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A COERCIVE COMBINED FIELD INTEGRAL EQUATION FOR ELECTROMAGNETIC SCATTERING*

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Abstract. Many boundary integral equation methods used in the simulation of direct electromagnetic scattering of a time-harmonic wave at a perfectly conducting obstacle break down when applied at frequencies close to a resonant frequency of the obstacle. A remedy is offered by special indirect boundary element methods based on the so-called combined field integral equation. However, hitherto no theoretical results about the convergence of discretized combined field integral equations have been available.

In this paper we propose a new combined field integral equation, convert it into variational form, establish its coercivity in the natural trace spaces for electromagnetic fields, and conclude existence and uniqueness of solutions for any frequency. Moreover, a conforming Galerkin discretization of the variational equations by means of div_Γ-conforming boundary elements can be shown to be asymptotically quasi-optimal. This permits us to derive quantitative convergence rates on sufficiently fine, uniformly shape-regular sequences of surface triangulations.

Key words. electromagnetic scattering, combined field integral equations, coercivity, boundary element methods, Galerkin scheme

AMS subject classifications. 65N30, 65N38, 83N50

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1. Introduction.

The numerical simulation of direct scattering at a perfect conductor, the so-called scatterer, is a central task in computational electromagnetism. The scatterer occupies a bounded domain $\Omega \subset \mathbb{R}^3$. In general, Ω will have Lipschitz-continuous boundary $\Gamma := \partial\Omega$, which can be equipped with an exterior unit normal vector field $\mathbf{n} \in \boldsymbol{L}^\infty(\Gamma)$. With boundary element methods in mind, we do not lose generality by admitting only scatterers Ω that are polyhedra with flat faces and a Lipschitz-continuous boundary. We emphasize that the extension of the results to curvilinear faces is straightforward.

Electromagnetic waves propagate outside the scatterer in the "air region" $\Omega' := \mathbb{R}^3 \setminus \Omega$. From an electrodynamic point of view, Ω' is filled with a homogeneous, isotropic, and linear material. Excitation is provided by the electric field $\mathbf{e}_i$ of an incident (plane) wave of angular frequency $\omega > 0$. Hence, we can switch to the frequency domain and are left with complex amplitudes (phasors) as unknown spatial functions. After suitable scaling, the complex amplitude $\mathbf{e}$ of the scattered field satisfies the following exterior Dirichlet problem for the *electric wave equation* [23, Chap. 6]:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{e} - \kappa^2\mathbf{e} = 0 \quad \text{in } \Omega', \tag{1.1}$$

$$\mathbf{e} \times \mathbf{n} = g := \mathbf{e}_i \times \mathbf{n} \quad \text{on } \Gamma. \tag{1.2}$$

The constant $\kappa := \omega\sqrt{\epsilon_0\mu_0} > 0$ is called the *wave number*, because $\kappa/2\pi$ tells us the number of wavelengths per unit length. Henceforth, κ will stand for a fixed

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positive wave number. These equations have to be supplemented with the *Silver–Müller radiation conditions*

$$\int_{\partial B_r} \left|\mathbf{curl}\,\mathbf{e}\times\mathbf{n} + i\kappa(\mathbf{n}\times\mathbf{e})\times\mathbf{n}\right|^2 \,\mathrm{d}S \to 0 \quad \text{for } r\to\infty, \tag{1.3}$$

where B_r is a ball around 0 with radius $r > 0$. Existence and uniqueness of solutions of (1.1) and (1.3) can be inferred from Rellich's lemma [18, 42].

Integral equation methods are a natural choice for the discretization of the direct electromagnetic scattering problem, which is posed on an unbounded domain. Prominent examples are the electric field integral equation (EFIE) and magnetic field integral equation; see [42, sect. 5.6] or [18, Chap. 3]. These indirect methods display a worrisome instability when κ^2 coincides with a Dirichlet or Neumann eigenvalue (resonant frequency) of the $\mathbf{curl}\,\mathbf{curl}$-operator inside Ω; then the integral equation may not have a solution. After discretization this manifests itself in extreme ill-conditioning of the resulting linear systems of equations if κ is close to a resonant frequency [20].

Two classes of integral equation methods are known to avoid this difficulty. The first is the method proposed in [26] and examined for electromagnetism in [30]. However, it entails constructing an auxiliary surface and can be haunted by stability problems, too. The second, vastly more popular class of methods are approaches based on combined field integral equations (CFIEs), as introduced in [3] and [17]. A particular representative will be the focus of this paper.

CFIEs owe their name to the presence of both single and double layer potentials in the ansatz for the electric field in Ω'. As a theoretical tool they were pioneered for acoustic scattering in [3] and for electromagnetics in [43]. These methods are widely used in computational electromagnetism [47]. For acoustics, existence and uniqueness of solutions can be shown for smooth scatterers [23]. Yet, in the case of electromagnetism even this remains elusive. Hence, mainly for the sake of theoretical treatment, regularized formulations have been introduced by Kress in [36]. However, the idea is applicable only for scattering at smooth objects, and it is not suitable for numerical implementation.

In this article we hark back to the idea of regularization in a different way. Based on recent advances in the understanding of boundary integral operators of electromagnetic scattering achieved in [15, 32, 33], we apply regularization to the double layer part of the integral operator. Reformulation as a mixed problem and subsequent Galerkin discretization pave the way to a practical computational scheme. It is the first method based on an electromagnetic CFIE that can be proven to converge quasi-optimally in relevant trace norms. Related techniques for the Helmholtz equation of acoustic scattering are covered in [14].

The developments in this paper rest on a huge body of previous work. We will restate the most important results. However, in order to maintain a reasonable length we cannot elaborate on most of the existing theory of boundary integral equations for electromagnetic scattering. However, we will try to give comprehensive references for all results we rely upon.

The plan of the paper is as follows: the next section will give a concise survey of relevant function spaces and trace theorems and prove some new results which are needed in the rest of the paper. Then we briefly recall the crucial integral operators of electromagnetic scattering. In the fourth section we will present and analyze the new CFIE and the variational problem associated with it. The fifth section will be devoted to proving the asymptotic quasi optimality of a Galerkin discretization. Based on it, the final section will give quantitative convergence estimates.

2. Function spaces and traces. Let $\Omega \subseteq \mathbb{R}^3$ be any of the sets $\Omega, \Omega', \mathbb{R}^3$. We define the Fréchet space $\boldsymbol{L}^2_{\mathrm{loc}}(\Omega) = \{\mathbf{u}_{|\Omega} : \mathbf{u} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)\}$, where $\boldsymbol{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ is the space of complex, vector-valued, locally square-integrable functions on $\mathbb{R}^3$. In a similar way, we define the Sobolev spaces $\boldsymbol{H}^s_{\mathrm{loc}}(\Omega)$, $s \geq 0$ (see, e.g., [1] for definitions), with the convention $\boldsymbol{H}^0 \equiv \boldsymbol{L}^2$. The subscript $_{\mathrm{loc}}$ will be dropped whenever Ω is bounded: in this case, $\boldsymbol{H}^s(\Omega)$ is a Hilbert space endowed with the natural graph norm $\|\mathbf{u}\|_{\boldsymbol{H}^s(\Omega)}$ and seminorm $|\mathbf{u}|_{\boldsymbol{H}^s(\Omega)}$, respectively [1]. Parentheses will consistently be used to express inner products.

With D a first order differential operator, for any $s \geq 0$ we define

$$\boldsymbol{H}^s_{\mathrm{loc}}(D, \Omega) := \{\mathbf{u} \in \boldsymbol{H}^s_{\mathrm{loc}}(\Omega) : \; D\mathbf{u} \in \boldsymbol{H}^s_{\mathrm{loc}}(\Omega)\}, \tag{2.1}$$

$$\boldsymbol{H}^s_{\mathrm{loc}}(D0, \Omega) := \{\mathbf{u} \in \boldsymbol{H}^s_{\mathrm{loc}}(\Omega) : \; D\mathbf{u} = 0\}. \tag{2.2}$$

When $s = 0$, we simplify the notation by setting $\boldsymbol{H}^0 = \boldsymbol{H}$. If Ω is bounded, $\boldsymbol{H}^s_{\mathrm{loc}}(D, \Omega)$ is endowed with the graph norm $\|\cdot\|^2_{\boldsymbol{H}^s(D,\Omega)} := \|\cdot\|^2_{\boldsymbol{H}^s(\Omega)} + \|D\cdot\|^2_{\boldsymbol{H}^s(\Omega)}$ and seminorm $|\cdot|^2_{\boldsymbol{H}^s(D,\Omega)} := |\cdot|^2_{\boldsymbol{H}^s(\Omega)} + |D\cdot|^2_{\boldsymbol{H}^s(\Omega)}$. This defines the spaces $\boldsymbol{H}^s(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^s(\mathrm{div}, \Omega)$ and $\boldsymbol{H}^s(\mathbf{curl}\, 0, \Omega)$, $\boldsymbol{H}^s(\mathrm{div}\, 0, \Omega)$, for which [28, Chap. 1] is the main reference.

The integration by parts formulae for the operators **curl** and div suggest that we define the tangential trace mapping $\gamma_{\mathbf{t}} : \mathbf{u} \mapsto \mathbf{u}_{|\Gamma} \times \mathbf{n}$ and the normal component trace $\gamma_{\mathbf{n}} : \mathbf{u} \mapsto \mathbf{u}_{|\Gamma} \cdot \mathbf{n}$. To begin with, they are defined for $\mathbf{u} \in C^\infty(\overline{\Omega})^3$.

The trace theorem for $\boldsymbol{H}^1(\Omega)$ [29, Thm. 1.5.1.1] shows that the tangential trace $\gamma_{\mathbf{t}} : \boldsymbol{C}^\infty(\bar{\Omega}) \mapsto \boldsymbol{L}^\infty(\Gamma)$ and the normal trace $\gamma_{\mathbf{n}} : \boldsymbol{C}^\infty(\bar{\Omega}) \mapsto L^\infty(\Gamma)$ are continuous as mappings $\boldsymbol{H}(\mathbf{curl}; \Omega) \mapsto \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}(\mathrm{div}; \Omega) \mapsto H^{-\frac{1}{2}}(\Gamma)$, respectively. Here, $H^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ are the dual spaces of $H^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma) := (H^{\frac{1}{2}}(\Gamma))^3$, respectively, with respect to the pivot spaces $L^2(\Gamma)$ and $\boldsymbol{L}^2(\Gamma)$. Consequently, the traces can be extended to $\boldsymbol{H}(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}(\mathrm{div}; \Omega)$, respectively. Moreover, if we define the antisymmetric pairing

$$\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle_{\boldsymbol{\tau},\Gamma} := \int_\Gamma (\boldsymbol{\mu} \times \mathbf{n}) \cdot \boldsymbol{\eta} \, \mathrm{d}S, \qquad \boldsymbol{\mu}, \boldsymbol{\eta} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) := \{\mathbf{u} \in (L^2(\Gamma))^3, \, \mathbf{u} \cdot \mathbf{n} = 0\}, \tag{2.3}$$

then we can state the integration by parts formula for the **curl**-operator as [9, sect. 4]

$$\int_\Omega (\mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v}) \, \mathrm{d}\mathbf{x} = \langle \gamma_{\mathbf{t}} \mathbf{v}, \gamma_{\mathbf{t}} \mathbf{u} \rangle_{\boldsymbol{\tau},\Gamma} \, . \tag{2.4}$$

A meaningful strong form of the electric wave equation (1.1) has to rely on yet another space: from the fact that a field $\mathbf{u}$ is a locally square-integrable function satisfying $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \mathbf{u} = 0$ we can conclude that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}$ is locally square-integrable, too. Hence, the space

$$\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega) := \{\mathbf{u} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega), \; \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \in \boldsymbol{L}^2_{\mathrm{loc}}(\Omega)\}$$

will play the role of the natural space for solutions of the electric wave equation with constant coefficients.

Trace spaces for electromagnetic fields are essential for stating the boundary integral equations and, in particular, their variational formulations. The corresponding results on nonsmooth boundaries are fairly recent: we refer the reader to [6, 9, 10] for the treatment of Lipschitz polyhedra. The issue of traces of $\boldsymbol{H}(\mathbf{curl}; \Omega)$ for general Lipschitz domains was settled in [13]. The results are summarized in the survey article [7].

Definition 2.1. *We introduce the Hilbert spaces $\boldsymbol{H}^s_\times(\Gamma) := \gamma_{\mathbf{t}}(\boldsymbol{H}^{s+1/2}(\Omega))$, $s \in (0,1)$, equipped with an inner product that renders $\gamma_{\mathbf{t}} : \boldsymbol{H}^{s+1/2}(\Omega) \mapsto \boldsymbol{H}^s_\times(\Gamma)$ continuous and surjective. For $s = 0$ we set $\mathbf{H}^0_\times(\Gamma) := \mathbf{L}^2_{\boldsymbol{t}}(\Gamma)$. The dual spaces with respect to the pairing $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$ are denoted by $\boldsymbol{H}^{-s}_\times(\Gamma)$.*

Remark 1. *When $s = 1$, the standard trace operator γ fails to map $H^{3/2}(\Omega)$ to $H^1(\Gamma)$, although $H^1(\Gamma)$ is well defined on the boundary Γ. In this case, we adopt the definition $\mathbf{H}^1_\times(\Gamma) := \gamma_{\mathbf{t}}(\gamma^{-1}H^1(\Gamma)^3)$, where γ^{-1} represents any continuous lifting from $H^1(\Gamma)$ to $H^{3/2}(\Omega)$ (see* [35]*).*

Next, we introduce the surface divergence operator div_Γ; cf. [9, sect. 2.1].

Let $\{\Gamma_1,\dots,\Gamma_P\}$, $P \in \mathbb{N}$, stand for the set of open flat faces of Γ, and write Σ_{ij} for the straight edge $\partial\Gamma_j \cap \partial\Gamma_i$. The vector $\boldsymbol{\nu}^{ij}$ lies in the plane of Γ_j, is perpendicular to Σ_{ij}, and points into the exterior of Γ_j. Then for $\mathbf{u} \in C^\infty(\overline{\Omega})$ we set

$$\mathrm{div}_\Gamma \gamma_{\mathbf{t}}\mathbf{u} := \begin{cases} \mathrm{div}_j(\gamma_{\mathbf{t}}\mathbf{u}_{|\Gamma_j}) & \text{on } \Gamma^j, \\ \Big((\gamma_{\mathbf{t}}\mathbf{u}_{|\Gamma_j})\cdot\boldsymbol{\nu}^{ij} + (\gamma_{\mathbf{t}}\mathbf{u}_{|\Gamma_i})\cdot\boldsymbol{\nu}^{ji}\Big)\delta_{ij} & \text{on } \overline{\Gamma^j}\cap\overline{\Gamma^i}, \end{cases} \tag{2.5}$$

where δ_{ij} is the delta distribution (in local coordinates) whose support is the edge $\overline{\Gamma^j}\cap\overline{\Gamma^i}$ and div_j denotes the two-dimensional divergence computed on the face Γ^j. By density, this differential operator can be extended to less regular distributions and, in particular, to functionals in $\boldsymbol{H}^{-\frac{1}{2}}_\times(\Gamma)$. We set

$$\boldsymbol{H}^s_\times(\mathrm{div}_\Gamma,\Gamma) := \{\boldsymbol{\mu} \in \boldsymbol{H}^s_\times(\Gamma),\ \mathrm{div}_\Gamma\boldsymbol{\mu} \in H^s(\Gamma)\}, \qquad s \in [-1/2, 0].$$

Finally, we denote by $\mathbf{curl}_\Gamma$ the operator adjoint to div_Γ with respect to the pairing $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$, i.e.,

$$\langle \mathbf{curl}_\Gamma q, \mathbf{p}\rangle_{\boldsymbol{\tau},\Gamma} = \langle \mathrm{div}_\Gamma \mathbf{p}, q\rangle_{\frac{1}{2},\Gamma}, \quad \mathbf{p} \in \boldsymbol{H}^{-\frac{1}{2}}_\times(\mathrm{div}_\Gamma,\Gamma), \quad q \in H^{\frac{1}{2}}(\Gamma). \tag{2.6}$$

It is known [7, sect. 1.2] that $\mathbf{curl}_\Gamma : H^s(\Gamma) \to \boldsymbol{H}^{s-1}_\times(\Gamma)$ is continuous for every s, $1/2 \le s \le 1$. The spaces just defined turn out to be the desired trace spaces; see [9, Prop. 1.7], [10, Thm. 5.4], and [13, sect. 2].

Theorem 2.2. *The operator $\gamma_{\mathbf{t}} : \boldsymbol{H}(\mathbf{curl};\Omega) \mapsto \boldsymbol{H}^{-\frac{1}{2}}_\times(\mathrm{div}_\Gamma,\Gamma)$ is continuous, is surjective, and possesses a continuous right inverse.*

The following self-duality of the electromagnetic trace space will be the foundation of weak formulations. The result was first given in [10].

Theorem 2.3. *The pairing $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$ can be extended to a continuous bilinear form on $\boldsymbol{H}^{-\frac{1}{2}}_\times(\mathrm{div}_\Gamma,\Gamma)$. With respect to $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$ the space $\boldsymbol{H}^{-\frac{1}{2}}_\times(\mathrm{div}_\Gamma,\Gamma)$ becomes its own dual.*

Piecewise smooth scatterers offer the possibility that some considerations can be done locally on the faces and thus become essentially two-dimensional. To provide a framework for such considerations we introduce the spaces $\boldsymbol{H}^s_\times(\Gamma_j)$, $s \ge 0$, defined locally on a face Γ_j in a straightforward fashion, and we denote by $\boldsymbol{H}^{-s}_\times(\Gamma_j)$, $s \in (0,\frac{1}{2})$, their duals (note that we adopt the notion introduced in [39] and not the one used in [29]).

In addition, we define the localized spaces $\boldsymbol{H}_{\times 0}(\mathrm{div}_\Gamma,\Gamma_j) := \{\mathbf{u} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma_j) : \widetilde{\mathbf{u}} \in \boldsymbol{H}_\times(\mathrm{div}_\Gamma,\Gamma)\}$, where $\widetilde{\ }$ denotes the trivial extension by zero to all of Γ. These spaces will be combined to

$$\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma,\Gamma) := \prod_{j=1}^{P} \boldsymbol{H}_{\times 0}(\mathrm{div}_\Gamma,\Gamma_j).$$

LEMMA 2.4. *The space* $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ *is dense in* $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

Proof. Let us adopt the notation Σ for the skeleton of the polyhedron, that is, the union of all edges Σ_{ij}, $1 \le i, j \le P$. Then we recall that regular functions compactly supported in $\bar{\Omega} \setminus \Sigma$ are dense in $H^1(\Omega)$ [41]. Of course, also the inclusion $\boldsymbol{H}^1(\Omega) \subset \boldsymbol{H}(\mathbf{curl}, \Omega)$ is dense. By continuity of the tangential trace operator $\gamma_{\mathbf{t}}$, we deduce that tangential vector fields in $\boldsymbol{H}_\times^{1/2}(\Gamma)$ compactly supported in $\Gamma \setminus \Sigma$ are dense in $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Since the set of fields in $\boldsymbol{H}_\times^{1/2}(\Gamma)$ compactly supported in $\Gamma \setminus \Sigma$ is a subset of $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$, the statement is proved. $\square$

LEMMA 2.5. *The embedding* $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *is compact.*

Proof. To begin with, since $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \subset \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$, we need to prove only that the injection $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma) \subset \boldsymbol{H}_\times^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is compact.

Let $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$ be a sequence such that $\|\mathbf{u}_n\|_{\boldsymbol{H}_\times(\mathrm{div}_\Gamma,\Gamma)} < 1$ for all n. Then, owing to the compact embedding $\mathbf{L}^2_{\mathbf{t}}(\Gamma) \hookrightarrow \boldsymbol{H}_\times^{-1/2}(\Gamma)$, there exists a subsequence $\mathbf{u}_{n_k}$ of $\mathbf{u}_n$ and a $\mathbf{u} \in \boldsymbol{H}_\times^{-1/2}(\Gamma)$ such that $\mathbf{u}_{n_k} \to \mathbf{u}$ strongly in $\boldsymbol{H}_\times^{-1/2}(\Gamma)$. The operator $\mathrm{div}_\Gamma : \boldsymbol{H}_\times^{-1/2}(\Gamma) \mapsto H^{-3/2}(\Gamma)$ is continuous (see [9] for a proof and the definition of $H^{-3/2}(\Gamma)$). Hence, $\mathrm{div}_\Gamma \mathbf{u}_{n_k} \to \mathrm{div}_\Gamma \mathbf{u}$ strongly in $H^{-3/2}(\Gamma)$.

On the other hand, we also know that $\|\mathrm{div}_\Gamma \mathbf{u}_{n_k}\|_{L^2(\Gamma)} < 1$, which implies that up to extraction of a subsequence $\mathrm{div}_\Gamma \mathbf{u}_{n_k}$ is strongly converging to an element in $H^{-1/2}(\Gamma)$. By uniqueness of the limit, we deduce that $\mathrm{div}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)$, and, up to selecting a subsequence, $\mathbf{u}_{n_k} \to \mathbf{u} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ strongly. $\square$

When we want to examine the convergence of boundary element methods quantitatively, extra smoothness of the functions to be approximated is indispensable. A convenient gauge for smoothness is offered by scales of Sobolev spaces. Again, localization is a handy tool: for any $s > \frac{1}{2}$, we define $\mathbf{H}^s_-(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2_{\mathbf{t}}(\Gamma) : \mathbf{u}_{|\Gamma^j} \in \boldsymbol{H}^s_{\mathbf{t}}(\Gamma^j)\}$. The corresponding space of scalar functions will be denoted by $H^s_-(\Gamma)$. Then, for $s \ge 1$, we set $\boldsymbol{H}^s_\times(\Gamma) := \boldsymbol{H}_\times^{\frac{1}{2}}(\Gamma) \cap \mathbf{H}^s_-(\Gamma)$.

To characterize extra smoothness of traces we resort to the family of Hilbert spaces

$$\boldsymbol{H}^s_\times(\mathrm{div}_\Gamma, \Gamma) := \begin{cases} \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) & \text{if } s = -\frac{1}{2}, \\ \{\boldsymbol{\mu} \in \boldsymbol{H}^s_\times(\Gamma),\ \mathrm{div}_\Gamma \boldsymbol{\mu} \in H^s(\Gamma)\} & \text{if } -\frac{1}{2} < s < \frac{1}{2}, \\ \{\boldsymbol{\mu} \in \boldsymbol{H}^s_\times(\Gamma),\ \mathrm{div}_\Gamma \boldsymbol{\mu} \in H^s_-(\Gamma)\} & \text{if } s \ge \frac{1}{2}. \end{cases}$$

The following trace theorem has been proved in the appendix of [8].

THEOREM 2.6. *Let* $\sigma \in \mathbb{R}$ *be the maximum real number such that* $\{p \in H^1(\Omega) : \Delta p \in L^2(\Omega),\ (\partial_{\mathbf{n}} p)_{|\Gamma} = 0\} \subset H^{1+\sigma'}(\Omega)$ *for all* $\sigma' < \sigma$. *For all* $0 \le s < \min\{\sigma, 1\}$ *the tangential trace mapping* $\gamma_{\mathbf{t}}$ *can be extended to a continuous and surjective mapping* $\gamma_{\mathbf{t}} : \boldsymbol{H}^s(\mathbf{curl}, \Omega) \mapsto \boldsymbol{H}_\times^{s-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, *which possesses a continuous right inverse.*

3. Potentials and integral operators.

Here we define the boundary integral operators relevant for electromagnetic scattering and recall a few of their properties. More details can be found in [42, Chap. 5], [23, Chap. 6], [15, sect. 3], and [34].

DEFINITION 3.1. *A distribution* $\mathbf{e} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega)$ *is called a* Maxwell solution *on some generic domain* Ω, *if it satisfies* (1.1) *in* Ω, *and the Silver–Müller radiation conditions at* ∞ *if* Ω *is not bounded.*

As far as the differential operator $\mathbf{curl}\,\mathbf{curl} - \kappa^2\,\mathrm{Id}$ is concerned, the integration by parts formula (2.4) suggests the distinction between *Dirichlet trace* $\gamma_{\mathbf{t}}$ and *Neumann* *trace* $\gamma_N := \kappa^{-1}\gamma_{\mathbf{t}}\circ\mathbf{curl}$. The trace γ_N can be labelled "magnetic," because it actually retrieves the tangential trace of the magnetic field solution. From the trace theorem, Theorem 2.2, we see that γ_N is meaningful on $\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega\cup\Omega')$.

LEMMA 3.2. *The trace* $\gamma_N : \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega'\cup\Omega) \to \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *is a continuous and surjective operator.*

The integral representation for Maxwell solutions relies on the famous Stratton–Chu representation formula for the electric field in $\Omega\cup\Omega'$ [46]. To state it we rely on the notion of a jump $[\cdot]_\Gamma$ across Γ defined by $[\gamma]_\Gamma := \gamma^+ - \gamma^-$ for some trace γ onto Γ. Here, superscripts $-$ and $+$ tag traces onto Γ from Ω and $\Omega' := \mathbb{R}^3\setminus\bar{\Omega}$, respectively. For notational simplicity, it is also useful to resort to the average $\{\gamma\}_\Gamma = \frac{1}{2}(\gamma^+ + \gamma^-)$. Both operators can be applied only to functions defined in $\Omega\cup\Omega'$.

As elaborated in [23, sect. 6.2], [42, sect. 5.5], and [18, Chap. 3, sect. 1.3.2], any Maxwell solution in $\Omega\cup\Omega'$ satisfies

$$\mathbf{u}(\mathbf{x}) = -\boldsymbol{\Psi}^\kappa_{DL}([\gamma_{\mathbf{t}}]_\Gamma(\mathbf{u}))(\mathbf{x}) - \boldsymbol{\Psi}^\kappa_{SL}([\gamma_N]_\Gamma(\mathbf{u}))(\mathbf{x}), \qquad \mathbf{x}\in\Omega\cup\Omega', \tag{3.1}$$

where we have introduced the (electric) *Maxwell single layer potential*

$$\boldsymbol{\Psi}^\kappa_{SL}(\boldsymbol{\mu})(\mathbf{x}) := \kappa\boldsymbol{\Psi}^\kappa_{\mathbf{A}}(\boldsymbol{\mu})(\mathbf{x}) + \frac{1}{\kappa}\,\mathbf{grad}_{\mathbf{x}}\,\Psi^\kappa_V(\mathrm{div}_\Gamma\boldsymbol{\mu})(\mathbf{x}), \qquad \mathbf{x}\notin\Gamma, \tag{3.2}$$

and the (electric) *Maxwell double layer potential*

$$\boldsymbol{\Psi}^\kappa_{DL}(\boldsymbol{\mu})(\mathbf{x}) := \mathbf{curl}_{\mathbf{x}}\,\boldsymbol{\Psi}^\kappa_{\mathbf{A}}(\boldsymbol{\mu})(\mathbf{x}), \qquad \mathbf{x}\notin\Gamma. \tag{3.3}$$

Here, Ψ^κ_V and $\boldsymbol{\Psi}^\kappa_{\mathbf{A}}$ are the scalar and the vectorial single layer potentials for the Helmholtz kernel $E_\kappa(\mathbf{x}) := \exp(i\kappa|\mathbf{x}|)/4\pi|\mathbf{x}|$, whose integral representation is given by ($\mathbf{x}\notin\Gamma$)

$$\Psi^\kappa_V(\phi)(\mathbf{x}) := \int_\Gamma \phi(\mathbf{y})E_\kappa(\mathbf{x}-\mathbf{y})\,dS(\mathbf{y}), \qquad \boldsymbol{\Psi}^\kappa_{\mathbf{A}}(\boldsymbol{\mu})(\mathbf{x}) := \int_\Gamma \boldsymbol{\mu}(\mathbf{y})E_\kappa(\mathbf{x}-\mathbf{y})\,dS(\mathbf{y}).$$

Both potentials $\boldsymbol{\Psi}^\kappa_{SL}$ and $\boldsymbol{\Psi}^\kappa_{DL}$ are Maxwell solutions; that is, for $\boldsymbol{\mu}\in\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$, they fulfill

$$(\mathbf{curl}\,\mathbf{curl} - \kappa^2\,\mathrm{Id})\boldsymbol{\Psi}^\kappa_{SL}(\boldsymbol{\mu}) = 0, \qquad (\mathbf{curl}\,\mathbf{curl} - \kappa^2\,\mathrm{Id})\boldsymbol{\Psi}^\kappa_{DL}(\boldsymbol{\mu}) = 0 \tag{3.4}$$

off the boundary Γ in a pointwise sense. In addition, they comply with the Silver–Müller radiation conditions.

From the well-known mapping properties of Ψ^κ_V and $\boldsymbol{\Psi}^\kappa_{\mathbf{A}}$ it is easy to get those for $\boldsymbol{\Psi}^\kappa_{SL}$ and $\boldsymbol{\Psi}^\kappa_{DL}$; see, e.g., [15, sect. 3].

THEOREM 3.3. *The following mappings are continuous:*

$$\begin{aligned}
\boldsymbol{\Psi}^\kappa_{SL} &: \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2,\Omega\cup\Omega')\cap\boldsymbol{H}_{\mathrm{loc}}(\mathrm{div}\,0;\Omega\cup\Omega'),\\
\boldsymbol{\Psi}^\kappa_{DL} &: \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2,\Omega\cup\Omega')\cap\boldsymbol{H}_{\mathrm{loc}}(\mathrm{div}\,0;\Omega\cup\Omega').
\end{aligned}$$

The fact that $\mathbf{curl}\circ\boldsymbol{\Psi}^\kappa_{SL} = \kappa\,\boldsymbol{\Psi}^\kappa_{DL}$ and $\mathbf{curl}\circ\boldsymbol{\Psi}^\kappa_{DL} = \kappa\,\boldsymbol{\Psi}^\kappa_{SL}$ implies

$$\gamma^\pm_N\boldsymbol{\Psi}^\kappa_{SL} = \gamma^\pm_{\mathbf{t}}\boldsymbol{\Psi}^\kappa_{DL}, \qquad \gamma^\pm_N\boldsymbol{\Psi}^\kappa_{DL} = \gamma^\pm_{\mathbf{t}}\boldsymbol{\Psi}^\kappa_{SL}. \tag{3.5}$$

This means that the following two *boundary integral operators* are sufficient for electromagnetic scattering:

$$\mathsf{S}_\kappa := \{\gamma_{\mathbf{t}}\}_\Gamma\circ\boldsymbol{\Psi}^\kappa_{SL} = \{\gamma_N\}_\Gamma\circ\boldsymbol{\Psi}^\kappa_{DL}, \qquad \mathsf{C}_\kappa := \{\gamma_{\mathbf{t}}\}_\Gamma\circ\boldsymbol{\Psi}^\kappa_{DL} = \{\gamma_N\}_\Gamma\circ\boldsymbol{\Psi}^\kappa_{SL}.$$

The continuity of S_κ and C_κ is immediate from Theorem 3.3, in conjunction with Lemma 3.2 and Theorem 2.2.

COROLLARY 3.4. *The operators* $\mathsf{S}_\kappa, \mathsf{C}_\kappa : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *are continuous.*

A fundamental tool for deriving boundary integral equations are *jump relations* describing the behavior of the potentials across Γ. For the Maxwell single and double layer potentials they closely resemble those for conventional single and double layer potentials for second order elliptic operators [40, Chap. 6]. For smooth domains these results are contained in [23, Thm. 6.11], [42, Thm. 5.5.1], and [45].

THEOREM 3.5. *The interior and exterior Dirichlet and Neumann traces of the potentials* $\boldsymbol{\Psi}_{SL}^\kappa$ *and* $\boldsymbol{\Psi}_{DL}^\kappa$ *are well defined and, on* $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$*, satisfy*

$$[\gamma_{\mathbf{t}}]_\Gamma \circ \boldsymbol{\Psi}_{SL}^\kappa = [\gamma_N]_\Gamma \circ \boldsymbol{\Psi}_{DL}^\kappa = 0, \qquad [\gamma_N]_\Gamma \circ \boldsymbol{\Psi}_{SL}^\kappa = [\gamma_{\mathbf{t}}]_\Gamma \circ \boldsymbol{\Psi}_{DL}^\kappa = -\,\mathrm{Id}\,.$$

As auxiliary boundary integral operators, which supply building blocks for S_κ and C_κ, we introduce the two single layer boundary integral operators

$$\mathsf{V}_\kappa := \{\gamma\}_\Gamma \circ \Psi_V^\kappa, \qquad \mathsf{A}_\kappa := \{\gamma_{\mathbf{t}}\}_\Gamma \circ \boldsymbol{\Psi}_{\mathbf{A}}^\kappa.$$

By inspecting the potential $\boldsymbol{\Psi}_{SL}^\kappa$ and recalling $\gamma_{\mathbf{t}} \circ \mathbf{grad} = \mathbf{curl}_\Gamma \circ \gamma$, it is clear that we can write

$$\mathsf{S}_\kappa = \kappa\, \mathsf{A}_\kappa + \kappa^{-1} \mathbf{curl}_\Gamma \circ \mathsf{V}_\kappa \circ \mathrm{div}_\Gamma. \tag{3.6}$$

It is easy to see that the bilinear form associated with S_κ is given by

$$\langle \mathsf{S}_\kappa \boldsymbol{\mu}, \boldsymbol{\xi} \rangle_{\boldsymbol{\tau},\Gamma} = \frac{1}{\kappa} \langle \mathrm{div}_\Gamma \boldsymbol{\mu}, \mathsf{V}_\kappa \mathrm{div}_\Gamma \boldsymbol{\mu} \rangle_{\frac{1}{2},\Gamma} - \kappa \langle \boldsymbol{\mu}, \mathsf{A}_\kappa \boldsymbol{\xi} \rangle_{\boldsymbol{\tau},\Gamma}\,. \tag{3.7}$$

Obviously, it involves two parts of different order, neither of which is a compact perturbation of the other. In recent years a very successful approach to variational problems of this kind has emerged; see [31, sect. 5.1], [15], and [8]. The idea is to consider the above bilinear form separately on the components of a suitable splitting

$$\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \boldsymbol{\mathcal{X}}(\Gamma) \oplus \boldsymbol{\mathcal{N}}(\Gamma), \tag{3.8}$$

where $\boldsymbol{\mathcal{N}}(\Gamma) = \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0; \Gamma)$, and $\boldsymbol{\mathcal{X}}(\Gamma) \subset \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is a closed subspace such that

1. the splitting (3.8) is direct, that is, $\boldsymbol{\mathcal{X}}(\Gamma) \cap \boldsymbol{\mathcal{N}}(\Gamma) = \emptyset$;
2. the embedding $\boldsymbol{\mathcal{X}}(\Gamma) \hookrightarrow \boldsymbol{H}_\times^{-\frac{1}{2}}(\Gamma)$ is compact.

Note that div_Γ has closed range in $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, and this implies that

$$\|\boldsymbol{\mu}\|_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} \le C\, \|\mathrm{div}_\Gamma \boldsymbol{\mu}\|_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{X}}(\Gamma).$$

By R^Γ and Z^Γ we denote the projectors onto $\boldsymbol{\mathcal{X}}(\Gamma)$ and $\boldsymbol{\mathcal{N}}(\Gamma)$, respectively, that are associated with the splitting (3.8). Examples of splittings satisfying these requirements are given by the "$\boldsymbol{L}_{\mathbf{t}}^2(\Gamma)$-orthogonal" Hodge decomposition [10] and the "projected regular splitting" [32, sect. 7].

To establish a generalized Gårding inequality for S_κ we employ the direct splitting (3.8) and two auxiliary lemmata; see [33, Lem. 3.2] and [12, Prop. 4.1].

LEMMA 3.6. *The integral operators* $\delta\mathsf{V}_\kappa := \mathsf{V}_\kappa - \mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ *and* $\delta\mathsf{A}_\kappa := \mathsf{A}_\kappa - \mathsf{A}_0 : \boldsymbol{H}_\times^{-\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{H}_\times^{\frac{1}{2}}(\Gamma)$ *are compact.*

LEMMA 3.7. *The operators* V_0 *and* A_0 *are continuous, are self-adjoint with respect to the bilinear pairings* $\langle\cdot,\cdot\rangle_{\frac{1}{2},\Gamma}$ *and* $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$*, respectively, and satisfy*

$$\langle\mu, \mathsf{V}_0\overline{\mu}\rangle_{\frac{1}{2},\Gamma} \geq C\,\|\mu\|^2_{H^{-\frac{1}{2}}(\Gamma)} \qquad \forall\mu \in H^{-\frac{1}{2}}(\Gamma),$$
$$\langle\boldsymbol{\mu}, \mathsf{A}_0\overline{\boldsymbol{\mu}}\rangle_{\boldsymbol{\tau},\Gamma} \geq C\,\|\boldsymbol{\mu}\|^2_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\Gamma)} \qquad \forall\boldsymbol{\mu} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0;\Gamma),$$

with constants $C > 0$ *depending only on* Γ*.*

The main result will be a generalized Gårding inequality for S_κ that involves the isomorphism

$$\mathsf{X}_\Gamma = \mathsf{R}^\Gamma - \mathsf{Z}^\Gamma : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma). \tag{3.9}$$

LEMMA 3.8 (cf. [12, 33]). *There is a compact bilinear form* $c_\Gamma : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \times \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \mathbb{C}$ *and a constant* $C > 0$ *such that*

$$|\langle\mathsf{S}_\kappa\boldsymbol{\mu}, \mathsf{X}_\Gamma\overline{\boldsymbol{\mu}}\rangle_{\boldsymbol{\tau},\Gamma} + c_\Gamma(\boldsymbol{\mu},\overline{\boldsymbol{\mu}})| \geq C\,\|\boldsymbol{\mu}\|^2_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \qquad \forall\boldsymbol{\mu} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma).$$

Remember that S_κ is the integral operator underlying the EFIE. Lemma 3.8 tells us that $\mathsf{S}_\kappa : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ is a Fredholm operator of index 0. This will ensure surjectivity as soon as injectivity holds. However, the very problem of instability at resonant frequencies is due to the failure of S_κ to be injective for certain discrete values of κ; see, e.g., [18], [42], or [15, sect. 5.2].

4. The CFIE.

The CFIEs arise from an indirect approach which aims to exploit that both $\boldsymbol{\Psi}^\kappa_{SL}$ and $\boldsymbol{\Psi}^\kappa_{DL}$ yield Maxwell solutions; see (3.4). The crudest variant starts from the trial expression

$$\mathbf{e} = -i\eta\boldsymbol{\Psi}^\kappa_{SL}(\boldsymbol{\zeta}) - \boldsymbol{\Psi}^\kappa_{DL}(\boldsymbol{\zeta}), \tag{4.1}$$

with some parameter $\eta > 0$. By the jump relations, taking the exterior Dirichlet trace $\gamma_{\mathbf{t}}^+$ results in the boundary integral equation

$$-i\eta\mathsf{S}_\kappa(\boldsymbol{\zeta}) + (\tfrac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa)(\boldsymbol{\zeta}) = \gamma_{\mathbf{t}}^+\mathbf{e}_i, \tag{4.2}$$

which is generically posed in $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$. At least on smooth surfaces the operator $\mathsf{C}_\kappa : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ is compact [42, sect. 5.5] and a generalized Gårding inequality for the sum $-i\eta\mathsf{S}_\kappa + \frac{1}{2}\,\mathrm{Id}$ is available. However, on nonsmooth surfaces C_κ cannot be dismissed as compact perturbation.

The bottom line is that existence of solutions of (4.2) cannot be established on nonsmooth surfaces, let alone any theory about discrete approximations. This dire state led Kress to propose the introduction of a smoothing operator into (4.1) in [36]. His analysis was set in Hölder spaces, and he targeted the single layer potential $\boldsymbol{\Psi}^\kappa_{SL}$, because, working on smooth surfaces, he could rely on the compactness of C_κ.

We cannot make this assumption, but we are aware of Lemma 3.8. This means that the Fredholm operator S_κ is not the problem, but it is the innocent looking identity Id in (4.2). Therefore, Kress's policy should be turned upside down, and regularization has to be aimed at the double layer potential $\boldsymbol{\Psi}^{\kappa}_{DL}$.

The crucial device for regularization is a *compact* "smoothing operator"

$$\mathsf{M} : \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

that satisfies

$$\boldsymbol{\mu} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) : \langle \mathsf{M}\boldsymbol{\mu}, \overline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\tau},\Gamma} > 0 \quad \Leftrightarrow \quad \boldsymbol{\mu} \neq 0.$$

According to the strategy outlined above it will enter the contribution of the double layer potential to the representation formula: we get the trial expression

$$\mathbf{e} = -i\eta \boldsymbol{\Psi}^{\kappa}_{SL}(\boldsymbol{\zeta}) - \boldsymbol{\Psi}^{\kappa}_{DL}(\mathsf{M}\boldsymbol{\zeta}), \tag{4.3}$$

where $\boldsymbol{\zeta} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, $\eta > 0$. By (3.4), this field is a Maxwell solution in $\Omega \cup \Omega'$. As above, the exterior Dirichlet trace applied to (4.3) results in the *new CFIE*

$$-i\eta \mathsf{S}_\kappa(\boldsymbol{\zeta}) + (\tfrac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa)(\mathsf{M}\boldsymbol{\zeta}) = \gamma^{+}_{\mathbf{t}} \mathbf{e}_i. \tag{4.4}$$

Since it is set in $\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, Theorem 2.3 hints at how to cast it into a variational form: find $\boldsymbol{\zeta} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ such that for all $\boldsymbol{\mu} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$,

$$-i\eta \left\langle \mathsf{S}_\kappa(\boldsymbol{\zeta}), \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau},\Gamma} + \left\langle (\tfrac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa)(\mathsf{M}\boldsymbol{\zeta}), \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau},\Gamma} = \left\langle \gamma^{+}_{\mathbf{t}} \mathbf{e}_i, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau},\Gamma}. \tag{4.5}$$

It shares the crucial uniqueness of solutions with other CFIEs.

THEOREM 4.1. *For all $\eta \neq 0$ and wave numbers $\kappa > 0$, the boundary integral equation* (4.5) *has a unique solution $\boldsymbol{\zeta} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.*

Proof. To demonstrate uniqueness, we assume that $\boldsymbol{\zeta} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ solves

$$-i\eta \mathsf{S}_\kappa(\boldsymbol{\zeta}) + (\tfrac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa)(\mathsf{M}\boldsymbol{\zeta}) = 0. \tag{4.6}$$

It is immediate from the jump relations that $\mathbf{e}$ given by (4.3) is an exterior Maxwell solution with $\gamma^{+}_{\mathbf{t}} \mathbf{e} = 0$. By their uniqueness we infer that $\mathbf{e} = 0$ in Ω'. Appealing to the jump relations from Theorem 3.5 once more, we find

$$\gamma^{-}_{\mathbf{t}} \mathbf{e} = -\mathsf{M}\boldsymbol{\zeta}, \qquad \gamma^{-}_{N} \mathbf{e} = -i\eta \boldsymbol{\zeta}.$$

Next, we use (2.4) and see that

$$i\mathbb{R} \ni i\eta \left\langle \boldsymbol{\zeta}, \overline{\mathsf{M}\boldsymbol{\zeta}} \right\rangle_{\boldsymbol{\tau},\Gamma} = \left\langle \gamma^{-}_{N} \mathbf{e}, \overline{\gamma^{-}_{\mathbf{t}} \mathbf{e}} \right\rangle_{\boldsymbol{\tau},\Gamma} = \int_{\Omega} \frac{1}{\kappa} |\,\mathbf{curl}\,\mathbf{e}|^2 \,\mathrm{d}\mathbf{x} - \kappa |\mathbf{e}|^2 \,\mathrm{d}\mathbf{x} \in \mathbb{R}.$$

Necessarily, $\left\langle \boldsymbol{\zeta}, \overline{\mathsf{M}\boldsymbol{\zeta}} \right\rangle_{\boldsymbol{\tau},\Gamma} = 0$ so that the requirements on M imply $\boldsymbol{\zeta} = 0$, which settles the issue of uniqueness.

Next, we know from Corollary 3.4 that $\mathsf{C}_\kappa : \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is continuous so that $(\frac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa) \circ \mathsf{M} : \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ turns out to be compact. Eventually, we conclude from Lemma 3.8 that the bilinear form of (4.5) satisfies a generalized Gårding inequality. Thus, a Fredholm alternative argument gives existence of a solution from its uniqueness. ∎

A simple eligible operator M can be introduced through a variational definition: for $\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and all $\mathbf{q} \in \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$, $\mathsf{M}\boldsymbol{\zeta} \in \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ is to satisfy

$$(\mathsf{M}\boldsymbol{\zeta}, \mathbf{q})_{0;\Gamma} + (\mathrm{div}_\Gamma \mathsf{M}\boldsymbol{\zeta}, \mathrm{div}_\Gamma \mathbf{q})_{0;\Gamma} = \langle \mathbf{q}, \boldsymbol{\zeta} \rangle_{\boldsymbol{\tau},\Gamma}\,, \tag{4.7}$$

where $(\cdot,\cdot)_{0;\Gamma}$ denotes the standard $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ scalar product. It becomes obvious that $\mathsf{M} : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ is a continuous linear operator. To prove injectivity, let $\boldsymbol{\zeta}$ be such that $\mathsf{M}\boldsymbol{\zeta} = 0$, and let $\boldsymbol{\eta} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ be the vector verifying

$$\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle_{\boldsymbol{\tau},\Gamma} = \|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)}.$$

Due to Lemma 2.4, there exists a sequence $\{\boldsymbol{\eta}_\ell\}_{\ell \in \mathbb{N}} \subset \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ converging to $\boldsymbol{\eta}$. Now choosing $\boldsymbol{\eta}_\ell$ as the test function in (4.7) and passing to the limit for $\ell \to \infty$, we obtain $\boldsymbol{\zeta} = 0$. The injectivity of M immediately implies

$$\left\langle \mathsf{M}\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}} \right\rangle_{\boldsymbol{\tau},\Gamma} = \|\mathsf{M}\boldsymbol{\zeta}\|^2_{\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)} > 0 \quad \Leftrightarrow \quad \boldsymbol{\zeta} \neq 0.$$

In addition, M inherits compactness from the embedding $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$; see Lemma 2.5: it meets all requirements listed above.

The composition of the integral operator C_κ and the smoothing operator M in (4.5) is not problematic. However, it cannot be handled in the context of Galerkin discretization, which we intend to apply; we have to find an equivalent weak form that can be discretized easily.

The usual trick to avoid operator products is to switch to a mixed formulation. Here, this amounts to introducing the new unknown $\mathbf{p} := \mathsf{M}\boldsymbol{\zeta}$. If we use the particular smoothing operator from (4.7), we get $\mathbf{p} \in \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ and may simply incorporate (4.7) into the eventual mixed variational problem: find $\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, $\mathbf{p} \in \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ such that for all $\boldsymbol{\mu} \in \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, $\mathbf{q} \in \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$,

$$\begin{array}{rclcl} -i\eta \langle \mathsf{S}_\kappa \boldsymbol{\zeta}, \boldsymbol{\mu} \rangle_{\boldsymbol{\tau},\Gamma} & + & \left\langle (\frac{1}{2}\,\mathrm{Id} - \mathsf{C}_\kappa)\mathbf{p}, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau},\Gamma} & = & \left\langle \gamma_{\mathbf{t}}^+ \mathbf{e}_i, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau},\Gamma}, \\ \langle \mathbf{q}, \boldsymbol{\zeta} \rangle_{\boldsymbol{\tau},\Gamma} & - & (\mathbf{p}, \mathbf{q})_{0;\Gamma} - (\mathrm{div}_\Gamma \mathbf{p}, \mathrm{div}_\Gamma \mathbf{q})_{0;\Gamma} & = & 0. \end{array} \tag{4.8}$$

The next lemma tells us that we need not worry about Id in (4.8).

LEMMA 4.2. *The bilinear forms* $\langle \cdot, \cdot \rangle_{\boldsymbol{\tau},\Gamma}$ *and* / $\langle \mathsf{C}_\kappa \cdot, \cdot \rangle_{\boldsymbol{\tau},\Gamma}$ *are compact as mapping* $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \times \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \mathbb{C}$.

Proof. It is enough to note that $\langle \cdot, \cdot \rangle_{\boldsymbol{\tau},\Gamma}$ / $\langle \mathsf{C}_\kappa \cdot, \cdot \rangle_{\boldsymbol{\tau},\Gamma} : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \times \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \to \mathbb{C}$ are continuous and the injection $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is compact due to Lemma 2.5. ∎

As an immediate consequence of this result we note that the off-diagonal terms in (4.8) represent compact bilinear forms. It remains to investigate the diagonal terms. First, $(\mathbf{p}, \mathbf{q})_{0;\Gamma} + (\mathrm{div}_\Gamma \mathbf{p}, \mathrm{div}_\Gamma \mathbf{q})_{0;\Gamma}$ is clearly elliptic in $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$, because it gives rise to its inner product. Second, the other bilinear form $\langle \mathsf{S}_\kappa \boldsymbol{\zeta}, \boldsymbol{\mu} \rangle_{\boldsymbol{\tau},\Gamma}$ has been found to verify a generalized Gårding inequality; see Lemma 3.8.

Let us summarize what we know about the entire variational problem (4.8). For the sake of brevity we write $\mathfrak{V} := \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \times \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ and denote by $\|\cdot\|_{\mathfrak{V}}$ its natural graph norm. We use the symbols $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \dots$ for pairs of functions in $\mathfrak{V}$. Let $\widetilde{a} : \mathfrak{V} \times \mathfrak{V} \mapsto \mathbb{C}$ be the bilinear form associated with (4.8). As an immediate consequence of the preceding considerations, it will also fulfill a generalized Gårding inequality. It can be stated using the isomorphism

$$\mathbb{X}_\Gamma : \mathfrak{V} \mapsto \mathfrak{V}, \qquad \mathbb{X}_\Gamma \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{q} \end{pmatrix} := \begin{pmatrix} \mathsf{X}_\Gamma \boldsymbol{\mu} \\ \mathbf{q} \end{pmatrix}.$$

COROLLARY 4.3. *There is a compact bilinear form* $\widetilde{c} : \mathfrak{V} \times \mathfrak{V} \mapsto \mathbb{C}$ *and a constant* $C_G > 0$ *such that*

$$|\widetilde{a}(\mathbb{X}_\Gamma \mathfrak{v}, \overline{\mathfrak{v}}) + \widetilde{c}(\mathfrak{v}, \overline{\mathfrak{v}})| \geq C_G \|\mathfrak{v}\|^2_{\mathfrak{V}} \qquad \forall \mathfrak{v} \in \mathfrak{V}.$$

Since we have confirmed the uniqueness of solutions of (4.8), a Fredholm alternative argument shows that $\widetilde{a}$ induces an isomorphism, in particular that the inf-sup condition

$$\sup_{\mathfrak{v} \in \mathfrak{V}} \frac{|\widetilde{a}(\mathfrak{u}, \mathfrak{v})|}{\|\mathfrak{v}\|_{\mathfrak{V}}} \geq C_S \|\mathfrak{u}\|_{\mathfrak{V}} \tag{4.9}$$

holds with $C_S > 0$ independent of $\mathfrak{u} \in \mathfrak{V}$.

REMARK 2. *Many choices of smoothing operators* M *are conceivable. For the following reasons we opted for the definition* (4.7)*.*

First, the operator M *is the inverse of* $-\,\mathbf{grad}_\Gamma \operatorname{div}_\Gamma + \mathrm{Id}$ *with Dirichlet boundary conditions on the skeleton* Σ*. We are anxious to use the inverse of a proper differential operator, because any nonlocal operator in the definition of* M *will be awkward to deal with in an implementation. We also aimed at making* M *local on each face of the polyhedron, which is satisfied by the concrete choice, since surface vector fields* $\boldsymbol{H}_\Sigma(\operatorname{div}_\Gamma, \Gamma)$ *have no flux across any edge in* Σ*.*

Second, we have to take great pains to ensure sufficient regularity of the solution for the new unknown $\mathbf{p}$*. If, in* (4.7)*, we used* $\boldsymbol{H}_\times(\operatorname{div}_\Gamma, \Gamma)$ *trial and test function spaces instead of* $\boldsymbol{H}_\Sigma(\operatorname{div}_\Gamma, \Gamma)$*, then the regularity of* $\mathbf{p}$ *would be impaired, because Laplace–Beltrami singularities* [12, *sect.* 5.2.1] *would sneak into* $\mathbf{p}$ *through the associated smoothing operator. We are going to resume the discussion at the end of section* 6*.*

5. Galerkin discretization. We equip the piecewise smooth compact two-dimensional surface Γ with an oriented triangulation Γ_h. This means that all its edges are endowed with a direction. We assume a perfect resolution of Γ; that is, $\Gamma = \bar{K}_1 \cup \cdots \cup \bar{K}_N$, where $\mathcal{K}_h := \{K_1, \ldots, K_N\}$ is the set of mutually disjoint open cells of Γ_h. Moreover, no cell may straddle boundaries of the smooth faces Γ^j of Γ. We will admit triangular and quadrilateral cells only: for each $K \in \mathcal{K}_h$ there is a diffeomorphism $\boldsymbol{\Phi}_K : \widehat{K} \mapsto \bar{K}$, where $\widehat{K}$ is the "unit triangle" or unit square in $\mathbb{R}^2$, depending on the shape of K [19, sect. 5].

This paves the way for a parametric construction of boundary elements: to begin with, choose finite-dimensional local spaces $\boldsymbol{\mathcal{W}}(\widehat{K}) \subset (C^\infty(\widehat{K}))^2$ of polynomial vector fields, together with a dual basis of so-called local degrees of freedom (d.o.f.). Possible choices for $\boldsymbol{\mathcal{W}}(\widehat{K})$ and related d.o.f. abound [5, Chap. III]: we may use the classical triangular Raviart–Thomas (RT_p) elements of polynomial order $p \in \mathbb{N}_0$ [44] that use

$$\boldsymbol{\mathcal{W}}(\widehat{K}) := \{\mathbf{x} \mapsto \mathbf{p}_1(\mathbf{x}) + p_2(\mathbf{x}) \cdot \mathbf{x},\ \mathbf{x} \in \widehat{K},\ \mathbf{p}_1 \in (\mathcal{P}_p(\widehat{K}))^2,\ p_2 \in \mathcal{P}_p(\widehat{K})\},$$

where $\mathcal{P}_p(\widehat{K})$ is the space of two-variable polynomials of total degree $\leq p$. Possible alternatives are the triangular BDM_p elements of degree p [4], $p \in \mathbb{N}_0$, which rely on $\boldsymbol{\mathcal{W}}(\widehat{K}) := (\mathcal{P}_{p+1}(\widehat{K}))^2$. In both cases, the usual d.o.f. involve certain polynomial moments of normal components on edges, together with interior vectorial moments for $p > 0$. For instance, in the case of RT_0, edge fluxes are the appropriate d.o.f.:

$$\boldsymbol{\mu}_h \in \boldsymbol{\mathcal{W}}(\widehat{K}) \mapsto \int_{\widehat{e}} \boldsymbol{\mu}_h \cdot \widehat{\mathbf{n}} \, \mathrm{d}S, \qquad \widehat{e} \text{ edge of } \widehat{K}.$$

Similar local spaces and d.o.f. are available for the unit square.

Using the pullback of 1-forms the local spaces can be lifted to the cells of Γ_h. In terms of vector fields this is equivalent to the *Piola transformation*

$$(\mathfrak{F}_K \boldsymbol{\mu})(\mathbf{x}) := \sqrt{\det(\mathrm{G})}\, \mathrm{G}^{-1}\, D\boldsymbol{\Phi}_K^T(\widehat{\mathbf{x}})\boldsymbol{\mu}(\widehat{\mathbf{x}}), \tag{5.1}$$

where $\mathrm{G} := D\boldsymbol{\Phi}(\widehat{\mathbf{x}})^T D\boldsymbol{\Phi}(\widehat{\mathbf{x}})$, $\mathbf{x} = \boldsymbol{\Phi}_K(\widehat{\mathbf{x}})$, $\widehat{\mathbf{x}} \in \widehat{K}$. Thus, we can introduce the global boundary element space

$$\boldsymbol{\mathcal{W}}_h := \{\boldsymbol{\mu} \in \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma) : \boldsymbol{\mu}_{|K} \in \mathfrak{F}_K(\boldsymbol{\mathcal{W}}(\widehat{K}))\, \forall K \in \mathcal{K}_h\}. \tag{5.2}$$

In practice, $\boldsymbol{\mathcal{W}}_h \subset \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$ is ensured by a suitable choice of d.o.f. Remember that d.o.f. have to be associated with individual edges of $\widehat{K}$ or the interior of $\widehat{K}$. It is crucial that the normal component of any $\widehat{\boldsymbol{\mu}}_h \in \boldsymbol{\mathcal{W}}(\widehat{K})$ on any edge $\widehat{e}$ of $\widehat{K}$ vanishes if and only if $\widehat{\boldsymbol{\mu}}_h$ belongs to the kernel of all local d.o.f. associated with $\widehat{e}$. In light of (2.5), this ensures $\boldsymbol{\mathcal{W}} \subset \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$. In the rest of the paper $\boldsymbol{\mathcal{W}}_h$ will designate a generic $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$-conforming boundary element space. It may arise from the RT_p family of elements, $p \in \mathbb{N}_0$, the BDM_p family, or a combination of both.

Based on the d.o.f. we can introduce *local interpolation operators* $\Pi_h : \mathrm{Dom}(\Pi_h) \mapsto \boldsymbol{\mathcal{W}}_h$. They are projectors onto $\boldsymbol{\mathcal{W}}_h$ and enjoy the fundamental *commuting diagram property* [5, Chap. III, sect. 3]

$$\mathrm{div}_\Gamma \circ \Pi_h = \mathsf{Q}_h \circ \mathrm{div}_\Gamma \quad \text{on } \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma) \cap \mathrm{Dom}(\Pi_h). \tag{5.3}$$

Here, Q_h is the $L^2(\Gamma)$-orthogonal projection onto a suitable space Q_h of Γ_h-piecewise polynomial discontinuous functions. It must be emphasized that the interpolation operators Π_h fail to be bounded on $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$; slightly more regularity of tangential vector fields in $\mathrm{Dom}(\Pi_h)$ is required [33, Lem. 5.1].

Next, we turn our attention to asymptotic properties of the boundary element spaces, in particular to estimates of interpolation errors and best approximation errors. We restrict ourselves to the h-version of boundary elements, which relies on uniformly shape-regular families $\{\Gamma_h\}_{h\in\mathbb{H}}$ of triangulations of Γ [22, Chap. 3, sect. 3.1]. Here, $\mathbb{H}$ stands for a decreasing sequence of meshwidths, and $\mathbb{H}$ is assumed to converge to zero.

By means of transformation to reference elements, the commuting diagram property, and Bramble–Hilbert arguments, interpolation error estimates can easily be obtained [5, Chap. III, sect. 3.3].

LEMMA 5.1 (interpolation error estimate). *For $0 < s \leq p+1$ we find constants $C > 0$, depending only on the shape regularity of the meshes, s and p, such that for all $\boldsymbol{\mu} \in \boldsymbol{H}^s_\times(\Gamma) \cap \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$, $h \in \mathbb{H}$,*

$$\|\boldsymbol{\mu} - \Pi_h \boldsymbol{\mu}\|_{\boldsymbol{L}^2(\Gamma)} \leq C h^s \left(\|\boldsymbol{\mu}\|_{\boldsymbol{H}^s_\times(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{\mu}\|_{L^2(\Gamma)} \right),$$

and such that for all $\boldsymbol{\mu} \in \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$, $\mathrm{div}_\Gamma \boldsymbol{\mu} \in H^s_-(\Gamma)$,

$$\|\mathrm{div}_\Gamma(\boldsymbol{\mu} - \Pi_h \boldsymbol{\mu})\|_{L^2(\Gamma)} \leq C h^s \|\mathrm{div}_\Gamma \boldsymbol{\mu}\|_{H^s_-(\Gamma)}.$$

COROLLARY 5.2. *The union of all boundary element spaces* $\boldsymbol{\mathcal{W}}_h$*,* $h \in \mathbb{H}$*, is dense in* $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

A particular variant of the above interpolation error estimate addresses vector fields with discrete surface divergence; cf. [33, Lem. 6.2].

LEMMA 5.3. *If* $\boldsymbol{\mu} \in \boldsymbol{H}_\times^s(\Gamma)$*,* $0 < s \le 1$*, and* $\mathrm{div}_\Gamma \boldsymbol{\mu} \in Q_h$*, then*

$$\|\boldsymbol{\mu} - \Pi_h \boldsymbol{\mu}\|_{\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)} \le C h^s \|\boldsymbol{\mu}\|_{\boldsymbol{H}_\times^s(\Gamma)},$$

where the constant $C > 0$ *depends only on the shape regularity of the meshes and the polynomial degree* p*.*

From the interpolation error estimates we instantly get best approximation estimates in terms of the $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$-norm. Yet, what we actually need is a result about approximation in the "energy norm" (trace norm) of the form

$$\inf_{\boldsymbol{\xi}_h} \|\boldsymbol{\mu}_h - \boldsymbol{\xi}_h\|_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} \le C h^{s+\frac{1}{2}} \|\boldsymbol{\mu}\|_{\boldsymbol{H}_\times^s(\mathrm{div}_\Gamma, \Gamma)}. \tag{5.4}$$

The estimate in $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$ does not directly provide (5.4). The question of obtaining (5.4) has been addressed in [8, sect. 4.4.2], and the idea is to use the duality argument face by face (each one seen as a regular open manifold), relying on the continuity of the normal components of vector fields in $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$. At the end of a technical procedure we obtain the following result [8, Thm. 4.9].

THEOREM 5.4. *Let* $\mathcal{P}_h : \boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \to \boldsymbol{\mathcal{W}}_h$ *be the orthogonal projection with respect to the* $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *inner product. Then for any* $-\frac{1}{2} \le s \le p+1$ *we have*

$$\|\boldsymbol{\mu} - \mathcal{P}_h \boldsymbol{\mu}\|_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} \le C h^{s+\frac{1}{2}} \|\boldsymbol{\mu}\|_{\boldsymbol{H}_\times^s(\mathrm{div}_\Gamma, \Gamma)} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{H}_\times^s(\mathrm{div}_\Gamma, \Gamma). \tag{5.5}$$

This theorem tells us that we can expect good approximation properties, much better than the estimates for the local interpolation error.

A finite-dimensional subspace of $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ is easily obtained from $\boldsymbol{\mathcal{W}}_h$ by setting all d.o.f. associated with edges on the skeleton Σ to zero. Let us write $\boldsymbol{\mathcal{W}}_{\Sigma,h}$ for the resulting space. By construction the estimates of Lemma 5.1 will carry over to $\boldsymbol{H}_\times^s(\Gamma) \cap \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$ and $\boldsymbol{\mathcal{W}}_{\Sigma,h}$.

Based on the boundary element spaces $\boldsymbol{\mathcal{W}}_h$ and $\boldsymbol{\mathcal{W}}_{\Sigma,h}$, which are contained in $\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $\boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma)$, respectively, we pursue a standard Galerkin discretization of (4.8). Writing $\mathfrak{V}_h = \boldsymbol{\mathcal{W}}_h \times \boldsymbol{\mathcal{W}}_{\Sigma,h}$, we end up with the following discrete problem:

$$\text{Find } \mathfrak{u}_h \in \mathfrak{V}_h \ : \ \widetilde{a}(\mathfrak{u}_h, \mathfrak{v}_h) = \left\langle \begin{pmatrix} \gamma_{\mathbf{t}}^+ \mathbf{e} \\ 0 \end{pmatrix}, \mathfrak{v}_h \right\rangle_{\boldsymbol{\tau},\Gamma} \qquad \forall\, \mathfrak{v}_h \in \mathfrak{V}_h. \tag{5.6}$$

We aim at establishing a *uniform discrete inf-sup condition* of the following form: there exists $C_D > 0$ such that

$$\sup_{\mathfrak{v}_h \in \mathfrak{V}_h} \frac{|\widetilde{a}(\mathfrak{u}_h, \mathfrak{v}_h)|}{\|\mathfrak{v}_h\|_{\mathfrak{V}}} \ge C_D \|\mathfrak{u}_h\|_{\mathfrak{V}} \qquad \forall \mathfrak{u}_h \in \mathfrak{V}_h, \qquad h \in \mathbb{H}. \tag{5.7}$$

According to [48] this guarantees existence of discrete solutions $\mathfrak{u}_h := (\boldsymbol{\zeta}_h, \mathbf{p}_h) \in \mathfrak{V}_h$ of (5.6) and translates into their quasi-optimal behavior:

$$\|\mathfrak{u} - \mathfrak{u}_h\|_{\mathfrak{V}} \le C_D^{-1} C_{\widetilde{a}} \inf_{\mathfrak{v}_h \in \mathfrak{V}_h} \|\mathfrak{u} - \mathfrak{v}_h\|_{\mathfrak{V}} \qquad \forall h \in \mathbb{H}, \tag{5.8}$$

where $C_{\widetilde{a}} > 0$ is the operator norm of $\widetilde{a}(\cdot,\cdot)$. We follow lines of reasoning laid out in [8, 15, 33]. As a first step towards a discrete inf-sup condition (5.7), we have to find a suitable candidate for $\mathfrak{v}$ in (4.9). To that end, introduce the operator $\mathsf{T} : \mathfrak{V} \mapsto \mathfrak{V}$ through

$$\widetilde{a}(\mathfrak{v}, \mathsf{T}\mathfrak{w}) = \widetilde{c}(\mathfrak{w}, \mathfrak{v}) \qquad \forall \mathfrak{v} \in \mathfrak{V},\ \mathfrak{w} \in \mathfrak{V},$$

where $\widetilde{c}$ is the compact bilinear form specified in Corollary 4.3. Owing to (4.9) this is a valid definition of a compact operator T. It is immediate from (4.9) and Lemma 3.8 that

$$|\widetilde{a}(\mathfrak{w}, (\mathbb{X}_\Gamma + \mathsf{T})\overline{\mathfrak{w}})| = |\widetilde{a}(\mathfrak{w}, \mathbb{X}_\Gamma\overline{\mathfrak{w}}) + c_\Gamma(\mathfrak{w}, \overline{\mathfrak{w}})| \geq C_G \, \|\mathfrak{w}\|^2_{\mathfrak{V}} \tag{5.9}$$

for all $\mathfrak{w} \in \mathfrak{V}_h$. Consequently, the choice $\mathfrak{v} := (\mathbb{X}_\Gamma + \mathsf{T})\overline{\mathfrak{w}}$ will make (4.9) hold with $C_S = C_G$.

Let $\mathfrak{w}_h \in \mathfrak{V}_h$, and $\mathfrak{v} := (\mathbb{X}_\Gamma + \mathsf{T})\mathfrak{w}_h$. In general, $\mathfrak{v} \notin \mathfrak{V}_h$ so that we have to use a projection. Write $\mathsf{P}_h : \boldsymbol{H}_\Sigma(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{\mathcal{W}}_{\Sigma,h}$ for the $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$-orthogonal projection and introduce

$$\mathbb{P} : \mathfrak{V} \mapsto \mathfrak{V}_h, \qquad \mathbb{P}\begin{pmatrix}\boldsymbol{\mu}\\ \mathbf{q}\end{pmatrix} := \begin{pmatrix}\mathcal{P}_h\boldsymbol{\mu}\\ \mathsf{P}_h\mathbf{q}\end{pmatrix}.$$

Then a promising candidate for the discrete inf-sup condition is the vector $\mathfrak{v}_h := \mathbb{P}_h\mathfrak{v} = (\mathbb{P}_h\mathbb{X}_\Gamma + \mathbb{P}_h\mathsf{T})\overline{\mathfrak{w}}_h$. The triangle inequality

$$|\widetilde{a}(\mathfrak{w}_h, \mathfrak{v}_h)| \geq |\widetilde{a}(\mathfrak{w}_h, \mathfrak{v})| - C_{\mathfrak{b}}\|\mathfrak{w}_h\|_{\mathfrak{V}}\|\mathfrak{v} - \mathfrak{v}_h\|_{\mathfrak{V}} \tag{5.10}$$

shows that (strong) convergence $\|\mathfrak{v} - \mathfrak{v}_h\|_{\mathfrak{V}} \to 0$ is needed. First, $\|(I - \mathbb{P}_h)\mathsf{T}\mathfrak{w}_h\|_{\mathfrak{V}} \to 0$ uniformly for all $\mathfrak{w}_h \in \mathfrak{V}_h$, since the composition of pointwise convergent and compact operators gives uniform convergence in operator norms [37, Cor. 10.4]. Second, it is important to note that X_Γ leaves the div_Γ of its argument function invariant, which means that the first component of $\mathbb{X}_\Gamma\mathfrak{w}_h$ has a surface divergence in a space of Γ_h-piecewise polynomials. This enables us to invoke Lemma 5.3, and we obtain (see [15, sect. 4.2]) that there exists an $s > 0$ such that

$$\|(I - \mathcal{P}_h)\mathbb{X}_\Gamma\mathfrak{w}_h\|_{\mathfrak{V}} \leq \|(\mathrm{Id} - \Pi_h)\mathsf{X}_\Gamma\boldsymbol{\mu}_h\|_{\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)} \leq Ch^s \, \|\mathrm{div}_\Gamma\mathsf{X}_\Gamma\boldsymbol{\mu}_h\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where $\boldsymbol{\mu}_h \in \boldsymbol{\mathcal{W}}_h$ stands for the first component of $\mathfrak{w}_h$.

Using these estimates in (5.10) and recalling that, by definition of $\mathfrak{v}$, $|\widetilde{a}(\mathfrak{w}_h, \mathfrak{v})| \geq C_G\|\mathfrak{w}_h\|^2_{\mathfrak{V}}$, we easily deduce the following theorem.

THEOREM 5.5. *There is an $h^* > 0$, depending on the parameters of the continuous problem and the shape regularity of the triangulation, such that a unique solution $\mathfrak{u}_h \in \mathfrak{V}_h$ of the discretized problem* (5.6) *exists, provided that $h < h_*$. It supplies an asymptotically optimal approximation to the continuous solution $\mathfrak{u} = (\boldsymbol{\zeta}, \mathbf{p})$ of* (4.8) *in the sense of* (5.8).

After choosing local bases of $\boldsymbol{\mathcal{W}}_h$ and $\boldsymbol{\mathcal{W}}_{\Sigma,h}$, we end up with a linear system of equations of the form

$$\begin{pmatrix} i\eta S & \frac{1}{2}B - C \\ B^T & -D \end{pmatrix} \begin{pmatrix} \underline{\zeta} \\ \underline{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{g}} \\ 0 \end{pmatrix}. \tag{5.11}$$

Here S and C will be dense square matrices arising from the discretized boundary integral operators S_κ and C_κ. The sparse, skew-symmetric matrix B is related to

$\langle\cdot,\cdot\rangle_{\boldsymbol{\tau},\Gamma}$, whereas the s.p.d. matrix D corresponds to the $\boldsymbol{H}_{\Sigma}(\mathrm{div}_\Gamma,\Gamma)$-inner product. The other symbols have obvious meanings.

Note that D is even block-diagonal with one sparse block for each face Γ_j, $j = 1,\dots,P$. Using advanced sparse Cholesky factorization techniques, it may be feasible to compute the application of D^{-1} to a vector directly. Then we face the linear system of equations

$$(i\eta S + (\tfrac{1}{2}B - C)D^{-1}B^T)\underline{\zeta} = \underline{\mathbf{g}}. \tag{5.12}$$

It can be solved only iteratively, because the actual matrix D^{-1} is not available. Besides, iterative solvers allow the use of fast summation techniques (multipole, $\mathcal{H}^2$-matrices) for the approximate application of S and C to a vector.

REMARK 3. *Of course, $\boldsymbol{\zeta}$ and $\mathbf{p}$ can be approximated in completely different boundary element spaces, as long as these are contained in $\boldsymbol{H}_\times(\mathrm{div}_\Gamma,\Gamma)$. The analysis can immediately be extended to this case.*

REMARK 4. *The iterative solution of* (5.12) *(e.g., by means of GMRES) requires a preconditioner, because the principal part of the related boundary integral operator is given by S_κ. As pointed out in* [21] *the condition number of S will deteriorate on fine meshes. Yet, the fact that S is related to the principal part also means that preconditioning needs only to target this matrix, which is the same matrix as in the Galerkin discretization of the EFIE. An elaborate preconditioning strategy has been devised in* [21].

Yet, if κ is close to a resonant frequency, S will become nearly singular, and preconditioning might suffer. This requires further investigation, which is beyond the scope of this paper.

REMARK 5. *The choice of η is another issue which has eluded theory so far. It is clear that η has a major impact on the spectral properties of the final linear system* (5.12)*, but it is not clear how to choose η to achieve good properties of the discrete problem. This situation is commonly faced with CFIE approaches. Some investigations in the case of two-dimensional acoustic scattering can be found in* [38]*; see also* [27, *sect.* 2.4.1] *and* [16].

REMARK 6. *For reasons explained in Remark* 2*, we have decided to use a localized version of M. One could argue that localization could be carried further by considering split faces. Of course, the theory will cover this, but it is important to keep in mind that the result of Theorem* 5.5 *is asymptotic in nature. The choice of M will affect the threshold h^*, and it may well be that certain choices of M will delay the onset of asymptotic convergence until unreasonably fine meshes. We acknowledge that this might also be true for our choice of M.*

6. Convergence estimates. In light of the asymptotic quasi optimality of the conforming Galerkin solutions expressed in Theorem 5.5, we have to investigate how well the solution $(\boldsymbol{\zeta},\mathbf{p})$ of (4.8) can be approximated in $\mathfrak{V}_h$. This entails knowledge about the regularity of both $\boldsymbol{\zeta}$ and $\mathbf{p}$.

Thanks to the localization of M onto the faces of Γ, studying the smoothness of $\mathbf{p}$ can chiefly rely on two-dimensional considerations.

LEMMA 6.1. *For a Lipschitz domain $\omega \subset \mathbb{R}^2$ we denote by α the maximum regularity exponent for the Laplace problem with Dirichlet or Neumann boundary conditions; i.e., if $\Delta u \in H^{\alpha-1}(\omega)$ and u verifies either the Dirichlet or Neumann homogeneous boundary condition, then $u \in H^{\alpha+1}(\omega)$.*

Let $\mathbf{f} \in (H^\sigma(\omega))^2$ and $\mathrm{curl}_{2D}\,\mathbf{f} \in H^\sigma(\omega)$, $\sigma \geq 0$. If $\mathbf{p} \in \boldsymbol{H}_0(\mathrm{div};\omega)$ satisfies

$$(\mathrm{div}\,\mathbf{p},\mathrm{div}\,\mathbf{v})_{0;\omega} + (\mathbf{p},\mathbf{v})_{0;\omega} = (\mathbf{f},\mathbf{v})_{0;\omega} \qquad \forall \mathbf{v} \in \boldsymbol{H}_0(\mathrm{div};\omega),$$

then $\mathbf{p} \in H^{\min\{\alpha,\sigma+1\}}(\omega)$ *and* $\operatorname{div}\mathbf{p} \in H^{\min\{\alpha+1,1+\sigma\}}(\omega)$.

Proof. It goes without saying that $\mathbf{p}$ is well defined. The main tool for the proof of the asserted regularity properties will be $\boldsymbol{L}^2(\omega)$-orthogonal Helmholtz decompositions (see [28, Chap. 1])

$$\boldsymbol{L}^2(\omega) = \mathbf{curl}_{2D}\, H_0^1(\omega) \oplus \mathbf{grad}\, H^1(\omega),$$
$$\boldsymbol{H}_0(\mathrm{div};\omega) = \mathbf{curl}_{2D}\, H_0^1(\omega) \oplus \mathbf{grad}\, H_0(\Delta,\omega),$$

where

$$H_0(\Delta,\omega) := \left\{\psi \in H^1(\omega) :\ \Delta\psi \in L^2(\omega),\ \frac{\partial\psi}{\partial\mathbf{n}} = 0 \text{ on } \partial\omega\right\}.$$

Accordingly, we decompose

$$\mathbf{p} = \mathbf{curl}_{2D}\,\varphi_1 + \mathbf{grad}\,\varphi_2, \qquad \mathbf{f} = \mathbf{curl}_{2D}\,\phi_1 + \mathbf{grad}\,\phi_2,$$

with $\varphi_1, \phi_1 \in H_0^1(\omega)$, $\varphi_2 \in H_0(\Delta,\omega)$, $\phi_2 \in H^1(\omega)$. A closer scrutiny reveals that

$$\mathrm{curl}_{2D}\,\mathbf{curl}_{2D}\,\phi_1 = -\Delta\phi_1 = \mathrm{curl}_{2D}\,\mathbf{f} \in H^\sigma(\omega) \quad \Rightarrow \quad \phi_1 \in H^{\min\{1+\alpha,2+\sigma\}}(\omega),$$

because of the $1+\alpha$-regularity of the Laplacian. Testing with $\mathbf{curl}_{2D}\,\nu$, $\nu \in H_0^1(\omega)$, in the definition of $\mathbf{p}$, we immediately see that $\varphi_1 = \phi_1$.

For $\nu_2 \in H_0(\Delta,\omega)$ we deduce from the variational equation that

$$(\operatorname{div}\mathbf{p}, \operatorname{div}\mathbf{grad}\,\nu_2)_{0;\omega} + (\mathbf{p}, \mathbf{grad}\,\nu_2)_{0;\omega} = (\mathbf{f}, \mathbf{grad}\,\nu_2)_{0;\omega}\,.$$

After integrating by parts, this means

$$(\operatorname{div}\mathbf{p}, \Delta\nu_2 - \nu_2)_{0;\omega} = (\mathbf{f}, \mathbf{grad}\,\nu_2)_{0;\omega}\,. \tag{6.1}$$

Now consider $\zeta \in H^1(\omega)$, solving

$$(\mathbf{grad}\,\zeta, \mathbf{grad}\,\nu)_{0;\omega} + (\zeta,\nu)_{0;\omega} = (\mathbf{f}, \mathbf{grad}\,\nu)_{0;\omega} \qquad \forall \nu \in H^1(\omega). \tag{6.2}$$

The regularity assumption implies that $\zeta \in H^{\min\{1+\alpha,1+\sigma\}}(\omega)$. We can pick $\nu \in H_0(\Delta,\omega)$ in this equation, carry out integration by parts, and subtract the result from (6.1). We end up with

$$(\operatorname{div}\mathbf{p} - \zeta, -\Delta\nu + \nu)_{0;\omega} = 0.$$

Since $(-\Delta + Id)(H_0(\Delta,\omega)) = L^2(\omega)$, we infer that $\operatorname{div}\mathbf{p} = \zeta$, i.e., $\operatorname{div}\mathbf{p} \in H^{\min\{1+\alpha,1+\sigma\}}(\omega)$. ∎

We point out that for a polygon ω the exponent α is directly related to the angles θ_i, $i = 1,\dots,n_c$, at its corners:

$$\alpha = \min\{1, \pi/\theta_i, i = 1,\dots,n_c\} \geq \tfrac{1}{2}.$$

This lemma can instantly be applied to all the smooth faces of Γ and supplies lifting properties of M, because there is no coupling between the faces.

COROLLARY 6.2. *If* $\boldsymbol{\mu} \in \boldsymbol{H}_\times^\sigma(\mathrm{div}_\Gamma,\Gamma)$*,* $\sigma \geq 0$*, then* $\mathsf{M}\boldsymbol{\mu} \in \boldsymbol{H}_\times^{\min\{\alpha,\sigma+1\}}(\mathrm{div}_\Gamma,\Gamma)$*, where* α *is the minimum of the* $\Delta_{\mathrm{Dir}}/\Delta_{\mathrm{Neu}}$*-regularity exponents on the flat faces* Γ_j*,* $j = 1,\dots,P$*.*

Assume that $\boldsymbol{\zeta} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is the unique solution of (4.5), and denote by $\mathbf{e} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega \cup \Omega')$ the Maxwell solution according to (4.3):

$$\mathbf{e} = -i\eta \boldsymbol{\Psi}_{SL}^{\kappa}(\boldsymbol{\zeta}) - \boldsymbol{\Psi}_{DL}^{\kappa}(\mathsf{M}\boldsymbol{\zeta}).$$

To study the regularity it is essential to recall that by the jump relations

$$\gamma_{\mathbf{t}}^{-}\mathbf{e} = -\mathsf{M}\boldsymbol{\zeta} - \mathbf{g}, \qquad \gamma_{N}^{-}\mathbf{e} = i\eta\boldsymbol{\zeta} - \mathbf{h}, \tag{6.3}$$

where we wrote $\mathbf{h} := \gamma_N^{+}\mathbf{e} \in \boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ for the exterior Neumann data of the scattered field. As $\mathbf{g} := \gamma_{\mathbf{t}}^{+}\mathbf{e}_i$ is the tangential trace of an incident wave, it will belong to $\boldsymbol{H}_{\times}^{s}(\mathrm{div}_\Gamma, \Gamma)$ for all $s > 0$. Additional information can be gleaned only from lifting properties of the Maxwell operator. Its regularity theory, elaborated in [25], justifies the following assumption.

Assumption 6.2.1. *There are two regularity indices $\sigma^-, \sigma^+ > \frac{1}{2}$ such that*

1. *any field $\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}^2, \Omega)$ solving*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \mathbf{grad}\,\mathrm{div}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \gamma_{\mathbf{t}}^{-}\mathbf{u} = 0, \quad \text{or} \quad \gamma_N^{-}\mathbf{u} = 0$$

   belongs to $\boldsymbol{H}^{\sigma}(\mathbf{curl}, \Omega)$ for all $\sigma \le \sigma^-$ if $\mathbf{f} \in H^{\sigma-1}(\Omega)$;

2. *any field $\mathbf{u} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega')$ satisfying the radiation condition and*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \mathbf{grad}\,\mathrm{div}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{f} \quad \text{in } \Omega', \quad \gamma_{\mathbf{t}}^{+}\mathbf{u} = 0, \quad \text{or} \quad \gamma_N^{+}\mathbf{u} = 0$$

   lies in $\boldsymbol{H}_{\mathrm{loc}}^{\sigma}(\mathbf{curl}, \Omega')$ for all $\sigma < \sigma^+$ if $\mathbf{f} \in H^{\sigma-1}(\Omega')$.

Owing to the trace theorem, Theorem 2.6, this assumption implies that $\mathbf{h} \in \boldsymbol{H}_{\times}^{\sigma^+-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Then we can resort to a "bootstrap argument."

Step 1. We remember a result by Costabel [24] confirming the existence of a constant $c > 0$ that depends only on Ω such that for all $\mathbf{u} \in \boldsymbol{H}(\mathrm{div}; \Omega) \cap \boldsymbol{H}(\mathbf{curl}; \Omega)$,

$$\left\|\gamma_{\mathbf{n}}^{-}\mathbf{u}\right\|_{L^2(\Gamma)} \le c\left\{\left\|\gamma_{\mathbf{t}}^{-}\mathbf{u}\right\|_{\boldsymbol{L}^2(\Gamma)} + \|\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\mathbf{u}\|_{L^2(\Omega)}\right\}, \tag{6.4}$$

$$\left\|\gamma_{\mathbf{t}}^{-}\mathbf{u}\right\|_{\boldsymbol{L}^2(\Gamma)} \le c\left\{\left\|\gamma_{\mathbf{n}}^{-}\mathbf{u}\right\|_{L^2(\Gamma)} + \|\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\mathbf{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\mathbf{u}\|_{L^2(\Omega)}\right\}. \tag{6.5}$$

Since $\mathbf{e}$ is a Maxwell solution in Ω, these estimates combined with Theorem 3.3 give

$$\begin{aligned}\left\|\gamma_N^{-}\mathbf{e}\right\|_{\boldsymbol{L}^2(\Gamma)} &\le C\left\{\left\|\gamma_{\mathbf{n}}^{-}\,\mathbf{curl}\,\mathbf{e}\right\|_{L^2(\Gamma)} + \kappa^2\|\mathbf{e}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\mathbf{e}\|_{\boldsymbol{L}^2(\Omega)}\right\}\\ &\le C\left\{\left\|\mathrm{div}_\Gamma\gamma_{\mathbf{t}}^{-}\mathbf{e}\right\|_{L^2(\Gamma)} + \|\boldsymbol{\zeta}\|_{\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} + \|\mathsf{M}\boldsymbol{\zeta}\|_{\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}\right\}.\end{aligned}$$

Similarly, we can use (6.4) and get

$$\begin{aligned}\left\|\mathrm{div}_\Gamma\gamma_N^{-}\mathbf{e}\right\|_{L^2(\Gamma)} = \kappa^2\left\|\gamma_{\mathbf{n}}^{-}\mathbf{e}\right\|_{L^2(\Gamma)} &\le C\left\{\left\|\gamma_{\mathbf{t}}^{-}\mathbf{e}\right\|_{\boldsymbol{L}^2(\Gamma)} + \|\mathbf{curl}\,\mathbf{e}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{e}\|_{\boldsymbol{L}^2(\Omega)}\right\}\\ &\le C\left\{\left\|\gamma_{\mathbf{t}}^{-}\mathbf{e}\right\|_{\boldsymbol{L}^2(\Gamma)} + \|\boldsymbol{\zeta}\|_{\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} + \|\mathsf{M}\boldsymbol{\zeta}\|_{\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}\right\}.\end{aligned}$$

The generic constants $C > 0$ may depend on Ω, κ, and η. The combined estimate reads

$$\left\|\gamma_N^{-}\mathbf{e}\right\|_{\boldsymbol{H}_{\times}(\mathrm{div}_\Gamma,\Gamma)} \le C\left\{\left\|\gamma_{\mathbf{t}}^{-}\mathbf{e}\right\|_{\boldsymbol{H}_{\times}(\mathrm{div}_\Gamma,\Gamma)} + \|\boldsymbol{\zeta}\|_{\boldsymbol{H}_{\times}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}\right\},$$

which means $\gamma_N^- \mathbf{e} \in \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$.

Step 2. Next, from (6.3) we infer that $\boldsymbol{\zeta} \in \boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$. Now, we can apply Corollary 6.2, we get $\mathsf{M}\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{\min\{1,\alpha\}}(\mathrm{div}_\Gamma, \Gamma)$, and (6.3) gives us $\gamma_{\mathbf{t}}^- \mathbf{e} \in \boldsymbol{H}_\times^{\min\{1,\alpha\}}(\mathrm{div}_\Gamma, \Gamma)$. Now, since $\mathbf{e}$ is a Maxwell solution verifying $\gamma_{\mathbf{t}}^- \mathbf{e} \in \boldsymbol{H}_\times^{\min\{1,\alpha\}}(\mathrm{div}_\Gamma, \Gamma)$, using Theorem 2.6 and Assumption 6.2.1, we have that $\mathbf{e}, \mathbf{curl}\,\mathbf{e} \in \boldsymbol{H}^{\min\{\sigma^-,1\}}(\mathbf{curl}, \Omega)$, i.e., $\gamma_N^- \mathbf{e} \in \boldsymbol{H}_\times^{\min\{\sigma^- - \frac{1}{2}, \frac{1}{2}\}}(\mathrm{div}_\Gamma, \Gamma)$.

Step 3. Finally, we can conclude that $\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{\min\{\sigma^- - \frac{1}{2}, \sigma^+ - \frac{1}{2}, \frac{1}{2}\}}(\mathrm{div}_\Gamma, \Gamma)$. On a polyhedron we can take for granted that either $\sigma^- < 1$ or $\sigma^+ < 1$. This gives us

$$\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{\min\{\sigma^-, \sigma^+\} - \frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma).$$

Besides, we have already seen that $\mathbf{p} = \mathsf{M}\boldsymbol{\zeta} \in \boldsymbol{H}_\times^{\min\{\alpha,1\}}(\mathrm{div}_\Gamma, \Gamma)$.

Now we can employ the best approximation estimates for div_Γ-conforming elements from Lemma 5.1 and Theorem 5.4 and get quantitative asymptotic convergence estimates.

THEOREM 6.3. *If we rely on $\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)$-conforming boundary elements for the discretization of both $\boldsymbol{\zeta}$ and $\mathbf{p}$, we are guaranteed to get*

$$\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\boldsymbol{H}_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} + \|\mathbf{p} - \mathbf{p}_h\|_{\boldsymbol{H}_\times(\mathrm{div}_\Gamma, \Gamma)} \le C(h^{\min\{\sigma^+, \sigma^-\}} + h^{\min\{\alpha, 1\}}),$$

with a constant $C > 0$ independent of the meshwidth h, but may depend upon the wave number k.

REMARK 7. *Since we are solving an indirect boundary integral equation, it is not surprising that the convergence is limited by singularities of both the interior and the exterior Maxwell problem. On the other hand, the main observation is that one can always have $\alpha > 1$ since it is enough to split nonconvex faces into convex ones. Thus, the rate of convergence is not affected by the introduction of the auxiliary unknown $\mathbf{p}$; i.e., $\mathbf{p}$ is always much more regular than the primal unknown $\boldsymbol{\zeta}$.*

REMARK 8. *The above estimate relies on global regularity of the exact solutions. However, we know that $\boldsymbol{\zeta}$ is a combination of traces of Maxwell solutions. Besides, $\mathbf{p}$ emerges as patched-together solutions of Dirichlet boundary value problems for Δ_Γ on the flat faces. In both cases, results on singularities of solutions of boundary value problems on nonsmooth domains reveal much detail about the behavior of $\boldsymbol{\zeta}$ and $\mathbf{p}$ close to edges and corners. We can make use of this knowledge in order to obtain significantly faster convergence on meshes that feature algebraically graded refinement towards the edges of* Γ [2, 11]. *In this case, making use again of the regularity of $\mathbf{p}$, one might need only to "resolve" the singularities of $\boldsymbol{\zeta}$ by mesh grading. The use of different meshes, on which $\boldsymbol{\zeta}$ and $\mathbf{p}$ are approximated, seems to be advisable in this case; cf. Remark* 3.

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