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Preconditioning in H(div) and Applications

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Abstract. Summarizing the work of [AFW97], we show how to construct preconditioners using domain decomposition and multigrid techniques for the system of linear algebraic equations which arises from the finite element discretization of boundary value problems associated to the differential operator $\boldsymbol{I} - \mathbf{grad}\,\mathrm{div}$. These preconditioners are shown to be spectrally equivalent to the inverse of the operator and thus may be used to precondition iterative methods so that any given error reduction may be achieved in a finite number of iterations independent of the mesh discretization. We describe applications of these results to the efficient solution of mixed and least squares finite element approximations of elliptic boundary value problems.

1.1 Introduction

This paper summarizes the work of [AFW97], in which we consider the solution of the system of linear algebraic equations which arises from the finite element discretization of boundary value problems in two space dimensions for the differential operator $\boldsymbol{I} - \mathbf{grad}\,\mathrm{div}$. The natural setting for the weak formulation of such problems is the space:

$$\boldsymbol{H}(\mathrm{div}) = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \,|\, \mathrm{div}\,\boldsymbol{u} \in L^2(\Omega) \right\}.$$

Let $(\,\cdot\,,\,\cdot\,)$ denote the $L^2(\Omega)$ inner product of both scalar and vector-valued functions and

$$J(\boldsymbol{u}, \boldsymbol{v}) := (\boldsymbol{u}, \boldsymbol{v}) + (\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{v})$$

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denote the innerproduct on $\boldsymbol{H}(\mathrm{div})$. If $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the weak formulation is to find $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div})$ such that for all $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div})$,

$$J(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}).$$

This corresponds to the boundary value problem

$$(\boldsymbol{I} - \mathbf{grad}\,\mathrm{div})\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \qquad \mathrm{div}\,\boldsymbol{u} = 0 \text{ on } \partial\Omega.$$

Note that if $\boldsymbol{u}$ is a gradient, then $(\boldsymbol{I} - \mathbf{grad}\,\mathrm{div})\boldsymbol{u} = -\boldsymbol{\Delta}\,\boldsymbol{u} + \boldsymbol{u}$, while if $\boldsymbol{u}$ is a curl, then $(\boldsymbol{I} - \mathbf{grad}\,\mathrm{div})\boldsymbol{u} = \boldsymbol{u}$. A simple situation in which the operator $\boldsymbol{I} - \mathbf{grad}\,\mathrm{div}$ arises occurs in the computation of $\boldsymbol{u} = \mathbf{grad}\,p$, where p is the solution of the Dirichlet problem

$$-\Delta\,p + p = g \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \partial\Omega.$$

Then $\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div})$ satisfies

$$J(\boldsymbol{u}, \boldsymbol{v}) = -(g, \mathrm{div}\,\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}).$$

Given a finite element subspace $\boldsymbol{V}_h$ of $\boldsymbol{H}(\mathrm{div})$, the natural finite element approximation scheme is: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$J(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

We shall consider the case when $\boldsymbol{V}_h$ consists of the Raviart–Thomas space of index $k \geq 0$, i.e., functions which on each triangle are of the form

$$\boldsymbol{v}(x, y) = \boldsymbol{p}(x, y) + (x, y)q(x, y), \qquad \boldsymbol{p} \in \mathcal{P}_k \times \mathcal{P}_k, \quad q \in \mathcal{P}_k,$$

(where $\mathcal{P}_k$ denotes the polynomials of degree $\leq k$) and for which $\boldsymbol{v}\cdot\boldsymbol{n}$ is continuous from triangle to triangle. The goal is to find an efficient procedure for solving the discrete linear system corresponding to this discretization, which we write as $\boldsymbol{J}_h\boldsymbol{u}_h = \boldsymbol{f}_h$. Denoting the eigenvalues of $\boldsymbol{J}_h$ by $\sigma(\boldsymbol{J}_h)$, since the spectral condition number

$$\kappa(\boldsymbol{J}_h) := \frac{\max|\sigma(\boldsymbol{J}_h)|}{\min|\sigma(\boldsymbol{J}_h)|}$$

of the operator $\boldsymbol{J}_h$ is $O(h^{-2})$, we will clearly need to precondition any standard iterative scheme if we want the number of iterations needed to achieve a given accuracy to be independent of h.

1.2 Preconditioning in the abstract

Let $X_h \subset L^2$ be a finite dimensional normed vectorspace. We identify X_h and X_h^* as sets, but put the dual norm on the latter (dual with respect to the L^2 inner product). Let $\mathcal{A}_h : X_h \to X_h$ be an L^2-symmetric linear isomorphism. We suppose that X_h is endowed with an appropriate (energy) norm, i.e., we suppose that

$$\|\mathcal{A}_h\|_{\mathcal{L}(X_h, X_h^*)}, \ \|\mathcal{A}_h^{-1}\|_{\mathcal{L}(X_h^*, X_h)} = O(1).$$

Given $f_h \in X_h$, we wish to solve $\mathcal{A}_h x_h = f_h$ by applying a standard iterative method such as CG or MINRES to the equation $\mathcal{B}_h \mathcal{A}_h x_h = \mathcal{B}_h f_h$, where $\mathcal{B}_h : X_h \to X_h$ is an L^2-symmetric, positive definite preconditioner. Our goal is to define $\mathcal{B}_h$ so that the action of $\mathcal{B}_h$ is easily computable and $\kappa(\mathcal{B}_h \mathcal{A}_h)$ is bounded uniformly with respect to h. Since

$$\max |\sigma(\mathcal{B}_h \mathcal{A}_h)| \le \|\mathcal{B}_h \mathcal{A}_h\|_{\mathcal{L}(X_h,X_h)} \le \|\mathcal{A}_h\|_{\mathcal{L}(X_h,X_h^*)} \|\mathcal{B}_h\|_{\mathcal{L}(X_h^*,X_h)}$$

and

$$\frac{1}{\min |\sigma(B_h \mathcal{A}_h)|} \le \|(\mathcal{B}_h \mathcal{A}_h)^{-1}\|_{\mathcal{L}(X_h,X_h)} \le \|\mathcal{A}_h^{-1}\|_{\mathcal{L}(X_h^*,X_h)} \|\mathcal{B}_h^{-1}\|_{\mathcal{L}(X_h,X_h^*)}$$

$\mathcal{B}_h$ is an effective preconditioner if

$$\|\mathcal{B}_h\|_{\mathcal{L}(X_h^*,X_h)},\ \|\mathcal{B}_h^{-1}\|_{\mathcal{L}(X_h,X_h^*)} = O(1).$$

In other words, $\mathcal{B}_h$ is an effective preconditioner if it has the same mapping properties as $\mathcal{A}_h^{-1}$. Note that the energy norm, and not the detailed structure of $\mathcal{A}_h$, determine these properties. Thus to solve the problem $\boldsymbol{J}_h \boldsymbol{u}_h = \boldsymbol{f}_h$, we need to construct an efficiently computable operator $\boldsymbol{K}_h : \boldsymbol{V}_h \to \boldsymbol{V}_h$ for which

$$\|\boldsymbol{K}_h\|_{\mathcal{L}(V_h^*,V_h)},\ \|\boldsymbol{K}_h^{-1}\|_{\mathcal{L}(V_h,V_h^*)} = O(1).$$

We will show how this can be done using domain decomposition and multigrid techniques.

1.3 Applications

We are interested in the operator $\boldsymbol{I} - \mathbf{grad}\,\mathrm{div}$ not for its own sake, but for its appearance in several important problems. Besides the example mentioned in the introduction, we will restrict our attention to two problems: the least squares formulation and the mixed formulation of second order scalar elliptic problems. Other examples are discussed in [AFW97]. We first discuss the least squares variational principle.

Consider the elliptic boundary value problem

$$\mathrm{div}(A\,\mathbf{grad}\,p) = g \text{ in } \Omega, \qquad p = 0 \text{ on } \partial\Omega,$$

where the coefficient matrix A is assumed measurable, bounded, symmetric, and uniformly positive definite on Ω. Introducing $\boldsymbol{u} = A\,\mathbf{grad}\,p$ leads to the first order system

$$\boldsymbol{u} - A\,\mathbf{grad}\,p = 0 \text{ in } \Omega, \quad \mathrm{div}\,\boldsymbol{u} = g \text{ in } \Omega, \quad p = 0 \text{ on } \partial\Omega.$$

The least squares variational principle characterizes the solution $(\boldsymbol{u}, p)$ as the minimizer of the functional

$$\|\boldsymbol{v} - A\,\mathbf{grad}\,q\|^2 + \|\,\mathrm{div}\,\boldsymbol{v} - g\|^2$$

over the space $\boldsymbol{H}(\mathrm{div}) \times \mathring{H}^1$, where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm and $\mathring{H}^1$ denotes the subspace of functions in $H^1(\Omega)$ which vanish on the boundary of Ω. Equivalently, we have the weak formulation

$$B(\boldsymbol{u}, p; \boldsymbol{v}, q) = (g, \mathrm{div}\,\boldsymbol{v}) \quad \text{for all } (\boldsymbol{v}, q) \in \boldsymbol{H}(\mathrm{div}) \times \mathring{H}^1,$$

where

$$B(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\boldsymbol{u} - A\, \mathbf{grad}\, p, \boldsymbol{v} - A\, \mathbf{grad}\, q) + (\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{v}).$$

To discretize the least squares formulation, we let $X_h = \boldsymbol{V}_h \times W_h$ be a finite-dimensional subspace of $\boldsymbol{H}(\mathrm{div}) \times \mathring{H}^1$. Then $x_h := (\boldsymbol{u}_h, p_h)$ is the minimizer over X_h of

$$\|\boldsymbol{v} - A\, \mathbf{grad}\, q\|^2 + \|\,\mathrm{div}\,\boldsymbol{v} - g\|^2,$$

or in weak form,

$$B(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = (g, \mathrm{div}\,\boldsymbol{v}) \quad \text{for all } (\boldsymbol{v}, q) \in X_h.$$

Defining $\mathcal{A}_h : X_h \to X_h$ by $(\mathcal{A}_h x, y) = B(x, y)$ and $f_h \in X_h$ by $(f_h, (\boldsymbol{v}, q)) = (g, \mathrm{div}\,\boldsymbol{v})$, we may rewrite our problem as $\mathcal{A}_h x_h = f_h$.

The key to the convergence theory for the least squares method is the following theorem (cf. Pehlivanov, Carey, Lazarov [PCL94] and Cai, Lazarov, Manteuffel, and McCormick [CLMM94]).

Theorem 1.1 *The bilinear form B is an inner product on $\boldsymbol{H}(\mathrm{div}) \times \mathring{H}^1$ equivalent to the usual one.*

A direct consequence of the theorem is that $\mathcal{A}_h : X_h \to X_h$ is symmetric, positive definite and satisfies

$$\|\mathcal{A}_h\|_{\mathcal{L}(X_h, X_h^*)},\ \|\mathcal{A}_h^{-1}\|_{\mathcal{L}(X_h^*, X_h)} = O(1).$$

Thus we need a preconditioner with the opposite mapping properties. Since $X_h = \boldsymbol{V}_h \times W_h$, we can choose a block diagonal preconditioner

$$\mathcal{B}_h = \begin{pmatrix} \boldsymbol{K}_h & 0 \\ 0 & M_h \end{pmatrix},$$

where $\boldsymbol{K}_h$ is a good preconditioner in $\boldsymbol{H}(\mathrm{div})$, i.e., it maps like $\boldsymbol{J}_h^{-1} : \boldsymbol{V}_h \to \boldsymbol{V}_h$, and M_h is a good preconditioner in $\mathring{H}^1$, i.e., it maps like $\Delta_h^{-1} : W_h \to W_h$. Hence we conclude that a good preconditioner for the discrete least squares system is obtained using an $\boldsymbol{H}(\mathrm{div})$ preconditioner for the vector variable and a standard $\mathring{H}^1$ preconditioner for the scalar variable.

We next consider a mixed variational formulation of this boundary value problem. The mixed variational principle characterizes $(\boldsymbol{u}, p)$ as a saddle point of

$$\frac{1}{2}(A^{-1}\boldsymbol{v}, \boldsymbol{v}) + (q, \mathrm{div}\,\boldsymbol{v}) - (g, q),$$

over $\boldsymbol{H}(\mathrm{div}) \times L^2$, or, in weak form,

$$(A^{-1}\boldsymbol{u}, \boldsymbol{v}) + (p, \mathrm{div}\,\boldsymbol{v}) = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}),$$

$$(\operatorname{div} \boldsymbol{u}, q) = (g, q) \qquad \text{for all } q \in L^2.$$

Choosing $X_h = \boldsymbol{V}_h \times S_h \subset \boldsymbol{H}(\operatorname{div}) \times L^2$, we can define a discrete solution $x_h = (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times S_h$ by restricting either the variational or weak formulation. This may be written $\mathcal{A}_h x_h = f_h$, with $\mathcal{A}_h : X_h \to X_h$ L^2-symmetric but indefinite, since $\mathcal{A}_h$ has the form

$$\mathcal{A}_h = \begin{pmatrix} a & b \\ b^t & 0 \end{pmatrix}.$$

The convergence of this method depends on the choice of $\boldsymbol{V}_h$ and S_h. The key hypotheses for the convergence analysis are the Brezzi conditions:

$$(A^{-1}\boldsymbol{v}, \boldsymbol{v}) \geq \gamma_1 \|\boldsymbol{v}\|_{H(\operatorname{div})} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_h \text{ with } \operatorname{div} \boldsymbol{v} \perp S_h,$$

$$\inf_{q \in S_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{v})}{\|q\| \, \|\boldsymbol{v}\|_{H(\operatorname{div})}} \geq \gamma_2.$$

These conditions are satisfied if $\boldsymbol{V}_h$ is the Raviart–Thomas space of index k and S_h the space of (discontinuous) piecewise polynomials of degree k. Brezzi's theorem states that if both hypotheses are satisfied, then $\mathcal{A}_h$ is an isomorphism and $\|\mathcal{A}_h^{-1}\|_{\mathcal{L}(X_h^*, X_h)}$ may be bounded in terms of the γ_i.

We thus base our choice of $\mathcal{B}_h$ on the discrete version of the isomorphism

$$\begin{pmatrix} A & -\textbf{grad} \\ \operatorname{div} & 0 \end{pmatrix} : \boldsymbol{H}(\operatorname{div}) \times L^2 \to \boldsymbol{H}(\operatorname{div})^* \times L^2.$$

We again use a simple block-diagonal preconditioner, which this time takes the form

$$\mathcal{B}_h = \begin{pmatrix} \boldsymbol{K}_h & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity on S_h and again $\boldsymbol{K}_h$ is a good preconditioner in $\boldsymbol{H}(\operatorname{div})$, i.e., it maps like $\boldsymbol{J}_h^{-1} : \boldsymbol{V}_h \to \boldsymbol{V}_h$.

We remark that most other work on preconditioning such mixed methods uses the alternate isomorphism

$$\begin{pmatrix} A & -\textbf{grad} \\ \operatorname{div} & 0 \end{pmatrix} : \boldsymbol{L}^2 \times \mathring{H}^1 \to \boldsymbol{L}^2 \times H^{-1},$$

which leads to a different (and less natural) choice of preconditioner.

1.4 An additive Schwarz preconditioner for $\boldsymbol{J}_h$

We let $\mathcal{T}_H = \{\Omega_n\}_{n=0}^N$, denote the coarse mesh and $\mathcal{T}_h$ a refinement (the fine mesh). We let $\{\Omega'_n\}_{n=1}^N$ be an overlapping covering aligned with the fine mesh such that $\Omega_n \subset \Omega'_n$. We make the standard assumption of sufficient but bounded overlap. Let $\boldsymbol{V}_n$ denote the Raviart–Thomas space approximating $\boldsymbol{H}(\operatorname{div}, \Omega'_n)$ with the boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega'_j \setminus \partial\Omega$. Let $\boldsymbol{V}_0$ denote the Raviart–Thomas approximation to $\boldsymbol{H}(\operatorname{div}, \Omega)$ using the coarse mesh.

Given $\boldsymbol{f} \in \boldsymbol{V}_h$, define $\boldsymbol{u}_n \in \boldsymbol{V}_n$ by $J(\boldsymbol{u}_n, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad$ for all $\boldsymbol{v} \in \boldsymbol{V}_n$. The additive Schwarz preconditioner is then defined by $\boldsymbol{K}_h \boldsymbol{f} := \sum_{n=0}^{N} \boldsymbol{u}_n$. Our main result for this domain decomposition preconditioner is the following theorem (cf. [AFW97] for the proof).

Theorem 1.2 *There exists a constant c independent of both h and H for which $\kappa(\boldsymbol{K}_h \boldsymbol{J}_h) \le c$.*

Following the theoretical framework of Dryja–Widlund [DW90] or Xu [Xu92], a critical step of the proof is the following decomposition lemma.

Lemma 1.1 *For all $\boldsymbol{v} \in \boldsymbol{V}_h$, there exist $\boldsymbol{v}_n \in \boldsymbol{V}_n$ with $\boldsymbol{v} = \sum_{n=0}^{N} \boldsymbol{v}_n$ and*

$$\sum_{n=0}^{N} \|\boldsymbol{v}_n\|_{H(\mathrm{div})}^2 \le c \|\boldsymbol{v}\|_{H(\mathrm{div})}.$$

The standard proof uses a partition of unity $\{\theta_n\}_{n=1}^{N}$ and takes $\boldsymbol{v}_0 \in \boldsymbol{V}_0$ a suitable approximation of $\boldsymbol{v}$ and $\boldsymbol{v}_n = \boldsymbol{\Pi}_h[\theta_n(\boldsymbol{v} - \boldsymbol{v}_0)]$ with $\boldsymbol{\Pi}_h$ a suitable local projection into $\boldsymbol{V}_h$. The analysis leads to the following estimates.

$$\begin{aligned} \|\operatorname{div} \boldsymbol{v}_n\| \le\ & c \|\operatorname{div}[\theta_n(\boldsymbol{v} - \boldsymbol{v}_0)]\| \\ \le\ & c \|\operatorname{\mathbf{grad}} \theta_n\|_{L^\infty} \|\boldsymbol{v} - \boldsymbol{v}_0\| + \|\theta_n\|_{L^\infty} \|\operatorname{div}(\boldsymbol{v} - \boldsymbol{v}_0)\| \\ \le\ & c H^{-1} \|\boldsymbol{v} - \boldsymbol{v}_0\| + \|\operatorname{div}(\boldsymbol{v} - \boldsymbol{v}_0)\|. \end{aligned}$$

In the standard elliptic case we bound the first term using $\|\boldsymbol{v} - \boldsymbol{v}_0\| \le CH \|\boldsymbol{v}\|_1$. However it is not true that $\|\boldsymbol{v} - \boldsymbol{v}_0\| \le CH \|\boldsymbol{v}\|_{H(\mathrm{div})}$, so this approach fails. We are able to get around this problem by using a discrete Helmholtz decomposition, which we now describe.

Let $\boldsymbol{V}_h$ denote the Raviart–Thomas space of index k, S_h the space of piecewise polynomials of degree k, and W_h the space of C^0 piecewise polynomials of degree $k+1$. Then we have the following discrete Helmholtz decomposition.

$$\boldsymbol{V}_h = \operatorname{\mathbf{curl}} W_h \oplus \operatorname{\mathbf{grad}}_h S_h,$$

where $\operatorname{\mathbf{grad}}_h : S_h \to \boldsymbol{V}_h$ is defined by $(\operatorname{\mathbf{grad}}_h s, \boldsymbol{v}) = -(s, \operatorname{div} \boldsymbol{v})$.

Returning to the decomposition lemma, we write $\boldsymbol{v} = \operatorname{\mathbf{curl}} w + \operatorname{\mathbf{grad}}_h s$ and observe that

$$\|\boldsymbol{v}\|_{H(\mathrm{div})}^2 = \|\operatorname{\mathbf{curl}} w\|^2 + \|\operatorname{\mathbf{grad}}_h s\|_{H(\mathrm{div})}^2.$$

We then decompose each term separately. Since $\|\operatorname{\mathbf{curl}} w\|_{H(\mathrm{div})} \approx \|w\|_1$, we can use the standard decomposition lemma on w to write

$$w = \sum_{j=0}^{n} w_j, \qquad \sum_{j=0}^{n} \|w_j\|_1^2 \le c \|w\|_1^2.$$

Taking curls gives us the desired result on the $\operatorname{\mathbf{curl}} w$ term.

For $\boldsymbol{v} = \operatorname{\mathbf{grad}}_h s$ and $\boldsymbol{v}_0 = \operatorname{\mathbf{grad}}_H s_0$, where $(s_0, \boldsymbol{v}_0)$ is the mixed method approximation to $(s, \boldsymbol{v})$ in the space $S_0 \times \boldsymbol{V}_0$, we can prove using standard results from the theory of mixed finite element approximations that

$$\|\boldsymbol{v} - \boldsymbol{v}_0\| \le CH \|\boldsymbol{v}\|_{H(\mathrm{div})},$$

and conclude the proof. The key is that although the above estimate does not hold for all $\boldsymbol{v} \in \boldsymbol{V}_h$, it does hold when $\boldsymbol{v} = \mathbf{grad}_h\, s$.

1.5 V-cycle preconditioner

We consider a nested sequence of meshes, $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_N$, and let $\boldsymbol{V}_n$ be the Raviart-Thomas space of some fixed order subordinate to the mesh $\mathcal{T}_n$. This gives a nested sequence of spaces $\boldsymbol{V}_1 \subset \boldsymbol{V}_2 \subset \cdots \subset \boldsymbol{V}_N = \boldsymbol{V}_h$ and corresponding operators $\boldsymbol{J}_n : \boldsymbol{V}_n \to \boldsymbol{V}_n$.

We also require smoothers $\boldsymbol{R}_n : \boldsymbol{V}_n \to \boldsymbol{V}_n$ which we discuss below and the $\boldsymbol{H}(\mathrm{div})$-projection operators $\boldsymbol{P}_n : \boldsymbol{H}(\mathrm{div}) \to \boldsymbol{V}_n$. Multigrid then defines $\boldsymbol{K}_n : \boldsymbol{V}_n \to \boldsymbol{V}_n$ recursively starting with $\boldsymbol{K}_1 = \boldsymbol{J}_1^{-1}$. We shall make use of the following multigrid convergence result.

Theorem 1.3 *Suppose that for each $n = 1, 2, \dots, N$ the smoother R_n is L^2-symmetric and positive semi-definite and satisfies the conditions*

$$J([\boldsymbol{I} - \boldsymbol{R}_n \boldsymbol{J}_n]\boldsymbol{v}, \boldsymbol{v}) \geq 0$$

$$(\boldsymbol{R}_n^{-1}[\boldsymbol{I} - \boldsymbol{P}_{n-1}]\boldsymbol{v}, [\boldsymbol{I} - \boldsymbol{P}_{n-1}]\boldsymbol{v}) \leq \alpha J([\boldsymbol{I} - \boldsymbol{P}_{n-1}]\boldsymbol{v}, [\boldsymbol{I} - \boldsymbol{P}_{n-1}]\boldsymbol{v}).$$

Then there exists a constant C independent of h and N such that the eigenvalues of $\boldsymbol{K}_h \boldsymbol{J}_h$ lie in the interval $[1 - \delta, 1]$ where $\delta = C/(C + 2m)$, m denoting the number of smoothings.

For standard elliptic operators many smoothers can be shown to satisfy the hypotheses, the simplest of which is the scalar smoother. However, the proof for the scalar smoother and some others fails in $\boldsymbol{H}(\mathrm{div})$ and the multigrid preconditioner constructed with these smoothers is not effective. We shall consider an additive Schwarz smoother, defined in the following way. For each vertex of the mesh, consider the patch of elements containing that vertex. These patches form an overlapping covering of Ω and so determine an additive Schwarz operator. We use this operator as our smoother. The verification of the first hypothesis is routine. The standard proof of the second fails, but the difficulty can be surmounted by again using the discrete Helmholtz decomposition in a manner similar to that used for the proof of domain decomposition. The complete proof is given in [AFW97].

1.6 Numerical Results

First we made a numerical study of the condition number of $\boldsymbol{J}_h$ and the effect of preconditioning. In Table 1.1, the level m mesh is a uniform triangulation of the unit square into 2^{2m-1} triangles and has mesh size $h = 1/2^{m-1}$. The space $\boldsymbol{V}_h$ is taken as the Raviart–Thomas space of index 0 on this mesh. The preconditioner $\boldsymbol{K}_h$ is the V-cycle multigrid preconditioner using one application of the standard additive Schwarz smoother with the scaling factor taken to be 1/2. The fifth column of the table clearly displays the expected growth of the condition number of $\boldsymbol{J}_h$ as $O(h^{-2})$,

Table 1.1 Condition numbers for the operator $\boldsymbol{J}_h$ and for the preconditioned operator $\boldsymbol{K}_h\boldsymbol{J}_h$, and iterations counts to achieve an error reduction factor of 10^6.

level	h	elements	dim $\boldsymbol{V}_h$	$\kappa(\boldsymbol{J}_h)$	$\kappa(\boldsymbol{K}_h\boldsymbol{J}_h)$	iterations
1	1	2	5	38	1.00	1
2	1/2	8	16	153	1.32	4
3	1/4	32	56	646	1.68	6
4	1/8	128	208	2,650	2.17	6
5	1/16	512	800	10,670	2.34	8
6	1/32	2,048	3,136	42,810	2.40	8
7	1/64	8,192	12,416	–	–	8

and the sixth column the boundedness of the condition number of the preconditioned operator $\boldsymbol{K}_h\boldsymbol{J}_h$.

As a second numerical study, we used the Raviart–Thomas mixed method to solve the factored Poisson equation

$$\boldsymbol{u} = \mathbf{grad}\, p, \quad \operatorname{div} \boldsymbol{u} = g \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \partial\Omega,$$

again on the unit square using the same sequence of meshes as in the first example. We chose $g = 2(x^2 + y^2 - x - y)$ so that $p = (x^2 - x)(y^2 - y)$. The discrete solution $(\boldsymbol{u}_h, p_h)$ belongs to the space $\boldsymbol{V}_h \times S_h$, with $\boldsymbol{V}_h$ the Raviart–Thomas space described above and S_h the space of piecewise constant functions on the same mesh. We solved the discrete equations both with a direct solver and by using the minimum residual method preconditioned with the block diagonal preconditioner having as diagonal blocks $\boldsymbol{K}_h$ and the identity (as discussed previously). Full multigrid was used to initialize the minimum residual algorithm. That is, the computed solution at each level was used as an initial guess at the next finer level, beginning with the exact solution on the coarsest (two element) mesh. In Table 1.2, we show the condition number of the discrete operator $\mathcal{A}_h$ and of the preconditioned operator $\mathcal{B}_h\mathcal{A}_h$. While the former quantity grows linearly with h^{-1} (since this is a first order system), the latter remains small.

Table 1.2 Condition numbers for the indefinite operator $\mathcal{A}_h$ corresponding to the mixed system and for the preconditioned operator $\mathcal{B}_h\mathcal{A}_h$.

level	h	dim $\boldsymbol{V}_h$	dim S_h	$\kappa(\mathcal{A}_h)$	$\kappa(\mathcal{B}_h\mathcal{A}_h)$
1	1	5	2	8.25	1.04
2	1/2	16	8	15.0	1.32
3	1/4	56	32	29.7	1.68
4	1/8	208	128	59.6	2.18
5	1/16	800	512	119	2.34

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