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The construction of some efficient preconditioners in the boundary element method

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Dedicated to Ian Sloan on the occasion of his 60th birthday

The discretization of first kind boundary integral equations leads in general to a dense system of linear equations, whose spectral condition number depends on the discretization used. Here we describe a general preconditioning technique based on a boundary integral operator of opposite order. The corresponding spectral equivalence inequalities are independent of the special discretization used, i.e., independent of the triangulations and of the trial functions. Since the proposed preconditioning form involves a (pseudo)inverse operator, one needs for its discretization only a stability condition for obtaining a spectrally equivalent approximation.

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1. Introduction

For a self-adjoint and bounded pseudodifferential operator of order 2α,

$$A : V^s(\Gamma, A) := H^s(\Gamma)_{/\ker A} \to H^{s-2\alpha}(\Gamma), \tag{1.1}$$

with

$$\langle Au, v\rangle_{H^{s-\alpha}(\Gamma)} \leqslant c_2^A \cdot \|u\|_{H^s(\Gamma)} \|v\|_{H^s(\Gamma)} \quad \forall u, v \in H^s(\Gamma),$$

we consider the variational equation: find $u \in V^s(\Gamma, A)$, such that

$$\langle Au, v\rangle_{H^{s-\alpha}(\Gamma)} = f(v) \tag{1.2}$$

is satisfied for all $v \in V^s(\Gamma, A)$. The given bounded linear form $f : V^s(\Gamma, A) \to \mathbb{R}$ satisfies the necessary compatibility conditions $f(v^0) = 0\ \forall v^0 \in \ker A^*$. Here, Γ is an $(n-1)$-dimensional Lipschitz continuous, closed or open surface in $\mathbb{R}^n$. If, in addition, we assume the $V^s(\Gamma, A)$-ellipticity of A, i.e.,

$$c_1^A \cdot \|v\|^2_{H^s(\Gamma)} \leqslant \langle Av, v\rangle_{H^{s-\alpha}(\Gamma)} \quad \forall v \in V^s(\Gamma, A), \tag{1.3}$$

then there exists a unique solution of (1.2) due to the Lax–Milgram theorem.

Let

$$V_h^s(\Gamma, A) = \operatorname{span}\{\varphi_k^\nu\}_{k=1}^N \subset V^s(\Gamma, A) \tag{1.4}$$

be a family of finite-dimensional subspaces over triangulations

$$\Gamma_h = \bigcup_{k=1}^N \Gamma_k \quad \text{with } h_k = \max_{x,y\in\Gamma_k} |x-y|, \tag{1.5}$$

of boundary elements Γ_k, with the global mesh size $h = \max h_k$. V_h may be spanned by a basis of trial functions φ_k^ν, e.g., by smoothest B-splines of piecewise polynomial degree ν. From the approximation property

$$\inf_{w_h\in V_h} \|v - w_h\|_{H^\tau(\Gamma)} \leqslant c_A \cdot h^{t-\tau} \cdot \|v\|_{H^t(\Gamma)} \tag{1.6}$$

for all $v \in H^t(\Gamma)$ and $\tau \leqslant t \leqslant \nu + 1$, $\tau < \nu + 1/2$ $(n = 2)$, $\tau \leqslant \nu$ $(n = 3)$ then follows the quasi-optimal convergence of the family of approximate solutions $u_h \to u$ in different Sobolev norms [10].

To construct the solutions u_h, we have to solve the equivalent system of linear equations

$$A_h \underline{u} = \underline{f} \tag{1.7}$$

for finding the coefficient vector $\underline{u}$ representing the solution by the isomorphism

$$\underline{u} \in \mathbb{R}^N \iff u_h(x) = \sum_{k=1}^N u_k \varphi_k^\nu(x) \in V_h^s(\Gamma, A). \tag{1.8}$$

The symmetric and positive definite matrix A_h in (1.7) is defined by

$$A_h[\ell, k] = \left\langle A\varphi_k^\nu, \varphi_\ell^\nu \right\rangle_{H^{s-\alpha}(\Gamma)}$$

for all $k, \ell = 1, \ldots, N$ and is in general dense due to the non-local definition of A. Since A is a pseudodifferential operator of order 2α and the Galerkin equations (1.2) are taken in the $H^{s-\alpha}(\Gamma)$ inner product, the condition number of A_h behaves as $h^{-|2s|}$, i.e., the iterative solution of (1.7) requires an efficient preconditioner to keep the numerical amount of work as low as possible. Such a preconditioner should not depend on the discretization, i.e., on the mesh and trial functions used. A symmetric preconditioning matrix C_h will be called efficient, if it satisfies the spectral equivalence inequalities

$$\gamma_1 \cdot (C_h \underline{u}, \underline{u}) \leqslant (A_h \underline{u}, \underline{u}) \leqslant \gamma_2 \cdot (C_h \underline{u}, \underline{u}) \tag{1.9}$$

for all $\underline{u} \in \mathbb{R}^N$ connected with V_h by the isomorphism (1.8), where the ratio γ_2/γ_1 does not depend on bad parameters such as the triangulation (1.5) with the mesh parameter h and the mesh ratio $h_{\max}/h_{\min}$. Note that in the sense of an efficient preconditioner also arbitrary (ill-shaped) boundary elements (e.g., triangles for $n = 3$) are allowed; however, to ensure the approximation property (1.6) and therefore the convergence of

the boundary element solution, some regularity requirements are necessary. In addition, the computational amount of work for executing $C_h^{-1}\underline{u}$ should be less or of same order as a matrix times vector multiplication with the stiffness matrix A_h itself. Hence, then the numerical amount of work to solve (1.7) for a dense stiffness matrix is of order $O(N^2)$. Note that this amount of work by dealing with dense matrices can be reduced to $O(N \log N)$ when using panel clustering methods as described in [7] or multipole algorithms as in [12].

The inequalities (1.9) imply an upper bound for the spectral condition number of the preconditioned system as

$$\kappa\big(C_h^{-1}A_h\big) \leqslant \frac{\gamma_2}{\gamma_1},$$

and the convergence of the preconditioned conjugate gradient iteration can be estimated (see [6]) by

$$\big\|\underline{u}^k - \underline{u}\big\|_{C_h^{-1}A_h} \leqslant 2 \cdot q_{\mathrm{CG}}^k \cdot \big\|\underline{u}^0 - \underline{u}\big\|_{C_h^{-1}A_h}, \quad q_{\mathrm{CG}} = \frac{\sqrt{\kappa(C_h^{-1}A_h)} - 1}{\sqrt{\kappa(C_h^{-1}A_h)} + 1}\,. \tag{1.10}$$

Then the number of iterations needed to get a relative error reduction ε is given by the smallest integer $k \geqslant \ln(\varepsilon/2)/\ln q_{\mathrm{CG}}$.

Due to the isomorphism (1.8), the inequalities (1.9) are equivalent to

$$\gamma_1 \cdot \langle Cu_h, u_h\rangle_{H^{s-\alpha}(\Gamma)} \leqslant \langle Au_h, u_h\rangle_{H^{s-\alpha}(\Gamma)} \leqslant \gamma_2 \cdot \langle Cu_h, u_h\rangle_{H^{s-\alpha}(\Gamma)} \tag{1.11}$$

for all $u_h \in V_h$, where C may be any pseudodifferential operator with the same properties as A. So the main task in our preconditioning technique is to find such an operator C satisfying (1.11) for all $v \in V$. Obviously, then the constants γ_i do not depend on the discretization used and, moreover, the matrix C_h defined by

$$C_h[\ell, k] = \big\langle C\varphi_k^\nu, \varphi_\ell^\nu\big\rangle_{H^{s-\alpha}(\Gamma)}$$

for all $k, \ell = 1, \ldots, N$, would be a good candidate for a preconditioner of A_h. Of course, we have to guarantee that the multiplication by the inverse C_h^{-1} can be done in an optimal manner.

Examples of pseudodifferential operators considered in this paper are all boundary integral operators appearing in the direct boundary element method for the numerical solution of mixed boundary value problems. Note that pseudodifferential operators of order zero include the class of classical Fredholm operators of the second kind with compact and smoothing integral operators, for which the stiffness matrices are well conditioned and can be inverted by iteratives methods [1] or, in particular, by Neumann series. Here, however, we consider a more general class of boundary integral equations such as those of the first kind involving the single layer potential and the hypersingular integral operator. Both operators also appear in the symmetric formulation of boundary integral equations [20,22] for solving mixed boundary value problems via the boundary element method.

There are different possibilities for deriving an optimal preconditioner for a stiffness matrix resulting from a boundary integral equation. In analogy to finite element methods one can use multigrid type preconditioners [8,18], where one has to distinguish between pseudodifferential operators of negative and of positive order. In the special case of a pseudodifferential operator of order -1, one can define the smoothing operator by using the H^{-1} inner product. The realization then corresponds to a difference operator on the discrete space. This approach, described in [2], is a special case of a multigrid preconditioner for the classical single layer potential.

A more general algebraic approach to construct preconditioners is based on additive Schwarz methods (ASM) in combination with appropriate splittings of the ansatz space. A detailed analysis of these methods and applications to boundary integral equations with the single layer potential and the hypersingular operator for $n = 2$ is given in [14,26,27] or in combination with multipole accelerated methods in [12].

A second possibility amounts to the construction of an operator with the same mapping properties as the original one, and to discretize it in such a way that the resulting matrix is structured, e.g., of Toeplitz type or circulant. Then the inversion can be performed via the Fast Fourier Transformation or via direct solvers for Toeplitz type matrices. However, these methods require a uniform grid, and moreover, an ordered numbering of the nodes and elements to get a structured matrix. Examples of such preconditioners in 2D and 3D are given, e.g., in [11,15,19].

In this paper we propose a preconditioning technique for discretized boundary integral equations which is independent of the space dimension and of the discretization used. The basic idea was already described in [24] without numerical examples. Here we give a general analysis of our proposed preconditioners including sufficient conditions for their stability. In [13] a similar preconditioner is analyzed for the case of operators on closed curves, i.e., for two-dimensional boundary value problems and uniform grids. There it is proved that the spectral condition number of the preconditioned system $D_h V_h$ is bounded, where V_h and D_h are the Galerkin matrices of the single layer potential and the hypersingular integral operator, respectively. The analysis in [13] is based on Fourier expansions of the integral operators on closed curves. For adaptive discretizations, however, additional scalings are required. In this paper we give a general approach to construct preconditioners applicable to boundary integral equations of the first kind and of arbitrary order. Moreover, the proposed preconditioners can be used in domain decomposition methods to precondition the global Schur complement system. Note that our construction works on general domain and interface boundaries and we do not use the Green function as in [29].

In section 2 we describe general preconditioning bilinear forms based on a pseudodifferential operator of opposite order to that of the original one and obtain corresponding spectral equivalence inequalities. Since these bilinear forms include the (pseudo)inverse operators, we discuss in section 3 the discretization of operators of the form $B'A^{-1}B$ and show related spectral equivalence inequalities. In section 4 we use these results to discretize our preconditioning forms. Applications and numerical ex-

amples for the construction of optimal preconditioners for boundary element methods are given in sections 5 and 6.

2. Preconditioning forms

To construct the preconditioning form

$$c(u,v) = \langle Cu, v\rangle_{H^{s-\alpha}(\Gamma)}$$

associated with an operator $C : H^s(\Gamma) \to H^{s-2\alpha}(\Gamma)$ by the Riesz theorem which satisfies the spectral equivalence inequalities (1.11), we consider a self-adjoint and bounded pseudodifferential operator of the order -2α, i.e.,

$$B : H^{s-2\alpha}(\Gamma) \to H^s(\Gamma). \tag{2.1}$$

We further assume the $H^{s-2\alpha}(\Gamma)$-ellipticity

$$c_1^B \cdot \|v\|^2_{H^{s-2\alpha}(\Gamma)} \leqslant \langle Bv, v\rangle_{H^{s-\alpha}(\Gamma)} \quad \forall v \in H^{s-2\alpha}(\Gamma)_{/\ker B} \tag{2.2}$$

with $c_1^B > 0$ and boundedness

$$\big|\langle Bu, v\rangle_{H^{s-\alpha}(\Gamma)}\big| \leqslant c_2^B \cdot \|u\|_{H^{s-2\alpha}(\Gamma)}\|v\|_{H^{s-2\alpha}(\Gamma)} \quad \forall u, v \in H^{s-2\alpha}(\Gamma). \tag{2.3}$$

To find such an operator B for a given pseudodifferential operator A, one may use the principal symbol of A, compute the inverse of this symbol and obtain a pseudodifferential operator B satisfying the above requirements. In section 5 we find such an appropriate operator B directly from the boundary integral equations used.

If the kernel of B is non-trivial then, for our class of operators, it is finite-dimensional and one may suppose a finite, orthonormal representation

$$\begin{aligned}\ker B &= \big\{v \in H^{s-2\alpha}(\Gamma):\ Bv = 0 \wedge v \not\equiv 0\big\}\\ &= \operatorname{span}\{v_p\}_{p=1}^m \quad \text{with } \langle v_p, v_q\rangle_{H^{s-2\alpha}(\Gamma)} = \delta_{pq}.\end{aligned}$$

Using the factor spaces

$$V^{s-2\alpha,0}(\Gamma, B) = H^{s-2\alpha}(\Gamma)_{/\ker B} = \big\{v \in H^{s-2\alpha}(\Gamma):\ \langle v, v_p\rangle_{H^{s-2\alpha}(\Gamma)} = 0\big\}, \tag{2.4}$$

$$V^{s,0}(\Gamma, B) = \big\{v \in H^s(\Gamma):\ \langle v, v_p\rangle_{H^{s-\alpha}(\Gamma)} = 0\ \forall v_p \in \ker B\big\}, \tag{2.5}$$

the uniquely determined Moore–Penrose pseudoinverse operator

$$\dot{B}^{-1} : V^{s,0}(\Gamma, B) \to V^{s-2\alpha,0}(\Gamma, B), \tag{2.6}$$

is bounded and $V^{s,0}(\Gamma, B)$-elliptic, i.e., there hold the inequalities

$$\frac{1}{c_2^B} \cdot \|u\|^2_{H^s(\Gamma)} \leqslant \big\langle \dot{B}^{-1}u, u\big\rangle_{H^{s-\alpha}(\Gamma)} \leqslant \frac{1}{c_1^B} \cdot \|u\|^2_{H^s(\Gamma)} \quad \text{for all } u \in V^{s,0}(\Gamma, B).$$

Hence, we obtain the spectral equivalence inequalities

$$c_1 \cdot \big\langle \dot{B}^{-1}u, u\big\rangle_{H^{s-\alpha}(\Gamma)} \leqslant \langle Au, u\rangle_{H^{s-\alpha}(\Gamma)} \leqslant c_2 \cdot \big\langle \dot{B}^{-1}u, u\big\rangle_{H^{s-\alpha}(\Gamma)} \tag{2.7}$$

for all $u \in V^s(\Gamma, A) \cap V^{s,0}(\Gamma, B)$ with some positive constants $c_i = c_i^A c_i^B$ $(i = 1, 2)$.

In general, we cannot expect the inclusion $V^s(\Gamma, A) \subseteq V^{s,0}(\Gamma, B)$, so we consider the decomposition

$$V^s(\Gamma, A) = V^{s,0}(\Gamma, B) \oplus \left[V^{s,0}(\Gamma, B)\right]^{\perp}.$$

To characterize the orthogonal complement $[V^{s,0}(\Gamma, B)]^{\perp}$, we introduce the Bessel potential operator $J : H^{s-2\alpha}(\Gamma) \to H^s(\Gamma)$ defined by the Riesz mapping

$$\langle Ju, v\rangle_{H^{s-\alpha}(\Gamma)} = \langle u, v\rangle_{H^{s-2\alpha}(\Gamma)}$$

for all $v \in H^{s-2\alpha}(\Gamma)$. To any given $u \in V$, let us associate the function

$$u_0(x) = u(x) - \sum_{p=1}^{m} \langle u, v_p\rangle_{H^{s-\alpha}(\Gamma)} (Jv_p)(x) \in V^{s,0}(\Gamma, B). \tag{2.8}$$

For the bilinear form defined by

$$c(u, w) = \left\langle \dot{B}^{-1} u_0, w_0 \right\rangle_{H^{s-\alpha}(\Gamma)} + \sum_{p=1}^{m} \langle u, v_p\rangle_{H^{s-\alpha}(\Gamma)} \langle w, v_p\rangle_{H^{s-\alpha}(\Gamma)} \tag{2.9}$$

we then have the following result:

Theorem 2.1. Let A and B be a pair of pseudodifferential operators with the properties (1.1)–(1.3) and (2.1)–(2.3), respectively. Let $\dot{B}^{-1}$ be its pseudoinverse operator as in (2.6). Then there hold the spectral equivalence inequalities

$$\gamma_1 \cdot c(u, u) \leqslant \langle Au, u\rangle_{H^{s-\alpha}(\Gamma)} \leqslant \gamma_2 \cdot c(u, u) \quad \text{for all } u \in V$$

with positive constants $\gamma_1 = c_1^A \cdot \min\{1, c_1^B\}$, $\gamma_2 = c_2^A \cdot \max\{1, c_2^B\}$.

Proof. Using the decomposition (2.8), the inequalities (1.3), (2.3) and the orthonormal representation of $\ker B$, we have

$$\begin{aligned}
\langle Au, u\rangle_{H^{s-\alpha}(\Gamma)} &\leqslant c_2^A \cdot \|u\|^2_{H^s(\Gamma)} \\
&= c_2^A \cdot \left\| u_0 + \sum_{p=1}^{m} \langle u, v_p\rangle^2_{H^{s-\alpha}(\Gamma)} \cdot Jv_p \right\|^2_{H^{s-\alpha}(\Gamma)} \\
&= c_2^A \cdot \left\{ \|u_0\|^2_{H^s(\Gamma)} + \sum_{p=1}^{m} \langle u, v_p\rangle^2_{H^{s-\alpha}(\Gamma)} \right\} \\
&\leqslant c_2^A \cdot \left\{ c_2^B \cdot \left\langle \dot{B}^{-1} u_0, u_0 \right\rangle_{H^{s-\alpha}(\Gamma)} + \sum_{p=1}^{m} \langle u, v_p\rangle^2_{H^{s-\alpha}(\Gamma)} \right\} \\
&\leqslant c_2^A \cdot \max\left\{1, c_2^B\right\} \cdot c(u, u).
\end{aligned}$$

The second inequality follows in the same manner. □

Note that in the special cases $V^s(\Gamma, A) \subseteq V^{s,0}(\Gamma, B)$ or $m = 0$, the spectral equivalence inequalities in theorem 2.1 coincide with (2.7).

Instead of the bilinear form (2.9), which is based on the pseudoinverse operator $\dot{B}^{-1}$, we may define an operator $\widetilde{B}$ via the Riesz representation theorem by using the bilinear form

$$\langle \widetilde{B}u, w\rangle_{H^{s-\alpha}(\Gamma)} := \langle Bu, w\rangle_{H^{s-\alpha}(\Gamma)} + \sum_{p=1}^{m} \langle u, v_p\rangle_{H^{s-2\alpha}(\Gamma)} \langle w, v_p\rangle_{H^{s-2\alpha}(\Gamma)} \qquad (2.10)$$

for $u, w \in H^{s-2\alpha}(\Gamma)$. Note that there holds

$$\langle Bu, w\rangle_{H^{s-\alpha}(\Gamma)} = \langle Bu_0, w_0\rangle_{H^{s-\alpha}(\Gamma)}$$

with

$$u_0(x) = u(x) - \sum_{p=1}^{m} \langle u, v_p\rangle_{H^{s-2\alpha}(\Gamma)} \cdot v_p(x) \in V^{s-2\alpha,0}(\Gamma, B) \qquad (2.11)$$

and $w_0 \in V^{s-2\alpha,0}(\Gamma, B)$, respectively.

Theorem 2.2. Let A and B be pseudodifferential operators with the properties (1.1)–(1.3) and (2.1)–(2.3), respectively. Let the operator $\widetilde{B}$ be defined by the bilinear form (2.10). Then there hold the spectral equivalence inequalities

$$c_1^{\widetilde{B}} \cdot \|u\|^2_{H^{s-2\alpha}(\Gamma)} \leqslant \langle \widetilde{B}u, u\rangle_{H^{s-\alpha}(\Gamma)} \leqslant c_2^{\widetilde{B}} \cdot \|u\|^2_{H^{s-2\alpha}(\Gamma)} \quad \text{for all } u \in V$$

with positive constants

$$c_1^{\widetilde{B}} = \min\{1, c_1^B\}, \qquad c_2^{\widetilde{B}} = \max\{1, c_2^B\}.$$

Proof. The proof is based on the orthogonal decomposition (2.11) which gives

$$\|u\|^2_{H^{s-2\alpha}(\Gamma)} = \|u_0\|^2_{H^{s-2\alpha}(\Gamma)} + \sum_{p=1}^{m} \langle u, v_p\rangle_{H^{s-2\alpha}(\Gamma)}.$$

Then the assertion follows by applying (2.2) and (2.3). □

Hence, the operator $\widetilde{B}$ is invertible and we can apply theorem 2.1 with $m = 0$.

It remains to characterize admissible discretizations of the bilinear form (2.9) and of the preconditioner C_h^{-1}. The problem lies in the fact that the bilinear form $c(\cdot,\cdot)$ includes the (pseudo)inverse operator $\dot{B}^{-1}$, whereas in general, only the operator B itself is given explicitly and hence it is impossible to compute the Galerkin discretization of the bilinear forms (2.9), (2.10) directly.

3. Basic spectral equivalence inequalities

Let $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces with the inner products $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{W}}$, respectively, and let $\mathcal{A}:\mathcal{V}\to\mathcal{V}^*$ be a $\mathcal{V}$-elliptic, self-adjoint and bounded operator satisfying

$$\begin{aligned}&\langle\mathcal{A}u,u\rangle_{\mathcal{V}}\geqslant c_1^{\mathcal{A}}\cdot\|u\|_{\mathcal{V}}^2,\\&\langle\mathcal{A}u,v\rangle_{\mathcal{V}}\leqslant c_2^{\mathcal{A}}\cdot\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}\quad\text{for all } u,v\in\mathcal{V}.\end{aligned}$$

Let $\mathcal{B}:\mathcal{W}\to\mathcal{V}^*$ be any bounded operator:

$$\langle Bw,v\rangle_{\mathcal{V}}\leqslant c_2^{\mathcal{B}}\cdot\|w\|_{\mathcal{W}}\|v\|_{\mathcal{V}}\quad\forall v\in\mathcal{V},\ w\in\mathcal{W}.$$

Obviously, then the operator defined by

$$\mathcal{T}=\mathcal{B}'\mathcal{A}^{-1}\mathcal{B}:\mathcal{W}\to\mathcal{W}^* \tag{3.1}$$

is bounded, too. Due to (3.1), in general we are not able to compute the Galerkin weights

$$\mathcal{T}_h[\ell,k]=\langle\mathcal{T}\varphi_k,\varphi_\ell\rangle_{\mathcal{W}}=\big\langle\mathcal{B}'\mathcal{A}^{-1}\mathcal{B}\varphi_k,\varphi_\ell\big\rangle_{\mathcal{W}}$$

for $k,\ell=1,\dots,N$ directly, where $\mathcal{W}_h=\operatorname{span}\{\varphi_k\}_{k=1}^N\subset\mathcal{W}$. Therefore we define an approximation

$$\widetilde{\mathcal{T}}_h=\mathcal{B}_h^{\mathrm{T}}\mathcal{A}_h^{-1}\mathcal{B}_h \tag{3.2}$$

with the help of the matrices

$$\begin{aligned}\mathcal{A}_h[j,i]&=\langle\mathcal{A}\psi_i,\psi_j\rangle_{\mathcal{V}},\\\mathcal{B}_h[j,k]&=\langle\mathcal{B}\varphi_k,\psi_j\rangle_{\mathcal{V}}\end{aligned}$$

for $i,j=1,\dots,M$; $k=1,\dots,N$, where $\mathcal{V}_h=\operatorname{span}\{\psi_i\}_{i=1}^M\subset\mathcal{V}$. Since the operator $\mathcal{A}$ is $\mathcal{V}$-elliptic and self-adjoint, the matrix $\mathcal{A}_h$ is symmetric and positive definite and, hence, invertible.

Theorem 3.1. For all $\underline{w}\in\mathbb{R}^N$ associated with $w_h\in\mathcal{W}_h$ by the isomorphism (1.8) there holds the upper spectral estimate

$$\big(\widetilde{\mathcal{T}}_h\underline{w},\underline{w}\big)\leqslant(\mathcal{T}_h\underline{w},\underline{w})$$

independent of the special choices of the trial spaces $\mathcal{V}_h$ and $\mathcal{W}_h$ used.

Proof. Let $\underline{w}\in\mathbb{R}^N$ be arbitrary, but fixed. Then, by the definition of the matrices $\mathcal{T}_h$ and $\widetilde{\mathcal{T}}_h$, we have

$$\begin{aligned}(\mathcal{T}_h\underline{w},\underline{w})&=\big\langle\mathcal{B}'\mathcal{A}^{-1}\mathcal{B}w_h,w_h\big\rangle_{\mathcal{W}}=\big\langle\mathcal{B}'u,w_h\big\rangle_{\mathcal{W}},\\\big(\widetilde{\mathcal{T}}_h\underline{w},\underline{w}\big)&=\big\langle\mathcal{B}'u_h,w_h\big\rangle_{\mathcal{W}}=\langle u_h,\mathcal{B}w_h\rangle_{\mathcal{V}},\end{aligned}$$

where $u \in \mathcal{V}$ and $u_h \in \mathcal{V}_h$ are defined by the variational problems

$$\begin{aligned} \langle \mathcal{A}u, v\rangle_{\mathcal{V}} &= \langle \mathcal{B}w_h, v\rangle_{\mathcal{V}} \qquad \text{for all } v \in \mathcal{V}, \\ \langle \mathcal{A}u_h, v_h\rangle_{\mathcal{V}} &= \langle \mathcal{B}w_h, v_h\rangle_{\mathcal{V}} \quad \text{for all } v_h \in \mathcal{V}_h. \end{aligned}$$

Setting $v = v_h$ and subtracting the first equation from the second one, we get

$$\big\langle \mathcal{A}(u - u_h), v_h \big\rangle_{\mathcal{V}} = 0$$

for all $v_h \in \mathcal{V}_h$. Hence, with the $\mathcal{V}$-ellipticity of $\mathcal{A}$ we find

$$\begin{aligned} 0 \leqslant c_1^{\mathcal{A}} \cdot \|u - u_h\|_{\mathcal{V}}^2 &\leqslant \big\langle \mathcal{A}(u - u_h), (u - u_h)\big\rangle_{\mathcal{V}} = \big\langle \mathcal{A}(u - u_h), u\big\rangle_{\mathcal{V}} \\ &= \langle \mathcal{A}u, u\rangle_{\mathcal{V}} - \langle \mathcal{A}u_h, u\rangle_{\mathcal{V}}, \end{aligned}$$

which implies the inequality

$$\langle \mathcal{A}u_h, u_h\rangle_{\mathcal{V}} = \langle \mathcal{A}u, u_h\rangle_{\mathcal{V}} \leqslant \langle \mathcal{A}u, u\rangle_{\mathcal{V}}$$

for the self-adjoint operator $\mathcal{A}$. Then it follows directly that

$$\begin{aligned} \big(\widetilde{\mathcal{T}}_h \underline{w}, \underline{w}\big) &= \langle u_h, \mathcal{B}w_h\rangle_{\mathcal{V}} = \langle \mathcal{A}u_h, u_h\rangle_{\mathcal{V}} \\ &\leqslant \langle \mathcal{A}u, u\rangle_{\mathcal{V}} = \langle u, \mathcal{B}w_h\rangle_{\mathcal{V}} = (\mathcal{T}_h \underline{w}, \underline{w}), \end{aligned}$$

which completes the proof. □

Remark 3.2. The upper spectral estimate in theorem 3.1 is sufficient for the spectral equivalence of the discrete Steklov–Poincaré operator in the symmetric representation to the continuous or, equivalently, to the discrete hypersingular boundary integral operator [21]. This is used for the construction of efficient preconditioners for boundary element methods corresponding to mixed boundary value problems in [22] or in [4,25] in domain decomposition methods based on the symmetric formulation.

Note that up to now nothing has been said about the invertibility of the matrix $\widetilde{\mathcal{T}}_h$, for which the following condition is imposed.

Theorem 3.3. If the stability condition

$$c_1^{\mathcal{B}} \cdot \|w_h\|_{\mathcal{W}} \leqslant \sup_{0 \neq v_h \in \mathcal{V}_h} \frac{|\langle \mathcal{B}w_h, v_h\rangle_{\mathcal{V}}|}{\|v_h\|_{\mathcal{V}}} \quad \text{for all } w_h \in \mathcal{W}_h \tag{3.3}$$

is satisfied, where the constant $c_1^{\mathcal{B}}$ is independent of the mesh size, then there holds the lower spectral inequality

$$\gamma_1 \cdot (\mathcal{T}_h \underline{w}, \underline{w}) \leqslant \big(\widetilde{\mathcal{T}}_h \underline{w}, \underline{w}\big)$$

for all $\underline{w} \in \mathbb{R}^N$ associated with $w_h \in \mathcal{W}_h$ by the isomorphism (1.8). Here, the constant γ_1 is given by

$$\gamma_1 = \frac{c_1^{\mathcal{A}}}{c_2^{\mathcal{T}}} \cdot \left(\frac{c_1^{\mathcal{B}}}{c_2^{\mathcal{A}}} \right)^2, \quad \text{where } c_2^{\mathcal{T}} = \frac{(c_2^{\mathcal{B}})^2}{c_1^{\mathcal{A}}}.$$

Proof. Since the operator $\mathcal{A}$ is $\mathcal{V}$-elliptic, the operator $\mathcal{A}^{-1}$ is bounded and we have

$$(\mathcal{T}_h \underline{w}, \underline{w}) = \langle \mathcal{T} w_h, w_h \rangle_{\mathcal{W}} \leqslant c_2^{\mathcal{T}} \cdot \|w_h\|_{\mathcal{W}}^2 \quad \text{with } c_2^{\mathcal{T}} = \frac{(c_2^{\mathcal{B}})^2}{c_1^{\mathcal{A}}}. \tag{3.4}$$

As in the proof of theorem 3.1 we obtain

$$\left| \langle \mathcal{B} w_h, v_h \rangle_{\mathcal{V}} \right| = \left| \langle A u_h, v_h \rangle_{\mathcal{V}} \right| \leqslant c_2^{\mathcal{A}} \cdot \|u_h\|_{\mathcal{V}} \|v_h\|_{\mathcal{W}}$$

and from the stability condition we conclude

$$c_1^{\mathcal{B}} \cdot \|w_h\|_{\mathcal{W}} \leqslant c_2^{\mathcal{A}} \cdot \|u_h\|_{\mathcal{V}}.$$

Then it follows directly that

$$\begin{aligned} \|w_h\|_{\mathcal{W}}^2 &\leqslant \left(\frac{c_2^{\mathcal{A}}}{c_1^{\mathcal{B}}} \right)^2 \cdot \|u_h\|_{\mathcal{V}}^2 \leqslant \frac{1}{c_1^{\mathcal{A}}} \cdot \left(\frac{c_2^{\mathcal{A}}}{c_1^{\mathcal{B}}} \right)^2 \cdot \langle \mathcal{A} u_h, u_h \rangle_{\mathcal{V}} \\ &= c \cdot \langle \mathcal{B} w_h, u_h \rangle_{\mathcal{V}} = c \cdot \big(\widetilde{\mathcal{T}}_h \underline{w}, \underline{w} \big). \end{aligned}$$

Combining this inequality with (3.4) completes the proof. □

4. Galerkin discretizations of preconditioners

The Galerkin discretization of the preconditioning form (2.9) requires the computation of all the Galerkin weights

$$C_h[\ell, k] = \big\langle \dot{B}^{-1} \varphi_{k,0}^{\nu}, \varphi_{\ell,0}^{\nu} \big\rangle_{H^{s-\alpha}(\Gamma)} + \sum_{p=1}^{m} \big\langle \varphi_k^{\nu}, v_p \big\rangle_{H^{s-\alpha}(\Gamma)} \big\langle \varphi_\ell^{\nu}, v_p \big\rangle_{H^{s-\alpha}(\Gamma)} \tag{4.1}$$

for the trial functions $\varphi_k^{\nu} \in V_h^s(\Gamma, A) \subset V^s(\Gamma, A)$ and $k, \ell = 1, \ldots, N$, where $\varphi_{k,0}^{\nu} \in V^{s,0}(\Gamma, B)$ is defined via (2.8).

For the orthogonal complement of $V^{s,0}(\Gamma, B)$ we suppose a representation in terms of the basis in V_h, i.e.,

$$(J v_p)(x) = \sum_{k=1}^{N} v_{p,k} \cdot \varphi_k^{\nu}(x) \tag{4.2}$$

for all $v_p \in \ker B$. In our applications discussed in section 5, condition (4.2) will be fulfilled. Otherwise, if $J v_p \notin V_h$, we consider the augmented trial space

$$\widetilde{V}_h = V_h \cup \operatorname{span}\{J v_p\}_{J v_p \notin V_h}.$$

Note that condition (4.2) is necessary for the definition of the pseudoinverse operator $\dot{B}^{-1}$ also on finite dimensional subspaces corresponding to the continuous case in section 2.

Now we give a series of technical lemmata to prepare the definition of a spectrally equivalent Galerkin approximation of the preconditioning form in appropriate factor spaces.

Lemma 4.1. Under assumption (4.2) there exists an orthogonal decomposition

$$V_h = V_h^0 \oplus \{Jv_p\}_{p=1}^m$$

with the $(N-m)$-dimensional subspace

$$V_h^0 = \operatorname{span}\{\varphi_k^{\nu,0}\}_{k=1}^{N-m} \subset V^{s,0}(\Gamma, B).$$

Proof. For $m = 0$, there follows immediately $V_h^0 = V_h$ and $V^{s,0}(\Gamma, B) = V$. For $m > 0$ and $(Jv_p) \in V_h$ we can assume $v_{p,N+1-p} \neq 0$, otherwise we reorder the indices in the representation (4.2). Then we have

$$\varphi_{N+1-p}^{\nu}(x) = \frac{1}{v_{p,N+1-p}} \left[(Jv_p)(x) - \sum_{\ell=1, \ell \neq N+1-p}^{N} v_{p,\ell} \cdot \varphi_\ell^\nu(x) \right]$$

and, recursively,

$$V_h = \operatorname{span}\{\varphi_1^\nu, \dots, \varphi_N^\nu\} = \operatorname{span}\{\varphi_1^\nu, \dots, \varphi_{N-m}^\nu, Jv_1, \dots, Jv_m\}.$$

For $\ell = 1, \dots, N-m$ we now use a partial Gram–Schmidt orthogonalization and find

$$\varphi_\ell^{\nu,0}(x) = \varphi_\ell^\nu(x) - \sum_{p=1}^{m} \langle \varphi_\ell^\nu, v_p \rangle_{H^{s-\alpha}(\Gamma)} (Jv_p)(x) \in V^{s,0}(\Gamma, B), \tag{4.3}$$

which completes the proof. □

Lemma 4.2. The basis transformation

$$\widetilde{T} : \operatorname{span}\{\varphi_1^{\nu,0}, \dots, \varphi_{N-m}^{\nu,0}, Jv_1, \dots, Jv_m\} \to \operatorname{span}\{\varphi_1^\nu, \dots, \varphi_N^\nu\}$$

is given by its matrix representation $\underline{u} = \widetilde{T}\underline{\widetilde{u}}$ defined by the components

$$u_j = \alpha_j \cdot \widetilde{u}_j + \sum_{p=1}^{m} v_{p,j} \cdot \left[\widetilde{u}_{N-m+p} - \sum_{\ell=1}^{N-m} \widetilde{u}_\ell \cdot \langle \varphi_\ell^\nu, v_p \rangle_{H^{s-\alpha}(\Gamma)} \right]$$

for $j = 1, \dots, N$ and

$$\alpha_j = \begin{cases} 1 & \text{for } j = 1, \dots, N-m, \\ 0 & \text{for } j = N-m+1, \dots, N. \end{cases}$$

Proof. Let an arbitrary function $u_h \in V_h$ be given in the form

$$u_h(x) = \sum_{\ell=1}^{N-m} \widetilde{u}_\ell \cdot \varphi_\ell^{\nu,0}(x) + \sum_{p=1}^{m} \widetilde{u}_{N-m+p} \cdot (Jv_p)(x).$$

Replacing the representations (4.3) for $\varphi_\ell^{\nu,0}(x)$ and (4.2) for $(Jv_p)(x)$ completes the proof, we omit the details. □

Let us denote by $f_k = f(\varphi_k^\nu)$ for $k = 1, \ldots, N$ the coefficients of a given linear form $f(\cdot) : H^s(\Gamma) \to \mathbb{R}$. Using the transformed basis functions

$$\widetilde{\varphi}_k^{\,\nu}(x) = \begin{cases} \varphi_k^{\nu,0}(x) & \text{for } k = 1, \ldots, N-m, \\ (Jv_{k-N+m})(x) & \text{for } k = N-m+1, \ldots, N, \end{cases} \tag{4.4}$$

the coefficients f_k are transformed via

$$\widetilde{f}_{N-m+p} = f(Jv_p) = f\left(\sum_{\ell=1}^{N} v_{p,\ell} \cdot \varphi_\ell^\nu \right) = \sum_{\ell=1}^{N} v_{p,\ell} \cdot f_\ell \tag{4.5}$$

for $p = 1, \ldots, m$ and via

$$\begin{aligned} \widetilde{f}_\ell = f\big(\varphi_\ell^{\nu,0}\big) &= f\left(\varphi_\ell^\nu - \sum_{p=1}^{m} \big\langle \varphi_\ell^\nu, v_p \big\rangle_{H^{s-\alpha}(\Gamma)} Jv_p \right) \\ &= f_\ell - \sum_{p=1}^{m} \big\langle \varphi_\ell^\nu, v_p \big\rangle_{H^{s-\alpha}(\Gamma)} \cdot \widetilde{f}_{N-m+p} \end{aligned} \tag{4.6}$$

for $\ell = 1, \ldots, N-m$.

Lemma 4.3. The transformation $\underline{\widetilde{f}} = T\underline{f}$ defined by (4.5), (4.6) is transposed to the basis transformation described in lemma 4.2, i.e., we have $\widetilde{T} = T^{\mathrm{T}}$.

Proof. The transformation matrix T is given by its entries

$$t_{k\ell} = \begin{cases} \delta_{k\ell} - \sum_{j=1}^{m} \big\langle \varphi_k^\nu, v_j \big\rangle_{H^{s-\alpha}(\Gamma)} v_{j,\ell} & \text{for } k = 1, \ldots, N-m, \\ v_{k-N+m,\ell} & \text{for } k = N-m+1, \ldots, N \end{cases}$$

and $\ell = 1, \ldots, N$. For the product $\underline{u} = T^{\mathrm{T}}\underline{\widetilde{u}}$ we get

$$u_k = \sum_{\ell=1}^{N} t_{\ell k} \widetilde{u}_\ell.$$

For $k = N-m+1, \ldots, N$ this is

$$u_k = \sum_{\ell=1}^{m} v_{\ell,k} \widetilde{u}_{N-m+\ell} - \sum_{\ell=1}^{N-m} \left[\sum_{j=1}^{m} \langle \varphi_\ell^\nu, v_j \rangle_{H^{s-\alpha}(\Gamma)} v_{j,k} \right] \widetilde{u}_\ell$$

$$= \sum_{\ell=1}^{m} v_{\ell,k} \left[\widetilde{u}_{N-m+\ell} - \sum_{j=1}^{N-m} \langle \varphi_j^\nu, v_\ell \rangle_{H^{s-\alpha}(\Gamma)} \widetilde{u}_j \right];$$

whereas for $k = 1, \ldots, N-m$ we get in an analogous manner

$$u_k = \widetilde{u}_k + \sum_{\ell=1}^{m} v_{\ell,k} \left[\widetilde{u}_{N-m+\ell} - \sum_{j=1}^{N-m} \langle \varphi_j^\nu, v_\ell \rangle_{H^{s-\alpha}(\Gamma)} \widetilde{u}_j \right],$$

which is just the basis transformation $\widetilde{T}$ defined in lemma 4.2. □

Now it is straightforward that the inverse matrix defined by (4.1) admits the representation

$$C_h^{-1} = T^{\mathrm{T}} \widetilde{C}_h^{-1} T \quad \text{with } \widetilde{C}_h[\ell, k] = c\big(\widetilde{\varphi}_k^{\,\nu}, \widetilde{\varphi}_\ell^{\,\nu}\big)$$

for $k, \ell = 1, \ldots, N$ and the transformed trial functions (4.4). Due to the definition (4.4), the matrix $\widetilde{C}_h$ has the block diagonal form

$$\widetilde{C}_h = \begin{pmatrix} \overline{C}_h & 0 \\ 0 & I_m \end{pmatrix},$$

where I_m denotes the m-dimensional identity matrix and where the matrix $\overline{C}_h$ is given by

$$\overline{C}_h[\ell, k] = \big\langle \dot{B}^{-1} \varphi_k^{\nu,0}, \varphi_\ell^{\nu,0} \big\rangle_{H^{s-\alpha}(\Gamma)} \tag{4.7}$$

for $k, \ell = 1, \ldots, N-m$. To define a spectrally equivalent approximation of $\overline{C}_h$, we consider a finite-dimensional subspace

$$W_h = \operatorname{span}\big\{\varphi_k^\mu\big\}_{k=1}^{M} \subset H^{s-2\alpha}(\Gamma),$$

where we assume $\ker B \subset W_h$. Then, according to lemma 4.1, there exists the orthogonal space decomposition

$$W_h = \operatorname{span}\big\{\varphi_1^{\mu,0}, \ldots, \varphi_{M-m}^{\mu,0}\big\} \oplus \{v_1, \ldots, v_m\} = W_h^0 \oplus \ker B$$

with the modified trial functions

$$\varphi_k^{\mu,0}(x) = \varphi_k^\mu(x) - \sum_{p=1}^{m} \big\langle \varphi_k^\mu, v_p \big\rangle_{H^{s-2\alpha}(\Gamma)} \cdot v_p(x) \in V^{s-2\alpha,0}(\Gamma, B).$$

Thus, an approximation of $\overline{C}_h$ is defined by

$$\overline{C}_h^A = \widetilde{M}_h^{\mathrm{T}} B_h^{-1} \widetilde{M}_h \tag{4.8}$$

with

$$B_h[j, i] = \big\langle B \varphi_i^{\mu,0}, \varphi_j^{\mu,0} \big\rangle_{H^{s-\alpha}(\Gamma)} = \big\langle B \varphi_i^\mu, \varphi_j^\mu \big\rangle_{H^{s-\alpha}(\Gamma)},$$

$$\widetilde{M}_h[j,k] = \langle \varphi_k^{\nu,0}, \varphi_j^{\mu,0} \rangle_{H^{s-\alpha}(\Gamma)}$$
$$= \langle \varphi_k^{\nu}, \varphi_j^{\mu} \rangle_{H^{s-\alpha}(\Gamma)} - \sum_{p=1}^{m} \langle \varphi_k^{\nu}, v_p \rangle_{H^{s-\alpha}(\Gamma)} \cdot \langle \varphi_j^{\mu}, v_p \rangle_{H^{s-2\alpha}(\Gamma)}$$

for $i,j = 1,\dots,M-m$; $k = 1,\dots,N-m$. The application of theorem 3.1 yields the upper spectral estimate

$$\big(\overline{C}_h^A \underline{u}, \underline{u}\big) \leqslant \big(\overline{C}_h \underline{u}, \underline{u}\big)$$

for all $\underline{u} \in \mathbb{R}^{N-m}$. Assuming in addition the stability condition

$$\gamma_1 \cdot \|u_h\|_{H^s(\Gamma)} \leqslant \sup_{0 \neq v_h \in W_h^0} \frac{|\langle u_h, v_h \rangle_{H^{s-\alpha}(\Gamma)}|}{\|v_h\|_{H^{s-2\alpha}(\Gamma)}} \quad \text{for all } u_h \in V_h^0, \tag{4.9}$$

we get from theorem 3.3 the spectral equivalence inequalities

$$c_1 \cdot \big(\overline{C}_h \underline{u}, \underline{u}\big) \leqslant \big(\overline{C}_h^A \underline{u}, \underline{u}\big) \leqslant \big(\overline{C}_h \underline{u}, \underline{u}\big) \quad \forall \underline{u} \in \mathbb{R}^{N-m}.$$

To ensure (4.9) and to get an applicable preconditioner, we choose $N = M$, i.e., the same dimension of the finite-dimensional subspaces V_h and W_h. Then, (4.9) follows from the invertibility of the modified mass matrix $\widetilde{M}_h$, since the basis transformation T defined in (4.5), (4.6) is regular. This is equivalent to the invertibility of the mass matrix defined by

$$M_h[\ell,k] = \langle \varphi_k^{\nu}, \varphi_\ell^{\mu} \rangle_{H^{s-\alpha}(\Gamma)}$$

for $k,\ell = 1,\dots,N$. From $N = M$ and due to the definition of the trial functions on the same triangulation grids we conclude that only the choices

$$\nu = \mu + 2k \quad \text{with } k \in \mathbb{N}_0 \tag{4.10}$$

are possible. Moreover, then the matrix M_h is strongly diagonally dominant and therefore invertible independently of N. Therefore, if condition (4.10) is satisfied, there holds the stability condition (4.9) with a constant γ_1 independent of the discretization used. In general, trial functions of even degree may be defined with respect to the boundary elements, whereas the trial functions of odd degree are defined corresponding to the boundary nodes, which then coincide with corner points of the triangulation. If, for the definition of the test functions φ_ℓ^{μ} the dual mesh is used as in [13], then condition (4.10) is to be replaced by

$$\nu = \mu + 2k + 1 \quad \text{with } k \in \mathbb{N}_0.$$

Summing up the results of this section, we obtain the following conclusion.

Theorem 4.4. Let $N = M$ and $\nu = \mu + 2k$ with $k \in \mathbb{N}_0$. Then the preconditioner $\widehat{C}_h$ defined by its inverse

$$\widehat{C}_h^{-1} = T^{\mathrm{T}} \begin{pmatrix} \widetilde{M}_h^{-1} B_h \widetilde{M}_h^{-\mathrm{T}} & 0 \\ 0 & I_m \end{pmatrix} T$$

satisfies the spectral equivalence inequalities (1.9) with constants independent of the discretization used.

The preconditioning technique proposed in this section requires only two basis transformations, one matrix times vector multiplication with B_h and two multiplications with the inverse of the modified mass matrix $\widetilde{M}_h^{-1}$ and $\widetilde{M}_h^{-\mathrm{T}}$, respectively.

Note that here the modified mass matrix $\widetilde{M}_h$ can be written in the form

$$\widetilde{M}_h = M_h - \sum_{p=1}^{m} \underline{m}_p^{\mu} \cdot \left(\underline{m}_p^{\nu}\right)^{\mathrm{T}}$$

with

$$\begin{aligned} M_h[\ell, k] &= \left\langle \varphi_k^{\nu}, \varphi_\ell^{\mu} \right\rangle_{H^{s-\alpha}(\Gamma)}, \\ \underline{m}_{p,k}^{\mu} &= \left\langle \varphi_k^{\mu}, v_p \right\rangle_{H^{s-2\alpha}(\Gamma)}, \\ \underline{m}_{p,\ell}^{\nu} &= \left\langle \varphi_\ell^{\nu}, v_p \right\rangle_{H^{s-\alpha}(\Gamma)} \end{aligned}$$

for $k, \ell = 1, \ldots, N-m$; $p = 1, \ldots, m$. Then the inverse matrix $\widetilde{M}_h^{-1}$ can be computed via the Sherman–Morrison–Woodbury formula [17] based on the inverse mass matrix M_h^{-1}. Note that the mass matrix M_h is diagonally dominant and sparse, so we can use the Jacobi iteration for executing the action of M_h^{-1}.

5. Applications to boundary integral operators

As an application of our proposed preconditioning technique we consider a homogeneous partial differential equation

$$L(x)u(x) = 0 \quad \text{for } x \in \Omega \tag{5.1}$$

with a self-adjoint elliptic differential operator $L(x)$ of second order. In addition to (5.1) there are given boundary conditions for $x \in \Gamma$. Here we consider conditions either of Dirichlet type

$$u(x) = g(x) \quad \text{for } x \in \Gamma \tag{5.2}$$

or of Neumann type

$$t(x) := (Tu)(x) = h(x) \quad \text{for } x \in \Gamma. \tag{5.3}$$

If a fundamental solution of (5.1) is known, these boundary value problems can be reduced to a system of boundary integral equations

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \mathcal{C} \begin{pmatrix} u \\ t \end{pmatrix} \tag{5.4}$$

between the Cauchy data u and t. Here, the boundary integral operators are defined as

$$\begin{aligned} (Vt)(x) &= \int_\Gamma U^*(x,y)t(y)\,\mathrm{d}s_y, \\ (Ku)(x) &= \int_\Gamma \big(T_y U^*(x,y)\big)^{\mathrm{T}} u(y)\,\mathrm{d}s_y, \\ \big(K't\big)(x) &= \int_\Gamma T_x U^*(x,y)t(y)\,\mathrm{d}s_y, \\ (Du)(x) &= -T_x \int_\Gamma \big(T_y U^*(x,y)\big)^{\mathrm{T}} u(y)\,\mathrm{d}s_y, \end{aligned}$$

where the kernel $U^*(\cdot,\cdot)$ is the given fundamental solution of (5.1) and T_x means the application of the conormal boundary operator at $x \in \Gamma$. The boundary integral operators V, K, D are pseudodifferential operators of orders -1, 0, 1, respectively; and their mapping properties on Lipschitz domains are well known [5]. For the Dirichlet problem (5.1), (5.2) we consider the boundary integral equation

$$(Vt)(x) = \big(\tfrac{1}{2}I + K\big)g(x) \quad \text{for } x \in \Gamma. \tag{5.5}$$

V is a bounded pseudodifferential operator of order $2\alpha = -1$, i.e.,

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma).$$

A particular case is the Laplacian $L = -\Delta$, where for $n = 2$ we assume $\mathrm{diam}(\Gamma) < 1$ [9] to ensure the $H^{-1/2}(\Gamma)$-ellipticity. Then there holds

$$c_1^V \cdot \|t\|^2_{H^{-1/2}(\Gamma)} \leqslant \langle Vt, t\rangle_{L^2(\Gamma)} \leqslant c_2^V \cdot \|t\|^2_{H^{-1/2}(\Gamma)} \quad \forall t \in H^{-1/2}(\Gamma) \tag{5.6}$$

and $n = 2$ or $n = 3$. To define a spectrally equivalent preconditioning form we need a pseudodifferential operator B of order $+1$ with

$$B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)\ .$$

From the Calderon projector $\mathcal{C}$ in (5.4) we find the relations

$$VD = \frac{1}{4}I - K^2, \qquad DV = \frac{1}{4}I - \big(K'\big)^2,$$

where both compositions are pseudodifferential operators of order 0. Therefore, an appropriate choice of B will be given by the hypersingular integral operator D. The

resulting preconditioning technique will be demonstrated for the model problem of the Laplace equation. There the fundamental solution is given by

$$U^*(x,y) = \frac{1}{2(n-1)\pi} \cdot \begin{cases} -\log|x-y| & \text{for } n=2; \\ \dfrac{1}{|x-y|} & \text{for } n=3. \end{cases}$$

In the following we describe three different preconditioners based on the hypersingular integral operator

$$(Du)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) u(y)\, \mathrm{d}s_y$$

with $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, i.e., D is a bounded pseudodifferential operator of order $2\alpha = 1$. Since for the Laplacian, D has a non-trivial kernel given by

$$\ker D = \operatorname{span}\{1\},$$

D satisfies the spectral equivalence inequalities

$$c_1^D \cdot \|u\|^2_{H^{1/2}(\Gamma)} \leqslant \langle Du, u\rangle_{L^2(\Gamma)} \leqslant c_2^D \cdot \|u\|^2_{H^{1/2}(\Gamma)} \tag{5.7}$$

on the subspace of all $u \in H^{1/2}(\Gamma)$ satisfying

$$\langle u, 1\rangle_{L^2(\Gamma)} = \int_\Gamma u(x)\, \mathrm{d}s_x = 0.$$

Hence we have $v_1(x) \equiv 1$ for $x \in \Gamma$. If we use the Sobolev–Slobodecki representation of the inner product in $H^{1/2}(\Gamma)$, given by

$$\langle u, v\rangle_{H^{1/2}(\Gamma)} = \langle u, v\rangle_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^n}\, \mathrm{d}s_x\, \mathrm{d}s_y,$$

the double integral vanishes for $u(x) \equiv \text{const}$. Hence, for the definition of the Bessel potential operators J, we get the variational problem of finding $Jv_1 \in H^{-1/2}(\Gamma)$ such that

$$\langle Jv_1, w\rangle_{L^2(\Gamma)} = \langle v_1, w\rangle_{H^{1/2}(\Gamma)} = \langle v_1, w\rangle_{L^2(\Gamma)}$$

is satisfied for all test functions $w \in H^{1/2}(\Gamma)$, which yields $Jv_1 \equiv 1$, too.

Now we apply theorem 2.1 with $m = 1$. Since for arbitrary polynomial degree ν, the B-spline basis functions $\{\varphi_k^\nu\}_{k=1}^N$ form a decomposition of the unity,

$$\sum_{k=1}^N \varphi_k^\nu(x) \equiv 1 \quad \text{for all } x \in \Gamma,$$

we get the transformed basis functions in the form

$$\widetilde{\varphi}_N^{\,\nu}(x) = \sum_{k=1}^N \varphi_k^\nu(x) \equiv 1,$$

$$\widetilde{\varphi}_k^{\nu,0}(x) = \varphi_k^\nu(x) - \frac{m_k^\nu}{|\Gamma|} \cdot v_1(x), \quad m_k^\nu = \int_\Gamma \varphi_k^\nu(x)\,\mathrm{d}s_x$$

for $k = 1, \ldots, N-1$, and the transformed basis functions for $\varphi_k^{\mu,0}(x)$, respectively.

The transformation T corresponding to (4.5), (4.6) is now given by

$$\widetilde{f}_N = f\big(\widetilde{\varphi}_N^\nu\big) = \sum_{\ell=1}^N f\big(\varphi_\ell^\nu\big) = \sum_{\ell=1}^N f_\ell,$$
$$\widetilde{f}_k = f\big(\widetilde{\varphi}_k^\nu\big) = f\big(\varphi_k^\nu\big) - \frac{m_k^\nu}{|\Gamma|} \cdot f\big(\widetilde{\varphi}_N^\nu\big) = f_k - \frac{m_k^\nu}{|\Gamma|} \cdot \widetilde{f}_N \quad \text{for } k = 1, \ldots, N-1.$$

For the transposed basis transformation we apply lemma 4.2 and obtain

$$f_N = \widetilde{f}_N - \frac{1}{|\Gamma|} \sum_{k=1}^{N-1} m_k^\nu \cdot \widetilde{f}_k,$$
$$f_k = \widetilde{f}_k + f_N \quad \text{for } k = 1, \ldots, N-1.$$

Hence, the preconditioner for the discrete single layer potential operator defined by the Galerkin–Bubnov matrix

$$V_h[\ell, k] = \big\langle V\varphi_k^\nu, \varphi_\ell^\nu \big\rangle_{L^2(\Gamma)} \quad \text{for } k, \ell = 1, \ldots, N$$

is given by

$$C_V^{-1} = T^{\mathrm{T}} \begin{pmatrix} \widetilde{M}_h^{-1} D_h \widetilde{M}_h^{-\mathrm{T}} & 0 \\ 0 & |\Gamma|^2 \end{pmatrix} T, \tag{5.8}$$

which includes the Galerkin discretization of the hypersingular integral operator

$$D_h[j, i] = \langle D\varphi_i^\mu, \varphi_j^\mu \rangle_{L^2(\Gamma)}$$

for $i, j = 1, \ldots, N-1$. Note that to satisfy the stability condition and to ensure $\varphi_k^\mu \in H^{1/2}(\Gamma)$, only the choices of smoothest splines with

$$1 \leqslant \mu = \nu + 2k \quad \text{with } k \in \mathbb{N}_0$$

are possible. The mass matrix is given by

$$M_h[\ell, i] = \big\langle \varphi_i^\nu, \varphi_\ell^\mu \big\rangle_{L^2(\Gamma)}$$

for $i, \ell = 1, \ldots, N-1$ and the modified mass matrix in (5.8) is then defined by the representation

$$\widetilde{M}_h = M_h - \frac{1}{|\Gamma|} \underline{m}^\mu \big(\underline{m}^\nu\big)^{\mathrm{T}}. \tag{5.9}$$

The inverse $\widetilde{M}_h^{-1}$ is given by the Sherman–Morrison formula [17]:

$$\widetilde{M}_h^{-1} = M_h^{-1} - \frac{1}{\alpha - |\Gamma|} M_h^{-1} \underline{m}^\nu \big(\underline{m}^\mu\big)^{\mathrm{T}} M_h^{-1}, \quad \alpha = \big(\underline{m}^\mu\big)^{\mathrm{T}} M_h^{-1} \underline{m}^\nu.$$

The above approach is independent of the space dimension n, but depends on the non-trivial kernel of the hypersingular integral operator D. For a generalization to more general partial differential equations, e.g., to the Lamé equations of linear elasticity, where the dimension of the kernel of D is $m = 3$ in two and $m = 6$ in three dimensions, it may be complicated to satisfy condition (4.2). So we are interested in an appropriate modification of our proposed preconditioning technique. Note that the hypersingular integral operator satisfies the norm equivalence

$$c_1^D \cdot |u|^2_{H^{1/2}(\Gamma)} \leqslant \langle Du, u\rangle_{L^2(\Gamma)} \leqslant c_2^D \cdot |u|^2_{H^{1/2}(\Gamma)} \tag{5.10}$$

for all $u \in H^{1/2}(\Gamma)$, where $|\cdot|_{H^{1/2}(\Gamma)}$ denotes the Slobodecki seminorm in the Sobolev space $H^{1/2}(\Gamma)$. Then the operator

$$B = \alpha \cdot I + D \quad \text{with } 0 < \alpha \in \mathbb{R} \tag{5.11}$$

satisfies the spectral equivalence inequalities

$$\min\big\{\alpha, c_1^D\big\} \cdot \|u\|^2_{H^{1/2}(\Gamma)} \leqslant \langle Bu, u\rangle_{L^2(\Gamma)} \leqslant \max\big\{\alpha, c_2^D\big\} \cdot \|u\|^2_{H^{1/2}(\Gamma)}$$

for all $u \in H^{1/2}(\Gamma)$; any choice $\alpha \in [c_1^D, c_2^D]$ yields optimal constants in the spectral equivalence inequalities above. Now we can apply theorem 2.1 with $m = 0$; and the resulting preconditioner is given by

$$C_V^{-1} = M_h^{-1}\big(\alpha \cdot \overline{M}_h + D_h\big) M_h^{-\mathrm{T}} \tag{5.12}$$

consisting of the matrices

$$\begin{aligned} D_h[j,i] &= \big\langle D\varphi_i^\mu, \varphi_j^\mu\big\rangle_{L^2(\Gamma)}, \\ \overline{M}_h[j,i] &= \big\langle \varphi_i^\mu, \varphi_j^\mu\big\rangle_{L^2(\Gamma)}, \\ M_h[\ell,i] &= \big\langle \varphi_i^\nu, \varphi_\ell^\mu\big\rangle_{L^2(\Gamma)} \quad \text{for } i,j,\ell = 1,\ldots,N. \end{aligned}$$

Both preconditioners in (5.8) and (5.12) are based on theorem 2.1; in addition we used an appropriate basis to discretize the bilinear form (2.9) directly or in combination with an additive correction to get a positive definite operator B, cf. (5.12).

For a third version of preconditioning, we use theorem 2.2 to define another positive definite appropriate operator B. Due to (2.10) we may define $\widetilde{B}$ via Wielandt's separation technique (see [3]) by the bilinear form

$$\big\langle \widetilde{B}u, w\big\rangle_{L^2(\Gamma)} = \langle Du, v\rangle_{L^2(\Gamma)} + \langle u, 1\rangle_{L^2(\Gamma)} \langle w, 1\rangle_{L^2(\Gamma)}.$$

The Galerkin discretization then leads to the matrix representation

$$\widetilde{B}_h = D_h + \underline{m}^\mu \cdot \big(\underline{m}^\mu\big)^{\mathrm{T}}$$

with the Galerkin weights

$$m_k^\mu = \int_\Gamma \varphi_k^\mu(x)\,\mathrm{d}s_x \quad \text{for } k = 1,\dots,N.$$

The resulting preconditioner is now given by

$$C_V^{-1} = M_h^{-1}\big[D_h + \underline{m}^\mu \cdot (\underline{m}^\mu)^{\mathrm{T}}\big] M_h^{-\mathrm{T}}. \tag{5.13}$$

For all three preconditioners, the numerical amount of work consists of a matrix times vector multiplication with the stiffness matrix D_h and of the application of M_h^{-1} and $M_h^{-\mathrm{T}}$, respectively. Note that M_h is sparse and diagonally dominant, so this action can be done, e.g., by the Jacobi iteration with O(N) multiplications. In the case of the preconditioner (5.8), two additional basis transformations appear, where the amount of work is again of the order O(N). Altogether, the numerical amount of work of our preconditioning technique is dominated by the multiplication with the dense stiffness matrix D_h and, therefore, of the same order as the multiplication with V_h.

In many applications, i.e., mixed or Neumann boundary value problems, one also needs a preconditioner for the discrete hypersingular integral operator D_h. To be more specific, let us consider the Neumann boundary value problem (5.1), (5.3) and the boundary integral equation

$$(Du)(x) = \big(\tfrac{1}{2}I - K'\big)h(x) \quad \text{for } x \in \Gamma. \tag{5.14}$$

Obviously, the given function h has to satisfy the compatibility condition $\langle h, 1\rangle_{L^2(\Gamma)} = 0$ to ensure solvability; and the solution is determined uniquely up to a constant.

To construct an optimal preconditioner for the Galerkin stiffness matrix D_h, we can use the same technique as demonstrated before; choosing $B = V$ we have $\ker B = \{0\}$, i.e., no basis transformation is needed. Applying the discretization with $\nu = \mu$, the resulting preconditioner is then given by

$$C_D^{-1} = M_h^{-\mathrm{T}} V_h M_h^{-1}. \tag{5.15}$$

6. Numerical examples

As a first numerical example we consider the Dirichlet problem for the Laplacian in the L-shaped domain

$$\Omega = [-0.25, 0.25]^2 \backslash [-0.25, 0]^2,$$

which is equivalent to the boundary integral equation

$$(Vt)(x) = \big(\tfrac{1}{2}I + K\big)g(x) \quad \text{for } x \in \Gamma,$$

where $\Gamma = \partial\Omega$ and where g is the given boundary potential on Γ. Let Γ be decomposed into an arbitrary partition of N boundary elements with local mesh sizes h_k. The single layer potential operator is discretized with piecewise constant trial functions ($\nu = 0$). Hence, the hypersingular boundary integral operator D as a preconditioner

Table 1
CG iterations for uniform mesh refinement.

	$C_V = \operatorname{diag} V_h$		$C_V^{-1} = (5.8)$		$C_V^{-1} = (5.12)$		$C_V^{-1} = (5.13)$	
N	Iter	sec	Iter	sec	Iter	sec	Iter	sec
32	17	0.01	7	0.04	8	0.01	7	0.01
64	24	0.01	8	0.10	8	0.03	8	0.02
128	35	0.04	8	0.23	9	0.08	8	0.07
256	43	0.47	8	0.55	9	0.27	8	0.23
512	59	2.46	8	1.44	9	0.93	8	0.81
1024	78	13.14	8	4.33	9	3.43	8	3.02

Table 2
Condition numbers in the uniform case.

N	$C_V = \operatorname{diag} V_h$	$C_V^{-1} = (5.8)$	$C_V^{-1} = (5.12)$	$C_V^{-1} = (5.13)$
32	40.72	1.69	2.00	1.68
64	81.03	1.70	2.06	1.69
128	162.52	1.71	2.10	1.71
256	325.91	1.71	2.13	1.71
512	653.12	1.72	2.16	1.72
1024	1307.98	1.72	2.18	1.72

is discretized with piecewise quadratic trials ($\mu = 2$) if the same break points in the case $n = 2$ or partitions in the case $= 3$ are used, respectively.

In our numerical example we compare our three proposed preconditioners and the simple Jacobi preconditioner with respect to the number of iterations and to the computing times needed for the preconditioned conjugate gradient method to get a relative reduction $\varepsilon = 10^{-8}$ of the residual in the C_V^{-1}-energetic ℓ_2-norm. In table 1, we give the results for the case of uniform mesh refinements.

Note that the number of iterations needed depends on the spectral condition number of the preconditioned system as given in (1.10). In table 2 we give estimates for the condition numbers according to the preconditioners used above. The minimal and maximal eigenvalue of the preconditioned system are approximated using a gradient method as described in [16].

All three preconditioners proposed provide constant bounds for the spectral condition number of the precondioned system and therefore only a constant number of iterations is needed; however, the preconditioner (5.8) requires twice the application of M_h^{-1}, respectively of $M_h^{-\mathrm{T}}$, and in addition two basis transformations. Hence, the preconditioner (5.8) is slightly more time consuming than the other ones.

In table 3, the results in the case of an adaptive refinement are given, where the mesh ratio is given by $h_{\max}/h_{\min}$. To construct the adaptive refinement we consider a Dirichlet problem with the given boundary data

$$g(x) = \log|x - x^*| \quad \text{with } x^* = (\varepsilon, \varepsilon)^{\mathrm{T}},\ \varepsilon = -0.0004.$$

Table 3
Adaptive refinement.

		$C_V = \operatorname{diag} V_h$		$C_V^{-1} = (5.8)$		$C_V^{-1} = (5.12)$		$C_V^{-1} = (5.13)$	
N	$h_{\max}/h_{\min}$	Iter	sec	Iter	sec	Iter	sec	Iter	sec
8	1	4	0.01	5	0.01	4	0.01	4	0.01
14	2	7	0.01	7	0.02	7	0.01	6	0.01
18	4	10	0.01	7	0.03	8	0.01	7	0.01
22	8	12	0.01	7	0.03	8	0.01	7	0.01
26	16	14	0.01	8	0.04	8	0.01	7	0.01
30	32	17	0.01	8	0.05	8	0.01	7	0.01
34	64	21	0.01	8	0.05	8	0.01	8	0.01
38	128	22	0.01	8	0.06	8	0.01	8	0.01
42	256	22	0.01	8	0.07	8	0.02	8	0.01
48	512	23	0.01	8	0.08	8	0.02	8	0.02
54	1024	24	0.01	8	0.09	8	0.02	8	0.02
68	2048	27	0.01	9	0.12	9	0.03	8	0.03
108	2048	31	0.03	9	0.21	9	0.06	8	0.05
210	2048	40	0.21	9	0.47	9	0.19	8	0.16
408	4096	52	1.40	10	1.28	9	0.61	9	0.59
796	4096	67	6.76	10	3.50	9	2.11	9	2.07

Table 4
Condition numbers in the adaptive case.

N	$h_{\max}/h_{\min}$	$C_V = \operatorname{diag} V_h$	$C_V^{-1} = (5.8)$	$C_V^{-1} = (5.12)$	$C_V^{-1} = (5.13)$
8	1	11.59	1.48	1.82	1.48
14	2	16.70	1.70	1.92	1.69
18	4	20.70	1.73	1.97	1.71
22	8	27.58	1.74	2.02	1.73
26	16	43.07	1.75	2.06	1.73
30	32	56.26	1.76	2.09	1.74
34	64	62.27	1.76	2.12	1.74
38	128	69.49	1.76	2.15	1.74
42	256	77.33	1.76	2.17	1.74
48	512	89.62	1.76	2.18	1.74
54	1024	101.99	1.76	2.20	1.74
68	2048	135.17	1.76	2.21	1.74
108	2048	263.98	1.74	2.22	1.84
210	2048	492.19	1.91	2.24	1.86
408	4096	938.12	2.02	2.24	1.87
796	4096	1968.64	2.93	2.25	1.88

Based on a local error estimator [23] we divide these elements into two parts, where the local L^2-error is larger than twenty percent of the maximum error.

These results confirm our theory, namely that all preconditioners proposed do not depend on the triangulation Γ_h. Moreover, there are no restrictions on the ordering of the boundary elements.

Again, in table 4 we give the corresponding estimates for the spectral condition numbers of the preconditioned systems.

To execute the action of M_h^{-1}, we use the Jacobi iteration; i.e., we replace M_h^{-1} by $M_{h,L}^{-1}$, where L denotes the number of iterations used. In both examples we set $L = 6$, only for the preconditioner (5.8) we used $L = 20$ when applying the Sherman–Morrison formula for the inversion of the modified mass matrix. For the preconditioner

$$C_V^{-1} = M_{h,L}^{-1}\big(D_h + \underline{m}^{\mu}\big(\underline{m}^{\mu}\big)^{\mathrm{T}}\big)M_{h,L}^{-\mathrm{T}}$$

we will demonstrate the influence of the choice of L. Note that also for $L = 0$, the spectral condition number of the preconditioned system is bounded due to the strong diagonal dominance of M_h. In the adaptive case we need some further requirements. Let us consider three neighbouring boundary elements Γ_{k-1}, Γ_k and Γ_{k+1} with local mesh sizes h_{k-1}, h_k, h_{k+1}, respectively. If we consider three corresponding trial functions of polynomial degree 2, say φ_{k-1}^2, φ_k^2, φ_{k+1}^2, we can compute the matrix entries of M_h for the piecewise constant test function φ_k^0 which vanishes outside Γ_k explicitly:

$$M_h^{\mathrm{T}}[k,\ell] = M_h[\ell,k] = \big\langle \varphi_k^0, \varphi_\ell^2 \big\rangle_{L^2(\Gamma)} \quad \text{for } \ell = k, k \pm 1,$$

$$M_h^{\mathrm{T}}[k,k] = \frac{h_k}{h_{k-1}+h_k}\left[\frac{1}{2}h_{k-1} + \frac{1}{6}h_k\right] + \frac{h_k}{h_{k+1}+h_k}\left[\frac{1}{2}h_{k+1} + \frac{1}{6}h_k\right],$$

$$M_h^{\mathrm{T}}[k,k-1] = \frac{1}{3}\frac{h_{k-1}^2}{h_{k-1}+h_k},$$

$$M_h^{\mathrm{T}}[k,k+1] = \frac{1}{3}\frac{h_{k+1}^2}{h_{k+1}+h_k}.$$

The worst case with respect to diagonal dominance is if $h_k < h_{k-1}$ and $h_k < h_{k+1}$. Then diagonal dominance can be guaranteed if we require $h_{k\pm1} < 3h_k/2$ for the refinement strategy in this particular situation. In all the other cases of our refinement strategy, e.g., $h_{k-1} = 2h_k$, $h_{k+1} = h_k$, the diagonal dominance of M_h is satisfied. This means that one has to avoid a local refinement where only one smallest element lies between two that are twice as large.

Finally we give some results concerning the preconditioner (5.15) for the discrete hypersingular integral operator. We consider the boundary integral equation (5.14) corresponding to the Neumann problem with the Laplacian. The Galerkin discretization of both boundary integral operators is done with piecewise linear trial functions ($\nu = \mu = 1$) on a family of uniform meshes. Note that the meshes are generated hierarchically, i.e., the ordering of the nodes is arbitrary; and therefore, circulant preconditioners cannot be used.

Note that the preconditioner (5.15) can also be used for the discrete Steklov–Poincaré operator, which appears in boundary element methods for mixed boundary value problems and in domain decomposition methods, too.

Table 5
Influence of the mass matrix.

	$L=20$		$L=6$		$L=3$		$L=0$	
N	Iter	sec	Iter	sec	Iter	sec	Iter	sec
32	7	0.02	7	0.01	8	0.01	16	0.01
64	8	0.05	8	0.02	8	0.02	22	0.03
128	8	0.11	8	0.07	8	0.06	23	0.12
256	8	0.32	8	0.23	9	0.24	23	0.51
512	8	0.99	8	0.81	9	0.86	23	1.98
1024	8	3.39	8	3.02	9	3.29	22	7.52

Table 6
Preconditioner for D_h.

	$C_D = I$			$C_D^{-1} =$ (5.15)		
N	$\kappa(C_D^{-1}D_h)$	Iter	sec	$\kappa(C_D^{-1}D_h)$	Iter	sec
32	8.21	12	0.01	1.57	7	0.01
64	16.30	17	0.01	1.59	7	0.02
128	32.55	26	0.03	1.60	7	0.05
256	65.09	37	0.40	1.60	7	0.19
512	130.19	54	2.26	1.61	7	0.69
1024	260.39	76	12.75	1.62	7	2.62

Table 7
CG iterations for uniform mesh refinement (linear elasticity).

		$C_V = \operatorname{diag} V_h$		$C_V^{-1} =$ (5.8)		$C_V^{-1} =$ (5.12)		$C_V^{-1} =$ (5.13)	
N	dim	Iter	sec	Iter	sec	Iter	sec	Iter	sec
32	64	30	0.01	14	0.15	14	0.04	14	0.03
64	128	41	0.05	14	0.31	14	0.09	14	0.08
128	256	55	0.59	15	0.82	14	0.31	15	0.30
256	512	69	2.89	15	2.14	14	1.11	15	1.14
512	1024	92	15.32	15	6.19	15	4.26	15	4.18

In a second example we demonstrate that all preconditioners proposed can be used for more general problems. There we consider a Dirichlet problem in linear elasticity, where the fundamental solution is given by the Kelvin tensor ($n = 2$)

$$U^*_{kl}(x,y) = \frac{1+\nu}{4\pi E(1-\nu)}\left[(4\nu-3)\log|x-y| + \frac{(x_k-y_k)(x_\ell-y_\ell)}{|x-y|^2}\right],$$

where E is the Young modulus and ν is the Poisson ratio.

As preconditioners for the discrete single layer potential we use the block-diagonal variants of all preconditioners derived from the Laplacian. The number of iterations are given in table 7 and corresponding estimates for the spectral condition number are given in table 8.

Table 8
Condition numbers in the uniform case (linear elasticity).

N	dim	$C_V = \operatorname{diag} V_h$	$C_V^{-1} =$ (5.8)	$C_V^{-1} =$ (5.12)	$C_V^{-1} =$ (5.13)
32	64	54.65	3.80	3.75	3.79
64	128	108.27	3.83	3.78	3.82
128	256	215.94	3.85	3.80	3.84
256	512	431.03	3.86	3.81	3.85
512	1024	860.81	3.86	3.82	3.86

7. Conclusions

All the preconditioners presented here are independent of the space dimension and of the underlying partial differential equation. Hence, we can extend this technique straightforwardly to the case $n = 3$ and to more general equations. Both preconditioners, for the discrete single layer potential operator as well as for the discrete hypersingular one, can be used to construct efficient iterative algorithms for mixed boundary value problems via boundary integral equations and in domain decomposition methods for parallelizing these methods [22,25]. In the present paper, the discrete forms of all preconditioners are given for the case of a closed surface. When dealing with mixed boundary value problems, the preconditioners have to be defined on open surfaces or, in domain decomposition, along the coupling boundaries. In these cases one has to introduce the correct trial spaces satisfying the stability condition in theorem 3.2. A detailed discussion of necessary modifications will be given in a forthcoming paper.

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