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On the numerical solution of the direct scattering problem for an open sound-hard arc[1]

Lars Mönch

Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany

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Abstract

We derive a hypersingular integral equation, which is equivalent to the scattering problem. Using the cosine substitution, we obtain an integral equation which is closely related to the integral equation for the case of a closed boundary. We solve this equation approximately by a quadrature formula method and obtain pointwise error estimates in Hölder norms by using a perturbation argument.

Keywords: Hypersingular integral equation on an open arc; Quadrature formula method

AMS classification: 35J05, 35P25, 45A05, 65R20

1. Introduction

The mathematical modeling of scattering phenomena for time-harmonic acoustic waves leads to boundary value problems for the Helmholtz equation. In this paper we want to describe a numerical solution method for the scattering of acoustic waves by a sound-hard open arc. Problems of this type arise in the analysis of cracks. We found a number of papers dealing with this problem, for example [9, 11, 14, 16, 20, 22, 23]. The papers [11] and [16] also describe efficient numerical methods. The corresponding problem for closed boundary curves is solved in [6, 15]. On the other hand, there exists no complete description of a numerical solution method including a rigorous convergence and error analysis for the case of an open arc. We want to use a boundary integral equation approach for solving the scattering problem. According to the singular behaviour of the gradient of the solution at the end points of the arc (cf. [20]) it seems to make sense to use the so-called cosine substitution to remove the singularities of the first derivative of the solution of the hypersingular integral equation. After the substitution, the integral equation turns out to be essentially the same as in the case

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of a closed boundary curve. The cosine substitution is also known from the study of singular integral equations and integro-differential equations in L^p-spaces with weight [17]. In recent times, analogous problems concerning the Laplace equation with Dirichlet boundary conditions and their efficient numerical treatment were stated and solved in a number of papers (cf. [19, 21]). These problems lead to integral equations with logarithmic kernel. For scattering from an open sound-soft arc, a numerical solution method is derived in [5]. In the context of Newton-type methods for solving the corresponding inverse problem, a very efficient method for solving the direct problem is necessary (cf. [5]), because in each Newton step one has to solve a number of direct problems with different boundary values.

The main purpose of this paper is to derive a numerical solution method for the direct scattering problem via a boundary integral equation approach and the presentation of the corresponding error and convergence analysis. This paper extends results derived in [15] for the case of a closed boundary curve. We avoid the discussion of the hypersingular integral equation as an equation involving finite-part integrals (cf. [14]). It seems to be possible to extend our analysis to systems of smooth open arcs and to piecewise smooth open arcs. The plan of the paper is as follows. In Section 2 we will describe the direct scattering problem and the way to reduce it to an one-dimensional hypersingular integral equation. By using the cosine transformation we obtain an existence proof for a solution of the scattering problem which is rather concise as compared with the work of [9]. In Section 3 we will develop a fully discrete numerical method for solving the hypersingular integral equation by combining quadrature and collocation methods. In Section 4 we will derive a convergence and error analysis by using a simple perturbation argument as suggested in [7]. For analytic arcs we obtain exponentially decreasing error rates so that it seems to be possible to use this quadrature method in each Newton step for the numerical solution of the corresponding inverse problem. Finally, in the last section we will present a numerical example which demonstrates the fast convergence rate of our method for smooth open arcs.

2. The direct scattering problem

Let Γ be an arc of class C^∞ with counterclockwise orientation, i.e.

$$\Gamma := \{\, \gamma(s) \mid s \in [-1,1] \,\},$$

where the parametrization γ is C^∞-smooth and injective. We denote the end points of the arc by x_{-1}^* and x_1^*.

The mathematical modeling of the scattering of acoustic waves by a sound-hard open arc Γ leads to the following boundary value problem for the Helmholtz equation: Given an incident wave $u^{\mathrm{i}}(x) := \mathrm{e}^{\mathrm{i}\kappa\langle x,d\rangle}$ with wave number $\kappa > 0$ and with a unit vector d giving the direction of propagation, find a function $u = u^{\mathrm{i}} + u^{\mathrm{s}} \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\overline{\mathbb{R}^2 \setminus \Gamma})$, which is continuous in x_i^*, $i = -1, 1$ with

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma \tag{1}$$

and which satisfies the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0 := \Gamma \setminus \{x_{-1}^*, x_1^*\}. \tag{2}$$

For the scattered wave u^s, we demand that the Sommerfeld radiation condition

$$\lim_{r\to\infty}\left(\frac{\partial u^s}{\partial r}-\mathrm{i}\kappa u^s\right)=\mathrm{o}\left(\frac{1}{\sqrt{r}}\right),\quad r=|x|, \tag{3}$$

holds uniformly in all directions $\hat{x}:=\frac{x}{|x|}$. Note that we do not require any "edge condition" for the behaviour of u at the end points of Γ. First we want to prove a uniqueness result. We need the following lemma stating that the application of Green's theorem is allowed in the present situation (for a more general situation also cf. [10]). In its proof we use an idea due to Chandler-Wilde [1] for the extension of the Heinz approximation approach to the case of an arc with Neumann boundary condition. By Ω_R we denote the closed disc of radius R with the center at the origin. Throughout this paper we will denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product on $\mathbb{R}^2$.

Lemma 2.1. *Let u be a solution to the direct scattering problem for an open sound-hard arc. Then we have for R large enough* $\operatorname{grad} u\in L^2(\Omega_R)$ *and the following version of Green's theorem holds*:

$$\int_{\Omega_R}|\operatorname{grad}u(y)|^2\,\mathrm{d}y-\kappa^2\int_{\Omega_R}|u(y)|^2\,\mathrm{d}y=\int_{\partial\Omega_R}u\frac{\partial\bar{u}(y)}{\partial n}\,\mathrm{d}s(y). \tag{4}$$

Proof. We choose a function $v\in C^2(\mathbb{R}^2)$ with $v(x_i^*)=u(x_i^*)$, $i=-1,1$ and consider $w:=u-v$ instead of u. The idea to incorporate such a function v is due to Chandler-Wilde [1]. Then we can proceed as in the case of a closed boundary curve (cf. [2, Lemma 3.8]). Due to space limits we only want to indicate the basic idea. We consider an odd, monotone increasing function $\psi\in C^1(\mathbb{R})$ with $\psi(s)=s$ for $s\geqslant 2$ and $\psi(s)=0$ for $0\leqslant s\leqslant 1$. We set $w_n:=\frac{\psi(n\operatorname{Re}w)}{n}+\mathrm{i}\frac{\psi(n\operatorname{Im}w)}{n}$. Note that because of our choice of v the functions w_n vanish near the end points of the arc. We use Green's theorem to obtain:

$$\int_{\Omega_R}\{\langle\operatorname{grad}w_n(y),\operatorname{grad}\bar{w}(y)\rangle+w_n(y)\Delta\bar{w}(y)\}\,\mathrm{d}y=\int_{\partial\Omega_R}w_n(y)\frac{\partial\bar{w}(y)}{\partial n}\,\mathrm{d}s(y).$$

With the help of Fatou's lemma and the dominated convergence theorem of Lebesgue we can pass to the limit. Since the application of Green's theorem is clearly valid for v we obtain the statement of the lemma. □

With the aid of Lemma 2.1 we are able to prove the following uniqueness theorem.

Theorem 2.2. *The direct scattering problem for an open sound-hard arc has at most one solution.*

Proof. The proof consists in a combination of Green's theorem, the Sommerfeld radiation condition and the Rellich lemma. We omit the details of the proof because in view of Lemma 2.1 it is the same as in the case of a closed boundary curve, which can be found in [2, p. 32]. □

We seek a solution as a double layer potential:

$$u^s(x):=\int_\Gamma\frac{\partial\Phi(x,y)}{\partial n(y)}\varphi(y)\,\mathrm{d}s(y),\quad x\in\mathbb{R}^2\backslash\Gamma, \tag{5}$$

where the unknown density φ is contained in the function space

$$C^{1,\alpha,*}(\Gamma) := \left\{ \varphi \mid \varphi(x_{-1}^*) = \varphi(x_1^*) = 0,\ \frac{\mathrm{d}\varphi(\gamma(s))}{\mathrm{d}s} = \frac{\tilde{\varphi}(\arccos s)}{\sqrt{1-s^2}},\ \tilde{\varphi} \in C^{0,\alpha}[0,\pi] \right\}$$

and where

$$\Phi(x,y) := \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|), \quad x \neq y,$$

denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ in terms of the Hankel function $H_0^{(1)} = J_0^{(1)} + \mathrm{i}N_0^{(1)}$ of the first kind and order zero with the Bessel function

$$J_0^{(1)}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

and the Neumann function

$$N_0^{(1)}(z) = \frac{2}{\pi}\left(\log\frac{z}{2} + C\right) J_0^{(1)}(z) + \frac{2}{\pi}\sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} \right\} \frac{(-1)^{n+1}}{(n!)^2} \left(\frac{z}{2}\right)^{2n}$$

with the Euler's constant $C = 0.57721\ldots$. By $C^{0,\alpha}[0,\pi]$ we denote the usual Hölder space of functions over $[0,\pi]$. Note that, because of our choice of $C^{1,\alpha,*}(\Gamma)$, the potential has the correct behaviour near the end points of Γ. This can be seen using a continuation of Γ to a closed curve $\tilde{\Gamma}$ with an injective and C^∞ parametrization. Then we can express u as a double layer potential over $\tilde{\Gamma}$ with continuous density $\tilde{\varphi} := \varphi\chi_\Gamma$. Here we denote by χ_U the characteristic function of a point set $U \subset \mathbb{R}^2$. Now we obtain the continuity of u in x_i^*, $i = -1, 1$ by using regularity properties of double layer potentials over a closed smooth boundary (cf. [2, p.38]). By applying the jump relations for double layer potentials it is easy to prove that the scattering problem is equivalent to the following hypersingular integral equation of the first kind:

$$\frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)} \varphi(y)\,\mathrm{d}s(y) = f(x), \quad x \in \Gamma_0, \tag{6}$$

where the right-hand side $f := -\frac{\partial u^{\mathrm{i}}}{\partial n}$ is C^∞. In order to reduce the degree of the singularity in the hypersingular integral we apply the following relation for $x \in \Gamma_0$:

$$\frac{\partial}{\partial n(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)} \varphi(y)\,\mathrm{d}s(y)$$

$$= \int_\Gamma \frac{\partial \Phi(x,y)}{\partial t(x)} \frac{\partial \varphi(y)}{\partial t}\,\mathrm{d}s(y) + \kappa^2 \int_\Gamma \Phi(x,y)\varphi(y)\langle n(x), n(y)\rangle\,\mathrm{d}s(y).$$

In the case of a closed curve and $\kappa = 0$, a proof of this relation can be found in [4, p. 101]. Here we essentially have used the fact that for the density $\varphi(x_1^*) = \varphi(x_{-1}^*) = 0$ holds. Now we use a parametrization of the arc Γ and the so-called cosine substitution to obtain a one-dimensional integral equation with a principal part closely related to the integral operator with Hilbert kernel.

After setting $s = \cos\tau$, $\tau \in (0,\pi)$, and parametrizing (6) we obtain the following equivalent integral equation:

$$\int_0^\pi \left\{ K_1(\tau,\sigma)\frac{\mathrm{d}}{\mathrm{d}\cos\sigma}\varphi(\tilde{\gamma}(\sigma))\sin\sigma + K_2(\tau,\sigma)\varphi(\tilde{\gamma}(\sigma)) \right\} \mathrm{d}\sigma = g(\tau) \tag{7}$$

for $\tau \in (0,\pi)$ with the right-hand side

$$g(\tau) := f(\tilde{\gamma}(\tau))|\dot{\gamma}(\cos\tau)|$$

and with the following kernels:

$$K_1(\tau,\sigma) := \frac{\mathrm{i}\kappa}{4}H_0^{(1)\prime}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|)\frac{\langle\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma),\dot{\gamma}(\cos\tau)\rangle}{|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|},$$

$$K_2(\tau,\sigma) := \frac{\mathrm{i}\kappa^2}{4}\langle n(\tilde{\gamma}(\tau)), n(\tilde{\gamma}(\sigma))\rangle \Phi(\tilde{\gamma}(\sigma),\tilde{\gamma}(\tau))|\dot{\gamma}(\cos\tau)||\dot{\gamma}(\cos\sigma)|\sin\sigma$$

for $\sigma \neq \tau$, $\sigma,\tau \in (0,\pi)$. Here for abbreviation we set $\tilde{\gamma}(\tau) := \gamma(\cos\tau), \tau \in [0,\pi]$. For the kernels K_1 and K_2 we derive the following explicit expressions. First, for K_2 we can write

$$K_2(\tau,\sigma) := M_1(\tau,\sigma)\log 4(\cos\sigma - \cos\tau)^2 + M_2(\tau,\sigma), \quad \sigma \neq \tau, \quad \sigma,\tau \in (0,\pi)$$

with the kernels M_1 and M_2:

$$M_1(\tau,\sigma) := -\frac{\kappa^2}{4\pi}J_0^{(1)}(\kappa|\tilde{\gamma}(\sigma)-\tilde{\gamma}(\tau)|)\sin\sigma\langle\dot{\gamma}(\cos\sigma),\dot{\gamma}(\cos\tau)\rangle,$$

$$M_2(\tau,\sigma) := K_2(\tau,\sigma) - M_1(\tau,\sigma)\log 4(\cos\sigma - \cos\tau)^2$$

and diagonal values

$$M_1(\tau,\tau) = -\frac{\kappa^2}{4\pi}|\dot{\gamma}(\cos\tau)|^2\sin\tau,$$

$$M_2(\tau,\tau) = \kappa^2\left(\frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\log\frac{\kappa}{4}|\dot{\gamma}(\cos\tau)|\right)|\dot{\gamma}(\cos\tau)|^2\sin\tau.$$

For the derivation of $M_1(\tau,\tau)$ and $M_2(\tau,\tau)$ we used the power series representation of $J_0^{(1)}$ and $N_0^{(1)}$ and the fact, that $\frac{|\gamma(\cos\sigma)-\gamma(\cos\tau)|^2}{(\cos\sigma-\cos\tau)^2}$ can be extended as a smooth positive function for $\sigma = \tau$ as can be seen by Taylor expansion. Hence M_1 and M_2 are smooth. The investigation of K_1 is more complicated. Here we have

$$K_1(\tau,\sigma) = \frac{\mathrm{i}}{4}\frac{\mathrm{d}}{\mathrm{d}\cos\tau}H_0^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|).$$

Our goal is to extract the strong singularity of the kernel K_1. To this end we define:

$$\tilde{M}(\tau,\sigma) := \frac{\mathrm{d}^2}{\mathrm{d}\cos\tau\,\mathrm{d}\sigma}\left\{\frac{\mathrm{i}}{4}H_0^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|) + \frac{1}{4\pi}\log 4(\cos\tau - \cos\sigma)^2\right\}.$$

According to the form of the Hankel function we split the kernel $\tilde{M}$ into two parts as follows:

$$\tilde{M}(\tau,\sigma) := \tilde{M}_1(\tau,\sigma)\log 4(\cos\sigma - \cos\tau)^2 + \tilde{M}_2(\tau,\sigma), \quad \sigma \neq \tau, \quad \sigma,\tau \in (0,\pi)$$

with

$$\tilde{M}_1(\tau,\sigma) := -\frac{1}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}\cos\tau\,\mathrm{d}\sigma} J_0^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|),$$
$$\tilde{M}_2(\tau,\sigma) := \tilde{M}(\tau,\sigma) - \tilde{M}_1(\tau,\sigma)\log 4(\cos\sigma-\cos\tau)^2$$

and diagonal terms

$$\tilde{M}_1(\tau,\tau) = \frac{1}{8\pi}\kappa^2|\dot{\gamma}(\cos\tau)|^2\sin\tau,$$
$$\tilde{M}_2(\tau,\tau) = -\frac{1}{4\pi}\left\{\left(\pi\mathrm{i} - 1 - 2C - 2\log\frac{\kappa|\dot{\gamma}(\cos\tau)|}{4}\right)\kappa^2\frac{|\dot{\gamma}(\cos\tau)|^2}{2}\right.$$
$$\left. + \frac{\langle\dot{\gamma}(\cos\tau),\ddot{\gamma}(\cos\tau)\rangle^2}{|\dot{\gamma}(\cos\tau)|^4} - \frac{|\ddot{\gamma}(\cos\tau)|^2}{2|\dot{\gamma}(\cos\tau)|^2} - \frac{\langle\dot{\gamma}(\cos\tau),\dddot{\gamma}(\cos\tau)\rangle}{3|\dot{\gamma}(\cos\tau)|^2}\right\}\sin\tau.$$

Here for the calculation of the smooth continuation $\tilde{M}_2(\tau,\tau)$ of $\tilde{M}_2(\tau,\sigma)$ we used for $s \to \tilde{s}$ the Taylor expansion

$$\gamma(s) = \gamma(\tilde{s}) + \dot{\gamma}(\tilde{s})(s-\tilde{s}) + \ddot{\gamma}(\tilde{s})\frac{(s-\tilde{s})^2}{2} + \dddot{\gamma}(\tilde{s})\frac{(s-\tilde{s})^3}{6} + \mathrm{O}(|s-\tilde{s}|^4)$$

of the smooth function γ, where we denote by $\mathrm{O}(|s-\tilde{s}|^k)$ a smooth function of order $|s-\tilde{s}|^k$ for $s \to \tilde{s}$. With the help of $\varphi(x_{-1}^*) = \varphi(x_1^*) = 0$ we get:

$$\int_0^\pi K_1(\tau,\sigma)\frac{\mathrm{d}}{\mathrm{d}\cos\sigma}\varphi(\tilde{\gamma}(\sigma))\sin\sigma\,\mathrm{d}\sigma$$
$$= \int_0^\pi \left\{\frac{1}{2\pi}\frac{1}{\cos\sigma-\cos\tau}\frac{\mathrm{d}}{\mathrm{d}\cos\sigma}\varphi(\tilde{\gamma}(\sigma))\sin\sigma + \tilde{M}(\tau,\sigma)\varphi(\tilde{\gamma}(\sigma))\right\}\mathrm{d}\sigma,$$

where the kernel $\tilde{M}(\tau,\sigma)$ is of the form:

$$\tilde{M}(\tau,\sigma) = \frac{\mathrm{i}r(\tau,\sigma)}{4}\left\{\kappa^2 H_0^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|) - \frac{2\kappa H_1^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|)}{|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|}\right\}$$
$$-\sin\sigma\frac{\mathrm{i}\kappa\langle\dot{\gamma}(\cos\tau),\dot{\gamma}(\cos\sigma)\rangle}{4|\gamma(\cos\tau)-\gamma(\cos\sigma)|}H_1^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|) - \frac{1}{2\pi}\frac{\sin\sigma}{(\cos\tau-\cos\sigma)^2}.$$

Here we used the well-known relation $(zH_1^{(1)}(z))' = zH_0^{(1)}(z)$ for $H_1^{(1)}(z) := -(H_0^{(1)}(z))'$. If we use a similar relation for Bessel functions we obtain

$$\tilde{M}_1(\tau,\sigma) = \frac{-r(\tau,\sigma)}{4\pi}\left\{\kappa^2 J_0^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|) - \frac{2\kappa J_1^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|)}{|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|}\right\}$$
$$+\frac{\kappa\sin\sigma\langle\dot{\gamma}(\cos\tau),\dot{\gamma}(\cos\sigma)\rangle}{4\pi|\gamma(\cos\tau)-\gamma(\cos\sigma)|}J_1^{(1)}(\kappa|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|),$$

where

$$r(\tau,\sigma) := -\sin\sigma\frac{\langle\dot{\gamma}(\cos\tau),\tilde{\gamma}(\sigma)-\tilde{\gamma}(\tau)\rangle\langle\dot{\gamma}(\cos\sigma),\tilde{\gamma}(\sigma)-\tilde{\gamma}(\tau)\rangle}{|\tilde{\gamma}(\tau)-\tilde{\gamma}(\sigma)|^2}.$$

For our convergence and error analysis we need an integral equation over the compact interval $[0,\pi]$. It is easy to see that (7) is equivalent to the following integral equation over $[0,\pi]$:

$$\int_0^{\pi} \sin\tau \left\{ K_1(\tau,\sigma)\dot{\psi}(\sigma) - K_2(\tau,\sigma)\psi(\sigma) \right\} \mathrm{d}\sigma = \tilde{f}(\tau), \tag{8}$$

with the right-hand side $\tilde{f}(\tau) := -f(\tilde{\gamma}(\tau))|\dot{\gamma}(\cos\tau)|\sin\tau$, $\tau\in[0,\pi]$, and the unknown density $\psi(\tau) := \varphi(\gamma(\cos\tau))$. In the following analysis we denote by $C_0^{k,\alpha}[0,\pi]$ the space of k-times Hölder continuously differentiable functions vanishing at the end points of the interval $[0,\pi]$. Later on we will prove the solvability of this integral equation for $\tilde{f}\in C_0^{0,\alpha}[0,\pi]$. In the following section we want to solve (8) numerically. Here, it is enough to look for solutions $\psi \in C_0^{1,\alpha}[0,\pi]$. We define one-dimensional integral operators $T,A,B: C_0^{1,\alpha}[0,\pi] \to C_0^{0,\alpha}[0,\pi]$ by

$$(T\psi)(\tau) := -\frac{1}{2\pi}\int_0^{\pi} \frac{\sin\tau}{\cos\sigma-\cos\tau}\dot{\psi}(\sigma)\,\mathrm{d}\sigma,$$

$$(A\psi)(\tau) := \frac{1}{2\pi}\int_0^{\pi} \tilde{K}_1(\tau,\sigma)\log 4(\cos\sigma-\cos\tau)^2\psi(\sigma)\,\mathrm{d}\sigma,$$

$$(B\psi)(\tau) := \frac{1}{2\pi}\int_0^{\pi} \tilde{K}_2(\tau,\sigma)\psi(\sigma)\,\mathrm{d}\sigma$$

with smooth and odd kernels

$$\tilde{K}_1(\tau,\sigma) := 2\pi\left\{\tilde{M}_1(\tau,\sigma) - M_1(\tau,\sigma)\right\}\sin\tau,$$

$$\tilde{K}_2(\tau,\sigma) := 2\pi\left\{\tilde{M}_2(\tau,\sigma) - M_2(\tau,\sigma)\right\}\sin\tau,$$

so that we can write the hypersingular integral equation (8) in the form:

$$T\psi + A\psi + B\psi = \tilde{f}. \tag{9}$$

Because we want to use error estimates for trigonometric interpolation in Hölder spaces over the interval $[0,2\pi]$, it is useful to extend the considered interval to $[0,2\pi]$. To this end we consider an odd continuation of the density ψ. By using the trigonometric identities $(\cos\sigma-\cos\tau)^2 = 4\sin^2\frac{\sigma-\tau}{2}\sin^2\frac{\sigma+\tau}{2}$ and $\frac{\sin\tau}{\cos\tau-\cos\sigma} = \frac{1}{2}\left\{\cot\frac{\sigma-\tau}{2} - \cot\frac{\sigma+\tau}{2}\right\}$ together with the oddness of the kernels $\tilde{K}_1$ and $\tilde{K}_2$ and the density ψ we get the following representations of T,A and $B: C_{\mathrm{odd}}^{k+1,\alpha}[0,2\pi] \to C_{\mathrm{odd}}^{k,\alpha}[0,2\pi]$

$$(T\psi)(\tau) = \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\sigma-\tau}{2}\dot{\psi}(\sigma)\,\mathrm{d}\sigma,$$

$$(A\psi)(\tau) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{K}_1(\tau,\sigma)\log 4\sin^2\frac{\sigma-\tau}{2}\psi(\sigma)\,\mathrm{d}\sigma,$$

$$(B\psi)(\tau) = \frac{1}{4\pi}\int_0^{2\pi} \tilde{K}_2(\tau,\sigma)\psi(\sigma)\,\mathrm{d}\sigma, \quad \tau\in[0,2\pi].$$

Here we denote by $C_{\mathrm{odd}}^{k,\alpha}[0,2\pi]$ the space of k-times Hölder continously differentiable odd functions over $[0,2\pi]$ vanishing at π and at the end points of the interval. The key observation for the following analysis is the fact that the operator $T: C^{k+1,\alpha}_{\text{odd}}[0,2\pi] \to C^{k,\alpha}_{\text{odd}}[0,2\pi]$ is bijective with a bounded inverse. This follows easily from the elementary integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\sigma-\tau}{2}\cos m\sigma \, d\sigma = -\sin m\tau, \quad \tau \in [0,2\pi] \tag{10}$$

and a representation of the odd function ψ as a sine-Fourier series. Note that $\tilde{f}$ is an odd function over $[0,2\pi]$. We obtain the following theorem:

Theorem 2.3. *The direct scattering problem for an open arc has a unique solution.*

Proof. Later we will see that A and B are compact in the considered spaces. So with the help of the Riesz–Fredholm theory applied to the equation of the second kind

$$\psi + T^{-1}(A+B)\psi = T^{-1}\tilde{f} \tag{11}$$

and due to the uniqueness of our boundary value problem we can assert the existence of a unique solution ψ of the hypersingular integral equation (8). □

Note that it is possible to extend Theorem 2.3 in a simple manner to systems of smooth disjoint open arcs by considering an obvious modification of the function space $C^{1,\alpha,*}(\Gamma)$, so that we get a considerably simplified proof for the results of [9] by using the cosine transformation.

3. Quadrature method

In this section we will present a fully discrete scheme for the solution of the direct scattering problem. We will use trigonometric polynomials as trial functions. First we want to develop some quadrature rules. For this we consider trigonometric interpolation with $2n$ nodal values $t_j^{(n)} := \frac{j\pi}{n}$, $j = 0,\dots,2n-1$. We interpolate with respect to the $2n$-dimensional space T_n of trigonometric polynomials over $[0,2\pi]$:

$$T_n := \left\{ v \in \mathrm{C}[0,2\pi] \mid v(t) = \sum_{m=0}^{n} a_m \cos mt + \sum_{m=1}^{n-1} b_m \sin mt, \quad a_m, b_m \in \mathbb{R} \right\}.$$

Let $P_n: C[0,2\pi] \to T_n$ be the corresponding interpolation operator. We will use the following interpolatory quadrature rules:

$$\frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\sigma-\tau}{2}\dot{\psi}(\sigma)\,d\sigma \sim \sum_{j=0}^{2n-1} \psi(t_j^{(n)})R_j^{(n)}(\tau),$$

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{K}_1(\tau,\sigma)\log 4\sin^2\frac{\sigma-\tau}{2}\psi(\sigma)\,d\sigma \sim \sum_{k=0}^{2n-1} F_k^{(n)}(\tau)\tilde{K}_1(\tau,t_k^{(n)})\psi(t_k^{(n)}),$$

$$\frac{1}{4\pi}\int_0^{2\pi} \tilde{K}_2(\tau,\sigma)\psi(\sigma)\,d\sigma \sim \frac{1}{4n}\sum_{j=0}^{2n-1} \tilde{K}_2(\tau,t_j^{(n)})\psi(t_j^{(n)}), \quad 0 \leqslant \tau \leqslant 2\pi,$$

where the weight functions R_j and F_k are given through the following integrals:

$$R_j^{(n)}(\tau) := \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\sigma-\tau}{2} L_j'^{(n)}(\sigma)\,\mathrm{d}\sigma,$$

$$F_k^{(n)}(\tau) := \frac{1}{2\pi}\int_0^{2\pi} \log 4\sin^2\frac{\sigma-\tau}{2} L_k^{(n)}(\sigma)\,\mathrm{d}\sigma, \quad 0\leqslant\tau\leqslant 2\pi.$$

Here $L_j^{(n)}$, $j=0,\dots,2n-1$, denote the Lagrange factors:

$$L_j^{(n)}(t) := \frac{1}{2n}\left\{1 + 2\sum_{m=1}^{2n-1} \cos m(t-t_j^{(n)}) + \cos n(t-t_j^{(n)})\right\}. \tag{12}$$

With the help of the integrals (10) we can derive the following explicit expressions for the weights, which we will need for the numerical implementation of our method:

$$R_j^{(n)}(t) = -\frac{1}{2n}\left\{\sum_{m=1}^{n-1} m\cos m(t-t_j^{(n)}) + \frac{n}{2}\cos n(t-t_j^{(n)})\right\},$$

$$F_k^{(n)}(t) = -\frac{1}{n}\left\{\sum_{m=1}^{n-1} \frac{1}{m}\cos m(t-t_k^{(n)}) + \frac{1}{2n}\cos n(t-t_k^{(n)})\right\}.$$

We introduce quadrature operators A_n and B_n by

$$(A_n\psi)(\tau) := \sum_{j=0}^{2n-1} F_j^{(n)}(\tau)\tilde{K}_1(\tau,t_j^{(n)})\psi(t_j^{(n)}),$$

$$(B_n\psi)(\tau) := \frac{1}{4n}\sum_{j=0}^{2n-1} \tilde{K}_2(\tau,t_j^{(n)})\psi(t_j^{(n)}), \quad \tau\in[0,2\pi].$$

Now we are able to formulate the following approximate version of the integral equation:

$$T\tilde{\psi}_n + A_n\tilde{\psi}_n + B_n\tilde{\psi}_n = \tilde{f}, \tag{13}$$

where we look for solutions $\tilde{\psi}_n \in T_{n,\text{odd}}$. Here by $T_{n,\text{odd}}$ we denote the following subspace of T_n:

$$T_{n,\text{odd}} := \left\{ v\in C[0,\pi] \mid v(\tau) = \sum_{j=1}^{n-1} a_j \sin j\tau,\ a_j\in\mathbb{R}\right\}.$$

Let $P_{n,\text{odd}}$ be the corresponding interpolation operator. For an odd function ψ vanishing at 0 and π we clearly have $P_n\psi = P_{n,\text{odd}}\psi$. Using a collocation method with respect to $P_{n,\text{odd}}$ we get the following fully discrete approximate version of the hypersingular integral equation:

$$T\tilde{\psi}_n + P_{n,\text{odd}}A_n\tilde{\psi}_n + P_{n,\text{odd}}B_n\tilde{\psi}_n = P_{n,\text{odd}}\tilde{f}. \tag{14}$$

Here we used the fact that $P_{n,\text{odd}}T\tilde{\psi}_n = T\tilde{\psi}_n$ for $\tilde{\psi}_n \in T_{n,\text{odd}}$, i.e. the quadrature rule for the principal part is exact for trigonometric polynomials, which we will need for the convergence analysis in the following section, too. Hence because of $\tilde{\psi}_n(t_j^{(n)}) = -\tilde{\psi}_n(t_{2n-1-j}^{(n)})$, $j=1,\dots,n-1$, and $\tilde{\psi}_n(0) =$

$\tilde{\psi}_n(\pi) = 0$ we only have to solve a $(n-1) \times (n-1)$ linear equation system for the unknown nodal values of $\tilde{\psi}_n$.

4. Convergence and error analysis

Our convergence proof is based on a perturbation argument as suggested in [7] in connection with pseudo-differential equations in Sobolev spaces of periodic functions. First we will prove some simple results on compactness and boundedness for the operators A and B. We omit the details for the operator B with smooth kernel. By $\|\cdot\|_{k,\alpha}$ we denote the usual Hölder norms for 2π-periodic functions. We obtain the following simple lemma:

Lemma 4.1. *The operator* $A : C^{m,\gamma}_{\text{odd}}[0,2\pi] \to C^{m,\alpha}_{\text{odd}}[0,2\pi]$ *is compact for each* $0 < \gamma, \alpha < 1$. *Moreover, A is a bounded operator from* $C^{m,\alpha}_{\text{odd}}[0,2\pi]$ *into* $C^{m+1,\alpha}_{\text{odd}}[0,2\pi]$.

Proof. For the second part of the lemma we simply use partial integration and then the well-known mapping properties of the integral operator with Hilbert kernel (cf. [12, p.63 and 69]). Because of this boundedness and the compactness of the embedding operator $I : C^{m+1,\gamma}_{\text{odd}}[0,2\pi] \to C^{m,\alpha}_{\text{odd}}[0,2\pi]$ (cf. [4, p.84]) we see the compactness of A in the considered spaces. $\square$

For our analysis we need some estimates for the error between the operators A and B and their approximations A_n and B_n for smooth functions. We get the following approximation theorem:

Theorem 4.2. *For all* $\psi \in T_{n,\text{odd}}$ *and* $k, m \in \mathbb{N}$, $m+\alpha \leqslant k+\beta$, $0 < \alpha, \beta < 1$ *and arbitrary* $0 < \varepsilon < \beta$ *we have the following estimates*:

$$\|A\psi - A_n\psi\|_{k,\beta} \leqslant c_n \|\psi\|_{k+1,\beta-\varepsilon} \tag{15}$$

with $\lim_{n\to\infty} c_n = 0$ *and*

$$\|A\psi - A_n\psi\|_{m,\alpha} \leqslant C \frac{\log n}{n^{k-m+\beta-\alpha}} \|\psi\|_{k+1,\beta-\varepsilon}. \tag{16}$$

Proof. Throughout this proof C denotes a generic constant. We only deal with estimating the Hölder semi-norm for $M\psi := ((A - A_n)\psi)^{(k)}$. The proof for the other terms of the Hölder norm is similar and simpler. We consider only one term of $M\psi$. With the help of differentiation under the integral sign (cf. [12]) and partial integration we obtain the following representation of this term:

$$(M_r\psi)(\tau) = \int_0^{2\pi} f_\tau(\sigma) \frac{\partial^{(r)}}{\partial\sigma^r} \left\{ P_n \left(\frac{\partial^{(k-r)}}{\partial\tau^{k-r}} K(\tau,\cdot)\psi(\cdot) \right)(\sigma) - \frac{\partial^{(k-r)}}{\partial\tau^{k-r}} K(\tau,\sigma)\psi(\sigma) \right\} \mathrm{d}\sigma$$

for $0 \leqslant r \leqslant k$. Here for abbreviation we set $f_\tau(\sigma) := \log 4\sin^2 \frac{\sigma-\tau}{2}, \sigma \neq \tau$. By using the Hölder inequality for $p = \frac{1}{\beta}$ and the boundedness of integral operators with Hilbert kernel in L^p-spaces (cf. [12, p. 63]) for

$$\mathrm{d}(\tau_1,\tau_2) := (M_r\psi)(\tau_1) - (M_r\psi)(\tau_2) = \int_{\tau_1}^{\tau_2} (M_r\psi)'(\sigma)\,\mathrm{d}\sigma \tag{17}$$

we obtain the following estimate:

$$|\mathrm{d}(\tau_1,\tau_2)| \leqslant C|\tau_1-\tau_2|^{\beta}\left(\int_0^{2\pi}|(M_r\psi)'(\sigma)|^{\frac{1}{1-\beta}}\,\mathrm{d}\sigma\right)^{1-\beta}$$
$$\leqslant C|\tau_1-\tau_2|^{\beta}\sum_{j=1}^{4}\max_{\tau\in[0,2\pi]}\left\|\frac{\partial^{(r)}}{\partial\sigma^r}\{P_n(K_j(\tau,\cdot)\psi(\cdot))(\cdot)-K_j(\tau,\cdot)\psi(\cdot)\}\right\|_{\infty}$$

with smooth kernels

$$K_1(\tau,\sigma):=\frac{\partial^{(k-r)}}{\partial\tau^{k-r}}K(\tau,\sigma),\qquad K_2(\tau,\sigma):=\frac{\partial^{(k+1-r)}}{\partial\tau^{k+1-r}}K(\tau,\sigma)$$

and

$$K_{2+i}(\tau,\sigma):=\frac{K_i(\tau,\sigma)-K_i(\sigma,\sigma)}{\tau-\sigma},\quad i=1,2.$$

Applying well-known error estimates for trigonometric interpolation in Hölder spaces (cf. [18, p. 78]), we derive

$$\frac{|\mathrm{d}(\tau_1,\tau_2)|}{|\tau_1-\tau_2|^{\beta}}\leqslant C\frac{\log n}{n^{\delta}}\sum_{j=1}^{4}\max_{\tau\in[0,2\pi]}\left\|\frac{\partial^{(k)}}{\partial\sigma^k}\{K_j(\tau,\cdot)\psi(\cdot)\}\right\|_{0,\beta+\delta}$$
$$\leqslant C\frac{\log n}{n^{\delta}}\sum_{i=1}^{4}\max_{\tau\in[0,2\pi]}\sum_{j=0}^{k}\binom{k}{j}\left\|\frac{\partial^{(k-j)}}{\partial\sigma^{k-j}}K_i(\tau,\cdot)\right\|_{0,\beta+\delta}\|\psi^{(j)}\|_{0,\beta+\delta}$$

for $\tau_1\neq\tau_2$ and δ sufficiently small. Then, by using the mean value theorem we find the final estimate

$$\frac{|\mathrm{d}(\tau_1,\tau_2)|}{|\tau_1-\tau_2|^{\beta}}\leqslant c_n\|\psi\|_{k+1,\beta-\varepsilon} \tag{18}$$

with $c_n:=C\frac{\log n}{n^{\delta}}\to 0$ for $n\to\infty$. The second estimate can be shown by the same techniques and by using the error estimates of [18]. □

Now we are able to establish our main convergence result:

Theorem 4.3. *The approximated equation*

$$T\tilde{\psi}_n+P_{n,\mathrm{odd}}(A_n+B_n)\tilde{\psi}_n=P_{n,\mathrm{odd}}\tilde{f} \tag{19}$$

has exactly one solution $\tilde{\psi}_n\in T_{n,\mathrm{odd}}$ *for any right-hand side* $\tilde{f}\in C^{k,\beta}_{\mathrm{odd}}[0,2\pi]$ *and for n large enough. Furthermore, we have the following pointwise error estimates*

$$\|\psi-\tilde{\psi}_n\|_{m+1,\alpha}\leqslant C\frac{\log n}{n^{k-m+\beta-\alpha}}\|\psi\|_{k+1,\beta}, \tag{20}$$

where ψ *is the solution of the exact integral equation and* $m+\alpha<k+\beta$, $0<\alpha,\beta<1$. *The constant C only depends on* α, β, m, k *and the parametrization of the arc.*

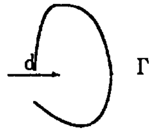

Fig. 1. Bowl shaped example for an open arc.

Proof. It is easy to extend the analysis of [7, Theorems 2.2 and 2.3] to our case. We only want to show the basic idea. In a first step we consider the unperturbed equation $T\tilde{\psi}_n = P_{n,\text{odd}}\tilde{f}$. The unique solution of this equation is $\tilde{\psi}_n = T^{-1}P_{n,\text{odd}}T\psi$ and an error estimate of the form (20) is valid. In a second step we obtain from Eqs. (9) and (19)

$$T\tilde{\psi}_n = P_{n,\text{odd}}T\left(\psi + T^{-1}(A+B)\psi - T^{-1}(A_n+B_n)\tilde{\psi}_n\right). \tag{21}$$

By using the error estimate obtained in the first step for Eq. (21) and by using Lemma 4.1 together with Theorem 4.2 we can now proceed as in the proof of Theorem 2.3 of [7]. □

If the considered arc is analytic we can replace (20) by the following sharper estimate:

$$\|\psi - \tilde{\psi}_n\|_\infty \leqslant C e^{-n\sigma}, \quad \sigma > 0, \tag{22}$$

i.e., the error decreases exponentially. We can prove this using error estimates for trigonometric interpolation of analytic functions (cf. [4, p. 161]).

5. Numerical example

We want to finish the paper with a numerical example. Therefore we consider a bowl-shaped open arc (Fig. 1) given by the following parametrization:

$$\gamma(s) = \left(2\sin\tfrac{3}{8}\pi\left(\tfrac{4}{3}+s\right), -\sin\tfrac{3}{4}\pi\left(\tfrac{4}{3}+s+\frac{2}{3\pi}\right)\right), \quad s\in[-1,1].$$

We are interested in the calculation of the far field pattern u_∞ of the scattered wave. The far field pattern describes the behavior of the scattered wave at infinity:

$$u(x) = \frac{e^{i\kappa|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x|\to\infty,$$

uniformly for all directions $\hat{x} = \frac{x}{|x|}$. From the asymptotic behavior of the Hankel functions of the first kind

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}}e^{i(z-\frac{\pi}{4}-n\frac{\pi}{2})}\left\{1+O\left(\frac{1}{z}\right)\right\}, \quad z\to\infty, \tag{23}$$

we obtain the far field pattern of the double layer potential for the open arc in parametrized form:

$$u_\infty(\hat{x}) = \sqrt{\frac{\kappa}{8\pi}}e^{-i\frac{\pi}{4}}\int_0^\pi \langle n(\gamma(\cos\sigma)),\hat{x}\rangle e^{-i\kappa\langle\hat{x},\gamma(\cos\sigma)\rangle}\psi(\sigma)\sin\sigma|\dot{\gamma}(\cos\sigma)|\,d\sigma.$$

Table 1
Numerical example for an open arc

	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty\ (d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
	8	−0.587910290	0.265774373	−0.282467346	−0.464603221
$\kappa = 1$	16	−0.611185696	0.287416611	−0.324791966	−0.470955188
	32	−0.611240963	0.287462269	−0.324885923	−0.470976200
	64	−0.611240963	0.287462269	−0.324885923	−0.470976201
	16	−0.886891788	1.187861198	−0.828215183	0.124239209
$\kappa = 5$	32	−0.888213424	1.189367204	−0.837872034	0.134310947
	64	−0.888213478	1.189367200	−0.837872294	0.134310996
	128	−0.888213478	1.189367200	−0.837872294	0.134310996
	16	−1.611509573	1.521448236	−0.102924375	0.671228116
	32	−1.479006855	1.852002669	0.535041419	1.239294531
$\kappa = 10$	64	−1.479463769	1.851328069	0.545414008	1.230052685
	128	−1.479463769	1.851328068	0.545414009	1.230052684
	256	−1.479463769	1.851328068	0.545414009	1.230052684

We solve the hypersingular integral equation in order to obtain approximate values for the density ψ. Then we use the trapezoidal rule to evaluate the integral for the far field expression. Table 1 shows some numerical values for the far field pattern $u_\infty(d)$ and $u_\infty(-d)$. Here d denotes the direction of the incident wave. In our example we choose $d = (1,0)$. We get the numerical results as listed in Table 1.

Clearly, the convergence is exponential. For higher values of κ the oscillation of the fundamental solution increases, hence the number of nodal points must be increased. Note that it is not possible to give the error explicitly because there is no arc for which we know the exact far field pattern.

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