Multilevel Additive Methods for Elliptic Finite Element Problems

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Abstract

An additive variant of the Schwarz alternating method is discussed. A general framework is developed, which is useful in the design and analysis of a variety of domain decomposition methods as well as certain multigrid methods. Three methods of this kind are then considered and estimates of their rates of convergence given. One is a Schwarz-type method using several levels of overlapping subregions. The others use multilevel, multigrid-like decompositions of finite element spaces and have previously been considered by Yserentant and Bramble, Pasciak and Xu. Throughout, we work with finite element approximations of linear, self-adjoint, elliptic problems.

Key words domain decomposition, elliptic equations, finite elements, Schwarz's alternating method
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1 Introduction.

In this paper, we discuss parallel algorithms for solving systems of linear algebraic equations, which result from finite element approximations of second order, elliptic problems. These algorithms are based on additive Schwarz methods (ASM), which form an important class of domain decomposition methods. One of our goals is to show that an ASM framework is quite

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useful when constructing and analyzing new as well as previously known iterative methods for solving elliptic finite element problems. This paper is a continuation of our previous studies; see [7],[8].

This paper is organized as follows. In Section 2, a general ASM framework is introduced for variational equations in Hilbert space. We work with projections, which are orthogonal with respect to the given bilinear form, and related subspaces; cf. Lions [9], where the classical Schwarz method is considered. We also show how the rate of convergence of Schwarz methods can be analyzed, and discuss briefly how they can be implemented.

In Section 3, a finite element approximation of second order, elliptic problems is considered. We construct and analyze iterative methods for the resulting linear systems using the results of Section 2. The original region Ω is divided into overlapping subregions. In the two-level algorithm, the restriction of the finite element model to these subregions, and a coarse finite element problem are solved in each iteration step. We also discuss a $\ell-$level version of the method, which is obtained by recursively refining the subregions, creating overlapping subregions on each level, and using projections related to all of them. We show that the rate of convergence of this method is independent of the parameters of triangulations and that the condition number grows at most quadratically with ℓ, the number of levels. We also prove that for problems on convex regions, the growth of the condition number is linear in ℓ. The two-level case has previously been considered in [7] and [8].

In Section 4, Yserentant's hierarchical basis method, see [12], is described inside our ASM framework. In Section 5, we consider a method due to Bramble, Pasciak and Xu [2] from the same point of view. Their method can be regarded as an extension of Yserentant's method and has the advantage that it also works well in three dimensions.

In this paper, we only consider symmetric, positive definite problems. We note that additive Schwarz methods for non-symmetric cases, including problems with some eigenvalues in the left half plane, have been considered recently by Cai and Widlund [5]. That work extends earlier work by Cai [3],[4].

2 The General Framework.

In this section, we present the additive Schwarz method in an abstract form. We consider a general variational problem in a Hilbert space V : Find $u \in V$, such that

$$a(u,v) = f(v) \ , \forall \ v \in V \ . \tag{2.1}$$

Here $a(u,v) : V \times V \to R$, is a bilinear form, which is symmetric, bounded and coercive. Therefore $a(u,v)$ can be used as a scalar product in V. $f(v) : V \to R$, is a continuous linear functional. By the Lax-Milgram theorem equation (2.1) has a unique solution.

To define an ASM for (1.1), we represent V as

$$V = V_0 + V_1 + \cdots + V_N, \tag{2.2}$$

where V_i are subspaces of V. Let $P_i : V \to V_i$, be the projection defined by

$$a(P_i w, v) = a(w,v) \ , \forall \ v \in V_i \ . \tag{2.3}$$

We note that if u is the solution of equation (2.1), then $P_i u$ can be computed by solving the equation

$$a(P_i u, v) = f(v) \ , \forall \ v \in V_i \ . \tag{2.4}$$

The problem (2.1) is replaced by an operator equation of the form

$$Pu \equiv (P_0 + P_1 + \cdots + P_N)u = g \ . \tag{2.5}$$

This equation must have the same solution as (2.1). For this to hold the right-hand side must be equal to $g = \sum_{i=0}^{N} g_i$, where $g_i = P_i u$. The element g_i can be constructed by solving (2.4) for $i = 0, \ldots, N$. We will obtain a strictly positive lower bound on the spectrum of P. This ensures that P is invertible and that the solution of equation (2.5) unique. We note that P is always automatically positive semi-definite and symmetric with respect to the bilinear form $a(u,v)$.

The equation (2.5) is solved by an iterative method usually the standard conjugate gradient method. It is well known that the number of iterations required, to decrease an appropriate norm of the error of this iterative method by a fixed factor, depends on $\kappa(P)$, the condition number of P. An estimate of $\kappa(P)$ reduces to obtaining estimates of the positive constants γ_i in the inequalities,

$$\gamma_0 a(u,v) \le a(Pv,v) \le \gamma_1 \, a(u,v) \ , \forall \ v \in V \ . \tag{2.6}$$

Ideally these bounds should be independent of the number of subspaces and the other parameters that are introduced in finite dimensional analogs of equation (2.1). We note that it is easy to show that $\gamma_1 \leq N+1$. Improved estimates of γ_1 can be obtained if many of the subspaces V_i are orthogonal, or almost orthogonal, to each other.

In order to establish the left inequality of (2.6), it is often convenient to use the following lemma, cf. [9],

Lemma 2.1. *If for every $u \in V$, there is a representation $u = \sum_{i=0}^{N} u_i$, $u_i \in V_i$ and a constant C_0, such that*

$$\sum_{i=0}^{N} a(u_i, u_i) \leq C_0^2\, a(u, u) \ , \tag{2.7}$$

then

$$C_0^{-2}\, a(u, u) \leq a(Pu, u) \ .$$

We now briefly discuss how this method can be implemented. For simplicity, we consider only the first Richardson method for solving equation (2.5). Thus,

$$u^{n+1} = u^n - \tau_{opt}(Pu^n - g) \ , \quad \tau_{\rm opt} = \frac{2}{(\gamma_0 + \gamma_1)} \ . \tag{2.8}$$

Here $r^n \equiv Pu^n - g = P(u^n - u) = \sum_{i=0}^{N} r_i^n$, $r_i^n = P_i(u^n - u)$. To find r_i^n, we solve

$$a(r_i^n, \phi) = a(u^n, \phi) - f(\phi) \ , \forall\ \phi \in V_i \tag{2.9}$$

for $i = 0, 1, \ldots, N$. These subproblems are independent and can therefore be solved in parallel.

One of the attractive features of iterative methods of this kind is the possibility of using inexact solvers to solve the subproblems (2.9). Let $b_i(u, v)$ be a bilinear, symmetric, positive definite form defined on $V_i \times V_i$, such that

$$\gamma_2 b_i(v, v) \leq a(v, v) \leq \gamma_3 b_i(v, v) \ , \forall\ v \in V_i \ , \tag{2.10}$$

with positive constants γ_2 and γ_3. We can then replace the operator P by $\tilde{P}$, where

$$\widetilde{P} = \widetilde{P}_0 + \widetilde{P}_1 + \cdots + \widetilde{P}_N \ ,$$

and $\tilde{P}_i : V \rightarrow V_i$, is defined by

$$b_i(\tilde{P}_i w, \phi) = a(w, \phi) \ , \forall\ \phi \in V_i \ . \tag{2.11}$$

It then follows from (2.6) and (2.10) that

$$\gamma_0 \gamma_2\, a(u, u) \leq a(\tilde{P} u, u) \leq \gamma_1 \gamma_3\, a(u, u) \ , \tag{2.12}$$

from which an estimate of the condition number of $\tilde{P}$ is obtained.

3 Schwarz Methods on Overlapping Subregions.

In this section, we construct and analyze an ASM for finite element approximations of second order elliptic problems in three dimensions, using the general framework of Section 2. (The method and analysis are equally valid for problems in the plane.) To simplify the presentation, we only discuss a standard Poisson equation with zero Dirichlet conditions approximated by continuous, piecewise linear finite elements using tetrahedral elements. Our results can be extended to more general finite element approximations and elliptic problems.

The continuous problem is of the form: Find $u \in H_0^1(\Omega)$, such that

$$a(u,v) = f(v) \ , \forall \ v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \, \nabla v \, dx \ , \quad f(v) = \int_\Omega f \, v \, dx \ .$$

The form $a(u,v)$ defines a seminorm in $H^1(\Omega)$ and a norm in $H_0^1(\Omega)$.

For simplicity, let Ω be a polyhedral region. The triangulation of Ω is introduced by dividing the region into nonoverlapping tetrahedrons Ω_i, $i = 1, \ldots, N$, which are also called substructures. This partitioning forms a coarse triangulation with a parameter $H = \max_i H_i$, where H_i is the diameter of Ω_i. All the substructures Ω_i are further divided into tetrahedral elements. We associate a parameter h with this finer triangulation, which is used in the overall finite element model of the problem (3.1). We assume that the coarse and fine triangulations are shape regular in the sense common to finite element theory; cf. Ciarlet [6].

Let $V^h = V^h(\Omega)$ be the finite element space of continuous, piecewise linear functions, defined on the fine triangulation, which vanish on $\partial\Omega$, the boundary of Ω. The discrete problem is of the form: Find $u_h \in V^h$, such that

$$a(u_h, v_h) = f(v_h) \ , \forall \ v_h \in V^h \ . \tag{3.2}$$

To define the ASM for equation (3.2), we construct overlapping subregions Ω_i', which cover Ω. Each substructure Ω_i is extended to a larger region Ω_i'. We assume that the distance between the boundaries $\delta\Omega_i$ and $\delta\Omega_i'$ is bounded from below by a fixed fraction of H_i, and that $\delta\Omega_i'$ does not cut through any element. If part of an extended subregion is outside of Ω, we cut off that part and denote the resulting subregion by Ω_i'.

With the subregion Ω_i', we associate a finite element subspace $V_i \subset V^h$ defined by

$$V_i = \{v \in V^h : v(x) = 0 \ , \ x \in C\Omega_i'\} \ , \ i \geq 1 \ .$$

The subspace V_0 is equal to V^H, the space of continuous, piecewise linear functions on the coarse triangulation generated by the substructures Ω_i.

Our finite element space V^h can be represented as the sum of $(N+1)$ subspaces, cf. (2.2),

$$V^h = V_0 + V_1 + \cdots + V_N \ . \tag{3.3}$$

The projections $P_i : V^h \to V_i$, are defined by

$$a(P_i w, \phi) = a(w, \phi) \ , \forall \ \phi \in V_i \ , \tag{3.4}$$

and the problem (3.2) is replaced by the equation,

$$Pu_h \equiv (P_0 + P_1 + \cdots + P_N)u_h = g_h, \tag{3.5}$$

where the right-hand side g_h is computed as $g_h = \sum_{i=0}^{N} g_{h,i}$, $g_{h,i} = P_i u_h$.

Theorem 3.1. *The following inequalities hold,*

$$\gamma_0 a(v,v) \leq a(Pv,v) \leq \gamma_1 a(v,v) \ , \forall \ v \in V^h \ . \tag{3.6}$$

Here γ_0 and γ_1 are positive constants independent of h, H and N.

Proof: The proof of the right inequality follows from the fact that, for $i = 1, \ldots, N$,

$$a(P_i v, v) = a_{\Omega_i'}(P_i v, v) \leq a_{\Omega_i'}(v,v) \ ,$$

and the fact that $\{\Omega_i'\}$ provides a finite covering of Ω. Here $a_{\Omega_i'}$ is defined the same way as the bilinear form a, but with the integration restricted to the subregion Ω_i'.

The proof of the left inequality is given in [7]. It is based on Lemma 2.1 and a partition of unity; cf. also our discussion below. □

We see that the rate of convergence of iterative methods, used for solving (3.5), can be regarded as optimal. If the first Richardson method is used, see (2.8),(2.9), $(N+1)$ independent subproblems are solved in each iteration. The first problem, $i = 0$, corresponds to the solution of the original problem on the coarse mesh; the remaining are local problems which are restrictions of problem (3.2) to the Ω_i'. We note that the global subproblem provides a mechanism for the global transportation of information. As shown in [11], the condition number of any method, without such a mechanism, grows at least as fast as $1/H^2$.

Remark. An ASM can also be viewed in terms of a preconditioner for problem (3.2). Let K be the matrix of the system corresponding to (3.2), i.e. $(K\underline{u}, \underline{v}) = a(u,v)$, $u, v \in V^h$. Let $K_{(i)}$ be the submatrices of K defined by $(K_{(i)}\underline{u}_{(i)}, \underline{v}_{(i)}) = a(u,v)$, $u, v \in V_i$. It can easily be shown that

$$P_i = \mathcal{P}_{(i)}^T \begin{pmatrix} K_{(i)}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{P}_{(i)} K = K_{(i)}^{\dagger} K \ ,$$

where $\mathcal{P}_{(i)}$ a permutation matrix. Let

$$\widehat{K}^{-1} \;=\; K_{(0)}^{\dagger} + K_{(1)}^{\dagger} + \cdots + K_{(N)}^{\dagger} \ .$$

It follows from Theorem 3.1 that the matrix $\widehat{K}$ is spectrally equivalent to K, i.e. $\kappa(\widehat{K}^{-1}K)$ is bounded independently of h, H and N.

The ASM, described above, uses two levels with a coarse and a fine mesh. If H is small, the local problems are small, but we then have to solve a large system of algebraic equations corresponding to the problem in $V_0 = V^H$. We can make the computation cheaper by applying a multilevel version. Let us consider ℓ rather than two levels of triangulations of Ω with substructures Ω_{ki} and a parameter h_k, $k = 0, 1, \ldots, \ell$. The triangulation on level k is a refinement of that of level $(k-1)$ with the level ℓ triangulation the finest and $h_\ell = h$. On each level of triangulation, we define a finite element space $V^{h_k}, k = 0, \ldots, \ell$, with $V^{h_\ell} = V^h$ and $V^{h_0} = V^H$. In turn each V^{h_k} , $k = 1, \ldots, \ell$, is represented as in the two-level ASM:

$$V^{h_k} = V_{k1} + \cdots + V_{kN_k} \ . \tag{3.7}$$

The subspace V_{ki} is associated with a subregion Ω'_{ki}, which is an extension of Ω_{ki} satisfying the same assumptions as in the case of two levels. In other words, $V_{ki} = V^{h_k}(\Omega) \cap H_0^1(\Omega'_{ik})$, extended by zero on and outside of $\partial\Omega'_{ik}$. The original finite element space V^h is now represented as

$$V^h = V^H + \sum_{k=1}^{\ell} \sum_{i=1}^{N_k} V_{ki} \ . \tag{3.8}$$

We introduce projections $P_{ki} : \ V^h \to V_{ki}$, by

$$a(P_{ki}v, \phi) = a(v, \phi) \ , \forall \ \phi \in V_{ki} \ ,$$

and replace the original problem (3.2) by

$$Pu_h \equiv (P_0 + \sum_{i=1}^{\ell} \sum_{i=1}^{N_k} P_{ki})\, u_h = g_h \ , \tag{3.9}$$

where

$$g_h = g_0 + \sum_{k=1}^{\ell} \sum_{i=1}^{N_k} g_{ki} \ , \quad g_0 = P_0 u_h \ , \quad g_{ki} = P_{ki} u_h \ .$$

Theorem 3.2. *The following inequalities hold,*

$$\hat{\gamma_0}(\ell+1)^{-1}\, a(v,v) \le a(Pv,v) \le \hat{\gamma_1}(\ell+1)\, a(v,v) \ , \forall\ v \in V^h \ . \tag{3.10}$$

Here $\hat{\gamma_0}$ and $\hat{\gamma_1}$ are positive constants independent of the mesh sizes and ℓ.

Proof: The right inequality is established by the same arguments as in the proof of Theorem 3.1.

The proof of the left inequality is based on Lemma 2.1 and a partition of unity. We introduce the L_2-projection $Q_k : V^h \to V^{h_k}$, by

$$(Q_k v, \phi)_{L^2(\Omega)} = (v, \phi)_{L^2(\Omega)} \ , \forall\ \phi \in V^{h_k} \ .$$

It is known, see for example Strang [10], that

$$\|v - Q_k v\|_{L^2(\Omega)} \le C h_k |v|_{H^1(\Omega)} \ , \forall\ v \in V^{h_k} \ , \tag{3.11}$$

and that these projections are uniformly bounded in H^1, i.e.

$$|Q_k v|_{H^1(\Omega)} \le C |v|_{H^1(\Omega)} \ , \forall\ v \in V^h \ . \tag{3.12}$$

For $v \in V^h$, let

$$v = v_0 + v_1 + \cdots + v_\ell \ , \tag{3.13}$$

where $v_0 = Q_0 v$, $v_k = (Q_k - Q_{k-1})v$, for $k = 1, \ldots, \ell - 1$, and $v_\ell = (I - Q_{\ell-1})v$. Let $v_{ki} = I_{h_k}(\Theta_{ki} v_k)$, where $\Theta_{ki} \in C_0^\infty(\Omega'_{ik})$, and Θ_{ki}, $i = 1, \cdots, N_k$, is a partition of unity with respect to the set $\{\Omega'_{ki}\}$. The interpolation operator $I_{h_k} : V^h \to V^{h_k}$, is associated with the triangulation defined by the substructures Ω_{ki}. It has previously been shown, cf. Dryja and Widlund [7], that a partition of unity can be chosen, such that

$$\sum_{i=1}^{N_k} a(v_{ki}, v_{ki}) \le C\{|v_k|^2_{H^1(\Omega)} + \frac{1}{h_{k-1}^2} \|v_k\|^2_{L^2(\Omega)}\} \ .$$

Using this inequality, (3.11) and (3.12), cf. [7], we obtain

$$\sum_{i=1}^{N_k} a(v_{ki}, v_{ki}) \le C\, a(v_k, v_k) \ , \tag{3.14}$$

for $k = 1, \ldots, \ell$, and

$$a(v_0, v_0) \leq C\, a(v, v) .$$

We obtain the left inequality of (3.10) by using (3.12), adding the resulting inequalities and using Lemma 2.1. $\square$

We do not know if the lower bound in Theorem 3.2 can be improved in the general case. In the case when Ω is convex the dependence on ℓ in the left inequality can be eliminated. This has been recently been proved by a Courant Institute graduate student, Xuejun Zhang. The following is the main idea of his proof. An alternative partition of an element $v \in V^h$ is used:

$$v = \tilde{v}_0 + \tilde{v}_1 + \cdots + \tilde{v}_\ell . \tag{3.15}$$

Here $\tilde{v}_0 = P_0 v$ and, for $k > 0$, $\tilde{v}_k = (P_k - P_{(k-1)})v$. The operator $P_k : V^h \to V^{h_k}$, is the projection defined using the bilinear form of (3.2). Since the region is convex, Nitsche's trick can be used, cf. Ciarlet [6], to show that

$$\|v - P_k v\|_{L^2(\Omega)} \leq C h_k |v|_{H^1(\Omega)} . \tag{3.16}$$

It is easy to show that, since $V^{h_{k-1}} \subset V^{h_k}$, the terms of equation (3.15) are orthogonal in H^1. The same partion of unity can now be used to further divide the individual terms of (3.15). The same estimate as in (3.14) results and the argument is completed by using the orthogonality of the $\tilde{v}_k$ from which follows that

$$\sum_{k=0}^{\ell} a(\tilde{v}_k, \tilde{v}_k) = a(v, v) .$$

We conclude this section by briefly discussing an implementation of the variant of the ASM that uses the first Richardson method; see (2.8) and (2.9). Let

$$r^n = r_0^n + \sum_{k=1}^{\ell} \sum_{i=1}^{N_k} r_{ki}^n ,$$

where

$$r_{ki}^n = P_{ki}(u_h^n - u_h) ,$$

and

$$r_0^n = P_0(u_h^n - u_h) .$$

Here $r_{ki}^n \in V_{ki}$ is defined by

$$a(r_{ki}^n, \phi) = a(u_h^n, \phi) - f(\phi) \ , \forall\ \phi \in V_{ki} .$$

In any ASM, contributions from the different subspaces are added, and since different sets of basis functions are used, this involves interpolation. Any basis function of a coarser space can be expressed as a linear combination of basis elements of a finer space. If so desired, this process can be carried out recursively; cf. Yserentant [12] for a discussion of similar issues. The process can be parallelized by dividing the work by subregions. The residuals, required for the right hand sides of the different equations corresponding to the different subspaces, can similarly be computed recursively from the values of $a(u_h^n, \phi) - f(\phi)$, where the ϕ are the standard nodal basis functions of V^h. We can think of this as multigrid operations.

The following numerical results have been obtained by Mr. Xuejun Zhang. The experiments are carried out on a unit square divided into square elements of different sizes. The triangulations are obtained by dividing each of these elements into two triangles. In these experiments only very small linear systems, of order 4, 9, 16 and 25 (and smaller) are solved. The overlap ratio measures the width of $\Omega_i' \setminus \Omega_i$ in terms of the side of the square region Ω_i. Here, in all cases, $\Omega_i' \setminus \Omega_i$ is one element wide. The last column of the table gives the number of iterations required to decrease the Euclidean norm of the residual by a factor 10^{-6}.

total # of elements	# of subdomains on lowest level	overlap ratio	# of levels levels	cond # $\kappa(P)$	# of iter. for $\epsilon = 10^{-6}$
8×8	2×2	1/2	3	7.2	11
16×16	2×2	1/2	4	9.3	17
32×32	2×2	1/2	5	10.7	20
64×64	2×2	1/2	6	11.7	21
9×9	3×3	1/3	2	4.6	9
27×27	3×3	1/3	3	7.1	16
81×81	3×3	1/3	4	8.4	19
243×243	3×3	1/3	5	9.4	21
16×16	4×4	1/4	2	5.2	13
64×64	4×4	1/4	3	7.3	17
256×256	4×4	1/4	4	8.4	20
25×25	5×5	1/5	2	5.7	14
125×125	5×5	1/5	3	7.6	17

4 Yserentant's Hierarchical Basis Method.

In this section, we describe the hierarchical basis method of Yserentant [12] as an ASM using the framework of Section 2.

Let $\Omega \subset R^2$ be a polygon. We consider a ℓ level triangulations of Ω, as in Section 3, with the difference that the level k triangles are obtained by dividing all level $(k-1)$ triangles into four congruent triangles. (As demonstrated in Yserentant [12], more general situations can also be considered.) Let $I_k v \equiv I_{h_k} v$ be the linear interpolant of $v \in V^h$ on the level k triangulation.

The following identity holds

$$v = I_0 v + (I_1 v - I_0 v) + \cdots + (I_\ell v - I_{\ell-1} v), \forall\ v \in V^h \ . \tag{4.1}$$

We represent V^h as a direct sum,

$$V^h = V_0 \oplus V_1 \oplus \cdots \oplus V_\ell \ , \tag{4.2}$$

where $V_0 = V^H$ and $V_k = R(I_k - I_{k-1})$, for $k > 0$, is the range of the operator $(I_k - I_{k-1})$. The elements of the finite element space V_k are functions defined on the level k triangulation, which vanish for all $x \in \mathcal{N}_{k-1}$. Here $\mathcal{N}_{k-1}$ is the set of nodal points of the level $k-1$ triangulation.

Let $P_i : V^h \rightarrow V_k$, be the projection defined by

$$a(P_i w, \phi) = a(w, \phi) \ , \forall\ \phi \in V_i \ . \tag{4.3}$$

We replace equation (3.2) by

$$P u_h \equiv (P_0 + P_1 + \cdots + P_\ell) u_h = g_h \ , \tag{4.4}$$

where $g_h = \sum_{i=0}^{\ell} P_i u_h$.

Theorem 4.1. *The following inequalities hold,*

$$\tilde{\gamma}_0 (\ell + 1)^{-2} a(u, u) \leq a(Pu, u) \leq \tilde{\gamma}_1 a(u, u) \ , \forall\ u \in V^h \ . \tag{4.5}$$

Here $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are positive constants independent of $\{h_k\}$ and ℓ.

Proof. The right inequality follows from Theorem 4.2 of [12].

The left inequality is proved using Lemma 2.1 with $v_k = (I_k - I_{k-1})v$ and using an inequality, given in [1], [12],

$$\|v_k\|^2_{L^\infty(\Omega)} \leq C(1 + \log(h_k / h_\ell)) |v|^2_{H^1(\Omega)} \ , \forall\ v \in V^h \ . \Box \tag{4.6}$$

We use inexact solvers for the subproblems in the implementation of this method. Let

$$b_k(v, w) \equiv \sum_{x \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} v(x) w(x) \ . \tag{4.7}$$

It can be shown, cf. [12], that,

$$\tilde{\gamma}_2 b_k(v, v) \le a(v, v) \le \tilde{\gamma}_3 b_k(v, v), \forall\ v \in V_k. \tag{4.8}$$

Here $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ are positive constants independent of h_k and ℓ. The operator P_k is replaced with $\widetilde{P}_k : V^h \to V_k$, which is defined by

$$b_k(\widetilde{P}_k w, \phi) = a(w, \phi)\ , \forall\ \phi \in V_k\ . \tag{4.9}$$

We obtain the result,

Theorem 4.2. *The following inequalities hold,*

$$\tilde{\gamma}_0 \tilde{\gamma}_2 (\ell + 1)^{-2} a(v, v) \le a(\widetilde{P} v, v) \le \tilde{\gamma}_1 \tilde{\gamma}_3 a(v, v)\ , \forall\ v \in V^h\ .$$

The proof of Theorem 4.2 follows from (4.5), (4.8) and (2.12).

5 Bramble, Pasciak and Xu's Method.

In this section, we describe the multilevel method recently introduced by Bramble, Pasciak and Xu [2]. It is a fast method for both two and three dimensional problems. It is based on L_2-projections, and it can be described as an ASM.

As in Section 4, we consider ℓ levels of triangulations of Ω, which can be a two or three dimensional region. On the level k triangulation, a conventional finite element space $V^{h_k}(\Omega)$ is defined. We note that $V^{h_{k-1}} \subset V^{h_k}$, $k = 1, \ldots, \ell$. We use the following identity

$$u = Q_0 u + (Q_1 - Q_0) u + \cdots + (Q_\ell - Q_{\ell-1}) u\ , \forall\ u \in V^h\ , \tag{5.1}$$

and represent the space V^h as

$$V^h = V_0 \oplus V_1 \oplus \cdots \oplus V_\ell\ . \tag{5.2}$$

Here $V_0 = Q_0 V^h$, $V_k = R(Q_k - Q_{k-1})$ and $Q_k : V^h \to V_k$, is the L_2-projections introduced in Section 3, i.e.

$$(Q_k w, \phi)_{L^2(\Omega)} = (w, \phi)_{L^2(\Omega)}\ , \forall\ \phi \in V_{h_k}\ . \tag{5.3}$$

We also use the projections P_k defined as in (4.3), using the subspaces V_k just introduced. We consider

$$P \equiv P_0 + P_1 + \cdots + P_\ell \ . \tag{5.4}$$

Theorem 5.1. *The following inequalities hold,*

$$\hat{\gamma}_0(\ell+1)^{-1}\, a(v,v) \le a(Pv,v) \le (\ell+1)\, a(v,v) \ , \forall\ v \in V^h \ . \tag{5.5}$$

Here $\hat{\gamma}_0$ is a positive constant independent of the $\{h_k\}$ and ℓ.

Proof. The right inequality is elementary since P is the sum of $\ell+1$ projections. To prove the left inequality, we use Lemma 2.1 with $v_0 = Q_0 v$ and $v_k = (Q_k - Q_{k-1})v, k = 1, \ldots, \ell$, and apply the estimate (3.12). □

Let $(Au, v)_{L^2(\Omega)} = a(u,v)$, $u, v \in V^h$, and let

$$(A_k u, v)_{L^2(\Omega)} = a(u,v) \ , \forall\ u, v \in V_k \ . \tag{5.6}$$

It is easy to see that $P_0 = A_0^{-1} Q_0 A$ and $P_k = A_k^{-1}(Q_k - Q_{k-1})A$, $k = 1, 2, \ldots, \ell$. Therefore,

$$P = (A_0^{-1} Q_0 + A_1^{-1}(Q_1 - Q_0) + \cdots + A_\ell^{-1}(Q_\ell - Q_{\ell-1}))A \ . \tag{5.7}$$

When using an iterative method to solve an equation with this operator, we need to compute $P_k v$ for given vectors v, i.e. solve

$$a(P_k v, \phi) = a(v, \phi) \ , \forall\ \phi \in V_k \ . \tag{5.8}$$

This is expensive. Therefore an inexact solver is used. Let

$$b_k(v_k, v_k) \equiv h_k^{-2}(v_k, v_k)_{L^2(\Omega)} \ , \forall\ v_k \in V_k \ . \tag{5.9}$$

Lemma 5.1. *For any $v_k \in V_k$, $k = 1, \ldots, \ell$, the following inequalities hold,*

$$\hat{\gamma}_2 b_k(v_k, v_k) \le a(v_k, v_k) \le \hat{\gamma}_3 b_k(v_k, v_k), \forall\ v_k \in V_k \ . \tag{5.10}$$

Here $\hat{\gamma}_2$ and $\hat{\gamma}_3$ are positive constants independent of h_k and k.

Proof. The left inequality follows from the estimate (3.11) and the fact that $v_k = (I - Q_{k-1})v_k$. To prove the right inequality, we use an inverse inequality. □

By Lemma 5.1, we have, for $k = 1, \ldots, \ell$,

$$\hat{\gamma}_2 h_k^{-2}(v_k, v_k)_{L^2} \le (A_k v_k, v_k)_{L^2} \le \hat{\gamma}_3 h_k^{-2}(v_k, v_k)_{L^2} \ , \forall\ v_k \in V_k \ ,$$

which shows that P_k can be replaced by $P_k^{(1)} = h_k^2(Q_k - Q_{k-1})A$ and that

$$P^{(1)} \equiv P_0 + P_1^{(1)} + \cdots + P_\ell^{(1)}$$

is spectrally equivalent to P. The operator $P^{(1)}$ can be rewritten as

$$P^{(1)} = (A_0^{-1}Q_0 - h_1^2 Q_0 + (h_1^2 - h_2^2)Q_1 + \cdots + (h_{\ell-1}^2 - h_\ell^2)Q_{\ell-1} + h_\ell^2 Q_\ell)A \,. \quad (5.11)$$

This means that

$$B^{(1)} \equiv A_0^{-1}Q_0 - h_1^2 Q_0 + (h_1^2 - h_2^2)Q_1 + \cdots + (h_{\ell-1}^2 - h_\ell^2)Q_{\ell-1} \qquad (5.12)$$

is spectrally equivalent to A^{-1}, to within the dependence on ℓ indicated in Theorem 5.1. This preconditioner is relatively expensive to use and it needs to be modified further. This is done in two steps, cf. [2]. We first introduce the operator

$$B^{(2)} = A_0^{-1}Q_0 + h_2^2 Q_1 + \cdots + h_\ell^2 Q_\ell \,.$$

If the refinement of the meshes satisfies the condition

$$h_{k+1}^{-2} \geq (1+\delta)h_k^{-2} \ , \quad k = 0, \ldots, \ell - 1 \ ,$$

then it is easy to show that

$$(B^{(1)}v, v)_{L^2} \leq (B^{(2)}v, v)_{L^2} \leq (1+\delta)(B^{(1)}v, v)_{L^2} \ , \forall \ v \in V^h \ .$$

More importantly, we also replace Q_k, $k = 1, \ldots, \ell$, by inexact solvers. Let

$$\tilde{b}_k(u, v) \equiv h_k^2 \sum_{x \in \mathcal{N}_k} u(x)\, v(x) \ ,$$

where $\mathcal{N}_k$ is the set of nodal points of the level k triangulation, and define $\widetilde{Q}_k : V_h \to V^{h_k}$, by

$$b(\widetilde{Q}_k v, \phi) = (v, \phi)_{L^2} \ , \forall \ \phi \in V^{h_k} \ .$$

It is easy to show that, in V^{h_k}, $\widetilde{Q}_k$ is spectrally equivalent to Q_k, with respect to the L_2-inner product. We thus obtain the final preconditioner B,

$$B \equiv A_0^{-1}Q_0 v + h_1^2 \widetilde{Q}_1 + \cdots + h_\ell^2 \widetilde{Q}_\ell \ ,$$

which is spectrally equivalent to $B^{(2)}$. It follows that

$$\kappa(BA) \leq C\ell^2 \ .$$

References

1. J. H. Bramble, *A Second Order Finite Difference Analogue of the First Biharmonic Boundary Value Problem.* Numer. Math. 9: 236-249, 1966.

2. J. H. Bramble, J. E. Pasciak and J. Xu, *Parallel Multilevel Preconditioners.* Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia 1990.

3. X.-C. Cai, *Some Domain Decomposition Algorithms for Nonselfadjoint Elliptic and Parabolic Partial Differential Equations.* Ph.D. thesis, Courant Institute, Sept. 1989.

4. X.-C. Cai, *An Additive Schwarz Algorithm for Nonselfadjoint Elliptic Equations.* Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia 1990.

5. X.-C. Cai and O. B. Widlund, *Domain Decomposition Algorithms for Indefinite Elliptic Problems.* Submitted to SIAM J. SISC.

6. P. Ciarlet, *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

7. M. Dryja and O. B. Widlund, *Some domain decomposition algorithms for elliptic problems.* Iterative Methods for Large Linear Systems, L. Hayes and D. Kincaid, eds., Academic Press, Orlando, Florida, 1989.

8. M. Dryja and O. B. Widlund, *Towards a unified theory of domain decomposition algorithms for elliptic problems.* Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia 1990.

9. P. L. Lions, *On the Schwarz alternating method. I.* First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. Meurant and J. Périaux, eds., SIAM, Philadelphia, 1988.

10. G. Strang, *Approximation in the finite element method.* Numer. Math. 19: 81-98, 1972.

11. O. B. Widlund, *Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane.* First International Symposium

on Domain Decomposition Methods for Partial Differential Equations. R. Glowinski, G. H. Golub, G. Meurant and J. Périaux, eds., SIAM, Philadelphia, 1988.

12. H. Yserentant, *On the multi-level splitting of finite element spaces.* Numer. Math. 49: 379-412, 1986.