Integral equations on open-curves : a new preconditioner

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April 3, 2018

Abstract

In this paper, we analyze preconditioners for the integral equations arising in the resolution of acoustic scattering by an open arc in 2D in the Galerkin setting.

1 Preliminaries

Globalement, il faut recopier les définitions de Bruno. Let Γ a smooth open simple curve in $\mathbb{R}^2$, and u_D and u_N two smooth functions on Γ. We consider the following boundary value problems, namely the Dirichlet problem (D):

$$\begin{cases} -\Delta u - k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma \\ u = u_D, & \text{on } \Gamma \end{cases}$$

and the Neumann problem (N):

$$\begin{cases} -\Delta u - k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma \\ \dfrac{\partial u}{\partial n} = u_N. & \text{on } \Gamma \end{cases}$$

These problems can be solved using integral equations. Let G the Green's function defined by

$$\begin{cases} G(z) = -\dfrac{1}{2\pi} \ln |z| \,, & \text{if } k = 0, \\ G(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0. \end{cases} \tag{1}$$

We consider the single-layer potential defined for $x \notin \Gamma$ by

$$\mathrm{SL}\lambda(x) = \int_\Gamma G(x-y)\lambda(y) d\sigma(y) \tag{2}$$

where σ is the arc measure on Γ.. Denoting by γ the trace operator on Γ and $S = \gamma\mathrm{SL}$, it is well-known that the solution u of (D) is given by

$$u = \mathrm{SL}\lambda$$

if λ is a solution of the integral equation

$$S\lambda = u_D. \tag{3}$$

The solution λ to the former problem is unique and well-defined. However, because of the edges of Γ, it is not smooth, leading in poor performance of numerical methods based on the discretization of λ itself. It is known that there exists a smooth function φ such that $\lambda = \frac{\varphi}{\omega(x)}$ whith

$$\omega(x) = \frac{1}{\sqrt{d(x, \partial\Gamma)}}$$

This is why, in [2], a weighted operator S_ω is introduced, defined by

$$S_\omega \varphi := S\left(\frac{\varphi}{\omega}\right).$$

This time, S_ω sends smooth functions on smooth functions, leading to improved convergence in numerical methods. Symmetrically, if we let

$$\mathrm{DL}\nu(x) = \int_\Gamma \frac{\partial}{\partial n_y} G(x-y)\nu(y)d\sigma(y) \tag{4}$$

the solution to problem (N) is obtained as

$$u = \mathrm{DL}\nu$$

where ν is the solution of the integral equation

$$N\nu = u_N, \tag{5}$$

and N is the so-called hypersingular operator defined by

$$N\nu = \lim_{z\to 0^+} \frac{\partial}{\partial z}\mathrm{DL}\nu(x + zn_x).$$

Similarly, if u_N is smooth, there exists a smooth function ψ such that

$$\nu = \psi\omega,$$

thus the corresponding weighted hyersingular operator is defined by

$$N_\omega \psi := N(\psi\omega)$$

In [2], it is shown that the operators S_ω and N_ω are inverse modulo a compact operator, justifying that they are good mutual preconditioners in the process of solving (3) and (5) numerically. Here study a new preconditioning technique based on a weighted version of the Laplace operator: for any function u defined on Γ

$$\Delta_\omega u := \omega\left(\omega u'\right)'$$

where the derivative is taken along the curvilinear abscissa. We analyze preconditioners given by that $S_\omega(\Delta_\omega - k^2\omega^2)$ for equation (3) and N_ω.

2 Functional framework

Definition 1. *Definition of the modified Sobolev spaces H^s_ω as complement of polynomials for the norm*

$$\|u\|_s^2 = \sum_{n=0}^{+\infty} (1+n^2)^{\frac{s}{2}} |u_n|^2$$

where $\hat{u}_n = \int_{-1}^{1} \frac{uT_n}{\omega}$ for all reak s.

Theorem 1. *For the bilinear form $(u,v) \mapsto \int_{-1}^{1} \frac{uv}{\omega}$, H^s_ω is the dual of H^{-s}_ω. We have compact injections from $H^s_\omega \to H^t_\omega$ when $s < t$.*

Theorem 2. *The space of (smooth functions) polynomials is dense in H^s_ω for all s.*

Theorem 3. *For all $s \in \mathbb{N}$, (we note $s = n$ in this case), we have*

$$\|u\|_n = \int_{-1}^{1} \frac{|(\omega dx)^n u|^2}{\omega} dx$$

For $n = 1$ in particular, one has

$$\|u\|_1 = \int_{-1}^{1} \omega \left|u'\right|^2$$

Theorem 4. *Eventuellement un résultat d'interpolation ? C'est facile à faire avec les suites, on peut en déduire le résultat sous forme "locale" qu'on avait tant cherché. Peut-être qu'on peut même en arriver à une interpolation autre part que sur le segment (pour un arc ouvert) par changement de variable.*

(par définition)

Theorem 5. *On peut définir de manière implicite les espaces de Sobolev pour $s = \pm\frac{1}{2}$. Par exemple*

$$\|u\|_{\frac{1}{2}} = \int_{\Gamma\times\Gamma} \ln|x-y|\, \omega(x)\omega(y)u'(x)u'(y)dxdy$$

Dans ce cas il faudrait montrer que les deux manières d'étendre la définition se raccordent bien (et ça permettrait de démontrer les résultats d'interpolation requis). On aurait également un vision Fourier discrète pour les courbes fermées.

2.1 Study on the segment with $k \neq 0$.

We write $H_0(z) = \frac{-1}{2\pi} \ln|z| J_0(z) + R(z)$ where R is an even entire function.

Proposition 1. *The functions $r \mapsto \frac{J_0(r)-1}{r^2}$ and $r \mapsto \frac{J_0'(r)}{r}$ are bounded on $\mathbb{R}$.*

Proof. We have for all $r \in \mathbb{R}$

$$\frac{J_0(r)-1}{r^2} = -\sum_{n=0}^{+\infty} \frac{(-1)^n}{(n-1)!^2}\left(\frac{r}{2}\right)^{2n}$$

which is easily shown to be absolutely convergent series for all $r > 0$. A similar argument gives the other result. $\square$

Pas sûr que ce soit utile.

Proposition 2. *For any k, we define $k_1 : r \mapsto (J_0(kr)-1)\ln(|r|)$ defined on $\mathbb{R}$. Then the function $-\Delta_\omega k_1$ is bounded on $\mathbb{R}$.*

Proof. We can write $-\Delta_\omega K_1 = (r^2-1)\partial_{rr}K_1 + r\partial_r K_1(r)$, yielding

$$-\Delta_\omega K_1 = \ln(|r|)\left(kJ_0'(kr) + k^2(r^2-1)J_0''(kr)\right) + (r^2-1)\left(2\frac{kJ_0'(kr)}{|r|} - \frac{J_0(kr)-1}{r^2}\right) + J_0(kr) - 1.$$

$\square$

Corollary 1. *The function*

$$k_1 : f \mapsto \int_{-1}^{1} \frac{k_1(k|x-y|)}{\omega(y)} f(y)dy$$

Is continuous from $H^s_\omega \to H^{s+2}_\omega$

Theorem 6. *An operator of the form*

$$Kf = \int_{-1}^{1} \frac{k(x,y)f(y)}{\omega(y)} \tag{6}$$

with $k \in C^\infty(-1,1)$ maps H^s_ω to $H^{+\infty}_\omega$ for all s.

2.2 Non-flat arc, non-zero frequency

We consider a smooth non-intersecting curve Γ in $\mathbb{R}^2$ and a smooth parametrization $\mathbf{r} : [-1,1] \to \Gamma$. We choose $\mathbf{r}$ such that $\left\|\frac{dr}{dt}\right\| = 1$. Indeed we can assume the curve has unit length by proper rescaling. Indeed, if u is solution of the Helmholtz equation outside Ω with some boundary conditions on Γ (Dirichlet or Neuman) and if we define $u^\lambda = u(\lambda r, \theta)$, we find $\Delta u^\lambda + k^2\lambda^2 u^\lambda = 0$ outside $\Omega_\lambda = \frac{\Omega}{\lambda}$. By choosing $\lambda = |\Gamma|$, the border of the new domain is of length 1.

Without the rescaling (but still assuming constant speed parametrization), we can write

$$\left|r(t) - r(t')\right|^2 = L^2\left|t-t'\right|^2 + \frac{C(t')^2}{2}\left|t-t'\right|^4$$

We note $G_k(t,t')$ the kernel of the non-zero non-flat arc operator.

Lemma 1. *We have the following expansion*

$$J_0(k\left|r(t)-r(t')\right|) = 1 - \frac{k^2}{4}L^2\left|t-t'\right|^2 + \left(\frac{k^4L^4}{64} - \frac{C(t')^2k^2}{8}\right)\left|t-t'\right|^4 + (t-t')^5F(t,t')$$

where F is a smooth bounded function.

Lemma 2. *If L is the length of the curve and $C(t')$ the curvature at a point t', one has*

$$G_k(t,t') = -\frac{1}{2\pi}\ln|t-t'|\left(1-\frac{k^2}{4}L^2\left|t-t'\right|^2+\left(\frac{k^4L^4}{64}-\frac{C(t')^2k^2}{8}\right)\left|t-t'\right|^4+(t-t')^5F(t,t')\right)$$
$$+R(t,t')$$

where R is in $C^\infty([-1,1]^2)$.

Lemma 3.

$$\begin{aligned}\Delta_\omega^{t'}\left((t-t')^2\ln\left|t-t'\right|\right) &= \omega^2(t')\frac{d^2}{dt'^2}\left((t-t')^2\ln\left|t-t'\right|\right)-t'\frac{d}{dt'}\left((t-t')^2\ln\left|t-t'\right|\right)\\ &= \omega^2(t')\left(2\ln\left|t-t'\right|+4-1\right)-t'\left(2(t'-t)\ln\left|t-t'\right|+2(t'-t)\right)\\ &= 2\omega^2(t)\ln\left|t-t'\right|+2(t-t')\ln\left|t-t'\right|(t+2t')+P(t,t')\end{aligned}$$

where P is a polynmial in t and t'.

Lemma 4. *The second term in this decomposition is the kernel of a bounded operator from H_ω^s to H_ω^{s+2}.*

Proof. Même raisonnement (on regarde une fois de plus l'action de Δ_ω et on conclut avec une borne L^2 de l'intégrale de Cauchy.) □

Lemma 5. *The application $f\mapsto\omega^2 f$ is continuous in H_ω^s for any s.*

Proof. This is obvious as $\left|\left\langle\omega^2 f,T_n\right\rangle_\omega\right|\le\left|\langle f,T_n\rangle_\omega\right|$. □

Lemma 6. *The application $f\mapsto\omega^2 f'$ is continuous from H_ω^s to H_ω^{s-1}. The proof involves the Cesaro thm.*

Lemma 7. *The operator $\Delta_\omega\omega^2-\omega^2\Delta_\omega$ is continuous from H_ω^s to H_ω^{s-1}.*

Proof. We use the formula : $\Delta_\omega\omega^2-\omega^2\Delta_\omega=4x\omega^2f'+(4x^2-2)f$ □

Lemma 8. *The operator $S_0\omega^2-\omega^2S_0$ is continuous from H_ω^s to H_ω^{s+2}.*

Proof. Use $\omega^2T_n=\frac{2T_n+T_{n+2}+T_{|n-2|}}{4}$. □

Be careful, the map $f\mapsto f'$ is not continuous from H_ω^1 to L_ω^2, as can be checked with the example $f=\omega$.

Les deux lemmes précédents : même preuve.

Lemma 9. $S_k-S_0=\omega(t)^2\frac{k^2}{2}L^2S_0\Delta_\omega^{-1}$

3 Application to preconditioning

In this section we propose two preconditioners for the Scattering problem, one based on the use of a method for the square root of an operator. We prove that the preconditioned systems have a conditioning that is independent of the mesh size. We show applications.

References

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[2] Oscar P Bruno and Stéphane K Lintner. Second-kind integral solvers for te and tm problems of diffraction by open arcs. *Radio Science*, 47(6), 2012.