New preconditioners for Laplace and Helmholtz integral equation on open curves:

II. Theoretical analysis.

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Abstract

We develop and apply a new kind of pseudo-differential operators on open curves to the study first-kind integral equations on open curves. We analyze two new preconditioners for the Laplace and Helmholtz integral equation and study the convergence orders of a Galerkin method on weighted L^2 spaces with a refined mesh.

Introduction

In [1] some new square-root preconditioners for the Laplace and Helmholtz integral equations have been introduced and their numerical efficiency has been demonstrated on several examples. Here, we develop the theory to prove the main results that were announced there. To this aim, we analyze the spaces $(T^s)_{s\in\mathbb{R}}$ and $(U^s)_{s\in\mathbb{R}}$, which are interlacing spaces of Chebyshev series defined on the unit segment. They provide two Hilbert interpolating scales suited to the definition of a new kind of pseudo-differential operators. The symbolic calculus available in those classes allows us to analyze the efficiency of the preconditioners of [1]. We also prove optimal approximation results for piecewise affine functions in a simple weighted Galerkin setting. In the first section, we establish some properties of the spaces T^s and U^s. After briefly collecting some facts on periodic pseudo-differential operators in the second section, we define the two new classes of pseudo-differential operators. In the third section, we apply this theory to the aforementioned preconditioners. Finally, the Galerkin analysis is exposed in the fourth section.

Parler d'abord des estimations d'erreurs. Ensuite expliquer à quoi servent les PPDO et parler de la nouveauté de cette classe d'opérateurs. Parler de la possibilité de calculer avec des logiciels. Dire qu'on ne rappelle pas le contexte applicatif qui est bien détaillé dans la partie I.

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1 Spaces T^s and U^s

1.1 Definitions

The Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$$

for $x \in [-1, 1]$, see [7]. Letting ∂_x the derivation operator and ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$, T_n and U_n satisfy the following identities:

$$-(\omega\partial_x)^2 T_n = n^2 T_n\,, \tag{1}$$

$$-(\partial_x\omega)^2 U_n = (n+1)^2 U_n\,. \tag{2}$$

Notice that here and in the following, $\partial_x\omega$ refers to the operator $f \mapsto \partial_x(\omega f)$ and not the function $\partial_x\omega(x)$. One can also check the identities

$$\partial_x T_n = n U_{n-1}\,, \tag{3}$$

$$-\omega\partial_x\omega U_n = (n+1)T_{n+1}\,. \tag{4}$$

The first one is obtained for example from the trigonometric definition of T_n. This combined with $-(\omega\partial_x)^2 T_n = n^2 T_n$ gives the second identity.

Both T_n and U_n are polynomials of degree n, and provide respectively a basis of the following Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{|u(x)|^2}{\sqrt{1-x^2}}dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} |u(x)|^2\sqrt{1-x^2}dx < +\infty \right\}.$$

Following the notations of [9], we denote the Banach duality products of $L^2_{\frac{1}{\omega}}$ and L^2_{ω} respectively by $\langle\cdot,\cdot\rangle_{\frac{1}{\omega}}$ and $\langle\cdot,\cdot\rangle_{\omega}$ and the inner products respectively by $(\cdot,\cdot)_{\frac{1}{\omega}}$ and $(\cdot,\cdot)_{\omega}$ normalized as follows

$$(u,v)_{\frac{1}{\omega}} = \langle u, \overline{v}\rangle_{\frac{1}{\omega}} := \frac{1}{\pi}\int_{-1}^{1} \frac{u(x)\overline{v(x)}}{\omega(x)}dx\,,$$

$$(u,v)_{\omega} = \langle u, \overline{v}\rangle_{\omega} := \frac{1}{\pi}\int_{-1}^{1} u(x)\overline{v(x)}\omega(x)dx\,.$$

The Chebyshev polynomials satisfy

$$(T_n, T_m)_{\frac{1}{\omega}} = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } m = n = 0 \\ 1/2 \text{ otherwise} \end{cases} \tag{5}$$

and

$$(U_n, U_m)_\omega = \begin{cases} 0 \text{ if } n \neq m \\ 1/2 \text{ otherwise,} \end{cases} \tag{6}$$

from which we obtain the so-called Fourier-Chebyshev decomposition: any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x)\,. \tag{7}$$

where the Fourier-Chebyshev coefficients of the first kind are given by $\hat{u}_n = \frac{(u,T_n)_{\frac{1}{\omega}}}{(T_n,T_n)_{\frac{1}{\omega}}}$ and satisfy the Parseval equality

$$\forall (u,v) \in L^2_{\frac{1}{\omega}} \quad (u,v)_{\frac{1}{\omega}} = \hat{u}_0 \overline{\hat{v}}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \overline{\hat{v}}_n\,.$$

When u is furthermore a smooth function, on can check that the series (7) converges uniformly to u. Similarly, any function $v \in L^2_\omega$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the Fourier-Chebyshev coefficients of the second kind $\check{v}_n$ are given by $\check{v}_n := \frac{(v,U_n)_\omega}{(U_n,U_n)_\omega}$ with the Parseval identity

$$(u,v)_\omega = \frac{1}{2} \sum_{n=0}^{+\infty} \check{u}_n \overline{\check{v}}_n\,.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *We define T^s as the set of formal series*

$$u = \sum_{n \in \mathbb{N}} \hat{u}_n T_n$$

where the coefficients $\hat{u}_n$ satisfy

$$\sum_{n \in \mathbb{N}} (1+n^2)^s \left|\hat{u}_n\right|^2 < +\infty\,.$$

Let $T^\infty = \cap_{s \geq 0} T^s$ and $T^{-\infty} = \cup_{s \in \mathbb{R}} T^s$. For $u \in T^s$ when $s \geq 0$, the series defining u converges in $L^2_{\frac{1}{\omega}}$ and the Fourier-Chebysehv coefficients of the first kind of u coincide with $\hat{u}_n$, allowing to identify T^s to a subspace of $L^2_{\frac{1}{\omega}}$ with $T^0 = L^2_{\frac{1}{\omega}}$. For all $u \in T^s$, we define the linear form $\langle u, \cdot \rangle_{\frac{1}{\omega}}$ by

$$\forall \varphi \in T^\infty, \langle u, \varphi \rangle_{\frac{1}{\omega}} = \frac{1}{2} \hat{u}_0 \hat{\varphi}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \hat{\varphi}_n\,. \tag{8}$$

This linear form has a unique continuous extension on T^{-s}, and the dual of T^s is the set of linear forms $\langle u, \cdot \rangle_{\frac{1}{\omega}}$ where $u \in T^{-s}$. Endowed with the scalar product

$$(u,v)_{T^s} := \hat{u}_0 \overline{\hat{v}}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \overline{\hat{v}}_n \,,$$

T^s is a Hilbert space for all s. A semi-norm on T^s can be defined as

$$|u|_{T^s}^2 := \frac{1}{2} \sum_{n=1}^{+\infty} n^{2s} \left|\hat{u}_n\right|^2 .$$

Definition 2. *In a similar fashion, we define U^s as the set of formal series*

$$u = \sum_{n \in \mathbb{N}} \check{u}_n U_n$$

where the coefficients $\check{u}_n$ satisfy

$$\sum_{n \in \mathbb{N}} (1+n^2)^s \left|\check{u}_n\right|^2 < +\infty .$$

Let $U^\infty = \cap_{s \in \mathbb{R}} U^s$ and $U^{-\infty} = \cup_{s \in \mathbb{R}} U^s$. For $u \in U^s$ when $s \geq 0$, the series defining u converges in L^2_ω and the Fourier-Chebyshev coefficients of the second kind of u coincide with $\check{u}_n$, allowing to identify U^s to a subspace of L^2_ω with $U^0 = L^2_\omega$. For all $u \in U^s$, we define the linear form $\langle u, \cdot \rangle_\omega$ by

$$\forall \varphi \in U^\infty, \langle u, \varphi \rangle_\omega := \frac{1}{2} \sum_{n=0}^{+\infty} \check{\varphi}_n \check{u}_n \,. \tag{9}$$

This linear form has a unique continuous extension on U^{-s}, and the dual of U^s may be identified to U^{-s} with respect to the bilinear form $\langle \cdot, \cdot \rangle_\omega$. Endowed with the scalar product

$$(u,v)_{U^s} := \frac{1}{2} \sum_{n \in \mathbb{N}} \left(1 + (n+1)^2\right)^s \check{u}_n \overline{\check{v}}_n \,,$$

U^s is a Hilbert space for all $s \in \mathbb{R}$.

Let $s_1, s_2 \in_R$, $\theta \in (0,1)$ and let $s = \theta s_1 + (1-\theta) s_2$. It is easy to check that

$$\forall u \in T^\infty, \|u\|_{T^s} \leq \|u\|_{T^{s_1}}^\theta \, \|u\|_{T^{s_2}}^{1-\theta}$$

and

$$\forall u \in U^\infty, \|u\|_{U^s} \leq \|u\|_{U^{s_1}}^\theta \, \|u\|_{U^{s_2}}^{1-\theta}$$

Therefore, $(T^s)_{s \in \mathbb{R}}$ and $(U^s)_{s \in \mathbb{R}}$ are interpolation scales.

1.2 Basic properties

For any real s, if $u \in T^s$ the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x)$$

converges to u in T^s. The same assertion holds for $u \in U^s$ when T_n is replaced by U_n. Therefore

Lemma 1. *$C^\infty([-1,1])$ is dense in T^s and U^s for all $s \in \mathbb{R}$.*

The polynomials T_n and U_n are connected by the following formulas:

$$T_0 = U_0, \quad T_1 = \frac{U_1}{2}, \quad \text{and } \forall n \geq 2, \quad T_n = \frac{1}{2}\left(U_n - U_{n-2}\right), \tag{10}$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}. \tag{11}$$

This leads to introduce the map

$$I : T^\infty \to U^\infty$$

defined by

$$\widecheck{I\varphi}_0 = \hat{\varphi}_0 - \frac{\hat{\varphi}_2}{2}, \quad \widecheck{I\varphi}_j = \frac{\hat{\varphi}_j - \hat{\varphi}_{j+2}}{2} \text{ for } j \geq 1.$$

I is bijective has the explicit inverse

$$\widehat{I^{-1}\varphi}_0 = \sum_{n=0}^{+\infty} \check{\varphi}_{2n}, \quad \widehat{I^{-1}\varphi}_j = 2\sum_{n=0}^{+\infty} \check{\varphi}_{j+2n} \text{ for } j \geq 1.$$

Lemma 2. *For all real s, I has a unique continuous extension from T^s to U^s and for $s > \frac{1}{2}$, I^{-1} has a continuous extension from U^s to T^{s-1}.*

Before starting the proof, we introduce the Cesàro operator C defined on $l^2(\mathbb{N}^*)$ by

$$(Cu)_n = \frac{1}{n}\sum_{k=1}^{n} u_k\,.$$

As is well-known, this is a linear continuous operator on the Hilbert space $l^2(\mathbb{N}^*)$. Its adjoint

$$(C^*u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k}\,,$$

is therefore also continuous on $l^2(\mathbb{N}^*)$. In other words, for all $(u_n)_n \in l^2(\mathbb{N})$,

$$\sum_{n=1}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{u_k}{k}\right)^2 \leq C\sum_{k=1}^{+\infty} u_k^2\,.$$

Proof. The first result is immediate from the definition of T^s, U^s and I. When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging thus $I^{-1}u$ is well defined. Since $u \in U^s$, the sequence $\left((1+n^2)^{s/2}\,|\check{u}_n|\right)_{n\geq 1}$ is in $l^2(\mathbb{N}^*)$. Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the sequence $(r_n)_n$ defined by

$$\forall n \geq 0, \quad r_n := \sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}\,|\check{u}_k|$$

is in $l^2(\mathbb{N})$ with a l^2 norm bounded by $\|u\|_{U^s}$. We now write

$$\begin{aligned}
\left\|I^{-1}u\right\|^2_{T^{s-1}} &= \sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left|\widehat{I^{-1}u_n}\right|^2 \\
&\leq 4\sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left(\sum_{k=n}^{+\infty}|\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}|\check{u}_k|\right)^2 . \\
&= 4\left\|(r_n)_n\right\|^2_{l^2}
\end{aligned}$$

We saw that the last quantity is controlled by $\|u\|^2_{U^s}$ so the result is proved. □

Début d'une digression à potentiellement enlever (mais garder dans le manuscrit).

Lemma 3. *Let $s > 1/2$ and let $u \in U^s$. Then there exists $0 < \varepsilon < 1$ such that $\omega^{-\frac{1+\varepsilon}{2}}u \in L^2_\omega$ with*

$$\left\|\omega^{-\frac{1+\varepsilon}{2}}u\right\|_\omega \leq C\,\|u\|_{U^s} .$$

Proof. We start by showing the following estimate

$$\forall\varepsilon \in (0,1),\ \exists C_\varepsilon : \forall n \in \mathbb{N},\quad I_n := \int_{-1}^{1} U_n^2\omega^{-\varepsilon} \leq C_\varepsilon(n+1)^\varepsilon .$$

Fix $\varepsilon \in (0,1)$. Using the variable change $x = \cos\theta$ and the symmetry of the integrand with respect to the change $\theta \to \pi - \theta$, we transform the quantity to be estimated to

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin((n+1)\theta)^2}{|\sin\theta|^{1+\varepsilon}}d\theta .$$

We split I_n into two parts. Let $I_{n,1} = \int_0^{\frac{\pi}{n+1}} \frac{\sin((n+1)\theta)^2}{|\sin\theta|^{1+\varepsilon}}d\theta$. On this interval, we use $\sin((n+1)\theta) \leq (n+1)\theta$ and $\sin\theta \geq \frac{2}{\pi}\theta$ to find

$$I_{n,1} \leq C(n+1)^2\int_0^{\frac{\pi}{n+1}} \theta^{1-\varepsilon} \leq C_{\varepsilon,1}(n+1)^\varepsilon .$$

Let $I_{n,2} = I - I_{n,1}$. On this interval, we estimate the numerator by

$$\sin((n+1)\theta) \leq 1$$

and use the same estimate as before for the denominator. One can check that this leads to $I_{n,2} \leq C_{\varepsilon,2}n^\varepsilon$. The proof of the main result is now as follows. Let $u \in U^s$ where $s > \frac{1}{2}$ and let $s = \frac{1}{2} + \varepsilon$. Then the series

$$\omega^{-\frac{1+\varepsilon}{2}}u = \sum_{n\in\mathbb{N}}\check{u}_n\frac{U_n}{\omega^{\frac{1+\varepsilon}{2}}}$$

converges in L^2_ω since

$$\sum_{n\in\mathbb{N}} \left\| \check{u}_n \frac{U_n}{\omega^{\frac{1+\varepsilon}{2}}} \right\|_{L^2_\omega} \leq C \sum_{n\in\mathbb{N}} |\check{u}_n| (n+1)^{\frac{\varepsilon}{2}}$$
$$\leq C \sqrt{\sum_{n\in\mathbb{N}} (n+1)^{2s} |\check{u}_n|^2} \sqrt{\sum_{n\in\mathbb{N}} (n+1)^{-1-\varepsilon}} \, .$$
$$\leq C \|u\|_{U^s}$$

Thus $\frac{u}{\omega^{\frac{1+\varepsilon}{2}}} \in L^2_\omega$ by normal convergence and the result is proved. □

Notice that for $\varphi \in C^\infty([-1,1])$, for all $\varepsilon > 0$, $\omega^{-\frac{3-\varepsilon}{2}} \varphi \in L^2_\omega$.

Corollary 1. *Let $u \in T^{-\infty}$. Then $Iu \in U^{-\infty}$ is characterized by*

$$\forall \varphi \in C^\infty([-1,1]), \quad \langle Iu, \varphi \rangle_\omega = \langle u, \omega^2 \varphi \rangle_{\frac{1}{\omega}} \, .$$

Let $u \in U^s$ with $s > \frac{1}{2}$. Let ε such that $\omega^{-\frac{1+\varepsilon}{2}} u \in L^2_\omega$. Then $I^{-1}u \in T^{-\infty}$ is characterized by

$$\forall \varphi \in C^\infty([-1,1]), \quad \langle I^{-1}u, \varphi \rangle_{\frac{1}{\omega}} = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega$$

Proof. We shall only treat the second statement, the first one being similar and simpler. By density of $C^\infty([-1,1])$ in U^s, we can fix a sequence of C^∞ functions u_N converging to u in U^s. Then, the sequence $\omega^{-\frac{1+\varepsilon}{2}} u_N$ converges to $\omega^{-\frac{1+\varepsilon}{2}} u$ in L^2_ω since, by the previous result,

$$\left\| \omega^{-\frac{1+\varepsilon}{2}} (u_N - u) \right\|_{L^2_\omega} \leq C \|u - u_N\|_{U^s} \, .$$

Thus, there holds

$$\lim_{N\to\infty} \left\langle \omega^{-\frac{1+\varepsilon}{2}} u_N, \omega^{-\frac{3-\varepsilon}{2}} \overline{\varphi} \right\rangle_\omega = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega \, .$$

By continuity of I^{-1} from U^s to T^{s-1}, we also have

$$\lim_{N\to\infty} \langle I^{-1}u_N, \varphi \rangle_{\frac{1}{\omega}} = \langle I^{-1}u, \varphi \rangle_{\frac{1}{\omega}} \, .$$

For all N, $I^{-1}u_N = u_N \in C^\infty([-1,1])$. Therefore, we obviously have

$$\langle I^{-1}u_N, \varphi \rangle_{\frac{1}{\omega}} = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u_N, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega \, ,$$

from which the result follows. □

Fin d'une digression potentiellement à enlever.
Let

$$u = \sum_{n\in\mathbb{N}} \hat{u}_n T_n, \quad v = \sum_{n\in\mathbb{N}} \check{v}_n U_n \, .$$

When $Iu = v$, we identify u and v in $T^{-\infty} \cap U^{-\infty}$. The previous results have shown that this identification is compatible with the equality of functions in $L^2_{\frac{1}{\omega}}$ or L^2_ω. The mapping property of I and I^{-1} can then be rephrased in the following continuous inclusions:

Corollary 2. *For all $s \in \mathbb{R}$, $T^s \subset U^s$ and for all $s > \frac{1}{2}$, $U^s \subset T^{s-1}$.*

One immediate consequence of the previous result is that $T^\infty = U^\infty$. Moreover, there holds

Lemma 4.
$$T^\infty = C^\infty([-1,1])\,.$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (1), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^{1} \frac{(\omega\partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, the function $(\omega\partial_x)^{2k}u$ is C^∞, and since $\|T_n\|_\infty = 1$, the integral is bounded independently of n. Thus, the coefficients $\hat{u}_n$ have a fast decay, proving that $C^\infty([-1,1]) \subset T^\infty$.

For the converse inclusion, if $u \in T^\infty$, the series

$$u(x) = \sum_{n=0} \hat{u}_n T_n(x)$$

is normally converging since $\|T_n\|_\infty = 1$, so u is a continuous function. This proves $T^\infty \subset C^0([-1,1])$. It suffices to show that $\partial_x u \in T^\infty$ and apply an induction argument. Applying term by term differentiation, since $\partial_x T_n = nU_{n-1}$ for all n (with the convention $U_{-1} = 0$),

$$\partial_x u(x) = \sum_{n=1}^{+\infty} n\hat{u}_n U_{n-1}(x).$$

Therefore, $\partial_x u$ is in $U^\infty = T^\infty$ which proves the result. $\square$

Lemma 5. *For $s \leq \frac{1}{2}$, the functions of U^s cannot be identified to functions in $T^{-\infty}$.*

Proof. Let $s \leq \frac{1}{2}$, and let us assume by contradiction that the functions of U^s can be identified to elements of $T^{-\infty}$. Then, there must exist a continuous map I from U^s to $T^{-\infty}$ with the property

$$\forall u \in C^\infty([-1,1]), \quad Iu = u.$$

We introduce the function u defined by $\check{u}_n = \frac{1}{n\ln(n)}$. One can check that $u \in U^{\frac{1}{2}} \subset U^s$, thus Iu must be element of $T^{-\infty}$. For all N, the function

$$u_N = \sum_{n=0}^{N} \check{u}_n U_n$$

is in U^∞ and $(u_N)_{N\in\mathbb{N}}$ converges to u in U^s. By continuity of I, the sequence $(\langle Iu_N, T_0\rangle_{\frac{1}{\omega}})_{N\in\mathbb{N}}$ must converge with limit $\langle Iu, T_0\rangle_{\frac{1}{\omega}}$. But since $Iu_N = u_N$,

$$\langle Iu_N, T_0\rangle_{\frac{1}{\omega}} = \langle u_N, T_0\rangle_{\frac{1}{\omega}} = \sum_{n=0}^{N} \check{u}_n \langle U_n, T_0\rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2}\rfloor} \frac{1}{2k\ln(2k)}\,.$$

This sum diverges to $+\infty$ when N goes to infinity, giving the contradiction. $\square$

We now extend the definition of the derivation operators ∂_x and $\omega\partial_x\omega$ appearing in eqs. (3) and (4).

Lemma 6. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s defined by*

$$\forall v \in C^\infty([-1,1]), \quad \langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}} \ .$$

In a similar fashion, the operator $\omega\partial_x\omega$ can be extended into a continuous map from U^{s+1} to T^s defined by

$$\forall v \in C^\infty([-1,1]), \quad \langle \omega\partial_x\omega u, v\rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v\rangle_\omega \,.$$

Proof. Using eqs. (3) and (4), one can check that the formulas indeed extend the usual definition of the two operators for smooth functions. We now show that the map ∂_x extended this way is continuous from T^{s+1} to U^s. The definition

$$\forall v \in U^\infty, \langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}}$$

gives a sense to $\partial_x u$ for all u in $T^{-\infty}$, as a duality $T^{-\infty}\times T^\infty$ product, because if $v\in U^\infty(= C^\infty([-1,1])$, then $\omega\partial_x\omega v = (1-x^2)v' - xv$ also lies in $C^\infty([-1,1])(= T^\infty)$. Letting $w = \partial_x u$, we have by definition for all n

$$\check{w}_n = \langle w, U_n\rangle_\omega = -\langle u, \omega\partial_x\omega U_n\rangle_{\frac{1}{\omega}} = n\,\langle u, T_{n+1}\rangle_{\frac{1}{\omega}} = n\hat{u}_{n+1}$$

Obviously, this implies the announced continuity with

$$\|w\|_{U^s} \le \|u\|_{T^{s+1}} \ .$$

The properties of $\omega\partial_x\omega$ on T^s are established similarly. □

Corollary 3. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$. On the other hand, $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s and from U^{s+1} to U^s for all $s\in\mathbb{R}$.*

Proof. For the continuity of ∂_x from T^{s+2} to T^s, we use the continuity of ∂_x from T^{s+2} to U^{s+1} and then of the identity from U^{s+1} to T^s. For the continuity of ∂_x from U^{s+2} to U^s, we use the same arguments in reverse order.
On the other hand, we have, for $n \ge 2$,

$$\omega\partial_x\omega T_n = \omega\partial_x\omega\frac{U_n - U_{n-2}}{2} = \frac{(n+1)T_{n+1} - (n-1)T_{n-1}}{2} \,.$$

Therefore $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s. Finally, $\omega\partial_x\omega$ is continuous from U^{s+1} to T^s and the inclusion $T^s\subset U^s$ is continuous thus $\omega\partial_x\omega$ is continuous from U^{s+1} to U^s. □

Lemma 7. *For all $\varepsilon > 0$, if $u\in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and*

$$\exists C : \forall x\in[-1,1], \quad |u(x)| \le C\,\|u\|_{T^{1/2+\varepsilon}} \,.$$

Similarly, if $u\in U^{3/2+\varepsilon}$, then u is continuous and

$$\exists C : \forall x\in[-1,1], \quad |u(x)| \le C\,\|u\|_{U^{3/2+\varepsilon}} \,.$$

Proof. Let $x \in [-1,1]$. Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\|T_n\|_{L^\infty} = 1$. Applying Cauchy-Schwarz's inequality, one gets

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \|u\|_{T^{\frac{1}{2}+\varepsilon}}.$$

The second statement is deduced from the fist and the continuous inclusion $U^s \subset T^{s-1}$ established in Lemma 2. $\square$

1.3 Link with Periodic Sobolev spaces

We briefly recall here the definition of the periodic Sobolev spaces on the torus $\mathbb{T}_{2\pi} := \mathbb{R}/2\pi\mathbb{Z}$. A smooth function u on $\mathbb{T}_{2\pi}$ can be decomposed in Fourier series

$$u(\theta) = \sum_{n\in\mathbb{Z}} \mathcal{F}u(n) e^{in\theta}$$

with the Fourier coefficients defined by

$$\mathcal{F}u(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta) e^{-in\theta} d\theta.$$

For $n \in \mathbb{Z}$, let $e_n : \theta \mapsto e^{in\theta}$. We define the Fourier coefficients of any periodic distribution u on $\mathbb{T}_{2\pi}$, by $\mathcal{F}u(n) := u(e_{-n})$. The space H^s is then defined for all s as the set of periodic distributions on $\mathbb{T}_{2\pi}$ for which

$$\|u\|_{H^s}^2 := \sum_{n\in\mathbb{Z}} (1+n^2)^s |\mathcal{F}u(n)|^2 < +\infty$$

Introducing the duality product

$$\langle u, v \rangle_{\mathbb{T}_{2\pi}} = \sum_{n\in\mathbb{Z}} \mathcal{F}u(n)\mathcal{F}v(-n)\,, \tag{12}$$

H^s is identified to the dual of H^{-s} and $H^0 = L^2(\mathbb{T}_{2\pi})$. For $u, v \in H^0$, $\langle u, v\rangle_{\mathbb{T}_{2\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} uv$. The space H^s is the direct sum $H^s_e + H^s_o$ where $H^s_e := \{u \in H^s \mid \mathcal{F}u(n) = \mathcal{F}u(-n)\}$ and $H^s_o := \{u \in H^s \mid \mathcal{F}u(n) = -\mathcal{F}u(-n)\}$. Note that when u is continuous, $u \in H^s_e \iff \forall\theta \in \mathbb{T}_{2\pi}, u(-\theta) = u(\theta)$ and $u \in H^s_o \iff \forall \theta \in \mathbb{T}_{2\pi}, u(-\theta) = -u(\theta)$.

Definition 3. *We define the operators $\mathcal{C} : T^{-\infty} \to H_e^{-\infty}$ by*

$$\forall n \in \mathbb{Z}, \quad \mathcal{F}(\mathcal{C}u)(n) = \begin{cases} \hat{u}_0 & \text{if } n = 0, \\ \frac{\hat{u}_{|n|}}{2} & \text{otherwise,} \end{cases}$$

and $\mathcal{S} : U^{-\infty} \to H_o^{-\infty}$ by

$$\forall n \in \mathbb{Z}, \quad \mathcal{F}(\mathcal{S}u)(n) = \begin{cases} 0 & \text{if } n = 0, \\ \operatorname{sign}(n)\frac{\hat{u}_{|n|-1}}{2} & \text{otherwise.} \end{cases}$$

Lemma 8. *The operators $\mathcal{C}$ and $\mathcal{S}$ map smooth functions to smooth functions. For all $(u, v) \in T^{-\infty} \times T^{\infty}$,*

$$\langle u, v \rangle_{\frac{1}{\omega}} = \langle \mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} \,.$$

For all $(u, v) \in U^{-\infty} \times U^{\infty}$,

$$\langle u, v \rangle_{\omega} = \langle \mathcal{S}u, \mathcal{S}v \rangle_{\mathbb{T}_{2\pi}}$$

Proof. The first assertion is obvious from the definition of $\mathcal{C}$ and $\mathcal{S}$. Let $(u, v) \in T^{-\infty} \times T^{\infty}$. By definition of $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{T}_{2\pi}}$ eqs. (8) and (12),

$$\begin{aligned}
\langle \mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} &= \sum_{n \in \mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n) \mathcal{F}(\mathcal{C}v)(-n) \\
&= \hat{u}_0 \hat{v}_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\hat{u}_{|n|}}{2} \frac{\hat{v}_{|n|}}{2} \\
&= \hat{u}_0 \hat{v}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \hat{v}_n \\
&= \langle u, v \rangle_{\frac{1}{\omega}} \,.
\end{aligned} \,.$$

The second identity is proved similarly. □

Lemma 9. *For all $s \in \mathbb{R}$, the operators $\mathcal{C}$ and $\mathcal{S}$ induce bijective isometries respectively from T^s to H^s_e and from U^s to H^s_o. For $u \in C^{\infty}([-1, 1])$,*

$$\mathcal{C}u(\theta) = u(\cos \theta) \quad \textit{and} \quad \mathcal{S}u(\theta) = \sin \theta u(\cos \theta) \,.$$

Let $v, w \in C^{\infty}(\mathbb{T}_{2\pi})$, an even and an odd function respectively. Then

$$\mathcal{C}^{-1}v(x) = v(\arccos x) \quad \textit{and} \quad \mathcal{S}^{-1}w(x) = \frac{w(\arccos x)}{\omega(x)} \,.$$

Proof. Let J^T_s, J^U_s and $\tilde{J}_s$ the linear continuous mappings defined respectively on $T^{-\infty}$, $U^{-\infty}$ and $H^{-\infty}$ by

$$J^T_s T_n = (1 + n^2)^{\frac{s}{2}} T_n, \quad J^U_s U_{n-1} = (1 + n^2)^{\frac{s}{2}} U_{n-1}, \quad \tilde{J}_s e_n = (1 + n^2)^{\frac{s}{2}} e_n$$

where we recall that e_n is the function $\theta \mapsto e^{in\theta}$. One can check easily that for $u \in T^s$ and $v \in U^s$

$$\|u\|^2_{T^s} = \left\langle J^T_s u, \overline{J^T_s u} \right\rangle_{\frac{1}{\omega}}, \quad \|v\|^2_{U^s} = \left\langle J^U_s v, \overline{J^U_s v} \right\rangle_{\omega}$$

while for $w \in H^s$,

$$\|w\|^2_{H^s} = \|u\|^2_{T^s} = \left\langle \tilde{J}_s u, \overline{\tilde{J}_s u} \right\rangle_{\mathbb{T}_{2\pi}} \,.$$

Moreover, the following identities hold:

$$\mathcal{C} J^T_s = \tilde{J}_s \mathcal{C}, \quad \mathcal{S} J^U_s = \tilde{J}_s \mathcal{S} \,.$$

The isometric property of $\mathcal{C}$ may now be deduced from Lemma 8 as follows. Let $u_N = \sum_{n=0}^{N} u_n T_n$. There holds

$$\begin{aligned}\left\langle J_s^T u, \overline{J_s^T u_N}\right\rangle_{\frac{1}{\omega}} &= \left\langle \mathcal{C} J_s^T u, \overline{\mathcal{C} J_s^T u_N}\right\rangle_{\mathbb{T}_{2\pi}}\\ &= \left\langle \tilde{J}_s \mathcal{C} u, \overline{\tilde{J}_s \mathcal{C} u_N}\right\rangle_{\mathbb{T}_{2\pi}}.\end{aligned}$$

Sending N to infinity, by continuity of J_s^T, $\tilde{J}^s$ and $\mathcal{C}$, this yields

$$\|u\|_{T^s}^2 = \|\mathcal{C}u\|_{H^s}^2$$

The isometric property of $\mathcal{S}$ is establish in a similar manner. Let u a smooth function. Since $\mathcal{C}u$ is smooth, the Fourier series of $\mathcal{C}u$ converge pointwise to $\mathcal{C}u$. Thus, for all $\theta \in \mathbb{T}_{2\pi}$,

$$\begin{aligned}\mathcal{C}u(\theta) &= \sum_{n\in\mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n) e^{in\theta}\\ &= \hat{u}_0 + \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\hat{u}_{|n|}}{2} e^{in\theta}\\ &= \hat{u}_0 + \frac{1}{2}\sum_{n=1}^{+\infty} \hat{u}_n \left(e^{in\theta} + e^{-in\theta}\right)\\ &= \sum_{n=0}^{+\infty} \hat{u}_n \cos(n\theta)\\ &= \sum_{n=0}^{+\infty} \hat{u}_n T_n(\cos\theta)\end{aligned}$$

The last sum also converges pointwise to $u(\cos\theta)$ since $u \in T^\infty$. Similar calculations show that $\mathcal{S}u(\theta) = \sin\theta u(\cos\theta)$, using this time $\sin((n+1)\theta) = \sin\theta U_n(\cos\theta)$. To prove the bijectivity of $\mathcal{S}$ and $\mathcal{C}$, one can check that they have the explicit inverses $\mathcal{C}^{-1}$ and $\mathcal{S}^{-1}$ respectively defined on H^s_e and H^s_o as

$$\forall n \in \mathbb{N}, \quad \left(\widehat{\mathcal{C}^{-1}u}\right)_n = \begin{cases}\mathcal{F}u(0) & \text{if } n = 0,\\ 2\mathcal{F}u(n) & \text{otherwise,}\end{cases}$$

and

$$\forall n \in \mathbb{N}, \quad \left(\widehat{\mathcal{S}^{-1}u}\right)_n = 2\mathcal{F}u(n+1)\,.$$

Finally, the expression of $\mathcal{C}^{-1}u$ (resp $\mathcal{S}^{-1}u$) when u is a smooth even (resp. odd) function on $\mathbb{T}_{2\pi}$ is deduced from the expression of $\mathcal{C}$ (resp. $\mathcal{S}$). □

1.4 Equivalent norms on T^n and U^n

We now provide a characterization of the spaces T^n and U^n in terms of weighted L^2 norms of the derivatives and give equivalent norms on those spaces when n is an integer.

Lemma 10. *The operator ω is a bijective isometry from U^0 to T^0 with inverse $\frac{1}{\omega}$.*

Proof. This result follows from

$$\|\omega u\|^2_{\frac{1}{\omega}} = \frac{1}{\pi}\int_{-1}^{1} \frac{|(\omega u)|^2}{\omega} = \frac{1}{\pi}\int_{-1}^{1} \omega\,|u|^2 = \|u\|_\omega \ ,$$

valid for all $u \in L^2_\omega$. $\square$

Definition 4. *For an even integer n, the operator $(\omega\partial_x)^n : T^{-\infty} \to T^{-\infty}$ is defined by*

$$(\omega\partial_x)^0 = I_d, \quad \forall k > 0,\ (\omega\partial_x)^{2k} := (\omega\partial_x\omega)\partial_x(\omega\partial_x)^{2k-2}$$

The operator $(\partial_x\omega)^n : U^{-\infty} \to U^{-\infty}$ is defined in an analogous way.

Lemma 11. *Let n an even integer. For all $s \in \mathbb{R}$, $(\omega\partial_x)^n$ is continuous from T^s to T^{s-n} and $(\partial_x\omega)^n$ is continuous from U^s to U^{s-n}.*

Proof. Those results follow from the definition of the operators and by induction using the mapping properties of ∂_x and $\omega\partial_x\omega$ established in Lemma 6. $\square$

Definition 5. *For an odd integer n, the operator $(\omega\partial_x)^n : T^n \to T^0$ is defined by*

$$(\omega\partial_x)^n := \omega\partial_x(\omega\partial_x)^{n-1}\,.$$

The operator $(\partial_x\omega)^n : T^n \to T^0$ is defined in an analogous way.

From Lemma 10, we deduce

Corollary 4. *When n is odd, the operators $(\omega\partial_x)^n$ and $(\partial_x\omega)^n$ are well defined and continuous respectively from T^n to T^0 and from U^n to U^0.*

Lemma 12. *Let $n \in \mathbb{N}$. If n is even,*

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; (\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}} \right\}$$

If n is odd,

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \partial_x(\omega\partial_x)^{n-1} u \in L^2_{\omega} \right\}$$

Moreover $u \mapsto \sqrt{\|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}}}$ defines an equivalent norm on T^n, and for all $u \in T^n$,

$$|u|_{T^n} = \|(\omega\partial_x)^n u\|_{L^2_{\frac{1}{\omega}}}\ .$$

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 11 and Corollary 4. For the converse inclusion, let u in $L^2_{\frac{1}{\omega}}$. If n is even, say $n = 2k$, we assume that $(\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}}$. The Fourier-Chebyshev coefficients of $a = (\omega\partial_x)^n u$ are given for $j > 0$ by

$$\hat{a}_j = 2\left((\omega\partial_x)^{2k} u(x), T_j\right)_{\frac{1}{\omega}} = 2\left(u(x), (\omega\partial_x)^{2k} T_j\right)_{\frac{1}{\omega}} = (-1)^k j^{2k} \hat{u}_j\,.$$

while for $j = 0$, $\hat{a}_j = 0$. Applying Parseval's equality to the function a, this gives

$$\frac{1}{2}\sum_{j>0} j^{2n}\left|\hat{u}_j\right|^2 = \left\|(\omega\partial_x)^n u\right\|^2_{\frac{1}{\omega}} \ . \tag{13}$$

On the other hand, if n is odd, say $n = 2k+1$, let $b := \partial_x(\omega\partial_x)^{2k}u$. The assumption is now $b \in L^2_\omega$, and by Lemma 10, $\omega b\,(= (\omega\partial_x)^n u) \in T^0$ with

$$\|\omega b\|_{\frac{1}{\omega}} = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}} = \|b\|_\omega \ .$$

One can write

$$\check{b}_j = 2\left(\partial_x(\omega\partial_x)^{2k}u, U_j\right) = -2\left(u, (\omega\partial_x)^{2k}(\omega\partial_x\omega)U_j\right) \ .$$

Using $-\omega\partial_x\omega U_j = (j+1)T_{j+1}$, we obtain

$$\check{b}_j = (-1)^k (j+1)^{2k+1}\hat{u}_{j+1} \, .$$

Parseval's equality then implies that (13) also hold for odd n. This establishes that $u \in T^n$ and $|u|_{T^n} = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}}$. For the norm equivalence, adding the Parseval equality for $u \in L^2_{\frac{1}{\omega}}$ to eq. (13), we get

$$\left|\hat{u}_0\right|^2 + \frac{1}{2}\sum_{j>0}(1+j^{2n})\left|\hat{u}_j\right|^2 = \left\|u + (\omega\partial_x)^n u\right\|^2 \ . \tag{14}$$

There are two constants c and C such that $c(1+j^2)^n \leq (1+j^{2n}) \leq C(1+j^2)^n$. Injecting this in (14), we obtain

$$\frac{c}{2}\|u\|^2_{T^n} \leq \|u + (\omega\partial_x)^n u\|^2_{\frac{1}{\omega}} \leq C\,\|u\|^2_{T^n} \ .$$

Moreover,

$$\|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}} \leq \|u + (\omega\partial_x)^n u\|^2_{\frac{1}{\omega}} \leq 2\left(\|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}}\right) ,$$

and the equivalence of the norms follows. □

Lemma 13. *Let $n \in \mathbb{N}$. If n is even, then*

$$U^n = \left\{u \in L^2_\omega \;\middle|\; (\partial_x\omega)^n u \in L^2_\omega\right\} \ .$$

If n is odd, then

$$U^n = \left\{u \in L^2_\omega \;\middle|\; \omega\partial_x\omega(\partial_x\omega)^{n-1}u \in L^2_{\frac{1}{\omega}}\right\} \ .$$

Moreover, $u \mapsto \sqrt{\int_{-1}^1 \omega|(\partial_x\omega)^n u|^2}$ defines an equivalent norm on U^n.

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 11 and Corollary 4. For the converse inclusion, if n is even, let $a = (\partial_x\omega)^n u$, we assume that $a \in L^2_\omega$. One has

$$\check{a}_j = (-1)^k\,(1+j)^n\,\check{u}_j$$

so by Parseval's equality,

$$\frac{1}{2}\sum_{j=0}^{+\infty}(j+1)^{2n}\left|\check{u}_j\right|^2 = \|(\partial_x\omega)^n u\|_\omega^2 \tag{15}$$

If n is odd, the assumption is that $b = \omega\partial_x\omega(\partial_x\omega)^{n-1}u$ is in $L^2_{\frac{1}{\omega}}$. One has, for $j > 0$,

$$\hat{b}_j = \left\langle \omega\partial_x\omega(\partial_x\omega)^{n-1}u, T_j\right\rangle = -\left\langle u, (\partial_x\omega)^{n-1}\partial_x T_j\right\rangle$$

Using $T_j' = jU_{j-1}$, this yields, for $j > 0$,

$$\hat{b}_j = j^{2n}\check{u}_{j-1}\,.$$

while $\hat{b}_0 = 0$. By Lemma 10, $\frac{b}{\omega}$ $(= (\partial_x\omega)^n u)$ $\in U^0$. Applying Parseval's equality to b in $L^2_{\frac{1}{\omega}}$ and using $\left\|\frac{b}{\omega}\right\|_\omega = \|b\|_{\frac{1}{\omega}}$, we find that (15) also holds for n odd, and thus the inclusion is proved. Finally, there exists two constants c and C such that for all $j \in \mathbb{N}$,

$$c(1+(j+1))^{2n} \leq (j+1)^{2n} \leq C(1+(j+1))^{2n}.$$

This implies the equivalence of the norms. □

1.5 Generalization to a curve

Parametrization of the curve

We start by introducing some notation that will be extensively used throughout all the remainder of this work. Let Γ a smooth open curve in $\mathbb{R}^2$ parametrized by a smooth C^∞ diffeomorphism $r : [-1,1] \to \Gamma$. We assume that $|r'(x)| = \frac{|\Gamma|}{2}$ for all $x \in [-1,1]$, where $|\Gamma|$ is the length of Γ. This parametrization is related to the curvilinear abscissa $M(s)$ through

$$r(x) = M\left(\frac{|\Gamma|}{2}(1+x)\right).$$

Let $R : C^\infty(\Gamma) \longrightarrow C^\infty(-1,1)$ defined by

$$Ru(x) = u(r(x))\,.$$

The tangent and normal vectors on the curve τ are respectively defined by

$$\tau(x) = \frac{\partial_x r(x)}{|\partial_x r(x)|}, \quad n(x) = \frac{\partial_x\tau(x)}{|\partial_x\tau'(x)|}$$

Let $N : \Gamma \to \mathbb{R}^2$ such that $N(r(x)) = n(x)$, that is, $N = R^{-1}n$. Let $\kappa(x)$ the signed curvature of Γ at the point $r(x)$. Frenet-Serret's formulas give

$$\begin{aligned} r(y) =& r(x) + (y-x)\frac{|\Gamma|}{2}\tau(x) + \frac{(y-x)^2}{2}\frac{|\Gamma|^2}{4}\kappa(x)n(x) \\ &+ \frac{(x-y)^3}{6}\frac{|\Gamma|^3}{8}(\kappa'(x)n(x) - \kappa(x)^2\tau(x)) + O\left((x-y)^4\right), \end{aligned}.$$

so that

$$|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4}(y-x)^2 - \frac{(y-x)^4}{192}\,|\Gamma|^4\,\kappa(x)^2 + O(x-y)^5\,. \tag{16}$$

For $u, v \in L^2(\Gamma)$, we have by change of variables in the integral

$$\langle u, v\rangle_{L^2(\Gamma)} = \frac{|\Gamma|}{2}\,\langle Ru, Rv\rangle_{L^2(-1,1)}\ .$$

The tangential derivative ∂_τ on Γ satisfies $\partial_\tau = \frac{2}{|\Gamma|}R^{-1}\partial_x R$. Moreover, let $\omega_\Gamma := \frac{|\Gamma|}{2}R^{-1}\omega(x)R$ the "weight" on the curve Γ. The uniform measure on Γ is denoted by $d\sigma$.

Spaces $T^s(\Gamma)$ and $U^s(\Gamma)$

The definition of the spaces T^s can be transported on the curve Γ, replacing the basis (T_n) and (U_n) by $(R^{-1}T_n)$ and $(R^{-1}U_n)$. The spaces $T^s(\Gamma)$ and $U^s(\Gamma)$ are thus defined as the sets of formal series respectively of the form

$$u = \sum_{n\in\mathbb{N}} \hat{u}_n R^{-1}T_n, \quad v = \sum_{n\in\mathbb{N}} \check{v}_n R^{-1}T_n$$

where $Ru = \sum \hat{u}_n T_n \in T^s$ and $Rv = \sum \check{v}_n U_n \in U^s$. To u and v are associated the linear forms

$$\forall\varphi \in C^\infty(\overline{\Gamma}), \quad \langle u, \varphi\rangle_{\frac{1}{\omega_\Gamma}} := \langle Ru, R\varphi\rangle_{\frac{1}{\omega}}\ ,$$

$$\forall\varphi \in C^\infty(\overline{\Gamma}), \quad \langle v, \varphi\rangle_{\omega_\Gamma} := \frac{|\Gamma|^2}{4}\,\langle Rv, R\varphi\rangle_{\omega}\ .$$

From the results of the previous section we deduce

Lemma 14. *For all $s \in \mathbb{R}$, $T^s(\Gamma)$ and $U^s(\Gamma)$ are Hilbert spaces for the scalar products*

$$(u, v)_{T^s(\Gamma)} = (Ru, Rv)_{T^s}\ ,$$

$$(u, v)_{U^s(\Gamma)} = \frac{|\Gamma|^2}{2}\,(Ru, Rv)_{U^s}\ .$$

With these definitions,

$$(u, v)_{T^0(\Gamma)} = \langle u, \overline{v}\rangle_{\frac{1}{\omega_\Gamma}} = \int_\Gamma \frac{u(x)\overline{v(x)}}{\omega_\Gamma(x)}dx$$

and

$$(u, v)_{U^0(\Gamma)} = \langle u, \overline{v}\rangle_{\omega_\Gamma} = \int_\Gamma \omega_\Gamma(x)u(x)\overline{v(x)}dx$$

thus $T^0(\Gamma) = L^2_{\frac{1}{\omega_\Gamma}}$ and $U^0(\Gamma) = L^2_{\omega_\Gamma}$. For $s \in \mathbb{R}$, the dual of $T^s(\Gamma)$ is the set of linear forms $\langle u, \cdot\rangle_{\frac{1}{\omega_\Gamma}}$ where $u \in T^{-s}$, and the dual of $U^s(\Gamma)$ is the set of linear forms $\langle u, \cdot\rangle_{\omega_\Gamma}$ where $u \in U^{-s}(\Gamma)$. For $s < t$, the injections $T^t(\Gamma) \subset T^s(\Gamma)$ and $U^t(\Gamma) \subset U^s(\Gamma)$ are compact. $(T^s(\Gamma))_{s\in\mathbb{R}}$ and $(U^s(\Gamma))_{s\in\mathbb{R}}$ are two

Hilbert interpolation scales. Equivalent scalar products on T^n and U^n are given respectively by

$$(u,v) \mapsto \int_\Gamma \frac{u(x)\overline{v(x)} + (\omega_\Gamma \partial_\tau)^n u(x) (\omega_\Gamma \partial_\tau)^n \overline{v(x)}}{\omega_\Gamma(x)} d\sigma(x)\,,$$

$$(u,v) \mapsto \int_\Gamma (\partial_\tau \omega_\Gamma)^n u(x) (\partial_\tau \omega_\Gamma \partial)^n \overline{v(x)} \omega_\Gamma(x) d\sigma(x)\,,$$

For all $s \in \mathbb{R}$, $T^s(\Gamma) \subset U^s(\Gamma)$ and for all $s > \frac{1}{2}$, $U^s(\Gamma) \subset T^{s-1}(\Gamma)$ with continuous inclusions. For $\varepsilon > 0$, $T^{1/2+\varepsilon}(\Gamma) \subset C^0(\Gamma)$ and $U^{3/2+\varepsilon} \subset C^0(\Gamma)$. Moreover, $T^\infty(\Gamma) = U^\infty(\Gamma) = C^\infty(\overline{\Gamma})$.

2 Application to Galerkin analysis

In this section, we introduce the Helmholtz scattering problem outside the curve Γ. We then apply the theory of the first section to this problem in the case of a zero wavenumber and flat geometry.

2.1 Helmholtz scattering by an open curve in $\mathbb{R}^2$

Recall the definition and parametrization of the curve Γ detailed in subsection 1.5. The problem of wave scattering by Γ is formalized by the bidimensional Helmholtz equation

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \tag{17}$$

when one considers furthermore Dirichlet (sound-soft) or Neumann (sound-hard) boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \tag{18}$$

or

$$\frac{\partial u}{\partial N} = u_N \text{ on } \Gamma \tag{19}$$

respectively. Suitable decay at infinity is also required, through Sommerfeld's radiation condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{20}$$

with $r = |x|$ for $x \in \mathbb{R}^2$. A classical method of resolution of (17) is that of integral formulation, which involves the resolution of the following boundary integral equations (BIEs)

$$S_k \lambda = u_D, \quad N_k \mu = u_N \tag{21}$$

where S_k and N_k are respectively the single-layer and hypersingular operators. We refer the reader to [1] and references therein for more details on the classical connection between eqs. (21) and the original problem (17). The operators S_k and N_k admit the integral representations

$$\begin{aligned} (S_k \lambda)(x) &= \int_\Gamma G_k(x-y) \lambda(y) d\sigma(y)\,, \\ (N_k \mu)(x) &= \lim_{\varepsilon \to 0^+} \int_\Gamma N(y) \cdot \nabla G_k(x + \varepsilon N(x) - y) \mu(y) d\sigma_y. \end{aligned} \tag{22}$$

for $x \in \Gamma$, with the Green function G_k is defined by

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi} \ln |z| \,, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{23}$$

where H_0 is the Hankel function of the first kind. It is known that S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ [14, Theorem 1.8] and N_k maps continuously $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ [14, Theorem 1.4]. In the case $k = 0$, the Helmholtz scattering reduces to the Laplace problem. The kernel of the hypersingular operator has a non-integrable singularity, but numerical calculations are facilitated by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned} \langle N_k \mu, \nu \rangle \quad = \quad & \int_{\Gamma \times \Gamma} G_k(x-y) \mu'(x) \nu'(y) \\ & -k^2 G_k(x,y) \mu(x) \nu(y) n(x) \cdot n(y) d\sigma_x d\sigma_y \,. \end{aligned} \tag{24}$$

For the geometry under consideration, the solutions λ and μ of the BIEs (21) have singularities (even for C^∞ data u_D and u_N) due to the presence of edges on the scatterer (see [1] and references therein). It is usual to introduce weighted versions of the usual layer potentials, known to enjoy better mapping properties than S_k and N_k. Namely, we define

$$S_{k,\omega_\Gamma} := S_k \frac{1}{\omega_\Gamma} \,, \quad N_{k,\omega_\Gamma} := N_k \omega_\Gamma \,, \tag{25}$$

and recast the BIEs (21) as $\lambda = \frac{\alpha}{\omega_\Gamma}$, $\mu = \omega_\Gamma \beta$ where α and β are found by solving the weighted BIEs

$$S_{k,\omega_\Gamma} \alpha = u_D \,, \quad N_{k,\omega_\Gamma} \beta = u_N \,. \tag{26}$$

When Γ is the segment $[-1,1] \times \{0\}$, the weight is ω (as in the previous section) and in this case, we simply denote the weighted integral operators by $S_{k,\omega}$ and $N_{k,\omega}$. The relation between N_k and N_k can be rewritten in terms of N_{k,ω_Γ} and S_{k,ω_Γ}:

Lemma 15. *There holds*

$$N_{k,\omega_\Gamma} = -\partial_\tau S_{k,\omega_\Gamma} \omega_\Gamma \partial_\tau \omega_\Gamma - k^2 V_k \omega_\Gamma^2$$

where V_k is the integral operator defined by

$$V_k u = \int_\Gamma \frac{G_k(x-y) N(x) \cdot N(y) u(y)}{\omega_\Gamma(y)} d\sigma(y) \,.$$

Proof. Eq. (24) can be rewritten equivalently as

$$N_k u = -\partial_\tau S_k \partial_\tau u - k^2 \int_\Gamma G_k(x-y) N(x) \cdot N(y) u(y) d\sigma(y) \,.$$

Using the definitions of N_{k,ω_Γ} and S_{k,ω_Γ}, the results follow from simple manipulations on this expression. ☐

2.2 Operators $S_{0,\omega}$ and $N_{0,\omega}$ on the flat segment

In this section, we consider the case where the wavenumber k is equal to 0 and $\Gamma = [-1,1] \times \{0\}$. The parametrization r is then the constant function equal to 1, $\partial_\tau = \partial_x$ and $\omega_\Gamma = \omega$. In this simple context, $S_{0,\omega}$ and $N_{0,\omega}$ have elementary properties that allow us to characterize T^s and U^s for $s = \pm\frac{1}{2}$.

Single layer potential The operator $S_{0,\omega}$ takes the form

$$S_{0,\omega}\alpha(x) = \int_{-1}^{1} \frac{\ln|x-y|\,\alpha(y)}{\sqrt{1-y^2}} dy\,.$$

There holds

$$S_{0,\omega}T_n = \sigma_n T_n \tag{27}$$

where

$$\sigma_n = \begin{cases} \dfrac{\ln(2)}{2} & \text{if } n = 0 \\ \dfrac{1}{2n} & \text{otherwise.} \end{cases}$$

Those identities are fundamental in our analysis. A proof can be found in [] reprendre ref. We deduce easily

Lemma 16. *The operator $S_{0,\omega}$ is a positive bicontinuous bijection from T^s to T^{s+1} for all $s \in \mathbb{R}$.*

In particular, $S_{0,\omega}$ maps T^∞ to itself, so the image of a smooth function by $S_{0,\omega}$ is a smooth function. We now proceed to show the following characterization of $T^{-1/2}$ and $T^{1/2}$. The next result, and Lemma 19 stated below are equivalent to results formulated in [4] (see equations (4.77-4.86), and Propositions 3.1 and 3.3 therein).

Lemma 17. *We have $T^{-1/2} = \omega\tilde{H}^{-1/2}(-1,1)$ and for all $u \in \tilde{H}^{-1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}\,.$$

Moreover, $T^{1/2} = H^{1/2}(-1,1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{T^{1/2}}$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the (unweighted) single-layer operator S_0 is positive and bounded from below on $\tilde{H}^{-1/2}(-1,1)$, (see [9] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1,1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle S_0 u, u\rangle}.$$

On the other hand, the explicit expression (27) imply that if $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_{0,\omega}\alpha, \alpha\rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since $\alpha = \omega u$, $\langle S_{0,\omega}\alpha, \alpha\rangle_{\frac{1}{\omega}} = \langle S_0 u, u\rangle$. This proves the first result. For the second result, we know that, $(H^{1/2}(-1,1))' = \tilde{H}^{-1/2}(-1,1)$

(taking the identification with respect to the usual L^2 duality denoted by $\langle \cdot, \cdot \rangle$, [8] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v \neq 0} \frac{\langle u, v \rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}} \,.$$

According to the previous result, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v \rangle = \langle u, \alpha \rangle_{\omega}$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha \neq 0} \frac{\langle u, \alpha \rangle_{\frac{1}{\omega}}}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$, showing the result. □

Hypersingular operator For $k = 0$ and when $\Gamma = [-1, 1] \times \{0\}$, the identity (15) becomes

$$\langle N_{0,\omega} \beta, \beta' \rangle_{\omega} = \langle S_{0,\omega} (\omega \partial_x \omega) \beta, (\omega \partial_x \omega) \beta' \rangle_{\frac{1}{\omega}}$$

Noticing that $(\omega \partial_x \omega) U_n = -(n+1) T_{n+1}$, we have for all $n \neq m$

$$\langle N_{0,\omega} U_n, U_m \rangle_{\omega} = 0 \,.$$

Therefore, we have

$$N_{0,\omega} U_n = \nu_n U_n$$

with $\nu_n \|U_n\|_{\omega}^2 = (n+1)^2 \sigma_{n+1} \|T_{n+1}\|_{\frac{1}{\omega}}^2$, that is, $\nu_n = \frac{(n+1)}{2}$. We deduce

Lemma 18. *The operator $N_{0,\omega}$ is a bicontinuous positive bijection from U^s to U^{s-1} for all $s \in \mathbb{R}$.*

In particular, $N_{0,\omega}$ maps smooth functions to smooth functions. As before, we obtain a characterization of U^s for $s = \pm \frac{1}{2}$ from the previous formula.

Lemma 19. *We have $U^{1/2} = \frac{1}{\omega} \tilde{H}^{1/2}(-1, 1)$ and for all $u \in \tilde{H}^{1/2}(-1, 1)$,*

$$\|u\|_{\tilde{H}^{1/2}} \sim \left\| \frac{u}{\omega} \right\|_{U^{1/2}} \,.$$

Moreover, $U^{-1/2} = H^{1/2}(-1, 1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{U^{1/2}} \,.$$

2.3 Galerkin method

A Galerkin method based on a refined mesh and weighted L^2 scalar products is described in [1] to solve eqs. (26) and orders of convergences are stated without proofs for the Laplace problem ($k = 0$) when $\Gamma = [-1, 1] \times \{0\}$. In this section we provide the proofs for those statements. Let us consider the following discretization of the segment $[-1, 1]$

$$-1 = x_0 < x_1 < \cdots < x_N = 1$$

where $x_i = \cos(i \frac{\pi}{N})$.

2.4 Dirichlet problem

Let V_h the Galerkin space of (discontinuous) piecewise affine functions on the mesh $(x_i)_{0\leq i\leq N}$ defined above, and α_h the unique solution in V_h to

$$(S_{0,\omega}\alpha_h, \alpha_h')_{\frac{1}{\omega}} = (u_D, \alpha_h')_{\frac{1}{\omega}}\,, \quad \forall \alpha_h' \in V_h\,. \tag{28}$$

Let $\lambda_h = \frac{\alpha_h}{\omega}$. Using the notation C to denote any constant that does not depend on the relevant parameters, we then have

Theorem 1. *If the data u_D is in T^{s+1} for some $-\frac{1}{2} \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2}\,\|\omega\lambda\|_{T^s} \leq Ch^{s+1/2}\,\|u_D\|_{T^{s+1}}\,.$$

Note that since $\mathcal{C}S_0 = S_{0,\omega}\mathcal{C}$, which is proved using the change of variables $\theta = \arccos(x)$, the variational problem (28) is equivalent, by Lemma 9, to

$$(S_0\varphi_h, \varphi_h')_{\mathbb{T}_{2\pi}} = (\mathcal{C}u_D, \varphi_h')_{\mathbb{T}_{2\pi}}\,, \quad \forall \alpha_h' \in \mathcal{C}V_h\,,$$

where $\varphi_h = \mathcal{C}\alpha_h$ and $(u,v)_{\mathbb{T}_{2\pi}} = \int_{-\pi}^{\pi} u(\theta)\overline{v(\theta)}dx$ is the usual L^2 scalar product on $\mathbb{T}_{2\pi}$. If instead of affine functions, V_h would contain affine functions of $\arccos(x)$, then φ_h and the test functions φ_h' would be piecewise affine functions of θ. The standard Galerkin theory then gives

$$\|\varphi_h - \varphi\|_{H^{-1/2}} \leq Ch^{s+\frac{1}{2}}\,\|\mathcal{C}u_D\|_{H^s}\,,$$

where $\varphi = \mathcal{C}\alpha$. This implies Theorem 1 since $\|\mathcal{C}u_D\|_{H^s} = \|u_D\|_{T^s}$ and

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} = \|\varphi - \varphi_h\|_{H^{-1/2}}\,.$$

Here instead, φ_h' are piecewise affine functions of $\cos\theta$ and to the best knowledge of the author, no approximation theory is available for this kind of basis functions in H^s_e. This is the main difficulty in the proof of Theorem 1.

Proof. Since $S_{0,\omega}$ is coercive in $T^{-1/2}$ by Lemma 16, we have an analog of Céa's lemma in our context:

$$\forall \alpha \in T^{-1/2}, \quad \|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha_h' \in V_h} \|\alpha - \alpha_h'\|_{T^{-1/2}}\,. \tag{29}$$

From this we deduce

$$\forall \alpha \in T^{-1/2} \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C\,\|\alpha\|_{T^{-1/2}}\,.$$

where Π_h is the Galerkin projection operator defined by $\Pi_h\alpha = \alpha_h$. In addition, we are going to show

$$\forall \alpha \in T^2 \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{5/2}\,\|\alpha\|_{T^2}\,. \tag{30}$$

By interpolation, this implies

$$\forall \alpha \in T^s\,, \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{s+1/2}\,\|\alpha\|_{T^s}\,.$$

The result then follows from this, since, on the one hand

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C \|\alpha - \alpha_h\|_{T^{-1/2}}$$

by Lemma 17 and, on the other hand, $\omega u = \alpha = S_{0,\omega}^{-1} u_D$ and therefore, by Lemma 16,

$$\|\alpha\|_{T^s} \leq C \|u_D\|_{T^{s+1}} .$$

As it is classical with piecewise affine discontinuous basis functions, we prove (30) by studying the properties of two particular operators: the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h and the interpolation operator I_h which maps a continuous function α to the function of V_h that matches α at the breakpoints x_i. Because of Céa's lemma, we have

$$\|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C \|(I_d - \mathbb{P}_h)\alpha\|_{T^{-1/2}} .$$

Therefore, it suffices to show (30) where Π_h is replaced by $\mathbb{P}_h$ to establish the result. We shall first show that

$$\forall \alpha \in T^s, \quad \|(I_d - \mathbb{P}_h)\alpha\| \leq Ch^s \|\alpha\|_{T^s} \tag{31}$$

for $s \in [0, 2]$. The estimate in $T^{-1/2}$ norm is then deduced by the classical duality method:

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h\alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}} ,$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N\alpha, \eta - \mathbb{P}_h\eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}} .$$

This, by Cauchy-Schwarz equality and (31), gives

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} \leq Ch^2 \|\alpha\|_{T^2} \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} ,$$

implying the desired etimate. The proof of (31) also follows the classical path: first, it is obvious that

$$\|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq C \|\alpha\|_{L^2_{\frac{1}{\omega}}} .$$

Using again an interpolation argument, we only need to show

$$\forall \alpha \in T^2, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} . \tag{32}$$

To this aim, we establish the following estimate:

$$\forall \alpha \in T^2, \quad \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} , \tag{33}$$

and conclude with $\|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}}$ since $\mathbb{P}_h$ minimizes the $L^2_{\frac{1}{\omega}}$ error. For $\alpha \in T^2$, $I_h\alpha$ is well-defined by Lemma 7. To prove (33), let us fix $\alpha \in T^2$. The function $\mathcal{C}(I_d - I_h)\alpha$ belongs to H^2 by Lemma 9 and vanishes on θ_i for all $i \in [0, N]$. It is well-known that for a function v in $H^2(a,b)$ which vanishes on a and b, there holds

$$\int_a^b |v(\theta)|^2 \, d\theta \leq C(b-a)^4 \int_a^b |\partial_{\theta\theta} v(\theta)|^2 \, d\theta \, . \tag{34}$$

Thus, applying this result on $[\theta_i, \theta_{i+1}]$ to $v = (\mathcal{C}(I_d - I_h)u)_{|[\theta_i,\theta_{i+1}]}$ and summing those inequalities for $i = 0$ to $N-1$, we obtain

$$\int_0^\pi |\mathcal{C}(I_d - I_h)\alpha(\theta)|^2 \, d\theta \leq h^4 \int_0^\pi |\partial_{\theta\theta}\mathcal{C}(I_d - I_h)\alpha(\theta)|^2 \ ,$$

in other words $\|\mathcal{C}(I_d - I_h)\alpha(\theta)\|_{\mathbb{T}_{2\pi}} \leq Ch^2 \, |\mathcal{C}(I_d - I_h)\alpha(\theta)|_{H^2_e}$. This, by Lemma 9, implies

$$\|(I_d - I_h)\alpha\|_{\frac{1}{\omega}} \leq Ch^2 \, |(I_d - I_h)\alpha|_{T^2} \ .$$

If $\mathcal{C}I_h\alpha$ were affine, the proof would end here since in this case $|I_h\alpha|_{T^2} = |\mathcal{C}I_h\alpha|_{H^2} = 0$. This does not hold in our case and is the main difference with the standard proof. Nevertheless, if we show that

$$|I_h\alpha|_{T^2} \leq C \, \|\alpha\|_{T^2} \ ,$$

eq. (33) follows. The expression of $\mathcal{C}I_h\alpha$ is

$$\mathcal{C}I_h\alpha(\theta) = \alpha(x_i) + \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)}(\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{C}I_h\alpha|^2 = \left| \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right|^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 = \left| \int_{x_i}^{x_{i+1}} \partial_x\alpha(x) dx \right|^2 ,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 \leq \int_{x_i}^{x_{i+1}} \frac{|\partial_x\alpha(x)|^2}{\omega(x)} dx \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

Notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. We deduce

$$\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{C}I_h\alpha|^2 \leq C \, \|\partial_x\alpha\|^2_{\frac{1}{\omega}}$$

Thus, $|I_h\alpha|_{T^2} \leq C \, \|\partial_x\alpha\|_{\frac{1}{\omega}}$. By Corollary 3, one has $\|\partial_x\alpha\|_{\frac{1}{\omega}} \leq C \, \|\alpha\|_{T^2}$, thus (33) is established, concluding the proof. □

Corollary 5. *For all* $(s,t) \in [-\frac{1}{2}, 2]$, *there exists a constant* $C_{t,s}$ *such that*

$$\|\alpha - \alpha_h\|_{T^s} \leq C_{t,s} h^{t-s} \|\alpha\|_{T^t} \,.$$

Proof. Let us first establish the inverse estimate

$$\forall \alpha_h' \in V_h, \, \forall s \in \mathbb{N}, \quad \|\alpha_h'\|_{T^s} \leq C_{t,s} h^{-s} \|\alpha_h'\|_{\frac{1}{\omega}} \,.$$

For this, we fix a function α_h' in V_h and restrict our attention on a fixed segment $[x_i, x_{i+1}]$. Let us first assume that s is an integer. Let $u_h(u) = \mathcal{C}\alpha_h(\theta_i + uh)$ defined on $[0,1]$. Notice that

$$\partial_u^k u_h(u) = h^k \partial_\theta^k \mathcal{C}\alpha_h'(\theta_i + hu) \,.$$

Since u_h belongs to a finite-dimensional space of dimension 2, all norms are equivalent, in particular there exists a constant $C_{t,s}$ such that

$$\|u_h\|_{H^s(0,1)} \leq C_{t,s} \|u_h\|_{L^2(0,1)}^2 \,.$$

Therefore

$$\begin{aligned}
|\mathcal{C}\alpha_h'|_{H^s(\theta_i,\theta_{i+1})}^2 &= h \int_0^1 h^{-2s} \left|\partial_u^s u_h(u)\right| du \\
&\leq C_{t,s} h^{-2s} \int_0^1 \left|\partial_u^t u_h(u)\right| h du \\
&\leq C_{t,s} h^{-2s} \|\mathcal{C}\alpha_h'\|_{L^2(\theta_i,\theta_{i+1})} \,.
\end{aligned}$$

as claimed. When s is not an integer, the result is deduced by interpolation. To establish the announced result, by interpolation, it suffices to show that $\|\alpha - \alpha_h\|_{T^2} \leq C \|\alpha\|_{T^2}$, and conclude by interpolation. This is achieved by writing

$$\|\alpha - \alpha_h\|_{T^2} \leq \|\alpha\|_{T^2} + \|I_h \alpha\|_{T^2} + \|I_h\alpha - \alpha_h\|_{T^2} \,.$$

We have seen in the previous proof that $\|I_h\alpha\|_{T^2} \leq C \|\alpha\|_{T^2}$. For the third term, we can use the inverse estimate:

$$\|I_h\alpha - \alpha_h\| \leq C h^{-2} \|I_h\alpha - \alpha_h\|_{\frac{1}{\omega}} \,.$$

Using triangular inequality

$$\|I_h\alpha - \alpha_h\| \leq C h^{-2} \left(\|\alpha - I_h\alpha\|_{\frac{1}{\omega}} + \|\alpha - \alpha_h\|_{\frac{1}{\omega}} \right) \,.$$

In the previous proof, we have shown that

$$\|\alpha - \alpha_h\|_{\frac{1}{\omega}} \leq C \|\alpha - I_h\alpha\|_{\frac{1}{\omega}} \leq C h^2 \|\alpha\|_{T^2}$$

and the proof is concluded. □

2.5 Neumann problem

Let W_h the Galerkin space of continuous piecewise affine functions on the mesh $(x_i)_{0\le i\le N}$ defined above, and β_h the unique solution in W_h to

$$(N_{0,\omega}\beta_h, \beta'_h)_\omega = (u_N, \beta'_h)_\omega\,, \quad \forall \beta'_h \in W_h\,. \tag{35}$$

Then, $\mu_h = \beta_h\omega$ is the proposed approximation for μ.

Theorem 2. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \le s \le 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \le Ch^{s-\frac{1}{2}} \left\|\frac{\mu}{\omega}\right\|_{U^s} \le Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}\,.$$

Proof. As a first step, one can easily establish the following analog of Céa's lemma:

$$\forall \beta \in U^{\frac{1}{2}}, \quad \|\beta - \beta_h\|_{U^{1/2}} \le C \inf_{\beta'_h \in W_h} \|\beta - \beta'_h\|_{U^{1/2}}\,.$$

In particular, denoting again by Π_h the Galerkin projection in this new context,

$$\forall \beta \in U^{1/2}, \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \le C\|\beta\|_{U^{1/2}}\,.$$

Once we prove

$$\forall \beta \in U^2, \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \le Ch^{\frac{3}{2}}\|\beta\|_{U^2}\,. \tag{36}$$

we get by interpolation

$$\forall \beta \in U^s, \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \le Ch^{s-\frac{1}{2}}\|\beta\|_{U^s}$$

for all $s \in [\frac{1}{2}, 2]$. The result follows since on the one hand

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} = \|\beta - \beta_h\|_{U^{1/2}}$$

by Lemma 19, and on the other hand, $\frac{\mu}{\omega} = \beta = N_{0,\omega}^{-1}u_N$, and therefore, by Lemma 18,

$$\|\beta\|_{U^s} \le C\|u_N\|_{U^{s-1}}\,.$$

Like before, the proof of (36) involves the study of the interpolation operator I_h. Namely, if we have

$$\forall \beta \in U^2, \|(I_d - I_h)\beta\|_\omega \le Ch^2\|\beta\|_{U^2}\,, \quad \|(I_d - I_h)\beta\|_{U^1} \le Ch\|\beta\|_{U^2}\,, \tag{37}$$

then, by interpolation, we obtain

$$\|(I_d - I_h)\beta\|_{U^{1/2}} \le Ch^{3/2}\|u\|_{U^2}\,,$$

which gives (36) after applying Céa's lemma. Let us show the first estimate in (37). Applying Lemma 9 and using again the property of H^2 functions vanishing at the boundary (34) one can write

$$\begin{aligned}\int_{x_i}^{x_{i+1}} \omega\,|(I_d - I_h)\beta|^2 &\le C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}(\mathcal{S}\beta - \mathcal{S}I_h\beta)|^2 \\ &\le Ch^4\left(2\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}\beta|^2 + 2\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}I_h\beta|^2\right).\end{aligned}$$

Summing for $i = 0, \cdots, N-1$, by Lemma 9, we get

$$\|(I_d - I_h)\beta\|_\omega \leq Ch^2 \left(\|\beta\|_{U^2} + |\mathcal{S}I_h\beta|_{H^2}\right) \tag{38}$$

As for the Dirichlet conditions, the proof would end here with another choice of basis functions, namely functions of the form

$$\phi_i(x) = \frac{a_i + b_i \arccos(x)}{\omega(x)},$$

because in this case, $\mathcal{S}I_h u$ would be affine and thus the second term in the right hand side would be 0. Here, we need to show that the second term is controlled by $\|\beta\|_{U^2}$. Using the expression of I_h, one can write

$$\begin{aligned}\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{S}I_h\beta|^2 \leq C \Bigg(&|\beta(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta d\theta \\ &+ \left|\frac{\beta(x_{i+1}) - \beta(x_i)}{\cos\theta_{i+1} - \cos\theta_i}\right|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta(1+\cos^2\theta)d\theta \Bigg)\end{aligned} \tag{39}$$

We can estimate the first term, thanks to Lemma 7:

$$|\beta(x_i)| \leq C \|\beta\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned}|\beta(x_{i+1}) - \beta(x_i)|^2 &= \left|\int_{x_i}^{x_{i+1}} \partial_x\beta\right|^2 \\ &\leq \int_{x_i}^{x_{i+1}} \omega\,|\partial_x\beta|^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\ &= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega\,|\partial_x\beta|^2.\end{aligned}$$

Observe that the quantity

$$\frac{|\theta_{i+1} - \theta_i| \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta(1+\cos^2\theta)d\theta}{(\cos\theta_i - \cos\theta_{i+1})^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2\theta_i$. Therefore, (38) becomes

$$\|(I_d - I_h)\beta\|_\omega \leq Ch^2 \left(\|\beta\|_{U^2} + \|\partial_x\beta\|_\omega\right).$$

Recalling Corollary 3, the second term in this estimate is controlled by $\|\beta\|_{U^2}$ and the first estimate of (37) is established. The second estimate of (37) can be shown in a similar manner, concluding the proof. □

Using Aubin-Nitsche's duality technique and inverse estimates as in the previous paragraph, one can check that the following result holds.

Corollary 6. *For all $(s,t) \in [0,2]$, there exists a constant $C_{s,t}$ such that*

$$\|\beta - \beta_h\|_{U^s} \leq C_{s,t} h^{t-s} \|\beta\|_{U^s}.$$

3 Pseudo-differential operators

Donner une raison pour laquelle on fait les pseudo-diffs. On veut analyser les préconditionneurs. On veut un calcul symbolique pour reproduire la preuve standard en termes d'op pseudo-diffs.

3.1 Periodic pseudo-differential operators

On the family of periodic Sobolev spaces H^s, a class of periodic pseudo differential operators (PPDO) is studied in [15]. We quickly review here the definitions and properties needed for our purposes. A PPDO of order α on H^s is an operator of the form

$$Au(\theta) = \sum_{n\in\mathbb{Z}} \sigma_A(\theta, n)\hat{u}_n e^{in\theta}\,.$$

for a "prolongated symbol" $\sigma_A \in C^\infty(\mathbb{T}_T \times \mathbb{R})$ satisfying

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \quad \left| D_\theta^j D_\xi^k \sigma_A(\theta,\xi)\right| \leq C_{j,k}(1+|\xi|)^{\alpha-k}\,. \tag{40}$$

Here, $\hat{u}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)e^{-in\theta}d\theta$ are the usual Fourier coefficients of u and

$$D_\theta := \frac{1}{i}\frac{\partial}{\partial\theta}, \quad D_\xi := \frac{1}{i}\frac{\partial}{\partial\xi},$$

with for $j \geq 1$, $D_\theta^{j+1} = D_\theta D_\theta^j$, and $D_\xi^{j+1} = D_\xi D_\xi^j$. The class of symbols that satisfy (40) is denoted by Σ^α, and $\Sigma^{-\infty} := \cup_{\alpha\in\mathbb{Z}}\Sigma^\alpha$. The operator defined by a symbol σ is denoted by $Op(\sigma)$ and the set of PPDOs of order α is denoted by $Op(\Sigma^\alpha)$.

The prolongated symbol is not unique but determined uniquely at the integer values of ξ by

$$\sigma_A(\theta, n) = e_{-n}(\theta)Ae_n(\theta)\,, \tag{41}$$

where $e_n : \theta \mapsto e^{in\theta}$, as shown in [15]. This justifies the terminology of "prolongated symbol". The operator A is in $Op(\Sigma^\alpha)$ if and only if

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \left| D_\theta^j \Delta_n^k \sigma_A(\theta,n)\right| \leq C_{j,k}(1+|n|)^{\alpha-k}\,,$$

where $\Delta_n\phi(\theta,n) = \phi(\theta,n+1) - \phi(\theta,n)$ and for $k \geq 1$, $\Delta_n^{k+1}\phi = \Delta_n(\Delta_n^k\phi)$. That is, if the symbol defined in (41) satisfies this condition, then there exists a prolongated symbol satisfying (40). Because of this, we write $\sigma \in \Sigma^p$ for a symbol $\sigma(\theta,n)$ that can be prolongated to a symbol $\sigma(\theta,\xi) \in \Sigma^p$. An operator in $Op(\Sigma^\alpha)$ maps continuously H^s to $H^{s+\alpha}$ for all $s \in \mathbb{R}$. The composition of two operators in $Op(\Sigma^\alpha)$ and $Op(\Sigma^\beta)$ gives rise to an operator in $Op(\Sigma^{\alpha+\beta})$. If two symbols a and b in $\Sigma^{-\infty}$ satisfy $a - b \in \Sigma^\alpha$, we write $a = b + \Sigma^\alpha$.

Definition 6. *Let $a \in \Sigma^{-\infty}$. If there exists a sequence of reals $(p_j)_{j\in\mathbb{N}}$ such that $p_j < p_{j+1}$ and a sequence of symbols $a_j \in \Sigma^{p_j}$ such that for all N, $a = \sum_{i=0}^N a_i + \Sigma^{p_j+1}$, we write*

$$a = \sum_{i=0}^{+\infty} a_j\,.$$

This is called an asymptotic expansion of the symbol a.

The symbol of the composition of two PPDOs A and B is denoted by $\sigma_A \# \sigma_B$ and satisfies the asymptotic expansion

$$\sigma_A \# \sigma_B(t,\xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \sigma_A(\theta,\xi) D_\theta^j \sigma_B(\theta,\xi)\,. \tag{42}$$

We will also use the following result, proved in [15]:

Theorem 1. *Consider an integral operator K of the form*

$$K : u \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} a(\theta,\theta')\kappa(\theta-\theta')u(\theta')d\theta'.$$

where a is 2π-periodic and C^∞ in both arguments and κ is a 2π-periodic distribution. Assume that the Fourier coefficients $\hat{\kappa}(n)$ of κ can be prolonged to a function $\hat{\kappa}(\xi)$ on $\mathbb{R}$ such that

$$\forall k \in \mathbb{N}, \exists C_k > 0 : \quad \left|\partial_\xi^k \hat{\kappa}(\xi)\right| \leq C_k (1+|\xi|)^{\alpha-k}\,.$$

for some α. Then K is in $Op(\Sigma^\alpha)$ with a symbol satisfying the asymptotic expansion

$$\sigma_K(\theta,\xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \hat{\kappa}(\xi) D_t^j a(t,\theta)_{|t=\theta}\,. \tag{43}$$

In particular, taking $\kappa = 1$, we see that for all functions $a \in C^\infty(\mathbb{T}_T^2)$

$$Ku = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(\theta,\theta')u(\theta')d\theta'$$

is in $Op\,(\Sigma^{-\infty})$.

3.2 Pseudo-differential operators on $T^s(\Gamma)$

Lemma 20. *Let A a PPDO that stabilizes the set of smooth even functions. Then A coincides on this set with the operator B defined by the symbol*

$$\sigma_B(\theta,n) = \frac{\sigma_A(\theta,n) + \sigma_A(-\theta,-n)}{2}\,.$$

Moreover, σ_B admits the following decomposition:

$$\sigma_B(\theta,n) = a_1(\cos\theta,n) + i\sin(\theta) a_2(\cos\theta,n)$$

with

$$a_1(x,n) = \frac{\sigma_B(\arccos(x),n) + \sigma_B(\arccos(x),-n)}{2}$$

$$a_2(x,n) = \frac{\sigma_B(\arccos(x),n) - \sigma_B(\arccos(x),-n)}{2i\sqrt{1-x^2}}$$

and a_1 and a_2 are C^∞ in x. The functions a_1 and a_2 thus defined are denoted by $a_1^T(A)$ and $a_2^T(A)$.

Proof. For a smooth even function u, one has

$$Au(\theta) = \frac{Au(\theta) + Au(-\theta)}{2}\,,$$

thus

$$Au(\theta) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \sigma_A(\theta, n)\hat{u}_n e^{in\theta} + \frac{1}{2}\sum_{n\in\mathbb{Z}} \sigma_A(-\theta, n)\hat{u}_n e^{-in\theta}\,.$$

Since u is even, $\hat{u}_n = \hat{u}_{-n}$, so that $Au(\theta) = Bu(\theta)$ where B is the operator with symbol $\sigma_B(\theta, n) = \frac{\sigma_A(\theta,n)+\sigma_A(-\theta,-n)}{2}$. In particular, it satisfies the following symmetry:

$$\sigma_B(-\theta, -n) = \sigma_B(\theta, n)\,.$$

We write $\sigma_B(\theta, n) = f_B(\theta, n) + g_B(\theta, n)$ where $f_B(\theta, n) = \frac{\sigma_B(\theta,n)+\sigma_B(\theta,-n)}{2}$ and $g_B(\theta, n) = \frac{\sigma_B(\theta,n)-\sigma_B(\theta,-n)}{2}$. Notice that f_B (resp. g_B) is even (resp. odd) in both θ and n. The functions a_1 and a_2 defined in the statement of the Lemma satisfy

$$a_1(x, n) = f_B(\arccos(x), n), \quad a_2(x, n) = \frac{g_B(\arccos(x), n)}{i\sqrt{1 - x^2}}\,,$$

thus

$$\sigma_B(\theta, n) = a_1(\cos\theta, n) + i\sin\theta a_2(\cos\theta, n)\,.$$

For fixed n, there holds $a_1(\cdot, n) = \mathcal{C}^{-1}f_B(\cdot, n)$ and $a_2(\cdot, n) = -i\mathcal{S}^{-1}g_B$. By Lemma 9, a_1 and a_2 are thus C^∞ in x since f_B (resp. g_B) is a smooth even (resp. odd) function. $\square$

We use this result to transport the notion of periodic pseudo-differential operators to the segment $[-1, 1]$ by the change of variable $x = \cos\theta$.

Definition 7. *Let A an operator on $T^{-\infty}$ and assume that there exists a couple of smooth functions a_1 and a_2 in $C^\infty([-1, 1] \times \mathbb{N})$ such that for all $n \in \mathbb{N}$,*

$$AT_n = a_1(x, n)T_n - \omega^2 a_2(x, n)U_{n-1}\,. \tag{44}$$

with by convention, $U_{-1} = 0$. Such a (non-unique) couple of functions is called a pair of symbols of A. For $n \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$, define the symbol $\tilde{\sigma}_T(a_1, a_2)$ by

$$\tilde{\sigma}_T(a_1, a_2)(\theta, n) := a_1(\cos\theta, |n|) + i\sin\theta\, sign(n) a_2(\cos\theta, |n|)\,.$$

We say that $(a_1, a_2) \in S^\alpha_T$ if $\tilde{\sigma}_T(a_1, a_2) \in \Sigma^\alpha$. We also take the notation $S^\infty_T := \cup_{\alpha\in\mathbb{R}} S^\alpha_T$. The operator defined by a pair of symbols (a_1, a_2) is denoted by $Op_T(a_1, a_2)$ and the set of pseudo-differential operators (of order α) in $T^{-\infty}$ by $Op(S^\infty_T)$ ($Op(S^\alpha_T)$).

Recall the definition of the isometric mapping $\mathcal{C}$ from Lemma 9.

Theorem 2. *Let $(a_1, a_2) \in S^\alpha_T$ and $A = Op_T(a_1, a_2)$. There holds*

$$\mathcal{C}A = \tilde{A}\mathcal{C}$$

where $\tilde{A} = Op(\tilde{\sigma}_T(a_1, a_2))$. Reciprocally, let $A : T^\infty \to T^{-\infty}$ a linear operator satisfying

$$\forall u \in T^\infty, \quad \mathcal{C}Au = \tilde{A}\mathcal{C}u$$

where $\tilde{A}$ is a PPDO of order α with a symbol $\sigma_{\tilde{A}}$. Then A has a unique linear continuous extension on $T^{-\infty}$ satisfying $\mathcal{C}A = \tilde{A}\mathcal{C}$. This extension does not depend on $\tilde{A}$ and is in $Op(S_T^\alpha)$. Moreover, A admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$.

Proof. For the direct result, we start by showing the equality for $u = T_n$ for all $n \in \mathbb{N}$. One has $\mathcal{C}T_n(\theta) = T_n(\cos(\theta)) = \cos(n\theta)$. Consequently,

$$\tilde{A}\left(\mathcal{C}T_n\right)(\theta) = \frac{\tilde{A}e^{in\theta} + \tilde{A}e^{-in\theta}}{2} \tag{45}$$

which, using the determination of the symbol (41), yields

$$\tilde{A}\left(\mathcal{C}T_n\right)(\theta) = \frac{\sigma(\theta, n)e^{in\theta} + \sigma(\theta, -n)e^{-in\theta}}{2}\,.$$

where $\sigma = \tilde{\sigma}_T(a_1, a_2)$. Replacing this definition in the former equation, one gets

$$\tilde{A}\left(\mathcal{C}u\right)(\theta) = a_1(\cos\theta, n)\cos(n\theta) - \sin\theta a_2(\cos\theta, n)\sin(n\theta).$$

Since $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n\theta) = \sin\theta U_{n-1}(\cos\theta)$,

$$\begin{aligned}\tilde{A}\left(\mathcal{C}T_n\right)(\theta) &= a_1(\cos\theta, n)T_n(\cos\theta) - (1-\cos^2\theta)a_2(\cos\theta, n)U_{n-1}(\cos\theta)\,,\\ &= \mathcal{C}(AT_n)\end{aligned} \tag{46}$$

as claimed. To show the general case, fix $u \in T^{-\infty}$ and $\tilde{v} \in H^{-\infty}$. One has, by linearity and continuity of A, $\tilde{A}$ and $\mathcal{C}$:

$$\begin{aligned}\langle \mathcal{C}Au, \tilde{v}\rangle_{\mathbb{T}_{2\pi}} &= \sum_{n=0}^{+\infty} \hat{u}_n \langle \mathcal{C}AT_n, \tilde{v}\rangle_{\mathbb{T}_{2\pi}}\\ &= \sum_{n=0}^{+\infty} \hat{u}_n \left\langle \tilde{A}\mathcal{C}T_n, \tilde{v}\right\rangle_{\mathbb{T}_{2\pi}}\\ &= \left\langle \tilde{A}\mathcal{C}u, \tilde{v}\right\rangle_{\mathbb{T}_{2\pi}}\,.\end{aligned}$$

The last identity shows that $\mathcal{C}Au = \tilde{A}\mathcal{C}u$ for all $u \in T^{-\infty}$, in other words, $\mathcal{C}A = \tilde{A}\mathcal{C}$. For the converse result, we now assume that for any $u \in T^\infty$, $\mathcal{C}Au = \tilde{A}\mathcal{C}u$ where $\tilde{A}$ is some PPDO of order α with a symbol $\sigma_{\tilde{A}}$. The previous computations show that any linear continuous extension of A satisfies

$$\mathcal{C}A = \tilde{A}\mathcal{C}\,. \tag{47}$$

This in turn defines uniquely the operator A on $T^{-\infty}$ since for $u \in T^{-\infty}$ and $v \in T^\infty$, one has, by Lemma 8,

$$\langle Au, v\rangle_{\frac{1}{\omega}} = \langle \mathcal{C}Au, \mathcal{C}v\rangle_{\mathbb{T}_{2\pi}}\,.$$

Obviously, this definition does not depend on $\tilde{A}$. Let us show that A sends T^∞ to T^∞. Let $u \in T^\infty$ and let $s \in \mathbb{R}$. Using (47), the continuity of $\tilde{A}$ from $H^{s+\alpha}$ to H^s and the isometric property of $\mathcal{C}$,

$$\begin{aligned}\left\|Au\right\|_{T^s} &= \left\|\mathcal{C}Au\right\|_{H^s}\\ &= \left\|\tilde{A}\mathcal{C}u\right\|_{H^s}\\ &\leq C\left\|\mathcal{C}u\right\|_{H^{s+\alpha}}\\ &\leq C\left\|u\right\|_{T^{s+\alpha}}\,.\end{aligned}$$

The last quantity is finite since $u \in T^\infty \subset T^{s+\alpha}$. This proves that A sends T^∞ to T^∞.
Eq (47) implies in particular that $\tilde{A}$ stabilizes the set of smooth even functions since $\mathcal{C}Au(\theta) = Au(\cos\theta)$ is even and A maps smooth functions to smooth functions. Lemma 20 can thus be applied. Let $a_1 = a_1^T(\tilde{A})$ and $a_2 = a_2^T(\tilde{A})$. Starting from eq. (47), the computations from eqs. (45) to (46) can be performed in reverse order to show

$$AT_n(\cos\theta) = a_1(n, \cos\theta)T_n(\cos\theta) - (1 - \cos^2\theta)a_2(n, \cos\theta)U_{n-1}(\cos\theta)\,,$$

which, taking $x = \cos\theta$, leads to $A = Op_T(a_1, a_2)$. To establish that A is in $Op(S_T^\alpha)$, we have to show $\tilde{\sigma}(a_1, a_2) \in \Sigma^\alpha$. By Lemma 20, this is exactly the symbol σ_B defined by

$$\sigma_B(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}\,.$$

This is indeed in Σ^α since it is the case for $\sigma_{\tilde{A}}$ by assumption. □

Remark 1. *When $A \in Op(S_T^\alpha)$, there is an infinity of operators $\tilde{A}$ satisfying $\mathcal{C}A = \tilde{A}\mathcal{C}$. Indeed, if this holds for some $\tilde{A}$, it also holds for $\tilde{A} + B$ where B is any PPDO of order α with the property that $Bu = 0$ when u is even. This non-uniqueness is also reflected by the fact that the couple of symbols of an operator A in $Op(S_T^\alpha)$ is not unique, or in other words, the null operator has non-trivial pair of symbols in $S_T^{-\infty}$. For example take a_1 and a_2 as follows: fix $n_0 \in \mathbb{N}$ and let*

$$a_1(x, n_0) = -\omega^2 U_{n_0-1}(x)\,, \quad a_2(x, n_0) = T_{n_0}(x)$$

while $a_1(x, n) = a_2(x, n) = 0$ for $n \neq 0$. Obviously, $(a_1, a_2) \in S_T^{-\infty}$ and $Op_T(a_1, a_2) \equiv 0$. One idea to enforce uniqueness would be to take for $\tilde{A}$ the operator $\tilde{A}^*$ satisfying $\tilde{A}^*u = 0$ whenever u is odd. Such a condition would demand the following symmetry on the symbol $\sigma_{\tilde{A}^*}$:

$$\sigma_{\tilde{A}^*}(\theta, -n) = e^{2in\theta}\sigma_{\tilde{A}^*}(\theta, n)\,.$$

One can show that if $\mathcal{C}A = \tilde{A}\mathcal{C}$ for some operator $\tilde{A}$, then the symbol of $\tilde{A}^*$ must be given by

$$\sigma_{\tilde{A}^*}(\theta, n) = \sigma_{\tilde{A}}(\theta, n) + e^{-2in\theta}\sigma_{\tilde{A}}(\theta, -n)\,.$$

However, in general, this symbol is not in Σ^α because of the oscillatory term $e^{-2in\theta}$. In other words, one cannot always construct an operator $\tilde{A}^*$ satisfying the following three conditions

- *$\tilde{A}^*$ coincides on the set of even functions with some given PPDO $\tilde{A}$ of order α.*
- *$\tilde{A}^*$ vanishes on the set of odd functions*
- *$\tilde{A}^*$ is a PPDO of order α.*

As a conclusion, it is not clear how to fix a natural representative in the class of pairs (a_1, a_2) that define the same operator A.

Definition 8. *Let $A : T^{-\infty}(\Gamma) \to T^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $T^{-\infty}(\Gamma)$ if $RAR^{-1} \in Op(S_T^\infty)$ ($\in Op(S_T^\alpha)$). The set of pseudo-differential operators of order α on $T^{-\infty}(\Gamma)$ is denoted by $Op(S_T^\alpha(\Gamma))$. We say that (a_1, a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1}.*

As a corollary of Theorem 2, we have he following properties

Corollary 7. *Let $A \in Op(S_T^\alpha(\Gamma))$. Then for all s, A is continuous from $T^s(\Gamma)$ to $T^{s-\alpha}(\Gamma)$. If B and C respectively belong to $Op(S_T^{\alpha_1}(\Gamma))$ and $Op(S_T^{\alpha_2}(\Gamma))$, with pairs of symbols (b_1, b_2) and (c_1, c_2), then BC is in $Op(S_T^{\alpha_1+\alpha_2}(\Gamma))$ and admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$ where*

$$\tilde{A} = Op(\tilde{\sigma}_T(b_1, b_2))Op(\tilde{\sigma}_T(c_1, c_2)) = Op(\tilde{\sigma}_T(b_1, b_2)\#\tilde{\sigma}_T(c_1, c_2))\,.$$

Proof. Let $A \in Op(S_T^\alpha(\Gamma)$ and $s \in \mathbb{R}$. By Theorem 2, there exists $\tilde{A} \in Op(\Sigma^\alpha)$ such that

$$\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$$

Using the definition of the norm on $T^s(\Gamma)$, the isometric property of $\mathcal{C}$ and the continuity of $\tilde{A}$ from H^s to $H^{s-\alpha}$, we have for all $u \in T^s(\Gamma)$,

$$\begin{aligned}\|Au\|_{T^{s-\alpha}(\Gamma)} = \|RAu\|_{T^{s-\alpha}} = \|\mathcal{C}RAu\|_{H^{s-\alpha}} = \left\|\tilde{A}\mathcal{C}Ru\right\|_{H^{s-\alpha}} \\ \leq C\,\|\mathcal{C}Ru\|_{H^s} = C\,\|Ru\|_{T^s} = C\,\|u\|_{T^s(\Gamma)}\,.\end{aligned}$$

très moche, non ? ça prend beaucoup de ligne si on fait une par une.
Let $B, C \in Op(S_T^{\alpha_1}(\Gamma)) \times Op(S_T^{\alpha_2}(\Gamma))$, with respective pairs of symbols (b_1, b_2) and (c_1, c_2). Let $\tilde{B} = Op(\tilde{\sigma}(b_1, b_2))$ and $\tilde{C} = Op(\tilde{\sigma}(c_1, c_2))$. We have

$$\mathcal{C}RBR^{-1} = \tilde{B}\mathcal{C}, \quad \text{and} \quad \mathcal{C}RCR^{-1} = \tilde{C}\mathcal{C}\,.$$

Therefore,

$$\mathcal{C}RBCR^{-1} = \tilde{A}\mathcal{C}.$$

where $\tilde{A} = \tilde{B}\tilde{C}$. One has $\tilde{A} \in Op(\Sigma^{\alpha_1+\alpha_2})$. By Theorem 2, $RBCR^{-1}$ is in $Op(S^{\alpha_1+\alpha_2})$ and admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$. By definition, this means that $BC \in Op(S^{\alpha_1+\alpha_2}(\Gamma))$ and admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$. □

Remark 2. *The previous result gives a method for a symbolic calculus on the class $S_T^\alpha(\Gamma)$ as follows. If B and C respectively admit the pair of symbols (b_1, b_2) and (c_1, c_2), then BC admits the pair of symbols*

$$(b_1, b_2)\#_T(c_1, c_2) := (a_1^T(\tilde{\sigma}_T(b_1, b_2)\#\tilde{\sigma}_T(c_1, c_2)), a_2^T(\tilde{\sigma}_T(b_1, b_2)\#\tilde{\sigma}_T(c_1, c_2)))\,.$$

One can use (42) to compute an asymptotic expansion of $\tilde{\sigma}(b_1, b_2)\#\tilde{\sigma}(c_1, c_2)$ which, in turn, gives an asymptotic expansion of $(b_1, b_2)\#_T(c_1, c_2)$. The proofs of Theorem 5 and Theorem 6 rely on this method, but the details of the computation are omitted as they are quite heavy. In compensation, a Maple code giving procedures for the symbolic calculus is provided in [6].

3.3 Pseudo-differential operators on $U^s(\Gamma)$

We define similarly a class of pseudo-differential operators on the spaces $U^s(\Gamma)$. One can show the following result:

Lemma 21. *Let A a PPDO that stabilizes the set of smooth odd functions. Then A coincides on this set with the operator B with symbol given by*

$$\sigma_B(n,\theta) = \frac{\sigma_A(\theta,n) + \sigma_A(-\theta,-n)}{2}\,.$$

Moreover, σ_B admits the following decomposition

$$\sigma_B(n,\theta) = ia_1(\cos\theta, n) + \sin\theta a_2(\cos\theta, n)$$

with

$$a_1(x,n) = \frac{\sigma_B(\arccos(x),n) + \sigma_B(\arccos(x),-n)}{2i}$$
$$a_2(n,x) = \frac{\sigma_B(\arccos(x),n) - \sigma_B(\arccos(x),-n)}{2\sqrt{1-x^2}}$$

and a_1 and a_2 are C^∞. The functions a_1 and a_2 thus defined are denoted by $a_1^U(A)$ and $a_2^U(A)$.

Let A an operator on $U^{-\infty}$ and assume that there exists a couple of smooth functions a_1 and a_2 in $C^\infty([-1,1]\times\mathbb{N})$ such that for all $n \in \mathbb{N}$,

$$AU_n = a_1(x,n)U_n + a_2(x,n)T_{n+1}\,.$$

Such a (non-unique) couple of functions is called a pair of symbols of A. For $n \in \mathbb{Z}$ and $\theta \in [0,2\pi]$, define the symbol $\tilde{\sigma}_U(a_1,a_2)$ by

$$\tilde{\sigma}_U(a_1,a_2)(\theta,n) = ia_1(\cos\theta,|n|) + \sin\theta\,\mathrm{sign}(n)a_2(\cos\theta,|n|)\,.$$

We say that $(a_1,a_2) \in S_U^\alpha$ if $\tilde{\sigma}_U(a_1,a_2) \in \Sigma^\alpha$, and $S_U^\infty := \cup_{\alpha\in\mathbb{Z}} S_U^\alpha$. The operator defined by a pair of symbols (a_1,a_2) is denoted by $Op_U(a_1,a_2)$ and the set of pseudo-differential operatos of order α in $U^{-\infty}$ by $Op(S_U^\alpha)$. Recall the definition of the isometric mapping $\mathcal{S}$ from Lemma 9. Adapting the proof of Theorem 2, oe can show

Theorem 3. *Let $(a_1,a_2) \in S_U^\alpha$ and $A = Op_U(a_1,a_2)$. There holds*

$$\mathcal{S}A = \tilde{A}\mathcal{S}$$

where $\tilde{A} = Op(\tilde{\sigma}_U(a_1,a_2))$. Reciprocally, let $A : T^\infty \to T^{-\infty}$ a linear operator satisfying

$$\forall u \in T^\infty, \quad \mathcal{S}Au = \tilde{A}\mathcal{S}u$$

where $\tilde{A}$ is a PPDO of order α with a symbol $\sigma_{\tilde{A}}$. Then A has a unique linear continuous extension on $T^{-\infty}$ satisfying $\mathcal{S}A = \tilde{A}\mathcal{S}$. This extension is in $Op(S_U^\alpha)$ and A admits the pair of symbols $\left(a_1^U(\tilde{A}), a_2^U(\tilde{A})\right)$.

Definition 9. *Let $A : U^{-\infty}(\Gamma) \to U^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $U^{-\infty}(\Gamma)$ if $RAR^{-1} \in Op(S_U^\infty)$ ($\in Op(S_U^\alpha)$). The set of pseudo-differential operators of order α on $U^{-\infty}(\Gamma)$ is denoted by $Op(S_U^\alpha(\Gamma))$. We say that (a_1,a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1}.*

Corollary 8. *Let $A \in Op(S^{\alpha}_{U}(\Gamma))$. Then for all s, A is continuous from U^s to $U^{s-\alpha}$. If B and C respectively belong to $Op(S^{\alpha_1}_{U}(\Gamma))$ and $Op(S^{\alpha_2}_{U}(\Gamma))$, with pairs of symbols (b_1, b_2) and (c_1, c_2), then BC is in $Op(S^{\alpha_1+\alpha_2}_{U}(\Gamma))$ and admits the pair of symbols $(a_1^U(\tilde{A}), a_2^U(\tilde{A}))$ where*

$$\tilde{A} = Op(\tilde{\sigma}_U(b_1, b_2))Op(\tilde{\sigma}_U(c_1, c_2)) = Op(\tilde{\sigma}_U(b_1, b_2)\#\tilde{\sigma}_U(c_1, c_2))\,.$$

Lemma 22. *Let $A \in Op(S^{\alpha}_{T}(\Gamma))$ and $B = -\partial_\tau A\omega_\Gamma\partial_\tau\omega_\Gamma$. Then $B \in Op(S^{\alpha+2}_{U}(\Gamma))$ and if $\tilde{A}$ is a PPDO such that $\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$, then $\mathcal{S}RBR^{-1} = -\partial_\theta\tilde{A}\partial_\theta\mathcal{S}$.*

Proof. On can check the following identities:

$$\begin{aligned}\partial_\theta\mathcal{S} &= -\mathcal{C}\omega\partial_x\omega\,,\\ \partial_\theta\mathcal{C} &= -\mathcal{S}\partial_x\,.\end{aligned}$$

Let $A' = RAR^{-1}$ and $B' = RBR^{-1}$. Assuming that $\mathcal{C}A' = \tilde{A}\mathcal{C}$, there holds

$$\begin{aligned}\mathcal{S}B' &= -\mathcal{S}R\partial_\Gamma A\omega_\Gamma\partial_\Gamma\omega_\Gamma R^{-1}\\ &= -\mathcal{S}\partial_x A'\omega\partial_x\omega\\ &= \partial_\theta\mathcal{C}A'\omega\partial_x\omega\\ &= \partial_\theta\tilde{A}\mathcal{C}\omega\partial_x\omega\\ &= -\partial_\theta\tilde{A}\partial_\theta\mathcal{S}\,.\end{aligned}$$

Since $\tilde{A}$ can be chosen as a PPDO of order α by Theorem 2, $\partial_\theta\tilde{A}\partial_\theta$ is then a PPDO of order $\alpha+2$ and by Theorem 3, we conclude that $B \in Op(S^{\alpha+2}_{U}(\Gamma))$. $\square$

Lemma 23. *Let $A \in Op(S^{\alpha}_{T}(\Gamma))$ and $B = A\omega_\Gamma^2$. Then $B \in Op(S^{\alpha}_{U}(\Gamma))$ and if $\tilde{A}$ is a PPDO such that $\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$, then $\mathcal{S}RBR^{-1} = \sin\tilde{A}\sin\mathcal{S}$ where* $\sin$ *denotes the operator $f(\theta) \mapsto \sin(\theta)f(\theta)$.*

Proof. This follows from the identities

$$\mathcal{S} = \sin\mathcal{C}, \quad \mathcal{C}\omega^2 = \sin\mathcal{S}$$

and the same arguments as in the proof of Lemma 22. $\square$

Definition 10. *Let A and B in $Op(S^{\infty}_{T}(\Gamma))$ (resp. $Op(S^{\infty}_{U}(\Gamma))$). If $A - B \in Op(S^{\alpha}_{T}(\Gamma))$ (resp. $Op(S^{\alpha}_{U}(\Gamma))$), we write $A = B + T_\alpha$ (resp. $A = B + U_\alpha$).*

4 Application to Helmholtz scattering

In this section, we apply the analytical tools introduced in the previous section to the study of the Helmholtz scattering problems. The object of this section is to prove Theorem 5 and Theorem 6. We start by introducing the notations, and characterize the spaces T^s and U^s for $s = \pm\frac{1}{2}$.

4.1 Layer potentials

Faire les mêmes modifs que sur le papier concis.

Recall the parametrization of the curve Γ **??**. We seek a solution to the two problems

$$-\Delta u_i - k^2 u_i = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma, \quad i = 1,2 \tag{48}$$

with the following additional conditions

- Dirichlet or Neumann boundary conditions, respectively

$$u_1 = u_D, \text{ and } \frac{\partial u_2}{\partial n} = u_N \text{ on } \Gamma \tag{49}$$

  where $\frac{\partial u}{\partial n} = n_\Gamma \cdot \nabla u$.

- Suitable decay at infinity, given for $k > 0$ by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{50}$$

  with $r = |x|$ for $x \in \mathbb{R}^2$.

When $k = 0$, the radiation condition must be replaced by an appropriate decay of u and ∇u at infinity, see for example [13, 14], or [9, Chap. 7] Vérifier le chapitre et la page. Existence and uniqueness results are available for those problems, but the solutions fail to be regular even with smooth data u_D and u_N. More precisely, let $\lambda = \left[\frac{\partial u_1}{\partial n}\right]_\Gamma$ and $\mu = [u_2]_\Gamma$ where $[\cdot]_\Gamma$ refers to the jump of a quantity across Γ, we have the following result.

Theorem 4. *(see e.g. [10, 13, 14]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (17,49,20) both possess a unique solution $u_i \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ, λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

while μ satisfies

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$.

For the definition of Sobolev spaces on smooth open curves, we follow [9] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma})\right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma})\right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential We define the single-layer potential by

$$\mathcal{S}_k\lambda(x) = \int_\Gamma G_k(x-y)\lambda(y)d\sigma(y) \tag{51}$$

where G_k is the Green's function

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi}\ln|z|\,, & \text{if } k=0,\\ G_k(z) = \dfrac{i}{4}H_0(k|z|), & \text{if } k>0, \end{cases} \tag{52}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here H_0 is the Hankel function of the first kind. For $k > 0$, the solution u_1 to the Dirichlet problem admits the representation

$$u_1 = \mathcal{S}_k\lambda \tag{53}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the jump of the normal derivative of u_1 across Γ and is the unique solution to

$$S_k\lambda = u_D\,. \tag{54}$$

Here, $S_k := \gamma\mathcal{S}_k$ where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. When $k = 0$, the computation of u_1 also involves the resolution of (54) but some subtleties arise in the representation of u_1 by (53). On this topic, see [13, Theorem 1.4].

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k\mu(x) = \int_\Gamma N(y)\cdot\nabla G_k(x-y)\mu(y)d\sigma(y), \quad x \in \mathbb{R}^2\setminus\Gamma$$

for any smooth function μ defined on Γ. The normal derivative of $\mathcal{D}_k\mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \frac{\partial}{\partial_n}\mathcal{D}_k$. This operator admits the following representation for $x \in \Gamma$

$$N_k\mu(x) = \lim_{\varepsilon\to 0^+}\frac{\partial}{\partial\varepsilon}\int_\Gamma N(y)\cdot\nabla G(x+\varepsilon N(x)-y)\mu(y)d\sigma(y). \tag{55}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned}\langle N_k\mu,\nu\rangle_{L^2(\Gamma)} = \int_{\Gamma\times\Gamma} & G_k(x-y)\partial_\tau\mu(x)\partial_\tau\nu(y)\\ & - k^2G_k(x-y)\mu(x)\nu(y)N(x)\cdot N(y)d\sigma(x)d\sigma(y)\,.\end{aligned} \tag{56}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u_2 to the Neumann problem can be written as

$$u_2 = \mathcal{D}_k\mu \tag{57}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ is the jump of u_2 across Γ and is the unique solution to

$$N_k\mu = u_N\,. \tag{58}$$

4.2 Non-flat arc and non-zero frequency

We now turn to the general case of a non-flat arc and non-zero frequency, and prove the results announced in [1].

Dirichlet problem

Lemma 24. *The operator S_{k,ω_Γ} is in $Op(S_T^{-1}(\Gamma))$, and satisfies*

$$\mathcal{C}RS_{k,\omega_\Gamma}R^{-1} = \tilde{S}_k\mathcal{C}$$

where the symbol of $\tilde{S}_k \in Op(\Sigma^{-1})$ has the asymptotic expansion

$$\begin{aligned}\sigma_{\tilde{S}_k}(\theta,\xi) = &\frac{1}{2\xi} + \frac{k^2\left|\Gamma\right|^2\sin(\theta)^2}{16\xi^3} + \frac{3ik^2|\Gamma|^2\sin\theta\cos\theta}{16\xi^4}\\ &+\frac{-768k^2\kappa(\theta)^2\sin^4\theta + 64k^2\left|\Gamma\right|^2\sin^2\theta - 48k^2\left|\Gamma\right|^2\cos^2\theta + 3k^4\left|\Gamma\right|^4\sin^4\theta}{128\xi^5}\\ &+\Sigma^{-6}\,.\end{aligned} \tag{59}$$

Proof. The Hankel function admits the following expansion

$$H_0(z) = \frac{-1}{2\pi}\ln|z|J_0(z) + F_1(z^2) \tag{60}$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. We fix a smooth function $u \in T^\infty$. One has

$$\left(S_{k,\omega}u\right)(x) = \int_{-1}^{1} H_0\left(k\left|r(x)-r(y)\right|\right)\frac{u(y)}{\omega(y)}dy\,.$$

Using the variable changes $x=\cos\theta$, $y=\cos\theta'$, we get

$$S_{k,\omega}u(\cos\theta) = \int_0^\pi H_0(k\left|r(\cos\theta)-r(\cos\theta')\right|)u(\cos(\theta))d\theta\,,$$

which, in view of (60), can be rewritten as

$$\begin{aligned}S_{k,\omega}u(\cos\theta) =&\frac{-1}{2\pi}\int_0^\pi \ln\left|\cos\theta-\cos\theta'\right| J_0(k\left|r(\cos\theta)-r(\cos\theta')\right|)\mathcal{C}u(\theta)d\theta\\ &+\int_0^\pi F_2(\cos\theta,\cos\theta')\mathcal{C}u(\theta)d\theta'\end{aligned}$$

where

$$F_2(x,y) = \ln\frac{|r(x)-r(y)|}{|x-y|} + F_1(k^2(x-y)^2)$$

is a C^∞ function. By parity, the second integral defines an operator

$$Ku(\theta) = \frac{1}{2}\int_{-\pi}^{\pi} F_2(\cos\theta,\cos\theta')\mathcal{C}u(\theta)d\theta.$$

There holds $K = \tilde{R}_1\mathcal{C}$ where, by Theorem 1, $R_1 \in Op(\Sigma^{-\infty})$. For the first integral, we make the following classical manipulations. We first write $\cos\theta-\cos\theta' =$

$-2\sin\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}$. Thus $\ln|\cos\theta-\cos\theta'| = \ln\left|\sqrt{2}\sin\frac{\theta+\theta'}{2}\right| + \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right|$. We then integrate and apply the change of variables $\theta\to-\theta$ for the second term, yielding

$$S_{k,\omega}u(\cos\theta) = \left(\tilde{S}_{k,1}+\tilde{R}_1\right)\mathcal{C}u(\theta)$$

where

$$\tilde{S}_{k,1}u(\theta) = \frac{-1}{2\pi}\int_{-\pi}^{\pi}\ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right| J_0(k\left|r(\cos\theta)-r(\cos\theta')\right|)u(\theta')d\theta'$$

Let $g := \theta\mapsto -\frac{1}{2\pi}\ln\left|\sqrt{2}\sin\frac{\theta}{2}\right|$. It is well-known that $\hat{g}(n) = \frac{1}{2n}$ for $n\neq 0$. We may prolong this by $g(\xi) = \frac{1}{2\xi}$ away from $\xi = 0$. Let $a(\theta,\theta') = J_0(k\left|r(\cos\theta)-r(\cos\theta')\right|)$, which is a smooth function. By Theorem 1, the operator

$$\tilde{S}_{k,1}u(\theta) := \int_{-\pi}^{\pi} g(\theta-\theta')a(\theta,\theta')u(\theta')d\theta'$$

is in $Op(\Sigma^{-1})$. In particular, $\tilde{S}_{k,1}u$ is a smooth function, thereby, $\theta\mapsto S_{k,\omega}u(\cos\theta)$ is a smooth even function. Lemma 9 then ensures

$$S_{k,\omega}u(\cos\theta) = \mathcal{C}S_{k,\omega}u(\theta)\,.$$

This establishes that $\mathcal{C}S_{k,\omega}u = \tilde{S}_k\mathcal{C}u$ for any smooth function u. By Theorem 2, this implies that $S_{k,\omega}\in Op\left(S_T^{-1}\right)$. We can compute the symbol of $\tilde{S}_{k,1}$ using the asymptotic expansion (43). The terms $\partial_s^j a(t,s)_{|t=s}$, can be related to the geometric characteristics of Γ through expansion (16). Using a computer calculator, we find that the the rhs of (59) is an asymptotic expansion of $\tilde{S}_{k,1}$. Obviously, this expansion also holds for $\tilde{S}_k := \tilde{S}_{k,1}+\tilde{R}_1$. The result is proved, recalling $S_{k,\omega_\Gamma} = R^{-1}S_{k,\omega}R$. □

In particular, by Corollary 7,

Corollary 9. *S_{k,ω_Γ} is continuous from $T^s(\Gamma)$ to $T^{s+1}(\Gamma)$ for all $s\in\mathbb{R}$ and thus maps $C^\infty(\Gamma)$ to itself.*

Lemma 25. *The operator $-(\omega_\tau\partial_\tau)^2 - k^2\omega_\Gamma^2$ is in $Op(S_T^2(\Gamma))$ and satisfies*

$$\mathcal{C}R\left[-(\omega_\tau\partial_\tau)^2 - k^2\omega_\Gamma^2\right]R^{-1} = \tilde{D}_k\mathcal{C}$$

where $\tilde{D}_k\in Op(\Sigma^2)$ has the following symbol

$$\sigma_{\tilde{D}_k}(\theta,\xi) = |\xi|^2 - k^2|\Gamma|^2\sin^2(\theta)\,. \tag{61}$$

Proof. Recalling equations (**??**), one has

$$-(\omega_\Gamma)^2 - k^2\omega_\Gamma^2 = R^{-1}\left[-(\omega\partial_x)^2 - k^2\omega^2\right]R\,.$$

Letting $D_k = -(\omega\partial_x)^2 - k^2\omega^2$,

$$D_kT_n = (n^2 - k^2|\Gamma|^2\omega^2)T_n\,.$$

The result is then a consequence of Theorem 2. □

Theorem 5. *The operators* $\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]$ *and* S_{k,ω_Γ} *are respectively in* $Op(S_T^2(\Gamma))$ *and* $Op(S_T^{-1}(\Gamma))$ *and satisfy*

$$\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right] S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + T_{-4}.$$

Proof. We have shown that $\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]$ and S_{k,ω_Γ} are respectively in $Op(S_T^2(\Gamma))$ and $Op(S_T^{-1}(\Gamma))$ in the previous two lemmas. Using the method described in Remark 2, we can compute an asymptotic expansion of the symbol of the pseudo-differential operator

$$\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right] S_{k,\omega_\Gamma}^2 - \frac{I_d}{4}\,.$$

The symbol of this operator is found to be in $S_T^{-4}(\Gamma)$, from which the result follows. □

Remark 3. *The previous theorem implies the following fact*

$$-(\omega_\Gamma\partial_\tau)^2 S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + R$$

where R *is in* $Op(S_T^2(\Gamma))$. *This is also a compact perturbation of the identity. Nevertheless, since* $R = k^2\omega_\Gamma^2 S_{k,\omega_\Gamma}^2 + T_{-4}$ *the term* $k^2\omega^2 S_{k,\omega}^2$ *can be viewed as the leading first order correction accounting for the wavenumber. The inclusion of this term in the preconditioner leads to a drastic reduction of the number of GMRES iterations in numerical applications, as demonstrated in [1].*

Neumann problem We saw in Lemma 15 that $N_{k,\omega_\Gamma} = N_1 - k^2 N_2$ where

$$N_1 = -\partial_\tau S_{k,\omega}\omega_\Gamma\partial_\tau\omega_\Gamma$$

and $N_2 = V_k\omega_\Gamma^2$ with

$$V_k u(x) = \int_\Gamma \frac{G_k(x-y)N(x)\cdot N(y)u(y)}{\omega_\Gamma(y)} d\sigma(y)\,.$$

Lemma 26. *The operator* N_1 *is in* $Op(S_U^2(\Gamma))$ *and*

$$\mathcal{S}RN_1R^{-1} = \tilde{N}_1\mathcal{S}$$

where $\tilde{N}_1$ *is a PPDO with a symbol* $\sigma_{\tilde{N}_1}$ *satisfying*

$$\sigma_{\tilde{N}_1}(\theta,\xi) = \frac{\xi}{2} + \frac{1}{16}\frac{k^2\left|\Gamma\right|^2\sin^2(\theta)}{\xi} + i\frac{k^2L^2\sin\theta\cos\theta}{16\xi^2} + \Sigma^{-3} \tag{62}$$

Proof. This result is obtained by symbolic calculus combining Lemma 24 and Lemma 22. □

A small adaptation of the proof of Lemma 24 yields the following result:

Lemma 27. *The operator V_k is in $Op(S_T^{-1}(\Gamma))$ and*

$$\mathcal{C}RV_kR^{-1} = \tilde{V}_k\mathcal{C}$$

where $\tilde{V}_k$ is a PPDO with a symbol $\sigma_{\tilde{N}_2}$ satisfying

$$\sigma_{\tilde{V}_k} = \frac{1}{2\xi} + \Sigma^{-3}$$

Applying Lemma 23, we deduce

Corollary 10. *The operator N_2 is in $Op(S_U^{-1}(\Gamma))$ and satisfies*

$$\mathcal{S}RN_2R^{-1} = \tilde{N}_2\mathcal{S}$$

where the symbol of $\tilde{N}_2$ has the asymptotic expansion

$$\sigma_{\tilde{N}_2} = \frac{\sin^2\theta}{2\xi} + i\frac{\sin\theta\cos\theta}{2\xi^2} + \Sigma^{-3}\,. \tag{63}$$

It is also easy to check that

Lemma 28. *The operator $\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$ is in $Op(S_U^2(\Gamma))$ and satisfies*

$$\mathcal{S}R\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]R^{-1} = \tilde{D}_k\mathcal{S}$$

where $\tilde{D}_k$ is the operator defined in Lemma 25.

Theorem 6. *The operators $\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$ and N_{k,ω_Γ} are respectively in $Op(S_U^2(\Gamma))$ and $Op(S_U^1(\Gamma))$and satisfy*

$$N_{k,\omega_\Gamma}^2 = \left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right] + U_{-2}.$$

Proof. Gathering the previous lemmas, we have asymptotic expansions available for the symbols of the operators N_{k,ω_Γ} and $\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$. We can thus, using the method of Remark 2, compute an asymptotic expansion of the symbol of the operator $N_{k,\omega_\Gamma}^2 - \left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$ which turns out to be in $S_U^{-2}(\Gamma)$, giving the result. ☐

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