Preconditionning the Helmholtz problem on curved arcs in dimension 2

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Abstract

We introduce new preconditioners for the first kind integral equations involved in the resolution of the Dirihclet and Neumann problems on open arcs in the plane, both for Laplace and Helmholtz equations. We use weighted versions of the layer potentials, and introduce our preconditioners as square roots of local operators, in close analogy to the case of closed curves.

Introduction

The problem of preconditionning the linear systems coming from the discretization of first kind integral equations has received considerable attention since two decades roughly. Among the possible strategies are the so-called pseudo-differential preconditionner, whose analysis uses tools from pseudo-differential calculus [**?**, **?**, **?**, **?**]. Roughly speaking, if the original problem is written in the abstract way

$$\mathcal{L}u = f\,, \tag{1}$$

the strategy consists in findind a suitable operator $\mathcal{K}$ such that, when left mulitplying the (1) by $\mathcal{K}$, one needs to solve

$$\mathcal{K}\mathcal{L}u = \mathcal{K}f\,. \tag{2}$$

Now if $\mathcal{K}\mathcal{L}$ is a compact perturbation of the identity, the condition number of the discretized underlying system is independant of the chosen size of the mesh, leading to optimal convergence rate of the numerical methods used to solve the system.

Several strategies, depending on the problem to solve have been studied in the literature [] to propose such operators $\mathcal{K}$, that often turn out to be very effective in practice, when numerical applications are considered. However, all the preceding results and theories are limited to smooth domains and very little is known when the integral equation is posed on domains with corners (in 2D), wedges or conical points (in 3D). One of the reason might be the fact that pseudo-differential calculus is difficult to generalize on such manifolds, and more problematic, the underlying operators are difficult to analyze on such domains.

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Nevertheless, a program similar to those given on smooth manifold was proposed a few years ago [?, ?, ?, ?] who tackled the problem of preconditionning the first or second layer potential for Laplace or Helmholtz equation in dimension 2 or 3 but for very particular domains: a straight and then curved segment in 2D and a unit disc in 3D. Here, we introduce a new kind of preconditioners for the weighted layer potentials studied in many other works such as [4]. We include numerical results to test their efficiency.

In the first section, we give an overview of the resolution of the Laplace and Helmholtz scattering problems by means of layer potentials on singular domains. In the second, we restrict our attention to the case of the Laplace operator in the case of the unit segment. We then generalize our results to non-zero frequency and non-flat arc in the third section. The fourth section describes a simple Galerkin setting to test our preconditioners. Numerical results are exposed in the last section.

Peaufiner l'intro

Throughout this article, C refers to a constant which specific value can change from line to line.

1 The scattering problem outside an open curve

Let Γ be a smooth non-intersecting open curve, and let $k \geq 0$ the wave number. We seek a solution of the problem

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \tag{3}$$

when one considers furthermore

- Dirichlet or Neumann boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \tag{4}$$

or

$$\frac{\partial u}{\partial n} = u_N \text{ on } \Gamma \tag{5}$$

respectively.

- Suitable decay at infinity, given by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{6}$$

with $r = |x|$ for $x \in \mathbb{R}^2$.

In the preceding equation n stands for a smooth unit normal vector to Γ.

Existence and uniqueness of solutions to the previous problems are guaranteed by the following theorem.

Theorem 1. *(see e.g. [11, 15, 16]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (3,4,6) and (3,5,6) both possess a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

Near an edge of the arc, μ is locally given by

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi.$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$ and C.

For the definition of Sobolev spaces on smooth open curves, we follow [10] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma}) \right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential We define the single-layer potential by

$$\mathcal{S}_k\lambda(x) = \int_\Gamma G_k(x-y)\lambda(y)d\sigma(y) \tag{7}$$

where G_k is the Green's function defined by

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi}\ln|z|, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4}H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{8}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here H_0 is the Hankel function of the first kind. The solution u of the Dirichlet problem can be computed from (3,4,6) as

$$u = \mathcal{S}_k\lambda \tag{9}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the unique solution to

$$S_k\lambda = u_D\,. \tag{10}$$

Here, $S_k := \gamma\mathcal{S}_k$ where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k\mu(x) = \int_\Gamma n(y)\cdot\nabla G_k(x-y)\mu(y)d\sigma(y).$$

for any smooth function μ defined on γ. The normal derivative of $\mathcal{D}_k\mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \partial_n\mathcal{D}_k$. This operator admits the representation for $x \in \Gamma$

$$N_k\mu = \lim_{\varepsilon\to 0^+}\int_\Gamma n(y)\cdot\nabla G(x+\varepsilon n(x)-y)\mu(y)d\sigma(y). \tag{11}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\langle N_k\mu,\nu\rangle = \int_{\Gamma\times\Gamma} G_k(x-y)\mu'(x)\nu'(y) - k^2 G_k(x,y)\mu(x)\nu(y)n(x)\cdot n(y)d\sigma(x)d\sigma(y)\,. \tag{12}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u to the Neumann problem (3,5,6) can be written as

$$u = \mathcal{D}_k\mu \tag{13}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ solves

$$N_k\mu = u_N\,. \tag{14}$$

2 Laplace equation on a flat segment

In this section, we restrict our attention to the case where Γ is the open segment $(-1,1)\times\{0\}$, and $k=0$. We study the properties of the equations

$$S\lambda = -u_D$$

and

$$N\mu = u_N$$

and show their invertibility in a range of Sobolev-like spaces. This problem has been already considered thoroughly, both in terms of analytical and numerical properties in the literature (see for instance [4, 8]), and it turns out that the Chebyshev polynomials of first and second kind play a very important role. However, we go further compared to the literature by by constructing a functional framework close to Sobolev spaces, based on Chebyshev polynomials that allows us to give a complete framework for the existence and uniqueness of the solutions to the preceding equations, as well as new preconditioners.

2.1 Analytical setting

Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n\arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}\,.$$

We let $\omega(x) = \sqrt{1-x^2}$. We also denote by ω the operator $u(x)\mapsto\omega(x)u(x)$. Moreover, let ∂_x the derivation operator, the Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)T_n'' - xT_n' + n^2T_n = 0 \text{ and } (1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0$$

that we rewrite under the form

$$(\omega\partial_x)^2 T_n = -n^2 T_n\,, \tag{15}$$

$$(\partial_x\omega)^2 U_n = -(n+1)^2 U_n\,. \tag{16}$$

(Notice that by $(\partial_x \omega) f$ we mean $(\omega f)'$.) As we shall see, the preceding equations are crucial in our analysis.

Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\mathrm{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\mathrm{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} f^2(x)\sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_{\omega}$ the inner products in $L^2_{\frac{1}{\omega}}$ and L^2_{ω} respectively. Chebyshev polynomials also satisfy

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } m = n = 0 \\ \pi/2 \text{ otherwise} \end{cases} \tag{17}$$

and

$$\int_{-1}^{1} U_n(x) U_m(x) \sqrt{1-x^2} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi/2 \text{ otherwise} \end{cases} \tag{18}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{19}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by

$$\hat{u}_n := \begin{cases} \dfrac{2}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x) T_n(x)}{\sqrt{1-x^2}} dx & \text{if } n \neq 0\,, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x)}{\sqrt{1-x^2}} dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^{1} \frac{u^2(x)}{\sqrt{1-x^2}} dx = \frac{\pi \hat{u}_0^2}{2} + \pi \sum_{n=1}^{+\infty} \hat{u}_n^2\,.$$

When u is furthermore a smooth function, the series (19) converges uniformly to u. Similarly, any function $v \in L^2_{\omega}$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^{1} v(x) U_n(x) \sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^{1} v^2(x)\sqrt{1-x^2}dx = \frac{\pi}{2}\sum_{n=0}^{+\infty} \check{v}_n^2 \,.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we may define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^s \hat{u}_n^2 < +\infty \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u, v\rangle_{T^s} = \frac{\pi}{2}\hat{u}_0\hat{v}_0 + \pi\sum_{n=1}^{+\infty}(1+n^2)^s\hat{u}_n\hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty} n^{2s}\left|\hat{u}_n\right|^2 .$$

We denote by T^∞ the Fréchet space $T^\infty := \cap_{s\in\mathbb{R}}T^s$, and by $T^{-\infty}$ the set of continuous linear forms on T^∞. For $l \in T^{-\infty}$, we note $\hat{l}_n = l(T_n)$, so that for $u \in T^\infty$,

$$l(u) = \frac{\pi}{2}\hat{l}_0\hat{u}_0 + \pi\sum_{n=1}^{+\infty}\hat{l}_n\hat{u}_n \,.$$

We choose to identify the dual of $L^2_{\frac{1}{\omega}}$ to itself using the previous bilinear form. With this identification, any element of T^s with $s \geq 0$ can also be seen as an element of $T^{-\infty}$. Furthermore, the space T^{-s} can be defined for all $s \geq 0$ as

$$T^{-s} = \left\{ u \in T^{-\infty} \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^{-s}\hat{u}_n^2 < \infty \right\}.$$

Using the preceding identification T^{-s} becomes the dual of T^s. Obviously, for any real s, if $u \in T^s$, the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N}\hat{u}_n T_n(x)$$

converges to u in T^s. Therefore, the space $C^\infty([-1,1])$ is dense in T^s for all $s \in \mathbb{R}$. For $s < t$, the inclusion $T^s \subset T^t$ is compact.

In a similar fashion, we define the following spaces:

Definition 2. *For all $s \geq 0$, we set*

$$U^s = \left\{ u \in L^2_{\omega} \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^s \check{u}_n^2 \right\}.$$

We extend as before the definition to negative indices by setting U^{-s} to be the dual of U^s for $s \geq 0$.

Lemma 1. *For all real s, $T^s \subset U^s$ and for all $s > 1/2$, $U^s \subset T^{s-1}$.*

Proof. The first property is immediate once it has been noticed that for $n \geq 2$, $T_n(x) = \frac{1}{2}(U_n - U_{n-2})$, while $T_0 = U_0$ and $T_1 = \frac{U_1}{2}$. The second comes from the expressions

$$U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}.$$

When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging, allowing to identify u to a function in $T^{-\infty}$, with

$$\hat{u}_0 = \sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2\sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since $u \in U^s$, $(1+n^2)^{s/2} |\check{u}|$ is in l^2 and by continuity of the adjoint of the Cesàro operator in l^2, the sequence $r_n := \left(\sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k|\right)_n$ is in l^2. But

$$\begin{aligned}
\|u\|^2_{T^{s-1}} &= \sum_{n=0}^{+\infty} (1+n^2)^{s-1} |\hat{u}_n|^2 \\
&\leq 4\sum_{n=0}^{+\infty} (1+n^2)^{s-1} \left(\sum_{k=n}^{+\infty} |\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty} \left(\sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k|)\right)^2. \\
&= 4\|r_n\|^2_{l^2}.
\end{aligned}$$

□

One immediate consequence is that $T^\infty = U^\infty$. Moreover, we have the following result:

Lemma 2.

$$T^\infty = C^\infty([-1,1]).$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (15), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^{1} \frac{(\omega\partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, this proves that $C^\infty([-1,1]) \subset T^\infty$.

To prove the converse inclusion, first notice that, by normal convergence of the series, $T^\infty \subset C^0([-1,1])$. Now, let $u \in T^\infty$, it suffices to show that $u' \in T^\infty$ and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty} n u_n U_{n-1}(x).$$

Therefore, u' is in $U^\infty = T^\infty$. □

Proposition 1. *If ψ is a C^∞ function on $[-1,1]$, then the operator*

$$u(t) \mapsto \psi(t)u(t)$$

is of order 0*, and for any $s \in \mathbb{R}$,*

$$\|\psi u\|_{T^s} \leq C2^{|s|/2} \|u\|_{T^s} \|\psi\|_{T^{|s|+1}} .$$

where C is independent of ψ and s.

Proof. Let $u \in T^s$, we rewrite u as

$$u = \sum_{n=-\infty}^{+\infty} u'_n T_n$$

where for $n < 0$ we define $T_n = T_{|n|}$, and with

$$u'_n = \begin{cases} u_0 & \text{if } n = 0 \\ \frac{u_{|n|}}{2} & \text{otherwise.} \end{cases}$$

We apply the same idea to ψ, and using $T_m T_n = T_{m+n} + T_{m-n}$,

$$\psi u = \sum_{m,n} u'_n \psi'_m (T_{m+n} + T_{m-n}) = \sum_m \left(\sum_n u'_n (\psi'_{n+m} + \psi'_{n-m}) \right) T_m$$

With the same convenstions as for ψ and u, let us compute $(\psi u)'_n$. For $n \neq 0$, we have

$$(\psi u)'_n = \sum_m u'_m \psi'_{m+n} + \sum_n u'_n \psi'_{m-n},$$

$$(\psi u)'_0 = 2 \sum_m u'_m \psi'_m.$$

Thus, we have, for all $n \in \mathbb{Z}$, $(\psi u)'_n = S^1_n + S^2_n$ with $S^n_1 = \sum_m u'_m \psi'_{n+m}$ and $S^n_2 = \sum_m u'_m \psi'_{n-m}$. Applying Peetre's inequality, we get, for all $n \in \mathbb{Z}$,

$$(1+n^2)^{s/2} \left|S^1_n\right| \leq 2^{|s|/2} \sum_m (1+m^2)^{s/2} \left|u'_m\right| (1+|m+n|^2)^{|s/2|} \left|\psi'_{m+n}\right|$$

and by Young's inequality with $r = 2, p = 2, q = 1$,

$$\sum_{n=-\infty}^{+\infty} (1+n^2)^s \left|S^1_n\right|^2 \leq 2^{|s|} \|u\|^2_s \left(\sum_{m=-\infty}^{+\infty} (1+m^2)^{|s|/2} \left|\psi'_m\right| \right)^2$$

The last sum is finite because $\psi \in T^{|s|+1}$ and

$$\sum_{m=-\infty}^{+\infty} (1+m^2)^{|s|/2} |\psi'_m| \leq \sqrt{\left(\sum_{m=-\infty}^{+\infty} \frac{1}{1+m^2} \right) \left(\sum_{m=-\infty}^{+\infty} (1+m^2)^{|s|+1} |\psi'|^2_m \right)}.$$

Similar computations lead to the same estimate for S^2_n and thus,

$$\|\psi u\|^2_{T^s} = \sum_{m=-\infty}^{+\infty} (1+m^2)^s \left|(\psi u)'_m\right|^2 \leq C2^{|s|} \|u\|^2_{T^s} \|\psi\|^2_{T^{|s|+1}} .$$

$\square$

Lemma 3. *Let $g(x,y)$ the kernel of an integral operator of order $\alpha \in \mathbb{R}$ in the spaces T^s, that is,*

$$G : u \mapsto \int_{-1}^{1} \frac{g(x,y)u(y)}{\omega(y)} dy$$

is continuous from T^s to $T^{s+\alpha}$ for all s. Let $R(x,y)$ a C^∞ function. Then the operator

$$K : \int_{-1}^{1} \frac{g(x,y)R(x,y)u(y)}{\omega(y)} dy$$

is of order α.

Proof. Since R is in C^∞, one can show that R admits the following expression:

$$R(x,y) = \sum_{m,n} r_{m,n} T_m(x) T_n(y) \tag{20}$$

Moreover, the regularity of R ensures $r_{m,n}$ satisfies for all $s,t \in \mathbb{R}$,

$$\sum_{m,n} (1+m^2)^s (1+n^2)^t \left| r_{m,n} \right|^2 < +\infty.$$

To prove this property, one can for example apply the operator $(\omega\partial_x)^2$ repeatedly in the two variables. The resulting function is C^∞, and in particular, square integrable on $[0,1] \times [0,1]$. We then write the Parseval's identity and the result follows. We can thus write

$$Ku = \sum_{m,n} r_{m,n} T_m G T_n u$$

where for each m,n, the operator $T_m G T_n$ is defined by

$$T_m G T_n u(x) = T_m(x) \int_{-1}^{1} \frac{G(x,y) T_n(y) u(y)}{\omega(y)} dy.$$

This operator is in $L(T^s, T^{s+\alpha})$ by the previous lemma, with

$$\left\| T_m G T_n \right\|_{T^s \to T^{s+\alpha}} \le \left\| G \right\|_{T^s \to T^{s+\alpha}} 2^{|s|+|s+\alpha|} (1+n^2)^{|s|+1} (1+m^2)^{|s+\alpha|+1}.$$

thus, the series in (20) is normally convergent in $L(T^s, T^{s+\alpha})$, which proves the claim. □

Lemma 4. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and there exists a constant C such that for all $x \in [-1,1]$,*

$$|u(x)| \le C \left\| u \right\|_{T^{1/2+\varepsilon}}.$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$|u(x)| \le C \left\| u \right\|_{U^{3/2+\varepsilon}}.$$

Proof. We write

$$|u(x)| \le \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\|T_n\|_{L^\infty} = 1$. Cauchy-Schwarz's inequality then yields

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \, \|u\|_{T^{\frac{1}{2}+\varepsilon}} \, .$$

For the second statement, we use the inclusion $U^s \subset T^{s-1}$ valid for $s > 1/2$, as established in Lemma 1. ☐

Lemma 5. *We have $L^2_\omega = \frac{1}{\omega} L^2_{\frac{1}{\omega}}$ and the pointwise multiplication by $\frac{1}{\omega}$ defines an bijective isometry from $L^2_{\frac{1}{\omega}}$ to L^2_ω with inverse ω.*

Lemma 6. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s as*

$$\langle \partial_x u, v \rangle_\omega := - \langle u, \omega \partial_x \omega v \rangle_{\frac{1}{\omega}} \, .$$

For all real s, the operator $\omega \partial_x \omega$ can be extended into a continuous operator from $U^{s+1} \to T^s$ as

$$\langle \omega \partial_x \omega u, v \rangle_{\frac{1}{\omega}} := - \langle u, \partial_x v \rangle_\omega \, .$$

Proof. First, note that if $v \in T^\infty$, then $\omega \partial_x \omega v = (1-x^2)v' - xv \in T^\infty$. The definition of ∂_x is thus correct. The claimed continuity is an immediate consequence of

$$\omega \partial_x \omega U_n = -(n+1) T_{n+1}.$$

The second assertion comes from $T_n' = n U_{n-1}$. ☐

The following corollary will be used in our Galerkin analysis:

Corollary 1. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$.*

Proof. For the first case we use ∂_x is continuous from T^{s+2} to U^{s+1} and then the identity is continuous from U^{s+1} to T^s. For the second, we use the same arguments in the reverse order. ☐

We now provide a characterization of the spaces T^s and U^s. For this, we need the following result:

Corollary 2. *The operator $\omega \partial_x$ defined on T^∞ as*

$$\omega \partial_x u = \omega u'$$

has a continuous extension from T^1 to T^0. Similarly, the operator $\partial_x \omega$ has a continuous extension from U^1 to U^0.

Proof. For the first part, we use the fact that ∂_x is continuous from T^1 to U^0, and, by Lemma 5, ω is continuous from U^0 to T^0. For the second part, we use the fact that $\omega \partial_x \omega$ is continuous from U^1 to T^0, and from the same lemma, the multiplication by $\frac{1}{\omega}$ is continuous from T^0 to U^0. ☐

For a function u defined on $[a,b]$, we denote by $\tilde{u}$ the function defined on $[\theta_a, \theta_b] := [\arccos(a), \arccos(b)]$ as

$$\tilde{u}(\theta) = u(\cos(\theta))$$

and by Vu the function defined as

$$Vu(\theta) := \sin(\theta)\tilde{u}(\theta).$$

We then have the following characterization:

Theorem 2. *A function u belongs to the space T^n if and only if $u = \tilde{u} \circ \arccos$ for some even function $\tilde{u}$ satisfying $\tilde{u} \in H^n(-\pi, \pi)$. In this case,*

$$\|u\|_{T_n} \sim \|\tilde{u}\|_{H^n}, |u|_{T^n} \sim |\tilde{u}|_{H^n}.$$

Similarly, u belongs to $\in U^n$ if and only if $u = \frac{1}{\sqrt{1-x^2}} Vu \circ \arccos$ for some odd function Vu in $H^n(-\pi, \pi)$. In this case,

$$\|u\|_{U_n} \sim |Vu|_{H^n}.$$

Moreover, of $u \in T^n$, then $(\omega\partial_x)^n u$ is in $L^2_{\frac{1}{\omega}}$ and

$$|u|^2_{T^n} \sim \int_{-1}^{1} \frac{((\omega\partial_x)^n u)^2}{\omega}.$$

If $u \in U^n$, $(\partial_x \omega)^n \in L^2_\omega$ and

$$\|u\|_{U_n} = \int_{-1}^{1} \omega((\partial_x\omega)^n u)^2.$$

Proof. The first two equivalences stem from the fact that

$$\hat{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{u}(\theta)\cos(n\theta), \check{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Vu(\theta)\sin((n+1)\theta)d\theta,$$

which can be verified by using the change of variables $x = \cos\theta$ in the definitions of $\hat{u}_n$ and $\check{u}_n$. Now, let us show that if $u \in T^n$, then $(\omega\partial_x)^n$ is in $L^2_{\frac{1}{\omega}}$. The operator $(\omega\partial_x)^2$ is continuous from T^s to T^{s-2} for all real s which implies the result if n is even. If n is odd, say $n = 2k+1$, we write $(\omega\partial_x)((\omega\partial_x)^2)^k$, and conclude using Corollary 2. The same kind of proof also shows that if $u \in U^n$, $(\partial_x\omega)^n u \in L^2_\omega$. The rest of the proof can be performed by computing the quantities for functions in T^∞, perform integrations by part and conclude with the density of C^∞ in T^s and U^s. □

Definition 3. *If $A : T^\infty \to T^{-\infty}$ can be extended into a continuous operator from T^s to T^{s+p} for any $s \in \mathbb{R}$, we shall say that it is of order p. When an operator is of order p for all $p \in \mathbb{N}$, we call it a smoothing operator. When A and B are such that there exists a smoothing operator R such that*

$$A = B + R,$$

we write $A = B \mod T^\infty$.

2.2 Single layer equation

In this section we focus on the equation $S\lambda = g$, that is we seek $\lambda \in \tilde{H}^{-1/2}$ such that

$$-\frac{1}{2\pi}\int_{-1}^{1} \log|x-y|\lambda(y) = -g(x), \quad \forall x \in (-1,1)\,. \tag{21}$$

This equation is sometimes called "Symm's integral equation" and its resolution has received a lot of attention in the 1990's. Numerical methods, using both collocation and Galerkin have been presented and analyzed [2, 14, 18–20].

Our analysis lies on the following formula. For a proof, see for example [9] Theorem 9.2. Note that this is also the main ingredient in several connected works, such as [8] and [4].

Proposition 2.

$$-\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln|x-y|}{\sqrt{1-y^2}} T_n(y)dy = s_n T_n(x)$$

where

$$s_n = \begin{cases} \frac{\ln(2)}{2} & \textit{if } n = 0 \\ \frac{1}{2n} & \textit{otherwise.} \end{cases}$$

Using the decomposition of g and of the logarithmic kernel on the basis T_n, we see that the solution λ to equation (21) admits the following expansion

$$\lambda(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \frac{\hat{g}_n}{s_n} T_n(x). \tag{22}$$

We deduce the following well-known fact:

Corollary 3. *If the data g is in $C^\infty([-1,1])$, the solution λ to the equation*

$$S\lambda = g$$

is of the form

$$\lambda = \frac{\alpha}{\sqrt{1-x^2}}$$

with $\alpha \in C^\infty([-1,1])$.

Proof. Let $\alpha = \sqrt{1-x^2}\lambda$ where λ is the solution of $S\lambda = g$ with $g \in C^\infty([-1,1])$. Lemma 2 implies that $g \in T^\infty$, and by equation (22),

$$\hat{\alpha}_n = \frac{\hat{g}_n}{s_n},$$

so α also belongs to $T^\infty = C^\infty([-1,1])$. □

We follow [4] by noticing that the behavior in $\frac{1}{\sqrt{1-x^2}}$ is consistent with the expected singularity near the edges and introduce the weighted single layer operator as the operator that appeared in Proposition 2.

Definition 4. *(See [4]) Let S_ω be the weighted single layer operator defined by*

$$S_\omega : \alpha \in C^\infty([-1,1]) \longrightarrow -\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln|x-y|}{\omega(y)}\alpha(y)dy$$

We also recall that the operator $(\omega\partial_x)^2$ is defined by

$$(\omega\partial_x)^2 : \alpha \in C^\infty([-1,1]) \longrightarrow (1-x^2)\alpha''(x) - x\alpha'(x).$$

The action of these operators on T^∞ is easy to analyze using (15) and Proposition 2. By density of T^∞ in T^s for all s, we get:

Proposition 3. *The operator S_ω is a self-adjoint, positive definite operator, and defines a continuous bijection from T^s to T^{s+1} for all real s. In particular, S_ω is of order 1 and is compact in T^s. Similarly, for any $s \in \mathbb{R}$, the operator $-(\omega\partial_x)^2$ is positive, self-adjoint, and of order -2.*

Proof. It suffices to remark that if $u = \sum_{n=0}^{\infty} \hat{u}_n T_n \in T^s$, then

$$S_\omega u = \frac{\ln(2)}{2}\hat{u}_0 T_0 + \sum_{n=1}^{\infty} \frac{\hat{u}_n}{2n} T_n$$

while

$$-(\omega\partial_x)^2 u = \sum_{n=0}^{\infty} n^2 \hat{u}_n T_n\,.$$

□

To obtain the solution of (21), we can thus solve

$$S_\omega \alpha = -u_D, \tag{23}$$

and let $\lambda = \frac{\alpha}{\omega}$.

We are now in a position to find an expression for the inverse of S_ω. An explicit inverse has already appeared in the literature. In particular, in [6, 17], explicit variational forms for this inverse operator are derived rigorously. (A similar method is also employed in the recent paper [5] in $\mathbb{R}^3$ for the case of the unit disk.) We just state here the following formal decomposition:

$$\frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|^2} = \frac{-1+xy}{2|x-y|^2} = \sum_{n=1}^{+\infty} nT_n(x)T_n(y)$$

with $M(x,y) = \frac{1}{2}\left((y-x)^2 + (\omega(x)+\omega(y))^2\right)$.

However, using the preceding analysis, we have an alternative way of defining this exact inverse, which leads to an expression in the form of the square root of a local operator. To state the next result, we define the operator π_0 as the $L^2_{1/\omega}$ orthogonal projector on T_0. Namely

$$\pi_0\alpha(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\alpha(y)}{\omega(y)}dy\,.$$

The preceding definition can be extended to $u \in T^s$ for any $s \in \mathbb{R}$ by setting $\pi_0 u$ as the solution of

$$\begin{cases} \langle \alpha - \pi_0\alpha, T_0\rangle_{\frac{1}{\omega}} = 0\,, \\ \pi_0\alpha \in \mathrm{Span}(T_0)\,, \end{cases}$$

since $T_0 \in T^\infty$. Of course, π_0 is continuous from T^s to T^∞ for any s.

Theorem 3. *The operators S_ω, $-(\omega\partial_x)^2$, and π_0 commute and satisfy*

$$S_\omega \left(-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0 \right) S_\omega = \frac{I}{4}\,. \tag{24}$$

Proof. The Chebyshev polynomials (T_n) are a common Hilbert basis of eigenvectors for the three operators $S\omega$, $-(\omega\partial_x)^2$ and π_0, so they all commute on $T^{-\infty}$. Moreover, one has using Proposition 2, equation (15) and the definition of π_0

$$S_\omega T_n = \frac{1}{2n} T_n\,,\ \pi_0 T_n = 0 \text{ and } -(\omega\partial_x)^2 T_n = n^2 T_n \text{ if } n \neq 0\,,$$

while

$$S_\omega T_0 = \frac{\ln(2)}{2} T_0\,,\ \pi_0 T_0 = T_0 \text{ and } -(\omega\partial_x)^2 T_0 = 0 \text{ otherwise.}$$

Equation (24) follows from the fact that (T_n) is a Hilbert basis of $T^{-\infty}$. □

Remark 1. *A direct proof of the commutation of S_ω and $-(\omega\partial_x)^2$ can be done. We give it in the more general case of Helmholtz equation (see Theorem 5).*

From the preceding formula, we can extract the explicit inverse of S_ω in terms of the square root of the inner operator.

Definition 5. *Let an operator $A : T^s \to T^{s+2p}$ such that*

$$AT_n = a_n T_n$$

with $a_n \geq 0$. We define $\sqrt{A} T^s \to T^{s+p}$ as the operator satisfying

$$\sqrt{A} T_n = \sqrt{a_n} T_n.$$

Corollary 4. *The inverse of S_ω can be equivalently expressed as*

$$S_\omega^{-1} = 2\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}\,. \tag{25}$$

The last result shows that $\sqrt{-(\omega\partial_x)^2}$ and S_ω can be thought as inverse operators (modulo constant terms) and that, at the very least, $\sqrt{-(\omega\partial_x)^2}$ can be used as a preconditioner for S_ω. Indeed, $2\sqrt{-(\omega\partial_x)^2} S_\omega$ is a compact perturbation of identity. This makes a clear link with the approximation of the Dirichlet to Neumann map proposed in [1] in terms of a square root operator. The link will be even clearer when Helmholtz equation will be considered. In Table 1, we compare the number of iterations for the numerical resolution of Equation (23) by the method detailed in section 5 without preconditioner, and with a preconditioner given by $M^{-1}\,[B]\,M^{-1}$ where M is the mass matrix and $[B]$ is the Galerkin matrix of the operator $\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$. The right hand side in (23) is chosen as $u_D(x) = (x^2 + 0.001)^{-1/2}, x \in (-1, 1)$.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
25	7	0.1	25	0.14
50	7	0.11	35	0.21
100	7	0.12	44	0.36
200	7	0.14	53	0.55
400	7	0.17	64	0.95
800	7	0.29	76	2.0
1600	7	0.51	90	4.3
3200	7	0.95	107	9.5

Table 1: Number of iteration and time needed for the numerical resolution of (23) using Galerkin finite elements with and without preconditioner.

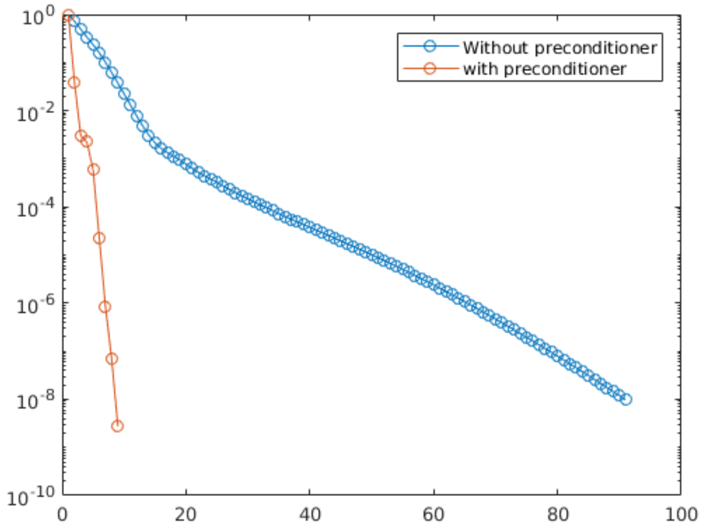

Figure 1: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N = 1600$.

2.3 Hypersingular equation

We now turn our attention to the equation

$$N\mu = g \tag{26}$$

Similarly to the previous section and following again the idea of [4], we consider a rescaled version of the hypersingular operator $N_\omega := N\omega$ defined by

$$N_\omega \mu = \lim_{\varepsilon \to 0} \int_{-1}^{1} n(y) \cdot \nabla G(x + \varepsilon n(x) - y) \sqrt{1 - y^2} dy$$

We can get the solution to equation (26) by solving

$$N_\omega \beta = u_N, \tag{27}$$

and letting $\mu = \omega\beta$. We now show that N_ω can also be analyzed in our functional framework, using this time the spaces U^s.

Lemma 7. *For any β, β', one has*

$$\langle N_\omega \beta, \beta' \rangle_\omega = \langle S_\omega \omega \partial_x \omega \beta, \omega \partial_x \omega \beta' \rangle_{\frac{1}{\omega}} .$$

Proof. It is sufficient to show this formula for β and β' in U^∞ by density. Indeed, for such β, β', both sides of the identity define continuous bilinear forms on T^∞. We use the well-known integration by part formula

$$\langle Nu, v \rangle = \langle S\partial_x u, \partial_x v \rangle ,$$

valid when u and v vanish at the extremities of the segment (see for example [4]). For a smooth β, we thus have

$$\langle N(\omega\beta), (\omega\beta') \rangle = \langle S\partial_x(\omega\beta), \partial_x(\omega\beta') \rangle$$

which obviously implies the announced identity. $\square$

Proposition 4. *N_ω is a positive definite, self-adjoint operator continuous from U^s to U^{s-1} for all real s. For all $n \in \mathbb{N}$, we have*

$$N_\omega U_n = \frac{n+1}{2} U_n.$$

Moreover, $-(\partial_x \omega)^2$ is also positive definite of order 2.

Proof. From identity $T'_{n+1} = (n+1)U_n$ and Equation (15) we obtain

$$\omega \partial_x \omega U_n = -(n+1)T_{n+1}.$$

Therefore, by Lemma 7

$$\begin{aligned} \langle N_\omega U_m, U_n \rangle_\omega &= (n+1)(m+1) \langle S_\omega T_{m+1}, T_{n+1} \rangle_{\frac{1}{\omega}} \\ &= \delta_{m=n} \frac{n+1}{2}. \end{aligned}$$

The fact that $-(\partial_x \omega)^2$ is self-adjoint positive definite of order 2 is a consequence of Equation (16). $\square$

As an application of this result, one can also derive the formal expansions as in [7]

$$\frac{1}{(x-y)^2} = \sum_{n=0}^{+\infty} 2(n+1) U_n(x) U_n(y) ,$$

that lead, by applying for $(\partial_x \omega)^{-2}$ on both sides, to the following explicit kernel for the inverse of N_ω:

$$\ln \left(\frac{(y-x)^2 + (\omega(x) + \omega(y))^2}{2|x-y|} \right) = \sum_{n=0}^{+\infty} \frac{2U_n(x)U_n(y)}{n+1}.$$

Here instead, we give a simple expression of the inverse of N_ω as the inverse square root of a local operator:

Theorem 4. *The operators N_ω and $-(\partial_x\omega)^2$ commute and*

$$-N_\omega(\partial_x\omega)^{-2}N_\omega = \frac{I}{4}\,.$$

The inverse of N_ω is therefore

$$N_\omega^{-1} = 2\sqrt{-(\partial_x\omega)^{-2}}\,. \tag{28}$$

In Table 2, we compare the number of iterations for the numerical resolution of Equation (27) by the method detailed in **??** without preconditioner, and with a preconditioner given by $M^{-1}\,[B]\,M^{-1}$ where M is the mass matrix and $[B]$ is the Galerkin matrix of the operator $\sqrt{-(\partial_x\omega)^{-2}}$. The right hand side in (27) is chosen as $u_N(x) = (x^2+0.001)^{1/2}, x \in (-1,1)$.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
25	4	0.07	25	0.14
50	4	0.09	50	0.31
100	4	0.10	100	0.74
200	4	0.12	200	2.0
400	4	0.21	400	7.3
800	4	0.56	799	30
1600	4	2.5	1579	120
3200	4	17.7	3007	630

Table 2: Number of iteration and time needed for the numerical resolution of (23) using Galerkin finite elements with and without preconditioner.

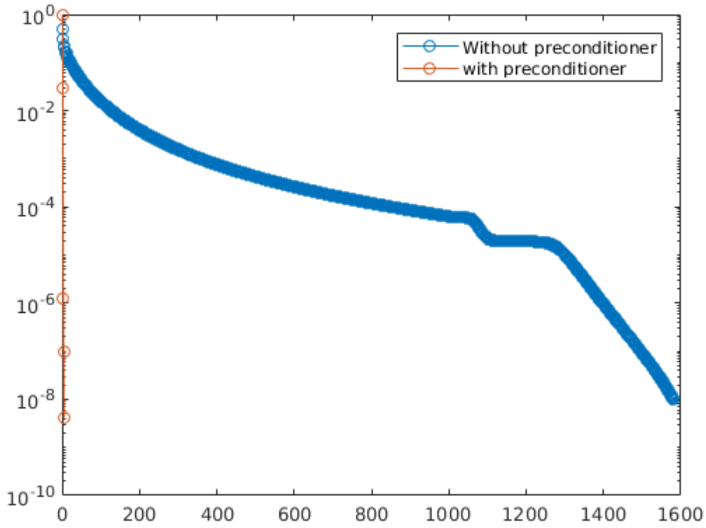

Figure 2: Number of iteration in the resolution of the hypersingular integral equation with a mesh of size $N = 1600$. The importance of preconditioning in this case is more obvious than in the case of the single-layer equation.

Remark 2. *In [4], the main theorem is equivalent to stating that $N_\omega S_\omega$ and $S_\omega N_\omega$ are bicontinuous operators in T^s, with a spectrum concentrated around $\frac{1}{4}$, which can be exploited for preconditioning purposes. This implies that N_ω is continuous from T^s to T^{s-1} for all $s > 1$ (in fact, this remains true for $s > \frac{1}{2}$). The two main arguments involved in the proof are the explicit expression of $N_\omega T_n$ and the continuity of the adjoint of the Cesaro operator in $l^2(\mathbb{N})$. The same arguments can be used to prove that S_ω is also a bicontinuous operator from U^s to U^{s+1} for all $s > 1/2$.*

3 Helmholtz equation on a segment

In this section, we introduce preconditioners for the integral equations on $\Gamma = [-1, 1]$ for the Helmholtz equation. Recall the definition of the single layer and hypersingular operators, S_k and N_k, given in (7) and (11), and the integral equations for the Dirichlet and Neumann problems, (10) and (14). As before, let $S_{k,\omega} := S_k \frac{1}{\omega}$ and $N_{k,\omega} := N_k \omega$. We begin by establishing the following result:

Theorem 5. *The following commutations hold:*

$$S_{k,\omega}\left[-(\omega\partial_x)^2 - k^2\omega^2\right] = \left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega},$$

$$N_{k,\omega}\left[-(\partial_x\omega)^2 - k^2\omega^2\right] = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] N_{k,\omega}.$$

Proof. We start with the first commutation. Since $(\omega\partial_x)^2$ is self adjoint and symmetric, we have

$$S_{k,\omega}(\omega\partial_x)^2 = \int_{-1}^{1} \frac{(\omega_y\partial_y)^2\left[G_k(x-y)\right]u(y)}{\omega(y)},$$

where we use the notation ω_y and ∂_y to emphasize the dependence in the variable y. Thus,

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = \int_{-1}^{1} \frac{D_k(x,y)u(y)}{\omega(y)},$$

where $D_k(x,y) := \left[(\omega_y\partial_y)^2 - (\omega_x\partial_x)^2\right]\left[G_k(x-y)\right]$. One has

$$D_k(x,y) = G_k''(x-y)(\omega_y^2 - \omega_x^2) + G_k'(x-y)(y+x).$$

Since G_k is a solution of the Helmholtz equation, we have for all $(x \neq y) \in \mathbb{R}^2$

$$G_k'(x-y) = (y-x)(G_k''(x-y) + k^2 G(x-y)),$$

thus

$$D_k(x,y) = G_k''(x-y)\left(\omega_y^2 - \omega_x^2 + y^2 - x^2\right) + k^2(y^2-x^2)G_k(x-y).$$

A careful analysis shows that no Dirac mass appears in the previous formula, that is, the previous formula is an equality of two functions in $T^{-\infty}$. Note that $y^2 - x^2 = \omega_x^2 - \omega_y^2$ so the first term vanishes and we find

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = k^2\left(\omega^2 S_{k,\omega} - S_{k,\omega}\omega^2\right).$$

The proof of the second commutation is postponed to Appendix A. □

This theorem implies that the operators $S_{k,\omega}$ and $N_{k,\omega}$ share the same eigenvectors as, respectively, $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$ and $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]$. We can look for eigenfunctions of the operator $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$, to find a diagonal basis for $S_{k,\omega}$. They are the solutions to the differential equation

$$(1-x^2)y'' - xy' = \lambda y.$$

Once we set $x = \cos\theta$, $\tilde{y}(\theta) = y(x)$, $q = \frac{k^2}{4}$, $a = \lambda + 2q$, $\tilde{y}$ is a solution of the standard Mathieu equation

$$\tilde{y}'' + (a - 2q\cos(2\theta))\tilde{y} = 0.$$

There are a discrete set of values $a_{2n}(q)$ for which this equation possesses an even and 2π periodic function. The corresponding solution is known as the Mathieu cosine, and usually denoted by CE_n. Here, we use the notation CE_n^k to emphasize the dependency in the parameter $k = \sqrt{2q}$ of those functions. The normalization is taken as

$$\int_0^{2\pi} \mathrm{CE}_n^k(\theta)^2 d\theta = \pi.$$

They satisfy

$$\int_{-\pi}^{\pi} \mathrm{CE}_n^k(\theta)\mathrm{CE}_m^k(\theta) = \pi\delta_{m,n}.$$

Any even 2π periodic function in $L^2(-\pi,\pi)$ can be expanded along the functions CE_n, with the coefficients obtained by orthonormal projection. Letting

$$T_n^k := \mathrm{CE}_n^k(\arccos(x)),$$

in analogy to the zero-frequency case, we have

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] T_n^k = \lambda_{n,k}^2 T_n^k.$$

For large n, using the general results from the theory of Hill's equations (see [12, eq. 28.29.21]) we have the following asymptotic formula for $\lambda_{n,k}$:

$$\lambda_{n,k}^2 = n^2 - \frac{k^4}{16n^2} + o\left(n^{-2}\right).$$

The first commutation established in Theorem 5 implies that the Matthieu cosines are also the eigenfunctions of the single-layer operator. An equivalent statement is given in [3, Thm 4.2], if we allow the degenerate case $\mu = 0$. Unfortunately, the lack of knowledge about the eigenvalues of $S_{k,\omega}$ prevents us from applying a similar analysis as that performed in the first part of this work. Instead, we will perform a perturbation analysis, much like [4].

3.1 Dirichlet

In this section, we focus on the Dirichlet problem with non-zero frequency, and the corresponding integral equation

$$S_k\lambda = u_D \tag{29}$$

Here again, we define a rescaled operator $S_{k,\omega} := S_k\frac{1}{\omega}$, i.e.

$$S_{k,\omega}\alpha : x \mapsto \int_{-1}^{1} \frac{H_0(k|x-y|)\alpha(y)}{\omega(y)}dy.$$

If we let $\lambda = \dfrac{\alpha}{\omega}$, then equation (29) is equivalent to

$$S_{k,\omega}\alpha = u_D.$$

The Hankel function can be written as

$$H_0(z) = \frac{-1}{2\pi}\ln|z|J_0(z) + F_1(z^2)$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. Using the power series definition of J_0,

$$\begin{aligned}\frac{i}{4}H_0(k\,|x-y|) = {} & \frac{-1}{2\pi}\ln|x-y| \\ & + \frac{1}{2\pi}\frac{k^2}{4}(x-y)^2\ln|x-y| \\ & + (x-y)^4\ln|x-y|F_2(x,y) + F_3(x,y)\end{aligned} \tag{30}$$

where F_2 and F_3 are C^∞. Let us introduce the operators O_n defined for $n \geq 1$ as

$$O_n : \alpha \mapsto \int_{-1}^{1} (x-y)^{n-1}\ln|x-y|\,\frac{\alpha(y)}{\omega(y)}.$$

In Appendix B, we prove that for all n, O_n is of order n.

Lemma 8. *The operator $S_{k,\omega}$ admits the following expansion*

$$S_{k,\omega} = S_\omega + \frac{1}{2\pi}\frac{k^2}{4}O_3 + R_5 + R_\infty$$

where R_5 is an operator of order 5 and R_∞ is a smoothing operator.

Proof. From equation (30), it suffices to show that the operator

$$R_5 : \alpha \mapsto \int_{-1}^{1} (x-y)^4 \ln|x-y| F_2(x,y)\frac{\alpha(y)}{\omega(y)}$$

is of order 5. Since O_5 is of order 5, this is true in view of Lemma 3. $\square$

In particular, the operator $S_{k,\omega}$ is well defined on $T^{-\infty}$, and is of order 1. Moreover, we see that $S_{k,\omega}$ is a compact perturbation of S_ω in T^s. Therefore, the product $\sqrt{(-\omega\partial_x)^2}S_{k,\omega}$ is a compact perturbation of identity. In fact, using Equation 30, we can see that $S_{k,\omega}(-\omega\partial_x)^2 S_{k,\omega} = \frac{I_d}{4} + K$ where K is a compact operator of order 2. However, the condition number of $\sqrt{-(\omega\partial_x)^2}S_{k,\omega}$ is certainly independent of the mesh size, but increases with the frequency k. It is desirable to include in the preconditioner a correction for the frequency number.

Theorem 6. *There holds*

$$S_{k,\omega}\left(-(\omega\partial_x)^2 - k^2\omega^2\right)S_{k,\omega} = \frac{I}{4} + K$$

where K is of order 4.

Proof. Using the expansion of Lemma 8, we can write

$$\begin{aligned} -S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} &= -S_\omega(\omega\partial_x)^2 S_\omega \\ &\quad -\frac{1}{2\pi}\frac{k^2}{4}\left(S_\omega(\omega\partial_x)^2 O_3 + O_3(\omega\partial_x)^2 S_\omega\right) + K' \end{aligned}$$

where K' is of order 4. By Theorem 3, the first term is $\frac{I}{4} + R$ where R is a smoothing operator. Then, simple calculations show that

$$(\omega\partial_x)^2\left((x-y)^2\ln|x-y|\right) = 2\omega^2(x)\ln|x-y| - 2x(x-y)\ln|x-y| + P(x,y)$$

where P is a polynomial in x and y. Dividing by $\omega(y)$ and integrating on both sides with respect to y, we get

$$\frac{1}{2\pi}(\omega\partial_x)^2 O_3 = 2\omega^2 S_\omega - \frac{1}{\pi}xO_2 + R$$

where R is a smoothing operator. Taking the adjoint operator in both sides (with respect to the T^0 scalar product), we get

$$\frac{1}{2\pi}O_3(\omega\partial_x)^2 = 2S_\omega\omega^2 + \frac{1}{\pi}O_2 x + R'$$

where R' is also a smoothing operator. Lemma 17 thus implies

$$-S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} = \frac{I}{4} - k^2 S_\omega\omega^2 S_\omega + K''$$

where K'' is of order 4. The announced result holds since, using Lemma 8,

$$S_\omega\omega^2 S_\omega - S_{k,\omega}\omega^2 S_{k,\omega}$$

is an operator of order 4. $\square$

We recall that $\lambda_{n,k}^2$ are the eigenvalues of $-(\omega\partial_x) - k^2\omega^2$. Let $s_{n,k}$ the eigenvalues of $S_{k,\omega}$ on the basis of Mathieu cosines, that is

$$S_{k,\omega}T_{n,k} = s_{n,k}T_n^k.$$

The previous has the following consequence:

Corollary 5. *One has*

$$s_{n,k}\lambda_{n,k} = \frac{1}{4} + r_{n,k}$$

where $r_{n,k}$ satisfies

$$\sum_{n=0}^{+\infty}(1+n^2)^4 \left|r_{n,k}\right|^2 < +\infty$$

The last result, when also in consideration of the commutation shown in Theorem 5, implies that $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$ is a compact perturbation of the inverse of $S_{k,\omega}$, prompting us to use it as a preconditioner in Equation (29).

4 Non-flat arc

- Restate the theorem, en disant que c'est la même preuve.

We return to the case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, and define a new preconditioner for thee corresponding integral equation. The main result of this section is Theorem 6 We fix a smooth, constant speed, parametrization $r : [-1,1] \to \mathbb{R}^2$ of Γ. The constant-speed assumption ensures $(x,y) \in [-1,1]$, one has

$$\left|r(x) - r(y)^2\right| = \frac{\left|\Gamma\right|^2}{4}\left|x-y\right|^2 + |x-y|^4 F_1(x,y) \tag{31}$$

where $|\Gamma|$ is the length of Γ and F_1 is a C^∞ function on $[-1,1]^2$.

4.1 Neumann

5 Numerical methods

In this section, we describe and analyze the Galerkin scheme used to solve the integral equations in this work. To keep matters simple, we focus on equations (21) and (26) on the flat strip. The results extend to the general case using standard arguments in the theory of boundary element methods. Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [13, Chap. 4,] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [**?**] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [4], and some results exist for piecewise linear functions in the colocation setting [**?**].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 7 and Theorem 8.

In what follows, we introduce a discretization of the segment $[-1,1]$ as $-1 = x_0 < x_1 < \cdots < x_N = 1$, and let $\theta_i := \arccos(x_i)$. We define the parameter h of the discretization as

$$h := \min_{i=0\cdots N-1} |\theta_{i+1} - \theta_i| \, .$$

In practice, one should use a mesh for which $|\theta_i - \theta_{i+1}|$ is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the $i-th$ interval is approximately $(ih)^2$. In comparison, in the h-BEM method with $p = 1$ polynomial order, this would only lead to a convergence in $O(h)$ (cf. [**?**, Theorem 1.3]).

5.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution λ of (21). To achieve it, we use a variational formulation of (23) to compute an approximation α_h of α, and set $\lambda_h = \frac{\alpha_h}{\omega}$. Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at x_i. Let α_h the unique solution in V_h to

$$\langle S_\omega \alpha_h, \alpha'_h \rangle_{\frac{1}{\omega}} = - \langle u_D, \alpha'_h \rangle_{\frac{1}{\omega}} \, , \quad \forall \alpha'_h \in V_h.$$

We shall prove the following result:

Theorem 7. *If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C h^{s+1/2} \|u_D\|_{T^{s+1}} \, .$$

In particular, when u_D is smooth, it belongs to T^∞ so the rate of convergence is $h^{5/2}$. We start by proving an equivalent of Céa's lemma:

Lemma 9. *There exists a constant C such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

Proof. In view of the properties of S_ω stated in Proposition 2, we have the equivalent norm

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \rangle \, .$$

Since $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = - \langle u_D, \alpha'_h \rangle$ for all $\alpha'_h \in V_h$, we deduce

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \rangle \, , \quad \forall \alpha'_h \in V_N.$$

By duality

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of S_ω from $T^{-1/2}$ to $T^{1/2}$. □

From this we can derive the rate of convergence for α_h to the true solution α. We use the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h, which satisfies the following properties:

Lemma 10. *For any function u,*

$$\|(\mathrm{I}-\mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C\|u\|_{L^2_{\frac{1}{\omega}}},$$

$$\|(\mathrm{I}-\mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2\|u\|_{T_2}.$$

The proof requires the following well-known result:

Lemma 11. *Let $\tilde{u}$ in the Sobolev space $H^2(\theta_1,\theta_2)$,such that $\tilde{u}(\theta_1)=\tilde{u}(\theta_2)=0$. Then there exists a constant C independent of θ_1 and θ_2 such that*

$$\int_{\theta_1}^{\theta_2}\tilde{u}(\theta)^2 \leq C(\theta_1-\theta_2)^4\int_{\theta_1}^{\theta_2}\tilde{u}''(\theta)^2 d\theta$$

Proof. The first inequality is obvious since $\mathbb{P}_h$ is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the $L^2_{\frac{1}{\omega}}$ norm,

$$\|I-\mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I-I_h u\|_{L^2_{\frac{1}{\omega}}}, \tag{32}$$

where $I_h u$ is the piecewise affine (continuous) function that matches the values of u at the breakpoints x_i. By Theorem 2, on each interval $[x_i, x_{i+1}]$, the function $\tilde{u}(\theta) := u(\cos(\theta))$ is in the Sobolev space $H^2(\theta_i,\theta_{i+1})$ so we can apply Lemma 11:

$$\int_{x_i}^{x_{i+1}}\frac{(u-I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}}(\tilde{u}-\tilde{I_h u})^2 \leq (\theta_{i+1}-\theta_i)^4\int_{\theta_i}^{\theta_{i+1}}(\tilde{u}-\tilde{I_h u})''^2.$$

This gives

$$\int_{x_i}^{x_{i+1}}\frac{(u-I_h u)^2}{\omega} \leq 2h^4\left(\int_{\theta_i}^{\theta_{i+1}}\tilde{u}''^2+\int_{\theta_i}^{\theta_{i+1}}\tilde{I_h u}''^2\right). \tag{33}$$

Before continuing, we need to establish the following result

Lemma 12. *There holds*

$$\int_{\theta_i}^{\theta_{i+1}}\tilde{I_h u}''^2 \leq C\int_{x_i}^{x_{i+1}}\frac{u'^2}{\omega}$$

Proof. The expression of $I_h u$ is given by

$$\tilde{I_h u}(\theta) = u(x_i)+\frac{u(x_i)-u(x_{i+1})}{\cos(\theta_{i+1})-\cos(\theta_i)}(\cos(\theta)-\cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}}\tilde{I_h u}''^2 = \left(\frac{u(x_i)-u(x_{i+1})}{\cos(\theta_{i+1})-\cos(\theta_i)}\right)^2\int_{\theta_i}^{\theta_{i+1}}\cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left(\int_{x_i}^{x_{i+1}} u'(t)dt\right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \leq \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. □

We can now conclude the proof of Lemma 10. Summing all inequalities (33) for $i = 0, \cdots N+1$, we get

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}}^2 \leq Ch^4 \left(\|u\|_{T^2}^2 + \|u'\|_{T_0}^2\right).$$

By Corollary 1, the operator ∂_x is continuous from T^2 to T^0 which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T^2}.$$

Thanks to (32), this concludes the proof. □

We obtain the following corollary by interpolation:

Corollary 6. *The operator* $\mathrm{I} - \mathbb{P}_N$ *is continuous from* $L^2_{\frac{1}{\omega}}$ *to* T^s *for* $0 \leq s \leq 2$ *with*

$$\|(\mathrm{I} - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s}.$$

We can now prove Theorem 7:

Proof. First, using **??**, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}}.$$

Moreover, if u_D is in T^{s+1}, then $\alpha = S_\omega^{-1} u_D$ is in T^s and $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$. By the analog of Céa's lemma, Lemma 9, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s}.$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} = \inf_{\eta\in T^{1/2},\eta\neq 0} \frac{(\alpha - \mathbb{P}_N\alpha, \eta - \mathbb{P}_h\eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

Using Cauchy-Schwarz's inequality and Corollary 6 ($s = \frac{1}{2}$),

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s}.$$

□

5.2 Neumann problem

We now turn to the numerical resolution of (26). We use a variational form for equation (27), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce W_h the space of continuous piecewise affine functions with breakpoints at x_i, and we denote by β_h the unique solution in W_h to the variational equation:

$$\langle N_\omega \beta_h, \beta_h' \rangle_\omega = \langle u_N, \beta_h' \rangle_\omega, \quad \forall \beta_h' \in W_h. \tag{34}$$

Then, $\mu_h = \omega\beta_h$ is the proposed approximation for μ. We shall prove the following:

Theorem 8. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Like before, we start with an analog of Céa's lemma:

Lemma 13. *There exists a constant C such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta_h' \in W_h} \|\beta - \beta_h'\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator I_h:

Lemma 14. *There holds*

$$\|u - I_h u\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_h u\|_{U^1} \leq Ch \|u\|_{U^2}$$

Proof. We only show the first estimation, the method of proof for the second being similar. Using again Lemma 11 on each segment $[x_i, x_{i+1}]$, one can write

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} \omega(u - I_h u)^2 \quad &\leq \quad C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_h u)''^2 \\
&\leq \quad Ch^4 \left(2\int_{\theta_i}^{\theta_{i+1}} Vu''^2 + 2\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \right)
\end{aligned}$$

where we recall that for any function u, Vu is defined as

$$Vu(\theta) = \sin(\theta)u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

Lemma 15.

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(\|u\|_{U_2}^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

Proof. Using the expression of I_h, one can write

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(|u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \right.$$
$$\left. + \left(\frac{u(x_{i+1}) - u(x_i)}{\cos\theta_{i+1} - \cos\theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2) \right) \quad (35)$$

We can estimate the first term, thanks to Lemma 4:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned} (u(x_{i+1}) - u(x_i))^2 &= \left(\int_{x_i}^{x_{i+1}} \partial_x u \right)^2 \\ &\leq \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\ &= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2. \end{aligned}$$

to conclude, it remains to observe that the quantity

$$\frac{|(\theta_{i+1} - \theta_i)| \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2)}{(\cos(\theta_i) - \cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2(\theta_i)$ $\square$

We now plug the estimate Lemma 15 in (35), and sum over i:

$$\|u - I_h u\|_{L^2_\omega}^2 \leq Ch^4(\|u\|_{U^2}^2 + \|u'\|_{L^2_\omega}^2).$$

This implies the claim once we use the continuity of ∂_x from U^2 to U^0, cf. Corollary 1. $\square$

We can now prove Theorem 8

Proof. Let us denote by Π_h the Galerkin projection operator defined by $\beta \mapsto \beta_h$. Since it is an orthogonal projection on W_h with respect to the scalar product $(\beta, \beta') := \langle N_\omega \beta, \beta' \rangle$, it is continuous from $U^{1/2}$ to itself, so we have for any u in $U^{1/2}$.

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C \|u\|_{U^{1/2}} .$$

We are now going to show the estimate

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2} .$$

By the analog of Céa's lemma Lemma 13, one has $\|(I - \Pi_h)u\|_{U^{1/2}} \leq \|(I - I_h)u\|_{U^{1/2}}$. By interpolation, this norm satisfies

$$\|(I - I_h)u\|_{U^{1/2}} \leq C \sqrt{\|(I - I_h)u\|_{U^0}} \sqrt{\|(I - I_h)u\|_{U^1}},$$

which yields, applying Lemma 14,

$$\|(I - I_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2} .$$

By interpolation, for all $s \in [1/2, 2]$, we get

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{s-1/2} \|u\|_{U^s} .$$

In view of **??**, we have $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$. Moreover, since N_ω is a continuous bijection from U^{s+1} to U^s for all s, there holds

$$\|\beta\|_{U^s} = \left\|N_\omega^{-1} u_N\right\|_{U^s} = \|u_N\|_{U^{s-1}} .$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-1/2} \|\beta\|_{U^s} \leq Ch^{s-1/2} \|u_N\|_{U^{s-1}} .$$

□

5.3 Numerical convergence rates

6 Conclu

Résumé de ce qu'on a fait, du lien qu'on a fait. Ouverture sur les singularités de type coin puis 3D. Beaucoup plus compliqué car pas de relations analytiques qui nous aident. Expliquer la beauté de l'approche numérique avec un poids. On propose le préconditioneur avec un test numérique ?

Possible analyse pseudo-diff ? En reparler ? Lien avec Antoine et Darbas.

A Commutation of N_ω and $(\partial_x \omega)^2 + k^2\omega^2$

To ease the computations, we take some notations: let $\Delta_\omega := (\omega\partial_x)^2$, $\Delta_\omega^T := (\partial_x \omega)^2$, $N_\omega := N_{k,\omega}$, and $S_\omega := S_{k,\omega}$. Using, Equation 12 we can write

$$N_\omega = -\partial_x S_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2.$$

To show that N_ω and $\Delta_\omega^T + k^2\omega^2$ commute, we compute their commutator C and show that it is null. We have

$$\begin{array}{rcl} C & := & N_\omega \Delta_\omega^T - \Delta_\omega^T N_\omega + k^2 N_\omega \omega^2 - k^2\omega^2 N_\omega \\ & = & -\partial_x S_\omega \Delta_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2 \Delta_\omega^T \\ & & +\partial_x \Delta_\omega S_\omega \omega \partial_x \omega + k^2 \Delta_\omega^T S_\omega \omega^2 \\ & & -k^2 \partial_x S_\omega \omega \partial_x \omega^3 - k^4 S_\omega \omega^4 \\ & & +k^2\omega^2 \partial_x S_\omega \omega \partial_x \omega + k^4 \omega^2 S_\omega \omega^2 \end{array}$$

where each term in the r.h.s. of the first equality gives rise to a line in the second. We rearrange the terms as follows:

$$\begin{array}{rcl} C & = & \partial_x(\Delta_\omega S_\omega - S_\omega \Delta_\omega)\omega\partial_x\omega - k^2\partial_x S_\omega \omega \partial_x \omega^3 + k^2\omega^2\partial_x S_\omega \omega \partial_x \omega \\ & & +k^4(\omega^2 S_\omega - S_\omega \omega^2)\omega^2 \\ & & +k^2(\Delta_\omega^T S_\omega \omega^2 - S_\omega \omega^2 \Delta_\omega^T) \end{array}$$

For the first term, we inject the commutation shown in Theorem 5. For the last line, we use the following identities:

$$\Delta_\omega^T = \Delta_\omega - 2x\partial_x - 1$$

$$\omega^2 \Delta_\omega^T = \Delta_\omega \omega^2 + \omega^2 + 2\omega x \partial_x \omega$$

Let $D = \frac{C}{k^2}$,

$$\begin{array}{rcl} D & = & \partial_x S_\omega \omega(\omega^2\partial_x - \partial_x\omega^2)\omega + (\omega^2\partial_x - \partial_x\omega^2)S_\omega\omega\partial_x\omega \\ & & +k^2(\omega^2 S_\omega - S_\omega\omega^2)\omega^2 \\ & & +(\Delta_\omega - 2x\partial_x - 1)S_\omega\omega^2 - S_\omega(\Delta_\omega\omega^2 + \omega^2 + 2\omega x\partial_x\omega) \end{array}$$

We use $\omega^2\partial_x - \partial_x\omega^2 = 2x$, and the relation $\partial_x S\omega^2 = S_\omega\omega\partial_x\omega$, obtained by integration by parts.

$$\begin{array}{rcl} D & = & 2S_\omega\omega\partial_x x\omega + 2xS_\omega\omega\partial_x\omega \\ & & +\left(k^2(\omega^2 S_\omega - S_\omega\omega^2) + \Delta_\omega S_\omega - S_\omega\Delta_\omega\right)\omega^2 \\ & & -2S_\omega\omega^2 - 2xS_\omega\omega\partial_x\omega - 2S_\omega\omega x\partial_x\omega \end{array}$$

Using again the commutation shown in Theorem 5, we are left with

$$D \quad = \quad 2S_\omega\omega(\partial_x x - x\partial_x)\omega - 2S_\omega\omega^2$$

This is null since $\partial_x x - x\partial_x = 1$.

B Order of the operators O_n

Lemma 16. *For all $n \geq 1$, there exists a function $o_n : \mathbb{N}^2 \to \mathbb{R}$ satisfying the following conditions*

(i) $$\forall k \in \mathbb{Z}, \quad O_n T_k = \sum_{i=-\infty}^{+\infty} o_n(k,i) T_{k-i} \tag{36}$$

$$\textit{(ii)} \qquad \forall \alpha,\beta \in \mathbb{N}, \forall i,k \in \mathbb{Z}, \quad \left|\Delta_i^\alpha \Delta_k^\beta o_n(k,i)\right| \leq C_{n,\alpha,\beta}(1+k^2)^{-n-1-\beta} \tag{37}$$

$$\textit{(iii)} \qquad \forall i,k \in \mathbb{Z}, \quad |i| \geq n \implies o_n(k,i) = 0 \tag{38}$$

where Δ_i *and* Δ_k *represent the discrete derivation operator in the variables* i *and* k *respectively, e.g.*

$$\Delta_i o_n(k,i) = o_n(k,i+1) - o_n(k,i),$$

and Δ_i^α *denotes the* α*-th iterate of* Δ_i*. We use the convention* $T_k := T_{|k|}$ *for* $k \in \mathbb{Z}$*.*

Proof. We shall prove this by induction. First for $n = 1$, $O_1 = 2\pi S_\omega$, and we simply have $o_1(k,i) = \delta_{i=0} s_k$ where s_k are the eigenvalues of S_ω defined in Proposition 2. Obviously, o_1 satisfies all the requirements. Second, notice that for $n \geq 1$,

$$O_{n+1} = xO_n - O_n x,$$

which combined with the identity

$$xT_n = \frac{T_{n-1} + T_{n+1}}{2}$$

valid for all $n \in \mathbb{Z}$, implies

$$O_{n+1}T_k = \sum_{i=-\infty}^{+\infty} o_n(k,i) \frac{T_{k-i+1} + T_{k-i-1}}{2} - \frac{1}{2} O_n T_{k+1} - \frac{1}{2} O_n T_{k-1}.$$

By the recurrence assumption (i), this implies

$$\begin{aligned} O_{n+1}T_k \quad &= \quad \frac{1}{2} \sum_{-\infty}^{+\infty} \left(o_n(k,i-1) - o_n(k+1,i-1)\right) T_{k-i} \\ &\quad + \frac{1}{2} \sum_{-\infty}^{+\infty} \left(o_n(k,i+1) - o_n(k-1,i+1)\right) T_{k-i}. \end{aligned}$$

Thus, condition (i) is satisfied with

$$o_{n+1}(k,i) := \frac{\left(o_n(k,i-1) - o_n(k+1,i-1) + o_n(k,i+1) - o_n(k-1,i+1)\right)}{2}.$$

If $|i| \geq n+1$ then by triangular inequality $|i-1| \geq n$ and $|i+1| \geq n$ so all terms in the rhs are null by the assumption (iii), which shows that (iii) also holds for o_{n+1}. Finally, the assumption (ii) is easily checked for o_{n+1} once we write

$$o_{n+1}(k,i) = \frac{-\Delta_k(k,i-1) + \Delta_k(k-1,i+1)}{2}.$$

□

Lemma 17. *The operator* O_n *is of order* n*.*

Proof. Since the sum in (36) is finite, by linearity, it is sufficient to show that the operator O_n^i defined by

$$\forall k \in \mathbb{Z}, \quad O_n^i T_k = o_n(k,i) T_{k-i}$$

is of order n. We treat the case $i > 0$, the opposite case being analogous. Let $u \in T^s$ for some s, there holds

$$O_n^i u = \sum_{k=0}^{+\infty} o_n(k+i,i)\hat{u}_{k+i} T_k + \sum_{k=0}^{i} o_n(i-k,i)\hat{u}_{i-k} T_k.$$

We let Vu and Ru respectively the two terms of the rhs. Obviously, R is a smoothing operator with

$$\|Ru\|_{T^{s+n}} \leq (1+i)^n \|u\|_{T^s}.$$

Now, for all $k \in \mathbb{N}$ let

$$\hat{v}_k := o_n(i+k,i)\hat{u}_{i+k}.$$

Applying Peetre's inequality, one has

$$(1+k^2)^{n+s} |\hat{v}_k|^2 \leq C\left(1+i^2\right)^{|n+s|} \left(1+(i+k)^2\right)^{n+s} |o_n(k+i,i)|^2 |\hat{u}_{k+i}|^2.$$

the condition (ii) in Lemma 16 with $\alpha = \beta = 0$ yields

$$|o_n(k+i,i)|^2 \leq C\left(1+(k+i)\right)^{-2n} \leq 2C\left(1+(k+i)^2\right)^{-n}.$$

Therefore, $\|Vu\|_{T^{s+n}} \leq C(1+i)^{|n+s|} \|u\|_{T^s}$ which concludes the proof. □

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