PSEUDODIFFERENTIAL OPERATORS

Four Lectures at MSRI, September 2008

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Contents

I. Pseudodifferential operators with smooth symbol on $\mathbb{R}^n$

This lecture covers a number of topics on pseudodifferential operators on Euclidean space. In §1 we give the Fourier integral representation of such an operator, with symbol $p(x,\xi)$. We take $p(x,\xi)$ in one of Hörmander's symbol classes $S^m_{\rho,\delta}$. For basic results in linear PDE, the emphasis is naturally on $(\rho,\delta)=(1,0)$, but other classes prove useful from time to time. (In Lecture II there will be a special emphasis on classes $S^m_{1,\delta}$, both for $\delta\in[0,1)$ and for $\delta=1$.) We proceed in §2 to discuss adjoints, products, and commutators of such operators. As seen there, we typically need $0\le\delta<\rho\le 1$. In §3 we discuss the behavior of such operators on L^2-Sobolev spaces, and (for $\rho=1$) on L^p-Sobolev spaces, and also Hardy and bmo-Sobolev spaces, and Zygmund spaces (which include Hölder spaces). Section 4 deals with elliptic operators, producing results on global and local regularity. In this section we derive further results for strongly elliptic operators, including particularly Gårding's inequality.

Section 5 treats hyperbolic equations, starting with first order symmetric hyperbolic systems. Energy estimates provide a tool for proving existence and uniqueness of solutions. We show how a certain class of second order hyperbolic PDE can be transformed to first order symmetric hyperbolic systems. Results on strongly elliptic operators from §4 are useful in this analysis. In §6 we conjugate a pseudodifferential operator P by the solution operator to a scalar hyperbolic equation and obtain a family $P(t)$ of pseudodifferential operators, whose principal symbols are related by a Hamiltonian flow; this is Egorov's theorem. In §7 we define the wave front set $WF(f)$ of a distribution f on $\mathbb{R}^n$. This is a conic subset of "phase space," (x,ξ)-space, which via $(x,\xi)\mapsto x$ projects onto the singular support of f. Results on propagation of singularities are naturally expressed in terms of wave front sets. We show how the solution operator to a first order scalar hyperbolic equation propagates wave front sets, as a consequence of Egorov's theorem. Section 8 discusses pseudodifferential operators on a compact manifold M. We show that $OPS^m_{\rho,\delta}(M)$ is well defined when $\rho\in(1/2,1]$ and $\delta=1-\rho$, as a consequence of Egorov's theorem.

The material presented here is necessarily a bit sketchy. Much more detailed presentations can be found in several monographs, such as [H5], [T2], and [Tr], as well as other sources listed in the references.

1. Representations of pseudodifferential operators

The following is the Fourier integral representation of a pseudodifferential operator on $\mathbb{R}^n$:

$$p(x,D)u=(2\pi)^{-n/2}\int_{\mathbb{R}^n} p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}\,d\xi, \tag{1.1}$$

where $\hat{u}(\xi)$ is the Fourier transform of u:

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(y) e^{-iy\cdot\xi}\, dy. \tag{1.2}$$

The function $p(x,\xi)$ in (1.1) is called the symbol of $p(x,D)$. Symbol classes will be discussed below. When $p(x,\xi) = \sum p_\alpha(x)\xi^\alpha$ is a polynomial in ξ, (1.1) defines the differential operator $p(x,D) = \sum p_\alpha(x) D^\alpha$, where $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j = (1/i)\partial/\partial x_j$. This follows by applying D_j to the Fourier inversion formula

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{1.3}$$

giving

$$D^\alpha u(x) = (2\pi)^{-n/2} \int \xi^\alpha \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{1.4}$$

The power of this representation is to convert differential operators to algebraic operators. As an illustration, the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad u(0,x) = f(x) \tag{1.5}$$

becomes

$$\frac{\partial \hat{u}}{\partial t} = -|\xi|^2 \hat{u}, \quad \hat{u}(0,\xi) = \hat{f}(\xi), \tag{1.6}$$

with solution

$$u(t,x) = e^{t\Delta} f(x) = (2\pi)^{-n/2} \int e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{1.7}$$

Another representation for pseudodifferential operators is the singular integral representation:

$$p(x,D)u = \int K(x, x-y) u(y)\, dy, \tag{1.8}$$

where

$$K(x,x-y) = (2\pi)^{-n} \int p(x,\xi) e^{i(x-y)\cdot\xi}\, d\xi. \tag{1.9}$$

We can apply this to (1.7), in concert with the formula

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{iz\cdot\xi}\, d\xi = (4\pi t)^{-n/2} e^{-|z|^2/4t}, \tag{1.10}$$

to obtain the formula

$$e^{t\Delta}f(x) = (4\pi t)^{-n/2}\int e^{-|x-y|^2/4t}f(y)\,dy. \tag{1.11}$$

The operator

$$\Delta^{-1}f(x) = -(2\pi)^{-n/2}\int |\xi|^{-2}\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi \tag{1.12}$$

can also be written

$$\begin{aligned}\Delta^{-1}f(x) &= \int G(x-y)f(y)\,dy,\\ G(x-y) &= C_n|x-y|^{-(n-2)}, \quad n\ge 3.\end{aligned} \tag{1.13}$$

One way to get this formula is via

$$\Delta^{-1}f(x) = \int_0^\infty e^{t\Delta}f(x)\,dt, \tag{1.14}$$

and an application of (1.11), along with a change of variable, $s = 1/t$.

Operators of the form (1.1) are called pseudodifferential operators provided $p(x,\xi)$ belongs to a symbol class, such as $S^m_{\rho,\delta}$, defined as follows. Given $m\in\mathbb{R}$, $0\le\delta,\rho\le 1$, we say

$$p(x,\xi)\in S^m_{\rho,\delta} \Longleftrightarrow |D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{1.15}$$

Here $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. We say

$$p(x,\xi)\in S^m_{\rho,\delta} \Longleftrightarrow p(x,D)\in OPS^m_{\rho,\delta}. \tag{1.16}$$

The most basic symbol class is $S^m_{1,0}$, introduced in [KN]. An important subclass, which we denote S^m (or sometimes S^m_{cl}) consists of symbols $p(x,\xi)\in S^m_{1,0}$ such that

$$p(x,\xi)\sim\sum_{j\ge 0}p_j(x,\xi), \tag{1.17}$$

where $p_j(x,\xi)\in S^{m-j}_{1,0}$ is homogeneous of degree $m-j$ in ξ for $|\xi|\ge 1$, and (1.17) means $p(x,\xi)-\sum_0^k p_j(x,\xi)\in S^{m-k-1}_{1,0}$. (We then say $p(x,D)\in OPS^m$.) The classes $S^m_{\rho,\delta}$ were introduced in [H1], and there are more general classes, arising in [BF] and [H4].

The term $p_0(x,\xi)$ in (1.17) is called the principal symbol of $p(x,D)\in OPS^m$. For general $p(x,\xi)\in S^m_{\rho,\delta}$, the principal symbol of $p(x,D)$ is the equivalence class of $p(x,\xi)$, mod $S^{m-(\rho-\delta)}_{\rho,\delta}$. For this to be meaningful, one needs $\rho > \delta$. Further clarification of this will arise in §2.

The function $|\xi|^{-2}$ in (1.12) does not belong to S^{-2}, due to the singularity at $\xi = 0$. One has $E \in OPS^{-2}$ for

$$Ef(x) = -(2\pi)^{-n/2} \int \frac{1-\varphi(\xi)}{|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{1.18}$$

given $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ for ξ in a neighborhood of 0. In such a case,

$$\begin{aligned} \Delta Ef(x) &= E\Delta f(x) = (I+R)f(x), \\ Rf(x) &= -(2\pi)^{-n/2} \int \varphi(\xi)\hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \end{aligned} \tag{1.19}$$

We have

$$R \in OPS^{-\infty} = \bigcap_{m>0} OPS^{-m}_{1,0}. \tag{1.20}$$

Such an operator will be shown to be a smoothing operator, and we say E is a *parametrix* for Δ. One important motivation for the theory of pseudodifferential operators is to construct parametrices for various classes of differential operators, starting with the class of elliptic operators. See §4 for more on this.

Returning to the singular integral representation (1.8)–(1.9), note that

$$D_x^\beta D_z^\gamma z^\alpha K(x,z) = (2\pi)^{-n} \int p_{\alpha\beta}(x,\xi)\xi^\gamma e^{iz\cdot\xi}\, d\xi, \tag{1.21}$$

where

$$\begin{aligned} &p_{\alpha\beta}(x,\xi) = D_x^\beta D_\xi^\alpha p(x,\xi) \\ &\Rightarrow p_{\alpha\beta}(x,\xi)\xi^\gamma \in S^{m-\rho|\alpha|+\delta|\beta|+|\gamma|}_{\rho,\delta}. \end{aligned} \tag{1.22}$$

Provided $\rho > 0$, we see that given β and γ, as long as $|\alpha|$ is large enough, $p_{\alpha\beta}(x,\xi)\xi^\gamma$ is integrable in ξ, so that the left side of (1.21) is bounded and continuous. Consequently, whenever $p(x,\xi) \in S^m_{\rho,\delta}$ and $\rho > 0$, the Schwartz kernel $K(x, x-y)$ of $p(x,D)$ is C^∞ off the diagonal $x = y$ and is rapidly decreasing as $|x-y| \to \infty$.

More can be said if $\rho = 1$. In fact

$$p(x,\xi) \in S^m_{1,\delta} \Rightarrow |D_x^\beta D_y^\gamma K(x,x-y)| \le C_{\beta\gamma}|x-y|^{-n-m-|\beta|-|\gamma|}, \tag{1.23}$$

provided

$$m + |\beta| + |\gamma| > -n. \tag{1.24}$$

This holds for $\delta \in [0,1]$. See, e.g., [T5], Chapter 7, §2, for a proof.

We end this section with a brief comment about a more general Fourier integral representation than (1.1), namely

$$Au(x) = (2\pi)^{-n} \iint a(x,y,\xi) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi. \tag{1.25}$$

Here x, y, and ξ each run over $\mathbb{R}^n$. We say

(1.26)
$$a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2} \Leftrightarrow |D_x^{\beta_1} D_y^{\beta_2} D_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta_1\beta_2} \langle\xi\rangle^{m-\rho|\alpha|+\delta_1|\beta_1|+\delta_2|\beta_2|}.$$

An operator of the form (1.25) can be rewritten in the form (1.1), with

$$\begin{aligned} p(x,\xi) &= (2\pi)^{-n} \int a(x,y,\eta) e^{i(x-y)\cdot(\eta-\xi)}\, dy\, d\eta \\ &= e^{iD_\xi\cdot D_y} a(x,y,\xi)\big|_{y=x}. \end{aligned} \tag{1.27}$$

A variant of the stationary phase method can be used to show that if $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$, then

$$\begin{gathered} 0 \le \delta_2 < \rho \le 1 \Longrightarrow p(x,\xi) \in S^m_{\rho,\delta},\ \ \delta = \max(\delta_1,\delta_2), \quad \text{and} \\ p(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)\big|_{y=x}, \end{gathered} \tag{1.28}$$

the asymptotic formula holding in the sense that the sum over $|\alpha| < N$ above differs from $p(x,\xi)$ by an element of $S^{m-N(\rho-\delta)}_{\rho,\delta}$. Proofs can be found in the sources cited above. We will see several applications in §2 and §8.

2. Symbol calculus: adjoints, products, and commutators

The adjoint of an operator $p(x,D)$ as in (1.1) has the form

$$p(x,D)^* u = (2\pi)^{-n} \iint p(y,\xi)^* e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi. \tag{2.1}$$

This is a special case of (1.25), with $a(x,y,\xi) = p(y,\xi)^*$. Hence (1.27) gives

$$p(x,D)^* = p^*(x,D), \quad p^*(x,\xi) = e^{iD_\xi\cdot D_x} p(x,\xi)^*, \tag{2.2}$$

and we deduce from (1.28) that

$$\begin{gathered} p(x,\xi) \in S^m_{\rho,\delta},\ 0 \le \delta < \rho \le 1 \Rightarrow p^*(x,\xi) \in S^m_{\rho,\delta}, \quad \text{and} \\ p^*(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha p(x,\xi)^*. \end{gathered} \tag{2.3}$$

For the composition of operators $p_j(x,D) \in OPS^{m_j}_{\rho_j,\delta_j}$, we have $p_1(x,D)p_2(x,D) = (p_1 \circ p_2)(x,D)$, with

$$
\begin{aligned}
(p_1 \circ p_2)(x,\xi) &= (2\pi)^{-n} \iint p_1(x,\eta)p_2(y,\xi)e^{i(x-y)\cdot(\eta-\xi)}\, d\eta\, dy \\
&= e^{iD_\eta \cdot D_y}\, p_1(x,\eta)p_2(y,\xi)\big|_{y=x,\eta=\xi},
\end{aligned} \tag{2.4}
$$

and the sort of stationary phase analysis that gives (1.28) also gives

(2.5)

$$
0 \le \delta_2 < \rho_1 \le 1 \Rightarrow (p_1 \circ p_2)(x,\xi) \in S^{m_1+m_2}_{\rho,\delta}, \quad \rho = \min\{\rho_k\},\ \delta = \max\{\delta_j\}, \quad \text{and}
$$

$$
(p_1 \circ p_2)(x,\xi) \sim \sum_{\alpha \ge 0} \frac{i^{|\alpha|}}{\alpha!}\, D^\alpha_\xi p_1(x,\xi) D^\alpha_x p_2(x,\xi).
$$

Proofs can be found in the references cited in §1. In particular, if $p_j \in S^{m_j}_{\rho,\delta},\ 0 \le \delta < \rho \le 1$,

$$
(p_1 \circ p_2)(x,\xi) = p_1(x,\xi)p_2(x,\xi) \mod S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}. \tag{2.6}
$$

Let us specialize to $\rho = 1,\ \delta = 0$. Taking the terms $|\alpha| = 0$ and 1 in (2.5) gives

$$
(p_1 \circ p_2)(x,\xi) = p_1(x,\xi)p_2(x,\xi) + \frac{1}{i} \sum_{j=1}^n \frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j} \mod S^{m_1+m_2-2}_{1,0}, \tag{2.7}
$$

assuming $p_j \in S^{m_j}_{1,0}$. In particular if $p_j(x,\xi)$ are scalar valued, or more generally are commuting matrices, we have for the commutator $[p_1(x,D), p_2(x,D)] = p_1(x,D)p_2(x,D) - p_2(x,D)p_1(x,D)$,

$$
[p_1(x,D), p_2(x,D)] = [p_1,p_2](x,D) \in OPS^{m_1+m_1-1}_{1,0}, \tag{2.8}
$$

with

$$
[p_1,p_2](x,\xi) = \frac{1}{i} \sum_{j=1}^n \Big(\frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j} - \frac{\partial p_1}{\partial x_j} \frac{\partial p_2}{\partial \xi_j} \Big) \mod S^{m_1+m_2-2}_{1,0}. \tag{2.9}
$$

We also write the symbol on the right side of (2.8) as

$$
\frac{1}{i}\{p_1,p_2\}(x,\xi) = \frac{1}{i} H_{p_1} p_2(x,\xi), \tag{2.10}
$$

where $\{p_1,p_2\}$ denotes the Poisson bracket and H_{p_1} the Hamiltonian vector field

$$
H_{p_1} = \sum_{j=1}^n \Big(\frac{\partial p_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p_1}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big) \tag{2.11}
$$

on $\mathbb{R}^{2n}$ associated to the function $p_1(x,\xi)$.

As we will see, (2.6) leads to elliptic regularity results while (2.8)–(2.11) lead to results on propagation of singularities.

3. Operator estimates on function spaces

The first fundamental L^2-operator estimate is

$$p(x,\xi) \in S^0_{\rho,\delta},\ 0 \le \delta < \rho \le 1 \Longrightarrow p(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{3.1}$$

An ingenious proof of [H5] proceeds in stages. First, it is elementary that $p(x,D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for each $p \in [1,\infty]$ if $p(x,\xi) \in S^{-m}_{\rho,\delta}$ with $\rho > 0$ and m large enough, by the integral kernel estimates arising from (1.21)–(1.22). Next, if $a > 0$ and $p(x,\xi) \in S^{-a}_{\rho,\delta}$, then $P = p(x,D)$ has the property $(P^*P)^k \in OPS^{-ka}_{\rho,\delta}$, as long as $0 \le \delta < \rho \le 1$, by results of §2. Consequently $(P^*P)^k$ is bounded on $L^2(\mathbb{R}^n)$ for k large, which implies $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Finally, if $p(x,\xi)$ is as in (3.1), then

$$q(x,D) = p(x,D)^* p(x,D) \in OPS^0_{\rho,\delta}. \tag{3.2}$$

Suppose

$$|q(x,\xi)| \le M - b, \quad b > 0, \tag{3.3}$$

so

$$A(x,\xi) = \bigl(M - \operatorname{Re} q(x,\xi)\bigr)^{1/2} \in S^0_{\rho,\delta}, \tag{3.4}$$

and, by results of §2,

$$A(x,D)^* A(x,D) = M - q(x,D) + r(x,D), \quad r(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}. \tag{3.5}$$

We have

$$\begin{aligned} M\|u\|^2_{L^2} - \|p(x,D)u\|^2_{L^2} &= \|A(x,D)u\|^2_{L^2} - (r(x,D)u,u) \\ &\ge -C\|u\|^2_{L^2}, \end{aligned} \tag{3.6}$$

the last inequality by the known boundedness of $r(x,D)$ on $L^2(\mathbb{R}^n)$. This implies

$$\|p(x,D)u\|^2_{L^2} \le (M+C)\|u\|^2_{L^2}, \tag{3.7}$$

giving (3.1).

The endpoint cases of (3.1), with $\delta = \rho \in [0,1)$, hold. This is the Calderon-Vaillancourt theorem. See [CV]; proofs can also be found in [H5] and [T2].

The result (3.1) extends to L^p-boundedness for $p \in (1, \infty)$ in case $\rho = 1$:

$$p(x,\xi) \in S^0_{1,\delta},\ \delta \in [0,1) \Longrightarrow p(x,D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad \forall\, p \in (1,\infty). \tag{3.8}$$

This follows from Calderon-Zygmund theory. Generally, if P has integral kernel $k(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$|k(x,y)| \le C|x-y|^{-n}, \quad |\nabla_{x,y} k(x,y)| \le C|x-y|^{-n-1}, \tag{3.9}$$

and if P is bounded on $L^2(\mathbb{R}^n)$, then P is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$. For details, see, e.g., [St2] or [T5], Chapter 13. In case $P = p(x,D) \in OPS^0_{1,\delta}$, the estimates in (3.9) follow from (1.23), actually for $\delta \in [0, 1]$. The restriction $\delta < 1$ is needed to apply (3.1).

An important extension of (3.8) allows one to estimate $p(x,D)u$ when u takes values in a Hilbert space $\mathcal{H}_1$ and $p(x,\xi)$ takes values in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{H}_2$ being also a Hilbert space. The following is a paradigmatic special case:

$$\begin{gathered} P(\xi) \in C^\infty(\mathbb{R}^n, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)), \quad \|D^\alpha_\xi P(\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le C_\alpha \langle \xi \rangle^{-|\alpha|} \\ \Longrightarrow P(D) : L^p(\mathbb{R}^n, \mathcal{H}_1) \to L^p(\mathbb{R}^n, \mathcal{H}_2), \quad \forall\, p \in (1,\infty). \end{gathered} \tag{3.10}$$

This leads to an important circle of results known as Littlewood-Paley Theory. To obtain this, start with a partition of unity $\{\psi_j : j \ge 0\}$:

$$1 = \sum_{j=0}^\infty \psi_j(\xi)^2 \tag{3.11}$$

where $\psi_j \in C^\infty$, $\psi_0(\xi)$ is supported on $|\xi| \le 1$, $\psi_1(\xi)$ is supported on $\frac{1}{2} \le |\xi| \le 2$, and $\psi_j(\xi) = \psi_1(2^{1-j}\xi)$ for $j \ge 2$. We take $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_2 = \ell^2$, and look at

$$\Phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, \ell^2) \tag{3.12}$$

given by

$$\Phi(f) = (\psi_0(D)f, \psi_1(D)f, \psi_2(D)f, \dots). \tag{3.13}$$

This is clearly an isometry, though of course it is not surjective. The adjoint

$$\Phi^* : L^2(\mathbb{R}^n, \ell^2) \longrightarrow L^2(\mathbb{R}^n), \tag{3.14}$$

given by

$$\Phi^*(g_0, g_1, g_2, \dots) = \sum \psi_j(D) g_j \tag{3.15}$$

satisfies

$$\Phi^*\Phi = I \tag{3.16}$$

on $L^2(\mathbb{R}^n)$. Note that $\Phi = \Phi(D)$, where

$$\Phi(\xi) = (\psi_0(\xi), \psi_1(\xi), \psi_2(\xi), \dots). \tag{3.17}$$

It is easy to see that the hypothesis in (3.10) is satisfied by both $\Phi(\xi)$ and $\Phi^*(\xi)$. Hence, for $1 < p < \infty$,

$$\begin{aligned} \Phi &: L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n, \ell^2) \\ \Phi^* &: L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n). \end{aligned} \tag{3.18}$$

In particular, Φ maps $L^p(\mathbb{R}^n)$ isomorphically onto a closed subspace of $L^p(\mathbb{R}^n, \ell^2)$, and we have compatibility of norms:

$$\|u\|_{L^p} \approx \|\Phi u\|_{L^p(\mathbb{R}^n,\ell^2)}. \tag{3.19}$$

In other words,

$$C_p'\|u\|_{L^p} \le \Big\| \Big\{ \sum_{j=0}^{\infty} |\psi_j(D)u|^2 \Big\}^{1/2} \Big\|_{L^p} \le C_p\|u\|_{L^p}, \tag{3.20}$$

for $1 < p < \infty$.

The result (3.8) fails at the endpoints $p = 1$ and $p = \infty$. They work when $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are replaced by $\mathfrak{h}^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$, spaces introduced in [G] as local versions of the spaces $\mathfrak{H}^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ studied in [FS]. These spaces can be defined as follows.

$$\mathfrak{h}^1(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{G}^b f \in L^1(\mathbb{R}^n)\}, \tag{3.21}$$

where

$$\mathcal{G}^b f(x) = \sup_{0<r\le 1} \sup_{\varphi\in\mathcal{F}} \Big| \int \varphi_r(x-y) f(y)\, dy \Big|, \tag{3.22}$$

with

$$\mathcal{F} = \{\varphi \in C_0^1(B_1) : \|\nabla\varphi\|_{L^\infty} \le 1\} \tag{3.23}$$

and $\varphi_r(x) = r^{-n}\varphi(r^{-1}x)$. Meanwhile

$$\mathrm{bmo}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{N} f \in L^\infty(\mathbb{R}^n)\}, \tag{3.24}$$

where

$$\mathcal{N}f(x) = \sup_{B \in \mathcal{B}_1(x)} \frac{1}{V(B)} \int_B |f(y) - f_B|\, dy + \frac{1}{V(B_1(x))} \int_{B_1(x)} |f(y)|\, dy, \tag{3.25}$$

with

$$\mathcal{B}_1(x) = \{B_r(x) : 0 < r \le 1\}. \tag{3.26}$$

L^p-Sobolev spaces can be characterized as

$$H^{s,p}(\mathbb{R}^n) = (1-\Delta)^{-s/2} L^p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{3.27}$$

where the right side is a priori a linear subspace of $\mathcal{S}'(\mathbb{R}^n)$. Note that $(1-\Delta)^{-s/2} \in OPS^{-s}_{1,0}$. The result (3.8) plus the results on products from §2 give

$$\begin{gathered} p(x,\xi) \in S^m_{1,\delta},\ \delta \in [0,1) \Longrightarrow \\ p(x,D) : H^{s,p}(\mathbb{R}^n) \to H^{s-m,p}(\mathbb{R}^n), \quad \forall\, s \in \mathbb{R},\ p \in (1,\infty). \end{gathered} \tag{3.28}$$

In addition, there are Hardy and bmo-Sobolev spaces:

$$\mathfrak{h}^{s,1}(\mathbb{R}^n) = (1-\Delta)^{-s/2} \mathfrak{h}^1(\mathbb{R}^n), \quad \mathfrak{h}^{s,\infty}(\mathbb{R}^n) = (1-\Delta)^{-s/2}\, \mathrm{bmo}(\mathbb{R}^n). \tag{3.29}$$

For $p(x,\xi)$ as in (3.28), we have

$$p(x,D) : \mathfrak{h}^{s,p}(\mathbb{R}^n) \longrightarrow \mathfrak{h}^{s-m,p}(\mathbb{R}^n), \quad s \in \mathbb{R},\ p = 1, \infty. \tag{3.30}$$

In case $p = 2$, we can use (3.1) in place of (3.8) to obtain

$$\begin{gathered} p(x,\xi) \in S^m_{\rho,\delta},\ 0 \le \delta < \rho \le 1 \Longrightarrow \\ p(x,D) : H^{s,2}(\mathbb{R}^n) \to H^{s-m,2}(\mathbb{R}^n). \end{gathered} \tag{3.31}$$

Another important class of function spaces consists of the Zygmund spaces:

$$C^s_*(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|\psi_j(D) f\|_{L^\infty} \le C 2^{-sj} \}, \tag{3.32}$$

where $\{\psi_j(\xi)\}$ is the Littlewood-Paley partition of unity given in (3.11). One has

$$k \in \mathbb{Z}^+,\ \alpha \in (0,1) \Longrightarrow C^{k+\alpha}_*(\mathbb{R}^n) = C^{k,\alpha}(\mathbb{R}^n), \tag{3.33}$$

the space of functions whose kth order derivatives are Hölder continuous of exponent α. Parallel to (3.28), one has

$$p(x,\xi) \in S^m_{1,\delta},\ \delta \in [0,1) \Longrightarrow p(x,D) : C^s_*(\mathbb{R}^n) \to C^{s-m}_*(\mathbb{R}^n), \quad \forall\, s, m \in \mathbb{R}. \tag{3.34}$$

Proofs can be found in [St2] and [T5], Chapter 13.

The case $\delta = 1$ is worthy of mention. The conclusions in (3.28) and (3.34) fail for general $p(x,\xi) \in S^m_{1,1}$ when $s - m \le 0$, but E. Stein showed that they hold when $s - m > 0$. These results play an important role in the study of paradifferential operators, as we will see in Lecture II. There we will also introduce an important subclass of $S^m_{1,1}$, for which the conclusions in (3.28) and (3.34) continue to hold for all $s, m \in \mathbb{R}$.

There are also local L^p-Sobolev spaces, etc. If $\Omega \subset \mathbb{R}^n$ is open and $f \in H^{\sigma,q}(\mathbb{R}^n)$ for some $\sigma \in \mathbb{R},\ q \in (1,\infty)$, and if $s \in \mathbb{R},\ p \in (1,\infty)$,

$$f \in H^{s,p}_{\rm loc}(\Omega) \Longleftrightarrow \varphi f \in H^{s,p}(\mathbb{R}^n),\ \forall\, \varphi \in C^\infty_0(\Omega). \tag{3.35}$$

Parallel to (3.28), we have for such f as in (3.35),

$$p(x,\xi) \in S^m_{1,\delta},\ \delta \in [0,1) \Longrightarrow p(x,D)f \in H^{s-m,p}_{\rm loc}(\Omega). \tag{3.36}$$

To see this, given $\varphi \in C^\infty_0(\Omega)$, take $\psi \in C^\infty_0(\Omega)$ such that $\psi \equiv 1$ on supp φ, and write

$$\varphi(x)p(x,D)f = \varphi(x)p(x,D)(\psi f) + \varphi(x)p(x,D)((1-\psi)f). \tag{3.37}$$

The first term on the right side of (3.37) belongs to $H^{s,p}(\mathbb{R}^n)$, by (3.28). As for the second term, results of §2 imply

$$\varphi(x)p(x,D)(1-\psi(x)) \in OPS^{-\infty}_{1,0}, \tag{3.38}$$

so this term belongs to $C^\infty_0(\mathbb{R}^n)$. There are analogues of this for $f \in \mathfrak{h}^{s,1}_{\rm loc}(\Omega), \mathfrak{h}^{s,\infty}_{\rm loc}(\Omega)$, and $C^s_{*,\rm loc}(\Omega)$. Also, in case $p = 2$, one can use (3.31) in place of (3.28) to deduce that if $f \in H^{\sigma,q}(\mathbb{R}^n)$ and $f \in H^{s,2}_{\rm loc}(\Omega)$, then

$$p(x,\xi) \in S^m_{\rho,\delta},\ 0 \le \delta < \rho < 1 \Longrightarrow p(x,D)f \in H^{s-m,2}_{\rm loc}(\Omega). \tag{3.39}$$

4. Elliptic equations

Given $p(x,\xi) \in S^m_{\rho,\delta},\ 0 \le \delta < \rho \le 1$, we say $P = p(x,D)$ is elliptic provided there exist $C, K \in (0,\infty)$ such that

$$|p(x,\xi)^{-1}| \le C\langle\xi\rangle^{-m}, \quad \text{for} \quad |\xi| \ge K. \tag{4.1}$$

In such a case, if we take $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ for $|\xi| \ge 2K,\ 0$ for $|\xi| \le K$, it follows that

$$q(x,\xi) = \varphi(\xi)p(x,\xi)^{-1} \in S^{-m}_{\rho,\delta}. \tag{4.2}$$

Then, with $Q = q(x,D) \in OPS^{-m}_{\rho,\delta}$, results of §2 give

$$PQ = I - R_1, \quad QP = I - S_1, \quad R_1, S_1 \in OPS^{-(\rho-\delta)}_{\rho,\delta}. \tag{4.3}$$

We can then take

$$E_r \sim Q\Bigl(\sum_{j\ge 0} R_1^j\Bigr), \;\; E_\ell \sim \Bigl(\sum_{j\ge 0} S_1^j\Bigr)Q \in OPS^{-m}_{\rho,\delta}, \tag{4.4}$$

and obtain

$$PE_r = I - R, \quad E_\ell P = I - S, \quad R, S \in OPS^{-\infty}_{1,0}. \tag{4.5}$$

The identity $E_\ell(PE_r) = (E_\ell P)E_r$ implies $E_\ell - E_r \in OPS^{-\infty}_{1,0}$, so we can set $E = E_\ell$ and obtain (with a slightly different R)

$$PE = I - R, \quad EP = I - S, \quad R, S \in OPS^{-\infty}_{1,0}. \tag{4.6}$$

One calls E_r in (4.5) a right parametrix of P and E_ℓ a left parametrix. The argument just given shows that if P has both a right and left parametrix (and it does if P is elliptic), these essentially coincide, yielding E as in (4.6), called a (two-sided) parametrix of P.

Given such a parametrix, we deduce various global regularity results. For example, suppose $u \in H^{\sigma,2}(\mathbb{R}^n)$ for some $\sigma \in \mathbb{R}$, and

$$Pu = f. \tag{4.7}$$

Applying E to both sides gives

$$u = Ef + Su. \tag{4.8}$$

Now $u \in H^{\sigma,2}(\mathbb{R}^n) \Rightarrow Su \in H^{k,2}(\mathbb{R}^n)$, $\forall\, k$, and $f \in H^{s,2}(\mathbb{R}^n) \Rightarrow Ef \in H^{s+m,2}(\mathbb{R}^n)$, by (3.31). Hence

$$f \in H^{s,2}(\mathbb{R}^n) \Longrightarrow u \in H^{s+m,2}(\mathbb{R}^n). \tag{4.9}$$

In case $\rho = 1$, i.e., $P \in OPS^m_{1,\delta}$ is elliptic, $\delta \in [0,1)$, use of (3.28) in place of (3.31) gives L^p-Sobolev regularity for $p \in (1,\infty)$:

$$u \in H^{\sigma,p}(\mathbb{R}^n), \;\; Pu = f \in H^{s,p}(\mathbb{R}^n) \Longrightarrow u \in H^{s+m,p}(\mathbb{R}^n). \tag{4.10}$$

Also, for $\rho = 1$, there are analogous regularity results in Hardy and bmo-Sobolev spaces and Zygmund spaces, via the mapping properties (3.30) and (3.34).

Bringing in the results (3.35)–(3.39) on local L^p-Sobolev spaces, we have various local elliptic regularity results. For example, if $P \in OPS^m_{\rho,\delta}$ is elliptic, $0 \le \delta < \rho \le 1$,

$$u \in H^{\sigma,2}(\mathbb{R}^n),\ Pu = f \in H^{s,2}_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{s+m,2}_{\mathrm{loc}}(\Omega), \tag{4.11}$$

and in case $\rho = 1$, given $p \in (1,\infty)$,

$$u \in H^{\sigma,q}(\mathbb{R}^n),\ Pu = f \in H^{s,p}_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{s+m,p}_{\mathrm{loc}}(\Omega), \tag{4.12}$$

with analogues for local regularity in Hardy and bmo-Sobolev spaces and Zygmund spaces.

We now discuss special properties of $p(x,D)$ when $p(x,\xi) \in S^m_{\rho,\delta}$ satisfies

$$P(x,\xi) = \frac{1}{2}\Bigl(p(x,\xi) + p(x,\xi)^*\Bigr) \ge C_0|\xi|^m, \quad \text{for} \quad |\xi| \ge K, \tag{4.13}$$

with $C_0 > 0$. (As usual, $0 \le \delta < \rho \le 1$.) We say such an operator is strongly elliptic. Taking $C_1 \in (0, C_0)$, we can pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$q(x,\xi) = (1 - \varphi(\xi))\Bigl(P(x,\xi) - C_1\langle\xi\rangle^m\Bigr)^{1/2} \in S^{m/2}_{\rho,\delta}. \tag{4.14}$$

Note that, with $Q = q(x,D)$ and $\Lambda = (1-\Delta)^{1/2}$,

$$Q^*Q = P(x,D) - C_1\Lambda^m - r(x,D), \quad r(x,D) \in OPS^{m-(\rho-\delta)}_{\rho,\delta}. \tag{4.15}$$

This gives

$$\begin{aligned} \mathrm{Re}(p(x,D)u,u) &= C_1(\Lambda^m u,u) + (Q^*Qu,u) + (r(x,D)u,u) \\ &\ge C_1\|u\|^2_{H^{m/2}} + (r(x,D)u,u) \\ &\ge C_1\|u\|^2_{H^{m/2}} - C_2\|u\|^2_{H^{(m-(\rho-\delta))/2}}. \end{aligned} \tag{4.16}$$

Here and below we set $H^s = H^{s,2}$. We can take arbitrary $\sigma < (m-(\rho-\delta))/2$ and use the estimate

$$\|u\|^2_{H^{(m-(\rho-\delta))/2}} \le \varepsilon\|u\|^2_{H^{m/2}} + C_{\varepsilon,\sigma}\|u\|^2_{H^\sigma}, \tag{4.17}$$

to produce the following result, known as Gårding's inequality, for strongly elliptic $p(x,D) \in OPS^m_{\rho,\delta}$, when $0 \le \delta < \rho \le 1$. Namely, with C_0 as in (4.13), $\sigma < m/2$, there exists C_2 such that

$$\mathrm{Re}(p(x,D)u,u) \ge C_1\|u\|^2_{H^{m/2}} - C_2\|u\|^2_{H^\sigma}. \tag{4.18}$$

Let us specialize to the case $m > 0$. Then we can take $\sigma = 0$ in (4.18), getting

$$\operatorname{Re}((p(x,D) + C_2)u, u) \geq C_1 \|u\|^2_{H^{m/2}}. \tag{4.19}$$

Standard Hilbert space theory gives

$$p(x,D) + C_2 : H^{m/2}(\mathbb{R}^n) \longrightarrow H^{-m/2}(\mathbb{R}^n), \quad \text{invertible}, \tag{4.20}$$

so we have

$$E = (p(x,D) + C_2)^{-1} : H^{-m/2}(\mathbb{R}^n) \longrightarrow H^{m/2}(\mathbb{R}^n). \tag{4.21}$$

It can also be shown that

$$E \in OPS^{-m}_{\rho,\delta}, \tag{4.22}$$

by taking a parametrix $F \in OPS^{-m}_{\rho,\delta}$ for $p(x,D) + C_2$ and comparing various formulas for $F(p(x,D) + C_2)E$ and $E(p(x,D) + C_2)F$.

Although we do not have the space to discuss them here, we mention the existence of "sharp Gårding inequalities," of Hörmander, Lax-Nirenberg, and Fefferman-Phong. Treatments can be found in [H5].

5. Hyperbolic equations

Let $K(t,x,\xi) \in S^1_{1,0}$ be a $k \times k$ matrix, depending smoothly on t. This defines $K(t) = K(t,x,D) \in OPS^1_{1,0}$. The equation

$$\frac{\partial u}{\partial t} = K(t)u, \quad u(0) = f \tag{5.1}$$

is said to be symmetric hyperbolic provided $K(t,x,\xi) + K(t,x,\xi)^* \in S^0_{1,0}$, or equivalently

$$K(t) + K(t)^* \in OPS^0_{1,0}. \tag{5.2}$$

In such a case, we have unique solvability, and the following regularity:

$$f \in H^{s,2}(\mathbb{R}^n) \Longrightarrow u \in C(\mathbb{R}, H^{s,2}(\mathbb{R}^n)). \tag{5.3}$$

This result follows from energy estimates of the following sort. Set $\Lambda = (1 - \Delta)^{1/2}$. If u satisfies (5.1) and is sufficiently smooth, then

$$\begin{aligned} \frac{d}{dt}\|u(t)\|^2_{H^{s,2}} &= 2\operatorname{Re}(\Lambda^s u_t, \Lambda^s u) \\ &= 2\operatorname{Re}(\Lambda^s K(t)u, \Lambda^s u) \\ &= 2\operatorname{Re}(K(t)\Lambda^s u, \Lambda^s u) + 2\operatorname{Re}(A(t)u, \Lambda^s u). \end{aligned} \tag{5.4}$$

where $A(t) = [\Lambda^s, K(t)] \in OPS^s_{1,0}$. This plus (5.2) and operator estimates from §3 gives

$$\frac{d}{dt}\|u(t)\|^2_{H^{s,2}} \le C\|u(t)\|^2_{H^{s,2}}, \tag{5.5}$$

which leads to

$$\|u(t)\|_{H^{s,2}} \le e^{Ct}\|f\|_{H^{s,2}}. \tag{5.6}$$

To actually establish existence, one convenient approach is to bring in a family of smoothing operators, such as

$$J_\varepsilon = e^{\varepsilon\Delta}, \tag{5.7}$$

modify (5.1) to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon K(t) J_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = J_\varepsilon f, \tag{5.8}$$

which has a global solution by ODE techniques, and estimate $u_\varepsilon(t)$ via arguments parallel to (5.4)–(5.6). Details cen be found in [T2], Chapter 4, for example.

We move on to a class of second order hyperbolic equations. Suppose $L(x, D)$ is a second order differential operator:

$$L(x, D) = \sum_{|\alpha|\le 2} a_\alpha(x) D^\alpha, \quad |D_x^\beta a_\alpha(x)| \le C_{\alpha\beta}, \tag{5.9}$$

and assume

$$L_2(x, \xi) \le -C|\xi|^2, \tag{5.10}$$

for some $C > 0$, where $L_2(x, \xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$. Then one can take C_0 sufficiently large that

$$A(x, \xi) = \bigl(C_0 - L(x, \xi)\bigr)^{1/2} + C_0 \in S^1, \tag{5.11}$$

and that

$$A = A(x, D) \in OPS^1, \quad A^{-1} \in OPS^{-1}, \tag{5.12}$$

the last result via (4.18)–(4.22). Then the second order equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = 0, \quad u(0) = f, \quad \partial_t u(0) = g \tag{5.13}$$

yields the following first order system for $v_1 = Au,\ v_2 = \partial_t u$:

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & A \\ LA^{-1} & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5.14}$$

Since $LA^{-1} = -A^*$ mod OPS^0, this has the form (5.1). We deduce global solvability of (5.13), with regularity

$$f \in H^s(\mathbb{R}^n),\ g \in H^{s-1}(\mathbb{R}^n) \Longrightarrow u \in C(\mathbb{R}, H^s(\mathbb{R}^n)). \tag{5.15}$$

We mention that solutions to (5.13) enjoy finite propagation speed:

$$\operatorname{supp} f, g \subset B_R(p) \Longrightarrow \operatorname{supp} u(t) \subset B_{R+K|t|}(p), \tag{5.16}$$

for $K < \infty$ depending on the top order coefficients of $L(x, D)$. If these coefficients are real analytic, this follows from Holmgren's theorem. In the general case (5.9), one can approximate by real analytic coefficients and apply a limiting argument.

We have no space here to discuss symmetrizable hyperbolic systems, strictly hyperbolic systems, and finite propagation speed for strictly hyperbolic PDE, which gives another approach to the proof of (5.16). These topics can be found in Chapter 4 of [T2].

6. Egorov's theorem

We want to examine the behavior of operators obtained by conjugating a pseudodifferential operator $P_0 \in OPS^m_{1,0}$ by the solution operator to a scalar hyperbolic equation of the form

$$\frac{\partial u}{\partial t} = iA(t, x, D_x)u, \tag{6.1}$$

where we assume $A = A_1 + A_0$ with

$$A_1(t, x, \xi) \in S^1_{cl} \text{ real}, \quad A_0(t, x, \xi) \in S^0_{cl}. \tag{6.2}$$

We suppose $A_1(t, x, \xi)$ is homogeneous in ξ, for $|\xi| \geq 1$. Denote by $S(t, s)$ the solution operator to (6.1), taking $u(s)$ to $u(t)$. This is a bounded operator on each Sobolev space H^σ, with inverse $S(s, t)$. Set

$$P(t) = S(t, 0)P_0 S(0, t). \tag{6.3}$$

We aim to prove the following result of Egorov.

Theorem 6.1. *If $P_0 = p_0(x, D) \in OPS^m_{1,0}$, then for each t, $P(t) \in OPS^m_{1,0}$, modulo a smoothing operator. The principal symbol of $P(t)$ (mod $S^{m-1}_{1,0}$) at a point (x_0, ξ_0) is equal to $p_0(y_0, \eta_0)$, where (y_0, η_0) is obtained from (x_0, ξ_0) by following the flow $\mathcal{C}(t)$ generated by the (time dependent) Hamiltonian vector field*

$$H_{A_1(t,x,\xi)} = \sum_{j=1}^n \Big(\frac{\partial A_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial A_1}{\partial x_j} \frac{\partial}{\partial \xi_j} \Big). \tag{6.4}$$

To start the proof, differentiating (6.3) with respect to t yields

$$P'(t) = i[A(t, x, D), P(t)], \quad P(0) = P_0. \tag{6.5}$$

We will construct an approximate solution $Q(t)$ to (6.5) and then show that $Q(t) - P(t)$ is a smoothing operator.

So we are looking for $Q(t) = q(t, x, D) \in OPS^m_{1,0}$, solving

$$Q'(t) = i[A(t, x, D), Q(t)] + R(t), \quad Q(0) = P_0, \tag{6.6}$$

where $R(t)$ is a smooth family of operators in $OPS^{-\infty}$. We do this by constructing the symbol $q(t, x, \xi)$ in the form

$$q(t, x, \xi) \sim q_0(t, x, \xi) + q_1(t, x, \xi) + \cdots . \tag{6.7}$$

Now the symbol of $i[A, Q(t)]$ is of the form

$$H_{A_1} q + \{A_0, q\} + i \sum_{|\alpha| \ge 2} \frac{i^{|\alpha|}}{\alpha!} \Big(A^{(\alpha)} q_{(\alpha)} - q^{(\alpha)} A_{(\alpha)} \Big), \tag{6.8}$$

where $A^{(\alpha)} = D^\alpha_\xi A$, $A_{(\alpha)} = D^\alpha_x A$, etc. Since we want the difference between this and $\partial q/\partial t$ to have order $-\infty$, this suggests defining $q_0(t, x, \xi)$ by

$$\Big(\frac{\partial}{\partial t} - H_{A_1} \Big) q_0(t, x, \xi) = 0, \quad q_0(0, x, \xi) = p_0(x, \xi). \tag{6.9}$$

Thus $q_0(t, x_0, \xi_0) = p_0(y_0, \eta_0)$, as in the statement of the Theorem; therefore $q_0(t, x, \xi) \in S^m_{1,0}$. The equation (6.9) is called a *transport equation.* Recursively we obtain transport equations

$$\Big(\frac{\partial}{\partial t} - H_{A_1} \Big) q_j(t, x, \xi) = b_j(t, x, \xi), \quad q_j(0, x, \xi) = 0, \tag{6.10}$$

for $j \ge 1$, with solutions in $S^{m-j}_{1,0}$, leading to a solution to (6.6).

Finally we show $P(t) - Q(t)$ is a smoothing operator. Equivalently, we show that, for any $f \in H^\sigma(\mathbb{R}^n)$,

$$v(t) - w(t) = S(t,0)P_0 f - Q(t)S(t,0)f \in H^\infty(\mathbb{R}^n), \tag{6.11}$$

where $H^\infty(\mathbb{R}^n) = \cap_s H^s(\mathbb{R}^n)$. (Here we deal only with L^2-Sobolev spaces; $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$.) Note that

$$\frac{\partial v}{\partial t} = iA(t,x,D)v, \quad v(0) = P_0 f, \tag{6.12}$$

while use of (6.6) gives

$$\frac{\partial w}{\partial t} = iA(t,x,D)w + g, \quad w(0) = P_0 f, \tag{6.13}$$

where

$$g = R(t)S(t,0)w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n)). \tag{6.14}$$

Hence

$$\frac{\partial}{\partial t}(v - w) = iA(t,x,D)(v - w) - g, \quad v(0) - w(0) = 0. \tag{6.15}$$

Thus energy estimates for hyperbolic equations from §5 yield $v(t) - w(t) \in H^\infty$ for any $f \in H^\sigma(\mathbb{R}^n)$, completing the proof.

A check of the proof shows that

$$P_0 \in OPS^m_{cl} \Longrightarrow P(t) \in OPS^m_{cl}. \tag{6.16}$$

Also the proof readily extends to yield the following:

Proposition 6.2. *With $A(t,x,D)$ as before,*

$$P_0 \in OPS^m_{\rho,\delta} \Longrightarrow P(t) \in OPS^m_{\rho,\delta} \tag{6.17}$$

provided

$$\rho > \frac{1}{2}, \quad \delta = 1 - \rho. \tag{6.18}$$

One needs $\delta = 1 - \rho$ to insure that $p(\mathcal{C}(t)(x,\xi)) \in S^m_{\rho,\delta}$, and one needs $\rho > \delta$ to insure that the transport equations generate $q_j(t,x,\xi)$ of progressively lower order.

7. Wave front set and microlocal regularity

We define the notion of wave front set of a distribution $u \in H^{-\infty}(\mathbb{R}^n) = \cup_s H^s(\mathbb{R}^n)$, which refines the notion of singular support. Throughout this section, $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$.

If $p(x,\xi) \in S^m$ has principal symbol $p_m(x,\xi)$, homogeneous in ξ, then the characteristic set of $P = p(x,D)$ is given by

$$\text{Char } P = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : p_m(x,\xi) = 0\}. \tag{7.1}$$

If $p_m(x,\xi)$ is a $K \times K$ matrix, take the determinant. Equivalently, (x_0,ξ_0) is noncharacteristic for P, or P is elliptic at (x_0,ξ_0), if $|p(x,\xi)^{-1}| \le C|\xi|^{-m}$, for (x,ξ) in a small conic neighborhood of (x_0,ξ_0), and $|\xi|$ large. By definition, a conic set is invariant under the dilations $(x,\xi) \mapsto (x, r\xi),\ r \in (0,\infty)$. The wave front set is defined by

$$WF(u) = \bigcap \{\text{Char } P : P \in OPS^0,\ Pu \in C^\infty\}. \tag{7.2}$$

Clearly $WF(u)$ is a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. If π is the projection $(x,\xi) \mapsto x$, we have:

Proposition 7.1. *$\pi(WF(u)) =$ sing supp u.*

Proof. If $x_0 \notin$ sing supp u, there is a $\varphi \in C_0^\infty(\mathbb{R}^n),\ \varphi = 1$ near x_0, such that $\varphi u \in C_0^\infty(\mathbb{R}^n)$. Clearly $(x_0,\xi) \notin$ Char φ for any $\xi \ne 0$, so $\pi(WF(u)) \subset$ sing supp u.

Conversely, if $x_0 \notin \pi(WF(u))$, then for any $\xi \ne 0$ there is a $Q \in OPS^0$ such that $(x_0,\xi) \notin$ Char Q and $Qu \in C^\infty$. Thus we can construct finitely many $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$ and each $(x_0,\xi),\ |\xi| = 1$ is noncharacteristic for some Q_j. Let $Q = \sum Q_j^* Q_j \in OPS^0$. Then Q is elliptic near x_0 and $Qu \in C^\infty$, so u is C^∞ near x_0.

We define the associated notion of $ES(P)$ for a pseudodifferential operator. Let U be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. We say $p(x,\xi) \in S^m_{\rho,\delta}$ has order $-\infty$ on U if for each closed conic set V of U we have estimates, for each N,

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta N V} \langle \xi \rangle^{-N}, \quad (x,\xi) \in V. \tag{7.3}$$

If $P = p(x,D) \in OPS^m_{\rho,\delta}$, we define the essential support of P (and of $p(x,\xi)$) to be the smallest closed conic set on the complement of which $p(x,\xi)$ has order $-\infty$. We denote this set by $ES(P)$.

From the symbol calculus of §2 it follows easily that

$$ES(P_1P_2) \subset ES(P_1) \cap ES(P_2) \tag{7.4}$$

provided $P_j \in OPS^{m_j}_{\rho_j,\delta_j}$ and $\rho_1 > \delta_2$. To relate $WF(Pu)$ to $WF(u)$ and $ES(P)$, we begin with the following.

Lemma 7.2. *Let $u \in H^{-\infty}(\mathbb{R}^n)$ and suppose U is a conic open set satisfying $WF(u) \cap U = \emptyset$. If $P \in OPS^m_{\rho,\delta}$, $\rho > 0$, and $\delta < 1$, and $ES(P) \subset U$, then $Pu \in C^\infty$.*

Proof. Taking $P_0 \in OPS^0$ with symbol identically 1 on a conic neighborhood of $ES(P)$, so $P = PP_0$ mod $OPS^{-\infty}$, it suffices to conclude that $P_0 u \in C^\infty$, so we can specialize the hypothesis to $P \in OPS^0$.

By hypothesis, we can find $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$ and each $(x, \xi) \in ES(P)$ is noncharacteristic for some Q_j, and if $Q = \sum Q_j^* Q_j$, then $Qu \in C^\infty$ and $\text{Char } Q \cap ES(P) = \emptyset$. We claim there exists $A \in OPS^0$ such that $AQ = P$ mod $OPS^{-\infty}$. Indeed, let $\tilde{Q}$ be an elliptic operator whose symbol equals that of Q on a conic neighborhood of $ES(P)$, and let $\tilde{Q}^{-1}$ denote a parametrix for $\tilde{Q}$. Now simply set set $A = P\tilde{Q}^{-1}$. Consequently (mod C^∞) $Pu = AQu \in C^\infty$, so the lemma is proved.

We are ready for the basic result on the preservation of wave front sets by a pseudodifferential operator.

Proposition 7.3. *If $u \in H^{-\infty}$ and $P \in OPS^m_{\rho,\delta}$, with $\rho > 0,\ \delta < 1$, then*

$$WF(Pu) \subset WF(u) \cap ES(P). \tag{7.5}$$

Proof. First we show $WF(Pu) \subset ES(P)$. Indeed, if $(x_0, \xi_0) \notin ES(P)$, choose $Q = q(x, D) \in OPS^0$ such that $q(x, \xi) = 1$ on a conic neighborhood of (x_0, ξ_0) and $ES(Q) \cap ES(P) = \emptyset$. Thus $QP \in OPS^{-\infty}$, so $QPu \in C^\infty$. Hence $(x_0, \xi_0) \notin WF(Pu)$.

In order to show that $WF(Pu) \subset WF(u)$, let Γ be any conic neighborhood of $WF(u)$ and write $P = P_1 + P_2,\ P_j \in OPS^m_{\rho,\delta}$, with $ES(P_1) \subset \Gamma$ and $ES(P_2) \cap WF(u) = \emptyset$. By Lemma 7.2, $P_2 u \in C^\infty$. Thus $WF(Pu) = WF(P_1 u) \subset \Gamma$, which shows $WF(Pu) \subset WF(u)$.

One says that a pseudodifferential operator of type (ρ, δ), with $\rho > 0$ and $\delta < 1$, is microlocal. As a corollary, we have the following sharper form of local regularity for elliptic operators, called microlocal regularity.

Corollary 7.4. *If $P \in OPS^m_{\rho,\delta}$ is elliptic, $0 \le \delta < \rho \le 1$, then*

$$WF(Pu) = WF(u). \tag{7.6}$$

Proof. We have seen that $WF(Pu) \subset WF(u)$. On the other hand, if $E \in OPS^{-m}_{\rho,\delta}$ is a parametrix for P, we see that $WF(u) = WF(EPu) \subset WF(Pu)$. In fact, by an argument close to the proof of Lemma 7.2, we have for general P that

$$WF(u) \subset WF(Pu) \cup \ \text{Char } P. \tag{7.7}$$

We next discuss how the solution operator e^{itA} to a scalar hyperbolic equation $\partial u/\partial t = iA(x,D)u$ propagates the wave front set. We assume $A(x,\xi) \in S^1_{cl}$, with real principal symbol. Suppose $WF(u) = \Sigma$. Then there is a countable family of operators $p_j(x,D) \in OPS^0$, each of whose complete symbols vanish in a neighborhood of Σ, but such that

$$\Sigma = \bigcap_j \{(x,\xi) : p_j(x,\xi) = 0\}. \tag{7.8}$$

We know that $p_j(x,D)u \in C^\infty$ for each j. Using Egorov's Theorem, we want to construct a family of pseudodifferential operators $q_j(x,D) \in OPS^0$ such that $q_j(x,D)e^{itA}u \in C^\infty$, this family being rich enough to describe the wave front set of $e^{itA}u$.

Indeed, let $q_j(x,D) = e^{itA}p_j(x,D)e^{-itA}$. Egorov's Theorem implies that $q_j(x,D) \in OPS^0$, (modulo a smoothing operator) and gives the principal symbol of $q_j(x,D)$. Since $p_j(x,D)u \in C^\infty$, we have $e^{itA}p_j(x,D)u \in C^\infty$, which in turn implies $q_j(x,D)e^{itA}u \in C^\infty$. From this it follows that $WF(e^{itA}u)$ is contained in the intersection of the characteristics of the $q_j(x,D)$, which is precisely $C(t)\Sigma$, the image of Σ under the canonical transformation $C(t)$, generated by H_{A_1}. In other words,

$$WF(e^{itA}u) \subset C(t)WF(u).$$

However, our argument is reversible; $u = e^{-itA}(e^{itA}u)$. Consequently, we have:

Proposition 7.5. *If $A = A(x,D) \in OPS^1$ is scalar with real principal symbol, then, for $u \in H^{-\infty}$,*

$$WF(e^{itA}u) = C(t)WF(u). \tag{7.9}$$

The same argument works for the solution operator $S(t,0)$ to a time-dependent scalar hyperbolic equation.

8. Pseudodifferential operators on compact manifolds

Let M be a smooth compact manifold. We aim to define the class $OPS^m_{\rho,\delta}(M)$, at least for a certain range of (ρ,δ). A first attempt would be to say that if $P : C^\infty(M) \to \mathcal{D}'(M)$, then $P \in OPS^m_{\rho,\delta}(M)$ provided its Schwartz kernel is C^∞ off the diagonal in $M \times M$, and there exists an open cover Ω_j of M, a subordinate partition of unity φ_j, and diffeomorphisms $F_j : \Omega_j \to \mathcal{O}_j \subset \mathbb{R}^n$ which transform the operators $\varphi_k P \varphi_j : C^\infty(\Omega_j) \to \mathcal{E}'(\Omega_k)$ into pseudodifferential operators in $OPS^m_{\rho,\delta}$, as defined in §1.

This is not entirely satisfactory as a definition. We need to know that if $P \in OPS^m_{\rho,\delta}(M)$, then P is so transformed by every coordinate cover. This comes down to demanding that the class $OPS^m_{\rho,\delta}$ defined in §1 is invariant under a coordinate transformation, i.e., a diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$. It would suffice to establish this for the case where F is the identity outside a compact set.

In case $\rho \in (1/2, 1]$ and $\delta = 1 - \rho$, this invariance is a special case of the Egorov Theorem established in §6. Indeed, one can find a time-dependent vector field $X(t)$ whose flow at $t = 1$ coincides with F and apply Theorem 6.1 to $iA(t, x, D) = X(t)$. This shows that the class $OPS^m_{\rho,\delta}(M)$ is well defined for a compact manifold M provided

$$\frac{1}{2} < \rho \le 1, \quad \text{and} \quad \delta = 1 - \rho. \tag{8.1}$$

Note that the formula for the principal symbol of the conjugated operator given there implies

$$p(1, F(x), \xi) = p_0(x, F'(x)^t \xi), \tag{8.2}$$

so that the principal symbol is well defined on the *cotangent bundle* of M.

An alternative approach to coordinate invariance is to insert the coordinate changes into the Fourier integral representation of P and work on that. This approach has the advantage of working for a larger set of symbol classes $S^m_{\rho,\delta}$ than the more general conjugation invariance applies to. In fact, one needs only

$$\frac{1}{2} < \rho \le 1, \quad \rho + \delta \ge 1. \tag{8.3}$$

Proofs can be found in [H5] and [T2].

Defining classes of pseudodifferential operators on noncompact M requires further considerations. Lecture IV will take this up for complete Riemannian manifolds with bounded geometry.

II. Pseudodifferential operators with rough symbols and paradifferential operators

In this second lecture we treat pseudodifferential operators associated to symbols with low regularity. In §1 we lay out the basic symbol classes we will consider. Basic examples start with symbols of differential operators with Hölder continuous coefficients. We introduce the process of symbol smoothing, writing a rough symbol $p(x,\xi)$ as $p^{\#}(x,\xi)+p^b(x,\xi)$, with $p^{\#}(x,\xi)\in S^m_{1,\delta}$, so that some symbol calculus applies to $p^{\#}(x,D)$, either such as discussed in Lecture I (when $\delta<1$) or in Proposition 1.1 in this section (dealing with some cases where $\delta=1$), while $p^b(x,D)$ is regarded as a remainder, to be estimated. Section 2 provides crucial operator estimates. Section 3 illustrates how results of §§1–2 apply to regularity for linear elliptic equations with rough coefficients.

Section 4 treats paradifferential operators, introduced in [Bon] to study nonlinear PDE. These are linear operators whose symbols are singular in the sense of belonging to $S^m_{1,1}$. Results of §§1–2 are set up to apply to these operators also. In §5 we illustrate how results of §§2 and 4 apply to regularity for nonlinear elliptic PDE, getting short proofs of Schauder type regularity results, though not the deeper results that rely on DeGiorgi-Nash-Moser theory or Krylov-Safanov theory.

Section 6 studies paraproducts, a particularly interesting class of paradifferential operators. A number of special paraproduct estimates are produced, and the theory is applied to an important estimate on $\|uv\|_{H^{s,p}}$ due to [CW], applicable to numerous problems in nonlinear PDE. We close with §7, analyzing estimates on commutators, $[P,f]u=Pfu-fPu$, for various classes of P and f, with emphasis on $f\in \mathrm{Lip}^1(\mathbb{R}^n)$ and $f\in \mathrm{bmo}(\mathbb{R}^n)$. Applications of the latter class to div-curl estimates and to regularity results for elliptic operators with coefficients in $L^\infty\cap \mathrm{vmo}$ are also discussed.

More detailed presentations of results discussed here can be found in [CM], [T2], [T5], [T6], and references given there.

1. Symbol classes and symbol smoothing

The most basic classes of symbols with limited smoothness we will deal with are $C^rS^m_{1,\delta}$ and $C^r_*S^m_{1,\delta}$, with $r\in\mathbb{R}^+$ and $\delta\in[0,1]$, defined as follows. First,

$$
\begin{aligned}
p(x,\xi)\in C^rS^m_{1,\delta}\Leftrightarrow |D^\alpha_\xi p(x,\xi)| &\le C_\alpha\langle\xi\rangle^{m-|\alpha|},\\
\|D^\alpha_\xi p(\cdot,\xi)\|_{C^r} &\le C_\alpha\langle\xi\rangle^{m-|\alpha|+r\delta},\ \text{and}\\
\|D^\alpha_\xi p(\cdot,\xi)\|_{C^j} &\le C_\alpha\langle\xi\rangle^{m-|\alpha|+j\delta},\quad \text{for}\ \ 0\le j\le r.
\end{aligned}
\tag{1.1}
$$

Next,

$$p(x,\xi) \in C_*^r S_{1,\delta}^m \Leftrightarrow |D_\xi^\alpha p(x,\xi)| \le C_\alpha \langle \xi \rangle^{m-|\alpha|}, \quad \text{and} \tag{1.2}$$
$$\|D_\xi^\alpha p(\cdot,\xi)\|_{C_*^r} \le C_\alpha \langle \xi \rangle^{m-|\alpha|+r\delta}.$$

Recall $C_*^r(\mathbb{R}^n)$ is the Zygmund space, defined by (I.3.32), while $C^r(\mathbb{R}^n) = C^{k,\alpha}(\mathbb{R}^n)$ if $r = k+\alpha,\ k \in \mathbb{Z}^+,\ \alpha \in [0,1)$. As stated in (I.3.33), $C_*^r(\mathbb{R}^n) = C^r(\mathbb{R}^n)$ if $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, but $C^r S_{1,\delta}^m$ and $C_*^r S_{1,\delta}^m$ differ slightly, according to the definitions (1.1) and (1.2).

Basic examples of such symbols include $p(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)\xi^\alpha$, symbols of differential operators with coefficients $p_\alpha(x)$ in Hölder spaces, or Zygmund spaces. In such cases, $\delta = 0$. Cases with nonzero δ will arise below.

In order to deal with the operator $p(x,D)$ associated to such $p(x,\xi)$, it is convenient to split $p(x,\xi)$ into two pieces:

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi), \tag{1.3}$$

where $p^\#(x,\xi)$ is obtained from $p(x,\xi)$ by "symbol smoothing" and $p^b(x,\xi)$ is the remainder. We define $p^\#(x,\xi)$ as follows. Let $\{\psi_k(\xi) : k \ge 0\}$ be a Littlewood-Paley partition of unity, as in (I.3.11). Set

$$p^\#(x,\xi) = \sum_{k=0}^\infty J_{\varepsilon_k} p(x,\xi)\, \psi_k(\xi), \tag{1.4}$$

where

$$J_\varepsilon f(x) = \psi_0(\varepsilon D) f(x). \tag{1.5}$$

Take $\delta \in (0,1]$, and set

$$\varepsilon_k = 2^{-k\delta}, \quad (\text{or, if } \ \delta = 1)\ \varepsilon_k = 2^{-(k-3)}. \tag{1.6}$$

The first symbol smoothing result is

$$p(x,\xi) \in C^r S_{1,0}^m \Rightarrow p^\#(x,\xi) \in S_{1,\delta}^m,\ p^b(x,\xi) \in C^r S_{1,\delta}^{m-r\delta}, \tag{1.7}$$

with a similar result for $p(x,\xi) \in C_*^r S_{1,0}^m$. The proof is a straightforward consequence of the following estimates, with $\varepsilon \in (0,1]$:

$$\begin{aligned} \|D_x^\beta J_\varepsilon f\|_{C_*^r} &\le C_\beta \varepsilon^{-|\beta|} \|f\|_{C_*^r}, \\ \|f - J_\varepsilon f\|_{C_*^{r-s}} &\le C\varepsilon^s \|f\|_{C_*^r}, \quad s \ge 0, \\ \|f - J_\varepsilon f\|_{L^\infty} &\le C\varepsilon^r \|f\|_{C_*^r}, \quad r > 0. \end{aligned} \tag{1.8}$$

The first estimate applies to $p^{\#}(x,\xi)$ and the next two to $p^b(x,\xi)$. Details on this and other symbol smoothing results given below can be found in Chapters 1–3 of [T4] and in Chapter 13, §§9–10, of [T5].

Here is one generalization of (1.7). Take $\delta < \gamma \le 1$ and apply symbol smoothing to $p(x,\xi) \in C^r S^m_{1,\delta}$, with $\varepsilon_k = 2^{-j(\gamma-\delta)}$. The result is

$$p(x,\xi) \in C^r S^m_{1,\delta} \Rightarrow p^{\#}(x,\xi) \in S^m_{1,\gamma},\ p^b(x,\xi) \in C^r S^{m-r(\gamma-\delta)}_{1,\gamma}. \tag{1.9}$$

The point of making the decomposition (1.3) is to be able to apply symbol calculus to the operator $p^{\#}(x,D)$, while $p^b(x,D)$ is a remainder to be estimated. In connection with this, it is useful to record further symbol properties of $p^{\#}(x,\xi)$. For example, one has

$$\begin{aligned} p(x,\xi) \in C^r S^m_{1,0} \Rightarrow D^\beta_x p^{\#}(x,\xi) \in S^m_{1,\delta}, &\qquad |\beta| \le r, \\ S^{m+\delta(|\beta|-r)}_{1,\delta}, &\qquad |\beta| > r. \end{aligned} \tag{1.10}$$

This is proven using the following complement to (1.8):

$$\begin{aligned} \|D^\beta_x J_\varepsilon f\|_{L^\infty} \le C\|f\|_{C^r}, &\qquad |\beta| \le r, \\ C\varepsilon^{-(|\beta|-r)}\|f\|_{C^r_*}, &\qquad |\beta| > r. \end{aligned} \tag{1.11}$$

The following symbol classes serve to record such symbol behavior and variants:

$$\mathcal{A}^r_0 S^m_{1,\delta} \subset \mathcal{A}^r S^m_{1,\delta} \subset {}^r S^m_{1,\delta}. \tag{1.12}$$

We say $p(x,\xi) \in {}^r S^m_{1,\delta}$ provided $D^\beta_x p(x,\xi)$ satisfies the conclusions in (1.10). We say

$$\begin{aligned} p(x,\xi) \in \mathcal{A}^r S^m_{1,\delta} \Leftrightarrow \|D^\alpha_\xi p(\cdot,\xi)\|_{C^r} \le C_\alpha \langle\xi\rangle^{m-|\alpha|}, &\quad \text{and} \\ |D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|+\delta(|\beta|-r)}, &\quad \text{for } |\beta| > r. \end{aligned} \tag{1.13}$$

Furthermore, we say

$$p(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta} \Leftrightarrow \|D^\alpha_\xi p(\cdot,\xi)\|_{C^{r+s}} \le C_{\alpha s}\langle\xi\rangle^{m-|\alpha|+\delta s}, \quad \text{for } s \ge 0. \tag{1.14}$$

The inclusions in (1.12) are easy to verify. The following result refines (1.10):

$$p(x,\xi) \in C^r S^m_{1,0} \Rightarrow p^{\#}(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta}. \tag{1.15}$$

This can be proven by supplementing (1.11) with

$$\|J_\varepsilon f\|_{C^{r+s}_*} \le C\varepsilon^{-s}\|f\|_{C^r_*}, \quad s \ge 0. \tag{1.16}$$

We next bring in special symbol classes that record the behavior of $p^{\#}(x,\xi)$ when one takes $\varepsilon_k = 2^{-(k-3)}$ in (1.4). Given $\rho \in (0,1)$, we say

$$\begin{aligned} p(x,\xi) \in \mathcal{B}_\rho S^m_{1,1} \Leftrightarrow p(x,\xi) \in S^m_{1,1}, &\quad \text{and} \\ \hat{p}(\eta,\xi) \ \text{ is supported on } \ |\eta| < \rho|\xi|&, \end{aligned} \tag{1.17}$$

and we set

$$\mathcal{B}S^m_{1,1} = \bigcup_{\rho<1} \mathcal{B}_\rho S^m_{1,1}. \tag{1.18}$$

We also set

$$\mathcal{B}^r S^m_{1,1} = \mathcal{B}S^m_{1,1} \cap \mathcal{A}^r S^m_{1,1}. \tag{1.19}$$

We have for this type of symbol smoothing

$$\begin{aligned} p(x,\xi) \in C^r S^m_{1,0} \Rightarrow p^{\#}(x,\xi) &\in \mathcal{B}^r S^m_{1,1}, \\ p^b(x,\xi) &\in C^r S^{m-r}_{1,1}. \end{aligned} \tag{1.20}$$

The collection of symbol classes listed in (1.12) and (1.17)–(1.19) might strike one as overabundant. However, these various classes arose in the papers [Bon], [Mey], and [H6] (with different notation), and it seems useful to keep all of them.

As stated, we want to apply symbol calculus to operators of the form $p^{\#}(x,D)$ arising in the decomposition (1.3). This includes cases when (1.20) holds. The following result on operator composition is a variant of results of [Bon] and [Mey], obtained in [AT]. A proof is also given in Chapter 13 of [T5].

Proposition 1.1. *Assume*

$$a(x,\xi) \in S^\mu_{1,1}, \quad b(x,\xi) \in \mathcal{B}S^m_{1,1}. \tag{1.21}$$

Then

$$a(x,D)b(x,D) = p(x,D) \in OPS^{\mu+m}_{1,1}, \tag{1.22}$$

and there is the following result on the remainder $r_\nu(x,\xi)$ *in*

$$p(x,\xi) = \sum_{|\alpha|\le\nu} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi a(x,\xi) D^\alpha_x b(x,\xi) + r_\nu(x,\xi). \tag{1.23}$$

Namely, if one has

$$\|D^\alpha_\xi b(\cdot,\xi)\|_{C^r_*} \le C_\alpha \langle\xi\rangle^{m_2-|\alpha|}, \tag{1.24}$$

then

$$\nu+1>r \Longrightarrow r_\nu(x,\xi)\in S^{\mu+m_2-r}_{1,1}. \tag{1.25}$$

In particular, this holds with $m_2=m$ *for all* $b(x,\xi)\in\mathcal{B}^r S^m_{1,1}$.

If one has (1.21), (1.24), and also, for some $\mu_2\le\mu$,

$$|D^\beta_x D^\alpha_\xi a(x,\xi)|\le C_{\alpha\beta}\langle\xi\rangle^{\mu_2-|\alpha|+|\beta|},\quad \text{for}\ \ |\alpha|\ge\nu+1, \tag{1.26}$$

then (1.28) improves to

$$\nu+1>r \Longrightarrow r_\nu(x,\xi)\in S^{\mu_2+m_2-r}_{1,1}. \tag{1.27}$$

Remark. Hypothesis (1.24) holds with $r=k$ if

$$D^\beta_x b(x,\xi)\in S^{m_2}_{1,1},\quad \text{for}\ \ |\beta|\le k. \tag{1.28}$$

Also, it is very significant to note that one can take $r=0$ in (1.24).

The family $OP\mathcal{B}S^*_{1,1}$ does not form an algebra, but the following result is useful.

Proposition 1.2. *Assume that for some* $\rho\in(0,1/4)$,

$$a(x,\xi)\in\mathcal{B}_\rho S^\mu_{1,1},\quad b(x,\xi)\in\mathcal{B}_\rho S^m_{1,1}. \tag{1.29}$$

Then

$$a(x,\xi)b(x,\xi)\in\mathcal{B}S^{\mu+m}_{1,1},\quad a(x,D)b(x,D)\in OP\mathcal{B}S^{\mu+m}_{1,1}. \tag{1.30}$$

2. Operator estimates on function spaces

The fundamental result of this section is the following result of [Bou], following earlier work of [St2]. A proof is also given in Chapter 13 of [T5].

Proposition 2.1. *If* $r>0$ *and* $p\in(1,\infty)$, *then for* $p(x,\xi)\in C^r_*S^m_{1,1}$,

$$p(x,D): H^{s+m,p}(\mathbb{R}^n)\longrightarrow H^{s,p}(\mathbb{R}^n), \tag{2.1}$$

provided $0<s<r$. *Furthermore, under these hypotheses,*

$$p(x,D): C^{s+m}_*(\mathbb{R}^n)\longrightarrow C^s_*(\mathbb{R}^n). \tag{2.2}$$

It suffices to treat the case $m = 0$. The proof begins by writing $p(x,\xi)$ as a rapidly convergent sum of elementary symbols, where by definition an elementary symbol in $C^r_* S^0_{1,1}$ is of the form

$$q(x,\xi) = \sum_{k=0}^{\infty} Q_k(x)\varphi_k(\xi), \tag{2.3}$$

where $\varphi_k(\xi)$ is supported on $\langle\xi\rangle \sim 2^k$ and bounded in $S^0_{1,0}$ (in fact $\varphi_k(\xi) = \varphi_1(2^{1-k}\xi)$ for $k \geq 2$), and Q_k satisfies

$$\|Q_k\|_{L^\infty} \leq C, \quad \|Q_k\|_{C^r_*} \leq C \cdot 2^{kr}. \tag{2.4}$$

One is reduced to proving (2.1) and (2.2) for $p(x,\xi) = q(x,\xi)$. One next decomposes $q(x,\xi)$ into three pieces:

$$\begin{aligned} q(x,\xi) &= \sum_k \Big\{ \sum_{j=0}^{k-4} Q_{kj}(x) + \sum_{j=k-3}^{k+3} Q_{kj}(x) + \sum_{j=k+4}^{\infty} Q_{kj}(x) \Big\} \varphi_k(\xi) \\ &= q_1(x,\xi) + q_2(x,\xi) + q_3(x,\xi), \end{aligned} \tag{2.5}$$

by a variant of symbol smoothing. Namely, if $\{\psi_j : j \geq 0\}$ is the Littlewood-Paley partition of unity used in (1.4), we take

$$Q_{kj}(x) = \psi_j(D) Q_k(x). \tag{2.6}$$

It is shown that (2.1)–(2.2) hold for $q_1(x,D)$ for all $s \in \mathbb{R}$, that they hold for $q_2(x,D)$ for $s > 0$, and that they hold for $q_3(x,D)$ for $s < r$. Estimates of the form (6.14) (see §6) are used in the analysis of each term $q_j(x,D)$. Details can be found in the references cited above.

In case $p(x,\xi) \in \mathcal{B}S^m_{1,1}$, only $q_1(x,\xi)$ arises in this analysis, and we have:

Proposition 2.2. *If $p(x,\xi) \in \mathcal{B}S^m_{1,1}$, then (2.1) and (2.2) hold for all $s \in \mathbb{R}$.*

For $S^m_{1,1}$ replaced by $S^m_{1,\delta}$, we have the following.

Proposition 2.3. *If $\delta \in [0,1)$ and $p(x,\xi) \in C^r_* S^m_{1,\delta}$, with $r > 0$, then*

$$\begin{aligned} p(x,D) &: H^{s+m,p}(\mathbb{R}^n) \longrightarrow H^{s,p}(\mathbb{R}^n), \\ p(x,D) &: C^{s+m}_*(\mathbb{R}^n) \longrightarrow C^s_*(\mathbb{R}^n), \end{aligned} \tag{2.7}$$

provided $p \in (1,\infty)$ and

$$-(1-\delta)r < s < r. \tag{2.8}$$

We already have (2.8) for $0 < s < r$. To get it for $-(1-\delta)r < s \le 0$, use the symbol smoothing (1.9); with $\gamma \in (\delta, 1)$,

$$p(x,\xi) = p^{\#}(x,\xi) + p^{b}(x,\xi), \quad p^{\#}(x,\xi) \in S^{m}_{1,\gamma}, \quad p^{b}(x,\xi) \in C^{r}_{*}S^{m-r(\gamma-\delta)}_{1,\gamma}. \tag{2.9}$$

The mapping properties in (2.7) hold for $p^{\#}(x,D)$ for all $s \in \mathbb{R}$, and the corresponding properties for $p^{b}(x,D)$ can be deduced from Proposition 2.1, which gives

$$p^{b}(x,D) : H^{s+m,p} \to H^{s+(\gamma-\delta)r,p} \subset H^{s,p}, \quad \forall\, s >\in (-(\gamma-\delta)r, 0]. \tag{2.10}$$

Then let $\gamma \nearrow 1$ to obtain (2.7) under the hypothesis (2.8).

3. Linear elliptic regularity

We illustrate how to use results developed so far to obtain global elliptic regularity results. Assume $r > 0$ and

$$p(x,\xi) \in C^{r}_{*}S^{m}_{1,0} \ \text{ is elliptic,} \tag{3.1}$$

that

$$-r < \sigma < s < r, \quad 1 < p < \infty, \tag{3.2}$$

and that

$$u \in H^{\sigma+m,p}(\mathbb{R}^n), \quad p(x,D)u = f \in H^{s,p}(\mathbb{R}^n). \tag{3.3}$$

We claim

$$u \in H^{s+m,p}(\mathbb{R}^n). \tag{3.4}$$

To get this, use the symbol smoothing $p(x,\xi) = p^{\#}(x,\xi) + p^{b}(x,\xi)$, with

$$p^{\#}(x,\xi) \in S^{m}_{1,\delta} \ \text{ elliptic}, \quad p^{b}(x,\xi) \in C^{r}_{*}S^{m-r\delta}_{1,\delta}. \tag{3.5}$$

Let $E \in OPS^{-m}_{1,\delta}$ be a parametrix for $p^{\#}(x,D)$. Then (3.3) yields

$$u = Ef - Ep^{b}(x,D)u \mod H^{\infty,p}(\mathbb{R}^n), \tag{3.6}$$

we have $Ef \in H^{s+m,p}(\mathbb{R}^n)$, and Proposition 2.1 gives

$$p^{b}(x,D)u \in H^{\sigma_1,p}(\mathbb{R}^n), \quad \sigma_1 = \min(\sigma + r\delta, s), \tag{3.7}$$

under the hypothesis on u in (3.3), hence $Ep^b(x,D)u \in H^{\sigma_1+m,p}(\mathbb{R}^n)$, hence

$$u \in H^{\sigma_1+m,p}(\mathbb{R}^n). \tag{3.8}$$

Iterating this argument yields the assertion (3.4).

4. Paradifferential operators

Paradifferential operators arise in the process of "linearizing" nonlinear operators. They originated in work of [Bon], with important complements by [Mey]. We begin with a construction of [Mey]. Assume

$$F \in C^\infty(\mathbb{R}), \quad F(0) = 0. \tag{4.1}$$

Let $\{\psi_k : k \geq 0\}$ be the Littlewood-Paley partition of unity we have used before, and set

$$\Psi_k(\xi) = \sum_{\ell \leq k} \psi_\ell(\xi), \quad u_k = \Psi_k(D)u, \quad u_{-1} = 0. \tag{4.2}$$

Then

$$\begin{aligned} F(u) &= \sum_{k \geq -1} \big\{ F(u_{k+1}) - F(u_k) \big\} \\ &= M(x,D)u, \end{aligned} \tag{4.3}$$

where

$$M(x,\xi) = \sum_k m_k(x)\psi_{k+1}(\xi), \quad m_k(x) = \int_0^1 F'(u_k + \tau\psi_{k+1}(D)u)\, d\tau. \tag{4.4}$$

When needed, we use the notation $M_F(u;x,\xi)$ in place of $M(x,\xi)$. To estimate $M(x,\xi)$, given $u \in L^\infty(\mathbb{R}^n)$, we have, by the chain rule

$$\|D_x^\ell m_k\|_{L^\infty} \leq C_\ell \sum_{1 \leq \nu \leq \ell} \|D^{\ell_1} u_{k+1}\|_{L^\infty} \cdots \|D^{\ell_\nu} u_{k+1}\|_{L^\infty} \cdot \|F''\|_{C^{\nu-1}}. \tag{4.5}$$

Also

$$\|D^{\ell_j} u_{k+1}\|_{L^\infty} \leq C_{\ell_j}\, 2^{k\ell_j}\, \|u\|_{L^\infty}. \tag{4.6}$$

Since $2^{k\ell} \sim \langle \xi \rangle^\ell$ on supp ψ_{k+1}, we have

$$u \in L^\infty(\mathbb{R}^n) \Longrightarrow M_F(u;x,\xi) \in S^0_{1,1}. \tag{4.7}$$

The operator estimates of Proposition 2.1 are available, and we see the operator norms depend on $\|u\|_{L^\infty}$ alone. This immediately gives an important circle of results known as Moser estimates: for $p \in (1,\infty),\ s > 0$,

$$\|F(u)\|_{H^{s,p}} \le C(\|u\|_{L^\infty})\,\|u\|_{H^{s,p}}, \tag{4.8}$$

with *linear* dependence on $\|u\|_{H^{s,p}}$. There are parallel estimates on C^s_* norms, for $s > 0$. Such estimates are of great use in nonlinear analysis.

If $u \in C^r(\mathbb{R}^n)$, the estimates (4.5)–(4.6) can be supplemented by

$$\|m_k\|_{C^{r+s}} \le C \cdot 2^{ks}, \tag{4.9}$$

from which we obtain

$$u \in C^r(\mathbb{R}^n),\ r \ge 0 \Longrightarrow M_F(u;x,\xi) \in \mathcal{A}^r_0 S^0_{1,1} \subset C^r S^0_{1,0}, \tag{4.10}$$

recalling the family of symbols in (1.12).

Next, we apply symbol smoothing, as in (1.3)–(1.6):

$$M_F(u;x,\xi) = M^\#(x,\xi) + M^b(x,\xi). \tag{4.11}$$

If we take $\delta \in (0,1)$ in (1.6),

$$\begin{aligned} u \in C^r(\mathbb{R}^n),\ r \ge 0 \Longrightarrow M^\#(x,\xi) &\in \mathcal{A}^r_0 S^0_{1,\delta}, \\ M^b(x,\xi) &\in S^{-r\delta}_{1,1}, \end{aligned} \tag{4.12}$$

the result on $M^\#(x,\xi)$ following from (4.10) and (1.15). If we take $\delta = 1$ and $\varepsilon_k = 2^{-(k-3)}$ in (1.6), we get

$$\begin{aligned} u \in C^r(\mathbb{R}^n),\ r \ge 0 \Longrightarrow M^\#(x,\xi) &\in \mathcal{B}^r S^0_{1,1}, \\ M^b(x,\xi) &\in S^{-r}_{1,1}, \end{aligned} \tag{4.13}$$

the result on $M^\#(x,\xi)$ following from (4.10) and (1.20). For more details, see [Mey], or [T2], or [T5], Chapter 13.

Results discussed above extend easily to the case of a function F of several variables, say $u = (u_1,\dots,u_L)$. Directly extending (4.3)–(4.4), we have for $F \in C^\infty(\mathbb{R}^L),\ F(0) = 0$,

$$F(u) = \sum_{j=1}^{L} M_j(x,D)u_j, \tag{4.14}$$

with

$$M_j(x,\xi) = \sum_k m^j_k(x)\psi_{k+1}(\xi), \tag{4.15}$$

where

$$m_k^j(x) = \int_0^1 (\partial_j F)\bigl(\Psi_k(D)u + \tau\psi_{k+1}(D)u\bigr)\, d\tau. \tag{4.16}$$

Clearly the results established above apply to the $M_j(x,\xi)$ here, e.g.,

$$u \in C^r \Longrightarrow M_j(x,\xi) \in \mathcal{A}_0^r S_{1,1}^m. \tag{4.17}$$

In the particular case $F(u,v) = uv$, we obtain

$$uv = A(u;x,D)v + A(v;x,D)u + \Psi_0(D)u \cdot \Psi_0(D)v \tag{4.18}$$

where

$$A(u;x,\xi) = \sum_{k=0}^{\infty} \Bigl[\Psi_k(D)u + \frac{1}{2}\psi_{k+1}(D)u\Bigr]\psi_{k+1}(\xi). \tag{4.19}$$

Since this symbol belongs to $S_{1,1}^0$ for $u \in L^\infty$, we obtain the following Moser estimate, valid for $p \in (1,\infty),\ s > 0$:

$$\|uv\|_{H^{s,p}} \le C\bigl[\|u\|_{L^\infty}\|v\|_{H^{s,p}} + \|u\|_{H^{s,p}}\|v\|_{L^\infty}\bigr]. \tag{4.20}$$

We now analyze a nonlinear differential operator in terms of a paradifferential operator. Denote $D^m u = \{D^\alpha u : |\alpha| \le m\}$. Say $\zeta = \{\zeta_\alpha : |\alpha| \le m\}$, and assume $F(x,\zeta)$ is smooth in (x,ζ) and $F(x,0) = 0$. In analogy with (4.14)–(4.16), we have

$$F(x, D^m u) = \sum_{|\alpha|\le m} M_\alpha(x,D) D^\alpha u, \tag{4.21}$$

where

$$M_\alpha(x,\xi) = \sum_k m_k^\alpha(x)\psi_{k+1}(\xi) \tag{4.22}$$

with

$$m_k^\alpha(x) = \int_0^1 \frac{\partial F}{\partial \zeta_\alpha}\bigl(\Psi_k(D)D^m u + \tau\psi_{k+1}(D)D^m u\bigr)\, d\tau. \tag{4.23}$$

As in (4.10), we have, for $r \ge 0$,

$$u \in C^{m+r} \Longrightarrow M_\alpha(x,\xi) \in \mathcal{A}_0^r S_{1,1}^0 \subset C^r S_{1,0}^0. \tag{4.24}$$

In other words, if we set

$$M(u;x,D) = \sum_{|\alpha|\le m} M_\alpha(x,D) D^\alpha, \tag{4.25}$$

we obtain

Proposition 4.1. *If* $u \in C^{m+r},\ r \geq 0,$ *then*

$$F(x, D^m u) = M(u; x, D)u, \tag{4.26}$$

with

$$M(u; x, \xi) \in \mathcal{A}_0^r S_{1,1}^m \subset C^r S_{1,0}^m. \tag{4.27}$$

As in (4.12)–(4.13), in this case, symbol smoothing yields

$$M(u; x, \xi) = M^\#(x, \xi) + M^b(x, \xi), \tag{4.28}$$

with

$$M^\#(x, \xi) \in \mathcal{A}_0^r S_{1,\delta}^m, \quad M^b(x, \xi) \in S_{1,1}^{m-r\delta}, \tag{4.29}$$

for $\varepsilon_k = 2^{-k\delta}$ in (1.6), and, for $\varepsilon_k = 2^{-(k-3)}$,

$$M^\#(x, \xi) \in \mathcal{B}^r S_{1,1}^m, \quad M^b(x, \xi) \in S_{1,1}^{m-r}. \tag{4.30}$$

Parallel to (4.8), we have from Proposition 4.1 and Proposition 2.1 the following Moser estimates: for $p \in (1, \infty),\ s > 0,$

$$\|F(x, D^m u)\|_{H^{s,p}} \leq C(\|u\|_{C^m})\, \|u\|_{H^{s+m,p}}. \tag{4.31}$$

5. Nonlinear elliptic regularity

Here we apply results of §§1, 2, and 4 to obtain global regularity results for a solution u to

$$F(x, D^m u) = f \in H^{s,p}(\mathbb{R}^n), \tag{5.1}$$

with $F(x, \zeta)$ as in (4.21). We assume $p \in (1, \infty)$ and

$$u \in C^{\sigma+m}(\mathbb{R}^n) \cap H^{\sigma+m,p}(\mathbb{R}^n), \quad 0 < \sigma < s, \tag{5.2}$$

and we assume $F(x, D^m u)$ is elliptic at u. We claim that

$$u \in H^{s+m,p}(\mathbb{R}^n). \tag{5.3}$$

To start, by Proposition 4.1 and (4.28)–(4.29), we have

$$F(x, D^m u) = M^\#(x, D)u + M^b(x, D)u, \tag{5.4}$$

where, with δ chosen in $(0,1)$,

$$M^{\#}(x,\xi) \in S^m_{1,\delta}, \text{ elliptic}, \quad M^b(x,\xi) \in S^{m-r\delta}_{1,1}. \tag{5.5}$$

Letting $E \in OPS^{-m}_{1,\delta}$ be a parametrix for $M^{\#}(x,D)$, we then have

$$u = Ef - EM^b(x,D)u, \text{ mod } H^{\infty,p}(\mathbb{R}^n). \tag{5.6}$$

We have $Ef \in H^{s+m.p}(\mathbb{R}^n)$, and Proposition 2.1 gives

$$M^b(x,D)u \in H^{\sigma+r\delta,p}(\mathbb{R}^n), \tag{5.7}$$

under the hypotheses on u in (5.2), hence $EM^b(x,D)u \in H^{\sigma+r\delta+m,P}(\mathbb{R}^n)$, so

$$u \in H^{\sigma_1+m,p}(\mathbb{R}^n), \quad \sigma_1 = \min(\sigma + r\delta, s). \tag{5.8}$$

Iterating this argument yields the assertion (5.3).

6. Paraproducts

Paraproducts, which arose independently in [Bon], [CM], and [Tri], are a particulerly interesting class of paradifferential operators. They arise in the following expansion of a product:

$$uv = T_u v + T_v u + R(u,v), \tag{6.1}$$

where, with ψ_k, Ψ_k as in (4.2),

$$T_u v = \sum_{k \geq 3} \Psi_{k-3}(D)u \cdot \psi_{k+1}(D)v, \tag{6.2}$$

so

$$R(u,v) = \sum_{j,k:|j-k|\leq 3} \psi_j(D)u \cdot \psi_k(D)v. \tag{6.3}$$

Note that T_u arises from the operator $A(u;x,D)$ in (4.18) via symbol smoothing. Clearly

$$\begin{aligned} u \in L^\infty(\mathbb{R}^n) \Longrightarrow T_u \in OP\mathcal{B}S^0_{1,1}, \quad &\text{and} \\ R_u \in OPS^0_{1,1}, \quad & \end{aligned} \tag{6.4}$$

where $R_u v = R(u,v)$. Results of §2 specialize to

$$\|T_u v\|_{H^{s,p}} \leq C\|u\|_{L^\infty}\|v\|_{H^{s,p}}, \tag{6.5}$$

for all $p \in (1,\infty),\ s \in \mathbb{R}$, and corresponding estimates for $v \in C^s_*(\mathbb{R}^n)$, while they imply

$$\|R(u,v)\|_{H^{s,p}} \le C\|u\|_{L^\infty}\|v\|_{H^{s,p}}, \tag{6.6}$$

for $p \in (1,\infty)$, but only for $s > 0$, with similar estimates for $v \in C^s_*(\mathbb{R}^n)$.

A remarkable paraproduct estimate of [CM] implies the following complement to (6.5)–(6.6): for $p \in (1,\infty)$,

$$\|T_u v\|_{L^p},\quad \|R(u,v)\|_{L^p} \le C\|u\|_{L^p}\|v\|_{\mathrm{bmo}}. \tag{6.7}$$

A proof is also given in Appendix D of [T2]. The following extensions of (6.7) are established in Propositions 3.5.D and 3.5.F of [T2]. Given $p \in (1,\infty)$,

$$\|R(u,v)\|_{H^{s,p}} \le C\|u\|_{H^{s-r,p}}\|v\|_{\mathfrak{h}^{r,\infty}},\quad r \in \mathbb{R},\ s \in [0,\infty), \tag{6.8}$$

$$\|T_u v\|_{H^{s,p}} \le C\|u\|_{H^{s-r,p}}\|v\|_{\mathfrak{h}^{r,\infty}},\quad 0 \le s \le r, \tag{6.9}$$

where $\mathfrak{h}^{r,\infty}(\mathbb{R}^n)$ is the bmo-Sobolev space defined in §I.3.

Other estimates arise from the implications

$$\begin{gathered} u \in C^{-\mu}_*(\mathbb{R}^n) \Longrightarrow T_u \in OP\mathcal{B}S^\mu_{1,1},\quad \mu > 0, \\ u \in C^r_*(\mathbb{R}^n) \Longrightarrow R_u \in OPS^{-r}_{1,1},\quad r \in \mathbb{R}. \end{gathered} \tag{6.10}$$

which follows readily from the definitions (6.2)–(6.3). The latter implication yields the following variant of (6.8) (since $R(u,v) = R(v,u)$):

$$\|R(u,v)\|_{H^{s,p}} \le C\|u\|_{H^{s-r,p}}\|v\|_{C^r_*},\quad r \in \mathbb{R},\ s > 0. \tag{6.11}$$

Note that (6.11) is stronger than (6.8) for $s > 0$, but it is not applicable in the important case $s = 0$.

Before presenting the next result, we mention additional tools that will be used. One involves the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\mathrm{Vol} B_r(x)} \int\limits_{B_r(x)} |f(y)|\, dy, \tag{6.12}$$

for which one has

$$\|Mf\|_{L^p} \le C_p\|f\|_{L^p},\quad 1 < p \le \infty. \tag{6.13}$$

The others involve Littlewood-Paley theory. First, if $\hat{f}_k(\xi)$ are supported on shells $\langle\xi\rangle \sim 2^k$, then for $p \in (1,\infty),\ s \in \mathbb{R}$,

$$\Big\|\sum_{k\ge0} f_k\Big\|_{H^{s,p}} \le C\Big\|\Big(\sum_{k\ge0} 2^{2ks}|f_k|^2\Big)^{1/2}\Big\|_{L^p}. \tag{6.14}$$

If $f_k = \psi_k(D)f$, the converse estimate also holds. Furthermore, if $\hat{f}_k(\xi)$ are supported on balls $|\xi| \le C2^k$, the (6.14) holds, for $p \in (1,\infty)$, $s > 0$. These results follow readily from the Littlewood-Paley results given in (I.3.11)–(I.3.18), and they also play a role in the proof of Proposition 2.1. (See, e.g., [T5], Chapter 13, §9 for details.) The significance here is that the terms

$$f_k = \Psi_{k-3}(D)u \cdot \psi_{k+1}(D)v \tag{6.15}$$

appearing in (6.2) have $\hat{f}_k$ supported in such shells, while the terms

$$f_k = \sum_{j=k-3}^{k+3} \psi_j(D)u \cdot \psi_k(D)v \tag{6.16}$$

appearing in (6.3) have $\hat{f}_k$ supported in such balls.

Our next goal is to use paraproducts to establish the following result of [CW], of frequent use in nonlinear PDE.

Proposition 6.1. *We have, for $s > 0$, $1 < p < \infty$,*

$$\|uv\|_{H^{s,p}} \le C\|u\|_{L^{q_1}}\|v\|_{H^{s,q_2}} + C\|v\|_{L^{r_1}}\|u\|_{H^{s,r_2}} \tag{6.17}$$

provided

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}, \quad q_2, r_2 \in (1,\infty),\ q_1, r_1 \in (1,\infty]. \tag{6.18}$$

Note that the Moser estimate (4.20), i.e.,

$$\|uv\|_{H^{s,p}} \le C\|u\|_{L^\infty}\|v\|_{H^{s,p}} + C\|u\|_{H^{s,p}}\|v\|_{L^\infty},$$

is the special case $q_1 = r_1 = \infty$ of (6.17).

To prove this, we write uv as in (6.1), and see that suffices to show that, under the hypotheses of Proposition 6.1,

$$\|T_u v\|_{H^{s,p}} \le C\|u\|_{L^{q_1}}\|v\|_{H^{s,q_2}}, \tag{6.19}$$

$$\|R(u,v)\|_{H^{s,p}} \le C\|u\|_{L^{q_1}}\|v\|_{H^{s,q_2}}. \tag{6.20}$$

In fact, we have, for all $s \in \mathbb{R}$,

$$\begin{aligned}
\|T_u v\|_{H^{s,p}} &\sim \Big\| \Big(\sum_k 2^{2ks} |\Psi_{k-3}u|^2 |\psi_k v|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\le C \Big\| Mu \Big(\sum_k 2^{2ks} |\psi_k v|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\le C \|Mu\|_{L^{q_1}} \Big\| \Big(\sum_k 2^{2ks} |\psi_k v|^2 \Big)^{1/2} \Big\|_{L^{q_2}} \\
&\le C \|u\|_{L^{q_1}} \|v\|_{H^{s,q_2}}.
\end{aligned} \tag{6.21}$$

Here, M is the Hardy-Littlewood maximal operator (6.12), and we have used the material around (6.14). This proves (6.19). Next, for $s > 0$,

$$\tag{6.22} \begin{aligned} \|R(u,v)\|_{H^{s,p}} &\le C \Big\| \Big(\sum_{|j-k|\le 4} 2^{2ks} |\psi_j u|^2 |\psi_k v|^2 \Big)^{1/2} \Big\|_{L^p} \\ &\le C \Big\| Mu \Big(\sum_k 2^{2ks} |\psi_k v|^2 \Big)^{1/2} \Big\|_{L^p} \end{aligned}$$

and, as in (6.21), this last quantity is $\le C\|Mu\|_{L^{q_1}} \|v\|_{H^{s,q_2}}$, so we have (6.20).

We now connect paraproducts with the operators that arose in §4 to treat compositions:

$$\tag{6.23} F(u) = M_F(u; x, D)u,$$

for F as in (4.1). We compare

$$\tag{6.24} M_F(u; x, D) = \sum m_k(x)\psi_{k+1}(D), \quad m_k(x) = \int_0^1 F'(\Psi_k(D)u + \tau\psi_{k+1}(D)u)\, d\tau,$$

with

$$\tag{6.25} T_{F'(u)} = \sum \Psi_{k-3}(D) F'(u)\, \psi_{k+1}(D).$$

It is fairly straightforward to obtain

$$\tag{6.26} u \in C^r(\mathbb{R}^n),\ r > 0 \Longrightarrow M_F(u; x, D) - T_{F'(u)} \in OPS^{-r}_{1,1}.$$

In other words, $T_{F'(u)}$ plays essentially the same role in approximating $M_F(u; x, D)$ as the symbol-smoothed operator $M^\#(x, D)$ given in (4.13).

7. Commutator estimates

The following three commutator estimates on $[P, f]u = P(fu) - f(Pu)$ are very useful in PDE.

Proposition 7.1. *Given* $p \in (1, \infty),\ s \ge 0,\ P \in OP\mathcal{B}S^m_{1,1},\ m > 0,$

$$\tag{7.1} \|[P, f]u\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1} \|u\|_{H^{m-1+s,p}} + C\|f\|_{H^{m+s,p}} \|u\|_{L^\infty}.$$

Proposition 7.2. *Given* $p \in (1, \infty),\ P \in OP\mathcal{B}S^1_{1,1},$

$$\tag{7.2} \|[P, f]u\|_{L^p} \le C\|f\|_{\mathrm{Lip}^1} \|u\|_{L^p}.$$

Proposition 7.3. *Given* $p \in (1, \infty)$, $P \in OP\mathcal{B}S^0_{1,1}$,

$$\|[P, f]u\|_{L^p} \le C\|f\|_{\mathrm{bmo}}\|u\|_{L^p}. \tag{7.3}$$

Proposition 7.1 is due to Moser when P is a differential operator and $s = 0$, to [KP] when $P = (1 - \Delta)^{m/2}$ and $s = 0$. In [T2] it was extended to $s \ge 0$ and $P \in OPS^m_{1,0}$. Proposition 7.2 is due to A. P. Calderon when $P \in OPS^1_{cl}$, and to [CM2] when $P \in OPS^1_{1,0}$. Note that (7.2) is sharper then the case $m = 1$, $s = 0$ of (7.1). Proposition 7.3 is due to [CRW] for $P \in OPS^0_{cl}$. The extensions of these results to the form stated above, with $P \in OP\mathcal{B}S^*_{1,1}$, were given in [AT].

A paraproduct approach to these commutator estimates starts with

$$\begin{aligned} P(fu) &= PT_f u + PT_u f + PR(f, u), \\ f(Pu) &= T_f Pu + T_{Pu} f + R(f, Pu), \end{aligned} \tag{7.4}$$

giving

$$[P, f]u = [P, T_f]u + \ 4 \text{ other terms}. \tag{7.5}$$

Symbol calculus is applied to $[P, T_f]$. For use in Propositions 7.1–7.2, we have (for ρ small)

$$f \in \mathrm{Lip}^1(\mathbb{R}^n),\ P \in OP\mathcal{B}_\rho S^m_{1,1} \Longrightarrow [T_f, P] \in OP\mathcal{B}S^{m-1}_{1,1}. \tag{7.6}$$

For use in Proposition 7.3, we have

$$f \in C^0_*(\mathbb{R}^n),\ P \in OP\mathcal{B}_\rho S^0_{1,1} \Longrightarrow [T_f, P] \in OP\mathcal{B}S^0_{1,1}. \tag{7.7}$$

These results follow from Proposition 1.1 and an examination of the symbol $F(x, \xi)$ of T_f, for which one has

$$|D^\beta_x D^\alpha_\xi F(x, \xi)| \le C_{\alpha\beta}\langle\xi\rangle^{-r-|\alpha|+|\beta|}\|f\|_{C^r_*}, \quad \text{for} \ \ |\alpha| \ge 1, \tag{7.8}$$

valid for each $r \in \mathbb{R}$. For Proposition 7.3, the following results play a role:

$$\begin{aligned} f \in C^0_*(\mathbb{R}^n) \Longrightarrow\ & F(x, \xi) \in \mathcal{B}S^\varepsilon_{1,1}, \quad \forall\, \varepsilon > 0, \\ & \|F(\cdot, \xi)\|_{C^0_*} \le C\|f\|_{C^0_*}, \\ & \nabla_x F(x, \xi) \in \mathcal{B}S^1_{1,1}. \end{aligned} \tag{7.9}$$

These results allow for effective use of (1.24)–(1.25) in the symbol analysis of PT_f and (1.26)–(1.27) in the symbol analysis of $T_f P$.

Remark. The results (7.6)–(7.7) hold for $\rho < 1/2$. If ρ is closer to 1, one can redefine T_f, replacing $\Psi_{k-3}(D)$ in (6.2) by $\Psi_{k-N}(D)$, for sufficiently large N, and restore (7.6)–(7.7), while slightly altering the remainder terms.

As for the four remainder terms in (7.5), estimates given in §6 handle them. To give a sample, the implication

$$u \in L^\infty(\mathbb{R}^n),\ P \in OP\mathcal{B}S^m_{1,1},\ m > 0 \Longrightarrow T_{Pu} \in OP\mathcal{B}S^m_{1,1} \tag{7.10}$$

follows from (6.10) and the containment $Pu \in C^{-m}_*(\mathbb{R}^n)$, the estimate

$$\|R(f, Pu)\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{H^{m-1+s,p}}, \quad s \ge 0, \tag{7.11}$$

follows from (6.8) and the inclusion $\mathrm{Lip}^1(\mathbb{R}^n) \subset \mathfrak{h}^{1,\infty}(\mathbb{R}^n)$ (and $R(u,v) = R(v,u)$), and the estimate

$$\|T_{Pu}f\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1}\|Pu\|_{H^{s-1,p}}, \quad 0 \le s \le 1, \tag{7.12}$$

follows from (6.9). In the setting of Proposition 7.3, the estimates in (6.7) handle the four remainder terms.

Proposition 7.3 (in the case $P \in OPS^0_{cl}$) was exploited in [CLMS] to produce important "div-curl" estimates. We mention an approach to such estimates, using a "super-commutator" estimate described in Proposition 7.4 below.

We will switch from working on Euclidean space $\mathbb{R}^n$ to a compact, oriented, n-dimensional Riemannian manifold M. So far in this second lecture we have not mentioned analysis on manifolds, but there is no difficulty extending Propositions 7.1–7.3 to this setting, at least for $P \in OPS^*_{1,\delta}(M)$ with $\delta \in [0,1)$, via partitions of unity and local coordinate charts.

To define the super-commutator, let f be an ℓ-form, on M, set $W_f u = f \wedge u$, and define

$$\begin{aligned} [[\Lambda^{-1}d, W_f]] = [\Lambda^{-1}d, W_f] &\quad \text{if } \ \ell \ \text{ is even}, \\ \{\Lambda^{-1}d, W_f\} &\quad \text{if } \ \ell \ \text{ is odd}, \end{aligned} \tag{7.13}$$

where $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$. Here, d is the exterior derivative and $\Lambda = (I - \Delta)^{1/2}$. We prove the following estimate.

Proposition 7.4. *For* $1 < p < \infty$,

$$\| \, [[\Lambda^{-1}d, W_f]]\beta\|_{L^p} \le C_p\|f\|_{\mathrm{bmo}}\|\beta\|_{L^p}. \tag{7.14}$$

Proof. Write $W_f = \sum M_{f_i} W_{e_i}$ where e_i are smooth ℓ-forms, f_i are real-valued functions, and $\sum \|f_i\|_{\mathrm{bmo}} \sim \|f\|_{\mathrm{bmo}}$. Then

$$[[\Lambda^{-1}d, W_f]]\beta = \sum_i [\Lambda^{-1}d, M_{f_i}]W_{e_i}\beta + \sum_i M_{f_i}[[\Lambda^{-1}d, W_{e_i}]]\beta. \tag{7.15}$$

Now the estimate (7.3) applies to the first sum on the right. Since the principal symbol of $\Lambda^{-1}d$ is wedge by $i|\xi|^{-1}\xi$, we have

$$[[\Lambda^{-1}d, W_{e_i}]] \in OPS^{-1}_{1,0}, \tag{7.16}$$

so the estimate on the second term on the right side of (7.15) is elementary.

We apply this result, first to an estimate of $du \wedge dv$. Let u be a j-form and v a k-form on M, $j + k \le n - 2$. Let f be an ℓ-form, $\ell = n - j - k - 2$. We set $u = \Lambda^{-1}\tilde{u}$, $v = \Lambda^{-1}\tilde{v}$, and desire to estimate

$$\int\limits_M f \wedge du \wedge dv = (W_f d\Lambda^{-1}\tilde{u}, \delta\Lambda^{-1} * \tilde{v}). \tag{7.17}$$

Here, δ is the adjoint of d, and $*$ is the Hodge star operator. Since $W_f\Lambda^{-1}dd\Lambda^{-1} = 0$, the right side of (7.17) is equal to

$$(\Lambda^{-1}dW_f d\Lambda^{-1}\tilde{u}, *\tilde{v}) = \bigl([[\Lambda^{-1}d, W_f]]d\Lambda^{-1}\tilde{u}, *\tilde{v}\bigr). \tag{7.18}$$

Applying Proposition 7.4, we deduce that

$$\Bigl|\int\limits_M f \wedge du \wedge dv\Bigr| \le C_p \|f\|_{\mathrm{bmo}} \|u\|_{H^{1,p}} \|v\|_{H^{1,p'}}. \tag{7.19}$$

Next, we estimate k-fold wedge products. Assume u_j are ℓ_j-forms, and that $\sum_{j=1}^k (\ell_j + 1) = m \le n$. Let f be an $(n-m)$-form. Then we will show that

$$\Bigl|\int\limits_M f \wedge du_1 \wedge \cdots \wedge du_k\Bigr| \le C_p \|f\|_{\mathrm{bmo}} \|u_1\|_{H^{1,p_1}} \cdots \|u_k\|_{H^{1,p_k}}, \tag{7.20}$$

provided $p_j \in (1, \infty]$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1, \quad p_k \in (1, \infty). \tag{7.21}$$

To prove this, note that, since $du_1 \wedge \cdots \wedge du_{k-1}$ is closed, we can use Hodge theory to write

$$du_1 \wedge \cdots \wedge du_{k-1} = du + h, \tag{7.22}$$

where h is a harmonic form and

$$\begin{gathered} \|u\|_{H^{1,p}} + \|h\|_{L^\infty} \le C \|u_1\|_{H^{1,p_1}} \cdots \|u_{k-1}\|_{H^{1,p_{k-1}}}, \\ \frac{1}{p_1} + \cdots + \frac{1}{p_{k-1}} = \frac{1}{p}, \quad p \in (1, \infty), \quad p_k = p'. \end{gathered} \tag{7.23}$$

Then, with $v = u_k$, we have

$$\int\limits_M f \wedge du_1 \wedge \cdots \wedge du_k = \int\limits_M f \wedge du \wedge dv + \int\limits_M f \wedge h \wedge dv. \tag{7.24}$$

The last integral in (7.24) is easy to estimate, and the estimate (7.19) applies to the other integral on the right side of (7.24). This proves the desired estimate (7.20). The case $k = n,\ \ell_j = 0$ yields a Jacobian determinant estimate, which played a particularly significant role in [CLMS].

We move to another consequence of Proposition 7.3. We continue to work on a compact Riemannian manifold M. Let

$$\mathrm{vmo}(M) = \text{ closure of } \ C^\infty(M) \ \text{ in } \ \mathrm{bmo}(M). \tag{7.25}$$

Proposition 7.5. *Given* $p \in (1, \infty),\ P \in OPS^0_{1,\delta}(M),\ \delta \in [0, 1),$

$$f \in \mathrm{vmo}(M) \Longrightarrow [P, f] : L^p(M) \to L^p(M) \ \ \textit{is compact.} \tag{7.26}$$

Proof. When $f \in C^\infty(M)$, $[P, f] \in OPS^{-(1-\delta)}_{1,\delta}(M)$, and the compactness on $L^p(M)$ is clear. The asserted compactness in (7.26) then follows by (7.3) and a standard limiting argument.

Using Proposition 7.5, one can deduce Fredholm results for operators in divergence form (in local coordinates, modulo lower order terms)

$$\begin{gathered} Lu = \sum \partial_j a^{jk}(x) \partial_k u, \\ L : H^{1,p}(M) \longrightarrow H^{-1,p}(M), \quad 1 < p < \infty, \end{gathered} \tag{7.27}$$

given L elliptic, with

$$a^{jk} \in L^\infty(M) \cap \mathrm{vmo}(M), \tag{7.28}$$

and Fredholm results for operators in non-divergence form (in local coordinates, modulo lower order terms)

$$\begin{gathered} Lu = \sum a^{jk}(x) \partial_j \partial_k u, \\ L : H^{2,p}(M) \longrightarrow L^p(M), \quad 1 < p < \infty, \end{gathered} \tag{7.29}$$

given L elliptic with a^{jk} as in (7.28). From there one can deduce global and local regularity results, obtained in [CFL]; see also [T6], §§I.11 and III.1. Along these lines we also mention a partial regularity result, applicable to foliation regularity results, in [RT].

III. Layer potentials on Lipschitz domains and other classes of uniformly rectifiable domains

This lecture covers material on singular integral operators on various fairly rough surfaces, which have applications to boundary problems for PDE on rough domains. The basic object has the form

$$\mathcal{K}f(x) = \int\limits_{\partial\Omega} k(x-y)f(y)\,d\sigma(y),$$

where Ω is an n-dimensional domain, σ is surface measure on $\partial\Omega$, $f \in L^p(\partial\Omega, d\sigma)$, and x is first taken in the complement of $\partial\Omega$. One desires maximal function estimates on $\mathcal{K}f(x)$, and results on the limiting behavior as x approaches $\partial\Omega$, in particular how that limit is related to $Kf = \lim_{\varepsilon\to 0} K_\varepsilon f$, where

$$K_\varepsilon f(x) = \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} k(x-y)f(y)\,d\sigma(y), \quad x \in \partial\Omega.$$

And one wants maximal function estimates on $K_\varepsilon f$. Here $k(z)$ is an odd function, smooth in $z \in \mathbb{R}^n \setminus 0$, and homogeneous of degree $-(n-1)$. More generally we can replace $k(z)$ by $k(x,z)$, or by $k(y,z)$, which allows extension of the results to the setting of $\Omega \subset M$, an n-dimensional manifold.

In §1 we treat the Cauchy integral on Lipschitz curves. Results obtained on this are extended to singular integrals on higher dimensional Lipschitz surfaces in §2, via the method of rotations. The scope is extended further in §3, to a class of domains called uniformly rectifiable. This is essentially the maximal class of domains on which such boundedness results hold. In §4 we consider an important subclass, the class of regular SKT domains (also called chord-arc domains with vanishing constant). Such domains for which $\partial\Omega$ is also compact form essentially the maximal class of domains for which such operators K are compact.

The last two sections connect this work to boundary problems for second order elliptic PDE, particularly of the form $Lu = 0$, with $L = \Delta - V$, where Δ is the Laplace operator on $\mathbb{R}^n$, or more generally the Laplace Beltrami operator on a Riemannian manifold. In §5 we construct double layer potentials and single layer potentials associated to such L and apply results of §§2–4 to these operators. In §6 we apply these results to the Dirichlet problem on Ω, with emphasis on the cases when Ω is either a regular SKT domain or a Lipschitz domain, in a compact Riemannian manifold with a Hölder continuous metric tensor.

1. Cauchy integral on Lipschitz curves

Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, with Lipschitz constant L, and consider the Lipschitz graph

$$\Gamma = \{t + iA(t) : t \in \mathbb{R}\}. \tag{1.1}$$

Let Ω_+ denote the region in $\mathbb{C}$ above Γ and Ω_- the region below Γ. We form the Cauchy integral

$$\mathcal{K}_\Gamma f(z) = \int\limits_\Gamma \frac{f(\zeta)}{z - \zeta}\, d\zeta, \quad z \in \Omega_\pm. \tag{1.2}$$

The following result was established in [CMM], following [Ca2].

Theorem 1.1. *The limits*

$$K_\Gamma^\pm f(z) = \lim_{\pm y \searrow 0} \mathcal{K}_\Gamma f(z + iy), \quad z \in \Gamma, \tag{1.3}$$

exist and define operators

$$K_\Gamma^\pm : L^2(\Gamma) \longrightarrow L^2(\Gamma), \tag{1.4}$$

satisfying

$$\|K_\Gamma^\pm f\|_{L^2(\Gamma)} \le C_0(1 + L)^2 \|f\|_{L^2(\Gamma)}, \tag{1.5}$$

for some absolute constant C_0.

In addition to [CMM], a number of other proofs have been given. We outline a proof from [CJS]. It exploits the behavior of

$$\mathcal{E}_\Gamma f(z) = \frac{d}{dz}\mathcal{K}_\Gamma f(z) = -\int\limits_\Gamma \frac{f(\zeta)}{(z - \zeta)^2}\, d\zeta, \tag{1.6}$$

on $\Omega_\pm$. The following plays a key role.

Lemma 1.2. *Suppose F is holomorphic on Ω_+ and vanishes at infinity. Then*

$$\|F\|_{L^2(\Gamma)} \le C_1(1 + L)\|F'\|_{\mathcal{H}_+}, \tag{1.7}$$

where

$$\mathcal{H}_+ = L^2(\Omega_+, d(z)\, dx\, dy), \quad d(z) = dist(z, \Gamma). \tag{1.8}$$

Analogous estimates hold for F holomorphic on Ω_-, with $\mathcal{H}_- = L^2(\Omega_-, d(z)\, dx\, dy)$.

To attack Theorem 1.1, we first assume $A \in C_0^\infty(\mathbb{R})$. Once we obtain the estimate (1.5), the case of general Lipschitz A follows by a limiting argument.

To proceed, with $A \in C_0^\infty(\mathbb{R})$, the existence of $K_\Gamma^\pm$ is classical, and Lemma 1.2 implies

$$\|K_\Gamma^\pm f\|_{L^2(\Gamma)} \le C(1+L)\|\mathcal{E}_\Gamma f\|_{\mathcal{H}_\pm}. \tag{1.9}$$

Next,

$$(\mathcal{E}_\Gamma f, g)_{\mathcal{H}_+} = -\int_\Gamma f(\zeta) T(\overline{g})\, d\zeta, \tag{1.10}$$

where, for $f \in \mathcal{H}_+$,

$$Tf(\zeta) = \iint_{\Omega_+} \frac{f(z)d(z)}{(z-\zeta)^2}\, dx\, dy, \quad \zeta \in \Gamma. \tag{1.11}$$

It follows that

$$|(\mathcal{E}_\Gamma f, g)_{\mathcal{H}_+}| \le \|f\|_{L^2(\Gamma)} \|T\overline{g}\|_{L^2(\Gamma)}, \tag{1.12}$$

so we get

$$\|\mathcal{E}_\Gamma f\|_{\mathcal{H}_+} \le C(1+L)\|f\|_{L^2(\Gamma)} \tag{1.13}$$

from the following:

Lemma 1.3. *For some absolute constant C_2,*

$$\|Tf\|_{L^2(\Gamma)} \le C_2(1+L)\|f\|_{\mathcal{H}_+}. \tag{1.14}$$

Lemma 1.3 is established by applying (the Ω_- version of) Lemma 1.2 to $F = Tf|_{\Omega_-}$:

$$\|Tf\|_{L^2(\Gamma)} \le C(1+L)\|(Tf)'\|_{\mathcal{H}_-}, \tag{1.15}$$

so it suffices to estimate $\|(Tf)'\|_{\mathcal{H}_-}$ in terms of $\|f\|_{\mathcal{H}_+}$. This estimate in turn can be deduced from an operator norm estimate on

$$\begin{aligned} &S : L^2(\Omega_+) \longrightarrow L^2(\Omega_-), \\ &SF(w) = d(w)^{1/2} \iint_{\Omega_+} \frac{F(z)d(z)^{1/2}}{(z-w)^3}\, dx\, dy, \end{aligned} \tag{1.16}$$

which follows from Schur's lemma.

Putting together (1.9) and (1.13) (and its counterpart for $\mathcal{H}_-$) gives the asserted estimate (1.5), at least for $A \in C_0^\infty(\mathbb{R})$, and as stated above the general case of Lipschitz A follows by a limiting argument.

A couple of comments about the proof of Lemma 1.2 given in [CJS]. It brings in a conformal diffeomorphism $\Phi : \mathbb{R}^2_+ \to \Omega_+$, with $\Phi(\mathbb{R}) = \Gamma$ and $\Phi(\infty) = \infty$, and uses the Koebe-Bieberbach theorem:

$$|\Phi'(z)|y \le \operatorname{dist}(\Phi(z), \partial\Omega) \le 4|\Phi'(z)|y, \tag{1.17}$$

to reduce the estimate (1.7) to

$$\int_{-\infty}^{\infty} |G(x)|^2 |\Phi'(x)|\, dx \le C(1+L)^2 \iint\limits_{\mathbb{R}^2_+} |G'(z)|^2 |\Phi'(z)| y\, dx\, dy, \tag{1.18}$$

for holomorphic functions G on $\mathbb{R}^2_+$ decaying at infinity. The estimate (1.18) is given an elementary (though clever) proof in [CJS]. Details can also be found in [T6], §IV.1.

Remark. The Koebe-Bieberbach theorem has the following geometric formulation. Let $\Omega \subset \mathbb{C}$ be a simply connected domain (not all of $\mathbb{C}$) and let $\gamma(z)^2\, |dz|^2$ denote its Poincare metric tensor. Then, for $z \in \Omega$,

$$\frac{1}{2} \operatorname{dist}(z, \partial\Omega) \le \frac{1}{\gamma(z)} \le 2 \operatorname{dist}(x, \partial\Omega). \tag{1.19}$$

We describe further properties of the Cauchy integral. Details can be found in [Jo], or in §IV.1 of [T6]. First, the integral kernels of $K^\pm_\Gamma$ satisfy the estimates (I.3.9), with $n = 1$, so Calderon-Zygmund theory applies. Once one has the L^2-estimates (1.4), it follows that

$$K^\pm_\Gamma : L^p(\Gamma) \longrightarrow L^p(\Gamma), \quad 1 < p < \infty. \tag{1.20}$$

Also, if one forms

$$K_{\Gamma,\delta} f(z) = \int\limits_{\Gamma \setminus B_\delta(z)} \frac{f(\zeta)}{z - \zeta}\, d\zeta, \quad z \in \Gamma, \tag{1.21}$$

one has estimates on the maximal function

$$K^\#_\Gamma f(z) = \sup_{0<\delta\le 1} |K_{\Gamma,\delta} f(z)|, \tag{1.22}$$

namely

$$\|K^\#_\Gamma f\|_{L^p(\Gamma)} \le C(p, L) \|f\|_{L^p(\Gamma)}, \quad 1 < p < \infty, \tag{1.23}$$

and there is a limit

$$\lim_{\delta\to 0} K_{\Gamma,\delta} f(z) = K_\Gamma f(z), \tag{1.24}$$

both in L^p-norm and pointwise a.e. on Γ; one writes

$$K_\Gamma f(z) = \mathrm{PV} \int_\Gamma \frac{f(z)}{z-\zeta}\, d\zeta. \tag{1.25}$$

There is also a nontangential maximal function, defined as follows. Pick $M > L$ and let

$$\mathcal{C}_M = \{(x,y) : |y| \le 1,\ |y| > M|x|\}. \tag{1.26}$$

Then set

$$\mathcal{K}_\Gamma^* f(z) = \sup_{\zeta\in\mathcal{C}_M} |\mathcal{K}_\Gamma f(z+\zeta)|, \quad z\in\Gamma. \tag{1.27}$$

One has

$$\|\mathcal{K}_\Gamma^* f\|_{L^p(\Gamma)} \le C(p,M)\|f\|_{L^p(\Gamma)}, \quad 1<p<\infty. \tag{1.28}$$

This is useful in showing that the limits (1.3) exist pointwise a.e., as well as in L^p-norm, and more generally

$$K_\Gamma^\pm f(z) = \lim_{\mathcal{C}_M^\pm \ni \zeta\to 0} \mathcal{K}_\Gamma f(z+\zeta), \quad \text{a.e. on } \Gamma, \tag{1.29}$$

where $\mathcal{C}_M^\pm = \{(x,y)\in\mathcal{C}_M : \pm y > 0\}$. The operators $K_\Gamma^\pm$ and K_Γ are related by the following formula, due for smooth Γ to Plemelj:

$$K_\Gamma^\pm f(z) = K_\Gamma f(z) \pm \pi i f(z). \tag{1.30}$$

It is useful to reformulate these results in terms of singular integral operators on $L^2(\mathbb{R})$. We can rewrite the Cauchy integral (1.2) as

$$\mathcal{K}_\Gamma f(z) = \int_{-\infty}^{\infty} \frac{f\circ\alpha(t)}{z-t-iA(t)} \alpha'(t)\, dt, \tag{1.31}$$

with $\alpha(t) = t + iA(t)$. Taking $z = s + iA(s) + i\sigma$ and $g(t) = f(\alpha(t))\alpha'(t)$, we have an essentially equivalent operator

$$\mathcal{K}_A^\sigma g(s) = \int_{-\infty}^{\infty} \frac{g(t)}{s-t+i(A(s)-A(t))+i\sigma}\, dt. \tag{1.32}$$

Theorem 1.1 yields limiting operators as $\pm\sigma \searrow 0$,

$$K_A^{\pm} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \tag{1.33}$$

of operator norm $\le C(1+L)^2$, with integral kernels

$$k_A(s,t) = \frac{1}{s-t+i(A(s)-A(t))}. \tag{1.34}$$

There are natural analogues of (1.20)–(1.30).

This analysis extends to the following variant. If $A : \mathbb{R} \to \mathbb{R}$ has Lipschitz constant L, let γ be a compact subset of $\mathbb{C} \setminus [-2L, 2L]$. For $\zeta \in \gamma$, set

$$A_\zeta(t) = \zeta t - A(t), \quad k_\zeta(s,t) = \frac{1}{A_\zeta(s) - A_\zeta(t)}. \tag{1.35}$$

Then one has

$$K_\zeta : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad K_\zeta f(z) = \mathrm{PV} \int k_\zeta(s,t) g(t)\, dt, \tag{1.36}$$

satisfying

$$\|K_\zeta f\|_{L^2(\mathbb{R})} \le C(1+L)^2 \|f\|_{L^2(\mathbb{R})}. \tag{1.37}$$

This leads to the following results of [CDM], which play an important role in the extension to several dimensions, which will be taken up in §2.

Lemma 1.4. *With $A : \mathbb{R} \to \mathbb{R}$ as above, set*

$$e_A(s,t) = \frac{1}{s-t}\ \mathit{exp}\, i\Big(\frac{A(s)-A(t)}{s-t}\Big). \tag{1.38}$$

Then PV e_A *is the kernel of a operator* E_A *on* $L^2(\mathbb{R})$*, satisfying*

$$\|E_A f\|_{L^2} \le C(1+L)^3 \|f\|_{L^2}. \tag{1.39}$$

Proof. If Ω is the region in $\mathbb{C}$ consisting of points of distance ≤ 1 from $[-2L, 2L]$, and $\gamma = \partial\Omega$, we have

$$e_A(s,t) = \frac{1}{2\pi i} \int_\gamma e^{i\zeta} k_\zeta(s,t)\, d\zeta, \tag{1.40}$$

and (1.39) then follows from (1.37).

Proposition 1.5. *Let $\varphi : \mathbb{R} \to \mathbb{R}^n$ be Lipschitz. Let $F \in C^N(\mathbb{R}^n)$, and assume $N > n+3$. Consider*

$$\tau(s,t) = \frac{1}{s-t} F\Bigl(\frac{\varphi(s)-\varphi(t)}{s-t}\Bigr). \tag{1.41}$$

Then PV $\tau(s,t)$ *is the kernel of a bounded operator on $L^2(\mathbb{R})$.*

Proof. If φ has Lipschitz constant L, you can alter F outside $B_L(0) \subset \mathbb{R}^n$, without altering τ, to be periodic of period $2\pi L$ in each argument. Then Proposition 1.5 follows from Lemma 1.4 and a Fourier series argument.

2. Singular integral operators on Lipschitz surfaces

One can pass from the one-dimensional result of §1 to a useful multi-dimensional result by the method of rotations. Our treatment of the next proposition follows [CDM] and [Dav].

Proposition 2.1. *Let $k \in C^N(\mathbb{R}^n \setminus 0)$ be odd and homogeneous of degree $-k$. Assume $N > n-k+3$. Let Γ be a k-dimensional Lipschitz graph, of the form*

$$\Gamma = \{(x, \varphi(x)) : x \in \mathbb{R}^k\}, \tag{2.1}$$

where $\varphi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is Lipschitz. Set $\psi(x) = (x, \varphi(x))$. Then

$$Kf(x) = P.V. \int\limits_{\mathbb{R}^k} k\bigl(\psi(x) - \psi(y)\bigr) f(y)\, dy \tag{2.2}$$

is a well defined operator satisfying

$$K : L^2(\mathbb{R}^k) \longrightarrow L^2(\mathbb{R}^k). \tag{2.3}$$

Proof. Write

$$Kf(x) = c_k \int\limits_{S^{k-1}} T_\omega f(x)\, dS(\omega), \tag{2.4}$$

where, for $\omega \in S^{k-1}$,

$$T_\omega f(x) = \int_{-\infty}^{\infty} k\bigl(\psi(x) - \psi(x+s\omega)\bigr) f(x+s\omega) |s|^{k-1}\, ds. \tag{2.5}$$

We estimate the operator norm of T_ω on $L^2(\mathbb{R}^k)$. To do this, let $V_\omega = (\omega)^\perp = \{x \in \mathbb{R}^k : x \cdot \omega = 0\}$, and note that

$$\|T_\omega f\|_{L^2}^2 = \int_{V_\omega} \|T_{\omega,\xi} f_\xi\|_{L^2}^2 \, d\xi, \tag{2.6}$$

where $f_\xi(t) = f(\xi + t\omega)$ and $T_{\omega,\xi}$ is the singular integral operator (acting on functions on $\mathbb{R}$) with kernel

$$\tau_{\omega,\xi}(s,t) = k\big(\psi(\xi + s\omega) - \psi(\xi + t\omega)\big)|s-t|^{k-1}. \tag{2.7}$$

Thus our task is to estimate the operator norm of $T_{\omega,\xi}$ on $L^2(\mathbb{R})$. Note that

$$\begin{aligned} \tau_{\omega,\xi}(s,t) &= \frac{1}{s-t}\, k\Big(\frac{\psi(\xi+s\omega) - \psi(\xi+t\omega)}{s-t}\Big) \\ &= \frac{1}{s-t}\, F_\omega\Big(\frac{\varphi(\xi+s\omega) - \varphi(\xi+t\omega)}{s-t}\Big), \end{aligned} \tag{2.8}$$

where $F_\omega \in C^N(\mathbb{R}^{n-k})$ is given by

$$F_\omega(x_{k+1}, \dots, x_n) = k(\omega_1, \dots, \omega_k, x_{k+1}, \dots, x_n). \tag{2.9}$$

Now the function $t \mapsto \varphi(\xi + t\omega)$ is Lipschitz, uniformly in ξ and ω. Hence the desired estimate on $\|T_{\omega,\xi}\|_{\mathcal{L}(L^2)}$ follows from Proposition 1.5, and the proof of Proposition 2.1 is complete.

As in §1, we see that the operator K in (2.2)–(2.3) is a Calderón-Zygmund operator, and we have an estimate on $K : L^p(\mathbb{R}^k) \to L^p(\mathbb{R}^k)$:

$$\|K\|_{\mathcal{L}(L^p)} \le C(p,\Gamma) \big\| k\big|_{S^{n-1}} \big\|_{C^N}, \quad 1 < p < \infty. \tag{2.10}$$

The operator K in (2.2) is closely related to the principal-value singular integral operator:

$$\begin{aligned} K_\Gamma f(x) &= \text{P.V.} \int_\Gamma k(x-y) f(y)\, d\sigma(y) \\ &= \lim_{\varepsilon \to 0} \int_{\{y \in \Gamma : |x-y| > \varepsilon\}} k(x-y) f(y)\, d\sigma(y), \end{aligned} \tag{2.11}$$

where $d\sigma$ is the area element of Γ, induced from the Euclidean structure of $\mathbb{R}^n$. This is also related to the following operator, defined for $x \in \mathbb{R}^n \setminus \Gamma$:

$$\mathcal{K}_\Gamma f(x) = \int_\Gamma k(x-y) f(y)\, d\sigma(y). \tag{2.12}$$

in a fashion parallel to analogues in §1.

We now restrict attention to Lipschitz graphs of dimension $n-1$ in $\mathbb{R}^n$, i.e., to the case $k = n-1$ of Proposition 2.1.

As in §1 we have estimates on nontangential maximal functions. If Γ is a Lipschitz graph, with Lipschitz constant $\le L$, and if $\vartheta > 1$ is chosen, then for each $x \in \Gamma$, consider the cone $\mathcal{C}_x = \mathcal{C} + x$, where $\mathcal{C} = \{x \in \mathbb{R}^n : \vartheta L|x'| \le |x_n| \le 1\}$. For a function u defined on $\mathbb{R}^n$, we define the nontangential maximal function:

$$u^*(x) = \sup_{y \in \mathcal{C}_x} |u(y)|, \quad x \in \Gamma. \tag{2.13}$$

From the analysis above and in §1 it follows that, with $\mathcal{K}_\Gamma$ as in (2.12), one has, for $f \in L^p(\Gamma),\ 1 < p < \infty$,

$$\|(\mathcal{K}_\Gamma f)^*\|_{L^p(\Gamma)} \le C\|f\|_{L^p(\Gamma)}, \tag{2.14}$$

with a bound on C of the form (2.10).

Extending jump relations given in (1.30), one can show that $\mathcal{K}_\Gamma f$ has nontangential boundary values a.e. on Γ, which are related to $K_\Gamma f$ by:

$$(\mathcal{K}_\Gamma f)_\pm(x) = \mp\tfrac{1}{2} i P_{-1}\big(n(x)\big) f(x) + K_\Gamma f(x). \tag{2.15}$$

Here, $(\mathcal{K}_\Gamma f)_+(x)$ is the limit from above Γ and $(\mathcal{K}_\Gamma f)_-(x)$ is the limit from below Γ (within the cone $\mathcal{C}_x$), $n(x)$ is the unit (downward-pointing) conormal to Γ at x (defined a.e. on Γ), and $P_{-1}(\xi)$ is the principal symbol (homogeneous of degree -1 in ξ) of the operator $Pu(x) = \int_{\mathbb{R}^n} k(x-y)u(y)\,dy$.

Our next goal is to extend the analysis of (2.11) to the variable-coefficient case. As above, let Γ be a Lipschitz graph in $\mathbb{R}^n$, of the form $x_n = \varphi(x_1, \dots, x_{n-1})$. Here we follow [MT1].

Proposition 2.2. *There exists $M = M(n)$ such that the following holds. Let $b(x,z)$ be odd in z and homogeneous of degree $-(n-1)$ in z, and assume $D_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$, for $|\alpha| \le M$. Then $b(x, x-y)$ is the kernel of an operator B, bounded on $L^p(\Gamma)$, for $1 < p < \infty$, and so is $b(y, x-y)$.*

Proof. The classical method of spherical harmonic decomposition due to Calderón and Zygmund works in this case. Thus, we can write

$$b(x,z) = \sum_{j \ge 1} b_j(x) \varphi_j\big(z/|z|\big)\, |z|^{-(n-1)}, \tag{2.16}$$

where $\{\varphi_j : j \ge 1\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator on the sphere S^{n-1}. Furthermore, we can assume that φ_j is odd whenever $b_j \ne 0$. With N as in Proposition 2.1 and M sufficiently larger than N, the regularity hypothesis implies

$$\|b_j\|_{L^\infty} \|\varphi_j\|_{C^N} \le C j^{-2}. \tag{2.17}$$

Note that, if $k_j(x) = \varphi_j(x/|x|)|x|^{-(n-1)}$ with φ_j odd, then the operator K_j on $L^p(\Gamma)$ with kernel $k_j(x-y)$ is estimable by (2.10), and, for $f \in L^p(\Gamma)$,

$$Bf(x) = \sum_{j\geq 1} b_j(x) K_j f(x). \tag{2.18}$$

Hence,

$$\begin{aligned} \|B\|_{\mathcal{L}(L^p)} &\leq C(p,\Gamma) \sum \|b_j\|_{L^\infty} \|\varphi_j\|_{C^N} \\ &\leq C(p,\Gamma) \sup_{|\alpha|\leq M} \|D_z^\alpha b(x,z)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})}. \end{aligned} \tag{2.19}$$

and the proof is done for $b(x,x-y)$. To treat $b(y,x-y)$, just replace $b_j(x)$ by $b_j(y)$ in the sum on the right side of (2.16).

Proposition 2.2 applies to the Schwartz kernels of certain pseudodifferential operators. In fact, operators in $OPC^0S^{-1}_{\mathrm{cl}}$ have Schwartz kernels that differ from those treated in Proposition 2.2 by kernels with weak singularities, and with a different asymptotic behavior far from the diagonal. For our purposes it is sufficient to use the elementary consequence that the conclusions of these propositions hold, provided one acts on functions with support on a given compact subset Γ_0 of Γ, and estimates the norm of the resulting function over Γ_0. In the rest of this section we will restrict attention to this case.

We now state the consequence of Proposition 2.2 most directly relevant for the analysis for elliptic boundary problems.

Proposition 2.3. *If $p(x,\xi) \in C^0S^{-1}_{\mathrm{cl}}$ has a principal symbol that is odd in ξ, then the Schwartz kernel of $p(x,D)$ is the kernel of an operator bounded on $L^p(\Gamma_0)$, for $1 < p < \infty$.*

The operator B in Proposition 2.2 is given by

$$Bf(x) = \text{P.V.} \int_\Gamma b(x,x-y) f(y)\, d\sigma(y). \tag{2.20}$$

This is related to the following operator, defined for $x \in \mathbb{R}^n \setminus \Gamma$:

$$\mathcal{B}f(x) = \int_\Gamma b(x,x-y) f(y)\, d\sigma(y). \tag{2.21}$$

There is an estimate on the nontangential maximal function for $\mathcal{B}f$. Under the hypotheses of Proposition 2.2, if $\mathcal{B}$ is as in (2.21) then, by (2.18), we have

$$(\mathcal{B}f)^*(x) \leq \sum_{j\geq 1} \|b_j\|_{L^\infty} (\mathcal{K}_j f)^*(x). \tag{2.22}$$

Thus, using estimates of the form (2.17) and (2.14), we have:

Proposition 2.4. *If $p(x,\xi) \in C^0 S^{-1}_{\mathrm{cl}}$ has a principal symbol that is odd in ξ, then its Schwartz kernel is the kernel of an operator $\mathcal{B}$, satisfying*

$$\|(\mathcal{B}f)^*\|_{L^p(\Gamma_0)} \le C_p \|f\|_{L^p(\Gamma_0)}, \tag{2.23}$$

for $1 < p < \infty$ (and f supported on Γ_0).

Given (2.15), the superposition arguments used above yield:

Proposition 2.5. *If $p(x,\xi)$ is as in Proposition 2.4, with principal symbol $p_{-1}(x,\xi)$, then, a.e. on Γ, we have nontangential limits*

$$(\mathcal{B}f)_\pm(x) = \mp \tfrac{1}{2} i p_{-1}\big(x, n(x)\big) f(x) + Bf(x). \tag{2.24}$$

3. Singular integral operators on uniformly rectifiable domains

Perhaps the maximal class of open domains $\Omega \subset \mathbb{R}^n$ for which it would make sense to discuss integral equations on the boundary is the class of domains with locally finite perimeter, i.e., domains Ω for which $\nabla \chi_\Omega = \mu$ is a locally finite $\mathbb{R}^n$-valued measure. In such a case, the Radon-Nikodym theorem implies $\mu = -\nu\sigma$, where σ is a locally finite positive measure, supported on $\partial\Omega$, and $\nu \in L^\infty(\partial\Omega, d\sigma)$ is an $\mathbb{R}^n$-valued function, satisfying $|\nu(x)| = 1$, σ-a.e. It then follows from the Besicovitch differentiation theorem that

$$\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu \, d\sigma = \nu(x), \tag{3.1}$$

for σ-a.e. x. Works of Federer and De Giorgi produced the following results on the structure of σ. First,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^*\Omega, \tag{3.2}$$

where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure and $\partial^*\Omega \subset \partial\Omega$ is the reduced boundary of Ω, defined as

$$\partial^*\Omega = \{x : (3.1) \text{ holds, with } |\nu(x)| = 1\}.$$

Second, $\partial^*\Omega$ is countably rectifiable, i.e., it is a countable disjoint union

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{3.3}$$

where each M_k is an $(n-1)$-dimensional Lipschitz surface and $\mathcal{H}^{n-1}(N) = 0$. In fact, M_k can be taken to be $(n-1)$-dimensional C^1 surfaces, to which ν is normal in the usual sense.

In general, $\partial\Omega$ can be much larger than $\partial^*\Omega$, and it is useful to have criteria for when

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0. \tag{3.4}$$

One class of examples is the class of domains whose boundaries are graphs of continuous functions A satisfying

$$A : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}, \quad \nabla A \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}). \tag{3.5}$$

Such a domain has locally finite perimeter, with

$$\sigma\bigl(\{(x, A(x)) : x \in \mathcal{O}\}\bigr) = \int\limits_{\mathcal{O}} \sqrt{1 + |\nabla A(x)|^2}\, dx, \tag{3.6}$$

and (3.4) holds; see [HMT], §2.2.

Remark. Given any open $\Omega \subset \mathbb{R}^n$, one defines the measure theoretic boundary $\partial_*\Omega$ to consist of $x \in \partial\Omega$ at which both Ω and $\mathbb{R}^n \setminus \Omega$ have positive density. It is known that if Ω has locally finite perimeter, then $\partial^*\Omega \subset \partial_*\Omega$ and $\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0$. Furthermore, Ω has locally finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_*\Omega \cap K) < \infty$ for each compact $K \subset \mathbb{R}^n$. Good sources for this material include [EG] and [Zie], or one could consult [Fed].

The class of domains with locally finite perimeter is too large for singular integrals such as arise in Proposition 2.1 to be bounded. David and Semmes [Dav], [DS], introduced the notion of uniformly rectifiable sets as a class on which one does have such bounded operators. To define this, we first bring in the concept of Ahlfors regularity.

Given a domain $\Omega \subset \mathbb{R}^n$, we say $\partial\Omega$ is Ahlfors regular provided there exist $a, b \in (0, \infty)$ such that

$$ar^{n-1} \le \mathcal{H}^{n-1}(B_r(x) \cap \partial\Omega) \le br^{n-1} \tag{3.7}$$

for each $x \in \partial\Omega,\ r \in (0, \infty)$. If $\partial\Omega$ is compact, we require (3.7) only for $r \in (0, 1]$.

Given this, we say $\partial\Omega \subset \mathbb{R}^n$ is uniformly rectifiable provided it is Ahlfors regular and the following property holds. There exist δ, $M \in (0, \infty)$, called the UR constants of $\partial\Omega$, such that for each $x \in \partial\Omega,\ R > 0$, there is a Lipschitz map

$$\varphi : B^{n-1}_R \longrightarrow \mathbb{R}^n, \quad B^{n-1}_R = \{x \in \mathbb{R}^{n-1} : |x| < R\}, \tag{3.8}$$

such that

$$\|\nabla\varphi\|_{L^\infty} \le M, \quad \mathcal{H}^{n-1}\bigl(\partial\Omega \cap B_R(x) \cap \varphi(B^{n-1}_R)\bigr) \ge \delta R^{n-1}. \tag{3.9}$$

If $\partial\Omega$ is compact, we require this only for $R \in (0, 1]$. It is readily verified that if $\partial\Omega$ is uniformly rectifiable then it is countably rectifiable. We call Ω a UR domain provided $\partial\Omega$ is uniformly rectifiable and (3.4) holds.

The following result is established in [Dav], Proposition 4 bis.

Proposition 3.1. *Assume $\Omega \subset \mathbb{R}^n$ is a UR domain. Take $p \in (1, \infty)$. Then there exist $N \in \mathbb{Z}^+,\ C \in (0, \infty)$, each depending only on p and the Ahlfors regularity and UR constants of $\partial\Omega$, with the following property.*

Assume $k \in C^N(\mathbb{R}^n \setminus 0)$ is odd and homogeneous of degree $-(n-1)$. Then, with

$$\begin{aligned} T_\varepsilon f(x) &= \int\limits_{\partial\Omega \setminus B_\varepsilon(x)} k(x-y) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \\ T_* f(x) &= \sup_{0<\varepsilon\le 1} |T_\varepsilon f(x)|, \end{aligned} \tag{3.10}$$

one has

$$\|T_* f\|_{L^p(\partial\Omega, d\sigma)} \le C \big\| k\big|_{S^{n-1}} \big\|_{C^N} \|f\|_{L^p(\partial\Omega, d\sigma)}. \tag{3.11}$$

In §§3.2–3.5 of [HMT], this estimate is supplemented by results on

$$\mathcal{T} f(x) = \int\limits_{\partial\Omega} k(x-y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{3.12}$$

for $k(x-y)$ as in Proposition 3.1. First, there is the nontangential maximal function,

$$\mathcal{N}(\mathcal{T} f)(x) = \sup\{|\mathcal{T} f(z)| : z \in \Gamma(x)\}, \quad x \in \partial\Omega\}, \tag{3.13}$$

where, fixing $\alpha > 0$, one sets

$$\Gamma(x) = \{z \in \Omega : |z - x| < (1+\alpha)\ \mathrm{dist}(z, \partial\Omega)\}. \tag{3.14}$$

It is shown that, for $p \in (1, \infty)$,

$$\|\mathcal{N}(\mathcal{T} f)\|_{L^p(\partial\Omega, d\sigma)} \le C \big\| k\big|_{S^{n-1}} \big\|_{C^N} \|f\|_{L^p(\partial\Omega, d\sigma)}. \tag{3.15}$$

It is also proven in [HMT] that, given $p \in (1, \infty),\ f \in L^p(\partial\Omega, d\sigma)$, the limit

$$T f(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x) \tag{3.16}$$

exists for σ-a.e. $x \in \partial\Omega$, that

$$T : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \tag{3.17}$$

and furthermore that

$$\lim_{\Gamma(x) \ni z \to x} \mathcal{T} f(z) = T f(x) + \frac{1}{2i} \hat{k}(\nu(x)) f(x), \quad \text{for } \ \sigma\text{-a.e. } x \in \partial\Omega. \tag{3.18}$$

The proof involves some Clifford analysis, amongst other things.

REMARK. Each UR domain, and more generally each Ahlfors regular domain $\Omega \subset \mathbb{R}^n$, has the weak accessibility property

$$x \in \overline{\Gamma(x)}, \quad \text{for } \ \sigma\text{-a.e. } x \in \partial\Omega. \tag{3.19}$$

This is proven in §2.3 of [HMT]. This helps make (3.18) meaningful.

There is also an extension of Proposition 3.1 and the results (3.15)–(3.18) to the variable coefficient setting, parallel to Propositions 2.2–2.5, given in §3.5 of [HMT].

The definition of UR domains is complicated, and it is valuable to have in hand results guaranteeing that more simply defined classes of domains are UR. The following result is proven in [DJ]. Assume that $\partial\Omega$ is Ahlfors regular and that the following "two disks" condition holds: There exists $C \in (0,\infty)$ such that for each $x \in \partial\Omega$ and $r > 0$, there are two $(n-1)$-dimensional disks, with centers at a distance $\le r$ from x, of radius r/C, one contained in Ω and the other contained in $\mathbb{R}^n \setminus \overline{\Omega}$. (If $\partial\Omega$ is compact, just take $r \in (0,1]$.) Then $\partial\Omega$ is uniformly rectifiable. The monograph [DS] studies a number of characterizations of the class of UR domains.

One class of UR domains is the class of domains whose boundaries are graphs of functions A satisfying

$$A : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}, \quad \nabla A \in \mathrm{bmo}(\mathbb{R}^{n-1}); \tag{3.20}$$

cf. [HMT], §§2.5 and 3.1. An open set $\Omega \subset \mathbb{R}^n$ is called a bmo_1 domain if $\partial\Omega$ is locally the graph of such functions. It is called a vmo_1 domain if it has such a form, with $\nabla A \in \mathrm{vmo}(\mathbb{R}^n)$, the closure of $C_0^\infty(\mathbb{R}^n)$ in $\mathrm{bmo}(\mathbb{R}^n)$.

4. Singular integral operators on SKT domains

We will introduce a class of domains called regular SKT domains in [HMT]. To do this, we first define the "John condition." We say an open set $\Omega \subset \mathbb{R}^n$ satisfies a John condition provided there exists $\theta \in (0,1)$ and $R > 0$ (we require $R = \infty$ if $\partial\Omega$ in unbounded), such that the following conditions hold. First, for each $p \in \partial\Omega,\ r \in (0,R)$, there exists

$$p_r \in B_r(p) \cap \Omega \ \text{ such that } \ B_{\theta r}(p_r) \subset \Omega. \tag{4.1}$$

Second, for each $x \in \partial\Omega \cap B_r(p)$, there exists a path γ_x from x to p_r in Ω (except for the endpoint x), such that

$$\text{length } \gamma_x \le \frac{r}{\theta}, \quad \text{and } \ \mathrm{dist}(\gamma_x(t), \partial\Omega) \ge \theta|\gamma_x(t) - x|, \ \forall\, t. \tag{4.2}$$

If both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a John condition, we say Ω satisfies a two-sided John condition.

By (4.1), the two-sided John condition is stronger than the two disk condition given in §3, so each $\Omega \subset \mathbb{R}^n$ with Ahlfors regular boundary that satisfies a two-sided John condition is a UR domain. It is shown in §3.1 of [HMT] that each bmo_1 domain satisfies a two-sided John condition.

We now characterize regular SKT domains.

Characterization. An open set $\Omega \subset \mathbb{R}^n$ is a regular SKT domain provided it has the following properties:

$$\partial\Omega \text{ is Ahlfors regular,} \tag{4.3}$$

$$\Omega \text{ satisfies a two-sided John condition,} \tag{4.4}$$

$$\nu \in \mathrm{vmo}(\partial\Omega, d\sigma). \tag{4.5}$$

Here $\mathrm{vmo}(\partial\Omega, d\sigma)$ is the closure of $C_0(\partial\Omega)$ in $\mathrm{bmo}(\partial\Omega, d\sigma)$, whose definition is parallel to that of $\mathrm{bmo}(\mathbb{R}^n)$ given in (I.3.24)–(I.3.25).

This is not the original definition. The class of domains under consideration was introduced in [Sem] and developed further in [KT]; see also [CKL] for further discussion. There they were called "chord-arc domains with vanishing constant," extending to higher dimensions a class of domains arising in complex function theory (for $n = 2$). Such a label does not capture the essence of the class of domains when $n \geq 3$, so these domains were called regular SKT domains in [HMT]. The definition given in [Sem] and [KT] is somewhat different from the characterization given above, bringing in particularly the notion of Reifenberg flatness. The equivalence of their definition and the characterization above was established in §4.2 of [HMT].

As stated above, each bmo_1 domain satisfies conditions (4.3) and (4.4). If Ω is a vmo_1 domain, then condition (4.5) is also satisfied. (This is not a tautology, but it is established in §2.5 of [HMT].) It follows that each vmo_1 domain is a regular SKT domain.

The following compactness result is established in §4.5 of [HMT].

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a regular SKT domain and assume $\partial\Omega$ is compact. Let $E \in C^\infty(\mathbb{R}^n \setminus 0)$ be even and homogeneous of degree $-n$. Consider*

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{\partial\Omega \setminus B_\varepsilon(x)} \langle x - y, \nu(y) \rangle E(x - y) f(y) \, d\sigma(y), \quad x \in \partial\Omega. \tag{4.6}$$

Then

$$T : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \ \text{ is compact, } \forall\, p \in (1, \infty). \tag{4.7}$$

Note that the boundedness of T on $L^p(\partial\Omega, d\sigma)$ for $p \in (1, \infty)$ follows from Proposition 3.1 (supplemented by (3.17)–(3.18)). The integral kernels appearing in (4.6) are less general than those appearing in (3.10), as would be necessary for compactness, even for smoothly bounded Ω. However, they include cases arising in the layer potential approach to second order elliptic PDE. The most basic example is the harmonic double layer:

$$Kf(x) = \lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int\limits_{\partial\Omega \setminus B_\varepsilon(x)} \langle x - y, \nu(y) \rangle |x - y|^{-n} f(y) \, d\sigma(y). \tag{4.8}$$

The following extension of Proposition 4.1 is also established in §4.5 of [HMT]. Here $\mathcal{K}(L^p)$ denotes the space of compact linear operators on $L^p(\partial\Omega, d\sigma)$.

Proposition 4.2. *Let Ω satisfy (4.3)–(4.4) and assume $\partial\Omega$ is compact. Pick $p \in (1, \infty)$, $\varepsilon > 0$. Then there exists $\delta > 0$, depending on ε, p, $k|_{S^{n-1}}$, and the geometric characteristics of Ω, with the property that, with T as in (4.6),*

$$\textit{dist}(\nu, \mathrm{vmo}(\partial\Omega, d\sigma)) < \delta \Longrightarrow \textit{dist}(T, \mathcal{K}(L^p)) < \varepsilon. \tag{4.9}$$

The proof of Propositions 4.1–4.2 occupies about 12 pages of [HMT], and it begins with a treatment in §4.4 of the result for vmo_1 domains. Estimates from [Hof] get the proof started.

As in §2, one has variable coefficient versions of these results, in which $k(x - y)$ in (4.6) is replaced by $k(x, x - y)$, where $k(x, z)$ is even and homogeneous of degree $-n$ in z, and $D_z^\alpha k(x, z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$ for all α. This is useful for the layer potential treatment of variable coefficient PDE, involving such operators as the Laplace operator on a Riemannian manifold.

There is a converse to the compactness result of Proposition 4.1, described as follows. In addition to the operator K given in (4.8), we bring in "Riesz transforms,"

$$\mathcal{R}_k f(x) = \lim_{\varepsilon \to 0} \int\limits_{\partial\Omega \setminus B_\varepsilon(x)} \frac{x_k - y_k}{|x - y|^n} f(y) \, d\sigma(y), \quad x \in \partial\Omega. \tag{4.10}$$

These operators are bounded on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, \infty)$, by the results of §3, whenever Ω is a UR domain. Next, with $\nu = (\nu_1, \dots, \nu_n)$ the unit normal to $\partial\Omega$, set

$$M_{\nu_j} f(x) = \nu_j(x) f(x). \tag{4.11}$$

The following is proven in §4.6 of [HMT].

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ satisfy (4.3)–(4.4), and assume $\partial\Omega$ is compact. Then Ω is a regular SKT domain if and only if*

$$K \ \ \textit{and each} \ \ [M_{\nu_j}, \mathcal{R}_k] \ \ \textit{are compact on} \ \ L^2(\partial\Omega, d\sigma). \tag{4.12}$$

The fact that K is compact if Ω is a regular SKT domain follows from Proposition 4.1. The fact that $[M_{\nu_j}, \mathcal{R}_k]$ is compact in such a case follows from a variant of Proposition II.7.3, though one requiring quite a different proof, given in §2.4 of [HMT]. The major point addressed in §4.6 of [HMT] is the converse part of this proposition.

Converses in this spirit to Proposition 4.2 are also established in [HMT], but we will not state them here.

5. Layer potentials

Let $\Omega \subset \mathbb{R}^n$ be a UR domain. Throughout this section we will also assume $\partial\Omega$ is compact. The double layer potential associated with the Laplace operator Δ on $\mathbb{R}^n$ is

$$\mathcal{D}f(x) = \int_{\partial\Omega} \partial_{\nu_y} G(x-y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{5.1}$$

where G is the fundamental solution to the Laplace equation, i.e., $\Delta G = \delta$, so

$$\mathcal{D}f(x) = C_n \int_{\partial\Omega} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} f(y)\, d\sigma(y). \tag{5.2}$$

Material developed in the preceding sections allows us to draw conclusions about the limiting behavior of $\mathcal{D}f(x)$ as x approaches $\partial\Omega$, relate the limits to

$$Kf(x) = \lim_{\varepsilon\to 0} C_n \int_{\partial\Omega\setminus B_\varepsilon(x)} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} f(y)\, d\sigma(y), \tag{5.3}$$

and analyze boundedness and possible compactness of K on $L^p(\partial\Omega, d\sigma)$. Once we discuss these issues, we will move on to double layer potentials arising for a class of second order elliptic operators with variable (perhaps rough) coefficients.

The special classes of UR domains we will emphasize are the classes of Lipschitz domains and of regular SKT domains, but to afford some perspective, we first point out an apparently maximal class of domains for which K in (5.3) has an elementary analysis, namely a class of $C^{1,\omega}$-domains, i.e., domains for which $\partial\Omega$ is locally the graph of a C^1 function A such that ∇A has modulus of continuity ω. In such a case,

$$|\langle x-y, \nu(y)\rangle| \le C|x-y|\, \omega(|x-y|), \tag{5.4}$$

and hence the integral kernel for K in (5.3) has the bound

$$|k(x,y)| \le C\frac{\omega(|x-y|)}{|x-y|^{n-1}}. \tag{5.5}$$

If ω satisfies the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty, \tag{5.6}$$

Schur's lemma applies to yield boundedness of K on $L^p(\partial\Omega, d\sigma)$, for each $p \in [1,\infty]$, and also to show that $K - K_\varepsilon$ tends to 0 in L^p-operator norm as $\varepsilon \to 0$, where K_ε is the operator defined by the right side of (5.3) (before passing to the limit). For each $\varepsilon > 0$, K_ε has bounded integral kernel, and the fact that K_ε is compact on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1,\infty)$ is elementary. One approach: K_ε is Hilbert-Schmidt on $L^2(\partial\Omega, d\sigma)$, and interpolation with boundedness on L^p for $p = 1$ and $p = \infty$ can be applied to yield compactness on L^p for $1 < p < \infty$.

If $\partial\Omega$ is C^1 but without a modulus of continuity satisfying (5.6), this elementary argument does not apply, but [FJR], making use of [Ca2], established compactness of K in such a case.

Now we record what results of §§3–4 imply for $\mathcal{D}$ and K. Results of §3 imply that, whenever Ω is a UR domain,

$$\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega,d\sigma)} \le C_p \|f\|_{L^p(\partial\Omega,d\sigma)}, \quad 1 < p < \infty, \tag{5.7}$$

where $\mathcal{N}(\mathcal{D}f)$ is the nontangential maximal function, defined as in (3.13), that

$$K : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \quad 1 < p < \infty, \tag{5.8}$$

and that

$$(\mathcal{D}f)_\pm(x) = \Bigl(\pm\frac{1}{2}I + K\Bigr)f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.9}$$

where $(\mathcal{D}f)_+(x)$ is the nontangential limit of $\mathcal{D}f(y)$ as $y \to x$ from within $\Omega_+ = \Omega$, and $(\mathcal{D}f)_-(x)$ the nontangential limit as $y \to x$ from within $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$.

Results of §4 imply that if Ω is a regular SKT domain (and $\partial\Omega$ is compact) then K is compact on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1,\infty)$. As we have noted, the class of regular SKT domains contains the class of vmo_1 domains, and this in turn contains the class of C^1 domains.

In case Ω is a Lipschitz domain, (5.7)–(5.9) follow from earlier results, discussed in §2. When Ω is Lipschitz, K need not be compact on $L^p(\partial\Omega, d\sigma)$; such compactness fails even for polygonal domains in $\mathbb{R}^2$. This influences results on the Dirichlet problem, as we discuss in §6.

Moving to a more general setting, let $\mathbb{R}^n$ have a metric tensor g_{jk}, let Δ denote the associated Laplace-Beltrami operator, let $V \in L^\infty(\mathbb{R}^n)$ be ≥ 0, and take $L = \Delta - V$. Assume $L : H^{1,2}(\mathbb{R}^n) \to H^{-1,2}(\mathbb{R}^n)$ is invertible, and say

$$L^{-1}u(x) = \int E(x,y)u(y)\,dV(y), \tag{5.10}$$

where dV is the volume element associated to g_{jk}. Then the double layer potential associated to a UR domain Ω is

$$\mathcal{D}f(x) = \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y) f(y)\, d\sigma_g(y), \tag{5.11}$$

where σ_g stands for the surface measure and ν the unit normal vector to $\partial\Omega$ induced by the metric tensor g_{jk}. One has

$$d\sigma_g = \rho\, d\sigma, \quad \rho(x) = \sqrt{g(x)}\, G(x, n(x))^{1/2}, \tag{5.12}$$

where $g(x) = \det(g_{jk}(x))$, $G(x,\xi) = g^{jk}\xi_j\xi_k$, and n is the unit conormal to $\partial\Omega$ with respect to the Euclidean metric. A parametrix construction, detailed for progressively rougher metric tensors in [MT1]–[MT4], gives

$$\sqrt{g(x)}\, E(x,y) = e_0(x-y, x) + e_1(y,x), \tag{5.13}$$

where the leading term has the form (for $n \geq 3$)

$$e_0(z,x) = C_n \Bigl(\sum g_{jk}(x) z_j z_k\Bigr)^{-(n-2)/2}, \tag{5.14}$$

and the remainder $e_1(y,x)$ satisfies the following estimates if the metric tensor is Hölder continuous, say $g_{jk} \in C^\alpha$ for some $\alpha \in (0,1)$:

$$|e_1(y,x)| \leq C|x-y|^{-(n-2-\alpha)}, \quad |\nabla_y e_1(y,x)| \leq C|x-y|^{-(n-1-\alpha)}. \tag{5.15}$$

Cf. Proposition 2.4 of [MT4], which improves (2.70)–(2.71) of [MT2]. (Tools discussed in §II.3 of these notes play a role in the analysis.) The contribution of $e_1(y,x)$ to $\mathcal{D}f$ in (5.11) can be handled by elementary means, and the results of §3 apply to the contribution of $e_0(x-y,x)$ to (5.11). We deduce that the results (5.7)–(5.9) continue to hold in this context, with

$$Kf(x) = \lim_{\varepsilon\to 0} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \partial_{\nu_y} E(x,y) f(y)\, d\sigma_g(y). \tag{5.16}$$

Furthermore, a calculation (see §5.2 of [HMT] for details) gives

$$K = K^\# + K_0, \tag{5.17}$$

where K_0 has a weakly singular integral kernel and is seen to be compact on $L^p(\partial\Omega, d\sigma)$ by elementary means, and

$$K^\# f(x) = \lim_{\varepsilon\to 0} C_n \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \frac{\langle x-y, n(y)\rangle}{\Gamma(x, x-y)^{n/2}} G(y, n(y))^{-1/2} f(y)\, d\sigma_g(y), \tag{5.18}$$

where $\Gamma(x, x-y) = \sum g_{jk}(x)(x_j - y_j)(x_k - y_k)$ and $\langle\ ,\ \rangle$ denotes the Euclidean inner product. One deduces from Proposition 4.1 that $K^{\#}$ is compact on $L^p(\partial\Omega, d\sigma)$ for $1 < p < \infty$, and hence so is K, when Ω is a regular SKT domain (and $\partial\Omega$ is compact).

In addition to the double layer potential, the single layer potential

$$\mathcal{S}f(x) = \int_{\partial\Omega} E(x,y) f(y)\, d\sigma_g(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{5.19}$$

also plays an important role. One wants to estimate $\nabla \mathcal{S} f$ and establish the limiting behavior of $\partial_\nu \mathcal{S} f$. To make this analysis for rough metric tensors, it is convenient to replace (5.13) by

$$\sqrt{g(y)}E(x,y) = e_0(x-y, y) + e_1(x,y) \tag{5.20}$$

(using the symmetry $E(x,y) = E(y,x)$), and apply results of §3 to the contribution of $e_0(x-y,y)$ to $\nabla\mathcal{S}$, and elementary consequences of (5.15) to the remainder. The result is that, when Ω is a UR domain,

$$\|\mathcal{N}(\nabla \mathcal{S} f)\|_{L^p(\partial\Omega, d\sigma)} \le C\|f\|_{L^p(\partial\Omega, d\sigma)}, \quad 1 < p < \infty, \tag{5.21}$$

and, for $f \in L^p(\partial\Omega, d\sigma),\ p \in (1,\infty)$,

$$(\partial_\nu \mathcal{S} f)_\pm(x) = \Big(\mp\frac{1}{2} I + K^*\Big) f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.22}$$

where

$$K^* : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \tag{5.23}$$

is the adjoint of K in (5.16). In addition, for $f \in L^p(\partial\Omega, d\sigma)$,

$$(\mathcal{S}f)_+(x) = (\mathcal{S}f)_-(x) = Sf(x), \quad \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.24}$$

where $Sf(x)$ is defined as in (5.19), with $x \in \partial\Omega$.

We end this section with the following remark. Now that we've worked with a non-flat Riemannian metric on $\mathbb{R}^n$, there are no difficulties in extending this setting, replacing $\mathbb{R}^n$ with an n-dimensional Riemannian manifold M. An open set $\Omega \subset M$ is a UR domain (respectively, a Lipschitz domain, or a regular SKT domain) provided $\partial\Omega$ can be covered by coordinate charts and in these coordinates satisfies the appropriate conditions. There is the issue of invariance of these properties under coordinate transformations, which can be happily ignored (or tackled, which has been done in [HMT2]). In the next section we will in fact take M to be a compact Riemannian manifold.

6. Dirichlet problem on rough domains

We work in the following setting. M is a compact Riemannian manifold with a Hölder continuous metric tensor, and $\Omega \subset M$ is a UR domain. Also set $\Omega_+ = \Omega$ and $\Omega_- = M \setminus \overline{\Omega}$. We assume Ω is connected, but we do not assume Ω_- is connected.

Let Δ be the Laplace Beltrami operator on M, and take $V \in L^\infty(M),\ V \geq 0$. We assume $V > 0$ on a set of positive measure in each connected component of Ω_-. Set

$$L = \Delta - V. \tag{6.1}$$

Then $L : H^{1,2}(M) \to H^{-1,2}(M)$ is invertible. Take $E,\ \mathcal{D}$, and $\mathcal{S}$ as in §5. Here we drop the g-subscripts, so σ_g is denoted σ.

We study the following Dirichlet problem. Given $f \in L^p(\partial\Omega, d\sigma)$, find $u \in C^1_{\mathrm{loc}}(\Omega)$ satisfying the conditions

$$Lu = 0 \ \text{ on } \ \Omega, \quad \mathcal{N}u \in L^p(\partial\Omega, d\sigma), \quad u\big|_{\partial\Omega} = f, \tag{6.2}$$

where the last condition means the nontangential limit of u exists for σ-a.e. $x \in \partial\Omega$ and is equal to $f(x)$. The method of layer potentials looks for a solution u in the form

$$u = \mathcal{D}g, \quad g \in L^p(\partial\Omega, d\sigma). \tag{6.3}$$

By (5.9), this works if and only if

$$\Bigl(\frac{1}{2}I + K\Bigr)g = f. \tag{6.4}$$

Thus we are led to seek conditions guaranteeing that

$$\frac{1}{2}I + K : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \ \text{ is invertible.} \tag{6.5}$$

This is a classical method when $\partial\Omega$ has smoothness somewhat better than C^1. For C^1 domains, this was carried through in [FJR] for $p \in (1, \infty)$, and for Lipschitz domains in [Ver], for $p = 2$, in both cases in the setting $M =$ flat $\mathbb{R}^n$ (and $V \equiv 0$). In such a setting, a topological restriction was needed; $\partial\Omega$ was required to be connected. One motivation for [MT1] was the realization that going from constant coefficients to variable coefficients afforded a convenient way to eliminate such a topological restriction.

Generalizing previous results, the attack on the validity of (6.5) made in [HMT] starts with the following.

Proposition 6.1. *If $\Omega \subset M$ is a UR domain, then the map*

$$\frac{1}{2}I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{6.6}$$

is injective.

Proof. Suppose $f \in L^2(\partial\Omega)$ and $(\frac{1}{2}I + K^*)f = 0$. Set $u = \mathcal{S}f$, so $(\Delta - V)u = 0$ on $M \setminus \partial\Omega$. The estimate (5.21) allows the use of Green's formula, to write

$$\int\limits_{\Omega_-} \{|\nabla u|^2 + V|u|^2\}\, dV = -\int\limits_{\partial\Omega} u\, \frac{\partial u}{\partial \nu}\, d\sigma. \tag{6.7}$$

By (5.22), the right side of (4.3) vanishes when $f \in \ker(\frac{1}{2}I + K^*)$. Thus u is constant on each connected component of Ω_- and $u = 0$ on supp V. Hence $u = 0$ on Ω_-. Hence, by (5.24), $Sf = 0$ a.e. on $\partial\Omega$, so, again by (5.24) and Green's formula, we have

$$\int\limits_{\Omega} \{|\nabla u|^2 + V|u|^2\}\, dV = \int\limits_{\partial\Omega} u\, \frac{\partial u}{\partial \nu}\, d\sigma = 0. \tag{6.8}$$

Hence u is constant on Ω, so $\partial_\nu u_+ = 0$ a.e. on $\partial\Omega$. Since, by (5.22), f is equal to the jump of $\partial_\nu u$ across $\partial\Omega$, we have $f = 0$, so Proposition 6.1 is proven.

Remark. Actually, in the setting of Proposition 6.1, to apply Green's formula takes a bit of work. We give further details. The asserted formula is

$$\int\limits_{\Omega} \operatorname{div} v\, dV = \int\limits_{\partial\Omega} \langle \nu, v|_{\partial\Omega}\rangle\, d\sigma, \tag{6.9}$$

with

$$v = u\,\nabla u, \tag{6.10}$$

which gives

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 + V u^2 \quad \text{on} \quad M \setminus \partial\Omega. \tag{6.11}$$

The following result is proven in §5.3 of [HMT].

Proposition 6.2. *If M has a continuous metric tensor and $\Omega \subset M$ is Ahlfors regular, then (6.9) holds whenever*

$$\operatorname{div} v \in L^1(\Omega) \quad \textit{and} \quad v \in \mathfrak{L}^p, \tag{6.12}$$

for some $p > 1$*, where*

$$\mathfrak{L}^p = \{v \in C(\Omega) : \mathcal{N}v \in L^p(\partial\Omega, d\sigma) \text{ and } \exists \text{ nontangential limit } v|_{\partial\Omega}, \sigma\text{-a.e.}\}. \tag{6.13}$$

Most of the work needed to prove this is done in §2.3 of [HMT], which treats the Euclidean case. As for the applicability of this result here, we have

$$\begin{aligned} f \in L^2(\partial\Omega, d\sigma) &\Longrightarrow \mathcal{N}\nabla u \in L^2(\partial\Omega, d\sigma) \\ &\Longrightarrow \mathcal{N}v \in L^p(\partial\Omega, d\sigma), \quad \text{for some } \ p > 1, \end{aligned} \tag{6.14}$$

whenever Ω is a UR domain, by (5.21) and (6.11). Also we have

$$\begin{aligned} \mathcal{N}\nabla u \in L^2(\partial\Omega, d\sigma) &\Longrightarrow |\nabla u|^2 \in L^q(\Omega) \\ &\Longrightarrow \operatorname{div} v \in L^q(\Omega), \quad \text{some } \ q > 1, \end{aligned} \tag{6.15}$$

whenever Ω is Ahlfors regular (as shown in [HMT], §3.2). Furthermore, local elliptic regularity results, obtainable via methods discussed in §II.3, imply u is C^1 on the interior regions Ω and Ω_-, so v is continuous there.

Versions of (6.9) valid for general finite perimeter domains can be found in [EG], [Fed], and [Zie], but they require much more regularity on v. We also mention that the justification of Green's formula in the proof of Proposition 6.1 is much easier if Ω is a Lipschitz domain. This was done (in the Euclidean context) in [Ver].

Using Proposition 6.1 and the compactness result given in §5 that

$$K : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \ \text{ is compact, for } \ p \in (1, \infty), \tag{6.16}$$

when Ω is a regular SKT domain, we have the following.

Proposition 6.3. *If M has a Hölder continuous metric tensor and $\Omega \subset M$ is a regular SKT domain, then the invertibility result (6.5) holds for each $p \in (1, \infty)$.*

Proof. Since K is compact on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, \infty)$, we have $(1/2)I + K$ Fredholm of index 0 on L^p. Proposition 6.1 implies $(1/2)I + K^*$ is injective on $L^p(\partial\Omega, d\sigma)$ for each $p \in [2, \infty)$, hence $(1/2)I + K$ is bijective on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, 2]$. In particular,

$$\frac{1}{2}I + K \ \text{ is injective on } \ L^p(\partial\Omega, d\sigma), \tag{6.17}$$

for each $p \in (1, 2]$, hence for each $p \in (1, \infty)$. This, together with Fredholmness of degree 0, gives the asserted invertibility.

Thus when $\Omega \subset M$ is a regular SKT domain the Dirichlet problem (6.2) is solvable for each $f \in L^p(\partial\Omega, d\sigma)$, for $p \in (1, \infty)$. (One can also let $p = \infty$.)

Uniqueness also holds. This takes an additional argument, given in §7.1 of [HMT]. It is also shown there that, for $p \in (1,\infty)$,

$$f \in H^{1,p}(\partial\Omega, d\sigma) \Longrightarrow \mathcal{N}(\nabla u) \in L^p(\partial\Omega, d\sigma). \tag{6.18}$$

In such a case, u is constructed in the form

$$u = \mathcal{S}(S^{-1}f). \tag{6.19}$$

The theory of L^p-Sobolev spaces $H^{1,p}(\partial\Omega, d\sigma)$ is less straightforward in this general context than it is for Lipschitz domains. It is developed in the context of Ahlfors regular domains in §3.6 of [HMT], and then further in §4.3, where comparison is made with other works on L^p-Sobolev spaces on metric measure spaces.

We turn to the setting of Lipschitz domains.

Proposition 6.4. *If M has a Hölder continuous metric tensor and $\Omega \subset M$ is a Lipschitz domain, then there exists $\varepsilon = \varepsilon(\Omega, L) > 0$ such that invertibility in (6.5) holds for each $p \in (2-\varepsilon, \infty)$.*

As a consequence, (6.2) is solvable for each $f \in L^p(\partial\Omega, d\sigma)$, as long as $p \in (2-\varepsilon, \infty)$. (One can also let $p = \infty$.)

The first key step is to establish invertibility on $L^2(\partial\Omega, d\sigma)$. This was done in [Ver], in the Euclidean context, and in [MT1]–[MT4] in the Riemannian manifold context. In light of Proposition 6.1, this invertibility follows from:

$$\pm\frac{1}{2}I + K^* \ \text{ are Fredholm of index 0 on } \ L^2(\partial\Omega, d\sigma). \tag{6.20}$$

(The result with the minus sign is useful in the study of the Neumann problem.) In the route taken in [MT1]–[MT4], (6.20) follows from:

$$\begin{aligned} \lambda \in \mathbb{R},\ |\lambda| \ge \frac{1}{2} \Longrightarrow \lambda I + K^* : L^2(\partial\Omega, d\sigma) &\to L^2(\partial\Omega, d\sigma) \\ &\text{has closed range and finite-dimensional kernel.} \end{aligned} \tag{6.21}$$

In fact, (6.21) implies that each such $\lambda I + K^*$ is semi-Fredholm, with a well defined index. Such an index is continuous in λ, hence constant on $(-\infty, -1/2]$ and on $[1/2, \infty)$. Invertibility is clear for large $|\lambda|$, so the index must be zero.

The key tool used to establish (6.20) in [Ver] and (6.21) in [MT1] is a Rellich identity, which can be stated as follows, in the Riemannian manifold context. Pick a Lipschitz vector field w, transverse to $\partial\Omega$, so that $\langle \nu, w\rangle \ge a > 0$ on $\partial\Omega$. Given $f \in L^2(\partial\Omega, d\sigma)$, set $u = \mathcal{S}f$ on $M \setminus \partial\Omega$. The identity is

$$\begin{aligned} &\int_{\partial\Omega} \langle \nu, w\rangle \bigl\{ |\nabla_T u|^2 - (\partial_\nu u)^2 \bigr\}\, d\sigma \\ &= 2\int_{\partial\Omega} (\nabla_{Tw} u)(\partial_\nu u)\, d\sigma - 2\int_{\Omega} (\nabla_w u) h\, dV \\ &\quad + \int_{\Omega} \bigl\{ (\operatorname{div} w)|\nabla u|^2 - 2(\mathcal{L}_w g)(\nabla u, \nabla u) \bigr\}\, dV, \end{aligned} \tag{6.22}$$

where Tw is the component of w tangent to $\partial\Omega$, and where

$$\Delta u = h, \tag{6.23}$$

so, for $Lu = (\Delta - V)u = 0$, $h = Vu$. To prove this identity, one can compute $\operatorname{div}(\langle \nabla u, \nabla u\rangle w)$ and $2\operatorname{div}(\nabla_w u \cdot \nabla u)$, and apply the divergence theorem to the difference. This is applicable when the metric tensor is C^1, or even Lipschitz, and we can immediately pass to the pair of estimates

$$\begin{aligned} \|\nabla_T u\|^2_{L^2(\partial\Omega)} &\le C\|\partial_\nu u\|^2_{L^2(\partial\Omega)} + C\{\|h\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\}, \\ \|\partial_\nu u\|^2_{L^2(\partial\Omega)} &\le C\|\nabla_T u\|^2_{L^2(\partial\Omega)} + C\{\|h\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\}, \end{aligned} \tag{6.24}$$

A similar argument is made with Ω replaced by Ω_-. Then the jump relations (5.22) are used to show that

$$\begin{aligned} \Big\|\Big(\pm\frac{1}{2}I + K^*\Big)f\Big\|_{L^2(\partial\Omega)} \le C\Big\|\Big(\mp\frac{1}{2}I + K^*\Big)f\Big\|_{L^2(\partial\Omega)} &+ C\Big|\int\limits_{\partial\Omega} Sf\, d\sigma\Big| \\ &+ C\|\mathcal{S}f\|_{L^2(M)}, \end{aligned} \tag{6.25}$$

and hence

$$\|f\|_{L^2(\partial\Omega)} \le C\Big\|\Big(\pm\frac{1}{2}I + K^*\Big)f\Big\|_{L^2(\partial\Omega)} + C\|\mathcal{S}f\|_{H^{1,2}(M)}. \tag{6.26}$$

Now using (5.21) one can show that

$$\mathcal{S} : L^2(\partial\Omega, d\sigma) \longrightarrow H^{1,2}(M) \ \text{ is compact.} \tag{6.27}$$

This implies semi-Fredholmness of $\pm(1/2)I + K^*$.

Next, the cases for which $|\lambda| > 1/2$ are treated by the following Rellich-type identity, (4.22) of [MT1]:

$$\begin{aligned} &\Big(\lambda^2 - \frac{1}{4}\Big)\int\limits_{\partial\Omega} \langle \nu, w\rangle |f|^2\, d\sigma + \int\limits_{\partial\Omega} \langle \nu, w\rangle |\nabla_T Sf|^2\, d\sigma \\ &= \int\limits_{\partial\Omega} \langle \nu, w\rangle |(\lambda I + K^*)f|^2\, d\sigma + 2\int\limits_{\partial\Omega} (\nabla_{Tw} Sf)\big((\lambda I + K^*)f\big)\, d\sigma + \mathcal{R}, \end{aligned} \tag{6.28}$$

where $\mathcal{R}$ denotes a quantity satisfying an estimate

$$|\mathcal{R}| \le C\|\mathcal{S}f\|^2_{H^{1,2}(M)}. \tag{6.29}$$

Hence we have

$$\|f\|_{L^2(\partial\Omega)} \le C\|(\lambda I + K^*)f\|_{L^2(\partial\Omega)} + C\|\mathcal{S}f\|_{H^{1,2}(M)}, \tag{6.30}$$

giving the asserted semi-Fredholmness (6.21).

This Rellich identity argument is not directly applicable when the metric tensor g on M is merely Hölder, and further arguments are developed in [MT2] to establish (6.21) in such a more general setting. These involve approximating g by a sequence of smooth metric tensors, uniformly and boundedly in C^α.

Invertibility in (6.5) for $p = 2$ implies solvability of (6.2) for $p = 2$. Once the uniqueness is established, one shows that the solution given by (6.3)–(6.4) coincides with the solution given by the variational method when $f \in H^{1/2,2}(\partial\Omega)$, and with the solution given by maximum principle arguments when $f \in C(\partial\Omega)$, or more generally $f \in L^\infty(\partial\Omega)$. Thus we have a solution operator

$$\mathrm{PI} : L^2(\partial\Omega, d\sigma) \longrightarrow C^1_{\mathrm{loc}}(\Omega), \tag{6.31}$$

and estimates

$$\|\mathcal{N}(\mathrm{PI}\, f)\|_{L^p(\partial\Omega,d\sigma)} \le C\|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{6.32}$$

valid for $p = 2$ and $p = \infty$, and hence, by interpolation, for all $p \in [2, \infty]$.

The invertibility of $(1/2)I + K$ in (6.5) for $p \in (2 - \varepsilon, 2 + \varepsilon)$ follows from the invertibility at $p = 2$ and a general functional analysis argument. The invertibility on the rest of $p \in (2 - \varepsilon, \infty)$ is suggested by the solvability results arising from (6.32), but the proof requires further arguments, given in [MT3].

To say (6.2) is solvable for each $f \in L^p(\partial\Omega, d\sigma)$ is to say that the harmonic measure belongs to $L^{p'}(\partial\Omega, d\sigma)$. Thus Proposition 6.4 implies that whenever $\Omega \subset M$ is a Lipschitz domain, there exists $\varepsilon > 0$ such that harmonic measure belongs to $L^q(\partial\Omega, d\sigma)$ for all $q \in [1, 2 + \varepsilon)$. In general this is sharp. It is easy to estimate harmonic measure on the boundary of a polygonal domain in $\mathbb{R}^2$ and, given $q_0 > 2$, construct such a domain whose harmonic measure does not belong to $L^{q_0}(\partial\Omega, d\sigma)$.

The following result, established in [MT2]–[MT3], applies to solvability of the Dirichlet problem with boundary data $f \in H^{1,p}(\partial\Omega, d\sigma)$.

Proposition 6.5. *If M has a Hölder continuous metric tensor and $\Omega \subset M$ is a Lipschitz domain, then there exists $\varepsilon = \varepsilon(\Omega, L) > 0$ such that*

$$S : L^p(\partial\Omega, d\sigma) \longrightarrow H^{1,p}(\partial\Omega, d\sigma), \quad 1 < p < 2 + \varepsilon, \tag{6.33}$$

is invertible.

To apply this to the Dirichlet problem, construct the solution u by (6.19) and make use of the estimate (5.21).

There are further results on the Dirichlet problem, such as Sobolev-Besov estimates, which we will not touch on here. It is clear from (5.22) that results on solvability of the equation

$$\Big(-\frac{1}{2}I + K^*\Big)f = g, \tag{6.34}$$

which by (6.20) is Fredholm of index zero on $L^2(\partial\Omega, d\sigma)$, apply to the Neumann problem. There are also results on natural boundary problems for the Hodge Laplacian, which include the Maxwell system of electromagnetism, results on the Stokes system, with applications to the Navier-Stokes equations, and other systems of elliptic PDE, which can be found in various references at the end of these notes.

IV. Pseudodifferential operators on noncompact manifolds with bounded geometry

In this lecture we introduce some classes of pseudodifferential operators on a Riemannian manifold M with bounded geometry. We define "bounded geometry" as follows. First we assume there exists $R_0 \in (0, \infty)$ such that for each $p \in M$, the exponential map

$$\mathrm{Exp}_p : T_p M \longrightarrow M \tag{0.1}$$

has the property

$$\mathrm{Exp}_p : B_{R_0}(0) \longrightarrow B_{R_0}(p) \ \text{ diffeomorphically,} \tag{0.2}$$

where $B_r(p) = \{x \in M : d(x,p) < r\}$, $d(x,p)$ denoting the distance from x to p. Furthermore, the pull-back of the metric tensor from $B_{R_0}(p) \subset M$ to $B_{R_0}(0) \subset T_p M$, identified with $B_{R_0}(0) \subset \mathbb{R}^n$ $(n = \dim M)$, uniquely up to an element of $O(n)$, furnishes a collection of $n \times n$ matrices $G_p(x) = (g^p_{jk}(x))$ satisfying

$$\{G_p : p \in M\} \ \text{ is bounded in } \ C^\infty(B_{R_0}(0), \mathrm{End}(\mathbb{R}^n)). \tag{0.3}$$

We also require that

$$\xi \cdot G_p(x)\xi \geq \frac{1}{2}|\xi|^2, \quad \forall\, p \in M, x \in B_{R_0}(0), \xi \in \mathbb{R}^n, \tag{0.4}$$

and that

$$B_{R_0}(p) \ \text{ is geodesically convex, } \forall\, p \in M. \tag{0.5}$$

Such is a complete Riemannian manifold with bounded geometry. Given this, we find it convenient to multiply the metric tensor of M by a constant, if necessary, so we can say the properties above hold with

$$R_0 = 4. \tag{0.6}$$

Having (0.2)–(0.6), we can pick $p_k \in M$, $k \in \mathbb{Z}^+$, such that

$$\{B_{1/2}(p_k) : k \in \mathbb{Z}^+\} \ \text{ covers } \ M, \tag{0.7}$$

while, for some $K = K(M) < \infty$,

$$\forall\, p \in M, \ \text{at most } K \text{ balls } B_2(p_k) \text{ contain } p. \tag{0.8}$$

We can then form a partition of unity $\sum_k \varphi_k = 1$ such that

$$\text{supp}\ \varphi_k \subset B_1(p_k), \quad \varphi_k \circ \text{Exp}_{p_k} \text{ is bounded in } C_0^\infty(B_1(0)). \tag{0.9}$$

We call such $\{\varphi_k : k \in \mathbb{Z}^+\}$ a *tame* partition of unity, and the collection $\{B_1(p_k) : k \in \mathbb{Z}^+\}$ a *tame cover* of M.

We mention that each Riemannian manifold with bounded geometry satisfies a volume bound of the form

$$\text{Vol}(B_r(p)) \leq C\langle r\rangle^\mu e^{\kappa r}, \quad \forall\, r \in (0,\infty), \tag{0.10}$$

for some constants C, μ, and κ, independent of $p \in M$.

Section 1 makes an analysis of certain families of functions of the Laplace Beltrami operator Δ, such as $(\lambda I - \Delta)^{m/2}$, for $\lambda > 0,\ m \in \mathbb{R}$, as pseudodifferential operators on M. Given $W > 0$, we define a class of operators $\Psi_W^m(M)$, whose Schwartz kernels behave like those of operators in $OPS_{1,0}^m(\mathbb{R}^n)$ near the diagonal, in a uniform fashion, and away from the diagonal decay like $\langle d(x,y)\rangle^{-k} e^{-W d(x,y)}$, $\forall\, k$, as do all derivatives. We also introduce a smaller class of operators, $\widetilde{\Psi}_W^m(M)$. It is shown that $(\lambda I - \Delta)^{m/2} \in \widetilde{\Psi}_W^m(M)$ when $\lambda > W^2$. More generally, we consider $\Phi(\sqrt{-\Delta})$ for Φ in a space $\mathcal{S}_W^m$, consisting of even functions (of ζ) smooth and satisfying symbolic estimates on the strip $\{|\text{Im}\,\zeta| \leq W\}$, and holomorphic on the interior. Even more generally, we consider $\Phi(\sqrt{-L})$, with $L = \Delta + B^2$, in case $\text{Spec}(-\Delta) \subset [B^2, \infty)$.

In §2 we show that $P : L^p(M) \to L^p(M)$ for all $p \in (1,\infty)$ whenever $P \in \Psi_W^0(M)$ with $W \geq \kappa$, and also whenever $P \in \widetilde{\Psi}_W^0(M)$ with $W \geq \kappa/2$, provided (0.10) holds. In §3 we show that if $P_1 \in \Psi_W^{m_1}(M)$ and $P_2 \in \Psi_{W+\kappa}^{m_2}(M)$, then P_1P_2 and P_2P_1 belong to $\Psi_W^{m_1+m_2}(M)$. In §4 we define L^p-Sobolev spaces $H^{s,p}(M)$, with $p \in (1,\infty)$. We give one definition for $s = k \in \mathbb{Z}^+$ and another for general $s \in \mathbb{R}$, and show they are equivalent when $s = k$. We discuss mapping properties $P : H^{s,p}(M) \to H^{s-m,p}(M)$, and show that $\{H^{s,p}(M) : s \in \mathbb{R}\}$ forms a complex interpolation scale, for each $p \in (1,\infty)$. Section 5 obtains a few additional results in the case that M is a symmetric space of noncompact type.

In §6 we define local Hardy space $\mathfrak{h}^1(M)$ and $\text{bmo}(M)$, extending to the setting of Riemannian manifolds with bounded geometry the definitions of $\mathfrak{h}^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$ given in §I.3. We also define Hardy and bmo-Sobolev spaces $\mathfrak{h}^{s,1}(M)$ and $\mathfrak{h}^{s,\infty}(M)$ and give results about the action of operators in $\Psi_W^m(M)$ on these spaces. This material was developed in [T7], and is currently being applied to the study of nonlinear wave equations on hyperbolic space, in joint work of the author and Jason Metcalfe.

1. Functions of the Laplace operator

A fruitful approach to study a function $\Phi(A)$ of a self-adjoint operator A is via the formula

$$\Phi(A)f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(t) e^{itA} f \, dt, \tag{1.1}$$

whose validity follows from the spectral theorem and the Fourier inversion formula. If Φ is even, we can rewrite this as

$$\Phi(A)f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(t) \cos tA \, f \, dt. \tag{1.2}$$

We can apply this to

$$A = \sqrt{-\Delta}, \tag{1.3}$$

when Δ is the Laplace-Beltrami operator on the Riemannian manifold M. In some important cases one has not merely $\operatorname{Spec}(-\Delta) \subset [0, \infty)$, but

$$\operatorname{Spec}(-\Delta) \subset [B^2, \infty), \tag{1.4}$$

and it is convenient to take

$$A = \sqrt{-L}, \quad L = \Delta + B^2. \tag{1.5}$$

Note that for $f \in L^2(M)$, $u(t,x) = \cos t\sqrt{-L}\, f(x)$ solves the wave equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = 0, \quad u(0,x) = f(x), \; u_t(0,x) = 0. \tag{1.6}$$

For an important class of functions Φ, defined below, one can obtain a great deal of information about $\Phi(\sqrt{-L})$ by exploiting two properties of the wave equation (1.6):

(a) Construction of a parametrix for small $|t|$;
(b) Use of finite propagation speed.

The parametrix is constructed by the method of geometrical optics. In local coordinates, one sets

$$U(t)f(x) = \sum_{\pm} (2\pi)^{-n/2} \int a^{\pm}(t,x,\xi) e^{i\varphi^{\pm}(t,x,\xi)} \hat{f}(\xi) \, d\xi, \tag{1.7}$$

where $\varphi^{\pm}$ is real valued, smooth for $\xi \neq 0$, and homogeneous of degree 1 in ξ, and

$$a^{\pm}(t,x,\xi) \sim \sum_{k \geq 0} a_k^{\pm}(t,x,\xi), \tag{1.8}$$

with $a_k(t,x,\xi) \in S^{-k}_{cl}$, homogeneous of degree $-k$ in $|\xi|$ for $|\xi| \geq 1$. To construct these functions, one calculates

$$(\partial_t^2 - L)(a^\pm e^{i\varphi^\pm}) = b^\pm e^{i\varphi^\pm}, \quad b^\pm(t,x,\xi) \sim \sum_{k\geq 0} b_k^\pm(t,x,\xi), \tag{1.9}$$

with $b_k^\pm \in S^{2-k}_{cl}$, homogeneous of degree $2-k$ in ξ for $|\xi| \geq 1$. One has

$$b_0^\pm(t,x,\xi) = -a_0^\pm\big(|\partial_t\varphi^\pm|^2 - |\nabla_x\varphi^\pm|^2\big). \tag{1.10}$$

We set this equal to zero by requiring $\varphi^\pm$ to satisfy the eikonal equations

$$\frac{\partial\varphi^\pm}{\partial t} = \pm|\nabla_x\varphi^\pm|. \tag{1.11}$$

We take initial data $\varphi^\pm(0,x,\xi) = x\cdot\xi$. The fact that M has bounded geometry implies that if we work in local exponential coordinate systems as described in this lecture's introduction, then there exists $a > 0$ such that there is a solution for $|t| \leq a$ on all unit balls, with good bounds. Note that $\varphi^+(-t,x,\xi) = \varphi^-(t,x,\xi)$.

Once (1.11) is achieved, the formula for $b_1^\pm$ simplifies to

$$b_1^\pm(t,x,\xi) = i\big(2\varphi_t^\pm\partial_t a_0^\pm - 2\langle\nabla_x\varphi^\pm, \nabla_x a_0^\pm\rangle + a_0^\pm(\partial_t^2 - \Delta)\varphi^\pm\big). \tag{1.12}$$

This vanishes provided $a_0^\pm(t,x,\xi)$ satisfies the first transport equation

$$2\varphi_t^\pm\frac{\partial a_0^\pm}{\partial t} = 2\langle\nabla_x\varphi^\pm, \nabla_x a_0^\pm\rangle - a_0^\pm(\partial_t^2 - \Delta)\varphi^\pm. \tag{1.13}$$

By (1.11), $\varphi_t^\pm$ is nonzero for $|t|$ small enough. Then one can impose the initial condition $a_0^\pm(0,x,\xi) = 1/2$. Higher order transport equations for $a_k^\pm(t,x,\xi)$ are obtained from similar formulas for $b_k^\pm(t,x,\xi)$. We have $a^+(-t,x,\xi)e^{i\varphi^+(-t,x,\xi)} = a^-(t,x,\xi)e^{i\varphi^-(t,x,\xi)}$, hence $U(-t) = U(t)$. The fact that $U(t) - \cos t\sqrt{-L}$ is a smoothing operator for $|t| \leq a$ follows via energy estimates from §I.5.

The role of this parametrix in the analysis of $\Phi(\sqrt{-L})$ arises as follows. Take $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $|t| \leq a/4$, 0 for $|t| \geq a/2$, and write (1.2) as

$$\begin{aligned}
&\Phi(\sqrt{-L})f \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \widehat{\Phi}(t)\psi(t)\cos t\sqrt{-L}\,f\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \widehat{\Phi}(t)(1-\psi(t))\cos t\sqrt{-L}\,f\,dt \\
&= \Phi^\#(\sqrt{-L})f + \Phi^b(\sqrt{-L})f.
\end{aligned} \tag{1.14}$$

Decreasing a if necessary, we assume $a \leq 1$. Note that the Schwartz kernel of $\Phi^\#(\sqrt{-L})$ is contained in

$$\{(x,y) \in M\times M : d(x,y) \leq a/2\}. \tag{1.15}$$

In local coordinates, and modulo a smooth remainder, we have

$$\Phi^{\#}(\sqrt{-L})f(x) = C_n \sum_{\pm} \int_{-\infty}^{\infty} \int \widehat{\Phi}^{\#}(t) a^{\pm}(t,x,\xi) e^{i\varphi^{\pm}(t,x,\xi)} \hat{f}(\xi)\, d\xi. \tag{1.16}$$

Now

$$\int \widehat{\Phi}^{\#}(t) a^{\pm}(t,x,\xi) e^{i\varphi^{\pm}(t,x,\xi)}\, dt = \Phi^{\#}(D_t)\Big(a^{\pm}(t,x,\xi)e^{i\varphi^{\pm}(t,x,\xi)}\Big)\Big|_{t=0}, \tag{1.17}$$

and if $\Phi^{\#} \in S^m_{1,0}(\mathbb{R})$, since $\varphi^{\pm} \neq 0$, there is an asymptotic expansion

$$\Phi^{\#}(D_t)(a^{\pm}e^{i\varphi^{\pm}}) = e^{\pm}(t,x,\xi)e^{i\varphi^{\pm}}, \tag{1.18}$$

with $e^{\pm} \in S^m_{1,0}$, satisfying

$$e^{\pm}(t,x,\xi) \sim \sum_{k\geq 0} e^{\pm}_k(t,x,\xi), \tag{1.19}$$

with $e^{\pm}_k(t,x,\xi) \in S^{m-k}_{1,0}$ and

$$e^{\pm}_0(t,x,\xi) = a^{\pm}(t,x,\xi)\Phi^{\#}(\varphi^{\pm}_t(t,x,\xi)). \tag{1.20}$$

In particular,

$$\Phi^{\#}(D_t)(a^{\pm}e^{i\varphi^{\pm}})\big|_{t=0} = e^{\pm}(0,x,\xi)e^{ix\cdot\xi}, \tag{1.21}$$

and

$$\begin{aligned} e^{\pm}(0,x,\xi) &\sim \sum_{k\geq 0} e^{\pm}_k(0,x,\xi) \\ e^{\pm}_0(0,x,\xi) &= \frac{1}{2}\Phi^{\#}(\pm|\xi|). \end{aligned} \tag{1.22}$$

We therefore have:

Proposition 1.1. *Assume $\Phi \in S^m_{1,0}(\mathbb{R})$ is even. Then*

$$\Phi^{\#}(\sqrt{-L}) \in \Psi^m_{\#}(M), \tag{1.22}$$

the class of operators on functions on M defined as follows.

Given an operator $P : C^\infty_0(M) \to \mathcal{D}'(M)$, we say $P \in \Psi^m_{\#}(M)$ provided the following conditions hold. First we assume its Schwartz kernel $K_P \in \mathcal{D}'(M\times M)$ satisfies

$$\begin{gathered} \operatorname{supp} K_P \subset \{(x,y) \in M\times M : d(x,y) \leq 1\}, \\ \operatorname{sing\,supp} K_P \subset \operatorname{diag}(M\times M) = \{(x,x) : x \in M\}. \end{gathered} \tag{1.23}$$

Next, we assume that, for each $p \in M$,

$$M_{\varphi_1} P M_{\varphi_2} \in OPS^m_{1,0}(\mathbb{R}^n), \tag{1.24}$$

with uniform bounds, independent of $p \in M$, where this statement has the following meaning.

For each $p \in M$, use $\mathrm{Exp}_p : T_pM \to M$, satisfying (0.1)–(0.6), to identify $B_4(p) \subset M$ with $B_4(0) \subset T_pM$, further identified with $B_4(0) \subset \mathbb{R}^n$, uniquely up to the action of $O(n)$. Thus functions supported on $B_4(p) \subset M$ are identified with functions supported on $B_4(0) \subset \mathbb{R}^n$. We pick $\varphi_j \in C_0^\infty(B_4(0)) \approx C_0^\infty(B_4(p))$, equal to 1 on $B_2(p)$, and set $M_{\varphi_j} f = \varphi_j f$, and use these identifications to regard $M_{\varphi_1} P M_{\varphi_2}$ as operating on functions on $\mathbb{R}^n$.

We turn to $\Phi^b(\sqrt{-L})$ in (1.14). Note that for an even function Φ,

$$\Phi \in S^m_{1,0}(\mathbb{R}) \Longrightarrow \widehat{\Phi}(t)(1-\psi(t)) \in \mathcal{S}(\mathbb{R}). \tag{1.25}$$

At this point, we may as well treat $\Phi(\sqrt{-L})$ when $\Phi \in \mathcal{S}(R)$ is even. We want to obtain estimates on the integral kernel $k_\Phi(x,y)$ of $\Phi(\sqrt{-L})$, given by

$$\Phi(\sqrt{-L})f(x) = \int\limits_M K_\Phi(x,y) f(y)\, dV(y). \tag{1.26}$$

Note that

$$k_\Phi(\cdot,y) = \Phi(\sqrt{-L})\delta_y. \tag{1.27}$$

Given that M has bounded geometry, we can find φ_y and ψ_y, supported in $B_1(y)$, such that, with $m = [n/4]+1$,

$$\delta_y = L^m \varphi_y + \psi_y, \quad \|\varphi_y\|_{L^2},\ \|\psi_y\|_{L^2} \le C. \tag{1.28}$$

Then

$$k_\Phi(\cdot,y) = \Phi_m(\sqrt{-L})\varphi_y + \Phi(\sqrt{-L})\psi_y, \quad \Phi_m(\lambda) = \lambda^{2m}\Phi(\lambda). \tag{1.29}$$

Also

$$L_y^j L_x^k k_\Phi(x,y) = k_{\Phi_{j+k}}(x,y). \tag{1.30}$$

Bringing in (1.2), we have

$$\begin{aligned} k_\Phi(\cdot,y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\Phi}(t)\, \cos t\sqrt{-L}(L^m\varphi_y + \psi_y)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big\{ (-1)^m \widehat{\Phi}^{(2m)}(t)\, \cos t\sqrt{-L}\,\varphi_y \\ &\qquad + \widehat{\Phi}(t)\, \cos t\sqrt{-L}\,\psi_y \Big\}\, dt, \end{aligned} \tag{1.33}$$

and more generally

$$L_y^j L_x^k k_\Phi(\cdot,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big\{ (-1)^m \widehat{\Phi}^{(2m+2j+2k)}(t) \, \cos t\sqrt{-L}\,\varphi_y + \widehat{\Phi}^{(2j+2k)}(t) \, \cos t\sqrt{-L}\,\psi_y \Big\} \, dt. \tag{1.34}$$

Since $\cos t\sqrt{-L}$ has L^2-operator norm ≤ 1, it readily follows for even functions Φ,

$$\Phi \in \mathcal{S}(\mathbb{R}) \Longrightarrow k_\Phi \in C^\infty(M \times M). \tag{1.35}$$

We can also estimate the rate of decay of $k_\Phi(x,y)$ when $d(x,y)$ is large. For this we bring in finite propagation speed to deduce from (1.33) that

$$\begin{aligned} &d(x,y) \ge R+1 \Longrightarrow \\ &k_\Phi(x,y) = \frac{1}{\sqrt{2\pi}} \int\limits_{|t|>R} \{ (-1)^m \widehat{\Phi}^{(2m)}(t) \, \cos t\sqrt{-L}\,\varphi_y(x) \\ &\qquad\qquad + \widehat{\Phi}(t) \, \cos t\sqrt{-L}\,\psi_y(x) \Big\} \, dt. \end{aligned} \tag{1.36}$$

We get estimates of the following sort. With C independent of $y \in M$,

$$\|k_\Phi(\cdot,y)\|_{L^2(M \setminus B_R(y))} \le C N_0(\Phi,R) + C N_m(\Phi,R), \tag{1.37}$$

where

$$N_j(\Phi,R) = \int\limits_{|t|>R} |\widehat{\Phi}^{(2j)}(t)| \, dt. \tag{1.38}$$

More generally, via (1.34),

$$\|L_y^j L_x^k k_\Phi(\cdot,y)\|_{L^2(M \setminus B_R(y))} \le C N_{j+k}(\Phi,R) + C N_{m+j+k}(\Phi,R). \tag{1.39}$$

Local elliptic regularity and Sobolev regularity give, also with C independent of $x, y \in M$,

$$|k_\Phi(x,y)| \le C \sum_{j=0}^{2} N_{jm}(\Phi, d(x,y)-2), \tag{1.40}$$

and similar estimates on its derivatives.

If $\Phi \in \mathcal{S}(\mathbb{R})$, then all quantities $N_m(\Phi,R)$ are rapidly decreasing as $R \to \infty$. However, for a Riemannian manifold M with bounded geometry, one has such volume estimates as

$$\mathrm{Vol}(B_r(p)) \le C \langle r \rangle^\mu e^{\kappa r}, \tag{1.41}$$

and if $\kappa > 0$ one desires bounds on $k_\Phi(x,y)$ involving exponential decay. The following has proven to be a useful class of functions Φ. Given $W > 0$, let

$$\Omega_W = \{\zeta \in \mathbb{C} : |\mathrm{Im}\,\zeta| < W\}. \tag{1.42}$$

We denote by $\mathcal{S}_W^m$ the space of even functions $\Phi \in C^\infty(\overline{\Omega}_W)$, holomorphic on Ω_W, and satisfying the symbol estimates

$$|\Phi^{(k)}(\zeta)| \le C_k \langle \zeta \rangle^{m-k}, \quad \zeta \in \overline{\Omega}_W. \tag{1.43}$$

The decomposition $\Phi = \Phi^\# + \Phi^b$ of (1.14) gives $\Phi^\#(\sqrt{-L}) \in \Psi_\#^m(M)$, as before, and $\Phi^b \in \mathcal{S}_W^{-\infty}$. One has

$$|\widehat{\Phi}^b(t)| \le C_j \langle t \rangle^{-j} e^{-W|t|}, \tag{1.44}$$

and similar estimates on derivatives, yielding the following results.

Given $W > 0$, we say

$$P^b \in \Psi_W^{-\infty}(M) \tag{1.45}$$

provided it has the form

$$P^b f(x) = \int\limits_M k^b(x,y) f(y)\, dV(y), \tag{1.46}$$

where $k^b \in C^\infty(M \times M)$ satisfies

$$|k^b(x,y)| \le C_j \langle d(x,y) \rangle^{-j} e^{-W d(x,y)}, \tag{1.47}$$

and also such estimates hold for all x and y-derivatives of $k^b(x,y)$ (say in local exponential coordinate systems). We set

$$\Psi_W^m(M) = \{P^\# + P^b : P^\# \in \Psi_\#^m(M),\ P^b \in \Psi_W^{-\infty}(M)\}. \tag{1.48}$$

The result (1.22) together with the estimate (1.40) and analogues, give the following conclusion.

Proposition 1.2. *We have*

$$\Phi \in \mathcal{S}_W^m \Longrightarrow \Phi(\sqrt{-L}) \in \Psi_W^m(M). \tag{1.49}$$

It is useful to record a slightly smaller class of operators. We say

$$P^b \in \widetilde{\Psi}_W^{-\infty}(M) \tag{1.50}$$

provided it has the form (1.46) and $k^b \in C^\infty(M \times M)$ satisfies, with $C_{jk\ell}$ independent of $y \in M$,

$$\|L_y^j L_x^k k^b(\cdot, y)\|_{L^2(M \setminus B_R(y))} \le C_{jk\ell} \langle R \rangle^{-\ell} e^{-WR}, \tag{1.51}$$

and

$$\|L_y^j L_x^k k^b(x, \cdot)\|_{L^2(M \setminus B_R(x))} \le C_{jk\ell} \langle R \rangle^{-\ell} e^{-WR}. \tag{1.52}$$

We then set

$$\widetilde{\Psi}_W^m(M) = \{P^\# + P^b : P^\# \in \Psi_\#^m(M),\ P^b \in \widetilde{\Psi}_W^{-\infty}(M)\}. \tag{1.53}$$

The derivation of (1.40) from (1.39) leads to

$$\widetilde{\Psi}_W^{-\infty}(M) \subset \Psi_W^{-\infty}(M), \quad \text{hence} \ \ \widetilde{\Psi}_W^m(M) \subset \Psi_W^m(M). \tag{1.54}$$

The estimates (1.39) themselves then imply the following improvement of Proposition 1.2. As will be seen in §2, this next result has somewhat stronger implications for L^p-operator norm bounds than Proposition 1.2.

Proposition 1.3. *We have*

$$\Phi \in \mathcal{S}_W^m \Longrightarrow \Phi(\sqrt{-L}) \in \widetilde{\Phi}_W^m(M). \tag{1.54}$$

Proof. The estimate (1.39) gives (1.51) directly. Since $\Phi(\sqrt{-L})^* = \Phi^*(\sqrt{-L})$, with $\Phi^*(\zeta) = \overline{\Phi(\overline{\zeta})}$, we also have (1.52).

Remark. A family of examples of functions of $\sqrt{-L}$ to keep in mind is the following, with $m, t, b \in \mathbb{R}$:

$$(-\Delta + b^2)^{(m+it)/2} = (-L + B^2 + b^2)^{(m+it)/2}. \tag{1.56}$$

This is of the form $\Phi(\sqrt{-L})$, with

$$\Phi(\zeta) = (\zeta^2 + B^2 + b^2)^{(m+it)/2}. \tag{1.57}$$

We have

$$\Phi \in \mathcal{S}_W^m, \quad \forall\, W < \sqrt{B^2 + b^2}. \tag{1.58}$$

2. L^p operator norm estimates

We seek results on when elements of $\Psi_W^0(M)$ and of $\widetilde{\Psi}_W^0(M)$ are bounded on $L^p(M)$. The first result is a straightforward consequence of (I.3.8).

Proposition 2.1. *If* $P \in \Psi^0_{\#}(M)$, *then* $P : L^p(M) \to L^p(M)$ *for each* $p \in (1, \infty)$.

It remains to see when elements of $\Psi^{-\infty}_W(M)$ and $\widetilde{\Psi}^{-\infty}_W(M)$ are bounded on $L^p(M)$. As is well known, given that M has bounded geometry, there exist constants C, μ, and κ such that for each $r \in (0, \infty)$, $p \in M$,

$$\mathrm{Vol}(B_r(p)) \le C \langle r \rangle^{\mu} e^{\kappa r}. \tag{2.1}$$

(Stronger results are proven in §4 of [CGT].) In case $M = \mathcal{H}^n$, n-dimensional hyperbolic space of constant sectional curvature -1, one has

$$\mathrm{Vol}(B_r(p)) = C_n \int_0^r \bigl(\sinh s\bigr)^{n-1}\, ds. \tag{2.2}$$

Here is a result for $\Psi^0_W(M)$, implicit in [CGT].

Proposition 2.2. *If* $W \ge \kappa$, *then*

$$P^b \in \Psi^{-\infty}_W(M) \Longrightarrow P^b : L^p(M) \to L^p(M), \quad \forall\, p \in [1, \infty]. \tag{2.3}$$

Hence

$$P \in \Psi^0_W(M) \Longrightarrow P : L^p(M) \to L^p(M), \quad \forall\, p \in (1, \infty). \tag{2.4}$$

Proof. If $k^b(x, y)$ is the integral kernel of P^b, to prove (2.3) it suffices to show that

$$\begin{aligned} &\sup_y \int\limits_M |k^b(x, y)|\, dV(x) < \infty, \quad \text{and} \\ &\sup_x \int\limits_M |k^b(x, y)|\, dV(y) < \infty. \end{aligned} \tag{2.5}$$

Our hypotheses yield

$$\int\limits_M |k^b(x, y)|\, dV(x) \le C_k \int_0^\infty \langle r \rangle^{\mu - k} e^{(\kappa - W) r}\, dr \le C < \infty, \tag{2.6}$$

independent of y, provided $W \ge \kappa$ and we pick $k > \mu + 1$. The other part of (2.5) works the same way. From (2.3) and Proposition 2.1 we have (2.4).

Here is the result for $\widetilde{\Psi}^0_W(M)$, implicit in [T3].

Proposition 2.3. *If $W \geq \kappa/2$, then*

$$P^b \in \widetilde{\Psi}_W^{-\infty}(M) \Longrightarrow P^b : L^p(M) \to L^p(M), \quad \forall\, p \in [1,\infty]. \tag{2.7}$$

Hence

$$P \in \widetilde{\Psi}_W^{0}(M) \Longrightarrow P : L^p(M) \to L^p(M), \quad \forall\, p \in (1,\infty). \tag{2.8}$$

Proof. Again it suffices to verify (2.5). We estimate the first integral in (2.5) by dividing M into shells

$$A_j(y) = \{x \in M : j \leq d(x,y) \leq j+1\}. \tag{2.7}$$

We have the following estimate:

$$\begin{aligned} \int\limits_M |k^b(x,y)|\, dV(x) &= \sum_{j\geq 0} \int\limits_{A_j(y)} |k^b(x,y)|\, dV(x) \\ &\leq \sum_{j\geq 0} \bigl(\operatorname{Vol} A_j(y)\bigr)^{1/2} \|k^b(\cdot,y)\|_{L^2(A_j(y))} \\ &\leq C \sum_{j\geq 0} \langle j\rangle^{\mu/2} e^{j\kappa/2} \|k^b(\cdot,y)\|_{L^2(A_j(y))}. \end{aligned} \tag{2.8}$$

Bringing in (1.51) (with $j = k = 0$) we have

$$\|k^b(\cdot,y)\|_{L^2(A_j(y))} \leq C_\ell \langle j\rangle^{-\ell} e^{-jW}, \tag{2.9}$$

and taking $\ell > \mu/2 + 1$ yields the first bound in (2.5), as long as $W \geq \kappa/2$. The second bound in (2.5) is proven similarly. Again, Proposition 2.1 and (2.7) yield (2.8).

Propositions 1.2 and 1.3 can now be applied. The result of applying Proposition 1.3 is better, so we record that conclusion.

Proposition 2.3. *If the volume estimate (2.1) holds, then*

$$\Phi \in \mathcal{S}_W^0,\ W \geq \frac{\kappa}{2} \Longrightarrow \Phi(\sqrt{-L}) : L^p(M) \to L^p(M), \quad \forall p \in (1,\infty). \tag{2.10}$$

Hilbert space theory implies $\Phi(\sqrt{-L})$ is bounded on $L^2(M)$ for each bounded Φ. A standard argument using the Stein interpolation theorem allows one to pass from Proposition 2.3 to the following (Theorem 1.6 of [T3]), which is sharp when M is a symmetric space of noncompact type.

Proposition 2.4. *If $\Phi \in \mathcal{S}^0_W$, then*

$$\Phi(\sqrt{-L}) : L^p(M) \longrightarrow L^p(M), \tag{2.11}$$

provided

$$p \in (1, \infty), \quad \textit{and} \ \ W \geq \left| \frac{1}{p} - \frac{1}{2} \right| \cdot \kappa. \tag{2.12}$$

3. Operator products

We want to analyze products $P_1 P_2$ when $P_j \in \Psi^{m_j}_{W_j}(M)$. As in §2, we assume M has bounded geometry and the volume bound

$$\mathrm{Vol}(B_r(p)) \leq C \langle r \rangle^\mu e^{\kappa r}. \tag{3.1}$$

To make straightforward sense of $P_1 P_2$, we define some classes of "test functions" f and give conditions under which $g = P_2 f$ is a test function and $P_1 g$ is a test function.

We define classes of test functions as follows. Fix $p \in M$ and say

$$f \in C^\infty_W(M) \Longleftrightarrow |L^k f(x)| \leq C_{jk} \langle d(x, p) \rangle^{-j} e^{-W d(x,y)}, \ \forall\, j, k \geq 0. \tag{3.2}$$

We start with the following result.

Proposition 3.1. *We have*

$$P \in \Psi^m_W(M), \ f \in C^\infty_{W+\kappa}(M) \Longrightarrow Pf \in C^\infty_W(M). \tag{3.3}$$

Proof. It follows readily from results on §§I.3–I.4 that

$$P^\# \in \Psi^m_\#(M), \ f \in C^\infty_W(M) \Longrightarrow P^\# f \in C^\infty_W(M), \tag{3.4}$$

so it remains to estimate $g = P^b f$, given $P^b \in \Psi^{-\infty}_W(M)$. We have

$$g(x) = \int_M k^b(x, z) f(z) \, dV(z), \tag{3.5}$$

and estimates on $k^b(x, z)$ and $f(z)$ give

$$|g(x)| \leq C_{jk} \int_M \langle d(x, z) \rangle^{-j} \langle d(z, p) \rangle^{-k} e^{-W[d(x,z)+d(z,p)]} e^{-\kappa d(z,p)} \, dV(z). \tag{3.6}$$

We have

$$\begin{aligned}\langle d(x,z)\rangle\langle d(z,p)\rangle &\approx (1+d(x,z))(1+d(z,p)) \\ &\ge 1+d(x,z)+d(z,p),\end{aligned} \tag{3.7}$$

and

$$d(x,z)+d(z,p)\ge d(x,p), \tag{3.8}$$

so taking $k\ge j+\mu+2$ in (3.6), we get

$$|g(x)|\le C_j\langle d(x,p)\rangle^{-j}e^{-Wd(x,p)}\int\limits_M \langle d(z,p)\rangle^{-\mu-2}e^{-\kappa d(z,p)}\,dV(z), \tag{3.9}$$

and the integral is finite. There are similar estimates on $L^k g(x)$, so (3.4) is established.

The following is a useful complement to Proposition 3.1.

Proposition 3.2. *We have*

$$P\in\Psi^m_{W+\kappa}(M),\ f\in C^\infty_W(M)\Longrightarrow Pf\in C^\infty_W(M). \tag{3.9}$$

Proof. The result (3.4) is applicable, and it remains to estimate (3.5). In place of (3.6), this time we have

$$|g(x)|\le C_{jk}\int\limits_M \langle d(x,z)\rangle^{-j}\langle d(z,p)\rangle^{-k}e^{-\kappa d(x,z)}e^{-W[d(x,z)+d(z,p]}\,dV(z). \tag{3.10}$$

Again use (3.7)–(3.8), this time with $j\ge k+\mu+2$, to get

$$|g(x)|\le C_k\langle d(x,p)\rangle^{-k}e^{-Wd(x,p)}\int\limits_M \langle d(x,z)\rangle^{-\mu-2}e^{-\kappa d(x,z)}\,dV(z), \tag{3.11}$$

and the integral is again finite, with a bound independent of $x\in M$. Similar estimates hold for $L^k g(x)$, so (3.9) is proven.

Here is the corresponding composition result.

Proposition 3.3. *We have*

$$P_1\in\Psi^{m_1}_W(M),\ P_2\in\Psi^{m_2}_{W+\kappa}(M)\Longrightarrow P_1P_2,\ P_2P_1\in\Psi^{m_1+m_2}_W(M). \tag{3.12}$$

Proof. As a preliminary, note the mapping properties

$$\begin{aligned}&C^\infty_{W+2\kappa}(M)\xrightarrow{P_2}C^\infty_{W+\kappa}(M)\xrightarrow{P_1}C^\infty_W(M),\\ &C^\infty_{W+\kappa}(M)\xrightarrow{P_1}C^\infty_W(M)\xrightarrow{P_2}C^\infty_W(M).\end{aligned} \tag{3.13}$$

To proceed, write $P_j = P_j^\# + P_j^b$ with $P_j^\# \in \Psi_\#^{m_j}(M)$, $P_1^b \in \Psi_W^{-\infty}(M)$, and $P_2^b \in \Psi_{W+\kappa}^{-\infty}(M)$. Furthermore, arrange that the Schwartz kernels of $P_j^\#$ are supported within distance $1/2$ of the diagonal in $M \times M$. We claim that the following hold:

$$P_1^\# P_2^\# \in \Psi_\#^{m_1+m_2}(M), \tag{3.14}$$

and

$$P_1^\# P_2^b,\ P_1^b P_2^\# \in \Phi_W^{-\infty}(M), \tag{3.15}$$

together with their counterparts with the subscripts 1 and 2 interchanged, and furthermore

$$P_1^b P_2^b,\ P_2^b P_1^b \in \Psi_W^{-\infty}(M). \tag{3.16}$$

Of these, (3.14) and its counterpart follow directly from results of §I.2 (plus coordinate invariance, from §I.8). The first result in (3.15) follows from the fact that

$$k_{P_1^\# P_2^b}(\cdot, y) = P_1^\# k_{P_2^b}(\cdot, y), \tag{3.17}$$

plus standard pseudodifferential operator estimates, and the second part from the first, by passing to the adjoint. Similarly for their counterparts.

This leaves (3.16). We have

$$k_{P_1^b P_2^b}(x,y) = \int\limits_M k_{P_1^b}(x,z) k_{P_2^b}(z,y)\, dV(z), \tag{3.18}$$

hence

$$|k_{P_1^b P_2^b}(x,y)| \le C_{jk} \int\limits_M \langle d(x,z)\rangle^{-j} \langle d(z,y)\rangle^{-k} e^{-W[d(x,z)+d(z,y)]} e^{-\kappa d(z,y)}\, dV(z). \tag{3.19}$$

This has the same form as (3.6), with y in place of p. Hence the estimates on (3.6) apply. Similarly,

$$|k_{P_2^b P_1^b}(x,y)| \le C_{jk} \int\limits_M \langle d(x,z)\rangle^{-j} \langle d(z,y)\rangle^{-k} e^{-\kappa d(x,z)} e^{-W[d(x,z)+d(z,y)]}\, dV(z), \tag{3.20}$$

which is estimated as in (3.10). There are similar estimates on derivatives, and (3.16) follows. This gives (3.12).

The following variant of Proposition 3.3 is proven in [T7].

Proposition 3.4. *Given $W > \kappa/2$, we have*

$$P_j \in \Psi^{m_j}_W(M) \Longrightarrow P_1 P_2 \in \Psi^{m_1+m_2}_{W-\kappa/2}(M). \tag{3.21}$$

4. Sobolev spaces

As usual, M is a Riemannian manifold with bounded geometry, satisfying (0.1)–(0.6). We want to define and study the spaces $H^{s,p}(M)$ of functions (or distributions) with s derivatives in $L^p(M)$. Here is one natural definition when $s = k$ is a positive integer. Let $\mathcal{V}^1(M)$ denote the space of smooth vector fields X on M with the property that, in each exponential coordinate system $\mathrm{Exp}_q : T_qM \supset B_1(0) \to B_1(q)$, there is a uniform bound (independent of q) on the coefficients of X and, for each k, a uniform bound on all the derivatives of these coefficients of order $\le k$. Let $\mathcal{V}^k(M)$ denote the set of linear combinations of operators of the form $\mathcal{L} = X_1 \cdots X_j$, with $X_\nu \in \mathcal{V}^1(M)$ and $j \le k$. Then we can define

$$H^{k,p}(M) = \{u \in L^p(M) : \mathcal{L}u \in L^p(M),\ \forall\, \mathcal{L} \in \mathcal{V}^k(M)\}. \tag{4.1}$$

There are alternative characterizations of these spaces. For one, let $\{B_1(p_\ell) : \ell \in \mathbb{Z}^+\}$ be a tame cover of M and $\{\varphi_\ell : \ell \in \mathbb{Z}^+\}$ a tame partition of unity, as defined in (0.7)–(0.9). Given a function u on M, set

$$u_\ell = (\varphi_\ell u) \circ \mathrm{Exp}_{p_\ell}, \tag{4.2}$$

a function supported on $B_1(0) \subset T_{p_\ell}M$, which we can identify with $B_1(0) \subset \mathbb{R}^n$, uniquely up to the action of an element of $O(n)$. Then (given $p < \infty$)

$$u \in H^{k,p}(M) \Leftrightarrow \sum_\ell \sum_{|\alpha| \le k} \|D^\alpha u_\ell\|^p_{L^p(B_1(0))} < \infty. \tag{4.3}$$

Thie equivalence is straightforward.

We next define Sobolev spaces for arbitrary index of regularity $s \in \mathbb{R}$, as

$$H^{s,p}(M) = (\lambda I - \Delta)^{-s/2} L^p(M). \tag{4.4}$$

where we take $\lambda > \kappa^2$, where κ is as in (3.1). From here on, we work under the condition

$$1 < p < \infty. \tag{4.5}$$

Of course, we need to show that when $s = k$ is a positive integer, (4.1) is equivalent to (4.4). Before tackling this, we first need to show that the right side of (4.4) is well defined. This will follow from results obtained in §§1–3. To begin, we write

$$(\lambda I-\Delta)^{-s/2}=\Phi_{s,\lambda}(\sqrt{-\Delta}), \tag{4.6}$$

where

$$\Phi_{s,\lambda}(\zeta)=(\zeta^2+\lambda)^{-s/2}. \tag{4.7}$$

With $\mathcal{S}^m_W$ defined as in (1.42)–(1.43), we have

$$\Phi_{s,\lambda}\in\mathcal{S}^{-s}_W,\quad \forall\, W<\sqrt{\lambda}. \tag{4.8}$$

Hence, by (1.56)–(1.58), given $\lambda>0$,

$$(\lambda I-\Delta)^{-s/2}\in\Psi^{-s}_W(M),\quad \forall\, W<\sqrt{\lambda}. \tag{4.9}$$

We can now establish the following.

Proposition 4.1. *Given $\lambda>\kappa^2$,*

$$(\lambda I-\Delta)^{-k/2}:L^p(M)\longrightarrow H^{k,p}(M). \tag{4.10}$$

where the spaces on the right are defined by (4.1).

Proof. Note that $\mathcal{V}^k(M)\subset\Psi^k_\#(M)$. Hence, by (3.12),

$$\mathcal{L}\in\mathcal{V}^k(M)\Longrightarrow\mathcal{L}(\lambda I-\Delta)^{-k/2}\in\Psi^0_W(M),\quad\forall\, W<\sqrt{\lambda}. \tag{4.11}$$

As long as we can take $W>\kappa$, we can apply Proposition 2.1 to conclude that such $\mathcal{L}(\lambda I-\Delta)^{-k/2}$ is bounded on $L^p(M),\ p\in(1,\infty)$, establishing (4.10).

At this point, we have the spaces defined on the right side of (4.4) contained in the spaces defined in (4.1), when $s=k$ is a positive integer.

To proceed, it will be convenient to know that

$$(\lambda I-\Delta)^{-r/2}(\lambda I-\Delta)^{-s/2}f=(\lambda I-\Delta)^{-(r+s)/2}f,\quad\forall\, r,s\in\mathbb{R}, \tag{4.12}$$

whenever $f\in L^p(M),\ 1<p<\infty$. The result (4.12) for $f\in L^2(M)$ is a well known consequence of Hilbert space spectral theory. In that case, the self-duality of $L^2(M)$ extends to produce the duality

$$\Bigl((\lambda I-\Delta)^{-s/2}L^2(M)\Bigr)'=(\lambda I-\Delta)^{s/2}L^2(M),\quad\forall\, s\in\mathbb{R}. \tag{4.13}$$

Now, given that $(\lambda I-\Delta)^{-k/2}L^2(M)$ is contained in $H^{k,2}(M)$ as defined by (4.1), or by (4.3.5), we have

$$(\lambda I-\Delta)^{-k/2}L^2(M)\subset L^\infty(M),\quad\forall\, k>\frac{n}{2}, \tag{4.14}$$

and hence, by duality,

$$L^1(M)\subset(\lambda I-\Delta)^{k/2}L^2(M),\quad\forall\, k>\frac{n}{2}, \tag{4.15}$$

from which it follows that whenever $k>n/2$,

$$L^p(M)\subset(\lambda I-\Delta)^{k/2}L^2(M),\quad\forall\, p\in(1,2]. \tag{4.16}$$

We can now prove:

Lemma 4.2. *The identity (4.12) holds for all $f \in L^p(M)$, $1 < p < \infty$.*

Proof. We have seen that (4.12) holds for all $f \in L^2(M)$. The result (4.16) implies (4.12) holds on $L^p(M)$ for $p \in (1,2]$. The facts that (4.12) holds on $L^p(M)$ for $p \in (2,\infty)$ follow by duality.

We are now prepared to prove:

Proposition 4.3. *If $s = k$ is a positive integer, the spaces defined by (4.1) coincide with those defined by (4.4) (assuming $p \in (1,\infty)$).*

Proof. We have one set of inclusions. For the converse, assume u has the property

$$u,\ X_1 \cdots X_j u \in L^p(M), \quad \forall\, j \le k,\ X_\nu \in \mathcal{V}^1(M), \tag{4.17}$$

We claim

$$f = (\lambda I - \Delta)^{k/2} u \in L^p(M). \tag{4.18}$$

If so, then, by (4.12),

$$u = (\lambda I - \Delta)^{-k/2} f, \tag{4.19}$$

and we are done.

The result (4.18) is elementary if $k = 2j$ is an *even* integer. Then $(\lambda I - \Delta)^j$ is a differential operator, and it is a finite linear combination of operators of the form appearing in (4.17). Now suppose $k = 2j + 1$. The same argument shows that

$$v = (\lambda I - \Delta)^j u \tag{4.20}$$

has the property

$$v,\ Xv \in L^p(M), \quad \forall\, X \in \mathcal{V}^1(M). \tag{4.21}$$

If we can show that for such v,

$$(\lambda I - \Delta)^{1/2} v \in L^p(M), \tag{4.22}$$

we will be done. To get this, write

$$(\lambda I - \Delta)^{1/2} v = \Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta})v + \Phi^{b}_{-1,\lambda}(\sqrt{-\Delta})v, \tag{4.23}$$

with $\Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^1_{\#}(M)$ and $\Phi^{b}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^{-\infty}_W(M)$, for all $W < \sqrt{\lambda}$. Estimates in (2.3) give

$$\Phi^{b}_{-1,\lambda}(\sqrt{-\Delta}) : L^p(M) \longrightarrow L^p(M), \tag{4.24}$$

for $p \in (1, \infty)$. It remains to show that

$$\Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta})v \in L^p(M), \tag{4.25}$$

whenever (4.21) holds. Indeed, since $P = \Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^1_{\#}(M)$, as defined after Proposition 1.1, pseudodifferential operator calculus allows us to write

$$P = Q_0 + \sum_{j=1}^{N} Q_j X_j, \tag{4.26}$$

with

$$X_1, \dots, X_N \in \mathcal{V}^1(M), \quad Q_j \in \Psi^0_{\#}(M). \tag{4.27}$$

Hence, if v satisfies (4.21),

$$Pv = Q_0 v + \sum_{j=1}^{N} Q_j (X_j v) \in L^p(M), \tag{4.28}$$

by Proposition 2.1.

Having identified the spaces (4.1) with their counterparts in (4.4) when $s = k \in \mathbb{N}$, we next show that for general $s \in \mathbb{R}$, the spaces (4.4) are independent of the choice of λ, as long as $\lambda > \kappa^2$.

Proposition 4.4. *For each $s \in \mathbb{R}$, $p \in (1, \infty)$,*

$$\mu, \lambda > \kappa^2 \Longrightarrow (\lambda I - \Delta)^{-s/2} L^p(M) = (\mu I - \Delta)^{-s/2} L^p(M). \tag{4.29}$$

Proof. Note that

$$(\mu I - \Delta)^{s/2} (\lambda I - \Delta)^{-s/2} = \psi_{s,\mu,\lambda}(\sqrt{-\Delta}), \quad \psi_{s,\mu\lambda}(\zeta) = \Big(\frac{\mu + \zeta^2}{\lambda + \zeta^2}\Big)^{s/2}, \tag{4.30}$$

and $\psi_{s,\mu,\lambda} \in \mathcal{S}^0_W$ for all $W < \min(\mu, \lambda)$. Hence

$$(\mu I - \Delta)^{s/2} (\lambda I - \Delta)^{-s/2} : L^p(M) \longrightarrow L^p(M), \tag{4.31}$$

by Proposition 2.2, with inverse $\psi_{s,\lambda,\mu}(\sqrt{-\Delta})$, so (4.31) is an isomorphism. This gives (4.29).

We next record how elements of $\Psi^m_W(M)$ act on these Sobolev spaces.

Proposition 4.5. *Take $m, s \in \mathbb{R}$ and assume $W \geq \kappa$. Then*

$$P \in \Psi^m_W(M) \Rightarrow \ P : H^{s,p}(M) \to H^{s-m,p}(M), \quad \forall\, p \in (1,\infty). \tag{4.32}$$

Proof. The results in (4.32) are equivalent to the existence of $\lambda > \kappa^2$ such that

$$Q = (\lambda I - \Delta)^{(s-m)/2} P (\lambda I - \Delta)^{-s/2} \tag{4.33}$$

has the mapping properties

$$Q : L^p(M) \to L^p(M), \quad p \in (1,\infty). \tag{4.34}$$

To get this, take $\lambda > (W+\kappa)^2$, so $(\lambda I - \Delta)^{\sigma/2} \in \Psi^\sigma_{W+\kappa}(M)$. An application of Proposition 3.3 gives $P(\lambda I - \Delta)^{-s/2} \in \Psi^{m-s}_W(M)$, and a second application gives $Q \in \Psi^0_W(M)$. Then the mapping properties in (4.34) follow from Proposition 2.2.

We next prove a density result.

Proposition 4.6. *For each $s \in \mathbb{R}$ and $p \in (1,\infty)$,*

$$C^\infty_0(M) \ \text{ is dense in } \ H^{s,p}(M). \tag{4.35}$$

Proof. Pick $W > \kappa$ and consider the family of spaces $C^\infty_W(M)$ defined by (3.2). Note that $C^\infty_{W+\kappa}(M)$ is dense in $L^p(M)$. Hence, by (4.4), with $\lambda > W^2$,

$$\{(\lambda I - \Delta)^{-s/2} f : f \in C^\infty_{W+\kappa}(M)\} \ \text{ is dense in } \ H^{s,p}(M). \tag{4.36}$$

By Proposition 3.1, this space is contained in $C^\infty_W(M)$, so $C^\infty_W(M)$ is dense in $H^{s,p}(M)$. It is elementary that $C^\infty_0(M)$ is dense in $C^\infty_W(M)$ in the $H^{k,p}(M)$-norm, for each $k \in \mathbb{Z}^+$, as long as $W > \kappa$. Choosing $k > s$ then gives (4.35).

We next show that for each $p \in (1,\infty)$, $\{H^{s,p}(M) : s \in \mathbb{R}\}$ forms a complex interpolation scale.

Proposition 4.7. *For $p \in (1,\infty)$, $s_0 < s_1 \in \mathbb{R}$, $\theta \in (0,1)$,*

$$[H^{s_0,p}(M), H^{s_1,p}(M)]_\theta = H^{(1-\theta)s_0+\theta s_1,p}(M). \tag{4.37}$$

Proof. With $\mathcal{O} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, we form $\mathcal{H}$, the space of functions $u : \overline{\mathcal{O}} \to H^{s_0,p}(M)$, bounded and continuous on $\overline{\mathcal{O}}$, holomorphic on $\mathcal{O}$, satisfying

$$\sup_{y\in\mathbb{R}} \|u(iy)\|_{H^{s_0,p}}, \ \sup_{y\in\mathbb{R}} \|u(1+iy)\|_{H^{s_1,p}} < \infty. \tag{4.38}$$

Then the left side of (4.37) is

$$\{u(\theta) : u \in \mathcal{H}\}. \tag{4.39}$$

To prove (4.37), first take $u \in \mathcal{H}$. Pick $\varphi \in C_0^\infty(M)$ such that $0 \le \varphi \le 1,\ \varepsilon > 0$, and $\lambda > \kappa^2$, and form

$$v_{\varepsilon,\varphi}(z) = e^{\varepsilon z^2}\varphi(x)e^{\varepsilon\Delta}(\lambda I - \Delta)^{-((1-z)s_0+zs_1)/2}u(z). \tag{4.40}$$

We have $v : \overline{\mathcal{O}} \to L^p(M)$, continuous, holomorphic on $\mathcal{O}$, vanishing at infinity. By Proposition 2.2,

$$\sup_{y\in\mathbb{R}} \|v_{\varepsilon,\varphi}(iy)\|_{L^p},\ \ \sup_{y\in\mathbb{R}} \|v_{\varepsilon,\varphi}(1+iy)\|_{L^p} \le C, \tag{4.41}$$

with C independent of $\varepsilon \in (0,1]$ and φ (given $0 \le \varphi \le 1$), and the maximum principle gives

$$\|v_{\varepsilon,\varphi}(\theta)\|_{L^p} \le C, \tag{4.42}$$

with the same C (hence independent of ε, φ). Taking $\varepsilon \searrow 0$ and $\varphi \nearrow 1$ gives

$$u(\theta) \in H^{(1-\theta)s_0+\theta s_1,p}(M), \tag{4.43}$$

so the left side of (4.37) is contained in the right side of (4.37).

For the reverse inclusion, take $f \in H^{(1-\theta)s_0+\theta s_1,p}(M)$. Then set

$$u(z) = T(z)f = e^{z^2-\theta^2}(\lambda I - \Delta)^{(\theta-z)(s_0-s_1)}f. \tag{4.44}$$

Then we have $u \in \mathcal{H}$. The bounds (4.38) again follow from Proposition 2.2. The assewtion that $T(z)f$ maps $\overline{\mathcal{O}} \to H^{s_0,p}(M)$ boundedly also follows from Proposition 2.2. That it maps continuously is readily verified for $f \in H^{\sigma,p}(M)$, as long as $\sigma > (1-\theta)s_0 + \theta s_1$. Then it follows for all $f \in H^{(1-\theta)s_0+\theta s_1,p}(M)$, by such uniform bounds, plus the fact that

$$H^{\sigma,p}(M) \ \text{ is dense in } \ H^{s,p}(M), \ \ \forall\, \sigma > s, \tag{4.45}$$

itself a corollary of Proposition 4.6.

5. Further results for symmetric spaces of noncompact type

A symmetric space of noncompact type is a Riemannian manifold $M = G/K$, where G is a semisimple Lie group of noncompact type and K a maximal compact subgroup. Examples include hyperbolic space $\mathcal{H}^n$, with constant sectional curvature -1, amongst others. (However, this definition excludes Euclidean space.) We refer to [Hel] for basic material; basic results are also summarized in §2 of [T3]. Without going into details, we mention the following key fact: there exists a positive quantity, denoted $|\rho|^2$, with the property that

$$\mathrm{Spec}\,(-\Delta) = [|\rho|^2, \infty) \quad \text{on} \quad L^2(M) \tag{5.1}$$

and

$$\mathrm{Vol}\, B_r(p) \sim C r^\beta e^{2|\rho| r}, \quad r \to \infty, \tag{5.2}$$

for some $\beta \in (0, \infty)$. Cf. [T3], (2.2) and (2.9). When $M = \mathcal{H}^n$, $|\rho| = (n-1)/2$. Now, if we set

$$L = \Delta + |\rho|^2, \tag{5.3}$$

so $\mathrm{Spec}\,(-L) = [0, \infty)$ on $L^2(M)$, we can apply Proposition 2.4 to deduce that, for $p \in (1, \infty)$,

$$\Phi \in \mathcal{S}^0_W,\ W > \left| \frac{2}{p} - 1 \right| \cdot |\rho| \Longrightarrow \Phi(\sqrt{-L}) : L^p(M) \to L^p(M). \tag{5.4}$$

Using this, we can establish the following variant of the fact that

$$(\lambda I - \Delta)^{m/2} : H^{s,p}(M) \longrightarrow H^{s-m,p}(M), \tag{5.5}$$

for $s, m \in \mathbb{R},\ p \in (1, \infty)$, given $\lambda > 0$ sufficiently large, which was proven in §4, in the setting of general manifolds with bounded geometry.

Proposition 5.1. *If M is a symmetric space of noncompact type, then for $s, m \in \mathbb{R},\ p \in (1, \infty)$,*

$$(-\Delta)^{m/2} : H^{s,p}(M) \longrightarrow H^{s-m,p}(M). \tag{5.6}$$

Remark. This fails when $M = \mathbb{R}^n$.

Proof. In light of the results of §4, (5.6) is equivalent to the assertion that, for $\lambda > 0$ sufficiently large,

$$(\lambda I - \Delta)^{(s-m)/2} (-\Delta)^{m/2} (\lambda I - \Delta)^{-s/2} : L^p(M) \longrightarrow L^p(M). \tag{5.7}$$

We can write this operator as

$$(\lambda I + |\rho|^2 - L)^{(s-m)/2} (|\rho|^2 - L)^{m/2} (\lambda I + |\rho|^2 - L)^{-s/2} = \Phi(\sqrt{-L}), \tag{5.8}$$

where $\Phi(\zeta) = (\lambda + |\rho|^2 + \zeta^2)^{-m/2}(|\rho|^2 + \zeta^2)^{m/2}$, and we see that

$$\Phi \in \mathcal{S}^0_W, \quad \forall\, W < |\rho|. \tag{5.9}$$

Now for each $p \in (1,\infty)$, $|2/p - 1| < 1$, so (5.7) follows from (5.4).

6. Hardy spaces, bmo, and associated Sobolev spaces

In this last section we define $\mathfrak{h}^1(M)$ and $\mathrm{bmo}(M)$ when M is a Riemannian manifold with bounded geometry and discuss some basic properties. Proofs can be found in [T7]. These spaces are defined in analogy with the spaces $\mathfrak{h}^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ in (I.3.21)–(I.3.26).

The spaces on $\mathbb{R}^n$ were introduced in [G], as variants of the spaces $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ treated in [FS]. One advantage of the spaces $\mathfrak{h}^1$ and bmo is that they are invariant under multiplication by bounded Lipschitz functions.

We mention that [I] introduced a variant of H^1 and BMO for functions on rank-one symmetric spaces of noncompact type, though the definition there differs from the Euclidean case in ways that make it problematic to give a unified treatment. This is perhaps another advantage of $\mathfrak{h}^1$ and bmo. We also mention the recent work [CaMM], extending the scope of [I] to a class of measured metric spaces.

To define $\mathfrak{h}^1(M)$, we set up the following maximal function. Given $f \in L^1_{\mathrm{loc}}(M)$, let

$$\mathcal{G}^b f(x) = \sup_{0<r\le 1} \mathcal{G}_r f(x), \tag{6.1}$$

where

$$\mathcal{G}_r f(x) = \sup\Big\{\Big|\int \varphi(y) f(y)\, dV(y)\Big| : \varphi \in \mathcal{F}(B_r(x))\Big\}, \tag{6.2}$$

with

$$\mathcal{F}(B_r(x)) = \Big\{\varphi \in C^1_0(B_r(x)) : \|\varphi\|_{\mathrm{Lip}} \le \frac{1}{r^{n+1}}\Big\}. \tag{6.3}$$

We then set

$$\mathfrak{h}^1(M) = \{f \in L^1_{\mathrm{loc}}(M) : \mathcal{G}^b f \in L^1(M)\}, \tag{6.4}$$

with norm

$$\|f\|_{\mathfrak{h}^1} = \|\mathcal{G}^b f\|_{L^1}. \tag{6.5}$$

One could replace $C_0^1(B_r(x))$ by $\{\varphi \in \text{Lip}(M) : \text{supp}\,\varphi \subset B_r(x)\}$ and get the same result. A comparison with (I.3.21)–(I.3.23) shows that when $M = \mathbb{R}^n$, the space $\mathfrak{h}^1(M)$ defined above coincides with the space $\mathfrak{h}^1(\mathbb{R}^n)$ defined in the §I.3.

To define bmo(M), we set up the following maximal functions. Given $f \in L^1_{\text{loc}}(M)$, let

$$f^\#(x) = \sup_{B\in\mathcal{B}(x)} \frac{1}{V(B)} \int\limits_B |f - f_B|\, dV, \quad f_B = \frac{1}{V(B)} \int\limits_B f\, dV, \tag{6.6}$$

where

$$\mathcal{B}(x) = \{B_r(x) : 0 < r \le 1\}. \tag{6.7}$$

Then define

$$\mathcal{N} f(x) = f^\#(x) + \mathcal{N}_0 f(x), \quad \mathcal{N}_0 f(x) = \frac{1}{V(B_1(x))} \int\limits_{B_1(x)} |f|\, dV. \tag{6.8}$$

We set

$$\text{bmo}(M) = \{f \in L^1_{\text{loc}}(M) : \mathcal{N} f \in L^\infty(M)\}, \tag{6.9}$$

with norm

$$\|f\|_{\text{bmo}} = \|\mathcal{N} f\|_{L^\infty}. \tag{6.10}$$

In case $M = \mathbb{R}^n$, the definition of bmo(M) given here is clearly equivalent to that of bmo$(\mathbb{R}^n)$ given in §I.3; cf. (I.3.24)–(I.3.26).

It is useful to make note of some equivalent norms. For example, in place of $f^\#$, consider

$$f^s(x) = \sup_{B\in\mathcal{B}(x)} \inf_{c_B\in\mathbb{C}} \frac{1}{V(B)} \int\limits_B |f - c_B|\, dV. \tag{6.11}$$

Given $B \in \mathcal{B}(x)$ and taking c_B to realize this infimum, we have

$$|f_B - c_B| = \Bigl|\frac{1}{V(B)} \int\limits_B (f - c_B)\, dV\Bigr| \le f^s(x), \tag{6.12}$$

which gives

$$f^s(x) \le f^\#(x) \le 2 f^s(x). \tag{6.13}$$

It is also useful to note that one can fix $a, b, c \in (0, \infty)$, with $a < b$, and replace $\mathcal{B}(x)$ by

$$\widetilde{\mathcal{B}}(x) = \{Q_r^\alpha(x) : 0 < r \le 1,\ \alpha \in \mathcal{A}\}, \tag{6.14}$$

where $Q_r^\alpha(x)$ is a family of measurable sets with the property that for each $r \in (0, 1]$,

$$\begin{aligned} V(Q_r^\alpha(x)) \ge cV(B_r(x)),\ \ Q_r^\alpha(x) \subset B_{br}(x), &\quad \text{for all } \alpha, \text{ and} \\ B_{ar}(x) \subset Q_r^\alpha(x), &\quad \text{for some } \alpha. \end{aligned} \tag{6.15}$$

One gets functions comparable in size in (6.11) and hence also in (6.6). In connection with this, we recall that the original treatments in [JN] and [FS] used cubes containing x in place of balls centered at x. One consequence of this observation is that the John-Nirenberg estimate, proven in [JN] for functions defined on a cube in $\mathbb{R}^n$, is applicable in our current situation. We have, for each ball $B \subset M$ of radius ≤ 1,

$$\frac{1}{V(B)} \int_B e^{\alpha|f - f_B|}\, dV \le \gamma, \tag{6.16}$$

with

$$\alpha = \frac{\beta}{\|f\|_{\mathrm{bmo}}}, \quad \beta, \gamma \text{ constants}. \tag{6.17}$$

Cf. (3′) of [JN].

It is convenient to know that $\mathfrak{h}^1(M)$ and $\mathrm{bmo}(M)$ are modules over $\mathrm{Lip}(M) \cap L^\infty(M)$. In fact, a more precise result holds. Let σ be a modulus of continuity, and say

$$a \in C^\sigma(M) \Longleftrightarrow |a(x) - a(y)| \le L\sigma(d(x, y)), \quad \text{for } \ d(x, y) \le 1, \tag{6.18}$$

for some $L \in [0, \infty)$. Define $\|a\|_{C^\sigma}$ to be the smallest L for which (6.18) holds (this is a seminorm). We then have the following result.

Proposition 6.1. *Let σ be a modulus of continuity satisfying the Dini condition*

$$D(\sigma) = \int_0^1 \frac{\sigma(r)}{r}\, dr < \infty. \tag{6.19}$$

We also assume $\sigma(r)/r$ is monotonically decreasing on $(0, 1]$ (or constant). Then

$$a \in L^\infty(M) \cap C^\sigma(M),\ f \in \mathfrak{h}^1(M) \Longrightarrow af \in \mathfrak{h}^1(M). \tag{6.20}$$

On the other hand, if $a \in L^\infty(M) \cap C^\sigma(M)$ with

$$\sigma(r) = \Big(\log \frac{1}{r}\Big)^{-1}, \quad 0 < r \le \frac{1}{2}, \tag{6.21}$$

then

$$f \in \mathrm{bmo}(M) \Longrightarrow af \in \mathrm{bmo}(M). \tag{6.22}$$

The proof of (6.20) is fairly straightforward; Schur's lemma is involved. The proof of (6.22) uses the John-Nirenberg estimate (6.16).

REMARK. Note that the Dini condition (6.19) just barely fails for $\sigma(r)$ given by (6.21). $\mathfrak{h}^1$-bmo duality, discussed below, allows one to amalgamate these results.

Using Proposition 6.1, we can establish the following.

Proposition 6.2. *Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity. Given $f \in L^1_{\mathrm{loc}}(M)$, we have*

$$f \in \mathfrak{h}^1(M) \Longleftrightarrow \sum_k \|\varphi_k f\|_{\mathfrak{h}^1} < \infty, \tag{6.23}$$

and

$$\|f\|_{\mathfrak{h}^1} \approx \sum_k \|\varphi_k f\|_{\mathfrak{h}^1}. \tag{6.24}$$

Furthermore,

$$f \in \mathrm{bmo}(M) \Longleftrightarrow \sup_k \|\varphi_k f\|_{\mathrm{bmo}} < \infty, \tag{6.25}$$

and

$$\|f\|_{\mathrm{bmo}} \approx \sup_k \|\varphi_k f\|_{\mathrm{bmo}}. \tag{6.26}$$

Proposition 6.2 combines nicely with the following elementary result. We recall that there is an isometric isomorphism of the n-dimensional inner product space T_pM with $\mathbb{R}^n$, determined uniquely up to the action of $O(n)$.

Proposition 6.3. *We have, uniformly in $k \in \mathbb{Z}^+$,*

$$\|\varphi_k f\|_{\mathfrak{h}^1(M)} \approx \|(\varphi_k f) \circ \mathrm{Exp}_{p_k}\|_{\mathfrak{h}^1(\mathbb{R}^n)}. \tag{6.27}$$

and

$$\|\varphi_k f\|_{\mathrm{bmo}(M)} \approx \|(\varphi_k f) \circ \mathrm{Exp}_{p_k}\|_{\mathrm{bmo}(\mathbb{R}^n)}. \tag{6.28}$$

Corollary 6.4. *In the setting of Proposition 6.2,*

$$\|f\|_{\mathfrak{h}^1(M)} \approx \sum_k \|(\varphi_k f)\circ \operatorname{Exp}_{p_k}\|_{\mathfrak{h}^1(\mathbb{R}^n)}. \tag{6.29}$$

and

$$\|f\|_{\mathrm{bmo}(M)} \approx \sup_k \|\varphi_k f)\circ \operatorname{Exp}_{p_k}\|_{\mathrm{bmo}(\mathbb{R}^n)}. \tag{6.30}$$

These results open the door to making use of Euclidean results of [G], and, by extension, results of [FS]. For example, we can prove the duality

$$\mathfrak{h}^1(M)' = \mathrm{bmo}(M), \tag{6.31}$$

using Corollary 6.4 and the result of [G] that (6.31) holds for $M=\mathbb{R}^n$, itself a consequence of the famous result

$$H^1(\mathbb{R}^n)' = \mathrm{BMO}(\mathbb{R}^n) \tag{6.32}$$

of [FS]. Furthermore, the result of [G] that

$$\begin{aligned} P\in OPS^0_{1,0}(\mathbb{R}^n) \Longrightarrow P &: \mathfrak{h}^1(\mathbb{R}^n)\to \mathfrak{h}^1(\mathbb{R}^n) \ \text{ and} \\ P &: \mathrm{bmo}(\mathbb{R}^n)\to \mathrm{bmo}(\mathbb{R}^n) \end{aligned} \tag{6.33}$$

can be used to prove the following.

Proposition 6.5. *Take κ as in (0.10). Then*

$$\begin{aligned} P\in \Psi^0_W(M),\ W\ge\kappa \ \Longrightarrow P &: \mathfrak{h}^1(M)\to \mathfrak{h}^1(M) \ \textit{and} \\ P &: \mathrm{bmo}(M)\to \mathrm{bmo}(M). \end{aligned} \tag{6.34}$$

We can define Hardy and bmo-Sobolev spaces. Parallel to (4.4), we can take $\lambda>\kappa^2$ and set

$$\begin{aligned} \mathfrak{h}^{s,1}(M) &= (\lambda I-\Delta)^{-s/2}\,\mathfrak{h}^1(M), \\ \mathfrak{h}^{s,\infty}(M) &= (\lambda I-\Delta)^{-s/2}\ \mathrm{bmo}(M). \end{aligned} \tag{6.35}$$

As in §4, these spaces are shown to be independent of the choice of λ, as long as $\lambda>\kappa^2$. As with (4.1), one can show that, for $k\in\mathbb{Z}^+$,

$$\begin{aligned} \mathfrak{h}^{k,1}(M) &= \{u\in \mathfrak{h}^1(M) : \mathcal{L}u\in \mathfrak{h}^1(M),\ \forall\,\mathcal{L}\in\mathcal{V}^k(M)\}, \\ \mathfrak{h}^{h,\infty}(M) &= \{u\in \mathrm{bmo}(M) : \mathcal{L}u\in \mathrm{bmo}(M),\ \forall\,\mathcal{L}\in\mathcal{V}^k(M)\}. \end{aligned} \tag{6.36}$$

Parallel to Proposition 4.5, we have:

Proposition 6.6. *Take $m, s \in \mathbb{R}$ and assume $W \geq \kappa$. Then*

$$P \in \Psi^m_W(M) \Longrightarrow P : \mathfrak{h}^{s,1}(M) \to \mathfrak{h}^{s-m,1}(M), \quad \text{and} \quad P : \mathfrak{h}^{s,\infty}(M) \to \mathfrak{h}^{s-m,\infty}(M). \tag{6.37}$$

Another result of [T7] is the following analogue of the sharp maximal function estimate in L^p of [FS]:

Proposition 6.7. *Assume $p \in (1,\infty)$, $f \in L^1_{\mathrm{loc}}(M)$, and $\mathcal{N}f \in L^p(M)$. Then $f \in L^p(M)$ and*

$$\|f\|_{L^p(M)} \leq C_p \|\mathcal{N}f\|_{L^p(M)}. \tag{6.38}$$

Using this, [T7] establishes the following interpolation result, a variant of Corollary 2 of [FS]:

Proposition 6.8. *Take $s \in \mathbb{R}$. Assume we have a bounded operator*

$$R : L^2(M) \to L^2(M), \quad R : L^1(M) \to \mathfrak{h}^{s,\infty}(M), \tag{6.39}$$

satisfying

$$\|Rf\|_{L^2} \leq M_1 \|f\|_{L^2}, \quad \|Rf\|_{\mathfrak{h}^{s,\infty}} \leq M_0 \|f\|_{L^1}. \tag{6.40}$$

Then, for $\theta \in (0,1)$,

$$R : L^{p(\theta)}(M) \to H^{(1-\theta)s,p(\theta)'}(M), \quad p(\theta) = \frac{2}{2-\theta}, \quad p(\theta)' = \frac{2}{\theta}, \tag{6.41}$$

and (with $C_\theta \in (0,\infty)$ independent of R and f),

$$\|Rf\|_{H^{(1-\theta)s,p(\theta)'}} \leq C_\theta M_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}. \tag{6.42}$$

Such a result can be applied to dispersive estimates and Strichartz estimates. Details will be presented elsewhere.

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