Multigrid Solvers and Preconditioners for First Kind Integral Equations

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We discuss multigrid methods and multilevel preconditioners for first kind boundary integral equations with weakly and hypersingular kernels. We find that the number of iterations needed is bounded or grows no worse than logarithmically in the numbers of unknowns. We also discuss the complexity for parallel implementations.

I. INTRODUCTION

We present multigrid methods for the boundary integral equation of the first kind which results from the Neumann problem for the Laplacian in $\mathbb{R}^2$. We obtain that the convergence factor of the V-cycle grows no worse than logarithimically towards 1, whereas the W-cycle and the variable V-cycle have uniformly bounded convergence factors. For the integral equation resulting from the Dirichlet problem a modified set of basis functions is needed for convergence of the multigrid method. Numerical experiments support the theoretical results. We also consider the multigrid method and a simpler multilevel procedure as preconditioners for the conjugate gradient method and discuss their complexity for both serial and parallel implementations.

The integral equations under consideration are, for $x \in P$ and given f,

$$Dv(x) := -\frac{1}{\pi}\,\text{f.p.}\int_\Gamma \frac{v(y)}{|x-y|^2}\,ds_y = f(x) \tag{1}$$

and

$$V\Psi(x) := -\frac{1}{\pi}\int_\Gamma \ln|x-y|\Psi(y)\,ds_y = f(x)\,. \tag{2}$$

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Here f.p. denotes a finite part integral and Γ is an open curve in $\mathbb{R}^2$. For closed curves we can obtain similar results. Equations (1) and (2), respectively, arise in solving the Neumann and Dirichlet problem for the Laplacian in $\mathbb{R}^2\backslash\overline{\Gamma}$. The operators D and V are strongly elliptic pseudodifferential operators of orders $+1$ and -1, respectively (see Costabel [1]). The Galerkin schemes for Eqs. (1) and (2), respectively, are

Find $v_h \in S_h^1(\Gamma)$ such that for all $w_h \in S_h^1(\Gamma)$

$$\langle Dv_h, w_h\rangle = \langle f, w_h\rangle . \tag{3}$$

Find $\Psi_h \in S_h^0(\Gamma)$ such that for all $\phi_h \in S_h^0(\Gamma)$

$$\langle V\Psi_h, \phi_h\rangle = \langle f, \phi_h\rangle . \tag{4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the L^2 duality on Γ given by $\int_\Gamma g(x)\, h(x)\, dx$ for smooth functions g, h. The boundary element spaces on Γ are introduced via a quasiuniform mesh as follows. $S_h^0(\Gamma)$ consists of piecewise constant functions whereas $S_h^1(\Gamma)$ consists of continuous, piecewise linear functions which vanish at the endpoints of Γ.

The operators D and V define continuous, positive-definite and symmetric bilinear forms $a(v,w) = \langle Dv, w\rangle$ and $a(\Psi,\phi) = \langle V\Psi, \phi\rangle$ for $v, w \in \tilde{H}^{1/2}(\Gamma)$ and $\Psi, \phi \in \tilde{H}^{-1/2}(\Gamma)$ (see Costabel [1]). (For the definition of the Sobolev spaces see Costabel [1] and von Petersdorff and Stephan [2].) Therefore the Galerkin solutions of Eqs. (3) and (4) converge quasiuniformly in the energy norm towards the exact solutions of the integral equations (1) and (2).

In the following we are concerned with solving efficiently the above Galerkin equations which—after choosing a basis $\{\varphi_h^1, \ldots, \varphi_h^N\}$ of $S_h^1(\Gamma)$ or $S_h^0(\Gamma)$—both lead to a symmetric, positive-definite linear system

$$A_h \mathbf{u}_h = \mathbf{f}_h \tag{5}$$

with

$$(A_h)_{ij} = a(\varphi_h^j, \varphi_h^i), (\mathbf{f}_h)_i = \langle f, \varphi_h^i\rangle .$$

Let $N := \dim S_h$ [with $S_h = S_h^0(\Gamma)$ or $S_h^1(\Gamma)$]; then Eq. (5) is an $N \times N$ linear system with a full matrix. Hence, a Gauss solver requires CN^3 operations. We may also use an iterative Jacobi method to solve Eq. (5) with an accuracy of the order of the Galerkin error. Since the condition of the Galerkin matrix is of order $O(N)$ the convergence factor of one Jacobi step behaves like $1 - C_0 N^{-1}$. Hence $CN \log N$ iteration steps of N^2 operations (A_h is a full matrix) are needed, giving a total number of iterations of $CN^3 \log N$. (For the corresponding problem in $\mathbb{R}^3$ one needs $CN^{5/2} \log N$ operations.)

A more efficient method is the conjugate gradient algorithm. Here the average convergence factor per step behaves like

$$\frac{\mathrm{cond}(A_h)^{1/2} - 1}{\mathrm{cond}(A_h)^{1/2} + 1} = 1 - 2C_0 N^{-1/2} + O(N^{-1}),$$

hence $CN^{1/2} \log N$ steps and therefore $CN^{5/2} \log N$ operations are needed ($CN^{9/4} \log N$ for the problem in $\mathbb{R}^3$). Numerical results for Eqs. (3) and (4) with $\Gamma = (-1,1) \times \{0\}$ and $f = 1$ are presented in Table I. The numbers confirm the theoretical result that the relative error after $\sqrt{N}$ iteration is bounded by a constant $q < 1$, independent of h.

The aim of the multigrid method is to give an iteration procedure which has a convergence factor $\delta \le q < 1$ per iteration with a constant q independent of N. Then the number of iterations needed to reduce the error to the order of the Galerkin error is inde-

TABLE I. Relative error with respect to energy norm after $\approx\sqrt{N}$ conjugate gradient iterations starting with 0.

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
operator D	1.823×10^{-4}	2.024×10^{-3}	2.562×10^{-3}
operator V	1.936×10^{-2}	1.797×10^{-2}	7.017×10^{-3}
h	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
operator D	5.516×10^{-3}	7.196×10^{-3}	5.975×10^{-3}
operator V	4.476×10^{-3}	2.170×10^{-3}	1.520×10^{-3}

pendent of N if the so-called full multigrid cycle is performed. Hence the system (5) can be solved with an accuracy of the order of the Galerkin error using a total number of $C'N^2$ operations.

II. MULTIGRID SOLVERS

In order to describe the multigrid method we introduce a sequence of nested finite-dimensional spaces of boundary elements $S_{h_0} \subset S_{h_1} \subset \cdots \subset S_{h_m}$ with $h_m = h$.

Let $\tilde{v}_{h_m} \in S_{h_m}$ be an approximation of the Galerkin solution $v_{h_m} \in S_{h_m}$ of (5), i.e.,

$$a(v_{h_m}, w_{h_m}) = \langle f, w_{h_m} \rangle \qquad \forall w_{h_m} \in S_{h_m}. \tag{6}$$

Then one multigrid step of level m for Eq. (6) is defined as follows:

1. *Presmoothing*: Apply $\nu_1(m)$ relaxation steps for (5) to v_{h_m}.
2. *If* $m \geq 1$: Starting with $\tilde{v}_{h_{m-1}} = 0$, execute μ multigrid steps of level $m-1$ to find an approximation $\tilde{v}_{h_{m-1}}$ to the solution $v_{h_{m-1}} \in S_{h_{m-1}}$ of the problem

$$a(v_{h_{m-1}}, w_{h_{m-1}}) = \langle f, w_{h_{m-1}} \rangle - a(\tilde{v}_{h_m}, w_{h_{m-1}}) \qquad \forall w_{h_{m-1}} \in S_{h_{m-1}}.$$

   Add $\tilde{v}_{h_{m-1}}$ to $\tilde{v}_{h_m}$.
3. *Postsmoothing*: Apply $\nu_2(m)$ relaxation steps for (5) to $\tilde{v}_{h_m}$.

Here a relaxation step can be a dampened Jacobi or a symmetric SOR (SSOR) step. The algorithm with $\mu = 1$ is called V-cycle, and with $\mu = 2$ it is called W-cycle. The numbers $\nu_1(m)$, $\nu_2(m)$ of presmoothing and postsmoothing steps are usually chosen independent of m. The case $\mu = 1$ and $\nu_j(k) = \tilde{\nu}_j 2^{m-k}$ with constants $\tilde{\nu}_j$ is called variable V-cycle [3].

For the hypersingular operator D we have the following result for ν_1, ν_2 independent of m:

Theorem 1. (a) [2] *Let the Galerkin equations* (3) *be given on a quasiuniform mesh. Then there holds for the convergence factor* δ_V (δ_W) *of the V-cycle* (*W-cycle*)

(*i*) $\delta_V \leq q < 1$ *if* Γ *is a closed curve,*
(*ii*) $\delta_W \leq q < 1$ *if* Γ *is an open or closed curve.*

(b) [4] *Let* (3) *be given on a graded mesh which is refined towards the endpoints of* Γ. *Then there holds*

$$\delta_V \leq q < 1.$$

The assertions in (*a*) *and* (*b*) *hold for any number of smoothing steps with* $\nu_1 + \nu_2 \geq 1$ *and the constant* q *is independent of* $N = \dim S_h^1(\Gamma)$.

Remark. The results of Theorem 1(a) are also valid when Γ is a surface or surface piece in $\mathbb{R}^3$ (see von Petersdorff and Stephan [2]).

The following numerical results in Table II illustrate Theorem 1. We consider the hypersingular equation

$$Dv = 1 \quad \text{on} \quad \Gamma = (-1,1) \times \{0\} \subset \mathbb{R}^2$$

and solve the Galerkin scheme (3) using continuous piecewise linear functions (vanishing at the endpoints of Γ) and the standard basis functions. We use either a uniform mesh with $h_k = 2^{-k}$ or a mesh refined toward the endpoints with grading parameter $\beta = 3$ (see von Petersdorff and Stephan [4]). We use dampened Jacobi with $\omega = \frac{1}{2}$ as a relaxation scheme.

The numbers in Table II clearly show that δ_w for a uniform mesh and δ_v for a graded mesh are bounded away from 1, as claimed in Theorem 1(a) (ii) and 1(b). For δ_v on a uniform mesh, Theorem 1 does not give any result, and the values seem to be growing very slowly. For the variable V-cycle the numbers suggest that δ_v is bounded away from 1. In order to obtain theoretical results for the V-cycle and the variable V-cycle on the uniform mesh we show that the multigrid theory by Bramble and Pasciak [3] applies to the present case. Here we have $a(u,v) = \langle Du, v\rangle$ for $u, v \in \tilde{H}^{1/2}(\Gamma)$ where D is the hypersingular operator in Eq. (1).

To this end we assume in the following a uniform mesh and introduce the operator $A_k: S_{h_k} \to S_{h_k}$ by

$$A_k w := h_k^{-1} \sum_j a(w, \phi^j_{h_k}) \phi^j_{h_k}.$$

Note that $A_k = h_k{}^{-1} B_k^{-1} D_{h_k}$ with B_k and D_{h_k} as defined in Ref. 2 if $R_k := I$ is used. Then the proof of Lemma 3 in Ref. 2 (with $d = 1$ instead of 2) implies

$$\rho(A_k) \le c h_k{}^{-1}. \tag{7}$$

For $u = \Sigma u^j \phi^j_{h_k}$ and $v = \Sigma v^j \phi^j_{h_k}$ let us define

$$(u,v)_k := h_k \sum_j u^j v^j \quad \text{and} \quad \|u\|_k := (u,u)_k^{1/2}.$$

Here $\{\phi^j_{h_k}\}$ is the standard basis of the space S_{h_k} of piecewise linears vanishing at the endpoints of Γ. We observe that

$$(A_k w, \phi)_k = a(w, \phi) \quad \text{for all} \quad \phi \in S_{h_k}.$$

For $k = 1, \ldots, m$ we define the projector $P_{k-1}: S_{h_k} \to S_{h_{k-1}}$ by

$$a(P_{k-1} w, \phi) = a(w, \phi) \quad \text{for all} \quad \phi \in S_{h_{k-1}}, \tag{8}$$

i.e., $P_{k-1} w$ is the Galerkin projection of $w \in S_{h_k}$ on $S_{h_{k-1}}$.

TABLE II. Multigrid results for the operator D.

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
δ_v, uniform mesh	0.188	0.208	0.213	0.217	0.219	0.220	0.222
δ_w, uniform mesh	0.188	0.197	0.197	0.196	0.196	0.196	0.196
δ_v for variable V-cycle (uniform mesh)	0.188	0.198	0.198	0.198	0.197	0.197	0.197
δ_v, graded mesh ($\beta = 3$)	0.271	0.261	0.264	0.266	0.266	0.266	0.266

In Ref. 3 the convergence analysis is based on a regularity and approximation assumption of the form [see (3.3) in Ref. 3]

$$a((I - P_{k-1})u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|_k^2}{\lambda_k} \right)^\alpha a(u,u)^{1-\alpha} \quad \text{for all} \quad u \in S_{h_k} \tag{9}$$

where λ_k is the largest eigenvalue of A_k for some α with $0 < \alpha \leq 1$.

Proposition 1: *The assumption (9) holds for $0 < \alpha < 1$ if Γ is open and for $0 < \alpha \leq 1$ if Γ is closed.*

Proof. First we note that for $0 \leq s \leq 1$ there are constants c_0 and c_1 independent of h_k and that for all $u \in S_{h_k}$

$$c_0 \|A_k^{s/2} u\|_k \leq \|u\|_{\tilde{H}^{s/2}(\Gamma)} \leq c_1 \|A_k^{s/2} u\|_k . \tag{10}$$

This is easy to see for $s = 0$. For $s = 1$ we have

$$\|A_k^{1/2} u\|_k^2 = (A_k u, u)_k = a(u,u) .$$

On the other hand there holds the norm equivalence

$$\|u\|_{\tilde{H}^{1/2}(\Gamma)} \simeq |||u||| ,$$

where $|||u||| := a(u,u)^{1/2}$. This yields (10) for $s = 1$. The general case follows then by interpolation.

Next we obtain as in the proof of Proposition 5.1 in Ref. 3

$$\begin{aligned} a((I - P_{k-1})u, u) &\leq |||A_k^{\alpha/2} u||| \quad |||A_k^{-\alpha/2}(I - P_{k-1})u||| \\ &= |||A_k^{\alpha/2} u||| \quad \|A_k^{(1-\alpha)/2}(I - P_{k-1})u\|_k \\ &\leq c_0^{-1} |||A_k^{\alpha/2} u||| \quad \|(I - P_{k-1})u\|_{\tilde{H}^{(1-\alpha)/2}(\Gamma)} . \end{aligned}$$

Furthermore, interpolation gives

$$|||A_k^{\alpha/2} u|||^2 \leq a(u,u)^{1-\alpha} \|A_k u\|_k^{2\alpha} .$$

By duality estimates [see (3.8) in Ref. 2], we have, using (8),

$$\|(I - P_{k-1})u\|_{\tilde{H}^{(1-\alpha)/2}(\Gamma)} \leq c h_{k-1}^{\alpha/2} a((I - P_{k-1})u, u)^{1/2} .$$

Finally, with (7) we observe $\lambda_k \leq c h_k^{-1}$ yielding (9). ∎

Remark. The proof of Prop. 1 is based on the estimate

$$\begin{aligned} |||A_k^{-\alpha/2}(I - P_{k-1})u||| &\leq c \cdot h_{k-1}^{\alpha/2} a((I - P_{k-1})u, u)^{1/2} \\ &= c \cdot h_{k-1}^{\alpha/2} |||(I - P_{k-1})u||| . \end{aligned} \tag{11}$$

Using the notation in Ref. 2, we have with $h_k A_k = B_k^{-1} D_{h_k}$

$$\begin{aligned} h_k^{-\alpha/2} |||A_k^{-\alpha/2} v||| &= a((h_k A_k)^{-\alpha} v, v)^{1/2} = \langle D_{h_k} (B_k^{-1} D_{h_k})^{-\alpha} v, v \rangle^{1/2} \\ &= \langle B_k (B_k^{-1} D_{h_k})^{1-\alpha} v, v \rangle^{1/2} =: |v|_{1-\alpha} . \end{aligned}$$

Hence (11) is equivalent to

$$|(I - P_{k-1})u|_{1-\alpha} \leq \tilde{C} |||(I - P_{k-1})u||| .$$

(using $h_k \sim h_{k-1}$), which is approximation property (A.2) in Ref. 2.

The second assumption (3.4) in Ref. 3 depends only on the choice of the relaxation scheme and is satisfied for, e.g., the dampened Jacobi method. Theorems 1 and 5 in Ref. 3 now yield

Theorem 2.

(i) *For the convergence factor δ_m of the m-level V-cycle with a fixed number $\nu_1 = \nu_2 = \nu$ of smoothing steps there holds $\delta_m \leq 1 - \sigma_m$ with*

$$\sigma_m = \frac{\nu^\alpha}{\nu^\alpha + M_\alpha (m_0 + m)^{(1-\alpha)/2}}$$

with positive constants M_α, m_0 as in Ref. 3.

(ii) *For the convergence factor δ_m of the m-level variable V-cycle with $\nu_1(k) = \nu_2(k) = \tilde{\nu} 2^{m-k}$ smoothing steps on level k there holds $\delta_k \leq 1 - \sigma_m$ and*

$$\sigma_m = \frac{\tilde{\nu}^\alpha}{\tilde{\nu}^\alpha + M_\alpha}$$

with a positive constant M_α as in Ref. 3.

From (i) we see that δ_m behaves like $1 - Cm^{(-1-\alpha)/\alpha}$ where m is the number of levels. Hence $C(\log N)^{(1-\alpha)/\alpha}$ iterations are needed using the full multigrid algorithm, resulting in $CN^2(\log N)^{(1-\alpha)/\alpha}$ operations. For the case of an open curve and a uniform mesh we can choose $\alpha = 1 - \epsilon$ and have that

$$\delta_m = 1 - C_{\tilde{\epsilon}} (\log N)^{-\tilde{\epsilon}} \quad \text{for every} \quad \tilde{\epsilon} > 0$$

resulting in $\tilde{C}_{\tilde{\epsilon}} (\log N)^{\tilde{\epsilon}} N^2$ operations. This is compatible with the numerical results in Table II.

From (ii) we obtain that the variable V-cycle has a convergence factor which is bounded away from 1. This is also apparent in the values in Table II which are very close to the values of the W-cycle. This is reasonable since the total numbers of smoothing steps in the W-cycle also grows like 2^{m-k}. However, the algorithm of the variable V-cycle has a simpler structure.

Remark. The results of Theorem 2 are also valid when Γ is a surface or a surface piece in $\mathbb{R}^3$.

For the case of the single layer potential operator the standard multigrid theory does not apply since V is an operator of negative order. Indeed, the numerical results in Table III show that the convergence factor δ approaches 1 for decreasing h. Here we consider the Galerkin scheme for the equation $V\Psi = x$ on $\Gamma = [-1, 1] \times \{0\}$, using piecewise constants with the standard basis functions. However, the analysis in Ref. 8 shows that using a modified set of basis functions leads to a convergent multigrid method. In this case, we can choose the derivatives $\{(\phi_h^l)'(x) | \phi_h^l \in S_h^1(\Gamma)\}$ of the piecewise linear functions, together with the linear function $\phi_h^0 \equiv 1$. This is motivated by the relation

$$\langle V\phi', \Psi' \rangle = \langle D\phi, \Psi \rangle, \quad \text{for all} \quad \phi, \Psi \in S_h^1(\Gamma),$$

which allows use of similar arguments as for the hypersingular operator D in (1). Therefore the results of Theorems 1 and 2 also hold for the weakly singular operator V in (2) if

TABLE III. Multigrid results for the operator V.

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
δ_W for standard basis	0.285	0.387	0.468	0.530	0.580	0.621	0.655
δ_W for modified basis	0.300	0.277	0.270	0.269	0.268	0.268	0.268

this modified basis is used. Table III gives the convergence factor for the W-cycle; here SOR with $\omega = 1.4$ was used as relaxation method. The numbers show that the convergence factor for the modified basis remains bounded away from 1.

III. MULTILEVEL PRECONDITIONERS AND PARALLEL ALGORITHMS

The multigrid algorithm can also be used as a preconditioner for the conjugate gradient algorithm. For $g \in S_{h_m}$, let $B_m g$ be the result of one multigrid step for

$$a(v_{h_m}, w_{h_m}) = (g, w_{h_m})$$

with initial guess 0 for v_{h_m}. The performance of the conjugate gradient algorithm preconditioned by B_m is determined by the condition of the matrix $B_m A_m$. The analysis in Ref. 6 shows that

$$(1 - \delta)a(v,v) \leq a(B_m A_m v, v) \leq a(v,v) \qquad \forall v \in S_{h_m} \tag{12}$$

and therefore

$$\text{cond}(B_m A_m) \leq \frac{1}{1-\delta} = \frac{1}{\sigma}, \qquad \sigma > 0, \quad \text{independent of } m.$$

Hence one conjugate gradient step yields a convergence factor of the order $1 - \sigma^{1/2}$ on average, and $c\sigma^{-1/2}|\log N|$ iterations are needed, with σ independent of m.

In Ref. 6 more general multilevel preconditioners $\tilde{B}_m$ are constructed which satisfy

$$Cm^{1-1/\alpha} a(v,v) \leq a(\tilde{B}_m A_m v, v) \leq \tilde{c}m \quad a(v,v) \qquad \forall v \in S_{h_m} \tag{13}$$

yielding

$$\text{cond}(\tilde{B}_m A_m) \leq cm^{1/\alpha}. \tag{14}$$

Hence $O(m^{1/2\alpha}) = O((\log N)^{1/2\alpha}))$ iterations are required. However, the evaluation of $\tilde{B}_m$ can be highly parallelized. In particular, Bramble *et al.* [6] consider $\tilde{B}_m$ defined by

$$\tilde{B}_m v = \sum_{k=1}^{m} \sum_{l} (v, \phi_{h_k}^l)\phi_{h_k}^l. \tag{15}$$

It is shown in Theorem 2 in Ref. 6 that the regularity and approximation assumption (9) implies (13). Since we proved assumption (9) for the hypersingular operator D with $\alpha = 1 - \epsilon$ for an open curve and $\alpha = 1$ for a closed curve, we also obtain convergence results for the conjugate gradient method preconditioned by $\tilde{B}_m$.

Note that the definition of $\tilde{B}_m$ depends only on the basis functions $\phi_{h_k}^l$ and does not involve the operator D. Hence the statements in Ref. 6 about implementation and complexity of $\tilde{B}_m$ remain valid. The computation of a product $\tilde{B}_m v$ involves similar projection and interpolation steps as the multigrid algorithm, but none of the relaxation steps. In a serial implementation, the projection and interpolation steps amount to $O(N)$ operations. A relaxation step involves $O(N^2)$ operations, due to the full matrix. Hence the computation of $\tilde{B}_m v$ costs $O(N)$ operations, but one multigrid preconditioning step $B_m v$ costs $O(N^2)$ operations. The multigrid preconditioning achieves a bounded condition number $\text{cond}(B_m A_m)$ whereas the multilevel preconditioning (15) results in a slowly growing condition number $\text{cond}(\tilde{B}_m A_m) \leq C(\log N)^{1+\epsilon}$ (with $\epsilon > 0$ for an open curve, $\epsilon = 0$ for a closed curve). But in each case one conjugate gradient iteration involves a product $A_m v$ taking $O(N^2)$ opera-

tions. Currently Hackbusch and others are investigating so-called clustering techniques [7] which aim to reduce the complexity of the product $A_k v$ from $O(N^2)$ to $O(N)$ for serial implementations.

For a parallel implementation of the algorithms we consider as in Ref. 6 a shared memory machine with an unlimited number of processors. Then a matrix-vector product $A_m v$ takes only a time of $C \log N$, even if the matrix A_m is dense. In a parallel implementation, the projection and interpolation steps can be computed in time of $C \log N$ (since a projection or interpolation from each level to the next one can be done in parallel in a fixed time).

In conclusion, the linear system can be solved in a time growing logrithmically in N for a parallel implementation. However, the full matrix A_m implies that $O(N^2)$ processors are required as compared to $O(N)$ processors for the finite element case. But here again clustering techniques might be used to reduce the number of processors to $O(N)$.

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