Boundary element preconditioners for a hypersingular integral equation on an interval *

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We propose an almost optimal preconditioner for the iterative solution of the Galerkin equations arising from a hypersingular integral equation on an interval. This preconditioning technique, which is based on the single layer potential, was already studied for closed curves [11,14]. For a boundary element trial space, we show that the condition number is of order $(1 + |\log h_{\text{min}}|)^2$, where h_{min} is the length of the smallest element. The proof requires only a mild assumption on the mesh, easily satisfied by adaptive refinement algorithms.

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1. Introduction

Consider the model hypersingular integral equation on an interval,

$$(Du)(x) := -\frac{1}{2\pi}\,\mathrm{fp}\int_{-1}^{1}\frac{u(y)}{(x-y)^2}\,\mathrm{d}y = f(x) \quad \text{for } -1 < x < 1, \tag{1.1}$$

where fp means the finite part in the sense of Hadamard (see below). Equations of the type (1.1) arise in two-dimensional screen and crack problems [16]. To simplify the exposition, we have taken the interval $\Gamma := (-1, 1)$ as the domain of integration, but our results are valid more generally for a Lipschitz arc. This paper analyses a preconditioner for the Galerkin equations arising from (1.1), based on the single layer potential. Much of our theory does not depend on the choice of the trial space, but our interest is mainly in standard, piecewise-polynomial, boundary element spaces.

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Let $-1 < x - \varepsilon < x + \varepsilon < 1$ and put $I_\varepsilon(x) = (-1, 1) \setminus (x - \varepsilon, x + \varepsilon)$. If $u(\pm 1) = u'(\pm 1) = 0$, then after integrating by parts twice and performing some simple Taylor expansions, we find that

$$-\int_{I_\varepsilon(x)} \frac{u(y)}{(x-y)^2}\,\mathrm{d}y = -\frac{2u(x)}{\varepsilon} + \int_{-1}^{1} u''(y)\log|y-x|\,\mathrm{d}y + \mathrm{O}\big(\varepsilon|\log\varepsilon|\big),$$

so, taking the finite part [9, p. 70] as $\varepsilon \to 0^+$, we have

$$Du(x) = \frac{1}{2\pi}\int_{-1}^{1} u''(y)\log|y-x|\,\mathrm{d}y.$$

Furthermore, it is not difficult to show that if u and v are, for instance, any C^∞ functions with compact support in $\mathbb{R}$, then

$$-\int_{-\infty}^{\infty}\left(\mathrm{fp}\int_{-\infty}^{\infty}\frac{u(y)}{(x-y)^2}\,\mathrm{d}y\right)v(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[u(x)-u(y)][v(x)-v(y)]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y;$$

cf. [5, theorem I-3]. Therefore, if the supports of u and v lie in the closed interval $\overline{\Gamma} = [-1, 1]$, then

$$\langle Du, v\rangle = \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{[u(x)-u(y)][v(x)-v(y)]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y + \frac{2}{\pi}\int_{-1}^{1}\frac{u(x)v(x)}{1-x^2}\,\mathrm{d}x, \quad (1.2)$$

where $\langle\cdot,\cdot\rangle$ denotes the bilinear pairing

$$\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,\mathrm{d}x. \quad (1.3)$$

We see from (1.2) that the bilinear form associated with D is symmetric and strictly positive-definite. Thus, if W_h is any conforming trial space, i.e., any finite-dimensional space of functions defined on Γ such that

$$\langle Du_h, u_h\rangle < \infty \quad \text{for all } u_h \in W_h, \quad (1.4)$$

then we can find a unique solution $u_h \in W_h$ to the Galerkin equations

$$\langle Du_h, v_h\rangle = \langle f, v_h\rangle \quad \text{for all } v_h \in W_h. \quad (1.5)$$

In view of (1.2) the trial functions, if they are to satisfy (1.4), must vanish at both end points:

$$u_h(\pm 1) = 0 \quad \text{for all } u_h \in W_h.$$

Given a basis $\{\phi_k\}_{k=1}^N$ for W_h, we introduce the corresponding $N \times N$ stiffness matrix D_h with entries

$$D_h[l, k] := \langle D\phi_k, \phi_l\rangle.$$

Obviously, D_h is symmetric and strictly positive-definite.

We take a partition Γ_h consisting of $N+1$ subintervals determined by the points

$$-1 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1,$$

with local mesh sizes $h_k = x_k - x_{k-1}$ ($1 \leqslant k \leqslant N+1$) and global mesh sizes

$$h_{\max} := \max_{1\leqslant k\leqslant N+1} h_k \quad \text{and} \quad h_{\min} = \min_{1\leqslant k\leqslant N+1} h_k.$$

For $r \geqslant 2$ and $0 \leqslant m \leqslant r-2$, let $S_0^{r,m}(\Gamma_h)$ denote the space of C^m splines of order r (i.e., degree at most $r-1$) that vanish at ± 1. Let us put $W_h = S_0^{r,m}(\Gamma_h)$, and suppose that the ϕ_k are standard nodal basis functions. For example, when $r=2$ and $m=0$, the ϕ_k are just the hat functions determined by Γ_h. The stability and convergence of the Galerkin boundary element method for the hypersingular equation (1.1) was proved in [16]. Also, it was shown in [1] that if the *local* mesh ratio is bounded, i.e., if $h_k \leqslant c h_l$ whenever $|k-l| = 1$, then

$$c_1 N^{-1}\bigl[1 + \bigl|\log(N h_{\min})\bigr|\bigr]^{-1}(\underline{u}, \underline{u}) \leqslant (D_h \underline{u}, \underline{u}) \leqslant c_2(\underline{u}, \underline{u}) \quad \text{for all } \underline{u} \in \mathbb{R}^N, \quad (1.6)$$

where $(\cdot,\cdot)$ is the Euclidean inner product. Hence, in this case the ℓ_2 condition number of the stiffness matrix satisfies the upper bound

$$\kappa(D_h) \leqslant cN\bigl[1 + \bigl|\log(N h_{\min})\bigr|\bigr].$$

Moreover, numerical experiments in [1] indicate that, although not always sharp, this bound is fairly realistic for typical meshes of practical interest, i.e., the condition number of D_h is roughly proportional to N.

Our strategy for preconditioning D_h is based on the weakly singular integral operator V defined by

$$(Vt)(x) := \frac{1}{2\pi}\int_{-1}^{1} \log\left(\frac{1}{|x-y|}\right) t(y)\,\mathrm{d}y \quad \text{for } -1 < x < 1. \quad (1.7)$$

It was shown in [11,14] that when Γ is a *closed* curve the inverse of V can be discretised to yield a matrix spectrally equivalent to D_h. This property of the matrices is connected with the fact that the operator VD is bounded on $L_2(\Gamma)$. (Actually, in our technical proofs it is the mapping properties of V and D in half-order Sobolev norms that are decisive.)

When Γ is the interval $(-1,1)$, or indeed any *open* arc, the operator VD is no longer bounded on $L_2(\Gamma)$, and the preconditioner is no longer spectrally equivalent to D_h. However, we are able to estimate the growth of the condition number of the preconditioned system in terms of the growth of an imbedding constant for the finite-dimensional subspace W_h.

We will show that for *any* trial space W_h satisfying (1.4), if $\Theta_h \geqslant 1$ and if

$$\int_{-1}^{1} \frac{[u_h(x)]^2}{1-x^2}\,\mathrm{d}x \leqslant c\Theta_h\left(\int_{-1}^{1}\bigl[u_h(x)\bigr]^2\,\mathrm{d}x + \int_{-1}^{1}\int_{-1}^{1}\frac{[u_h(x)-u_h(y)]^2}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y\right) \quad (1.8)$$

for all $u_h \in W_h$, then

$$c_1\left\langle V^{-1}w, w\right\rangle \leqslant \langle Dw, w\rangle \leqslant c_2\Theta_h\left\langle V^{-1}w, w\right\rangle \quad \text{for all } w \in W_h. \tag{1.9}$$

Hence, defining the matrix C_h by

$$C_h[l,k] = \left\langle V^{-1}\phi_k, \phi_l\right\rangle, \tag{1.10}$$

we have

$$c_1(C_h\underline{u}, \underline{u}) \leqslant (D_h\underline{u}, \underline{u}) \leqslant c_2\Theta_h(C_h\underline{u}, \underline{u}) \quad \text{for all } \underline{u} \in \mathbb{R}^N, \tag{1.11}$$

and, therefore, $\kappa(C_h^{-1}D_h) \leqslant c\Theta_h$. In cases of interest Θ_h grows slowly in comparison to $\kappa(D_h)$, so C_h would be an effective preconditioner for D_h, except for the fact that the matrix entries (1.10) are not readily computable. Thus, instead of C_h we shall use as our preconditioner the matrix

$$\widetilde{C}_h := M_h V_h^{-1} M_h, \tag{1.12}$$

where V_h is just the stiffness matrix of V, and M_h is the mass matrix:

$$V_h[l,k] := \langle V\phi_k, \phi_l\rangle \quad \text{and} \quad M_h[l,k] := \langle \phi_k, \phi_l\rangle.$$

Both V_h and M_h are symmetric and strictly positive-definite. For a nodal finite element basis, M_h is also sparse. In the case of the hat functions, M_h is even strictly diagonally dominant. In a standard preconditioned conjugate gradient method [6, section 10.3] the cost of applying the preconditioner $\widetilde{C}_h$ at each step, i.e., the cost of multiplying a vector by

$$\widetilde{C}_h^{-1} = M_h^{-1} V_h M_h^{-1},$$

is of the same order as the cost of multiplying a vector by the original stiffness matrix D_h. We shall prove that if, in addition to (1.4) and (1.8), the subspace W_h has a certain stability property, then C_h and $\widetilde{C}_h$ are spectrally equivalent to each other and, therefore,

$$c_1\left(\widetilde{C}_h\underline{u}, \underline{u}\right) \leqslant (D_h\underline{u}, \underline{u}) \leqslant c_2\Theta_h\left(\widetilde{C}_h\underline{u}, \underline{u}\right) \quad \text{for all } \underline{u} \in \mathbb{R}^N, \tag{1.13}$$

implying that

$$\kappa\left(\widetilde{C}_h^{-1}D_h\right) \leqslant c\Theta_h. \tag{1.14}$$

We will also show that (1.8) holds with

$$\Theta_h = \left[1 + |\log h_{\min}|\right]^2 \quad \text{when } W_h = S_0^{r,m}(\Gamma_h). \tag{1.15}$$

When Γ is a closed curve as in our earlier papers [11,14], no factor Θ_h is needed in (1.14), i.e., $\kappa(\widetilde{C}_h^{-1}D_h)$ is bounded independently of h. In fact, for a closed curve the second term in (1.2) is absent so we have no need for an estimate like (1.8).

The organisation of the paper is as follows. In section 2, we summarise some standard facts about Sobolev spaces and the mapping properties of the operators D and V, and prove some technical estimates needed for our analysis. The main result (1.13) is proved in section 3, after which, in section 4, we verify that the boundary element space $S_0^{r,m}(\Gamma_h)$ satisfies (1.15), and discuss the required stability property. Finally, section 5 confirms the predicted logarithmic growth of $\kappa(\widetilde{C}_h^{-1} D_h)$ with some numerical results for piecewise-linear boundary elements.

2. Sobolev spaces

Our analysis of the preconditioner $\widetilde{C}_h$ uses as a key tool fractional- and negative-order Sobolev norms. Below we summarise the relevant definitions and results; for further details see [2, chapter 2; 7,10].

For $s \in \mathbb{R}$, we define $H^s(\mathbb{R})$, the Sobolev space on the whole real line, to be the space of real-valued temperate distributions u for which the Fourier transform $\hat{u}$ exists as a locally square-integrable function, and for which the norm

$$\|u\|_{H^s(\mathbb{R})} := \left(\int_{-\infty}^{\infty} \left(1 + \xi^2\right)^s \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi \right)^{1/2}$$

is finite. Throughout this section, we let Γ denote *any* open subinterval of $\mathbb{R}$, not just the interval $(-1, 1)$. We define two kinds of Sobolev spaces on Γ,

$$H^s(\Gamma) := \left\{ u\colon\ u = U|_\Gamma \text{ for some } U \in H^s(\mathbb{R}) \right\},$$
$$\widetilde{H}^s(\Gamma) := \left\{ u \in H^s(\mathbb{R})\colon\ \operatorname{supp} u \subseteq \overline{\Gamma} \right\},$$

with norms

$$\|u\|_{H^s(\Gamma)} := \inf_{u = U|_\Gamma} \|U\|_{H^s(\mathbb{R})} \quad \text{and} \quad \|u\|_{\widetilde{H}^s(\Gamma)} := \|u\|_{H^s(\mathbb{R})}.$$

These two families of spaces are related by duality with respect to the bilinear pairing $\langle u, v\rangle := \int_\Gamma u(x)v(x)\,\mathrm{d}x$. In fact,

$$\left[H^s(\Gamma)\right]' = \widetilde{H}^{-s}(\Gamma) \quad \text{and} \quad \left[\widetilde{H}^s(\Gamma)\right]' = H^{-s}(\Gamma) \quad \text{for } -\infty < s < \infty.$$

Also, if $u \in \widetilde{H}^s(\Gamma)$, then $u|_\Gamma \in H^s(\Gamma)$ and

$$\|u|_\Gamma\|_{H^s(\Gamma)} \leqslant c\|u\|_{\widetilde{H}^s(\Gamma)}.$$

In particular, if $s \geqslant 0$ then the elements of $\widetilde{H}^s(\Gamma)$ can be viewed as functions in $L_2(\Gamma)$, and we have an imbedding $\widetilde{H}^s(\Gamma) \subseteq H^s(\Gamma)$. Moreover, by the Sobolev imbedding theorem, the pointwise values of $u \in H^s(\Gamma)$ are meaningful if $s > 1/2$.

We now focus on the case when $0 < s < 1$.

Equivalent norms for $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ can be defined in terms of the semi-norm $|\cdot|_{s,\Gamma}$ defined by

$$|u|_{s,\Gamma}^2 := \int_\Gamma \int_\Gamma \frac{[u(x) - u(y)]^2}{|x - y|^{1+2s}} \,\mathrm{d}x\,\mathrm{d}y \quad \text{for } 0 < s < 1.$$

In fact, it is not difficult to show using Plancherel's theorem that

$$|u|^2_{s,\mathbb{R}} = a_s \int_{-\infty}^{\infty} |\xi|^{2s} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi, \quad \text{where} \quad a_s = 2\int_0^\infty \frac{|\mathrm{e}^{\mathrm{i}2\pi t} - 1|^2}{t^{2s+1}} \,\mathrm{d}t \sim \frac{1}{s(1-s)},$$

so when $\Gamma = \mathbb{R}$,

$$c_1\|u\|^2_{H^s(\Gamma)} \leqslant \|u\|^2_{L_2(\Gamma)} + s(1-s)|u|^2_{s,\Gamma} \leqslant c_2\|u\|^2_{H^s(\Gamma)} \quad \text{for } 0 < s < 1, \tag{2.1}$$

where the constants c_1 and c_2 are independent of s. In fact, (2.1) holds for any subinterval $\Gamma \subseteq \mathbb{R}$, as one sees with the help of a suitable extension operator from Γ to $\mathbb{R}$ (local reflection about the end points would suffice). If $u \in \widetilde{H}^s(\Gamma)$, so that $\operatorname{supp} u \subseteq \overline{\Gamma}$, then an equivalent norm to $\|u\|_{\widetilde{H}^s(\Gamma)} = \|u\|_{H^s(\mathbb{R})}$ is

$$\|u\|^2_{L_2(\mathbb{R})} + s(1-s)|u|^2_{s,\mathbb{R}} = \|u\|^2_{L_2(\Gamma)} + s(1-s)\left(|u|^2_{s,\Gamma} + \int_\Gamma \big[u(x)\big]^2 w_{s,\Gamma}(x)\,\mathrm{d}x\right),$$

where the weight $w_{s,\Gamma}$ is given by

$$w_{s,\Gamma}(x) = \int_{\mathbb{R}\setminus\Gamma} |y-x|^{-1-2s}\,\mathrm{d}y \quad \text{for } x \in \Gamma.$$

For instance, in the case of the half-line $\Gamma = \mathbb{R}^+ = (0,\infty)$, we see that $w_{s,\mathbb{R}^+}(x) = x^{-2s}/(2s)$. The next theorem summarises the relationship between $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ for $0 < s < 1$.

Theorem 2.1. If $0 < s < 1/2$, then $\widetilde{H}^s(\Gamma) = H^s(\Gamma)$ with equivalent norms. If $1/2 < s < 1$, then $\widetilde{H}^s(\Gamma)$ is the closed subspace of $H^s(\Gamma)$ consisting of those $u \in H^s(\Gamma)$ that vanish at the end points of Γ. However, $\widetilde{H}^{1/2}(\Gamma)$ is a proper subspace of $H^{1/2}(\Gamma)$, with a strictly finer topology. Thus, there exist unbounded sequences in $\widetilde{H}^{1/2}(\Gamma)$ that are bounded in $H^{1/2}(\Gamma)$.

For a proof of this theorem, see [10, pp. 58–66]. In fact, the next lemma gives one of the key steps in the proof, which we repeat here in order to show how the constant depends on s.

Lemma 2.2. Let $u\colon [0,\infty) \to \mathbb{R}$ be a C^∞ function with compact support. If $0 < s < 1/2$, then

$$\int_0^\infty x^{-2s}\big[u(x)\big]^2\,\mathrm{d}x \leqslant \frac{3}{(s-1/2)^2}\int_0^\infty\int_0^\infty \frac{[u(x)-u(y)]^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

If, in addition, $u(0) = 0$, then this estimate is valid also for $1/2 < s < 1$.

Proof. Note that the double integral converges if $0 < s < 1$. For $x > 0$, define

$$v(x) = \frac{1}{x}\int_0^x \big[u(x) - u(y)\big]\,\mathrm{d}y = u(x) - \frac{1}{x}\int_0^x u(y)\,\mathrm{d}y$$

and

$$w(x) = \int_x^\infty \frac{v(y)}{y}\,\mathrm{d}y.$$

Since $v'(x) = u'(x) - x^{-1}v(x)$ and $w'(x) = -x^{-1}v(x)$, we see that

$$u'(x) = v'(x) - w'(x).$$

Furthermore, u has compact support, and both $v(x)$ and $w(x)$ tend to zero as x tends to infinity, so

$$u(x) = v(x) - w(x).$$

By the Cauchy–Schwarz inequality,

$$\bigl[v(x)\bigr]^2 \leqslant \frac{1}{x}\int_0^x \bigl[u(y) - u(x)\bigr]^2\,\mathrm{d}y,$$

implying

$$\begin{aligned}\int_0^\infty x^{-2s}\bigl[v(x)\bigr]^2\,\mathrm{d}x &\leqslant \int_0^\infty x^{-1-2s}\int_0^x \bigl[u(y) - u(x)\bigr]^2\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_0^\infty \int_y^\infty x^{-1-2s}\bigl[u(y) - u(x)\bigr]^2\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant \int_0^\infty \int_0^\infty \frac{[u(x) - u(y)]^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for $0 < s < 1$. Also, Hardy's inequality shows that if $s < 1/2$ then

$$\begin{aligned}\int_0^\infty x^{-2s}\bigl[w(x)\bigr]^2\,\mathrm{d}x &\leqslant \int_0^\infty \left(x^{1/2-s}\int_x^\infty |v(y)|\frac{\mathrm{d}y}{y}\right)^2\frac{\mathrm{d}x}{x}\\ &\leqslant \frac{1}{(1/2-s)^2}\int_0^\infty \bigl[x^{1/2-s}v(x)\bigr]^2\frac{\mathrm{d}x}{x},\end{aligned}$$

and the first part of the lemma follows.

One easily verifies by Taylor expansion that $v(x) \to 0$ as $x \to 0^+$, so $w(x) = v(x) - u(x) \to -u(0)$ as $x \to 0^+$. Thus, if $u(0) = 0$ then $w(0) = 0$, implying that

$$w(x) = w(x) - w(0) = w(x) - \int_0^\infty \frac{v(y)}{y}\,\mathrm{d}y = -\int_0^x \frac{v(y)}{y}\,\mathrm{d}y.$$

Hence, by Hardy's inequality, if $s > 1/2$ then

$$\begin{aligned}\int_0^\infty x^{-2s}\bigl[w(x)\bigr]^2\,\mathrm{d}x &\leqslant \int_0^\infty \left(x^{-(s-1/2)}\int_0^x \bigl|v(y)\bigr|\frac{\mathrm{d}y}{y}\right)^2\frac{\mathrm{d}x}{x}\\ &\leqslant \frac{1}{(s-1/2)^2}\int_0^\infty \bigl[x^{-(s-1/2)}v(x)\bigr]^2\frac{\mathrm{d}x}{x},\end{aligned}$$

giving the second part of the lemma. $\square$

Later, in the proof of theorem 4.1, we will use the following estimate.

Lemma 2.3. If $1/2 < s \leqslant 1$, and if $u \in H^s(\Gamma)$ vanishes at the end points of Γ, then

$$\|u\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant \frac{c}{s-1/2}\|u\|_{H^s(\Gamma)},$$

where the constant c does not depend on s.

Proof. After localising via suitable cutoff functions, we may assume that $\Gamma = \mathbb{R}^+$. By lemma 2.2, if $1/2 < s < 1$ then

$$\int_\Gamma \left[u(x)\right]^2 w_{s,\Gamma}(x)\,\mathrm{d}x = \frac{1}{2s}\int_0^\infty x^{-2s}\left[u(x)\right]^2\mathrm{d}x \leqslant \frac{3}{2s(s-1/2)^2}|u|^2_{s,\Gamma},$$

so, recalling (2.1) and the associated discussion of equivalent norms for $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$, we have

$$\begin{aligned}\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leqslant \|u\|^2_{\widetilde{H}^s(\Gamma)} &\leqslant c\|u\|^2_{L_2(\Gamma)} + cs(1-s)\left(|u|^2_{s,\Gamma} + \int_\Gamma \left[u(x)\right]^2 w_{s,\Gamma}(x)\,\mathrm{d}x\right)\\ &\leqslant c\left(\|u\|^2_{L_2(\Gamma)} + \frac{s(1-s)}{(s-1/2)^2}|u|^2_{s,\Gamma}\right) \leqslant \frac{c}{(s-1/2)^2}\|u\|^2_{H^s(\Gamma)},\end{aligned}$$

as claimed. The case $s = 1$ is now trivial. □

Crucial for our analysis are the following properties of the integral operators D and V defined in (1.1) and (1.7).

Theorem 2.4. Let Γ be a bounded subinterval of the real line. The operators

$$D: \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \quad \text{and} \quad V: \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

are bounded. Furthermore,

$$c_1\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leqslant \langle Du, u\rangle \leqslant c_2\|u\|^2_{\widetilde{H}^{1/2}(\Gamma)} \quad \text{for all } u \in \widetilde{H}^{1/2}(\Gamma),$$

and, provided the length of Γ is less than 4,

$$c_1\|t\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \leqslant \langle Vt, t\rangle \leqslant c_2\|t\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \quad \text{for all } t \in \widetilde{H}^{-1/2}(\Gamma).$$

We can prove this theorem by extending Γ to a closed, Lipschitz curve and applying the results of Costabel [4]. The positive-definiteness of D can be seen from (1.2). The positive-definiteness of V depends on the fact that the logarithmic capacity of a line segment is one-quarter of its length [8, pp. 280–289].

3. Estimating the condition number

In this section, we prove the estimate (1.14) for the condition number of the preconditioned matrix $\widetilde{C}_h^{-1} D_h$. The analysis in [14], covering the case when Γ is a closed curve, carries over to the present setting with a few modifications. A key assumption is that the trial space W_h has the following *stability property*:

$$\|u_h\|_{H^{1/2}(\Gamma)} \leqslant c \sup_{0 \neq v_h \in W_h} \frac{|\langle u_h, v_h \rangle|}{\|v_h\|_{\widetilde{H}^{-1/2}(\Gamma)}} \quad \text{for all } u_h \in W_h. \tag{3.1}$$

This estimate is used in the second part of the next theorem.

Theorem 3.1. Let $\{\phi_k\}_{k=1}^N$ be any basis for a trial space $W_h \subseteq \widetilde{H}^{1/2}(\Gamma)$.

(1) The matrices C_h and $\widetilde{C}_h$, defined by (1.10) and (1.12), satisfy

$$\left(\widetilde{C}_h \underline{u}, \underline{u}\right) \leqslant (C_h \underline{u}, \underline{u}) \quad \text{for all } \underline{u} \in \mathbb{R}^N.$$

(2) If the trial space has the stability property (3.1), then

$$(C_h \underline{u}, \underline{u}) \leqslant c\left(\widetilde{C}_h \underline{u}, \underline{u}\right) \quad \text{for all } \underline{u} \in \mathbb{R}^N.$$

Proof. Given $\underline{u} \in \mathbb{R}^N$, we put

$$u_h = \sum_{k=1}^N u_k \phi_k \in \widetilde{H}^{1/2}(\Gamma) \subseteq H^{1/2}(\Gamma) \quad \text{and} \quad t = V^{-1} u_h \in \widetilde{H}^{-1/2}(\Gamma).$$

In this way,

$$(C_h \underline{u}, \underline{u}) = \left\langle V^{-1} u_h, u_h \right\rangle = \langle t, u_h \rangle = \langle t, Vt \rangle.$$

Next, let $t_h \in W_h$ be the unique solution of

$$\langle V t_h, v_h \rangle = \langle u_h, v_h \rangle \quad \text{for all } v_h \in W_h, \tag{3.2}$$

or equivalently, $t_h = \sum_{k=1}^N t_k \phi_k$ where $\underline{t}$ is the unique solution of the linear system

$$V_h \underline{t} = M_h \underline{u}.$$

The matrices M_h and V_h are both symmetric, so

$$\begin{aligned} \left(\widetilde{C}_h \underline{u}, \underline{u}\right) &= \left(M_h V_h^{-1} M_h \underline{u}, \underline{u}\right) = \left(V_h^{-1} M_h \underline{u}, M_h \underline{u}\right) \\ &= \left(V_h^{-1} M_h \underline{u}, V_h \underline{t}\right) = (M_h \underline{u}, \underline{t}) = \langle u_h, t_h \rangle = \langle V t, t_h \rangle, \end{aligned}$$

and since $\langle V t, t_h \rangle = \langle u_h, t_h \rangle = \langle V t_h, t_h \rangle$, we find that

$$0 \leqslant \left\langle V(t - t_h), t - t_h \right\rangle = \langle V t, t \rangle - \langle V t_h, t_h \rangle = (C_h \underline{u}, \underline{u}) - \left(\widetilde{C}_h \underline{u}, \underline{u}\right).$$

This completes the proof of (1).

To prove (2), we retain the notation above, and deduce from the stability property (3.1) and the Galerkin equations (3.2) for t_h that

$$\|u_h\|_{H^{1/2}(\Gamma)} \leqslant c \sup_{0\neq v_h\in W_h} \frac{|\langle Vt_h, v_h\rangle|}{\|v_h\|_{\widetilde{H}^{-1/2}(\Gamma)}} \leqslant c\|Vt_h\|_{H^{1/2}(\Gamma)} \leqslant c\|t_h\|_{\widetilde{H}^{-1/2}(\Gamma)}.$$

Thus,

$$(C_h\underline{u}, \underline{u}) = \langle V^{-1}u_h, u_h\rangle \leqslant c\|u_h\|^2_{H^{1/2}(\Gamma)} \leqslant c\|t_h\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \leqslant c\langle Vt_h, t_h\rangle = c\big(\widetilde{C}_h\underline{u}, \underline{u}\big),$$

as claimed. □

The spectral equivalence of C_h and $\widetilde{C}_h$, established in theorem 3.1 above, implies that $\widetilde{C}_h$ is almost spectrally equivalent to D_h.

Theorem 3.2. Let $\{\phi_k\}_{k=1}^N$ be any basis for a trial space $W_h \subseteq \widetilde{H}^{1/2}(\Gamma)$. If $\Theta_h \geqslant 1$ is such that (1.8) holds, and if W_h has the stability property (3.1), then

$$c_1\big(\widetilde{C}_h\underline{u}, \underline{u}\big) \leqslant (D_h\underline{u}, \underline{u}) \leqslant c_2\Theta_h\big(\widetilde{C}_h\underline{u}, \underline{u}\big) \quad \text{for all } \underline{u} \in \mathbb{R}^N,$$

and hence $\kappa(\widetilde{C}_h^{-1}D_h) \leqslant c\Theta_h$.

Proof. As explained in section 1, it suffices to establish (1.9). By theorem 2.4, the operator V has a bounded inverse $V^{-1}: H^{1/2}(\Gamma) \to \widetilde{H}^{-1/2}(\Gamma)$ satisfying

$$c_1\|w\|^2_{H^{1/2}(\Gamma)} \leqslant \langle V^{-1}w, w\rangle \leqslant c_2\|w\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } w \in H^{1/2}(\Gamma).$$

Since $\|w\|^2_{H^{1/2}(\Gamma)} \leqslant \|w\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leqslant c\langle Dw, w\rangle$, the first inequality in (1.9) follows. Also, since $\Theta_h \geqslant 1$ satisfies (1.8), if $w \in W_h$ then

$$\langle Dw, w\rangle \leqslant c\|w\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leqslant c\|w\|^2_{H^{1/2}(\Gamma)} + c\int_{-1}^{1} \frac{[u(x)]^2}{1-x^2}\,\mathrm{d}x \leqslant c\Theta_h\|w\|^2_{H^{1/2}(\Gamma)},$$

proving the second inequality in (1.9). □

We conclude this section by formulating a sufficient condition for the stability property (3.1), in terms of the L_2-projection $Q_h: L_2(\Gamma) \to W_h$ defined by

$$\langle Q_hu, v_h\rangle = \langle u, v_h\rangle \quad \text{for all } v_h \in W_h. \tag{3.3}$$

Lemma 3.3. If the L_2-projection onto W_h is stable in $H^{1/2}(\Gamma)$, i.e., if

$$\|Q_hu\|_{H^{1/2}(\Gamma)} \leqslant c\|u\|_{H^{1/2}(\Gamma)} \quad \text{for all } u \in H^{1/2}(\Gamma), \tag{3.4}$$

then W_h has the stability property (3.1).

Proof. We introduce a linear operator $P_h: H^{1/2}(\Gamma) \to W_h \subseteq \widetilde{H}^{-1/2}(\Gamma)$ defined by

$$\langle P_hv, w_h\rangle = \langle v, w_h\rangle_{H^{1/2}(\Gamma)} \quad \text{for all } w_h \in W_h.$$

Since

$$\|P_h v\|_{\widetilde{H}^{-1/2}(\Gamma)} = \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{|\langle P_h v, w\rangle|}{\|w\|_{H^{1/2}(\Gamma)}} = \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{|\langle P_h v, Q_h w\rangle|}{\|w\|_{H^{1/2}(\Gamma)}}$$
$$= \sup_{0\neq w\in H^{1/2}(\Gamma)} \frac{|\langle v, Q_h w\rangle_{H^{1/2}(\Gamma)}|}{\|w\|_{H^{1/2}(\Gamma)}},$$

our assumption on Q_h implies that P_h is stable:

$$\|P_h v\|_{\widetilde{H}^{-1/2}(\Gamma)} \leqslant c\|v\|_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Hence, given $u_h \in W_h$, we see by putting $v_h^* = P_h u_h$ that

$$\frac{|\langle v_h^*, u_h\rangle|}{\|v_h^*\|_{\widetilde{H}^{-1/2}(\Gamma)}} = \frac{\|u_h\|^2_{H^{1/2}(\Gamma)}}{\|P_h u_h\|_{\widetilde{H}^{-1/2}(\Gamma)}} \geqslant c\|u_h\|_{H^{1/2}(\Gamma)},$$

and so (3.1) holds. □

4. Boundary element trial spaces

We now wish to verify the two assumptions of our main result, theorem 3.2, in the case when the trial functions are splines, i.e., when $W_h = S_0^{r,m}(\Gamma_h)$.

Theorem 4.1. If $W_h = S_0^{r,m}(\Gamma_h)$, then the estimate (1.8) holds with

$$\Theta_h = \left[1 + |\log h_{\min}|\right]^2.$$

Proof. We use the following inverse inequality [12]: if $0 < \varepsilon < 1$, then

$$\|u_h\|_{H^s(\Gamma)} \leqslant c_\varepsilon h_{\min}^{1/2-s}\|u_h\|_{H^{1/2}(\Gamma)} \quad \text{for } u_h \in W_h \text{ and } 1/2 \leqslant s \leqslant 3/2-\varepsilon.$$

Hence, if $1/2 < s \leqslant 1$, then by lemma 2.3,

$$\|u_h\|_{\widetilde{H}^{1/2}(\Gamma)} \leqslant \frac{c}{s-1/2}\|u_h\|_{H^s(\Gamma)} \leqslant \frac{c h_{\min}^{1/2-s}}{s-1/2}\|u_h\|_{H^{1/2}(\Gamma)},$$

for all $u_h \in W_h$. Since the constant c is independent of s, and since elementary calculus shows that if $h_{\min} \leqslant \mathrm{e}^{-2}$ then

$$\min_{1/2<s\leqslant 1} \frac{h_{\min}^{1/2-s}}{s-1/2} = \frac{h_{\min}^{1/2-s^\star}}{s^\star-1/2} = \mathrm{e}|\log h_{\min}|, \quad \text{where} \quad s^\star = \frac{1}{2} + \frac{1}{|\log h_{\min}|} \in (1/2, 1],$$

we obtain the bound

$$\int_{-1}^{1} \frac{[u(x)]^2}{1-x^2}\,\mathrm{d}x \leqslant c\|u_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \leqslant c\left[1 + |\log h_{\min}|\right]^2\|u_h\|^2_{H^{1/2}(\Gamma)}$$

for all $u_h \in W_h$. The estimate for Θ_h follows at once. □

Turning to the second assumption of theorem 3.2, we recall from lemma 3.3 that it suffices to show the stability of the L_2-projection in $H^{1/2}(\Gamma)$. If $m = 0$, then we can make use of the following result of Crouzeix and Thomée [3, theorem 2].

Theorem 4.2. If the family of partitions Γ_h satisfies

$$\frac{h_i}{h_j} \leqslant c\alpha^{|i-j|} \quad \text{with} \quad 1 \leqslant \alpha < r^2, \tag{4.1}$$

then the L_2-projection onto $W_h = S_0^{r,0}(\Gamma_h)$ is stable in $H^1(\Gamma)$, i.e.,

$$\|Q_h u\|_{H^1(\Gamma)} \leqslant c\|u\|_{H^1(\Gamma)} \quad \text{for all } u \in H^1(\Gamma). \tag{4.2}$$

In fact, since Q_h is trivially stable in $L_2(\Gamma)$, it follows at once by simple interpolation and duality arguments that if (4.2) holds, then the L_2-projection is stable in $H^s(\Gamma)$ for $-1 \leqslant s \leqslant 1$, and in particular for $s = 1/2$. We remark that the mesh assumption (4.1) still allows strong local refinement, and can be satisfied in practice by adaptive algorithms.

A future paper [13] by the second author will prove that (3.4) can be satisfied for $m > 0$, as well as in the case when a different trial space is used to define Q_h via (3.3). The latter result is of interest if, in (1.12), the discrete weakly singular operator V_h is constructed with different trial functions from the ϕ_k used for the Galerkin approximation of the hypersingular integral equation (1.1). However, in all of these cases, the mesh condition (4.1) needs to be replaced by some appropriate condition which depends on the trial functions used.

5. Numerical results

We consider (1.1) with a constant right-hand side,

$$-\frac{1}{2\pi}\,\mathrm{fp}\int_{-1}^{1}\frac{u(y)}{(x-y)^2}\,\mathrm{d}y = 1 \quad \text{for } -1 < x < 1.$$

The exact solution is then $u(x) = 2\sqrt{1-x^2}$; cf. [15]. We solve the Galerkin equations (1.5) by a preconditioned conjugate gradient method with a relative residual reduction of $\varepsilon = 10^{-16}$, using as the preconditioner the matrix $\widetilde{C}_h$ defined in (1.12).

Table 1 includes the results for a uniform partition of the interval $(-1, +1)$ into $N + 1$ boundary elements. In this case, $h_{\min} = h_{\max} = 2/(N + 1)$ so the mesh assumption (4.1) is satisfied with $\alpha = 1$. Note that the logarithmic growth of the largest eigenvalue $\lambda_{\max}$ corresponds to the upper estimate in (1.13), whereas the smallest eigenvalue is bounded below by a constant.

Table 2 gives the results in the case when no preconditioner is used. Here, the smallest eigenvalue behaves like N^{-1}, whereas the largest one tends to a constant value, as expected from (1.6).

Table 1
Uniform mesh. Preconditioning by (1.12).

$N+1$	$\lambda_{\min}(\widetilde{C}_h^{-1}D_h)$	$\lambda_{\max}(\widetilde{C}_h^{-1}D_h)$	$\kappa(\widetilde{C}_h^{-1}D_h)$	Iterations
4	1.70E−1	3.23E−1	1.90	2
8	1.67E−1	3.83E−1	2.30	4
16	1.65E−1	4.51E−1	2.73	8
32	1.64E−1	5.28E−1	3.23	10
64	1.62E−1	6.16E−1	3.80	10
128	1.62E−1	7.14E−1	4.42	11
256	1.61E−1	8.23E−1	5.10	12
512	1.61E−1	9.41E−1	5.85	12
Theory	$\geqslant c$	$\leqslant c(\log N)^2$	$\leqslant c(\log N)^2$	

Table 2
Uniform mesh. No preconditioning.

$N+1$	$\lambda_{\min}(D_h)$	$\lambda_{\max}(D_h)$	$\kappa(D_h)$	Iterations
4	2.74E−1	5.50E−1	2.01	2
8	1.44E−1	5.55E−1	3.85	4
16	7.28E−2	5.54E−1	7.61	8
32	3.64E−2	5.62E−1	15.46	15
64	1.81E−2	5.64E−1	31.08	24
128	9.06E−3	5.65E−1	62.39	38
256	4.53E−3	5.66E−1	125.08	57
512	2.26E−3	5.66E−1	250.26	88
Theory	$\geqslant cN^{-1}$	$\leqslant c$	$\leqslant cN$	

To investigate the effect of using a non-uniform partition, we define the mesh points

$$x_k = \begin{cases} -1 + \left(\dfrac{2k}{N+1}\right)^3, & k = 0, \ldots, \dfrac{N+1}{2}, \\ +1 - \left(2 - \dfrac{2k}{N+1}\right)^3, & k = \dfrac{N+1}{2} + 1, \ldots, N+1. \end{cases}$$

In this way, we achieve a local refinement around the two end points, where the exact solution has square root singularities. Observe that $h_{\min} \sim N^{-3}$, $h_{\max} \sim N^{-1}$ and $h_i/h_j \leqslant c(i-j)^2$.

The results in tables 3 and 4 confirm that the preconditioner $\widetilde{C}_h$ is effective for non-uniform meshes. Observe that if *no* preconditioning is used, then the iteration counts for the non-uniform mesh (table 4) are almost the same as those for the uniform one (table 2). The mesh refinement has more impact on the preconditioned system, but the worsening of the condition number is relatively minor.

Table 3
Non-uniform mesh. Preconditioning by (1.12).

$N+1$	$h_{\max}/h_{\min}$	$\lambda_{\min}(\widetilde{C}_h^{-1}D_h)$	$\lambda_{\max}(\widetilde{C}_h^{-1}D_h)$	$\kappa(\widetilde{C}_h^{-1}D_h)$	Iterations
4	7	1.84E−1	3.89E−1	2.12	3
8	37	1.65E−1	5.65E−1	3.43	7
16	169	1.61E−1	7.69E−1	4.79	13
32	721	1.60E−1	1.11	6.95	15
64	2977	1.60E−1	1.61	10.02	16
128	12097	1.60E−1	2.20	13.73	16
256	48769	1.61E−1	3.22	20.03	17
512	195841	1.60E−1	4.00	24.93	18
1024	784897	1.61E−1	4.88	30.38	19
Theory	N^2	$\geqslant c$	$\leqslant c(\log N)^2$	$\leqslant c(\log N)^2$	

Table 4
Non-uniform mesh. No preconditioning.

$N+1$	$h_{\max}/h_{\min}$	$\lambda_{\min}(D_h)$	$\lambda_{\max}(D_h)$	$\kappa(D_h)$	Iterations
4	7	4.19E−1	5.35E−1	1.28	2
8	37	2.53E−1	5.86E−1	2.32	4
16	169	1.32E−1	5.87E−1	4.46	8
32	721	6.66E−1	5.87E−1	8.82	14
64	2977	3.34E−2	5.87E−1	17.58	21
128	12097	1.67E−2	5.87E−1	35.12	36
256	48769	8.36E−3	5.87E−1	70.23	54
512	195841	4.18E−3	5.87E−1	140.44	88
1024	784897	2.09E−3	5.87E−1	280.88	122
Theory	N^2	$\geqslant N^{-1}(\log N)^{-1}$	$\leqslant c$	$\leqslant cN \log N$	

For our final example, we consider a geometric mesh refinement with $h_{\min} = 2^{1-N}$, $h_{\max} = 1$ and $h_i/h_j \leqslant c2^{|i-j|}$. Although one would not use such a mesh in practice, the results are interesting from a theoretical point of view. In table 5, the growth in $\kappa(\widetilde{C}_h^{-1}D_h)$ is a little slower than predicted by our theory. More surprising are the results in table 6, which indicate that with *no* preconditioning the condition number is nearly bounded.

In conclusion, we remark that for a uniform mesh our preconditioner (1.12) gives comparable results to the additive Schwarz preconditioner considered in [15]. Furthermore, the preconditioner (1.12) can be used even for adaptively refined meshes, and no hierarchy of the element ordering is needed. Also, our preconditioner can be applied directly when using higher-order boundary elements, instead of linear ones.

Table 5
Geometric refinement with $h_{\text{min}} = 2^{1-N}$, $h_{\text{max}} = 1$. Preconditioning by (1.12).

$N+1$	$h_{\text{max}}/h_{\text{min}}$	$\lambda_{\text{min}}(\widetilde{C}_h^{-1} D_h)$	$\lambda_{\text{max}}(\widetilde{C}_h^{-1} D_h)$	$\kappa(\widetilde{C}_h^{-1} D_h)$
4	8	1.67E−1	3.43E−1	2.95
8	128	1.63E−1	7.63E−1	4.67
12	2048	1.62E−1	1.28	7.79
16	32768	1.62E−1	1.93	11.90
20	524288	1.62E−1	2.76	17.01
Theory	2^N	$\geqslant c$	$\leqslant cN^2$	$\leqslant cN^2$

Table 6
Geometric refinement with $h_{\text{min}} = 2^{1-N}$, $h_{\text{max}} = 1$. No preconditioning.

$N+1$	$h_{\text{max}}/h_{\text{min}}$	$\lambda_{\text{min}}(D_h)$	$\lambda_{\text{max}}(D_h)$	$\kappa(D_h)$
4	8	3.34E−1	5.85E−1	1.75
8	128	2.76E−1	6.00E−1	2.17
12	2048	2.61E−1	6.05E−1	2.31
16	32768	2.46E−1	6.01E−1	2.44
20	524288	2.47E−1	6.02E−1	2.44
Theory	2^N	$\geqslant cN^{-2}$	$\leqslant c$	$\leqslant cN^2$

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