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A PRECONDITIONER FOR THE ELECTRIC FIELD INTEGRAL EQUATION BASED ON CALDERON FORMULAS*

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Abstract. We describe a preconditioning technique for the Galerkin approximation of the electric field integral equation (EFIE), which arises in the scattering theory for harmonic electromagnetic waves. It is based on a discretization of the Calderon formulas and the Helmholtz decomposition. We prove several properties of the method, in particular that it produces a variational solution on a subspace of the Galerkin space for which we have an LBB inf-sup condition. When the Krylov spaces associated with the continuous operators are nondegenerate we prove that the discrete Krylov spaces converge as the mesh refinement goes to zero; when, moreover, the EFIE is nondegenerate on the continuous Krylov spaces, the discrete Krylov iterates converge towards the continuous ones. We also argue that one might expect the continuous Krylov iterates to exhibit superlinear convergence of the form $n \mapsto C^n (n!)^{-\alpha}$ for some $C > 0$ and $\alpha > 0$. Finally, we illustrate the theory with numerical experiments.

Key words. electric field integral equation, Calderon formula, preconditioning, Krylov subspace

AMS subject classifications. 65N38, 78M15

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Introduction. In [51] Steinbach and Wendland described several strategies for the preconditioning of some boundary integral equations of the first kind, based on the knowledge of an operator of the opposite order. On several examples of symmetric positive definite (SPD) integral operators, they provided a discretization of the operators to construct a preconditioner such that the extreme eigenvalues of the preconditioned matrix remain bounded away from 0 and $+\infty$ independently of the mesh refinement. When iterative algorithms are used to solve the matrix equations, this in turn is well known to yield convergence estimates that are also independent of the mesh refinement, of the form $\|U^n - U^\star\| \leq C\alpha^n \|U^\star\|$ for some $C > 0$, $0 < \alpha < 1$ (so-called linear convergence).

In [17] we adapted the theory to some non-SPD problems. We showed that in these cases one can still prove that the spectral condition number of the preconditioned matrix remains bounded independently of the mesh refinement, as long as all the bilinear forms involved satisfy uniform inf-sup estimates on the Galerkin spaces. This was applied to some problems of three-dimensional acoustic scattering. Complementary results, in particular close to resonant frequencies, were exposed in [18].

We restricted our attention to preconditioners deduced from Calderon formulas—for some scalar operators—and remarked that they provide an inverse up to a compact endomorphism. One would expect this to produce a matrix with a spectrum clustered around 1. This would imply very fast, perhaps in some sense superlinear, convergence of Krylov subspace methods. However, we presented no formal proof of this intuition.

In this paper we use the Calderon formulas—for some operators on tangential vector fields—to construct a preconditioner for the electric field integral equation (EFIE). Compared with the scalar case there is a major pitfall: the involved bilinear forms do not all satisfy uniform inf-sup conditions on the standard Galerkin spaces

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(Proposition 3.1). We nevertheless construct a numerically efficient preconditioner for which we can identify and prove satisfactory properties. In effect the Galerkin problem is solved on a subspace of the standard space, and we show that this subspace has all the properties required to ensure the well-posedness of the EFIE on it (Theorem 3.15). We also prove that the discretized preconditioner is stable and approximating in the sense of Proposition 4.6. The discrete Krylov spaces converge as the mesh refinement h decreases to 0 (Theorem 4.7) and from this we deduce that, under some natural hypotheses, the approximate solution at iteration n converges as $h \to 0$, towards a vector u^n (Corollary 4.10). We also explain why we expect (u^n) to converge superlinearly, perhaps even at the rate $C^n(n!)^{-\alpha}$, for some $C > 0$, $\alpha > 0$ (section 4.3).

We first officially detailed this discretization technique in [16] and announced the theorems justifying it in [19]. Proofs of one of them (Theorem 1.2) can be found in [15]. (This paper also contains complementary results on the behavior of the equation at resonant frequencies.) In the present paper we explicitate and prove—and sometimes improve—the remaining announced results, as well as a few others that sustain the construction.

At about the same time, in [21], another research team announced progress on what also amounts to the use of the Calderon formulas for the construction of a stable method. However, discretizations were proposed only for Nyström schemes, and it seems that for these they did not encounter a difficulty comparable to Proposition 3.1. Since this was the main problem we had to solve in the Galerkin setting, in extending the method from acoustics to electromagnetics, we believe there to be no overlap for the techniques involved.

Outline. The paper is organized as follows:

Section 1. We describe the continuous problem we are dealing with and state sufficient conditions for the discretization to satisfy uniform inf-sup estimates. We also prove the Calderon formulas.

Section 2. We define the Galerkin spaces we will use and give some useful properties: negative norm estimates, approximation properties for harmonic tangent fields, and properties of the discrete Helmholtz decomposition.

Section 3. We introduce the preconditioner we propose, after having described the main difficulty. Then we give a first interpretation of the system we solve and the projections we use. We give several characterizations of the range of the discrete rotation operator and deduce from these that the EFIE is well posed on this subspace. We also devise an intrinsic stopping criterion for the algorithm.

Section 4. We show the convergence of the Krylov subspaces and explain why this should lead to superlinear convergence of the iterates.

Section 5. We illustrate the theory with numerical results for diffraction by a sphere, a cavity, and a singular geometry.

1. The EFIE.

We briefly recall the setting for exterior boundary value problems for the harmonic Maxwell equations, the integral representation of exterior electromagnetic fields, and the related integral equation known as the EFIE, as presented, for instance, in Cessenat [14] or Nédélec [43]. Then we turn to the discretization of this equation by the Galerkin method and state some new sufficient conditions (obtained in Christiansen [15]) for its well-posedness, in the sense of satisfying a uniform inf-sup condition. This in turn is well known to guarantee *quasi-optimal* convergence of the approximate solutions. Finally, we include a proof of the Calderon formulas and explain why they should lead to an efficient preconditioning technique.

1.1. The continuous problem. Let Ω_- be a smooth, bounded, and open subset of $\mathbb{R}^3$, denote by Γ its surface, and denote by Ω_+ the complement of $\Omega_- \cup \Gamma$. We refer to Ω_- as the interior domain and to Ω_+ as the exterior domain. The unit-length orthogonal vector field on Γ, pointing into Ω_+, is denoted n. We suppose throughout that Ω_+ is connected. The free-space harmonic Maxwell equations for vector fields E and H in Ω_+ are

$$\operatorname{curl} E = +i\omega\mu H, \tag{1.1}$$

$$\operatorname{curl} H = -i\omega\epsilon E, \tag{1.2}$$

where μ is the magnetic permeability, ϵ is the electric permittivity, and ω is the pulsation. Define the wavenumber k and the impedance Z by

$$k = \omega(\mu\epsilon)^{1/2}, \tag{1.3}$$

$$Z = (\mu/\epsilon)^{1/2}. \tag{1.4}$$

Then we have $+i\omega\mu = +ikZ$ and $-i\omega\epsilon = -ik/Z$.

Spaces of functions. On any smooth Riemannian manifold M which is either an open subset of an Euclidean space or is compact without boundary, the usual Sobolev spaces of scalar and tangential fields of regularity order $s \in \mathbb{R}$ are denoted $\mathrm{H}^s(M)$ and $\mathrm{H}^s_{\mathrm{T}}(M)$, respectively (see, e.g., Taylor [52, Chap. 4]), and the corresponding norms are both written

$$u \mapsto |u|_s. \tag{1.5}$$

On any open subset Ω of $\mathbb{R}^3$, define the Sobolev spaces $\mathrm{H}^s_{\mathrm{curl}}(\Omega)$ of vector fields by

$$\mathrm{H}^s_{\mathrm{curl}}(\Omega) = \{v \in \mathrm{H}^s_{\mathrm{T}}(\Omega) \ : \ \operatorname{curl} v \in \mathrm{H}^s_{\mathrm{T}}(\Omega)\}. \tag{1.6}$$

We will use also the Hilbert spaces $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$ of tangent fields on Γ defined by

$$\mathrm{H}^s_{\mathrm{div}}(\Gamma) = \{u \in \mathrm{H}^s_{\mathrm{T}}(\Gamma) \ : \ \operatorname{div} u \in \mathrm{H}^s(\Gamma)\}. \tag{1.7}$$

The spaces $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$ are equipped with the norms

$$u \mapsto \|u\|_s \ : \ \|u\|_s^2 = |u|_s^2 + |\operatorname{div} u|_s^2. \tag{1.8}$$

We define $\mathrm{H}^s_{\mathrm{rot}}(\Gamma)$ in a similar way, but we do not introduce any notation for the corresponding norm. Notice that $u \mapsto u \times n$ induces isomorphisms $\mathrm{H}^s_{\mathrm{rot}}(\Gamma) \to \mathrm{H}^s_{\mathrm{div}}(\Gamma)$ and $\mathrm{H}^s_{\mathrm{div}}(\Gamma) \to \mathrm{H}^s_{\mathrm{rot}}(\Gamma)$.

For any space of (scalar) distributions X on Γ, we denote by $X^\bullet$ the subspace of elements u such that for all v that are constant on each connected component of Γ (i.e., satisfying $\Delta v = 0$) $\langle u, v\rangle = 0$. Δ has a meaning even for vector distributions (currents). As usual, the Hilbert spaces we consider are vector spaces over $\mathbb{C}$ obtained as complexifications of real Hilbert spaces of scalar and tangent fields. In particular, they are equipped with conjugations $u \mapsto \overline{u}$, induced by the standard conjugation in $\mathbb{C}$.

Recall the result of Paquet [45] (see [43, Thm. 5.4.2, p. 209]) that we have well-defined continuous and surjective tangential trace operators for arbitrary large enough $R > 0$ (with $B_R = \{x \in \mathbb{R}^3 \ : \ |x| < R\}$):

$$\gamma_{\mathrm{T}} : \left\{ \begin{array}{rcl} \mathrm{H}^0_{\mathrm{curl}}(\Omega_+ \cap B_R) & \to & \mathrm{H}^{-1/2}_{\mathrm{rot}}(\Gamma), \\ v & \mapsto & v_{\mathrm{T}} = v - (v \cdot n)n. \end{array} \right. \tag{1.9}$$

For simplicity we denote by $\mathrm{H}^0_{\mathrm{curl}}(\Omega_+)_{loc}$ the Fréchet space of vector fields in Ω_+ whose restrictions are in $\mathrm{H}^0_{\mathrm{curl}}(\Omega_+ \cap B_R)$ for all $R > 0$. We caution the reader that this notation is sometimes (but not in this article) used for a different space consisting of the fields whose restrictions are in $\mathrm{H}^0_{\mathrm{curl}}(\Omega_+ \cap U)$ for all open U such that $\overline{U}$ is a compact subset of Ω_+. This space is larger, since it allows rather wild behavior close to Γ. In particular, there is no trace operator on this space.

A technique we shall use many times is to write tangential vector fields u in the form of their Helmholtz decomposition:

$$u = \operatorname{grad} p + \operatorname{rot} q + \alpha, \tag{1.10}$$

with $\Delta\alpha = 0$, and to use regularity theorems of the Laplacian to characterize p and q. We shall refer to this technique as the *HDRL*. It was used to study electromagnetic scattering by DeLaBourdonnaye [24].

Integral representation for exterior scattering problems. The basic property for exterior boundary value problems for the harmonic Maxwell equations is (see [43, Thm. 5.4.6, p. 220]) that for any $k > 0$, and any $v \in \mathrm{H}^{-1/2}_{\mathrm{rot}}(\Gamma)$, there is a unique $(E, H) \in \mathrm{H}^0_{\mathrm{curl}}(\Omega_+)^2_{loc}$ such that

- (E, H) solves the harmonic Maxwell equations in Ω_+,
- (E, H) satisfies the Silver–Müller radiation conditions,
- and $\gamma_{\mathrm{T}} E = v$.

Let G_k be the fundamental solution of the Helmholtz operator $-\Delta - k^2$ satisfying the Sommerfeld radiation condition

$$G_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{1.11}$$

and let Φ_k be the potential, mapping any sufficiently smooth tangent field u on Γ to the field in $\mathbb{R}^3$ defined away from Γ by

$$(\Phi_k u)(y) = \int_\Gamma G_k(x, y) u(x) \mathrm{d}x. \tag{1.12}$$

Returning to the above boundary value problem, if k is not a resonance of the *interior* Maxwell equations there is a unique $u \in \mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$ such that for all $y \in \Omega_+$

$$E = (1 + (1/k^2) \operatorname{grad} \operatorname{div}) \Phi_k u, \tag{1.13}$$

$$H = 1/(ikZ) \operatorname{curl} \Phi_k u. \tag{1.14}$$

This formula is a special case of the Stratton–Chu integral representation (see [43, Thm. 5.5.1, p. 234]). For any $k \neq 0$ we define the electric field integral operator A_k by

$$A_k u = \gamma_{\mathrm{T}}(1 + (1/k^2) \operatorname{grad} \operatorname{div}) \Phi_k u. \tag{1.15}$$

One shows that the EFIO is continuous $A_k : \mathrm{H}^s_{\mathrm{div}} \to \mathrm{H}^s_{\mathrm{rot}}$ and that if k is not a resonance of the interior problem it is an isomorphism (see [43, Thm. 5.6.2, p. 247]).

Thus if k is not a resonant frequency the exterior problem for a given $v \in \mathrm{H}^{-1/2}_{\mathrm{rot}}(\Gamma)$ is reduced to the problem of solving the integral equation $A_k u = v$, called EFIE.

Variational formulation. From the HDRL it follows that the bilinear form on smooth tangent fields,

$$(u,v) \mapsto \langle u,v\rangle = \int_\Gamma u\cdot v, \tag{1.16}$$

extends continuously to a duality between $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$ and $\mathrm{H}^{-1-s}_{\mathrm{rot}}(\Gamma)$ (see also [43, Lem. 4.5.1, p. 208]). Thus one obtains the variational formulation of the EFIE. For a given $v \in \mathrm{H}^s_{\mathrm{rot}}(\Gamma)$, solve

$$u \in \mathrm{H}^s_{\mathrm{div}}(\Gamma) \quad \text{and} \quad \forall u' \in \mathrm{H}^{-1-s}_{\mathrm{div}}(\Gamma) \quad \langle A_k u, u'\rangle = \langle v, u'\rangle. \tag{1.17}$$

Remark that the case $s = -1/2$ is symmetric. From a practical point of view it is important that we have the following expression, with only weakly singular integrals (all integrals are on Γ):

$$\begin{aligned}\langle A_k u, u'\rangle = &\iint G_k(x,y)u(x)\cdot u'(y)\mathrm{d}x\mathrm{d}y \\ &-(1/k^2)\iint G_k(x,y)\operatorname{div} u(x)\operatorname{div} u'(y)\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.18}$$

1.2. Discretization. Put $X = \mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$. Given some sequence (X_h) of closed (finite-dimensional) subspaces of X, the Galerkin method to solve (1.17) is to consider the problems

$$u \in X_h \quad \text{and} \quad \forall u' \in X_h \quad \langle A_k u, u'\rangle = \langle v, u'\rangle. \tag{1.19}$$

When it is uniquely solvable for each h one obtains a sequence (u_h), and it is of fundamental importance to know to which extent it converges towards $A_k^{-1}v$.

In this context we have the following fundamental theorem due to Babuska [2].

THEOREM 1.1. *Let X be a reflexive Banach space and $\mathcal{A} : X \to X^\star$ be linear and continuous. Suppose we have a closed subspace X_h of X and that for some $\alpha > 0$ we have*

$$\inf_{u\in X_h} \sup_{u'\in X_h} \frac{|(\mathcal{A}u)(u')|}{\|u\|\,\|u'\|} \geq \alpha \tag{1.20}$$

$$\forall u' \in X_h \quad \big(\forall u \in X_h \ (\mathcal{A}u)(u') = 0\big) \Rightarrow \big(u' = 0\big). \tag{1.21}$$

Then the induced map $\mathcal{A}_h : X_h \to X_h^\star$ is invertible (with an inverse of norm less than α^{-1}). Moreover, for all $l \in X^\star$, if we have a solution $u \in X$ to $\mathcal{A}u = l$, then

$$\|\mathcal{A}_h^{-1}(l|_{X_h}) - u\| \leq \big(1 + \alpha^{-1}\|\mathcal{A}\|\big)\inf\{\|u' - u\| \ : \ u' \in X_h\}. \tag{1.22}$$

Notice that finite-dimensional subspaces are closed and that for these condition (1.21) is implied by (1.20).

Now suppose we have a family (X_h) of closed subspaces of X. When there is an α that holds for all h in estimate (1.20), and (1.21) holds for all h, we say that we have a *uniform discrete inf-sup condition*. Then the only remaining point is whether the spaces X_h are *approximating*, in the sense that

$$\forall u \in X \quad \lim_h \inf\{\|u - u'\| \ : \ u' \in X_h\} = 0. \tag{1.23}$$

In general this question is well studied in the literature (with improved convergence estimates on some dense subspaces of X). However, in order to justify the preconditioning technique we shall describe in this paper, we will need to study this question for some nonstandard spaces.

Inf-sup conditions for the EFIE. The Galerkin discretization of the EFIE by Raviart–Thomas-type vector fields was studied by Bendali [4, 5]. More generally, we consider the following hypotheses for the Galerkin spaces (X_h).

(H0) The spaces X_h are finite-dimensional subspaces of $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$, which are stable under the conjugation $u \mapsto \overline{u}$ (*conj.-stable* for short).

(H1) There is $C > 0$ such that for all $u \in \mathrm{H}^1_{\mathrm{div}}(\Gamma)$

$$\inf_{u' \in X_h} \|u - u'\|_0 \le Ch\|u\|_1. \tag{1.24}$$

(H2) There is $C > 0$ such that, for all $u \in X_h$, $\|u\|_0 \le Ch^{-1}\|u\|_{-1}$.

(H3) Putting $W_h = \{u \in X_h \ : \ \operatorname{div} u = 0\}$, there is $C > 0$ such that for all $u \in X_h$, if

$$\forall w \in W_h \quad \langle u, w\rangle = 0, \tag{1.25}$$

then the solution ϕ of

$$\phi \in \mathrm{H}^1(\Gamma)^\bullet \quad \text{and} \quad \Delta\phi = \operatorname{div} u \tag{1.26}$$

satisfies

$$|u - \operatorname{grad}\phi|_0 \le Ch|\operatorname{div} u|_0. \tag{1.27}$$

Notice that (H3) implies the usual inf-sup estimate: There is $C > 0$ such that

$$\inf_{q \in \operatorname{div} X_h} \sup_{u \in X_h} \frac{|\langle q, \operatorname{div} u\rangle|}{|q|_0 \|u\|_0} \ge \frac{1}{C}. \tag{1.28}$$

The following theorem was proved in Christiansen [15].

THEOREM 1.2. *If k is not a resonance of the interior problem and a family (X_h) of Galerkin spaces satisfies the four conditions* (H0), ..., (H3)*, then the bilinear form induced by A_k on X satisfies a uniform inf-sup condition on X_h.*

Of course these hypotheses also guarantee that in addition the (X_h) are approximating, so the approximate solution converges to the exact one (see section 2.2 for some details on this question).

The fact that the fields obtained by suitable transportation of Raviart–Thomas fields onto Γ satisfy these hypotheses is also checked in Christiansen [15], relying mostly on classical results that can be found, for instance, in Brezzi and Fortin [12]. Variants of the estimate on discrete Helmholtz decompositions appearing in (H3) have been used to study eigenvalue problems in mixed form and related discrete compactness results (see Kikuchi [37], Boffi [7], Boffi, Brezzi, and Gastaldi [8], and Demkowicz et al. [27]). Here, as already indicated, we will need to prove the hypothesis for some new spaces, in order to justify our preconditioning technique.

Solving the matrix equation. To solve the Galerkin problem one chooses a basis $e_h = (e_h(i))$ of X_h and defines the matrix $A_h(k)$ and the tuple V_h by

$$A_h(k)_{ij} = \langle A(k)e_h(j), e_h(i)\rangle, \qquad (V_h)_i = \langle v, e_h(i)\rangle. \tag{1.29}$$

In other words $A_h(k)$ is the matrix, from e_h to its dual basis, of the induced map

$$\mathcal{A}_h(k) : \left\{ \begin{array}{ccc} X_h & \to & X_h^\star, \\ u & \mapsto & \langle A(k)u, \cdot\rangle, \end{array} \right. \tag{1.30}$$

whereas V_h is the coordinate vector, in the dual basis of e_h, of the linear form $\langle v, \cdot\rangle$ restricted to X_h.

Then the discrete Galerkin problem (1.19) is stated in matrix terms as

$$A_h(k)U = V_h. \tag{1.31}$$

When this equation is solved iteratively one usually observes very slow convergence, if one observes it at all. Loosely speaking this is due to the fact that the operator A_k, via the Helmholtz decomposition, is seen to have one term of order 1 and another of order -1 acting on supplementary infinite-dimensional subspaces and with different signs. Thus, at least if the basis e_h is such that the canonical scalar product on $\mathbb{C}^{\dim X_h}$ corresponds to the $\mathrm{H}^0_{\mathrm{T}}(\Gamma)$ scalar product, the spectrum of $A_h(k)$ accumulates both at 0 and ∞. The presence of resonant frequencies further deteriorates the conditioning of the matrix.

This motivates our search for a preconditioner, that is, a matrix Z_h, such that, when Z_h is incorporated in an iterative solver, the reduction in the number of iterations required to obtain a satisfactory approximate solution outweighs the overhead of multiplying by Z_h. It is well known that this is achieved whenever Z_h is some easily computable approximate inverse of $A_h(k)$.

For ease of interpretation we will drop the matrix point of view and look instead for some easily computable $\mathcal{Z}_h : X_h^\star \to X_h$ which approximately inverts $\mathcal{A}_h(k)$. However, it should be kept in mind that the method is effective only in as far as it can be translated into a matrix manipulating algorithm.

1.3. Calderon formulas. The preconditioning technique we study in this paper is based on the Calderon formulas which we start by recalling. They are detailed in the electromagnetic setting in both Cessenat [14] and Nédélec [43]. We include a derivation of them mainly because the notations adopted here are not quite the same. Of course many of the arguments developed in this section were implicitly assumed while we introduced the EFIE and should be placed earlier in a strictly logical development.

Denote by B the operator on tangent fields on Γ defined by

$$Bu = u \times n. \tag{1.32}$$

Let $\wp$ be the orthogonal projection onto Γ, which is defined and smooth on a tubular neighborhood of Γ. Extending n to this neighborhood by $\wp$, we can define at any point x of this neighborhood, the tangential component of any vector v, by $\mathrm{T}_x v = v-(v\cdot n(x))n(x)$. Define an operator C_k on tangent fields on Γ, by taking the principal value of the tangential component in the exterior and interior domains (with respect to shrinking balls centered on Γ), of the following potential:

$$C_k u = \mathrm{PV}\,\mathrm{T}\,\mathrm{curl}\,\Phi_k u. \tag{1.33}$$

In fact, for smooth u the field $\mathrm{curl}\,\Phi_k u$ has different interior and exterior tangential traces which are both finite. More precisely, denoting γ^+_{T} and γ^-_{T} the exterior and interior trace operators one has

$$\gamma^+_{\mathrm{T}}\,\mathrm{curl}\,\Phi_k u = +(1/2)Bu + C_k u, \tag{1.34}$$

$$\gamma^-_{\mathrm{T}}\,\mathrm{curl}\,\Phi_k u = -(1/2)Bu + C_k u. \tag{1.35}$$

In particular, one has the familiar *jump formula*

$$u = [(\mathrm{curl}\,\Phi_k u) \times n] = B(\gamma^-\,\mathrm{curl}\,\Phi_k u - \gamma^+\,\mathrm{curl}\,\Phi_k u). \tag{1.36}$$

We also remind the reader that for potentials of the form

$$(1 + (1/k^2)\,\mathrm{grad}\,\mathrm{div})\Phi_k u, \tag{1.37}$$

the exterior and interior tangential traces are equal (and given by the EFIO), thus there is no tangential "jump" for these.

To derive the Calderon formulas the last ingredient we need is the *representation theorem*.

THEOREM 1.3. *Suppose* (E, H) *is a field whose restrictions to* Ω_- *and* Ω_+ *are in* $\mathrm{H}^0_{\mathrm{curl}}(\Omega_-)^2$ *and* $\mathrm{H}^0_{\mathrm{curl}}(\Omega_+)^2_{loc}$ *and solve Maxwell's equations for a given wavenumber* k. *Suppose also that it verifies the Silver–Müller radiation conditions. Define the electric and magnetic currents* j *and* m *on* Γ *by the jump formulas*

$$j = [H \times n] = (\gamma_{\mathrm{T}}^- H - \gamma_{\mathrm{T}}^+ H) \times n, \tag{1.38}$$

$$m = [E \times n] = (\gamma_{\mathrm{T}}^- E - \gamma_{\mathrm{T}}^+ E) \times n. \tag{1.39}$$

Then in Ω_- *and* Ω_+ *we have*

$$E = (+ikZ)(1 + (1/k^2)\,\mathrm{grad}\,\mathrm{div})\Phi_k j + \mathrm{curl}\,\Phi_k m, \tag{1.40}$$

$$H = (-ik/Z)(1 + (1/k^2)\,\mathrm{grad}\,\mathrm{div})\Phi_k m + \mathrm{curl}\,\Phi_k j. \tag{1.41}$$

Now the theorem we are interested in is the following.

THEOREM 1.4. *The following operator is a projector:*

$$\begin{bmatrix} 1/2 - BC_k & -(-ik/Z)BA_k \\ -(+ikZ)BA_k & 1/2 - BC_k \end{bmatrix}. \tag{1.42}$$

More explicitly, we have

$$BC_k BC_k + k^2 BA_k BA_k = 1/4, \tag{1.43}$$

$$BC_k BA_k + BA_k BC_k = 0. \tag{1.44}$$

Proof. Choose two (smooth enough) tangent fields u and v on Γ. Define fields E and H by putting, *in the exterior domain*,

$$E = (+ikZ)(1 + (1/k^2)\,\mathrm{grad}\,\mathrm{div})\Phi_k u + \mathrm{curl}\,\Phi_k v, \tag{1.45}$$

$$H = (-ik/Z)(1 + (1/k^2)\,\mathrm{grad}\,\mathrm{div})\Phi_k v + \mathrm{curl}\,\Phi_k u. \tag{1.46}$$

Then we have

$$-\gamma_{\mathrm{T}}^+ H \times n = -(-ik/Z)BA_k v + (1/2 - BC_k)u, \tag{1.47}$$

$$-\gamma_{\mathrm{T}}^+ E \times n = -(+ikZ)BA_k u + (1/2 - BC_k)v. \tag{1.48}$$

In the interior domain put $E = 0$ and $H = 0$. Now we have

$$[H \times n] = -\gamma_{\mathrm{T}}^+ H \times n, \tag{1.49}$$

$$[E \times n] = -\gamma_{\mathrm{T}}^+ E \times n. \tag{1.50}$$

Using the representation theorem, these tangent fields give rise to new integral representations for E and H. Now to say that the announced operator is a projector just expresses that taking the jumps (or the appropriate exterior traces) of these new integral representations for the same fields (E, H) should yield the same jumps (or exterior traces). □

The operator appearing in (1.42) is called the *exterior Calderon projector*.

The crucial remark is now that the operator BC_kBC_k is a compact endomorphism of $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$, thus $4k^2BA_kB$ inverts A_k up to a compact operator. The scalar coefficient $4k^2$ is unimportant for preconditioning purposes, so our aim will be to discretize the operator BA_kB. Since we deal with variational formulations we express our goal in terms of bilinear forms, for which it is preferable to have symmetric formulations, so, remarking that $B = -B^\star = B^{\star-1}$, we set out to discretize the map

$$\mathcal{Z}_k = \mathcal{B}^{\star-1}\mathcal{A}_k\mathcal{B}^{-1}, \tag{1.51}$$

where $\mathcal{B}$ is the isomorphism

$$\left\{\begin{array}{rcl} \mathrm{H}^s_{\mathrm{div}}(\Gamma) & \rightarrow & \mathrm{H}^{-1-s}_{\mathrm{div}}(\Gamma)^\star, \\ u & \mapsto & \langle Bu, \cdot\rangle. \end{array}\right. \tag{1.52}$$

2. Some properties of some Galerkin spaces.

We recall the definition and basic properties of the Galerkin spaces on Γ that we will use in this article, including the null sequences relating spaces of scalar and tangent finite elements, as well as some negative norm estimates. Of particular importance will be the approximation of harmonic tangent fields and the structure of the discrete Helmholtz decomposition that largely follows from it.

2.1. Surface finite element spaces.

Recall that we denote by $\wp$ the orthogonal projection onto Γ, which is defined and smooth on a tubular neighborhood of Γ. Let $(\mathcal{T}_h)$ be a family of triangulations of Γ, where for all h the largest diameter of a triangle of $\mathcal{T}_h$ is h. We will always suppose that $(\mathcal{T}_h)$ is regular and most often that it is (globally) quasi-uniform. Let Γ_h be the affine polyhedron determined by $\mathcal{T}_h$, considered as a Lipschitz manifold. For small enough h, $\wp$ induces Lipschitz isomorphisms $\Gamma_h \rightarrow \Gamma$, and we denote by Ξ_h the inverse mappings.

Fix a nonzero $m \in \mathbb{N}$. On Γ_h we consider the space $\mathsf{S}^0(\mathcal{T}_h)$ of continuous scalar functions whose restriction to any triangle is P^m (a polynomial of degree m), the space $\mathsf{S}^1(\mathcal{T}_h)$ of Raviart–Thomas $\mathrm{H}^0_{\mathrm{div}}$-conforming vector fields of degree m, and the space $\mathsf{S}^2(\mathcal{T}_h)$ of scalar functions whose restriction to any triangle is P^{m-1}.

From these finite element spaces on Γ_h we deduce finite element spaces on Γ by the transport formulas

$$\begin{array}{rcll} \mathsf{S}^0_h & = & \{x \mapsto & u(\Xi_h(x)) : u \in \mathsf{S}^0(\mathcal{T}_h)\}, \\ \mathsf{S}^1_h & = & \{x \mapsto & \mathrm{Jac}\,\Xi_h(x)\mathrm{D}\Xi_h(x)^{-1}u(\Xi_h(x)) : u \in \mathsf{S}^1(\mathcal{T}_h)\}, \\ \mathsf{S}^2_h & = & \{x \mapsto & \mathrm{Jac}\,\Xi_h(x)u(\Xi_h(x)) : u \in \mathsf{S}^2(\mathcal{T}_h)\}. \end{array} \tag{2.1}$$

These transport formulas were chosen to make the following diagram commute. The horizontal arrows are the differential operators rot and div, whereas the vertical ones are the above transport formulas.

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathsf{S}^0(\mathcal{T}_h) & \rightarrow & \mathsf{S}^1(\mathcal{T}_h) & \rightarrow & \mathsf{S}^2(\mathcal{T}_h) & \rightarrow & 0 \\ \downarrow & 0 & \downarrow & \mathrm{rot} & \downarrow & \mathrm{div} & \downarrow & 0 & \downarrow \\ 0 & \rightarrow & \mathsf{S}^0_h & \rightarrow & \mathsf{S}^1_h & \rightarrow & \mathsf{S}^2_h & \rightarrow & 0 \end{array} \tag{2.2}$$

Of course this diagram is a realization of a corresponding diagram on differential forms, on which the exterior derivative act, transported by the standard pull-back of differential forms determined by Ξ_h. The connection between finite elements and differential forms, especially Whitney forms, was stressed by Bossavit [9] and further explicitated in the affine case in Hiptmair [32]. While this is useful to keep in mind we stick to tangent vector fields and scalar fields on Γ, since in accordance with widespread conventions we have chosen to represent the exterior electromagnetic fields as fields of vectors, not alternate forms. The relevance of commuting diagrams to the study of finite elements is noted in Boffi [7].

Remark also that when studying the approximation of the boundary Γ by piecewise polynomial triangulations as in Nédélec [40], one is led to consider Galerkin spaces defined by pull-backs by maps that are slightly different from Ξ_h.

2.2. Basic negative norm estimates. Since negative (and noninteger) Sobolev norms and corresponding approximation results pervade this article, we now recall rather informally the results needed. Of course we do not claim any originality for these results, and we have included them mainly for the convenience of the exposition.

Let (X_h) be a family of Galerkin spaces satisfying (H0) and (H1). Let $\mathcal{Q}_h$ be the $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$-orthogonal projection onto X_h.

It is well known that

$$\|\mathcal{Q}_h u\|_0 \le C\|u\|_0 \quad \text{and} \quad \|u - \mathcal{Q}_h u\|_0 \le Ch\|u\|_1. \tag{2.3}$$

From the HDRL (section 1.1) it follows that the spaces $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$ for $0 \le s \le 1$ can be obtained by interpolation. Hence interpolation on the operator $\mathcal{I} - \mathcal{Q}_h$, for $0 \le s \le 1$, gives

$$\|u - \mathcal{Q}_h u\|_0 \le Ch^s\|u\|_s. \tag{2.4}$$

Then one uses the regularity of the $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$-inner product (written $(\cdot|\cdot)_0$) on various Sobolev spaces. This technique is the familiar Aubin–Nitsche trick. That $\mathrm{H}^s_{\mathrm{div}}(\Gamma)$ and $\mathrm{H}^{-s}_{\mathrm{div}}(\Gamma)$ are dual with respect to the $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$-inner product can be deduced from the fact that the operator $I - \operatorname{grad}\operatorname{div}$ is an isomorphism $\mathrm{H}^s_{\mathrm{div}}(\Gamma) \to \mathrm{H}^{s-1}_{\mathrm{rot}}(\Gamma)$ and that this space, as already remarked, is the $\mathrm{L}^2_{\mathrm{T}}$-dual of $\mathrm{H}^{-s}_{\mathrm{div}}(\Gamma)$. Both of these facts can be proved using the HDRL. For $0 \le s \le 1$ we have

$$\|u - \mathcal{Q}_h u\|_{-s} \le C \sup_{v \in \mathrm{H}^s_{\mathrm{div}}} \frac{|(u - \mathcal{Q}_h u|v)_0|}{\|v\|_s} \tag{2.5}$$

$$\le C \sup_{v \in \mathrm{H}^s_{\mathrm{div}}} \frac{|(u - \mathcal{Q}_h u|v - \mathcal{Q}_h v)_0|}{\|v\|_s} \tag{2.6}$$

$$\le C\|u - \mathcal{Q}_h u\|_0 \; \|\mathcal{I} - \mathcal{Q}_h\|_{0,s}. \tag{2.7}$$

Here $\|\mathcal{I} - \mathcal{Q}_h\|_{0,s}$ is of course the norm of the induced map

$$\mathcal{I} - \mathcal{Q}_h : \mathrm{H}^s_{\mathrm{div}}(\Gamma) \to \mathrm{H}^0_{\mathrm{div}}(\Gamma). \tag{2.8}$$

This gives for $0 \le s, s' \le 1$

$$\|u - \mathcal{Q}_h u\|_{-s} \le Ch^{s+s'}\|u\|_{s'}. \tag{2.9}$$

If in addition we have the inverse inequality (H2) we obtain

$$\|\mathcal{Q}_h u\|_{-1} \le \|u\|_{-1} + \|u - \mathcal{Q}_h u\|_{-1} \le \|u\|_{-1} + Ch\|u\|_0 \le C\|u\|_{-1}. \tag{2.10}$$

This is proved first for $u \in \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ and then extended to $u \in \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)$ by a density argument. By interpolation on $\mathcal{Q}_h$ one then extends this stability result to all $-1 \leq s \leq 0$:

$$\|\mathcal{Q}_h u\|_s \leq C\|u\|_s. \tag{2.11}$$

2.3. Approximation of harmonic fields. Since we deal with surfaces which we do not require to be simply connected, a useful construct is that of the associated cohomology groups of which we give the realizations in terms of vector and scalar fields, the so-called harmonic fields. (This notion of harmonicity is only remotely related to the harmonicity of the electromagnetic waves we consider.) We denote these spaces by $\mathbb{G}^i$ for $i = 0, 1, 2$. For instance $\mathbb{G}^1$ can be characterized as the L_2-orthogonal of the range of the rot operator on smooth scalar fields (or $\mathrm{H}^1(\Gamma)$), in the kernel of the div operator on smooth tangent fields (resp., $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$).

Noticing that the two rows in the diagram (2.2) are null sequences, we consider, for each horizontal pair of consecutive arrows, the L_2-orthogonal of the range of the left arrow, in the kernel of the right arrow. For the second row we denote these vector spaces by G^0_h, G^1_h, and G^2_h.

It is a remarkable fact that these "discrete" cohomology groups have the "right" dimension, i.e., the dimension of their continuous analogues $\mathbb{G}^i$. This is either elementary or can be deduced from the *Euler–Poincaré formula*. The use of this formula should not come as a surprise, since it is one of the main reasons for the effectiveness of *simplical* triangulations in algebraic topology. It has been used in finite element theory for quite some time, even at the textbook level; see, for instance, Nédélec [42].

For each h, let N^0_h be the number of vertices, N^1_h the number of edges (segments), and N^2_h the number of faces (triangles) in $\mathcal{T}_h$. Let N^C be the number of connected components of Γ.

We leave it as an exercise to check that for G^0_h and G^2_h the dimension is the number N^C of connected components of Γ. Remark also that $\mathsf{G}^0_h = \mathbb{G}^0$, whereas $\mathsf{G}^2_h \neq \mathbb{G}^2$. However, the elements of $\mathbb{G}^2$, which are the functions that are constant on each connected component of Γ, are of course well approximated by the elements of G^2_h.

We now turn to the more interesting case of G^1_h. We have

$$\begin{aligned}
\dim \mathsf{G}^1_h &= \big(\dim \mathsf{S}^1_h - (\dim \mathsf{S}^2_h - N^C)\big) - \big(\dim \mathsf{S}^0_h - N^C\big) && (2.12)\\
&= -\dim \mathsf{S}^0_h + \dim \mathsf{S}^1_h - \dim \mathsf{S}^2_h + 2N^C && (2.13)\\
&= -\big(N^0_h + (m-1)N^1_h + (m-2)(m-1)/2N^2_h\big) && (2.14)\\
&\quad + \big(mN^1_h + m(m-1)N^2_h\big) - \big(m(m+1)/2N^2_h\big) + 2N^C && (2.15)\\
&= -N^0_h + N^1_h - N^2_h + 2N^C && (2.16)\\
&= \dim \mathbb{G}^1. && (2.17)
\end{aligned}$$

To see that G^1_h converges in some sense to $\mathbb{G}^1$, consider the map Ω_h, called a Fortin operator in Boffi [6], which to any $u_0 \in \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ associates the first component u of the solution (u, q) of

$$\left\{\begin{array}{l} u \in \mathsf{S}^1_h \\ q \in \mathsf{S}^{2\bullet}_h \end{array}\right. \quad \left\{\begin{array}{lll} \forall u' \in \mathsf{S}^1_h & \langle u, u'\rangle + \langle q, \operatorname{div} u'\rangle & = \langle u_0, u'\rangle; \\ \forall q' \in \mathsf{S}^{2\bullet}_h & \langle q', \operatorname{div} u\rangle & = \langle q', \operatorname{div} u_0\rangle. \end{array}\right. \tag{2.18}$$

This saddle-point problem satisfies the LBB inf-sup conditions; therefore there is a $C > 0$ such that for all h and all $u \in \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ we have

$$\|u - \Omega_h u\|_0 \leq C \inf\{\|u - u'\|_0 \ : \ u' \in \mathsf{S}^1_h\}. \tag{2.19}$$

Notice also that Ω_h maps divergence-free fields to divergence-free fields and that if $u \in \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ is such that $\operatorname{rot} u = 0$ (as elements of $\mathrm{H}^{-1}(\Gamma)$), then $\Omega_h u$ is L_2-orthogonal to $\operatorname{rot} \mathsf{S}^0_h$. In particular, Ω_h maps $\mathbb{G}^1$ into G^1_h. Since all norms on $\mathbb{G}^1$ are equivalent, we therefore have an estimate of the form

$$\forall u \in \mathbb{G}^1 \quad \|u - \Omega_h u\|_0 \le C h^m \|u\|_0 \tag{2.20}$$

($m = 1 > 0$ for lowest order elements[1]) so that, for sufficiently small h, Ω_h determines injections $\mathbb{G}^1 \to \mathsf{G}^1_h$ which are arbitrarily close in norm to the identity mapping on $\mathbb{G}^1$. Since the spaces have the same dimension these induced maps are in fact isomorphisms, and the inverse mappings are Ch^m-close to the identity mapping on G^1_h.

Remark. For reasons of dimension, for the above system (2.18) to satisfy the LBB inf-sup conditions it is necessary that, *among all spaces that contain* $\operatorname{div} \mathsf{S}^1_h$, $\mathsf{S}^{2\bullet}_h$ be minimal. (Of course the LBB condition can be verified for some smaller spaces that do not contain $\operatorname{div} \mathsf{S}^1_h$.) On the other hand, for the above constructed injection $\mathbb{G}^1 \to \mathsf{G}^1_h$ to be *onto* it is necessary that $\mathsf{S}^{0\bullet}_h$ be maximal *among all spaces that* rot *maps into* S^1_h. It is remarkable that these algebraic optimality conditions (which were our guide for the choice of spaces) are also sufficient for convergence purposes.

Given a tangent field u one may now ask how the field a_0 defined by

$$a_0 \in \mathbb{G}^1 \quad \text{and} \quad \forall a' \in \mathbb{G}^1 \quad \langle a_0, a' \rangle = \langle u, a' \rangle \tag{2.21}$$

relates to its discrete analogue a_h defined by

$$a_h \in \mathsf{G}^1_h \quad \text{and} \quad \forall a' \in \mathsf{G}^1_h \quad \langle a_h, a' \rangle = \langle u, a' \rangle. \tag{2.22}$$

Since G^1_h is not a subspace of $\mathbb{G}^1$ one can view this as a nonconforming Galerkin problem. We have already proved that G^1_h converges in a sense to $\mathbb{G}^1$. Later, in Proposition 4.8 we provide the necessary variant of Theorem 1.1 to deduce from this the existence of a_h and its convergence to a_0.

Proposition 2.1. *For the exact and approximate harmonic tangent fields a_0 and a_h obtained as solutions of* (2.21) *and* (2.22)*, for a given tangent field u, we have the estimates*

$$\|a_h - a_0\|_0 \le C h^m \|u\|_{\mathrm{H}^{-1}_{\mathrm{rot}}(\Gamma)}. \tag{2.23}$$

Another useful observation is that, parallel to the fact that all norms on the finite-dimensional space $\mathbb{G}^1$ are equivalent, we have easily obtained the following.

Lemma 2.2. *There is $C > 0$ such that for all $-1 \le s \le 0$ and all h*

$$\forall u \in \mathsf{G}^1_h \quad \|u\|_0 \le C \|u\|_s. \tag{2.24}$$

2.4. Discrete Helmholtz decomposition. We will frequently use the fact that each $u \in \mathsf{S}^1_h$ can be written in a unique way:

$$u = \operatorname{rot} p + a + g, \tag{2.25}$$

with $p \in \mathsf{S}^{0\bullet}_h$, $a \in \mathsf{G}^1_h$, and g in the L_2-orthogonal of the kernel of the divergence operator in S^1_h. Notice that this decomposition expresses that S^1_h is split into a direct

[1] In fact, to derive the inf-sup estimates and related stability results we need, it will not be necessary to know that higher order elements yield higher order estimates.

sum of three subspaces which are orthogonal both for the $\mathrm{H}^0_{\mathrm{T}}$ and the $\mathrm{H}^0_{\mathrm{div}}$ scalar products.

What makes this decomposition useful is that it has the following continuity and approximation properties, compared with the "exact" Helmholtz decomposition, which we write

$$u = \operatorname{rot} p_0 + a_0 + g_0, \tag{2.26}$$

with $p_0 \in \mathrm{H}^{1/2}(\Gamma)^\bullet$, $a_0 \in \mathbb{G}^1$, and $g_0 \in \operatorname{grad} \mathrm{H}^{3/2}(\Gamma)$.

PROPOSITION 2.3. *There is $C > 0$ such that for all h, all $u \in \mathsf{S}^1_h$, the above decompositions* (2.25) *and* (2.26) *are related by*

$$\|g - g_0\|_0 \le Ch\|u\|_0, \qquad \|g\|_{-1} \le C\|u\|_{-1}, \tag{2.27}$$

$$\|a - a_0\|_0 \le Ch\|u\|_0, \qquad \|a\|_{-1} \le C\|u\|_{-1}, \tag{2.28}$$

$$\|\operatorname{rot} p - \operatorname{rot} p_0\|_0 \le Ch\|u\|_0, \qquad \|\operatorname{rot} p\|_{-1} \le C\|u\|_{-1}. \tag{2.29}$$

Proof. (i) As already remarked, by hypothesis (H3), we have

$$|g - g_0|_0 \le Ch|\operatorname{div} u|_0. \tag{2.30}$$

This immediately gives $\|g - g_0\|_0 \le Ch\|u\|_0$. Then we remark that

$$\|g\|_{-1} \le \|g - g_0\|_{-1} + \|g_0\|_{-1} \le Ch\|u\|_0 + \|u\|_{-1} \le C\|u\|_{-1}. \tag{2.31}$$

(ii) The fact that $\|a - a_0\|_0 \le Ch\|u\|_0$ was proved in the preceding section and gives $\|a\|_{-1} \le C\|u\|_{-1}$ just as above.

(iii) The last part of the proposition is deduced from the two preceding ones writing

$$\operatorname{rot} p - \operatorname{rot} p_0 = -(g + a) + (g_0 + a_0) = (g_0 - g) + (a_0 - a). \qquad \square \tag{2.32}$$

3. Stable discretizations of the Calderon formulas. First we explain why the most natural idea (at least to us, for quite some time) is actually flawed. Then we define the discretization which we propose to use for preconditioning. It is associated with a subspace of the usual Galerkin space for which we prove an LBB inf-sup condition and some basic approximation properties.

3.1. A flawed idea. Given a family of Galerkin spaces X_h in X the most straightforward idea is to introduce the maps

$$\mathcal{B}_h : \left\{ \begin{array}{ccc} X_h & \to & X_h^\star, \\ u & \mapsto & \langle Bu, \cdot \rangle \end{array} \right. \tag{3.1}$$

and then to put

$$\mathcal{Z}_h(k) = \mathcal{B}_h^{\star -1} \mathcal{A}_h(k) \mathcal{B}_h^{-1}. \tag{3.2}$$

As remarked in Christiansen and Nédélec [17], if not only $\mathcal{A}_h(k)$ but also $\mathcal{B}_h$ satisfies a uniform discrete inf-sup condition on X_h, then the spectral condition number of $\mathcal{Z}_h(k)\mathcal{A}_h(k)$ is bounded independently of h. Of course, since the operators we deal with are not SPD, this is not enough to guarantee convergence of Krylov subspace algorithms, but it is nevertheless a significant progress compared with the lack of a preconditioner. Unfortunately, for the standard Galerkin spaces, this last inf-sup condition does not hold. Indeed, throughout this paragraph let X_h denote the Raviart–Thomas spaces of degree m (with our conventions the minimal degree is $m = 1$) on Γ. We will use the fact that X_h satisfies the hypotheses (H0), . . . ,(H3).

PROPOSITION 3.1. *Let $X_h = \mathsf{S}_h^1$. Let K_h be the space*

$$\begin{aligned} \{u \in \mathsf{S}_h^1 \ : \ &\forall v \in \mathsf{S}_h^1 \quad \operatorname{div} v = 0 \Rightarrow \langle u, v\rangle = 0 \quad \textit{and} \\ &\forall v \in \mathsf{S}_h^0 \quad \langle u, \operatorname{grad} v\rangle = 0\}. \end{aligned} \tag{3.3}$$

Then we have

$$\liminf_h \frac{\dim K_h}{\dim X_h} \geq \frac{1}{2m+1} \tag{3.4}$$

and

$$\limsup_h \sup_{u \in K_h} \sup_{u' \in X_h} \frac{b(u,u')}{\|u\|_{-1/2}\|u'\|_{-1/2}} h^{-1/2} < +\infty. \tag{3.5}$$

Proof. (i). We have

$$\dim K_h \geq (\dim \mathsf{S}_h^2 - N^C) - (\dim \mathsf{S}_h^0 - N^C) \tag{3.6}$$

$$\begin{aligned} &\geq (m(m+1)/2)N_h^2 \\ &\qquad -N_h^0 + (m-1)N_h^1 + ((m-2)(m-1)/2)N_h^2 \end{aligned} \tag{3.7}$$

$$\geq -N_h^0 - (m-1)N_h^1 + (2m-1)N_h^2. \tag{3.8}$$

Recall that since each segment is shared by exactly two triangles, $2N_h^1 = 3N_h^2$, which together with the Euler–Poincaré formula gives

$$N_h^1 \sim 3N_h^0 \quad \text{and} \quad N_h^2 \sim 2N_h^0. \tag{3.9}$$

This gives

$$-N_h^0 - (m-1)N_h^1 + (2m-1)N_h^2 \sim mN_h^0. \tag{3.10}$$

One also checks that

$$\dim X_h \sim m(2m+1)N_h^0. \tag{3.11}$$

This gives the first inequality.

(ii). To prove the second part of the theorem we use the fact that X_h satisfies (H3). For any $u \in X_h$ we denote by ϕ_u the unique $\phi \in \mathrm{H}^1(\Gamma)^\bullet$ such that $\Delta\phi = \operatorname{div} u$. Then (H3) asserts that if $u \in X_h$ is L_2-orthogonal to the kernel of the divergence operator on X_h, then we have an estimate of the form $|u - \operatorname{grad}\phi_u|_0 \leq Ch|\operatorname{div} u|_0$. From Proposition 2.3, using an inverse inequality, one can deduce that

$$\|u - \operatorname{grad}\phi_u\|_{-1/2} \leq Ch^{1/2}\|u\|_{-1/2} \tag{3.12}$$

and

$$\|\operatorname{grad}\phi_u\|_{-1/2} \leq C\|u\|_{-1/2}. \tag{3.13}$$

Choose $u \in K_h$ and $u' \in X_h$. Put $u' = \operatorname{rot} p' + a' + g'$ as in (2.25). Remark first that

$$\langle u \times n, \operatorname{rot} p' \rangle = -\langle \operatorname{div} u, p' \rangle = 0. \tag{3.14}$$

Then write

$$\langle u \times n, g' \rangle = \langle (u - \operatorname{grad} \phi_u + \operatorname{grad} \phi_u) \times n, (g' - \operatorname{grad} \phi_{g'} + \operatorname{grad} \phi_{g'}) \rangle. \tag{3.15}$$

Developing and using the continuity of b as well as the fact that

$$\langle \operatorname{grad} \phi_u \times n, \operatorname{grad} \phi_{g'} \rangle = 0, \tag{3.16}$$

we obtain

$$|\langle u \times n, g' \rangle| \leq C \|u - \operatorname{grad} \phi_u\|_{-1/2} \|g' - \operatorname{grad} \phi_{g'}\|_{-1/2} \tag{3.17}$$

$$+ C \|u - \operatorname{grad} \phi_u\|_{-1/2} \|\operatorname{grad} \phi_{g'}\|_{-1/2} \tag{3.18}$$

$$+ C \|\operatorname{grad} \phi_u\|_{-1/2} \|g' - \operatorname{grad} \phi_{g'}\|_{-1/2}. \tag{3.19}$$

By the above estimates (3.12) and (3.13) it follows that

$$|\langle u \times n, g' \rangle| \leq C h^{1/2} \|u\|_{-1/2} \|g'\|_{-1/2}. \tag{3.20}$$

Therefore

$$|\langle u \times n, g' \rangle| \leq C h^{1/2} \|u\|_{-1/2} \|u'\|_{-1/2}. \tag{3.21}$$

Finally, for the last part, for any $\tilde{a}'$, $\langle u \times n, a' \rangle$ equals

$$\langle (u - \operatorname{grad} \phi_u) \times n, \tilde{a}' \rangle + \langle \operatorname{grad} \phi_u \times n, \tilde{a}' \rangle + \langle u \times n, (a' - \tilde{a}') \rangle. \tag{3.22}$$

Choosing $\tilde{a}' \in \mathbb{G}^1$ to be an approximation of a', one immediately obtains the proposition. ∎

Thus one sees that the reason for the degeneracy is that the subspace of X_h of elements which are in a sense discrete gradients does not have the same dimension as the subspace of rotationals.

3.2. Auxiliary spaces. Let $(\mathsf{S}_h^0, \mathsf{S}_h^1, \mathsf{S}_h^2)$, and $(\mathsf{S}_h'^0, \mathsf{S}_h'^1, \mathsf{S}_h'^2)$ be two triples of spaces of the type we discussed; more precisely, they should satisfy the null sequence condition, the discrete cohomology groups should have the "right" dimension, and S_h^1 and $\mathsf{S}_h'^1$ should satisfy the hypotheses (H0), ..., (H3).

Two examples to keep in mind (the first one detailed and the second one mentioned in Christiansen and Nédélec [19]) are

- the case where S_h^1 and $\mathsf{S}_h'^1$ are equal and consist of lowest order Raviart–Thomas fields (then S_h^0 consists of continuous P^1 FE and S_h^2 of P^0 FE);
- the case where $(\mathsf{S}_h^0, \mathsf{S}_h^1, \mathsf{S}_h^2)$ corresponds to lowest order Raviart–Thomas fields, whereas $(\mathsf{S}_h'^0, \mathsf{S}_h'^1, \mathsf{S}_h'^2)$ corresponds to lowest order Brezzi–Douglas–Marini fields on the same mesh (then $\mathsf{S}_h'^0$ consists of continuous P^2 FE and $\mathsf{S}_h'^2$ of P^0 FE).

More generally (though it is not necessary) one might want to choose spaces such that the L_2-projections $\mathsf{S}_h^{2\bullet} \to \mathsf{S}_h'^{0\bullet}$, and $(\ker_{\operatorname{div}} \mathsf{S}_h^1) \times n \to \mathsf{S}_h'^1$ have kernels which are small in some sense (for instance have dimensions bounded by some small integer independently of h). Anticipating what follows, this would guarantee that the subspace $(\mathsf{S}_h^1)^\wedge$ of S_h^1, to be defined later, is almost all of S_h^1. If the first triple of spaces is based on Raviart–Thomas fields of any order, one can take for this purpose Brezzi–Douglas–Marini fields *of the same order* on the same mesh in the second triple.

In fact, in addition to the hypothesis H0, … ,H3 we will use some L^2 estimates for the spaces S_h^0 and S_h^2 (in the proofs of Lemmas 3.11 and 3.12) and an additional L^2 estimate for S_h^1 (proof of Proposition 4.2). Therefore we require in what follows that the spaces $(\mathsf{S}_h^0, \mathsf{S}_h^1, \mathsf{S}_h^2)$ and $(\mathsf{S}_h'^0, \mathsf{S}_h'^1, \mathsf{S}_h'^2)$ are the standard finite element spaces based on Raviart–Thomas or Brezzi–Douglas–Marini finite elements. The two triples can, however, have different orders m and m'. They can even be constructed on different meshes (with associated parameters h and h') as long as $(1/C)h \leq h' \leq Ch$. The most useful cases are $m' \geq m$ and $h' \leq h$.

Other Galerkin spaces are commonly used to solve boundary integral equations, and the method might work in the present state for such Galerkin spaces also. For instance, finite elements based on meshes with both triangular and quadrilateral elements pose no additional problem, once one has identified the appropriate null sequences called microlocal discretizations which are currently being developed.

3.3. Definition. Our starting point is to try to construct a preconditioner for the variational formulation of the EFIE on S_h^1. For this purpose we will use the auxiliary spaces $(\mathsf{S}_h'^0, \mathsf{S}_h'^1, \mathsf{S}_h'^2)$. As it turns out, with this preconditioner the EFIE is actually solved variationally on a subspace of S_h^1. However, we shall prove that this subspace (it will be denoted $(\mathsf{S}_h^1)^\wedge$) satisfies the hypotheses (H0),…,(H3), which ensures the well-posedness of the discrete problem.

Starting with a linear form $l \in \mathsf{S}_h^{1\star}$, determine the solution (u, q) of

$$\left\{ \begin{array}{l} u \in \mathsf{S}_h^1 \\ q \in \mathsf{S}_h^{2\bullet} \end{array} \right. \quad \left\{ \begin{array}{lll} \forall u' \in \mathsf{S}_h^1 & \langle u, u' \rangle + \langle q, \operatorname{div} u' \rangle & = \quad l(u'); \\ \forall q' \in \mathsf{S}_h^{2\bullet} & \langle q', \operatorname{div} u \rangle & = \quad 0. \end{array} \right. \tag{3.23}$$

Then to (u, q) associate the following element of $\mathsf{S}_h'^1$:

$$v = \mathcal{P}_{\mathsf{S}_h'^1}(u \times n) - \operatorname{rot} \mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q), \tag{3.24}$$

where for any space X_h, $\mathcal{P}_{\mathsf{X}_h}$ denotes the L_2-orthogonal projections onto X_h.

Let $\Theta_h : \mathsf{S}_h^{1\star} \to \mathsf{S}_h'^1, l \mapsto v$ be the composition of these two maps (defined by (3.23) and (3.24)), and let $\Theta_h^\star : \mathsf{S}_h'^{1\star} \to \mathsf{S}_h^{1\star\star} \approx \mathsf{S}_h^1$ be its adjoint. Then we put

$$\mathcal{Z}_h = \Theta_h^\star \mathcal{A}_h'(k) \Theta_h, \tag{3.25}$$

where $\mathcal{A}_h'(k) : \mathsf{S}_h'^1 \to \mathsf{S}_h'^{1\star}$ is the map induced by $\mathcal{A}(k)$.

Remark. In some cases it might be of interest to replace $\mathcal{A}_h'(k)$ by $\mathcal{A}_h'(k')$ for some different, possibly complex, wavenumber k'. In particular, a small perturbation $k' = k + i\epsilon$ guarantees invertibility even at resonant frequencies and is related to the limiting absorption principle. However, we will not discuss this possibility here.

3.4. Interpretation of the system. The invertibility in the sense of Babuska–Brezzi of the system (3.23) can be reinterpreted as the fact that the bilinear form b satisfies a uniform LBB inf-sup estimate on the spaces $\mathsf{S}_h^{1\#} \times \mathsf{S}_h^1$, where we have used the notation

$$\mathsf{S}_h^{1\#} = \{u \times n : u \in \mathsf{S}_h^1 \quad \text{and} \quad \operatorname{div} u = 0\} + \{\operatorname{rot} q : q \in \mathsf{S}_h^{0\bullet}\}. \tag{3.26}$$

To give a precise meaning to and prove this statement, notice that $\mathsf{S}_h^{1\#}$ is a subspace of $\mathrm{H}_{\mathrm{div}}^{-1}(\Gamma)$ (but contains vector-valued measures concentrated on the curved lines $\Xi_h^{-1}([S])$, where S is a segment in $\mathcal{T}_h$). We will also need the following lemma.

LEMMA 3.2. *Any $v \in \mathrm{H}^{-1+s}_{\mathrm{div}}(\Gamma)$, can be written in a unique way:*

$$v = u \times n - \operatorname{rot} q, \tag{3.27}$$

with $u \in \mathrm{H}^s_{\mathrm{div}}(\Gamma)$, $\operatorname{div} u = 0$, and $q \in \mathrm{H}^s(\Gamma)^\bullet$, and we have the equivalence of norms[2]

$$\|v\|^2_{-1+s} \approx |u|^2_s + |q|^2_s. \tag{3.28}$$

Proof. This can be proved using the HDRL (section 1.1). ∎

The lemma expresses that we have exhibited isomorphisms (for each s):

$$\{u \in \mathrm{H}^s_{\mathrm{div}}(\Gamma) \ : \ \operatorname{div} u = 0\} \times \mathrm{H}^s(\Gamma)^\bullet \to \mathrm{H}^{s-1}_{\mathrm{div}}(\Gamma). \tag{3.29}$$

In particular, the sum appearing in the definition of $\mathsf{S}^{1\#}_h$ is direct. Furthermore, we notice that

$$b(u \times n - \operatorname{rot} q, v') = \langle B(u \times n - \operatorname{rot} q), v'\rangle = -\langle u, v'\rangle - \langle q, \operatorname{div} v'\rangle, \tag{3.30}$$

where we have used the notation $\langle \cdot, \cdot \rangle$ for the three different standard dualities on

$$\mathrm{H}^{-1}_{\mathrm{rot}} \times \mathrm{H}^0_{\mathrm{div}}\ ,\ \ \mathrm{H}^0_{\mathrm{T}} \times \mathrm{H}^0_{\mathrm{T}}, \quad \text{and} \quad \mathrm{H}^0 \times \mathrm{H}^0. \tag{3.31}$$

Therefore, given $l \in \mathsf{S}^{1\star}_h$, if (u, q) solves system (3.23), then $v = u \times n - \operatorname{rot} q$ solves

$$v \in \mathsf{S}^{1\#}_h \quad \forall v' \in \mathsf{S}^1_h \quad -b(v, v') = l(v'), \tag{3.32}$$

and if v solves this equation, then, writing $v = u \times n - \operatorname{rot} q$ as in (3.26), (u, q) is also given by the *continuous* inverse of the map (3.29) and solves system (3.23).

Using the well-known properties of this system, one immediately obtains the following.

PROPOSITION 3.3. *There is $C > 0$ such that for all h*

$$\inf_{v \in \mathsf{S}^{1\#}_h} \sup_{v' \in \mathsf{S}^1_h} \frac{|b(v, v')|}{\|v\|_{-1}\|v'\|_0} \geq 1/C. \tag{3.33}$$

One also checks directly that these spaces have the same dimension.

3.5. Interpretation of the projections. According to Lemma 3.2 the projections defined by (3.24) correspond to a projection in the $\mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)$-norm. Lemma 3.16 and Proposition 3.18 further justify this interpretation.

3.6. A characterization of the kernel of Θ_h. In this paragraph, for any space X_h, $\mathcal{P}_{\mathsf{X}_h}$ is the orthogonal projection onto X_h for the usual L_2-inner product (on scalar or vector fields). The symbol $\perp$ is also relative to these inner products.

We introduce the following auxiliary spaces:

$$\mathsf{s}^0_h = \{\ p \in \mathsf{S}^{0\bullet}_h \ : \ p \perp \mathsf{S}'^{2\bullet}_h\} \quad \text{and} \quad \mathsf{s}^2_h = \{\ q \in \mathsf{S}^{2\bullet}_h \ : \ q \perp \mathsf{S}'^{0\bullet}_h\}. \tag{3.34}$$

Define also

$$(\mathsf{S}^1_h)^\wedge = \{v \in \mathsf{S}^1_h \ : \ v \perp \operatorname{rot} \mathsf{s}^0_h \quad \text{and} \quad \operatorname{div} v \perp \mathsf{s}^2_h\}. \tag{3.35}$$

[2]There is an obvious misprint in [19].

The introduction of s_h^0 and s_h^2 is justified by the two following lemmas, whereas that of $(\mathsf{S}_h^1)^\wedge$ is justified by Proposition 3.7. It shows that (for h small enough—we will not always repeat this condition—) $(\mathsf{S}_h^1)^\wedge$ is the range of $\Theta_h^\star$.

LEMMA 3.4. *There is h_0 such that for all $h < h_0$ and all divergence-free $u \in \mathsf{S}_h^1$*

$$\mathcal{P}_{\mathsf{S}_h'^1}(u \times n) = 0 \iff u \in \operatorname{rot} \mathsf{s}_h^0. \tag{3.36}$$

Proof. We use the fact that for all $\epsilon > 0$ there is $h_0 > 0$ such that for $h < h_0$

$$\forall a \in \mathsf{G}_h^1 \quad |a \times n - \mathcal{P}_{\mathsf{G}_h'^1}(a \times n)|_0 \leq \epsilon |a|_0. \tag{3.37}$$

Choosing a h_0 relative to a $\epsilon < 1$ we suppose from now on that $h < h_0$.

Pick $u \in \ker_{\mathsf{S}_h^1} \operatorname{div}$. Put $u = \operatorname{rot} p + a$, with $p \in \mathsf{S}_h^{0\bullet}$ and $a \in \mathsf{G}_h^1$.

(i). Suppose that the projection of $u \times n$ is zero. For all divergence-free $a' \in \mathsf{S}_h'^1$, we have

$$0 = \langle \mathcal{P}_{\mathsf{S}_h'^1}(-\operatorname{grad} p + a \times n), \overline{a'} \rangle = \langle (-\operatorname{grad} p + a \times n), \overline{a'} \rangle = \langle a \times n, \overline{a'} \rangle. \tag{3.38}$$

Put $a' = \mathcal{P}_{\mathsf{G}_h'^1}(a \times n)$. Then

$$|a|_0^2 = |\langle a \times n, \overline{a \times n} \rangle| = |\langle a \times n, (\overline{a \times n - a'}) \rangle| \leq \epsilon |a|_0^2. \tag{3.39}$$

Hence $a = 0$. Moreover, for all $v \in \mathsf{S}_h'^1$,

$$\langle \operatorname{rot} p \times n, v \rangle = \langle p, \operatorname{div} v \rangle. \tag{3.40}$$

It follows that $p \perp \mathsf{S}_h'^{2\bullet}$ and $u \in \operatorname{rot} \mathsf{s}_h^0$.

(ii). Conversely, if $u \in \operatorname{rot} \mathsf{s}_h^0$, then the above equality (3.40) shows that $u \times n \perp \mathsf{S}_h'^1$, and hence its projection is zero. ◻

LEMMA 3.5. *For all $q \in \mathsf{S}_h^{2\bullet}$ we have*

$$\operatorname{rot} \mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q) = 0 \iff \mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q) = 0 \iff q \in \mathsf{s}_h^2. \tag{3.41}$$

Proof. The proof is trivial. ◻

LEMMA 3.6. *The spaces $\mathcal{P}_{\mathsf{S}_h'^1}((\ker_{\mathrm{H}^0_{\mathrm{div}}(\Gamma)} \operatorname{div}) \times n)$ and $\operatorname{rot} \mathsf{S}_h'^0$ are L_2-orthogonal.*

Proof. Indeed, if $u \in \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ is divergence-free and $p \in \mathsf{S}_h'^0$, then (since $\operatorname{rot} p \in \mathsf{S}_h'^1$)

$$\langle \mathcal{P}_{\mathsf{S}_h'^1}(u \times n), \operatorname{rot} p \rangle = \langle u \times n, \operatorname{rot} p \rangle = -\langle \operatorname{div} u, p \rangle = 0. \qquad ◻ \tag{3.42}$$

We now prove the following.

PROPOSITION 3.7. *For all $l \in \mathsf{S}_h^{1\star}$ we have*

$$\Theta_h(l) = 0 \iff \forall u' \in (\mathsf{S}_h^1)^\wedge \quad l(u') = 0. \tag{3.43}$$

Proof. Pick $l \in \mathsf{S}_h^{1\star}$. Let (u, q) be the solution of

$$\left\{ \begin{array}{l} u \in \mathsf{S}_h^1 \\ q \in \mathsf{S}_h^{2\bullet} \end{array} \right. \quad \left\{ \begin{array}{llcl} \forall u' \in \mathsf{S}_h^1 & \langle u, u' \rangle + \langle q, \operatorname{div} u' \rangle & = & l(u'); \\ \forall q' \in \mathsf{S}_h^{2\bullet} & \langle q', \operatorname{div} u \rangle & = & 0. \end{array} \right. \tag{3.44}$$

With these definitions we have $\Theta_h(l) = 0$ if and only if

$$\mathcal{P}_{\mathsf{S}_h'^1}(u \times n) - \operatorname{rot} \mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q) = 0. \tag{3.45}$$

According to Lemma 3.6, this is in turn equivalent to

$$\mathcal{P}_{\mathsf{S}_h'^1}(u \times n) = 0 \quad \text{and} \quad \operatorname{rot} \mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q) = 0. \tag{3.46}$$

In Lemmas 3.4 and 3.5 we gave equivalent statements for these two conditions.

(i). If $\Theta_h(l) = 0$, then $u \in \operatorname{rot} \mathsf{s}_h^0$ and $q \in \mathsf{s}_h^2$; hence, for all $u' \in (\mathsf{S}_h^1)^\wedge$,

$$l(u') = \langle u, u' \rangle + \langle q, \operatorname{div} u' \rangle = 0. \tag{3.47}$$

That is to say, l vanishes on $(\mathsf{S}_h^1)^\wedge$.

(ii). If l vanishes on $(\mathsf{S}_h^1)^\wedge$, then

- for all $u' \in \ker_{\mathsf{S}_h^1} \operatorname{div}$ such that $u' \perp \operatorname{rot} \mathsf{s}_h^0$ we have (since $u' \in (\mathsf{S}_h^1)^\wedge$)

$$\langle u, u' \rangle = l(u') = 0, \tag{3.48}$$

  so $u \in \operatorname{rot} \mathsf{s}_h^0$;
- for all $q' \in \mathsf{S}_h^{2\bullet}$ such that $q' \perp \mathsf{s}_h^2$, picking $u' \in \mathsf{S}_h^1$ such that $\operatorname{div} u' = q'$ and $u' \perp \ker_{\mathsf{S}_h^1} \operatorname{div}$, we have (since $u' \in (\mathsf{S}_h^1)^\wedge$)

$$\langle q, q' \rangle = \langle q, \operatorname{div} u' \rangle = l(u') = 0, \tag{3.49}$$

  and hence $q \in \mathsf{s}_h^2$.

The proof is complete. □

3.7. Approximation properties of the range of $\Theta_h^\star$. We give yet another characterization of $(\mathsf{S}_h^1)^\wedge$, which will enable us to deduce its approximation properties.

LEMMA 3.8. *Pick $u \in \mathsf{S}_h^1$. Put $u = \operatorname{rot} p + a + g$ with $p \in \mathsf{S}_h^{0\bullet}$, $a \in \mathsf{G}_h^1$, and $g \in \mathsf{S}_h^1$ such that $g \perp \ker_{\mathsf{S}_h^1} \operatorname{div}$. Then we have*

$$u \in (\mathsf{S}_h^1)^\wedge \iff \operatorname{rot} p \perp \operatorname{rot} \mathsf{s}_h^0 \quad \textit{and} \quad \operatorname{div} g \perp \mathsf{s}_h^2. \tag{3.50}$$

Proof. The proof is trivial. □

Now we give equivalent expressions for the above two conditions.

LEMMA 3.9. *Choose $q \in \mathsf{S}_h^{2\bullet}$. We have*

$$q \perp \mathsf{s}_h^2 \iff \exists p \in \mathsf{S}_h'^{0\bullet}, \qquad q = \mathcal{P}_{\mathsf{S}_h^{2\bullet}}(p). \tag{3.51}$$

Proof. More generally, we have the following result: Let X be a Hilbert space. The orthogonal projection onto a closed subspace Y of X is written P_Y, and orthogonality is denoted by $\perp$. If Y and Z are two closed subspaces of X, we put

$$Y_{\perp Z} = \{x \in Y \ : \ x \perp Z\}. \tag{3.52}$$

Let A and B be two closed subspaces of X. Then

$$A_{\perp (A_{\perp B})} = P_A(B). \qquad \square \tag{3.53}$$

LEMMA 3.10. *Choose $p \in \mathsf{S}_h^{0\bullet}$. We have*

$$\operatorname{rot} p \perp \operatorname{rot} \mathsf{s}_h^0 \iff \exists q \in \mathsf{S}_h'^{2\bullet}, \qquad \operatorname{rot} p = \mathcal{P}_{\operatorname{rot} \mathsf{S}_h^{0\bullet}}(\operatorname{rot} \Delta^{-1} q). \tag{3.54}$$

Proof. We apply the same technique once again. For all $p \in \mathsf{S}_h^{0\bullet}$ and all $q \in \mathsf{S}_h'^{2\bullet}$ we have

$$\langle p, q \rangle = -\langle \operatorname{rot} p, \operatorname{rot} \Delta^{-1} q \rangle. \tag{3.55}$$

Hence

$$\operatorname{rot} p \in \operatorname{rot} \mathsf{s}_h^0 \iff p \in \mathsf{s}_h^0 \tag{3.56}$$
$$\iff p \perp \mathsf{S}_h^{\prime 2\bullet} \tag{3.57}$$
$$\iff \operatorname{rot} p \perp \operatorname{rot} \Delta^{-1} \mathsf{S}_h^{\prime 2\bullet} \tag{3.58}$$
$$\iff \operatorname{rot} p \perp \mathcal{P}_{\operatorname{rot} \mathsf{S}_h^{0\bullet}}(\operatorname{rot} \Delta^{-1} \mathsf{S}_h^{\prime 2\bullet}). \tag{3.59}$$

So

$$\operatorname{rot} p \perp \operatorname{rot} \mathsf{s}_h^0 \iff \operatorname{rot} p \in \mathcal{P}_{\operatorname{rot} \mathsf{S}_h^{0\bullet}}(\operatorname{rot} \Delta^{-1} \mathsf{S}_h^{\prime 2\bullet}). \qquad \square \tag{3.60}$$

We will also need the following two approximation results.

LEMMA 3.11. *There is $C > 0$ such that for all h and all $\phi \in \mathrm{H}^1(\Gamma)^\bullet$*

$$\inf\{|\phi - q|_0 \ : \ q \in \mathcal{P}_{\mathsf{S}_h^{2\bullet}}(\mathsf{S}_h^{\prime 0\bullet})\} \leq Ch|\phi|_1. \tag{3.61}$$

Proof. Notice that

$$|\phi - \mathcal{P}_{\mathsf{S}_h^{2\bullet}}\mathcal{P}_{\mathsf{S}_h^{\prime 0\bullet}}\phi| \leq |\phi - \mathcal{P}_{\mathsf{S}_h^{\prime 0\bullet}}\phi| + |\mathcal{P}_{\mathsf{S}_h^{\prime 0\bullet}}\phi - \mathcal{P}_{\mathsf{S}_h^{2\bullet}}\mathcal{P}_{\mathsf{S}_h^{\prime 0\bullet}}\phi| \tag{3.62}$$
$$\leq Ch|\phi|_1 + Ch|\mathcal{P}_{\mathsf{S}_h^{\prime 0\bullet}}\phi|_1 \tag{3.63}$$
$$\leq Ch|\phi|_1. \qquad \square \tag{3.64}$$

LEMMA 3.12. *There is $C > 0$ such that for all h and all $\phi \in \mathrm{H}^2(\Gamma)^\bullet$*

$$\inf\{|\operatorname{rot}\phi - u|_0 \ : \ u \in \mathcal{P}_{\operatorname{rot} \mathsf{S}_h^{0\bullet}}(\operatorname{rot} \Delta^{-1} \mathsf{S}_h^{\prime 2\bullet})\} \leq Ch|\operatorname{rot}\phi|_1. \tag{3.65}$$

Proof. We have

$$|\Delta^{-1}\mathcal{P}_{\mathsf{S}_h^{\prime 2\bullet}}\Delta\phi - \phi|_1 \leq C|\mathcal{P}_{\mathsf{S}_h^{\prime 2\bullet}}\Delta\phi - \Delta\phi|_{-1} \leq Ch|\Delta\phi|_0. \tag{3.66}$$

So, with $\psi = \Delta^{-1}\mathcal{P}_{\mathsf{S}_h^{2\bullet}}\Delta\phi$, we have

$$|\operatorname{rot}\psi - \operatorname{rot}\phi|_0 \leq Ch|\operatorname{rot}\phi|_1 \tag{3.67}$$

and

$$|\mathcal{P}_{\operatorname{rot} \mathsf{S}_h^{0\bullet}} \operatorname{rot}\psi - \operatorname{rot}\psi|_0 \leq Ch|\operatorname{rot}\psi|_1. \tag{3.68}$$

However,

$$|\operatorname{rot}\psi|_1 \leq C|\Delta^{-1}\mathcal{P}_{\mathsf{S}_h^{\prime 2\bullet}}\Delta\phi|_2 \leq C|\mathcal{P}_{\mathsf{S}_h^{\prime 2\bullet}}\Delta\phi|_0 \leq C|\Delta\phi|_0 \leq C|\operatorname{rot}\phi|_1. \tag{3.69}$$

Now combine inequalities (3.68) and (3.69) and conclude using (3.67). $\square$

From the above results we deduce the following fundamental theorem.

THEOREM 3.13. *The spaces $X_h = (\mathsf{S}_h^1)^\wedge$ satisfy hypothesis* (H1).

Proof. Pick $u \in \mathrm{H}^1_{\mathrm{div}}(\Gamma)$. Consider its Helmholtz decomposition

$$u = \operatorname{rot}\phi + \alpha + \operatorname{grad}\psi. \tag{3.70}$$

(i) The field $\operatorname{rot}\phi$ is approximated using Lemma 3.12, which gives an element of X_h according to Lemmas 3.8 and 3.10.

(ii) The field α is approximated by $a = \Omega_h\alpha$, where Ω_h is defined by system (2.18).

(iii) The field $\Delta\psi$ is approximated by a $q \in \mathcal{P}_{\mathsf{S}_h^{2\bullet}}(\mathsf{S}_h^{\prime 0\bullet})$ following Lemma 3.11. Then we consider $\Omega_h \operatorname{grad}\Delta^{-1}q$ which is in X_h according to Lemmas 3.8 and 3.9. $\square$

3.8. Well-posedness.

PROPOSITION 3.14. *The spaces* $X_h = (\mathsf{S}^1_h)^\wedge$ *satisfy hypothesis* (H3).

Proof. Choose $u \in X_h$ such that

$$u \perp \ker_{X_h} \operatorname{div}. \tag{3.71}$$

Trivially we have $u \in \mathsf{S}^1_h$. Moreover, if u' is a divergence-free element of S^1_h it can be written $\operatorname{rot} p + u''$ with $p \in \mathsf{s}^0_h$ and a divergence-free $u'' \in X_h$. (To see this just remark that the L_2-orthogonal of $\operatorname{rot} \mathsf{s}^0_h$ in $\ker_{\mathsf{S}^1_h} \operatorname{div}$ is a subspace of X_h.) And since $u \perp \operatorname{rot} \mathsf{s}^0_h$ and $u \perp u''$ we therefore have

$$u \perp \ker_{\mathsf{S}^1_h} \operatorname{div}. \tag{3.72}$$

Then the proposition follows from the result known to hold for S^1_h. □

We have therefore reached the main goal of this section.

THEOREM 3.15. *The spaces* $(\mathsf{S}^1_h)^\wedge$ *satisfy the four hypothesis* (H0), ..., (H3).

If $\mathcal{A}(k)$ were symmetric positive definite, then $\mathcal{Z}_h\mathcal{A}_h$ would determine an isomorphism $(\mathsf{S}^1_h)^\wedge \to (\mathsf{S}^1_h)^\wedge$, and, for a given h, the preconditioned conjugate gradients (PCG) algorithm would converge towards the variational solution on $(\mathsf{S}^1_h)^\wedge$, which by the above theorem is a good one.

In our indefinite case it might be that $\mathcal{Z}_h\mathcal{A}_h$ does not determine an isomorphism $(\mathsf{S}^1_h)^\wedge \to (\mathsf{S}^1_h)^\wedge$; this would be the case if the bilinear form induced by $\mathcal{A}$ on the range of Θ_h (as opposed to $\Theta^\star_h$) were degenerate, a question we have not settled. However, we are sure that the PCG algorithm constructs iterates that are in $(\mathsf{S}^1_h)^\wedge$, and later—in section 3.9—we will show how to construct stopping criteria that guarantee that the residual is small as a linear form on $(\mathsf{S}^1_h)^\wedge$. Thus one can *check* that the approximate solution given by the PCG algorithm is close to the variational solution on $(\mathsf{S}^1_h)^\wedge$. Theorem 3.15 ensures that this variational solution (exists, is unique, and) is close to the *best* (for any chosen norm) approximation on $(\mathsf{S}^1_h)^\wedge$ of the exact solution and that for small h this best approximation is a *good* approximation.

3.9. Stopping criterion. Proposition 3.7 shows that, for all $u \in (\mathsf{S}^1_h)^\wedge$, u solves the variational problem

$$u \in (\mathsf{S}^1_h)^\wedge \quad \text{and} \quad \forall v \in (\mathsf{S}^1_h)^\wedge \quad a(u,v) = l(v) \tag{3.73}$$

if and only if

$$\Theta_h(\mathcal{A}_h u - l_h) = 0, \tag{3.74}$$

which is in turn equivalent to

$$\|\Theta_h(\mathcal{A}_h u - l_h)\| = 0, \tag{3.75}$$

for any norm $\|\cdot\|$ on S'^1_h. We now set out to define norms on S'^1_h that are uniformly equivalent to the $\mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)$-norm but more readily computable.

The following lemma should be seen in relation to Lemma 3.2.

LEMMA 3.16. *There is* $C > 0$ *such that for all* $u \in \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)$, *if* $u = \operatorname{rot} p + v$ *with* $v \in \mathrm{H}^0_{\mathrm{T}}(\Gamma)$ *and* $p \in \mathrm{H}^0(\Gamma)$, *we have*

$$\|u\|^2_{-1} \le C\big(|p|^2_0 + |v|^2_0\big). \tag{3.76}$$

Proof. It holds

$$\|\operatorname{rot} p + v\|_{-1}^2 = |\operatorname{rot} p + v|_{-1}^2 + |\operatorname{div} v|_{-1}^2. \tag{3.77}$$

However,

$$|\operatorname{rot} p + v|_{-1}^2 \leq 2\big(|\operatorname{rot} p|_{-1}^2 + |v|_{-1}^2\big) \leq C\big(|p|_0^2 + |v|_{-1}^2\big). \tag{3.78}$$

Then one immediately concludes using $|\operatorname{div} v|_{-1} \leq C|v|_0$. ∎

For convenience we state as a separate lemma the following fact which follows from (H3).

LEMMA 3.17. *There is $C > 0$ such that for all $u \in X_h$, if*

$$\forall u' \in X_h \quad \operatorname{div} u' = 0 \ \Rightarrow\ \langle u, u'\rangle = 0, \tag{3.79}$$

then

$$|u|_0 \leq C|\operatorname{div} u|_{-1}. \tag{3.80}$$

Proof. Let ϕ be the solution of

$$\phi \in \mathrm{H}^1(\Gamma)^\bullet \quad \text{and} \quad \Delta\phi = \operatorname{div} u. \tag{3.81}$$

We have

$$|u|_0 \leq |u - \operatorname{grad}\phi|_0 + |\operatorname{grad}\phi|_0 \leq Ch|\operatorname{div} u|_0 + C|\operatorname{div} u|_{-1}. \tag{3.82}$$

The lemma then follows from an inverse inequality. ∎

It is a particular case of Lemma 3.2 that the converse inequality of Lemma 3.16 holds whenever v is such that $\operatorname{rot} v = 0$. This fact has the following discrete analogue.

PROPOSITION 3.18. *There is $C > 0$, such that for all h and all $u \in \mathsf{S}_h^1$, if $u = \operatorname{rot} p + v$ with $p \in \mathsf{S}_h^{0\bullet}$ and $v \perp \operatorname{rot} \mathsf{S}_h^{0\bullet}$, then*

$$|p|_0^2 + |v|_0^2 \leq C\|u\|_{-1}^2. \tag{3.83}$$

Proof. Put $v = a + w$ with $a \in \mathsf{G}_h^1$ and $w \perp \ker_{\mathsf{S}_h^1} \operatorname{div}$. We have

$$|v|_0^2 = |a|_0^2 + |w|_0^2. \tag{3.84}$$

However, by Lemma 2.2 and Proposition 2.3 we have

$$|a|_0 \leq C|a|_{-1} \leq C\|u\|_{-1}, \tag{3.85}$$

and by Lemma 3.17 we have

$$|w|_0 \leq C|\operatorname{div} w|_{-1} = |\operatorname{div} u|_{-1} \leq \|u\|_{-1}. \tag{3.86}$$

So

$$|v|_0^2 \leq C\|u\|_{-1}^2. \tag{3.87}$$

Moreover,

$$|p|_0^2 \leq C|\operatorname{rot} p|_{-1}^2 \leq C\big(|\operatorname{rot} p + v|_{-1}^2 + |v|_{-1}^2\big). \tag{3.88}$$

The proposition follows. ∎

Of course the same holds true for the spaces with a prime, which is in fact what will be of interest to us.

Let l be a linear form on S^1_h, and, as in the definition of the preconditioner, let (u, q) be the solution of system (3.23) so that

$$\Theta_h l = \mathcal{P}_{\mathsf{S}'^1_h}(u \times n) - \operatorname{rot} \mathcal{P}_{\mathsf{S}'^0_h\bullet}(q). \tag{3.89}$$

Lemma 3.16 together with Proposition 3.18 now prove that we have the uniform (i.e., independent of h) equivalence of norms

$$\|\Theta_h l\|^2_{-1} \approx |\mathcal{P}_{\mathsf{S}'^1_h}(u \times n)|^2_0 + |\mathcal{P}_{\mathsf{S}'^0_h\bullet}(q)|^2_0. \tag{3.90}$$

A stopping criterion can therefore be a sufficient reduction of this norm. It is important to notice that to effectively compute these norms in the course of a conjugate gradients algorithm is a negligible task compared with the other ones, requiring only two sparse matrix products (at each iteration).

Another norm which is both natural and easily computable is $\|\Theta_h l\|_0$. Also, the quantity $(\|\Theta_h l\|_0 \|\Theta_h l\|_{-1})^{1/2}$, though not a norm, satisfies the interpolation inequality

$$\|\Theta_h l\|_{1/2} \leq C\big(\|\Theta_h l\|_0 \|\Theta_h l\|_{-1}\big)^{1/2} \tag{3.91}$$

and is therefore another good candidate for the construction of a stopping criterion. We have not determined to which extent the choice between these candidates really produces any significant differences on industrial problems.

4. Behavior of the iterates. We denote by Θ the continuous analogue of Θ_h, that is, the map $(\mathrm{H}^0_{\mathrm{div}}(\Gamma))^\star \to \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)$ which to l associates $u \times n - \operatorname{rot} q$, where u and q are the solutions of the continuous saddle-point problem. We also have $\mathcal{Z} = \Theta^\star \mathcal{A} \Theta$.

The Krylov subspaces are defined to be

$$\mathfrak{K}^n_h = \{P(\mathcal{Z}_h \mathcal{A}_h)\mathcal{Z}_h l|_{\mathsf{S}^1_h} \;:\; P \in \mathbb{C}[X],\ \deg P \leq n\}. \tag{4.1}$$

Their importance stems from the fact that—for fixed h—the PCG algorithm attempts to determine (by short recurrences) the sequence of solutions (u^n_h) of the problems

$$u \in \mathfrak{K}^n_h \quad \text{and} \quad \forall v \in \mathfrak{K}^n_h \quad a(u, v) = l(v). \tag{4.2}$$

For generalities about the PCG and related algorithms we refer to Barrett et al. [3] or Kelley [36]. Since we deal with complex-symmetric matrices, see also Freund [29]. In this section we investigate the convergence of the spaces $\mathfrak{K}^n_h$ towards their continuous analogues $\mathfrak{K}^n$, for fixed n as $h \to 0$, where naturally we have put

$$\mathfrak{K}^n = \{P(\mathcal{Z}\mathcal{A})\mathcal{Z}l \;:\; P \in \mathbb{C}[X],\ \deg P \leq n\}. \tag{4.3}$$

We will deduce from this results on the convergence as $h \to 0$ of the iterate u^n_h towards the solution u^n of

$$u \in \mathfrak{K}^n \quad \text{and} \quad \forall v \in \mathfrak{K}^n \quad a(u, v) = l(v). \tag{4.4}$$

We emphasize that for non-SPD problems the convergence or breakdown of the Lanczos process is as of today not completely understood. Here our point of view is to suppose that the continuous Lanczos process is well-defined up to iteration n, and then show that for small enough h, the discrete one is also well-defined, and yields arbitrarily close iterates. If the ideal Lanczos process breaks down one should not expect the discrete one to behave well. We have not observed this pathology yet but should it occur one can consider in addition to various restart and look-ahead techniques perturbating the preconditioning operator.

Finally, we argue that the sequence (u^n) might converge superlinearly, as we show it to be the case for SPD operators, when the preconditioner is an inverse modulo a compact endomorphism.

4.1. Stability and convergence of Krylov subspaces.

PROPOSITION 4.1. *There is $C > 0$ such that for all h and all $l \in (\mathsf{S}_h^1)^\star$*

$$\|\Theta_h(l)\|_{-1} \leq C \sup_{v \in \mathsf{S}_h^1} \frac{|l(v)|}{\|v\|_0}. \tag{4.5}$$

Proof. Lemma 3.16 gives

$$\|\Theta_h(l)\|_{-1}^2 \leq C\big(|\mathcal{P}_{\mathsf{S}_h'^1}(u \times n)|_0^2 + |\mathcal{P}_{\mathsf{S}_h'^{0\bullet}}(q)|_0^2\big) \leq C\big(|u|_0^2 + |q|_0^2\big). \tag{4.6}$$

This gives the announced estimate. □

PROPOSITION 4.2. *There is $C > 0$ such that for all $l \in (\mathrm{H}_{\mathrm{div}}^{-1}(\Gamma))^\star$ and all h*

$$\|\Theta_h l|_{\mathsf{S}_h^1} - \Theta l\|_{-1} \leq Ch\|l\|_{-1\star}. \tag{4.7}$$

Proof. Let (u_h, q_h) be the solutions of the discrete saddle-point problem, and let (u_0, q_0) be the solutions of the continuous one. The well-known properties of this problem (in particular, Propositions 2.13 (p. 64) and 3.9 (p. 132) in Brezzi and Fortin [12]) yield

$$|u_h - u_0|_0^2 + |q_h - q_0|_0^2 \leq Ch^2(|u_0|_1^2 + |q_0|_1^2) \leq Ch^2\|l\|_{-1\star}^2. \tag{4.8}$$

Denoting for simplicity the L_2-orthogonal projections onto the appropriate spaces by $\mathcal{P}_h$, we have

$$|\mathcal{P}_h(u_h \times n) - (u_0 \times n)|_0 \tag{4.9}$$
$$\leq |\mathcal{P}_h(u_h \times n) - \mathcal{P}_h(u_0 \times n)|_0 + |\mathcal{P}_h(u_0 \times n) - u_0 \times n|_0 \tag{4.10}$$
$$\leq |(u_h \times n) - (u_0 \times n)|_0 + Ch|u_0 \times n|_1 \tag{4.11}$$
$$\leq Ch|u_0|_1. \tag{4.12}$$

Using the same technique, we also obtain

$$|\mathcal{P}_h(q_h) - q_0|_0 \leq Ch|q_0|_1. \tag{4.13}$$

This completes the proof, using Lemma 3.16. □

From these two propositions we deduce stability and convergence estimates for Θ_h in half-integer Sobolev norms.

COROLLARY 4.3. *There is $C > 0$ such that for all h and all $l \in \mathrm{H}_{\mathrm{div}}^{-1}(\Gamma)^\star$*

$$\|\Theta_h l|_{\mathsf{S}_h^1} - \Theta l\|_{-1/2} \leq Ch^{1/2}\|l\|_{-1\star}, \tag{4.14}$$

and there is $C > 0$ such that for all h and all $l \in (\mathsf{S}_h^1)^\star$

$$\|\Theta_h l\|_{-1/2} \leq C \sup_{v \in \mathsf{S}_h^1} \frac{|l(v)|}{\|v\|_{-1/2}}. \tag{4.15}$$

Proof. Let $\mathcal{Q}_h$ be the $\mathrm{H}^0_{\mathrm{div}}(\Gamma)$-orthogonal projection onto S'^1_h. The required properties of this projection were summarized in section 2.2. We have for all $l \in \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)^\star$

$$\begin{aligned}
&(4.16) \quad \|\Theta_h l|_{\mathsf{S}^1_h} - \Theta l\|_{-1/2} \leq \|\Theta_h l|_{\mathsf{S}^1_h} - \mathcal{Q}_h \Theta l\|_{-1/2} + \|\mathcal{Q}_h \Theta l - \Theta l\|_{-1/2}\\
&(4.17) \quad \leq C h^{-1/2} \|\Theta_h l|_{\mathsf{S}^1_h} - \mathcal{Q}_h \Theta l\|_{-1} + C h^{1/2} \|\Theta l\|_0\\
&(4.18) \quad \leq C h^{-1/2} \big(\|\Theta_h l|_{\mathsf{S}^1_h} - \Theta l\|_{-1} + \|\Theta l - \mathcal{Q}_h \Theta l\|_{-1}\big) + \cdots\\
&(4.19) \quad \leq C h^{1/2} \|l\|_{-1\star},
\end{aligned}$$

and repeating the same sort of arguments, still supposing that $l \in \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)^\star$,

$$\begin{aligned}
&(4.20) \quad \|\Theta_h l|_{\mathsf{S}^1_h}\|_0 \leq \|\Theta_h l - \mathcal{Q}_h \Theta l\|_0 + \|\mathcal{Q}_h \Theta l\|_0\\
&(4.21) \quad \leq C h^{-1} \|\Theta_h l - \mathcal{Q}_h \Theta l\|_{-1} + \|\Theta l\|_0\\
&(4.22) \quad \leq C h^{-1} \big(\|\Theta_h l - \Theta l\|_{-1} + \|\Theta l - \mathcal{Q}_h \Theta l\|_{-1}\big) + \cdots\\
&(4.23) \quad \leq C \|l\|_{-1\star}.
\end{aligned}$$

Combining this estimate with Proposition 4.1 by interpolation, we obtain, for $l \in \mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)^\star$,

$$\|\Theta_h l|_{\mathsf{S}^1_h}\|_{-1/2} \leq C \|l\|_{-1/2\star}. \tag{4.24}$$

The apparently more refined version announced can then be deduced from the existence of an extension operator with norm one (the adjoint of the $\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$-orthogonal projection onto S^1_h) or a Hahn–Banach theorem. ∎

We have similar estimates for $\Theta^\star_h$.

COROLLARY 4.4. *There is $C > 0$ such that for all h and all $l \in \mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)^\star$*

$$\|\Theta^\star_h l|_{\mathsf{S}'^1_h} - \Theta^\star l\|_{-1/2} \leq C h^{1/2} \|l\|_{-1\star}, \tag{4.25}$$

and there is $C > 0$ such that for all h and all $l \in (\mathsf{S}'^1_h)^\star$

$$\|\Theta^\star_h l\|_{-1/2} \leq C \sup_{v \in \mathsf{S}'^1_h} \frac{|l(v)|}{\|v\|_{-1/2}}. \tag{4.26}$$

Proof. Using the fact that a (bounded) operator has the same norm as its adjoint we first obtain from Proposition 4.2 that

$$\|\Theta^\star_h l|_{\mathsf{S}'^1_h} - \Theta^\star l\|_{-1} \leq C h \|l\|_{-1\star}. \tag{4.27}$$

As in the proof of Corollary 4.3 this yields the first equation. The second one follows trivially from the second estimate of the same corollary. ∎

From this one deduces the following.

COROLLARY 4.5. *Let (l_h) be a sequence of linear forms on $\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$ which converges to l (in the norm sense in the dual of $\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$). Then $\Theta_h l_h|_{\mathsf{S}^1_h}$ converges to Θl in $\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)$. Similarly, $\Theta^\star_h l_h|_{\mathsf{S}'^1_h}$ converges to $\Theta^\star l$.*

Proof. The technique of proof is very classical (see, for instance, Folland [28, Prop. 5.17, p. 169] for the just as easy case of constant l_h) and relies on Corollary 4.3 (and Corollary 4.4 for the second part) using the density of $\mathrm{H}^{-1}_{\mathrm{div}}(\Gamma)^\star$ in $\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)^\star$. ∎

Then we immediately obtain stability and approximation properties for the preconditioner $\mathcal{Z}_h$.

PROPOSITION 4.6. *There is $C > 0$ such that for all h and all $l \in \mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)^\star$*

$$\|\mathcal{Z}_h l|_{\mathsf{S}_h^1}\|_{-1/2} \le C\|l\|_{-1/2\star}. \tag{4.28}$$

If a sequence of linear forms $l_h \in \mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)^\star$ converges to l in $\mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)^\star$, then $\mathcal{Z}_h l_h|_{\mathsf{S}_h^1}$ converges to $\mathcal{Z}l$.

We are now ready to prove the announced theorem.

THEOREM 4.7. *For all $l \in (\mathrm{H}_{\mathrm{div}}^{-1/2})^\star$ and all $n \in \mathbb{N}$, $(\mathcal{Z}_h\mathcal{A}_h)^n\mathcal{Z}_h l|_{\mathsf{S}_h^1}$ converges to $(\mathcal{Z}\mathcal{A})^n\mathcal{Z}l$ in $\mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)$.*

Proof. This follows from the above results using a simple recursion argument. □

Remark. We have not derived any optimal orders of convergence, for smoother than necessary data and perhaps higher order finite elements, though we do not expect this to yield any serious difficulties or require methods of proof different from the above ones. Nor have we tried to determine the minimum hypotheses on the regularity of the triangulations under which our conclusions hold; in particular, we have not determined to which extent the quasi-uniformity hypothesis (which is used for the inverse inequalities) can be relaxed. Of course also working on nonsmooth surfaces would put severe limitations on the range of Sobolev spaces we could use.

4.2. Convergence of the iterates. Let X be a Banach space and X_1, X_0 two closed subspaces. When these are nonzero, the *gap* from X_1 to X_0, denoted $\delta(X_1, X_0)$, is defined to be

$$\delta(X_1, X_0) = \sup_{u_1 \in X_1} \inf_{u_0 \in X_0} \|u_1 - u_0\|/\|u_1\|. \tag{4.29}$$

This definition is extended straightforwardly to the case when X_1 or X_0 is zero. For a thorough discussion of the gap we refer to Kato [35] but for us the definition is enough.

Suppose that X_0 *splits*, i.e., has a closed supplementary (for instance finite-dimensional spaces automatically split, as do closed subspaces of Hilbert spaces), so that we have a continuous projector $P : X \to X$ with range X_0. For all $u \in X$, one has

$$\forall u' \in X_0 \quad \|u - Pu\| = \|(u - u') - (Pu - u')\| = \|(u - u') - P(u - u')\|. \tag{4.30}$$

Hence

$$\|u - Pu\| \le \|I - P\| \left(\inf_{u' \in X_0} \|u - u'\|/\|u\| \right) \|u\|. \tag{4.31}$$

In particular,

$$\forall u \in X_1 \quad \|u - Pu\| \le \|I - P\|\delta(X_1, X_0)\|u\|. \tag{4.32}$$

Thus if (X_h) is a family of closed subspaces such that $\lim_h \delta(X_h, X_0) = 0$, then for sufficiently small h the spaces PX_h are closed in X_0 and P induces isomorphisms $X_h \to PX_h$ which are arbitrarily close in norm to the identity mapping on X_h.

PROPOSITION 4.8. *Let X be a reflexive Banach space and $\mathcal{A} : X \to X^\star$ a continuous linear map. Suppose X_0 is a closed subspace that splits yielding a projector*

P. Suppose X_h is another closed subspace, and that the induced maps $\mathcal{A}_0 : X_0 \to X_0^\star$, $\mathcal{A}_h : X_h \to X_h^\star$ satisfy the inf-sup conditions (1.20), (1.21), *with constants α_0 and α_h. Also put $\delta_h = \delta(X_h, X_0)$.*

Then $\mathcal{A}_0$ and $\mathcal{A}_h$ are invertible ; moreover, for any $l \in X^\star$, if we put $u_h = \mathcal{A}_h^{-1} l|_{X_h}$ and $u_0 = \mathcal{A}_0^{-1} l|_{X_0}$, we have for all $u' \in X_h$

$$\|u_h - u_0\| \leq \alpha_h^{-1}(1 + \alpha_0^{-1}\|\mathcal{A}\|)\|I - P\|\delta_h\|l\| + (1 + \alpha_h^{-1}\|\mathcal{A}\|)\|u_0 - u'\|. \tag{4.33}$$

Proof. That $\mathcal{A}_0$ and $\mathcal{A}_h$ are invertible is part of Theorem 1.1. Concerning the approximation property, we have (as usual we denote by a the bilinear form corresponding to $\mathcal{A}$)

$$\|u_h - u_0\| \leq \|u_h - u'\| + \|u' - u_0\| \leq \alpha_h^{-1} \sup_{v \in X_h} \frac{|a(u_h - u', v)|}{\|v\|} + \|u' - u_0\|. \tag{4.34}$$

Now (for $v \in X_h$) write

$$a(u_h - u', v) = a(u_h, v) + a(u_0 - u', v) - a(u_0, Pv) - a(u_0, v - Pv) \tag{4.35}$$

$$= l(v - Pv) + a(u_0 - u', v) - a(u_0, v - Pv). \tag{4.36}$$

Therefore,

$$|a(u_h - u', v)|/\|v\| \leq (1 + \alpha_0^{-1}\|a\|)(\|I - P\|)\delta_h\|l\| + \|a\|\,\|u_0 - u'\|. \tag{4.37}$$

This proves the proposition. □

Remark. Just as Theorem 1.1 this proposition can easily be extended to the more general setting of a continuous map $\mathcal{A} : X \to Y^\star$ and subspaces $X_0 \subset X$ and $Y_0 \subset Y$.

Suppose now (and this is the case for both the approximation of harmonic fields and the approximation of Krylov subspaces we were discussing) that we have a family (X_h) of subspaces of X and *surjective* linear maps $\Lambda_h : X_0 \to X_h$ such that $\lim_h \|\Lambda_h - I\| = 0$. When $\|\Lambda_h - I\| < 1$, Λ_h is invertible (so X_h is closed), and $\|\Lambda_h^{-1}\| \leq (1 - \|\Lambda_h - I\|)^{-1}$, and $\Lambda_h^{-1} - I$ considered as a map $X_h \to X$ has norm $\|\Lambda_h^{-1} - I\| \leq (1 - \|\Lambda_h - I\|)^{-1}\|\Lambda_h - I\|$. In particular,

$$\delta(X_h, X_0) \leq (1 - \|\Lambda_h - I\|)^{-1}\|\Lambda_h - I\|. \tag{4.38}$$

We also trivially have

$$\delta(X_0, X_h) \leq \|\Lambda_h - I\|. \tag{4.39}$$

Given some continuous bilinear form a on X, we define is_h and is_0 by

$$\mathrm{is}_h = \inf_{u \in X_h} \sup_{v \in X_h} \frac{|a(u, v)|}{\|u\|\,\|v\|} \quad \text{and} \quad \mathrm{is}_0 = \inf_{u \in X_0} \sup_{v \in X_0} \frac{|a(u, v)|}{\|u\|\,\|v\|}. \tag{4.40}$$

Some tedious elementary manipulations yield (independently of the existence of the map Λ_h) the inequality

$$\mathrm{is}_h \geq \mathrm{is}_0 \frac{\big(1 - \delta(X_h, X_0)\big)}{\big(1 + \delta(X_0, X_h)\big)} - \|a\| \left(\frac{\big(1 + \delta(X_h, X_0)\big)}{\big(1 - \delta(X_0, X_h)\big)} \delta(X_0, X_h) + \delta(X_h, X_0) \right). \tag{4.41}$$

Now if we plug estimates (4.38) and (4.39) into (4.41) we can conclude that as $h \to 0$, is_h becomes greater than $\mathrm{is}_0 - \epsilon$ for any $\epsilon > 0$. Combining this fact with Proposition 4.8, we obtain the following.

PROPOSITION 4.9. *Let X be a reflexive Banach space and $\mathcal{A} : X \to X^\star$ be a continuous linear map. Suppose that X_0 is a closed linear subspace that splits yielding a projector P and that $\mathcal{A}$ induces an isomorphism $X_0 \to X_0^\star$, with an inf-sup estimate α_0. Suppose we have a family (X_h) of subspaces of X, equipped with surjective linear continuous maps $\Lambda_h : X_0 \to X_h$, such that $\lim_h \|I - \Lambda_h\| = 0$.*

Then for any $0 < \alpha < \alpha_0$ there is $h_0 > 0$ such that, for all $h < h_0$, $\mathcal{A}$ induces isomorphisms $X_h \to X_h^\star$, and, for all $l \in X^\star$, with the notations of Proposition 4.8, *we have*

$$\|u_h - u_0\| \le \alpha^{-1}(1 + \alpha^{-1}\|\mathcal{A}\|)(1 + \|I - P\|)\,\|\Lambda_h - I\|\,\|l\|. \tag{4.42}$$

Applying this proposition to the discrete and continuous Krylov spaces yields the following.

COROLLARY 4.10. *Fix an $n \in \mathbb{N}$. Suppose that $\dim \mathfrak{K}^n = n+1$ and that the map $\mathfrak{K}^n \to \mathfrak{K}^{n\star}$ induced by $\mathcal{A}$ is invertible. Then there is $h_n > 0$ such that for all $h < h_n$ the map $\mathfrak{K}_h^n \to \mathfrak{K}_h^{n\star}$ induced by $\mathcal{A}$ is invertible; moreover, given $l \in \mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)^\star$, the solutions u_h^n and u^n of* (4.2) *and* (4.4) *satisfy an estimate of the form*

$$\|u_h^n - u^n\|_{-1/2} \le C\|\Lambda_h^n - I\|, \tag{4.43}$$

where $\Lambda_h^n : \mathfrak{K}^n \to \mathfrak{K}_h^n$ is the unique linear map that satisfies, for $0 \le i \le n$,

$$\Lambda_h^n : (\mathcal{Z}\mathcal{A})^i \mathcal{Z} l \mapsto (\mathcal{Z}_h\mathcal{A}_h)^i \mathcal{Z}_h l|_{\mathsf{S}_h^1}. \tag{4.44}$$

Of course Theorem 4.7 shows that (for fixed n) $\|\Lambda_h^n - I\| \to 0$, and the above mentioned question of regularity is whether for smooth l we have estimates of the form $\|\Lambda_h^n - I\| \le Ch^s$ for $s > 0$.

It is also possible to give a slightly different and more algorithmic variant of this corollary. Namely, define an "abstract" conjugate gradients algorithm by skipping all the h indices in some implementation in terms of $\mathcal{Z}_h$ and $\mathcal{A}_h$ of the conjugate gradient algorithm on S_h^1. (It should be checked that this is actually possible.) Then if the abstract algorithm is well defined up to iteration n there is $h_n > 0$ such that for all $h < h_n$ the discrete algorithm is well defined up to iteration n. Convergence of the iterates is described by the above corollary. Notice that it covers the case of algorithms that can skip full rank Krylov subspaces on which $\mathcal{A}$ is degenerate, as long as the next Krylov subspace is also full rank and $\mathcal{A}$ is nondegenerate on it, so-called Look-ahead algorithms described in Parlett, Taylor, and Liu [46]. Abstract conjugate gradient algorithms are common folklore even for non-Hermitian operators and described for instance in Gutknecht [30].

Remark. We have not proved that the spectral condition number of the restriction of $\mathcal{Z}_h\mathcal{A}_h$ to $(\mathsf{S}_h^1)^\wedge$ is uniformly bounded, and we even suspect that this may not be true. More precisely, the spectral radius of the endomorphism induced by $\mathcal{Z}_h\mathcal{A}_h$ on $(\mathsf{S}_h^1)^\wedge$ is uniformly bounded but perhaps not that of its inverse. The lack of this property (which was the guide and main focus of Steinbach and Wendland [51] and Christiansen and Nédélec [17]) was our principal motivation for proving the convergence of the Krylov spaces and the corresponding approximate solutions.

4.3. Evidence of superlinear convergence. The above discussion (in particular Corollary 4.10) suggests that the behavior of the approximate solutions (u_h^n) is similar to the behavior of (u^n), at least for moderate n (compared with $\dim \mathsf{S}_h^1 \approx h^{-2}$). Concerning the behavior of (u^n), the convergence theory is more satisfactory in the SPD case than in the non-SPD case we are dealing with.

Here we remind the reader how, in the infinite-dimensional SPD case, the property of inversion up to a compact operator leads to *superlinear* convergence. In other words, with the proposed preconditioner for the first kind integral equation EFIE, one recovers the kind of convergence usually associated with second kind integral equations. Though preconditioning was not a focus at the time, such estimates seem to date back to Hayes [31]. The theory does not directly apply to the studied case, but (for smooth surfaces) it does give complementary convergence estimates on some related preconditioning techniques, in particular those described in Steinbach and Wendland [51], which were the starting point of the method we have described here. We also believe these developments to give good *indication* on the behavior we can expect for our present problem.

Suppose X is a real Hilbert space, $\mathcal{A} : X \to X^\star$ is linear continuous, and induces a symmetric positive definite bilinear form. Suppose $\mathcal{Z} : X^\star \to X$ is linear continuous and symmetric. Given $l = \mathcal{A}u^\star \in X^\star$ define the Krylov spaces as above with real polynomials only. We suppose that we do not provide an approximate solution—other than 0—to start the algorithm, though this can easily be accounted for.

Remark first that $\mathcal{A}$ induces a scalar product on X with associated norm $\|\cdot\|_\mathcal{A}$ and that u^n solves (4.4) if and only if $u \mapsto \|u - u^\star\|_\mathcal{A}$ is minimal on $\mathfrak{K}^n$ at u^n. Thus for all real polynomials P such that $\deg P \leq n$,

$$\|u^n - u^\star\|_\mathcal{A} \leq \|P(\mathcal{Z}\mathcal{A})\mathcal{Z}\mathcal{A}u^\star - u^\star\|_\mathcal{A}. \tag{4.45}$$

Hence for all P such that $\deg P \leq n+1$, and $P(0) = 1$,

$$\|u^n - u^\star\|_\mathcal{A} \leq \|P(\mathcal{Z}\mathcal{A})u^\star\|_\mathcal{A}. \tag{4.46}$$

Then remark that $\mathcal{Z}\mathcal{A}$ is continuous, and symmetric with respect to the bilinear form induced by $\mathcal{A}$, and therefore has a resolution of the identity E on the spectrum $\sigma = \sigma(\mathcal{Z}\mathcal{A}) \subset \mathbb{R}$ such that we can write (we refer to Rudin [50, Chap. 12] for definitions and notations)

$$\|P(\mathcal{Z}\mathcal{A})u^\star\|_\mathcal{A}^2 \leq \int_\sigma |P(\lambda)|^2 \mathrm{d}E_{u^\star,u^\star} \tag{4.47}$$

$$\leq \sup\{|P(\lambda)|^2 \ : \ \lambda \in \sigma \cap \operatorname{supp}\, \mathrm{d}E_{u^\star,u^\star}\}\, \|u^\star\|_\mathcal{A}^2. \tag{4.48}$$

Finally in the case were $\mathcal{Z}\mathcal{A} - I$ is compact we can put $\sigma = \{1\} \cup \{\lambda_i : i \in \mathbb{N}\}$, where $(|\lambda_i - 1|)$ is a decreasing sequence converging to 0. To be sure that the algorithm is well-defined for all n we suppose that $\mathcal{Z}$ is positive definite. Then $\lambda_i \neq 0$, and we can define polynomials P_n by

$$P_n \ : \ P_n(\lambda) = \Pi_{i=0}^n (1 - \lambda/\lambda_i). \tag{4.49}$$

Remark that for any i and any λ such that $|1 - \lambda| \leq |1 - \lambda_i|$

$$|1 - \lambda/\lambda_i| = |(\lambda_i - 1 + 1 - \lambda)/\lambda_i| \leq 2|1 - 1/\lambda_i|, \tag{4.50}$$

which gives

$$\sup_{\lambda \in \sigma} |P_n(\lambda)| \leq 2^{n+1} \Pi_{i=0}^n |1 - 1/\lambda_i|. \tag{4.51}$$

Since $|1 - 1/\lambda_i| \to 0$, this immediately implies superlinear convergence. More generally, if (λ_i) is a sequence of nonzero *complex* numbers, such that $(|\lambda_i - 1|)$ decreases to 0, the estimates (4.50) (for complex λ) and (4.51) (with the same definition of P_n, and still with $\sigma = \{1\} \cup \{\lambda_i \ : \ i \in \mathbb{N}\}$) remain true, which might be of interest to other Krylov-subspace algorithms applied to non-SPD operators. Moreover, if the asymptotic behavior of the eigenvalues of the residual $\mathcal{Z}\mathcal{A} - I$ is known and we have an estimate of the form $|1-\lambda_i| \leq Ci^{-\alpha}$ for some $C > 0$, $\alpha > 0$, we get the convergence estimate (for a larger C and the same α)

$$\|u^n - u^\star\|_{\mathcal{A}} \leq C^n (n!)^{-\alpha} \|u^\star\|_{\mathcal{A}}. \tag{4.52}$$

More explicitly, returning to the case of operators on a smooth compact Riemannian manifold Γ, if the dimension of the manifold is N and the residual is an operator of order $-s$ for some $s > 0$ (i.e., in terms of Sobolev spaces, continuous $\mathrm{H}^{s'}(\Gamma) \to \mathrm{H}^{s'+s}(\Gamma)$ for all s') which commutes with the Laplacian on Γ, this holds with $\alpha = s/N$. This can be deduced from the eigenvalue asymptotics of the Laplacian, for which we refer to Taylor [52, Vol. II, Chap. 8]. An alternative and more general approach based on trace-class theory is exposed in Winther [56] and also gives a factor of the form $C^n(n!)^{-\alpha}$. When Γ has symmetries an even larger α might hold in the estimate (4.52) due to the degeneracy of eigenvalues. For the determination of the orders of different integral operators we refer to Nédélec [43]; in particular, the order of the residual in our preconditioning strategy for the EFIE is -2 (though we do not even claim to have proved that the PCG does not break down in this case).

5. Numerical results. In order to evaluate the performance of the preconditioner it is customary to show the convergence graphs. We use the notations of (1.29), and we denote by U_h^n the tuple of coordinates of the approximate solution u_h^n at iteration n in the chosen basis. The convergence graphs are then of the form

$$n \mapsto \log_{10}(\|A_h(k)U_h^n - V_h\|/\|V_h\|) \tag{5.1}$$

for a given choice of norms on the tuples. The standard norm is the ℓ_2 norm on tuples. From a functional point of view this is not very natural, but on the other hand functional norms are not readily computable—with a notable exception for those we defined in section 3.9.

5.1. Sphere. We start by showing convergence graphs for the canonical example of diffraction of a plane wave by the unit sphere for the wave lengths $\lambda = 8,\ 4,\ 2,\ 1$ ($k = 2\pi/\lambda$); see Figure 5.1. The discretization of the sphere has 2252 vertices and 4500 triangles, leading to 6750 degrees of freedom for lowest order Raviart–Thomas finite elements.

We consider here the case of the preconditioner using the same Galerkin space as the original and use a complex symmetric conjugate gradients algorithm.

Each graphic displays three curves. The thin line (upper graph) is obtained without preconditioning for the ℓ_2 norm on tuples; the dotted line (which stagnates) is obtained with the proposed preconditioner for the ℓ_2 norm on tuples; the third line (the bottom graph) is obtained with the proposed preconditioner for the natural norm defined in section 3.9.

We make the following comments:

- With the preconditioner, each iteration is a little more than twice as slow as without any preconditioner: we apply the Galerkin matrix once more, and do a considerable amount of sparse matrix manipulations.

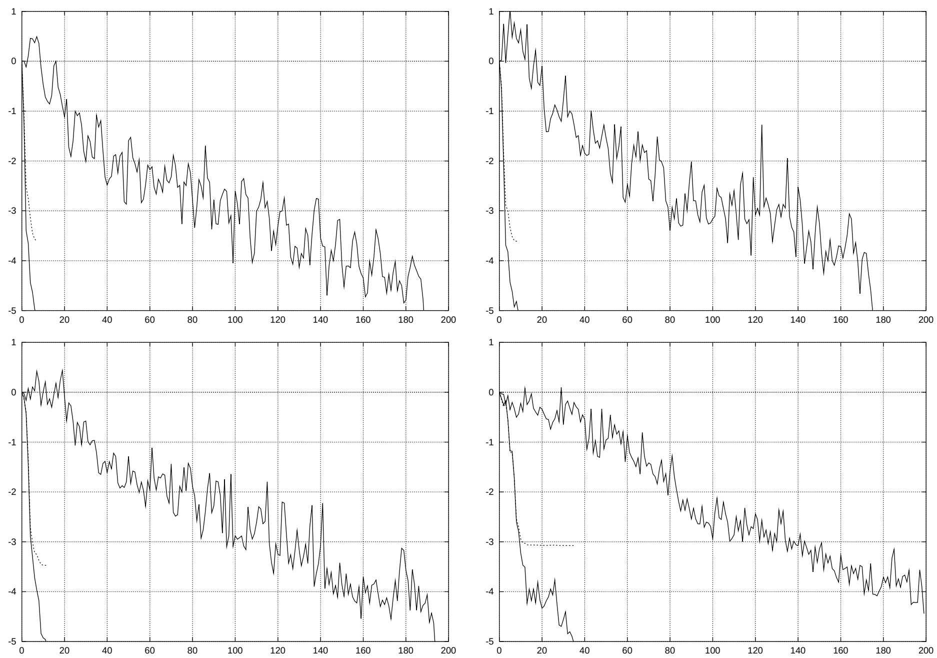


FIG. 5.1. *Convergence graphs for the unit sphere at $\lambda = 8,\ 4,\ 2,\ 1$ (top left to bottom right).*

- In the preconditioned case and in particular at $\lambda = 1$, the ℓ_2 norm (dotted line) of the residual stagnates, whereas the natural norm continues to decrease, confirming that the variational problem is indeed solved on a strict subspace.
- The preconditioner is particularly efficient at low frequencies—which is the only really ill-conditioned case on spheres.
- Usual stopping criteria vary from 10^{-2} to 10^{-5}, depending on the accuracy of the result required; for all these the preconditioned algorithm is several times faster than the algorithm without preconditioning.
- The auxiliary problems in the preconditioner were solved iteratively with a tolerance of 10^{-7} (saddle-point problem) and 10^{-8} (L^2 projections); accumulation of such errors and other round-off errors could also partly explain the instabilities observed at the last iterations at $\lambda = 1$.
- The far-field patterns deduced from the electric currents computed iteratively with and without preconditioner were not graphically distinguishable from those computed by standard factorization.
- Using the Brezzi–Douglas–Marini finite elements in the preconditioner (while still using Raviart–Thomas for the original problem) yields slightly better accuracy and requires slightly fewer iterations. Also the dotted line does not stagnate, confirming that the equation is then solved variationally on a much larger subspace. However, each iteration is much slower, since there are twice as many degrees of freedom in the preconditioner (at the lowest level).

5.2. Cavity. We consider now diffraction by a cavity. In cavities trapped rays create long range nontrivial interactions. This is a considerably more challenging problem than scattering by convex objects, since, without preconditioning, it happens that the iterates do not converge.

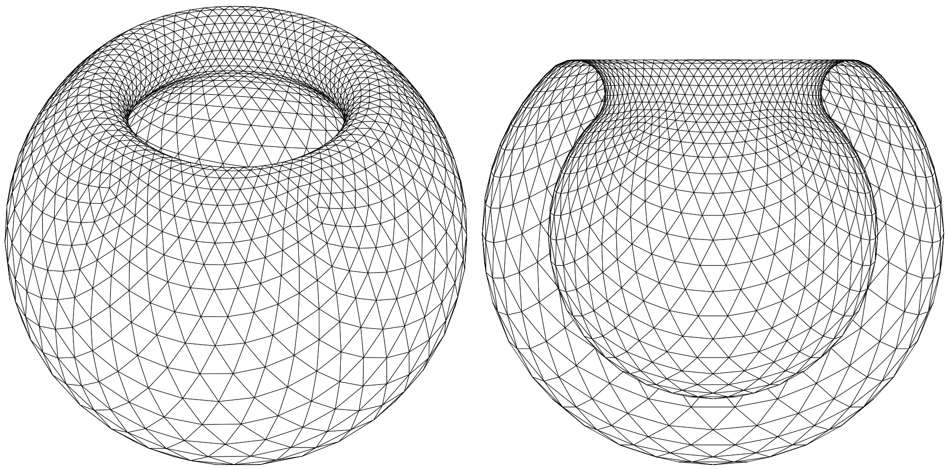


FIG. 5.2. *Cavity seen from outside and vertical section.*

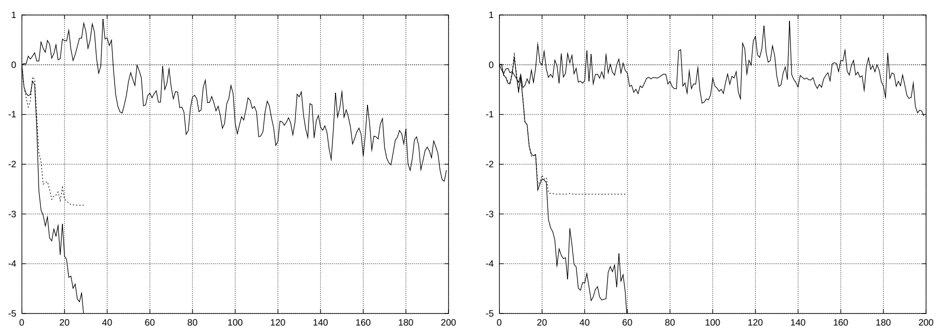


FIG. 5.3. *Convergence graphs for the cavity at* $\lambda = 8,\ 2$ *(left and right).*

The particular geometry we consider can be described as follows: take two concentric spheres, one with radius 5/6 and one with radius 7/6; excavate a cone with top at the origin and half-angle $\pi/4$ and join the interior surface with the exterior one with half a torus. See Figure 5.2. The mesh of the cavity was constructed from a mesh of the unit sphere (in fact the same one as in the preceding example) by successive deformations.

The cavity is lit by a horizontally polarized plane wave entering the cavity tangentially to its walls. (The wave vector makes an angle of $3\pi/4$ with the vertical direction.) The convergence graphs obtained for the cavity are displayed in Figure 5.3.

Notice that at $\lambda = 2$, the preconditioner not only speeds up convergence, it actually enables it.

In the preconditioned case, we observe a slow-down in the convergence, which we interpret as stemming from the fact that the Calderon formulas are less well represented on a discrete level. (The discrete iterates depart from the continuous ones.) It seems also that the slow-down always occurs slightly after the ℓ_2 norm of the residual stagnates. This stagnation could be indication that the current iterate is as close to the exact continuous solution as the exact Galerkin solution. Thus the

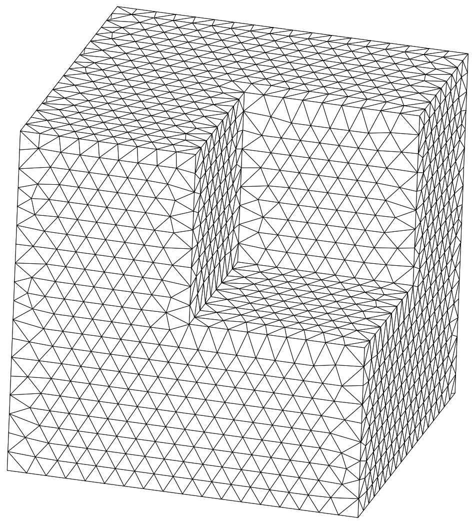


FIG. 5.4. *Indented cube seen from outside.*

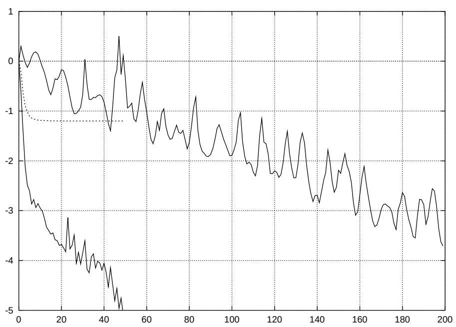
stagnation in the ℓ_2 norm of the residual would be a good stopping criterion and then the slow-down would never be observed.

5.3. Indented cube. The Galerkin discretization of the EFIE is well known to perform well not only on smooth surfaces but also for the polyhedral ones that often occur in applications. Though we have justified the preconditioning technique only for smooth surfaces, it appears to perform well also on such nonsmooth ones. However, when the preconditioning operator $Z : X' \to X$ does not invert the operator $A : X \to X'$ up to a compact residual, but rather is such that ZA is an automorphism of X, one expects the ideal conjugate gradients algorithm to converge not superlinearly, but rather linearly, as is easily proven for SPD operators.

We show numerical results for the following geometry. The scattering object is the indented cube $[-1,1]^3\backslash]0,1]^3$. The interior domain contains several types of singularities, both convex and nonconvex. Also when a plane wave with wave-vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_i < 0$, hits the reentrant corner, geometrical optics predicts that it should be scattered mainly in the direction $-\sigma$, after three reflections. The mesh used for the numerical experiments is shown in Figure 5.4. It has 2164 vertices and 4324 triangles, leading to 6486 edges (and degrees of freedom for Raviart–Thomas finite elements).

In Figure 5.5 we show the convergence graphs obtained for an incident plane wave with wave-vector positively proportional to $(-1,-1,-1)$, with wavelength $\lambda = 8$ and $\lambda = 4$, and with horizontal polarization. Contrary to the case of smooth surfaces there might be significant loss of accuracy when solving the Galerkin problem on $(\mathsf{S}_h^1)^\wedge$ rather than S_h^1 when lowest degree Raviart–Thomas fields are used both in the problem formulation and the preconditioner. This problem would be remedied using Brezzi–Douglas–Marini fields in the preconditioner, since then $(\mathsf{S}_h^1)^\wedge$ has very low codimension in S_h^1. However, for the wavelengths used here the far-field patterns were not graphically distinguishable.

Perspectives. The study of the method on nonsmooth surfaces is still ongoing. In particular, the evaluation of the impact of singularities in the surface and corre-



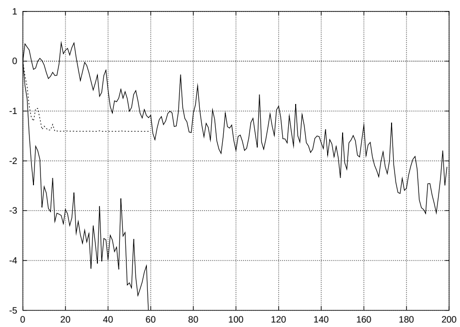


FIG. 5.5. *Convergence graphs for the indented cube at $\lambda = 8,\ 4$ (left and right).*

sponding mesh refinement strategies is important for many applications and could perhaps be achieved using recent results from Buffa, Costabel, and Schwab [13] and Hiptmair and Schwab [33].

As remarked on the numerical experiments, the preconditioning technique displays good stability at low frequencies. This stability can be enhanced by making the discrete Helmholtz decompositions still more explicit. For instance, focusing on the preconditioner, one applies separately the two terms of the operator in (1.18) to the two terms of the vector in (3.24). We shall come back elsewhere to this point, which is important for simulating semiconductor devices.

The method can also be extended to treat scattering by perfectly conducting simplical complexes (including open surfaces as well as branched ones, where more than two surfaces meet at an edge). In these cases one can no longer keep the same type of Galerkin spaces in the preconditioner as in the variational formulation of the EFIE, and the Calderon formulas, which require the surface to be orientable, need to be adapted. For some generalizations of the method described here we have obtained speed-ups comparable to the above ones, though the justifications are as of today at best intuitive.

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