Regularization in 3D for Anisotropic Elastodynamic Crack and Obstacle Problems

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I - INTRODUCTION

The problems of wave scattering by obstacles or cracks appear very often in geophysics and in mechanics. In particular the linearized theory of elastodynamics for 3 dimensional elastic material is used frequently, because this theory keeps the analysis relatively simple. Even with this theory, however, a practical analysis is possible only with the use of some numerical methods. This has been the raison d'être of many numerical experiments carried out in the engineering community. Among those numerical methods tested so far, the boundary integral equation (BIE) method has been accepted favourably by engineers, presumably because it can deal with scattered waves effectively in external problems. In particular the double layer potential representation is considered to be an efficient tool of numerical analysis for wave problems including cracks. The only inconvenience of the double layer potential approach, however, is the hypersingularity of the kernel, which does not permit the use of conventional numerical integration techniques. Hence we can take advantage of this approach only after weakening the hypersingularity of the kernel, or only after 'regularizing' it. As a matter of fact, some of such attemps can be found in the articles by Sladek & Sladek [11], Bui [5], Bonnet [4], Polch et.al [10], Nishimura & Kobayashi [8], [9] who used the collocation method and in Nedelec [7], Bamberger [1] where the variational method has been used.

As the number of the publications on this subject tells, there exist various different possibilities of the regularization. However, not all of these regularizations are universal because some of them work only with collocation methods, and because others may destroy the causality in the time domain. For example the authors of [11], [5], [4], [10] seem to have devised their techniques mainly with collocation methods in their minds. Also the formulation in [7] may not be very useful in time domain because it will produce kernels which violate the causality. In addition, there is no guarantee that the generalizations of the formulations in [1] and [8] preserve the causality in the general anisotropic case, although they do in the isotropic case. In view of this we shall investigate a unified method of generating 'good' regularized

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integral equations in the double layer potential approach for the general 3 dimensional anisotropic elastodynamics.

This paper begins by recapitulating the governing equations of the 3 dimensional elastodynamics. We then discuss the structure of the hypersingular kernel Σ which appears in the integral equation obtained from the double layer potential in the frequency domain. Specifically, we shall show that Σ allows a decomposition into a sum of a hypersingular kernel 'rot rot rot rot Φ' and a regular kernel R in a way that Φ and R maintain the causality possessed by Σ. Also we shall present an explicit form of Φ for the general 3 dimensional anisotropic case. We then proceed to demonstrate that this decomposition readily produces a variational form in terms of Φ and R. The kernels included in this variational form will be seen to inherit the correct causality possessed by Φ and R. We then discuss the isotropic case. A few remarks concerning collocation conclude this paper.

II - GOVERNING EQUATIONS

Let Ω^- be an open bounded domain of $\mathbf{R}^3$ whose boundary is a regular closed surface Γ and let Ω^+ be the open domain complement to $\overline{\Omega^-}$. We are interested in solving wave propagation problems for anisotropics materials by B.I.E method, in time domain or in frequency domain. The governing equations for the wave scattering problems in the time domain are :

$$\text{(II-1)} \qquad \begin{cases} \operatorname{div} \sigma - \rho \dfrac{\partial^2 \vec{u}}{\partial t^2} = 0 & \text{in } \Omega^+ \times \mathbf{R}_+ \\ \vec{u}(x,0^+) = 0 & x \in \Omega^+ \\ \dfrac{\partial \vec{u}}{\partial t}(x,0^+) = 0 & x \in \Omega^+ \\ \sigma \vec{n} = \vec{g} & \text{on } \Gamma \times \mathbf{R}_+ \end{cases}$$

where $\vec{u}$, ρ, σ, $\vec{n}$ are the displacement, mass density, stress tensor, the outward unit normal to the boundary respectively and $\vec{g}$ is a given function.

The stress σ in (II-1) is related to the displacement $\vec{u}$ by Hooke's law given by :

$$\text{(II-2)} \qquad \sigma_{ij} = C_{ijkl}\, \varepsilon_{kl} = C_{ijkl}\, u_{k,l}$$

where the summation convention is used, $\varepsilon_{ij}(\vec{u})$ is the strain defined by

$$\text{(II-3)} \qquad \varepsilon_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and C is the elasticity tensor which is positive definite and has the symmetry given by

$$C_{ijkl} = C_{jikl} = C_{klij}$$

We also have

(II-4) $$\varepsilon_{ij} = A_{ijkl}\,\sigma_{kl}$$

where $A = C^{-1}$.

We now take the Fourier Transforms (FT) of equations (II-1) with respect to time to obtain the problem in frequency domain

(II-5) $$\begin{cases} \operatorname{div}\sigma + \rho\omega^2\vec{u} = 0 & \text{in } \Omega^+ \\ \sigma\vec{n} = \vec{g} & \text{on } \Gamma \\ \varepsilon(\vec{u}) = A\sigma & \text{in } \Omega^+ \\ \vec{u} \text{ satisfies the radiation condition} & \end{cases}$$

In (II-5) we have used the same symbols $\vec{u}$, σ ... for the time Fourier transforms of $\vec{u}(x,t)$, $\sigma(x,t)$ ... because we will be mainly concerned with the frequency domain.

In order to solve this problem we introduce the double layer potential which satisfies the following equations in addition to the radiation condition :

(II-6) $$\begin{cases} \operatorname{div}\sigma + \rho\omega^2\vec{u} = 0 & \text{in } \Omega^- \cup \Omega^+ \\ [\sigma\vec{n}] = 0 & \\ [\vec{u}] = \vec{\varphi} & \\ \varepsilon(\vec{u}) = A\sigma & \text{in } \Omega^- \cup \Omega^+ \end{cases}$$

where $[f] = f_- - f_+$ is the difference between the interior limit f_- and the exterior limit f_+ of a function f regular in $\overline{\Omega^-}$ and $\overline{\Omega^+}$. If we consider the derivatives in (II-2) in the distributional sense we obtain

(II-7) $$\varepsilon = \tilde{\varepsilon} - t\,\delta_\Gamma$$

where t is defined as

(II-8) $$t_{ij} = \frac{1}{2}(\varphi_i n_j + \varphi_j n_i)$$

δ_Γ is the surface Dirac measure associated with Γ and $\tilde{\varepsilon}$ is the function part of ε defined as

(II-9) $$\tilde{\varepsilon} = \varepsilon_{/\Omega^+} + \varepsilon_{/\Omega^-}$$

In (II-9) $\varepsilon_{/\Omega}$ stands for the extension by 0 to $\mathbf{R}^3$ of the restriction of ε to Ω. Equation (II-6) can then be rewritten in the distributional sense as

(II-10) $$\begin{cases} \operatorname{div} \sigma + \rho\omega^2\vec{u} = 0 \\ \varepsilon(\vec{u}) - A\sigma = -t\,\delta_\Gamma \end{cases}$$

We now introduce the fundamental solution U, Σ defined by

(II-11) $$\begin{cases} \operatorname{div} \Sigma + \rho\omega^2 U = 0 \\ \varepsilon(U) - A\Sigma = \delta(x)\,\mathbb{I} \end{cases}$$

where

$$(\mathbb{I})^{kl}_{ij} = \frac{1}{2}\left(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li} \right)$$

With U and Σ we can write the double layer potential in (II-6) as

(II-12) $$\begin{cases} \vec{u} = - U * t\,\delta_\Gamma = - \displaystyle\int_\Gamma U(x-y)\, t(y)\, d\gamma_y \\ \sigma = -\Sigma * t\,\delta_\Gamma = - \displaystyle\int_\Gamma \Sigma(x-y)\, t(y)\, d\gamma_y \end{cases}$$

The original boundary value problem is then reduced to an integral equation on Γ given by

(II-13) $$\lim_{\substack{x = x_0 + \varepsilon\vec{n}(x_0) \\ \varepsilon \to 0}} \left\{ -\frac{1}{2}\int_\Gamma \Sigma^{kl}_{ij}(x-y)\, (\varphi_k(y)n_l(y) + \varphi_l(y)n_k(y))\, n_j(x_0)\, d\gamma_y \right\} = g_i(x_0)$$

After solving (II-13) for the unknown φ, we determine $\vec{u}$ and σ by (II-12).

As can be shown, however, the kernel Σ in (II-13) is asymptotically proportional to $1/|x-y|^3$ as $|x-y|$ approaches 0, and we have to give a sense to the limit in (II-13). Because of this strong singularity, we cannot solve (II-13) directly by using conventional numerical methods. In fact, one usually uses integration by parts, or "regularization", to reduce the singularity to an integrable one.and it allows us to define $\sigma(x)\vec{n}(x)$ in a distributional sense as we will see in (IV-1). As we have seen in the introduction, however, the existing regularization techniques may not always be very convenient because some of them are useful only with collocation methods, while others may destroy the causality in time domain. For example, Nedelec [7] proposed to regularize Σ by using the following identity :

$$\hat{\Sigma}_{ij}^{kl} = \frac{1}{|\xi|^4} e_{ipq}\xi_p e_{lab}\xi_a e_{qrs}\xi_r e_{bcd}\xi_c \hat{\Sigma}_{sj}^{kd}$$
$$- \frac{1}{|\xi|^4}\left(e_{ipq}\xi_p e_{qrs}\xi_r\xi_l \hat{\Sigma}_{sj}^{kd} \xi_d + e_{lab}\xi_a e_{bcd}\xi_c\xi_i \hat{\Sigma}_{sj}^{kd} \xi_s - \xi_i\xi_l \hat{\Sigma}_{sj}^{kd} \xi_s\xi_d\right)$$

where

$$e_{imr} = \begin{cases} 1 & \text{if (i,m,r) is obtained by a circular permutation of (1,2,3)} \\ 0 & \text{if 2 indices are equal} \\ -1 & \text{otherwise} \end{cases}$$

He then obtained a variational formulation in a manner analogous to the one to be given in IV. His method works in frequency domain. Unfortunately, however, it destroys the causality in time domain. Indeed, we notice that the Fourier inverse transform of the above expression is given in terms of the convolution of $|x|$ and the derivatives of Σ. Since the former (the fundamental solution of the double laplacian) violates the causality, so does the resulting convolution. Hence we propose in the next chapter a formulation which does not have this inconvenience.

Before closing this chapter we notice that the same integral equation (II-13) solves crack problems, where Γ is a surface in $\mathbf{R}^3$ which is identified as a crack. Bearing this application in mind we shall henceforth drop the assumption that Γ is the boundary of Ω^-.

III - COMPUTATION OF Σ

In this chapter we shall discuss the structure of Σ. The analysis will be in frequency domain unless stated otherwise.To begin with we compute Σ explicitly. From the FT in space of (II-11)(denoted by ^) we deduce that

(III-1) $$\hat{\Sigma}_{ij}^{kl} = \frac{i}{2} C_{ijmn} (\hat{U}_m^{kl} \xi_n + \hat{U}_n^{kl} \xi_m) - C_{ijkl},$$

(III-2) $$-\frac{\xi_j}{2} C_{ijmn} (\hat{U}_m^{kl} \xi_n + \hat{U}_n^{kl} \xi_m) + \rho\omega^2 \hat{U}_i^{kl} = i\xi_j C_{ijkl},$$

where ξ is the parameter of the spatial FT. The inversion of (III-2) leads to :

(III-3) $$\hat{U}_i^{kl} = -\Lambda_{im}^{-1} i\xi_j C_{mjkl}$$

where

(III-4) $$\Lambda = \hat{\Gamma} - \rho\omega^2 I$$

(III-5) $$\hat{\Gamma}_{ik} = C_{ijkl}\xi_j\xi_l$$

(III-6) $$I_{ik} = \delta_{ik}$$

Finally, we substitute U in (III-1) to get the expression of $\hat{\Sigma}$:

(III-7)
$$\begin{cases} \hat{\Sigma}_{ij}^{kl} = C_{ijmn}\,\xi_m\,\Lambda^{-1}_{nr}\,\xi_s\,C_{rskl} - C_{ijkl} \\ \quad = \dfrac{C_{ijmn}\,\xi_m\,(\mathrm{cof}\,\Lambda)_{rn}\,\xi_s\,C_{rskl} - C_{ijkl}\det\Lambda}{\det\Lambda} \end{cases}$$

which satisfies : $\hat{\Sigma}_{ij}^{kl} = \hat{\Sigma}_{ji}^{kl} = \hat{\Sigma}_{kl}^{ij}$ because of the symmetries of C and Λ.

From (III-7) we see that the causality of Σ in time domain is determined by the inverse Fourier transform of $L = (\det \Lambda)^{-1}$. In other words L is linked only to the wave velocities of the material. In fact the numerator in (III-7) is a polynomial in ξ and ω, which implies that Σ is given in terms of certain derivatives of L. Conversely, we see that a function whose FT is written as

(polynomials of ξ and ω).L

possesses the same causality as Σ.

We now proceed to show the existence of a decomposition of Σ which facilitates the regularization without destroying the causality :

THEOREM 1 : There exists a stress function Φ and a kernel R such that :

(III-8) $$\Sigma_{ij}^{kl} = rot_i rot_j rot_k rot_l \Phi^{\cdot\cdot}_{\cdot\cdot} + \rho\omega^2 R_{ij}^{kl}$$

holds. These kernels have the same symmetry as Σ and take the following forms:

$$\hat{\Phi} = \frac{P(\xi)}{\det\Lambda} \quad ; \quad \hat{R} = \frac{Q(\omega,\xi)}{\det\Lambda}$$

where P and Q are polynomials of the respective arguments homogeneous of degree 2 and 4, respectively . Hence $rot_i rot_k \Phi^{\cdot l}_{j}$ and R are locally integrable. P is not determined uniquely, although Q is. More precisely, P is written as

$$P_{ij}^{kl} = \overset{o}{P}{}_{ij}^{kl} + \xi_i\,E_{mjkl}\,\xi_m + \xi_j\,E_{mikl}\,\xi_m + \xi_k\,E_{mlij}\,\xi_m + \xi_l\,E_{mkij}\,\xi_m$$

where $\overset{o}{P}{}_{ij}^{kl}$ is a 'particular solution' and E_{mjkl} is a constant tensor symmetric with respect to k and l.

Remark : We define $\mathrm{rot}_i\Phi^{kl}_{.j}$ as the rotation of Φ^{kl}_{ij} considered as a vector in i, with j,k,l being fixed. We therefore have

$$(\mathrm{rot}_i\Phi^{kl}_{.j})_i = e_{imr}\partial_m\Phi^{kl}_{rj}$$

Proof : see [3]

As a matter of fact we can write down the stress function explicitly, as shown in the following

***THEOREM 2** : An expression for $\hat{\Phi}$ is :*

$$\hat{\Phi}^{kl}_{ij} = \frac{A_{ktib}\,A_{lnjd} + A_{ltib}\,A_{knjd}}{\det(K-\rho\omega^2 A)/(\rho\omega^2)^3}\, e_{tun}\, e_{bvd}\, \xi_u\, \xi_v$$

where

$$K = \begin{pmatrix} \xi_1^2 & 0 & 0 & 0 & \xi_1\xi_3 & \xi_1\xi_2 \\ 0 & \xi_2^2 & 0 & \xi_3\xi_2 & 0 & \xi_1\xi_2 \\ 0 & 0 & \xi_3^2 & \xi_3\xi_2 & \xi_1\xi_3 & 0 \\ 0 & \xi_3\xi_2 & \xi_3\xi_2 & \xi_2^2+\xi_3^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_3 & 0 & \xi_1\xi_3 & \xi_1\xi_2 & \xi_1^2+\xi_3^2 & \xi_3\xi_2 \\ \xi_1\xi_2 & \xi_1\xi_2 & 0 & \xi_1\xi_3 & \xi_3\xi_2 & \xi_1^2+\xi_2^2 \end{pmatrix}$$

Proof : see [3]

IV - VARATIONAL FORM

We rewrite the integral equation (II-13) as

(II-13) $$\Sigma^{kl}_{ij} * (\varphi_k n_l \delta_\Gamma)\, n_j \delta_\Gamma = -g_i \delta_\Gamma$$

The variational formulation corresponding to (II-13) is

$$b(\varphi,\psi) = -\langle g, \psi\delta_\Gamma \rangle$$

where

$$b(\varphi,\psi) = \left\langle \Sigma_{ij}^{kl} * (\varphi_k n_l \delta_\Gamma), \psi_i n_j \delta_\Gamma \right\rangle ,$$

We now use the decomposition of Σ in theorem 1 to get

$$b(\varphi,\psi) = \left\langle (\mathrm{rot}_i \mathrm{rot}_j \mathrm{rot}_k \mathrm{rot}_l) \Phi_{ij}^{kl} * (\varphi_k n_l \delta_\Gamma), \psi_i n_j \delta_\Gamma \right\rangle + \rho\omega^2 \left\langle R_{ij}^{kl} * (\varphi_k n_l \delta_\Gamma), \psi_i n_j \delta_\Gamma \right\rangle$$

Repeated use of integration by parts removes the singularity in this form to yield

$$\text{(IV-1)} \qquad b(\varphi,\psi) = \left\langle F_{ij}^{kl} * (\overrightarrow{\mathrm{rot}}_\Gamma \varphi_k)_l \delta_\Gamma, (\overrightarrow{\mathrm{rot}}_\Gamma \psi_i)_j \delta_\Gamma \right\rangle + \rho\omega^2 \left\langle R_{ij}^{kl} * (\varphi_k n_l \delta_\Gamma), \psi_i n_j \delta_\Gamma \right\rangle$$

with

$$F_{ij}^{kl} = - e_{ima} e_{knb}\, \partial_{mn}^2 \Phi_{aj}^{bl}$$
$$\left(\overrightarrow{\mathrm{rot}}_\Gamma \varphi_k\right) \delta_\Gamma = \overrightarrow{\mathrm{rot}}\, (\varphi_k\, \vec{n}\, \delta_\Gamma) .$$

Equation (IV-1) gives a regularized bilinear form for (II-13) in frequency domain because F and R are locally integrable by virtue of theorem 1. We then obtain a variational form in time domain by taking the inverse Fourier transform of (IV-1). The factor ω^2 will give rise to some time derivatives of φ and ψ. For example a bilinear form symmetric, with respect to φ and ψ, is obtained as

$$\text{(IV-2)} \qquad b(\varphi,\psi) = \left\langle F_{ij}^{kl} * (\overrightarrow{\mathrm{rot}}_\Gamma \varphi_k)_l\, \delta_\Gamma, (\overrightarrow{\mathrm{rot}}_\Gamma \psi_i)_j\, \delta_\Gamma \right\rangle - \rho \left\langle R_{ij}^{kl} * (\dot{\varphi}_k n_l \delta_\Gamma), \dot{\psi}_i n_j \delta_\Gamma \right\rangle$$

Again, we have used the same symbols F and R for the Fourier inverse transforms with respect to ω of the frequency domain versions of F and R. Of course $\langle\, ,\, \rangle$ and $*$ in (IV-2) are those for $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$. For more details about the variational form in the isotropic case, one can see [2].

V - ISOTROPIC CASE

In this case the compliance tensor A is given in terms of the Lamé 's constants (λ,μ) as

$$A_{ijkl} = \frac{1}{4\mu} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2\lambda}{3\lambda + 2\mu} \delta_{ij}\delta_{kl} \right)$$

With this formula one easily shows

$$\text{(V-1)} \quad \left[\begin{array}{l} \left(A_{ktib}\, A_{lnjd} + A_{ltib}\, A_{knjd} \right) e_{tun}\, e_{bvd}\, \xi_u\, \xi_v \sim \dfrac{|\xi|^2 \tilde{\Phi}_{klij}}{4\mu^2(3\lambda + 2\mu)} \\ \dfrac{\det(K - \rho\omega^2\ A)}{(\rho\omega^2)^3} = \dfrac{D_P D_S^2}{4\mu^5(3\lambda + 2\mu)} \end{array}\right.$$

where

$$\text{(V-2)} \quad \begin{array}{l} \tilde{\Phi}_{klij} = (\lambda + 2\mu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + 2\lambda\ \delta_{kl}\delta_{ij} \\ D_S = \mu|\xi|^2 - \rho\omega^2 ;\ D_P = (\lambda + 2\mu)|\xi|^2 - \rho\omega^2 \end{array}$$

and $\sim$ indicates an equality modulo $\xi_{i,j,k,l}$. These formulae and Theorem 2 determine Φ. Because of the isotropy, however, one can obtain a simpler expression for the stress function :

$$\text{(V-3)} \quad \left\{\begin{array}{l} \Phi_{ij}^{kl}(x) = -\mu^2\, \tilde{\Phi}_{ijkl}\, h(x) \\ R_{ij}^{kl}(x) = -\, \tilde{R}_{ij}^{kl}(\partial)\, h(x) \end{array}\right.$$

where $\tilde{\Phi}$ is given in (V-2) and

$$\text{(V-4)} \quad \left\{\begin{array}{l} \tilde{R}_{ij}^{kl}(\xi) = \mu\Big\{|\xi|^2\big(3\mu\delta_{ij}\delta_{kl} + (\lambda + 3\mu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\big) \\ \quad - 2\lambda(\delta_{ij}\xi_k\xi_l + \delta_{kl}\xi_i\xi_j) - \mu\left(\delta_{jk}\xi_i\xi_l + \delta_{jl}\xi_i\xi_k + \delta_{ik}\xi_j\xi_l + \delta_{il}\xi_j\xi_k\right)\Big\} \\ \quad - \rho\omega^2\left(\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) \\ h(x) = \dfrac{1}{4\pi(\lambda + \mu)\rho\omega^2}\left(\dfrac{e^{ik_T|x|}}{|x|} - \dfrac{e^{ik_L|x|}}{|x|}\right) \\ k_T = \sqrt{\dfrac{\rho\omega^2}{\mu}}\ ;\ k_L = \sqrt{\dfrac{\rho\omega^2}{\lambda + 2\mu}} \end{array}\right.$$

These results coincide with the expressions obtained by Nishimura & Kobayashi [8]. Notice that the stress function given here has the same symmetry as possessed by the elasticity constant. A lengthy but straightforward calculation shows, however, that

$$\begin{array}{ll} e_{i_1 j_1 k_1}\cdots e_{i_4 j_4 k_4}\, \xi_{j_1}\cdots\xi_{j_4}\, a_{k_1\ldots k_4} = 0 & \\ a_{ijkl} = a_{jikl} = a_{klij} & a_{ijkl} : \text{constant} \end{array}$$

imply $a_{ijkl} = 0$. In this sense the symmetric decomposition given above is unique. This decomposition and the general formulae in (IV-1,2) yield a variational form for the present case.

VI - CONCLUDING REMARKS

The decomposition of Σ into a sum of a stress function part and a weakly singular part R is useful with the variational method as well as with the collocation one. It is noted that the collocation requires a C^1 element, while the variational formulation works with a finite element of class C^0.

In the isotropic case, all the kernels are explicit. An application of the present formulation, in this case, to the collocation method in the frequency domain and in the time domain can be found in [8], [9]. A variational formulation in the time domain has been derived from this decomposition, again in the isotropic case, in [2].

There are attempts at the use of non-regularized kernels with a numerical integration formula for hypersingular functions. For references see Martin & Rizzo [6].

An explicit expression for $\hat{\Phi}$ is given in [3]. However in the general anisotropic case we need some extra works to obtain explicitly Φ and R.

REFERENCES

[1] Bamberger, A. (1983) Approximation de la diffraction d'ondes élastiques : une nouvelle approche (I), (II), (III), Internal report n°91, 96, 98 of Centre de Mathématiques appliquées, Ecole Polytechnique, France.
[2] Becache, E & Ha Duong, T (1989) Formulation Variationnelle Espace - Temps Associée au Potentiel de Double Couche des Ondes Elastiques, Internal report n°199 of Centre de Mathématiques appliquées, Ecole Polytechnique, France.
[3] Becache, E , Nedelec , J-C & Nishimura, N (1989) Regularization in 3D for anisotropic elastodynamic crack and obstacle problems, Internal report n°205 of Centre de Mathématiques appliquées, Ecole Polytechnique, France.
[4] Bonnet, M. (1987) Méthode des équations intégrales régularisées en élastodynamique, Bulletin de la direction des études et recherches, EDF, France.
[5] Bui, H.D.(1977) An integral equations method for solving the problems of a plane crack of arbitrary shape, J. Mech. Phys. Solids, 25, 29-39.
[6] Martin, P.A. & Rizzo, F.J. (1989) On boundary integral equations for crack problems, Proc. Roy. Soc. London (A) 421, 341-355.
[7] Nedelec, J.-C. (1983) Le potentiel de double couche pour les ondes élastiques, Internal report n°99 of Centre de Mathématiques appliquées, Ecole Polytechnique, France.
[8] Nishimura, N. & Kobayashi, S. (1989) A regularized boundary integral integral equation method for elastodynamic crack problems, Comp. Mech. ,4, 319-328.
[9] Nishimura, N., Guo, C & Kobayashi, S. (1987) Boundary Integral Equation Methods in Elastodynamic Crack Problems , Proc. 9th Int Conf BEM, vol2, ed Brebbia, Wendland, Kuhn, Springer Verlag, 279-291.
[10] Polch, E.Z. , Cruse, T.A & Huang, C.-J (1987) Traction BIE solutions for flat cracks, Comp. Mech., 2, 253-267.
[11] Sladek, V. & Sladek, J. (1984) Transient elastodynamic three-dimensional problems in cracked bodies, Appl. Math. Model., 8,2-10.