A New Well-Conditioned Integral Formulation for Maxwell Equations in Three Dimensions

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***Abstract*—We present a new boundary integral equation dedicated to the solution of the boundary problem of a perfectly electrically conducting surface for the harmonic Maxwell equations in unbounded domains. Any solution of the harmonic Maxwell equations is represented as the electromagnetic field generated by a combination of electric and magnetic potentials. These potentials are those appearing in the classical combined field integral equation (CFIE), but their coupling is realized by an operator $\tilde{Y}^+$ instead of a coefficient. Therefore, the integral equation obtained can be viewed as a generalization of the CFIE. In this paper, we propose an explicit construction of the coupling operator $\tilde{Y}^+$ which is designed to approximate the exterior admittance operator of the scattering obstacle. A local approximation by the admittance operator of the tangential plane seems to be relevant thanks to the localization effects related to high-frequency phenomena. The provided numerical simulations show that this formulation leads to linear systems that are better conditioned compared to more classical integral equations, which speeds up the resolution when solved with iterative techniques.**

***Index Terms*—Combined field integral equation (CFIE), electromagnetic scattering, preconditioner.**

I. Introduction

INTEGRAL equation methods are very popular for solving electromagnetic scattering problems. They can size the equations on the surface of the scattering obstacle which lowers the dimension of the problem. Consequently, the equations are non local and lead after discretization to dense linear systems. As the frequency increases, one of the known difficulties comes from the fact that the scattering surface must be discretized more and more finely, and the size of the resulting linear systems becomes prohibitive. Typically scattering by realistic objects like an aircraft at several gigahertz yields to systems that exceed one million of degrees of freedom. Therefore, direct solvers are too costly and iterative methods are usually prefered. Even though, for systems of such sizes, the matrix vector products arising in iterative techniques may yet be too expensive and methods have been developed to get a sparse approximation of the linear system (see, for instance, the impedance matrix localization (IML) method of Canning [1] or the fast multipole method (FMM) of Rokhlin [2]).

The resolution of the linear systems deriving from boundary integral equations also suffers from the convergence slowness of the iterative algorithms, which is directly linked to the conditioning of the linear systems. Algebraic preconditioners (like the sparse pattern approximate inverse (SPAI) [3]) have been used to improve the situation. However, these preconditioners, are constructed directly from the matrix of the linear system obtained after discretization, without taking care of mathematical properties of the continuous operator.

On the other hand, efficient preconditioners can also be built from a regularization, at the continuous level, of the boundary integral equations. The determination of a parametrix B (in a weak sense) of the underlying operator A of the integral equation, i.e., an operator B so that the composition $B \circ A$ is a compact perturbation of the identity, is a way to achieve this goal. Actually, compact perturbations of the identity have the well-known property to produce after discretization well-conditioned linear systems (in particular whose conditioning is invariant of the space step of the mesh). Therefore, it seems natural to expect that the discretization of the parametrix B will lead to a good preconditioner for the matrix deriving from the operator A. This preconditioning technique has been investigated for some classical boundary integral equations. For instance, McLean and Tran [4] have worked on two integral equations of the first kind obtained for solving the Dirichlet and Neumann problems for the two-dimensional (2-D) Laplacian. These equations involve the single-layer and the hypersingular operators respectively for which the matrices obtained after discretization are both ill-conditioned. To overcome this default, the authors propose to regularize the single-layer operator by the hypersingular one and vice versa, their composition being a compact perturbation of the identity. In [5], Steinbach and Wendland develop a more general technique for first kind integral equations in2-D and 3-D. This technique is based on a boundary integral operator of opposite order to that of the original one. The resulting spectral condition number of the preconditioned system is proved to be bounded independently of the triangulations and of the trial functions. Some regularization strategies have also been established for more classical formulations used in the electromagnetic framework. [6] and [7] give answers for the electric field integral equation (EFIE). Both stabilizations of this equation are based on the Calderon fomulas [8]. As far as the combined field integral equation (CFIE) is concerned (this equation is solved in most of the industrial codes modeling electromagnetic scattering phenomena), a weak parametrix of the underlying operator is exhibited by Levadoux in [9], which enables the author to construct two efficient preconditioners for systems deriving from the integral equation.

Another approach consists in establishing new integral formulations for the considered boundary value problems in order

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that their discretization directly produces a well-conditioned linear system. The analysis of the so-called Calderon projectors [8] leads Levadoux in [10] to construct a new boundary integral equation dedicated to the Dirichlet problem for the 2-D Helmholtz equation. This formalism is based on the construction of a coupling operator which is designed to approximate the admittance of the scatterer. Different approximations are proposed, for which the deriving equation is well-posed at any frequency. The numerical results allow to conclude that the system deriving from this equation is well-conditioned. In this framework, other approximations of the admittance are tested in [11] which speed up the iterative resolution. A new integral formulation for electromagnetic scattering problems is also studied in [12] and [13] which combines the standard magnetic field integral operator (MFIO) and the electric field integral operator (EFIO) composed with itself. The two EFIO are taken at different frequencies to avoid spurious resonances with the MFIO. The good conditioning of this equation lies, as in [7], on the regularization effects of the EFIO on itself. The formulations proposed in [14] are likewise connected to this kind of techniques. A new system of integral equations for the harmonic Maxwell equations deriving from elementary manipulations of classical equations is introduced and proved to be well-posed. An iterative algorithm is proposed to solve the system and proved to be convergent.

We would like to carry on the development and to begin the numerical implementation of the technique employed to reformulate the boundary value problems, established in acoustics in [10], and for which an extension to the electromagnetic framework is for the first time mentionned in [15]. The principle is to choose a potential, which is used to represent a solution of the harmonic Maxwell equations in a 3-D unbounded domain satisfying a radiation condition at infinity. The determination of this potential is governed by the desire to make the deriving system well-conditioned. Thanks to the Stratton–Chu formula, this potential can be chosen as the combination of the magnetic and electric potentials, the same as the ones used in the CFIE, where the electric potential is being composed with an operator (denoted by Y^+) called admittance of the scattering obstacle. On non canonical surfaces where the exact admittance can not be explicitly given, the strategy consists in approximating the coupling operator Y^+ via the explicit construction of an operator denoted by $\tilde{Y}^+$. If the exact admittance of the considered scattering surface was known, taking $\tilde{Y}^+ = Y^+$ would give a boundary integral equation for which the operator would be the identity. Therefore, the approximation $\tilde{Y}^+$ is wanted to be as close as possible to the exact admittance Y^+ in order that the operator of the boundary equation remains close to identity and produces after discretization a system easy to solve. The construction of this coupling operator is based on the localization effects related to high-frequency phenomena. It seems relevant to build $\tilde{Y}^+$ as a local approximation of the exact admittance, splitting the boundary into subdomains comparable to canonical surfaces for which the admittance is well-known.

The paper is divided as follows. In Section II we introduce the new boundary integral equation for electromagnetic scattering in 3-D taking into account these principles. We discuss in Section III the well-posedness of this equation and in Section IV we describe the Galerkin discretization of the equation and the construction of the matrix of the system. Eventually some numerical results on simple scattering surfaces are reported in Section V. They allow to conclude that the equation leads to a well-conditionned system.

II. A New Boundary Integral Formulation

Let $D = \overline{\Omega^-}$ be a compact in $\mathbb{R}^3$. Its complement, denoted by Ω^+, is supposed to be connected and its boundary Γ of $\mathcal{C}^\infty$ regularity. Let ν be the unit normal vector to Γ directed to the exterior of D. The electromagnetic trace operators are defined according to

$$\sigma_0 = \nu\times, \quad \sigma_1 = \frac{1}{ik}\nu \times \overrightarrow{\nabla} \times \quad \text{on } \Gamma. \tag{1}$$

They are provided with a sign $+$ or $-$ whether they are exterior or interior traces.

Scattering of time-harmonic electromagnetic waves by a perfectly electrically conducting (PEC) surface is governed by the harmonic Maxwell problem: $\mathbf{E}^i$ being a given incoming electromagnetic wave, find the electric field $\mathbf{E}$ satisfying

$$\overrightarrow{\nabla} \times \overrightarrow{\nabla} \times \mathbf{E} - k^2\mathbf{E} = 0 \text{ in } \Omega^+ \tag{2}$$

$$\sigma_0^+\mathbf{E} = E_0 = -\nu \times \mathbf{E}^i \text{ on } \Gamma \tag{3}$$

$$\lim_{|x|\to\infty} |x| \left((\overrightarrow{\nabla} \times \mathbf{E}) \times \frac{x}{|x|} - ik\mathbf{E} \right) = 0. \tag{4}$$

A field satisfying the Maxwell equation (2) in Ω^- is called in the later an internal wave whereas a field satisfying the Maxwell equation (2) in Ω^+ and the Silver-Müller radiation condition (4) an external wave.

Let $G(k,x,y)$ be the fundamental solution of the Helmholtz equation

$$G(k,x,y) = -\frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y$$

and let $\mathcal{G}_k$ be the potential mapping any sufficiently smooth current u on Γ to the field defined in $\mathbb{R}^3 \setminus \Gamma$ by

$$(\mathcal{G}_k u)(x) = \int_\Gamma G(k,x,y)u(y)\mathrm{d}y.$$

Thanks to the Stratton–Chu formula, every external (resp. internal) wave can be represented as

$$\begin{aligned} \mathbf{E} &= \mathcal{C}^+(\sigma_0^+\mathbf{E}, \sigma_1^+\mathbf{E}) \text{ in } \Omega^+ \\ (\text{resp. } \mathbf{E} &= \mathcal{C}^-(\sigma_0^-\mathbf{E}, \sigma_1^-\mathbf{E}) \text{ in } \Omega^-) \end{aligned} \tag{5}$$

with the following notations:

$$\mathcal{C}^+(f,g) = \mathcal{T}g - \mathcal{K}f, \quad (\text{resp. } \mathcal{C}^-(f,g) = -\mathcal{T}g + \mathcal{K}f) \tag{6}$$

where the potentials $\mathcal{T}$ and $\mathcal{K}$ are defined, respectively, as

$$\begin{cases} \mathcal{T}u(x) = \frac{1}{ik}\overrightarrow{\nabla} \times \overrightarrow{\nabla} \times (\mathcal{G}_k u)(x) \text{ in } \mathbb{R}^3 \setminus \Gamma \\ \mathcal{K}u(x) = \overrightarrow{\nabla} \times (\mathcal{G}_k u)(x) \text{ in } \mathbb{R}^3 \setminus \Gamma. \end{cases} \tag{7}$$

Since the exterior PEC boundary problem (2)–(4) is well-posed, there exists a unique field $\mathbf{E}$ such that $\sigma_0^+\mathbf{E} = E_0$ and

the mapping $E_0 \mapsto \sigma_1^+\mathbf{E}$ induces an operator denoted by Y^+ and called admittance. Due to (5), the external wave $\mathbf{E}$ can then be expressed only in function of its trace $\sigma_0^+\mathbf{E}$

$$\mathbf{E} = \mathcal{C}^+(\sigma_0^+\mathbf{E}, Y^+\sigma_0^+\mathbf{E}) \text{ in } \Omega^+. \tag{8}$$

Applying the trace σ_0^+ to this representation yields $\sigma_0^+\mathcal{C}^+(\sigma_0^+\mathbf{E}, Y^+\sigma_0^+\mathbf{E}) = \sigma_0^+\mathbf{E}$ and then the operator $\sigma_0^+\mathcal{C}^+(Id \oplus Y^+)$ is the identity. Nevertheless an explicit formulation of the admittance operator Y^+ on non canonical scattering surfaces can obviously not be expected. Therefore, the aim of this study is to build an approximation of the potential $\mathcal{C}^+(Id \oplus Y^+)$. We suggest to represent the external wave $\mathbf{E}$ in function of an unknown vector field u (distribution) defined on Γ with the help of a potential of the kind

$$\mathbf{E} = \mathcal{C}^+(u, \tilde{Y}^+u)$$

where the operator denoted by $\tilde{Y}^+$, which must be constructed in an explicit way, is intended to approximate the exact admittance of the scattering obstacle. The underlying current u of this representation is determined, using the boundary condition of a PEC surface (3), as a solution of the problem

$$\text{Find } u \text{ such that } \sigma_0^+\mathcal{C}^+(u, \tilde{Y}^+u) = E_0. \tag{9}$$

Of course the closer $\tilde{Y}^+$ to Y^+, the closer $\sigma_0^+\mathcal{C}^+(Id \oplus \tilde{Y}^+)$ to the identity and, therefore, the better conditioned the resulting system.

As the EFIO and the MFIO operators T and $K - 1/2$ are respectively defined by

$$\sigma_0^+\mathcal{T} = T \quad \sigma_0^+\mathcal{K} = K - \frac{1}{2} \tag{10}$$

the boundary integral equation (9) can be rewritten as

$$T\tilde{Y}^+u + \left(\frac{1}{2} - K\right)u = E_0. \tag{11}$$

This integral formulation may be interpreted as a generalization of the CFIE since the coupling of the EFIO and MFIO operators, usually given by scalar coefficients, is now realized in (11) by an operator. For this reason, we suggest to call the formulation (11) *Generalized Combined Field Integral Equation* (GCFIE) [10].

Thanks to the hypothesis of localization of the admittance at high-frequency, we propose to approximate the operator Y^+ by pieces. If the localization is fine enough, the boundary can be split into subdomains comparable to their tangential plane and we suggest to approximate the admittance on each subdomain by the admittance of the infinite plane.

Let us determine this admittance operator. Let u be a compact supported trace on the infinite plane. Since the trace σ_0^+ of the field $-2\mathcal{K}u$ is $(I - 2K)u = u$, the field $-2\mathcal{K}u$ is the solution of the PEC problem with boundary condition u. Then the admittance on the infinite plane is the trace σ_1^+ of this field, i.e., $\sigma_1^+(-2\mathcal{K}u) = -2Tu$, and the approximation of the admittance $\tilde{Y}^+$ is built as a localization of the EFIO.

Then we propose to approximate the admittance by the operator of order one defined by

$$\tilde{Y}^+ = -2\sum_n \chi_n T \chi_n \tag{12}$$

with $\{\chi_n\}_{n=1,\ldots,N}$ a $\mathcal{C}^\infty$ quadratic partition of the unity associated with a covering of Γ by a family of open surfaces $\{\mathcal{U}_n\}_{n=1,\ldots,N}$

$$\sum_{n=1}^{N} \chi_n^2 = 1 \quad \text{and} \quad \text{supp}\chi_n \subset \mathcal{U}_n \qquad \forall n = 1, \ldots, N.$$

The need to localize the operator $-2T$ is suggested not only by the physical properties of the diffraction phenomena but also by mathematical observations, and confirmed by the numerical experiments (see Section V). In fact, if we took $\tilde{Y}^+ = -2T$, thanks to the Calderon formula $T^2 = K^2 - 1/4$ [8], we would have

$$T\tilde{Y}^+ + \frac{1}{2} - K = 2\left(\frac{1}{2} - K\right)(K + 1).$$

Therefore, the integral formulation (11) would possess every false internal resonance of the operator $((1/2) - K)$.

In the next section, we study the well-posedness of the formulation (11) with the coupling operator $\tilde{Y}^+$ being a localization of the operator $-2T$ (12).

III. Well-Posedness of the Integral Equation

We denote by $H^s(\Gamma)$ and $H_{\mathrm{T}}^s(\Gamma)$ the usual Sobolev spaces of scalar and tangential fields respectively of regularity order $s \in \mathbb{R}$. Let X^+ (resp. X^-) be the space composed by the couples of traces

$$(\sigma_0^+\mathbf{E}, \sigma_1^+\mathbf{E}) \in \mathcal{C}^\infty(\Gamma) \times \mathcal{C}^\infty(\Gamma)$$

(resp. $(\sigma_0^-\mathbf{E}, \sigma_1^-\mathbf{E})$) with $\mathbf{E}$ an external (resp. internal) wave. We recall that the operators

$$C^+ = \sigma_0^+ \oplus \sigma_1^+(\mathcal{C}^+) \quad C^- = \sigma_0^- \oplus \sigma_1^-(\mathcal{C}^-)$$

known as Calderon projectors [8], split the space of traces $X = \mathcal{C}^\infty(\Gamma) \times \mathcal{C}^\infty(\Gamma)$ into the direct sum $X = X^+ \oplus X^-$.

Then every couple of traces $(0, v) \in X$ can be decomposed according to

$$(0, v) = C^+(0, v) + C^-(0, v).$$

For the first trace: $\sigma_0^+\mathcal{C}^+(0, v) + \sigma_0^-\mathcal{C}^-(0, v) = 0$, and we get the following continuity relation:

$$\sigma_0^+\mathcal{T}v = \sigma_0^-\mathcal{T}v = Tv. \tag{13}$$

For the second trace: $\sigma_1^+\mathcal{C}^+(0, v) + \sigma_1^-\mathcal{C}^-(0, v) = v$, which gives the following jump relation:

$$\sigma_1^+\mathcal{T}v - \sigma_1^-\mathcal{T}v = v. \tag{14}$$

In a similar way, $(u,0) = C^+(u,0) + C^-(u,0)$ for every $(u,0) \in X$. Then $-\sigma_0^+\mathcal{K}u + \sigma_0^-\mathcal{K}u = u$ and, according to (10)

$$\sigma_0^-\mathcal{K}u = \left(K + \frac{1}{2}\right)u. \tag{15}$$

Moreover, using the definitions (1) and (7), we can easily check after an elementary calculus that $\sigma_1^-\mathcal{T}u = -\sigma_0^-\mathcal{K}u$. Therefore

$$\sigma_1^-\mathcal{T}u = -\left(K + \frac{1}{2}\right)u. \tag{16}$$

We first give some properties of X^+ and X^-.

Lemma III.1 Traces Properties: The internal and external traces verify the following properties:

P1) if $(u,v) \in X^-$, then $\mathcal{R}e(\langle v, \nu \times \bar{u}\rangle) = 0$;

P2) if $(u,v) \in X^+ \setminus \{(0,0)\}$, then $\mathcal{R}e\,(\langle v, \nu \times \bar{u}\rangle) > 0$.

$\langle .,. \rangle$ denotes the $H_{\mathrm{T}}^0(\Gamma)$-inner product and $\bar{u}$ the complex conjugate of u.

Proof: Let $\mathbf{E}$ and $\mathbf{F}$ be internal waves such that $(\sigma_0^-\mathbf{E}, \sigma_1^-\mathbf{E}) \in X^-$ and $(\sigma_0^-\mathbf{F}, \sigma_1^-\mathbf{F}) \in X^-$. The Green's formula applied to the couple $(\mathbf{E},\mathbf{F})$ on Ω^- gives

$$\int_{\Omega^-} \mathbf{E}\Delta\mathbf{F} - \mathbf{F}\Delta\mathbf{E}\mathrm{d}\Omega = \int_\Gamma \{(\nu\cdot\mathbf{E})\overrightarrow{\nabla}\cdot\mathbf{F} - \mathbf{E}\cdot(\nu\times\overrightarrow{\nabla}\times\mathbf{F}) - (\nu\cdot\mathbf{F})\overrightarrow{\nabla}\cdot\mathbf{E} + \mathbf{F}\cdot(\nu\times\overrightarrow{\nabla}\times\mathbf{E})\}d\Gamma.$$

As $\mathbf{E}$ and $\mathbf{F}$ are solutions of the Maxwell equation (2), they are divergence free and satisfy the vector Helmholtz equations $\Delta\mathbf{E} + k^2\mathbf{E} = 0$ and $\Delta\mathbf{F} + k^2\mathbf{F} = 0$. Hence, the previous equation reduces to

$$0 = \int_\Gamma \{\mathbf{F}\cdot(\nu\times\overrightarrow{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\nu\times\overrightarrow{\nabla}\times\mathbf{F})\}\,\mathrm{d}\Gamma.$$

Taking $\mathbf{F} = \bar{\mathbf{E}}$ yields $\mathcal{R}e(\int_\Gamma(\nu\times\overrightarrow{\nabla}\times\mathbf{E})\cdot\bar{\mathbf{E}}\,\mathrm{d}\Gamma) = 0$ and *(P1)* immediately comes from the definitions of the traces. □

Let $\mathbf{E}$ and $\mathbf{F}$ be external waves such that $(\sigma_0^+\mathbf{E}, \sigma_1^+\mathbf{E}) \in X^+$ and $(\sigma_0^+\mathbf{F}, \sigma_1^+\mathbf{F}) \in X^+$. Let Γ_r be the sphere centered at the origin of radius r large enough so that D is contained in the interior of Γ_r. We denote by D_r the domain $\{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < r\}$. Applying Green's first vector theorem to $(\mathbf{E},\mathbf{F})$ on the domain D_r, we obtain

$$\int_{D_r} \{\mathbf{F}\cdot\Delta\mathbf{E} + (\overrightarrow{\nabla}\times\mathbf{F})\cdot(\overrightarrow{\nabla}\times\mathbf{E}) + (\overrightarrow{\nabla}\cdot\mathbf{F})\cdot(\overrightarrow{\nabla}\cdot\mathbf{E})\}\mathrm{d}\Omega = \int_{\partial D_r} \{(\nu\cdot\mathbf{F})\overrightarrow{\nabla}\cdot\mathbf{E} - \mathbf{F}\cdot(\nu\times\overrightarrow{\nabla}\times\mathbf{E})\}\,\mathrm{d}\Gamma.$$

Taking $\mathbf{F} = \bar{\mathbf{E}}$, denoting $\mathbf{H} = (1/ik)\overrightarrow{\nabla}\times\mathbf{E}$ and using the properties of the solutions of the Maxwell equation stated in the first part of the proof, we get

$$\int_\Gamma \bar{\mathbf{E}}\cdot(\nu\times\mathbf{H})\mathrm{d}\Gamma = \int_{\Gamma_r} \bar{\mathbf{E}}\cdot(\nu\times\mathbf{H})\,\mathrm{d}\Gamma + ik\int_{D_r}\{|\mathbf{E}|^2 - |\mathbf{H}|^2\}\,\mathrm{d}\Omega.$$

Taking the real part yields to

$$\mathcal{R}e\left(\int_\Gamma \bar{\mathbf{E}}\cdot(\nu\times\mathbf{H})\,\mathrm{d}\Gamma\right) = \mathcal{R}e\left(\int_{\Gamma_r} \bar{\mathbf{E}}\cdot(\nu\times\mathbf{H})\mathrm{d}\Gamma\right) = \frac{1}{2}\int_{\Gamma_r}|\nu\times\mathbf{H}+\mathbf{E}|^2\,\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_r}\{|\nu\times\mathbf{H}|^2 + |\mathbf{E}|^2\}\,\mathrm{d}\Gamma.$$

From the Silver–Müller condition, $\int_{\Gamma_r}|\nu\times\mathbf{H}+\mathbf{E}|^2\mathrm{d}\Gamma \to 0, r\to\infty$, and we obtain $\mathcal{R}e(\int_\Gamma(\nu\times\bar{\mathbf{H}})\cdot\bar{\mathbf{E}}\,\mathrm{d}\Gamma) \le 0$. If it vanishes, $\int_{\Gamma_r}|\mathbf{E}|^2\mathrm{d}\Gamma \to 0, r\to\infty$, and, by the Rellich's lemma [16], $\mathbf{E} = 0$ in Ω^+, which establishes *(P2)*. □

Then we can show that the admittance operator $\tilde{Y}^+$ satisfies the following positivity property:

$$\text{if } u \in \mathcal{C}^\infty(\Gamma)\setminus\{0\}, \quad \mathcal{R}e(\langle\tilde{Y}^+u, \nu\times\bar{u}\rangle) > 0.$$

Let $\mathbf{E}_n = \mathcal{T}(\chi_n u)$. Using the relation (13) and the jump relation (14)

$$\mathcal{R}e(\langle\tilde{Y}^+u, \nu\times\bar{u}\rangle) = -2\sum_n \mathcal{R}e(\langle T\chi_n u, \nu\times\chi_n\bar{u}\rangle) = -2\sum_n \mathcal{R}e(\langle\sigma_0^+\mathbf{E}_n, \nu\times\overline{(\sigma_1^+ - \sigma_1^-)\mathbf{E}_n}\rangle).$$

Inserting the continuity relation (13) into this equation and applying the criterion *(P1)* yields

$$\mathcal{R}e(\langle\tilde{Y}^+u, \nu\times\bar{u}\rangle) = -2\sum_n \mathcal{R}e(\langle\sigma_0^+\mathbf{E}_n, \nu\times\overline{\sigma_1^+\mathbf{E}_n}\rangle).$$

The criterion *(P2)* enables us to conclude that each term of the sum is positive. If the sum vanishes, by the criterion *(P2)*, $\sigma_0^+\mathbf{E}_n = 0$ for all n. Thanks to the uniqueness of the PEC boundary problem solutions, $\mathbf{E}_n = 0$ in Ω^+. Due to the analyticity of $\mathbf{E}_n$ in $\mathbb{R}^3\setminus\mathcal{U}_n$, $\mathcal{U}_n$ being an open surface, $\mathbf{E}_n = 0$ in $\mathbb{R}^3\setminus\mathcal{U}_n$. Then $\sigma_1^+\mathbf{E}_n = \sigma_1^-\mathbf{E}_n = 0$. By the jump relation (14), $\chi_n u = 0$ for all n and $u \equiv 0$, which establishes the announced positivity property of $\tilde{Y}^+$. This immediately implies that the operator $\tilde{Y}^+$ is one-to-one on $\mathcal{C}^\infty(\Gamma)$.

Therefore, the underlying operator $T\tilde{Y}^+ + (1/2) - K$ of the (11) with the operator $\tilde{Y}^+$ defined in (12) is also one-to-one on $\mathcal{C}^\infty(\Gamma)$. Let $u \in \mathcal{C}^\infty(\Gamma)\setminus\{0\}$. Hence, $\mathcal{R}e(\langle\tilde{Y}^+u, \nu\times\bar{u}\rangle) > 0$. Using the criterion *(P1)*, $(u, \tilde{Y}^+u) \notin X^-$ and $(u, \tilde{Y}^+u)$ has a component $C^+(u, \tilde{Y}^+u) \neq (0,0)$ in X^+. Therefore, the criterion *(P2)* implies that

$$\mathcal{R}e(\langle\sigma_1^+\mathcal{C}^+(u,\tilde{Y}^+u), \nu\times\overline{\sigma_0^+\mathcal{C}^+(u,\tilde{Y}^+u)}\rangle) > 0.$$

Since $(T\tilde{Y}^+ + (1/2) - K)u = \sigma_0^+\mathcal{C}^+(u,\tilde{Y}^+u)$ and $\sigma_0^+\mathcal{C}^+(u,\tilde{Y}^+u) \neq 0$ thanks to the last inequality, the operator $T\tilde{Y}^+ + (1/2) - K$ is one-to-one on $\mathcal{C}^\infty(\Gamma)$.

Due to the Calderon formula $T^2 = K^2 - 1/4$, if we had simply taken $\tilde{Y}^+ = -2T$, the composition $T\tilde{Y}^+$ and then the underlying operator $T\tilde{Y}^+ + (1)/(2) - K$ in (11) would be perturbations of the identity by operators of order ≤ -1, the operator K being of order -1. It is reasonable to think that this property

is preserved when the admittance $\tilde{Y}^+$ is a localization of the operator $-2T$.

This hypothesis implies that the operator $T\tilde{Y}^+ + (1/2) - K$ is hypoelliptic. Indeed, if $(T\tilde{Y}^+ + (1/2) - K)u = v \in \mathcal{C}^\infty(\Gamma)$, there exists an operator C of order ≤ -1 such that $u = Cu + v$. As $Cu \in H_T^{s+1}(\Gamma)$ if $u \in H_T^s(\Gamma)$, we deduce that $u \in \mathcal{C}^\infty(\Gamma)$.

As the kernel of the operator $T\tilde{Y}^+ + (1/2) - K$ is trivial in $\mathcal{C}^\infty(\Gamma)$, the hypoellipticity of $T\tilde{Y}^+ + (1/2) - K$ enables to conclude that its kernel is also trivial in $H_{\mathrm{T}}^s(\Gamma)$.

Moreover, since every operator of order ≤ -1 is clearly compact in $H_{\mathrm{T}}^s(\Gamma)$, the operator $T\tilde{Y}^+ + (1/2) - K$ is a compact perturbation of the identity in $H_{\mathrm{T}}^s(\Gamma)$. Therefore, the Fredholm theory yields the well-posedness of the (11) in $H_{\mathrm{T}}^s(\Gamma)$ at all frequencies.

However, it is difficult to prove in the general case that the composition $T\tilde{Y}^+$ with the operator $\tilde{Y}^+$ defined in (12) is a compact perturbation of the identity in $H_{\mathrm{T}}^s(\Gamma)$. We just treat here a simple example. We can consider the case of a scattering surface composed by several connected components and localize the operator $-2T$ on each of these connected components. In this way, the symbol of $\tilde{Y}^+$ is the same as the one of $-2T$. Then the composition $T\tilde{Y}^+$ is a compact perturbation of the identity in $H_{\mathrm{T}}^s(\Gamma)$ and the (11) is well-posed at all frequencies. The problem we have to face with arbitrary surfaces and localization techniques like (12) is that the truncations χ_n deeply modify the symbolic structure of the operator T on the transition domains where χ_n is not equal to 1. Then the proof of the regularization of T by the operator $\tilde{Y}^+$ defined in (12) requires a finer symbolic analysis which will be given in a future paper.

IV. Discretization of the Boundary Equation

We now focus our attention on the discretization of the boundary integral equation (11). We propose a discretization technique and the numerical results obtained after implementation in an iterative solver are given in the next section. The theoretical justification of this construction will be the subject of a future communication.

Let $(\Gamma_h)_h$ be a family of triangulations of Γ. Let X_h be the well-known space of Rao–Wilton–Glisson (RWG) finite elements [17] with basis functions $\{\phi_k\}_k$. Note that the consistency between the continuous variational problem modeling the exterior boundary value problem for the time-harmonic Maxwell equations and the discrete one, obtained with these finite elements, has been proved in [18], [23].

The $H^0(\Gamma_h)$- and $H_{\mathrm{T}}^0(\Gamma_h)$-inner products are both written

$$\langle u_h, v_h \rangle = \int_{\Gamma_h} u_h \cdot v_h.$$

We also set $H_{\mathrm{div}}^0(\Gamma_h)$ the Hilbert space of tangential fields on Γ_h defined as

$$H_{\mathrm{div}}^0(\Gamma_h) = \left\{ u_h \in H_T^0(\Gamma_h); \overrightarrow{\nabla} \cdot u \in H^0(\Gamma_h) \right\}.$$

The $H_{\mathrm{div}}^0(\Gamma_h)$-inner product is written

$$\langle\langle u_h, v_h \rangle\rangle = \langle u_h, v_h \rangle + \langle \overrightarrow{\nabla} \cdot u_h, \overrightarrow{\nabla} \cdot v_h \rangle.$$

The difficulty with the (11) is the composition of the two pseudo-differential operators $\tilde{Y}^+$ and T. The large number of unknowns makes a straightforward matrix composition impossible at high frequencies. To remedy this problem, we apply fast multipole methods (FMMs) ([2], [19]–[21]). These techniques yield very efficient matrix vector products and are perfectly suited for the discretization of the operator (12).

Another problem is that the image of the $\tilde{Y}^+$ operator is not included in the finite element space X_h and so must be projected in it before applying the operator T. In order to obtain the best possible approximation of this image in X_h, the operator $\tilde{Y}^+$ is represented by its $H_{\mathrm{div}}^0(\Gamma_h)$ Galerkin matrix composed with an, iteratively computed, inverse of the Gram–Schmidt matrix of the degrees of freedom in X_h.

A. Galerkin Matrices

The discretization of the operators involved in (11) is realized by Galerkin method in the following way.

In the sequel we will denote by $[A]$ and $[[A]]$ the Galerkin matrices of the operator A with respect to the $H_{\mathrm{T}}^0(\Gamma_h)$- and $H_{\mathrm{div}}^0(\Gamma_h)$-inner products respectively on the X_h basis functions. The Galerkin matrices of the operators T, $I/2$, and K appearing in (11) are built with the $H_{\mathrm{T}}^0(\Gamma_h)$-inner product

$$\begin{aligned} \left[\frac{1}{2}I\right]_{i,j} &= \left\langle \frac{1}{2}\phi_j, \phi_i \right\rangle \\ [K]_{i,j} &= \langle K\phi_j, \phi_i \rangle \\ [T]_{i,j} &= \langle T\phi_j, \phi_i \rangle. \end{aligned} \tag{17}$$

The discretization of the operator $\tilde{Y}^+$ defined in (12) as a localization of the operator T includes the Galerkin matrix of T built with the $H_{\mathrm{div}}^0(\Gamma_h)$-inner product

$$[[T]]_{i,j} = [T]_{i,j} + \langle \overrightarrow{\nabla} \cdot T\phi_j, \overrightarrow{\nabla} \cdot \phi_i \rangle. \tag{18}$$

The product of divergences can be rewritten, using the Gauss theorem, as

$$\begin{aligned} \langle \overrightarrow{\nabla} \cdot T\phi_j, \overrightarrow{\nabla} \cdot \phi_i \rangle = \sum_{K \in \mathfrak{T}_h} \sum_{a \in \mathfrak{A}(K)} \int_a \left(T\phi_j \cdot n_a^K \right) \overrightarrow{\nabla} \cdot \phi_i \\ - \sum_{K \in \mathfrak{T}_h} \int_K T\phi_j \cdot \overrightarrow{\nabla} \left(\overrightarrow{\nabla} \cdot \phi_i \right) \end{aligned}$$

with $\mathfrak{T}_h$ denoting the triangular elements of the mesh, $\mathfrak{A}_(K)$ the edges of the triangle K and n_a^K the unit normal vector of the edge a directed toward the exterior of K.

Since the divergence of the RWG basis functions is constant over each triangle of the mesh, the last integral vanishes on each triangle K and the matrix of the underlying operator T of the coupling operator is implemented as

$$[[T]]_{i,j} = [T]_{i,j} + \sum_{K \in T_h} \sum_{a \in \mathfrak{A}(K)} \int_a \left(T\phi_j \cdot n_a^K \right) \overrightarrow{\nabla} \cdot \phi_i. \tag{19}$$

The discretization of the operator $\tilde{Y}^+$ also requires the construction of both a covering of the scattering surface Γ by open subsurfaces and a quadratic partition of the unity associated to this covering.

B. Construction of the Localization

The localization technique exposed here is based on a splitting of the scattering surface Γ into height subsurfaces but it can easily be extended to a splitting into $8^N, N = 2, 3, \ldots$, subsurfaces.

Let χ^+ be the function defined on $\mathbb{R}$ according to

$$\chi^+(s) = \begin{cases} 0, & \text{if } s < -\varepsilon \\ 1, & \text{if } s > +\varepsilon \\ \sin\left(\pi \frac{s+\varepsilon}{4\varepsilon}\right), & \text{if } -\varepsilon < s < +\varepsilon. \end{cases} \tag{20}$$

This function can be interpreted as the truncation of order zero of the positive real axis. The truncation for the negative axis is given by $\chi^-(s) = \chi^+(-s)$ and $\{\chi^+, \chi^-\}$ is a quadratic partition of the real axis.

This strategy is generalized in $\mathbb{R}^3$ using a tensor product $\chi^{s_1} \otimes \chi^{s_2} \otimes \chi^{s_3}$ with s_i a sign $+$ or $-$. The family of all these functions creates a quadratic partition of the unity on $\mathbb{R}^3$, and after restriction on Γ. The associated subsurfaces are given by $\{(x, y, z) \in \Gamma; s_1 x > -\varepsilon, s_2 y > -\varepsilon, s_3 z > -\varepsilon\}$.

Discretization of $T\tilde{Y}^+ u$

Let us describe the algorithm used to discretize $T\tilde{Y}^+ u$.

The first step in the discretization of $\tilde{Y}^+ u$ consists in evaluating the vector related to each localization $\chi_n u$. If $u = \sum_p u_p \phi_p \in X_h$, each localization is represented by

$$\chi_n u = \sum_q X_n^q u_q \phi_q$$

with X_n^q the value of the truncation χ_n at the center of the edge associated with the basis function ϕ_q. Then the vector $[\chi_n u]$ is composed by the coefficients $[\chi_n u]_q = X_n^q u_q$ and the matrix vector product $[[T]] \cdot [\chi_n u]$ with the matrix $[[T]]$ defined in (19) can be realized.

Since the image of T is not included in X_h, the vector resulting from the product $[[T]] \cdot [\chi_n u]$ does not represent the amplitude coefficients of a X_h element on the X_h basis functions. In view of doing the matrix vector product of the matrix $[T]$ defined in (17) with this vector, $[[T]] \cdot [\chi_n u]$ must be converted in an amplitude vector V_n on X_h. The evaluation of this vector is achieved via the resolution of the linear system

$$[[I]] \cdot V_n = [[T]] \cdot [\chi_n u] \tag{21}$$

where $[[I]]$ is the Galerkin matrix of the identity with respect to the $H^0_{\text{div}}(\Gamma_h)$-inner product. That can be interpreted as the $H^0_{\text{div}}(\Gamma_h)$-projection of $T\chi_n u$ on X_h. Since the matrix $[[I]]$ is sparse, the system (21) is simply solved iteratively.

The discretization of $\tilde{Y}^+ u$ is given after localizing the vector V_n on the nth subsurface and summing the contributions of each subsurface.

Then we can do the matrix vector product with the matrix $[T]$ defined in (17) and the vector resulting from the discretization of $\tilde{Y}^+ u$ stated above.

V. Numerical Results

Some numerical results are reported concerning the resolution of the linear system deriving from the discretization of (11) described in the previous section. They are compared with the results obtained for a discretization of the classical formulations CFIE and EFIE. The linear systems are solved via the generalized conjugate residual (GCR) algorithm [22]. The iterations are stopped when the residue

$$(\text{residue})_n = \frac{|AU_n - S|}{|S|}$$

is less than 10^{-4}, with S the vector composed by the coefficients $S_i = \langle E_0, \phi_i \rangle$, A the matrix of the linear system and U_n the nth iterate.

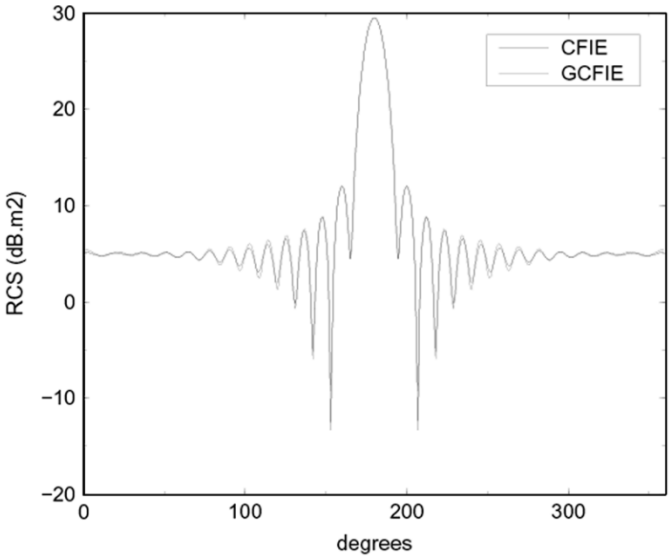

Fig. 1. RCS curves in the plane $(k, \mathbf{E}^i)$ for the PEC sphere.

A. Validation

The validation of the proposed discretization is achieved by comparing the bistatic radar cross section (RCS) produced by the algorithms modeling the GCFIE obtained with the coupling coefficient defined in (12) and the CFIE. The PEC surface used is a sphere of radius 1 m centered at the origin and meshed with 12 288 degrees of freedom (see Fig. 3). The incoming electromagnetic wave $(\mathbf{E}^i, \mathbf{H}^i)$ is a plane wave of frequency 800 MHz. We evaluate the total RCS on two orthogonal planes: the first one spaned by $\overrightarrow{k}$ and $\mathbf{E}^i$, the second one by $\overrightarrow{k}$ and $\mathbf{H}^i$. Figs. 1 and 2 allow to conclude that the fields calculated by iterative resolution of either a CFIE or a GCFIE are similar. We now have to investigate the numerical performances of the linear system deriving from the new formulation.

B. Numerical Performances

We compare the numerical performances for the linear systems associated with the four following formulations: the EFIE, the CFIE, the GCFIE (11) with $\tilde{Y}^+ = -2T$ (denoted by GCFIE1) and the GCFIE (11) with $\tilde{Y}^+$ defined in (12) (denoted by GCFIE2).

Three PEC surfaces (see Figs. 3–5) are considered for the numerical experiments. The first one is a sphere of radius 1 m centered at the origin. The second one is obtained by symmetry

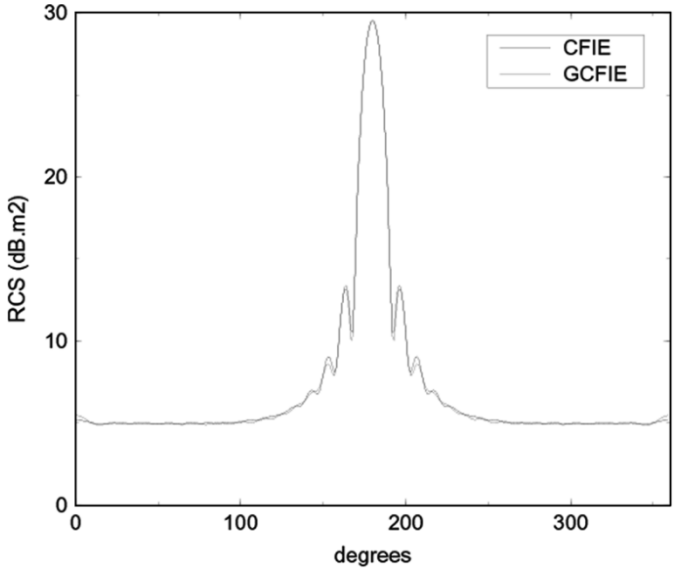

Fig. 2. RCS curves in the plane $(k, \mathbf{H}^i)$ for the PEC sphere.

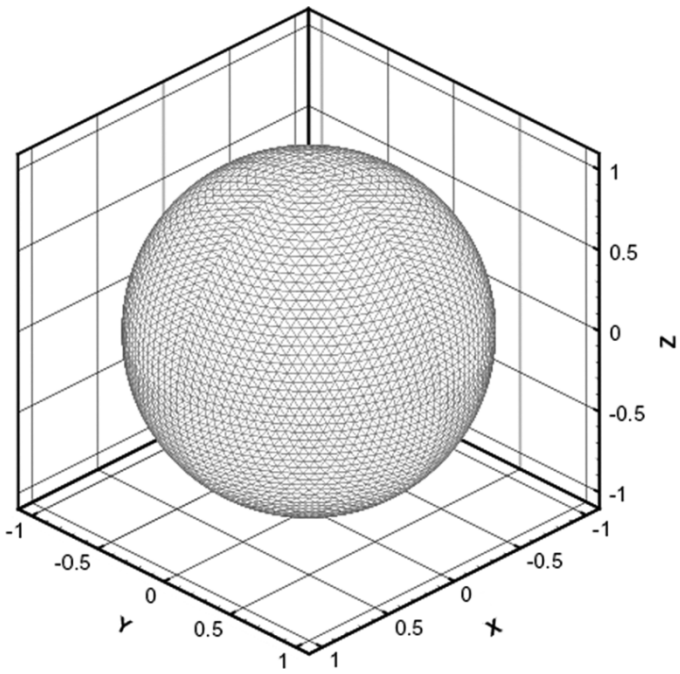

Fig. 3. Mesh of the sphere with 12288 degrees of freedom.

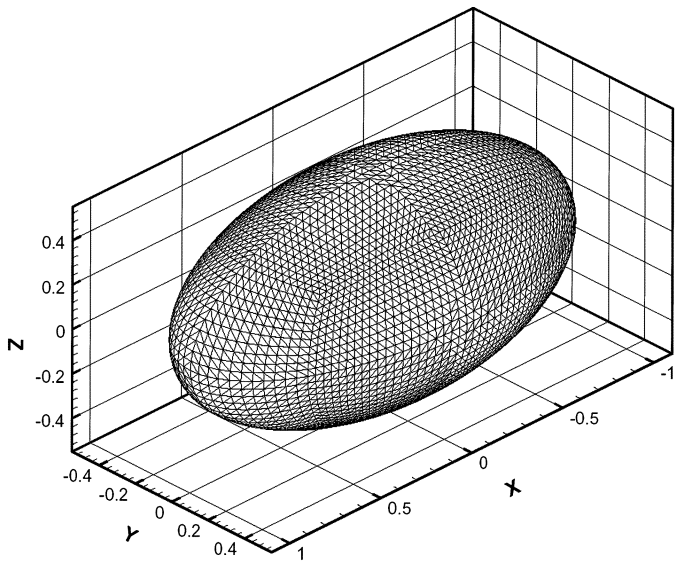

Fig. 4. Mesh of the ellipsoid with 13200 degrees of freedom.

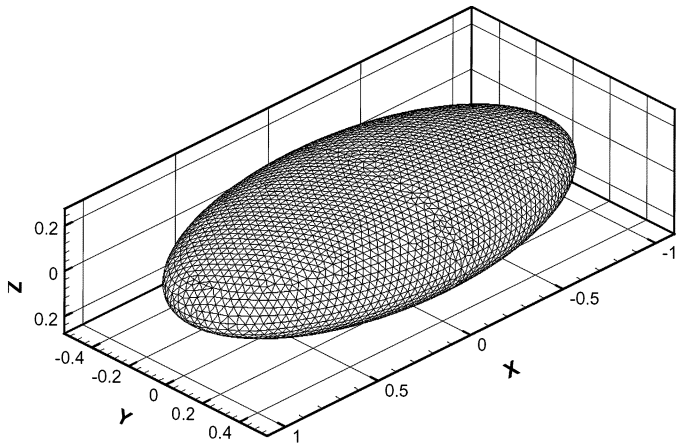

Fig. 5. Mesh of the almond with 12000 degrees of freedom.

of revolution around the big axis of an ellipse with big axis of length 2 m and small axis of length 1 m. The third one is an almond with axes of length 2, 1, and 0.5 m. (We warn the reader that this is not the usual NASA almond with a singularity in one tip.)

We solve the linear systems associated with these formulations for incoming plane waves of different frequencies. To each frequency corresponds a mesh of Γ for which the size of the edges is of order $\lambda/6$ with λ the wave length. The matrix vector products appearing in the GCR algorithm have been compressed thanks to FMM ([2], [19]–[21]). Using the same vocabulary as in [21], the FMM level -1 only involves one cube, the initial one including the surface Γ. The level 0 is obtained dividing the cube of the level -1 into 8 new cubes. This process is reproduced to obtain the levels of higher order. At the frequency 100 MHz, we use the FMM level 0 (8 cubes), at 200 MHz the level 1 (64 cubes), at 400 MHz and 800 MHz the level 2 (512 cubes).

The first test compares the iterative solver performance for the formulations EFIE, CFIE, GCFIE1, and GCFIE2. In Figs. 6–8 are drawn the convergence curves for each formulation on the three surfaces meshed with about 12 000–13 000 degrees of freedom at the frequency 800 MHz. The curves represent the GCR residual with respect to iterations. The three figures show an extremely slow convergence of the EFIE, which is already well-known. The convergence of the equation GCFIE1 is almost the same as the one of the EFIE and is much slower than the one of the GCFIE2, which confirms the need of a localized coupling operator. Moreover the equation GCFIE2 converges even faster than the CFIE. To obtain a residual of order 10^{-4}, it requires half iterations than the CFIE on the three meshes.

The second test evaluates the condition number, defined as the ratio of the largest to smallest eigenvalues, of the matrices representing the EFIE, CFIE, GCFIE1, and GCFIE2 operators.

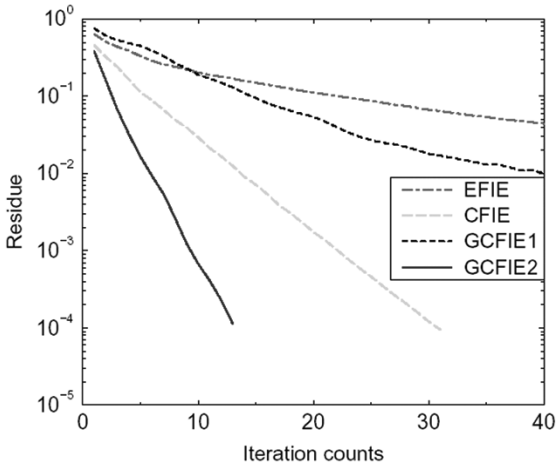

Fig. 6. Convergence curves for the sphere with 12 288 degrees of freedom.

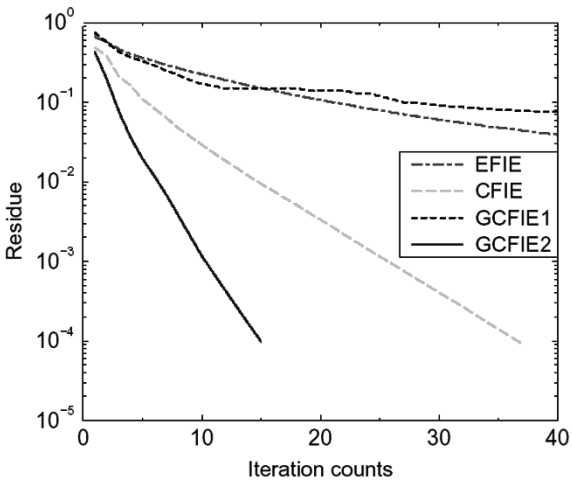

Fig. 7. Convergence curves for the ellipsoid with 13 200 degrees of freedom.

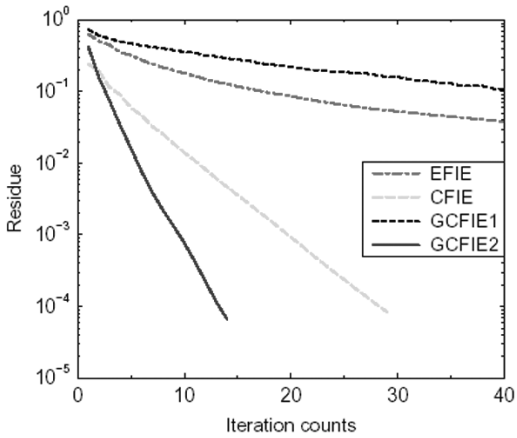

Fig. 8. Convergence curves for the almond with 12 000 degrees of freedom.

These eigenvalues are numerically evaluated thanks to the power method. The scattering surface used is the sphere of radius 1 m. Table I contains the condition number of each matrix for increasing frequencies of the incoming plane wave. Since the iterative evaluation of the smallest eigenvalue of a matrix by the power method requires the inversion of this matrix at each iteration, and since the iterative inversion of the EFIE matrix is performed with very large iteration counts, the condition number of the EFIE matrix is only reported for the two first frequencies. We notice that the condition number of the EFIE is very large even for the lowest frequencies. Contrarily to the GCFIE2, the condition number of the GCFIE1 matrix rapidly grows up with the frequency increase. This proves that the localization of the coupling operator enables to stabilize the condition number of the deriving matrix. Finally, the condition number of the GCFIE2 matrix stabilizes at about 2.4, whereas the condition number of the CFIE matrix continues to grow up. Then the GCFIE2 matrix is better-conditioned than the CFIE matrix.

TABLE I
CONDITION NUMBERS FOR THE SPHERE

Degrees of freedom	192	768	3072	12288
Frequency (MHz)	100	200	400	800
GCFIE1	3.09	2.64	6.28	32.16
GCFIE2	3.60	2.12	2.39	2.41
CFIE	3.28	4.06	5.19	7.05
EFIE	18.49	21.31	×	×

TABLE II
ITERATION COUNTS FOR THE SPHERE

Degrees of freedom	192	768	3072	12288
Frequency (MHz)	100	200	400	800
GCFIE1	10	18	35	93
GCFIE2	11	11	12	14
CFIE	30	21	26	31
EFIE	30	77	181	>200

TABLE III
ITERATION COUNTS FOR THE ELLIPSOID

Degrees of freedom	240	726	3234	13200
Frequency (MHz)	100	200	400	800
GCFIE1	13	25	35	147
GCFIE2	21	18	14	15
CFIE	21	31	31	37
EFIE	53	169	179	>200

TABLE IV
ITERATION COUNTS FOR THE ALMOND

Degrees of freedom	204	768	3012	12000
Frequency (MHz)	100	200	400	800
GCFIE1	17	32	37	98
GCFIE2	22	30	14	14
CFIE	23	39	29	29
EFIE	52	196	>200	>200

The third test studies the influence of the frequency increase on the convergence of the formulations EFIE, CFIE, GCFIE1 and GCFIE2. Tables II–IV, list for each scattering surface the number of iterations required to reach a residual of order 10^{-4} for the four studied frequencies. First of all, we show that, contrarily to the GCFIE1, the number of iterations requiered for the GCFIE2 stabilizes when the frequency grows up. This proves the relevance of the localization of the coupling operator. In all cases, the GCFIE2 results are better than the one of the EFIE which is the formulation the most affected by the frequency increase. On the meshes of the sphere, the GCFIE2 always requires half iterations than the CFIE to reach a residual of order 10^{-4}. On the meshes of the ellipsoid and the almond, its performance is equal to the one of the CFIE at the lowest frequency.

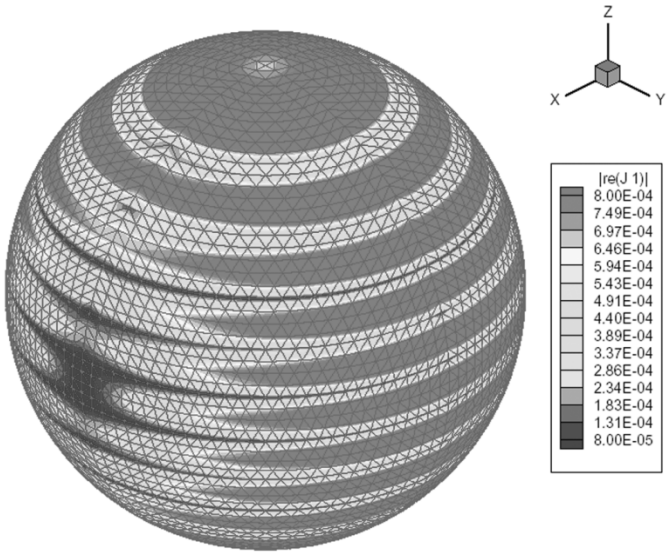

Fig. 9. Real part of the tangential trace $-\nu \times \mathbf{E}^i$.

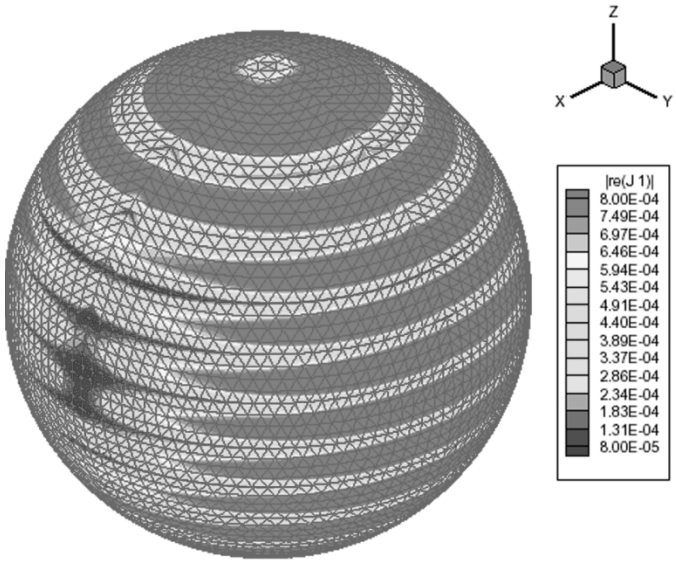

Fig. 10. Real part of the current solution of the GCFIE2.

For the other studied frequencies, the GCFIE2 convergence is clearly faster than the CFIE. In fact, the residual of order 10^{-4} is attained for the GCFIE2 on these geometries with two times less iterations than for the CFIE at the highest frequencies. Moreover this test indicates that the approximations related to the FMM formulations, which generate additional errors, do not perturb the stabilization process working in the GCFIE2 formulation.

In Figs. 9 and 10 are visualized respectively the real part of the tangential trace of the incoming electric field $-\nu \times \mathbf{E}^i$ and the real part of the current solution of the GCFIE2 on the sphere with 12 288 degrees of freedom at 800 MHz. We have previously noticed that the underlying operator of the GCFIE2 would be the identity if $\tilde{Y}^+$ was the exact admittance of the scattering obstacle. Then the solution of this limit formulation would be the tangential trace $-\nu \times \mathbf{E}^i$ of the incoming electric field. Therefore, the similarity between the current solution and this trace enables to evaluate the quality of the approximation $\tilde{Y}^+$ of the exact admittance. We observe that the real part of the current solution has almost the same structure as the real part of the trace of the incoming field. This similarity proves that the proposed construction of $\tilde{Y}^+$ is relevant.

The drawback of the proposed discretization scheme is the computational cost generated by the Gram–Schmidt inversions. In fact, at each iteration of the resolution algorithm, N iterative inversions of the Gram–Schmidt matrix need to be performed for a splitting of the scattering surface into N subsurfaces. Moreover, the localization of the scattering phenomena becomes finer and finer with the frequency increase, allowing a splitting in a growing number of subsurfaces. These subsurfaces become closer and closer to their tangential plane and then the approximation of the admittance improves. Nevertheless, increasing the number of subsurfaces also increases the number of Gram–Schmidt inversions. Although the Gram–Schmidt matrix is sparse, the computational cost related to these inversions becomes prohibitively big. An alternative discretization scheme, which does not contain any projection and, therefore, which reduces the computational cost, will be detailed in a future paper.

VI. Conclusion

We have given a new boundary integral equation for the electromagnetic scattering by sufficiently smooth perfectly conducting closed surfaces in 3-D. The equation combines the magnetic field integral operator and the electric one composed with an approximation of the admittance of the scattering surface. The technique of localization of the admittance with a quadratic partition of the unity is for the first time considered in the electromagnetic framework and in 3-D. The exact admittance is approximated by the localization of the admittance of the infinite plane. This approach can easily be incorporated in fast solvers using FMM. The linear systems deriving from this new formulation clearly have a better behavior than the one associated with the other classical integral formulations tested when solved with iterative techniques. The numerical results show that these systems are better conditioned which leads to a speed up of the iterative convergence.

References

[1] F. Canning, "Improved impedance matrix localization method," *IEEE Trans. Antennas Propag.*, vol. 41, no. 5, pp. 659–667, May 1993.
[2] V. Rokhlin, "Rapid solution of integral equations of classical potential theory," *J. Comp. Phys.*, vol. 60, no. 2, pp. 187–207, 1985.
[3] M. Benzi and M. Tuma, "A sparse approximate inverse preconditioner for nonsymetric linear systems," *SIAM J. Sci. Comput.*, vol. 19, no. 3, pp. 968–994, 1998.
[4] W. McLean and T. Tran, "A preconditioning strategy for boundary element Galerkin methods," *Numer. Methods Partial Differential Equations (13)*, pp. 283–301, 1997.
[5] O. Steinbach and W. L. Wendland, "The construction of some efficient preconditioners in the boundary element method," *Adv. Comput. Math.*, vol. 9, no. 1–2, pp. 191–216, 1998.
[6] R. J. Adams and G. S. Brown, "Stabilization procedure for electric field integral equation," *Electron. Lett.*, vol. 35, no. 23, 1999.
[7] S. Christiansen, "A preconditioner for the electric field integral equation based on Calderon formulas," *SIAM J. Numer. Anal.*, vol. 40, no. 3, pp. 1100–1135, 2002.
[8] M. Cessenat, *Mathematical Methods in Electromagnetism, Linear Theory and Applications*. River Edge, NJ: World Scientific, 1996.
[9] D. P. Levadoux, Proposition de préconditionneurs pour léquation CFIE de lélectromagnétisme, in *Math. Model. Numer. Anal.*, to be published.

[10] D. P. Levadoux, "Etude d'uneéquation intégrale adaptée à la résolution hautes fréquences de l'équation de Helmholtz," Ph.D. dissertation, University Paris-VI, Paris, France, 2001.
[11] X. Antoine and M. Darbas, "Alternative integral equations for the iterative solution of acoustic scattering problems," *Quart. J. Mech. Appl. Math.*, 2005, to be published.
[12] H. Contopanagos, B. Dembart, M. Epton, J. Ottusch, V. Rokhlin, J. Visher, and S. Wandzura, "Well-conditioned boundary integral equations for three-dimensional electromagnetic scattering," *IEEE Trans. Antennas Propag.*, vol. 50, no. 12, pp. 1824–1830, Dec. 2002.
[13] Z. Liu, R. J. Adams, and L. Carin, "Well-conditioned MLFMA formulation for closed PEC targets in the vicinity of a half space," *IEEE Trans. Antennas Propag.*, vol. 51, no. 10, pp. 2822–2829, Oct. 2003.
[14] F. Collino and B. Després, "Integral equations via saddle point problems for time-hamonic Maxwell's equations," *J. Comp. Appl. Math.*, vol. 150, pp. 157–192, 2003.
[15] D. P. Levadoux and B. L. Michielsen, "A new class of integral equations for scattering problems," presented at the 6th Int. Conf. Mathematical and Numerical Aspects of Wave Propagation, Jyvaskyla, Finland, Jun. 30–Jul. 4 2003.
[16] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*. Germany: Springer-Verlag, 1992.
[17] S. Rao, D. Wilton, and A. Glisson, "Electromagnetic scattering by surfaces of arbitrary shape," *IEEE Trans. Antennas Propag.*, vol. AP-30, no. 3, pp. 409–418, May 1982.
[18] A. Bendali, "Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element method; Part 1: The continuous problem," *Math. Comp.*, vol. 43, no. 167, pp. 29–46, 1984.
[19] W. C. Chew, J. M. Lin, C. C. Lu, E. Michielssen, and J. M. Song, "Fast solution methods in electromagnetics," *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 533–543, Mar. 1997.
[20] E. Darve, "The fast multipole method: Numerical implementation," *J. Comput. Phys.*, vol. 160, pp. 195–240, 2000.
[21] J. Simon, "Extension des méthodes multipoles rapides: résolution pour des seconds membres multiples et applications aux objets diélectriques," Ph.D. dissertation, Versailles Saint-Quentin-en-Yvelines University, France, 2003.
[22] S. C. Eisenstat, H. C. Elman, and M. H. Schultz, "Variational iteration methods for non symmetric systems of linear equations," *SIAM J. Num. Anal.*, vol. 20, pp. 345–357, 1983.
[23] A. Bendali, "Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element method; Part 2: The discrete problem," *Math. Comp.*, vol. 43, no. 167, pp. 47–68, 1984.

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