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Mathematics

Cauchy integrals on Lipschitz curves and related operators

(commutators/singular integrals/weighted inequalities)

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ABSTRACT In this note, we establish certain properties of the Cauchy integral on Lipschitz curves and prove the L^p-boundedness of some related operators. In particular, we obtain the recent results of R. R. Coifman and Y. Meyer [(1976) "Commutateurs d'intégrales singulières:" *Analyse harmonique d'Orsay* n° 211, Université Paris XI] on the continuity of the so-called commutator operators.

1. The Cauchy integral

THEOREM 1. *Let* Γ *be a curve in the complex plane given by the equation* $z(t) = t + i\varphi(t)$, *where* $\varphi(t)$ *is a real-valued function on the real line with a bounded derivative, and let*

$$A_{\varphi,\epsilon}f = \frac{1}{2\pi i}\int_{|s-t|>\epsilon} \frac{f(s)}{z(s)-z(t)}\,dz(s),\ \epsilon > 0.$$

Then there exists a positive number α *such that* $\|\varphi'\|_\infty < \alpha$ *implies that the operator* $\sup_\epsilon |A_{\varphi,\epsilon}f|$ *is of weak type (1,1) and bounded in* L^p, $1 < p < \infty$, *and that* $\lim_{\epsilon\to 0} A_{\varphi,\epsilon}f$ *exists pointwise almost everywhere for* f *in* L^p, $1 \le p < \infty$.

Proof: We shall first consider the case in which $\varphi(t)$ is infinitely differentiable, and has compact support and shall show that the operator $A_\varphi f = \lim_{\epsilon\to 0} A_{\varphi,\epsilon}f$ is bounded in L^2, and has a norm which can be estimated in terms of a bound for $\varphi'(t)$ alone provided that $\|\varphi'\|_\infty < \alpha$. Once this is established the results stated above will follow by applying standard results and techniques.

Let then $\varphi(t) \in C_0^\infty$, $|\varphi'(t)| \le M$, and consider the operators

$$A_{\lambda\varphi}f = A(\lambda)f = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{|s-t|>\epsilon}\frac{f(s)}{z_\lambda(s)-z_\lambda(t)}\,dz_\lambda(s),$$

where $z_\lambda(t) = t + i\lambda\varphi(t)$, $0 \le \lambda \le 1$, and

$$B(\lambda)f = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{|t-s|>\epsilon}\left[\frac{i[\varphi(t)-\varphi(s)]}{[z_\lambda(s)-z_\lambda(t)]^2} + \frac{i}{[z_\lambda(s)-z_\lambda(t)]}\frac{\varphi'(s)}{z_\lambda'(s)}\right]f(s)\,dz_\lambda(s).$$

They are well-defined, at least for $f \in C_0^\infty$, and in this case $A(\lambda)f$ and $B(\lambda)f$ are continuous functions of t. The operator $B(\lambda)$ is obtained as the formal derivative of $A(\lambda)$ with respect to λ, but since for $\epsilon > 0$ the integral in the definition of $B(\lambda)$ is the derivative of the one in the definition of $A(\lambda)$ and they converge uniformly as $\epsilon \to 0$, we have indeed

$$A(\lambda)f = A(0)f + \int_0^\lambda B(s)f\,ds,\quad f \in C_0^\infty. \qquad [1]$$

On the other hand, $A(\lambda)$ is uniformly bounded in L^2. To see this, we write the kernel of $A(\lambda)$ as

$$\frac{1}{s-t}[1 + k(\lambda,s,t)]$$

where, as is readily verified, $k(\lambda,s,t)$ is infinitely differentiable and has a double Fourier transform $h(\lambda,u,v)$ which is integrable uniformly in λ. Expressing k in terms of h, and using the uniform boundedness in L^2 of the truncated Hilbert transform and Minkowski's integral inequality, we obtain the desired result.

Our goal is to estimate the norm of $B(\lambda)$ in terms of $A(\lambda)$ and M, that is, a bound for $\varphi'(t)$. This in conjunction with Eq. 1 will give us an estimate for the norm of $A(\lambda)$ in terms of M alone.

Let O_1 and O_2 be the open subsets of the complex plane consisting of the points lying above and below the curve Γ, respectively. With a function $f(t)$ in C_0^∞, we associate the functions $F_1(w)$ and $F_2(w)$, analytic in O_1 and O_2, respectively, given by

$$F_j(w) = \frac{(-1)^{j+1}}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(s)}{z(s)-w}\,dz(s),\quad w \in O_j. \qquad [2]$$

It is not difficult to see that these functions extend as C^∞ functions to the curve Γ and that

$$F_j(z(t)) = \frac{1}{2}f(t) \pm A(1)f. \qquad [3]$$

Consequently we have

$$f(t) = F_1(z(t)) + F_2(z(t))$$
$$\|F_j(z(t))\|_2 \le \left(\frac{1}{2} + \|A(1)\|\right)\|f\|_2. \qquad [4]$$

We now introduce the operator

$$C_\delta f = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{i[\varphi(t)-\varphi(s)]}{[z(s)-z(t)-i\delta]^2}f(s)\,dz(s),\qquad \delta > 0,\quad f \in C_0^\infty.$$

As $\delta \to 0$, $C_\delta f$ converges in the mean of order 2 to

$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{|s-t|>\epsilon}\frac{i[\varphi(t)-\varphi(s)]}{[z(s)-z(t)]^2}f(s)\,dz(s) - \frac{i}{2}\varphi'(t)z'(t)^{-1}f(t)$$

so that

$$B(1)f = \lim_{\delta\to 0} C_\delta f + \frac{1}{2}\varphi'(t)z'(t)^{-1}f(t) + A(1)i\varphi'(t)z'(t)^{-1}f(t). \qquad [5]$$

To estimate the norm of the operator $Cf = \lim_{\delta\to 0} C_\delta f$, we consider the bilinear form

$$L(f,g) = \int_{-\infty}^{+\infty} gCf\,dz(t) = \lim_{\delta\to 0}\int_{-\infty}^{+\infty} gC_\delta f\,dz(t),\qquad g \in C_0^\infty.$$

Setting

$$\varphi(t) = \int_{-\infty}^{+\infty} e(t-u)\varphi'(u)\,du$$

where $e(t)$ is the characteristic function of $t \geq 0$, substituting in the expression for $C_\delta f$ above and interchanging the order of integration we obtain

$$2\pi i L(f,g) = \lim_{\delta \to 0} \int_{-\infty}^{+\infty} \varphi'(u) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \times \frac{i[e(t-u) - e(s-u)]}{[z(s) - z(t) - i\delta]^2} f(s)g(t)dz(s)dz(t)du.$$

To calculate the inner double integral, we consider the functions $F_j(w)$ and $G_j(w)$ associated with f and g as in Eq. 2 and replace f and g by

$$f(s) = F_1(z(s)) + F_2(z(s))$$
$$g(t) = G_1(z(t)) + G_2(z(t)).$$

Observing that

$$\int_{-\infty}^{+\infty} \frac{1}{[z(s) - z(t) - i\delta]^2} F_j(z(s))dz(s) = \begin{cases} 2\pi i F_1'(z(t) + i\delta) & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases}$$

and similarly for G_j, and denoting by Γ_u the path $z = t + i\varphi(t)$, $t \geq u$, we obtain

$$L(f,g) = \lim_{\delta \to 0} \times \int_{-\infty}^{+\infty} \varphi'(u) i \int_{\Gamma_u} [(G_1(z) + G_2(z))F_1'(z + i\delta) + (F_1(z) + F_2(z))G_2'(z - i\delta)]dzdu.$$

Because $\varphi'(u)$ has compact support, and the functions F_j and G_j and their derivatives are continuous, and $O(z^{-1})$ as $z \to \infty$, we can pass to the limit under the integral signs above. Thus, denoting by $H_1(z)$ and $H_2(z)$ the analytic functions in O_1 and O_2, respectively, such that $H_1' = F_1'G_1$, $H_2' = F_2G_2'$, and tending to zero at infinity, we find that

$$L(f,g) = -i \int_{-\infty}^{+\infty} \varphi'(u)[F_1(z(u))G_2(z(u)) + H_1(z(u)) + H_2(z(u))]du. \quad [6]$$

We proceed now to estimate the functions H_j. For this purpose we let $\phi(z)$ be a function mapping the upper halfplane $R(z) > 0$ conformally onto O_1 and such that $\phi(z) \simeq z$ as $z \to \infty$. Because the boundary of O_1 is a C^∞ curve, $\phi(z)$ and $\phi'(z)$ extend continuously to $R(z) \geq 0$ and $\phi'(z)$ does not vanish in $R(z) \geq 0$ and is bounded there. Furthermore, the vector $\phi'(t)$ is tangent to the curve Γ at the point $\phi(t)$ and therefore

$$|\arg \phi'(t)| \leq \arctan M.$$

Taking the logarithm of $\phi'(z)$ and using the maximum principle, we find that the preceding inequality is satisfied at all points of $R(z) \geq 0$. As is readily seen, this implies that

$$0 \leq R(\phi'(z)) \leq |\phi'(z)| \leq R(\phi'(z))(1 + M^2)^{1/2}.$$

Now let I be an interval $(s - a, s + a)$. Then, since $R(\phi'(t)) \geq 0$, by comparing with the Poisson integral we find that

$$\frac{1}{|I|} \int_I R(\phi'(t))dt \leq \pi R(\phi'(s + ia)),$$

and this combined with the preceding inequality gives

$$\frac{1}{|I|} \int_I |\phi'(t)|dt \leq \pi |\phi'(s + ia)|(1 + M^2)^{1/2}.$$

Evidently, the same inequality holds for $\phi'(z)^{-1}$, and multiplying we find that

$$\left[\int_I |\phi'(t)|dt\right]\left[\int_I |\phi'(t)|^{-1}dt\right] \leq \pi^2|I|^2(1 + M^2)$$

that is, the function $|\phi'(t)|$ belongs to the class A_2 of Muckenhoupt with constant $\pi^2(M^2 + 1)$.

Returning to the function H_1, let us consider the functions $G_1(\phi(z))$, $F_1(\phi(z))$, $H_1(\phi(z))$ in the upper halfplane and the associated maximal and Lusin functions

$$m(t) = \sup_{|u| \leq v} |G_1(\phi(t + u + iv))|$$
$$S(t)^2 = \int_{|u| \leq v} \left|\frac{d}{dz} F_1(\phi(t + u + iv))\right|^2 dudv$$
$$\bar{S}(t)^2 = \int_{|u| \leq v} \left[\frac{d}{dz} H_1(\phi(t + u + iv))\right|^2 dudv.$$

Because

$$\frac{d}{dz} H_1(\phi(z)) = G_1(\phi(z)) \frac{d}{dz} F_1(\phi(z))$$

we have $\bar{S}(t) \leq m(t)S(t)$ and

$$\int_{-\infty}^{+\infty} \bar{S}(t)|\phi'(t)|dt \leq \left[\int_{-\infty}^{+\infty} m(t)^2|\phi'(t)|dt\right]^{1/2}\left[\int_{-\infty}^{+\infty} S(t)^2|\phi'(t)|dt\right]^{1/2}.$$

But the function $|\phi'(t)|$ is in the class A_2 and consequently (see ref. 5)

$$\int_{-\infty}^{+\infty} |H_1(z(t))||z'(t)|dt = \int_{-\infty}^{+\infty} |H_1(\phi(t))||\phi'(t)|dt \leq c_M \int_{-\infty}^{+\infty} \bar{S}(t)|\phi'(t)|dt$$

$$\int_{-\infty}^{+\infty} m(t)^2|\phi'(t)|dt \leq c_M \int_{-\infty}^{+\infty} |G_1(\phi(t))|^2|\phi'(t)|dt = c_M \int_{-\infty}^{+\infty} |G_1(z(t))|^2|z'(t)|dt$$

$$\int_{-\infty}^{+\infty} S(t)^2|\phi'(t)|dt \leq c_M \int_{-\infty}^{+\infty} |F_1(\phi(t))|^2|\phi'(t)|dt = c_M \int_{-\infty}^{+\infty} |F_1(z(t))|^2|z'(t)|dt$$

in which c_M is a constant depending on M and, as above, $z(t) = t + i\varphi(t)$. Because $1 \leq |z'(t)| \leq (1 + M^2)^{1/2}$, combining these inequalities we obtain

$$\|H_1(z(t))\| \leq c_M \|F_1(z(t))\|_2 \|G_1(z(t))\|_2,$$

where c_M is another constant depending only on M. Clearly, a similar inequality is valid for H_2. Thus Eqs. 4 and 6 yield

$$|L(f,g)| \leq c_M \left(\frac{1}{2} + \|A(1)\|\right)^2 \|f\|_2 \|g\|_2,$$

which implies that

$$\|C\| \leq c_M \left(\frac{1}{2} + \|A(1)\|\right)^2,$$

and from this and Eq. 5 it follows that

$$\|B(1)\| \leq c_M \left(\frac{1}{2} + \|A(1)\|\right)^2,$$

in which c_M is still another constant depending only on M.

Evidently, the preceding argument and this last inequality are valid for $A(\lambda)$ and $B(\lambda)$, $0 \le \lambda \le 1$. Thus, Eq. **1** implies that

$$\|A(\lambda)\| \le \|A(0)\| + c_M \int_0^\lambda \left(\frac{1}{2} + \|A(s)\|^2\right) ds, \qquad [7]$$

where, as above, the norms are norms of operators in L^2. But $A(0)$ is just half the ordinary Hilbert transform so that $\|A(0)\| = \frac{1}{2}$. Consequently, the function $\|A(\lambda)\|$, $0 \le \lambda \le 1$, is majorized by the solution of the differential equation

$$y' = c_M \left(\frac{1}{2} + y\right)^2, \quad y(0) = \frac{1}{2},$$

that is

$$\|A(\lambda)\| \le (1 - c_M\lambda)^{-1} - \frac{1}{2}, \quad 0 \le \lambda < c_M{}^{-1}, \lambda \le 1.$$

Let now $\alpha = \sup M c_M{}^{-1}$, where the supremum is taken over all $M \ge 0$ and the corresponding constants c_M, $c_M \ge 1$, for which Eq. **7** holds with $\|\varphi'\|_\infty \le M$. Then if $\|\overline{\varphi}'\|_\infty < \alpha$, and $\|\overline{\varphi}'\|_\infty < M c_M{}^{-1}$, setting $\lambda = \|\overline{\varphi}'\|_\infty M^{-1} < c_M{}^{-1} \le 1$ and $\overline{\varphi} = \lambda\varphi$, the preceding inequality becomes

$$\|A_{\overline{\varphi}}\| \le (1 - c_M\lambda)^{-1} - \frac{1}{2} = (1 - \|\overline{\varphi}'\|_\infty M^{-1} c_M) - \frac{1}{2},$$

and letting $M c_M{}^{-1}$ tend to α we finally obtain

$$\|A_{\overline{\varphi}}\| \le (1 - \|\overline{\varphi}'\|_\infty \alpha^{-1})^{-1} - \frac{1}{2}, \quad \|\overline{\varphi}'\|_\infty < \alpha. \qquad [8]$$

To complete the proof of our theorem, we shall henceforth consider only functions φ with $\|\varphi'\|_\infty < \alpha$. Because the techniques we shall employ are standard we will merely outline our argument.

First, we show that A_φ is continuous in L^p for $1 < p < \infty$. For this purpose, we let f be a function with vanishing integral and supported in the interval $|t - t_0| \le \frac{1}{2}\delta$. Then a simple calculation shows that

$$\int_{|t-t_0|>\delta} |A_\varphi(f/z')(t)|\, dt \le c\|f\|_1.$$

Hence, from Eq. **8** and *Theorem 1* in ref. 1 we conclude that

$$\|A_\varphi(f/z')\|_p \le c\|f\|_p, \quad 1 < p < 2,$$

with c depending only on p and $\|\varphi'\|_\infty$. Now, the standard duality argument shows that the same result holds for $2 < p < \infty$. Consequently, A_φ is bounded in L^p, $1 < p < \infty$.

Next consider the following operators with φ still in $C_0{}^\infty$:

$$A^{(1)}f = \int \delta(t)^{-1}\eta((t-s)\epsilon(t)^{-1})(A_\varphi f/z')(s)dz(s)$$
$$A^{(2)}f = \int [1 - \eta((t-s)\epsilon^{-1}(t))](z(s) - z(t))^{-1}f(s)ds$$
$$A^{(3)}f = \int_{|s-t|\ge\epsilon(t)} (z(s) - z(t))^{-1}f(s)ds,$$

where $\eta(t)$ is an even non-negative function in $C_0{}^\infty$ which equals 1 near the origin, $\epsilon(t)$ is an arbitrary positive measurable function of t and

$$\delta(t) = \int \eta((t-s)\epsilon(t)^{-1})dz(s).$$

Then the following inequalities are readily verified:

$$\int_{|t-t_0|>\rho} |A^{(2)}f|dt \le c\|f\|_1, \qquad [9]$$

provided that $\int f dt = 0$ and $f(t) = 0$ for $|t - t_0| > \frac{1}{2}\rho$

$$c\epsilon(t) \le |\delta(t)| \le c^{-1}\epsilon(t) \qquad [10]$$

$$\|A^{(1)}f\|_p \le c\|f\|_p, \quad 1 < p < \infty \qquad [11]$$

$$|A^{(2)}f - A^{(3)}f| \le cm(f), \qquad [12]$$

where the constants c in Eqs. **9**, **10**, and **12** can be taken so as to depend only on the function η, and the one in Eq. **11** so as to depend on this function and on $\|\varphi'\|_\infty$, and where $m(f)$ is the ordinary Hardy-Littlewood maximal function of f.

Furthermore, we have

$$\|A^{(1)}f - A^{(3)}f\|_p \le c\|f\|_p, \quad 1 < p < \infty \qquad [13]$$

with c depending only on p and the function $\eta(t)$. To see this, we write

$$(A^{(1)}f - A^{(3)}f)(t) = \int k(t,u)f(u)du$$

and for simplicity assume that $\eta(t) = 0$ for $|t| \ge 1$ and $\eta(t) = 1$ for $|t| \le \frac{1}{2}$. Then if $|t - u| > 2\epsilon(t)$,

$$\begin{aligned} k(t,u) &= \int \delta(t)^{-1}\eta((t-s)\epsilon(t)^{-1})(z(u) - z(s))^{-1}dz(s) - (z(u) - z(t))^{-1} \\ &= \int \delta(t)^{-1}\eta((t-s)\epsilon(t)^{-1})[(z(u) - z(s))^{-1} - (z(u) - z(t))^{-1}]dz(s). \end{aligned}$$

Because $|z'| \le 1 + \alpha$ and α is finite (see the remark below), this last integral is readily seen to be majorized by $c\epsilon(t)(t - u)^{-2}$ with c depending only on the function η. If on the other hand, $\epsilon(t) \le |t - u| \le 2\epsilon(t)$, then

$$\begin{aligned} k(t,u) = &\int_{|s-t|<2\epsilon(t)} \delta(t)^{-1}[\eta((t-s)\epsilon(t)^{-1}) - \eta((t-u)\epsilon(t)^{-1})](z(u) - z(t))^{-1}dz(s) \\ &+ \delta(t)^{-1}\eta((t-u)\epsilon(t)^{-1}) \lim_{\delta\to 0} \int_{\delta \le |s-t| < 2\epsilon(t)} \\ &\times (z(u) - z(s))^{-1}dz(s) + (z(u) - z(t))^{-1} \end{aligned}$$

and estimating the first integral in terms of η' and taking Eq. **10** into account, we find that in this case $k(t,u)$ is majorized by $c\epsilon(t)^{-1}$. Finally, if $|t - u| < \epsilon(t)$ we have the same preceding expression for $k(t,u)$ with the last term omitted so that the same estimate as in the preceding case holds. From all this there follows

$$|k(t,u)| \le c\epsilon(t)((t-u)^2 + \epsilon(t)^2)^{-1},$$

which implies that

$$|A^{(1)}f - A^{(3)}f| \le cm(f)$$

where $m(f)$ is the Hardy-Littlewood maximal function of f, and this in turn clearly implies Eq. **13**.

We are now near the completion of our proof. From Eqs. **11**, **12**, and **13** there follows that $A^{(2)}$ is bounded in L^p, $1 < p < \infty$ with a norm that can be estimated in terms of p and $\|\varphi'\|_\infty$ only, and this combined with Eq. **9** and Theorem 1 in ref. 1 implies that $A^{(2)}$ is also of weak type (1,1). But then Eq. **12** allows us to conclude that the same result holds for $A^{(3)}$.

So far we have assumed that φ is a function in $C_0{}^\infty$, but since all preceding estimates depend on $\|\varphi'\|_\infty$ only, as far as their dependence on φ is concerned, a passage to the limit shows that the same results and estimates hold for operators involving general functions φ with $\|\varphi'\|_\infty < \alpha$. Furthermore, the estimates for $A^{(3)}$ are independent of the function $\epsilon(t)$, which is positive measurable but otherwise arbitrary, and this implies that sup $|A_{\varphi,\epsilon}f|$ is of weak type (1,1) and strong type (p,p), $1 < p < \infty$, whenever φ is a Lipschitz function with $\|\varphi'\|_\infty < \alpha$.

Finally, to prove the pointwise existence of $\lim_{\epsilon\to 0} A_{\varphi,\epsilon}f$, we observe that this limit clearly exists at t_0 if f is in C_0^∞ and $f(t_0) = 0$, or if f is in C_0^∞, f is constant near t_0 and $\varphi'(t_0)$ exists. From this we conclude that if f is in C_0^∞ then $\lim_{\epsilon\to 0} A_{\varphi,\epsilon}f$ exists almost everywhere, whence the general result follows from the fact that sup $|A_{\varphi,\epsilon}f|$ is of weak type (1,1) and strong type (p,p), $1 < p < \infty$.

Remark: The condition $\|\varphi'\|_\infty < \alpha$ for the validity of our theorem could be removed by showing that $\sup_{c_M \geq 1} Mc_M^{-1} = \alpha = \infty$. Our method, however, cannot possibly yield this result. Indeed, the inequality

$$\|B(\lambda)\| \leq c_M \left(\frac{1}{2} + \|A(\lambda)\|\right)^2$$

must hold, and setting $\lambda = 0$, $\varphi = M\psi$ with ψ fixed and $\|\psi'\|_\infty = 1$ we find that $c_M \geq \|B(0)\| > cM$, $c > 0$.

2. Related integral operators

THEOREM 2. *Let* F(z) *be analytic in the disc* $|z| < R$ *and* φ *a real Lipschitz function on the real line such that* $\|\varphi'\|_\infty < R\alpha(1 + \alpha^2)^{-1/2}$, *where* α *is as in Theorem 1. Let*

$$L_\epsilon f = \int_{|s-t|>\epsilon} \frac{1}{s-t} F\left(\frac{\varphi(s) - \varphi(t)}{s-t}\right) f(s)ds.$$

Then the operator $\sup_\epsilon |L_\epsilon f|$ *is of weak type* (1,1) *and strong type* (p,p), $1 < p < \infty$, *and* $\lim_{\epsilon\to 0} (L_\epsilon f)(t)$ *exists almost everywhere for* f *in* L^p, $1 \leq p < \infty$.

For lack of space, we merely outline the proof. Setting

$$A_{z,\epsilon}f = \int_{|s-t|>\epsilon} [s - t - z^{-1}(\varphi(s) - \varphi(t))]^{-1} f(s)ds, \quad |z| = \rho < R, \qquad [14]$$

where ρ is sufficiently close to R, we have

$$(L_\epsilon f)(t) = \int_{|z|=\rho} F(z)(A_{z,\epsilon}f)(t)\frac{dz}{z}. \qquad [15]$$

If $z^{-1} = u + iv$, setting $\bar{t} = t - u\varphi(t)$, $\bar{s} = s - u\varphi(s)$ in Eq. **1**, we find that $A_{z,\epsilon}f$ can be expressed by an integral like the one in *Theorem 1* plus a term which is majorized by the maximal function $m(f)$ of f and tends to zero at every point where $m(f)$ is finite and φ' exists (the presence of this term is due to the fact that the inverse image of an interval centered at $\bar{t}$ under the mapping $s \to \bar{s}$ is not necessarily centered at t). From this there follows that if f is in L^p, $1 < p < \infty$, then $\|\sup|A_{z,\epsilon}f|\|_p \leq c\|f\|_p$ with c independent of z and $\lim_{\epsilon\to 0} A_{z,\epsilon}f$ exists almost everywhere, and this combined with Eq. **2** shows that $\sup_\epsilon |L_\epsilon f|$ is of strong type (p,p) and that $\lim_{\epsilon\to 0} L_\epsilon f$ exists almost everywhere for f in L^p, $1 < p < \infty$. To complete the proof, one now argues as in the last part of the proof of *Theorem 1*. For additional references see refs. 2–4.

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1. Benedek, A., Calderón, A. P. & Panzone, R. (1962) "Convolution operators on Banach space valued functions," *Proc. Natl. Acad. Sci. USA* **48,** 356–365.
2. Calderón, A. P. (1965) "Commutators of singular integral operators," *Proc. Natl. Acad. Sci. USA* **53,** 1092–1099.
3. Coifman, R. R. & Meyer, Y. (1975) "On commutators of singular integrals and bilinear singular integrals," *Trans. Am. Math. Soc.* **212,** 315–331.
4. Coifman, R. R. & Meyer, Y. (1976) "*Commutateurs d'intégrales singulières:*" *Analyse harmonique d'Orsay,* n° 211., Université Paris XI.
5. Gundy, R. P. & Wheeden, R. L. (1975) "Weighted integral inequalities for the non-tangential maximal function, Lusin area integral, and Walsh-Paley series," *Studia Math.* **49,** 107–124.