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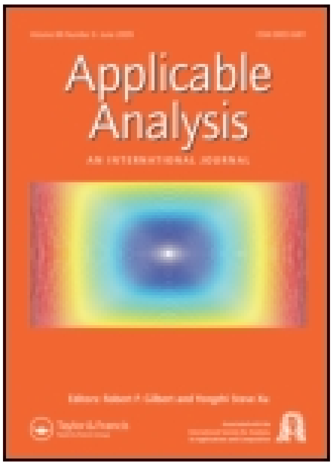

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Additive Schwarz Methods for the H-Version Boundary Element Method

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Abstract

We study additive Schwarz methods (two level and multilevel) for the h-version boundary element method. Both weakly singular and hypersingular integral equations of the first kind are considered. We prove that the condition numbers of the additive Schwarz operators are bounded independently of the number of levels and number of mesh points. Thus we show that the additive Schwarz method as a parallel preconditioner, which was originally designed for finite element discretisation of differential equations, is also an efficient solver for boundary integral operators, which are non-local operators.

KEY WORDS: h-version boundary integral equation method, additive Schwarz operator, preconditioned conjugate gradient algorithm

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1 Introduction.

In this paper we use domain decomposition purely at the integral equation level. We study the additive Schwarz method which is to be used as a preconditioner to solve the matrix systems arising from the h-version Galerkin scheme for hypersingular and weakly singular integral equations.

The boundary integral operators we deal with in this paper are positive definite and symmetric. Thus the application of the h-version Galerkin method yields linear systems of the form

$$A_N \psi_N = g_N, \tag{1}$$

where the coefficient matrix A_N is a dense, positive-definite, and symmetric matrix. (Here N denotes the number of unknowns, which is also the number of mesh points.) Since the condition number of A_N grows like N the conjugate gradient algorithm when used to solve (1) yields a rate of convergence

$$\rho = \frac{\kappa(A_N)^{1/2} - 1}{\kappa(A_N)^{1/2} + 1} = \frac{N^{1/2} - 1}{N^{1/2} + 1} = 1 - O(\frac{1}{N^{1/2}}) \quad \text{as } N \to \infty,$$

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which approaches 1 (the non-convergent status of the iterative method) at the rate of $1/N^{1/2}$ as $N \to \infty$. Therefore a preconditioner is necessary.

The additive Schwarz method was proved to be efficient for finite element discretisations of elliptic differential equations. A multilevel version of this method was first introduced in [5], and the proof for the bounds of the eigenvalues was improved in [16]. An alternative to this multilevel method is the so-called BPX preconditioner introduced in [3].

We shall prove in this paper that the additive Schwarz method is also efficient for matrix systems arising from boundary integral operators in the sense that it yields preconditioned systems having bounded condition numbers. More precisely, we shall prove that the 2-level method gives a preconditioned system having condition number bounded independently of the number of mesh points, and the multilevel method gives a preconditioned system that has condition number bounded independently of the number of levels and number of mesh points. These results will be proved for both hypersingular and weakly singular operators. With these preconditioners, the preconditioned conjugate gradient algorithm then has a rate of convergence which is a constant less than 1. Hence after a fixed number of iterations, the method converges.

The paper is organised as follows. Section 2 gives the analysis of the additive Schwarz method for the hypersingular integral operator. Section 3 treats the weakly singular integral operator in the same manner. Numerical results are given in § 4 to underline our theoretical results.

In this paper, c denotes a generic constant and may take different values at different occurrences. For the ease of the presentation we consider integral equations on the unit interval $\Gamma = (-1, 1)$. The extension to the general case of a polygonal curve Γ is straight forward.

2 Hypersingular integral equation.

We consider the hypersingular integral equation

$$Dv(x) := -\frac{1}{\pi} f.p. \int_\Gamma \frac{v(y)}{|x-y|^2}\, ds_y = f(x), \quad x \in \Gamma = (-1, 1), \tag{2}$$

where f.p. denotes a finite part integral in the sense of Hadamard. Let $\widetilde{\Gamma}$ be an arbitrary closed curve containing Γ. We define, as in [7, 9], the Sobolev spaces

$$\widetilde{H}^{1/2}(\Gamma) = \left\{ v|_{\widetilde{\Gamma}} : \ v \in H^1_{loc}(\mathbb{R}^2) \text{ and } v|_{\widetilde{\Gamma}\setminus\Gamma} = 0 \right\},$$

and $H^{-1/2}(\Gamma)$ being the dual space of $\widetilde{H}^{1/2}(\Gamma)$ with respect to the L^2 inner product on Γ. As was shown in [4], D is continuous and invertible from $\widetilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$. Moreover, D is strongly elliptic, i.e., there exists a constant $\gamma > 0$ such that

$$\langle Dv, v \rangle \geq \gamma \|v\|^2_{\widetilde{H}^{1/2}(\Gamma)},$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 duality on Γ. Hence D defines a continuous, positive-definite and symmetric bilinear form $a(v, w) = \langle Dv, w \rangle$ for $v, w \in \widetilde{H}^{1/2}(\Gamma)$.

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We consider a uniform mesh of size h on Γ as follows

$$x_j = -1 + jh, \quad h = \frac{2}{N_h}, \quad j = 0, \dots, N_h. \tag{3}$$

We then define on this mesh the space V_h of continuous piecewise-linear functions on Γ which vanish at the endpoints of Γ. Then V_h is a subset of $\widetilde{H}^{1/2}(\Gamma)$. The h-version boundary element method for Equation (2) reads as:

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \textit{for any } v_h \in V_h. \tag{4}$$

The stability and convergence of the scheme (4) was proved in [13]. It is known that the condition number of the matrix system derived from (4) is cN. We show in this section that the additive Schwarz methods yield preconditioned systems which have bounded condition numbers.

2.1 2-level method.

Let $\phi_{h,i}$, $i = 1, \dots, N_h - 1$ denote the hat functions forming a basis for V_h, i.e., $\phi_{h,i}$ is a piecewise-linear function which takes value 1 at x_i and value 0 at other mesh points. We then decompose V_h as

$$V_h = V_H + V_{h,1} + \cdots + V_{h,N_h-1}, \tag{5}$$

where V_H, also written as $V_{h,0}$, is defined as V_h but on the uniform mesh with mesh size $H = 2h$, and where $V_{h,i} =$ span $\{\phi_{h,i}\}$ for $i = 1, \dots, N_h - 1$. For $j = 0, \dots, N_h - 1$, let P_j be a projection from V_h onto $V_{h,j}$ defined for any $v_h \in V_h$ by

$$a(P_j v_h, w_{h,j}) = a(v_h, w_{h,j}) \quad \text{for any } w_{h,j} \in V_{h,j}. \tag{6}$$

The additive Schwarz operator P is then defined as

$$P = \sum_{j=0}^{N_h-1} P_j. \tag{7}$$

We note that P is symmetric and positive definite with respect to the bilinear form $a(\cdot,\cdot)$. The additive Schwarz method for equation (4) consists in solving, by an iterative method, the equation

$$P u_h = g_h, \tag{8}$$

where the right hand side is given by

$$g_h = \sum_{j=0}^{N_h-1} g_{h,j}, \tag{9}$$

with $g_{h,j}$ being solutions of

$$a(g_{h,j}, w_{h,j}) = \langle f, w_{h,j} \rangle \quad \text{for any } w_{h,j} \in V_{h,j}. \tag{10}$$

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With g_h defined by (9) and (10), equations (4) and (8) have the same solutions. In fact, if u_h is a solution of (4) then from the definitions of P_j and $g_{h,j}$ we deduce

$$\begin{aligned} a(P_j u_h, w_{h,j}) &= a(u_h, w_{h,j}) \\ &= \langle f, w_{h,j}\rangle = a(g_{h,j}, w_{h,j}) \quad \text{for any } w_{h,j} \in V_{h,j}, \end{aligned}$$

i.e., $P_j u_h = g_{h,j}$. Hence $P u_h = g_h$. On the other hand, if P is invertible and u_h is the solution to (8) then by using successively the symmetry of P, Eqns (9), (6), (10), and (7) we obtain

$$\begin{aligned} a(u_h, v_h) &= a(P^{-1} g_h, v_h) = a(g_h, P^{-1} v_h) \\ &= \sum_{j=0}^{N_h-1} a(g_{h,j}, P^{-1} v_h) = \sum_{j=0}^{N_h-1} a(g_{h,j}, P_j P^{-1} v_h) \\ &= \sum_{j=0}^{N_h-1} \langle f, P_j P^{-1} v_h\rangle = \langle f, v_h\rangle \quad \text{for any } v_h \in V_h. \end{aligned}$$

It is noted that an explicit form for P is not necessary in solving (8). In fact, if we use the first Richardson method to solve (8), then given an iterate u_h^n we compute u_h^{n+1} by

$$u_h^{n+1} = u_h^n - \tau(P u_h^n - g_h),$$

where the residual $r^n := (P u_h^n - g_h)$ is computed as $r^n = \sum_{j=0}^{N_h-1} r_j^n$ with $r_j^n := P_j u_h^n - g_{h,j}$ being solutions of

$$a(r_j^n, w_{h,j}) = a(u_h^n, w_{h,j}) - \langle f, w_{h,j}\rangle \quad \text{for any } w_{h,j} \in V_{h,j}. \tag{11}$$

Here τ is a damping parameter whose optimal value is given by

$$\tau_{opt} = \frac{2}{\lambda_{\max}(P) + \lambda_{\min}(P)}.$$

A more practicable method to solve (8) is the conjugate gradient method. In fact the additive Schwarz method can be viewed as a preconditioned conjugate gradient method. More precisely, let D_h be the discrete form of the hypersingular operator D, i.e. $D_h : V_h \to V_h$ be defined for any $v_h \in V_h$ as

$$\langle D_h v_h, w_h\rangle = a(v_h, w_h) \quad \text{for any } w_h \in V_h. \tag{12}$$

Then the additive Schwarz operator P can be written as $P = B D_h$, where the preconditioner B is given by

$$B = \sum_{j=0}^{N_h-1} D_{h,j}^{-1} Q_{h,j}.$$

Here $Q_{h,j}$ is the L^2-projection from V_h onto $V_{h,j}$, and $D_{h,j}$ is the restriction of D_h onto $V_{h,j}$, i.e.

$$\langle Q_{h,j} v_h, w_{h,j}\rangle = \langle v_h, w_{h,j}\rangle \quad \text{for any } v_h \in V_h \text{ and } w_{h,j} \in V_{h,j},$$

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and

$$\langle D_{h,j}v_{h,j}, w_{h,j}\rangle = a(v_{h,j}, w_{h,j}) \quad \text{for any } v_{h,j}, w_{h,j} \in V_{h,j}.$$

This representation of P can be seen from the fact that $P_j = D_{h,j}^{-1}Q_{h,j}D_h$. In the implementation of the preconditioned conjugate gradient method, an explicit form for B is not important. Indeed, we only need to know the acting of B on any $v_h \in V_h$. We will give in § 4 an example of how to compute Bv_h from v_h.

It is known that the rates of convergence of the above-mentioned methods to solve (8) depend on the condition number $\kappa(P)$ of P. Moreover, if there exists positive constants c_0 and c_1 such that

$$c_0 a(v_h, v_h) \leq a(Pv_h, v_h) \leq c_1 a(v_h, v_h) \quad \text{for any } v_h \in V_h,$$

then $\kappa(P) \leq c_1/c_0$. We shall prove that these constants are independent of the number of mesh points N_h. A bound for the minimum eigenvalue of P, which also ensures the invertibility of P, can be obtained by using the following lemma (see [10, 14])

Lemma 2.1 *If there exists a positive constant C_0 such that for any $v_h \in V_h$ there exists $v_H \in V_H$ and $v_{h,j} \in V_{h,j}$, $j = 1, \ldots, N_h - 1$ satisfying $v_h = v_H + \sum_{j=1}^{N_h-1} v_{h,j}$ and*

$$a(v_H, v_H) + \sum_{j=1}^{N_h-1} a(v_{h,i}, v_{h,i}) \leq C_0^{-1} a(v_h, v_h), \tag{13}$$

then

$$\lambda_{\min}(P) \geq C_0.$$

In view of this lemma we shall prove (13) in order to get good bound for $\lambda_{\min}(P)$. It is necessary to define an appropriate decomposition for any $v_h \in V_h$. We need a partition of unity $\{\theta_j\}_{j=1,\ldots,N_h-1}$ defined as follows: $\{\theta_j\}$ are piecewise-linear functions such that

$$\theta_1 = \begin{cases} 1 & \text{at } x_0 \text{ and } x_1, \\ 0 & \text{at other mesh points,} \end{cases} \qquad \theta_{N_h-1} = \begin{cases} 1 & \text{at } x_{N_h-1} \text{ and } x_{N_h}, \\ 0 & \text{at other mesh points,} \end{cases}$$

and

$$\theta_j = \begin{cases} 1 & \text{at } x_j, \\ 0 & \text{at other mesh points,} \end{cases}$$

for $j = 2, \ldots, N_h - 2$. We note that our partition of unity has the following properties:

$$\operatorname{supp} \theta_j = \overline{\Gamma}_j' := [x_{j-1}, x_{j+1}] \quad \text{and} \quad \left|\frac{d\theta_j}{dx}\right| \leq \frac{c}{h}. \tag{14}$$

We can now decompose $v_h \in V_h$ as follows:

$$v_h = v_H + v_{h,1} + \cdots + v_{h,N_h-1}, \tag{15}$$

where $v_H := \widetilde{P}_{V_H} v_h$ and $v_{h,j} := \Pi_h(\theta_j w_h)$ with $w_h := v_h - v_H$. Here $\widetilde{P}_{V_H}$ is the Galerkin projection from $\widetilde{H}^{1/2}(\Gamma)$ onto V_H, and Π_h is the interpolation operator from $C(\Gamma)$ onto V_h. It

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is well known that (see e.g. [13]) for any $\epsilon > 0$ there exists a constant c independent of H such that

$$\|w_h\|_{L^2(\Gamma)} = \|v_h - \widetilde{P}_{V_H} v_h\|_{L^2(\Gamma)} \le cH^{1/2-\epsilon/2}\|v_h\|_{\widetilde{H}^{1/2}(\Gamma)}, \tag{16}$$

and

$$\|v_H\|_{\widetilde{H}^{1/2}(\Gamma)} \le c\|v_h\|_{\widetilde{H}^{1/2}(\Gamma)}, \quad \|w_h\|_{\widetilde{H}^{1/2}(\Gamma)} \le c\|v_h\|_{\widetilde{H}^{1/2}(\Gamma)}. \tag{17}$$

With the decomposition (15) we can now prove a lower bound for the minimum eigenvalue of P.

Lemma 2.2 *For any $\epsilon > 0$ there exists a positive constant C_0 independent of h such that*

$$\lambda_{\min}(P) \ge C_0 h^\epsilon.$$

Proof. For any $j = 1, \ldots, N_h - 1$ we have by using the boundedness of Π_h, the interpolation property of the Sobolev norms, the properties of the partition of unity (14), and the inequality $ab \le (a^2 + b^2)/2$

$$\begin{aligned}
\|v_{h,j}\|^2_{\widetilde{H}^{1/2}(\Gamma'_j)} &\le c\|\theta_j w_h\|^2_{\widetilde{H}^{1/2}(\Gamma'_j)} \\
&\le c\|\theta_j w_h\|_{L^2(\Gamma'_j)}\|\theta_j w_h\|_{\widetilde{H}^1(\Gamma'_j)} \\
&\le c\|w_h\|_{L^2(\Gamma'_j)}\left(\|w_h\|_{H^1(\Gamma'_j)} + h^{-1}\|w_h\|_{L^2(\Gamma'_j)}\right) \\
&\le c\left(h^{-1}\|w_h\|^2_{L^2(\Gamma'_j)} + h^{-1/2}\|w_h\|_{L^2(\Gamma'_j)} h^{1/2}\|w_h\|_{H^1(\Gamma'_j)}\right) \\
&\le c\left(h^{-1}\|w_h\|^2_{L^2(\Gamma'_j)} + h\|w_h\|^2_{H^1(\Gamma'_j)}\right).
\end{aligned}$$

Summing over j, and using (16), (17) and the inverse property we then have

$$\begin{aligned}
\sum_{j=1}^{N_h-1}\|v_{h,j}\|^2_{\widetilde{H}^{1/2}(\Gamma'_j)} &\le c\left(h^{-1}\|w_h\|^2_{L^2(\Gamma)} + h\|w_h\|^2_{\widetilde{H}^1(\Gamma)}\right) \\
&\le c\left(h^{-\epsilon}\|v_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \|w_h\|^2_{\widetilde{H}^{1/2}(\Gamma)}\right) \\
&\le ch^{-\epsilon}\|v_h\|^2_{\widetilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

Hence

$$\begin{aligned}
a(v_H, v_H) + \sum_{j=1}^{N_h-1} a(v_{h,j}, v_{h,j}) &\simeq \|v_H\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \sum_{j=1}^{N_h-1}\|v_{h,j}\|^2_{\widetilde{H}^{1/2}(\Gamma'_j)} \\
&\le ch^{-\epsilon}\|v_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \simeq C_0^{-1}h^{-\epsilon}a(v_h, v_h),
\end{aligned}$$

and therefore by Lemma 2.1, $\lambda_{\min}(P) \ge C_0 h^\epsilon$. The lemma is proved. □

Lemma 2.3 *There exists a positive constant C_1 independent of h such that*

$$\lambda_{\max}(P) \le C_1.$$

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Proof. In order to prove the lemma we prove the following equivalent inequality

$$a(Pv_h, v_h) \leq C_1 a(v_h, v_h) \quad \text{for any } v_h \in V_h. \tag{18}$$

Let $T = \sum_{j=1}^{N_h-1} P_j$. Then $P = P_0 + T$, cf. (7), and therefore

$$a(Pv_h, v_h) \leq a(P_0 v_h, v_h) + a(Tv_h, v_h).$$

Since P_0 is bounded in $\widetilde{H}^{1/2}(\Gamma)$ we have

$$a(P_0 v_h, v_h) \leq a(P_0 v_h, P_0 v_h)^{1/2} a(v_h, v_h)^{1/2} \leq c a(v_h, v_h). \tag{19}$$

On the other hand, since $V_{h,i} =$ span $\{\phi_{h,i}\}$ we have for any $v_h \in V_h$

$$P_i v_h = \frac{a(v_h, \phi_{h,i})}{a(\phi_{h,i}, \phi_{h,i})} \phi_{h,i},$$

and therefore

$$T v_h = \sum_{i=1}^{N_h-1} \frac{a(v_h, \phi_{h,i})}{a(\phi_{h,i}, \phi_{h,i})} \phi_{h,i}.$$

Hence

$$a(Tv_h, v_h) = \sum_{i=1}^{N_h-1} \frac{a(v_h, \phi_{h,i})^2}{a(\phi_{h,i}, \phi_{h,i})}. \tag{20}$$

It is necessary to estimate $a(v_h, \phi_{h,i})$. By noting that supp $\phi_{h,i} = \overline{\Gamma}'_i = [x_{i-1}, x_{i+1}]$ and using the definition (12) we have

$$a(v_h, \phi_{h,i})^2 = \langle D_h v_h, \phi_{h,i} \rangle^2 \leq \|D_h v_h\|^2_{L^2(\Gamma'_i)} \|\phi_{h,i}\|^2_{L^2(\Gamma'_i)}. \tag{21}$$

Simple calculations give $\|\phi_{h,i}\|^2_{L^2(\Gamma'_i)} = 2h/3$ and $\|\phi_{h,i}\|^2_{\widetilde{H}^1(\Gamma'_i)} = 2h/3 + 2/h$. Hence $\|\phi_{h,i}\|^2_{L^2(\Gamma'_i)} \leq h^2 \|\phi_{h,i}\|^2_{\widetilde{H}^1(\Gamma'_i)}$. Using interpolation, we then obtain

$$\|\phi_{h,i}\|^2_{L^2(\Gamma'_i)} \leq h \|\phi_{h,i}\|^2_{\widetilde{H}^{1/2}(\Gamma'_i)}. \tag{22}$$

Noting that $a(\phi_{h,i}, \phi_{h,i}) \simeq \|\phi_{h,i}\|^2_{\widetilde{H}^{1/2}(\Gamma'_i)}$, we infer from (20), (21) and (22)

$$a(Tv_h, v_h) \leq ch \sum_{i=1}^{N_h-1} \|D_h v_h\|^2_{L^2(\Gamma'_i)} \leq ch \|D_h v_h\|^2_{L^2(\Gamma)}. \tag{23}$$

By using the definition (12) of D_h, the mapping property of D, and the inverse property we obtain

$$\begin{aligned}
\|D_h v_h\|^2_{L^2(\Gamma)} &= \langle D_h v_h, D_h v_h \rangle = \langle D v_h, D_h v_h \rangle \\
&\leq \|D_h v_h\|_{\widetilde{H}^\epsilon(\Gamma)} \|D v_h\|_{H^{-\epsilon}(\Gamma)} \leq c \|D_h v_h\|_{\widetilde{H}^\epsilon(\Gamma)} \|v_h\|_{\widetilde{H}^{1-\epsilon}(\Gamma)} \\
&\leq c h^{-1/2} \|D_h v_h\|_{L^2(\Gamma)} \|v_h\|_{\widetilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

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Hence
$$\|D_h v_h\|^2_{L^2(\Gamma)} \le ch^{-1}\|v_h\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{24}$$
Inequalities (23) and (24) then give
$$a(Tv_h, v_h) \le ca(v_h, v_h).$$
This inequality together with (19) yields (18); therefore the lemma is proved. □

Combining Lemmas 2.2 and 2.3 we then have

Theorem 2.4 *For any $\epsilon > 0$ there exist positive constants C_0 and C_1 independent of the number of mesh points such that the condition number of the 2-level additive Schwarz operator P defined by (7) is bounded as*
$$\kappa(P) \le (C_1/C_0)h^{-\epsilon}.$$

Remark 2.5 *The term $h^{-\epsilon}$ is due to the singularity of the exact solution of the integral equation at the endpoints of the open curve Γ (cf. (16)). When Γ is a closed curve, such a term reduces to 1 and therefore the condition number of P is bounded independently of h.*

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2.2 Multilevel method.

When applying the 2-level method discussed above we have to solve a global problem on the coarse space V_H (cf. (10) and (11)). The inclusion of this global problem in our algorithm is necessary because otherwise we will have the diagonal preconditioner, which cannot work for this kind of operator since the diagonal entries are constant. (See also [5, 14] for other comments.) However, when H is small this global problem is large. Therefore it is natural to carry out the same algorithm for this global problem, which leads to a multilevel method.

The multilevel method is defined as follows. Starting with a coarse mesh
$$\mathcal{N}^1 : -1 = x_1^1 < x_2^1 = 0 < x_3^1 = 1,$$
we divide each subinterval into two equal intervals. Hence, if h_l is the meshstep of $\mathcal{N}^l$, $l = 1, \ldots, L-1$, then $h_l = 2h_{l+1}$. For $l = 1, \ldots, L$, let V^l be the spline space associated with $\mathcal{N}^l$ which contains continuous and piecewise-linear functions vanishing at the endpoints ± 1. Let $\left\{\phi_1^l, \ldots, \phi_{N_l}^l\right\}$ be the nodal basis for V^l, where $N_l = 2^l - 1$ is the dimension of V^l. We then decompose V^l as
$$V^l = \sum_{i=1}^{N_l} V_i^l,$$
with $V_i^l = \operatorname{span}\{\phi_i^l\}$ for $i = 1, \ldots, N_l$. Eventually, V^L is decomposed as
$$V^L = \sum_{l=1}^{L}\sum_{i=1}^{N_l} V_i^l.$$
The multilevel additive Schwarz operator is now defined as
$$P_{MAS} = \sum_{l=1}^{L}\sum_{i=1}^{N_l} P_{V_i^l}, \tag{25}$$

where $P_{V_i^l} : V^L \to V_i^l$ is defined for any $v \in V^L$ by

$$a(P_{V_i^l}v, w) = a(v, w) \quad \text{for all } w \in V_i^l.$$

This multilevel method was originally designed in [5] for finite element discretisations of elliptic differential equations. An alternative discussion was given in [3] with the so-called BPX preconditioner. We shall prove in this section that this method is also a good preconditioner for the hypersingular integral equation.

We will first prove a bound for the minimum eigenvalue of P_{MAS}.

Lemma 2.6 *For any $\epsilon > 0$ there exists a positive constant C_0 independent of the number of levels and the number of mesh points such that*

$$\lambda_{\min}(P_{MAS}) \geq C_0 h_L^\epsilon.$$

Proof. In view of Lemma 2.1, we will prove that

$$\sum_{l=1}^{L}\sum_{i=1}^{N_l} a(v_i^l, v_i^l) \leq C_0^{-1} h_L^{-\epsilon} a(v, v), \tag{26}$$

for some decomposition $v = \sum_{l=1}^{L}\sum_{i=1}^{N_l} v_i^l$. Similarly to the case of the 2-level method, we define for each $l = 1, \ldots, L$ a projector $\widetilde{P}_{V^l} : \widetilde{H}^{1/2}(\Gamma) \to V^l$ by: for any $v \in \widetilde{H}^{1/2}(\Gamma)$,

$$a(\widetilde{P}_{V^l}v, \phi) = a(v, \phi) \quad \text{for any } \phi \in V^l,$$

i.e., $\widetilde{P}_{V^l}$ is the Galerkin projection on V^l. Similar to (16) we now have

$$\|\widetilde{P}_{V^l}v - v\|_{L^2(\Gamma)} \leq c h_l^{1/2-\epsilon/2} \|v\|_{\widetilde{H}^{1/2}(\Gamma)}, \tag{27}$$

for any $\epsilon > 0$ and any $v \in \widetilde{H}^{1/2}(\Gamma)$. With the introduction of $\widetilde{P}_{V^l}$, we can first decompose $v \in V^L$ as $v = \sum_{l=1}^{L} v^l$ by letting

$$v^l := \left(\widetilde{P}_{V^l} - \widetilde{P}_{V^{l-1}}\right) v \quad \text{for any } l = 1, \ldots, L, \quad \text{with } \widetilde{P}_{V^0} \equiv 0. \tag{28}$$

We then further decompose v^l as

$$v^l = \sum_{i=1}^{N_l} v_i^l, \quad \text{with } v_i^l := \Pi^l(\theta_i^l v^l) \in V_i^l. \tag{29}$$

Here Π^l is the standard nodal value interpolation operator from $C(\Gamma)$ onto V^l and, for each $l = 1, \ldots, L$, $\left\{\theta_i^l\right\}_{i=1}^{N_l}$ is a partition of unity defined as in (14). We note that from the definition (28) of v^l there holds

$$a(v^l, v^k) = 0 \quad \text{for } 1 \leq l \neq k \leq L. \tag{30}$$

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In fact, assume that $l < k$. Then $v^l \in V^l \subset V^{k-1} \subset V^k$, and therefore

$$\begin{aligned} a(v^k, v^l) &= a(\widetilde{P}_{V^k} v, v^l) - a(\widetilde{P}_{V^{k-1}} v, v^l) \\ &= a(v, v^l) - a(v, v^l) = 0. \end{aligned}$$

There also holds from the definition (28) of v^l that

$$\widetilde{P}_{V^{l-1}} v^l = 0 \quad \text{for any } l = 1, \dots, L. \tag{31}$$

With the above decomposition of $v \in V^L$, $v = \sum_{l=1}^{L} \sum_{j=1}^{N_l} v_j^l$, we can now prove (26). As in the proof of Lemma 2.2 we have

$$\begin{aligned} \|v_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} &\leq c\|\theta_i^l v^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} \leq c\|\theta_i^l v^l\|_{\widetilde{H}^0(\Gamma_i^l)} \|\theta_i^l v^l\|_{\widetilde{H}^1(\Gamma_i^l)} \\ &\leq c\|v^l\|_{L^2(\Gamma_i^l)} \left(h_l^{-1}\|v^l\|_{L^2(\Gamma_i^l)} + \|v^l\|_{H^1(\Gamma_i^l)}\right) \\ &\leq c\left(h_l^{-1}\|v^l\|^2_{L^2(\Gamma_i^l)} + h_l^{-1/2}\|v^l\|_{L^2(\Gamma_i^l)} h_l^{1/2}\|v^l\|_{H^1(\Gamma_i^l)}\right) \\ &\leq c\left(h_l^{-1}\|v^l\|^2_{L^2(\Gamma_i^l)} + h_l\|v^l\|^2_{H^1(\Gamma_i^l)}\right). \end{aligned}$$

Summing over $i = 1, \dots, N_l$ we achieve

$$\sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} \leq c\left(h_l^{-1}\|v^l\|^2_{L^2(\Gamma)} + h_l\|v^l\|^2_{\widetilde{H}^1(\Gamma)}\right). \tag{32}$$

It is now necessary to estimate $\|v^l\|^2_{L^2(\Gamma)}$. By writing (see (31))

$$v^l = (I - \widetilde{P}_{V^{l-1}})v^l,$$

and by using (27), we obtain

$$\|v^l\|_{L^2(\Gamma)} = \|(I - \widetilde{P}_{V^{l-1}})v^l\|_{L^2(\Gamma)} \leq c h_{l-1}^{1/2-\epsilon/2}\|v^l\|_{\widetilde{H}^{1/2}(\Gamma)}. \tag{33}$$

Inequalites (32), (33) and the inverse property of the spline functions now yield

$$\begin{aligned} \sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} &\leq c\left(h_{l-1}^{-\epsilon}\|v^l\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \|v^l\|^2_{\widetilde{H}^{1/2}(\Gamma)}\right) \\ &\leq c h_L^{-\epsilon}\|v^l\|^2_{\widetilde{H}^{1/2}(\Gamma)}. \end{aligned}$$

Summing over $l = 1, \dots, L$ and using the orthogonality of v^l, see (30), we obtain

$$\sum_{l=1}^{L}\sum_{i=1}^{N_l} a(v_i^l, v_i^l) \simeq \sum_{l=1}^{L}\sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} \leq c h_L^{-\epsilon} \sum_{l=1}^{L} \|v^l\|^2_{\widetilde{H}^{1/2}(\Gamma)} = c h_L^{-\epsilon}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)} \simeq c h_L^{-\epsilon} a(v, v).$$

Inequality (26) is proved, and therefore the lemma is proved. □

We will next prove a bound for $\lambda_{\max}(P_{MAS})$ which is independent of the number of levels and the number of mesh points. Let $T_l = \sum_{i=1}^{N_l} P_{V_i^l}$. Then we can write $P_{MAS} = \sum_{l=1}^{L} T_l$. The following strengthened Cauchy-Schwarz inequality is essential to obtain a bound for the $\lambda_{\max}(P_{MAS})$:

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Lemma 2.7 *There exist constants $c > 0$ and $\gamma \in (0,1)$ such that for any $v \in V^k$ where $1 \le k \le l \le L$ there holds*

$$a(T_l v, v) \le c\gamma^{2(l-k)} a(v,v).$$

Proof. Since $V_i^l = \text{span}\left\{\phi_i^l\right\}$ we have for any $v \in V^L$

$$P_{V_i^l} v = \frac{a(v,\phi_i^l)}{a(\phi_i^l,\phi_i^l)}\phi_i^l,$$

and therefore

$$T_l v = \sum_{i=1}^{N_l} \frac{a(v,\phi_i^l)}{a(\phi_i^l,\phi_i^l)}\phi_i^l.$$

Hence

$$a(T_l v, v) = \sum_{i=1}^{N_l} \frac{a(v,\phi_i^l)^2}{a(\phi_i^l,\phi_i^l)}. \tag{34}$$

It is necessary to estimate $a(v,\phi_i^l)$. In order to do so we define, for any $l = 1,\dots,L$, $D_l : V^l \to V^l$ as

$$\langle D_l\phi, \psi\rangle = a(\phi,\psi) \quad \text{for any } \phi,\psi \in V^l.$$

Then, by noting that supp $\phi_i^l = \overline{\Gamma}'_{l,i} := [x_{i-1}^l, x_{i+1}^l]$, we have

$$a(v,\phi_i^l) = \langle D_l v, \phi_i^l\rangle \le \|D_l v\|_{L^2(\Gamma'_{l,i})}\|\phi_i^l\|_{L^2(\Gamma'_{l,i})}. \tag{35}$$

As given in the proof of Lemma 2.3 we have (cf. (22))

$$\|\phi_i^l\|^2_{L^2(\Gamma'_{l,i})} \le h_l \|\phi_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma'_{l,i})}. \tag{36}$$

Inequalities (34), (35) and (36) together with the equivalence $a(\phi_i^l,\phi_i^l) \simeq \|\phi_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma'_{l,i})}$ imply

$$a(T_l v, v) \le ch_l \sum_{i=1}^{N_l} \|D_l v\|^2_{L^2(\Gamma'_{l,i})} \le ch_l\|D_l v\|^2_{L^2(\Gamma)}. \tag{37}$$

By using the definition of D_l, the mapping property of D, and the inverse property we obtain

$$\begin{aligned}
\|D_l v\|^2_{L^2(\Gamma)} &= \langle D_l v, D_l v\rangle = \langle Dv, D_l v\rangle \\
&\le \|D_l v\|_{\widetilde{H}^{\epsilon}(\Gamma)}\|Dv\|_{H^{-\epsilon}(\Gamma)} \le c\|D_l v\|_{\widetilde{H}^{\epsilon}(\Gamma)}\|v\|_{\widetilde{H}^{1-\epsilon}(\Gamma)} \\
&\le ch_l^{-\epsilon}h_k^{-1/2+\epsilon}\|D_l v\|_{L^2(\Gamma)}\|v\|_{\widetilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

Hence

$$\|D_l v\|^2_{L^2(\Gamma)} \le ch_l^{-2\epsilon}h_k^{-1+2\epsilon}\|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}. \tag{38}$$

Inequalities (37) and (38) then give

$$a(T_l v, v) \le c\gamma^{2(l-k)}a(v,v),$$

where $\gamma = (1/2)^{(1-2\epsilon)/2}$; therefore the lemma is proved. □

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Lemma 2.8 *There exists a positive constant C_1 independent of the number of levels and the number of mesh points such that*

$$\lambda_{\max}(P_{MAS}) \le C_1.$$

Proof. The lemma is proved if one proves

$$a(P_{MAS}v, v) \le C_1 a(v, v) \quad \text{for any } v \in V^L. \tag{39}$$

The proof is similar to [2, Theorem 3.1] and is included here for completeness. For $l = 1, \dots, L$ let $P_l : V^L \to V^l$ be defined for any $v \in V^L$ as

$$a(P_l v, w) = a(v, w) \quad \text{for any } w \in V^l.$$

Then for any $v \in V^L$ we can write

$$P_l v = \sum_{k=1}^{l} (P_k - P_{k-1})v,$$

where $P_0 \equiv 0$. Hence

$$\begin{aligned} a(P_{MAS}v, v) &= \sum_{l=1}^{L} a(T_l v, v) = \sum_{l=1}^{L} a(T_l v, P_l v) \\ &= \sum_{l=1}^{L}\sum_{k=1}^{l} a(T_l v, (P_k - P_{k-1})v). \end{aligned}$$

Since $a(T_l\cdot, \cdot)$ is symmetric and positive definite we can use Cauchy-Schwarz's inequality with respect to that quadratic form and then use Lemma 2.7 to obtain, for any $\eta > 0$ and for $S := \sum_{i=0}^{\infty} \gamma^i$,

$$\begin{aligned} a(P_{MAS}v, v) &\le \sum_{l=1}^{L}\sum_{k=1}^{l} a(T_l v, v)^{1/2} a(T_l(P_k - P_{k-1})v, (P_k - P_{k-1})v)^{1/2} \\ &\le c\sum_{l=1}^{L}\sum_{k=1}^{l} \gamma^{l-k} a(T_l v, v)^{1/2} a((P_k - P_{k-1})v, (P_k - P_{k-1})v)^{1/2} \\ &= c\sum_{l=1}^{L}\sum_{k=1}^{l} \gamma^{l-k} a(T_l v, v)^{1/2} a((P_k - P_{k-1})v, v)^{1/2} \\ &\le c\left(\frac{\eta}{S}\sum_{l=1}^{L}\sum_{k=1}^{l} \gamma^{l-k} a(T_l v, v) + \frac{S}{4\eta}\sum_{l=1}^{L}\sum_{k=1}^{l} \gamma^{l-k} a((P_k - P_{k-1})v, v)\right) \\ &\le c\left(\eta\sum_{l=1}^{L} a(T_l v, v) + \frac{S^2}{4\eta}\sum_{k=1}^{L} a((P_k - P_{k-1})v, v)\right) \\ &= c\left(\eta a(P_{MAS}v, v) + \frac{S^2}{4\eta} a(v, v)\right). \end{aligned}$$

By choosing η sufficiently small we obtain (39); therefore the lemma is proved. □

Lemmas 2.6 and 2.8 now imply

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Theorem 2.9 *For any $\epsilon > 0$ there exist positive constants C_0 and C_1 independent of the number of levels and the number of mesh points such that the condition number of the multilevel additive Schwarz method defined by (25) has condition number bounded as*

$$\kappa(P_{MAS}) \leq (C_1/C_0)h_L^{-\epsilon}.$$

Remark 2.10 *See Remark 2.5.*

3 Weakly singular integral equation.

We now consider the weakly singular integral equation

$$Vu(x) := -\frac{1}{\pi}\int_\Gamma \log|x-y|u(y)\,ds_y = g(x) \quad \text{for } x \in \Gamma = (-1,1). \tag{40}$$

It was proved in [4] that V is continuous and invertible from $\widetilde{H}^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. Here the Sobolev space $H^{1/2}(\Gamma)$ is defined as the space of functions which are the traces of functions in $H^1_{loc}(\mathbb{R}^2)$ and $\widetilde{H}^{-1/2}(\Gamma)$ is its dual (see e.g. [7, 9]). Since $cap(\Gamma) < 1$, it is known that there exists a constant $\gamma > 0$ such that

$$\langle Vv, v\rangle \geq \gamma\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Hence V defines a continuous, positive-definite and symmetric bilinear form $a(v,w) = \langle Vv, w\rangle$ for $v, w \in \widetilde{H}^{-1/2}(\Gamma)$. On the mesh (3) we now define $\overline{V}_h$ as the space of piecewise-constant functions. The h-version boundary element method for Equation (40) reads as

Find $u_h \in \overline{V}_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \quad \textit{for any } v_h \in \overline{V}_h. \tag{41}$$

The stability and convergence of the scheme (41) was proved in [12]. As in the case of the hypersingular integral operator, we now define the additive Schwarz methods to solve, instead of the matrix system derived from (41), preconditioned systems which have condition numbers bounded independently of the number of mesh points.

We follow the same approach as in § 2 to prove constant bounds for the maximum and minimum eigenvalues of the additive Schwarz operators. However, now that our bilinear form $a(\cdot,\cdot)$ defines a norm equivalent to the $\widetilde{H}^{-1/2}$ norm, we need to create a mechanism to go back and forth between the $\widetilde{H}^{-1/2}$ norm and the $\widetilde{H}^{1/2}$ norm. This can be done by introducing a generalised antiderivative operator, which was discussed in [8]. We recall here the results on that operator.

Lemma 3.1 *Let Γ' be an arbitrary interval in $\mathbb{R}$. For all $0 \leq s \leq 1$ the following statements are valid*

(i) The generalised differential operator $D_s : \widetilde{H}^s(\Gamma') \to \widetilde{H}^{s-1}(\Gamma')$ is bounded.

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(ii) There exists a bounded linear operator $I_s : \widetilde{H}_0^{s-1}(\Gamma') \to \widetilde{H}^s(\Gamma')$ satisfying

$$D_s I_s \chi = \chi \text{ for any } \chi \in \widetilde{H}_0^{s-1}(\Gamma'),$$

where $\widetilde{H}_0^{s-1}(\Gamma') = \{\chi \in \widetilde{H}^{s-1}(\Gamma') : \langle 1, \chi \rangle_{L^2(\Gamma')} = 0\}$.

Remark 3.2 *We note that for any $s \in [0,1)$, I_s is the extension of I_1 onto $\widetilde{H}_0^{s-1}(\Gamma')$, i.e., $I_s v = I_1 v$ for any $v \in L_0^2(\Gamma')$.*

3.1 2-level method.

Let $\overline{V}_H$ be the space of piecewise-constant functions defined on a coarser mesh with mesh size $H = 2h$. For $j = 1, \dots, N_h - 1$, let $\overline{V}_{h,j} =$ span $\{\chi_{h,j}\}$ where $\chi_{h,j}$ is the derivative of the hat function $\phi_{h,j}$ defined in §§ 2.1. More precisely, $\chi_{h,j}$ is given by

$$\chi_{h,j}(x) := \begin{cases} 1/h & \text{for } x_{j-1} < x < x_j, \\ -1/h & \text{for } x_j < x < x_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that supp $\chi_{h,j} = \overline{\Gamma}'_j = [x_{j-1}, x_{j+1}]$, and that $\chi_{h,j} \in \widetilde{H}_0^{-1/2}(\Gamma'_j)$. We also define $\chi_{h,0} \equiv 1$. We then decompose $\overline{V}_h$ as

$$\overline{V}_h = \overline{V}_H + \overline{V}_{h,1} + \cdots + \overline{V}_{h,N_h}. \tag{42}$$

With the additive Schwarz operator P defined as in §§ 2.1 with V_h, V_H and $V_{h,j}$ replaced by $\overline{V}_h$, $\overline{V}_H$ and $\overline{V}_{h,j}$ respectively, we now prove that the additive Schwarz method also gives a good preconditioner for boundary element Galerkin method for the weakly singular integral equation.

We will first use Lemmas 2.1 and 3.1 to prove a bound for the minimum eigenvalue of P.

Lemma 3.3 *For any $\epsilon > 0$ there exists a positive constant C_0 independent of h such that*

$$\lambda_{\min}(P) \geq C_0 h^{\epsilon}.$$

Proof. In view of Lemma 2.1 we will find a decomposition $v_h = v_H + \sum_{j=1}^{N_h-1} v_{h,j}$ such that

$$a(v_H, v_H) + \sum_{j=1}^{N_h-1} a(v_{h,j}, v_{h,j}) \leq C_0^{-1} h^{-\epsilon} a(v_h, v_h). \tag{43}$$

Let $\widetilde{P}_{\overline{V}_H}$ be the Galerkin projection from $\widetilde{H}^{-1/2}(\Gamma)$ on $\overline{V}_H$. For any $v_h \in \overline{V}_h$ we define $\tilde{v}_H := \widetilde{P}_{\overline{V}_H} v_h$, and $w_h := v_h - \tilde{v}_H$. Similarly to (16) and (17) we now have

$$\|w_h\|_{\widetilde{H}^{-1}(\Gamma)} = \|v_h - \widetilde{P}_{\overline{V}_H} v_h\|_{\widetilde{H}^{-1}(\Gamma)} \leq c H^{1/2-\epsilon/2} \|v_h\|_{\widetilde{H}^{-1/2}(\Gamma)}, \tag{44}$$

and

$$\|\tilde{v}_H\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq c\|v_h\|_{\widetilde{H}^{-1/2}(\Gamma)}, \quad \|w_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq c\|v_h\|_{\widetilde{H}^{-1/2}(\Gamma)}. \tag{45}$$

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Since $\{\chi_{h,0},\ldots,\chi_{h,N_h-1}\}$ forms a basis for $\overline{V}_h$ we can write w_h uniquely as

$$w_h = w_{h,0} + \cdots + w_{h,N_h-1},$$

where $w_{h,j} = c_j\chi_{h,j}$ for some constant c_j. It is easy to check that $w_{h,0} = \langle w_h, \chi_{h,0}\rangle/|\Gamma|$, where $|\Gamma|$ is the length of Γ. Therefore, for $0 \le s \le 1$ we have

$$\|w_{h,0}\|_{\widetilde{H}^{-s}(\Gamma)} \le \frac{1}{|\Gamma|}\|1\|_{\widetilde{H}^{-s}(\Gamma)}\|1\|_{H^s(\Gamma)}\|w_h\|_{\widetilde{H}^{-s}(\Gamma)}. \tag{46}$$

Finally, we write v_h as

$$v_h = v_H + v_{h,1} + \cdots + v_{h,N_h-1},$$

where

$$v_H := \tilde{v}_H + w_{h,0} \quad \text{and} \quad v_{h,j} := w_{h,j}.$$

With this decomposition we first deduce from (45) and (46)

$$a(v_H, v_H) \simeq \|v_H\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \le c\|v_h\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \simeq ca(v_h, v_h). \tag{47}$$

Since $v_{h,j} \in \widetilde{H}_0^{-1/2}(\Gamma'_j)$, we can use Lemma 3.1 and the inverse property to obtain

$$\begin{aligned}\sum_{j=1}^{N_h-1} a(v_{h,j}, v_{h,j}) &\simeq \sum_{j=1}^{N_h-1} \|v_{h,j}\|^2_{\widetilde{H}^{-1/2}(\Gamma'_j)} \le c\sum_{j=1}^{N_h-1} \|I_1 v_{h,j}\|^2_{\widetilde{H}^{1/2}(\Gamma'_j)} \\ &\le c\sum_{j=1}^{N_h-1} h^{-1}\|I_1 v_{h,j}\|^2_{\widetilde{H}^0(\Gamma'_j)}.\end{aligned} \tag{48}$$

Let $\tilde{w}_h := \sum_{j=1}^{N_h-1} v_{h,j} = w_h - w_{h,0}$. By noting that $I_1 v_{h,j}$ is a hat function having support Γ'_j one can easily check that

$$\sum_{j=1}^{N_h-1} \|I_1 v_{h,j}\|^2_{\widetilde{H}^0(\Gamma'_j)} \le 2\|I_1\tilde{w}_h\|^2_{\widetilde{H}^0(\Gamma)}. \tag{49}$$

Hence by using Lemma 3.1 we infer from (48), (49), (46), and (44)

$$\begin{aligned}\sum_{j=1}^{N_h-1} a(v_{h,j}, v_{h,j}) &\le ch^{-1}\|I_1\tilde{w}_h\|^2_{\widetilde{H}^0(\Gamma)} \le ch^{-1}\|\tilde{w}_h\|^2_{\widetilde{H}^{-1}(\Gamma)} \le ch^{-1}\|w_h\|^2_{\widetilde{H}^{-1}(\Gamma)} \\ &\le ch^{-\epsilon}\|v_h\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \simeq ch^{-\epsilon}a(v_h, v_h).\end{aligned} \tag{50}$$

Inequalities (47) and (50) yield (43); therefore the lemma is proved. □

In the same manner, an upper bound for the maximum eigenvalue of P can be proved by using Lemma 3.1 and following the lines of the proof of Lemma 2.3.

Lemma 3.4 *There exists a positive constant C_1 independent of h such that*

$$\lambda_{\max}(P) \le C_1.$$

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Proof. In view of the proof of Lemma 2.3 it suffices to prove

$$a(Tv_h, v_h) \le a(v_h, v_h). \tag{51}$$

Similar to (20), we now have

$$a(Tv_h, v_h) = \sum_{i=1}^{N_h-1} \frac{a(v_h, \chi_{h,i})^2}{a(\chi_{h,i}, \chi_{h,i})}. \tag{52}$$

To estimate $a(v_h, \chi_{h,i})^2$ we note that

$$a(v_h, \chi_{h,i})^2 = \langle V v_h, \chi_{h,i}\rangle^2 \le \|V v_h\|^2_{H^{1-\epsilon}(\Gamma_i')} \|\chi_{h,i}\|^2_{\widetilde{H}^{-1+\epsilon}(\Gamma_i')}. \tag{53}$$

An inequality analogous to (22) can be proved by using again Lemma 3.1 and (22) itself (note that $I_1\chi_{h,i} = \phi_{h,i}$)

$$\|\chi_{h,i}\|^2_{\widetilde{H}^{-1}(\Gamma_i')} \le c\|I_1\chi_{h,i}\|^2_{\widetilde{H}^0(\Gamma_i')} \le ch\|I_1\chi_{h,i}\|^2_{\widetilde{H}^{1/2}(\Gamma_i')} \le ch\|\chi_{h,i}\|^2_{\widetilde{H}^{-1/2}(\Gamma_i')}. \tag{54}$$

Inequalities (52), (53), and (54), the equivalence $a(\chi_{h,i}, \chi_{h,i}) \simeq \|\chi_{h,i}\|^2_{\widetilde{H}^{-1/2}(\Gamma_i')}$, the inverse property, and the mapping property of V imply

$$\begin{aligned} a(Tv_h, v_h) &\le ch^{1-2\epsilon} \sum_{i=1}^{N_h-1} \|V v_h\|^2_{H^{1-\epsilon}(\Gamma_i')} \le ch^{1-2\epsilon}\|V v_h\|^2_{H^{1-\epsilon}(\Gamma)} \\ &\le ch^{1-2\epsilon}\|v_h\|^2_{\widetilde{H}^{-\epsilon}(\Gamma)} \le c\|v_h\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \simeq ca(v_h, v_h). \end{aligned}$$

The second inequality above is a known property of the H^s-norm (see e.g. [11]). Inequality (51) is proved and so is the lemma. □

Combining Lemmas 3.3 and 3.4 we then have

Theorem 3.5 *For any $\epsilon > 0$ there exist positive constants C_0 and C_1 independent of the number of mesh points such that the 2-level additive Schwarz operator P associated with (42) has condition number bounded as*

$$\kappa(P) \le (C_1/C_0)h^{-\epsilon}.$$

Remark 3.6 *See Remark 2.5.*

3.2 Multilevel method.

We define the P_{MAS} operator in the same manner as in §§ 2.2 with subspaces of piecewise-linear functions replaced by subspaces of piecewise-constant functions and with the operator D replaced by V. However, as in the case of the 2-level method the use of Haar's basis functions is necessary. More precisely, we now consider the following space decomposition. Consider a nested sequence of meshes

$$\mathcal{N}^1 \subset \cdots \subset \mathcal{N}^L,$$

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where each mesh $\mathcal{N}^l$ has mesh size $h_l = 2^{-l}$ and is defined by

$$x_j^l = -1 + jh_l, \quad j = 0, \ldots, 2^{l+1}.$$

On each mesh $\mathcal{N}^l$ we again define Haar's basis functions

$$\chi_j^l(x) := \begin{cases} 1/h_l & \text{for } x_{j-1}^l < x < x_j^l, \\ -1/h_l & \text{for } x_j^l < x < x_{j+1}^l, \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, N_l$, where $N_l = 2^l - 1$. Let $\overline{V}_0 = \text{span }\{1\}$ (the constant 1 function), and let $\overline{V}_j^l = \text{span }\{\chi_j^l\}$. We then decompose $\overline{V}^L$ as

$$\overline{V}^L = \overline{V}_0 + \sum_{l=1}^{L}\sum_{j=1}^{N_l} \overline{V}_j^l.$$

The multilevel additive Schwarz operator is now defined as

$$P_{MAS} = P_{\overline{V}_0} + \sum_{l=1}^{L}\sum_{i=1}^{N_l} P_{\overline{V}_i^l}, \tag{55}$$

where $P_{\overline{V}_i^l} : \overline{V}^L \to \overline{V}_i^l$ is defined for any $v \in \overline{V}^L$ by

$$a(P_{\overline{V}_i^l} v, w) = a(v, w) \quad \text{for all } w \in \overline{V}_i^l.$$

We note that this multilevel method is a variant of the BPX preconditioner introduced in [3]. For the application of the latter method as well as multigrid methods to the weakly singular operator, it is necessary to apply first the difference star preconditioner to change the spectrum of the operator (see [1, 6]). Our use of Haar's basis functions for the decomposition is an alternative to the difference star scheme.

A bound for the minimum eigenvalue of P_{MAS} can be obtained if one can find an appropriate decomposition for each $v \in \overline{V}^L$.

Lemma 3.7 *For any $\epsilon > 0$ there exists a positive constant C_0 independent of the number of levels and the number of mesh points such that*

$$\lambda_{\min}(P_{MAS}) \geq C_0 h_L^{\epsilon}.$$

Proof. In view of Lemma 2.1, we will prove that for any $v \in \overline{V}^L$ there exists $v_0 \in \overline{V}_0$ and $v_i^l \in \overline{V}_i^l$ such that $v = v_0 + \sum_{l=1}^{L}\sum_{i=1}^{N_l} v_i^l$ and

$$a(v_0, v_0) + \sum_{l=1}^{L}\sum_{i=1}^{N_l} a(v_i^l, v_i^l) \leq C_0^{-1} h_L^{-\epsilon} a(v, v), \tag{56}$$

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Let $\overline{V}^L$. We first decompose v as $v = \sum_{l=1}^{L} v^l$ by letting

$$v^l := \left(\widetilde{P}_{\overline{V}^l} - \widetilde{P}_{\overline{V}^{l-1}}\right) v \quad \text{for any } l = 1, \ldots, L, \quad \text{with } \widetilde{P}_{\overline{V}^0} \equiv 0,$$

where $\widetilde{P}_{\overline{V}^l}$ is the Galerkin projection from $\widetilde{H}^{-1/2}(\Gamma)$ onto $\overline{V}^l$. Analogous to (33) we now have

$$\|v^l\|_{\widetilde{H}^{-1}(\Gamma)} \leq c h_{l-1}^{1/2-\epsilon/2} \|v^l\|_{\widetilde{H}^{-1/2}(\Gamma)}. \tag{57}$$

We have (cf. (30))

$$a(v^l, v^k) = 0 \quad \text{for } 1 \leq l \neq k \leq L. \tag{58}$$

On each level, v^l can be uniquely written as

$$v^l = v_0^l + v_1^l + \cdots + v_{N_l}^l,$$

where $v_j^l \in \overline{V}_j^l$. Here, $V_0^l = \text{span}\,\{1\}$. Let

$$v_0 = \sum_{l=1}^{L} v_0^l.$$

Then we decompose $v \in \overline{V}^L$ as

$$v = v_0 + \sum_{l=1}^{L} \sum_{i=1}^{N_l} v_i^l.$$

Firstly, similar to (46), we have

$$\|v_0^l\|^2_{\widetilde{H}^{-s}(\Gamma)} \leq c \|v^l\|^2_{\widetilde{H}^{-s}(\Gamma)}, \quad 0 \leq s \leq 1. \tag{59}$$

Also, since $v_0 = \langle v, 1 \rangle / |\Gamma|$ we have

$$\|v_0\|^2_{\widetilde{H}^{-s}(\Gamma)} \leq c \|v\|^2_{\widetilde{H}^{-s}(\Gamma)}, \quad 0 \leq s \leq 1. \tag{60}$$

Next, by using Lemma 3.1 and the inverse property we obtain

$$\sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{-1/2}(\Gamma_i^l)} \quad \leq \quad c \sum_{i=1}^{N_l} \|I_1 v_i^l\|^2_{\widetilde{H}^{1/2}(\Gamma_i^l)} \leq c \sum_{i=1}^{N_l} h_l^{-1} \|I_1 v_i^l\|^2_{\widetilde{H}^0(\Gamma_i^l)}. \tag{61}$$

As in the proof of Lemma 3.3, if $\tilde{v}^l := \sum_{i=1}^{N_l} v_i^l = v^l - v_0^l$ then (cf. (49))

$$\sum_{i=1}^{N_l} \|I_1 v_i^l\|^2_{\widetilde{H}^0(\Gamma_i^l)} \leq 2 \|I_1 \tilde{v}^l\|^2_{\widetilde{H}^0(\Gamma)}. \tag{62}$$

From (59) we deduce

$$\|\tilde{v}^l\|^2_{\widetilde{H}^{-s}(\Gamma)} \leq c \|v^l\|^2_{\widetilde{H}^{-s}(\Gamma)} \quad 0 \leq s \leq 1. \tag{63}$$

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Hence (61), (62), (63), and (57) together with Lemma 3.1 imply

$$\begin{aligned}\sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{-1/2}(\Gamma_i^l)} &\le ch_l^{-1}\|I_1\tilde{v}^l\|^2_{\widetilde{H}^0(\Gamma)} \le ch_l^{-1}\|\tilde{v}^l\|^2_{\widetilde{H}^{-1}(\Gamma)} \le ch_l^{-1}\|v^l\|^2_{\widetilde{H}^{-1}(\Gamma)}\\ &\le ch_l^{-\epsilon}\|v^l\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \le ch_L^{-\epsilon}\|v^l\|^2_{\widetilde{H}^{-1/2}(\Gamma)}.\end{aligned}$$

Summing over $l = 1, \ldots, L$ and using the orthogonality of v^l, see (58), we obtain

$$\begin{aligned}\sum_{l=1}^{L}\sum_{i=1}^{N_l} a(v_i^l, v_i^l) &\simeq \sum_{l=1}^{L}\sum_{i=1}^{N_l} \|v_i^l\|^2_{\widetilde{H}^{-1/2}(\Gamma_i^l)} \le ch_L^{-\epsilon}\sum_{l=1}^{L}\|v^l\|^2_{\widetilde{H}^{-1/2}(\Gamma)}\\ &= ch_L^{-\epsilon}\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \simeq ch_L^{-\epsilon}a(v,v).\end{aligned}$$

This inequality together with (60) yield (56); therefore the lemma is proved. □

An upper bound of the maximum eigenvalue of P_{MAS} can be proved in the same manner as in the case of the hypersingular integral operator if we have the following strengthened Cauchy-Schwarz inequality:

Lemma 3.8 *Let $T_l = \sum_{i=1}^{N_l} P_{\overline{V}_i^l}$. Then there exist constants $c > 0$ and $\gamma \in (0,1)$ such that for any $v \in \overline{V}^k$ where $1 \le k \le l \le L$ there holds*

$$a(T_l v, v) \le c\gamma^{2(l-k)}a(v,v).$$

Proof. The proof goes along the lines of the proofs of Lemmas 2.7 and 3.4, and is omitted. □

Therefore we have the following result

Lemma 3.9 *There exists a positive constant C_1 independent of the number of levels and the number of mesh points such that*

$$\lambda_{\max}(P_{MAS}) \le C_1.$$

Combining Lemmas 3.7 and 3.9 we obtain

Theorem 3.10 *For any $\epsilon > 0$ there exist positive constants C_0 and C_1 independent of the number of levels and number of mesh points such that the multilevel additive Schwarz method defined by (55) has condition number bounded as*

$$\kappa(P_{MAS}) \le (C_1/C_0)h_L^{-\epsilon}.$$

Remark 3.11 *See Remark 2.5.*

4 Numerical results.

We consider the equation (2) with the right hand side $f(x) \equiv 2$, and the equation (40) with the right hand side $g(x) = 2x$. We solve the Galerkin equations (4) and (41) by the conjugate gradient (CG) method, and by the preconditioned conjugate gradient method with preconditioners B given by the 2-level and multilevel methods.

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		Condition numbers			Numbers of iterations		
L	N_L	CG	2-level	MAS	CG	2-level	MAS
2	3	2.01	1.74	1.64	2	2	2
3	7	3.86	2.10	2.41	4	4	4
4	15	7.69	2.20	3.04	6	6	6
5	31	15.35	2.18	3.46	9	7	9
6	63	30.95	2.18	3.76	14	7	10
7	127	62.24	2.17	3.97	20	7	10
8	255	124.93	2.17	4.13	29	7	11
9	511	250.28	2.17	4.26	40	7	11

Table 1: Hypersingular integral equation

The numbers in Tables 1 and 2 show that without preconditioning the condition numbers of the Galerkin matrices grow as $O(N_L)$. With the 2-level or multilevel (MAS) methods, an asymptotic behaviour (to a constant) can be observed. The numbers of iterations for preconditioned methods can also be seen to be bounded.

As mentioned in § 2, it is not necessary to know the explicit form of B. What is important for the implementation is to compute the acting of B on a spline. For example, in the case of the multilevel method for the hypersingular integral operator, for any $v_L \in V^L$, $\hat{v}_L := Bv_L$ can be computed as follows (cf. [15, 16]):

1. Restrict v_L to each level: for $l = L-1, \dots, 1$ and $i = 1, \dots, N_l$,
$$\begin{aligned} v_l(ih_l) &:= \Pi^l_{l+1} v_{l+1}(ih_l) \\ &= v_{l+1}((2i-1)h_{l+1})/2 + v_{l+1}(2ih_{l+1}) + v_{l+1}((2i+1)h_{l+1})/2. \end{aligned}$$

2. Scale v_l on each level: for $l = 1, \dots, L$ and $i = 1, \dots, N_l$,
$$v_l^*(ih_l) := D_l^{-1} v_l(ih_l) = v_l(ih_l)/a(\phi_i^l, \phi_i^l).$$

3. Interpolate to finest level: for $l = 2, \dots, L$,
$$\begin{aligned} \hat{v}_1 &:= v_1^*, \\ \hat{v}_l &:= v_l^* + \Pi^l_{l-1}\hat{v}_{l-1}, \end{aligned}$$
where for $i = 1, \dots, N_{l-1}$,
$$\begin{aligned} \Pi^l_{l-1}\hat{v}_{l-1}((2i-1)h_l) &= \left(\hat{v}_{l-1}((i-1)h_{l-1}) + \hat{v}_{l-1}(ih_{l-1})\right)/2, \\ \Pi^l_{l-1}\hat{v}_{l-1}(2ih_l) &= \hat{v}_{l-1}(ih_{l-1}). \end{aligned}$$

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		Condition numbers		Numbers of iterations	
L	N_L	CG	2-level	CG	2-level
4	16	15.75	2.36	8	8
5	32	32.68	2.37	20	11
6	64	65.11	2.32	33	14
7	128	130.07	2.29	48	14
8	256	259.18	2.26	64	14
9	512	517.30	2.25	89	15
10	1024	1035.07	2.24	126	15

Table 2: Weakly singular integral equation

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