MINIMIZING THE CONDITION NUMBER OF BOUNDARY INTEGRAL OPERATORS IN ACOUSTIC AND ELECTROMAGNETIC SCATTERING

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SUMMARY

The question of non-uniqueness in boundary integral equation formulations of exterior boundary-value problems in time-harmonic acoustic and electromagnetic scattering can be resolved by seeking the solutions in the form of a combined single- and double-layer potential in acoustics or a combined electric- and magnetic-dipole field in electromagnetics. We present an analysis of the appropriate choice of the coupling parameters which is optimal in the sense of minimizing the condition number of the boundary integral operators.

1. Introduction

LET $D \subset \mathbb{R}^3$ be a bounded domain with a connected boundary surface ∂D of class C^2. By ν we denote the unit normal to ∂D directed into the exterior $\mathbb{R}^3 \setminus D$. Consider the exterior Dirichlet problem for the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{1.1}$$

with boundary condition

$$u = f \qquad \text{on } \partial D \tag{1.2}$$

and the Sommerfeld radiation condition

$$\partial u(\mathbf{x})/\partial r - iku(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \qquad r = |\mathbf{x}| \to \infty \tag{1.3}$$

uniformly for all directions $\mathbf{x}/|\mathbf{x}|$. Here the wavenumber k is a given positive number and $f \in C(\partial D)$ is a given function. This boundary-value problem can be reduced to an integral equation of the second kind which is uniquely solvable for all wavenumbers $k > 0$ by seeking the solution in the form

$$u(\mathbf{x}) = \int_{\partial D} \left\{ \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{y})} - i\eta \Phi(\mathbf{x}, \mathbf{y}) \right\} \psi(\mathbf{y}) \, ds(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3 \setminus \bar{D} \tag{1.4}$$

with a positive coupling parameter η. Here

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi \, |\mathbf{x} - \mathbf{y}|} \tag{1.5}$$

denotes the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$. The combined double- and single-layer potential (1.4) solves the exterior Dirichlet problem provided the density $\psi \in C(\partial D)$ is a solution of the integral equation

$$\psi + K\psi - i\eta S\psi = 2f, \tag{1.6}$$

where $K, S: C(\partial D) \to C(\partial D)$ denote the compact integral operators defined by

$$(K\psi)(\mathbf{x}) = 2\int_{\partial D} \frac{\partial\Phi(\mathbf{x}, \mathbf{y})}{\partial\nu(\mathbf{y})} \psi(\mathbf{y})\, ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D, \tag{1.7}$$

$$(S\psi)(\mathbf{x}) = 2\int_{\partial D} \Phi(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D. \tag{1.8}$$

This integral equation is uniquely solvable for all $k > 0$ and all $\eta > 0$. Note that for $\eta = 0$ the integral equation becomes non-unique if k^2 is an eigenvalue of the interior Neumann problem in D, a so-called irregular frequency. The approach (1.4) with $\eta > 0$ was introduced independently by Leis (**1**), Brakhage and Werner (**2**) and Panich (**3**) in order to overcome this non-uniqueness problem. For a detailed discussion see (**4**).

Instead of studying the integral equation (1.6) in the classical Banach space $C(\partial D)$ of continuous functions it can also be treated in the Sobolev spaces $H^r(\partial D)$, $r \geqslant 0$, as worked out by Giroire (**5**). We note that $K: H^r(\partial D) \to H^r(\partial D)$ is a compact operator and $S: H^r(\partial D) \to H^{r+1}(\partial D)$ is a bounded operator. In the rest of this paper it will be important that the Sobolev spaces are Hilbert spaces.

Numerical implementations of the combined double- and single-layer approach have been given by Greenspan and Werner (**6**) and various other authors (see (**4**)), mostly for the special choice of the coupling parameter $\eta = 1$. In this paper we shall investigate the problem of how to choose the parameter η in (1.4) in order to minimize the condition number of the integral equation (1.6) in the Sobolev spaces $H^r(\partial D)$. We define

$$A = I + K - i\eta S, \tag{1.9}$$

where I denotes the identity operator. Then (1.6) becomes

$$A\psi = 2f. \tag{1.10}$$

The condition number (**7**) of A is given by

$$\operatorname{cond}(A) = \|A\|\,\|A^{-1}\|, \tag{1.11}$$

and for numerical purposes it is desirable to have $\operatorname{cond}(A)$ as small as possible. Provided the condition number is small, small perturbations of the operator A and of f on the right will only lead to small changes in the solution ψ of (1.10). Since we may consider any numerical approximation method for solving the integral equation (1.10) as a perturbation we can therefore expect good error estimates if the condition number of A is small. For an example of how to express the error for projection methods in solving (1.10) by the condition number see (**8**).

Our presentation simplifies and extends parts of the results obtained by Kress and Spassov (**9**). It also covers the exterior Neumann problem for the Helmholtz equation and the boundary-value problem for the time-harmonic Maxwell equations describing the reflection of electromagnetic waves at the perfect conductor D.

2. Condition numbers for the combined double- and single-layer approach

Assume that the eigenfunctions

$$A\varphi_n = \lambda_n \varphi_n, \qquad n \in \mathbb{N}, \tag{2.1}$$

form a complete orthonormal system in $H^r(\partial D)$. Then from

$$\varphi = \sum_{n\in\mathbb{N}} (\varphi, \varphi_n)_{H^r} \varphi_n$$

and

$$A\varphi = \sum_{n\in\mathbb{N}} \lambda_n (\varphi, \varphi_n)_{H^r} \varphi_n$$

we conclude that

$$\begin{aligned}\|A\varphi\|_{H^r}^2 &= \sum_{n\in\mathbb{N}} |(A\varphi, \varphi_n)_{H^r}|^2 = \sum_{n\in\mathbb{N}} |\lambda_n|^2 \, |(\varphi, \varphi_n)_{H^r}|^2 \\ &\leq (\lambda_{\max})^2 \sum_{n\in\mathbb{N}} |(\varphi, \varphi_n)_{H^r}|^2 = (\lambda_{\max})^2 \, \|\varphi\|_{H^r}^2,\end{aligned}$$

where

$$\lambda_{\max} = \sup_{n\in\mathbb{N}} |\lambda_n|. \tag{2.2}$$

Hence, $\|A\| = \lambda_{\max}$ and similarly $\|A^{-1}\| = 1/\lambda_{\min}$ where

$$\lambda_{\min} = \inf_{n\in\mathbb{N}} |\lambda_n|. \tag{2.3}$$

Thus, in the Hilbert space $H^r(\partial D)$ we can express the condition number in terms of the eigenvalues as

$$\operatorname{cond}(A) = \lambda_{\max}/\lambda_{\min}. \tag{2.4}$$

In order to obtain the approximate solution to the exterior Dirichlet problem as well as numerically solving the integral equation (1.6) we also have to evaluate the combined potential (1.4) for the density ψ. As a typical problem consider the evaluation of the normal derivative

$$\frac{\partial u}{\partial \nu} = \tfrac{1}{2}[i\eta(I - K') + T]\psi, \tag{2.5}$$

where $K': H^r(\partial D) \to H^r(\partial D)$ denotes the compact integral operator defined by

$$(K'\psi)(\mathbf{x}) = 2\int_{\partial D} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{x})} \psi(\mathbf{y})\, ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D, \tag{2.6}$$

and where $T: H^r(\partial D) \to H^{r-1}(\partial D)$ denotes the bounded operator (see (5)) defined by

$$(T\psi)(\mathbf{x}) = 2\frac{\partial}{\partial \nu(\mathbf{x})} \int_{\partial D} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{y})} \psi(\mathbf{y})\, ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D. \tag{2.7}$$

We write

$$B = i\eta(I - K') + T, \tag{2.8}$$

and note that we also have to approximate B. For the same reasons as those above for the operator A we want to choose η such that the condition number of B is also small.

We have to consider B as a bijective operator from $H^r(\partial D)$ onto $H^{r-1}(\partial D)$. Assume that the eigenfunctions

$$B\psi_n = \mu_n \psi_n, \qquad n \in \mathbb{N}, \tag{2.9}$$

simultaneously form a complete orthonormal system in $H^{r-1}(\partial D)$ and a complete orthogonal system in $H^r(\partial D)$. Then the

$$\chi_n = \frac{1}{\alpha_n} \psi_n, \qquad n \in \mathbb{N},$$

form a complete orthonormal system in $H^r(\partial D)$ where

$$\alpha_n = (\psi_n, \psi_n)_{H^r}^{\frac{1}{2}}, \qquad n \in \mathbb{N}. \tag{2.10}$$

Then since $\psi = \sum_{n \in \mathbb{N}} (\psi, \chi_n)_{H^r} \chi_n$ and $B\psi = \sum_{n \in \mathbb{N}} \mu_n (\psi, \chi_n)_{H^r} \chi_n$ we conclude that

$$(B\psi, \psi_n)_{H^{r-1}} = \frac{\mu_n}{\alpha_n} (\psi, \chi_n)_{H^r}$$

and

$$\begin{aligned} \|B\psi\|_{H^{r-1}}^2 &= \sum_{n \in \mathbb{N}} |(B\psi, \psi_n)_{H^{r-1}}|^2 \leqq (\mu_{\max})^2 \sum_{n \in \mathbb{N}} |(\psi, \chi_n)_{H^r}|^2 \\ &= (\mu_{\max})^2 \|\psi\|_{H^r}^2, \end{aligned}$$

where

$$\mu_{\max} = \sup_{n \in \mathbb{N}} |\mu_n|/\alpha_n. \tag{2.11}$$

Hence, $\|B\| = \mu_{\max}$ and as before $\|B^{-1}\| = 1/\mu_{\min}$, where

$$\mu_{\min} = \inf_{n \in \mathbb{N}} |\mu_n|/\alpha_n. \tag{2.12}$$

Thus, we can express the condition number in terms of the eigenvalues as

$$\operatorname{cond}(B) = \mu_{\max}/\mu_{\min}. \tag{2.13}$$

3. Condition numbers for the exterior of the unit sphere

Let u be a solution to the Helmholtz equation in $\mathbb{R}^3\backslash\bar{D}$ satisfying the radiation condition. Then from the Green's representation formula we have (see (4))

$$-u + Ku - S\frac{\partial u}{\partial \nu} = 0 \tag{3.1}$$

for the boundary values and the normal derivative on ∂D. Similarly, for a solution v to the Helmholtz equation in D we have

$$v + Kv - S\frac{\partial v}{\partial \nu} = 0. \tag{3.2}$$

Now consider the special case of the unit sphere $\partial D = \Omega$ and choose

$$\left.\begin{aligned} u(\mathbf{x}) &= h_n(kr)Y_n^m(\theta, \varphi), \\ v(\mathbf{x}) &= j_n(kr)Y_n^m(\theta, \varphi), \end{aligned}\right\} \tag{3.3}$$

where (r, θ, φ) denote the spherical coordinates of $\mathbf{x}$, Y_n^m, $m = -n, \ldots, n$, denote the linearly-independent orthonormal spherical harmonics of order n, j_n denotes the spherical Bessel function of order n, y_n denotes the spherical Neumann function of order n and $h_n = j_n + iy_n$ denotes the spherical Hankel function of the first kind. Then, inserting (3.3) into (3.1) and (3.2) respectively, we obtain

$$-h_n(k) + h_n(k)KY_n^m - kh_n'(k)SY_n^m = 0$$

and

$$j_n(k) + j_n(k)KY_n^m - kj_n'(k)SY_n^m = 0.$$

From this we can eliminate

$$KY_n^m = \left(-1 - \frac{2h_n(k)j_n'(k)}{w_n(k)}\right)Y_n^m, \qquad SY_n^m = -\frac{2h_n(k)j_n(k)}{kw_n(k)}\,Y_n^m,$$

where $w_n(k) = j_n(k)h_n'(k) - h_n(k)j_n'(k) = i/k^2$ denotes the Wronskian. Hence,

$$AY_n^m = \lambda_n Y_n^m, \tag{3.4}$$

where

$$\lambda_n(k) = 2ikh_n(k)[kj_n'(k) - i\eta j_n(k)], \qquad n = 0, 1, 2, \ldots. \tag{3.5}$$

Since the Y_n^m form a complete orthogonal set in $H^r(\Omega)$ we can apply (2.4). From (3.5) we observe that our restriction to the unit sphere is not essential since the case of a sphere of radius R can be reduced to the case of the unit sphere by scaling $Rk \to k^*$ and $R\eta \to \eta^*$.

Using the power-series expansions for the spherical Bessel and Neumann functions we find that

$$[\lambda_n(k)]^2 = \frac{4(n^2+\eta^2)}{(2n+1)^2} + O(k^2), \qquad n = 0, 1, 2, \dots, \tag{3.6}$$

and from this it can be deduced (see (9)) that for small k the optimal parameter is given by

$$\eta_{opt}(k) = \tfrac{1}{2} + O(k^2) \tag{3.7}$$

with

$$\operatorname{cond}(A_{opt}(k)) = 3/\sqrt{5} + O(k^2). \tag{3.8}$$

For general k it seems to be impossible to obtain any explicit results on the optimal parameter. Hence, using (2.4) and (3.5) we have numerically computed the optimal parameter η_{opt} and the optimal cond (A_{opt}) depending on k. The results are given by the broken curve in Fig. 1. For numerical

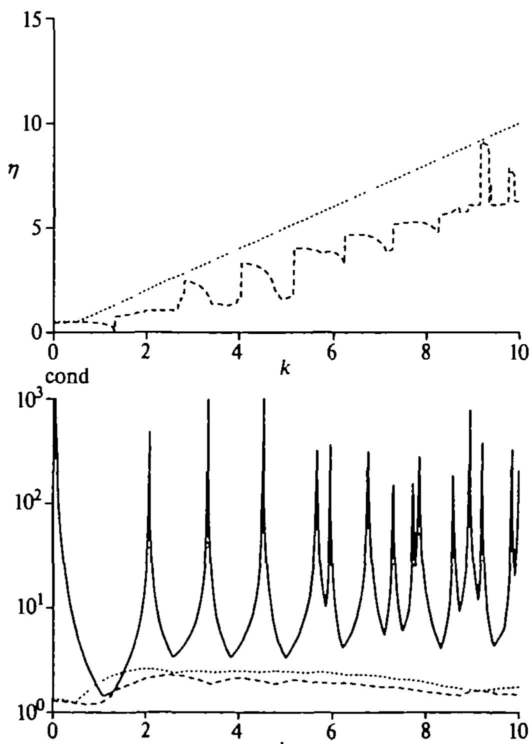

FIG. 1. cond (A) for $\eta = 0$ (full line), $\eta = \eta_{opt}$ (broken line) and $\eta = \max(0{\cdot}5, k)$ (dotted line)

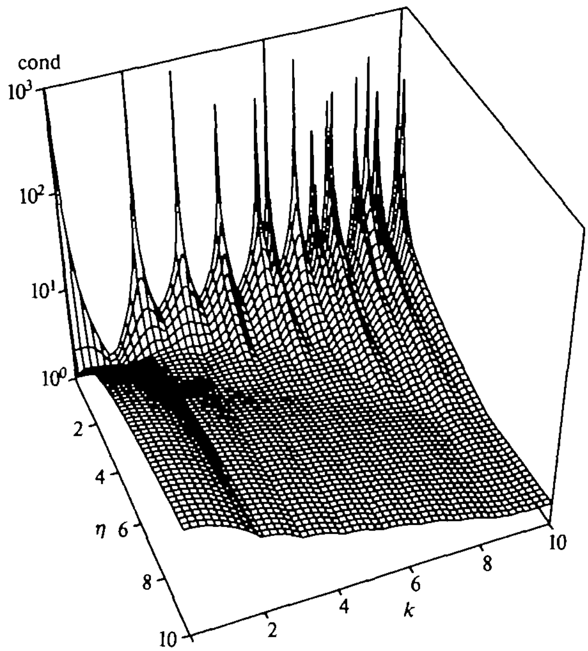

FIG. 2. cond (A) depending on k and η

purposes, of course, it is not essential to choose the optimal parameter η_{opt}; it suffices to choose η in such a way that the integral equation (1.6) becomes reasonably well conditioned. From Fig. 2 it is kind of obvious to try η proportional to k: for instance

$$\eta = \begin{cases} \frac{1}{2}, & k \leq \frac{1}{2}, \\ k, & k \geq \frac{1}{2}. \end{cases} \tag{3.9}$$

Then the dotted curve in Fig. 1 indicates that the condition number of A for η given by (3.9) is close to the optimal condition number. Figure 2 also demonstrates that any constant parameter η for high wavenumbers k will lead eventually to high condition numbers.

As was pointed out by R. E. Kleinman in a private communication (1983), there is also some heuristic argument for the choice $\eta = k$. In this case the kernel of the integral operator $K - ikS$ becomes

$$\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \nu(\mathbf{y})} - ik\Phi(\mathbf{x}, \mathbf{y}),$$

which by the radiation condition for the fundamental solution is concentrated around the diagonal $\mathbf{x} = \mathbf{y}$.

Taking the normal derivative in the Green's representation formula yields (see (4))

$$-\frac{\partial u}{\partial \nu}+Tu-K'\frac{\partial u}{\partial \nu}=0 \tag{3.10}$$

and

$$\frac{\partial v}{\partial \nu}+Tv-K'\frac{\partial v}{\partial \nu}=0, \tag{3.11}$$

for the functions u and v described by (3.3). From this we can eliminate

$$K'Y_n^m=\left(1-\frac{2h_n'(k)j_n(k)}{w_n(k)}\right)Y_n^m, \qquad TY_n^m=-\frac{2kh_n'(k)j_n'(k)}{w_n(k)}Y_n^m.$$

Hence

$$BY_n^m=\mu_n Y_n^m, \tag{3.12}$$

where

$$\mu_n(k)=-2ik^2h_n'(k)[kj_n'(k)-i\eta j_n(k)], \qquad n=0,1,2,\ldots. \tag{3.13}$$

On $H^r(\Omega)$ we use the norm given by

$$\|f\|_{H^r}=\left[\sum_{n=0}^{\infty}(1+n(n+1))^r\sum_{m=-n}^{n}\left|\int_\Omega f\bar{Y}_n^m\,ds\right|^2\right]^{\frac{1}{2}}, \tag{3.14}$$

and note that for $r=0$ this coincides with

$$\|f\|_{H^0}=\left(\int_\Omega |f|^2\,ds\right)^{\frac{1}{2}},$$

and for $r=1$, because

$$\int_\Omega \nabla Y_n^m\,.\,\nabla \bar{Y}_{n'}^{m'}\,ds=n(n+1)\delta_{nn'}\delta^{mm'},$$

it coincides with

$$\|f\|_{H^1}=\left(\int_\Omega (|f|^2+|\nabla f|^2)\,ds\right)^{\frac{1}{2}}.$$

Then,

$$\alpha_n=[1+n(n+1)]^{\frac{1}{2}}. \tag{3.15}$$

Using the power-series expansions for the spherical Bessel and Neumann functions we find that

$$[\mu_n(k)]^2=\frac{4(n+1)^2(n^2+\eta^2)}{(2n+1)^2}+O(k^2), \qquad n=0,1,2,\ldots \tag{3.16}$$

and from this, proceeding as in (9) for the operator A, it can be deduced that for small k the optimal parameter for B is also given by (3.7) with

$$\operatorname{cond}(B_{\mathrm{opt}}(k))=\tfrac{3}{10}\sqrt{15}+O(k^2). \tag{3.17}$$

For general k the numerical results based on (2.13), (3.13) and (3.15) are contained in Figs 3 and 4. They indicate that choosing η as in (3.9) gives a

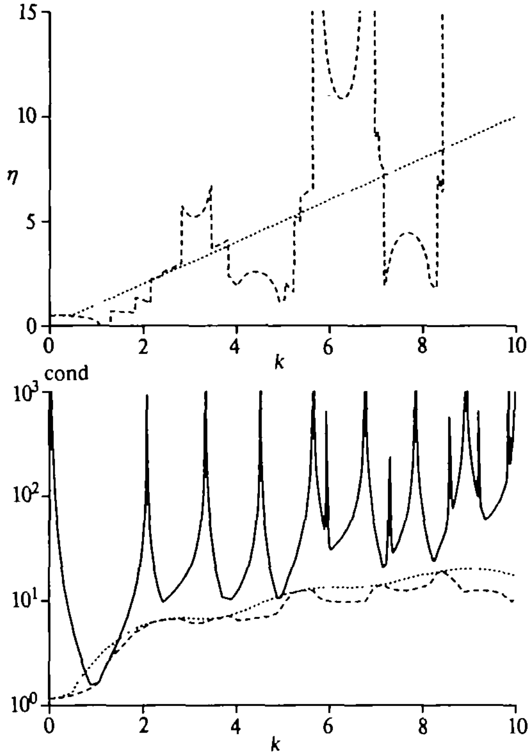

FIG. 3. cond (B) for $\eta = 0$ (full line), $\eta = \eta_{opt}$ (broken line) and $\eta = \max(0\cdot5, k)$ (dotted line)

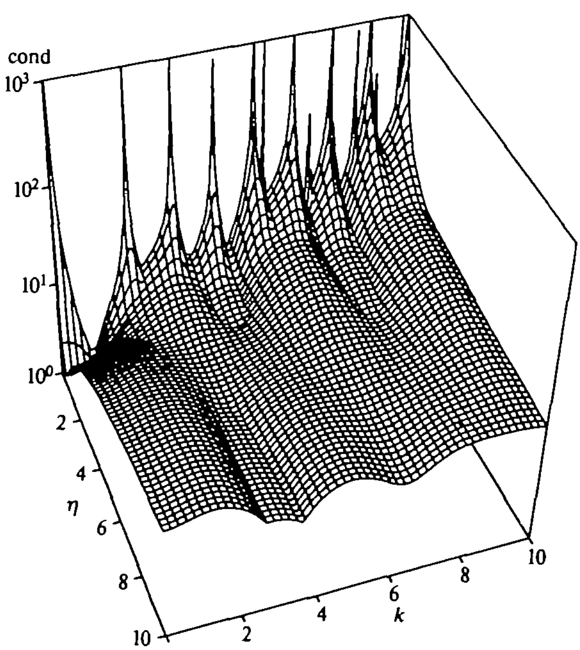

FIG. 4. cond (B) depending on k and η

condition number for B which, again, is close to the optimal condition number.

It seems to be difficult to carry out the above analysis for arbitrary boundaries ∂D. Nevertheless, the special results for the exterior of the unit sphere $\partial D=\Omega$ can serve as a guide to the choice of the parameter η. In particular, the choice (3.9) will lead to well-conditioned operators A and B for small perturbations (in a proper norm) of spherical domains.

Our analysis also covers the treatment of the exterior Neumann boundary-value problem with boundary condition

$$\frac{\partial u}{\partial \nu}=g \qquad \text{on } \partial D. \tag{3.18}$$

The combined single- and double-layer potential

$$u(\mathbf{x})=\int_{\partial D}\left\{\Phi(\mathbf{x},\mathbf{y})+i\xi\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\nu(\mathbf{y})}\right\}\varphi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x}\in\mathbb{R}^3\backslash\bar{D}, \tag{3.19}$$

solves the exterior Neumann problem provided the density φ is a solution of the integral equation

$$\varphi-K'\varphi-i\xi T\varphi=-2g. \tag{3.20}$$

The operator $I-K'-i\xi T$ is obtained from B essentially by substituting ξ for $1/\eta$. Therefore the almost-optimal choice corresponding to (3.9) is given by

$$\xi=\begin{cases}2, & k\leqslant\frac{1}{2},\\ 1/k, & k\geqslant\frac{1}{2}.\end{cases} \tag{3.21}$$

This result coincides with an observation made by Meyer *et al.* (**10**). Their numerical experiments for various geometries and various wavenumbers k indicate that of the three choices $\xi=0$, $\xi=1$ and $\xi=1/k$ the last gives the most accurate results.

4. Condition numbers for the exterior of the unit circle

Our analysis can be extended to the two-dimensional case. In $\mathbb{R}^2$ the fundamental solution (1.5) has to be replaced by

$$\Phi(\mathbf{x},\mathbf{y})=\tfrac{1}{4}iH_0(k\,|\mathbf{x}-\mathbf{y}|), \tag{4.1}$$

where H_0 denotes the Hankel function of the first kind and of order zero. For the unit circle $\partial D=\Omega$, instead of the special solutions given by (3.3) we use

$$\left.\begin{aligned}u(\mathbf{x})&=H_n(k\rho)e^{\pm in\varphi},\\ v(\mathbf{x})&=J_n(k\rho)e^{\pm in\varphi},\end{aligned}\right\} \tag{4.2}$$

where ρ, φ are the polar coordinates of $\mathbf{x}$, J_n denotes the Bessel function of order n, Y_n denotes the Neumann function of order n and $H_n=J_n+iY_n$ denotes the Hankel function of the first kind. Proceeding as in the case of the unit sphere, and using the Wronskian $w_n(k) = J_n(k)H_n'(k) - H_n(k)J_n'(k) = 2i/\pi k$, we find the eigenfunctions

$$Ae^{\pm in\varphi} = \lambda_n e^{\pm in\varphi}, \qquad Be^{\pm in\varphi} = \mu_n e^{\pm in\varphi}, \tag{4.3}$$

with eigenvalues

$$\lambda_n(k) = \pi i H_n(k)[kJ_n'(k) - i\eta J_n(k)] \tag{4.4}$$

and

$$\mu_n(k) = -\pi ikH_n'(k)[kJ_n'(k) - i\eta J_n(k)], \tag{4.5}$$

where $n = 0, 1, 2, \dots$.

On the unit circle, for $H^r(\Omega)$ we use the norm given by

$$\|f\|_{H^r} = \left(\sum_{n=-\infty}^{\infty} (1+n^2)^r \, |f_n|^2\right)^{\frac{1}{2}}, \tag{4.6}$$

where the f_n denote the Fourier coefficients of f. Then,

$$\alpha_n = (1+n^2)^{\frac{1}{2}}. \tag{4.7}$$

Using the power-series expansions for the Bessel functions we find that

$$[\lambda_n(k)]^2 = \begin{cases} \left(1+\dfrac{\eta^2}{n^2}\right) + O(k^2), & n = 1, 2, \dots, \\ [\pi^2 + 4(\ln \frac{1}{2}k + C)^2]\eta^2 + O(k^2 \ln^2 k), & n = 0, \end{cases} \tag{4.8}$$

where $C = 0{\cdot}5772\dots$ is Euler's constant, and

$$[\mu_n(k)]^2 = \begin{cases} (n^2+\eta^2) + O(k^2), & n = 1, 2, \dots, \\ 4\eta^2 + O(k^2 \ln^2 k), & n = 0. \end{cases} \tag{4.9}$$

From this it can be deduced (see (**9**)) that for small k the optimal parameter for the operator A is given by

$$\eta_{\mathrm{opt}}(k) = \{\pi^2 + 4(\ln \tfrac{1}{2}k + C)^2\}^{-\frac{1}{4}}\{1 + O(k^2 \ln^2 k)\}, \tag{4.10}$$

and for the operator B by

$$\eta_{\mathrm{opt}}(k) = \tfrac{1}{2} + O(k^2 \ln^2 k). \tag{4.11}$$

The logarithmic term in (4.10) reflects the logarithmic behaviour of the fundamental solution (4.1) as $k \to 0$.

The numerical results contained in Figs 5 to 8 again indicate that choosing η proportional to k as in (3.9) gives condition numbers for A and B which are very close to the optimal condition number.

Similar numerical investigations have been carried out by Sachs (**11**) in the case of an ellipse with axis ratio varying from $0{\cdot}1$ to 1. The main conclusion is that again a parameter η proportional to the wavenumber k gives almost optimal condition numbers for the operator A. In addition,

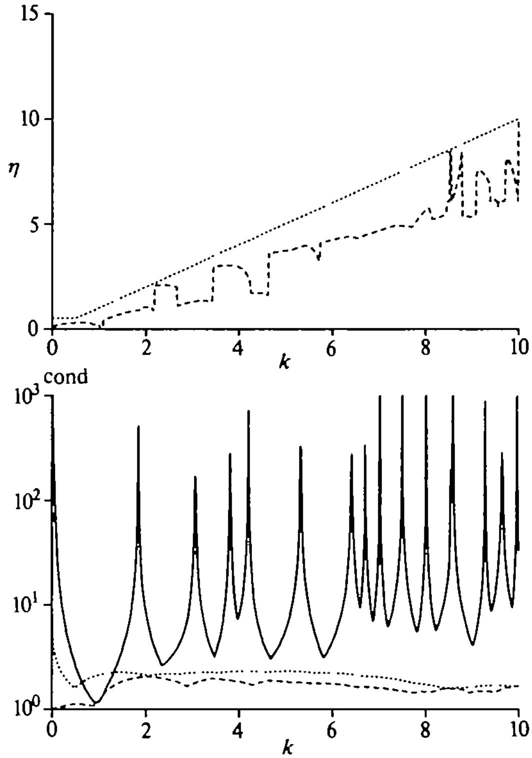

FIG. 5. cond (A) in $\mathbb{R}^2$ for $\eta = 0$ (full line), $\eta = \eta_{opt}$ (broken line) and $\eta = \max(0{\cdot}5, k)$ (dotted line)

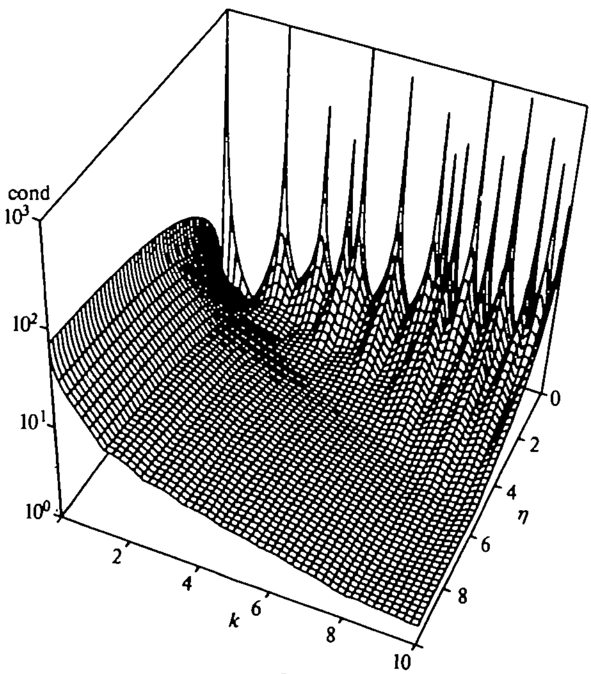

FIG. 6. cond (A) in $\mathbb{R}^2$ depending on k and η

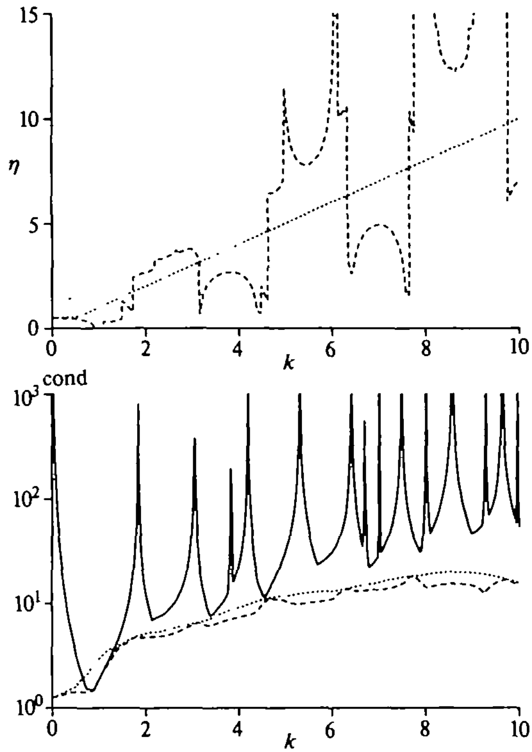

FIG. 7. cond (B) in $\mathbb{R}^2$ for $\eta = 0$ (full line), $\eta = \eta_{opt}$ (broken line) and $\eta = \max(0{\cdot}5, k)$ (dotted line)

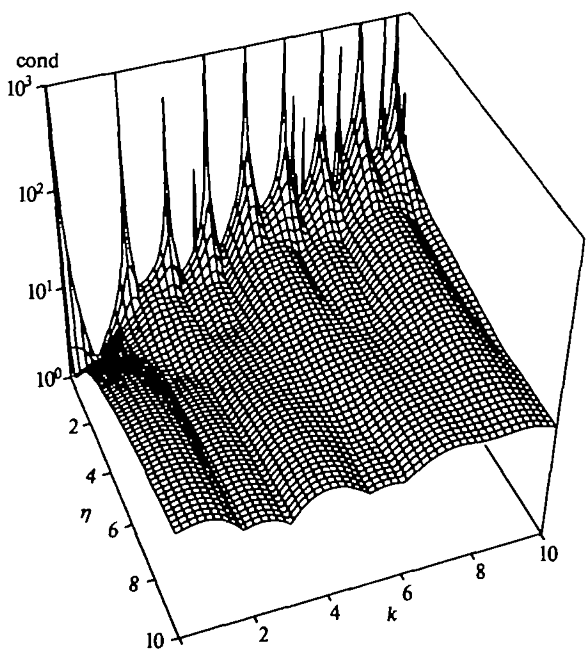

FIG. 8. cond (B) in $\mathbb{R}^2$ depending on k and η

numerical experiments carried out by Lamprecht (**12**), suggest that, for the unit circle, choosing η proportional to k yields almost optimal condition numbers for the operator A also in the L_∞- and L_1-norm.

5. Condition numbers for the combined magnetic- and electric-dipole approach

Consider the exterior boundary-value problem for the Maxwell equations

$$\operatorname{curl}\mathbf{E}-ik\mathbf{H}=\mathbf{0},\quad \operatorname{curl}\mathbf{H}+ik\mathbf{E}=\mathbf{0}\qquad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{5.1}$$

with boundary condition

$$\boldsymbol{\nu}\wedge\mathbf{E}=\mathbf{c}\qquad \text{on } \partial D \tag{5.2}$$

and the Silver–Müller radiation condition

$$\mathbf{H}(\mathbf{x})\wedge\hat{\mathbf{x}}-\mathbf{E}(\mathbf{x})=o(|\mathbf{x}|^{-1}),\qquad |\mathbf{x}|\to\infty, \tag{5.3}$$

uniformly for all directions $\hat{\mathbf{x}}=\mathbf{x}/|\mathbf{x}|$. Here the wavenumber k is a given positive number and $\mathbf{c}\in C^{0,\alpha}(\partial D)$ is a given tangential field with the additional property that its surface divergence $\operatorname{Div}\mathbf{c}$ exists in the sense of the limit integral definition and is of class $C^{0,\alpha}(\partial D)$, $0<\alpha<1$. This Maxwell boundary-value problem describing the reflection of electromagnetic waves at the perfect conductor D can be reduced to a singular integral equation of the second kind which is uniquely solvable for all wavenumbers $k>0$ by seeking the solution in the form of an electromagnetic field generated by a combined magnetic- and electric-dipole distribution. To describe this approach we introduce the following normed spaces of tangential fields

$$\left.\begin{aligned}
\mathcal{T}^{0,\alpha}(\partial D)&=\{\mathbf{a}:\partial D\to\mathbb{C}^3\mid \mathbf{a}\cdot\boldsymbol{\nu}=0,\mathbf{a}\in C^{0,\alpha}(\partial D)\},\\
\mathcal{S}^{0,\alpha}(\partial D)&=\{\mathbf{a}\in\mathcal{T}^{0,\alpha}(\partial D)\mid \operatorname{Div}\mathbf{a}\in C^{0,\alpha}(\partial D)\},\\
\tilde{\mathcal{S}}^{0,\alpha}(\partial D)&=\{\mathbf{a}\in\mathcal{S}^{0,\alpha}(\partial D)\mid \operatorname{Div}(\boldsymbol{\nu}\wedge\mathbf{a})\in C^{0,\alpha}(\partial D)\},
\end{aligned}\right\} \tag{5.4}$$

with norms

$$\left.\begin{aligned}
\|\mathbf{a}\|_{\mathcal{T}^{0,\alpha}}&=\|\mathbf{a}\|_{C^{0,\alpha}},\\
\|\mathbf{a}\|_{\mathcal{S}^{0,\alpha}}&=\|\mathbf{a}\|_{C^{0,\alpha}}+\|\operatorname{Div}\mathbf{a}\|_{C^{0,\alpha}},\\
\|\mathbf{a}\|_{\tilde{\mathcal{S}}^{0,\alpha}}&=\|\mathbf{a}\|_{C^{0,\alpha}}+\|\operatorname{Div}\mathbf{a}\|_{C^{0,\alpha}}+\|\operatorname{Div}\boldsymbol{\nu}\wedge\mathbf{a}\|_{C^{0,\alpha}}.
\end{aligned}\right\} \tag{5.5}$$

The electromagnetic field

$$\left.\begin{aligned}
\mathbf{E}(\mathbf{x})&=\operatorname{curl}\int_{\partial D}\Phi(\mathbf{x},\mathbf{y})\mathbf{a}(\mathbf{y})\,ds(\mathbf{y})+\\
&\qquad+i\xi\operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(\mathbf{x},\mathbf{y})(\nu(\mathbf{y})\wedge\mathbf{a}(\mathbf{y}))\,ds(\mathbf{y}),\\
\mathbf{H}(\mathbf{x})&=\frac{1}{ik}\operatorname{curl}\mathbf{E}(\mathbf{x}),\qquad \mathbf{x}\in\mathbb{R}^3\backslash\bar{D},
\end{aligned}\right\} \tag{5.6}$$

with a positive coupling parameter ξ, solves the exterior Maxwell boundary-value problem provided the density $\mathbf{a}\in\tilde{\mathcal{T}}^{0,\alpha}(\partial D)$ is a solution of the integral equation

$$\mathbf{a}+M\mathbf{a}+i\xi N\mathbf{a}=2\mathbf{c}, \tag{5.7}$$

where $M:\mathcal{T}^{0,\alpha}(\partial D)\to\mathcal{T}^{0,\alpha}(\partial D)$ denotes the compact integral operator defined by

$$(M\mathbf{a})(\mathbf{x})=2\int_{\partial D}\boldsymbol{\nu}(\mathbf{x})\wedge\operatorname{curl}_{\mathbf{x}}\{\Phi(\mathbf{x},\mathbf{y})\mathbf{a}(y)\}\,ds(\mathbf{y}),\qquad \mathbf{x}\in\partial D, \tag{5.8}$$

and where $N:\tilde{\mathcal{T}}^{0,\alpha}(\partial D)\to\mathcal{T}^{0,\alpha}(\partial D)$ denotes the bounded operator defined by

$$(N\mathbf{a})(\mathbf{x})=2\boldsymbol{\nu}(\mathbf{x})\wedge\operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(\mathbf{x},\mathbf{y})(\boldsymbol{\nu}(\mathbf{y})\wedge\mathbf{a}(\mathbf{y}))\,ds(\mathbf{y}),\qquad \mathbf{x}\in\partial D. \tag{5.9}$$

As shown in (**4**) this integral equation is uniquely solvable for all $k>0$ and all $\xi>0$. Note that for $\xi=0$ the integral equation becomes non-unique if k is an eigenvalue for the interior Maxwell problem in D. Introduce the operator

$$C=I+M+i\xi N. \tag{5.10}$$

Then, $C:\tilde{\mathcal{T}}^{0,\alpha}(\partial D)\to\mathcal{T}^{0,\alpha}(\partial D)$ is bounded and from the analysis in (**4**) it can be concluded that C has a bounded inverse $C^{-1}:\mathcal{T}^{0,\alpha}(\partial D)\to\tilde{\mathcal{T}}^{0,\alpha}(\partial D)$. This result in the classical Hölder-space structure leads to a conjecture that corresponding results hold in an appropriate Sobolev-space setting. Indeed, in the following analysis, as a first step we shall show that this is true for the special case of a sphere $\partial D=\Omega$.

In order to obtain the approximate solution to the Maxwell boundary-value problem in addition to numerically solving the integral equation (5.7) we also have to evaluate the combined electromagnetic field (5.6) for the density $\mathbf{a}$. As a typical problem consider the evaluation of the magnetic field

$$\boldsymbol{\nu}\wedge\mathbf{H}=\tfrac{1}{2}[\xi k(I+M)+(i/k)N](\boldsymbol{\nu}\wedge\mathbf{a}). \tag{5.11}$$

We define

$$\tilde{C}=\xi k(I+M)+(i/k)N, \tag{5.12}$$

and observe that $\tilde{C}$ is obtained from C essentially by replacing ξ by $1/\xi k^2$. Therefore, it suffices to discuss the condition number of C.

6. Condition numbers for the exterior of the unit sphere in electromagnetics

Let $\mathbf{E},\mathbf{H}$ be a solution to the Maxwell equations in $\mathbb{R}^3\setminus\bar{D}$ satisfying the radiation condition. Then from the Stratton and Chue representation formula we have (see (4))

$$\boldsymbol{\nu}\wedge\mathbf{E}-M(\boldsymbol{\nu}\wedge\mathbf{E})-(1/ik)N(\boldsymbol{\nu}\wedge(\boldsymbol{\nu}\wedge\mathbf{H}))=\mathbf{0} \tag{6.1}$$

for the tangential components on the boundary. Similarly, for a solution $\mathbf{F},\mathbf{G}$ to the Maxwell equations in D we have

$$-\boldsymbol{\nu}\wedge\mathbf{F}-M(\boldsymbol{\nu}\wedge\mathbf{F})-(1/ik)N(\boldsymbol{\nu}\wedge(\boldsymbol{\nu}\wedge\mathbf{G}))=\mathbf{0}. \tag{6.2}$$

Now consider the special case of the unit sphere $\partial D=\Omega$ and choose

$$\left.\begin{aligned}\mathbf{E}&=\operatorname{curl}(\mathbf{x}u), &\quad \mathbf{H}&=(1/ik)\operatorname{curl}\mathbf{E},\\ \mathbf{F}&=\operatorname{curl}(\mathbf{x}v), &\quad \mathbf{G}&=(1/ik)\operatorname{curl}\mathbf{F},\end{aligned}\right\} \tag{6.3}$$

where u and v are given by (3.3). Then from (6.1) and (6.2) we can eliminate

$$M\,\nabla Y_n^m=\left(-1-\frac{2h_n(k)[j_n(k)+kj_n'(k)]}{kw_n(k)}\right)\nabla Y_n^m,$$

$$N\,\nabla Y_n^m=\frac{2kh_n(k)j_n(k)}{w_n(k)}\nabla Y_n^m.$$

Hence,

$$C\,\nabla Y_n^m=\gamma_n(k)\,\nabla Y_n^m, \tag{6.4}$$

where

$$\gamma_n(k)=2ikh_n(k)[j_n(k)+kj_n'(k)-i\eta k^2 j_n(k)],\qquad n=1,2,\dots. \tag{6.5}$$

Analogously, from

$$\left.\begin{aligned}\mathbf{H}&=\operatorname{curl}(\mathbf{x}u), &\quad \mathbf{E}&=-(1/ik)\operatorname{curl}\mathbf{H},\\ \mathbf{G}&=\operatorname{curl}(\mathbf{x}v), &\quad \mathbf{F}&=-(1/ik)\operatorname{curl}\mathbf{G},\end{aligned}\right\} \tag{6.6}$$

we deduce that

$$M(\boldsymbol{\nu}\wedge\nabla Y_n^m)=\left(-1+\frac{2[h_n(k)+kh_n'(k)]j_n(k)}{kw_n(k)}\right)(\boldsymbol{\nu}\wedge\nabla Y_n^m),$$

$$N(\boldsymbol{\nu}\wedge\nabla Y_n^m)=\frac{2[h_n(k)+kh_n'(k)][j_n(k)+kj_n'(k)]}{kw_n(k)}(\boldsymbol{\nu}\wedge\nabla Y_n^m).$$

Hence,

$$C(\boldsymbol{\nu}\wedge\nabla Y_n^m)=\delta_n(k)(\boldsymbol{\nu}\wedge\nabla Y_n^m), \tag{6.7}$$

where

$$\delta_n(k)=-2ik[h_n(k)+kh_n'(k)][j_n(k)+i\eta\{j_n'(k)+kj_n'(k)\}],\qquad n=1,2,\dots. \tag{6.8}$$

The tangential fields $\{\nabla Y_n^m, \boldsymbol{\nu}\wedge\nabla Y_n^m,\ n=1,2,\dots,\ m=-n,\dots,n\}$ form a complete orthogonal set on the space of L^2-tangential fields

$$\mathcal{T}^2(\Omega)=\{\mathbf{a}\in L^2(\Omega)\mid \boldsymbol{\nu}\cdot\mathbf{a}=0\}$$

endowed with the usual L^2-scalar product. Corresponding to the Hölder spaces given by (5.4) and (5.5) for $r \geqslant 0$ we introduce the following Sobolev spaces

$$\left.\begin{aligned} \mathcal{S}^r(\Omega) &= \{\mathbf{a} \in \mathcal{T}^2(\Omega) \mid \mathbf{a}, \operatorname{Div} \mathbf{a} \in H^r(\Omega)\}, \\ \tilde{\mathcal{S}}^r(\Omega) &= \{\mathbf{a} \in \mathcal{S}^r(\Omega) \mid \operatorname{Div}(\boldsymbol{\nu} \wedge \mathbf{a}) \in H^r(\Omega)\}, \end{aligned}\right\} \tag{6.9}$$

with norms

$$\left.\begin{aligned} \|\mathbf{a}\|_{\mathcal{S}^r} &= \left\{\sum_{n=1}^{\infty} (1+n(n+1))^{r+1} \sum_{m=-n}^{n} |a_n^m|^2 + \right. \\ &\qquad \left. + (1+n(n+1))^r \sum_{m=-n}^{n} |b_n^m|^2\right\}^{\frac{1}{2}}, \\ \|\mathbf{a}\|_{\tilde{\mathcal{S}}^r} &= \left\{\sum_{n=1}^{\infty} (1+n(n+1))^{r+1} \sum_{m=-n}^{n} (|a_n^m|^2 + |b_n^m|^2)\right\}^{\frac{1}{2}} \end{aligned}\right\} \tag{6.10}$$

where

$$a_n^m = \frac{1}{[n(n+1)]^{\frac{1}{2}}} \int_\Omega \mathbf{a} \cdot \nabla \bar{Y}_n^m \, ds, \qquad b_n^m = \frac{1}{[n(n+1)]^{\frac{1}{2}}} \int_\Omega \mathbf{a} \cdot (\boldsymbol{\nu} \wedge \nabla \bar{Y}_n^m) \, ds.$$

We note that for $r = 0$ these norms coincide with

$$\|\mathbf{a}\|_{\mathcal{S}^0} = \left(\int_\Omega \{|\mathbf{a}|^2 + |\operatorname{Div} \mathbf{a}|^2\} \, ds\right)^{\frac{1}{2}},$$

$$\|\mathbf{a}\|_{\tilde{\mathcal{S}}^0} = \left(\int_\Omega \{|\mathbf{a}|^2 + |\operatorname{Div} \mathbf{a}|^2 + |\operatorname{Div}(\boldsymbol{\nu} \wedge \mathbf{a})|^2\} \, ds\right)^{\frac{1}{2}}.$$

Then, since

$$\mathbf{a} = \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^{\frac{1}{2}}} \sum_{m=-n}^{n} \{a_n^m \nabla Y_n^m + b_n^m (\boldsymbol{\nu} \wedge \nabla Y_n^m)\}$$

and

$$C\mathbf{a} = \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^{\frac{1}{2}}} \sum_{m=-n}^{n} \{\gamma_n a_n^m \nabla Y_n^m + \delta_n b_n^m (\boldsymbol{\nu} \wedge \nabla Y_n^m)\},$$

we have, for $C: \tilde{\mathcal{S}}^r(\Omega) \to \mathcal{S}^r(\Omega)$, the norm

$$\|C\| = \sup_{n \in \mathbb{N}} \left\{|\gamma_n|, \frac{|\delta_n|}{[1+n(n+1)]^{\frac{1}{2}}}\right\}. \tag{6.11}$$

Similarly,

$$\|C^{-1}\| = 1 \Big/ \left(\inf_{n \in \mathbb{N}} \left\{|\gamma_n|, \frac{|\delta_n|}{[1+n(n+1)]^{\frac{1}{2}}}\right\}\right). \tag{6.12}$$

The numerical results based on this analysis are contained in Figs 9 and 10. We have set $\eta = \xi^{-1}$ and our results again indicate that for $k \geqslant \frac{1}{2}$ the

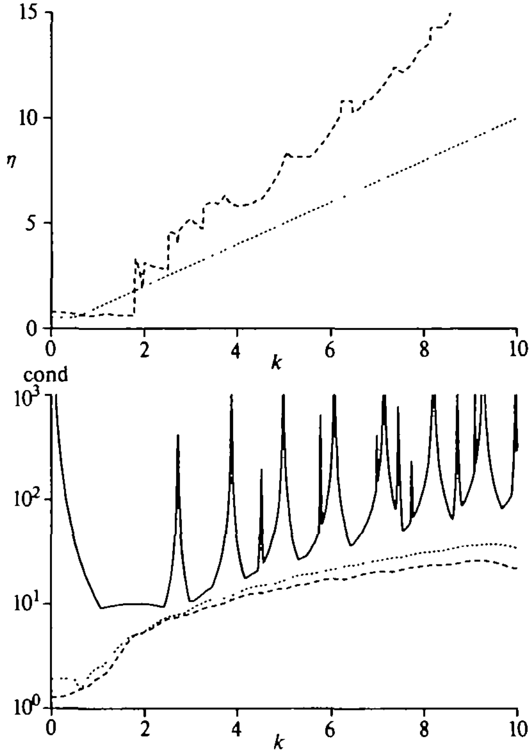

FIG. 9. cond (C) for $\eta = 0$ (full line), $\eta = \eta_{opt}$ (broken line) and $\eta = \max(0{\cdot}5, k)$ (dotted line)

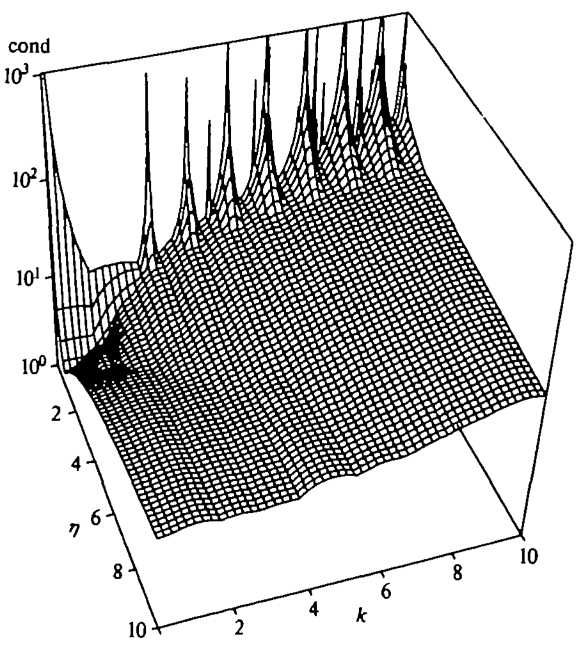

FIG. 10. cond (C) depending on k and η

choice $\eta = k$, which means that $\xi = 1/k$, gives a condition number for C which is close to the optimal condition number. Note that because of the remark following (5.12) this choice also gives almost-optimal condition number for the operator $\tilde{C}$.

For small k, as opposed to the exterior Dirichlet problem in acoustics, it is not necessary to use the combined approach for the Maxwell boundary-value problem since, in the limiting potential theoretic case when $k = 0$, equation (5.7) is uniquely solvable in $\mathcal{T}^{0,\alpha}(\partial D)$ for $\eta = 0$. Hence, we do not include a discussion of the optimal parameter for small k.

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