
  - Default of one bond does not influence the other.
- Simplifies calculations:
  - Default probabilities remain uncorrelated.
  - Example: $P(D_1 \cap D_2) = P(D_1) \cdot P(D_2)$.
- Historically justified by:
  - Diversification.
  - Uncorrelated defaults under normal conditions.

Systemic Risks: Challenges to Independence

Systemic Risks: Challenges to Independence

- Systemic risks disrupt independence:
  - Defaults become correlated during crises.
  - Shared macroeconomic factors increase joint defaults.
- Example: 2008 financial crisis:
  - Rising mortgage defaults driven by economic downturn.
  - Increased correlation in bond defaults.
- Critical thinking reveals:
  - Diversification alone cannot guarantee safety.
  - Assumption of independence is fragile.

Event Tree for Two Bonds

Event Tree for Two Bonds

- Two independent bonds:
  - Combine event trees of individual bonds.
  - Visualized as a double event tree.

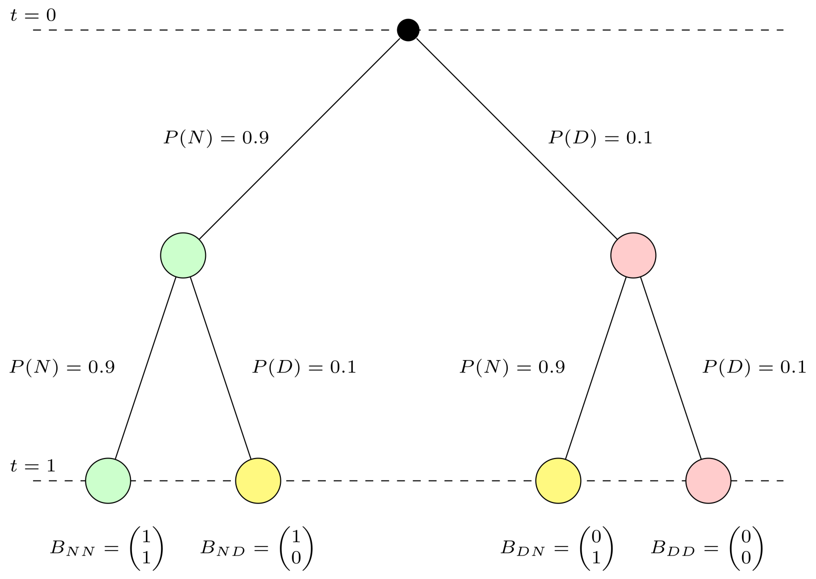
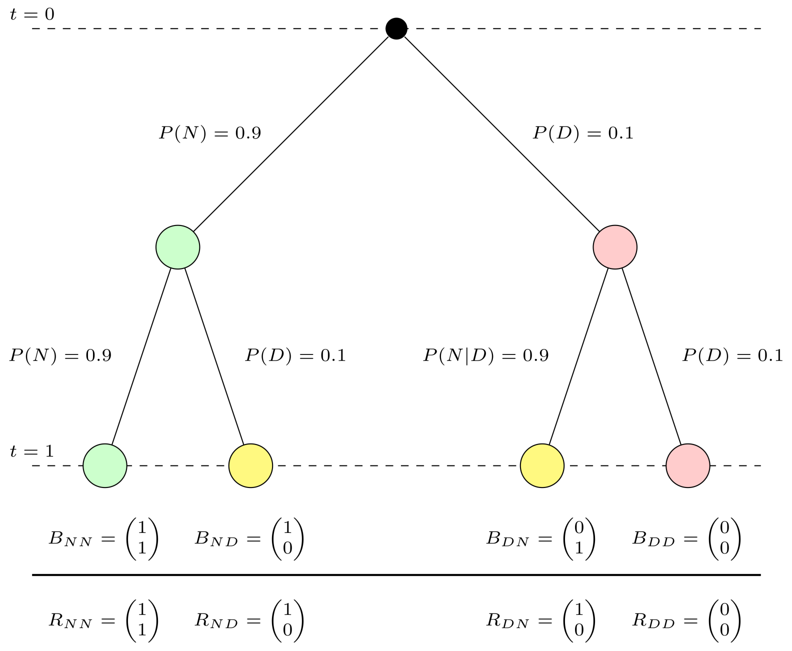

$t = 0$
$P(N) = 0.9$
$P(D) = 0.1$
$P(N) = 0.9$
$P(D) = 0.1$
$P(N) = 0.9$
$P(D) = 0.1$
$t = 1$
$B_{NN} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$B_{ND} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$B_{DN} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$B_{DD} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Figure 2

Pooling and Tranching: Independent Risks

Pooling and Tranching: Independent Risks

- Re-engineering risk profiles:
  - Pool payoffs of two bonds.
  - Create two new securities:
    1. Pays €1 except when both bonds default.
    2. Pays €0 except when both bonds do not default.
- Probabilities:
  - Both bonds default: $P(D_1 \cap D_2) = 0.1 \cdot 0.1 = 0.01$.
  - Both bonds do not default: $P(N_1 \cap N_2) = 0.9 \cdot 0.9 = 0.81$.



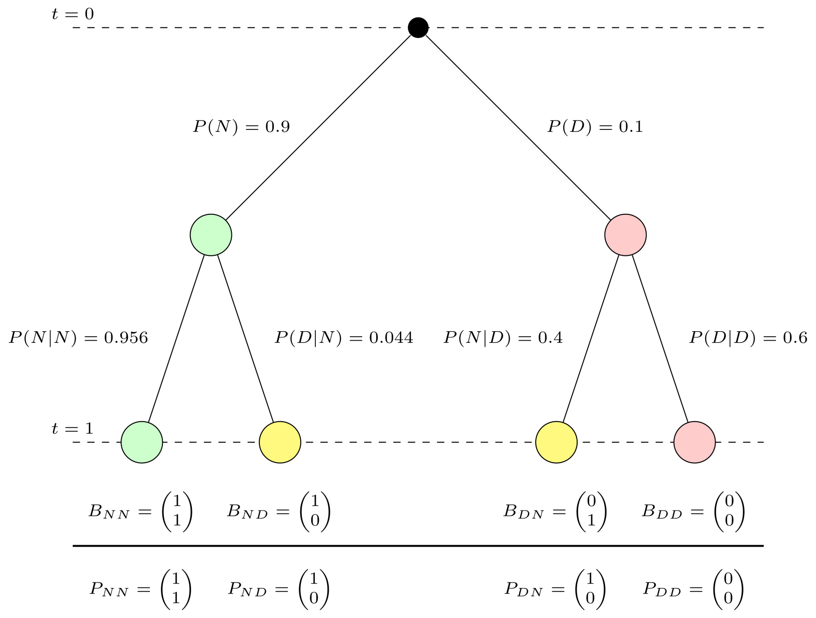
$t = 0$
$P(N) = 0.9$
$P(D) = 0.1$
$P(N) = 0.9$
$P(D) = 0.1$
$P(N|D) = 0.9$
$P(D) = 0.1$
$t = 1$
$B_{NN} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $B_{ND} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $B_{DN} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $B_{DD} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$R_{NN} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $R_{ND} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $R_{DN} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $R_{DD} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Figure 3

Pooling and Tranching: Dependent Risks

Pooling and Tranching: Dependent Risks

- What happens if independence does not hold?
  - Default probabilities change:
    - $P(D_2 \mid D_1) = 0.6$.
    - $P(D_2 \mid N_1) = 0.044$.
- Increased correlation during systemic events:
  - Shared risks drive joint defaults.



$t = 0$
$P(N) = 0.9$
$P(D) = 0.1$
$P(N|N) = 0.956$
$P(D|N) = 0.044$
$P(N|D) = 0.4$
$P(D|D) = 0.6$
$t = 1$
$B_{NN} = \begin{pmatrix}1\\1\end{pmatrix}$
$B_{ND} = \begin{pmatrix}1\\0\end{pmatrix}$
$B_{DN} = \begin{pmatrix}0\\1\end{pmatrix}$
$B_{DD} = \begin{pmatrix}0\\0\end{pmatrix}$
$P_{NN} = \begin{pmatrix}1\\1\end{pmatrix}$
$P_{ND} = \begin{pmatrix}1\\0\end{pmatrix}$
$P_{DN} = \begin{pmatrix}1\\0\end{pmatrix}$
$P_{DD} = \begin{pmatrix}0\\0\end{pmatrix}$

Figure 4

Impact of Dependence on Risk Profiles

Impact of Dependence on Risk Profiles

- Dependent risks change probabilities:
  - Both bonds default: $P(D_1) \cdot P(D_2 \mid D_1) = 0.1 \cdot 0.6 = 0.06$.
  - Six times higher than under independence.
- Structured finance products fail:
  - Investment-grade tranches lose their safety.
  - Junk plus junk remains junk.

Lessons from Structured Finance

Lessons from Structured Finance

1. **Diversification**:
   - Assets must be from independent sectors.
2. **Macroeconomic Stability**:
   - Low systemic risk is crucial.
3. **Transparent Modeling**:
   - Dependencies must be accounted for.

- Neglecting these led to flawed models and systemic failures during the 2008 crisis.

Conditional Probability

Conditional Probability

- Conditional probability formalizes how the probability of one event changes when another event is known to occur.
- It provides a framework for understanding dependencies quantitatively.

Definition: Conditional Probability

Definition: Conditional Probability

Definition: Conditional Probability

Let B be an event with positive probability. For an arbitrary event A, the **conditional probability** of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) \neq 0$$

Undefined Conditional Probabilities

Undefined Conditional Probabilities

- Conditional probabilities are **undefined** when the conditioning event B has $P(B) = 0$.
- This distinction is:
  - Irrelevant for **discrete sample spaces**.
  - Crucial in the **general theory**.

Clarifying Conditional Probabilities

Clarifying Conditional Probabilities

- Conditional probabilities represent a **notation** change:
  - Probabilities adjust to reflect known conditions.
- Example: Revisit the financial crisis scenario:
  - Highlighted how dependencies can amplify systemic risk.

The Probability Tree and Conditional Probabilities

The Probability Tree and Conditional Probabilities

- A **probability tree** is labeled with **edge probabilities**:
  - Representing marginal and conditional probabilities at each level.
  - At $t = 0$
    - $P(B_1 = N) = 0.9$, $P(B_1 = D) = 0.1$
  - At $t = 1$
    - $P(B_2 = N \mid B_1 = N) = 0.956$,
      $P(B_2 = D \mid B_1 = N) = 0.044$
    - $P(B_2 = N \mid B_1 = D) = 0.4$, $P(B_2 = D \mid B_1 = D) = 0.6$.

Defining Probabilities in R

Defining Probabilities in R

- Probabilities defined from the **probability tree**:

```
# Define the probabilities

# Marginal probabilities for B_1
P_N <- 0.9  # Probability that B_1 does not default
P_D <- 0.1  # Probability that B_1 defaults

# Conditional probabilities for B_2 given B_1
P_N_given_N <- 0.8604/0.9
# B_2 does not default given B_1 does not default
P_D_given_N <- 0.0396/0.9
# B_2 defaults given B_1 does not default
P_N_given_D <- 0.4
# B_2 does not default given B_1 defaults
P_D_given_D <- 0.6
```

Computing Joint Probabilities

Computing Joint Probabilities

- Joint probabilities, $P(A \cap B)$, are calculated using the **multiplication rule**:

$$P(A \cap B) = P(A \mid B) \cdot P(B).$$

```
# Calculate joint probabilities

P_NN <- P_N * P_N_given_N  # Both bonds do not default
P_ND <- P_N * P_D_given_N  # B_1 does not default, B_2 defa
P_DN <- P_D * P_N_given_D  # B_1 defaults, B_2 does not def
P_DD <- P_D * P_D_given_D  # Both bonds default
```

Simulating Bond Portfolio Defaults

Simulating Bond Portfolio Defaults

Goal:

- Simulate a bond portfolio with two types of bonds (`B1` and `B2`).
- Reproduce a portfolio where default probabilities align with the given contingency table.

Key Concepts:

- Joint probabilities from the contingency table.
- Unconditional and conditional probabilities.

Set up simulation parameters

Set up simulation parameters

```r
N <- 5000  # Total number of bonds
P_DD <- 0.06  # P(X = D, Y = D)
P_DN <- 0.04  # P(X = D, Y = N)
P_ND <- 0.04  # P(X = N, Y = D)
P_NN <- 0.86  # P(X = N, Y = N)

# Verify probabilities sum to 1
stopifnot(abs(P_DD + P_DN + P_ND + P_NN - 1) < 1e-6)
```

```r
# Simulate joint outcomes based on the contingency table
simulate_defaults <- function(N, probs) {
  sample(
    c("DD", "DN", "ND", "NN"),
    size = N,
    replace = TRUE,
    prob = probs
  )
}
```

```r
# Generate portfolio data
portfolio <- data.frame(
  BondID = 1:N,
  BondType = sample(c("B1", "B2"), N,
                    replace = TRUE, prob = c(0.5, 0.5))
)

# Assign joint default outcomes
portfolio$JointOutcome <-
  simulate_defaults(N, c(P_DD, P_DN, P_ND, P_NN))
portfolio$X_Defaulted <-
  portfolio$JointOutcome %in% c("DD", "DN")
portfolio$Y_Defaulted <-
  portfolio$JointOutcome %in% c("DD", "ND")
```

```r
# Compute unconditional probabilities
P_X_D <- mean(portfolio$X_Defaulted)  # P(X = D)
P_Y_D <- mean(portfolio$Y_Defaulted)  # P(Y = D)

# Compute conditional probabilities
P_X_given_Y_D <-
  mean(portfolio$X_Defaulted[portfolio$Y_Defaulted])
# P(X = D | Y = D)
P_Y_given_X_D <-
  mean(portfolio$Y_Defaulted[portfolio$X_Defaulted])
# P(Y = D | X = D)
```

```r
# Display results
cat("Unconditional Probabilities:\n")
cat("P(X = D):", round(P_X_D, 4), "\n")
cat("P(Y = D):", round(P_Y_D, 4), "\n\n")

cat("Conditional Probabilities:\n")
cat("P(X = D | Y = D):", round(P_X_given_Y_D, 4), "\n")
cat("P(Y = D | X = D):", round(P_Y_given_X_D, 4), "\n")
```

```r
# Verify calibration matches input probabilities
calibration_check <- table(portfolio$JointOutcome) / N
expected_probs <- c(P_DD, P_DN, P_ND, P_NN)

calibration_result <- data.frame(
  JointOutcome = names(calibration_check),
  Frequency = as.numeric(calibration_check),
  Expected = expected_probs
)

print(calibration_result)
```

Key Takeaways

- **Conditional Probability**:
  - Computed as the relative frequency in a subset of rows where the condition holds.
  - E.g., $P(X = D \,|\, Y = D)$ focuses only on rows where `Y_Defaulted` is `TRUE`.
- **Calibration**:
  - Simulated frequencies align closely with expected probabilities from the contingency table.
- **Practical Application**:
  - Demonstrates how dependency structures in default risks are modeled.

Advanced R Concepts

Advanced R Concepts

In this section, we explore advanced R programming concepts:

1. **Environments**: How R evaluates and stores variables.
2. **Scoping Rules**: How R resolves variable names.
3. **Closures**: Functions that retain the environment where they were created.

Environments

Environments

- An **environment** in R stores objects (variables, functions, etc.).
- The **global environment** stores user-created objects.
- Local variables can override global ones in specific functions.

Example: Global and Local Variables

```
# Global interest rate
interest_rate <- 0.05

# Function to calculate interest
calculate_interest <- function(principal, rate = interest_r
  interest <- principal * rate
  return(interest)
}

# Global calculation
global_interest <- calculate_interest(1000)

# Local override
local_interest <- calculate_interest(1000, rate = 0.07)

cat("Global Interest:", global_interest, "\n")
```

Scoping Rules

Scoping Rules

- R uses **lexical scoping** to find variables:
  - Searches the closest environment first.
  - Moves outward to enclosing environments.

Example: Nested Functions

```
# Global default rates
default_rates <- c(AAA = 0.01, BBB = 0.02, Junk = 0.05)

# Function for conditional default
conditional_default <- function(rating) {
  local_default_rates <- c(
    AAA = unname(default_rates["AAA"]),
    BBB = unname(default_rates["BBB"]),
    Junk = unname(default_rates["Junk"])
  )
  return(local_default_rates[rating])
```

Lookup Tables

Lookup Tables

- **Lookup tables** map inputs to outputs.
- Example: Default probabilities for credit ratings.

Benefits:

1. Centralizes data for easy updates.
2. Avoids repetitive conditional statements.

Closures

Closures

- **Closures** are functions that retain their creation environment.
- Used to create dynamic and reusable functions.

Example: Probability Calculator Factory

```
# Function factory
probability_calculator_factory <- function(event_probabilit
  function(conditional_probability) {
    joint_probability <- event_probability * conditional_pr
    return(joint_probability)
  }
}

# Create calculators
junk_calculator <- probability_calculator_factory(0.05)
bbb_calculator <- probability_calculator_factory(0.02)

# Calculate joint probabilities
junk_joint <- junk_calculator(0.1)
bbb_joint <- bbb_calculator(0.2)
```

Analyzing Closures

Analyzing Closures

1. **Function Factory**:
   - Takes `event_probability` as input.
   - Returns a function for calculating joint probabilities.
2. **Reusable Calculators**:
   - `junk_calculator` for Junk bonds.
   - `bbb_calculator` for BBB bonds.
3. **Encapsulation**:
   - Parameters are “locked in” during function creation.

Bayes' Rule: A Cornerstone of Probability

Bayes' Rule: A Cornerstone of Probability

- Bayes' Rule provides a systematic method for updating probabilities based on new evidence.
- Formula:
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$
- Explanation:
  - A: Observed evidence.
  - B: Hypothesis or prior belief.

Deriving Bayes' Rule

Deriving Bayes' Rule

- From the Multiplication Rule:
  1. $P(B|A)P(A) = P(A \cap B)$
  2. $P(A|B)P(B) = P(A \cap B)$
  - Equating and dividing by $P(A)$:

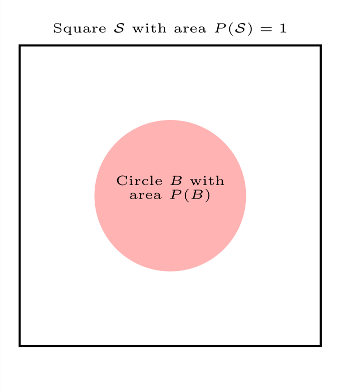
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Significance: Ties prior beliefs to evidence using conditional probabilities.

Speck of Sand: An Intuitive Illustration

Speck of Sand: An Intuitive Illustration

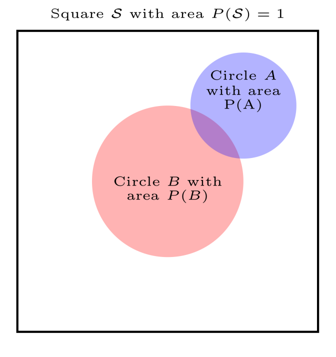
- A square of area 1 represents the sample space.
- Circle B represents the event with area $P(B)$.
- A speck of sand falls randomly in the square.



Updating Beliefs

Updating Beliefs

- New Information:
  - The speck is known to land in another circle A inside the square.
- Question:
  - What is $P(B|A)$, the probability that the speck is in B, given it is inside A?



Overlap Between A and B

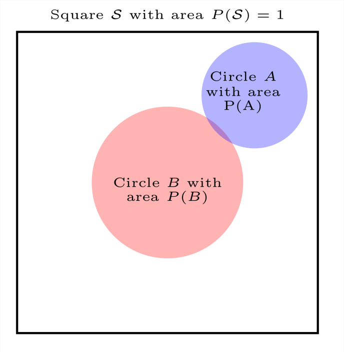
Overlap Between A and B

- The updated probability $P(B|A)$ depends on the overlap of B and A:

$$P(B|A) = \frac{\text{Area of } A \cap B}{\text{Area of } A} = \frac{P(A \cap B)}{P(A)}$$

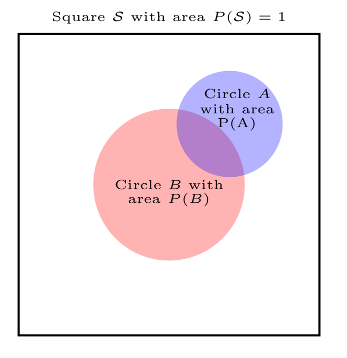
Small Overlap: Low Probability

Small Overlap: Low Probability



Large Overlap: High Probability

Large Overlap: High Probability



Bayesian Interpretation

Bayesian Interpretation

- Terms:
  - $P(B)$: Prior probability.
  - $P(A|B)$: Likelihood.
  - $P(A)$: Normalizing constant.
- Process:
  1. Start with prior $P(B)$.
  2. Add evidence $P(A|B)$.
  3. Compute updated belief $P(B|A)$.

Project Overview

Problem Description

- A bank is evaluating a loan application.
- Goal: Estimate the likelihood of default using historical data.
- Key probabilities provided:
  1. **Default Rates**:
     - $P(D) = 0.04$ (default probability).
     - $P(ND) = 0.96$ (non-default probability).
  2. **Low Credit Score**:
     - $P(L|D) = 0.7$ (probability of low credit score given default).
     - $P(L|ND) = 0.1$ (probability of low credit score given non-default).
- Objective:
  - Compute the posterior probability of default given a low credit score, $P(D|L)$.

Key Questions

1. Compute $P(D|L)$ Theoretically:
   - Use Bayes' Rule:

$$P(D|L) = \frac{P(L|D) \cdot P(D)}{P(L|D) \cdot P(D) + P(L|ND) \cdot P(ND)}$$

2. Simulate the Scenario in R:
   - Create a dataset of 10,000 customers.
   - Assign default status based on $P(D)$ and simulate credit scores.
3. Compute $P(D|L)$ Empirically:
   - Calculate $P(D|L)$ using simulated data and compare with the theoretical result.
4. Visualize Results:
   - Plot theoretical vs. simulated probabilities.