Formulaire d'intégrales singulières

Martin AVERSENG

October 2, 2017

1 $I_0(M) = \int_{[AB]} \ln|M - Y| d\Gamma(Y)$

A and B are two points in the plane. We parametrize the integral according to the figure. A coordinate system is chosen so that M lies on the $y-axis$, with coordinate $M:(0,d)$, and A and B both lie on the $x-axis$, with respective coordinates $A:(a,0)$ and $B:(b,0)$. We let $\overrightarrow{u} = \frac{\overrightarrow{AB}}{|AB|}$ and $\overrightarrow{n}$ a couple of unit tangent and normal vector of the segment $[AB]$. This way, we have $a = \overrightarrow{MA} \cdot \overrightarrow{u}$, $b = \overrightarrow{MB} \cdot \overrightarrow{u}$, and $d = \left|\overrightarrow{AM} \cdot \overrightarrow{n}\right|$ (the integral is unchanged by changing the orientation of $\overrightarrow{n}$).

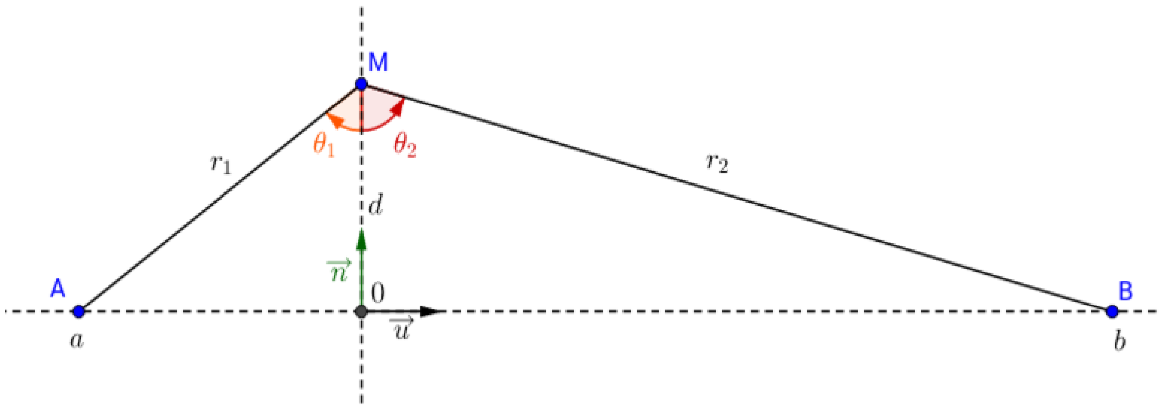

Figure 1: Parametrization for the computation of $I_0(M)$

According to this choice of parametrization,

$$\begin{aligned} I_0(M) &= \int_a^b \ln\sqrt{y^2+d^2}\,dy \\ &= \left[y \ln\sqrt{d^2+y^2} - y + d \arctan\frac{y}{d}\right]_a^b \\ &= b\ln r_2 - a\ln r_1 - |AB| + d(\theta_2 - \theta_1) \end{aligned}$$

By continuity of the single layer potential, the value of $I_0(M)$ for M aligned with A and B is obtained by sending d to 0. If M is in the segment, $\theta_2 - \theta_1 = \pi$, while it is 0 otherwise.

2 $I_1(M) = \int_{[AB]} \ln|M - Y| \phi_1(Y) d\Gamma(Y)$

The function ϕ_1 is an affine function of Y. Using the same parametrization as previously, and taking $\phi_1(y) = \alpha y + \beta$, we need only compute the integral $I_1 = \int_a^b y \ln\sqrt{d^2 + y^2} y$. We have

$$\begin{aligned} I_1 &= \left[\frac{1}{2} \ln\sqrt{d^2 + y^2}(d^2 + y^2) - \frac{1}{4}(d^2 + y^2) \right]_a^b \\ &= \frac{1}{2}\left(r_2^2 \ln r_2 - r_1^2 \ln r_1 \right) + \frac{1}{4}\left(r_2^2 - r_1^2 \right) \end{aligned}$$

Accordingly,

$$\begin{aligned} I_1(M) &= \alpha I_1 + \beta I_0(M) \\ &= \frac{\alpha}{2}\left(r_2^2 \ln r_2 - r_1^2 \ln r_1 \right) + \frac{\alpha}{4}\left(r_2^2 - r_1^2 \right) + \beta \left[b \ln r_2 - a \ln r_1 - |AB| + d(\theta_2 - \theta_1) \right] \end{aligned}$$