Project details

1 Introduction

Scattering problems

Assume that an incident harmonic wave of frequency k is sent to an obstacle, or a material heterogeneity (an airplane in the sky, a crack in a material, a tumor in the body...). The presence of the obstacle generates a scattered field propagating away from this heterogeneity. The shape and intensity of this scattered field depends essentially on the location, geometry and material properties of the scatterer. In many contexts it is fundamental to predict by numerical computation, with high fidelity and in a short time, the scattered field from the *a priori* knowledge of the scattering object and incident wave.

Scattering problems arise for several different types of waves, by increasing level of complexity:

- Acoustics, sound waves in dimensions 2 and 3.
- Electromagnetism, Maxwell equations
- Linear elasticity.

Usually, practical applications involve the last two kinds of waves, at high frequency, and obstacles with complex geometries.

Integral equations

Among the standard mathematical tools most adapted to this task are the formalism of integral equations and the Galerkin boundary element method (BEM). Reference monographs include e.g. [23, 26]). The discretization of integral equations by BEM leads to dense linear systems, which, in real-life applications, often have more than 10^6 unknowns. This makes it prohibitive to solve the linear system by a direct method such as LU factorization, and instead, it is classical to use iterative methods such as GMRES [25].

For the so-called first-kind integral equations, powerful mesh-related convergence results are well-known, but the linear systems are notoriously ill-conditioned, causing the iterative solvers to progress very slowly. Active research has been devoted to this issue in the past 30 years. On the one hand, some teams have targeted the iteration (matrix-vector product) time by developing compression and acceleration methods e.g. Fast Multipole Methods [18] and H-matrices [19]. On the other hand, other teams have focused on finding suitable preconditioners. In this branch, [28] was a pioneer work.

Second-kind integral equations on the other hand are known to enjoy better stability with respect to discretization parameters, and for this reason, in many instances, researchers have tried to avoid the use of first-kind integral equations despite their simplicity. Nevertheless, the approximation properties of the second-kind formulations are less clear, and they degenerate for complex scatterers such as screens.

2 Recent advances and open problems

Let us survey some recent advances in the field of integral equations, and related open problems.

Scattering by complex obstacles

The obstacles become complex when they are not smooth domains or Lipschitz domains, a class of scatterers that has become very classical with [23]. Problems arise when singularities are present (edges, conical points). There is a particular attention devoted to screens and cracks, which are the first example of non-Lipschitz domains [8, 17, 21, 22, 24]. In this case, the formulation of well-posed integral equations in a proper mathematical framework is well-understood since [27, 29], and the recent advances lie in the proposition of more efficient numerical methods, including high-order approximation schemes and provably "optimal preconditioners". Additionally, some more challenging geometries are considered, including multi-screens [15, 16] and fractal screens [13]. In those cases, the achievement was to obtain well-posed integral equations and study their theoretical properties. For those complex geometries, many natural questions remain open, e.g. convergence of the numerical methods, singularity analysis and efficient preconditioning.

Dependance in the frequency

Although optimal preconditioners are by now available for scattering problems by Lipschitz domains and screens, [28, 20], it is a difficult matter to analyze the preconditioning performance in terms of the frequency number. The English school has been concerned with theoretical asymptotic analysis of the condition number of the classical integral equations without preconditioners [14] and the use of high-frequency solutions as trial functions in the Galerkin setting to improve the convergence rate [12]. Besides the popular Calderón preconditioners [10, 11], which have recently been adapted to the screen context [7] "analytic preconditioners" based on pseudo-differential calculus which exhibit good frequency robustness [2] were proposed, which entertain a close link with regularized combined-field integral equations [9]. Although the frequency robustness properties are clear and extensively surveyed e.g. [5], there is no mathematical theory adapted to the study of this asymptotic robustness as to now. There is a feeling in the community that pseudo-differential analysis, which is often involved in constructing those robust preconditioners, should also play a role in providing quantitative estimates for this frequency dependence buch such desirable analysis has not been achieved yet.

Own work

During my PhD thesis, I have contributed to the research on compression and acceleration methods for the linear systems coming from boundary element methods [3]. On the other hand, I worked on new high-frequency preconditioners for acoustics integral equations on screens in dimension 2. The first publication on this topic [1] presents the analytical framework and numerical testing. The new preconditioners are analogs of those of [2] and present similar frequency robustness, but this time on a screen context. A significant advantage of the method is that the preconditioners obtained are "quasi-local" in the sense that their use involve only sparse matrix, as opposed to Calderón preconditioners which are dense and require an additional compression steps. On the other hand, [4] is a more in-depth theoretical paper in which I develop a new type of pseudo-differential calculus on open curves that was used to design the preconditioners introduced in the first paper. This pseudo-differential calculus features the possibility of computing symbols of the layer potentials up to arbitrary orders and is probably a first step in assessing the frequency robustness of this approach.

3 Goals and research plan

Frequency-robust preconditioners on 3D screens

Although the numerical method published in [1] is only adapted to 2-dimensional contexts, generalizations to 3-dimensions seem possible and have been attempted in the last chapter of my thesis, with promising results. I will continue my research on this topic in order to make a publication for scalar scattering problems by screens in 3D. In turn, a natural extension to electromagnetic scattering will be possible, as a natural combination of my work and the host laboratory [22].

Conformal preconditioners

Another very natural generalization of my thesis work is the treatment of polygonal scatterers in 2D. We have recently found a mathematical identity that draws a close link between preconditioners and conformal mappings, which hints at completely new and powerful preconditioners adapted to corner singularities. A middle-term objective of this research project will be to analyze rigorously this idea and implement it efficiently. The expected output is a method that completely solves the question of polygons in 2D at any frequency, which would be a significant achievement in the domain. In addition, it might be possible to transport the pseudo-differential to the polygon via the conformal mapping, which would be a very interesting novelty.

This work would also open some possible breakthrough for 3D singularities such as wedges, which are omnipresent in practical applications. Exploring this idea is a long-term objective of this project.

Integral equations on multi-screens

One of the most promising ideas for this research is the possibility of formulating "combined-field" integral equations for multi-screens. Combined field formulations are well-known for smooth scatterers and have been introduced more than 50 years ago [6] as a way to guarantee the well-posedness of the equation at any wave frequency. For screens, combined field integral equations have been abandoned, because they degenerate when the obstacle no longer has interior. However, we have recently discovered that the formalism of multi-screens restores a "virtual" interior to the obstacle, and a combined-field equation can be formulated mathematically without difficulty. Preliminary numerical simulations have shown the power of this idea and it will be an immediate objective of this project to implement this idea and analyze it rigorously. This would lead without doubts to an important publication in our field.

Lastly, after his thesis, the applicant has done a 6 month post-doctorate with Xavier Claeys, a former student of the host professor, on the topic of preconditioning integral equations on multi-screens using a domain decomposition approach. The resulting method is expected to be very robust and versatile. This collaboration will be continued during the post-doc and eventually lead to a publication.

4 Timetable

A timetable of the research project can be proposed as follows:

- Submission of the paper on preconditioners for 3D screens for scalar waves by December 2020.
- Submission of the paper on preconditioners for 3D screens for electromagnetic waves by March 2021.
- Submission of the conformal preconditioners paper by July 2021.
- In parallel of those, work on the combined-field multi-screens formulations, expected publication by end of 2021.
- Also in parallel, continued work on the additive Schwarz preconditioners and publication before end of 2021.

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