

Practica 11(2021)

Preambulo:

Para una distribución binomial:

Parameters: $n \in \{0, 1, 2, \dots\}$, number of trials

Support: $k \in \{0, 1, \dots, n\}$, number of successes

$$\text{PMF: } \binom{n}{k} p^k q^{n-k}$$

$$\begin{aligned} E(X) &= \sum_{k=0}^n k \binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} \left(k=0, k \binom{n}{k} p^k q^{n-k} = 0 \right) \\ &= \sum_{k=1}^n n \binom{n-1}{k-1} p^k q^{n-k} \left(k \binom{n}{k} = n \binom{n-1}{k-1} \right) \\ &= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)} \\ &\quad (\text{sacar } np \text{ y } (n-1) - (k-1) = n-k) \\ &= np \sum_{j=0}^m \binom{m}{j} p^j q^{m-j} \\ &\quad (\text{putting } m = n-1, j = k-1) \\ &= np \quad (\text{Binomial Theorem and } p+q=1) \end{aligned}$$

Problema 1

Enunciado: Resolucion: ## Inciso a

$$\begin{aligned} V &= \frac{\partial}{\partial \theta} \ln \left(\frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}} \right) \\ &= \frac{\partial}{\partial \theta} \left\{ \frac{-x^2}{2\theta} \ln(e) - \ln(\sqrt{2\pi\theta}) \right\} \\ &= \frac{\partial}{\partial \theta} \left\{ \frac{-x^2}{2\theta} - \frac{1}{2} \ln \theta + \frac{1}{2} \ln(2\pi) \right\} \\ &= \frac{-x^2}{2} \cdot \frac{\partial}{\partial \theta} \frac{1}{\theta} - \frac{1}{2} \frac{\partial}{\partial \theta} \ln \theta \\ &= \frac{1}{\theta^2} (x^2 - \theta) \end{aligned}$$

$$\langle V \rangle = 0$$

Inciso b

$$\begin{aligned} V &= \frac{\partial}{\partial \theta} \ln \left(\frac{x^\theta}{1+2^\theta} \right) = \partial_\theta (\ln x^\theta - \ln(1+2^\theta)) \\ &= \ln x - \partial_\theta \ln(1+2^\theta) = \ln x - \frac{1}{1+2^\theta} \cdot \partial_\theta (1+2^\theta) \\ &= \ln x - \frac{1}{1+2^\theta} \partial_\theta e^{\theta \ln 2} = \ln x - \frac{1}{1+2^\theta} \cdot \ln 2 \cdot e^{\theta \ln 2} \\ &= \ln x - \frac{\ln 2}{1+2^\theta} 2^\theta \end{aligned}$$

$$\langle V \rangle = \left(\frac{1+2^\theta}{1+2^\theta} \right) = 1$$

Problema 2

Enunciado:

Resolucion: Demostrar:

$$\mathcal{I}(\theta) = -E \left[\frac{\partial^2}{\partial \theta^2} \log f(X; \theta) \mid \theta \right]$$

se tiene que

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} \log f(X; \theta) &= \frac{\frac{\partial^2}{\partial \theta^2} f(X; \theta)}{f(X; \theta)} - \left(\frac{\frac{\partial}{\partial \theta} f(X; \theta)}{f(X; \theta)} \right)^2 \\ &= \frac{\frac{\partial^2}{\partial \theta^2} f(X; \theta)}{f(X; \theta)} - \left(\frac{\partial}{\partial \theta} \log f(X; \theta) \right)^2 \end{aligned}$$

y que

$$E \left[\frac{\frac{\partial^2}{\partial \theta^2} f(X; \theta)}{f(X; \theta)} \mid \theta \right] = \frac{\partial^2}{\partial \theta^2} \int f(x; \theta) dx = \frac{\partial^2}{\partial \theta^2} 1 = 0$$

por lo que

$$\begin{aligned} E \left[\frac{\partial^2}{\partial \theta^2} \log f(X; \theta) \right] &= E \left[\frac{\frac{\partial^2}{\partial \theta^2} f(X; \theta)}{f(X; \theta)} \right] - E \left[\left(\frac{\partial}{\partial \theta} \log f(X; \theta) \right)^2 \right] \\ &= -E \left[\left(\frac{\partial}{\partial \theta} \log f(X; \theta) \right)^2 \right] = -J(\theta) \end{aligned}$$

Problema 3

Enunciado:

Resolucion: Sabiendo que

$$P(x_1, \dots, x_n | \theta) = \prod_{i=0}^n P(x_i | \theta)$$

$$\log \prod_{i=0}^n P(x_i | \theta) = \sum_{i=0}^n \log P(x_i | \theta)$$

Se tiene que

$$\begin{aligned} J(\theta) &= -E \left[\frac{\partial^2}{\partial \theta^2} \log P(x_1, \dots, x_n | \theta) \right] \\ &= -E \left[\sum_{i=0}^n \frac{\partial^2}{\partial \theta^2} \log P(x_i | \theta) \right] \\ &= \sum_{i=0}^n -E \left[\frac{\partial^2}{\partial \theta^2} \log P(x_i | \theta) \right] \\ &= \sum_{i=0}^n J_i(\theta) \\ &= nJ(\theta) \end{aligned}$$

Problema 4

Enunciado:

Resolucion: Primer caso

$$\begin{aligned} \partial_\theta^2 \log(\theta e^{-\theta x}) &= \partial_\theta \left(\frac{1}{\theta} - x \right) = \frac{-1}{\theta^2} \\ J_1(\theta) &= -E[\partial_\theta^2 \log(\theta e^{-\theta x})] = -E\left[\frac{-1}{\theta^2}\right] \\ &= \frac{1}{\theta^2} E[1] = \frac{1}{\theta^2} \end{aligned}$$

Segundo caso:

Usando que:

$$\begin{aligned} \partial_\theta^2 \log(\theta^2 e^{-\theta^2 x}) &= -\frac{2}{\theta^2} - 2x \\ E[x] &= \int_{\mathbb{R}^+} x e^{-\theta^2 x} dx = \frac{1}{\theta^2} \end{aligned}$$

Se tiene que:

$$\begin{aligned} J_2(\theta) &= -E[\partial_\theta^2 (\theta^2 e^{-\theta^2 x})] \\ &= -E\left[-\frac{2}{\theta^2} - 2x\right] \\ &= \frac{2}{\theta^2} + 2E[x] = \frac{4}{\theta^2} \end{aligned}$$

Problema 5

Enunciado:

Resolucion: Si hay dos formas de parametrizar:

$X \sim f(x | \theta) = g(x | \eta)$, con $\theta = \phi(\eta)$, $\eta = \psi(\theta)$
se pueden relacionar las informaciones de Fisher.

$$\begin{aligned} J^\theta(\theta) &= E \left\{ \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 \right\} \\ &= E \left\{ \left(\frac{\partial}{\partial \theta} \log g(x | \psi(\theta)) \right)^2 \right\} \\ &= E \left\{ \left(\frac{\partial}{\partial \eta} \log g(x | \eta) \frac{\partial \eta}{\partial \theta} \right)^2 \right\} \\ &= E \left\{ \left(\frac{\partial}{\partial \eta} \log g(x | \eta) \right)^2 \right\} \left(\frac{\partial \eta}{\partial \theta} \right)^2 \\ &= J^\eta(\eta) (J_\theta^\eta)^2 \\ &= J^\eta(\eta) \left(\frac{\partial \eta}{\partial \theta} \right)^2 \quad (1D) \end{aligned}$$

Se puede del otro lado:

$$I^\eta(\eta) = I^\theta(\theta) \left(\frac{\partial \theta}{\partial \eta} \right)^2 \quad (1D)$$

Problema 6

Enunciado:

Resolucion: Statement

La cota Cramér-Rao va a ser incrementalmente generalizada. Todas las versiones de la cota requieren ciertas condiciones de regularidad que se mencionan al final.

Caso 1

Se empieza por que el parámetro es escalar y tiene un estimador sin bias, se supone que θ es un desconocido parámetro determinista que se estima desde n observaciones independientes de x , cada uno desde una distribución de acuerdo a una densidad de probabilidad $f(x; \theta)$.

$$\text{var}(\hat{\theta}) \geq \frac{1}{J(\theta)}$$

donde la $J(\theta)$ viene dada por

$$J(\theta) = nE_{\theta} \left[\left(\frac{\partial \ell(X; \theta)}{\partial \theta} \right)^2 \right]$$

y $\ell(x; \theta) = \log(f(x; \theta))$. Bajo ciertas condiciones es posible:

$$J(\theta) = -nE_{\theta} \left[\frac{\partial^2 \ell(X; \theta)}{\partial \theta^2} \right]$$

Definimos la eficiencia como:

$$e(\hat{\theta}) = \frac{J(\theta)^{-1}}{\text{var}(\hat{\theta})}$$

y la cota se puede reescribir como:

$$e(\hat{\theta}) \leq 1$$

Caso 2

Considerando que hay bias, y el estimador con bias $T(X)$, no tiene valor de expectacion θ sino que es una $\psi(\theta)$. Se tiene $E\{T(X)\} - \theta = \psi(\theta) - \theta$ que puede ser no nula. Esto implica que

$$\text{var}(T) \geq \frac{[\psi'(\theta)]^2}{J(\theta)}$$

donde $\psi'(\theta)$ es la derivada de $\psi(\theta)$ (respecto a θ). Bound on the variance of biased estimators Apart from being a bound on estimators of functions of the parameter, this approach can be used to derive a bound on the variance of biased estimators with a given bias, as follows. Consider an estimator $\hat{\theta}$

with bias $b(\theta) = E\{\hat{\theta}\} - \theta$, and let $\psi(\theta) = b(\theta) + \theta$. By the result above, any unbiased estimator whose expectation is $\psi(\theta)$ has variance greater than or equal to $(\psi'(\theta))^2 / I(\theta)$. Thus, any estimator $\hat{\theta}$ whose bias is given by a function $b(\theta)$ satisfies

$$\text{var}(\hat{\theta}) \geq \frac{[1 + b'(\theta)]^2}{I(\theta)}$$

The unbiased version of the bound is a special case of this result, with $b(\theta) = 0$.

It's trivial to have a small variance - an "estimator" that is constant has a variance of zero. But from the above equation we find that the mean squared error of a biased estimator is bounded by

$$E((\hat{\theta} - \theta)^2) \geq \frac{[1 + b'(\theta)]^2}{I(\theta)} + b(\theta)^2$$

using the standard decomposition of the MSE. Note, however, that if $1 + b'(\theta) < 1$ this bound might be less than the unbiased Cramér-Rao bound $1/I(\theta)$. For instance, in the example of estimating variance below, $1 + b'(\theta) = \frac{n}{n+2} < 1$. Multivariate case Extending the Cramér-Rao bound to multiple parameters, define a parameter column vector

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_d]^T \in \mathbb{R}^d$$

with probability density function $f(x; \boldsymbol{\theta})$ which satisfies the two regularity conditions below. The Fisher information matrix is a $d \times d$ matrix with element $I_{m,k}$ defined as

$$I_{m,k} = E \left[\frac{\partial}{\partial \theta_m} \log f(x; \boldsymbol{\theta}) \frac{\partial}{\partial \theta_k} \log f(x; \boldsymbol{\theta}) \right] = -E \left[\frac{\partial^2}{\partial \theta_m \partial \theta_k} \log f(x; \boldsymbol{\theta}) \right]$$

Let $\mathbf{T}(X)$ be an estimator of any vector function of parameters, $\mathbf{T}(X) = (T_1(X), \dots, T_d(X))^T$, and denote its expectation vector $E[\mathbf{T}(X)]$ by $\boldsymbol{\psi}(\boldsymbol{\theta})$. The Cramér-Rao bound then states that the covariance matrix of $\mathbf{T}(X)$ satisfies

$$\text{cov}_{\boldsymbol{\theta}}(\mathbf{T}(X)) \geq \frac{\partial \boldsymbol{\psi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} [I(\boldsymbol{\theta})]^{-1} \left(\frac{\partial \boldsymbol{\psi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T$$

Problema 8

Enunciado: Demostrar:

$$J_{ik}(\vec{\theta}) = E \left[\frac{\partial}{\partial \theta_i} \log p(x | \vec{\theta}) \frac{\partial}{\partial \theta_k} \log p(x | \vec{\theta}) \right]$$

$$= -E \left[\frac{\partial^2}{\partial \theta_i \partial \theta_k} \log p(x | \vec{\theta}) \right]$$

Resolucion: se tiene que

$$\frac{\partial^2}{\partial \theta_i \partial \theta_k} \log p(x | \vec{\theta}) = \frac{\partial_{\theta_i} \partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})}$$

$$- \left(\frac{\partial_{\theta_i} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right) \left(\frac{\partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right)$$

y que

$$E \left[\frac{\partial_{\theta_i} \partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right] = \partial_{\theta_i} \partial_{\theta_k} \int_{\mathbb{R}} p(x | \vec{\theta}) dx = \partial_{\theta_i} \partial_{\theta_k} 1 = 0$$

por lo que

$$E \left[\frac{\partial^2}{\partial \theta_i \partial \theta_k} \log p(x | \vec{\theta}) \right] = E \left[\frac{\partial_{\theta_i} \partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right]$$

$$- E \left[\left(\frac{\partial_{\theta_i} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right) \left(\frac{\partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right) \right]$$

$$= -E \left[\left(\frac{\partial_{\theta_i} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right) \left(\frac{\partial_{\theta_k} p(x | \vec{\theta})}{p(x | \vec{\theta})} \right) \right]$$

$$= -E \left[\left(\partial_{\theta_i} \log p(x | \vec{\theta}) \right) \left(\partial_{\theta_k} \log p(x | \vec{\theta}) \right) \right]$$

$$= -J_{ik}(\vec{\theta})$$

Problema 9

Enunciado: **Resolucion:**

$$\vec{\vartheta}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} \frac{t}{2} \\ \frac{t}{2} \end{pmatrix}, \quad \dot{\vec{\vartheta}}(t) = \begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$Dist(\vec{\vartheta}^1, \vec{\vartheta}^2) = \int_0^1 \sqrt{\vec{\vartheta}^T J(\vec{\vartheta}) \vec{\vartheta}} dt$$

$$J_{ik}(\vartheta) = -E [\partial_{q_1} \log P(x | q_1, q_2) \partial_{q_2} \log P(x | q_1, q_2)]$$

$$\partial_{q_1} \log P(x | q_1, q_2) = \frac{\partial_{q_1} P(x | q_1, q_2)}{P(x | q_1, q_2)} = \begin{cases} \frac{-1}{1 - q_1 - q_2}, x = 0 \\ \frac{1}{q_1}, x = 1 \\ 0, x = 2 \end{cases}$$

$$\partial_{q_2} \log P(x | q_1, q_2) = \frac{\partial_{q_2} P(x | q_1, q_2)}{P(x | q_1, q_2)} = \begin{cases} \frac{-1}{1 - q_1 - q_2}, x = 0 \\ 0, x = 1 \\ \frac{1}{q_2}, x = 2 \end{cases}$$

$$J_{11}(\vartheta) = \frac{1}{1 - q_1 - q_2} + \frac{1}{q_1}$$

$$J_{22}(\vartheta) = \frac{1}{1 - q_1 - q_2} + \frac{1}{q_2}$$

$$J_{12}(\vartheta) = J_{21}(\vartheta) = \frac{1}{1 - q_1 - q_2}$$

$$J \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{bmatrix} \frac{1}{1 - q_1 - q_2} + \frac{1}{q_1} & \frac{1}{1 - q_1 - q_2} \\ \frac{1}{1 - q_1 - q_2} & \frac{1}{1 - q_1 - q_2} + \frac{1}{q_2} \end{bmatrix}$$

$$dist = \frac{1}{4} \sqrt{\frac{4}{1 - q_1 - q_2} + \frac{1}{q_1} + \frac{1}{q_2}}$$