AN INTEGRAL EQUATION FORMULATION FOR A BOUNDARY VALUE PROBLEM OF ELASTICITY IN THE DOMAIN EXTERIOR TO AN ARC*

George C. Hsiao, Ernst P. Stephan and Wolfgang L. Wendland

Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19711, U.S.A.
Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, U.S.A., and
Fachbereich Mathematik, Technische Hochschule Darmstadt, D-6100 Darmstadt, FRG.

Abstract: We consider here a Dirichlet problem for the two-dimensional linear elasticity equation in the domain exterior to an open arc in the plane. It is shown that the problem can be reduced to a system of boundary integral equations with the unknown density function being the jump of stresses across the arc. Existence, uniqueness as well as regularity results for the solution to the boundary integral equations are established in appropriate Sobolev spaces. In particular, asymptotic expansions concerning the singular behavior for the solution near the tips of the arc are obtained. By adding special singular elements to the regular splines as test and trial functions, an augmented Galerkin procedure is used for the corresponding boundary integral equations to obtain a quasi-optimal rate of convergence for the approximate solutions.

1. Introduction

This paper extends the results of the recent works [17][23][25][26] by the authors on the crack and screen problems for the Laplace and Helmholtz equation as well as on the exterior elasticity problems with a regular smooth boundary. Throughout the paper, let Γ be an open arc in the plane $\mathbb{R}^2$. We consider here the boundary value problem consisting of the linear elasticity equation for the displacement field $\underset{\sim}{u}$:

$$\mu\Delta\underset{\sim}{u} + (\lambda+\mu) \text{ grad div } \underset{\sim}{u} = \underset{\sim}{0} \quad \text{in} \quad \Omega_\Gamma = \mathbb{R}^2 \setminus \bar{\Gamma} \tag{E}$$

together with the boundary condition

$$\underset{\sim}{u}|_\Gamma = \underset{\sim}{g} \,, \tag{B}$$

* This work was supported by the "Alexander von Humboldt-Stiftung", FRG.

where $\mu > 0$ and $\lambda > -\mu$ are given Lâme constants [10], and $\underset{\sim}{g}$ is a prescribed smooth function. In addition we assume that $\underset{\sim}{u}-\underset{\sim}{r}$ is *regular* at infinity. Following [17] [19], by this we mean

$$D^{\alpha}(\underset{\sim}{u}-\underset{\sim}{r}) = O(|\underset{\sim}{x}|^{-\alpha-1}) \qquad \alpha = 0,1, \quad \text{as} \quad |x| \to \infty \tag{C}$$

with $D = \frac{\partial}{\partial x_i}$. More precisely, let us represent the rigid motion $\underset{\sim}{r}(x)$ in the form:

$$\underset{\sim}{r}(x) = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3(x_2\hat{e}_1 - x_1\hat{e}_2)$$

where $\hat{e}_i$ denotes the unit vectors in $\mathbb{R}^2$ and $\omega_i^{\prime}s$ are unknown constants. As indicated in [14] [17], condition (C) implies that

$$\int_{\Gamma} [T(\underset{\sim}{u})]ds_y = \underset{\sim}{0} \tag{C_1}$$

and in order to ensure the uniqueness, we further impose the equilibrium condition of vanishing total momentum

$$\int_{\Gamma} (y_2\hat{e}_1 - y_1\hat{e}_2) \cdot [T(\underset{\sim}{u})]ds_y = 0 \tag{C_2}$$

which will become transparent later (see (2.12)). We note that condition (C_2) will not be needed in the case when ω_3 is given [17]. Here in the formulation, as will be seen, $[T(\underset{\sim}{u})]$ stands for the jump of traction $T(\underset{\sim}{u})$ across Γ ,

$$T(\underset{\sim}{u}) := 2\mu \frac{\partial}{\partial\hat{n}} \underset{\sim}{u} + \lambda\hat{n} \operatorname{div} \underset{\sim}{u} + \mu\hat{n} \times \operatorname{curl} \underset{\sim}{u} \tag{1.1}$$

with $\hat{n}$ being the unit normal to Γ , and $\operatorname{curl} \underset{\sim}{u} := \operatorname{curl} (u_1,u_2,0)$.

In the following we shall refer to the problem defined by (E),(B), (C_1) and (C_2) as the crack problem (P). Such crack problems arise if e.g. an inlet of rigid material is immersed at Γ into the elastic material occupying Ω_Γ .

Our aim is to develop a solution procedure for (P) by making use of an integral equation method which allows us to obtain the explicit singular behavior of the "stress" near the tips of Γ . Following [17] [26], we reduce the problem (P) to a system of boundary integral equations of the first kind [11][13][28] with the jump of traction across Γ as the unknown. These boundary integral equations are derived by the "direct approach" based on the Betti formula. By using the method of local Mellin transform as in [3]-[8], and the calculus of pseudodifferential operators [9][22], we establish existence, uniqueness and regularity results for the solution of our boundary integral equations. In particular, we are able to obtain appropriate asymptotic expansions for the jump of tractions near the tips of Γ . The latter provides us useful informations concerning numerical treatment such as the Galerkin scheme for our boundary integral equations. In fact, in our augmented boundary element method, we use, as in [20] [26] [27] [29] in addition to the regular finite elements, appropriate singular elements concentrated near the tips and improve significantly the asymptotic convergence rates of our approximate solutions [15].

It should be emphasized that since our boundary integral equations are derived directly from the Betti formula, physically, the boundary charges are precisely the jumps of tractions across Γ . From our boundary element method using augmented test and trial function spaces with the appropriate singular elements, we are able to compute both approximate boundary charges and the stress intensity factors simultaneously. Hence, our asymptotic error estimates in [15] include explicit estimates for the stress intensity factors, as well.

In this paper, we shall present only the main idea and some of the results and leave the details to [15].

2. Integral Representation

We begin with the variational formulation for the problem (P). We then derive the integral representation for the variational solution by the direct method based on the Betti formula. In order to characterize the variational solution of (P), we introduce the function space $H_c^1(\Omega_\Gamma)$, the completion of all C^∞-functions $\underset{\sim}{f}(x)$ of the form

$$\underset{\sim}{f}(x) = \underset{\sim}{f}_o(x) + \underset{\sim}{r}(x) \tag{2.1}$$

with respect to the norm $\|\cdot\|_{1,c}$ defined by

$$\|\underset{\sim}{f}\|_{1,c} := \{\int_{\Omega_\Gamma} E(\underset{\sim}{f},\underset{\sim}{f})dx + \int_\Gamma |\underset{\sim}{f}|^2 ds\}^{1/2} \quad , \tag{2.2}$$

where $\underset{\sim}{f}_o$ is regular at infinity and $\underset{\sim}{r}$ denotes a rigid motion of the form:

$$\underset{\sim}{r}(x) = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3(x_2\hat{e}_1 - x_1\hat{e}_2) =: M(x)\underset{\sim}{\omega} \tag{2.3}$$

with $\underset{\sim}{\omega} = (\omega_1,\omega_2,\omega_3)^T$ and $M(x)$ the corresponding 2×3 matrix. Here

$$\begin{aligned} E(\underset{\sim}{f},\underset{\sim}{g}) := (\lambda+\mu)\, \operatorname{div} \underset{\sim}{f}\, \operatorname{div} \underset{\sim}{g} &+ \frac{1}{2}\mu \sum_{j\neq k}^{2} \left(\frac{\partial f_j}{\partial x_k} + \frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial g_j}{\partial x_k} + \frac{\partial g_k}{\partial x_j}\right) \\ &+ \frac{1}{2}\mu \sum_{j,k=1}^{2} \left(\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j}\right) \end{aligned} \tag{2.4}$$

is a bilinear form for the derivatives of $\underset{\sim}{f}$ and $\underset{\sim}{g}$ (see [19]). In addition, we denote by $\mathring{H}_c^1(\Omega_\Gamma)$ the closed suspace of $H_c^1(\Omega_\Gamma)$ such that

$$\mathring{H}_c^1(\Omega_\Gamma) = \{\underset{\sim}{f} \in H_c^1(\Omega_\Gamma) \mid \underset{\sim}{f}|_\Gamma = \underset{\sim}{0}\} .$$

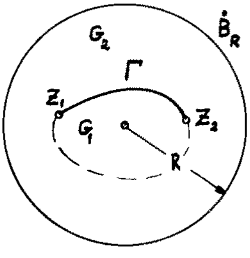
For the equivalent variational formulation of the crack problem (P), let $\underset{\sim}{h} \in H_c^1(\Omega_\Gamma)$ be an extension of the boundary values of $\underset{\sim}{g} \in H^{1/2}(\Gamma)$ with $\underset{\sim}{h}|_\Gamma = \underset{\sim}{g}$, and be regular at infinity. Then the problem (P) is equivalent to the *variational problem: for given* $\underset{\sim}{h} \in H_c^1(\Omega_\Gamma)$, *find a function* $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ *such that* $\underset{\sim}{u}-\underset{\sim}{h} \in \mathring{H}_c^1(\Omega_\Gamma)$ *and*

$$B(\underset{\sim}{u},\underset{\sim}{\phi}) := \int_{\Omega_\Gamma} E(\underset{\sim}{u},\underset{\sim}{\phi})\,dx = 0 \tag{2.5}$$

for all $\underset{\sim}{\phi} \in \mathring{H}_c^1(\Omega_\Gamma)$. In view of Korn's inequality [10] and the Riesz-Fréchet representation theorem it is easy to see that there exists exactly one solution $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ of the problem (P) given by the variational problem.

In order to derive an integral representation of the variational solution of the problem (P), as in [26], we extend Γ to an arbitrary smooth simple closed curve $\dot{G}_1$, and denote by G_1 the bounded domain inside $\dot{G}_1$. We use the notation $[v]$ to denote the jump $v_- - v_+$ of a function v across $\dot{G}_1$. Here the subscripts $-, +$ denote the limits taken from G_1 and $\mathbb{R}^2\setminus\overline{G}_1$ respectively. For later use, let B_R be a circle with radius R sufficiently large enough to enclose $\overline{G}_1$. The domain bounded by $\dot{G}_1$ and the boundary $\dot{B}_R$ of B_R will be denoted by G_2. The boundary $\dot{G}_2$ of G_2 consists of Γ together with $\dot{G}_1\setminus\Gamma$ and $\dot{B}_R$ (see Figure 1).

Figure 1:



In what follows, let $H^s(\dot{G}_1)$ be defined as the trace of $H^{s+1/2}(\mathbb{R}^2)$ for $s > 0$, as $L^2(\dot{G}_1)$ for $s = 0$, and as the dual space of $H^{-s}(\dot{G}_1)$ for $s < 0$. For $s \geq 0$, $H^s(\Gamma)$ denotes the usual trace space of $H^s(\dot{G}_1)$ on Γ and $\tilde{H}^s(\Gamma)$ is defined by

$$\tilde{H}^s(\Gamma) := \{\underset{\sim}{f} = \underset{\sim}{f}'|_\Gamma : \underset{\sim}{f}' \in H^s(\dot{G}_1) \ , \ \underset{\sim}{f}'|_{\dot{G}_1\setminus\Gamma} = \underset{\sim}{0}\}$$

equipped with the topology of $H^s(\dot{G}_1)$. For $s < 0$, we define

$$H^s(\Gamma) := (\tilde{H}^{-s}(\Gamma))' \text{ and } \tilde{H}^s(\Gamma) := (H^{-s}(\Gamma))'$$

by duality with respect to the $L^2(\Gamma)$ scalar product. It is clear that from the definition, for $s > 0$

$$\tilde{H}^{-s}(\Gamma) = \{\underset{\sim}{f} \in H^{-s}(\dot{G}_1) \mid \operatorname{supp}(\underset{\sim}{f}) \subset \bar{\Gamma}\}$$

which is also the completion of $C_0^\infty(\Gamma)$ with respect to the norm of $H^{-s}(\dot{G}_1)$ (see [1], [12, Theorem 2.5.1, p. 51 ff]).

We now state some properties concerning the solution $\underset{\sim}{u} \in H_c^2(\Omega_\Gamma)$ of (2.5):

<u>Lemma 2.1.</u> *Let* $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ *be the solution of* (2.5). *Then* $\underset{\sim}{u}|_{\dot{G}_1} \in H^{1/2}(\dot{G}_1)$ *and* $\underset{\sim}{u}|_\Gamma \in H^{1/2}(\Gamma)$. *For the traction* $T(\underset{\sim}{u})$, *we have*

$$T(\underset{\sim}{u})|_{\dot{G}} \in H^{-1/2}(\dot{G}_1) \quad \textit{and} \quad T(\underset{\sim}{u})|_\Gamma \in (\tilde{H}^{1/2}(\Gamma))' \ ;$$

moreover, if we denote by $[T(\underset{\sim}{u})] = T(\underset{\sim}{u})_- - T(\underset{\sim}{u})_+$ *the jump of the traction acoross* Γ, *then we have*

$$[T(\underset{\sim}{u})]|_\Gamma \in \tilde{H}^{-1/2}(\Gamma) \ .$$

Here in the definition of $T(\underset{\sim}{u})$ *(see* (1.1)*) the exterior normal derivative to* $\dot{G}_1$ *is used.*

The proof of Lemma 2.1 is similar to the two-dimensional screen problem of the Laplacian. It is based on the trace theorem [21] and Weyl's lemma; and we omit the details here [15].

For the integral representation of the solution $\underset{\sim}{u}$, we need the fundamental solution of (E), the Kelvin matrix (see e.g. [2])

$$\underset{\approx}{\gamma}(y,x) = \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)} \{\log \frac{1}{|x-y|} \underset{\approx}{I} + \frac{\lambda+\mu}{\lambda+3\mu} \frac{(x-y)(x-y)^T}{|x-y|^2}\} , \qquad (2.6)$$

and the corresponding stress matrix on $\dot{G}_j$ [17]

$$\underset{\approx}{\gamma}_1(y,x) := (T_y \underset{\approx}{\gamma}(y,x))^T$$
$$= \frac{\mu}{8\pi(\lambda+2\mu)} \{I + \frac{2(\lambda+\mu)}{\mu|x-y|^2}(x-y)(x-y)^T) \frac{\partial}{\partial n_y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial s_y}\} \log \frac{1}{|x-y|} , \tag{2.7}$$

where ^{T} stands for the transpose. By applying the Betti formula to the variational solution $\underset{\sim}{u}$ in G_i , $i = 1$ and 2 , we obtain

$$\alpha_j \underset{\sim}{u}(x) = - \int_{\dot{G}_j} \underset{\approx}{\gamma}_1(y,x)\underset{\sim}{u}(y)ds_y + \int_{\dot{G}_j} \underset{\approx}{\gamma}(y,x) \; T(\underset{\sim}{u})(y)ds_y \tag{2.8}$$

for fixed $x \in G_1$ with $\alpha_1 = 1$ and $\alpha_2 = 0$. These representations hold for $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$, since from Lemma 2.1 , we have $\underset{\sim}{u}_{|\dot{G}_j} \in H^{1/2}(\dot{G}_j)$ and $T(\underset{\sim}{u})_{|\dot{G}_j} \in H^{-1/2}(\dot{G}_j)$. Now from (2.8) it follows easily that

$$\underset{\sim}{u}(x) = - \int_{\dot{B}_R} \underset{\approx}{\gamma}_1(y,x)\underset{\sim}{u}(y)ds_y + \int_{\dot{B}_R} \underset{\approx}{\gamma}(y,x)T(\underset{\sim}{u})(y)ds_y + \int_\Gamma \underset{\approx}{\gamma}(y,x)[T(\underset{\sim}{u})](y)ds_y \tag{2.9}$$

for fixed $x \in G_1$. We note that $[T(\underset{\sim}{u})]_{|\dot{G}_1 \setminus \Gamma} = 0$.
If $\underset{\sim}{u} - M(x)\underset{\sim}{\omega}$ is regular at infinity, one can show that the first two terms tend to $M(x)\underset{\sim}{\omega}$ as $R \to \infty$ [17]. Thus, we arrive at the representation:

$$\underset{\sim}{u}(x) = \int_\Gamma \underset{\approx}{\gamma}(y,x)[T(\underset{\sim}{u})](y)ds_y + M(x)\underset{\sim}{\omega} \quad , \qquad x \in G_1 . \tag{2.10}$$

Clearly in a similar manner, one can show that the same representation holds for $x \in G_2$ with arbitrary R .

We summarize the foregoing results in the following theorem.

Theorem 2.2. *Suppose* $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ *is a variational solution of* (P) . *Then* $\underset{\sim}{u}$ *admits the integral representation:*

$$\underset{\sim}{u}(x) = \int_\Gamma \underset{\approx}{\gamma}(y,x)[T(\underset{\sim}{u})]ds_y + M(x)\underset{\sim}{\omega} , \qquad x \in \mathbb{R}^2 \setminus \overline{\Gamma} \tag{2.11}$$

where $[T(\underset{\sim}{u})]_{|\Gamma} \in \tilde{H}^{-1/2}(\Gamma)$ *is the jump of traction across* Γ *, satis-*

fying the condition (C_1) *, and* $M(x)\underset{\sim}{\omega}$ *corresponds to the rigid motion of* $\underset{\sim}{u}$ *at infinity.*

We remark that from the representation (2.11), one may derive the work formula as in [17]:

$$\int_{\Omega_\Gamma} E(\underset{\sim}{w},\underset{\sim}{u})\,dx = -\int_\Gamma \underset{\sim}{w}\cdot[T(\underset{\sim}{u})]ds + \int_\Gamma ((M(y)\underset{\sim}{\Omega}(\underset{\sim}{w}))\cdot[T(\underset{\sim}{u})]ds_y \tag{2.12}$$

for $\underset{\sim}{w} \in H_c^1(\Omega_\Gamma)$, where $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ is a variational solution of (P) and $M(y)\underset{\sim}{\Omega}(\underset{\sim}{w})$ corresponds to the rigid motion of $\underset{\sim}{w}$ at infinity. This work formula indicates that one indeed needs the additional condition such as (C_2) in order to ensure the uniqueness of the solution of the problem (P) .

3. Boundary Integral Equations.

We now reduce the variational boundary value problem of section 2 to equivalent boundary integral equations for the jump of traction, $[T(\underset{\sim}{u})]$ across Γ . This can be achieved from the integral representation (2.11) by letting x tend to Γ . In fact, the following result can be established.

Theorem 3.1. *Let* $\underset{\sim}{g} \in H^{1/2}(\Gamma)$ *be given. Then* $\underset{\sim}{u} \in H_c^1(\Omega_\Gamma)$ *is the variational solution of* (P) *if and only if* $[T(\underset{\sim}{u})]_{|\Gamma} \in \tilde{H}^{-1/2}(\Gamma)$, $\underset{\sim}{\omega} \in \mathbb{R}^3$ *solve the integral equations*

$$\begin{aligned} &\int_\Gamma \underset{\approx}{\gamma}(y,x)[T(\underset{\sim}{u})](y)ds_y + M(x)\underset{\sim}{\omega} = \underset{\sim}{g} \\ &\int_\Gamma [T(\underset{\sim}{u})]ds = \underset{\sim}{0} \quad \textit{and} \quad \int_\Gamma \underset{\sim}{m}_3(y)\cdot[T(\underset{\sim}{u})]ds = \underset{\sim}{0}\ . \end{aligned} \tag{3.1}$$

for $x \in \Gamma$ *, where* $\underset{\sim}{m}_3(y) := y_2\hat{e}_1 - y_1\hat{e}_2$.

The necessity follows clearly from the derivation of the integral representation (2.11). It remains only to show the last condition in (3.1). This follows immediately from (2.12) together with the condition $\int_\Gamma [T(\underset{\sim}{u})]ds = \underset{\sim}{0}$, if $\underset{\sim}{w}$ in (2.12) is replaced by $\underset{\sim}{\phi} \in \mathring{H}_c^1(\Omega_\Gamma)$. For the sufficiency, we refer to [15] for the details.

In order to guarantee that the system (1.3) is always solvable, we now need some properties concerning the integral operators in (3.1). For convenience, let us denote by V_Γ the boundary integral operator defined by

$$V_\Gamma \underset{\sim}{\phi}(x) := \int_\Gamma \underset{\approx}{\gamma}(y,x)\underset{\sim}{\phi}(y)ds_y , \qquad x \in \Gamma \tag{3.2}$$

and by Λ_Γ , the functional defined by

$$\Lambda_\Gamma \underset{\sim}{\phi} := \int_\Gamma M^T(y)\underset{\sim}{\phi}(y)ds_y \tag{3.3}$$

where $M^T(y)$ is the transpose of the matrix $M(y)$ in (2.3) , that is,

$$M^T(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y_2 & -y_1 \end{pmatrix} . \tag{3.4}$$

We also introduce the matrix operator A_Γ :

$$A_\Gamma(\underset{\sim}{\phi},\underset{\sim}{\omega}) := \begin{pmatrix} V_\Gamma & M \\ \Lambda_\Gamma & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\phi} \\ \underset{\approx}{\omega} \end{pmatrix} . \tag{3.5}$$

Then, as one expected from [26] and [29], the operator A_Γ here also satisfies a Gårding inequality in the energy space $\tilde{H}^{-1/2}(\Gamma)\times\mathbb{R}^3$. This means that A_Γ is a Fredholm operator of index zero [16] [18] [24] and hence together with the uniqueness of the solution of (3.1), it implies that (3.1) is always *uniquely solvable.* In the theorem below, the relevant mapping properties of the boundary integral operator A_Γ are included. These properties will be established in [15].

<u>Theorem 3.2.</u> *The matrix operator* A_Γ *and its adjoint* A_Γ^* *with respect to the duality*

$$<(\underset{\sim}{\psi},\underset{\sim}{\kappa}),(\underset{\sim}{\phi},\underset{\sim}{\omega})>_{L^2(\Gamma)\times\mathbb{R}^3} := (\underset{\sim}{\psi},\underset{\sim}{\phi})_{L^2(\Gamma)} + \underset{\approx}{\kappa}\cdot\underset{\approx}{\omega}$$

both are continuous and bijective mappings:

$$\tilde{H}^s(\Gamma) \times \mathbb{R}^3 \to H^{s+1}(\Gamma) \times \mathbb{R}^3 \quad \text{for} \quad -1 < s < 0 .$$

Moreover, the operator A_Γ *satisfies a Gårding inequality in* $\tilde{H}^{-1/2}(\Gamma)\times\mathbb{R}^3$, i.e. *there exists a constant* $\gamma > 0$ *and a continuous mapping*

$$C_\Gamma : \tilde{H}^{-1/2}(\Gamma)\times\mathbb{R}^3 \to H^{1/2+\varepsilon}(\Gamma)\times\mathbb{R}^3 \qquad \textit{for some} \quad \varepsilon > 0$$

such that the inequality

$$<(A_\Gamma + C_\Gamma)(\underset{\sim}{\phi},\underset{\sim}{\omega}),(\underset{\sim}{\phi},\underset{\sim}{\omega})>_{L^2(\Gamma)\times\mathbb{R}^3} \geq \gamma\{\|\underset{\sim}{\phi}\|^2_{\tilde{H}^{-1/2}(\Gamma)} + |\underset{\sim}{\omega}|^2\}$$

holds for all $(\underset{\sim}{\phi},\underset{\sim}{\omega}) \in \tilde{H}^{-1/2}(\Gamma) \times \mathbb{R}^3$.

We now come to the point of our main concern - the singularity of $[T(\underset{\sim}{u})]$ near the crack tips z_i . As in the case of potential theory [23] [25] [26], the variational solution $\underset{\sim}{u}$ of (2.5) has in general umbounded traction $[T(\underset{\sim}{u})]$ even for C^∞-data. Hence, one will not be able to improve the approximation of $[T(\underset{\sim}{u})]$ by the conventional constructive methods such as finite element or boundary element methods without any modifications, since for better approximations, one generally requires higher regularity of the exact solution $[T(\underset{\sim}{u})]$. For this purpose, it is best to decompose $[T(\underset{\sim}{u})]$ into special singular terms concentrated near the tips z_i and a regular remainder. In this way, as in [4] [20] [23] [25] [26] and [29], one may then augment the finite element spaces of test and trial functions with appropriate global singular elements, according to the special forms of singular terms in $[T(\underset{\sim}{u})]$, to improve the order of convergence of the approximations.

The following result concerns the asymptotic behavior of the exact solution $[T(\underset{\sim}{u})]$ of (3.1) near the crack tips. The analysis here follows [3]-[8] by using the Mellin transform technique, which gives us the the appropriate function spaces together with the exact form of the singularities at the tips.

<u>Theorem 3.3.</u> *For* $|\sigma| < \frac{1}{2}$ *, let* $g \in H^{s+\sigma}(\Gamma)$ *,* $s = \frac{3}{2}$ *or* $\frac{5}{2}$ *be given. Then the solution* $[T(\underset{\sim}{u})]_{|\Gamma} \in \tilde{H}^{-1/2}(\Gamma)$ *(and* $\underset{\sim}{\omega} \in \mathbb{R}^3$*) of the integral equations* (3.1) *admits the asymptotic representation form near the endpoints* $z_i \in \bar{\Gamma}$ *:*

(a) $$[T(\underset{\sim}{u})]_{|\Gamma} = \sum_{i=1}^{2} \underset{\sim}{\alpha}_i\, \rho_i^{-1/2}\chi_i + \underset{\sim}{\psi}_o \qquad \textit{for} \quad s = \frac{3}{2}$$

with $\psi_o \in \tilde{H}^{1/2+\sigma}(\Gamma)$, $\underset{\sim}{\alpha}_i \in \mathbb{R}^2$,

(b) $$[T(\underset{\sim}{u})]_{|\Gamma} = \sum_{i=1}^{2} (\underset{\sim}{\alpha}_i\rho_i^{-1/2} + \underset{\sim}{\beta}_i\rho_i^{1/2})\chi_i + \underset{\sim}{\psi}_i \qquad \textit{for} \quad s = \frac{5}{2}$$

with $\underset{\sim}{\psi}_1 = \tilde{H}^{3/2+\sigma}(\Gamma)$, $\underset{\sim}{\alpha}_i, \underset{\sim}{\beta}_i \in \mathbb{R}^2$.

Here ρ_i *denotes the distance between* $x \in \Gamma$ *and* z_i *, while* χ_i *is a* C^∞ *cut-off function such that* $0 \leq \chi_1 \leq 1$, $\chi_1 = 1$ *near* z_i

and $\chi_i = 0$ *elsewhere* , $i = 1,2$.

We remark that the coefficients $\underset{\sim}{\alpha}_i$ and $\underset{\sim}{\beta}_i$ in the above theorem are indeed the *stress intensity factors* as in the crack problem, since $T(\underset{\sim}{u})$ is the traction. We further comment that the regularity of the remainder term in $[T(\underset{\sim}{u})]_{|\Gamma}$ may be improved as the given data $\underset{\sim}{g}$ becomes smoother, however, in general the solution $[T(\underset{\sim}{u})]_{|\Gamma}$ possesses singularities and is always *unbounded* at the tips.

To incorporate the expansions of $[T(\underset{\sim}{u})]_{|\Gamma}$ into the augmented Sobolev spaces for the purpose of boundary element approximation, we need mapping properties of $[T(\underset{\sim}{u})]$ such as in Theorem 3.2. Let us begin with the following defintion:

<u>Definition 3.4.</u>

a) *For* $s < 1$ *, we define*

$$Z^s(\Gamma) := \{\underset{\sim}{\psi} = \sum_{i=1}^{2} \underset{\sim}{\alpha}_i \rho_i^{-1/2} \chi_i + \underset{\sim}{\psi}_o \mid \underset{\sim}{\alpha}_i \in \mathbb{R}^2 , \underset{\sim}{\psi}_o \in \tilde{H}^s(\Gamma)\}$$

equipped with

$$\|\underset{\sim}{\psi}\|_{Z^s(\Gamma)} := \begin{cases} \sum_{i=1}^{2} |\underset{\sim}{\alpha}_i| + \|\underset{\sim}{\psi}_o\|_{\tilde{H}^s(\Gamma)} & \text{for } 0 \le s < 1 \\ \|\underset{\sim}{\psi}\|_{\tilde{H}^s(\Gamma)} & \text{for } s < 0 \end{cases}$$

b) *For* $1 \le s < 2$ *, we define*

$$Z^s(\Gamma) := \{\underset{\sim}{\psi} = \sum_{i=1}^{2} (\underset{\sim}{\alpha}_i \rho_i^{-1/2} + \underset{\sim}{\beta}_i \rho_i^{1/2}) \chi_i + \underset{\sim}{\psi}_1 \mid \underset{\sim}{\alpha}_i, \underset{\sim}{\beta}_i \in \mathbb{R}^2 , \underset{\sim}{\psi}_1 \in \tilde{H}^s(\Gamma)\}$$

equipped with

$$\|\underset{\sim}{\psi}\|_{Z^s(\Gamma)} := \sum_{i=1}^{2} |\underset{\sim}{\alpha}_i| + |\underset{\sim}{\beta}_i| + \|\underset{\sim}{\psi}_1\|_{\tilde{H}^s(\Gamma)} .$$

These augmented spaces allow us to extend the mapping properties in Theorem 3.2 to higher order spaces. The following results are similar to those in [8],[26] and [29].

<u>Theorem 3.4.</u> *For fixed* $\sigma, |\sigma| < \frac{1}{2}$ *, the operator* A_Γ *defined by* (3.5) *possesses the following mapping properties:*

$$A_\Gamma : Z^{1/2+\sigma}(\Gamma)\times\mathbb{R}^3 \to H^{3/2+\sigma}(\Gamma)\times\mathbb{R}^3 \quad \textit{and}$$

$$A_\Gamma : Z^{3/2+\sigma}(\Gamma)\times\mathbb{R}^3 \to H^{5/2+\sigma}(\Gamma)\times\mathbb{R}^3$$

with

$$\{\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2,\underset{\sim}{\psi}_o,\underset{\sim}{\omega}\} \mapsto \begin{pmatrix} V_\Gamma(\sum_{i=1}^{2}\underset{\sim}{\alpha}_i\rho_i^{-1/2}\chi_i+\underset{\sim}{\psi}_o) + M\underset{\sim}{\omega} \\ \Lambda_\Gamma(\sum_{i=1}^{2}\underset{\sim}{\alpha}_i\rho_i^{-1/2}\chi_i+\underset{\sim}{\psi}_o) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{g} \\ \underset{\sim}{b} \end{pmatrix}$$

and (3.6)

$$\{\underset{\sim}{\alpha}_1,\underset{\sim}{\alpha}_2,\underset{\sim}{\beta}_1,\underset{\sim}{\beta}_2,\underset{\sim}{\psi}_1,\underset{\sim}{\omega}\} \mapsto \begin{pmatrix} V_\Gamma(\sum_{i=1}^{2}(\underset{\sim}{\alpha}_i\rho_i^{-1/2}+\underset{\sim}{\beta}_i\rho_i^{1/2})\chi_i+\underset{\sim}{\psi}_1) + M\underset{\sim}{\omega} \\ \Lambda_\Gamma(\sum_{i=1}^{2}(\underset{\sim}{\alpha}_i\rho_i^{-1/2}+\underset{\sim}{\beta}_i\rho_i^{1/2})\chi_i+\underset{\sim}{\psi}_1) \end{pmatrix} = \begin{pmatrix} \underset{\sim}{g} \\ \underset{\sim}{b} \end{pmatrix},$$

respectively, are continuous and bijective. Furthermore, there hold the corresponding à priori estimates:

$$\|\underset{\sim}{\psi}\|_{Z^{1/2+\sigma}(\Gamma)} + |\underset{\sim}{\omega}| \le c\{\|\underset{\sim}{g}\|_{H^{3/2+\sigma}(\Gamma)} + |\underset{\sim}{b}|\},$$

and

$$\|\underset{\sim}{\psi}\|_{Z^{3/2+\sigma}(\Gamma)} + |\underset{\sim}{\omega}| \le c\{\|\underset{\sim}{g}\|_{H^{5/2+\sigma}(\Gamma)} + |\underset{\sim}{b}|\} .$$

We remark that in particular, one may take $\underset{\sim}{b} = \underset{\sim}{0}$ in the above theorem. Then (3.6) coincides with the integral equation (3.1) with $[T(\underset{\sim}{u})]_{|\Gamma} = \sum_{i=1}^{2}\underset{\sim}{\alpha}_i\rho_i^{-1/2}\chi_i+\underset{\sim}{\psi}_o$ or $[T(\underset{\sim}{u})]_{|\Gamma} = \sum_{i=1}^{2}(\underset{\sim}{\alpha}_i\rho_i^{-1/2}+\underset{\sim}{\beta}_i\rho^{1/2})\chi_i+\underset{\sim}{\psi}_1$, depending on the given data $\underset{\sim}{g} \in H^{3/2+\sigma}(\Gamma)$ or $\underset{\sim}{g} \in H^{5/2+\sigma}(\Gamma)$. It is this system (3.6) that we have solved in [15] for $\underset{\sim}{\alpha}_1,\underset{\sim}{\psi}_o,\underset{\sim}{\omega}$ and $\underset{\sim}{\alpha}_i,\underset{\sim}{\beta}_i,\underset{\sim}{\psi}_1,\underset{\sim}{\omega}$ by an augmented boundary element method originally employed for a closed smooth curve [20] [27] [29]. In our augmented method we use besides regular splines for $\underset{\sim}{\psi}_o,\underset{\sim}{\psi}_1$, the special singular elements $\rho_i^{-1/2}\chi_i$, $\rho_i^{1/2}\chi_i$ as in Theorem 3.3 and 3.4. Our procedure has the advantage that we are able not only to obtain higher rates of convergence but also to compute the stress intensity factors simultaneously together with the approximate desired boundary charges $[T(\underset{\sim}{u})]_{|\Gamma}$. The details of our procedure and error estimates are available in [15] .

REFERENCES.

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York 1975.

[2] J.F. Ahner and G.C. Hsiao, On the two-dimensional exterior boundary-value problems of elasticity, SIAM J. Appl. Math. 31 (1976) pp. 677-685.

[3] M. Costabel, Boundary integral operators on curved polygons, Ann. Mat. Pura Appl. 33 (1983) pp. 305-326.

[4] M. Costabel and E. Stephan, Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation, Banach Center Publications, Warsaw, to appear (Preprint 593, FB Mathematik, TH Darmstadt 1981).

[5] M. Costabel and E. Stephan, Curvature terms in the asymptotic expansion for solutions of boundary integral equations on curved polygons, J. Integral Equations 5 (1983) pp. 353-371.

[6] M. Costabel and E. Stephan, A direct boundary integral equation method for transmission problems, to appear in J. Math. Anal. Appl. (Preprint 753, FB Mathematik, TH Darmstadt 1982).

[7] M. Costabel and E. Stephan, The method of Mellin transformation for boundary integral equations on curves with corners, in: Numerical Solution of Singular Integral Equations (ed. A. Gerasoulis,R.Vichnevetsky) IMACS, Rutgers Univ. (1984) pp. 95-102.

[8] M. Costabel, E. Stephan and W.L. Wendland, On the boundary integral equations of the first kind for the bi-Laplacian in a polygonal plane domain, Ann. Scuola Norm. Sup. Pisa, Ser. IV 10 (1983) pp 197-242.

[9] G.I. Eskin, Boundary Problems for Elliptic Pseudo-Differential Operators, Transl. of Math. Mon., American Math. Soc. 52, Providence, Rhode Island (1981).

[10] G. Fichera, Existence theorems in elasticity. Unilateral constraints in elasticity. - Handbuch der Physik (S. Flügge ed.) Berlin - Heidelberg - New York, 1972, VI a/2, pp. 347-424.

[11] G. Fichera, Linear elliptic equations of higher order in two independent variables and singular integral equations,in: Proc. Conf. Partial Differential Equations and Conf. Mechanics. University Wisconsin Press (1961) pp. 55-80.

[12] L. Hörmander, Linear Partial Differential Operators, Springer-Verlag, Berlin (1969).

[13] G.C. Hsiao and R.C. MacCamy, Solution of boundary value problems by integral equations of the first kind, SIAM Rev. 15 (1973) pp. 687-705.

[14] G.C. Hsiao, P. Kopp and W.L. Wendland, Some applications of a Galerkin-collocation method for boundary integral equations of the first kind, Math. Meth. Appl. Sci., 6 (1984), to appear.

[15] G.C. Hsiao, E. Stephan and W.L. Wendland, A boundary element approach to fracture mechanics, in preparation.

[16] G.C. Hsiao and W.L. Wendland, A finite element method for some integral equations of the first kind, J. Math. Anal. Appl. 58 (1977) pp. 449-481.

[17] G.C. Hsiao, W.L. Wendland, On a boundary integral method for some exterior problems in elasticity, to appear in Akad. Dokl. Nauk SSSR (Preprint 769, FB Mathematik, TH Darmstadt 1983).

[18] G.C. Hsiao, W.L. Wendland, The Aubin-Nitsche lemma for integral equations, J. Integral Equations 3 (1981) pp. 299-315.

[19] V.D. Kupradze, Potential methods in the Theory of Elasticity. Jerusalem, Israel Program Scientific Transl., 1965.

[20] U. Lamp, K,-T. Schleicher, E. Stephan and W.L. Wendland, Galerkin collocation for an improved boundary element method for a plane mixed boundary value problem, to appear in Computing.

[21] I.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications I, Berlin - Heidelberg - New York, Springer (1972).

[22] R. Seeley, Topics in pseudo-differential operators, in: Pseudo-Differential Operators (ed. L. Nirenberg) C.I.M.E., Cremonese, Roma (1969) pp. 169-305.

[23] E. Stephan, Boundary integral equations for mixed boundary value problems, screen and transmission problems in $\mathbb{R}^3$, Habilitationsschrift (Darmstadt) (1984).

[24] E. Stephan, W.L. Wendland, Remarks to Galerkin and least squares methods with finite elements for general elliptic problems, Lecture Notes Math. 564 Springer, Berlin (1976) pp. 461-471, Manuscripta Geodaesica 1 (1976) pp. 93-123.

[25] E. Stephan and W.L. Wendland, Boundary element method for membrane and torsion crack problems, Computer Meth. in Appl. Eng., 36 (1983) pp. 331-358.

[26] E. Stephan and W.L. Wendland, An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems, Applicable Analysis, to appear (Preprint 802, FB Mathematik, TH Darmstadt 1984).

[27] W.L. Wendland, I. Asymptotic convergence of boundary element methods; II. Integral equation methods for mixed boundary value problems, in: Lectures on the Numerical Solution of Partial Differential Equations (ed. I. Babuška, I.-P. Liu, J. Osborn) Lecture Notes, University of Maryland, Dept. Mathematics, College Park, Md. USA #20 (1981) pp. 435-528.

[28] W.L. Wendland, On applications and the convergence of boundary integral methods, in: Treatment of Integral Equations by Numerical Methods (ed. C.T. Baker and G.F. Miller) Academic Press, London (1982) pp. 465-476.

[29] W.L. Wendland, E. Stephan and G.C. Hsiao, On the integral equation method for the plane mixed boundary value problem of the Laplacian, Math. Meth. in Appl. Sci. 1 (1979) pp. 265-321.