7. We remark, finally, that for the (1, 3) form $A_x a_y^3$ the set of forms derived by convolution from the irreducible types consists of

$$A_x a_y^3, \quad (ABu)(ab)\, a_y^2 b_y^2, \quad (ABu)(ab)^3, \quad (ABC)(ab)^3\, C_y^3$$

(the other forms derived from $(ABC)\, a_x^3 b_x^3 c_x^3$ all being equivalent to multiples of the last one), so that the determination of the complete system of f, and therefore of the general rational class-cubic, reduces to the discussion of the simultaneous system of one quartic and two cubic binary forms, a heavy but perhaps not impossible task.

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ON THE COMPLETENESS OF DINI'S SERIES*

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1. *Introduction.* A criterion for the completeness of a set of functions $\phi_n(x)$, $1 \leqslant n < \infty$, normal and orthogonal in the interval (a, b), has been obtained in a form which states that, if the terms A_n of a critical series are constructed according to the rule

$$A_n = \frac{2}{(b-a)^2} \int_a^b \left[\int_a^\xi \phi_n(x)\, dx \right]^2 d\xi, \tag{1}$$

the set is complete if, and only if, $\sum_1^\infty A_n = 1$. We shall here consider the application of a criterion of this kind to Dini's series‡ of Bessel functions. The interval of orthogonality is then taken to be (0, 1), and in order advantageously to utilise the properties of the Bessel functions a modification of the rule (1) for constructing the constants A_n is required. Let then $f(x)$ and $g(x)$ be continuous and positive when $0 < x < 1$, and suppose that the orthogonal functions $\phi_n(x)$ are not null and not necessarily

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‡ Watson, 580.

normalised. Write

$$P_n = \int_0^1 g(\xi) \left[\int_0^{\xi} f(x)\, \phi_n(x)\, dx \right]^2 d\xi, \tag{2}$$

$$Q_n = \int_0^1 \{\phi_n(x)\}^2 dx \int_0^1 g(\xi) \left[\int_0^{\xi} \{f(\eta)\}^2 d\eta \right] d\xi. \tag{3}$$

Then the terms A_n of a critical series are given by

$$A_n = \frac{P_n}{Q_n}, \tag{4}$$

and a necessary and sufficient condition that the set of orthogonal functions $\phi_n(x)$ be complete is that the sum of the critical series $\sum_1^{\infty} A_n$ be unity.

2. *Dini's series.* For Bessel functions of the first kind, and of real order $\nu > -1$, we have*, for any values of λ and μ,

$$(\lambda^2 - \mu^2) \int_0^{\xi} x J_\nu(\lambda x)\, J_\nu(\mu x)\, dx = \mu \xi J_\nu(\lambda \xi)\, J_\nu'(\mu \xi) - \lambda \xi J_\nu'(\lambda \xi)\, J_\nu(\mu \xi), \tag{5}$$

$$\lambda^2 \int_0^{\xi} x^{\nu+1} J_\nu(\lambda x)\, dx = \nu \xi^{\nu} J_\nu(\lambda \xi) - \lambda \xi^{\nu+1} J_\nu'(\lambda \xi). \tag{6}$$

Let H be a real constant; then the roots of the equation

$$\lambda^{-\nu} \{\lambda J_\nu'(\lambda) + H J_\nu(\lambda)\} = 0 \tag{7}$$

occur in pairs symmetrical about $\lambda = 0$ and are all real or pure imaginary. When $H + \nu > 0$, all the roots are real; when $H + \nu < 0$, there is one, and only one, pair of imaginary roots $\lambda = \pm i\kappa$, $\kappa > 0$. When $H + \nu > 0$, we write k_1 for the least positive root of (7) and k_2, k_3, ... for the other positive roots in increasing order. When $H + \nu \leqslant 0$, the least positive root of (7) is taken to be k_2, and then k_3, k_4, ... are the other positive roots in increasing order; when $H + \nu < 0$, we write $k_1 = i\kappa$. For fixed values of ν and H, the real functions $\phi_n(x)$ are defined by the equations

$$\left.\begin{aligned} \phi_n(x) &= x^{\frac{1}{2}} k_n^{-\nu} J_\nu(k_n x), && (n \geqslant 2), \\ \phi_1(x) &= x^{\frac{1}{2}} k_1^{-\nu} J_\nu(k_1 x) && (H + \nu \neq 0), \\ \phi_1(x) &= x^{\nu + \frac{1}{2}}. && (H + \nu = 0). \end{aligned}\right\} \tag{8}$$

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* Watson, 134.

Thus, according to the definition of $\phi_1(x)$, we have, when $i\kappa$ is written for k_1,

$$\phi_1(x) = x^{\frac{1}{2}} \kappa^{-\nu} I_\nu(\kappa x) \quad (H+\nu < 0).$$

Then (5) and (6) show that, for any fixed values of H and of $\nu > -1$, the functions $\phi_n(x)$, $1 \leqslant n < \infty$, are members of a set orthogonal in (0, 1).

To investigate completeness we shall employ formulae (2) and (3) with

$$f(x) = x^{\nu+\frac{1}{2}}, \quad g(x) = x^{-2\nu-1}.$$

Then, in order to evaluate the terms of the critical series, we require the relation

$$\lambda \int_0^1 x^{\nu+1} J_\nu(\lambda x)\, dx = \xi^{\nu+1} J_{\nu+1}(\lambda\xi),$$

which is equivalent to (6) above, and we also* have

$$2\lambda^2 \int_0^1 \xi^{-2\nu-1} \left[\int_0^\xi x^{\nu+1} J_\nu(\lambda x)\, dx\right]^2 d\xi = 2\int_0^1 \xi \{J_{\nu+1}(\lambda\xi)\}^2 d\xi$$

$$= \{J'_{\nu+1}(\lambda)\}^2 + \left\{1 - \left(\frac{\nu+1}{\lambda}\right)^2\right\} \{J_{\nu+1}(\lambda)\}^2. \quad (9)$$

The recurrence relations† satisfied by the Bessel functions, taken with (7), also give

$$k_n J_{\nu+1}(k_n) = \nu J_\nu(k_n) - k_n J_\nu'(k_n) = (\nu+H) J_\nu(k_n),$$

$$k_n^2 J'_{\nu+1}(k_n) = k_n^2 J_\nu(k_n) - k_n(\nu+1) J_{\nu+1}(k_n) = \{k_n^2 - (\nu+1)(\nu+H)\} J_\nu(k_n),$$

which with (9) yield the value of P_n,

$$2P_n = k_n^{-2\nu-4}\{k_n^2 - (\nu+H)(\nu-H+2)\}\{J_\nu(k_n)\}^2 \quad (n > 1). \quad (10)$$

Similarly

$$8(\nu+1) Q_n = k^{-2\nu-2}(k_n^2 + H^2 - \nu^2)\{J_\nu(k_n)\}^2 \quad (n > 1). \quad (11)$$

Thus, by (4), (10) and (11), the terms of the critical series are

$$A_n = \frac{4(\nu+1)}{H^2+k_n^2-\nu^2}\left\{1 - \frac{(\nu+H)(\nu-H+2)}{k_n^2}\right\}$$

$$= \frac{4(\nu+1)}{k_n^2}\left\{1 - \frac{2(\nu+H)}{H^2+k_n^2-\nu^2}\right\} \quad (n > 1),$$

* Watson, 135.

† Watson, 45.

and also

$$A_1=\frac{4(\nu+1)}{k_1^2}\left\{1-\frac{2(\nu+H)}{H^2+k_1^2-\nu^2}\right\}\quad (H+\nu\neq 0),$$

$$A_1=\frac{1}{\nu+2}\qquad\qquad (H+\nu=0).$$

Thus the sum of the critical series is

$$\sum_1^\infty A_n=4(\nu+1)\sum_1^\infty\frac{1}{k_n^2}\left\{1-\frac{2(\nu+H)}{H^2+k_n^2-\nu^2}\right\}\quad (H+\nu\neq 0),\quad (12)$$

$$\sum_1^\infty A_n=\frac{1}{\nu+2}+4(\nu+1)\sum_2^\infty\frac{1}{k_n^2},\qquad (H+\nu=0).\quad (13)$$

Now consider the functions $\Phi(\lambda)$ and $\psi(\lambda)$ of the complex variable λ,

$$\Phi(\lambda)=\frac{4(\nu+1)}{\lambda^2}\left\{1-\frac{2(H+\nu)}{H^2+\lambda^2-\nu^2}\right\},$$

$$\psi(\lambda)=\lambda J_\nu'(\lambda)-HJ_\nu(\lambda).$$

Clearly, when $H+\nu\neq 0$, the sum of the critical series is half the sum of the values of $\Phi(\lambda)$ at the zeros of $\lambda^{-\nu}\psi(\lambda)$. When $H+\nu=0$, the sum of the critical series exceeds half the sum of the values of $\Phi(\lambda)$ at the zeros of $\lambda^{-\nu-2}\psi(\lambda)$ by $1/(\nu+2.)$ We may thus consider the contour integral

$$\frac{1}{2\pi i}\int\Phi(\lambda)\,\frac{\psi'(\lambda)}{\psi(\lambda)}\,d\lambda.$$

The asymptotic series for the Bessel functions taken in conjunction with the expressions for $\Phi(\lambda)$ and $\psi(\lambda)$ show that the integral tends to zero when the contour is a suitable circle whose radius tends to infinity. The sum of the residues at all the poles of the integrand is therefore zero. We observe that, by equation (11), the roots of $\lambda^2+H^2=\nu^2$ are distinct from any of the zeros of $\psi(\lambda)$ other than $\lambda=0$, and we infer that, if K is defined by the integral

$$-2\pi iK=\int_C 2(\nu+1)\left\{1-\frac{2(H+\nu)}{H^2+\lambda^2-\nu^2}\right\}\frac{\lambda HJ_\nu'(\lambda)+(\nu^2-\lambda^2)J_\nu(\lambda)}{HJ_\nu(\lambda)+\lambda J_\nu'(\lambda)}\,\frac{d\lambda}{\lambda^3},\quad (14)$$

where C is a contour which encloses the point $\lambda=0$ and the roots of $\lambda^2+H^2=\nu^2$, but no other poles of the integrand, then K is equal to the sum of the terms of the critical series associated with the non-zero roots of equation (7).

First suppose that $H^2-\nu^2 \neq 0$; then we find that, near $\lambda = 0$,

$$\frac{\lambda H J_\nu'(\lambda)+(\nu^2-\lambda^2)\,J_\nu(\lambda)}{HJ_\nu(\lambda)+\lambda J_\nu'(\lambda)} = \nu+\frac{\nu+H+2}{2(\nu+1)(\nu+H)}\lambda^2+O(\lambda^4),$$

and therefore that the residue of the integrand in (14) at $\lambda = 0$ is

$$\frac{4\nu(\nu+1)}{(\nu+H)(\nu-H)^2}-\frac{(\nu-H+2)(\nu+H+2)}{\nu^2-H^2} = -1+\frac{4H(\nu+1)}{(\nu+H)(\nu-H)^2}.$$

Each of the residues at a root of $\lambda^2+H^2 = \nu^2$ is $-2H(\nu+1)/(\nu+H)(\nu-H)^2$; and hence, when $H^2 \neq \nu^2$, the value of K is unity.

When $H = \nu \neq 0$, the expression for K becomes

$$-2\pi i K = \int_C 2(\nu+1)\left(1-\frac{4\nu}{\lambda^2}\right)\left\{1+\frac{\lambda J'_{\nu-1}(\lambda)}{J_{\nu-1}(\lambda)}\right\}\frac{d\lambda}{\lambda^3}.$$

In the neighbourhood of $\lambda = 0$,

$$1+\frac{\lambda J'_{\nu-1}(\lambda)}{J_{\nu-1}(\lambda)} = \nu-\frac{\lambda^2}{2\nu}-\frac{\lambda^4}{8\nu^2(\nu+1)}+O(\lambda^6),$$

and hence, by evaluating the residue at the unique pole within C at $\lambda = 0$, the value of K is again found to be unity.

Now let $H+\nu = 0$. The expression for K becomes

$$-2\pi i K = \int_C 2(\nu+1)\left\{1+\frac{\lambda J'_{\nu+1}(\lambda)}{J_{\nu+1}(\lambda)}\right\}\frac{d\lambda}{\lambda^3}.$$

In the neighbourhood of $\lambda = 0$,

$$1+\frac{\lambda J'_{\nu+1}(\lambda)}{J_{\nu+1}(\lambda)} = \nu+2-\frac{\lambda^2}{2(\nu+2)}+O(\lambda^4),$$

and hence we find that in this case the value of K is $(\nu+1)/(\nu+2)$; but by taking account of the value of the first term of the critical series (13) we can again infer completeness.

We arrive therefore at the known* conclusion that, if the values $\lambda = k_n > 0$ are the positive roots of the equation

$$\lambda^{-\nu}\{\lambda J_\nu'(\lambda)+HJ_\nu(\lambda)\} = 0 \quad (H \text{ real}, \ \nu > -1),$$

the set of orthogonal functions $x^{\frac{1}{2}} J_\nu(k_n x)$ is complete if, and only if, $H+\nu > 0$; that, when $H+\nu < 0$, this set is completed by the addition to

* Watson, 597.

it of the function $x^{\frac{1}{2}} I_\nu(\kappa x)$, where κ is the unique positive root of the equation

$$\kappa I_\nu'(\kappa)+HI_\nu(\kappa)=0 \quad (H+\nu<0),$$

and that this additional function reduces to $x^{\nu+\frac{1}{2}}$ when $H+\nu=0$.

3. *Particular cases.* Two well-known cases are of sufficient interest for explicit mention. When $\nu=\frac{1}{2}$, the orthogonal functions are of the form $\sin k_n x$, where the numbers k_n are the roots of

$$\lambda \cot \lambda = \tfrac{1}{2}-H.$$

Thus the set of orthogonal functions $\sin k_n x$, $k_n>0$, is complete only when $H>-\frac{1}{2}$; when $H<-\frac{1}{2}$, an additional function $\sinh \kappa x$ is required for completeness, where κ is the unique positive root of the equation

$$\kappa \coth \kappa = \tfrac{1}{2}-H \quad (H<-\tfrac{1}{2});$$

and, when $H=-\frac{1}{2}$, the additional function reduces to x.

When $\nu=-\frac{1}{2}$, the orthogonal functions are of the form $\cos k_n x$, where the numbers k_n are the roots of

$$\lambda \tan \lambda = H-\tfrac{1}{2}.$$

Thus the set of orthogonal functions $\cos k_n x$, $k_n>0$, is complete only when $H>\frac{1}{2}$; when $H\leqslant\frac{1}{2}$, a non-negative number κ is determined by the equation

$$\kappa \tanh \kappa = \tfrac{1}{2}-H \quad (H\leqslant\tfrac{1}{2}),$$

and the additional function $\cosh \kappa x$ is required for completeness.

In conclusion, I should like to express my acknowledgments to Prof. G. H. Hardy for his interest in this paper and for his provocative criticism of another, related to this, which I published recently*.

Reference.

G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd Ed. (Cambridge, 1944).

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* "On the completeness of a series of normal orthogonal functions", *Journal London Math. Soc.*, 20 (1945), 87–93.