Preconditionning the Helmholtz problem on curved arcs in dimension 2

François Alouges and Martin Averseng*

October 25, 2018

Abstract

1 Introduction

The problem of preconditionning the linear systems coming from the discretization of integral equations has received considerable attention since two decades roughly. Among the possible strategies are the so-called pseudo-differential preconditionner, whose analysis uses tools from pseudo-differential calculus [?, ?, ?, ?]. Roughly speaking, if the original problem is written in the abstract way

$$\mathcal{L}u = f\,, \tag{1}$$

the strategy consists in findind a suitable operator $\mathcal{K}$ such that, when left mulitplying the (1) by $\mathcal{K}$, one needs to solve

$$\mathcal{K}\mathcal{L}u = \mathcal{K}f\,. \tag{2}$$

Now if $\mathcal{K}\mathcal{L}$ is a compact perturbation of the identity, the condition number of the discretized underlying system is independant of the chosen size of the mesh, leading to optimal convergence rate of the numerical methods used to solve the system.

Several strategies, depending on the problem to solve have been studied in the literature [] to propose such operators $\mathcal{K}$, that often turn out to be very effective in practice, when numerical applications are considered. However, all the preceding results and theories are limited to smooth domains and very little is known when the integral equation is posed on domains with corners (in 2D), wedges or conical points (in 3D). One of the reason might be the fact that pseudo-differential calculus is difficult to generalize on such manifolds, and more problematic, the underlying operators are difficult to analyze on such domains.

Nevertheless, a program similar to those given on smooth manifold was proposed a few years ago [?, ?, ?, ?] who tackled the problem of preconditionning the first or second layer potential for Laplace or Helmholtz equation in dimension 2 or 3 but for very particular domains: a straight and then curved segment in 2D and a unit disc in 3D.

*CMAP, Ecole polytechnique, Route de Saclay, 91128 Palaiseau Cedex.

finir l'intro et mettre le plan

The remainder of the paper focuses on the numerical solution of the integral equations (**??**) and (**??**) using Galerkin finite elements. Though this will not lead to spectral convergence as when using trigonometric polynomial, we aim to generalize our method to cases where the eigenvectors of the operators are not explicitly known. In the first section, we treat the two equations corresponding to the case $k = 0$ and when Γ is the segment. We introduce the same rescaled operators as in [3]. We derive a simple preconditioner for both integral equation in the form of the square root of the differential operators (15) and (16), and verify its efficiency on numerical experiments using Galerkin discretization with piecewise polynomial finite elements. In the next section, we treat the general case using standard compact perturbation arguments, leading to a new preconditioner. Numerical tests are again provided.

2 The scattering problem in 2D

We focus here on the 2D case and consider the problem of the scattering of a wave by an open line. Let Γ be a smooth non-intersecting open curve, and let $k \geq 0$ the wave number. We seek for a solution of the problem

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \tag{3}$$

when one considers furthermore

- Dirichlet or Neumann boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \tag{4}$$

  or

$$\frac{\partial u}{\partial n} = u_N \text{ on } \Gamma \tag{5}$$

  respectively.

- Suitable decay at infinity, given by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{6}$$

  with $r = |x|$ for $x \in \mathbb{R}^2$.

In the preceding equation n stands for a smooth unit normal vector to Γ.

Existence and uniqueness of solutions to the previous problems are guaranteed by the following theorem.

Theorem 1. *(see e.g. [11, 15, 16]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (3,4,6) and (3,5,6) both possess a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ, for the Dirichlet problem, the solution has an unbounded gradient, with for $x \in \gamma$*

$$\frac{\partial u}{\partial n}(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right),$$

while for the Neumann problem, one has pas joli Qu'est-ce qui n'est pas joli exactement ? Je ne sais pas trop quoi changer.

$$u(x) = \sum_{i=1}^{2} \alpha_i \sqrt{\rho_i} \chi_i + \psi$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$. Here, χ_1 and χ_2 are two functions concentrated near each edge of the arc, ρ_i is the distance between x and the tip i, and α_i are two real numbers.

For the definition of Sobolev spaces on smooth open curves, we follow [10] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma}) \right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential The solution to the preceding problems can be expressed through integral formulations. Namely, we define for any smooth function λ defined on Γ the single-layer potential $\mathcal{S}_k$ by

$$\mathcal{S}_k \lambda(x) = \int_\Gamma G_k(x-y)\lambda(y) d\sigma(y) \tag{7}$$

where G_k is the Green's function defined by

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi} \ln|z|, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{8}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here $H_k^{(0)}$ is the Hankel function of the first kind. It is well known that $\mathcal{S}_k$ can be extended to a continuous mapping

$$\mathcal{S}_k : \tilde{H}^{-1/2}(\Gamma) \to H^1_{\text{loc}}(\mathbb{R}^2)$$

and calling $S_k = \gamma \mathcal{S}_k$ where γ is the trace operator on Γ, we may write the solution u of the Dirichlet problem (3,4,6) as

$$u = \mathcal{S}_k \lambda \tag{9}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ solves

$$S_k \lambda = u_D\,. \tag{10}$$

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k\mu(x) = \int_\Gamma n(y)\cdot\nabla G_k(x-y)\mu(y)d\sigma(y).$$

for any smooth function μ defined on γ. This time, $\mathcal{D}_k$ extends to a continuous mapping

$$\mathcal{D}_k : \tilde{H}^{1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$$

and the function $\mathcal{D}_k\mu$ has a jump across Γ. We define

$$D_k\mu = \{\mathcal{D}_k\mu\}$$

where $\{u\}$ denotes the average value of u on each side of Γ. D_k is then a continuous operator from $\tilde{H}^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

It is well-known that the normal derivative of $\mathcal{D}_k\mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \partial_n\mathcal{D}_k$. This operator admits the representation for $x \in \Gamma$

$$N_k\mu = \lim_{\varepsilon\to 0^+}\int_\Gamma n(y)\cdot\nabla G(x+\varepsilon n(x)-y)\mu(y)d\sigma(y). \tag{11}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\langle N_k\mu,\nu\rangle = \int_{\Gamma\times\Gamma} G_k(x-y)\mu'(x)\nu'(y) - k^2G_k(x,y)\mu(x)\nu(y)n(x)\cdot n(y)d\sigma(x)d\sigma(y)\,. \tag{12}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u to the Neumann problem (3,5,6) can be written as

$$u = \mathcal{D}_k\mu \tag{13}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ solves

$$N_k\mu = u_N\,. \tag{14}$$

3 The case of Laplace equation on a flat segment

In this section, we restrict our attention to the case where Γ is the open segment $(-1,1)\times\{0\}$, and $k=0$. We study the properties of the equations

$$S\lambda = -u_D$$

and

$$N\mu = u_N$$

and show their invertibility in a range of Sobolev-like spaces. This problem has been already considered thoroughly, both in terms of analytical and numerical properties in the literature (see for instance [**?**, **?**, **?**, **?**]), and it turns out that the Chebyshev polynomials of first and second kind play a very important role. However, we go further compared to the literature by constructing a functional framework close to Sobolev spaces, based on Chebyshev polynomials that allows us to give a complete framework for the existence and uniqueness of the solutions to the preceding equations, as well as new preconditioners.

3.1 Analytical setting

Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}.$$

We let $\omega(x) = \sqrt{1-x^2}$. We also denote by ω the operator $u(x) \mapsto \omega(x)u(x)$. Moreover, let ∂_x the derivation operator, the Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0 \text{ and } (1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0$$

that we rewrite under the form

$$(\omega\partial_x)^2 T_n = -n^2 T_n\,, \tag{15}$$

$$(\partial_x\omega)^2 U_n = -(n+1)^2 U_n\,. \tag{16}$$

(Notice that by $(\partial_x\omega)^2 U_n$ we mean $\partial_x(\omega\partial_x(\omega U_n))$.) As we shall see, the preceding equations are crucial in our analysis.

Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} f^2(x)\sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by $\langle\cdot,\cdot\rangle_{\frac{1}{\omega}}$ and $\langle\cdot,\cdot\rangle_{\omega}$ the inner products in $L^2_{\frac{1}{\omega}}$ and L^2_{ω} respectively. Chebyshev polynomials also satisfy

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } m = n = 0 \\ \pi/2 \text{ otherwise} \end{cases} \tag{17}$$

and

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi/2 \text{ otherwise} \end{cases} \tag{18}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{19}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by

$$\hat{u}_n := \begin{cases} \dfrac{2}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x)T_n(x)}{\sqrt{1-x^2}} dx & \text{if } n \neq 0\,, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x)}{\sqrt{1-x^2}} dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^{1} \frac{u^2(x)}{\sqrt{1-x^2}} dx = \frac{\pi \hat{u}_0^2}{2} + \pi \sum_{n=1}^{+\infty} \hat{u}_n^2 \,.$$

When u is furthermore a smooth function, the series (19) converges uniformly to u. Similarly, any function $v \in L^2_\omega$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^{1} v(x) U_n(x) \sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^{1} v^2(x) \sqrt{1-x^2} dx = \frac{\pi}{2} \sum_{n=0}^{+\infty} \check{v}_n^2 \,.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we may define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^s \hat{u}_n^2 < +\infty \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u, v \rangle_{T^s} = \frac{\pi}{2} \hat{u}_0 \hat{v}_0 + \pi \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty} n^{2s} \left| \hat{u}_n \right|^2 .$$

We denote by T^∞ the Fréchet space $T^\infty := \cap_{s \in \mathbb{R}} T^s$, and by $T^{-\infty}$ the set of continuous linear forms on T^∞. For $l \in T^{-\infty}$, we note $\hat{l}_n = l(T_n)$, so that for $u \in T^\infty$,

$$l(u) = \frac{\pi}{2} \hat{l}_0 \hat{u}_0 + \pi \sum_{n=1}^{+\infty} \hat{l}_n \hat{u}_n \,.$$

We choose to identify the dual of $L^2_{\frac{1}{\omega}}$ to itself using the previous bilinear form. With this identification, any element of T^s with $s \geq 0$ can also be seen as an element of $T^{-\infty}$. Furthermore, the space T^{-s} can be defined for all $s \geq 0$ as

$$T^{-s} = \left\{ u \in T^{-\infty} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^{-s} \hat{u}_n^2 < \infty \right\}.$$

Using the preceding identification T^{-s} becomes the dual of T^s. Obviously, for any real s, if $u \in T^s$, the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x)$$

converges to u in T^s. Therefore, the space $C^\infty([-1,1])$ is dense in T^s for all $s \in \mathbb{R}$. For $s < t$, the inclusion $T^s \subset T^t$ is compact.

In a similar fashion, we define the following spaces:

Definition 2. *For all $s \geq 0$, we set*

$$U^s = \left\{ u \in L^2_\omega \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^s \check{u}_n^2 \right\}.$$

We extend as before the definition to negative indices by setting U^{-s} to be the dual of U^s for $s \geq 0$.

Lemma 1. *For all real s, $T^s \subset U^s$ and for all $s > 1/2$, $U^s \subset T^{s-1}$.*

Proof. The first property is immediate once it has been noticed that for $n \geq 2$, $T_n(x) = \frac{1}{2}(U_n - U_{n-2})$, while $T_0 = U_0$ and $T_1 = \frac{U_1}{2}$. The second comes from the expressions

$$U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}.$$

When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging, allowing to identify u to a function in $T^{-\infty}$, with

$$\hat{u}_0 = \sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2\sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since $u \in U^s$, $(1+n^2)^{s/2} |\check{u}|$ is in l^2 and by continuity of the adjoint of the Cesàro operator in l^2, the sequence $r_n := \left(\sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k|\right)_n$ is in l^2. But

$$\begin{aligned}
\|u\|^2_{T^{s-1}} &= \sum_{n=0}^{+\infty} (1+n^2)^{s-1} |\hat{u}_n|^2 \\
&\leq 4\sum_{n=0}^{+\infty} (1+n^2)^{s-1} \left(\sum_{k=n}^{+\infty} |\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty} \left(\sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k|\right)^2. \\
&= 4\|r_n\|^2_{l^2}.
\end{aligned}$$

□

One immediate consequence is that $T^\infty = U^\infty$. Moreover, we have the following result:

Lemma 2.

$$T^\infty = C^\infty([-1,1])\,.$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (15), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^{1} \frac{(\omega\partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, this proves that $C^\infty([-1,1]) \subset T^\infty$.

To prove the converse inclusion, first notice that, by normal convergence of the series, $T^\infty \subset C^0([-1,1])$. Now, let $u \in T^\infty$, it suffices to show that $u' \in T^\infty$ and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty} n u_n U_{n-1}(x).$$

Therefore, u' is in $U^\infty = T^\infty$. $\square$

3.2 Single layer equation

In this section we focus on the equation $S\lambda = g$, that is we seek $\lambda \in \tilde{H}^{-1/2}$ such that

$$-\frac{1}{2\pi} \int_{-1}^{1} \log|x-y| \lambda(y) = -g(x), \quad \forall x \in (-1,1)\,. \tag{20}$$

This equation is sometimes called "Symm's integral equation" and its resolution has received a lot of attention in the 1990's. Numerical methods, using both collocation and Galerkin have been presented and analyzed [2, 14, 18–20].

Our analysis lies on the following formula. For a proof, see for example [8] Theorem 9.2. Note that this is also the main ingredient in several connected works, as [7] and [3].

Proposition 1.

$$-\frac{1}{2\pi} \int_{-1}^{1} \frac{\ln|x-y|}{\sqrt{1-y^2}} T_n(y) dy = \lambda_n T_n(x)$$

where

$$\lambda_n = \begin{cases} \dfrac{\ln(2)}{2} & \textit{if } n = 0 \\ \dfrac{1}{2n} & \textit{otherwise.} \end{cases}$$

Using the decomposition of g and of the logarithmic kernel on the basis T_n, we see that the solution λ to equation (20) admits the following expansion

$$\lambda(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \frac{\hat{g}_n}{\lambda_n} T_n(x). \tag{21}$$

We deduce the following well-known fact:

Corollary 1. *If the data g is in $C^\infty([-1,1])$, the solution λ to the equation*

$$S\lambda = g$$

is of the form

$$\lambda = \frac{\alpha}{\sqrt{1-x^2}}$$

with $\alpha \in C^\infty([-1,1])$.

Proof. Let $\alpha = \sqrt{1-x^2}\lambda$ where λ is the solution of $S\lambda = g$ with $g \in C^\infty([-1,1])$. Lemma 2 implies that $g \in T^\infty$, and by equation (21),

$$\hat{\alpha}_n = \frac{\hat{g}_n}{\lambda_n},$$

so α also belongs to $T^\infty = C^\infty([-1,1])$. □

We follow [3] by noticing that the behavior in $\frac{1}{\sqrt{1-x^2}}$ is consistent with the expected singularity near the edges and introduce the weighted single layer operator as the operator that appeared in Proposition 1.

Definition 3. *(See [3]) Let S_ω be the weighted single layer operator defined by*

$$S_\omega : \alpha \in C^\infty([-1,1]) \longrightarrow -\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln|x-y|}{\omega(y)}\alpha(y)dy$$

We also recall that the operator $(\omega\partial_x)^2$ is defined by

$$(\omega\partial_x)^2 : \alpha \in C^\infty([-1,1]) \longrightarrow (1-x^2)\alpha''(x) - x\alpha'(x).$$

The action of these operators on T^∞ is easy to analyze using (15) and Proposition 1. By density of T^∞ in T^s for all s, we get:

Proposition 2. *The operator S_ω is a self-adjoint, positive definite operator, and defines a continuous bijection from T^s to T^{s+1} for all real s. In particular, S_ω is of order 1 and is compact in T^s. Similarly, for any $s \in \mathbb{R}$, the operator $-(\omega\partial_x)^2$ is positive, self-adjoint, and of order -2.*

Proof. It suffices to remark that if $u = \sum_{n=0}^{\infty} \hat{u}_n T_n \in T^s$, then

$$S_\omega u = \frac{\ln(2)}{2}\hat{u}_0 T_0 + \sum_{n=1}^{\infty} \frac{\hat{u}_n}{2n} T_n$$

while

$$-(\omega\partial_x)^2 u = \sum_{n=0}^{\infty} n^2 \hat{u}_n T_n\,.$$

□

To obtain the solution of (20), we can thus solve

$$S_\omega \alpha = -u_D, \tag{22}$$

and let $\lambda = \frac{\alpha}{\omega}$.

We are now in a position to find the inverse of S_ω. An explicit inverse has already appeared in the literature. In particular, in [5, 17], explicit variational forms for this inverse operator are derived rigorously. (A similar method is also employed in the recent paper [4] in $\mathbb{R}^3$ for the case of the unit disk.) We just state here the following formal decomposition:

$$\frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|^2} = \frac{-1+xy}{2|x-y|^2} = \sum_{n=1}^{+\infty} nT_n(x)T_n(y)$$

with $M(x,y) = \frac{1}{2}\left((y-x)^2 + (\omega(x)+\omega(y))^2\right)$.

However, using the preceding analysis, we have an alternative way of defining this exact inverse, which leads to an inverse in the form of the square root of a local operator. To state the next result, we define the operator π_0 as the $L^2_{1/\omega}$ orthogonal projector on T_0. Namely

$$\pi_0\alpha(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\alpha(y)}{\omega(y)}dy\,.$$

The preceding definition can be extended to $u \in T^s$ for any $s \in \mathbb{R}$ by setting $\pi_0 u$ as the solution of

$$\begin{cases} \langle \alpha - \pi_0\alpha, T_0\rangle_{\frac{1}{\omega}} = 0\,, \\ \pi_0\alpha \in \mathrm{Span}(T_0)\,, \end{cases}$$

since $T_0 \in T^\infty$. Of course, π_0 is continuous from T^s to T^∞ for any s.

Theorem 2. *The operators S_ω, $-(\omega\partial_x)^2$, and π_0 commute and satisfy*

$$S_\omega\left(-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0\right)S_\omega = \frac{I}{4}\,. \tag{23}$$

Proof. The Chebyshev polynomials (T_n) are a common Hilbert basis of eigenvectors for the three operators $S\omega$, $-(\omega\partial_x)^2$ and π_0, so they all commute on $T^{-\infty}$. Moreover, one has using Proposition 1, equation (15) and the definition of π_0

$$S_\omega T_n = \frac{1}{2n}T_n\,,\ \pi_0 T_n = 0 \text{ and } -(\omega\partial_x)^2 T_n = n^2 T_n \text{ if } n \neq 0\,,$$

while

$$S_\omega T_0 = \frac{\ln(2)}{2}T_0\,,\ \pi_0 T_0 = T_0 \text{ and } -(\omega\partial_x)^2 T_0 = 0 \text{ otherwise.}$$

Equation (23) follows from the fact that (T_n) is a Hilbert basis of $T^{-\infty}$. □

Remark 1. *A direct proof of the commutation of S_ω and $-(\omega\partial_x)^2$ can be done. We give it in the more general case of Helmholtz equation (see Theorem 6).*

From the preceding formula, we can extract the explicit inverse of S_ω in terms of the square root of the inner operator.

Definition 4. *Let an operator $A : T^s \to T^{s+2p}$ such that*

$$AT_n = a_n T_n$$

with $a_n \geq 0$. We define $\sqrt{A}T^s \to T^{s+p}$ as the operator satisfying

$$\sqrt{A}T_n = \sqrt{a_n}T_n.$$

Corollary 2. *The inverse of S_ω can be equivalently expressed as*

$$S_\omega^{-1} = 2\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}\,. \tag{24}$$

The last result shows that $\sqrt{-(\omega\partial_x)^2}$ and S_ω can be thought as inverse operators (modulo constant terms) and that, at the very least, $\sqrt{-(\omega\partial_x)^2}$ can be used as a preconditioner for S_ω. Indeed, $2\sqrt{-(\omega\partial_x)^2}S_\omega$ is a compact perturbation of identity. This makes a clear link with the approximation of the Dirichlet to Neumann map proposed in [1] in terms of a square root operator. The link will be even clearer when Helmholtz equation will be considered.

3.3 Hypersingular equation

We now turn our attention to the equation

$$N\mu = g \tag{25}$$

Similarly to the previous section and following again the idea of [3], we consider a rescaled version of the hypersingular operator $N_\omega := N\omega$ defined by

$$N_\omega\mu = \lim_{\varepsilon\to 0}\int_{-1}^{1} n(y)\cdot\nabla G(x+\varepsilon n(x)-y)\sqrt{1-y^2}dy$$

Analogous to the previous section, we can get the solution to (25) by solving

$$N_\omega\beta = u_N, \tag{26}$$

and letting $\mu = \omega\beta$. We show that N_ω can also be analyzed in our functional framework, using now the spaces U^s.

Lemma 3. *For any β, β', one has*

$$\langle N_\omega\beta, \beta'\rangle_\omega = \langle S_\omega\omega\partial_x\omega\beta, \omega\partial_x\omega\beta'\rangle_{\frac{1}{\omega}}\,.$$

Proof. It is sufficient to show this formula for β and β' in U^∞ by density. Indeed, for such β, β', both sides of the identity define continuous bilinear forms on T^∞. We use the well-known integration by part formula

$$\langle Nu, v\rangle = \langle S\partial_x u, \partial_x v\rangle\,,$$

valid when u and v vanish at the extremities of the segment (see for example [3]). For a smooth β, we thus have

$$\langle N(\omega\beta), (\omega\beta')\rangle = \langle S\partial_x(\omega\beta), \partial_x(\omega\beta')\rangle$$

which obviously implies the announced identity. □

Proposition 3. *N_ω is a positive definite, self-adjoint operator continuous from U^s to U^{s-1} for all real s. For all $n\in\mathbb{N}$, we have*

$$N_\omega U_n = \frac{n+1}{2}U_n.$$

Moreover, $-(\partial_x\omega)^2$ is also positive definite of order 2.

Proof. From identity $T'_{n+1} = (n+1)U_n$ and Equation (15) we obtain

$$\omega\partial_x\omega U_n = -(n+1)T_{n+1}.$$

Therefore, by Lemma 3

$$\begin{aligned}\langle N_\omega U_m, U_n\rangle_\omega &= (n+1)(m+1)\,\langle S_\omega T_{m+1}, T_{n+1}\rangle_{\frac{1}{\omega}} \\ &= \delta_{m=n}\frac{n+1}{2}.\end{aligned}$$

The fact that $-(\partial_x\omega)^2$ is self-adjoint positive definite of order 2 is a consequence of Equation (16). □

As a consequence, we have the following result:

Theorem 3. *The operators N_ω and $-(\partial_x\omega)^2$ commute and*

$$-N_\omega(\partial_x\omega)^{-2}N_\omega = \frac{I}{4}\,.$$

The inverse of N_ω is therefore

$$N_\omega^{-1} = 2\sqrt{-(\partial_x\omega)^{-2}}\,. \tag{27}$$

In [3], it is shown that $N_\omega S_\omega$ and $S_\omega N_\omega$ are order zero operators with a spectrum concentrated around $\frac{1}{4}$, which can be exploited for preconditioning purposes.

Analogously to the previous, one can also derive the formal expansions as in [6]

$$\frac{1}{(x-y)^2} = \sum_{n=0}^{+\infty} 2(n+1)U_n(x)U_n(y)\,,$$

that lead, by applying for $(\partial_x\omega)^{-2}$ on both sides, to the following explicit kernel for the inverse of N_ω:

$$\ln\left(\frac{(y-x)^2 + (\omega(x)+\omega(y))^2}{2|x-y|}\right) = \sum_{n=0}^{+\infty}\frac{2U_n(x)U_n(y)}{n+1}.$$

3.4 Galerkin analysis

In this section, we describe and analyze a new Galerkin scheme to solve the integral equations (20) and (25). Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [13, Chap. 4,] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [**?**] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [3], and some results exist for piecewise linear functions in the colocation setting [**?**].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 4 and Theorem 5.

In what follows, we introduce a discretization of the segment $[-1,1]$ as $-1 = x_0 < x_1 < \cdots < x_N = 1$, and let $\theta_i := \arccos(x_i)$. We define the parameter h of the discretization as

$$h := \min_{i=0\cdots N-1} |\theta_{i+1} - \theta_i| .$$

In practice, one should use a mesh for which $|\theta_i - \theta_{i+1}|$ is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the $i-th$ interval is approximately $(ih)^2$. In comparison, in the h-BEM method with $p = 1$ polynomial order, this would only lead to a convergence in $O(h)$ (cf. [**?**, Theorem 1.3]).

3.4.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution λ of (20). To achieve it, we use a variational formulation of (22) to compute an approximation α_h of α, and set $\lambda_h = \frac{\alpha_h}{\omega}$. Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at x_i. Let α_h the unique solution in V_h to

$$\langle S_\omega \alpha_h, \alpha'_h \rangle_{\frac{1}{\omega}} = -\langle u_D, \alpha'_h \rangle_{\frac{1}{\omega}}, \quad \forall \alpha'_h \in V_h.$$

We shall prove the following result:

Theorem 4. *If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C h^{s+1/2} \|u_D\|_{T^{s+1}} .$$

In particular, when u_D is smooth, it belongs to T^∞ so the rate of convergence is $h^{5/2}$. We start by proving an equivalent of Céa's lemma:

Lemma 4. *There exists a constant C such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

Proof. In view of the properties of S_ω stated in Proposition 1, we have the equivalent norm

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \rangle .$$

Since $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = -\langle u_D, \alpha'_h \rangle$ for all $\alpha'_h \in V_h$, we deduce

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \rangle , \quad \forall \alpha'_h \in V_N.$$

By duality

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of S_ω from $T^{-1/2}$ to $T^{1/2}$. □

From this we can derive the rate of convergence for α_h to the true solution α. We use the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h, which satisfies the following properties:

Lemma 5. *For any function u,*

$$\|(\mathrm{I}-\mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C\,\|u\|_{L^2_{\frac{1}{\omega}}}\,,$$

$$\|(\mathrm{I}-\mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2\,\|u\|_{T_2}\,.$$

The proof requires the following well-known result:

Lemma 6. *Let $\tilde{u}$ in the Sobolev space $H^2(\theta_1,\theta_2)$,such that $\tilde{u}(\theta_1)=\tilde{u}(\theta_2)=0$. Then there exists a constant C independent of θ_1 and θ_2 such that*

$$\int_{\theta_1}^{\theta_2} \tilde{u}(\theta)^2 \leq C(\theta_1-\theta_2)^4 \int_{\theta_1}^{\theta_2} \tilde{u}''(\theta)^2 d\theta$$

Proof. The first inequality is obvious since $\mathbb{P}_h$ is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the $L^2_{\frac{1}{\omega}}$ norm,

$$\|I-\mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I-I_h u\|_{L^2_{\frac{1}{\omega}}}\,, \tag{28}$$

where $I_h u$ is the piecewise affine (continuous) function that matches the values of u at the breakpoints x_i. By **??**, on each interval $[x_i,x_{i+1}]$, the function $\tilde{u}(\theta):=u(\cos(\theta))$ is in the Sobolev space $H^2(\theta_i,\theta_{i+1})$ so we can apply Lemma 6:

$$\int_{x_i}^{x_{i+1}} \frac{(u-I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}} (\tilde{u}-\tilde{I_h}u)^2 \leq (\theta_{i+1}-\theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (\tilde{u}-\tilde{I_h u})''^2.$$

This gives

$$\int_{x_i}^{x_{i+1}} \frac{(u-I_h u)^2}{\omega} \leq 2h^4 \left(\int_{\theta_i}^{\theta_{i+1}} \tilde{u}''^2 + \int_{\theta_i}^{\theta_{i+1}} \tilde{I_h}u''^2 \right). \tag{29}$$

Before continuing, we need to establish the following result

Lemma 7. *There holds*

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 \leq C \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega}$$

Proof. The expression of $I_h u$ is given by

$$\tilde{I_h u}(\theta) = u(x_i) + \frac{u(x_i)-u(x_{i+1})}{\cos(\theta_{i+1})-\cos(\theta_i)}(\cos(\theta)-\cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 = \left(\frac{u(x_i)-u(x_{i+1})}{\cos(\theta_{i+1})-\cos(\theta_i)}\right)^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left(\int_{x_i}^{x_{i+1}} u'(t)dt\right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \leq \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. □

We can now conclude the proof of Lemma 5. Summing all inequalities (29) for $i = 0, \cdots N+1$, we get

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}}^2 \leq Ch^4 \left(\|u\|_{T^2}^2 + \|u'\|_{T_0}^2\right).$$

By **??**, the operator ∂_x is continuous from T^2 to T^0 which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T^2}.$$

Thanks to (28), this concludes the proof. □

We obtain the following corollary by interpolation:

Corollary 3. *The operator* $\mathrm{I} - \mathbb{P}_N$ *is continuous from* $L^2_{\frac{1}{\omega}}$ *to* T^s *for* $0 \leq s \leq 2$ *with*

$$\|(\mathrm{I} - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s}.$$

We can now prove Theorem 4:

Proof. First, using **??**, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}}.$$

Moreover, if u_D is in T^{s+1}, then $\alpha = S_\omega^{-1} u_D$ is in T^s and $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$. By the analog of Céa's lemma, Lemma 4, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s}.$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} = \inf_{\eta\in T^{1/2},\eta\neq 0} \frac{(\alpha - \mathbb{P}_N\alpha, \eta - \mathbb{P}_h\eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

Using Cauchy-Schwarz's inequality and Corollary 3 ($s = \frac{1}{2}$),

$$\|\alpha - \mathbb{P}_h\alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s}.$$

□

3.4.2 Neumann problem

We now turn to the numerical resolution of (25). We use a variational form for equation (26), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce W_h the space of continuous piecewise affine functions with breakpoints at x_i, and we denote by β_h the unique solution in W_h to the variational equation:

$$\langle N_\omega\beta_h, \beta'_h\rangle_\omega = \langle u_N, \beta'_h\rangle_\omega\,, \quad \forall\beta'_h \in W_h. \tag{30}$$

Then, $\mu_h = \omega\beta_h$ is the proposed approximation for μ. We shall prove the following:

Theorem 5. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Like before, we start with an analog of Céa's lemma:

Lemma 8. *There exists a constant C such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta'_h\in W_h} \|\beta - \beta'_h\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator I_h:

Lemma 9. *There holds*

$$\|u - I_hu\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_hu\|_{U^1} \leq Ch \|u\|_{U^2}$$

Proof. We only show the first estimation, the method of proof for the second being similar. Using again Lemma 6 on each segment $[x_i, x_{i+1}]$, one can write

$$\begin{aligned}\int_{x_i}^{x_{i+1}} \omega(u - I_hu)^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_hu)''^2 \\ &\leq Ch^4 \left(2\int_{\theta_i}^{\theta_{i+1}} Vu''^2 + 2\int_{\theta_i}^{\theta_{i+1}} (VI_hu)''^2\right)\end{aligned}$$

where we recall that for any function u, Vu is defined as

$$Vu(\theta) = \sin(\theta)u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

Lemma 10.

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(\|u\|_{U_2}^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

Proof. Using the expression of I_h, one can write

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(|u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \right.$$
$$\left. + \left(\frac{u(x_{i+1}) - u(x_i)}{\cos\theta_{i+1} - \cos\theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2(1+\cos^2) \right) \quad (31)$$

We can estimate the first term, thanks to **??**:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned}
(u(x_{i+1}) - u(x_i))^2 &= \left(\int_{x_i}^{x_{i+1}} \partial_x u \right)^2 \\
&\leq \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\
&= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2.
\end{aligned}$$

to conclude, it remains to observe that the quantity

$$\frac{|(\theta_{i+1} - \theta_i)| \int_{\theta_i}^{\theta_{i+1}} \sin^2(1+\cos^2)}{(\cos(\theta_i) - \cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2(\theta_i)$ □

We now plug the estimate Lemma 10 in (31), and sum over i:

$$\|u - I_h u\|_{L^2_\omega}^2 \leq Ch^4(\|u\|_{U^2}^2 + \|u'\|_{L^2_\omega}^2).$$

This implies the claim once we use the continuity of ∂_x from U^2 to U^0, cf. **??**. □

We can now prove Theorem 5

Proof. Let us denote by Π_h the Galerkin projection operator defined by $\beta \mapsto \beta_h$. Since it is an orthogonal projection on W_h with respect to the scalar product $(\beta, \beta') := \langle N_\omega \beta, \beta' \rangle$, it is continuous from $U^{1/2}$ to itself, so we have for any u in $U^{1/2}$.

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C \|u\|_{U^{1/2}} .$$

We are now going to show the estimate

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2} .$$

By the analog of Céa's lemma Lemma 8, one has $\|(I - \Pi_h)u\|_{U^{1/2}} \leq \|(I - I_h)u\|_{U^{1/2}}$. By interpolation, this norm satisfies

$$\|(I - I_h)u\|_{U^{1/2}} \leq C\sqrt{\|(I - I_h)u\|_{U^0}}\sqrt{\|(I - I_h)u\|_{U^1}},$$

which yields, applying Lemma 9,

$$\|(I - I_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2} .$$

By interpolation, for all $s \in [1/2, 2]$, we get

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{s-1/2} \|u\|_{U^s} .$$

In view of **??**, we have $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$. Moreover, since N_ω is a continuous bijection from U^{s+1} to U^s for all s, there holds

$$\|\beta\|_{U^s} = \left\|N_\omega^{-1} u_N\right\|_{U^s} = \|u_N\|_{U^{s-1}} .$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-1/2} \|\beta\|_{U^s} \leq Ch^{s-1/2} \|u_N\|_{U^{s-1}} .$$

□

We run some numerical tests to confirm the efficiency of the preconditioners we introduced. In the first test, we solve the Laplace equation with Dirichlet boundary conditions. The chosen function u_D is

$$u_D(x) = \frac{1}{\sqrt{(x - x_0)^2 + c^2}}$$

where $x_0 = 0.5$, $c = 1e^{-5}$. In Table 1, we show the evolution of the number of iterations as the size of the mesh increases. This data confirms that the number of iterations is independant of the mesh size. In Figure 1, we show the relative residual history for the resolution of this linear system with $N = 500$.

3.5 Numerical results

In this section, we provide some numerical results to accompany the previous analysis.

Inclure les figures d'ordre de convergence.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
25	7	0.07	25	0.19
50	7	0.10	37	0.45
100	7	0.14	48	0.80
200	7	0.18	62	1.21
400	7	0.33	78	2.51
800	7	0.78	99	5.48
1600	7	2.24	124	12.88

Table 1: Number of iteration and time needed for the computation of the solution to (20) using Galerkin finite elements with and without preconditioner.

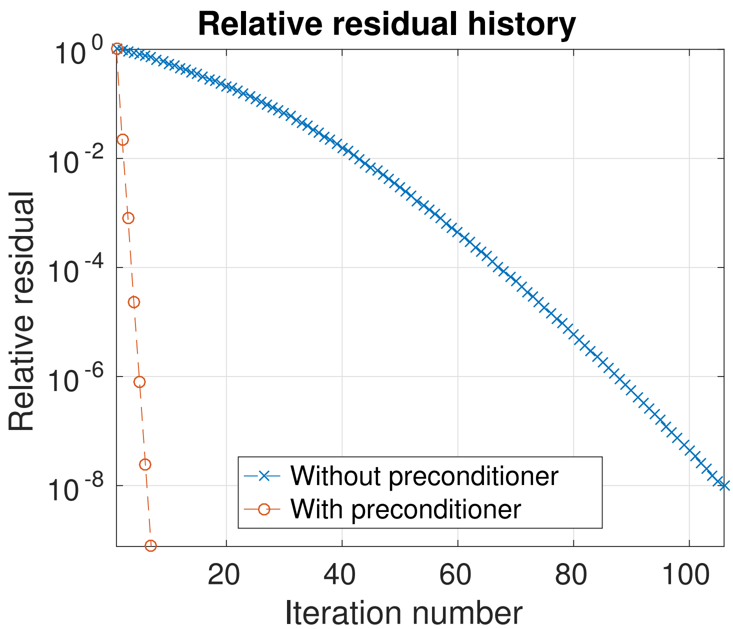

Figure 1: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N = 500$.

3.5.1 Convergence rates

3.5.2 Efficiency of the preconditioners

4 The case of the Helmholtz equation on a segment

In tis section, we keep studying the flat segment $\Gamma = [-1, 1]$, but tackle the non-zero frequency. Recall the definition of the single layer and hypersingular operators, S_k and N_k, given in (7) and (11), and the integral equations for the Dirichlet and Neumann problems, (10) and (14). We introduce new preconditioners for the numerical resolution of those two equations that generalize the results of the previous section. Let $S_{k,\omega} := S_k \frac{1}{\omega}$ and $N_{k,\omega} := N_k \omega$. We begin by establishing the following result:

Theorem 6. *The following commutations hold:*

$$S_{k,\omega} \left[-(\omega \partial_x)^2 - k^2 \omega^2 \right] = \left[-(\omega \partial_x)^2 - k^2 \omega^2 \right] S_{k,\omega},$$

$$N_{k,\omega} \left[-(\partial_x \omega)^2 - k^2 \omega^2 \right] = \left[-(\partial_x \omega)^2 - k^2 \omega^2 \right] N_{k,\omega}.$$

Proof. We start with the first commutation. Since $(\omega \partial_x)^2$ is self adjoint and symmetric, we have

$$S_{k,\omega} (\omega \partial_x)^2 = \int_{-1}^{1} \frac{(\omega_y \partial_y)^2 \left[G_k(x-y) \right] u(y)}{\omega(y)},$$

where we use the notation ω_y and ∂_y to emphasize the dependence in the variable y. Thus,

$$S_{k,\omega} (\omega \partial_x)^2 - (\omega \partial_x)^2 S_{k,\omega} = \int_{-1}^{1} \frac{D_k(x,y) u(y)}{\omega(y)},$$

where $D_k(x,y) := \left[(\omega_y \partial_y)^2 - (\omega_x \partial_x)^2 \right] \left[G_k(x-y) \right]$. One has

$$D_k(x,y) = G_k''(x-y)(\omega_y^2 - \omega_x^2) + G_k'(x-y)(y+x).$$

Since G_k is a solution of the Helmholtz equation, we have for all $(x \neq y) \in \mathbb{R}^2$

$$G_k'(x-y) = (y-x)(G_k''(x-y) + k^2 G(x-y)),$$

thus

$$D_k(x,y) = G_k''(x-y) \left(\omega_y^2 - \omega_x^2 + y^2 - x^2 \right) + k^2 (y^2 - x^2) G_k(x-y).$$

Note that $y^2 - x^2 = \omega_x^2 - \omega_y^2$ so the first term vanishes and we find

$$S_{k,\omega} (\omega \partial_x)^2 - (\omega \partial_x)^2 S_{k,\omega} = k^2 \left(\omega^2 S_{k,\omega} - S_{k,\omega} \omega^2 \right).$$

The proof of the second commutation is quite heavy and is postponed to annexes. $\square$

This theorem shows that the operators S_k and N_k share the same eigenvectors as, respectively, $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$ and $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]$. We can look for eigenfunctions of the operator $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$, to find a diagonal basis for $S_{k,\omega}$. They are the solutions to the differential equation

$$(1-x^2)y'' - xy' = \lambda y.$$

Once we set $x = \cos\theta$, $\tilde{y}(\theta) = y(x)$, $q = \frac{k^2}{4}$, $a = \lambda + 2q$, $\tilde{y}$ is a solution of the standard Mathieu equation

$$\tilde{y}'' + (a - 2q\cos(2\theta))\tilde{y} = 0.$$

There are a discrete set of values $a_{2n}(q)$ for which this equation possesses an even and 2π periodic function. The corresponding solution is known as the Mathieu cosine, and usually denoted by CE_n, with the normalization taken as

$$\int_0^{2\pi} \mathrm{CE}_n(\theta)^2 d\theta = \pi.$$

Those functions satisfy

$$\int_{-\pi}^{\pi} \mathrm{CE}_n(\theta)\mathrm{CE}_m(\theta) = \pi\delta_{m,n},$$

thus, any even 2π periodic function in $L^2(-\pi,\pi)$ can be expanded along the functions CE_n, with the coefficients obtained by orthonormal projection. Letting

$$T_n^k := \mathrm{CE}_n(\arccos(x),$$

in analogy to the zero-frequency case, we have

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] T_{n,k} = \lambda_{n,k}^2 T_{n,k}.$$

For large n, using the general results from the theory of Hill's equations (see [12, eq. 28.29.21]) we have the following asymptotic formula for $\lambda_{n,k}$:

$$\lambda_{n,k}^2 = n^2 - \frac{k^4}{16n^2} + o\left(n^{-2}\right).$$

The first commutation established in Theorem 6 implies that the Matthieu cosines are also the eigenfunctions of the single-layer operator. An equivalent statement is given in [**?**, Thm 4.2], if we allow the degenerate case $\mu = 0$. Unfortunately, the lack of explicit eigenvalues for $S_{k,\omega}$ prevents us from applying a similar analysis as that performed in the first part of this work. Instead, we will perform a perturbation analysis, much like [3].

4.1 Dirichlet

In this section, we focus on the Dirichlet problem with non-zero frequency, and the corresponding integral equation

$$S_k\lambda = u_D \tag{32}$$

Here again, we define a rescaled operator $S_{k,\omega} := S_k \frac{1}{\omega}$, i.e.

$$S_{k,\omega_\Gamma}\alpha : x \mapsto \int_{-1}^{1} \frac{H_0(k|x-y|)\alpha(y)}{\omega(y)} dy.$$

If we let $\lambda = \dfrac{\alpha}{\omega}$, then equation (32) is equivalent to

$$S_{k,\omega}\alpha = u_D.$$

The Hankel function can be written as

$$H_0(z) = \frac{-1}{2\pi} \ln|z| J_0(z) + F_1(z^2)$$

where J_0 is the Bessel function of first kind and order 0 and where F_2 is analytic. Using the power series definition of J_0,

$$\begin{aligned}\frac{i}{4}H_0(k\,|x-y|) \quad &= \quad \frac{-1}{2\pi}\ln|x-y| \\ &\quad + \frac{1}{2\pi}\frac{k^2}{4}(x-y)^2 \ln|x-y| \\ &\quad + (x-y)^4 \ln|x-y| F_2(x,y) + F_3(x,y)\end{aligned}$$

where F_1 and F_3 are C^∞. We define, for $n \geq 1$,

$$O_n : \alpha \mapsto \int_{-1}^{1} (x-y)^{n-1} \ln|x-y| \frac{\alpha(y)}{\omega(y)}.$$

Lemma 11. *The operator O_n is of order n.*

Proof. To show this, we will prove by reccurence that there exist functions $o_n : \mathbb{N}^2 \to \mathbb{R}$ such that

$$O_n T_k = \sum_{i=-\infty}^{+\infty} o_n(k,i) T_{k-i}$$

satisfying that for all α, β in $\mathbb{N}$,

$$\forall (k,i), \quad \left| \Delta_i^\alpha \Delta_k^\beta o_n(k,i) \right| \leq C_{n,\alpha,\beta}(1+k^2)^{-n-1-\beta} \tag{33}$$

where Δ_i and Δ_k are the discrete differentiation operators in the variables i and k respectively, e.g.

$$\Delta_i o_n(k,i) = o_n(k,i+1) - o_n(k,i),$$

and Δ_i^α is the α-th iterate of Δ_i. We use the convention $T_k := T_{|k|}$ for negative k. First $O_1 = 2\pi S_\omega$ is of order 1, and we have simply:

$$o_0(k,i) = \delta_{i=0} \times \begin{cases} \pi \ln(2) & \text{if } k = 0 \\ \frac{\pi}{k} & \text{otherwise,} \end{cases}$$

which obviously satisfies (33). Second, notice that

$$O_{n+1} = xO_n - O_n x,$$

which combined with the identity

$$xT_n = \frac{T_{n+1}+T_{n-1}}{2}$$

valid for all $n \in \mathbb{Z}$, implies

$$O_{n+1}T_k = \sum_{i=-\infty}^{\infty} o_n(k,i)\frac{T_{k-i+1}+T_{k-i-1}}{2} - \frac{1}{2}O_nT_{k+1} - \frac{1}{2}O_nT_{k-1}$$

Using our recurrence assumption,

$$\begin{aligned} O_{n+1}T_k &= \frac{1}{2}\sum_{i=-\infty}^{\infty}(o_n(k,i-1)-o_n(k+1,i-1))T_{k-i} \\ &\quad +\frac{1}{2}\sum_{i=-\infty}^{\infty}(o_n(k,i+1)-o_n(k-1,i+1))T_{k-i}. \end{aligned}$$

Hence

$$o_{n+1}(k,i) = -\Delta_k o_n(k,i-1) + \Delta_k o_n(k-1,i+1),$$

which concludes the recurrence. □

Lemma 12. *The operator $S_{k,\omega}$ admits the following expansion*

$$S_{k,\omega} = S_\omega + \frac{1}{2\pi}\frac{k^2}{4}O_3 + R_5 + R_\infty$$

where R_5 is an operator of order 5 and R_∞ is a smoothing operator.

In particular, the operator $S_{k,\omega}$ is well defined on $T^{-\infty}$, and is of order 1.

The expansion obtained in Lemma 12 together with the previous lemma imply the following

Theorem 7. *There holds:*

$$S_{k,\omega}(\omega\partial_x)^2S_{k,\omega} = \frac{I}{4} + K$$

where K is of order 2.

In fact, we will show a more precise result, that includes the first correction due to frequency number:

Theorem 8. *There holds*

$$S_{k,\omega}\left(-(\omega\partial_x)^2 - k^2\omega^2\right)S_{k,\omega} = \frac{I}{4} + K'$$

where K' is of order 3.

Proof. Using the expansion of Lemma 12, we can write

$$\begin{aligned} -S_{k,\omega}(\omega\partial_x)^2S_{k,\omega} &= -S_\omega(\omega\partial_x)^2S_\omega \\ &\quad -\frac{1}{2\pi}\frac{k^2}{4}\left(S_\omega(\omega\partial_x)^2O_3 + O_3(\omega\partial_x)^2S_\omega\right) + K' \end{aligned}$$

where K' is of order 4. By Theorem 2, the first term is $\frac{I}{4} + R$ where R is a smoothing operator. Then, simple calculations show that

$$(\omega\partial_x)^2\left((x-y)^2\ln|x-y|\right) = 2\omega^2(x)\ln|x-y| - 2x(x-y)\ln|x-y| + P(x,y)$$

where P is a polynomial in x and y. Dividing by $\omega(y)$ and integrating on both sides with respect to y, we get

$$\frac{1}{2\pi}(\omega\partial_x)^2 O_3 = 2\omega^2 S_\omega - \frac{1}{\pi}xO_2 + R$$

where R is a smoothing operator. Taking the adjoint operator in both sides (with respect to the T^0 scalar product), we get

$$\frac{1}{2\pi}O_3(\omega\partial_x)^2 = 2S_\omega\omega^2 - \frac{1}{\pi}O_2 x + R'$$

where R' is also a smoothing operator. Lemma 11 thus implies

$$-S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} = \frac{I}{4} - k^2 S_\omega\omega^2 S_\omega + K''$$

where K'' is of order 3. The announced result holds since, using Lemma 12,

$$S_\omega\omega^2 S_\omega - S_{k,\omega}\omega^2 S_{k,\omega}$$

is an operator of order 6. $\square$

Reporter ça après le non flat arc, puis parler du cas courbe fermée et poids $\omega = 1$ avec l'analogie Darbas Antoine. The last result, when also in consideration of the commutation shown in Theorem 6, implies that $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$ is a compact perturbation of the inverse of $S_{k,\omega}$, prompting us to use it as a preconditioner in Equation (32).

5 Non-flat arc, Helmholtz

- Restate the theorem, en disant que c'est la même preuve.

We return to the case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, and define a new preconditioner for thee corresponding integral equation. The main result of this section is Theorem 8 We fix a smooth, constant speed, parametrization $r : [-1,1] \to \mathbb{R}^2$ of Γ. The constant-speed assumption ensures $(x,y) \in [-1,1]$, one has

$$\left|r(x) - r(y)^2\right| = \frac{|\Gamma|^2}{4}|x-y|^2 + |x-y|^4 F_1(x,y) \tag{34}$$

where $|\Gamma|$ is the length of Γ and F_1 is a C^∞ function on $[-1,1]^2$.

5.1 Neumann

5.2 Numerical results

6 Conclu

Résumé de ce qu'on a fait, du lien qu'on a fait. Ouverture sur les singularités de type coin puis 3D. Beaucoup plus compliqué car pas de relations analytiques qui nous aident. Expliquer la beauté de l'approche numérique avec un poids. On propose le préconditioneur avec un test numérique ?

Possible analyse pseudo-diff ? En reparler ? Lien avec Antoine et Darbas.

7 Commutation of N_ω and $(\partial_x\omega)^2 + k^2\omega^2$

To ease the computations, we take some notations: let $\Delta_\omega := (\omega\partial_x)^2$, $\Delta_\omega^T := (\partial_x\omega)^2$, $N_\omega := N_{k,\omega}$, and $S_\omega := S_{k,\omega}$. Using, Equation 12 we can write

$$N_\omega = -\partial_x S_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2.$$

To show that N_ω and $\Delta_\omega^T + k^2\omega^2$ commute, we compute their commutator C and show that it is null. We have

$$\begin{aligned} C &:= N_\omega \Delta_\omega^T - \Delta_\omega^T N_\omega + k^2 N_\omega \omega^2 - k^2\omega^2 N_\omega \\ &= -\partial_x S_\omega \Delta_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2 \Delta_\omega^T \\ &\quad +\partial_x \Delta_\omega S_\omega \omega \partial_x \omega + k^2 \Delta_\omega^T S_\omega \omega^2 \\ &\quad -k^2 \partial_x S_\omega \omega \partial_x \omega^3 - k^4 S_\omega \omega^4 \\ &\quad +k^2\omega^2 \partial_x S_\omega \omega \partial_x \omega + k^4 \omega^2 S_\omega \omega^2 \end{aligned}$$

where each term in the r.h.s. of the first equality gives rise to a line in the second. We rearrange the terms as follows:

$$\begin{aligned} C &= \partial_x(\Delta_\omega S_\omega - S_\omega \Delta_\omega)\omega\partial_x\omega - k^2\partial_x S_\omega \omega \partial_x \omega^3 + k^2\omega^2 \partial_x S_\omega \omega \partial_x \omega \\ &\quad +k^4(\omega^2 S_\omega - S_\omega \omega^2)\omega^2 \\ &\quad +k^2(\Delta_\omega^T S_\omega \omega^2 - S_\omega \omega^2 \Delta_\omega^T) \end{aligned}$$

For the first term, we inject the commutation shown in Theorem 6. For the last line, we use the following identities:

$$\Delta_\omega^T = \Delta_\omega - 2x\partial_x - 1$$

$$\omega^2 \Delta_\omega^T = \Delta_\omega \omega^2 + \omega^2 + 2\omega x \partial_x \omega$$

Let $D = \frac{C}{k^2}$,

$$\begin{aligned} D &= \partial_x S_\omega \omega(\omega^2\partial_x - \partial_x \omega^2)\omega + (\omega^2\partial_x - \partial_x\omega^2)S_\omega \omega \partial_x \omega \\ &\quad +k^2(\omega^2 S_\omega - S_\omega \omega^2)\omega^2 \\ &\quad +(\Delta_\omega - 2x\partial_x - 1)S_\omega \omega^2 - S_\omega(\Delta_\omega \omega^2 + \omega^2 + 2\omega x \partial_x \omega) \end{aligned}$$

We use $\omega^2\partial_x - \partial_x\omega^2 = 2x$, and the relation $\partial_x S\omega^2 = S_\omega\omega\partial_x\omega$, obtained by integration by parts.

$$
\begin{aligned}
D &= 2S_\omega\omega\partial_x x\omega + 2xS_\omega\omega\partial_x\omega \\
&\quad + \left(k^2(\omega^2 S_\omega - S_\omega\omega^2) + \Delta_\omega S_\omega - S_\omega\Delta_\omega\right)\omega^2 \\
&\quad -2S_\omega\omega^2 - 2xS_\omega\omega\partial_x\omega - 2S_\omega\omega x\partial_x\omega
\end{aligned}
$$

Using again the commutation shown in Theorem 6, we are left with

$$
D = 2S_\omega\omega(\partial_x x - x\partial_x)\omega - 2S_\omega\omega^2
$$

This is null since $\partial_x x - x\partial_x = 1$.

8 Plus

References

[1] Xavier Antoine and Marion Darbas. Generalized combined field integral equations for the iterative solution of the three-dimensional helmholtz equation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 41(1):147–167, 2007.

[2] Kendall E Atkinson and Ian H Sloan. The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs. *mathematics of computation*, 56(193):119–139, 1991.

[3] Oscar P Bruno and Stéphane K Lintner. Second-kind integral solvers for te and tm problems of diffraction by open arcs. *Radio Science*, 47(6), 2012.

[4] Ralf Hiptmair, Carlos Jerez-Hanckes, and Carolina Urzua-Torres. Closed-form exact inverses of the weakly singular and hypersingular operators on disks. *arXiv preprint arXiv:1703.08556*, 2017.

[5] Carlos Jerez-Hanckes and Jean-Claude Nédélec. Variational forms for the inverses of integral logarithmic operators over an interval. *Comptes Rendus Mathematique*, 349(9-10):547–552, 2011.

[6] Carlos Jerez-Hanckes and Jean-Claude Nédélec. Explicit variational forms for the inverses of integral logarithmic operators over an interval. *SIAM Journal on Mathematical Analysis*, 44(4):2666–2694, 2012.

[7] Shidong Jiang and Vladimir Rokhlin. Second kind integral equations for the classical potential theory on open surfaces ii. *Journal of Computational Physics*, 195(1):1–16, 2004.

[8] John C Mason and David C Handscomb. *Chebyshev polynomials*. CRC Press, 2002.

[9] W McLean. A spectral galerkin method for a boundary integral equation. *Mathematics of computation*, 47(176):597–607, 1986.

[10] William Charles Hector McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge university press, 2000.

[11] Lars Mönch. On the numerical solution of the direct scattering problem for an open sound-hard arc. *Journal of computational and applied mathematics*, 71(2):343–356, 1996.

[12] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders. *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.16 of 2017-09-18.

[13] Stefan A Sauter and Christoph Schwab. Boundary element methods. *Boundary Element Methods*, pages 183–287, 2011.

[14] Ian H Sloan and EP Stephan. Collocation with chebyshev polynomials for symm's integral equation on an interval. *The ANZIAM Journal*, 34(2):199–211, 1992.

[15] Ernst P Stephan and Wolfgang L Wendland. An augmented galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. *Applicable Analysis*, 18(3):183–219, 1984.

[16] Ernst P Stephan and Wolfgang L Wendland. A hypersingular boundary integral method for two-dimensional screen and crack problems. *Archive for Rational Mechanics and Analysis*, 112(4):363–390, 1990.

[17] Carolina Alejandra Urzúa Torres et al. Optimal preconditioners for solving two-dimensional fractures and screens using boundary elements. 2014.

[18] Yeli Yan and Ian H Sloan. Mesh grading for integral equations of the first kind with logarithmic kernel. *SIAM journal on numerical analysis*, 26(3):574–587, 1989.

[19] Yeli Yan, Ian H Sloan, et al. *On integral equations of the first kind with logarithmic kernels*. University of NSW, 1988.

[20] Yi Yan. Cosine change of variable for symm's integral equation on open arcs. *IMA Journal of Numerical Analysis*, 10(4):521–535, 1990.