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Recent Progress on Boundary-Value Problems on Lipschitz Domains

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Introduction. In this note we will describe and sketch the proofs of some recent developments on boundary-value problems on Lipschitz domains.

In 1977 B. E. J. Dahlberg was able to show the solvability of the Dirichlet problem for Laplace's equation on a Lipschitz domain D with $L^2(\partial D, d\sigma)$ data and optimal estimates. In fact, he proved that, given a Lipschitz domain D, there exists $\varepsilon = \varepsilon(D)$ such that this can be done for data in $L^p(\partial D, d\sigma)$, $2 - \varepsilon \leqslant p \leqslant \infty$ (see [**6**, **7**] and [**8**]). Also, simple examples show that, given $p < 2$, there exists a Lipschitz domain D where this fails in $L^p(\partial D, d\sigma)$. Dahlberg's method consisted of a careful analysis of the harmonic measure. His techniques relied on positivity, Harnack's inequality, and the maximum principle, and, thus, they were not applicable to the Neumann problem, to systems of equations, or to higher-order equations. In 1978 E. Fabes, M. Jodeit, Jr., and N. Riviere [**15**] were able to utilize A. P. Calderón's theorem [**1**] on the boundedness of the Cauchy integral on C^1 curves to extend the classical method of layer potentials to C^1 domains. They were thus able to resolve the Dirichlet and Neumann problems for Laplace's equation, with $L^p(\partial D, d\sigma)$ data and optimal estimates for C^1 domains. They relied on Fredholm theory, exploiting the compactness of the layer potentials in the C^1 case. In 1979 D. Jerison and C. Kenig [**20**, **21**] were able to give a simplified proof of Dahlberg's results, using an integral identity that goes back to Rellich [**33**]. However, the method still relied on positivity. Shortly afterwards, Jerison and Kenig [**22**] were also able to treat the Neumann problem on Lipschitz domains, with $L^2(\partial D, d\sigma)$ data and optimal estimates. To do so they combined the Rellich-type formulas with Dahlberg's results on the Dirichlet problem. This still relied on positivity and dealt only with the L^2 case, leaving the corresponding L^p theory open.

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In 1981 R. Coifman, A. McIntosh, and Y. Meyer [**3**] established the boundedness of the Cauchy integral on any Lipschitz curve, opening the door to the applicability of the method of layer potentials to Lipschitz domains. This method is very flexible, does not rely on positivity, and does not in principle differentiate between a single equation or a system of equations. The difficulty then becomes the solvability of the integral equations, since, unlike the C^1 case, Fredholm theory is not applicable, because, on a Lipschitz domain, operators like the double-layer potential are not compact.

For the case of the Laplace equation, with $L^2(\partial D, d\sigma)$ data, this difficulty was overcome by G. C. Verchota [**36**] in 1982 in his doctoral dissertation. He made the key observation that the Rellich identities mentioned before are the appropriate substitutes for compactness in the case of Lipschitz domains. Thus, Verchota was able to recover the L^2 results of Dahlberg [**7**] and Jerison and Kenig [**22**] for Laplace's equation on a Lipschitz domain by using the method of layer potentials.

This paper is divided into two sections. The first, which consists of two parts, deals with Laplace's equation on Lipschitz domains: The first part explains the L^2 results of Verchota. The second part deals with a sketch of recent joint work of Dahlberg and Kenig (1984) [**9**]. We were able to show that, given a Lipschitz domain $D \subset \mathbb{R}^n$, there exists $\varepsilon = \varepsilon(D)$ such that one can solve the Neumann problem for Laplace's equation with data in $L^p(\partial D, d\sigma)$, $1 < p \leqslant 2 + \varepsilon$. Easy examples show that this range of p's is optimal. Moreover, we showed that the solution can be obtained by the method of layer potentials, and that Dahlberg's solution of the L^p Dirichlet problem can also be obtained by the method of layer potentials. We also obtained endpoint estimates for the Hardy space $H^1(\partial D, d\sigma)$, which generalize the results for $n = 2$ in [**25**] and [**26**] and for C^1 domains in [**16**]. The key idea in this work is that one can estimate the regularity of the so-called Neumann function for D by using the De Giorgi–Nash regularity theory for elliptic equations with bounded measurable coefficients. This, combined with the use of the so-called 'atoms', yields the desired results.

The second section, which consists of three parts, deals with higher-order problems. In Parts 1 and 2 we treat L^2 boundary-value problems for systems of equations. Part 1 deals with the systems of elastostatics; Part 2, with the Stokes system of hydrostatics. The results in Part 1 are joint work of Dahlberg, Kenig, and Verchota (see [**12**]); the results in Part 2 are joint work of E. Fabes, C. Kenig, and G. Verchota (see [**17**]). The results obtained had not been previously available for general Lipschitz domains, although a lot of work has been devoted to the case of piecewise linear domains (see [**27**, **28**] and their bibliographies). For the case of C^1 domains, our results for the systems of elastostatics had been previously obtained by A. Gutierrez [**19**], using compactness and Fredholm theory. This is, of course, not available for the case of Lipschitz domains. We are able to use once more the method of layer potentials. Invertibility is shown again by means of Rellich-type formulas. This works very well in the Dirichlet problem for the Stokes system (see Part 2), but serious difficulties occur for systems of elastostatics (see Part 1). These difficulties are overcome by proving a Korn-type inequality at the boundary. The proof of this inequality proceeds in three steps. One first establishes it for the case of small Lipschitz constant. One then proves an analogous inequality for nontangential maximal functions on any Lipschitz domain, by using the ideas of G. David [**13**] on increasing the Lipschitz constant. Finally, one can remove the nontangential maximal function, using the results on the Dirichlet problem for the Stokes system, which are established in Part 2. See Parts 1 and 2 for the details. Some partial results in this direction were previously announced in [**26**]. The third part of §2 deals with the Dirichlet problem for the biharmonic equation Δ^2 (a fourth-order elliptic equation) on an arbitrary Lipschitz domain in $\mathbb{R}^n$. This sketches joint work of Dahlberg, Kenig, and Verchota [**11**]. The case of C^1 domains in the plane was previously treated by J. Cohen and J. Gosselin [**2**], using layer potentials and compactness. We are able to reduce the problem, for an arbitrary Lipschitz domain in $\mathbb{R}^n$, to a bilinear estimate for harmonic functions. This is a Lipschitz-domain version of the paraproduct of J. M. Bony. See Part 3 of §2 for further details.

Complete proofs of the results explained in §1, Part 2 and §2 will appear in future publications.

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1. Laplace's equation.

Part 1. *L^2 theory on a Lipschitz domain for Laplace's equation by the method of layer potentials*. A bounded Lipschitz domain $D \subset \mathbb{R}^n$ is one which is locally given by the domain above the graph of a Lipschitz function. Such domains satisfy both the interior and exterior cone conditions. For such a domain D the nontangential region of opening α at a point $Q \in \partial D$ is

$$\Gamma_\alpha(Q) = \{ X \in D \colon |X - Q| < (1 + \alpha)\operatorname{dist}(X, \partial D) \}.$$

All results in this paper are valid when suitably interpreted for all bounded Lipschitz domains in $\mathbb{R}^n$, $n \geqslant 2$, with the nontangential approach regions defined above. For simplicity in this exposition, we restrict ourselves to the case $n \geqslant 3$ (and sometimes even to the case $n = 3$) and to domains $D \subset \mathbb{R}^n$, $D = \{(x, y): y > \varphi(x)\}$, where $\varphi: \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant M; i.e., $|\varphi(x) - \varphi(x')| \leqslant M|x - x'|$, $D^- = \{(x, y): y < \varphi(x)\}$. For fixed $M' < M$,

$$\Gamma_e(x) = \{(x, y): (y - \varphi(x)) < -M'|z - x|\} \subset D^-,$$

and

$$\Gamma_i(x) = \{(z, y): (y - \varphi(x)) > M'|z - x|\} \subset D.$$

Points in D will usually be denoted by X; points on ∂D, by $Q = (x, \varphi(x))$ or simply by x. N_x or N_Q will denote the unit normal to $\partial D = \Lambda$ at $Q = (x, \varphi(x))$. If u is a function defined on $\mathbb{R}^n \setminus \Lambda$, and $Q \in \partial D$, $u^{\pm}(Q)$ will denote $\lim_{X \to Q;\ X \in \Gamma_i(Q)} u(X)$ or $\lim_{X \to Q;\ X \in \Gamma_e(Q)} u(X)$, respectively. If u is a function defined on D, $N(u)(Q) = \sup_{X \in \Gamma_i(Q)} |u(X)|$.

We wish to solve the problems

$$\text{(D)} \quad \begin{cases} \Delta u = 0 & \text{in } D, \\ u|_{\partial D} = f \in L^2(\partial D, d\sigma), \end{cases} \qquad \text{(N)} \quad \begin{cases} \Delta u = 0 & \text{in } D, \\ \partial u/\partial N|_{\partial D} = f \in L^2(\partial D, d\sigma). \end{cases}$$

The results here are

THEOREM 1.1.1. *There exists a unique u solving (D), such that $N(u) \in L^2(\partial D, d\sigma)$, where the boundary values are taken nontangentially a.e. Moreover, u has the form*

$$u(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X - Q, N_Q \rangle}{|Q - X|^n} g(Q)\, d\sigma(Q)$$

for some $g \in L^2(\partial D, d\sigma)$.

THEOREM 1.1.2. *There exists a unique u tending to 0 at ∞, such that $N(\nabla u) \in L^2(\partial D, d\sigma)$, solving (N) in the sense that $N_Q \cdot \nabla u(X) \to f(Q)$ as $X \to Q$ nontangentially a.e. Moreover, u has the form*

$$u(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X - Q|^{n-2}} g(Q)\, d\sigma(Q)$$

for some $g \in L^2(\partial D, d\sigma)$.

In order to prove the above theorems, we introduce

$$\mathscr{K} g(X) = \frac{1}{\omega_n} \int_{\partial D} \frac{\langle X - Q, N_Q \rangle}{|X - Q|^n} g(Q)\, d\sigma(Q)$$

and

$$Sg(X) = \frac{-1}{\omega_n(n-2)} \int_{\partial D} \frac{1}{|X - Q|^{n-2}} g(Q)\, d\sigma(Q).$$

If $Q = (x, \varphi(x))$, $X = (z, y)$, then

$$\mathscr{K}g(z, y) = \frac{1}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{y - \varphi(x) - (z - x) \cdot \nabla\varphi(x)}{\left[|x - z|^2 + [\varphi(x) - \varphi(z)]^2\right]^{n/2}} g(x)\, dx,$$

$$Sg(z, y) = \frac{-1}{\omega_n(n-2)} \int_{\mathbb{R}^{n-1}} \frac{\sqrt{1 + |\nabla\varphi(x)|^2}}{\left[|x - z|^2 + [\varphi(x) - y]^2\right]^{(n-2)/2}} g(x)\, dx.$$

THEOREM 1.1.3. (a) *If $g \in L^p(\partial D, d\sigma)$, $1 < p < \infty$, then $N(\nabla Sg)$, $N(\mathscr{K}g)$ also belong to $L^p(\partial D, d\sigma)$, and their norms are bounded by $C\|g\|_{L^p(\partial D, d\sigma)}$.*

(b) $$\lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int_{|x-z|>\varepsilon} \frac{\varphi(z) - \varphi(x) - (z - x) \cdot \nabla\varphi(x)}{\left[|x - z|^2 + [\varphi(x) - \varphi(z)]^2\right]^{n/2}} g(x)\, dx = Kg(z)$$

exists a.e., and

$$\|Kg\|_{L^p(\partial D, d\sigma)} \leqslant C\|g\|_{L^p(\partial D, d\sigma)}, \qquad 1 < p < \infty;$$

$$\lim_{\varepsilon \to 0} \frac{-1}{\omega_n} \int_{|z-x|>\varepsilon} \frac{(z - x, \varphi(z) - \varphi(x))\sqrt{1 + |\nabla\varphi(x)|^2}}{\left[|2 - x|^2 + [\varphi(z) - \varphi(x)]^2\right]^{n/2}} g(x)\, dx$$

exists a.e. and in $L^p(\partial D, d\sigma)$, and its L^p norm is bounded by $C\|g\|_{L^p(\partial D, d\sigma)}$, $1 < p < \infty$.

(c) $$(\mathscr{K}g)^{\pm}(Q) = \pm\tfrac{1}{2}g(Q) + Kg(Q),$$

$$(\nabla Sg)^{\pm}(z) = \pm\tfrac{1}{2}g(z)N_z$$
$$+ \frac{1}{\omega_n} \lim_{\varepsilon \to 0} \int_{|z-x|>\varepsilon} \frac{(z - x, \varphi(z) - \varphi(x))\sqrt{1 + |\nabla\varphi(x)|^2}}{\left[|z - x|^2 + [\varphi(z) - \varphi(x)]^2\right]^{n/2}} g(x)\, dx.$$

COROLLARY 1.1.4. *$(N_z \nabla Sg)^{\pm}(z) = \pm\tfrac{1}{2}g(z) - K^*g(z)$, where K^* is the $L^2(\partial D, d\sigma)$-adjoint of K.*

The proof of Theorem 1.1.3 is an easy consequence of the deep results of Coifman–McIntosh–Meyer [**3**].

It is easy to see that (at least the existence part of) Theorems 1.1.1 and 1.1.2 will follow immediately if we can show that $\tfrac{1}{2}I + K$ and $\tfrac{1}{2}I + K^*$ are invertible on $L^2(\partial D, d\sigma)$. This is the result of G. Verchota [**36**].

THEOREM 1.1.5. *$\pm\tfrac{1}{2}I + K$, $\pm\tfrac{1}{2}I + K^*$ are invertible on $L^2(\partial D, d\sigma)$.*

In order to prove this theorem, it suffices to show that $\pm\tfrac{1}{2}I + K^*$ are invertible. In order to do so, we show that if $f \in L^2(\partial D, d\sigma)$, then

$$\left\|\left(\tfrac{1}{2}I + K^*\right)f\right\|_{L^2(\partial D, d\sigma)} \approx \left\|\left(\tfrac{1}{2}I - K^*\right)f\right\|_{L^2(\partial D, d\sigma)},$$

where the constants of equivalence depend only on the Lipschitz constant M. Let us take this for granted and show, for example, that $\tfrac{1}{2}I + K^*$ is invertible. To do this, note first that if $T = \frac{1}{2}I + K^*$, then $\|Tf\|_{L^2} \geqslant C\|f\|_{L^2}$, where C depends only on the Lipschitz constant M. For $0 \leqslant t \leqslant 1$ consider the operator $T_t = \frac{1}{2}I + K_t^*$, where K_t^* is the operator corresponding to the domain defined by $t\varphi$. Then $T_0 = \frac{1}{2}I$, $T_1 = T$, and $(\partial/\partial t)T_t: L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^{n-1})$, $1 < p < \infty$, with bound independent of t, by the theorem of Coifman–McIntosh–Meyer. Moreover, for each t, $\|T_t f\|_{L^2} \geqslant C\|f\|_{L^2}$, C independent of t. The invertibility of T now follows from the continuity method.

LEMMA 1.1.6. *Suppose that* T_t: $L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ *satisfy*
(a) $\|T_t f\|_{L^2} \geqslant C_1\|f\|_{L^2}$,
(b) $\|T_t f - T_s f\|_{L^2} \leqslant C_2|t - s|\,\|f\|_{L^2}$, $0 \leqslant t, s \leqslant 1$,
(c) T_0: $L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ *is invertible.*
Then T_1 *is invertible.*

The proof of 1.1.6 is very simple. We are thus reduced to proving

$$\left\|\left(\tfrac{1}{2}I + K^*\right)f\right\|_{L^2(\partial D,\, d\sigma)} \approx \left\|\left(\tfrac{1}{2}I - K^*\right)f\right\|_{L^2(\partial D,\, d\sigma)}. \tag{1.1.7}$$

In order to prove (1.1.7) we use the following formula, which goes back to Rellich [**33**] (see also [**31**, **30**, **22**]).

LEMMA 1.1.8. *Assume that* $u \in \operatorname{Lip}(\overline{D})$, $\Delta u = 0$ *in* D, *and* u *and its derivatives are suitably small at* ∞. *Then if* e_n *is the unit vector in the direction of the* y-*axis*,

$$\int_{\partial D} \langle N_Q, e_n\rangle |\nabla u|^2\, d\sigma = 2\int_{\partial D} \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial N}\, d\sigma.$$

PROOF. Observe that

$$\operatorname{div}\left(e_n|\nabla u|^2\right) = \frac{\partial}{\partial y}|\nabla u|^2 = 2\frac{\partial}{\partial y}\nabla u \cdot \nabla u,$$

and

$$\operatorname{div}\left(\frac{\partial u}{\partial y}\nabla u\right) = \frac{\partial}{\partial y}\nabla u \cdot \nabla u + \frac{\partial u}{\partial y} \cdot \operatorname{div}\nabla u = \frac{\partial}{\partial y}\nabla u \nabla u.$$

Stokes' theorem now gives the lemma.

We now deduce a few consequences of the Rellich identity. Recall that $N_x = (-\nabla\varphi(x), 1)/\sqrt{1 + |\nabla\varphi(x)|^2}$, so that $(1 + M^2)^{-1/2} \leqslant \langle N_x, e_n\rangle \leqslant 1$.

COROLLARY 1.1.9. *Let* u *be as in* 1.1.8, *and let* $T_1(x), T_2(x), \ldots, T_{n-1}(x)$ *be an orthogonal basis for the tangent plane to* ∂D *at* $(x, \varphi(x))$. *Let* $|\nabla_t u(x)|^2 = \sum_{j=1}^{n-1}|\langle \nabla u(x), T_j(x)\rangle|^2$. *Then*

$$\int_{\partial D}\left(\frac{\partial u}{\partial N}\right)^2 d\sigma \leqslant C\int_{\partial D}|\nabla_t u|^2\, d\sigma.$$

PROOF. Let $\alpha = e_n - \langle N_x, e_n\rangle N_x$, so that α is a linear combination of $T_1(x)$, $T_2(x), \ldots, T_{n-1}(x)$. Then

$$\partial u/\partial y = \langle N_x, e_n\rangle(\partial u/\partial N) + \langle \alpha, \nabla u\rangle.$$

Also,

$$|\nabla u|^2 = (\partial u/\partial N)^2 + |\nabla_t u|^2,$$

so

$$\int_{\partial D} \langle N_x, e_n\rangle \left(\frac{\partial u}{\partial N}\right)^2 d\sigma + \int_{\partial D} \langle N_x, e_n\rangle |\nabla_t u|^2 \, d\sigma$$
$$= 2\int_{\partial D} \langle N_x, e_n\rangle \left(\frac{\partial u}{\partial N}\right)^2 + 2\int_{\partial D} \langle \alpha, \nabla u\rangle \left(\frac{\partial u}{\partial N}\right) d\sigma.$$

Hence,

$$\int_{\partial D} \langle N_x, e_n\rangle \left(\frac{\partial u}{\partial N}\right)^2 d\sigma = \int_{\partial D} \langle N_x, e_n\rangle |\nabla_t u|^2 \, d\sigma - 2\int_{\partial D} \langle \alpha, \nabla u\rangle \frac{\partial u}{\partial N} d\sigma.$$

So

$$\int_{\partial D} \left(\frac{\partial u}{\partial N}\right)^2 d\sigma \leqslant C\int_{\partial D} |\nabla_t u|^2 \, d\sigma + C\left(\int_{\partial D} |\nabla_t u|^2 \, d\sigma\right)^{1/2} \left(\int_{\partial D} \left(\frac{\partial u}{\partial N}\right)^2 d\sigma\right)^{1/2},$$

and the corollary follows.

COROLLARY 1.1.10. *Let u be as in* 1.1.8. *Then*

$$\int_{\partial D} |\nabla_t u|^2 \, d\sigma \leqslant C\int_{\partial D} \left(\frac{\partial u}{\partial N}\right)^2 d\sigma.$$

PROOF.

$$\int_{\partial D} |\nabla u|^2 \, d\sigma \leqslant 2\left(\int_{\partial D} |\nabla u|^2 \, d\sigma\right)^{1/2} \left(\int_{\partial D} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma\right)^{1/2},$$

and the corollary follows.

COROLLARY 1.1.11. *Let u be as in* 1.1.8. *Then*

$$\int_{\partial D} |\nabla_t u|^2 \, d\sigma \approx \int_{\partial D} \left|\frac{\partial u}{\partial N}\right|^2 d\sigma.$$

In order to prove (1.1.7) let $u = Sg$. Because of 1.1.3(c), $\nabla_t u$ is continuous across the boundary, and by 1.1.4,

$$(\partial u/\partial N)^{\pm} = \left(\pm \tfrac{1}{2}I - K^*\right)g.$$

We now apply 1.1.11 in D and D^- to obtain (1.1.7). This finishes the proofs of 1.1.1 and 1.1.2.

We now turn our attention to L^2 regularity in the Dirichlet problem.

DEFINITION 1.1.12. $f \in L_1^p(\Lambda)$, $1 < p < \infty$, if $f(x, \varphi(x))$ has a distributional gradient in $L^p(\mathbb{R}^{n-1})$. It is easy to check that if F is any extension to $\mathbb{R}^n$ of f, then $\nabla_x F(x, \varphi(x))$ is well defined and belongs to $L^p(\Lambda)$. We call this $\nabla_t f$. The norm in $L_1^p(\Lambda)$ will be $\|\nabla_t f\|_{L^p(\Lambda)}$.

THEOREM 1.1.13. *The single-layer potential S maps $L^2(\Lambda)$ into $L_1^2(\Lambda)$ boundedly and has a bounded inverse.*

PROOF. The boundedness follows from 1.1.3(a). L^2-Neumann theory and 1.1.11 give

$$\|\nabla_t S(F)\|_{L^2(\Lambda)} \geqslant C\left\|\frac{\partial S(f)}{\partial N}\right\|_{L^2(\Lambda)} \geqslant C\|f\|_{L^2(\Lambda)}.$$

The argument used in the proof of 1.1.5 now proves 1.1.13.

THEOREM 1.1.14. Given $f \in L_1^2(\Lambda)$, there exists a harmonic function u, with $\|N(\nabla u)\|_{L^2(\Lambda)} \leqslant C\|\nabla_t f\|_{L^2(\Lambda)}$, such that $\nabla_t u = \nabla_t f$ (a.e.) nontangentially on Λ. u is unique (modulo constants), and we can chose $u = S(g)$, where $g \in L^2(\Lambda)$.

The existence part of 1.1.14 follows directly from 1.1.13.

Part 2. *The L^p theory for Laplace's equation on a Lipschitz domain.* The main results in this section are the following:

THEOREM 1.2.1. *There exists $\varepsilon = \varepsilon(M) > 0$ such that, given $f \in L^p(\partial D, d\sigma)$, $2 - \varepsilon \leqslant p < \infty$, there exists a unique u harmonic in D, with $N(u) \in L^p(\partial D, d\sigma)$, such that u converges nontangentially a.e. to f. Moreover, the solution u has the form*

$$u(x) = \frac{1}{\omega_n}\int_{\partial D} \frac{\langle X - Q, N_Q\rangle}{|X - Q|^n} g(Q)\, d\sigma(Q)$$

for some $g \in L^p(\partial D, d\sigma)$.

THEOREM 1.2.2. *There exists $\varepsilon = \varepsilon(M) > 0$ such that, given $f \in L^p(\partial D, d\sigma)$, $1 < p \leqslant 2 + \varepsilon$, there exists a unique u harmonic in D, tending to 0 at ∞, with $N(\nabla u) \in L^p(\partial D, d\sigma)$, such that $N_Q \cdot \nabla u(X)$ converges nontangentially a.e. to $f(Q)$. Moreover, u has the form*

$$u(X) = \frac{-1}{\omega_n(n-2)}\int_{\partial D} \frac{1}{|X - Q|^{n-2}} g(Q)\, d\sigma(Q)$$

for some $g \in L^p(\partial D, d\sigma)$.

THEOREM 1.2.3. *There exists $\varepsilon = \varepsilon(M) > 0$ such that, given $f \in L_1^p(\Lambda)$, $1 < p \leqslant 2 + \varepsilon$, there exists a harmonic function u, with*

$$\|N(\nabla u)\|_{L^p(\Lambda)} \leqslant C\|\nabla_t f\|_{L^p(\Lambda)}$$

and $\nabla_t u = \nabla_t f$ (a.e.) nontangentially on Λ. u is unique (modulo constants). Moreover, u has the form

$$u(x) = \frac{-1}{\omega_n(n-2)}\int_{\partial D} \frac{1}{|X - Q|^{n-2}} g(Q)\, d\sigma(Q)$$

for some $g \in L^p(\partial D, d\sigma)$.

The case $p = 2$ of the above theorems was discussed in Part 1. The first part of 1.2.1 (i.e., without the representation formula) is due to Dahlberg (1977) [**7**]. Theorem 1.2.3 was first proved by Verchota (1982) [**36**]. The representation formula in 1.2.1, Theorem 1.2.2, and the proof that we present of 1.2.3 are due to Dahlberg and Kenig (1984) **[9]**. Just as in §1, 1.2.1–1.2.3 follow from

THEOREM 1.2.4. *There exists* $\varepsilon = \varepsilon(M) > 0$ *such that* $\pm \frac{1}{2}I - K^*$ *is invertible in* $L^p(\partial D, d\sigma)$, $1 < p \leqslant 2 + \varepsilon$, $\pm \frac{1}{2}I - K$ *is invertible in* $L^p(\partial D, d\sigma)$, $2 - \varepsilon \leqslant p < \infty$, *and* $S\colon L^p(\partial D, d\sigma) \to L_1^p(\partial D, d\sigma)$, $1 < p \leqslant 2 + \varepsilon$, *is invertible.*

In order to prove Theorem 1.2.4, just as in Part 1, it is enough to show that if $u = Sf$, f nice, then, for $1 < p \leqslant 2 + \varepsilon$,

$$\|\nabla_t u\|_{L^p(\partial D, d\sigma)} \approx \|\partial u/\partial N\|_{L^p(\partial D, d\sigma)}.$$

This will be done by proving the following two theorems:

THEOREM 1.2.5. *Let* $\Delta u = 0$ *in D. Then*

$$\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leqslant C\|\partial u/\partial N\|_{L^p(\partial D, d\sigma)}, \qquad 1 < p \leqslant 2 + \varepsilon.$$

THEOREM 1.2.6. *Let* $\Delta u = 0$ *in D. Then*

$$\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leqslant C\|\nabla_t u\|_{L^p(\partial D, d\sigma)}, \qquad 1 < p \leqslant 2 + \varepsilon.$$

We first turn our attention to the case $1 < p < 2$ of Theorem 1.2.5. In order to do so we introduce some definitions. A surface ball B in Λ is a set of the form $(x, \varphi(x))$, where x belongs to a ball in $\mathbb{R}^{n-1}$.

DEFINITION 1.2.7. An atom a on Λ is a function supported in a surface ball B, with $\|a\|_{L^\infty} \leqslant 1/\sigma(B)$ and $\int_\Lambda a\, d\sigma = 0$. Notice that atoms are, in particular, L^2 functions.

The following interpolation theorem is of importance to us.

THEOREM 1.2.8. *Let T be a linear operator such that* $\|Tf\|_{L^2(\Lambda)} \leqslant C\|f\|_{L^2(\Lambda)}$ *and for all atoms a*, $\|Ta\|_{L^1(\Lambda)} \leqslant C$. *Then, for* $1 < p < 2$, $\|Tf\|_{L^p(\Lambda)} \leqslant C\|f\|_{L^p(\Lambda)}$.

For a proof see **[5]**. Thus in order to establish the case $1 < p < 2$ of 1.2.5, it suffices to show that if $a = \partial u/\partial N$ is an atom, then $\|N(\nabla u)\|_{L^1(\Lambda)} \leqslant C$. By dilation and translation invariance we can assume that $\varphi(0) = 0$ and supp $a \subset B_1 = \{(x, \varphi(x))\colon |x| < 1\}$. Let B^* be a large ball centered at $(0, 0)$ in $\mathbb{R}^n$ that contains $(x, \varphi(x))$, $|x| < 2$. The diameter of B^* depends only on M. Since $\|a\|_{L^p(\Lambda)} \leqslant 1/\sigma(B_1)^{1/2} = C$, by L^2-Neumann theory,

$$\int_{\partial D \cap B^*} N(\nabla u) \leqslant C\int_{\partial D \cap B^*} N(\nabla u)^2\, d\sigma \leqslant C.$$

Thus we need only estimate $\int_{B^* \cap \partial D} (\nabla u)\, d\sigma$. We do so by appealing to the regularity theory for divergence-form elliptic equations. Consider the bi-Lipschitzian mapping $\Phi\colon D \to D^-$ given by $\Phi(x, y) = (x, \varphi(x) - [y - \varphi(x)])$. Define u^* on D^- by $u^* = u \circ \Phi^{-1}$. A simple calculation shows that, in D^-, u^* verifies (in the weak sense) the equation $\operatorname{div}(A(x, y)\nabla u^*) = 0$, where

$$A(x, y) = \frac{1}{J\varphi(X)} \cdot (\Phi')^t(X) \cdot (\Phi')(X),$$

where $X = \Phi^{-1}(x, y)$. It is easy to see that $A \in L^\infty(D_-)$ and $\langle A(x, y)\xi, \xi\rangle \geqslant C|\xi|^2$. Notice also that supp $\partial u/\partial N \subset B_1 \subset B^* \cap \partial D$. Now define

$$B(x, y) = \begin{cases} I & \text{for } (x, y) \in D, \\ A(x, y) & \text{for } (x, y) \in D^-, \end{cases}$$

and

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{for } (x, y) \in D, \\ u^*(x, y) & \text{for } (x, y) \in D^-. \end{cases}$$

Because $\partial u/\partial N = 0$ in $\partial D \setminus B^*$, it is easy to see that $\tilde{u}$ is a (weak) solution in $\mathbb{R}^n \setminus B^*$ of the divergence-form elliptic equation with bounded measurable coefficients, $L\tilde{u} = \operatorname{div} B(x, y)\nabla\tilde{u} = 0$. In order to estimate u (and hence ∇u) at ∞, we use the following theorem of J. Serrin and H. Weinberger [**34**].

THEOREM 1.2.9. *Let $\tilde{u}$ solve $L\tilde{u} = 0$ in $\mathbb{R}^n \setminus B^*$ and suppose $\|\tilde{u}\|_{L^\infty(\mathbb{R}^n \setminus B^*)} < \infty$. Let $g(X)$ solve $Lg = 0$ in $|X| > 1$, with $g(X) \approx |X|^{2-n}$. Then $\tilde{u}(X) = \tilde{u}_\infty + \alpha g(X) + v(X)$, where $Lv = 0$ in $\mathbb{R}^n \setminus B^*$ and $|v(X)| \leqslant C\|\tilde{u}\|_{L^\infty(\mathbb{R}^n \setminus B^*)} \cdot |X|^{2-n-\nu}$, where $\nu > 0$, $C > 0$ depend only on the ellipticity constants of L. Moreover, $\alpha = c\int B(X)\nabla\tilde{u}(X) \cdot \nabla\Psi(X)$, where $\Psi \in C^\infty(\mathbb{R}^n)$, $\Psi = 0$ for X in $2B^*$, and $\Psi \equiv 1$ for large X.*

Let us assume for now that u is bounded, and let us show that if α is as in 1.2.9, then $\alpha = 0$. Pick a Ψ as in 1.2.9. In D, $B(X) = I$, so

$$\int_D B\nabla u \nabla\Psi = \int_D \nabla u \cdot \nabla\Psi = \lim_{\varepsilon \to 0} \int_{D_\delta^\varepsilon} \nabla u \cdot \nabla\Psi,$$

where

$$D_\delta^\varepsilon = \{(x, y) \colon |(x, y)| < \rho,\ y > \varphi(x) + \varepsilon\},$$

and ρ is large. The right side equals

$$\lim_{\varepsilon \to 0} \int_{\partial D_\rho^\varepsilon} \Psi \cdot \frac{\partial u}{\partial N} = \lim_{\varepsilon \to 0} \int_{\partial D_\rho^\varepsilon} [\Psi - 1] \frac{\partial u}{\partial N},$$

since, by the harmonicity of u, $\int_{\partial D_\rho^\varepsilon} \partial u/\partial N = 0$. Let

$$\partial D_{\rho,1}^\varepsilon = \{(x, y) \in \partial D_\rho^\varepsilon \colon y > \varphi(x) + \varepsilon\}$$

and $\partial D_{\rho,2}^\varepsilon = \partial D_\rho^\varepsilon \setminus \partial D_{\rho,1}^\varepsilon$. Then

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\partial D_\rho^\varepsilon} [\Psi - 1] \frac{\partial u}{\partial N} &= \lim_{\varepsilon \to 0} \int_{\partial D_{\rho,1}^\varepsilon} [\Psi - 1] \frac{\partial u}{\partial N} + \lim_{\varepsilon \to 0} \int_{\partial D_{\rho,2}^\varepsilon} [\Psi - 1] \frac{\partial u}{\partial N} \\ &= \int_{\partial D} [\Psi - 1] a = \int_{\partial D} \Psi a - \int_{\partial D} a = \int_{\partial D} \Psi a = 0, \end{aligned}$$

since $\Psi = 0$ on supp a. Moreover, $\int_{D_-} B\nabla\tilde{u}\nabla\Psi = \int_D \nabla u \cdot \nabla\Psi_*$, where $\Psi_* = \Psi \circ \Phi$ by our construction of B. The last term is also 0 by the same argument, so

$\alpha = 0$. We now show that u (and hence $\tilde{u}$) is bounded. We assume for simplicity that $n \geqslant 4$. Since $\|a\|_{L^2(\Lambda)} \leqslant C$, we know that

$$u(X) = C_n \int_{\partial D} \frac{f(Q)}{|X - Q|^{n-2}} \, d\sigma(Q),$$

with $\|f\|_{L^2(\Lambda)} \leqslant C$. Now, for $X \in D_1 = \{(x, y) \colon y > \varphi(x) + 1\}$,

$$\frac{1}{|X - Q|^{n-2}} \leqslant \frac{C}{1 + |Q|^{n-2}} \in L^2(\Lambda),$$

so $u \in L^\infty(D_1)$. Now let B be any ball in $\mathbb{R}^n$ so that $2B \subset \mathbb{R}^n \setminus B^*$, B is of unit size, and such that a fixed fraction of B is contained in D_1. Since $N(\nabla u) \in L^2(\Lambda)$, with norm less than C, $\int_{2B \cap D} |\nabla u|^2 \leqslant C$, and, moreover, on $B \cap D_1$, $|u(x)| \leqslant C$. Therefore, by the Poincaré inequality, $\int_{2B} \tilde{u}^2 \leqslant C$. But, since $\tilde{u}$ solves $L\tilde{u} = 0$,

$$\max_B |\tilde{u}| \leqslant C\left(\int_{2B} |\tilde{u}|^2\right)^{1/2} \leqslant C$$

[**29**]. Therefore, $\tilde{u} \in L^\infty(\mathbb{R}^n \setminus B^*)$, $\|\tilde{u}\|_{L^\infty(\mathbb{R}^n \setminus B^*)} \leqslant C$. Hence, since $\alpha = 0$, $\nabla u = \nabla v$, and $|v(x, y)| \leqslant C/(|x| + |y|)^{n-2+\nu}$, $\nu > 0$. For $R \geqslant R_0 = \operatorname{diam} B^*$, set $b(R) = \int_{A_R} N(\nabla u)^2$, where $A_R = \{(x, \varphi(x)) \colon R < |x| < 2R\}$. For each fixed R let

$$N_1(\nabla u)(x) = \sup\{|\nabla u(z, y)| \colon (z, y) \in \Gamma_i(x), \operatorname{dist}((z, y), \partial D) \leqslant \delta R\},$$

$$N_2(\nabla u)(x) = \sup\{|\nabla u(z, y)| \colon (z, y) \in \Gamma_i(x), \operatorname{dist}((z, y), \partial D) \geqslant \delta R\}.$$

In the set where the sup in N_2 is taken, u is harmonic, and the distance of any point X to the boundary is comparable to $|X|$. Thus, using our bound on v, we see that $N_2(\nabla u)(x) \leqslant C/|X|^{n-1+\nu} \approx C/R^{n-1+\nu}$, so $\int_{A_R} N_2(\nabla u)^2 \leqslant CR^{1-n-2\nu}$. Now let

$$\Omega_\tau = \{(x, y) \colon \varphi(x) < y < \varphi(x) + CR, \tau R < |X| < \tau^{-1} R\}, \qquad \tau \in \left(\tfrac{1}{4}, \tfrac{1}{2}\right).$$

By L^2-Neumann theory in Ω_τ, $\int_{A_R} N_1(\nabla u)^2 \, d\sigma \leqslant C \int_{\partial\Omega_\tau} |\nabla u|^2 \, d\sigma$. Integrating in τ from $\frac{1}{4}$ to $\frac{1}{2}$ gives

$$\int_{A_R} N_1(\nabla u)^2 \, d\sigma \leqslant \frac{C}{R} \int_{\Omega_{1/4} \setminus \Omega_{1/2}} |\nabla u|^2 \, dX \leqslant \frac{C}{R^3} \int_{C_1 R < |X| < C_2 R} u^2,$$

since $L\tilde{u} = 0$ (see [**29**] for example). The right side is bounded by $(C/R^3)(1/R^{2(n-2)-2\nu})$. Then

$$\int_{A_R} N(\nabla u) \leqslant C\left(\int_{A_R} N(\nabla u)^2\right)^{1/2} R^{(n-1)/2} \leqslant CR^{-\nu}.$$

Choosing $R = 2^j$ and adding in j, we obtain the desired estimate.

We now turn to the case $1 < p < 2$ of 1.2.6. We need a further definition.

Definition 1.2.10. A function a is an H_1^1 atom if $A = \nabla_t a$ satisfies (a) $\operatorname{supp} A \subset B$, a surface ball, (b) $\|A\|_{L^\infty} \leqslant 1/\sigma(B)$, (c) $\int A \, d\sigma = 0$.

We use the following interpolation result:

Theorem 1.2.11. *Let T be a linear operator such that $\|Tf\|_{L^2(\Lambda)} \leqslant C\|f\|_{L_1^2(\Lambda)}$ and $\|Ta\|_{L^1(\Lambda)} < C$ for all H_1^1 atoms a. Then for $1 < p < 2$,*

$$\|Tf\|_{L^p(\Lambda)} \leqslant C\|f\|_{L_1^p(\Lambda)}.$$

Hence, all we need to show is that if $\Delta u = 0$, $\nabla_t u = \nabla_t a$, and a is a unit size H_1^1 atom, $N(\nabla u) \in L^1(\Lambda)$. But note that if we let

$$\tilde{u}(x, y) = \begin{cases} u(x, y), & (x, y) \in D, \\ -u^*(x, y), & (x, y) \in D_-, \end{cases}$$

then $\tilde{u}$ is a weak solution of $L\tilde{u} = 0$ in $\mathbb{R}^n \setminus B^*$ since $u|_{\partial D \setminus B^*} = 0$. Then $\tilde{u} = \tilde{u}_\infty + \alpha g + v$, but $\alpha = 0$ since $\tilde{u} - \tilde{u}_\infty$ must change sign at ∞. The argument is then identical to the one given before.

Before we pass to the case $2 < p < 2 + \varepsilon$, we would like to point out that, using the techniques described above, one can develop the Stein–Weiss Hardy-space theory on an arbitrary Lipschitz domain in $\mathbb{R}^n$. This generalizes the results for $n = 2$ obtained in **[24]** and **[25]** and the results for C^1 domains in **[16]**.

Some of the results one can obtain are the following: Let

$$H^1_{\mathrm{at}}(\partial D) = \left\{ \sum \lambda_i a_i \colon \sum |\lambda_i| < +\infty,\ a_i \text{ is an atom} \right\},$$

$$H^1_{1,\mathrm{at}}(\partial D) = \left\{ \sum \lambda_i a_i \colon \sum |\lambda_i| < +\infty,\ a_i \text{ is an } H_1^1 \text{ atom} \right\}.$$

THEOREM 1.2.12. (a) *Given* $f \in H^1_{\mathrm{at}}(\partial D)$, *there exists a unique harmonic function* u, *which tends to* 0 *at* ∞, *such that* $N(\nabla u) \in L^1(\partial D)$ *and* $N_Q \nabla u(X) \to f(Q)$ *nontangentially a.e. Moreover*, $u(X) = S(g)(X)$, $g \in H^1_{\mathrm{at}}$. *Also*, $u|_{\partial D} \in H^1_{1,\mathrm{at}}(\partial D)$. (b) *Given* $f \in H^1_{1,\mathrm{at}}$, *there exists a unique* (*modulo constants*) *harmonic function* u *such that* $N(\nabla u) \in L^1(\partial D)$ *and* $\nabla_t u|_{\partial D} = \nabla_t f$ *a.e. Moreover*, $u = S(g)$, $g \in H^1_{\mathrm{at}}$, *and* $\partial u/\partial N \in H^1_{\mathrm{at}}(\partial D)$. (c) *If* u *is harmonic, and* $N(\nabla u) \in L^1(\partial D)$, *then* $\partial u/\partial N \in H^1_{\mathrm{at}}(\partial D)$, $u|_{\partial D} \in H^1_{1,\mathrm{at}}(\partial D)$. (d) $f \in H^1_{\mathrm{at}}(\partial D)$ *if and only if* $N(\nabla Sf) \in L^1(\partial D)$ *if and only if* $(\frac{1}{2}I - K^*)f \in H^1_{\mathrm{at}}(\partial D)$.

We turn now to L^p theory, $2 < p < 2 + \varepsilon$. In this case the results are obtained as automatic real-variable consequences of the fact that the L^2 results hold for all Lipschitz domains. We now show that $\|N(\nabla u)\|_{L^p(\Lambda)} \leqslant C\|\partial u/\partial N\|_{L^2(\Lambda)}$ for $2 < p < 2 + \varepsilon$.

The geometry will be clearer if we do it in $\mathbb{R}^n_+$ and transfer it to D by the bi-Lipschitzian mapping

$$\Phi \colon \mathbb{R}^n_+ \to D, \qquad \Phi(x, y) = (x, y + \varphi(x)).$$

We systematically ignore the distinction between sets in $\mathbb{R}^n_+$ and their images under Φ.

Let

$$\gamma = \{(x, y) \in \mathbb{R}^n_+ \colon |x| < y\}, \qquad \gamma^* = \{(x, y) \in \mathbb{R}^n_+ \colon \alpha|x| < y\},$$

where α is a small constant to be chosen. Let

$$m(x) = \sup_{(z, y) \in x + \gamma} |\nabla u(z, y)|, \qquad m^*(x) = \sup_{(z, y) \in x + \gamma^*} |\nabla u(z, y)|.$$

Our aim is to show that there is a small $\varepsilon_0 > 0$ such that

$$\int m^{2+\varepsilon}\, dx \leqslant C \int |f|^{2+\varepsilon}\, dx$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, where $f = \partial u/\partial N$. Let $h = M(f^2)^{1/2}$, where M denotes the Hardy–Littlewood maximal operator. Let

$$E_\lambda = \{x \in \mathbb{R}^{n-1}: m^*(x) > \lambda\}.$$

We claim that

$$\int_{\{m^*>\lambda;\, h\leqslant\lambda\}} m^2 \leqslant C\lambda^2|E_\lambda| + C\alpha\int_{\{m^*>\lambda\}} m^2.$$

Let us assume the claim and prove the desired estimate. First note that

$$\int_{E_\lambda} m^2 \leqslant \int_{\{m^*>\lambda;\, h\leqslant\lambda\}} m^2 + \int_{\{h>\lambda\}} m^2 \leqslant C\lambda^2|E_\lambda|$$

$$+ C\alpha\int_{\{m^*>\lambda\}} m^2 + \int_{\{h>\lambda\}} m^2,$$

by the claim. Now choose and fix α so that $C\cdot\alpha < \frac{1}{2}$. Then

$$\int_{E_\lambda} m^2 \leqslant C\lambda^2|E_\lambda| + C\int_{\{h\geqslant\lambda\}} m^2.$$

For $\varepsilon > 0$

$$\int m^{2+\varepsilon} = \varepsilon\int_0^\infty \lambda^{\varepsilon-1}\int_{\{m>\lambda\}} m^2\,d\lambda \leqslant \varepsilon\int_0^\infty \lambda^{\varepsilon-1}\int_{E_\lambda} m^2\,d\lambda$$

$$\leqslant C\varepsilon\int_0^\infty \lambda^{1+\varepsilon}|\{m^*>\lambda\}|\,d\lambda + C\varepsilon\int_0^\infty \lambda^{\varepsilon-1}\left(\int_{h>\lambda} m^2\right)d\lambda.$$

By a well-known inequality (see **[18]** for example), $|E_\lambda| \leqslant C_\alpha|\{m>\lambda\}|$. Thus

$$\int m^{2+\varepsilon} \leqslant C\varepsilon\int_0^\infty \lambda^{1+\varepsilon}|\{m>\lambda\}|\,d\lambda + C\varepsilon\int_0^\infty \lambda^{\varepsilon-1}\left(\int_{h>\lambda} m^2\right)d\lambda$$

$$\leqslant C\varepsilon\int m^{2+\varepsilon} + c\int m^2h^\varepsilon.$$

If we now choose ε_0 so that $C\varepsilon_0 < 1/2$, for $\varepsilon < \varepsilon_0$, $\int m^{2+\varepsilon} \leqslant C\int m^2h^\varepsilon$. If we now use Hölder's inequality with exponents $(2+\varepsilon)/2$ and $(2+\varepsilon)/\varepsilon$, we see that

$$\int m^{2+\varepsilon} \leqslant C\left(\int m^{2+\varepsilon}\right)^{2/(2+\varepsilon)}\left(\int M(f^2)^{(2+\varepsilon)/2}\right)^{\varepsilon/(2+\varepsilon)},$$

and the desired inequality follows from the Hardy–Littlewood maximal theorem.

It remains to establish the claim. Let $\{Q_k\}$ be a Whitney decomposition of the set $E_\lambda = \{m^* > \lambda\}$ such that $3Q_k \subset E_\lambda$ and $\{3Q_k\}$ has bounded overlap. Fix k; we can assume there exists $x \in Q_k$ such that $h(x) \leqslant \lambda$, and, hence $\int_{2Q_k} f^2 \leqslant C\lambda^2|Q_k|$. For $1 \leqslant \tau \leqslant 2$, let $Q_{k,\tau} = \tau Q_k$ and

$$\tilde{Q}_{k,\tau} = \{(x,y): x \in \tau Q_k, 0 < y < \tau\,\text{length}(Q_k)\}.$$

$Q_{k,\tau}$ (and $\Phi(\tilde{Q}_{k,\tau})$) is a Lipschitz domain uniformly in k, τ. Also, by construction of Q_k, there exists x_k with $\text{dist}(x_k, Q_k) \approx \text{length}(Q_k)$ such that $m^*(x_k) \leqslant \lambda$. Let

$$A_{k,\tau} = \partial\tilde{Q}_{k,\tau} \cap x_k + \gamma^*, \qquad B_{k,\tau} = \partial\tilde{Q}_{k,\tau} \cap \mathbb{R}^n_+ \setminus A_{k,\tau},$$

so that

$$\partial \tilde{Q}_{k,\tau} = Q_{k,\tau} \cup A_{k,\tau} \cup B_{k,\tau}.$$

Note that the height of $B_{k,\tau}$ is dominated by $C\alpha\,\text{length}(Q_k)$, and that $|\nabla u| \leqslant \lambda$ on $A_{k,\tau}$. Let m_1 be the maximal function of ∇u corresponding to the domain $\tilde{Q}_{k,\tau}$ (i.e., where the cones are truncated at height $\approx l(Q_k)$). Then for $x \in Q_k$, $m(x) \leqslant m_1(x) + \lambda$. Also,

$$\begin{aligned}\int_{Q_k} m_1^2 \leqslant \int_{\partial \tilde{Q}_{k,\tau}} m_1^2 \leqslant &\ \left(\text{using } L^2\text{-theory on } \tilde{Q}_{k,\tau}\right) \\ &\leqslant C\int_{B_{k,\tau}} |\nabla u|^2\,d\sigma + c\int_{A_{k,\tau}} |\nabla u|^2\,d\sigma + c\int_{2Q_k} f^2 \\ &\leqslant C\int_{B_{k,\tau}} |\nabla u|^2\,d\sigma + C\lambda^2|Q_k|.\end{aligned}$$

Integrating in τ between 1 and 2 we see that

$$\int_{Q_k} m_1^2 \leqslant \frac{C}{l(Q_k)} \int_0^{\alpha l(Q_k)} \int_{2Q_k} |\nabla u|^2 + C\lambda^2|Q_k| \leqslant C\alpha \int_{2Q_k} m^2 + C\lambda^2|Q_k|.$$

Thus

$$\int_{Q_k} m^2 \leqslant C\alpha \int_{2Q_k} m^2 + C\lambda^2|Q_k|.$$

Adding in k we see that

$$\int_{\{m^* > \lambda,\, h \leqslant \lambda\}} m^2 \leqslant C\lambda^2|E_\lambda| + C\alpha \int_{\{m^* > \lambda\}} m^2,$$

which is the claim. Note also that the same argument gives the estimate $\|N(\nabla u)\|_p \leqslant C\|\nabla_t u\|_p$, $2 < p < 2 + \varepsilon$, and the L^p theory is thus completed.

2. Higher-order boundary-value problems.

Part 1. *Systems of elastostatics*. We sketch the extension of the L^2 results for the Laplace equation to the systems of linear elastostatics on Lipschitz domains. These results are joint work of Dahlberg, Kenig, and Verchota and will be discussed in detail in a forthcoming paper **[12]**. Here we describe some of the main ideas in that work. For simplicity, we restrict our attention to domains D above the graph of a Lipschitz function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$.

Let $\lambda, \mu \geqslant 0$ be constants (Lamé moduli). We seek to solve the following boundary-value problems, where $\mathbf{u} = (u^1, u^2, u^3)$

$$\text{(2.1.1)} \qquad \begin{gathered}\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} = 0 \quad \text{in } D, \\ \mathbf{u}|_{\partial D} = \mathbf{f} \in L^2(\partial D, d\sigma);\end{gathered}$$

$$\text{(2.1.2)} \qquad \begin{gathered}\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} = 0 \quad \text{in } D, \\ \lambda(\operatorname{div}\mathbf{u})N + \mu\left\{\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right\}N\Big|_{\partial D} = \mathbf{f} \in L^2(\partial D, d\sigma).\end{gathered}$$

(2.1.1) corresponds to knowing the displacement vector $\mathbf{u}$ on the boundary of D, and (2.1.2) corresponds to knowing the surface stresses on the boundary of D. We seek to solve (2.1.1) and (2.1.2) by the method of layer potentials. In order to do so we introduce the Kelvin matrix of fundamental solutions (see **[27]** for example),

$$\Gamma(X) = \left(\Gamma_{ij}(X)\right),$$

where

$$\Gamma_{ij}(X) = \frac{A}{4\pi}\frac{\delta_{ij}}{|X|} + \frac{C}{4\pi}\frac{X_i X_j}{|X|^3},$$

and

$$A = \frac{1}{2}\left[\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right], \qquad C = \frac{1}{2}\left[\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right].$$

We also introduce the stress operator T, where

$$T\mathbf{u} = \lambda(\operatorname{div}\mathbf{u})N + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}N.$$

The double-layer potential of a density $\mathbf{g}(Q)$ is then given by

$$\mathbf{u}(X) = \mathscr{K}\mathbf{g}(X) = \int_{\partial D}\{T(Q)\Gamma(X-Q)\}^t\mathbf{g}(Q)\,d\sigma(Q),$$

where the operator T is applied to each column of the matrix Γ.

The single-layer potential of a density $\mathbf{g}(Q)$ is

$$\mathbf{u}(X) = S\mathbf{g}(X) = \int_{\partial D}\Gamma(X-Q)\cdot\mathbf{g}(Q)\,d\sigma(Q).$$

Our main results here parallel those of §1, Part 1. They are

THEOREM 2.1.3. (a) *There exists a unique solution of problem* (2.1.1) *in D with $N(\mathbf{u}) \in L^2(\partial D, d\sigma)$. Moreover, the solution u has the form* $\mathbf{u}(x) = \mathscr{K}\mathbf{g}(x)$, $\mathbf{g} \in L^2(\partial D, d\sigma)$.

(b) *There exists a unique solution of* (2.1.2) *in D that is* 0 *at infinity, with $N(\nabla\mathbf{u}) \in L^2(\partial D, d\sigma)$. Moreover, the solution* $\mathbf{u}$ *has the form* $\mathbf{u}(X) = S\mathbf{g}(X)$, $\mathbf{g} \in L^2(\partial D, d\sigma)$.

(c) If the data $\mathbf{f}$ in (2.1.1) belongs to $L_1^2(\partial D, d\sigma)$, we can solve (2.1.1) with $N(\nabla\mathbf{u}) \in L^2(\partial D, d\sigma)$.

The proof of Theorem 2.1.3 starts out following the pattern we used to prove 1.1.1, 1.1.2, and 1.1.14. We first show, as in Theorem 1.1.3, that the following lemma holds:

LEMMA 2.1.4. *Let $\mathscr{K}\mathbf{g}$, $S\mathbf{g}$ be defined as above, so that they both solve $\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = 0$ in $\mathbb{R}^3\setminus\partial D$. Then*

(a)
$$\|N(\mathscr{K}\mathbf{g})\|_{L^p(\partial D, d\sigma)} \leqslant C\|\mathbf{g}\|_{L^p(\partial D, d\sigma)},$$
$$\|N(\nabla S\mathbf{g})\|_{L^p(\partial D, d\sigma)} \leqslant C\|\mathbf{g}\|_{L^p(\partial D, d\sigma)} \quad \textit{for } 1 < p < \infty.$$

(b) $(\mathscr{K}\mathbf{g})^{\pm}(P) = \pm\mathbf{g}(P) + K\mathbf{g}(P)$,

$$\left(\frac{\partial}{\partial X_i}(S\mathbf{g})_j\right)^{\pm}(P) = \pm\left\{\frac{A+C}{2}n_i(P)g_j(P) - n_i(P)\cdot n_j(P)\langle N_P, g(P)\rangle\right\} + \left(\text{p.v.}\int_{\partial D}\frac{\partial}{\partial P_i}\Gamma(P-Q)\mathbf{g}(Q)\,d\sigma(Q)\right)_j,$$

where

$$K\mathbf{g}(P) = \text{p.v.}\int_{\partial D}\{T(Q)\Gamma(P-Q)\}^t\mathbf{g}(Q)\,d\sigma(Q),$$

and A, C are the constants in the definition of the fundamental solution.

Thus, just as in §1, Part 1 reduces to proving the invertibility of $\pm\frac{1}{2}I + K$, $\pm\frac{1}{2}I + K^*$ on $L^2(\partial D, d\sigma)$ and the invertibility of S from $L^2(\partial D, d\sigma)$ onto $L_1^2(\partial D, d\sigma)$. As before, using the jump relations, it suffices to show that if $\mathbf{u}(X) = S\mathbf{g}(X)$, then

$$\|T\mathbf{u}\|_{L^2(\partial D,\, d\sigma)} \approx \|\nabla_t\mathbf{u}\|_{L^2(\partial D,\, d\sigma)}.$$

Before explaining the difficulties in doing so, it is very useful to explain the stress operator T (and thus the boundary-value problem (2.1.2)), from the point of view of the theory of constant-coefficient, second-order, elliptic systems. We go back to working on $\mathbb{R}^n$ and use the summation convention.

Let a_{ij}^{rs}, $1 \leqslant r,\ s \leqslant m$, $1 \leqslant i,\ j \leqslant n$, be constants satisfying the ellipticity condition

$$a_{ij}^{rs}\xi_i\xi_j\eta^r\eta^s \geqslant C|\xi|^2|\eta|^2$$

and the symmetry condition $a_{ij}^{rs} = a_{ji}^{sr}$. Consider vector-valued functions $\mathbf{u} = (u^1, \ldots, u^m)$ on $\mathbb{R}^n$ satisfying the divergence-form system

$$\frac{\partial}{\partial X_i}a_{ij}^{rs}\frac{\partial}{\partial X_j}u^s = 0 \quad \text{in } D.$$

From variational considerations the most natural boundary conditions are Dirichlet conditions ($\mathbf{u}|_{\partial D} = \mathbf{f}$) or Neumann-type conditions, $\partial\mathbf{u}/\partial\nu = n_i a_{ij}^{rs}(\partial u^s/\partial X_j) = f_r$. The interpretation of problem (2.1.2) in this context is that we can find constants a_{ij}^{rs}, $1 \leqslant i, j \leqslant 3$, $1 \leqslant r,\ s \leqslant 3$, satisfying the ellipticity and symmetry conditions such that $\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = 0$ in D if and only if $(\partial/\partial X_i)a_{ij}^{rs}(\partial u^s/\partial X_j) = 0$ in D, and with $T\mathbf{u} = \partial\mathbf{u}/\partial\nu$. In order to obtain the equivalence between the tangential derivatives and the stress operator, we need an identity of Rellich type. Such identities are available for general, constant-coefficient systems (see **[32, 30]**).

LEMMA 2.1.5 (THE RELLICH, PAYNE–WEINBERGER, NEČAS IDENTITIES). *Suppose that $(\partial/\partial X_i)a_{ij}^{rs}(\partial/\partial X_j)u^s = 0$ in D, $a_{ij}^{rs} = a_{ji}^{sr}$, $\mathbf{h}$ is a constant vector in $\mathbb{R}^n$, and $\mathbf{u}$ and its derivatives are suitably small at ∞. Then*

$$\int_{\partial D} h_l n_l a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j}\,d\sigma = 2\int_{\partial D} h_i\frac{\partial u^r}{\partial X_i}n_l a_{lj}^{rs}\frac{\partial u^s}{\partial X_j}\,d\sigma.$$

PROOF. Apply the divergence theorem to the formula

$$\frac{\partial}{\partial X_l}\left[\left(h_l a_{ij}^{rs} - h_i a_{lj}^{rs} - h_j a_{il}^{rs}\right)\frac{\partial u^r}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j}\right] = 0.$$

REMARK 1. Note that if we are dealing with the case $m = 1$, $a_{ij} = I$, and we choose $\mathbf{h} = e_n$, we recover the identity we previously used for Laplace's equation.

REMARK 2. Note that if we had the stronger ellipticity assumption that $a_{ij}^{rs}\xi_i^r\xi_j^s \geqslant C\sum_{l,t}|\xi_l^t|^2$, we would have, if $\partial D = \{(x, \varphi(x))\colon \varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}, \|\nabla\varphi\|_\infty \leqslant M\}$, that $\|\nabla_t u\|_{L^2(\partial D, d\sigma)} \approx \|\partial u/\partial\nu\|_{L^2(\partial D, d\sigma)}$. In fact, if we take $\mathbf{h} = e_n$, then

$$\sum_r \int_{\partial D} |\nabla u^r|^2\, d\sigma \leqslant C\int_{\partial D} h_l n_l a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j}\, d\sigma = 2C\int_{\partial D} h_i\frac{\partial u^r}{\partial X_i}\cdot n_l a_{lj}^{rs}\frac{\partial u^s}{\partial X_j} d\sigma$$

$$\leqslant 2C\left(\sum_r\int_{\partial D}|\nabla u^r|^2\, d\sigma\right)^{1/2}\left(\int_{\partial D}\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma\right)^{1/2}.$$

Thus,

$$\sum_r\int_{\partial D}|\nabla u^r|^2\, d\sigma \leqslant C\int_{\partial D}\left|\frac{\partial u}{\partial\nu}\right|^2\, d\sigma.$$

For the opposite inequality observe that, for each r, s, j fixed, the vector $h_i n_l a_{lj}^{rs} - h_l n_l a_{ij}^{rs}$ is perpendicular to N. Because of Lemma 2.1.5,

$$\int_{\partial D} h_l n_l a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j}\, d\sigma = 2\int_{\partial D}\left(h_l n_l a_{ij}^{rs} - h_i n_l a_{lj}^{rs}\right)\frac{\partial u^l}{\partial X_i}\cdot\frac{\partial u^s}{\partial X_j}\, d\sigma.$$

Hence

$$\int_{\partial D}|\nabla u|^2\, d\sigma \leqslant C\left(\int_{\partial D}|\nabla_t u|^2\, d\sigma\right)^{1/2}\left(\int_{\partial D}|\nabla u|^2\, d\sigma\right)^{1/2},$$

so

$$\int_{\partial D}\left|\frac{\partial u}{\partial\nu}\right|^2\, d\sigma \leqslant C\int_{\partial D}|\nabla u|^2\, d\sigma \leqslant C\int_{\partial D}|\nabla_t u|^2.$$

REMARK 3. In the case in which we are interested, i.e., the case of systems of elastostatics

$$a_{ij}^{rs}\frac{\partial u^s}{\partial X_i}\cdot\frac{\partial u^r}{\partial X_j} = \lambda(\operatorname{div}\mathbf{u})^2 + \frac{\mu}{2}\sum_{i,j}\left(\frac{\partial u^j}{\partial X_i} + \frac{\partial u^i}{\partial X_j}\right)^2,$$

which clearly does not satisfy

$$a_{ij}^{rs}\xi_i^r\xi_j^s \geqslant C\sum_{l,t}|\xi_l^t|^2,$$

since the quadratic form involves only the symmetric part of the matrix (ξ_i^r). In this case, of course,

$$\frac{\partial\mathbf{u}}{\partial\nu} = T\mathbf{u} = \lambda(\operatorname{div}\mathbf{u})N + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}N.$$

REMARK 4. The inequality

$$\|\nabla \mathbf{u}\|_{L^2(\partial D,\, d\sigma)} \leqslant C\|\nabla_t \mathbf{u}\|_{L^2(\partial D,\, d\sigma)}$$

holds in the general case, directly from Lemma 2.1.5, by a more complicated algebraic argument. In fact, as in Remark 2,

$$\int_{\partial D} h_l n_l a_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j}\, d\sigma = 2\int_{\partial D} \left(h_l n_l a_{ij}^{rs} - h_i n_l a_{lj}^{rs}\right) \frac{\partial u^l}{\partial X_i} \cdot \frac{\partial u^s}{\partial X_j}\, d\sigma,$$

and, for fixed r, s, j, $(h_l n_l a_{ij}^{rs} - h_i n_l a_{lj}^{rs})$ is a tangential vector.

Thus,

$$\int_{\partial D} h_l n_l a_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j}\, d\sigma \leqslant C\left(\int_{\partial D} |\nabla_t \mathbf{u}|^2\, d\sigma\right)^{1/2} \left(\int_{\partial D} |\nabla \mathbf{u}|^2\, d\sigma\right)^{1/2}.$$

Consider now the matrix $d_{rs} = (a_{ij}^{rs} n_i n_j)^{-1}$. This is a strictly positive matrix, since $a_{ij}^{rs}\xi_i\xi_j\eta^r\eta^s \geqslant C|\xi|^2|\eta|^2$. Moreover,

$$\begin{aligned}
d_{rs}\left(\frac{\partial u}{\partial \nu}\right)_r\left(\frac{\partial u}{\partial \nu}\right)_s - a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} &= d_{rs} n_i a_{ij}^{rt}\frac{\partial u^t}{\partial X_j}\cdot n_l a_{lk}^{sm}\frac{\partial u^m}{\partial X_k} - a_{ij}^{rs}\frac{\partial u^r}{\partial X_i}\frac{\partial u^s}{\partial X_j} \\
&= d_{rs} n_k a_{kl}^{rt}\frac{\partial u^t}{\partial X_l}\cdot n_m a_{mv}^{s\tau}\frac{\partial u^\tau}{\partial X_v} - a_{vl}^{t\tau}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_l} \\
&= d_{rs} n_k a_{kv}^{rt}\frac{\partial u^t}{\partial X_v}\cdot n_m a_{ml}^{s\tau}\frac{\partial u^\tau}{\partial X_l} - a_{vl}^{t\tau}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_l} \\
&= \left\{d_{rs} n_k a_{kv}^{rt} n_m a_{ml}^{s\tau} - a_{vl}^{t\tau}\right\}\frac{\partial u^t}{\partial X_v}\frac{\partial u^\tau}{\partial X_l}.
\end{aligned}$$

Now, note that for t, τ, l fixed, $\{d_{rs} n_k a_{kv}^{rt} n_m a_{ml}^{s\tau} - a_{vl}^{t\tau}\}$ is perpendicular to N by our definition of d_{rs} and the symmetry of a_{ij}^{rs}:

$$\begin{aligned}
d_{rs} n_k a_{kv}^{rt} n_m a_{ml}^{st} n_v - a_{vl}^{t\tau} n_v &= a_{kv}^{rt} n_k n_v d_{rs} a_{ml}^{st} n_m - a_{ml}^{t\tau} n_m \\
&= a_{vk}^{tr} n_v n_k d_{rs} a_{ml}^{st} n_m - a_{ml}^{t\tau} n_m = \delta_{ts} a_{ml}^{st} n_m - a_{ml}^{t\tau} n_m \\
&= a_{ml}^{t\tau} n_m - a_{ml}^{t\tau} n_m = 0.
\end{aligned}$$

Therefore,

$$\int_{\partial D} h_l n_l d_{rs}\left(\frac{\partial \mathbf{u}}{\partial \nu}\right)_r\left(\frac{\partial \mathbf{u}}{\partial \nu}\right)_s \leqslant C\left(\int_{\partial D} |\nabla_t \mathbf{u}|^2\, d\sigma\right)^{1/2}\left(\int_{\partial D} |\nabla \mathbf{u}|^2\, d\sigma\right)^{1/2}.$$

Now,

$$\begin{aligned}
\left(\frac{\partial \mathbf{u}}{\partial \nu}\right)_r - a_{kj}^{rs} n_k n_j \frac{\partial u^s}{\partial N} &= n_i a_{ij}^{rs}\frac{\partial u^s}{\partial X_j} - a_{kj}^{rs} n_k n_j n_i \frac{\partial u^s}{\partial X_i} \\
&= n_i a_{ij}^{rs}\frac{\partial u^s}{\partial X_j} - a_{ki}^{rs} n_k n_j n_i \frac{\partial u^s}{\partial X_j} = \left\{n_i a_{ij}^{rs} - a_{ki}^{rs} n_k n_i n_j\right\}\frac{\partial u^s}{\partial X_j} \\
&= \left\{n_i a_{ij}^{rs} - a_{ik}^{rs} n_k n_i n_j\right\}\frac{\partial u^s}{\partial X_j}.
\end{aligned}$$

But, for i, r, s fixed, $a_{ij}^{rs} - a_{ik}^{rs} n_k n_j$ is perpendicular to N, so

$$\int_{\partial D} h_l n_l d_{rs} \left\{ a_{kj}^{rt} n_k n_j \frac{\partial u^t}{\partial N} \right\} \left\{ a_{il}^{s\tau} n_i n_l \frac{\partial u^\tau}{\partial N} \right\} d\sigma$$

$$\leqslant C \left\{ \left(\int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma \right)^{1/2} \left(\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \right)^{1/2} + \int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma \right\}.$$

We now choose $\mathbf{h} = e_n$, so that $h_l n_l \geqslant C$, and recall that (d_{rs}) and $(a_{kj}^{rt} n_k n_j)$ are strictly positive-definite matrices. We then see that

$$\int_{\partial D} \left| \frac{\partial \mathbf{u}}{\partial N} \right|^2 d\sigma \leqslant C \left\{ \left(\int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma \right)^{1/2} \left(\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \right)^{1/2} + \int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma \right\}.$$

Now, since $|\nabla \mathbf{u}|^2 = |\nabla_t \mathbf{u}|^2 + |\partial \mathbf{u}/\partial N|^2$, the remark follows.

REMARK 5. In order to show that $\int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma \leqslant C \int_{\partial D} |T\mathbf{u}|^2 \, d\sigma$, it suffices to show that

$$\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C \int_{\partial D} \left| \lambda (\operatorname{div} \mathbf{u}) I + \mu \{ \nabla \mathbf{u} + \nabla \mathbf{u}^t \} \right|^2 d\sigma.$$

In fact, if this inequality holds, we would clearly have

$$\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C \int_{\partial D} |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 \, d\sigma$$

(Korn-type inequality at the boundary). The Rellich–Payne–Weinberger–Nečas identity is, in this case (with $\mathbf{h} = e_n$),

$$\int_{\partial D} n_n \left\{ \frac{\mu}{2} |\nabla \mathbf{u} + \nabla \mathbf{u}^t|^2 + \lambda (\operatorname{div} \mathbf{u})^2 \right\} d\sigma$$

$$= 2 \int_{\partial D} \frac{\partial \mathbf{u}}{\partial y} \cdot \left\{ \lambda (\operatorname{div} \mathbf{u}) N + \mu \{ \nabla \mathbf{u} + \nabla \mathbf{u}^t \} N \right\} d\sigma.$$

But then

$$\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C \left(\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \right)^{1/2} \times \left(\int_{\partial D} \left| \lambda (\operatorname{div} \mathbf{u}) N + \mu \{ \nabla \mathbf{u} + \nabla \mathbf{u}^t \} N \right|^2 d\sigma \right)^{1/2}.$$

The rest of Part 1 is devoted to sketching the proof of the above inequality.

THEOREM 2.1.6. *Let* $\mathbf{u}$ *solve* $\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} = 0$ *in* D, $\mathbf{u} = S(\mathbf{g})$, *where* $\mathbf{g}$ *is nice. Then there exists a constant C depending only on the Lipschitz constant of* φ *so that*

$$\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C \int_{\partial D} \left| \lambda (\operatorname{div} \mathbf{u}) I + \mu \{ \nabla \mathbf{u} + \nabla \mathbf{u}^t \} \right|^2 d\sigma.$$

The proof proceeds in two steps. They are

LEMMA 2.1.7. *Let* $\mathbf{u}$ *be as in Theorem* 2.1.6. *Then*

$$\int_{\partial D} N(\nabla \mathbf{u})^2 \, d\sigma \leqslant C \int_{\partial D} N\left(\lambda (\operatorname{div} \mathbf{u}) I + \mu \{ \nabla \mathbf{u} + \nabla \mathbf{u}^t \} \right)^2 d\sigma.$$

LEMMA 2.1.8. *Let* **u** *be as in Theorem* 2.1.6. *Then*

$$\int_{\partial D} N\big(\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big)^2\, d\sigma$$
$$\leqslant C\int_{\partial D} \big|\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big|^2\, d\sigma.$$

Lemma 2.1.7 is proved by first doing so in the case when the Lipschitz constant is small, and then passing to the general case by using the ideas of David [**13**]. Lemma 2.1.8 is proved by observing that if **v** is any row of the matrix $\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}$, then **v** is a solution of the Stokes system

$$\text{(S)} \qquad \begin{cases} \Delta\mathbf{v} = \nabla p & \text{in } D, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } D, \\ \mathbf{v}|_{\partial D} = \mathbf{f} \in L^2(\partial D, d\sigma). \end{cases}$$

This is checked directly by using the system of equations $\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} = 0$. One then invokes the following theorem of Fabes, Kenig, and Verchota, whose proof is presented in the next section.

THEOREM 2.1.9. *Given* $\mathbf{f} \in L^2(\partial D, d\sigma)$, *there exists a unique solution* $(\mathbf{v}, p)$ *to* (S) *with* p *tending to* 0 *at* ∞ *and* $N(\mathbf{v}) \in L^2(\partial D, d\sigma)$. *Moreover*,

$$\|N(\mathbf{v})\|_{L^2(\partial D, d\sigma)} \leqslant C\|\mathbf{f}\|_{L^2(\partial D, d\sigma)}.$$

We now turn to a sketch of the proof of Lemma 2.1.7. We will need the following, unpublished, real-variable lemma of David [**14**].

LEMMA 2.1.10. *Let* $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be a function of two variables*, $t \in \mathbb{R}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. *Assume that for each* x, *the function* $t \to F(t, x)$ *is Lipschitz, with Lipschitz constant less than or equal to* M, *and for each* i, $1 \leqslant i \leqslant n$, *the function* $x_i \to F(t, x)$ *is Lipschitz, with Lipschitz constant less than or equal to* M_i, *for any choice of the other variables. Given an interval* $I \times J = I \times J_1 \times \cdots \times J_n$, *where the* J_i*'s and* I *are* 1-*dimensional compact intervals, there exists a function* $G(t, x)\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *with the following properties*:

(a) $G(t, x) \geqslant F(t, x)$ *on* $I \times J$.

(b) *If* $E = \{(t, x) \in I \times J\colon F(t, x) = G(t, x)\}$ *then* $|E| \geqslant \frac{3}{8}|I|\,|J|$.

(c) *For each* i *the function* $G(t, x_1, x_2, \ldots, x_{i-1}, \ldots, x_{i+1}, \ldots, x_n)$ *is Lipschitz, with Lipschitz constant less than or equal to* M_i, *and one of the following statements is true*: *For each* x *either* $-M \leqslant (\partial G/\partial t)(t, x) \leqslant 4M/5$ *or* $-4M/5 \leqslant (\partial G/\partial t)(t, x) \leqslant M$.

The proof of this lemma is the same as in the 1-dimensional case, treating x as a parameter (see [**13**]).

Before we proceed with the proof of Lemma 2.1.7, we would like to point out that in the analogue of Lemma 2.1.7 for bounded domains, a normalization is necessary, since if $\mathbf{u}(X)$ solves the systems of elastostatics, so does $\mathbf{u}(X) + \mathbf{a} + BX$, where **a** is a constant vector and B is any antisymmetric 3×3 matrix. The right side of the inequality in the lemma, of course, remains unchanged, while the left side increases if B 'increases'. The most convenient normalization is that, for some fixed point X^* in the domain, $\nabla\mathbf{u}(X^*) - \nabla\mathbf{u}(X^*)^t = 0$. This also gives uniqueness modulo constants to problem (2.1.2) in bounded domains.

We now need to introduce some definitions. Let $D_0 \subset \mathbb{R}^n_+$ be a fixed C^∞ domain with

$$\{(x,0): |\|x\|| = \max|x_i| \leqslant 1\} \subset \partial D_0,$$
$$\{(x,y): 0 < y < 1, |\|x\|| \leqslant 1\} \subset D_0 \subset \{(x,y): 0 < y < 2, |\|x\|| < 2\}.$$

If $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz, with $|\|\nabla\varphi\|| \leqslant M$, we construct the mapping $T_\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n$ by $T_\varphi(x,y) = (x, cy + \eta_y * \varphi(x))$, where $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ is radial, $\int \eta = 1$, and $c = c(M)$ is chosen so that $T_\varphi(\mathbb{R}^n_+) \subset \{(x,y): y > \varphi(x)\}$, and so that T_φ is a bi-Lipschitzian mapping. Also, it is clear that T_φ is smooth for (x,y) with $y > 0$, and $T_\varphi(x,0) = (x,\varphi(x))$. We denote the point $T_\varphi(0,1)$ by A_φ. Lemma 2.1.7 is an easy consequence of

LEMMA 2.1.11. *Given $M > 0$ and φ with $|\|\nabla\varphi\|| \leqslant M$, there exists a constant $C = C(M)$ such that, for all functions $\mathbf{u}$ in D_φ that are Lipschitz in $\overline{D}_\varphi$ and satisfy $\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = 0$ in D_φ and $\nabla\mathbf{u}(A_\varphi) = \nabla\mathbf{u}(A_\varphi)^t$, we have*

$$\left\|N_\varphi(\nabla\mathbf{u})\right\|_{L^2(\partial D,\,d\sigma)} \leqslant C\left\|N_\varphi\big(\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big)\right\|_{L^2(\partial D,\,d\sigma)}.$$

Here N_φ is the nontangential maximal operator corresponding to the domain D_φ.

This lemma will be proved by a series of propositions. Before we proceed we need to introduce one more definition. We say that proposition (M,ε) holds if, whenever φ is such that $|\|\nabla\varphi\|| \leqslant M$ and there exists a constant vector $\mathbf{a}$ with $|\|\mathbf{a}\|| \leqslant M$ so that $|\|\nabla\varphi - \mathbf{a}\|| \leqslant \varepsilon$, then for all Lipschitz functions $\mathbf{u}$ on $\overline{D}_\varphi$ with $\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = 0$ in D_φ and $\nabla\mathbf{u}(A_\varphi) = \nabla\mathbf{u}^t(A_\varphi)$, we have

$$\left\|N_\varphi(\nabla\mathbf{u})\right\|_{L^2(\partial D_\varphi,\,d\sigma)} \leqslant C\left\|N_\varphi\big(\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big)\right\|_{L^2(\partial D_\varphi,\,d\sigma)},$$

where $C = C(M,\varepsilon)$.

Note that is proposition (M,ε) holds, then the corresponding estimates automatically hold for all translates, rotates, or dilates of the domains D_φ when φ satisfies the conditions in proposition (M,ε). In the rest of this section, a coordinate chart will be a translate, rotate, or dilate of a domain D_φ. The bottom B_φ of ∂D_φ will be $T_\varphi(\partial D_0 \cap (x,0)\colon x \in \mathbb{R}^{n-1})$.

PROPOSITION 2.1.12. *Given $M > 0$, there exists $\varepsilon = \varepsilon(M)$ so that proposition (M,ε) holds.*

We will not give the proof but will just make a few remarks about it. First, in this case the stronger estimate

$$\left\|N_\varphi(\nabla\mathbf{u})\right\|_{L^2(\partial D_\varphi,\,d\sigma)} \leqslant C\left\|\lambda(\operatorname{div}\mathbf{u})N + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}N\right\|_{L^2(\partial D_\varphi,\,d\sigma)}$$

holds, because in this case the domain D_φ is a small perturbation of the smooth domain $D_{\mathbf{ax}}$. For the smooth domain $D_{\mathbf{ax}}$, we can solve problem (2.1.2) by the method of layer potentials (see [**27**], for example). If ε is small, a perturbation analysis based on the theorem of Coifman–McIntosh–Meyer [**13**] shows that this is still the case. This easily gives the estimate claimed above.

PROPOSITION 2.1.13. *For all* M, $\varepsilon > 0$, $\alpha \in (0, 0.1)$, *if proposition* (M, ε) *holds, then proposition* $(1 - \alpha M, 1.1\varepsilon)$ *holds.*

We postpone the proof and show first how Propositions 2.1.12 and 2.1.13 yield lemma 2.1.11.

PROOF OF LEMMA 2.1.11. We show that proposition (M, ε) holds for any M, ε. Fix M, ε and choose N so large that if $\varepsilon(10M)$ is as in Proposition 2.1.12, then $(1.1)^N\varepsilon(10M) \geqslant \varepsilon$. Now pick $\alpha_j > 0$ so that $\prod_{j=1}^N(1 - \alpha_j) = 1/10$. Then, since proposition $(10M, \varepsilon(10M))$ holds by Proposition 2.1.12, applying Proposition 2.1.13 N times we see that proposition (M, ε) holds.

We now sketch the proof of Proposition 2.1.13. We first note that it suffices to show that

$$\left\|N_\varphi(\nabla\mathbf{u})\right\|_{L^2(\partial D,\, d\sigma)} \leqslant C\left\|\tilde{N}_\varphi\big(\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big)\right\|_{L^2(\partial D,\, d\sigma)},$$

where $\tilde{N}_\varphi$ is the nontangential maximal operator with a wider opening of the nontangential region. This follows because of classical arguments relating nontangential maximal functions to different openings (see [**18**], for example). Now pick φ with $|\,\|\nabla\varphi\|\,| \leqslant (1 - \alpha)M$ and such that there exists $\mathbf{a}$ with

$$|\,\|\nabla\varphi - \mathbf{a}\|\,| \leqslant 1.1\varepsilon, \qquad |\,\|\mathbf{a}\|\,| \leqslant (1 - \alpha)M.$$

We choose $\tilde{N}_\varphi$ as follows: Since $\partial D_\varphi \setminus B_\varphi$ is smooth, it is easy to see that we can find a finite number of coordinate charts (i.e., rotates, translates, and dilates of D_Ψ) that are entirely contained in D_φ such that their bottoms B_Ψ are contained in ∂D_φ, $T_\Psi((x,0)\colon |\,\|x\|\,| < 1/2)$ cover ∂D_φ, and the Ψ's involved satisfy $|\,\|\nabla\Psi\|\,| \leqslant (1 - \alpha/2)M$, and there exist $\mathbf{a}_\Psi$ such that

$$|\,\|\mathbf{a}_\Psi\|\,| \leqslant (1 - \alpha/2)M \quad \text{and} \quad |\,\|\nabla\Psi - \mathbf{a}_\Psi\|\,| \leqslant 1.11\varepsilon.$$

The nontangential region defining $\tilde{N}_\varphi$ on $T_\Psi((x,0)\colon |\,\|x\|\,| < 1/2)$ is defined as follows: let $F \subset \{(x,0)\colon |\,\|x\|\,| < 1/2\}$ be a closed set. Consider the cone on $\mathbb{R}^n_+$, $\gamma = \{(x, y) \in \mathbb{R}^n_+\colon b|x| < y\}$, where b is a small constant. Now consider the domain D_F on $\mathbb{R}^n_+$ given by $D_F = \bigcup_{x\in F}((x,0) + \gamma)$. Then D_F is the domain above the graph of a Lipschitz function θ for which $|\,\|\nabla\theta\|\,| \leqslant cb$ for some absolute constant c (independent of F). It is also easy to see that we can now take b so small, depending only on M and ε, such that $T_\Psi(D_F)$ is the domain above the graph of a Lipschitz function $\tilde{\Psi}$, with $\tilde{\Psi} \geqslant \Psi$, that satisfies

$$|\,\|\nabla\tilde{\Psi}\|\,| \leqslant (1 - \alpha/10)M, \qquad \left|\,\|\nabla\tilde{\Psi} - \mathbf{a}_\Psi\|\,\right| \leqslant 1.111\varepsilon.$$

The nontangential region defining $\tilde{N}_\varphi$ for $Q \in T_\Psi((x,0)\colon |\,\|x\|\,| < 1/2)$ is then the image under T_Ψ of $(x,0) + \gamma$, with b chosen as above, suitably truncated, where $Q = T_\Psi((x,0))$. To simplify notation let

$$m = N_\varphi(\nabla\mathbf{u}), \qquad \overline{m} = \tilde{N}_\varphi\big(\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\big).$$

For $t > 0$ consider the open set $E_t = \{m > t\}$. We now produce a Whitney-type decomposition of E_t into a family of disjoint sets $\{U_j\}$ with the property that each U_j is contained in $T_\Psi((x,0)\colon |\|x\|| < 1/2)$ for a coordinate chart D_Ψ, and each U_j contains $T_\Psi(I_j)$, where I_j is a cube in $|\|x\|| < 1/2$, and is contained in $T_\Psi(\bar{I}_j)$, where $\bar{I}_j$ is a fixed multiple of I_j. Finally, we can also assume that there exists a constant η_0 such that, if $\operatorname{diam}(U_j) \leqslant \eta_0$, there exists a point Q_j in ∂D_φ, with $\operatorname{dist}(Q_j, U_j) \approx \operatorname{diam} U_j$, such that $m(Q_j) \leqslant t$. Now let $\beta > 1$ be given. We claim that there exists $\delta > 0$ so small that if $E_j = U_j \cap \{m > \beta t, \overline{m} \leqslant \delta t\}$, then

$$\sigma(E_j) \leqslant (1 - \eta_M)\sigma(U_j),$$

where $\eta_M > 0$. Assume the claim for now. Then

$$\begin{aligned}
\int_{\partial D_\varphi} m^2\, d\sigma &= 2\int_0^\infty t\sigma(E_t)\, dt = 2\beta^2 \int_0^\infty t\sigma(E_{\beta t})\, dt \\
&= \sum_j 2\beta^2 \int_0^\infty t\sigma(U_j \cap E_{\beta t})\, dt \\
&\leqslant \sum_j 2\beta^2 \int_0^\infty t\sigma(E_j)\, dt + 2\beta^2 \int_0^\infty t\sigma(\overline{m} > \delta t)\, dt \\
&\leqslant \sum_j 2\beta^2(1 - \eta_M) \int_0^\infty t\sigma(U_j)\, dt + 2\frac{\beta^2}{\delta^2} \int_0^\infty t\sigma\{\overline{m} > t\}\, dt \\
&= \beta^2(1 - \eta_M) \int_{\partial D_\varphi} m^2\, d\sigma + \frac{\beta^2}{\delta^2} \int_{\partial D_\varphi} \overline{m}^2\, d\sigma.
\end{aligned}$$

Thus if we choose $\beta > 1$, but so that $\beta^2 \cdot (1 - \eta_M) < 1$, the desired result follows. It remains to establish the claim. We argue by contradiction. Suppose the claim is false; then $\sigma(E_j) > (1 - \eta_M)\sigma(U_j)$. Let $\tilde{E}_j = T_\Psi^{-1}(E_j)$. If η_M is chosen sufficiently small, we can guarantee that $|\tilde{E}_j \cap I_j| \geqslant .99|I_j|$. Now let $F_j = \tilde{E}_j \cap I_j$ and construct the Lipschitz function $\tilde{\Psi}$ corresponding to it, as in the definition of $\tilde{N}_\varphi$. Thus $\tilde{\Psi} \geqslant \Psi$, $|\|\nabla\tilde{\Psi}\|| \leqslant (1 - \alpha/10)M$, and $|\|\nabla\tilde{\Psi} - \mathbf{a}_\Psi\|| \leqslant 1.111\varepsilon$. We now apply Lemma 2.1.10 to $\tilde{\Psi}$, one variable at a time, to find a Lipschitz function f, with $f \geqslant \tilde{\Psi}$ on I_j, such that if $\bar{F}_j = \{x \in I_j\colon f = \tilde{\Psi}\}$, then $|\bar{F}_j \cap F_j| \geqslant c\sigma(U_j)$, with $|\|\nabla f\|| \leqslant (1 - \alpha/10)M$, and such that there exists $\mathbf{a}_f$, with $|\|\mathbf{a}_f\|| \leqslant (1 - \alpha/10)M$ so that

$$|\|\nabla f - \mathbf{a}_f\|| \leqslant \tfrac{4}{5}(1.111\varepsilon) < \varepsilon.$$

We can also arrange the truncation of our nontangential regions in such a way that on the appropriate rotate, translate, and dilate of D_f (which, of course, is contained in the corresponding coordinate chart associated to D_Ψ, which is contained in D_φ),

$$\left|\lambda(\operatorname{div}\mathbf{u})I + \mu\{\nabla\mathbf{u} + \nabla\mathbf{u}^t\}\right| \leqslant \delta t.$$

To lighten the exposition we still denote the translate, rotate, and dilate of D_f by D_f. Note that proposition (M, ε) applies to it. We divide the sets U_j into two types. Type I sets are those with $\operatorname{diam} U_j \geqslant \eta_0$; type II sets are those for which $\operatorname{diam} U_j \leqslant \eta_0$. We first deal with type I. In this case D_f has diameter of order 1. Because of the solvability of problem (2.1.2) for balls, and our normalization, we see that on a ball $B \subset D_\varphi$, diam $B \approx 1$, $A_\varphi \in B$, we have

$$\int_B |\nabla \mathbf{u}|^2 \leqslant C \int_B |\lambda \operatorname{div} \mathbf{u} I + \mu\{\nabla \mathbf{u} + \nabla \mathbf{u}^t\}|^2.$$

Joining A_f to A_φ by a finite number of balls and using interior regularity results for the system $\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} = 0$, we see that $|\nabla \mathbf{u}(A_f)| \leqslant C\delta t$ for some absolute constant C. Then

$$\begin{aligned}
C\sigma(U_j)\beta^2 t^2 &\leqslant \int_{T_\Psi(\bar{F}_j \cap F_j)} m^2 \, d\sigma \leqslant C \int_{\partial D_f} N_f^2(\nabla u)\, d\sigma \\
&\leqslant C\sigma(U_j)\delta^2 t^2 + C\int_{\partial D_f} N_f^2\left(\nabla \mathbf{u} - \frac{[\nabla \mathbf{u}(A_f) - \nabla \mathbf{u}^t(A_f)]}{2}\right) d\sigma \\
&\leqslant C\sigma(U_j)\delta^2 t^2 + C\int_{\partial D_f} N_f^2\left(\lambda(\operatorname{div} \mathbf{u}) I + \mu\{\nabla \mathbf{u} + \nabla \mathbf{u}^t\}\right)^2 d\sigma,
\end{aligned}$$

by (M, ε). The last quantity is also bounded by $C\sigma(U_j)\delta^2 t^2$, which is a contradiction for small δ. Now assume that U_j is on type II. Note that, in this case, there exists $Q_j \in \partial D_\varphi$ with $\operatorname{dist}(Q_j, U_j) \approx \operatorname{diam} U_j$, and $|\nabla \mathbf{u}(x)| \leqslant t$ for all x in the nontangential region associated to Q_j. Therefore, it is easy to see, using the arguments we used to bound $|\nabla \mathbf{u}(A_f)|$ in case I, that for all X in a neighborhood of A_f and also on the top part of D_f, we have $|\nabla \mathbf{u}(X)| \leqslant t + C\delta t$. Since, for $Q \in T_\Psi(\bar{F}_j \cap F_j)$, $m(Q) \geqslant \beta t$ and $\beta > 1$, if δ is small enough, we see that we must have $N_f(\nabla \mathbf{u})(Q) \geqslant m(Q)$. Hence,

$$N_f\left(\nabla \mathbf{u} - \left[\frac{\nabla \mathbf{u}(A_f) - \nabla \mathbf{u}^t(A_f)}{2}\right]\right)(Q) \geqslant (\beta - 1 - C\delta)t \geqslant \frac{\beta - 1}{2} t$$

if δ is small and $Q \in T_\Psi(\bar{F}_j \cap F_j)$. Thus, applying (M, ε) to D_f, we see that

$$\begin{aligned}
C(\beta - 1)^2 t^2 \sigma(U_j) &\leqslant \int_{T_\Psi(\bar{F}_j \cap F_j)} N_f\left(\nabla \mathbf{u} - \left[\frac{\nabla \mathbf{u}(A_f) - \nabla \mathbf{u}^t(A_f)}{2}\right]\right)^2 d\sigma \\
&\leqslant \int_{\partial D_f} N_f\left(\nabla \mathbf{u} - \left[\frac{\nabla \mathbf{u}(A_f) - \nabla \mathbf{u}^t(A_f)}{2}\right]\right)^2 d\sigma \\
&\leqslant C\sigma(U_j)\delta^2 t^2,
\end{aligned}$$

a contradiction if δ is small. This finishes the proofs of Proposition 2.1.13 and, hence, Lemma 2.1.11.

Part 2. *The Stokes system of linear hydrostatics*. In this part I will sketch the proof of the L^2 results for the Stokes system of hydrostatics. These results are joint work of E. Fabes, C. Kenig, and G. Verchota [**17**]. We keep the notation introduced in Part 1.

We seek a vector-valued function $\mathbf{u} = (u^1, u^2, u^3)$ and a scalar-valued function p satisfying

$$(2.2.1)\qquad \begin{cases} \Delta \mathbf{u} = \nabla p & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u}|_{\partial D} = \mathbf{f} \in L^2(\partial D, d\sigma) & \text{in the nontangential sense.} \end{cases}$$

THEOREM 2.2.2 (ALSO THEOREM 2.1.9). *Given* $\mathbf{f} \in L^2(\partial D, d\sigma)$, *there exists a unique solution* $(\mathbf{u}, p)$ *to* (2.2.1), *with p tending to* 0 *at* ∞, *and* $N(\mathbf{u}) \in L^2(\partial D, d\sigma)$. *Moreover*, $\mathbf{u}(X) = \mathscr{K}\mathbf{g}(X)$, *with* $\mathbf{g} \in L^2(\partial D, d\sigma)$. ($\mathscr{K}$ *will be defined below*.)

In order to sketch the proof of 2.2.2, we introduce the matrix $\Gamma(X) = (\Gamma_{ij}(X))$ of fundamental solutions (see the book of Ladyzhenskaya [**28**]), where

$$\Gamma_{ij}(X) = \frac{1}{8\pi}\frac{\delta_{ij}}{|X|} + \frac{1}{8\pi}\frac{X_i X_j}{|X|^3},$$

and its corresponding pressure vector

$$q(X) = \left(q^i(X)\right), \quad \text{where } q^i(X) = X_i/4\pi|X|^3.$$

Our solution of (2.2.2) will be given in the form of a double-layer potential

$$\mathbf{u}(X) = \mathscr{K}\mathbf{g}(X) = -\int_{\partial D} \{H'(Q)\Gamma(X - Q)\}\mathbf{g}(Q)\, d\sigma(Q),$$

where

$$(H'(Q)\Gamma(X - Q))_{il} = \delta_{ij} q^l(X - Q) n_j(Q) + \left(\partial\Gamma_{il}/\partial Q_j\right)(X - Q) n_j(Q).$$

We also use the single-layer potential

$$\mathbf{u}(X) = S\mathbf{g}(X) = \int_{\partial D} \Gamma(X - Q)\mathbf{g}(Q)\, d\sigma(Q).$$

In the same way as one establishes 2.1.4, we have

LEMMA 2.2.3. *Let* $\mathscr{K}\mathbf{g}$, $S\mathbf{g}$ *be defined as above, with* $\mathbf{g} \in L^2(\partial D, d\sigma)$. *Then they both solve* $\Delta \mathbf{u} = \nabla p$ *in D and* D_-, $\operatorname{div} \mathbf{u} = 0$ *in D and* D_-. *Also*,

(a) $\|N(\mathscr{K}\mathbf{g})\|_{L^2(\partial D, d\sigma)} \leqslant C\|\mathbf{g}\|_{L^2(\partial D, d\sigma)}$,

(b) $(\mathscr{K}\mathbf{g})^{\pm}(P) = \pm\frac{1}{2}\mathbf{g}(P) - \text{p.v.} \int_{\partial D} \{H'(Q)\Gamma(P - Q)\}\mathbf{g}(Q)\, d\sigma(Q)$,

(d) $\|N(\nabla S\mathbf{g})\|_{L^2(\partial D, d\sigma)} \leqslant C\|\mathbf{g}\|_{L^2(\partial D, d\sigma)}$,

(d) $$\left(\frac{\partial}{\partial X_i}(S\mathbf{g})_j\right)^{\pm}(P) = \pm\left\{\frac{n_i(P)g_j(P)}{2} - \frac{n_i(P)n_j(P)}{2}\langle N_P, \mathbf{g}(P)\rangle\right\} + \text{p.v.} \int_{\partial D} \frac{\partial}{\partial p_i}\Gamma(P, Q)\mathbf{g}(Q)\, d\sigma(Q),$$

(e) $(HS\mathbf{g})^{\pm}(P) = \pm\frac{1}{2}\mathbf{g}(P) + \text{p.v.} \int_{\partial D} \{H(P)\Gamma(P - Q)\}\mathbf{g}(Q)\, d\sigma(Q)$, *where*

$$(H(X)\Gamma(X-Q))_{il} = n_j(x)\frac{\partial\Gamma_{il}}{\partial X_j}(X-Q) - \delta_{ij}q^l(X-Q)n_j(X).$$

For the proof in the case of smooth domains, see **[28]**.

The proof of Theorem 2.2.2 (at least the existence part of it) reduces to the invertibility in $L^2(\partial D, d\sigma)$ of the operator $\frac{1}{2}I + K$, where

$$K\mathbf{g}(P) = -\text{p.v.}\int_{\partial D}\{H'(Q)\Gamma(P-Q)\}\mathbf{g}(Q)\,d\sigma(Q).$$

As in previous cases, it is enough to show

$$\|(\tfrac{1}{2}I - K^*)\mathbf{g}\|_{L^2(\partial D, d\sigma)} \approx \|(\tfrac{1}{2}I + K^*)\mathbf{g}\|_{L^2(\partial D, d\sigma)}. \tag{2.2.4}$$

This is shown by using the following two integral identities:

LEMMA 2.2.5. *Let* $\mathbf{h}$ *be a constant vector in* $\mathbb{R}^n$, *and suppose that* $\Delta\mathbf{u} = \nabla p$, $\operatorname{div}\mathbf{u} = 0$, D, *and* $\mathbf{u}$, p *and their derivatives are suitably small at* ∞. *Then*

$$\int_{\partial D} h_l n_l \frac{\partial u^s}{\partial X_j}\cdot\frac{\partial u^s}{\partial X_j}\,d\sigma = 2\int_{\partial D}\frac{\partial u^s}{\partial N}\cdot h_l\frac{\partial u^s}{\partial X_l}\,d\sigma - 2\int_{\partial D} p n_s h_l \frac{\partial u^s}{\partial X_l}\,d\sigma.$$

LEMMA 2.2.6. *Let* $\mathbf{h}$, p *and* $\mathbf{u}$ *be as in* 2.2.5. *Then*

$$\int_{\partial D} h_l n_l p^2\,d\sigma = 2\int_{\partial D} h_r\frac{\partial u^r}{\partial N}p\,d\sigma - 2\int_{\partial D} h_r\frac{\partial u^r}{\partial X_i}\frac{\partial u^i}{\partial N}\,d\sigma + 2\int_{\partial D} h_r n_s\frac{\partial u^s}{\partial X_j}\frac{\partial u^r}{\partial X_i}\,d\sigma.$$

The proofs of 2.2.5 and 2.2.6 are simple applications of the properties of $\mathbf{u}$, p and the divergence theorem.

Choosing $\mathbf{h} = e_3$, we see that, from 2.2.6, we obtain

COROLLARY 2.2.7. *Let* $\mathbf{u}$, p *be as in* 2.2.5. *Then* $\int_{\partial D} p^2\,d\sigma \leqslant C\int_{\partial D}|\nabla\mathbf{u}|^2\,d\sigma$, *where* C *depends only on* M.

A consequence of Corollary 2.2.7 and Lemma 2.2.5 is that if $\partial\mathbf{u}/\partial\nu = \partial\mathbf{u}/\partial N - p\cdot N$, then we have

COROLLARY 2.2.8. *Let* $\mathbf{u}$, p *be as in* 2.2.5. *Then*

$$\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial\nu}\right|^2 d\sigma \approx \int_{\partial D}|\nabla_t\mathbf{u}|^2\,d\sigma + \sum_j\int_{\partial D}\left|n_s\frac{\partial u^s}{\partial X_j}\right|^2 d\sigma,$$

where the constants of equivalence depend only on M.

PROOF. Lemma 2.2.5 clearly implies, by Schwartz's inequality, that

$$\int_{\partial D}|\nabla\mathbf{u}|^2\,d\sigma \leqslant C\int_{\partial D}\left|\frac{\partial\mathbf{u}}{\partial\nu}\right|^2 d\sigma.$$

Moreover, arguing as in the second part of Remark 2 after 2.1.5, we see that 2.2.5 shows that

$$\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma \leqslant C\int_{\partial D} |\nabla_t \mathbf{u}|^2 \, d\sigma + \left|\int_{\partial D} p n_s h_l \frac{\partial u^s}{\partial X_l} \, d\sigma\right|.$$

By Corollary 2.2.7 the right side is bounded by

$$C\left(\int_{\partial D} |\nabla \mathbf{u}|^2 \, d\sigma\right)^{1/2}\left(\sum_j \int_{\partial D} \left|n_s \frac{\partial u^s}{\partial X_j}\right|^2 d\sigma\right)^{1/2} + c\int_{\partial D} |\nabla_t u|^2 \, d\sigma.$$

Corollary 2.2.8 now follows, using 2.2.7 once more.

To prove 2.2.4 let $\mathbf{u} = S(\mathbf{g})$. By 2.2.3(d), $\nabla_t \mathbf{u}$ and $n_s(\partial u^s/\partial X_j)$ are continuous across ∂D. Using this fact, 2.2.3(e), and Corollary 2.2.8, (2.2.4) follows.

In closing this part we would like to point out another boundary-value problem for the Stokes system that is of physical significance: the so-called slip boundary condition

$$\text{(2.2.9)} \qquad \begin{cases} \Delta \mathbf{u} = \nabla p & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \left((\nabla \mathbf{u} + \nabla \mathbf{u}^t)N - p \cdot N\right)\big|_{\partial D} = \mathbf{f} \in L^2(\partial D, d\sigma). \end{cases}$$

This problem is very similar to (2.1.2). Using the techniques introduced in Part 1, together with the observation that if $\Delta \mathbf{u} = \nabla p$, $\operatorname{div} \mathbf{u} = 0$ in D, the same is true for each row $\mathbf{v}$ of the matrix $[\nabla \mathbf{u} + \nabla \mathbf{u}^t - pI]$, we have obtained

THEOREM 2.2.10. *Given* $\mathbf{f} \in L^2(\partial D, d\sigma)$ *there exists a unique solution* $(\mathbf{u}, p)$ *to* (2.2.9) *tending to* 0 *at* ∞, *with* $N(\nabla \mathbf{u}) \in L^2(\partial D, d\sigma)$. *Moreover*, $\mathbf{u}(X) = S(\mathbf{g})(X)$, *with* $\mathbf{g} \in L^2(\partial D, d\sigma)$.

Part 3. *The Dirichlet problem for the biharmonic equation on Lipschitz domains.* This part deals with the Dirichlet problem for Δ^2 on an arbitrary Lipschitz domain in $\mathbb{R}^n$. The results are joint work of B. Dahlberg, C. Kenig, and G. Verchota [**11**]. We continue using the notation previously introduced.

We seek a function u defined in D such that

$$\text{(2.3.1)} \qquad \begin{cases} \Delta^2 u = 0 & \text{in } D, \\ u|_{\partial D} = f \in L_1^2(\partial D, d\sigma), \\ \partial u/\partial N|_{\partial D} = g \in L^2(\partial D, d\sigma), \end{cases}$$

where the boundary values are taken nontangentially a.e.

THEOREM 2.3.2. *There exists a unique u solving* (2.3.1), *with* $N(\nabla u) \in L^2(\partial D, d\sigma)$ *and*

$$\|N(\nabla u)\|_{L^2(\partial D, d\sigma)} \leqslant C\left\{\|g\|_{L^2(\partial D, d\sigma)} + \|f\|_{L_1^2(\partial D, d\sigma)}\right\},$$

where C depends only on M.

We only discuss existence. By 1.1.14 we may assume $f = 0$ on ∂D. Let $G(X, Y)$ be the Green function for Δ on D. Then, since $u|_{\partial D} = 0$, we have $u(X) = \int_D G(X, Y)\Delta u(y)\, dy$. Notice that $w(y) = \Delta u(y)$ is harmonic in D. We claim that $w(Y) = (\partial/\partial y)v(Y)$, where v is a harmonic function in D with $L^2(\partial D, d\sigma)$ Dirichlet data, and that the operator T: $v|_{\partial D} \to \partial u/\partial N|_{\partial D}$ is an invertible map from $L^2(\partial D, d\sigma)$ onto $L^2(\partial D, d\sigma)$. This would establish 2.3.2. In fact, by using the Green's potential representation, Fubini's theorem, and the fact that $(\partial/\partial N)G(-, Y)$ is the density of harmonic measure at $Y \in D$,

$$\int_{\partial D} vTv\, d\sigma = \int_D v(Y)\frac{\partial}{\partial y}v(Y)\, dY = \frac{1}{2}\int_{\mathbb{R}^{n-1}} v(x, \varphi(x))^2\, dx \geqslant C\int_{\partial D} v^2\, d\sigma.$$

This shows that if T: $L^2(\partial D, d\sigma) \to L^2(\partial D, d\sigma)$ is bounded, it will have a bounded inverse. To establish the boundedness of T, note that if h is harmonic in D, then the argument given above shows that

$$\int_{\partial D} hTv\, d\sigma = \int_D \frac{\partial v}{\partial y}(Y)h(Y)\, dY.$$

All we need, therefore, is the following bilinear estimate.

THEOREM 2.3.3. *If v, h are harmonic in D and tend to* 0 *at* ∞, *then*

$$\left|\int_D \frac{\partial v}{\partial y}(Y)\cdot h(Y)\, dY\right| \leqslant C\|v\|_{L^2(\partial D, d\sigma)}\cdot\|h\|_{L^2(\partial D, d\sigma)}.$$

PROOF. This theorem is a generalization to Lipschitz domains of the fact that the paraproduct of two L^2 functions is in L^1 (see [**4**]).

To establish the inequality we can assume, due to the invertibility of the double-layer potential (the representation formula in Theorem 1.1.1), that

$$h(Y) = \frac{1}{\omega_n}\int_{\partial D} \frac{\langle Y - Q, N_Q\rangle}{|Y - Q|^{n-2}} g(Q)\, d\sigma(Q),$$

with

$$\|g\|_{L^2(\partial D, d\sigma)} \leqslant C\|h\|_{L^2(\partial D, d\sigma)}.$$

Thus, since

$$\frac{\langle Y - Q, N_Q\rangle}{|Y - Q|^{n-2}} = C_n\frac{\partial}{\partial N_Q}\left(\frac{1}{|Y - Q|^{n-2}}\right),$$

it suffices to show that

$$\left\|\frac{\partial}{\partial N_Q}\int_D \frac{1}{|Y - Q|^{n-2}}\frac{\partial v}{\partial y}(Y)\, dY\right\|_{L^2(\partial D, d\sigma)} \leqslant C\|v\|_{L^2(\partial D, d\sigma)}.$$

In order to do so we obtain a representation formula for

$$\frac{\partial}{\partial N_Q}\int_D \frac{1}{|Y - Q|^{n-2}}\frac{\partial v}{\partial y}(Y)\, dY.$$

Fix $Q \in \partial D$ and let B satisfy $\Delta_Y B(Y-Q) = 1/|Y-Q|^{n-2}$; i.e., B is the fundamental solution for Δ^2 (for example, if $n \geqslant 5$, $B(Y) = C_n|Y|^{4-n}$). We recall the definition of the Riesz transforms $v_j = R_j v$, $j = 1,\dots,n-1$. They are harmonic functions that together with v satisfy the generalized Cauchy–Riemann equations (see [**35**]); i.e.,

$$\frac{\partial v}{\partial X_j} = \frac{\partial}{\partial y} R_j v \quad \text{and} \quad \frac{\partial v}{\partial y} = -\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} R_j v.$$

If $Y = (x, y)$, then

$$\frac{1}{|Y-Q|^{n-2}} \frac{\partial}{\partial y} v(Y) = \Delta_Y B \frac{\partial v}{\partial y}.$$

Using the summation convention, we have

$$\begin{aligned}\Delta_Y B \frac{\partial v}{\partial y} &= \left(\frac{\partial^2}{\partial x_j^2} B + \frac{\partial^2}{\partial y^2} B \right) \frac{\partial}{\partial y} v \\ &= \frac{\partial^2}{\partial x_j^2} B \frac{\partial}{\partial y} v - \frac{\partial^2 B}{\partial x_j \partial y} \frac{\partial v}{\partial y} + \frac{\partial^2}{\partial x_j \partial y} B \frac{\partial R_j v}{\partial y} - \frac{\partial^2 B}{\partial y^2} \cdot \frac{\partial}{\partial x_j} R_j v.\end{aligned}$$

Now let $e_1, e_2, \dots, e_{n-1}, e_n$ be the standard basis of $\mathbb{R}^n$, with e_n pointing in the direction of the y-axis. Then we can rewrite the right side as

$$\left\langle \left(-\frac{\partial^2}{\partial x_1 \partial y} B, \frac{-\partial^2 B}{\partial x_2 \partial y}, \dots, \frac{-\partial^2 B}{\partial x_{n-1} \partial y}, \sum_{j=1}^{n-1} \frac{\partial^2 B}{\partial x_j^2} \right), \nabla v \right\rangle$$
$$+ \sum_{j=1}^{n-1} \left\langle \frac{\partial^2 B}{\partial x_j \partial y} e_n, \nabla R_j v \right\rangle - \sum_{j=1}^{n-1} \left\langle \frac{\partial^2 B}{\partial y^2} e_j, \nabla R_j v \right\rangle.$$

Let

$$\boldsymbol{\alpha} = \left(-\frac{\partial^2 B}{\partial x_1 \partial y}, \frac{-\partial^2 B}{\partial x_2 \partial y}, \dots, \frac{-\partial^2 B}{\partial x_{n-1} \partial y}, \sum_{j=1}^{n-1} \frac{\partial^2 B}{\partial x_j^2} \right), \qquad \boldsymbol{\beta}_j = \frac{\partial^2 B}{\partial x_j \partial y} e_n - \frac{\partial^2 B}{\partial y^2} e_j.$$

Note that div $\boldsymbol{\beta}_j = 0$, div $\boldsymbol{\alpha} = 0$, and

$$\begin{aligned}\int_D \frac{1}{|Y-Q|^{n-2}} \frac{\partial}{\partial y} v(Y)\, dY &= \int_D \langle \boldsymbol{\alpha}, \nabla v \rangle + \sum_{j=1}^{n-1} \int_D \langle \boldsymbol{\beta}_j, \nabla R_j v \rangle \\ &= \int_{\partial D} v(P) \cdot \langle \boldsymbol{\alpha}(P), N_P \rangle\, d\sigma(P) + \sum_{j=1}^{n-1} \int_{\partial D} R_j v(P) \cdot \langle \boldsymbol{\beta}_j(P), N_P \rangle\, d\sigma(p)\end{aligned}$$

by the divergence theorem. This can be rewritten as

$$\int_{\partial D} \left[-n_j(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} B(P-Q) + n_n(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_j} B(P-Q) \right] v(P)\, d\sigma(P)$$
$$+ \sum_{j=1}^{n-1} \int_{\partial D} \left[n_n(P) \frac{\partial}{\partial P_j} \frac{\partial}{\partial P_n} B(P-Q) - n_j(P) \frac{\partial^2}{\partial P_n^2} B(P-Q) \right] R_j v(P)\, d\sigma(P).$$

Hence,

$$\frac{\partial}{\partial N_Q}\int_D \frac{1}{|Y-Q|^{n-2}}\frac{\partial}{\partial y}v(Y)\,dY$$

$$= \int_{\partial D}\left[-n_j(P)\frac{\partial}{\partial P_j}\frac{\partial}{\partial P_n}\langle \nabla B(P-Q), N_Q\rangle + n_n(P)\frac{\partial^2}{\partial P_n^2}\langle \nabla B(P-Q), N_Q\rangle\right] v(P)\,d\sigma(P)$$

$$+ \sum_{j=1}^{n-1}\int_{\partial D}\left[n_n(P)\frac{\partial}{\partial P_j}\frac{\partial}{\partial P_n}\langle \nabla B(P-Q), N_Q\rangle - n_j(P)\frac{\partial^2}{\partial P_n^2}\langle \nabla B(P-Q), N_Q\rangle\right] R_j v(P)\,d\sigma(P).$$

But, by the Coifman–McIntosh–Meyer theorem [**3**],

$$\frac{\partial}{\partial P_j}\frac{\partial}{\partial P_i}\frac{\partial}{\partial Q_k}B(P-Q)$$

is the kernel of a bounded operator in $L^2(\partial D, d\sigma)$. Thus,

$$\left\|\frac{\partial}{\partial N_Q}\int_D \frac{1}{|Y-Q|^{n-2}}\frac{\partial v}{\partial y}(Y)\,dY\right\| \leqslant C\left\{\|v\|_{L^2(\partial D, d\sigma)} + \sum_{j=1}^{n-1}\|R_j v\|_{L^2(\partial D, d\sigma)}\right\}.$$

Finally, we invoke the result of Dahlberg [**8**] that

$$\|R_j v\|_{L^2(\partial D, d\sigma)} \leqslant C\|v\|_{L^2(\partial D, d\sigma)}.$$

This concludes the proof of Theorem 2.3.3.

As a final comment we would like to point out that in this exposition we have emphasized nontangential maximal function estimates, but that optimal Sobolev space estimates also hold. For example, the solution **u** of (2.1.1) is in the Sobolev space $H^{1/2}(D)$, that of (2.1.2) in the Sobolev space $H^{3/2}(D)$, and the same is true for **u** in 2.1.3(c). The solution of (2.2.1) is in $H^{1/2}(D)$, while that of (2.2.9) is in $H^{3/2}(D)$. Finally, the solution u of (2.3.1) is in $H^{3/2}(D)$. All of these results can be proved in a unified fashion, using a variant of the proof of Lemma 2.1.11. The details will appear in a forthcoming paper of Dahlberg and Kenig [**10**].

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