Integral equations on open-curves : a new preconditioner

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Abstract

In this paper, we analyze preconditioners for the integral equations arising in the resolution of acoustic scattering by an open arc in 2D in the Galerkin setting.

1 Preliminaries

Globalement, il faut recopier les définitions de Bruno. Let Γ a smooth open simple curve in $\mathbb{R}^2$, and u_D and u_N two smooth functions on Γ. We consider the following boundary value problems, namely the Dirichlet problem (D):

$$\begin{cases} -\Delta u - k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma \\ u = u_D, & \text{on } \Gamma \end{cases}$$

and the Neumann problem (N):

$$\begin{cases} -\Delta u - k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \Gamma \\ \dfrac{\partial u}{\partial n} = u_N. & \text{on } \Gamma \end{cases}$$

These problems can be solved using integral equations. Let G the Green's function defined by

$$\begin{cases} G(z) = -\dfrac{1}{2\pi} \ln |z| \,, & \text{if } k = 0, \\ G(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0. \end{cases} \tag{1}$$

We consider the single-layer potential defined for $x \notin \Gamma$ by

$$\mathrm{SL}\lambda(x) = \int_\Gamma G(x-y)\lambda(y)d\sigma(y) \tag{2}$$

where σ is the arc measure on Γ.. Denoting by γ the trace operator on Γ and $S = \gamma\mathrm{SL}$, it is well-known that the solution u of (D) is given by

$$u = \mathrm{SL}\lambda$$

if λ is a solution of the integral equation

$$S\lambda = u_D. \tag{3}$$

The solution λ to the former problem is unique and well-defined. However, because of the edges of Γ, it is not smooth, leading in poor performance of numerical methods based on the discretization of λ itself. It is known that there exists a smooth function φ such that $\lambda = \frac{\varphi}{\omega(x)}$ whith

$$\omega(x) = \frac{1}{\sqrt{d(x, \partial\Gamma)}}$$

This is why, in [2], a weighted operator S_ω is introduced, defined by

$$S_\omega \varphi := S\left(\frac{\varphi}{\omega}\right).$$

This time, S_ω sends smooth functions on smooth functions, leading to improved convergence in numerical methods. Symmetrically, if we let

$$\mathrm{DL}\nu(x) = \int_\Gamma \frac{\partial}{\partial n_y} G(x-y)\nu(y)d\sigma(y) \tag{4}$$

the solution to problem (N) is obtained as

$$u = \mathrm{DL}\nu$$

where ν is the solution of the integral equation

$$N\nu = u_N, \tag{5}$$

and N is the so-called hypersingular operator defined by

$$N\nu = \lim_{z \to 0^+} \frac{\partial}{\partial z} \mathrm{DL}\nu(x + z n_x).$$

Similarly, if u_N is smooth, there exists a smooth function ψ such that

$$\nu = \psi\omega,$$

thus the corresponding weighted hyersingular operator is defined by

$$N_\omega \psi := N(\psi\omega)$$

In [2], it is shown that the operators S_ω and N_ω are inverse modulo a compact operator, justifying that they are good mutual preconditioners in the process of solving (3) and (5) numerically. Here study a new preconditioning technique based on a weighted version of the Laplace operator: for any function u defined on Γ

$$\Delta_\omega u := \omega\left(\omega u'\right)'$$

where the derivative is taken along the curvilinear abscissa. We analyze preconditioners given by that $S_\omega(\Delta_\omega - k^2\omega^2)$ for equation (3) and N_ω.

2 Analysis

Definition 1. *Definition of the modified Sobolev spaces H^s_ω.*

Theorem 1. *For the bilinear form $(u,v) \mapsto \int_{-1}^{1} \frac{uv}{\omega}$, H^s_ω is the dual of H^{-s}_ω. We have compact injections from $H^s_\omega \to H^t_\omega$ when $s < t$.*

2.1 Study on the segment with $k \neq 0$.

We write $H_0(z) = \frac{-1}{2\pi} \ln|z| J_0(z) + R(z)$ where R is an even entire function.

Proposition 1. *The functions $r \mapsto \frac{J_0(r)-1}{r^2}$ and $r \mapsto \frac{J_0'(r)}{r}$ are bounded on $\mathbb{R}$.*

Proof. We have for all $r \in \mathbb{R}$

$$\frac{J_0(r) - 1}{r^2} = -\sum_{n=0}^{+\infty} \frac{(-1)^n}{(n-1)!^2} \left(\frac{r}{2}\right)^{2n}$$

which is easily shown to be absolutely convergent series for all $r > 0$. A similar argument gives the other result. □

Proposition 2. *For any k, we define $k_1 : r \mapsto (J_0(kr) - 1)\ln(|r|)$ defined on $\mathbb{R}$. Then the function $-\Delta_\omega k_1$ is bounded on $\mathbb{R}$.*

Proof. We can write $-\Delta_\omega K_1 = (r^2 - 1)\partial_{rr} K_1 + r\partial_r K_1(r)$, yielding

$$-\Delta_\omega K_1 = \ln(|r|)\left(kJ_0'(kr) + k^2(r^2-1)J_0''(kr)\right) + (r^2-1)\left(2\frac{kJ_0'(kr)}{|r|} - \frac{J_0(kr)-1}{r^2}\right) + J_0(kr) - 1.$$

□

Corollary 1. *The function*

$$k_1 : f \mapsto \int_{-1}^{1} \frac{k_1(k|x-y|)}{\omega(y)} f(y)dy$$

Is continuous from $H^s_\omega \to H^{s+2}_\omega$

Theorem 2. *An operator of the form*

$$Kf = \int_{-1}^{1} \frac{k(x,y)f(y)}{\omega(y)} \tag{6}$$

with $k \in C^\infty(-1,1)$ maps H^s_ω to $H^{+\infty}_\omega$ for all s.

2.2 Non-flat arc, non-zero frequency

We consider a smooth non-intersecting curve Γ in $\mathbb{R}^2$ and a smooth parametrization $\mathbf{r} : [-1,1] \to \Gamma$. We choose $\mathbf{r}$ such that $\left\|\frac{dr}{dt}\right\| = 1$. Indeed we can assume the curve has unit length by proper rescaling. Indeed, if u is solution of the Helmholtz equation outside Ω with some boundary conditions on Γ (Dirichlet or Neuman) and if we define $u^\lambda = u(\lambda r, \theta)$, we find $\Delta u^\lambda + k^2\lambda^2 u^\lambda = 0$ outside $\Omega_\lambda = \frac{\Omega}{\lambda}$. By choosing $\lambda = |\Gamma|$, the border of the new domain is of length 1.

Without the rescaling (but still assuming constant speed parametrization), we can write

$$\left|r(t) - r(t')\right|^2 = L^2 \left|t - t'\right|^2 + \frac{C(t')^2}{2} \left|t - t'\right|^4$$

We note $G_k(t, t')$ the kernel of the non-zero non-flat arc operator.

Lemma 1. *We have the following expansion*

$$J_0(k\left|r(t) - r(t')\right|) = 1 - \frac{k^2}{4} L^2 \left|t - t'\right|^2 + \left(\frac{k^4 L^4}{64} - \frac{C(t')^2 k^2}{8}\right) \left|t - t'\right|^4 + (t-t')^5 F(t, t')$$

where F is a smooth bounded function.

Lemma 2. *If L is the length of the curve and $C(t')$ the curvature at a point t', one has*

$$\begin{aligned} G_k(t, t') = &-\frac{1}{2\pi} \ln|t - t'| \left(1 - \frac{k^2}{4} L^2 \left|t - t'\right|^2 + \left(\frac{k^4 L^4}{64} - \frac{C(t')^2 k^2}{8}\right) \left|t - t'\right|^4 + (t - t')^5 F(t, t')\right) \\ &+ R(t, t') \end{aligned}$$

where R is in $C^\infty([-1,1]^2)$.

Lemma 3.

$$\begin{aligned} \Delta_\omega^{t'} \left((t - t')^2 \ln\left|t - t'\right|\right) &= \omega^2(t') \frac{d^2}{dt'^2} \left((t - t')^2 \ln\left|t - t'\right|\right) - t' \frac{d}{dt'} \left((t - t')^2 \ln\left|t - t'\right|\right) \\ &= \omega^2(t') \left(2 \ln\left|t - t'\right| + 4 - 1\right) - t' \left(2(t' - t) \ln\left|t - t'\right| + 2(t' - t)\right) \\ &= 2\omega^2(t) \ln\left|t - t'\right| + 2(t - t') \ln\left|t - t'\right| (t + 2t') + P(t, t') \end{aligned}$$

where P is a polynmial in t and t'.

Lemma 4. *The application $f \mapsto \omega^2 f$ is continuous in H^s_ω for any s.*

Proof. This is obvious as $\left|\left\langle \omega^2 f, T_n \right\rangle_\omega\right| \leq \left|\langle f, T_n \rangle_\omega\right|$. ∎

Lemma 5. *The application $f \mapsto \omega^2 f'$ is continuous from H^s_ω to H^{s-1}_ω. The proof involves the Cesaro thm.*

Lemma 6. *The operator $\Delta_\omega \omega^2 - \omega^2 \Delta_\omega$ is continuous from H^s_ω to H^{s-1}_ω.*

Proof. We use the formula : $\Delta_\omega \omega^2 - \omega^2 \Delta_\omega = 4x\omega^2 f' + (4x^2 - 2)f$ ∎

Lemma 7. *The operator $S_0 \omega^2 - \omega^2 S_0$ is continuous from H^s_ω to H^{s+2}_ω.*

Proof. Use $\omega^2 T_n = \frac{2T_n + T_{n+2} + T_{|n-2|}}{4}$. ∎

Be careful, the map $f \mapsto f'$ is not continuous from H^1_ω to L^2_ω, as can be checked with the example $f = \omega$.

Lemma 8. $S_k - S_0 = \omega(t)^2 \frac{k^2}{2} L^2 S_0 \Delta_\omega^{-1}$

References

[1] Xavier Antoine and Marion Darbas. Generalized combined field integral equations for the iterative solution of the three-dimensional helmholtz equation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 41(1):147–167, 2007.

[2] Oscar P Bruno and Stéphane K Lintner. Second-kind integral solvers for te and tm problems of diffraction by open arcs. *Radio Science*, 47(6), 2012.