Effective Iterative Solution Methods for Boundary Element Equations

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ABSTRACT

The precondition techniques are developed for matrices arising from Galerkin method with different trial functions (piecewise constant, linear and quadratic) for two-dimensional and three-dimensional, simple and multiple connected domains with Dirichlet and Neumann boundary conditions.

INTRODUCTION

The boundary element method (BEM) leads to an algebraic system of equations with a full dense matrix (Brebbia,Walker [1], Wendland [2]). The Gaussian elimination is usually used for solving such linear systems. If we denote the discretization parameter in one space direction as h, we obtain matrices having the dimension $O(h^{-1})$ in two- and $O(h^{-2})$ in three-dimensional case. The number of arithmetical operations are $O(h^{-3})$ and $O(h^{-6})$ respectively. Since the given matrices are generally large, more efficient methods for the solution of BEM linear systems are necessary.

The BEM for the Dirichlet or Neumann problem in a two-dimensional circular domain and in a three-dimensional rotational domain leads to a system with a matrix having a special structure. It is a circulant or block-circulant structure. Such systems can be solved directly using Fast Fourier Transform (FFT), and the number of arithmetical operations would be only $O(h^{-1}\log h^{-1})$ in two- and $O(h^{-3}\log h^{-1})$ in three-dimensional case. Such matrices can be used for preconditioned iterative solution methods for the systems arising from general arbitrary domains. The precondition and system matrices are spectral equivalent. The rate of convergence of the Conjugate Gradient Method (CGM) is independent of h and the numerical work will be drastically decreasing to $O(h^{-2})$ in two- and $O(h^{-4})$ in three-dimensional case.

BOUNDARY INTEGRAL EQUATIONS

In this paper we consider the Dirichlet and Neumann problems for Poisson equation in two- and three-dimensional domain Ω:

$$\Delta u(x) = f(x),\ x \in \Omega, \tag{1}$$

$$(\mathcal{P}_1 u)(x) + (\mathcal{P}_2 v)(x) = g(x),\ x \in \Gamma = \partial\Omega, \tag{2}$$

where, $v(x)$ is the normal derivative of the function $u(x)$:

$$v(x) = \frac{\partial u(x)}{\partial n_x},$$

and n_x ($x \in \Gamma$) is the unit outward normal of the boundary Γ at x. The Operators $\mathcal{P}_1$ and $\mathcal{P}_2$ are

$$\mathcal{P}_1 = \mathcal{I},\ \mathcal{P}_2 = \mathcal{O}$$

for Dirichlet and

$$\mathcal{P}_1 = \mathcal{O},\ \mathcal{P}_2 = \mathcal{I}$$

for Neumann problem.

It is known (Wendland [2], Costabel [3]) that there are two relations on Γ between $u(x)$ and $v(x)$

$$(\mathcal{A}_1 v)(y) - \left(\left(\frac{1}{2}\mathcal{I} + \mathcal{B}\right) u\right)(y) = f_1(y),\ y \in \Gamma; \tag{3}$$

$$-\left(\left(\frac{1}{2}\mathcal{I} - \mathcal{B}'\right) v\right)(y) + (\mathcal{A}_2 u)(y) = f_2(y),\ y \in \Gamma. \tag{4}$$

The equation (3) is well known Green's third identity for potential theory. The operators $\mathcal{A}_1$ and $\mathcal{B}$ are the simple and double layer potentials:

$$(\mathcal{A}_1 v)(y) = \int_\Gamma u^*(x,y) v(x) ds_x;$$

$$(\mathcal{B} u)(y) = \int_\Gamma \frac{\partial u^*(x,y)}{\partial n_x} u(x) ds_x.$$

The equation (4) can be obtained (Blue [4]) from the second Green's identity. Here, $\mathcal{B}'$ is the adjoint operator of $\mathcal{B}$:

$$(\mathcal{B}' v)(y) = \int_\Gamma \frac{\partial u^*(x,y)}{\partial n_y} v(x) ds_x,$$

and $\mathcal{A}_2$ defined by

$$(\mathcal{A}_2 u)(y) = \int_\Gamma \frac{\partial^2 u^*(x,y)}{\partial n_x \partial n_y} (u(x) - u(y)) ds_x$$

is the operator with the hypersingular kernel, where $u^*(x,y)$ is the known fundamental solution of the Laplace equation:

$$u^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & for\ \Omega \subset R^2; \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & for\ \Omega \subset R^3. \end{cases}$$

Any one of the integral equations (3),(4) can be used for the numerical solution of the boundary value problem (1),(2).

Now we shall consider (Costabel [3], Costabel, Wendland [5]) the properties of the operators $\mathcal{A}_1, \mathcal{A}_2$, $\frac{1}{2}\mathcal{I} + \mathcal{B}$ and $\frac{1}{2}\mathcal{I} - \mathcal{B}'$. Let Ω be a bounded simple connected domain

whose boundary Γ is sufficiently smooth. All four operators are pseudo-differential operators of integer order on Γ:

$$\begin{aligned} \mathcal{A}_1 &: H^s(\Gamma) \mapsto H^{s+1}(\Gamma); \\ \mathcal{A}_2 &: H^s(\Gamma) \mapsto H^{s-1}(\Gamma); \\ \frac{1}{2}\mathcal{I} + \mathcal{B},\ \frac{1}{2}\mathcal{I} - \mathcal{B}' &: H^s(\Gamma) \mapsto H^s(\Gamma); \\ & s \in R, \end{aligned}$$

where $H^s(\Gamma)$ is the Sobolev space (Adams [6]) on boundary Γ. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are selfadjoint, i.e.

$$\begin{aligned} < \mathcal{A}_1 u, v >_0 = < u, \mathcal{A}_1 v >_0,\ \forall u, v \in H^{-1/2}(\Gamma); \\ < \mathcal{A}_2 u, v >_0 = < u, \mathcal{A}_2 v >_0,\ \forall u, v \in H^{1/2}(\Gamma), \end{aligned}$$

where $< \cdot, \cdot >_0$ denotes the duality pairing between $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, i.e. $L_2(\Gamma)$ inner product

$$< u, v >_0 = \int_\Gamma u(x)v(x)ds_x.$$

The operators $\mathcal{A}_1$ and $\frac{1}{2}\mathcal{I} - \mathcal{B}'$ satisfy a Gårding inequality, which can be formulated for an operator $\mathcal{A}: H^s(\Gamma) \mapsto H^{-s}(\Gamma)$: there exists a constant $\gamma > 0$ such that

$$< \mathcal{A}u, u >_0 \geq \gamma \|u\|_s^2, \quad \forall u \in H^s(\Gamma), \tag{5}$$

where $\|\cdot\|_s$ denotes the Sobolev norm in $H^s(\Gamma)$. The operators $\mathcal{A}_2$ and $\frac{1}{2}\mathcal{I} + \mathcal{B}$ satisfy (5) for all functions u with $< u, 1 >_0 = 0$, since both are singular operators:

$$(\mathcal{A} \cdot 1)(y) = 0.$$

The most effective iterative methods (for example CGM) need symmetric and positive definite matrix. These conditions can be satisfied if we choose equation (3) for the numerical solution of Dirichlet problem :

$$(\mathcal{A}_1 v)(y) = f_1(y),\ y \in \Gamma, \tag{6}$$

and the equation (4) for Neumann problem

$$(\mathcal{A}_2 v)(y) = f_2(y),\ y \in \Gamma. \tag{7}$$

TWO-DIMENSIONAL CASE

We begin with a parametrization of Γ by 1-periodic representation

$$\Gamma = \left\{ x \in R^2 : x = x(t),\ 0 \leq t < 1,\ |\dot{x}(t)| \geq \kappa > 0 \right\}. \tag{8}$$

Now we can rewrite the equations (6) and (7) as follows:

$$(\mathcal{A}_1 v)(\tau) = \int_0^1 u^*(x(t), x(\tau))v(x(t))|\dot{x}(t)|dt = f_1(\tau),$$

$$(\mathcal{A}_2 u)(\tau) = \int_0^1 \frac{\partial^2 u^*(x(t), x(\tau))}{\partial n_{x(t)} \partial n_{x(\tau)}}(u(x(t) - u(x(\tau))|\dot{x}(t)|dt = f_2(\tau),$$

$$0 \leq \tau < 1.$$

For the circular boundary Γ

$$\Gamma = \left\{ x \in R^2 : \; x = \frac{1}{2} \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}, \; 0 \le t < 1 \right\}$$

this yields:

$$(\mathcal{A}_1 v)(\tau) = -\frac{1}{2} \int_0^1 \log |\sin \pi(t-\tau)| v(t) dt = f_1(\tau), \tag{9}$$

$$(\mathcal{A}_2 u)(\tau) = -\frac{1}{2} \int_0^1 \frac{u(t)-u(\tau)}{\sin^2(t-\tau)} dt = f_2(\tau), \tag{10}$$

$$0 \le \tau < 1.$$

The following Lemma is proved in (Rjasanow [8],[9]).

Lemma 1 *The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ from (9),(10) have the eigenfunctions and eigenvalues:*

$$\begin{aligned} \mathcal{A}_1 v_k &= \lambda_k^{(1)} v_k, \\ \mathcal{A}_2 v_k &= \lambda_k^{(2)} v_k, \\ v_k &= e^{i2\pi kt}, \\ k &\in Z, \\ \lambda_0^{(1)} = \frac{1}{2}\log 2, \; \lambda_0^{(2)} &= 0, \\ \lambda_k^{(1)} = \frac{1}{4|k|}, \; \lambda_k^{(2)} &= |k|, \\ k &\in Z \setminus \{0\}. \end{aligned}$$

DISCRETIZATION

For the discretization of (6),(7) we begin with the approximation of $v(t)$ and $u(t)$ by piecewise polynomials $\phi_l^{(\nu)}(t)$, $l = 1, \ldots, n$ of degree $\nu = 0, 1, 2$ (B-Splines). We divide the basic interval $[0, 1)$ into $n > \nu + 1$ subintervals

$$[0,1) = \bigcup_{l=1}^{n} [t_l, t_{l+1}), \; t_l = (l-1)h, \; h = 1/n,$$

and introduce a n-dimensional subspace H_n of 1-periodic functions (see Rjasanow [9])

$$H_n = \{u_h : \; u_h = \Phi_n^\nu(t) y, \; y \in R^n\} = span(\phi_1^{(\nu)}(t), \ldots, \phi_n^{(\nu)}(t)).$$

GALERKIN METHOD

The Galerkin method for the equations (6),(7) leads to:

Find $v_h(t) \in H_n$ such that Galerkin equations

$$< \mathcal{A} v_h, w >_0 = < f, w >_0 \tag{11}$$

are satisfied for all functions $w \in H_n$.

We obtain from (11) the following systems of linear equations:

$$A_1 y_1 = b_1,$$
$$A_2 y_2 = b_2,$$
$$A_1,\ A_2 \in R^{n\times n},$$
$$y_1, y_2, b_1, b_2 \in R^n.$$

The matrices A_1 and A_2 have a special structure (Reichel [7], Rjasanow [8]).

Lemma 2 *The matrices A_1 and A_2 are symmetric, positive definite (A_2 is semi-definite) and circulant.*

We introduce the most simple circulant matrix as

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad J \in R^{n\times n} \tag{12}$$

It is not difficult to see that each circulant matrix $A \in R^{n\times n}$ satisfies

$$A = \sum_{l=1}^{n} a_{1l} J^{l-1} = a(J), \tag{13}$$

thus A is a polynomial function of the matrix J, furthermore (Woewodin, Tyrtyshnikow [10], Davis [11]) each circulant matrix A can be written as

$$A = n^{-1} F \Lambda F^*, \tag{14}$$

where F denotes the matrix of discrete Fourier transform

$$f_{kl} = e^{i\frac{2\pi}{n}(k-1)(l-1)}, \quad k,l = 1,\ldots,n$$

and Λ is the diogonal matrix with the eigenvalues of A:

$$\Lambda = diag(\lambda_1, \ldots \lambda_n).$$

The eigenvalues of A_1 and A_2 can be obtained by the following Lemma (see Rjasanow [8],[9])

Lemma 3 *The eigenvalues of the matrices A_1 and A_2 have the form*

$$\lambda_j^{(1)} = \begin{cases} \frac{1}{2} h \log 2, \ j = 1; \\ \frac{1}{4} h^2 \left(\frac{\sin \pi s}{\pi}\right)^{2\nu+2} \sum_{k=0}^{\infty} \frac{1}{(k+s)^{2\nu+3}} + \frac{1}{(k+1-s)^{2\nu+3}}, \ j > 1; \end{cases}$$

$$\lambda_j^{(2)} = \begin{cases} 0, \ j = 1; \\ \frac{2}{h} \left(\frac{\sin \pi s}{\pi}\right)^{2\nu+2} \sum_{k=0}^{\infty} \frac{1}{(k+s)^{2\nu+1}} + \frac{1}{(k+1-s)^{2\nu+1}}, \ j > 1; \end{cases}$$

$$s = \frac{j-1}{n}.$$

It is very easy to solve a system of linear equations

$$Aw = r$$

with the circulant matrix A. The solution w can be written with the help of (14) as

$$w = A^{-1}r = n^{-1}F\Lambda^{-1}F^*r,$$

and w is computed using FFT (Cooley, Tukey [12], Henrici [13]) needs only $O(n\log n) = O(h^{-1}\log h^{-1})$ arithmetical operations. We use the pseudo inverse matrix A_2^+ to solve a linear system with the matrix A_2 :

$$A_2^+ = n^{-1}F\Lambda^+F^*,\ \Lambda^+ = diag(0, 1/\lambda_2^{(2)}, \ldots, 1/\lambda_n^{(2)}).$$

PRECONDITIONED CONJUGATE GRADIENT METHOD

One of the most effective iterative solution methods for the symmetric, positive definite system of linear equations

$$Ay = b,\ A \in R^{n\times n},\ A = A^T > 0,\ y, b, \in R^n$$

is the conjugate gradient method (Hestenes,Stiefel [14]):

1. $y_0 \in R^n$;
   $r_0 = Ay_0 - b,\ w_0 = B^{-1}r_0$;
   $s_0 = w_0$;

2. for $k = 0, 1, \ldots$
   $$y_{k+1} = y_k - \alpha_{k+1}s_k,\ \alpha_{k+1} = \frac{(r_k, w_k)}{(As_k, s_k)};$$
   $$r_{k+1} = r_k - \alpha_{k+1}As_k,\ w_{k+1} = B^{-1}r_{k+1};$$
   $$w_{k+1} = w_{k+1} + \beta_{k+1}s_k,\ \beta_{k+1} = \frac{(r_{k+1}, w_{k+1})}{(r_k, w_k)},$$

where $B = B^T > 0$ is a preconditioning matrix.

For general arbitrary domains Ω (not being circular), the Galerkin approach still yields (Wendland [2], Costabel [3], Rjasanow [9]) a symmetric, positive definite matrix A (for the Neumann problem positive semi-definite) without having the additional special circulant property. In order to solve the resulting system of equations, we are able to use the preconditioned conjugate gradient method with the preconditioning matrix $B = A_1$ or $B = A_2$. Here, one step of the iterations requires the matrix-vector multiplication ($O(n^2)$ Operations), two scalar products ($O(n)$ Operations) and the solution of the preconditioning system ($O(n\log n)$ Operations). Hence, the work per iteration is mainly dominated by the work for the matrix-vector multiplication. The total number of operations is dependent on the spectral condition number $\kappa(B^{-1}A)$, which is bounded, the bound is independent of n (Rjasanow [8],[9]). In this case we need only $O(n^2) = O(h^{-2})$ arithmetical operations for the entire process.

MULTIPLE CONNECTED DOMAIN

Now let Γ be a multiple connected boundary

$$\Gamma = \bigcup_{l=1}^{m} \Gamma_l,$$

where each boundary Γ_l, $l = 1, \ldots, m$ is simple connected and can be parametized (see (8)). The Galerkin approach for the Dirichlet problem yields in this case a symmetric positive definite matrix A having the following block structure

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}, \quad A \in R^{N \times N}, \; N = \sum_{l=1}^{m} n_l,$$

where the blocks $A_{kl} \in R^{n_k \times n_l}$ are Galerkin matrices for the operators

$$\int_{\Gamma_l} u^*(x,y) v(x) ds_x, \quad y \in \Gamma_k,$$

and n_k denotes the number of discretization points on Γ_k. The preconditioning matrix B can be constructed as follows

$$B = diag(B_1, \ldots, B_m), \quad B \in R^{N \times N},$$

where each diagonal block $B_k \in R^{n_k \times n_k}$ is the Galerkin matrix for the operator (9). The number of iterations of CGM is independent of $h = \max h_k$, but is dependent of the geometry of the boundary Γ. This is shown in (Staude [15]) .

THREE-DIMENSIONAL CASE

Let $\Gamma = \partial\Omega$, $\Omega \in R^3$ be a bounded simple connected surface given by a parametric representation

$$\Gamma = \left\{ x \in R^3, \; x = x(t,z), \; 0 \le t < 1, \; 0 \le z \le 1 \right\}. \tag{15}$$

With the help of (15) we can write the boundary integral equation (6)

$$\frac{1}{4\pi} \int_{\Gamma} \frac{v(x)}{|x-y|} ds_x = f(y), \quad y \in \Gamma$$

as an integral equation over $[0,1) \times [0,1]$:

$$\frac{1}{4\pi} \int_0^1 \int_0^1 \frac{v(t,z)}{|x(t,z) - x(\tau,w)|} J(t,z) dt dz = f(\tau,w), \quad (\tau,w) \in [0,1) \times [0,1], \tag{16}$$

where $y = x(\tau,w), v(t,z) = v(x(t,z)), f(\tau,w) = f(x(\tau,w))$ and

$$J(t,z) = \sqrt{\left|\frac{\partial x}{\partial t}\right|^2 \left|\frac{\partial x}{\partial z}\right|^2 - \left(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z}\right)^2} \tag{17}$$

is the Jacobian of (15).

DISCRETIZATION

The functions on Γ are identified with functions on $[0,1) \times [0,1]$, 1-periodic in the first argument t. We devide the domain $[0,1) \times [0,1]$ into rectangles using nodes

$$\{(t_k, z_l) = (h_t(k-1), h_z(l-1)),\ k = 1,\dots,n,\ l = 1,\dots,m\},$$

where $h_t = 1/n$ and $h_z = 1/(m-1)$. For the approximation of $v(t,z)$ we introduce an N-dimensional subspace of functions H_N :

$$H_N = \left\{ v_N(t,z) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{m} y_{(i_1,i_2)} \phi_{(i_1,i_2)}(t,z) = \Phi y,\ y \in R^N,\ N = n \cdot m \right\}.$$

As $\phi_{(i_1,i_2)}(t,z)$ one can use the piecewise constant, linear or quadratic functions. The Galerkin method for (16) leads to:

Find $v_N \in H_N$ such that the Galerkin equations

$$< Av_N, w >_0 = < f, w >_0, \tag{18}$$

hold for all $w \in H_N$.

The equations (18) are replaced by the algebraic system

$$Ay = b,\ A \in R^{N \times N},\ y, b \in R^N, \tag{19}$$

where the matrix A is symmetric and positive definite having a block structure

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \cdots\cdots\cdots\cdots & & \\ A_{m1} & \cdots & A_{mm} \end{pmatrix},\ A_{i_2 j_2} \in R^{n \times n}.$$

ROTATIONAL DOMAIN

Let Γ be given by a parametric representation

$$\Gamma = \left\{ x \in R^3,\ x = \begin{pmatrix} R(z)\cos 2\pi t \\ R(z)\sin 2\pi t \\ z \end{pmatrix},\ 0 \le t < 1,\ 0 \le z \le 1 \right\}.$$

That is, Γ originates from the rotation of the curve

$$\{R(z),\ 0 \le z \le 1\}$$

about the z-axis. in this case the matrix A has some special properties: each block $A_{i_2 j_2}$ of A is symmetric and circulant, i.e. the matrix A is symmetric, positive definite, block circulant matrix with symmetric blocks. Such matrices are investigated in (Meyer, Rjasanow [16], Rjasanow [9]), where very efficient algorithms for the numerical solution of linear systems of equations are constructed.

The matrix A has the following form

$$A = A(J) = \begin{pmatrix} a_{11}(J) & \cdots & a_{1m}(J) \\ \cdots\cdots\cdots\cdots & & \\ a_{m1}(J) & \cdots & a_{mm}(J) \end{pmatrix},$$

where each block is a polynomial function (13) of the matrix J defined in (12). If we solve n eigenvalue problems

$$A(\omega_k)x_l(\omega_k) = \lambda_l(\omega_k)x_l(\omega_k), \quad l = 1,\ldots,m \qquad (20)$$
$$\omega_k = e^{i\frac{2\pi}{n}(k-1)}, \quad k = 1,\ldots,n$$

with symmetric matrices $A(\omega_k) \in R^{m\times m}$ (Meyer, Rjasanow [16]), the matrix $A(J)$ can be given by the formula:

$$A = \frac{1}{n}\sum_{k,l}\frac{1}{\lambda_l(\omega_k)}\left(x_l(\omega_k)\otimes f_k\right)\left(x_l(\omega_k)\otimes f_k\right)^*, \qquad (21)$$

where f_k are the eigenvectors of the matrix J:

$$Jf_k = \omega_k f_k, \; f_k = (\omega_k^0, \omega_k^1, \ldots, \omega_k^{n-1})^T,$$

and $\otimes$ is the Kronecker product.

The algorithm to compute the solution y is the following:

1. $C := (F^*B)^T \in C^{m\times n}$;
2. $Y := 0$;
3. for $l = 1,\ldots,m$

   3.1 $D_l = diag(d_1,\ldots,d_n)$ with $d_k = \dfrac{x_l^T(\omega_k)Ce_k}{n\lambda_l(\omega_k)}$;

   3.2 $Y := Y + Re(FD_lX_l^T)$,

where

$$X_l = (x_l(\omega_1)\vdots\ldots\vdots x_l(\omega_n)) \in R^{m\times n} \quad - \text{eigenvectors of (20)};$$
$$B = (b_1\vdots\ldots\vdots b_m) \in R^{n\times m} \quad - \text{given right side of (19)};$$
$$Y = (y_1\vdots\ldots\vdots y_m) \in R^{n\times m} \quad - \text{solution of (19)};$$
$$e_k = (0,\ldots,0,1,0,\ldots,0)^T \in R^m.$$

We use FFT for the steps 1. and 3.2 which requires $O(m^2 n\log n) = O(h^{-3}\log h^{-1})$ arithmetical operations for the entire algorithm. Furtheremore we need only $O(nm^3) = O(h^{-4})$ arithmetical operations (Garbow, Boyle, Dongarra, Moler [17], Smith, Boyle, Garbow, Ikebe, Klema, Moler [18]) to solve all eigenvalue problems (20).

NUMERICAL EXPERIMENTS

The presented methods for the iterative solution of systems of linear equations arising from boundary element equations have been used for the Dirichlet problem in two-dimensional domain Ω with the boundary

$$\Gamma = \left\{x \in R^2, \; x = \frac{1}{2}\begin{pmatrix}\cos 2\pi t \\ \sin 2\pi t(2 - \frac{3}{2}\sin 2\pi t)\end{pmatrix}, \; 0 \le t < 1\right\}$$

and in three-dimensional domain Ω with the boundary

$$\Gamma = \left\{ x \in R^2,\ x = \frac{1}{2} \begin{pmatrix} R(z)\cos 2\pi t \\ R(z)\sin 2\pi t(2 - \frac{3}{2}\sin 2\pi t) \\ z \end{pmatrix},\ 0 \le t < 1,\ 0 \le z \le 1 \right\}$$

with

$$R(z) = \sqrt{z(1-z)}.$$

We have used the **piecewise** linear ($\nu = 1$) in two- and piecewise constant ($\nu = 0$) functions in three-dimensional case for approximation $v(x)$. We have used the Galerkin method for the discretization of two-dimensional problem and collocation method (panel method) for three-dimensional problem.
The tables show the number of iterations (It) required for the accuracy $\epsilon = 10^{-8}$ and the corresponding computing times (CPU) measured in CPU seconds on the INMOS T800-transputer for the conjugate gradient method in two- and gradient method in three-dimensional case. We consider both methods with and without ($B = I$) preconditioning. The following tables also show the computing times for Cholesky and Gaussian elimination.

Table 1 *Two-dimensional case*

	without		with		Cholesky
n	It	CPU	It	CPU	CPU
16	18	0.07	10	0.16	0.02
32	30	0.39	10	0.40	0.08
64	44	1.94	11	1.16	0.46
128	61	9.97	11	3.35	3.03
256	82	51.42	11	10.59	21.83
512	114	279.56	11	36.12	165.16

Table 2 *Three-dimensional case*

	without		with		Gaussian
n	It	CPU	It	CPU	CPU
32	50	0.58	14	1.03	0.11
64	69	2.02	19	3.45	0.75
128	75	11.43	19	12.98	5.50
256	140	81.76	31	54.67	42.13
512	167	382.72	32	159.73	329.55

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