SIAM J. NUMER. ANAL.
Vol. 26, No. 3, pp. 574-587, June 1989

005

MESH GRADING FOR INTEGRAL EQUATIONS OF THE FIRST KIND WITH LOGARITHMIC KERNEL*

Y. YAN† AND I. H. SLOAN†

Abstract. When the Galerkin method is used to solve the first kind integral equations with logarithmic kernel on polygons or open arcs, the singularities of the solution degrade the rates of convergence. The aim of this paper is to show, for the particular case of piecewise-constant approximation spaces, that in some cases the $O(h^3)$ order of convergence can be restored by a suitable grading of the mesh.

Key words. mesh grading, Galerkin's method, first kind integral equations

AMS(MOS) subject classifications. primary 65N30, 65R20; secondary 45L10

1. Introduction. In the indirect boundary element method (BEM) for the two-dimensional Dirichlet problem, an unknown harmonic function u in a domain Ω with boundary Γ is represented as a single-layer potential of the form

$$u(\tau) = -\int_\Gamma \log|\tau - y| g(y)\, dy, \qquad \tau \in \Omega, \tag{1.1}$$

where dy is the element of arclength at a point $y \in \Gamma$. The reformulated problem then becomes a first kind boundary integral equation with logarithmic kernel as follows:

$$-\int_\Gamma \log|x-y| g(y)\, dy = f(x), \qquad x \in \Gamma \subset R^2, \tag{1.2}$$

where $f(x)$ is the known value of u on Γ (see [9], [8], [2], [12]). Hsiao and Wendland [8] were the first to give a rigorous error analysis for the Galerkin method applied to this boundary integral equation where Γ is smooth and closed curve. Later Costabel and Stephan [6] extended this analysis to treat the more difficult case where Γ is a polygon. More generally, Sloan and Spence [12] have investigated (1.2) for Γ either a closed contour or an open arc. The theory is particularly simple if the "transfinite-diameter" C_Γ is less than one, which is the case, for example, if the contour Γ lies within a circle of radius 1. A special role is played in [12] by the particular case

$$-\int_\Gamma \log|x-y| g_\tau(y)\, dy = -\log|x-\tau|, \qquad x \in \Gamma, \tag{1.3}$$

with solution g_τ, $\tau \in R^2$, for which the corresponding potential is

$$u_\tau(x) = -\int_\Gamma \log|x-y| g_\tau(y)\, dy, \qquad x \in R^2. \tag{1.4}$$

If Γ is a polygon or an open arc, singularities in the distribution $g(x)$ are produced at corners and ends (see [6], [12], [13]). This leads to a relatively poor convergence rate when the Galerkin method is applied with uniform meshes; the particular case of piecewise-constant approximation spaces is analyzed in [12]. In this paper, we will show that the order of convergence of quantities such as $u(\tau)$ may be restored by grading the mesh in a suitable way at the corners, and at the endpoints in the case of an open contour.

This approach is simple to implement and has been used extensively in finite-element computations for differential equations and weakly singular integral equations

* Received by the editors September 9, 1987; accepted for publication April 14, 1988.

† School of Mathematics, University of New South Wales, Sydney, New South Wales 2033, Australia.

(see [1], [7], [11], [13]). It has also been used for second kind boundary integral equations arising from the Dirichlet problem on a polygon through the double-layer potential formulation (see [4], [5]).

For convenience, we suppose that $C_\Gamma < 1$ and that Γ is parametrized by arclength, i.e., there exists a function ν of the arclength,

$$\nu : [0, d(\Gamma)] \to \Gamma \subset R^2,$$

where $d(\Gamma)$ is the length of Γ and $|d\nu/ds| = 1$. Then a transformed form of (1.2) is obtained:

$$-\int_0^{d(\Gamma)} \log |\nu(s) - \nu(\sigma)| g(\nu(\sigma))\, d\sigma = f(\nu(s)), \qquad s \in [0, d(\Gamma)],$$

which in operator form becomes

$$Kg = f. \tag{1.5}$$

Further, we suppose that the corner points are at $\nu_1 = \nu(s_1), \cdots, \nu_m = \nu(s_m)$, where $0 = s_1 < s_2 < \cdots < s_m \leqq d(\Gamma)$, with $s_m < d(\Gamma)$ for Γ a polygon and $s_m = d(\Gamma)$ for Γ an open contour. At each corner point ν_i, the number $\chi_i \in (-1, 1)$ is defined by requiring $(1 - \chi_i)\pi$ to be one of the angles $\nu_{i-1}\hat{\nu}_i\nu_{i+1}$, where, in the case of a polygon, the interior angle is taken, and $\nu_0 = \nu_m$, $\nu_{m+1} = \nu_1$, corresponding to $s_{m+1} = d(\Gamma)$. At the endpoints of an open contour we define $\chi_1 = \chi_m = -1$, corresponding to an angle of 2π. Under these assumptions, it is known that near the corner ν_i the solution $g(\nu(s))$ can generally be expected to have a singularity of the form $|s - s_i|^{\beta_i}$, where $\beta_i = -|\chi_i|/(1 + |\chi_i|) \geqq -\frac{1}{2}$ (see [9], [12]). If Γ is a polygon, however, the singularity may be weaker than this. Since the singularity in g may be traced to the singularities in the potential $u(\tau)$ (defined by (1.1) on the two sides of Γ (see [9]), it turns out that if $u(\tau)$ is nonsingular in the exterior region then the singularity in g becomes $|s - s_i|^{\chi_i/(1-\chi_i)}$, and if it is nonsingular in the interior the singularity in g becomes $|s - s_i|^{-\chi_i/(1+\chi_i)}$.

Now we adopt the graded meshes proposed by Rice [10] and Chandler [5]. Thus letting $q_1, \cdots, q_m$ be positive real numbers, for any integer n we define the side

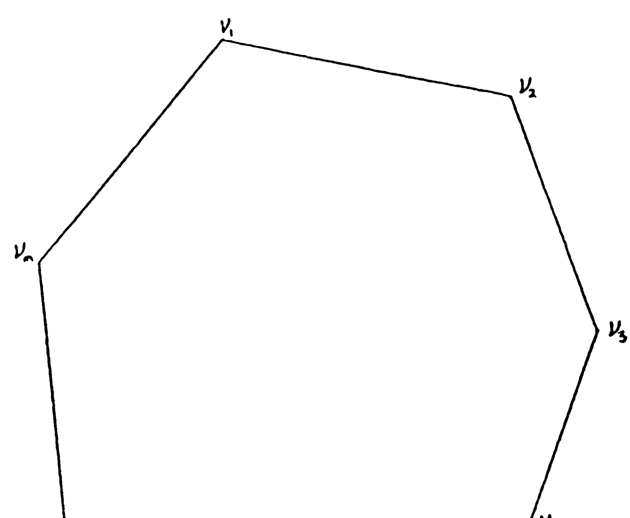

FIG. 1. *Example of closed contour.*

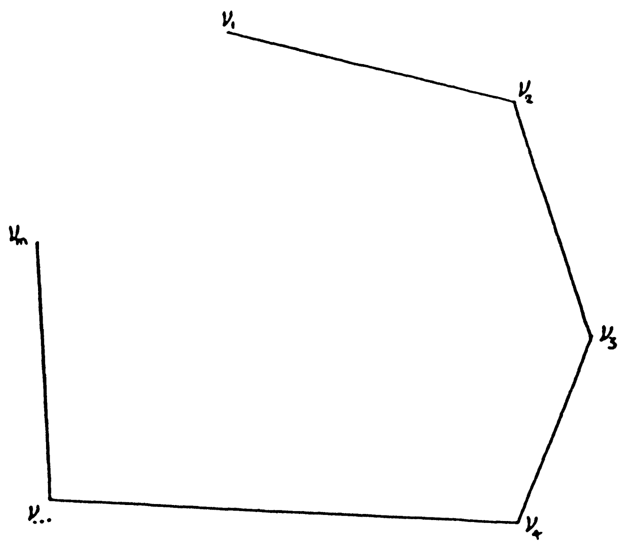

FIG. 2. *Example of open contour.*

$\Gamma_i = \{\nu(s)\colon s_i \leqq s \leqq s_{i+1}\}$ of Γ, where for the polygon $1 \leqq i \leqq m$, and for the open arc $1 \leqq i \leqq m-1$. Then on the side Γ_i the meshpoints $\{\sigma_{ij}\}$ are defined by

$$\sigma_{ij} = \begin{cases} s_i + \dfrac{1}{2}\left(\dfrac{j}{n}\right)^{q_i}(s_{i+1} - s_i), & 0 \leqq j \leqq n, \\ s_{i+1} - \dfrac{1}{2}\left(\dfrac{2n-j}{n}\right)^{q_{i+1}}(s_{i+1} - s_i), & n \leqq j \leqq 2n, \end{cases} \tag{1.6}$$

with $q_{m+1} = q_1$. Thus the grading parameter q_i corresponds to the corner ν_i rather than to the side Γ_i, because the singularity at each corner may be different. Let $S^h(\Gamma)$ (or similarly $S^h([0, d(\Gamma)])$) denote the piecewise-constant space with breakpoints $\{\nu(\sigma_{ij})\}$ (or $\{\sigma_{ij}\}$), where $h = 1/n$. The Galerkin method approximates g by $g^h \in S^h(\Gamma)$, such that

$$(Kg^h, x^h) = (f, x^h), \qquad x^h \in S^h(\Gamma),$$

where

$$(w, v) = \int_\Gamma w(y)v(y)\,dy = \int_0^{d(\Gamma)} w(\nu(\sigma))v(\nu(\sigma))\,d\sigma.$$

If we choose $q_1 = q_2 = \cdots = q_m = 1$, the meshes are uniform, and we have (see [12])

$$\|g^h - g\|^2 = \begin{cases} O(h^3), & \frac{1}{2} \leqq \beta, \\ O(h^{2(1+\beta)}), & -\frac{1}{2} < \beta < \frac{1}{2}, \\ O(h \log h^{-1}), & \beta = -\frac{1}{2}, \end{cases}$$

where $\beta = \min_i \beta_i$, and $\|v\|^2 = (Kv, v)$, which is nonnegative because $C_\Gamma < 1$.

These "energy" error norms play a key role in the error analysis in [12], and influence the rates of convergence obtained for the Galerkin approximation

$$u^h(\tau) = -\int_\Gamma \log|\tau - y| g^h(y)\,dy, \qquad \tau \in R^2,$$

corresponding to the exact potential

$$u(\tau) = -\int_\Gamma \log|\tau - y| g(y)\,dy, \qquad \tau \in R^2. \tag{1.7}$$

For $\beta \geqq \frac{1}{2}$ the $O(h^3)$ rate of convergence of the error norm is the best attainable with a piecewise-constant approximation space, even for a smooth contour Γ, but for $-\frac{1}{2} \leqq \beta < \frac{1}{2}$ the rate is less than optimal. In the latter case, we wish to choose grading parameters $q_1, \cdots, q_m$ that will restore the $O(h^3)$ convergence. For this purpose we will first establish some lemmas on the approximation of the function $w(s) = s^\beta$, with $\beta \geqq -\frac{1}{2}$, on the interval $[0, 1]$.

2. Several lemmas. Given $q \geqq 1$ and $t_i = (i/n)^q$, $i = 0, 1, \cdots, n$, define a subdivision of $[0, 1]$ as $\Pi = \{t_i \mid i = 0, 1, \cdots, n\}$, and let $S_n(\Pi)$ be the space of functions on $[0, 1]$ that are constant on each open interval (t_i, t_{i+1}), $i = 0, \cdots, n-1$. Suppose that P_n is the orthogonal projection from $L_2[0, 1]$ to $S_n(\Pi)$. Then we know that

$$P_n f = \sum_{i=0}^{n-1} h_i^{-1}(f, X_i) X_i,$$

where $h_i = t_{i+1} - t_i$ and

$$X_i(s) = \begin{cases} 1 & s \in (t_i, t_{i+1}), \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that the domain of P_n can be enlarged to $L_1[0, 1]$, i.e., $P_n : L_1[0, 1] \to S_n(\Pi)$, and

$$(P_n u, v) = (P_n u, P_n v) = (u, P_n v), \qquad u, v \in L_1[0, 1].$$

Denoting $P_n' = I - P_n$ and $w(s) = s^\beta$, we want to estimate in the following the value of $\|P_n' w\|_1$. Since $\|P_n' w\|_1 \leqq c \operatorname{dist}_1(w, S_n(\Pi))$, the estimates of $\operatorname{dist}_1(s^\beta, S_n(\Pi))$ given in [10] could be used to obtain results for $\|P_n' w\|_1$. However, more precise results may be obtained by our carrying out the analysis in a slightly different way.

LEMMA 2.1. *For $\beta > -1$ and n sufficiently large,*

$$\|P_n' w\|_1 \leqq \begin{cases} \dfrac{q^2 2^{q-1+|\beta q-1|} c_\beta}{q(1+\beta)-1} h, & q > \dfrac{1}{1+\beta}, \\ q^2 2^{q-1+|\beta q-1|} c_\beta h(1 + \log h^{-1}), & q = \dfrac{1}{1+\beta}, \\ \dfrac{q^2 2^{q+|\beta q-1|} c_\beta}{1-q(1+\beta)} h^{q(1+\beta)}, & q < \dfrac{1}{1+\beta}, \end{cases}$$

where $h = 1/n$, and $c_\beta = |\beta| + \int_0^1 |t^\beta - 1/(1+\beta)|\, dt$.

Remark. Note that if $\beta = 0$ then c_β vanishes, thus ensuring $\|P_n' w\|_1 = 0$.

Proof. By definition,

$$\begin{aligned} \|P_n' w\|_1 &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |s^\beta - h_i^{-1}(w, X_i)|\, ds \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left| s^\beta - h_i^{-1} \int_{t_i}^{t_{i+1}} \sigma^\beta\, d\sigma \right| ds \\ &\leqq \int_0^{t_1} \left| s^\beta - \frac{t_1^\beta}{1+\beta} \right| ds + \sum_{i=1}^{n-1} (t_{i+1} - t_i) |t_{i+1}^\beta - t_i^\beta|. \end{aligned}$$

But

$$\int_0^{t_1} \left| s^\beta - \frac{t_1^\beta}{1+\beta} \right| ds = t_1^{1+\beta} \int_0^1 \left| t^\beta - \frac{1}{1+\beta} \right| dt = h^{q(1+\beta)} \int_0^1 \left| t^\beta - \frac{1}{1+\beta} \right| dt,$$

$$\begin{aligned}\sum_{i=1}^{n-1}(t_{i+1}-t_i)|t_{i+1}^{\beta}-t_i^{\beta}| &= \sum_{i=1}^{n-1} t_i^{1+\beta}\left(\frac{t_{i+1}}{t_i}-1\right)\left|\left(\frac{t_{i+1}}{t_i}\right)^{\beta}-1\right| \\ &= \sum_{i=1}^{n-1} t_i^{1+\beta}\left[\left(1+\frac{1}{i}\right)^{q}-1\right]\left|\left(1+\frac{1}{i}\right)^{\beta q}-1\right| \\ &\leqq \sum_{i=1}^{n-1} t_i^{1+\beta}(q2^{q-1}i^{-1})(|\beta|q2^{|\beta q-1|}i^{-1}) \\ &= q^2|\beta|2^{q-1+|\beta q-1|}h^{q(1+\beta)}\sum_{i=1}^{n-1} i^{q(1+\beta)-2},\end{aligned}$$

where the second to the last step follows from an inequality to which we will frequently appeal:

$$|(1+\mu)^r-1|\leqq|r|2^{|r-1|}|\mu|,\quad \mu\in[-\tfrac{1}{2},1],\quad r\in R. \tag{2.1}$$

By a careful estimate we find that

$$\sum_{i=1}^{n-1} i^{q(1+\beta)-2}\leqq\begin{cases}\dfrac{h^{1-q(1+\beta)}}{q(1+\beta)-1}, & q>\dfrac{1}{1+\beta},\\ 1+\log h^{-1}, & q=\dfrac{1}{1+\beta},\\ \dfrac{2}{1-q(1+\beta)}, & q<\dfrac{1}{1+\beta}.\end{cases}$$

Thus by noting $q\geqq 1$,

$$\|P_n'w\|_1\leqq\begin{cases}\dfrac{q^2 2^{q-1+|\beta q-1|}c_\beta}{q(1+\beta)-1}h, & q>\dfrac{1}{1+\beta},\\ q^2 2^{q-1+|\beta q-1|}c_\beta h(1+\log h^{-1}), & q=\dfrac{1}{1+\beta},\\ \dfrac{q^2 2^{q+|\beta q-1|}c_\beta}{1-q(1+\beta)}h^{q(1+\beta)}, & q<\dfrac{1}{1+\beta}.\end{cases}$$

LEMMA 2.2. *Let* $x\in R^2$, $y_\sigma=(\sigma,0)\in R^2$, $R_x(\sigma)=\log|x-y_\sigma|$, $Rw(x)=\int_0^1 R_x(\sigma)w(\sigma)\,d\sigma$. *Then for* $\beta>-1$ *and n sufficiently large*

$$\sup_{x\in R^2}|RP_n'w(x)|\leqq cq^3 2^{2q+|\beta q-1|}[|\beta|+c(\beta)]h^{\min\{2,q(1+\beta)\}}\log h^{-1},$$

where c is a constant independent of n, q, and β, *and*

$$c(\beta)=\sup_{\mu\in R^2}\int_0^1\left|\log|\mu-y_\sigma|-\int_0^1\log|\mu-y_t|\,dt\right|\left|\sigma^\beta-\frac{1}{1+\beta}\right|d\sigma<\infty.$$

Remark. If $\beta=0$ then $c(\beta)$ vanishes, thus ensuring $RP_n'w(x)\equiv 0$.

Proof. We have that

$$\begin{aligned}|RP_n'w(x)| &= \left|\int_0^1 R_x(\sigma)P_n'w(\sigma)\,d\sigma\right| = \left|\int_0^1 P_n'R_x(\sigma)P_n'w(\sigma)\,d\sigma\right| \\ &\leqq \int_0^{t_1}|P_n'R_x(\sigma)||P_n'w(\sigma)|\,d\sigma+\int_{t_1}^1|P_n'R_x(\sigma)||P_n'w(\sigma)|\,d\sigma \\ &= I_1(x)+I_2(x).\end{aligned}$$

Then

$$I_1(x)=\int_0^{t_1}\left|\log|x-y_\sigma|-\frac{1}{t_1}\int_0^{t_1}\log|x-y_t|\,dt\right|\left|\sigma^\beta-\frac{1}{t_1}\int_0^{t_1}t^\beta\,dt\right|d\sigma$$
$$=t_1^{1+\beta}\int_0^1\left|\log|xt_1^{-1}-y_\sigma|-\int_0^1\log|xt_1^{-1}-y_t|\,dt\right|\left|\sigma^\beta-\frac{1}{1+\beta}\right|d\sigma$$
$$\leqq c(\beta)h^{q(1+\beta)}.$$

Thus,

$$\sup_{x\in R^2} I_1(x)\leqq c(\beta)h^{q(1+\beta)}.$$

On the other hand,

$$I_2(x)=\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}|P_n'R_x(\sigma)||P_n'w(\sigma)|\,d\sigma$$
$$\leqq\sum_{i=1}^{n-1}|t_{i+1}^\beta-t_i^\beta|\int_{t_i}^{t_{i+1}}|P_n'R_x(\sigma)|\,d\sigma.$$

Let $G_i(x)=\int_{t_i}^{t_{i+1}}|P_n'R_x(\sigma)|\,d\sigma$. Then if $x=(x_1,x_2)$ with $x_1\in[t_j,t_{j+1}]$ (where $[t_{-1},0]$ denotes $(-\infty,0]$ and $[t_n,t_{n+1}]$ denotes $[1,\infty)$), we may write

$$I_2(x)\leqq\sum_{i=1}^{n-1}|t_{i+1}^\beta-t_i^\beta|G_i(x)=F_1+F_2+F_3+F_4,$$

where for $j=-1,0,\cdots,n-1,n$,

$$F_1(x)=\sum_{i\notin\{j-1,j,j+1\}}|t_{i+1}^\beta-t_i^\beta|G_i(x),$$

$$F_2(x)=\begin{cases}0, & j=-1,0,1,\\ |t_j^\beta-t_{j-1}^\beta|G_{j-1}(x) & \text{otherwise,}\end{cases}$$

$$F_3(x)=\begin{cases}0, & j=-1,0,n,\\ |t_{j+1}^\beta-t_j^\beta|G_j(x) & \text{otherwise,}\end{cases}$$

$$F_4(x)=\begin{cases}0, & j=-1,n-1,n,\\ |t_{j+2}^\beta-t_{j+1}^\beta|G_{j+1}(x) & \text{otherwise.}\end{cases}$$

Now for $j=1,2,\cdots,n-1$,

$$|t_{j+1}^\beta-t_j^\beta|G_j(x)$$
$$=|t_{j+1}^\beta-t_j^\beta|(t_{j+1}-t_j)\int_0^1\left|\log|h_j^{-1}(x-y_{t_j})-y_\sigma|-\int_0^1\log|h_j^{-1}(x-y_{t_j})-y_t|\,dt\right|d\sigma$$
$$\leqq\bar c|t_{j+1}^\beta-t_j^\beta|(t_{j+1}-t_j)$$
$$\leqq\bar c q^2|\beta|2^{q-1+|\beta q-1|}h^{q(1+\beta)}j^{q(1+\beta)-2}\quad\text{(see (2.1))}$$
$$\leqq\bar c q^2|\beta|2^{q-1+|\beta q-1|}h^{\min\{2,q(1+\beta)\}},$$

where

$$\bar c=\sup_{\mu\in R^2}\int_0^1\left|\log|\mu-y_\sigma|-\int_0^1\log|\mu-y_t|\,dt\right|d\sigma<\infty.$$

Thus

$$\sup_{x\in R^2} F_3(x) \leqq q^2 2^{q-1+|\beta q-1|}|\beta|\bar{c}h^{\min\{2,q(1+\beta)\}}.$$

Similarly,

$$\sup_{x\in R^2} F_2(x) \leqq q^2 2^{q-1+|\beta q-1|}|\beta|\bar{c}h^{\min\{2,q(1+\beta)\}},$$

$$\sup_{x\in R^2} F_4(x) \leqq q^2 2^{q-1+|\beta q-1|}|\beta|\bar{c}h^{\min\{2,q(1+\beta)\}}.$$

At the same time, for $x_1 \in [t_j, t_{j+1}]$, $j=-1, 0, \cdots, n-1, n$,

$$F_1(x) = \sum_{j+1<i\leqq n-1} |t_{i+1}^\beta - t_i^\beta| G_i(x) + \sum_{1\leqq i<j-1} |t_{i+1}^\beta - t_i^\beta| G_i(x),$$

the first term of which is bounded by

$$\begin{aligned}
&\sum_{i=j+2}^{n-1} |t_{i+1}^\beta - t_i^\beta| (t_{i+1}-t_i) | \log|y_{t_{i+1}}-x| - \log|y_{t_i}-x||\\
&\quad\leqq \sum_{i=j+2}^{n-1} |t_{i+1}^\beta - t_i^\beta|(t_{i+1}-t_i)^2 \frac{1}{|t_{j+1}-t_i|}\\
&\quad\leqq q^3 2^{2q-2+|\beta q-1|}|\beta| \sum_{i=j+2}^{n-1} t_i^{1+\beta} \frac{i^{-3}}{1-((j+1)/i)^q} \quad \text{(see (2.1))}\\
&\quad\leqq q^3 2^{2q-2+|\beta q-1|}|\beta| h^{q(1+\beta)} \sum_{i=j+2}^{n-1} \frac{i^{q(1+\beta)-3}}{1-(j+1)/i}\\
&\quad= q^3 2^{2q-2+|\beta q-1|}|\beta| h^{q(1+\beta)} \sum_{i=j+2}^{n-1} \frac{i^{q(1+\beta)-2}}{i-(j+1)}\\
&\quad= q^3 2^{2q-2+|\beta q-1|}|\beta| h^{q(1+\beta)} \sum_{r=1}^{n-j-2} \frac{(j+1+r)^{q(1+\beta)-2}}{r}.
\end{aligned}$$

But

$$\begin{aligned}
\sum_{r=1}^{n-j-2} \frac{(j+1+r)^{q(1+\beta)-2}}{r} &\leqq (j+2)^{q(1+\beta)-2} + \int_1^{n-j-1} (j+1+t)^{q(1+\beta)-2} t^{-1}\,dt\\
&\leqq (j+2)^{q(1+\beta)-2} + [(j+2)^{q(1+\beta)-2} + n^{q(1+\beta)-2}] \log h^{-1};
\end{aligned}$$

thus,

$$\sum_{i=j+2}^{n-1} |t_{i+1}^\beta - t_i^\beta| G_i(x) \leqq 3q^3 2^{2q-2+|\beta q-1|}|\beta| h^{\min\{2,q(1+\beta)\}} \log h^{-1}.$$

Since the second term of $F_1(x)$ has the same bound, we obtain

$$\sup_{x\in R^2} F_1(x) \leqq 3q^3 2^{2q-1+|\beta q-1|}|\beta| h^{\min\{2,q(1+\beta)\}} \log h^{-1}$$

Thus,

$$\sup_{x\in R^2} I_2(x) \leqq cq^3 2^{2q+|\beta q-1|}|\beta| h^{\min\{2,q(1+\beta)\}} \log h^{-1},$$

with c a constant independent of n, q, and β, so that

$$\sup_{x\in R^2} |RP_n' w(x)| \leqq cq^3 2^{2q+|\beta q-1|}(c(\beta)+|\beta|) h^{\min\{2,q(1+\beta)\}} \log h^{-1}.$$

What we are concerned with in Lemmas 2.1 and 2.2 is the function $w(s)=s^\beta$, $\beta>-1$, on the interval $[0,1]$. It is worthwhile stating a corresponding result for a better-behaved function $z(s)\in C^1[0,1]$. The proof is similar to those above with β set equal to 1.

LEMMA 2.3. *If $z\in C^1[0,1]$, then*

$$\|P'_n z\|_1 \leqq cq^2 2^{2q}h,$$
$$\sup_{x\in R^2} |RP'_n z(x)| \leqq cq^3 2^{3q} h^2 \log h^{-1},$$

where c is a constant independent of n and q.

3. Error estimate for the graded mesh. From the discussion in § 2, we should not choose the index q too large, since otherwise the coefficients of the estimates will be unnecessarily large. And in practice, if we choose the index q too large the method may become unstable (because σ_{i1} then lies extremely close to the ith vertex), and could also be unnecessarily expensive (because points near the middle of a side become too widely separated, leading to a need to increase n). In the following we seek the minimum values of $q_1,\cdots,q_m$ that will give optimal convergence. We also explore the convergence for a moderate suboptimal value $q=2$.

For convenience we let $d_i=s_{i+1}-s_i$, $s_{i+1/2}=s_i+d_i/2$.

THEOREM 3.1. *Suppose that for $i=1,\cdots,m$ (in the case of an open arc $i=1,\cdots,m-1$)*

$$g(\nu(s))=\begin{cases} a_i^-|s-s_i|^{\beta_i}+\sum_j b_{i,j}^-|s-s_i|^{\alpha_{i,j}}+g_i^-(s), & s_i<s\leqq s_{i+1/2},\\ a_i^+|s-s_{i+1}|^{\beta_{i+1}}+\sum_j b_{i,j}^+|s-s_{i+1}|^{\alpha_{i+1,j}}+g_i^+(s), & s_{i+1/2}\leqq s<s_{i+1},\end{cases} \tag{3.1}$$

where $g_i^-\in C^1[s_i,s_{i+1/2}]$, $g_i^+\in C^1[s_{i+1/2},s_{i+1}]$, for each i $\sum_j$ only a finite sum, and $a_i^{-(+)}$, $b_{i,j}^{-(+)}$, and $\alpha_{i,j}$ are constants with $\alpha_{i,j}>\beta_i$ for all j. Let $\beta=\min\beta_i\geqq-\frac12$.

(i) *If we select $q_i=2$ for all i in (1.6), then*

$$\|g^h-g\|^2\leqq cB(h,\beta)h^{1+\min\{2,2(1+\beta)\}}\log h^{-1}, \tag{3.2}$$

where c is a constant independent of n, and

$$B(h,\beta)=\begin{cases}1, & \beta>-\frac12,\\ \log h^{-1}, & \beta=-\frac12.\end{cases}$$

(ii) *If we select $q_i\geqq 2/(1+\beta_i)$ for all i in (1.6), then*

$$\|g^h-g\|^2\leqq ch^3\log h^{-1}, \tag{3.3}$$

where c is a constant independent of n.

Proof. For convenience in the proof we let P^h (or correspondingly Λ^h) denote the orthogonal projection from $L_2(\Gamma)$ to $S^h(\Gamma)$ (or from $L_2[0,d(\Gamma)]$ to $S^h([0,d(\Gamma)])$), and let $g_\nu(s)$ denote the function $g(\nu(s))$. Moreover, we denote $\bar{x}=\overline{(x_1,x_2)}=(x_1,-x_2)$, $x^1x^2=(x_1^1,x_2^1)(x_1^2,x_2^2)=(x_1^1x_1^2-x_2^1x_2^2,\ x_1^1x_2^2+x_2^1x_1^2)$, which in effect allows us to consider points of the plane as complex numbers.

Because g^h is the optimal approximation to g in the sense of the energy norm,

$$\|g^h-g\|^2\leqq (K(P^hg-g),\quad (P^hg-g))\leqq \|K(P^hg-g)\|_{L_\infty(\Gamma)}\|P^hg-g\|_{L_1(\Gamma)}.$$

But

$$\begin{aligned}\|P^hg-g\|_{L_1(\Gamma)}&\leqq m\max_i\|P^hg-g\|_{L_1(\Gamma_i)}\\ &=m\max_i(\|\Lambda^hg_\nu-g_\nu\|_{L_1[s_i,s_{i+1/2}]}+\|\Lambda^hg_\nu-g_\nu\|_{L_1[s_{i+1/2},s_{i+1}]})\\ &\leqq c\max_i(\|P'_nw_i\|_1+\sum_j\|P'_nw_{i,j}\|_1+\|P'_nz_i^-\|_1+\|P'_nz_i^+\|_1),\end{aligned}$$

where $w_i(s)=s^{\beta_i}$, $w_{i,j}(s)=s^{\alpha_{i,j}}$, both $z_i^-(s)=g_i^-(s_i+d_is/2)$ and $z_i^+(s)=g_i^+(s_{i+1}-d_is/2)\in C^1[0,1]$ (see (3.1)), and c from now on denotes a constant independent of n that may take different values at its different occurrences.

On the other hand,

$$\|K(P^hg-g)\|_{L_\infty(\Gamma)}=\sup_{x\in\Gamma}\left|\sum_i\int_{\Gamma_i}\log|x-y|(P^hg-g)\,dy\right|$$
$$\leqq m\max_i\sup_{x\in\Gamma}\left|\int_{\Gamma_i}\log|x-y|(P^hg-g)\,dy\right|.$$

From the assumptions in § 1, $\nu(s)=\nu_i+(s-s_i)\eta$ for $s\in[s_i,s_{i+1}]$, where $\eta=(\nu_{i+1}-\nu_i)/d_i$, with $|\eta|=1$. Then

$$\tag{3.4}\begin{aligned}&\sup_{x\in\Gamma}\left|\int_{\Gamma_i}\log|x-y|(P^hg-g)\,dy\right|\\&=\sup_{s\in[0,d(\Gamma)]}\left|\int_{s_i}^{s_{i+1}}\log|\nu(s)-\nu_i-(\sigma-s_i)\eta|(\Lambda^hg_\nu-g_\nu)\,d\sigma\right|.\end{aligned}$$

Since

$$|\nu(s)-\nu_i-(\sigma-s_i)\eta|=|(\nu(s)-\nu_i)\bar\eta-(\sigma-s_i,0)|=|(\nu(s)-\nu_i)\bar\eta-y_{\sigma-s_i}|,$$

it follows that (3.4) is bounded by the value

$$\begin{aligned}\sup_{x\in R^2}\left|\int_{s_i}^{s_{i+1}}\log|x-y_{\sigma-s_i}|(\Lambda^h-I)g_\nu\,d\sigma\right|&\leqq\sup_{x\in R^2}\left|\int_{s_i}^{s_{i+1/2}}\log|x-y_{\sigma-s_i}|(\Lambda^h-I)g_\nu\,d\sigma\right|\\&\quad+\sup_{x\in R^2}\left|\int_{s_{i+1/2}}^{s_{i+1}}\log|x-y_{s_{i+1}-\sigma}|(\Lambda^h-I)g_\nu\,d\sigma\right|\\&\leqq c\sup_{x\in R^2}\left|\int_0^1\log|x-y_\sigma|P_n'\Big(w_i+\sum_j w_{i,j}+z_i^-\Big)\,d\sigma\right|\\&\quad+c\sup_{x\in R^2}\left|\int_0^1\log|x-y_\sigma|P_n'\Big(w_{i+1}+\sum_j w_{i+1,j}+z_i^+\Big)\,d\sigma\right|.\end{aligned}$$

Then by application of Lemmas 2.1–2.3, the results in Theorem 3.1 hold. This completes the proof.

In Theorem 3.1, assumption (3.1) for the solution $g(\nu(s))$ is based on the discussion in [9] and [12]. From the theorem, we find that by choosing $q_i=2$, rather than $q_i=1$, the precision of the approximation is increased by $h\log h^{-1}$ if $\beta<0$, and that for $\beta\geqq0$ the full order of convergence $O(h^3)$ is restored, apart perhaps from a logarithmic factor. Moreover, by choosing $q_i\geqq2/(1+\beta_i)$, the $O(h^3)$ precision, perhaps with an additional logarithmic factor, can be achieved all the time. If $q_i>2/(1+\beta_i)$, the logarithmic factor $\log h^{-1}$ can be omitted, but a more elaborate proof is needed. In the following examples, we will find that the factor $\log h^{-1}$ is not apparent numerically.

4. Applications. In this section we will test the effectiveness of the graded mesh for some of the examples that have been investigated in [12].

Example 1. Here the equation is

$$-\int_\Gamma \log|x-y|g(y)\,dy = 1, \qquad x\in\Gamma, \tag{4.1}$$

with Γ a straight-line segment of length 2, specifically

$$\Gamma = \{(s,0)\colon -1 \leqq s \leqq 1\}.$$

Following [12], the exact solution for the case $f(x)\equiv 1$ is denoted by λ_Γ; thus in this example $g(x)=\lambda_\Gamma(x)$. The quantities shown in Tables 1.1 and 1.2 are Galerkin approximations based on the graded mesh in (1.6), with $q_1=q_2=2$ in Table 1.1 and $q_1=q_2=4$ in Table 1.2. The quantities calculated follow.

(a) $C_\Gamma = \exp(-(\int_\Gamma \lambda_\Gamma)^{-1})$;

and the potential is defined by (1.7):

(b) $u(0,1)$;

(c) $u(1.2,0)$.

As discussed in [12], the exact value of C_Γ is $\frac{1}{2}$. Since Γ is an interval, the exact solution $g(x)=\lambda_\Gamma(x)$ is expected to have singularities of the form $|s-s_i|^{-1/2}$ at the two ends, and the known exact solution (see [12], [13]) does indeed have singularities of this form.

For the case of the grading parameters $q_1=q_2=2$. Theorem 3.1(i) gives

$$\|\lambda_\Gamma^h - \lambda_\Gamma\|^2 = O(h^2\log^2 h^{-1}).$$

But from [12] we have

$$\left|\int_\Gamma \lambda_\Gamma - \int_\Gamma \lambda_\Gamma^h\right| = |(\lambda_\Gamma,1)-(\lambda_\Gamma^h,1)| = \|\lambda_\Gamma^h-\lambda_\Gamma\|^2;$$

TABLE 1.1
$q=2$.

$n=2/h$	$C_\Gamma^h - C_\Gamma$	γ^h	$u^h(0,1)$	γ^h	$u^h(1.2,0)$	γ^h
2	−5.37 (−2)		−0.16356		−0.03501	
4	−1.41 (−2)		−0.24722		0.06155	
8	−3.69 (−3)	1.9231	−0.26535	2.2057	0.09189	1.6707
16	−9.43 (−4)	1.9265	−0.26999	1.9668	0.09958	1.9793
32	−2.38 (−4)	1.9625	−0.27116	1.9868	0.10149	2.0113
64	−5.99 (−5)	1.9815	−0.27146	1.9949	0.10196	2.0030
128	−1.50 (−5)	1.9909	−0.27153	1.9980	0.10208	2.0011
256	−3.76 (−6)	1.9955	−0.27155	1.9992	0.10211	2.0006

TABLE 1.2
$q=4$.

$n=2/h$	$C_\Gamma^h - C_\Gamma$	γ^h	$u^h(0,1)$	γ^h	$u^h(1.2,0)$	γ^h
2	−5.37 (−2)		−0.16356		−0.03501	
4	−8.36 (−3)		−0.25010		0.08121	
8	−1.11 (−3)	2.6465	−0.26882	2.2088	0.09915	2.6955
16	−1.47 (−4)	2.9051	−0.27122	2.9597	0.10172	2.8015
32	−1.91 (−5)	2.9203	−0.27151	3.0598	0.10207	2.8844
64	−2.44 (−6)	2.9395	−0.27155	2.9860	0.10211	2.9332
128	−3.09 (−7)	2.9641	−0.27155	2.9799	0.10212	2.9631
256	−3.90 (−8)	2.9832	−0.27155	2.9903	0.10212	2.9850

hence

(a) $C_\Gamma^h - C_\Gamma = O(h^2 \log^2 h^{-1})$.

Also from [12] we know that

$$|u^h(\tau) - u(\tau)| \leqq \|\lambda_\Gamma^h - \lambda_\Gamma\| \, \|g_\tau^h - g_\tau\| \quad \text{for } \tau \notin \Gamma,$$

where g_τ is the solution of (1.3). Since g_τ is expected to have the same kind of singularity as λ_Γ at the two ends, (3.2) applies with $\beta = -\frac{1}{2}$, giving

$$\|g_\tau^h - g_\tau\|^2 = O(h^2 \log^2 h^{-1}).$$

Hence for (b) and (c),

$$|u^h(\tau) - u(\tau)| = O(h^2 \log^2 h^{-1}).$$

A similar discussion leads to an estimate for the optimal case $q_1 = q_2 = 4$, as follows:

(a) $C_\Gamma^h - C_\Gamma = O(h^3 \log h^{-1})$;

(b), (c) $|u^h(\tau) - u(\tau)| = O(h^3 \log h^{-1})$.

In Tables 1.1 and 1.2 the convergence exponent γ^h is calculated, for example, for C_Γ^h by

$$\gamma^h = \log_2 [(C_\Gamma^{4h} - C_\Gamma^{2h})/(C_\Gamma^{2h} - C_\Gamma^h)].$$

The numerical results for (a)-(c) in Tables 1.1 and 1.2, with their corresponding apparent $O(h^2)$ and $O(h^3)$ orders of convergence, are in good agreement with these predictions, if logarithmic factors can be ignored.

Example 2. Here Γ is taken to be a closed contour, namely the boundary of the unit square:

$$\Gamma = \{(s_1, 0): 0 \leqq s_1 \leqq 1\} \cup \{(1, s_2): 0 \leqq s_2 \leqq 1\} \cup \{(s_1, 1): 0 \leqq s_1 \leqq 1\} \cup \{(0, s_2): 0 \leqq s_2 \leqq 1\},$$

and the integral equation is taken to be

$$-\int_\Gamma \log|x - y| g(y)\, dy = x_1^2 + x_2^2, \qquad x = (x_1, x_2) \in \Gamma. \tag{4.2}$$

The quantities shown in Tables 2.1 and 2.2 are the Galerkin approximations based on the graded mesh (1.6), with, respectively, $q_i = 2$ and $q_i = 3$, $i = 1, 2, 3, 4$. The calculated quantities are the following:

(a) $\int_\Gamma g(y)\, dy$;

(b) $u(1.2, 1.2)$;

(c) $u(0.6, 0.6)$.

For Table 2.1, to estimate the errors in $\int_\Gamma g^h(y)\, dy$, we use the formula in [12]:

$$\left| \int_\Gamma g^h - \int_\Gamma g \right| = |(g^h, 1) - (g, 1)| \leqq \|g^h - g\| \, \|\lambda_\Gamma^h - \lambda_\Gamma)\|.$$

TABLE 2.1
$q = 2$.

$n = 4/h$	$\int g^h$	γ^h	$u^h(1.2, 1.2)$	γ^h	$u^h(0.6, 0.6)$	γ^h
8	1.53946		0.54697		1.07064	
16	1.61495		0.60398		0.99971	
32	1.62521	2.8784	0.61090	3.0425	0.99543	4.0508
64	1.62688	2.6200	0.61195	2.7132	0.99503	3.4331
128	1.62716	2.5928	0.61212	2.6409	0.99498	3.0871
256	1.62721	2.6052	0.61215	2.6431	0.99498	3.0119

TABLE 2.2
$q=3$.

$n=4/h$	$\int g^h$	γ^h	$u^h(1.2, 1.2)$	γ^h	$u^h(0.6, 0.6)$	γ^h
8	1.53946		0.54697		1.07064	
16	1.61789		0.60413		1.00803	
32	1.62618	3.2412	0.61124	3.0078	0.99610	2.3907
64	1.62709	2.1906	0.61204	3.1470	0.99508	3.5612
128	1.62720	3.0443	0.61214	3.0247	0.99499	3.4180
256	1.62721	3.0005	0.61215	2.9994	0.99498	3.0885

Since all corners of Γ are right-angled, it follows from the discussion in § 1 that $g(x)$ and $\lambda_\Gamma(x)$ are both expected to have dominant singularities of the form $|s-s_i|^{-1/3}$, and therefore (3.2) gives the predicted orders of convergence:

$$\|g^h - g\|^2 = O(h^{7/3} \log h^{-1}),$$
$$\|\lambda_\Gamma^h - \lambda_\Gamma\|^2 = O(h^{7/3} \log h^{-1}).$$

It follows that

(a) $|\int_\Gamma g^h - \int_\Gamma g| = O(h^{7/3} \log h^{-1})$.

As before,

$$|u^h(\tau) - u(\tau)| \leqq \|g^h - g\| \, \|g_\tau^h - g_\tau\| \quad \text{for } \tau \notin \Gamma.$$

The behavior of $g_\tau^h - g_\tau$ will be rather different, as discussed in [12], depending on whether τ is in the exterior or the interior of the square. When τ is in the exterior, the singularities of $g_\tau(x)$ are of the form $|s-s_i|^{-1/3}$ at the corners; thus for (b) we have

$$|u^h(\tau) - u(\tau)| = O(h^{7/3} \log h^{-1}).$$

But when τ is in the interior of the square, the singularities of $g_\tau(x)$ are of the form $|s-s_i|$ at the corners; therefore for (c) we have $\|g_\tau^h - g_\tau\| = O(h^3 \log h^{-1})$, and hence

$$|u^h(\tau) - u(\tau)| = O(h^{8/3} \log h^{-1}).$$

A similar but easier discussion leads to results for Table 2.2:

(a) $|\int_\Gamma g^h - \int_\Gamma g| = O(h^3 \log h^{-1})$;

(b), (c) $|u^h(\tau) - u(\tau)| = O(h^3 \log h^{-1})$.

The numerical results for (a), (b), and (c) in Tables 2.1 and 2.2 are consistent with these predictions, although the results in Table 2.1 suggest that the convergence may be faster than predicted, perhaps being $O(h^{8/3})$ for (a) and (b), and $O(h^3)$ for (c).

Example 3. In this example we take Γ to be the closed contour bounding an L-shaped region

$$\Gamma = \{(s_1, 0): 0 \leqq s_1 \leqq 1\} \cup \{(1, s_2): 0 \leqq s_2 \leqq \tfrac{1}{2}\} \cup \{(s_1, \tfrac{1}{2}): \tfrac{1}{2} \leqq s_1 \leqq 1\}$$
$$\cup \{(\tfrac{1}{2}, s_2): \tfrac{1}{2} \leqq s_2 \leqq 1\} \cup \{(s_1, 1): 0 \leqq s_1 \leqq \tfrac{1}{2}\} \cup \{(0, s_2): 0 \leqq s_2 \leqq 1\}.$$

Thus there is a re-entrant corner at $(\frac{1}{2}, \frac{1}{2})$. The integral equation is again given by (4.2). The quantities shown in Tables 3.1 and 3.2 are the Galerkin approximations based on the graded mesh (1.6) with, respectively, $q_i = 2$ and $q_i = 3$, $i = 1, \cdots, 6$. The calculated quantities are the following:

(a) $\int_\Gamma g(y)\, dy$;

(b) $u(-0.25, -0.25)$;

(c) $u(0.25, 0.25)$.

TABLE 3.1
$q=2$.

$n=4/h$	$\int g^h$	γ^h	$u^h(-0.25,-0.25)$	γ^h	$u^h(0.25,0.25)$	γ^h
16	1.11066		−3.39932		2.60186	
32	1.12950		−3.45705		2.56767	
64	1.13238	2.7057	−3.46613	2.6683	2.56155	2.4830
128	1.13286	2.5864	−3.46765	2.5793	2.56067	2.7949
256	1.13294	2.5976	−3.46790	2.5988	2.56055	2.8360

TABLE 3.2
$q=3$.

$n=4/h$	$\int g^h$	γ^h	$u^h(-0.25,-0.25)$	γ^h	$u^h(0.25,0.25)$	γ^h
16	1.11281		−3.40525		2.61647	
32	1.13036		−3.45874		2.57222	
64	1.13263	2.9523	−3.46673	2.7429	2.56243	2.1757
128	1.13292	2.9676	−3.46779	2.9031	2.56077	2.5630
256	1.13295	2.9750	−3.46793	2.9633	2.56056	2.9513

For Table 3.1, as in the discussion of Example 2, we have
(a) $|\int_\Gamma g^h - \int_\Gamma g| = O(h^{7/3}\log h^{-1})$,
and for τ in the exterior,

$$|u^h(\tau)-u(\tau)| = O(h^{7/3}\log h^{-1}).$$

But for τ in the interior, because of the re-entrant corner at (0.5, 0.5), the dominant singularity of $g_\tau(x)$ is of the form $|s-s_i|^{-1/3}$. Thus

$$\|g_\tau^h - g_\tau\|^2 = O(h^{7/3}\log h^{-1}),$$

so that for both (b) and (c)

$$|u^h(\tau)-u(\tau)| \leqq \|g^h-g\|\,\|g_\tau^h-g_\tau\| = O(h^{7/3}\log h^{-1}).$$

In the same way, for Table 3.2 we have the following:
(a) $|\int_\Gamma g^h - \int_\Gamma g| = O(h^3\log h^{-1})$;
(b), (c) $|u^h(\tau)-u(\tau)| = O(h^3\log h^{-1})$.
The numerical results in Tables 3.1 and 3.2 for (a)-(c) are consistent with these predictions, though again the numerical results for the case $q_i=2$ suggest that the theoretical results can be further improved.

REFERENCES

[1] I. BABUŠKA AND W. C. RHEINBOLDT, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[2] C. A. BREBBIA, *Boundary Element Techniques in Computer-Aided Engineering*, NATO ASI Series, 1984.

[3] H. BRUNNER, *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Math. Comp., 45 (1985), pp. 417–437.

[4] G. A. CHANDLER, *Galerkin's method for boundary integral equations on polygonal domains*, J. Austral. Math. Soc. Ser. B., 26 (1984), pp. 1–13.

[5] ———, *Mesh grading for boundary integral equations*, in Computational Techniques and Applications, J. Noye and C. Fletcher, eds., Elsevier Science Publishers, North-Holland, Amsterdam, New York, 1984, pp. 289–296.

[6] M. Costabel and E. Stephan, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, in Mathematical Models and Methods in Mechanics, Banach Center Publication Volume 15, PWN-Polish Scientific Publications, Warsaw, 1985, pp. 175-251.

[7] I. G. Graham, *The numerical solution of Fredholm integral equations of the second kind*, Ph.D. thesis, University of New South Wales, New South Wales, Australia, 1980.

[8] G. C. Hsiao and W. L. Wendland, *A finite element method for some integral equations of the first kind*, J. Math. Anal. Appl., 58 (1977), pp. 449-481.

[9] M. A. Jaswon and G. T. Symm, *Integral Equation Methods in Potential Theory and Elastostatics*, Academic Press, London, 1977.

[10] J. R. Rice, *On the degree of convergence of nonlinear spline approximation*, in Approximation with Special Emphasis on Spline Functions, I. J. Schoenberg, ed., Academic Press, New York, 1969.

[11] C. Schneider, *Product integration for weakly singular integral equations*, Math. Comp., 36 (1981), pp. 207-213.

[12] I. H. Sloan and A. Spence, *The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory and application*, IMA J. Numer. Anal., 8 (1988), pp. 105-140.

[13] Y. Yan and I. H. Sloan, *On integral equations of the first kind with logarithmic kernel*, J. Integral Equations Appl., 1 (1988), pp. 1-31.