The Journal of Fourier Analysis and Applications
Volume 8, Issue 5, 2002

On the Spectra of Elastostatic and Hydrostatic Layer Potentials on Curvilinear Polygons

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Communicated by Carlos Kenig

ABSTRACT. We give a complete description of the spectra of certain elastostatic and hydrostatic boundary layer potentials in L^p, $1 < p < \infty$, on bounded curvilinear polygons. In particular, our analysis shows that the spectral radii of these operators on L^p, $2 \leq p < \infty$ are less than one. Such results are relevant in the context of constructively solving boundary value problems for the Lamé system of elasticity, the Stokes system of hydrostatics as well as the two dimensional Laplacian on curvilinear polygons. Our approach is based on Mellin transform techniques and Calderón–Zygmund theory.

1. Introduction

One of the classical approaches to solving boundary problems for (strongly) elliptic systems of equations is via the method of layer potentials. Typically, one seeks a reduction of the original problem to that of inverting an operator of the form "identity$+K$" on appropriate (boundary) function spaces. In recent years, considerable attention has been given to the case when the underlying domain Ω has a Lipschitz boundary and the boundary data are from $L^p(\partial\Omega)$, $1 < p < \infty$.

In [4], the layer potentials operators associated with the Lamé system of elastostatics

$$\mu\Delta + (\lambda + \mu)\nabla \operatorname{div}$$

in an arbitrary Lipschitz domain $\Omega \subseteq \mathrm{R}^n$ have been shown to be Fredholm with index zero on $L^p(\partial\Omega)$ for $2-\varepsilon < p < 2+\varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$. Similar results for the linearized

Math Subject Classifications. primary: 45E05, 47A10; secondary: 35J25, 42B20.
Keywords and Phrases. layer potentials, Lamé system, Stokes system, spectral radius, polygons, curvilinear polygons.
Acknowledgements and Notes. Supported in part by an Alfred P. Sloan Doctoral Dissertation Fellowship.

ISSN 1069-5869

Stokes system of hydrostatics have been obtained in [10]. In lower (Euclidean) dimensions more precise results are available. Consider, for example, $I + K_{\text{Lame}}$, the double layer potential operator relevant for the solution of the Dirichlet problem for the Lamé system. Using a symbolic calculus of pseudodifferential operators of Mellin type, it has been shown in [17] that $I + K_{\text{Lame}}$ is Fredholm with index zero on $L^p(\partial\Omega)$ for each $2 \le p < \infty$ whenever Ω is a (bounded) curvilinear polygonal domain in R^2. We also refer the reader to the work of Maz'ya and collaborators (cf., e. g., [19] and the references therein), as well as to [8] and [23] for related results in the two-dimensional polygonal domains setting.

For arbitrary Lipschitz domains in R^3, Dahlberg and Kenig [3] were able to prove 'atomic estimates' for $I + K^*_{\text{Lame}}$. By interpolating with the L^2 theory from [4], this yields the same conclusion as before but, this time, at the level of arbitrary Lipschitz domains in R^3. The range $2 \le p < \infty$ is sharp in the class of Lipschitz domains but the situation in dimension $n \ge 4$ is still very much open.

The main purpose of this work is to continue this line of research and study spectral properties for elastostatic and hydrostatic layer potentials on $L^p(\partial\Omega)$, $2 \le p < \infty$, in the case when Ω is an arbitrary bounded, two-dimensional curvilinear polygon. The aim is to provide an explicit description of the spectra of the operators K_{Lame}, K_{Stokes}, i. e.,

$$\sigma\left(K; L^p(\partial\Omega)\right) := \left\{w \in \mathrm{C};\ wI - K \ \text{ is not invertible in } \ L^p(\partial\Omega)\right\}\ , \tag{1.1}$$

where K stands for K_{Lame}, K_{Stokes}. Our main result in this direction (cf., Theorem 8) roughly states the following. Let $1 < p < \infty$ and consider a bounded, curvilinear polygon $\Omega \subset \mathrm{R}^2$ with angles $\{\theta_i;\ 1 \le i \le n\}$. For each $1 \le i \le n$, denote by $\Sigma_{\theta_i}(p)$ a certain bow-tie-shaped curve [for a precise definition see (4.15)] associated with the angle θ_i and the integrability exponent p. Also, set $\widehat{\Sigma_{\theta_i}(p)}$ for the closure of its interior. Then,

$$\sigma\left(K_{\text{Lame}}; L^p(\partial\Omega)\right) = \left(\bigcup_{1\le i\le n} \widehat{\Sigma_{\theta_i}(p)}\right) \bigcup \{\lambda_j\}_j\ , \tag{1.2}$$

where $\{\lambda_j\}_j$ is a finite subset of $(-1, 1]$ consisting of eigenvalues of K_{Lame} on $L^p(\partial\Omega)$. In addition, $wI - K_{\text{Lame}}$ is Fredholm whenever $w \notin \cup_{1\le i\le n}\Sigma_{\theta_i}(p)$ and its index can be expressed in terms of the winding numbers of w relative to each $\Sigma_{\theta_i}(p)$. For example, in the case of an angle $\theta = \frac{\pi}{3}$ and $p = 6$, the bow-tie curve alluded to above looks as in Figure 1. For the same angle $\theta = \frac{\pi}{3}$ but for $p = 10$ we have the result in Figure 2.

Furthermore, when Ω is a curvilinear triangle with angles $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$, then, the spectrum of K_{Lame} on $L^6(\partial\Omega)$ is shown in Figure 3. When Ω is the curvilinear triangle described above, but $p = 10$, we obtain the results in Figures 3–4.
The $*$'s denote the generic location of the eigenvalues $\{\lambda_j\}_j$. A similar discussion is valid in the case of K_{Stokes} and K_{Laplace}.

For numerical applications, it is generally of interest to expand the inverses $(I + K_{\text{Lame}})^{-1}$, $(I + K_{\text{Stokes}})^{-1}$ in Neumann series convergent in $L^p(\partial\Omega)$, $2 \le p < \infty$. Denoting by $\rho(T; \mathcal{X})$ the spectral radius of an operator T on the Banach space $\mathcal{X}$, this becomes equivalent to

$$\rho(K_{\text{Lame}}; L^p(\partial\Omega)) < 1 \quad \text{and} \quad \rho\left(K_{\text{Stokes}}; L^p(\partial\Omega)\right) < 1 \quad \text{for} \quad 2 \le p < \infty\ . \tag{1.3}$$

Estimates like (1.3) in the context of arbitrary Lipschitz domains in R^n are occasionally referred to as the 'spectral radius conjecture'. This conjecture makes the object of the open problem 3.2.10 on p. 119 in Kenig's book [12].

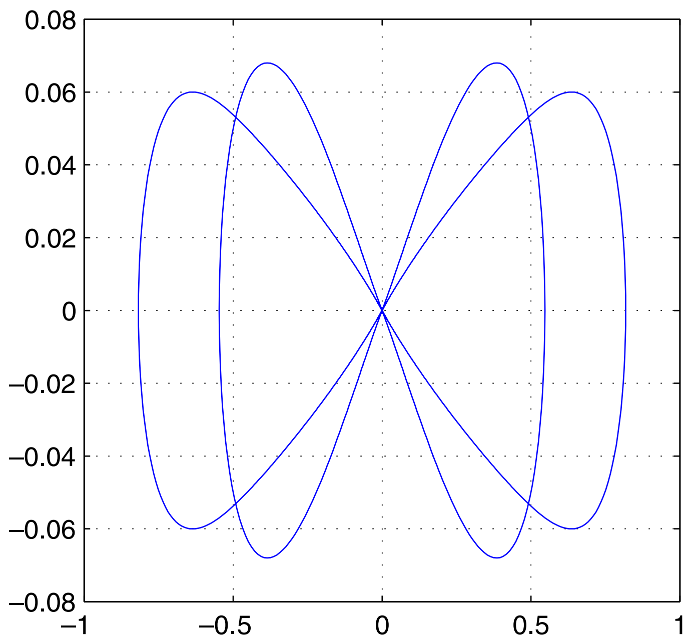

FIGURE 1 $\Sigma_{\pi/3}(6)$ - or the L^6 Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a sector of angle $\frac{\pi}{3}$.

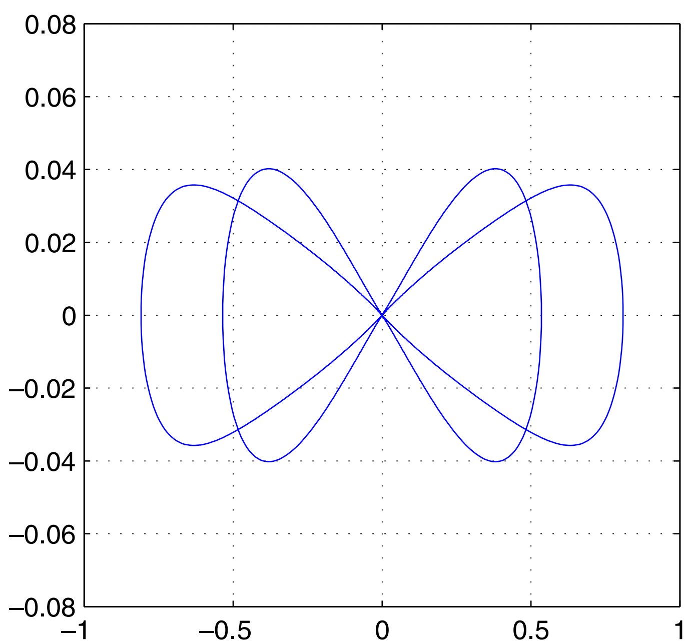

FIGURE 2 $\Sigma_{\pi/3}(10)$ - or the L^{10} Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a sector of angle $\frac{\pi}{3}$.

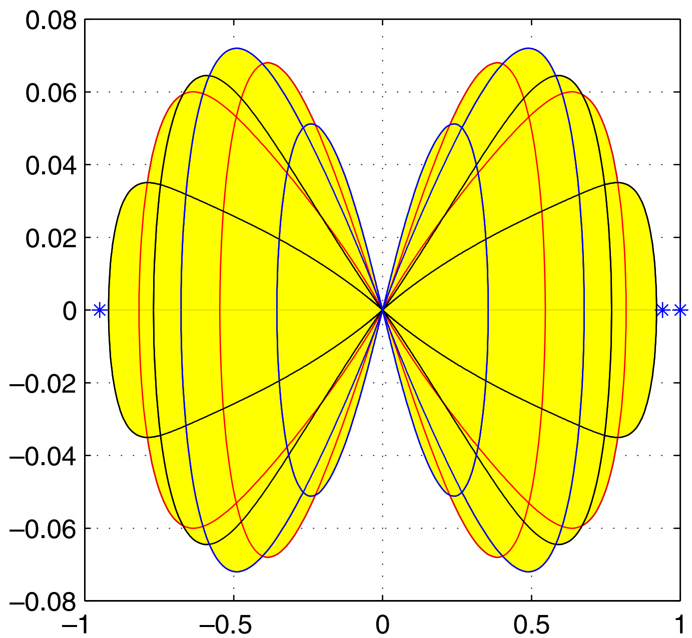

FIGURE 3 The L^6 Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a curvilinear triangle of angles $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

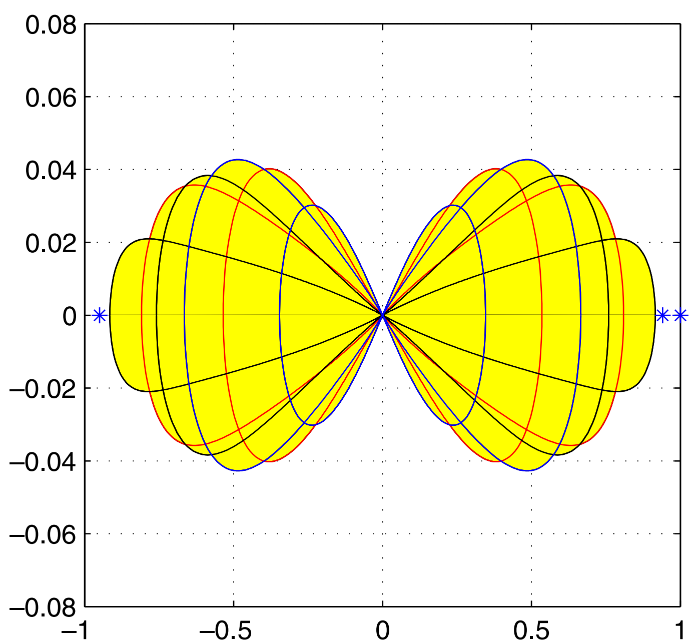

FIGURE 4 The L^{10} Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a curvilinear triangle of angles $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

Here, we are able to prove (1.3) in the case of curvilinear polygonal domains $\Omega \subseteq \mathrm{R}^2$. For certain partial results in the three dimensional setting see also [20]. The proof consists of two steps: the case of an infinite plane sector, and a localization argument. For the purpose of this introduction, let us point out that, in the former step, we produce an explicit formula, namely

$$\rho\left(K_{\mathrm{Lame}}; L^p(\partial\Omega)\right) = \frac{\left|\frac{\upsilon}{p}\sin\theta\cos\left(\frac{\pi-\theta}{p}\right) + \sin\left(\frac{\pi-\theta}{p}\right)\sqrt{1-\upsilon^2\frac{\sin^2\theta}{p^2}}\right|}{\sin\left(\frac{\pi}{p}\right)} . \tag{1.4}$$

In (1.4), $\theta \in (0, 2\pi)$ is the opening of the sector Ω, $\upsilon := (\mu+\lambda)/(3\mu+\lambda)$, and $2 \leq p < \infty$. For the case of hydrostatics (and with the same notation as before) we have

$$\rho\left(K_{\mathrm{Stokes}}; L^p(\partial\Omega)\right) = \frac{\left|\frac{1}{p}\sin\theta\cos\left(\frac{\pi-\theta}{p}\right) + \sin\left(\frac{\pi-\theta}{p}\right)\sqrt{1-\frac{\sin^2\theta}{p^2}}\right|}{\sin\left(\frac{\pi}{p}\right)} . \tag{1.5}$$

This corresponds to the formal choice $\lambda = \infty$ which forces $\upsilon = 1$ in (1.4). Also, corresponding to the choice of the Lamé moduli $\lambda + \mu = 0$, the Lamé system reduces to the vector Laplacian. Note that this entails $\upsilon = 0$ so that (1.4) yields

$$\rho\left(K_{\mathrm{Laplace}}; L^p(\partial\Omega)\right) = \left|\sin\left(\frac{\pi-\theta}{p}\right)\right| \Big/ \sin\left(\frac{\pi}{p}\right) \tag{1.6}$$

for each $2 \leq p < \infty$.

These formulas extend naturally to the case of bounded curvilinear polygons. For example, if $2 \leq p < \infty$, then,

$$\begin{aligned} &\max\left\{|w|;\ w \in \sigma\left(K_{\mathrm{Lame}}; L^p(\partial\Omega)\right),\ w \neq \lambda_j,\ \forall\, j\right\} \qquad (1.7)\\ &= \max_{1\leq i\leq n}\left\{\frac{\left|\frac{\upsilon}{p}\sin\theta_i\cos\left(\frac{\pi-\theta_i}{p}\right) + \sin\left(\frac{\pi-\theta_i}{p}\right)\sqrt{1-\upsilon^2\frac{\sin^2\theta_i}{p^2}}\right|}{\sin\left(\frac{\pi}{p}\right)}\right\} , \end{aligned}$$

where λ_j's and θ_i's are associated with Ω as stated in Theorem 8. Analogous formulas [based on (1.5) and (1.6)] hold in the case of the Stokes system and the Laplace equation. Furthermore, in all cases, the spectral radius is an element of the spectrum, i. e.,

$$\rho\left(K; L^p(\partial\Omega)\right) \in \sigma\left(K; L^p(\partial\Omega)\right), \quad \text{for } 2 \leq p < \infty ,$$

where K stands for K_{Lame}, K_{Stokes}, and K_{Laplace}, respectively.

Our approach employs Calderón Zygmund theory and Mellin transform techniques together with a careful analysis of the symbols of the operators involved. The layout of this article is as follows. In Section 2 we define the elastostatic and hydrostatic layer potentials in the two dimensional case corresponding, respectively, to the pseudostress and stress conormal derivatives. Section 3 contains a spectral theorem for Hardy kernels. In Section 4 we use this result in order to provide an explicit description of the spectrum of the aforementioned layer potentials in an infinite plane sector. In Section 5 we obtain formulas for the spectral radii of these operators on the L^p spaces, for $2 \leq p < \infty$. Section 6 deals with the Dirichlet problems for the Lamé and Stokes systems on curvilinear polygons. In Section 7, the main result of the article, dealing with the structure of the spectrum of K_{Lame} is stated and proved. Finally, the Appendix 8 contains the proof of a technical result needed in Section 5.

2. Elastostatic and Hydrostatic Layer Potentials

We start by reviewing *elastostatic layer potentials.* Concretely, consider the system of linear elastostatics $\mathcal{L}\vec{u} = 0$ in an open subset of R^2, where

$$\mathcal{L}\vec{u} := \mu \triangle \vec{u} + (\lambda + \mu)\nabla \operatorname{div} \vec{u} \,.$$

The displacement $\vec{u}$ has two components and μ and λ are the Lamé moduli which are assumed to satisfy $\mu > 0$ and $-\mu \leq \lambda$. The operator $\mathcal{L}$ can be represented in the following notation

$$\mathcal{L} = A(D) = \left(a_{ij}^{kl}\partial_i\partial_j\right)_{k,l} \,, \tag{2.1}$$

where $a_{ij}^{kl} = a_{ij}^{kl}(r) := \mu\,\delta_{ij}\delta_{kl} + (\mu+\lambda-r)\delta_{ik}\delta_{jl} + r\,\delta_{il}\delta_{jk}$. Here, $r \in \mathrm{R}$ is arbitrary, δ_{ij} is the Kronecker symbol, and $i, j, k, l \in \{1, 2\}$. Hereafter we shall use Einstein's convention for summation, i. e., an index repeating in the same expression means summation with respect to that index.

Let $\Omega \subset \mathrm{R}^2$ be a bounded Lipschitz domain, i. e., a domain whose boundary is locally given by graphs of Lipschitz functions in appropriate systems of coordinates (for a more detailed account see, e. g., [12]). Then, the outward unit normal vector N to $\partial\Omega$ exists almost everywhere with respect to the surface measure on $\partial\Omega$. Corresponding to $A := (a_{ij}^{kl})_{i,j,k,l}$, the conormal derivative for the operator $\mathcal{L}$ in (2.1) is given by

$$\left(\frac{\partial\vec{u}}{\partial N_A}\right)^j := N_i a_{ik}^{jl}(r)\partial_k u^l = \mu\frac{\partial u^j}{\partial N} + (\mu+\lambda-r)N_j \operatorname{div} \vec{u} + rN_i\partial_j u^i \,,$$

where $j = 1, 2$. The special choice $r := \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$ gives rise to the so called pseudostress conormal derivative which has the form

$$\frac{\partial\vec{u}}{\partial\nu} := \mu\nabla\vec{u}\cdot N + \frac{\mu(\mu+\lambda)}{3\mu+\lambda}(\nabla\vec{u})^t\cdot N + \frac{(2\mu+\lambda)(\mu+\lambda)}{3\mu+\lambda}(\operatorname{div}\vec{u})\,N \,,$$

where the superscript t indicates transposition of matrices and $\nabla\vec{u} = \left(\partial_j u^i\right)_{1\leq i,j\leq 2}$.

Let $G = (G_{ij})_{i,j}$ be the Kelvin matrix valued fundamental solution for the system of elastostatics, i. e.,

$$G_{ij}(X) := \frac{1}{2\mu(2\mu+\lambda)\pi}\left[\frac{3\mu+\lambda}{2}\delta_{ij}\log|X|^2 - (\mu+\lambda)\frac{X_iX_j}{|X|^2}\right], \quad X \in \mathrm{R}^2\setminus\{0\}\,,$$

where $i, j = 1, 2$ and δ_{ij} is the usual Kronecker symbol (cf., (9.2) in Chapter 9 of [14]). Also, denote by $K_{\mathcal{L}}$ the double layer operator corresponding to the pseudostress conormal derivative on the boundary of Ω. Specifically, if we denote by G^j the j-th column in the fundamental matrix G, then,

$$\left(K_{\mathcal{L}}\left(\vec{f}\right)\right)^i(P) := \int_{\partial\Omega}\left(\frac{\partial G^j}{\partial\nu}(P-\cdot)\right)^i(Q)f_j(Q)\,d\sigma(Q), \quad P \in \partial\Omega\,, \tag{2.2}$$

where $\vec{f} = (f_1, f_2) : \partial\Omega \to \mathrm{R}^2$ and $i = 1, 2$. Here $d\sigma$ stands for the canonical surface measure on $\partial\Omega$.

A straightforward computation gives that the i-th component of $\frac{\partial G^j}{\partial \nu}(X)$, denoted by $k_{\mathcal{L}}^{ij}(X)$, is

$$k_{\mathcal{L}}^{ij}(X) := \frac{-2\mu\delta_{ij}}{\pi(3\mu+\lambda)} \cdot \frac{\langle X, N(X)\rangle}{|X|^2} - \frac{2(\mu+\lambda)}{\pi(3\mu+\lambda)} \cdot \frac{X_i X_j \langle X, N(X)\rangle}{|X|^4}, \quad X \in \mathrm{R}^2 \setminus \{0\} . \tag{2.3}$$

Next, we briefly discuss *hydrostatic layer potentials.* To this end, consider the linearized, homogeneous, time independent Navier–Stokes equations, i. e., the Stokes system

$$\begin{cases} \triangle \vec{u} = \nabla p , \\ \operatorname{div} \vec{u} = 0 , \end{cases} \tag{2.4}$$

in an open set in R^2, where $\vec{u}$ is the velocity field and p is the pressure function. If we define the matrix $A = A(r) := (a_{ij}^{kl}(r))_{i,j,k,l}$ by

$$a_{ij}^{kl} = a_{ij}^{kl}(r) := \delta_{ij}\delta_{kl} + r\,\delta_{il}\delta_{jk}, \qquad \text{for } r \in \mathrm{R} ,$$

then, $a_{ij}^{kl}\partial_i\partial_j u^l = \triangle u^k + r\partial_k(\operatorname{div} \vec{u})$. Hence, any solution $\vec{u}$, p of the Stokes system (2.4) satisfies

$$a_{ij}^{kl}\partial_i\partial_j u^l = \partial_k p .$$

As before, let $\Omega \subset \mathrm{R}^2$ be a Lipschitz domain and denote by N the outward unit normal vector a.e. on $\partial\Omega$. The conormal derivative that corresponds to the matrix $A := (a_{ij}^{kl})_{i,j,k,l}$ is

$$\left(\frac{\partial \vec{u}}{\partial N_A}\right)^j := N_i a_{ik}^{jl}(r)\partial_k u^l - N_j p, \qquad \text{where} \quad j = 1, 2 .$$

The special choice $r := 1$ gives rise to the so called stress conormal derivative (see also, e. g., [15, 4]). This derivative has a physical interpretation and it is known as the *slip condition* when imposed at the boundary.

Going further, denote by $G = (G_{ij})_{i,j}$ the Kelvin matrix valued fundamental solution for the system of hydrostatics (see, e. g., [15]),

$$G_{ij}(X) := \frac{1}{2\pi}\left(\delta_{ij}\log|X|^2 - 2\frac{X_i X_j}{|X|^2}\right), \quad X \in \mathrm{R}^2 \setminus \{0\} .$$

Let $K_{\mathcal{S}}$ be the double layer hydrostatic operator corresponding to the stress conormal derivative on the boundary of Ω. Also, set G^j for the j-th column in the fundamental matrix. Then,

$$\left(K_{\mathcal{S}}\left(\vec{f}\right)\right)^i(P) := \int_{\partial\Omega}\left(\frac{\partial G^j}{\partial \nu}(P-\cdot)\right)^i(Q) f^j(Q)\, d\sigma(Q), \quad P \in \partial\Omega , \tag{2.5}$$

where $i = 1, 2$, and $\frac{\partial G^j}{\partial \nu} := \frac{\partial G^j}{\partial N_{A(1)}}$. The i-th component of $\frac{\partial G^j}{\partial \nu}(X)$, denoted by $k_{\mathcal{S}}^{ij}(X)$, is

$$k_{\mathcal{S}}^{ij}(X) := \frac{-2}{\pi} \cdot \frac{X_i X_j \langle X, N(X)\rangle}{|X|^4}, \qquad X \in \mathrm{R}^2 \setminus \{0\} . \tag{2.6}$$

The operators $K_{\mathcal{L}} = (K^{ij}_{\mathcal{L}})_{i,j}$ and $K_{\mathcal{S}} = (K^{ij}_{\mathcal{S}})_{i,j}$, with $i, j = 1, 2$, acting from $(L^p(\partial\Omega))^2$ to $(L^p(\partial\Omega))^2$, $1 \leq p < \infty$, are 2×2 matrices whose entries are linear operators on $L^p(\partial\Omega)$. Specifically, for any $f \in (L^p(\partial\Omega))^2$, $f = (f^1, f^2)$,

$$(K_{\#} f)^i(P) = K^{ij}_{\#} f^j(P) := \int_{\partial\Omega} k^{ij}_{\#}(P - Q) f^j(Q)\, d\sigma(Q), \quad P \in \partial\Omega,$$

where # stands for $\mathcal{L}$ or $\mathcal{S}$.

In closing, let us point out that, based on the results in [2], the operators $K_{\mathcal{L}}$ and $K_{\mathcal{S}}$ are bounded on $(L^p(\partial\Omega))^2$ for each $1 < p < \infty$, for any Lipschitz domain $\Omega \subseteq \mathrm{R}^2$.

3. Hardy Kernels on $(L^p(\mathrm{R}_+))^2$

The goal of this section is to give an explicit description of the spectrum of the *Hardy kernel operators* on the spaces $(L^p(0, \infty))^2$, $1 < p < \infty$. To this end, we start with some notation and preliminaries on the Mellin transform.

Recall first that if $\mathcal{X}$ is a Banach space, then, $\mathcal{B}(\mathcal{X})$ stands for the space of linear and bounded operators $T : \mathcal{X} \longrightarrow \mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$ we denote by $\sigma(T; \mathcal{X})$ the spectrum of the operator T, i. e., $\sigma(T; \mathcal{X}) := \{w \in \mathrm{C}\,;\ wI - T \text{ is not invertible on } \mathcal{X}\}$.

A vector-valued function $f = (f_i)_{1 \leq i \leq 2}$ belongs to the space $(L^p(\mathrm{R}_+))^2$, $1 \leq p < \infty$, provided that $f_i \in L^p(\mathrm{R}_+)$ for $i = 1, 2$, i. e.,

$$\|f_i\|_p = \left(\int_0^\infty |f_i(x)|^p\, dx \right)^{1/p} < \infty, \ \ i = 1, 2\ .$$

Let $\mathcal{C}^\infty_0(\mathrm{R}_+)$ be the space of infinitely many times differentiable functions, compactly supported on $[0, \infty)$. The Mellin transform of a function $f \in \mathcal{C}^\infty_0(\mathrm{R}_+)$ is defined as

$$\tilde{f}(z) := \int_0^\infty x^{z-1} f(x)\, dx, \quad z \in \mathrm{C}\ . \tag{3.1}$$

For any $f \in \mathcal{C}^\infty_0(\mathrm{R}_+)$, $\tilde{f}(z)$ is an entire function and the following inversion formula holds:

$$f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + \infty} x^{-z} \tilde{f}(z)\, dz\ , \tag{3.2}$$

where the above path integral is taken over the contour $z = 1/p + i\xi$, $-\infty < \xi < \infty$. For any $\alpha, \beta \in \mathrm{R}$, $\alpha < \beta$, define the strip $\Gamma_{\alpha,\beta} := \{z \in \mathrm{C}\,; \alpha < \mathrm{Re}\, z < \beta\}$, and let $\Gamma_\alpha := \{z \in \mathrm{C}\,; \mathrm{Re}\, z = \alpha\}$. If f is measurable on R_+ and the integral in (3.1) converges absolutely for all z in some strip $\Gamma_{\alpha,\beta}$ we shall call the integral $\tilde{f}(z)$ the *Mellin transform* of the function f. Note that $\tilde{f}$ is a holomorphic function in the strip $\Gamma_{\alpha,\beta}$. When dealing with a vector function $g = (g_i)_{1 \leq i \leq 2}$ such that g_i is defined on $[0, \infty)$ and $\tilde{g}_i(z)$ makes sense for each $1 \leq i \leq 2$, we let $\tilde{g}(z) := (\tilde{g}_i(z))_{1 \leq i \leq 2}$.

We make the following definition.

Definition 1. Let $k(x, y)$ be a measurable function defined on $\mathrm{R}_+ \times \mathrm{R}_+$. Then, k is a Hardy kernel for $L^p(\mathrm{R}_+)$, $1 \leq p < \infty$, provided that the following hold:

1. $k(x, y)$ is homogeneous of degree -1, i. e., for any $\lambda > 0$ we have $k(\lambda x, \lambda y) = \lambda^{-1} k(x, y)$;

2. There holds

$$\int_0^\infty |k(1,y)|\, y^{-1/p}\, dy \left(= \int_0^\infty |k(x,1)|\, x^{1/p-1}\, dx \right) < \infty\ .$$

Also, a matrix $k = \left(k_{ij}\right)_{i,j=1,2}$ of measurable functions on $\mathrm{R}_+ \times \mathrm{R}_+$ is called a Hardy kernel for $(L^p(\mathrm{R}_+))^2$, provided that each individual entry k_{ij} is a Hardy kernel for $L^p(\mathrm{R}_+)$.

For any $1 \le p < \infty$, consider the space $L^p_*(\mathrm{R}_+)$ as the set of measurable functions g on R_+ such that

$$\|g\|_{p,*} := \left(\int_0^\infty |g(x)|^p \frac{dx}{x} \right)^{1/p} < \infty\ .$$

Then, $g \mapsto g_p$ with $g_p(x) := x^{1/p} g(x)$ is an isometry from $L^p(\mathrm{R}_+)$ to $L^p_*(\mathrm{R}_+)$ and the condition (2) in the Definition 1 becomes, therefore, equivalent to $k_p(x,1) \in L^1_*(\mathrm{R}_+)$.

For the remaining part of the section consider $k = \left(k_{ij}\right)_{i,j=1,2}$ a Hardy kernel for $(L^p(\mathrm{R}_+))^2$, $1 \le p < \infty$, and for each $f \in (L^p(\mathrm{R}_+))^2$ let

$$Kf(x) := \int_0^\infty k(x,y) f(y)\, dy, \qquad x \in \mathrm{R}_+\ . \tag{3.3}$$

The operator K can be converted to a convolution operator on $\left(L^p_*(\mathrm{R}_+)\right)^2$ via the isometry $L^p(\mathrm{R}_+) \cong L^p_*(\mathrm{R}_+)$. By the homogeneity of the kernel k one has

$$x^{1/p}(Kf)(x) = \int_0^\infty (x/y)^{1/p} k(x/y,1) y^{1/p} f(y)(dy/y) = k_p * \left(y^{1/p} f(y) \right)(x)\ ,$$

where the convolution is relative to the multiplicative group of the positive real numbers (called also Mellin convolution). We include next an explicit description of the spectrum of K as an operator on $(L^p(\mathrm{R}_+))^2$, $1 \le p < \infty$. This can be obtained by adapting the argument in [9] or [1] to the matrix setting described above.

Theorem 1.

If k is a Hardy kernel for $(L^p(R_+))^2$, $1 < p < \infty$, then, the operator K defined in (3.3) is a bounded operator on $(L^p(R_+))^2$. The spectrum of K as an operator on $(L^p(R_+))^2$ consists of the closure of the set in the plane consisting of all points $w \in C$ such that

$$\det\left(wI - \tilde{k}\right)(1/p + i\xi) = 0 \quad \textit{for some } \xi \in R\ . \tag{3.4}$$

Above, I is the identity matrix operator and $\tilde{k} := (\tilde{k}_{ij})_{i,j=1,2}$.

We shall call the matrix $\tilde{k}$ the *matrix of the Mellin symbols* of the operator K on $(L^p(\mathrm{R}_+))^2$, $1 \le p < \infty$. In the notation of [16, 18, 7] of the algebra of pseudodifferential operators of Mellin type, the condition (3.4) reads $\det \mathrm{Smbl}^{1/p}(wI - K)(0,z) = 0$ for some $z = 1/p + i\xi$, $\xi \in \mathrm{R}$.

4. The Determinant of Matrix Mellin Symbols in a Plane Sector

Let Ω be the unbounded domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$. Let $w \in \mathrm{C}$ and recall the operators $K_{\mathcal{L}}$ and $K_{\mathcal{S}}$ from (2.2), (2.5), acting on the space $(L^p(\partial\Omega))^2$, $1 \le p < \infty$. With an eye toward employing Theorem 1 to explicitly describe the spectra of the operators $wI - K_{\mathcal{L}}$ and $wI - K_{\mathcal{S}}$ we compute the determinants of the matrices of their Mellin symbols. Their zeros are, in the language of the algebra of pseudodifferential operators of Mellin type, the "singularity types" (the zeros of $\det \mathrm{Smbl}^{1/p}(wI - K)$), for the system

$$(wI - K)\, f = g, \qquad f, g \in \left(L^p(\partial\Omega)\right)^2 ,$$

where K stands for $K_{\mathcal{L}}$ and $K_{\mathcal{S}}$, respectively. Actually, matters can be reduced to analyzing just the operator $K_{\mathcal{L}}$ by observing that the operator $K_{\mathcal{S}}$ formally corresponds to $K_{\mathcal{L}}$ for the particular choice of the Lamé modulus $\lambda = \infty$.

By rotation and translation invariance we can think of the domain Ω as the region above the graph of the function

$$h(x) := \cot(\theta/2)|x|, \quad x \in \mathrm{R} .$$

We regard each function on $\partial\Omega$ as a 2 dimensional vector with components the restrictions of the function to each of the two rays (left and right, respectively) of the angle. In particular, this identification allows us to represent $wI - K_{\mathcal{L}}$ as a 4×4 operator-valued matrix.

Let

$$\upsilon := \frac{\mu + \lambda}{3\mu + \lambda} . \tag{4.1}$$

Note that for any choice of the Lamé moduli μ, λ as in Section 2 we have $0 < \upsilon \le 1$. By (2.2) and a straightforward computation, the kernel of the operator $K_{\mathcal{L}}$ can be written in the form

$$k_{\mathcal{L}}(P, Q) = \frac{\langle Q - P, N(Q)\rangle}{\pi |P - Q|^2} \times \begin{pmatrix} 1 + \upsilon \frac{(P_1 - Q_1)^2 - (P_2 - Q_2)^2}{|P-Q|^2} & 2\upsilon \frac{(P_1 - Q_1)(P_2 - Q_2)}{|P-Q|^2} \\ 2\upsilon \frac{(P_1 - Q_1)(P_2 - Q_2)}{|P-Q|^2} & 1 - \upsilon \frac{(P_1 - Q_1)^2 - (P_2 - Q_2)^2}{|P-Q|^2} \end{pmatrix} ,$$

where $P = (P_1, P_2)$, $Q = (Q_1, Q_2)$ are points on $\partial\Omega \subseteq \mathrm{R}^2$.

Going further, let us note that in the case when P and Q are both on the same ray of the angle the above kernel vanishes, as $\langle Q - P, N(Q)\rangle = 0$. Next, consider the case when P is on the right ray and Q is on the left ray of the angle. Set $s = |P|$ and $t = |Q|$. We have $P = \left(s \sin\frac{\theta}{2}, s \cos\frac{\theta}{2}\right)$, $Q = \left(-t \sin\frac{\theta}{2}, t \cos\frac{\theta}{2}\right)$, $N(Q) = \left(-\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$, $\langle Q - P, N(Q)\rangle = s \sin\theta$, $|P - Q|^2 = s^2 - 2st \cos\theta + t^2$, $(P_1 - Q_1)(P_2 - Q_2) = \frac{s^2 - t^2}{2} \sin\theta$, and, respectively $(P_1 - Q_1)^2 - (P_2 - Q_2)^2 = -s^2 \cos\theta + 2st - t^2 \cos\theta$. Thus, in the case under discussion, $k_{\mathcal{L}}(P, Q)$ takes the form

$$\begin{aligned} k_{\mathcal{L}}(s, t) = {} & \frac{1}{\pi} \frac{s \sin\theta}{s^2 - 2st \cos\theta + t^2} \\ & \times \begin{pmatrix} 1 + \upsilon \dfrac{-s^2 \cos\theta + 2st - t^2 \cos\theta}{s^2 - 2st \cos\theta + t^2} & \upsilon \dfrac{\left(s^2 - t^2\right) \sin\theta}{s^2 - 2st \cos\theta + t^2} \\ \upsilon \dfrac{\left(s^2 - t^2\right) \sin\theta}{s^2 - 2st \cos\theta + t^2} & 1 - \upsilon \dfrac{-s^2 \cos\theta + 2st - t^2 \cos\theta}{s^2 - 2st \cos\theta + t^2} \end{pmatrix} . \end{aligned} \tag{4.2}$$

Finally, when P is on the left ray and Q is on the right ray of the angle, similar calculations give $P = \left(-s \sin\frac{\theta}{2}, s \cos\frac{\theta}{2}\right)$, $Q = \left(t \sin\frac{\theta}{2}, t \cos\frac{\theta}{2}\right)$, $N(Q) = \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$,

$\langle Q-P, N(Q)\rangle = s\ \sin\theta$, $|P-Q|^2 = s^2 - 2st\ \cos\theta + t^2$, $(P_1-Q_1)(P_2-Q_2) = -\frac{s^2-t^2}{2}\sin\theta$, and, respectively $(P_1-Q_1)^2-(P_2-Q_2)^2 = -s^2\ \cos\theta + 2st - t^2\ \cos\theta$. Consequently, in this case, $k_{\mathcal{L}}(P, Q)$ takes the form

$$k_{\mathcal{L}}(s,t) = \frac{1}{\pi}\frac{s\ \sin\theta}{s^2-2st\ \cos\theta+t^2} \times \begin{pmatrix} 1+\upsilon\dfrac{-s^2\ \cos\theta+2st-t^2\ \cos\theta}{s^2-2st\ \cos\theta+t^2} & -\upsilon\dfrac{\left(s^2-t^2\right)\sin\theta}{s^2-2st\ \cos\theta+t^2} \\ -\upsilon\dfrac{\left(s^2-t^2\right)\sin\theta}{s^2-2st\ \cos\theta+t^2} & 1-\upsilon\dfrac{-s^2\ \cos\theta+2st-t^2\ \cos\theta}{s^2-2st\ \cos\theta+t^2} \end{pmatrix}. \tag{4.3}$$

Define the kernels

$$\begin{aligned} k^1_{\mathcal{L}}(s,t) &:= \frac{1}{\pi}\frac{s\ \sin\theta}{s^2-2st\ \cos\theta+t^2}\ , \\ k^2_{\mathcal{L}}(s,t) &:= \frac{1}{\pi}\frac{(s\ \sin\theta)(-s^2\ \cos\theta+2st-t^2)}{\left(s^2-2st\ \cos\theta+t^2\right)^2}\ , \\ k^3_{\mathcal{L}}(s,t) &:= \frac{1}{\pi}\frac{s\left(s^2-t^2\right)\sin^2\theta}{\left(s^2-2st\ \cos\theta+t^2\right)^2}\ . \end{aligned} \tag{4.4}$$

In this notation (4.2) and (4.3) become, respectively,

$$k_{\mathcal{L}}(s,t) = \begin{pmatrix} k^1_{\mathcal{L}}(s,t)+\upsilon k^2_{\mathcal{L}}(s,t) & \pm\upsilon k^3_{\mathcal{L}}(s,t) \\ \pm\upsilon k^3_{\mathcal{L}}(s,t) & k^1_{\mathcal{L}}(s,t)-\upsilon k^2_{\mathcal{L}}(s,t) \end{pmatrix}. \tag{4.5}$$

Since

$$k^i_{\mathcal{L}}(s,1) = \begin{cases} s^{1-1/p}, & \text{as } t\to 0\ , \\ s^{-1/p-1}, & \text{as } t\to\infty\ , \end{cases}$$

for $i = 1, 2, 3$, the operators with kernels (4.4) are Hardy kernels operators (in the sense of Definition 1). The Mellin transforms of their kernels $k^i := k^i_{\mathcal{L}}(\cdot, 1)$, $i = 1, 2, 3$, are as follows:

$$\begin{aligned} \tilde{k}^1(z) &= \frac{\sin(z(\pi-\theta))}{\sin(z\pi)}\ , \\ \tilde{k}^2(z) &= \frac{z\ \sin\theta\ \cos(z(\pi-\theta))}{\sin(z\pi)}\ , \\ \tilde{k}^3(z) &= \frac{z\ \sin\theta\ \sin(z(\pi-\theta))}{\sin(z\pi)}\ , \end{aligned} \tag{4.6}$$

for each $z \in \mathrm{C}$ with $\operatorname{Re} z \notin \mathrm{Z}$. In order to make the subsequent computations somewhat easier to follow, we need more notation. Introduce:

$$\begin{aligned} &A := \upsilon z\ \sin\theta, \quad B := \sin(z(\pi-\theta))\ , \\ &C := \cos(z(\pi-\theta)), \quad D := \sin(z\pi)\ . \end{aligned} \tag{4.7}$$

Also, let $k_{\mathcal{L}}(s) := k_{\mathcal{L}}(s, 1)$. By taking the Mellin transform in (4.5) we obtain

$$\tilde{k}_{\mathcal{L}}(z) = \frac{1}{D} \begin{pmatrix} B + AC & \pm AB \\ \pm AB & B - AC \end{pmatrix} . \tag{4.8}$$

Therefore, recalling the convention that the double layer potential is a 4×4 operator-valued matrix, we see that the 4×4 matrix of Mellin multipliers is given by

$$\tilde{k}_{\mathcal{L}}(z) = \frac{1}{D} \begin{pmatrix} 0 & 0 & B + AC & AB \\ 0 & 0 & AB & B - AC \\ B + AC & -AB & 0 & 0 \\ -AB & B - AC & 0 & 0 \end{pmatrix} . \tag{4.9}$$

Hence, the matrix of the Mellin symbols of the operator $wI - K_{\mathcal{L}}$ on $(L^p(\partial\Omega))^2$ is

$$\left(wI - \tilde{k}_{\mathcal{L}}\right)(z) = \frac{1}{D} \begin{pmatrix} wD & 0 & -B - AC & -AB \\ 0 & wD & -AB & -B + AC \\ -B - AC & AB & wD & 0 \\ AB & -B + AC & 0 & wD \end{pmatrix} , \tag{4.10}$$

where $z = 1/p + i\xi$, $\xi \in \mathrm{R}$. In the following lemma, the determinant of the matrix in (4.10) is written as a product of two factors. This factorization follows from a tedious but elementary calculation which we omit.

Lemma 1.

The determinant of the matrix (4.10) can be written in the following form:

$$\det \left(wI - \tilde{k}_{\mathcal{L}}\right)(z) = \frac{\left((wD + AC)^2 - B^2 + (AB)^2\right)\left((wD - AC)^2 - B^2 + (AB)^2\right)}{D^4} . \tag{4.11}$$

Now, Lemma 1 together with Theorem 1 allow us to obtain the following explicit characterization of the spectrum of the operator $K_{\mathcal{L}}$ on the space $L^p(\partial\Omega)$ for $1 < p < \infty$.

Theorem 2.

Let Ω be the unbounded domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$. Also, consider the equation

$$\left((wD + AC)^2 - B^2 + (AB)^2\right)\left((wD - AC)^2 - B^2 + (AB)^2\right) = 0 . \tag{4.12}$$

Then, for each $1 < p < \infty$,

$$\sigma\left(K_{\mathcal{L}}; \left(L^p(\partial\Omega)\right)^2\right) = \left\{w \in C; \ \textit{such that (4.12) holds for some } z \in \Gamma_{1/p}\right\} \cup \{0\} ,$$

where $\Gamma_{1/p} := \{z \in C; \ \mathrm{Re}\, z = 1/p\}$.

Let us note here that if $M \in \mathrm{C}$ is given by

$$M := \pm\sqrt{1 - A^2} , \tag{4.13}$$

then, any solution $w \in \mathrm{C}$ of the equation (4.12) satisfies

$$w = \frac{BM + AC}{D} \quad \text{or} \quad w = \frac{BM - AC}{D} . \tag{4.14}$$

This follows easily from rewriting (4.12) as $((wD + AC)^2 - (BM)^2)((wD - AC)^2 - (BM)^2) = 0$. Let us also point out that the left hand side in (4.12) does not change if we replace θ by $2\pi - \theta$. This allows us to restrict (without loss of generality) our further analysis to $\theta \in (0, \pi]$, and assume that the complex number M has positive real part.

Finally, consider

$$\Sigma_\theta(p) := \left\{\pm\frac{BM \pm AC}{D}\right\} \cup \{0\}\ , \tag{4.15}$$

with A, B, C, M, and D as in (4.7) and (4.13) evaluated at θ and $z = 1/p + iy$, $y \in \mathrm{R}$. Since for a fixed p we have $\lim_{y\to\pm\infty} \frac{BM \pm AC}{D} = 0$, the set $\Sigma_\theta(p)$ consists of the union of the four closed curves $\Sigma_\pm^\pm(\theta, p)$ parametrically given by $y \mapsto w = \pm\frac{BM \pm AC}{D}$, $y \in [-\infty, \infty]$. The choice of the subscript in $\Sigma_\pm^\pm(\theta, p)$ corresponds to the choice of the sign $\pm$ in front of the fraction, while the choice of the superscript corresponds to the choice of the sign of AC in the numerator. Clearly, Σ_-^+ and Σ_-^- are the symmetric with respect to the y-axis of Σ_+^+ and Σ_+^-, respectively. In this notation Theorem 2 states that $\sigma(K_{\mathcal{L}}; (L^p(\partial\Omega))^2) = \Sigma_\theta(p)$.

We have included pictures of $\Sigma_\theta(p)$ for different values of θ and p in the introduction as well as the end of the next section. For a geometrical insight we present below (Figure 5) the graph of $\Sigma_+^+(\pi/4, 6)$ for $\upsilon = 1/2$.

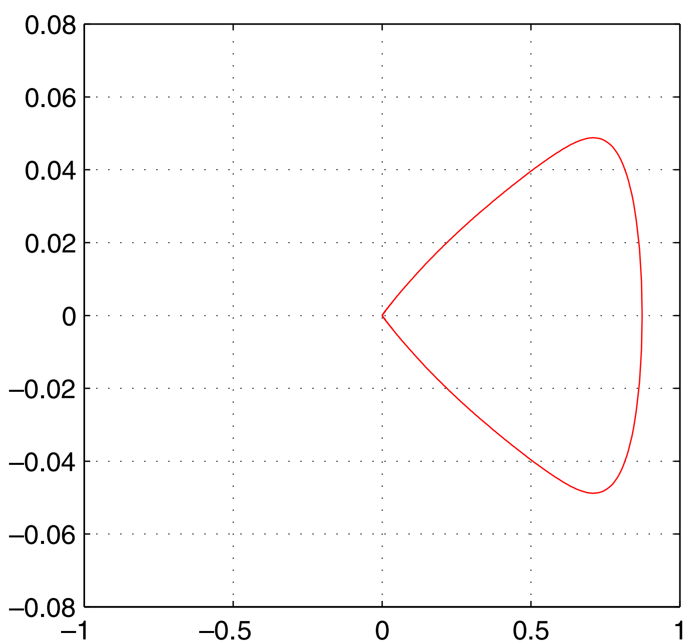

FIGURE 5 The curve $\Sigma_+^+(\pi/4, 6)$ corresponding to a glass material.

5. The Spectrum and the Spectral Radius in a Plane Sector

In this section we will be concerned with finding the spectral radius (i. e., the radius of the smallest closed disc centered at the origin that contains the spectrum) of the operators $K_{\mathcal{L}}$ and respectively $K_{\mathcal{S}}$ on $(L^p(\partial\Omega))^2$ for $2 \le p < \infty$. Throughout the section, Ω will be the domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$. In this context, our main result is as follows.

Theorem 3.

Let Ω be the unbounded domain consisting of the interior of an angle of measure $\theta \in (0, 2\pi)$. In the case of the elastostatic double layer potential, for any $2 \le p < \infty$, we have

$$\rho\left(K_{\mathcal{L}}; \left(L^p(\partial\Omega)\right)^2\right) = \frac{\left|\frac{\upsilon}{p}\sin\theta\cos\left(\frac{\pi-\theta}{p}\right) + \sin\left(\frac{\pi-\theta}{p}\right)\sqrt{1-\upsilon^2\frac{\sin^2\theta}{p^2}}\right|}{\sin\left(\frac{\pi}{p}\right)} . \tag{5.1}$$

Also, in the hydrostatic case we have

$$\rho\left(K_{\mathcal{S}}; \left(L^p(\partial\Omega)\right)^2\right) = \frac{\left|\frac{1}{p}\sin\theta\cos\left(\frac{\pi-\theta}{p}\right) + \sin\left(\frac{\pi-\theta}{p}\right)\sqrt{1-\frac{\sin^2\theta}{p^2}}\right|}{\sin\left(\frac{\pi}{p}\right)} . \tag{5.2}$$

Furthermore, the spectral radii belong to the corresponding spectra, i. e.,

$$\rho\left(K_{\#}; \left(L^p(\partial\Omega)\right)^2\right) \in \sigma\left(K_{\#}; \left(L^p(\partial\Omega\right)^2\right) ,$$

where the subscript # is either $\mathcal{L}$ or $\mathcal{S}$, and

$$\rho\left(K_{\mathcal{L}}; \left(L^p(\partial\Omega)\right)^2\right) < 1, \qquad \rho\left(K_{\mathcal{S}}; \left(L^p(\partial\Omega)\right)^2\right) < 1 . \tag{5.3}$$

As alluded to before, (5.2) follows from (5.1) by taking $\lambda = \infty$ which, in turn, implies $\upsilon = 1$. For the remaining of the argument we drop the subscript $\mathcal{L}$. Let us introduce

$$R(\theta, 1/p) := \left|\frac{BM + AC}{D}\right| \quad \text{evaluated at} \quad z = 1/p , \tag{5.4}$$

where M is as in (4.13) with positive real part. Since the right hand side in (5.1) is actually $\left|\frac{BM+AC}{D}\right|$ evaluated at $z = 1/p$, (5.4) can also be regarded as

$$R(\theta, x) := \frac{\left|\upsilon x\sin\theta\cos((\pi-\theta)x) + \sin((\pi-\theta)x)\sqrt{1-\upsilon^2x^2\sin^2\theta}\right|}{\sin(\pi x)}, \quad x \in (0, 1) . \tag{5.5}$$

In the light of (4.14), the proof of (5.1) reduces to showing that for any fixed $2 \le p < \infty$ and any $z \in \Gamma_{0,1/2} \cup \Gamma_{1/2}$, (that is, $z = x + iy$ with $x \in (0, 1/2]$ and $y \in$ R), one has

$$\left|\frac{BM \pm AC}{D}\right| \le R(\theta, x) . \tag{5.6}$$

The proof of (5.6) will follow from a sequence of technical lemmas which we now present. To state the first one, recall $\upsilon \in (0, 1]$ as in (4.1).

Lemma 2.

Let $\theta \in (0, \pi]$ and $a(x, y)$, $b(x, y) \in R$, $a(x, y) \geq 0$ be, respectively, the real and the imaginary parts of M at $z = x + iy$, $x \in [0, 1]$, $y \in R$. Consider

$$\begin{aligned} &c := \upsilon \sin\theta \ , \\ &f(x, y) := 2c(ax + by) \quad \textit{and} \quad g(x, y) := a^2 + b^2 - c^2\left(x^2 + y^2\right) \ , \\ &h(x, y) := (a + cy)^2 + (b - cx)^2 \quad \textit{and} \quad l(x, y) := (a - cy)^2 + (b + cx)^2 \ . \end{aligned} \tag{5.7}$$

Then, we have:

$$\begin{aligned} &a^2 - b^2 = 1 - c^2(x^2 - y^2) \quad \textit{and} \quad ab = -c^2 xy \ , \\ &b^2 = \frac{1}{2}\left(-1 + c^2\left(x^2 - y^2\right) + \sqrt{\left(1 - c^2\left(x^2 - y^2\right)\right)^2 + 4c^4x^2y^2}\right) \ , \\ &a(x, y) \ \textit{and} \ b(x, y) \ \textit{are, respectively, even and odd in the variable} \ y \ , \\ &\frac{\partial a}{\partial y} = \frac{c^2(ay - bx)}{a^2 + b^2} \quad \textit{and} \quad \frac{\partial b}{\partial y} = -\frac{c^2(by + ax)}{a^2 + b^2} \ , \\ &\frac{\partial h}{\partial y}(x, y) = \frac{2ca}{a^2 + b^2}\, h(x, y) \quad \textit{and} \quad \frac{\partial l}{\partial y}(x, y) = \frac{-2ca}{a^2 + b^2}\, l(x, y) \ , \\ &\frac{\partial f}{\partial y}(x, y) = \frac{2bc}{a^2 + b^2}\, g(x, y) \quad \textit{and} \quad \frac{\partial g}{\partial y}(x, y) = \frac{-2bc}{a^2 + b^2}\, f(x, y) \ . \end{aligned} \tag{5.8}$$

Above, a, b stand for $a(x, y)$ and $b(x, y)$, respectively.

Parenthetically, note that $a^2 + b^2$ never vanishes since $a^2 + b^2 = |1 - A^2|$ with $|A| < 1$. The above equalities follow from the definitions of a and b and elementary computations which we omit.

Corollary 1.

With the same notation as in the Lemma 2, the following hold for any $x \in [0, 1]$:

$$\begin{aligned} f^2(x, y) + g^2(x, y) \equiv 1 \quad \textit{and} \quad h(x, y)l(x, y) \equiv 1 \ , \\ \log(h(x, Y)) = \int_0^Y \frac{2ca}{a^2 + b^2}\, dy \ . \end{aligned} \tag{5.9}$$

Proof. It follows from Lemma 2 and some straightforward algebra manipulations that the identity $f(x, y)\frac{\partial f}{\partial y}(x, y) + g(x, y)\frac{\partial g}{\partial y}(x, y) = 0$ holds. In particular, this implies that, for each x fixed, $f^2 + g^2$ is constant when regarded as a function of y. Now let $y = 0$ in the first two rows of equalities in (5.8). Since $c^2x^2 \leq 1$ for any $x \in [0, 1]$ we get $b(x, 0) = 0$ and $a^2(x, 0) = 1 - c^2x^2$. Also, letting $y = 0$ in the definitions of the functions f and g in (5.7) we obtain $f^2 = 4c^2a^2x^2$ and $g^2 = (a^2 - c^2x^2)^2$. This finally gives that $f^2(x, 0) + g^2(x, 0) = (a^2(x, 0) + c^2x^2)^2 = 1$ and proves the first identity in (5.9).

Appealing again to Lemma 2 for the formulas for the derivatives of the functions h and l, we have $h(x, y)\dfrac{\partial l}{\partial y}(x, y) + l(x, y)\dfrac{\partial h}{\partial y}(x, y) = 0$. This implies that for a fixed x, the function $h(x,y)l(x,y)$ is constant in y. As before, we obtain that this constant is 1 by letting $y=0$. Regarding the third assertion of the Corollary 1, we simply integrate from 0 to Y in $\frac{\frac{\partial h}{\partial y}}{h} = \frac{2ca}{a^2+b^2}$ from (5.8) and use the fact that $h(x,0) = a^2(x,0) + c^2x^2 = 1$. □

Next we describe the quantities $|BM \pm AC|$.

Lemma 3.

Let $\theta \in (0,\pi]$. With the same notation as before we have

$$
\begin{aligned}
|BM \pm AC|^2 &= \frac{\cosh(2(\pi-\theta)y)}{2}\left[a^2+b^2+c^2\left(x^2+y^2\right)\right] \\
&\pm c\sinh(2(\pi-\theta)y)[ay-bx] \\
&- \frac{\cos(2(\pi-\theta)x)}{2}\left[a^2+b^2-c^2\left(x^2+y^2\right)\right] \pm c\sin(2(\pi-\theta)x)[ax+by]\,.
\end{aligned} \tag{5.10}
$$

Moreover, for $y \geq 0$ and $x \in [0,1/2]$, there holds

$$
|BM - AC| \leq |BM + AC|\,. \tag{5.11}
$$

Proof. Let

$$
R_\pm := \mathrm{Re}(BM \pm AC) \quad \text{and} \quad I_\pm := \mathrm{Im}(BM \pm AC)\,.
$$

A direct calculation based on the definitions of A, B, C and M shows that

$$
\begin{aligned}
R_\pm &= \cosh((\pi-\theta)y)[a\sin((\pi-\theta)x) \pm cx\cos((\pi-\theta)x)] \\
&\quad + \sinh((\pi-\theta)y)[\pm cy\sin((\pi-\theta)x) - b\cos((\pi-\theta)x)]\,,
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
I_\pm &= \cosh((\pi-\theta)y)[b\sin((\pi-\theta)x) \pm cy\cos((\pi-\theta)x)] \\
&\quad + \sinh((\pi-\theta)y)[\mp cx\sin((\pi-\theta)x) + a\cos((\pi-\theta)x)]\,,
\end{aligned} \tag{5.13}
$$

where $a := a(x,y)$ and $b := b(x,y)$ are as in Lemma 2. Then, (5.10) follows from the fact that $|BM \pm AC|^2 = R_\pm^2 + I_\pm^2$. As for (5.11), let us first note that

$$
\begin{aligned}
|BM + AC|^2 - |BM - AC|^2 &= 2c(\sinh(2(\pi-\theta)y)[ay-bx] \\
&\quad + [ax+by]\sin(2(\pi-\theta)x))\,.
\end{aligned} \tag{5.14}
$$

Since $ab = -c^2xy$ and $a \geq 0$ we obtain that, for $y \geq 0$, the function b takes only negative values. This entails $ay - bx \geq 0$ for $y \geq 0$.

Next, we claim that if $x \in [0,1]$ is fixed, then, $ax + by$, regarded as a function in the variable y, does not vanish on $(0,\infty)$. Assume by contradiction that $ax = -by$ for some $y > 0$. Then, since $ab = -c^2xy$ and b is negative, we get $b = -cx$ and, consequently, $a = cy$. Then, $a^2 - b^2 = -c^2(x^2 - y^2)$ and the first equality in the conclusion of Lemma 2 is violated. This shows that $ax + by$ does not change sign on $(0,\infty)$. Since when $y = 0$ it takes the positive value $a(x,0)x$, it follows by continuity that $ax + by \geq 0$ for any $x \in [0,1]$ and $y \geq 0$. Finally, if $x \in [0,1/2]$, then, $2(\pi-\theta)x \leq (\pi-\theta) \leq \pi$ and, thus, $\sin(2(\pi-\theta)x)$ is positive. This completes the proof of (5.11). □

For a fixed $x \in [0, 1/2]$, consider the functions

$$F_{\pm}(y) := |BM \pm AC|^2 - R(\theta, x)^2 |D|^2 , \tag{5.15}$$

where B, M, A, C, D are evaluated at $z := x + iy$.

Remark 1. The functions $F_{\pm}$ are even.

Proof. Indeed, note that $|D|^2 = (\cosh(\pi y)\sin(\pi x))^2 + (\sinh(\pi y)\cos(\pi x))^2 = \frac{\cosh(2\pi y)}{2} - \frac{\cos(2\pi x)}{2}$, which is an even function in the variable y. Then, the remark follows from (5.10) together with the fact that $a(x, y)$ is even and $b(x, y)$ is respectively odd in the variable y (see Lemma 2). $\square$

Remark 1 together with (5.11) imply that

$$\left|\frac{BM - AC}{D}\right| \le \left|\frac{BM + AC}{D}\right| \quad \text{for any} \quad y \in \mathrm{R} \quad \text{and} \quad x \in (0, 1/2] . \tag{5.16}$$

Therefore, (5.6) reduces to showing that

$$\left|\frac{BM + AC}{D}\right| \le R(\theta, x), \quad \text{for} \ \ z = x + iy, \ \ x, y \in \mathrm{R}, \ \ x = \frac{1}{p}, \ \ 2 \le p < \infty , \tag{5.17}$$

or, equivalently,

$$|BM + AC|^2 - R(\theta, x)^2 |D|^2 \le 0 , \tag{5.18}$$

for $z = x + iy$, $x, y \in \mathrm{R}$, $x = \frac{1}{p}$, $2 \le p < \infty$. Note the above forces $x \in (0, 1/2]$. Also, since $F_+(0) = 0$, one can read (5.18) as $F_+(y) \le F_+(0)$. Thus, verifying (5.18) reduces to proving the following.

Proposition 1.
For any fixed $x \in (0, 1/2]$, the function $F_+(y)$ attains its maximal value at $y = 0$.

Proof. Fix $x \in (0, 1/2]$ and let $p = 1/x \in [2, \infty)$. We rewrite $4F_+(y)$ in the more convenient form

$$\begin{aligned} 4F_+(y) = {} & e^{2(\pi-\theta)y} h(x, y) + e^{-2(\pi-\theta)y} l(x, y) - 2g(x, y)\cos(2(\pi-\theta)x) \\ & + 2f(x, y)\sin(2(\pi-\theta)x) - R(\theta, x)^2 \left(e^{2\pi y} + e^{-2\pi y} - 2\cos(2\pi x)\right), \end{aligned} \tag{5.19}$$

where the functions f, g, h, l have been defined in (5.7). Using the fact that $F_+(y)$ is even, it suffices to show that $\frac{\partial F_+}{\partial y}(y) \le 0$ for any $y \in [0, \infty)$. Differentiating with respect to y in (5.19) and using the identities from Lemma 2 one is able to obtain that

$$4\frac{\partial F_+}{\partial y}(y) = I + II ,$$

where

$$\begin{aligned} I := {} & 2\left(\pi - \theta + \frac{ca}{a^2 + b^2}\right)\left(e^{2(\pi-\theta)y} h(x, y) - e^{-2(\pi-\theta)y} l(x, y)\right) \\ & - 2\pi R(\theta, x)^2 \left(e^{2\pi y} - e^{-2\pi y}\right) , \\ II := {} & 4\frac{bc}{a^2 + b^2}\left(f(x, y)\cos(2(\pi-\theta)x) + g(x, y)\sin(2(\pi-\theta)x)\right) . \end{aligned} \tag{5.20}$$

The idea is to show that both I and II are ≤ 0. Note that at $y = 0$ we have $I = 0$. Hence, for $I \leq 0$, it suffices to show $\frac{\partial}{\partial y} I \leq 0$ for $y \in [0, \infty)$. Differentiating with respect to y using (5.8) in the formula for I given above, we get

$$\frac{\partial}{\partial y} I(y) = I_1 + I_2 \,, \tag{5.21}$$

where

$$\begin{aligned} I_1 &:= 4\left(\pi - \theta + \frac{ca}{a^2+b^2}\right)^2 \left(e^{2(\pi-\theta)y} h(x,y) + e^{-2(\pi-\theta)y} l(x,y)\right) \\ &\quad - 4\pi^2 R(\theta, x)^2 \left(e^{2\pi y} + e^{-2\pi y}\right) \,, \\ I_2 &:= 2\frac{\partial s}{\partial y}(x,y) \left(e^{2(\pi-\theta)y} h(x,y) - e^{-2(\pi-\theta)y} l(x,y)\right) \,, \end{aligned} \tag{5.22}$$

with

$$s(x,y) := \pi - \theta + \frac{c\,a(x,y)}{a^2(x,y) + b^2(x,y)} \tag{5.23}$$

in (5.22). Differentiating s with respect to y gives

$$\frac{\partial s}{\partial y} = \frac{c}{\left(a^2+b^2\right)^2} \left(\left(b^2 - a^2\right)\frac{\partial a}{\partial y} - 2ab\frac{\partial b}{\partial y}\right) \,. \tag{5.24}$$

We recall now the formulas for $\frac{\partial a}{\partial y}$ and $\frac{\partial b}{\partial y}$ in (5.8). Proceeding as in the proof of Lemma 3 we can conclude that $ax + by \geq 0$ and $ay - bx \geq 0$ for $y \in [0, \infty)$. This implies that $\frac{\partial a}{\partial y}(y) \geq 0$ and $\frac{\partial b}{\partial y}(y) \leq 0$. We also have that $a \geq 0$, $b \leq 0$ and $b^2 - a^2 \leq 0$. Thus, the second factor in (5.24) is negative. Therefore, for any $x \in [0, 1]$ and $y \in [0, \infty)$ we have

$$\frac{\partial s}{\partial y}(x,y) \leq 0 \,. \tag{5.25}$$

Next, for any $x \in (0, 1/2]$ fixed, let $H(y) := e^{2(\pi-\theta)y} h(x,y)$. From Corollary 1 we obtain that $e^{-2(\pi-\theta)y} l(x,y) = \frac{1}{H(y)}$. Hence,

$$e^{2(\pi-\theta)y} h(x,y) - e^{-2(\pi-\theta)y} l(x,y) = \frac{H^2(y) - 1}{H(y)} \,. \tag{5.26}$$

Now, for $y \geq 0$, we have that $e^{2(\pi-\theta)y} \geq 1$. Using the expression of $\frac{\partial h}{\partial y}(x,y)$ given in (5.8) together with the fact that h is positive, we conclude that h is monotonically increasing in the variable y. Since $h(x, 0) = 1$ it follows that $h(x,y) \geq 1$ for $y \in [0, \infty)$. This gives $H(y) \geq 1$ on $[0, \infty)$. Finally, taking this in (5.26) and (5.25) give

$$I_2(y) \leq 0 \quad \text{for} \quad \text{any} \quad y \in [0, \infty) \,. \tag{5.27}$$

We are left with showing that $I_1 \leq 0$ and $II \leq 0$. We will do this in four steps, contained in the Lemmas 4–7 below. This concludes the proof of Proposition 1, granted Lemmas 4–7. $\square$

Lemma 4.

Let $\theta \in (0, \pi]$, $p \in [2, \infty)$ and $\upsilon \in [0, 1]$. Then,

$$R(\theta, 1/p) = \frac{\frac{\upsilon}{p}\sin\theta\cos\left(\frac{\pi-\theta}{p}\right) + \sin\left(\frac{\pi-\theta}{p}\right)\sqrt{1-\upsilon^2\frac{\sin^2\theta}{p^2}}}{\sin\left(\frac{\pi}{p}\right)} < 1 \ , \tag{5.28}$$

Proof. The equality in (5.28) follows easily from (5.4) together with the observation that when $\theta \in (0, \pi]$ and $p \in [2, \infty)$ all the trigonometric functions in (5.4) take positive values and consequently the absolute value sign can be dropped. We are left with showing $R(\theta, 1/p) < 1$. To this end, rewrite the inequality in (5.28) as follows.

$$\upsilon x \sin\theta\cos((\pi-\theta)x) + \sin((\pi-\theta)x)\sqrt{1-\upsilon^2 x^2\sin^2\theta} < \sin(\pi x) \ , \tag{5.29}$$

where $x \in (0, 1/2]$, $x := 1/p$. Let us assume for the moment that for $x \in (0, 1/2]$ and $\theta \in (0, \pi]$ one has

$$\upsilon x \sin\theta < \sin(x\theta) \tag{5.30}$$

and continue the argument. Let $\tilde{x} \in [0, \pi/2]$ be such that

$$\sin\tilde{x} = \upsilon x \sin\theta \quad \text{and} \quad \cos\tilde{x} = \sqrt{1-\upsilon^2x^2\sin^2\theta} \ . \tag{5.31}$$

With this notation, the left hand side of (5.29) becomes $\sin((\pi-\theta)x + \tilde{x})$ and (5.30) simply reads $\tilde{x} < \theta x$. Thus,

$$0 \le (\pi-\theta)x + \tilde{x} < \pi x \le \pi/2 \ , \tag{5.32}$$

and, further, $\sin((\pi-\theta)x + \tilde{x}) < \sin(\pi x)$. This is (5.29). We are, therefore, left with proving (5.30). For this purpose, for a fixed $x \in (0, 1/2]$, consider the function

$$J(\theta) := \upsilon x \sin\theta - \sin(x\theta), \qquad \theta \in [0, \pi] \ .$$

Note that $J(0) = 0$ and, hence, it suffices to show that

$$J'(\theta) = x\left[\upsilon\cos\theta - \cos(x\theta)\right] < 0, \qquad \forall\,\theta \in (0, \pi] \ . \tag{5.33}$$

We have $\cos(x\theta) \ge \upsilon\cos(x\theta) > \upsilon\cos\theta$, as $\cos(x\theta) \ge 0$ for $x \in (0, 1/2]$ and $\theta \in [0, \pi]$. This takes care of (5.30) and the proof of Lemma 4 is now complete. □

Lemma 5.

Let $\theta \in (0, \pi]$, $x \in (0, 1/2]$ and $y \ge 0$. Then,

$$\pi - \theta + \frac{ca}{a^2+b^2} \le \pi R(\theta, x) \ , \tag{5.34}$$

where $a := a(x, y)$, $b := b(x, y)$ are as before. In particular we have

$$\pi - \theta + \frac{ca}{a^2+b^2} \le \pi \ , \tag{5.35}$$

Proof. First note that (5.34) is equivalent to

$$s(x, y) \le \pi R(\theta, x) \ , \tag{5.36}$$

where $s(x, y)$ is as in (5.23). From (5.25) we have that $s(x, \cdot)$ is monotonically decreasing on $[0, \infty)$ and, therefore, it suffices to prove (5.34) when $y = 0$. Since $a(x, 0) = \sqrt{1 - c^2x^2}$ and $b(x, 0) = 0$ we are left with showing that for any $x \in (0, 1/2]$, $\theta \in (0, \pi]$ and $\upsilon \in [0, 1]$ there holds

$$\pi - \theta + \frac{\upsilon \sin\theta}{\sqrt{1 - \upsilon^2 x^2 \sin^2\theta}} \leq \pi R(\theta, x) \ . \tag{5.37}$$

Before proceeding further let us point out that in the case $\upsilon = 0$ (which corresponds to the classical double layer potential operator for the Laplacian), the inequality (5.37) reduces to $\pi - \theta \leq \pi \frac{\sin((\pi-\theta)x)}{\sin \pi x}$, which follows easily form the fact that $0 \leq (\pi - \theta)x \leq \pi x$ and the fact that $\frac{\sin t}{t}$ is monotonically decreasing for $t \in [0, \pi]$. Returning now to (5.37), we proceed now as in the proof of Lemma 4 by rewriting

$$R(\theta, x) = \frac{\sin\left((\pi - \theta)x + \tilde{x}\right)}{\sin(\pi x)} \ , \tag{5.38}$$

where $\tilde{x} = \tilde{x}(\upsilon, \theta, x)$ is defined in (5.31). Next, we rewrite the left hand side of (5.37) as

$$\pi - \theta + \frac{\upsilon \sin\theta}{\sqrt{1 - \upsilon^2 x^2 \sin^2\theta}} = \pi - \theta + \frac{\tan\tilde{x}}{x} \ . \tag{5.39}$$

In this notation (5.37) is equivalent to having

$$\frac{\sin(\pi x)}{\pi} \leq \frac{\sin\left((\pi - \theta)x + \tilde{x}\right)}{\pi - \theta + \frac{\tan\tilde{x}}{x}} \ , \tag{5.40}$$

for all $x \in (0, 1/2]$, $\upsilon \in [0, 1]$ and $\theta \in (0, \pi]$.

For fixed $\theta \in (0, \pi]$ and $x \in (0, 1/2]$, we study the behavior of the right hand side of (5.40) with respect to the variable υ, for $\upsilon \in [0, 1]$. We have

$$\begin{aligned} &\frac{\partial}{\partial\upsilon}\left(\sin\left((\pi - \theta)x + \tilde{x}\right)\right) = \left(\cos\left((\pi - \theta)x + \tilde{x}\right)\right)\frac{\partial\tilde{x}}{\partial\upsilon}, \quad \text{and} \\ &\frac{\partial}{\partial\upsilon}\left(\pi - \theta + \frac{\tan\tilde{x}}{x}\right) = \frac{1}{x}\frac{1}{\cos^2\tilde{x}}\frac{\partial\tilde{x}}{\partial\upsilon} \ . \end{aligned} \tag{5.41}$$

Hence, the sign of $\frac{\partial}{\partial\upsilon}\left(\frac{\sin((\pi - \theta)x + \tilde{x})}{\pi - \theta + \frac{\tan\tilde{x}}{x}}\right)$ is the same as that one of

$$\left[\left(\pi - \theta + \frac{\tan\tilde{x}}{x}\right)\cos\left((\pi - \theta)x + \tilde{x}\right) - \frac{1}{x}\frac{1}{\cos^2\tilde{x}}\sin\left((\pi - \theta)x + \tilde{x}\right)\right]\frac{\partial\tilde{x}}{\partial\upsilon} \ . \tag{5.42}$$

Since $\tilde{x}$ is increasing in υ, we have $\frac{\partial\tilde{x}}{\partial\upsilon} \geq 0$. This shows that the sign of the expression in (5.42) is the same as that of

$$\left((\pi - \theta)x + \tan\tilde{x}\right)\cos^2\tilde{x} - \tan\left((\pi - \theta)x + \tilde{x}\right) \ . \tag{5.43}$$

The expression in (5.43) is negative as one has

$$\tan\left((\pi - \theta)x + \tilde{x}\right) \geq (\pi - \theta)x + \tan\tilde{x} \ , \tag{5.44}$$

for $x \in (0, 1/2]$ and $\theta \in (0, \pi]$. The latter inequality can be justified in the following manner. We have

$$\tan\left((\pi-\theta)x+\tilde{x}\right) = \frac{\tan((\pi-\theta)x)+\tan(\tilde{x})}{1-\tan((\pi-\theta)x)\tan(\tilde{x})}\,. \tag{5.45}$$

Since $(\pi-\theta)x, \tilde{x} \in [0, \pi/2]$ and also $(\pi-\theta)x+\tilde{x} \in [0, \pi/2]$, [from (5.32)], all the tangent functions in (5.45) take positive values. This implies $\tan((\pi-\theta)x+\tilde{x}) \geq \tan((\pi-\theta)x)+\tan(\tilde{x})$. Finally, using $\tan t \geq t$ for $t \in [0, \pi/2]$ and applying it for $t = (\pi-\theta)x$, we obtain that (5.44) holds.

In conclusion, the right hand side of (5.40) is monotonically decreasing in υ for $\upsilon \in [0, 1]$. Consequently, matters have been reduced to the case $\upsilon = 1$. Hence (5.37) becomes

$$\frac{\sin(\pi x)}{\pi} \leq \frac{\sin((\pi-\theta)x+\tilde{x}(1,\theta,x))}{\pi-\theta+\dfrac{\tan(\tilde{x}(1,\theta,x))}{x}}, \quad \forall x \in (0, 1/2],\ \theta \in (0, \pi]\,. \tag{5.46}$$

Let $\tilde{x}$ stand for $\tilde{x}(1,\theta,x)$ in the remaining part of the proof. Taking the derivative with respect to θ in the right hand side of (5.46) and using that by (5.32) we have $\cos((\pi-\theta)x+\tilde{x}) > 0$, we obtain that the sign of $\dfrac{\partial}{\partial\theta}\left(\dfrac{\sin((\pi-\theta)x+\tilde{x})}{\pi-\theta+\dfrac{\tan\tilde{x}}{x}}\right)$ is the same as that of

$$-\left[\left(x-\frac{\partial\tilde{x}}{\partial\theta}\right)\left(\pi-\theta+\frac{\tan\tilde{x}}{x}\right)-\left(1-\frac{1}{x}\frac{1}{\cos^2\tilde{x}}\frac{\partial\tilde{x}}{\partial\theta}\right)\tan((\pi-\theta)x+\tilde{x})\right]\,. \tag{5.47}$$

Differentiating implicitly with respect to θ in (5.31) we obtain $\dfrac{\partial\tilde{x}}{\partial\theta} = \cot\theta\tan\tilde{x}$. Therefore, $\frac{\partial\tilde{x}}{\partial\theta} \leq 0$ for $\theta \in \left[\frac{\pi}{2}, \pi\right]$. As a consequence

$$0 \leq \left(x-\frac{\partial\tilde{x}}{\partial\theta}\right) \leq \left(x-\frac{1}{\cos^2\tilde{x}}\frac{\partial\tilde{x}}{\partial\theta}\right)\,. \tag{5.48}$$

Now, using (5.44), we obtain

$$\left(x-\frac{\partial\tilde{x}}{\partial\theta}\right)\left(\pi-\theta+\frac{\tan\tilde{x}}{x}\right)-\left(1-\frac{1}{x}\frac{1}{\cos^2\tilde{x}}\frac{\partial\tilde{x}}{\partial\theta}\right)\tan((\pi-\theta)x+\tilde{x}) \leq 0\,. \tag{5.49}$$

Consequently, by (5.49) we have that the expression in (5.47) is positive. This proves that the right hand side of (5.46) is monotonically increasing in $\theta \in [\pi/2, \pi]$. Next, tedious and rather long considerations based on Taylor series expansions show that

$$\frac{\sin(\pi x)}{\pi} \leq \frac{\sin((\pi-\theta)x+\tilde{x}(1,\theta,x))}{\pi-\theta+\dfrac{\tan(\tilde{x}(1,\theta,x))}{x}}, \quad \forall\theta \in (0, \pi/2],\ x \in (0, 1/2]\,. \tag{5.50}$$

In order not to interrupt the flow of the presentation at the moment we postpone the proof of (5.50) for the Appendix. This completely finishes the proof of (5.46) and consequently (5.37) holds.

Finally, (5.35) follows from (5.34) and (5.28) in Lemma 4. □

The third step in the proof of Proposition 1 is presented below.

Lemma 6.
Let $\theta \in (0, \pi]$, $x \in (0, 1/2]$ *and* $y \in [0, \infty)$. *Then,*

$$e^{2(\pi-\theta)y}h(x, y) + e^{-2(\pi-\theta)y}l(x, y) \le e^{2\pi y} + e^{-2\pi y} , \tag{5.51}$$

where the functions h and l are as in (5.7).

Proof. Fix $x \in [0, 1/2]$ and let $H(y) := e^{2(\pi-\theta)y}h(x, y)$ and $K(y) := e^{2\pi y}$. Corollary 1 gives that $e^{-2(\pi-\theta)y}l(x, y) = 1/H(y)$ and, hence, (5.51) takes the form $H(y) + \dfrac{1}{H(y)} - K(y) - \dfrac{1}{K(y)} \le 0$ or, equivalently, $(H(y) - K(y))\left(1 - \frac{1}{H(y)K(y)}\right) \le 0$. Since H, K are positive functions and $H(y), K(y) \ge 1$, this comes down to proving $H(y) \le K(y)$ for $y \in [0, \infty)$. Note that we can rewrite this as $h(x, y) \le e^{2\theta y}$. In this form, this follows from (5.35) in Lemma 5 and the last equality in the conclusion of Corollary 1. □

At this stage, Lemma 4, Lemma 5, and Lemma 6 give $I_1 \le 0$ with I_1 as in (5.22). We are now finally ready to complete the proof of Proposition 1 by showing $II \le 0$.

Lemma 7.
Assume $\theta \in (0, \pi]$, $x \in (0, 1/2]$ *and* $y \in [0, \infty)$. *Then,*

$$II := 4\frac{cb}{a^2 + b^2}(f(x, y)\cos(2(\pi-\theta)x) + g(x, y)\sin(2(\pi-\theta)x)) \le 0 , \tag{5.52}$$

where $a, b, c, f(x, y), g(x, y)$ *are as in (5.7).*

Proof. Fix $x \in [0, 1/2]$. Since $b(x, y) \le 0$ for $y \ge 0$, (5.52) reduces to

$$f(x, y)\cos(2(\pi-\theta)x) + g(x, y)\sin(2(\pi-\theta)x) \ge 0 . \tag{5.53}$$

Straightforward manipulations based on (5.7) and (5.8) give that $f(x, y),\ g(x, y) \ge 0$ for $x \in (0, 1/2]$ and $y \ge 0$. Also, from Corollary 1 we have that $f^2(x, y) + g^2(x, y) = 1$. Let $\omega \in [0, \pi/2]$, $\omega := \omega(x, y)$ be such that

$$\sin(\omega(x, y)) = f(x, y) \quad \text{and} \quad \cos(\omega(x, y)) = g(x, y) . \tag{5.54}$$

In this notation, (5.53) becomes $\sin(\omega(x, y) + 2(\pi-\theta)x) \ge 0$. Therefore, it suffices to show that

$$\omega(x, y) + 2(\pi-\theta)x \le \pi . \tag{5.55}$$

To this end, let $T(y) := \omega(x, y) + 2(\pi-\theta)x$. Since $\omega(x, y) = \arcsin(2c(ax + by))$ we obtain that

$$\frac{\partial T}{\partial y}(y) = \frac{\frac{\partial f}{\partial y}(x, y)}{\sqrt{1 - f^2(x, y)}} = \frac{2cb}{a^2 + b^2} \le 0 . \tag{5.56}$$

The second equality in (5.56) follows from Lemma 2 and Corollary 1, whereas the inequality comes from the fact that $b(x, y) \le 0$ for $y \in [0, \infty)$. This means that T is monotonically decreasing as a function of y so that it is enough to check (5.55) when $y = 0$. That is, we need to prove that

$$U(x) := \arcsin\left(2cx\sqrt{1 - c^2x^2}\right) + 2(\pi-\theta)x \le \pi, \quad \forall\, x \in (0, 1/2] . \tag{5.57}$$

Note that

$$U'(x) = \frac{2c\left[\sqrt{1-c^2x^2} - \frac{c^2x^2}{\sqrt{1-c^2x^2}}\right]}{\sqrt{1-4c^2x^2\left(1-c^2x^2\right)}} + 2(\pi-\theta) \geq 0, \quad \text{for } \ x \in (0, 1/2]\ ,$$

i. e., U is monotonically increasing in x. Thus, it is enough to check (5.57) when $x = 1/2$, which comes down to showing

$$\arcsin\left(c\sqrt{1-\frac{c^2}{4}}\right) \leq \theta \quad \text{or, equivalently,} \quad c\sqrt{1-\frac{c^2}{4}} \leq \sin\theta\ . \tag{5.58}$$

However, this is obviously true as $c = \upsilon\sin\theta$ with $\upsilon \in [0, 1]$. This finishes the proof of Lemma 7. □

As explained before, Proposition 1 proves (5.1)–(5.2). Also, Lemma 4 gives (5.3). The proof of Theorem 3 is now complete.

Remark. The results stated in Theorem 3 can be sharpened in the following manner. For any $\theta \in (0, 2\pi)$ there exists $\varepsilon = \varepsilon(\theta, \upsilon)$ depending on the opening of the sector and the Lamé moduli via υ [defined in (4.1)] such that (5.1) holds for any $2 - \varepsilon < p < \infty$. Therefore, by Theorem 3 [more specifically by (5.3)], and simple continuity considerations, we obtain that there exists $\varepsilon > 0$ such that

$$\rho\left(K_{\mathcal{L}};\,\left(L^p(\partial\Omega)\right)^2\right) < 1 \quad \text{and} \quad \rho\left(K_{\mathcal{S}};\,\left(L^p(\partial\Omega)\right)^2\right) < 1\ ,$$

for $2 - \varepsilon < p < \infty$.

We conclude this section by presenting the L^2 spectra of the operators K_{Lame}, corresponding to the pseudostress conormal derivative, (see Figure 6 and Figure 7) K_{Stokes}, corresponding to the stress conormal derivative, (see Figure 8) and K_{Laplace}, the classical double layer potential operator, (see Figure 9) on a sector of angle $\frac{\pi}{5}$. We also include the behavior of the spectral radius of the pseudostress elastostatic layer potential with respect to the opening of the sector (see Figure 10) for fixed values of the Lamé moduli μ and λ.

To start with, for the particular choice of the Lamé constants $\lambda = \mu = 3.2$ (corresponding to a glass material cf., e. g., p. 70 in [22]) we have $\upsilon = 0.5$, where υ is as in (4.1). Below (Figure 6) we present the L^2 spectrum of the pseudostress elastostatic layer potential with $\upsilon = 0.5$ on a sector of angle $\frac{\pi}{5}$. One can notice that the spectral radius is real, less than one, and belongs to the spectrum.

In general, for most structural materials, the value of the quotient $\frac{\lambda}{\mu}$ does not deviate much from 2 and, hence υ does not deviate much from 0.6. In the case of rubber (highly incompressible material) one has $\frac{\lambda}{\mu} = 33$ and, therefore, $\upsilon = \frac{17}{18}$ (Figure 7). In the hydrostatic case (corresponding to the choice $\upsilon = 1$) we obtain a curve very similar to the highly incompressible material case discussed above (Figure 8). In Figure 9 we present the L^2 spectrum of the classical double layer potential operator for the Laplacian. Note that this case corresponds formally to the choice $\upsilon = 0$. As explained before, the vector Laplacian is a limiting case of the Lamé system corresponding to the choice of the Lamé constants $\lambda + \mu = 0$, i. e., when $\upsilon = 0$.

Finally, we present the behavior of the L^2 spectral radius of the elastostatic (pseudostress) double layer potential on glass (i. e., when $\upsilon = 0.5$) with respect to the opening of the sector. We point out that the sharper the angle, the closer to 1 the spectral radius is depicted in Figure 10.

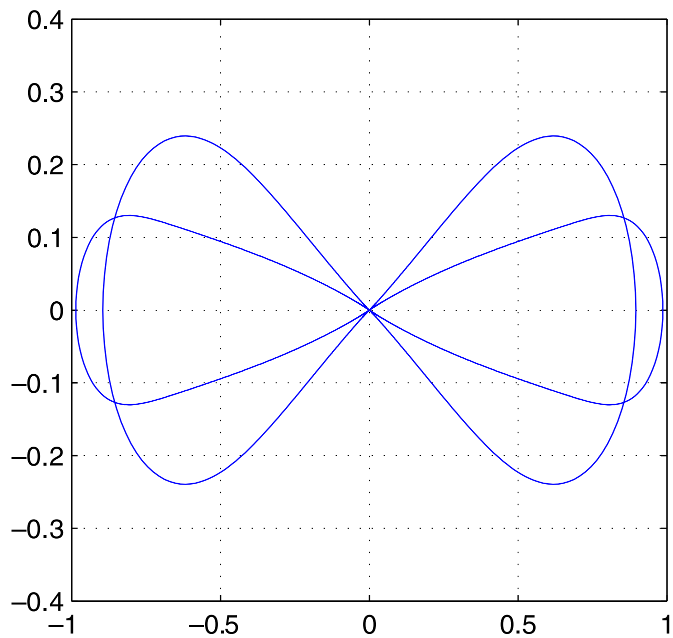

FIGURE 6 The L^2 Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a sector of angle $\frac{\pi}{5}$.

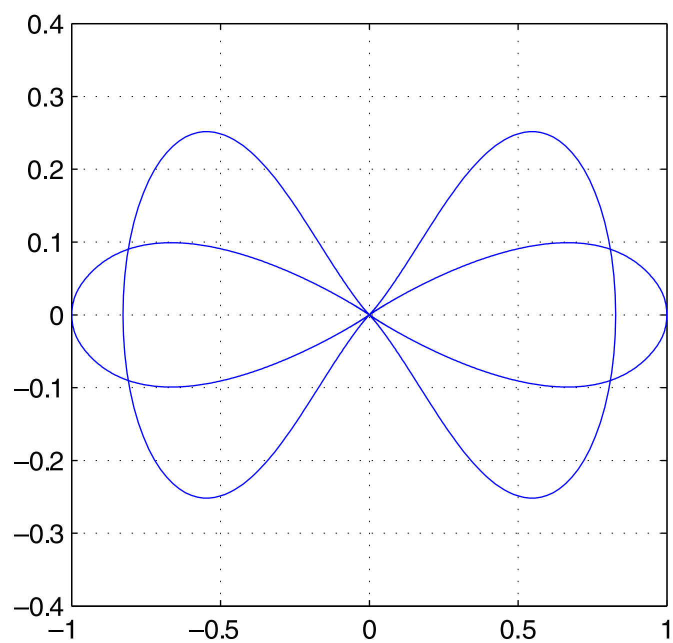

FIGURE 7 The L^2 Spectrum of the elastostatic (pseudostress) double layer potential operator on rubber for a sector of angle $\frac{\pi}{5}$.

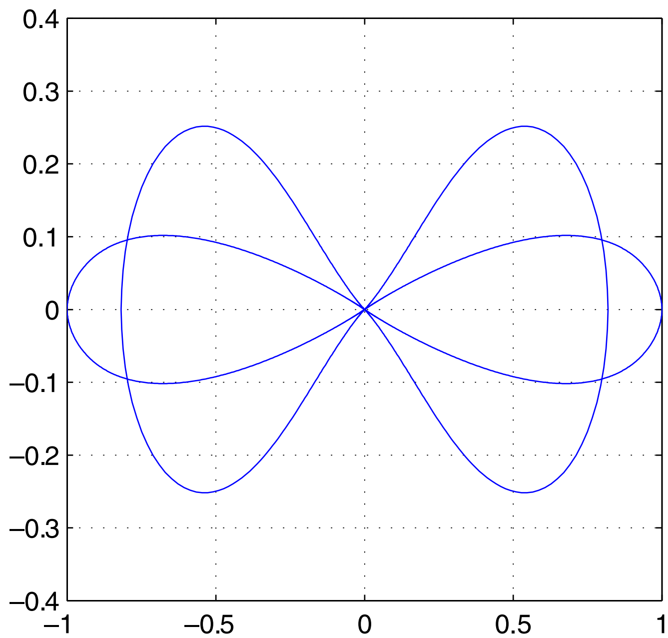

FIGURE 8 The L^2 Spectrum of the hydrostatic (stress) double layer potential for a sector of angle $\frac{\pi}{5}$.

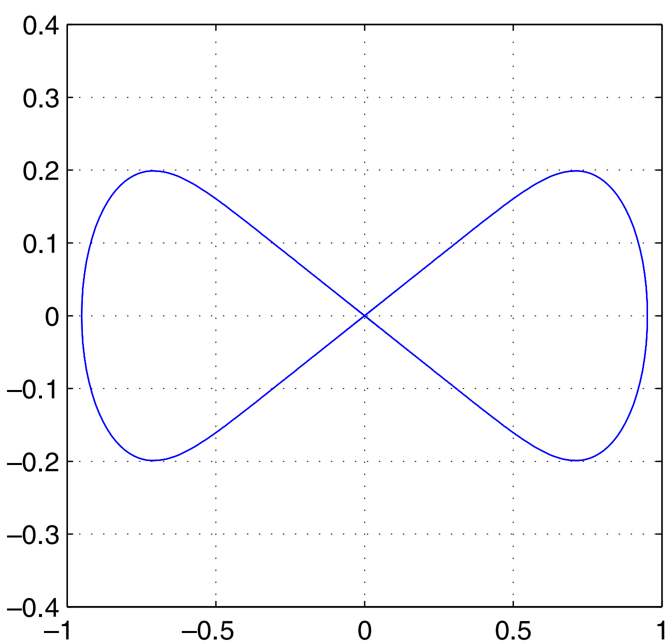

FIGURE 9 The L^2 Spectrum of the classical double layer potential operator for the Laplacian on a sector of angle $\frac{\pi}{5}$.

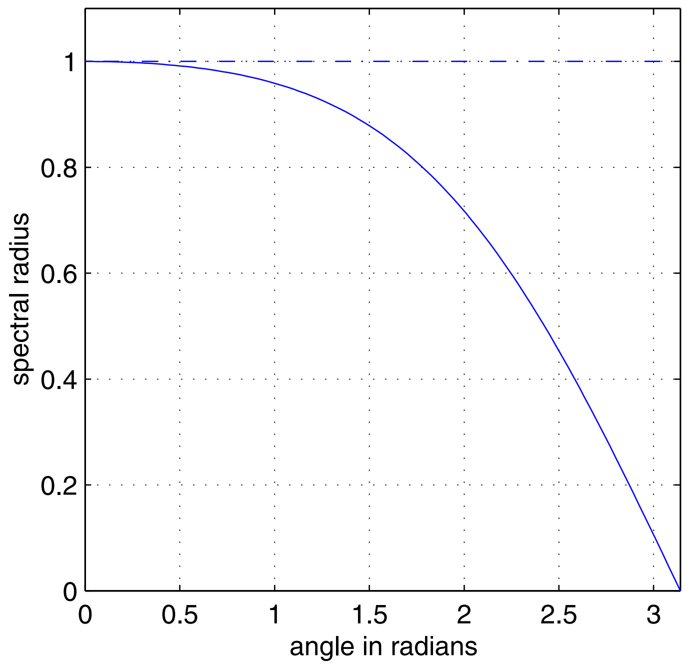

FIGURE 10 The L^2 Spectral radius of the elastostatic (pseudostress) double layer potential operator on glass versus the opening of the sector.

6. The Dirichlet Problems for the Lamé and Stokes Systems on Curvilinear Polygons

The aim of this section is to provide spectral radius estimates for the operators $K_{\mathcal{L}}$ and $K_{\mathcal{S}}$ on the L^p spaces of the boundary of two dimensional curvilinear polygons. This information is relevant for constructively solving the Dirichlet problems for the Lamé and Stokes systems on domains whose boundaries are simple closed curvilinear polygons.

To state our main result let us introduce the following notation. For $1 \leq p < \infty$, consider

$$L^p_0(\partial\Omega) := \left\{ f \in L^p(\partial\Omega)\,;\ \int_{\partial\Omega} f\, d\sigma = 0 \right\} .$$

Also, let Ψ denote the space of vector valued functions ψ on R^2 satisfying the three equations $\partial_i \psi^j + \partial_j \psi^i = 0,\quad 1 \leq i, j \leq 2$, and define

$$L^p_\Psi(\partial\Omega) := \left\{ f \in \left(L^p(\partial\Omega)\right)^2\,;\ \int_{\partial\Omega} f \cdot \psi\, d\sigma = 0, \quad \text{for all } \psi \in \Psi \right\} .$$

Note that $L^p_\Psi(\partial\Omega)$ is a subspace of codimension 3 of $(L^p(\partial\Omega))^2$. Also, using the Hahn–Banach and Riesz representation theorems it is easy to see that for any $1 < p < \infty$, the dual space of $L^p_\Psi(\partial\Omega)$ is $(L^q(\partial\Omega))^2 / \Psi$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have:

Theorem 4.

Let $\Omega \subseteq R^2$ be a bounded, simply connected domain whose boundary is a simple closed curvilinear polygon. For each $2 \le p < \infty$ we have

$$\sigma\Big(K_{\mathcal{L}};\, \big(L^p(\partial\Omega)/R\big)^2\Big) \subset D_r(0) \quad \textit{for some} \quad 0 < r < 1\ .$$

Thus, the spectral radius of $K_{\mathcal{L}}$ on $(L^p(\partial\Omega)/R)^2$ is strictly less than one. Consequently,

$$(I \pm K_{\mathcal{L}})^{-1} = \sum_{j=0}^{\infty} (\mp K_{\mathcal{L}})^j\ , \tag{6.1}$$

where the series converges absolutely in the operator norm on $(L^p(\partial\Omega)/R)^2$.

Moreover, for each $2 \le p < \infty$ it follows that

$$\sigma\Big(K_{\mathcal{S}};\, \big(L^p(\partial\Omega)\big)^2/\Psi\Big) \subset D_r(0) \quad \textit{for some} \quad 0 < r < 1\ .$$

Hence, the spectral radius of $K_{\mathcal{S}}$ on $(L^p(\partial\Omega))^2/\Psi$ is strictly less than one. In particular,

$$(I \pm K_{\mathcal{S}})^{-1} = \sum_{j=0}^{\infty} (\mp K_{\mathcal{S}})^j\ , \tag{6.2}$$

where the series converges absolutely in the operator norm on $(L^p(\partial\Omega))^2/\Psi$.

We postpone the proof of the Theorem 4 for the moment to discuss some important consequences. First, let us consider the interior Dirichlet problem for the Lamé system with data in $(L^p(\partial\Omega))^2$, $2 \le p < \infty$. That is,

$$(\mathcal{D}_{\mathcal{L}}) \quad \begin{cases} \vec{u} \in \big(\mathcal{C}^2(\Omega)\big)^2\ , \\ \mu\triangle\vec{u} + (\lambda+\mu)\nabla\ \operatorname{div}\vec{u} = 0 \quad \text{in}\ \ \Omega\ , \\ \vec{u}\big|_{\partial\Omega} = \vec{g} \in (L^p(\partial\Omega))^2\ , \\ \big\|(\vec{u})^*\big\|_{L^p(\partial\Omega)} < \infty\ . \end{cases} \tag{6.3}$$

Here $(\vec{u})^*$ is the non-tangential maximal function of $\vec{u}$ defined by $(\vec{u})^*(P) := \sup_{X\in\Upsilon(P)} |\vec{u}(X)|$, $P \in \partial\Omega$, where $\{\Upsilon(P)\}_{P\in\partial\Omega}$ is a family of nontangential approach regions of conical type (as in, e. g., [24]).

Next we consider the interior Dirichlet problem for the Stokes system with data in $(L^p(\partial\Omega))^2$. That is,

$$(\mathcal{D}_{\mathcal{S}}) \quad \begin{cases} \vec{u} \in \big(\mathcal{C}^2(\Omega)\big)^2,\ \ p \in \mathcal{C}^1(\Omega)\ , \\ \triangle\vec{u} = \nabla p \quad \text{in}\ \ \Omega\ , \\ \vec{u}\big|_{\partial\Omega} = \vec{g} \in (L^p(\partial\Omega))^2\ , \\ \big\|(\vec{u})^*\big\|_{L^p(\partial\Omega)} < \infty\ . \end{cases} \tag{6.4}$$

The exterior Dirichlet problems associated to the Lamé and Stokes systems are defined similarly but with the decay conditions

$$|\vec{u}(X)| + |\nabla\vec{u}(X)|\,|X| = \mathcal{O}(1) \quad \text{as} \quad |X| \to \infty\,, \tag{6.5}$$

and respectively

$$|\vec{u}(X)| + |\nabla\vec{u}(X)|\,|X| + |p(X)|\,|X| = \mathcal{O}(1) \quad \text{as} \quad |X| \to \infty\,. \tag{6.6}$$

Now, the Theorem 4 implies the following.

Corollary 2.

Let Ω be a bounded, simply connected domain whose boundary is a simple closed curvilinear polygon. Then, for any $2 \le p < \infty$ there exists unique solution for the Dirichlet problem $\mathcal{D}_{\mathcal{L}}$ given by

$$\vec{u}(X) = \sum_{j=0}^{\infty} \int_{\partial\Omega} k_{\mathcal{L}}(X, Q)(-K_{\mathcal{L}})^j \left(\left[\vec{g}\right]\right)(Q)\, d\sigma(Q) + \vec{C}, \quad X \in \Omega\,, \tag{6.7}$$

where $[\vec{g}]$ is the class of $\vec{g}$ in $(L^p(\partial\Omega)/R)^2$ and $\vec{C} \in R^2$.

Also, for any $2 \le p < \infty$ there exists unique solution for the Dirichlet problem $\mathcal{D}_{\mathcal{S}}$ given by

$$\vec{u}(X) = \sum_{j=0}^{\infty} \int_{\partial\Omega} k_{\mathcal{S}}(X, Q)(-K_{\mathcal{S}})^j \left(\left[\vec{g}\right]\right)(Q)\, d\sigma(Q) + \vec{h}(X), \quad X \in \Omega\,, \tag{6.8}$$

where $[\vec{g}]$ is the class of $\vec{g}$ in $(L^p(\partial\Omega))^2/\Psi$ and $\vec{h} \in \Psi$. Similar integral formula representations also hold for the solutions of the exterior Dirichlet problems.

With an eye toward proving Theorem 4 we record the following result important in the sequel.

Theorem 5.

Let $\Omega \subseteq R^2$ be a bounded, simply connected curvilinear polygon. For any $w \in C$, $|w| \ge 1$ and $w \neq 1$, the operators

$$\begin{aligned} wI - K_{\mathcal{L}} &: (L^p(\partial\Omega))^2 \longrightarrow (L^p(\partial\Omega))^2\,, \;\; 2 \le p < \infty\,, \\ wI - K_{\mathcal{S}} &: (L^p(\partial\Omega))^2 \longrightarrow (L^p(\partial\Omega))^2\,, \;\; 2 \le p < \infty\,, \end{aligned} \tag{6.9}$$

are invertible. Moreover, the operators $I - K_{\mathcal{L}}$, $I - K_{\mathcal{S}} \in \mathcal{B}\left((L^p(\partial\Omega))^2\right)$ are Fredholm with index zero. Furthermore,

$$\begin{aligned} I - K^*_{\mathcal{L}} &: \left(L^q_0(\partial\Omega)\right)^2 \longrightarrow \left(L^q_0(\partial\Omega)\right)^2\,, \\ I - K^*_{\mathcal{S}} &: L^q_\Psi(\partial\Omega) \longrightarrow L^q_\Psi(\partial\Omega)\,, \end{aligned} \tag{6.10}$$

are invertible, where $\frac{1}{p} + \frac{1}{q} = 1$ so that $1 < q \le 2$.

Proof. We start by showing that the operators (6.9) are Fredholm. To this end, we first parameterize $\partial\Omega$ and rewrite $wI - K$ in the corresponding parametric coordinates.

We can assume without loss of generality that the boundary of Ω has only one corner at $P_0 := (0,0)$. Hence Ω is a simply connected domain in R^2 whose boundary is a simple closed curve $\vec{\gamma}(t)$, $t \in [0,2]$, $\vec{\gamma}$ being of class $\mathcal{C}^1$ on $(0,2)$. Without loss of generality we may assume that $\vec{\gamma}(0) = \vec{\gamma}(2)$, $\frac{d\vec{\gamma}}{dt} \neq \vec{0}$ and $\frac{d\vec{\gamma}}{dt}\Big|_+(0) \neq \frac{d\vec{\gamma}}{dt}\Big|_-(2)$, where $\pm$ indicates the corresponding lateral derivatives. Next, introduce the 'false' vertices $P_1 := \vec{\gamma}(1)$ and $P_2 := P_0$. We parameterize the oriented portion from P_0 to P_1 of $\partial\Omega$ by $\vec{\gamma}_1(t) = \vec{\gamma}(t)$, $t \in [0,1]$ and the oriented portion from P_2 to P_1 of $\partial\Omega$ by $\vec{\gamma}_2(t) = \vec{\gamma}(2-t)$, $t \in [0,1]$. For each $\vec{f}$ vector function on Ω we consider the restriction of $\vec{f}$ to each 'ray' starting from the vertex P_0, $\vec{f}^i(t) := \vec{f}(\gamma_i(t))$, $t \in [0,1]$, $i = 1,2$. Let K stand for the operator $K_{\mathcal{L}}$ or $K_{\mathcal{S}}$ and denote its matrix kernel by k. We interpret the operator K as a 4×4 matrix of operators on $(L^p([0,1]))^4$,

$$K = \begin{pmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{pmatrix}, \tag{6.11}$$

where for each $i, j = 1,2$, K^{ij} is the 2×2 matrix of operators given by

$$K^{ij} \vec{f}(t) = \int_0^1 k\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) \vec{f}^j(t) \left|\frac{d\vec{\gamma}_j}{ds}\right| ds\,. \tag{6.12}$$

Now, for each $i, j \in \{1,2\}$ fixed, the kernel $k(\vec{\gamma}_i(t) - \vec{\gamma}_j(s))$ is a 2×2 matrix $k(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)) := \left(k_{lm}(\vec{\gamma}_i(t) - \vec{\gamma}_j(s))\right)_{l,m=1,2}$. For each entry, we can write

$$k_{lm}\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) = \sum_{r=1}^{2} k_{lm}^r \left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) N_r\left(\vec{\gamma}_j(s)\right)\,,$$

where N_r is the r-th component of the outward unit normal vector to $\partial\Omega$. For fixed $i, j, r \in \{1,2\}$ consider the matrix

$$k^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) := \left(k_{lm}^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right)\right)_{l,m=1,2}\,. \tag{6.13}$$

Using (6.13) we can write

$$k\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) = \sum_{r=1}^{2} N_r\left(\vec{\gamma}_j(s)\right) k^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right)\,. \tag{6.14}$$

Now, from (6.14) and (6.12) we obtain

$$K^{ij} \vec{f}(t) = \int_0^1 \sum_{r=1}^{2} k^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) \vec{f}^j(s) N_r\left(\vec{\gamma}_j(s)\right) \left|\frac{d\vec{\gamma}_j}{ds}\right| ds\,, \tag{6.15}$$

for any $i, j \in \{1,2\}$. Using the parameterization of $\partial\Omega$ introduced above, for any $w \in \mathrm{C}$, the operator $wI - K$ can be realized as a matrix $\left(A^{ij}\right)_{i,j=1,2}$, (each A^{ij} is a 2×2 matrix itself), with

$$\begin{aligned} A^{ij} \vec{f}(t) &= -\int_0^1 \sum_{r=1}^{2} k^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) \vec{f}^j(s) N_r\left(\vec{\gamma}_j(s)\right) \left|\frac{d\vec{\gamma}_j}{ds}\right| ds,\ i \neq j\,, \\ A^{ii} \vec{f}(t) &= w \left|\frac{d\vec{\gamma}_i}{dt}\right| I \vec{f}^i - \int_0^1 \sum_{r=1}^{2} k^r\left(\vec{\gamma}_i(t) - \vec{\gamma}_i(s)\right) \vec{f}^i(s) N_r\left(\vec{\gamma}_i(s)\right) \left|\frac{d\vec{\gamma}_i}{ds}\right| ds\,. \end{aligned} \tag{6.16}$$

Recall that we are left with proving that, for any $w \in \mathrm{C}$, $|w| \geq 1$, the operator $wI - K$ is Fredholm on $L^p(\partial\Omega)$, for $2 \leq p < \infty$. At this stage, we employ the symbolic calculus of pseudodifferential operators of Mellin type developed by J. Lewis, C. Parenti and J. Elschner (see [16, 18, 7]). To make the exposition self-contained we include a description of their results which are relevant for us here. We follow closely the notation in [16] and we refer the reader to this work for further details. $\square$

First recall the definition of the Mellin transform (3.1), the inversion formula (3.2) and its properties as described in Section 3. Next, we introduce the function space $\Theta^m_{\alpha,\beta}$.

Definition 2. For $0 \leq \alpha < \beta \leq 1$, let $\Theta^m_{\alpha,\beta}$ be the Frechet space of functions $f(z)$ holomorphic in the strip $\Gamma_{\alpha,\beta}$ such that for every l and for every $[a, b] \subset (\alpha, \beta)$,

$$|f|_{l,[a,b]} = \sup_{\Gamma_{a,b}} \left|(1+|z|)^{l-m} \frac{d^l}{d\,z^l} f(z)\right| < \infty\,. \tag{6.17}$$

For any $\tilde{k}(z) \in \Theta^{-1}_{\alpha,\beta}$ we shall call the operator

$$Kf(t) := \int_0^1 k\left(\frac{t}{s}\right) f(s) \frac{ds}{s} = \frac{1}{2\pi i} \int_{\Gamma_\gamma} t^{-z} \tilde{k}(z) \tilde{f}(z)\, dz \tag{6.18}$$

the *Hardy operator with kernel* $k(t)$. Let $C_{\alpha,\beta}(J)$ be the class of compact operators on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$. For any $f \in L^p(J)$ let Rf be the reflection

$$Rf(t) = f(1-t)\,.$$

Fix now a cut off function $\chi(t) \in \mathcal{C}^\infty_0\left(\left[0, \frac{1}{3}\right)\right)$, $\chi(t) \equiv 1$ on $\left[0, \frac{1}{4}\right]$. We are ready to describe the algebra $\mathrm{Op}\,\Sigma_{\alpha,\beta} \otimes C(J)$, where $C(J)$ is the space of continuous functions on J.

Definition 3. An operator A on $L^p(J)$, $\frac{1}{p} \in (\alpha, \beta)$, is in $\mathrm{Op}\,\Sigma_{\alpha,\beta} \otimes C(J)$ if and only if

$$A = c(t)I + d(t)H + \chi K_0 \chi + R\chi K_1 \chi R + \mathrm{Comp} \tag{6.19}$$

where

(1) $c(t), d(t) \in C(J)$;

(2) H is the finite Hilbert transform, i. e., $Hf(t) = p.v. \frac{1}{\pi} \int_0^1 \frac{1}{t-s} f(s)\, ds$;

(3) K_0 and K_1 are Hardy operators with kernels k_0 and respectively k_1;

(4) $\mathrm{Comp} \in C_{\alpha,\beta}(J)$.

If $A \in \mathrm{Op}\,\Sigma_{\alpha,\beta} \otimes C(J)$ has the structure (6.19), define

$$\begin{aligned} a_0(z) &:= c(0) + d(0)(-\cot \pi z) + \tilde{k}_0(z)\,, \\ a_{0\pm}(t) &:= a_{1\mp}(1-t) = c(t) \pm i\, d(t)\,, \\ a_1(z) &:= c(1) + d(1)(\cot \pi z) + \tilde{k}_1(z)\,. \end{aligned} \tag{6.20}$$

Definition 4. Let $A \in \mathrm{Op}\,\Sigma_{\alpha,\beta} \otimes C(J)$, $\frac{1}{p} \in (\alpha, \beta)$. The principal symbol of A as an operator on $L^p(J)$ denoted by $\mathrm{Smbl}^{1/p} A$, is the quadruple of functions $a_0(\frac{1}{p} + i\xi)$,

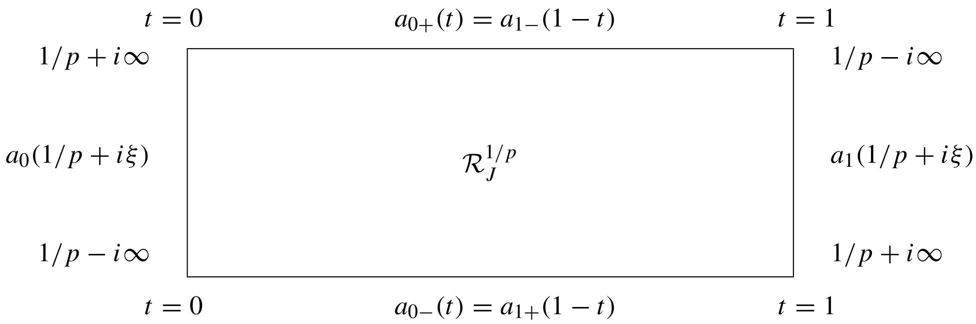

FIGURE 11 The symbol of the operator A.

$a_{0+}(t) = a_{1-}(1-t)$, $a_1(\frac{1}{p} + i\xi)$, $a_{0-}(t) = a_{1+}(1-t)$, considered as a continuous function on the clockwise oriented boundary $\mathcal{R}_J^{1/p}$ of the rectangle in Figure 11.

It is straightforward to see, using Definitions 3 and 4, that the Hardy operator K in (6.18) belongs to Op $\Sigma_{\alpha,\beta} \otimes C(J)$ and $\mathrm{Smbl}^{1/p}\, K$ is as in Figure 12.

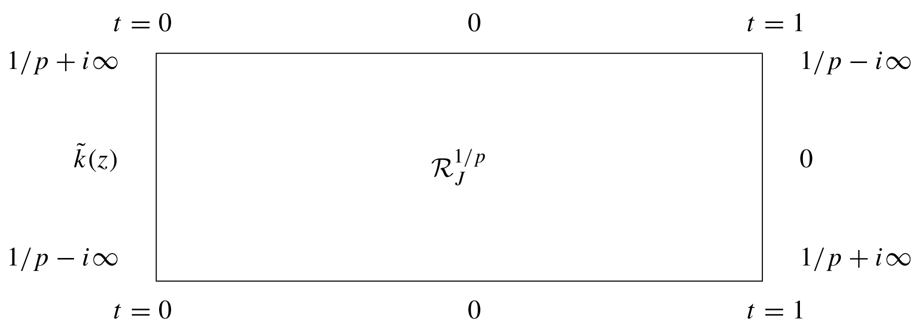

FIGURE 12 The symbol of a Hardy kernel operator.

Next we recall the notion of ellipticity for a system of Mellin operators on $L^p(J)$.

Definition 5. Let $A = (A_{ij})_{i,j=1,\ldots,N}$ be an $N \times N$ matrix of operators in Op $\Sigma_{\alpha,\beta} \otimes C(J)$. A is called elliptic on $(L^p(J))^N$ if and only if $\mathrm{Smbl}^{1/p}\, A$ is a nonsingular matrix on $\mathcal{R}_J^{1/p}$.

The following theorem, proved in [16], is going to be useful in the sequel.

Theorem 6.

Let $A = (A_{ij})_{i,j=1,\ldots,N}$ be an $N \times N$ matrix of operators in Op $\Sigma_{\alpha,\beta} \otimes C(J)$. *$A$ is a Fredholm operator on $(L^p(J))^N$ if and only if A is elliptic on $(L^p(J))^N$, (i. e.,* $\mathrm{Smbl}^{1/p}\, A$ *is a nonsingular matrix on $\mathcal{R}_J^{1/p}$).*

If A is Fredholm on $(L^p(J))^N$, then, its index is given by

$$\mathit{index}\left(A; \left(L^p(J)\right)^N\right) = \frac{1}{2\pi} \triangle_{\mathcal{R}_J^{1/p}} \left(\arg\left(\det\left(\mathrm{Smbl}^{1/p}\, A\right)\right)\right\} . \tag{6.21}$$

Following [16], one important example of an operator in Op $\Sigma_{\alpha,\beta} \otimes C(J)$ is the double layer potential operator on $\mathcal{C}^1$ curves. To define this concept we first make the following.

Definition 6. A function $k(x, y)$ is an analytic double layer kernel if k is real analytic in $\mathbb{R}^2 \setminus \{0\}$, odd and homogeneous of degree -1, (i. e., for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $k(\lambda x, \lambda y) = \lambda^{-1} k(x, y)$).

Let $\vec{\gamma}_i(t)$, $i = 1, 2$, $t \in J = [0, 1]$, be simple $\mathcal{C}^1$ curves with $\frac{d\vec{\gamma}_i}{dt} \neq \vec{0}$. Assume that $\vec{\gamma}_i(0) = (0, 0)$, $i = 1, 2$ and $\frac{d\vec{\gamma}_i}{dt}(0) = \vec{u}_i$, $\vec{u}_1 \neq \vec{u}_2$. For b continuous function and k analytic double layer kernel, we define

$$K^{ij} f(t) := \int_0^1 k\left(\vec{\gamma}_i(t) - \vec{\gamma}_j(s)\right) b\left(\vec{\gamma}_j(s)\right) f(s) \left|\frac{d\vec{\gamma}_j}{ds}\right| ds\,. \tag{6.22}$$

We have (see [16]).

Theorem 7.

The operators K^{ij} defined in (6.22) belong to $\mathrm{Op}\,\Sigma_{\alpha,\beta} \otimes C(J)$. *Moreover,*

$$\mathrm{Smbl}^{1/p}\, K^{ii} = \pi k\left(\vec{\gamma}_i{}'(t)\right) b\left(\vec{\gamma}_i(t)\right) \left|\frac{d\vec{\gamma}_i}{dt}\right| \mathrm{Smbl}^{1/p}(H), \quad i = 1, 2\,, \tag{6.23}$$

and K^{ij}, $i \neq j$, has the same symbol as the Hardy operator with kernel

$$k^{ij}(t) = \left|\vec{u}_j\right| b\left(\vec{\gamma}_j(0)\right) k\left(t\vec{u}_i - \vec{u}_j\right), \quad i, j = 1, 2\,. \tag{6.24}$$

With these at hand we are ready now to present the end of the following proof.

Proof of Theorem 5. It is not hard to see that the kernels $k^r(\vec{\gamma}_i(t) - \vec{\gamma}_j(s))$ from (6.15) are actually analytic double layer kernels [they can be determined explicitly from (2.3) and (2.6)]. Next, since $N_r(\gamma_j(\cdot))$ extends continuously to the interval [0, 1], it follows that $wI - K = \left(A^{ij}\right)_{i,j=1,2}$ with A^{ij} given in (6.16) is a double layer potential operator on $\mathcal{C}^1$ curves in the sense of (6.22). Applying Theorem 7 to the operator K^{ii} in (6.15) we obtain

$$\mathrm{Smbl}^{1/p}\left(K^{ii}\right) = \pi \sum_{r=1}^{2} k^r\left(\vec{\gamma}_i{}'(t)\right) N_r(\gamma_i(t)) \left|\frac{d\vec{\gamma}_i}{dt}\right| \mathrm{Smbl}^{1/p}(H), \quad i = 1, 2\,. \tag{6.25}$$

Recall from (2.3) and (2.6) that the kernel k contains the factor $\langle X, N(X)\rangle$. In turn, this implies $\sum_{r=1}^{2} k^r(\vec{\gamma}_i{}'(t)) N_r(\vec{\gamma}_i(t)) = 0$, as $\left\langle \vec{\gamma}_i{}'(t), N(\vec{\gamma}_i(t))\right\rangle = 0$. Since by Theorem 7, for $i \neq j$ the operator K^{ij} has the same symbol as a Hardy kernel operator (see Figure 6) its symbol vanishes at $t = 1$ and $z = \frac{1}{p} \pm \infty$. This implies that for $w \neq 0$ the symbol of the operator $A = wI - K$ does not vanish at $t = 1$ and $z = \frac{1}{p} \pm \infty$ as we have

$$\mathrm{Smbl}^{1/p}\, A(t, z) = \begin{pmatrix} w\left|\frac{d\vec{\gamma}_1}{dt}\right| & 0 \\ 0 & w\left|\frac{d\vec{\gamma}_2}{dt}\right| \end{pmatrix}. \tag{6.26}$$

In the case $t = 0$ we have that

$$\mathrm{Smbl}^{1/p}\, A(0, z) = \left|\frac{d\vec{\gamma}_1}{dt}(0)\right| \cdot \left|\frac{d\vec{\gamma}_2}{dt}(0)\right| \left(wI - \tilde{k}_{\mathcal{L}}\right)(z)\,, \tag{6.27}$$

where $(wI - \tilde{k}_{\mathcal{L}})(z)$ is as in (4.10), i. e., the sector case. Then, as a result of Lemma 1, the determinant of $\mathrm{Smbl}^{1/p}\, A(0, z)$ vanishes if and only if $w \in \mathbb{C}$ is as in (4.14). Then, since $|w| \geq 1$, Theorem 3 [more precisely (5.3)] gives that $\det \mathrm{Smbl}^{1/p}\, A(t, z)$ does not vanish at $t = 0$. This finishes the proof of the fact that $\mathrm{Smbl}^{1/p}(wI - K)$ is a nonsingular matrix on $\mathcal{R}_J^{1/p}$ for any $w \in \mathbb{C}$, $|w| \geq 1$ and $2 \leq p < \infty$. By Theorem 6 we obtain that the operators (6.9) in Theorem 5 are Fredholm with index zero.

The arguments necessary to conclude the injectivity of the operators $wI - K_{\mathcal{L}}$ and $wI - K_{\mathcal{S}}$ on $(L^p(\partial\Omega))^2$, $2 \leq p < \infty$, for $w \in \mathbb{C}$, $|w| \geq 1$ and $w \neq 1$, as well as the invertibility of the operators $I - K_{\mathcal{L}}^*$ on $\left(L_0^p(\partial\Omega)\right)^2$ and $I - K_{\mathcal{S}}^*$ on $L_\Psi^q(\partial\Omega)$ are as in [21], *Step II* of the proof of Theorem 1.2.1 of Chapter 1. See also Theorem 1 in [20] for a proof in the case $p = q = 2$. This completes the proof of Theorem 5. □

Finally, the conclusion of Theorem 4 follows from Theorem 5 and standard functional analysis arguments (see e. g., p. 16–17 in [21]).

7. The Spectrum on Curvilinear Polygons

In this section we provide an explicit description of the spectra of the operators $K_{\mathcal{L}}$ and $K_{\mathcal{S}}$ on L^p $(1 < p < \infty)$ spaces of the boundary of bounded curvilinear polygons. Similar results hold in the case of the Laplacian. As a consequence, we obtain that the spectral radii belong to the corresponding spectra. Our main result is the following.

Theorem 8.

Consider $\Omega \subseteq R^2$ a bounded, simply connected curvilinear polygon with angles θ_i, $i = 1, \ldots, n$ and let $p \in (1, \infty)$. For each $1 \leq i \leq n$ consider the bow-tie-shaped curve $\Sigma_{\theta_i}(p)$,as in (4.15), associated with the angle θ_i and the integrability exponent p. Set $\widehat{\Sigma_{\theta_i}(p)}$ for the closure of its interior. Finally, let $K_\#$ stand for either $K_{\mathcal{L}}$ or $K_{\mathcal{S}}$. Then,

$$\sigma\left(K_\#; \left(L^p(\partial\Omega)\right)^2\right) = \left(\bigcup_{1 \leq i \leq n} \widehat{\Sigma_{\theta_i}(p)}\right) \bigcup \{\lambda_j\}_j \,, \tag{7.1}$$

where $\{\lambda_j\}_j$ is a finite subset of $(-1, 1]$ consisting of eigenvalues of $K_\#$ on $(L^p(\partial\Omega))^2$. We have $1 \in \{\lambda_j\}_j$.

For $w \in C$, $w \in \bigcup_{1 \leq i \leq n} \Sigma_{\theta_i}(p)$ the operator $wI - K_\#$ is not Fredholm on $(L^p(\partial\Omega))^2$.

For $w \in C$, $w \notin \bigcup_{1 \leq i \leq n} \Sigma_{\theta_i}(p)$ the operator $wI - K_\#$ is Fredholm on $(L^p(\partial\Omega))^2$. Moreover, its index is given by

$$\mathit{index}\left(wI - K_\#; \left(L^p(\partial\Omega)\right)^2\right) = \sum_{i=1}^{n} W(w, \Sigma_{\theta_i}(p)) \,, \tag{7.2}$$

where $W(w, \Sigma_{\theta_i}(p))$ stands for the sum of the winding numbers of the point $w \notin \Sigma_{\theta_i}(p)$ with respect to each one of the four closed curves $\Sigma_{\pm}^{\pm}(\theta_i, p)$ constituting $\Sigma_{\theta_i}(p)$.

Proof. We drop the subscripts $\mathcal{L}$ and $\mathcal{S}$. Parameterize $\partial\Omega$ and rewrite the operator K in the corresponding parametric coordinates as we did in Section 7. Then, as a first step, we employ the symbolic calculus of Mellin pseudodifferential operators to obtain an explicit description of the essential spectrum of the operator K on $(L^p(\partial\Omega))^2$, $1 \leq p < \infty$. We follow closely the notation in [17].

We label the n successive vertices of $\partial\Omega$ as $P_2, P_4, \dots, P_{2n} = P_0$ when traversing $\partial\Omega$ in the counterclockwise direction. Suppose that the oriented piece $P_{2i}P_{2i+2}$ of $\partial\Omega$ is parameterized by $\vec{\gamma}_{2i}(t)$, $t \in [0,2]$, $i = 1, \dots, n$. For each i we introduce the false vertex $P_{2i-1} := \vec{\gamma}_{2i-2}(1)$ and, then, reparametrize the oriented piece of the boundary $P_{2i}P_{2i-1}$ by $\vec{\gamma}_{2i-1} := \vec{\gamma}_{2i-2}(2-t)$, $t \in [0,1]$. If $t = 0$, each parameterization is at one of the original vertices whereas if $t = 1$, each parameterization is at a midpoint. Denoting by Θ_i, $i = 1, \dots, 2n$ the angle interior to Ω at P_i we have

$$\Theta_{2i} = \theta_i, \quad \text{and} \quad \Theta_{2i-1} = \pi, \quad i = 1, \dots, n\,. \tag{7.3}$$

Without loss of generality we assume that $\left|\dfrac{d\vec{\gamma}_i}{dt}\right| = 1$ at $t = 0$ and $t = 1$ for $i = 1, \dots, 2n$. The arclength on each oriented piece P_iP_{i+1} is given by $d\sigma = (-1)^i \left|\dfrac{d\vec{\gamma}_i}{dt}\right| dt$. This finishes the parameterization of $\partial\Omega$.

Next, to write the operator K in the corresponding parametric coordinates we proceed as described in Section 7, (6.11) and (6.12). This time (having n vertices) we interpret K as a $2n \times 2n$ matrix of operators on $(L^p([0,1]))^{2n}$,

$$K = \begin{pmatrix} K^{11} & K^{12} & \dots & K^{1\,2n} \\ K^{21} & K^{22} & \dots & K^{2\,2n} \\ \dots & \dots & \dots & \dots \\ K^{2n\,1} & K^{2n\,2} & \dots & K^{2n\,2n} \end{pmatrix}, \tag{7.4}$$

with K^{ij} as in (6.12), $i, j = 1, \dots, 2n$. Except the cases $j = i-1, i, i+1 (\mathrm{mod}\ 2n)$, the operators K^{ij} are weakly singular and, thus, are compact on $(L^p([0,1]))^2$ and their symbol is identically zero. Also, when $j = i$ for all i, $j = i-1$ for odd i and $j = i+1$ for even i, the operator K^{ij} (since its kernel has $\langle X - Y, N(Y)\rangle$ as a factor) is weakly singular on smooth curves (in all these cases we have $\vec{\gamma}_i(t)$ and $\vec{\gamma}_j(t)$ belonging to the same 'side' of the curvilinear polygon $\partial\Omega$). Therefore, K^{ij} is compact and it has symbol zero for i, j as described above. In conclusion, $\mathrm{Smbl}^{1/p}(K^{ij})(t,z) = 0$ for all $j \neq i+1$ when i is odd and $\mathrm{Smbl}^{1/p}(K^{ij})(t,z) = 0$ for all $j \neq i-1$ when i is even. This implies that for any $w \in \mathrm{C}$, the symbol of the operator $wI - K$ is given by the following 2×2 block diagonal matrix of symbols:

$$-\begin{pmatrix} -wI & \mathrm{Smbl}^{1/p}\,K^{12} & 0 & 0 & \dots & 0 & 0 \\ \mathrm{Smbl}^{1/p}\,K^{21} & -wI & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -wI & \mathrm{Smbl}^{1/p}\,K^{2n-1\,2n} \\ 0 & 0 & 0 & 0 & \dots & \mathrm{Smbl}^{1/p}\,K^{2n\,2n-1} & -wI \end{pmatrix},$$

where all the symbols are evaluated at (t,z), $t \in [0,1]$, $z \in \Gamma_{1/p}$. This implies

$$\det\ \mathrm{Smbl}^{1/p}(wI - K)(t,z) = \prod_{i=1}^{n} \det \begin{pmatrix} -wI & \mathrm{Smbl}^{1/p}\,K^{2i-1\,2i} \\ \mathrm{Smbl}^{1/p}\,K^{2i\,2i-1} & -wI \end{pmatrix}(t,z). \tag{7.5}$$

Each factor in (7.5) is now of the type (4.11). When $t \neq 0$ we, therefore, obtain

$\det \mathrm{Smbl}^{1/p}(wI - K)(t, z) = w^{4n}$, When $t = 0$ we have that

$$\begin{aligned}
&\det \begin{pmatrix} wI & -\,\mathrm{Smbl}^{1/p}\, K^{2i-1\,2i} \\ -\,\mathrm{Smbl}^{1/p}\, K^{2i\,2i-1} & wI \end{pmatrix}(t, z) \\
&\quad = \frac{1}{D^4}\left((wD + A_{2i-1}C_{2i-1})^2 - B_{2i-1}^2 + (A_{2i-1}B_{2i-1})^2\right) \\
&\qquad \times \left((wD - A_{2i-1}C_{2i-1})^2 - B_{2i-1}^2 + (A_{2i-1}B_{2i-1})^2\right) , \qquad (7.6)
\end{aligned}$$

where $A_{2i-1}, B_{2i-1}, C_{2i-1}$ are the quantities A, B, C evaluated at $z \in \Gamma_{1/p}$, and $\theta = \Theta_{2i-1}$. Recall now that $\Theta_{2i-1} = \theta_i$. This is

$$\begin{aligned}
&\det \begin{pmatrix} wI & -\,\mathrm{Smbl}^{1/p}\, K^{2i-1\,2i} \\ -\,\mathrm{Smbl}^{1/p}\, K^{2i\,2i-1} & wI \end{pmatrix}(t, z) \\
&\quad = \left(w + \frac{B_{2i-1}M_{2i-1} + A_{2i-1}C_{2i-1}}{D}\right)\left(w - \frac{B_{2i-1}M_{2i-1} + A_{2i-1}C_{2i-1}}{D}\right) \\
&\qquad \times \left(w + \frac{B_{2i-1}M_{2i-1} - A_{2i-1}C_{2i-1}}{D}\right)\left(w - \frac{B_{2i-1}M_{2i-1} - A_{2i-1}C_{2i-1}}{D}\right), \qquad (7.7)
\end{aligned}$$

with M_{2i-1} being M as in (4.13) with positive real part, evaluated at $z \in \Gamma_{1/p}$ and $\theta = \Theta_{2i-1}$, $i = 1, \ldots, n$. The graph $y \mapsto w$ of the zeroes of the right hand side of (7.6) is the union of the curves $\Sigma_{\theta_i}(p)$ as introduced in (4.15). Now, employing Theorem 6 we conclude that the operator $wI - K$ is Fredholm iff $w \notin \bigcup_{1\le i\le n} \Sigma_{\theta_i}(p)$. In particular, the essential spectrum of the operator K on $(L^p(\partial\Omega))^2$, $1 < p < \infty$ is

$$\sigma_e\left(K, \left(L^p(\partial\Omega)\right)^2\right) = \mathrm{C} \setminus \bigcup_{1\le i\le n} \Sigma_{\theta_i}(p) , \qquad (7.8)$$

where

$$\sigma_e\left(K, \left(L^p(\partial\Omega)\right)^2\right) := \left\{w \in \mathrm{C};\ wI - K \ \text{ is not Fredholm on } \left(L^p(\partial\Omega)\right)^2\right\} .$$

Let us accept (7.2) for the moment and proceed with the proof of (7.1). Take $w \in \mathrm{C}$, $w \notin \bigcup_{1\le i\le n} \Sigma_{\theta_i}(p)$ and such that $w \in \bigcup_{1\le i\le n} \widehat{\Sigma_{\theta_i}(p)}$. Since w belongs to the interior of at least one curve $\Sigma_{\theta_i}(p)$, the index formula gives that index $\left(wI - K; (L^p(\partial\Omega))^2\right) \neq 0$. This implies

$$\bigcup_{1\le i\le n} \widehat{\Sigma_{\theta_i}(p)} \subseteq \sigma\left(K, \left(L^p(\partial\Omega)\right)^2\right) . \qquad (7.9)$$

To finish the proof of (7.1) we need to show that the only other possible elements of the spectrum must be isolated eigenvalues of the operator K on $(L^p(\partial\Omega))^2$ located in the interval $(-1, 1]$. Consider $w \in \mathrm{C}$, $w \in \sigma\left(K, (L^p(\partial\Omega))^2\right)$ such that $w \notin \bigcup_{1\le i\le n} \widehat{\Sigma_{\theta_i}(p)}$. The index formula gives that $wI - K$ is Fredholm with index zero on $(L^p(\partial\Omega))^2$. By passing to the dual we obtain that $\bar{w} \in \sigma\left(K^*, (L^q(\partial\Omega))^2\right)$ and $\bar{w}I - K^*$ is Fredholm with index zero on $(L^q(\partial\Omega))^2$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The operator $\bar{w}I - K^*$ in particular must fail to be injective, and, therefore, $\bar{w}$ is an eigenvalue of K^* regarded as an operator on $(L^q(\partial\Omega))^2$. When $q \geq 2$ the inclusion $L^q(\partial\Omega) \subset L^2(\partial\Omega)$ easily implies that $\bar{w}$ is actually an eigenvalue of K^* regarded as an operator on $\left(L^2(\partial\Omega)\right)^2$. As a consequence of an adaptation of an old argument of Kellogg [13] all L^2 eigenvalues of K^* belong to $(-1, 1]$ (cf., e. g., [20]), we have $\bar{w} \in (-1, 1]$ and, therefore, $w \in (-1, 1]$. The essence of the argument resides in the validity of the following integration by parts formula

$$\int_{\partial\Omega} S\bar{f}(Q)\left(\pm I - K^*\right) f(Q)\, d\sigma(Q) = \pm \int_{\Omega_\pm} (A\nabla\mathcal{S} f(X)) \cdot (\nabla\mathcal{S}\bar{f}(X))\, dX \qquad (7.10)$$

where $f \in \left(L^2(\partial\Omega)\right)^2$ and A is the particular matrix of coefficients that gives rise to the pseudo-stress conormal derivative in the elastostatic case. Our next step is to show that both sides of (7.10) make sense and, therefore, the equality holds for any $f \in (L^q(\partial\Omega))^2$, where $1 < q < 2$.

We have $(\pm I - K^*) f \in (L^q(\partial\Omega))^2$ and $S\bar{f} \in \left(L^q_1(\partial\Omega)\right)^2$. Appealing to the Sobolev embedding theorem we obtain $S\bar{f}$ belongs actually to any $(L^p(\partial\Omega))^2$, $1 < p < \infty$ and in particular to $(L^p(\partial\Omega))^2$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. This shows that $S\bar{f}(Q)(\pm I - K^*) f(Q) \in \left(L^1(\partial\Omega)\right)^2$ and, therefore, the left hand side of (7.10) makes sense.

Next, since $\nabla\mathcal{S} f$ belongs to the Sobolev space $\left(H^{\frac{1}{q},q}(\Omega)\right)^2$, by the Sobolev embedding theorem we obtain that $\nabla\mathcal{S} f \in (L^p(\Omega))^2$ for $\frac{1}{p} = \frac{1}{q} - \frac{1}{2q}$. Since $q > 1$ we have $p > 2$, and in particular $\nabla\mathcal{S} f \in \left(L^2(\partial\Omega)\right)^2$. This takes care of the right hand side of (7.10). Arguing as in [20] we obtain, then, that all L^q eigenvalues of K^* belong to $(-1, 1]$. This concludes the proof of the fact that all $w \in \sigma\left(K, (L^p(\partial\Omega))^2\right)$ such that $w \notin \bigcup_{1 \leq i \leq n} \widehat{\Sigma_{\theta_i}(p)}$ belong actually to the interval $(-1, 1]$. Moreover each such w belongs to $\partial\sigma\left(K, (L^p(\partial\Omega))^2\right) \setminus \sigma_e\left(K, (L^p(\partial\Omega))^2\right)$, which contains only isolated points (see e. g., p. 102 in [11]). This finishes the proof of (7.1).

We are left with proving the index formula (7.2). Consider $w \notin \bigcup_{1 \leq i \leq n} \widehat{\Sigma_{\theta_i}(p)}$. By Theorem 6, (6.21), (7.7), and the argument principle we have that

$$\operatorname{index}\left(wI - K; \left(L^p(\partial\Omega)\right)^2\right) = \frac{1}{2\pi} \times \sum_{i=1}^{n} \triangle_{\Gamma_{1/p}} \quad \left\{\arg[\text{right hand side of } (7.7)](z)\right\}. \qquad (7.11)$$

Note that we have used above that the change in the argument of $\mathrm{Smbl}^{1/p}(wI - K)$ on $\mathcal{R}_J^{\frac{1}{p}}$ occurs entirely on the contour $\Gamma_{1/p}$ which corresponds to $t = 0$. Since D does not change argument on $\Gamma_{1/p}$, (7.2) would follow from (7.11) and the argument principle provided that

$$\frac{1}{2\pi} \triangle_{\Gamma_{1/p}} \left\{\arg\left(\frac{B_{2i-1}M_{2i-1} + A_{2i-1}C_{2i-1}}{D} - w\right)\right\} = W\left(w, \Sigma_+^+(\theta_i, p)\right), \qquad (7.12)$$

for $i = 1, \ldots, n$ and all the other similar relations corresponding to the different choices $\pm$ in the right hand side of (7.7). To fix ideas, choose the signs as in (7.12). Define

$$f(z) := \frac{B_{2i-1}M_{2i-1} + A_{2i-1}C_{2i-1}}{D}, \quad z \in \Gamma_{1/p}. \qquad (7.13)$$

We point out that since $w \notin \bigcup_{1\le i\le n} \Sigma_{\theta_i}(p)$ the function f does not vanish on $\Gamma_{1/p}$ and, therefore, its change in argument is given by

$$\frac{1}{2\pi}\triangle_{\Gamma_{1/p}}\{\arg(f(z)-w)\} = \int_{-\infty}^{+\infty} \frac{\frac{\partial f}{\partial z}\left(\frac{1}{p}+iy\right)}{f\left(\frac{1}{p}+iy\right)-w}\,dy\,. \tag{7.14}$$

Using the Cauchy–Riemann equations to write $\frac{\partial f}{\partial z}$ as $\frac{1}{i}\frac{\partial f}{\partial y}$ and changing variables $\xi = f\left(\frac{1}{p}+iy\right)$ in (7.14) we obtain $\frac{1}{2\pi}\triangle_{\Gamma_{1/p}}\{\arg(f(z)-w)\} = \frac{1}{2\pi i}\int_{\Sigma_+^+(\theta_i,p)} \frac{1}{\xi - w}d\xi = W(w, \Sigma_+^+(\theta_i, p))$. This gives (7.12) and completes the proof of Theorem 8. $\square$

Remark 2. One can show that, regarding $y \in [-\infty, +\infty]$ as the parameter on the curve $\Sigma_+^+(\theta, p)$ the quantity $\frac{\text{Im }\Sigma_+^+(\theta,p)}{\text{Re }\Sigma_+^+(\theta,p)}(y)$ is decreasing for $y \in [0, \infty]$. Using the symmetry with respect to x axis of the curve $\Sigma_+^+(\theta, p)(y)$, this implies that for any $w \in \widehat{\Sigma_+^+(\theta, p)} \backslash \Sigma_+^+(\theta, p)$ we have that $W(w, \Sigma_+^+(\theta, p)) = 1$ while for $w \notin \widehat{\Sigma}_+^+(\theta, p)$ we have $W(w, \Sigma_+^+(\theta, p)) = 0$. Similar results hold for the other choices $\pm$ as subscript and superscripts of $\Sigma(\theta, p)$. In the notation of Theorem 8 this implies that for any $w \in \mathrm{C}$, $w \notin \bigcup_{1\le i\le n} \Sigma_{\theta_i}(p)$ the index of the operator $wI - K$ is equal to the number of curves $\Sigma_{\theta_i}(p)$ that w is an interior point of.

Remark 3. We would like to point out that Theorem 8 holds also in the case of the classical double layer potential for the Laplacian, i. e., for K_{Laplace}. This is an immediate consequence of Theorem 8.

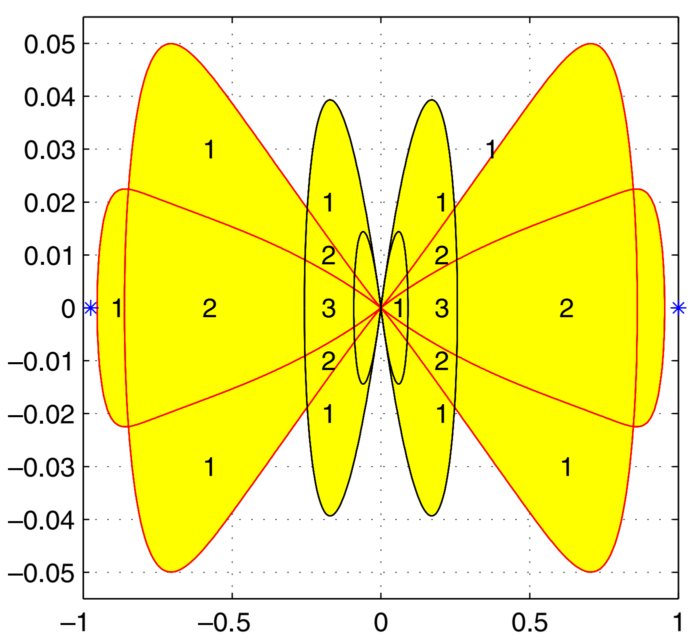

FIGURE 13 The L^6 Spectrum of the elastostatic (pseudostress) double layer potential operator on glass for a curvilinear polygon of angles $\frac{\pi}{10}$ and $\frac{5\pi}{6}$.

We conclude this section by presenting the spectra of the operator K_{Lame}, K_{Stokes}, and K_{Laplace} on a two-dimensional curvilinear polygon with two vertices P_1, P_2, and angles $\theta_1 = \pi/10$, $\theta_2 = 5\pi/6$. The Lamé case is presented in Figure 13. The $*$'s denote the generic position of the eigenvalues $\{\lambda_j\}_j$ of the operator K_{Lame} while the numbers show the values of the index of the operator $wI - K_{\text{Lame}}$ on $\left(L^6(\partial\Omega)\right)^2$ for w belonging to that particular region of the spectrum. In the hydrostatic case ($\upsilon = 1$) we obtain:

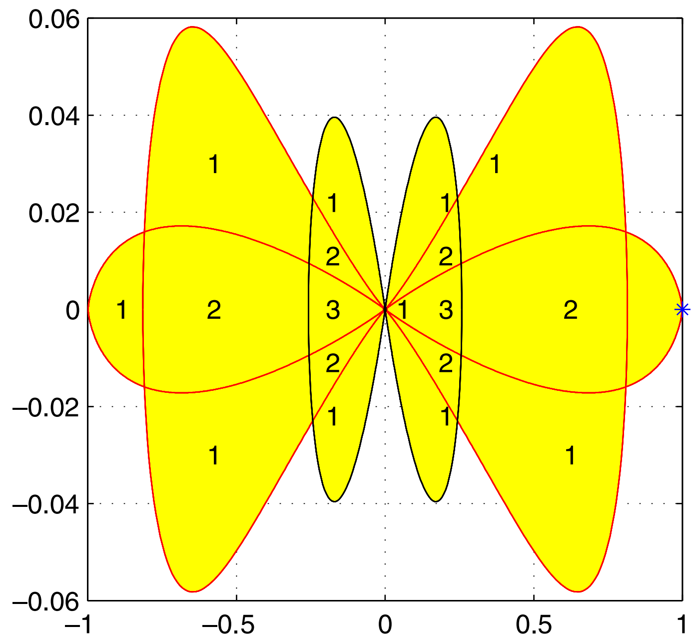

FIGURE 14 The L^6 Spectrum of the hydrostatic (stress) double layer potential operator on glass for a curvilinear polygon of angles $\frac{\pi}{10}$ and $\frac{5\pi}{6}$.

Since the coordinates of the curves $\Sigma_+^-(5\pi/6, 6)$ and $\Sigma_-^-(5\pi/6, 6)$ are of order 10^{-4} they can be seen in Figure 14 only by zooming in. Finally, we present the case of the classical double layer potential operator for the Laplacian. Recall that this case corresponds to the choice $\upsilon = 0$. The curve $\Sigma_\theta(p)$ is now '∞'-shaped. The reason for this is that $\upsilon = 0$ forces $A = 0$ [with A as in (4.7)], and, therefore, the curves $\Sigma_\pm^+(\theta, p)$ and $\Sigma_\pm^-(\theta, p)$ overlap. The spectrum of the operator K_{Laplace} on the space $\left(L^6(\partial\Omega)\right)^2$ is presented in Figure 15.

8. Appendix

In this Appendix we present the proof of the inequality (5.50) in Lemma 5. For the convenience of the reader we start by recalling an equivalent form of (5.50) in the proposition below.

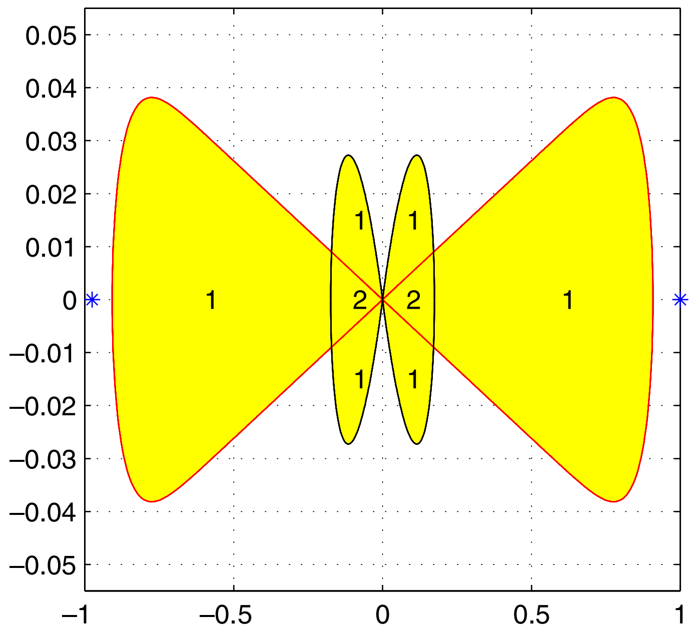

FIGURE 15 The L^6 Spectrum of the classical double layer potential operator for the Laplacian on glass for a curvilinear polygon of angles $\frac{\pi}{10}$ and $\frac{5\pi}{6}$.

Proposition 2.

For any $\theta \in (0, \pi/2]$ and $x \in (0, 1/2]$ there holds

$$\frac{\sin(\pi x)}{\pi x} \leq \frac{\sin((\pi - \theta)x + \tilde{x})}{(\pi - \theta)x + \tan \tilde{x}} , \tag{8.1}$$

where $\tilde{x} = \tilde{x}(\theta, x) = \arcsin(x \sin \theta)$.

Proof. We rewrite (8.1) in the form

$$1 - \frac{\sin((\pi - \theta)x + \tilde{x})}{\sin(\pi x)} \leq 1 - \frac{(\pi - \theta)x + \tan \tilde{x}}{\pi x} , \tag{8.2}$$

for $\theta \in (0, \pi/2]$, $x \in (0, 1/2]$. Next introduce

$$U(x, \theta) := \theta x - \tilde{x}(\theta, x), \quad \text{and} \quad V(x, \theta) := \theta x - \tan(\tilde{x}(\theta, x)) . \tag{8.3}$$

In this notation $(\pi - \theta)x + \tilde{x} = \pi x - U(x, \theta)$ and $(\pi - \theta)x + \tan \tilde{x} = \pi x - V(x, \theta)$. Note that by (5.32) we have $\tilde{x} \leq \theta x$ and, therefore, $U(x, \theta) \geq 0$. Hence, (8.2) becomes

$$\frac{\sin(\pi x) - \sin(\pi x - U(x, \theta))}{\sin(\pi x)} \leq \frac{V(x, \theta)}{\pi x} . \tag{8.4}$$

We have $\sin(\pi x) - \sin(\pi x - U(x, \theta)) = 2 \sin\left(\frac{U(x,\theta)}{2}\right) \cos\left(\pi x - \frac{U(x,\theta)}{2}\right)$. This together with $\sin t \leq t$ for $t \geq 0$ implies

$$\sin(\pi x) - \sin(\pi x - U(x, \theta)) \leq U(x, \theta) \cos\left(\pi x - \frac{U(x, \theta)}{2}\right) . \tag{8.5}$$

It is easy to see that $\arcsin(x\sin\theta) \geq x\sin\theta$ for $x \in (0, 1/2]$ and $\theta \in (0, \pi/2]$. Utilizing this back in the definition of the function $U(x,\theta)$ in (8.3) we obtain

$$U(x,\theta) \leq (\theta - \sin\theta)x\ . \tag{8.6}$$

Thus, by the monotonicity of the function cosine on the interval $[0, \pi/2]$,

$$\cos\left(\pi x - \frac{U(x,\theta)}{2}\right) \leq \cos\left(\left(\pi - \frac{\theta - \sin\theta}{2}\right)x\right)\ . \tag{8.7}$$

In the light of (8.4) and (8.5), proving (8.2) comes down to showing

$$U(x,\theta)\cos(mx) \leq \frac{\sin(\pi x)}{\pi x}V(x,\theta), \quad \text{where} \quad m := \pi - \frac{\theta - \sin\theta}{2}\ . \tag{8.8}$$

It is elementary that, as a function of θ, m is monotonically decreasing on $[0, \pi/2]$. This implies

$$m \in \left[\frac{3\pi}{4} + \frac{1}{2}, \pi\right]\ . \tag{8.9}$$

For $\theta \in [0, \pi/2]$ let $s := \sin\theta \in [0, 1]$ so that $\theta = \arcsin s$. We, then, have

$$\tilde{x} = \arcsin(xs) \quad \text{and} \quad \tan\tilde{x} = \frac{xs}{\sqrt{1-(xs)^2}}\ . \tag{8.10}$$

From the Taylor series expansion we have that

$$\frac{1}{\sqrt{1-t}} = 1 + \sum_{k=1}^{\infty} c_k t^k, \quad c_k = \frac{(2k-1)!!}{(2k)!!}, \quad \forall\ |t| < 1\ . \tag{8.11}$$

where $(2k-1)!!$ is the product of all positive odd numbers that are less than or equal to $(2k-1)$. Similarly, $(2k)!!$ stands for the product of all strictly positive even numbers that are less than or equal to $2k$. Since $xs \leq \frac{1}{2}$ and using (8.10) and (8.11), we obtain the following representation for $\tan\tilde{x}$:

$$\tan\tilde{x} = \frac{xs}{\sqrt{1-(xs)^2}} = xs + \sum_{k=1}^{\infty} c_k (xs)^{2k+1}\ . \tag{8.12}$$

Also, by integrating term by term the Taylor series expansion of $\dfrac{1}{\sqrt{1-(xs)^2}}$ we get

$$\tilde{x} = \arcsin(xs) = xs + \sum_{k=1}^{\infty} \frac{c_k}{2k+1}(xs)^{2k+1}\ . \tag{8.13}$$

Utilizing (8.12) in (8.3) gives

$$U(x,\theta) = x\arcsin s - \tilde{x} = x(\arcsin s - s) - \sum_{k=1}^{\infty} \frac{c_k}{2k+1} s^{2k+1} x^{2k+1}\ . \tag{8.14}$$

Similarly, (8.13), in concert with (8.4) implies

$$V(x,\theta) = x \arcsin s - \tilde{x} = x(\arcsin s - s) - \sum_{k=1}^{\infty} c_k s^{2k+1} x^{2k+1} . \tag{8.15}$$

Recall that $s = \sin\theta$ in (8.14) and (8.15). Now, expanding $\cos(mx)$ in Taylor series, we have

$$\cos(mx) = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{m^{2k}}{(2k)!} x^{2k} \leq 1 - K_0 x^2 , \tag{8.16}$$

where

$$K_0 = \frac{m^2}{2!} - \frac{m^4 x^2}{4!} . \tag{8.17}$$

Notice that K_0 regarded as a function of m^2 is increasing on the interval $[0, \frac{6}{x^2}]$. Furthermore, for any $x \in (0, 1/2]$ and m as in (8.9) we have that $m^2 \in [0, 24] \subseteq [0, \frac{6}{x^2}]$. In turn, this implies

$$\pi^2/2 \geq K_0 . \tag{8.18}$$

Going further, it is useful to point out here that from (8.14) we have

$$U(x,\theta) \leq x(\arcsin s - s) - x^3 s^3/6 . \tag{8.19}$$

Then, (8.19) and (8.16) give

$$U(x,\theta)\cos(mx) \leq \left(x(\arcsin s - s) - \frac{x^3 s^3}{6} \right) \left(1 - K_0 x^2 \right) , \tag{8.20}$$

with K_0 as in (8.17). Next, recall that

$$\frac{\sin(\pi x)}{\pi x} \geq 1 - \frac{\pi^2 x^2}{6} . \tag{8.21}$$

Returning to (8.15) notice that

$$V(x,\theta) = x(\arcsin s - s) - K_1 x^3 , \tag{8.22}$$

with

$$K_1 := s^3 \sum_{k=1}^{\infty} c_k (xs)^{2k-2} = s^3 \frac{\left(1 - (xs)^2\right)^{-\frac{1}{2}} - 1}{(xs)^2} . \tag{8.23}$$

The last equality in (8.23) follows from (8.11). Simple inspection of the definition of K_1 reveals that

$$K_1 \geq 1/2\, s^3 . \tag{8.24}$$

Now, (8.21) and (8.22) imply

$$\left(1 - \frac{\pi^2 x^2}{6} \right) \left(x(\arcsin s - s) - K_1 x^3 \right) \leq \frac{\sin(\pi x)}{\pi x} V(x,\theta) . \tag{8.25}$$

In the light of (8.25) and (8.20) to show that (8.8) holds it is sufficient to prove that

$$\begin{aligned}\left(1-\frac{\pi^2x^2}{6}\right)&\left[x(\arcsin s-s)-K_1x^3\right]\\&\geq\left[x(\arcsin s-s)-\frac{x^3s^3}{6}\right]\left(1-K_0x^2\right)\ .\end{aligned}\tag{8.26}$$

After some straightforward algebra (8.26) reduces to

$$(\arcsin s-s)\left[K_0-\frac{\pi^2}{6}\right]-K_1+\frac{s^3}{6}+\frac{x^2}{6}\left[K_1\pi^2-K_0s^3\right]\geq 0\ ,\tag{8.27}$$

for $x\in(0,1/2]$ and $s\in[0,1]$. From (8.24) and (8.18) we have that

$$\frac{x^2}{6}\left[K_1\pi^2-K_0s^3\right]\geq 0\ .\tag{8.28}$$

Thus, at this stage it suffices to prove the estimate

$$(\arcsin s-s)\left[K_0-\frac{\pi^2}{6}\right]-K_1+\frac{s^3}{6}\geq 0,\ \ \forall\ x\in(0,1/2]\ \ s\in[0,1]\ .\tag{8.29}$$

Clearly this finishes the proof of (8.27). When $s=0$ the inequality (8.29) is obviously satisfied. When $s>0$ we divide both sides of (8.29) by s^3. Consequently, (8.29) becomes equivalent to

$$\frac{\arcsin s-s}{s^3}\left[K_0-\frac{\pi^2}{6}\right]-\frac{K_1}{s^3}+\frac{1}{6}\geq 0,\ \ \forall\ x\in(0,1/2]\ \ s\in(0,1]\ .\tag{8.30}$$

We divide our analysis of (8.30) in four distinct cases.

Case 1: Assume $x\in(0,1/2]$ and $s\in(0,1/2]$.

Recall that $s=\sin\theta$ and, therefore, we have $\theta\in[0,\pi/6]$. Relying on the monotonicity of m, defined in (8.8), in the variable θ we obtain

$$m\geq 11\pi/12+1/4>3.1297\ .\tag{8.31}$$

Since K_0, as defined in (8.17), is increasing in m^2 and decreasing in x we have

$$K_0\geq\frac{3.1297^2}{2}-\frac{3.1297^4}{96}\geq 3.8981\ .\tag{8.32}$$

In order to continue let us recall (8.23); that is (8.23). Since $\dfrac{\left(1-y^2\right)^{-\frac{1}{2}}-1}{y^2}$ is increasing in y on $[0,\infty)$ and in this case $xs\leq 1/4$, we obtain

$$K_1\leq s^3\frac{\left(1-(1/4)^2\right)^{-\frac{1}{2}}-1}{(1/4)^2}\leq 0.5248s^3\ .\tag{8.33}$$

Finally, using (8.33), (8.32), plus the fact that $\arcsin s-s\geq\dfrac{1}{6}s^3$, we arrive at

$$\begin{aligned}\frac{\arcsin s-s}{s^3}\left[K_0-\frac{\pi^2}{6}\right]-\frac{K_1}{s^3}+\frac{1}{6}&\geq\frac{1}{6}\left(3.8981-\frac{\pi^2}{6}\right)\\&\quad-0.5248+\frac{1}{6}\geq 0.0173\geq 0\ .\end{aligned}\tag{8.34}$$

That is (8.30) holds. The analysis of Case 1 is, therefore, completed.

Case 2: Assume $x \in (0, 1/2]$ and $s \in [1/2, \sqrt{2}/2]$.
Note that $s \in [1/2, \sqrt{2}/2]$ forces $\theta = \arcsin s \in [\pi/6, \pi/4]$. Hence

$$m \geq \pi - \frac{\pi}{8} + \frac{1}{2}\sin\left(\frac{\pi}{4}\right) \geq 3.1024\,. \tag{8.35}$$

Reasoning as before this gives the lower bound for K_0

$$K_0 \geq \frac{3.1024^2}{2} - \frac{3.1024^4}{96} \geq 3.8474\,. \tag{8.36}$$

Producing an upper bound on K_1 is done much as before. Since $s \leq \sqrt{2}/2$ and $x \in (0, 1/2]$, we have $xs \leq \sqrt{2}/4$ and, consequently,

$$K_1 \leq s^3 \frac{\left(1 - \left(\frac{\sqrt{2}}{4}\right)^2\right)^{-\frac{1}{2}} - 1}{\left(\frac{\sqrt{2}}{4}\right)^2} \leq 0.5524 s^3\,. \tag{8.37}$$

It is easy to see, by appealing to the Taylor series representation for instance, that the function $\dfrac{\arcsin s - s}{s^3}$ is increasing in s on the interval $(0, 1]$. This implies in our case

$$\frac{\arcsin s - s}{s^3} \geq \frac{\arcsin(1/2) - 1/2}{(1/2)^3} \geq 0.1887\,. \tag{8.38}$$

Using (8.36), (8.37), and (8.38) in (8.30) gives

$$\begin{aligned}\frac{\arcsin s - s}{s^3}\left[K_0 - \frac{\pi^2}{6}\right] - \frac{K_1}{s^3} + \frac{1}{6} &\geq 0.1887\left(3.8474 - \frac{\pi^2}{6}\right)\\ &\quad - 0.5524 + \frac{1}{6} = 0.0298 \geq 0\,.\end{aligned} \tag{8.39}$$

Therefore, (8.30) holds in this case.

Case 3: Assume $x \in (0, 1/2]$ and $s \in [\sqrt{2}/2, \sqrt{3}/2]$.
This case corresponds to $\theta = \arcsin s \in [\pi/4, \pi/3]$. Much as before

$$m \geq \pi - \frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) \geq 3.0510\,. \tag{8.40}$$

In turn, this implies

$$K_0 \geq \frac{3.0510^2}{2} - \frac{3.0510^4}{96} > 3.7516\,. \tag{8.41}$$

Since $xs \leq \sqrt{3}/4$, an upper bound for K_1 is given by

$$K_1 \leq s^3 \frac{\left(1 - \left(\frac{\sqrt{3}}{4}\right)^2\right)^{-\frac{1}{2}} - 1}{\left(\frac{\sqrt{3}}{4}\right)^2} \leq 0.5835 s^3\,. \tag{8.42}$$

Also, using the monotonicity of the function $\dfrac{\arcsin s - s}{s^3}$ we obtain

$$\frac{\arcsin s - s}{s^3} \geq \frac{\arcsin(\sqrt{2}/2) - \sqrt{2}/2}{(\sqrt{2}/2)^3} \geq 0.2214 \,. \tag{8.43}$$

Finally, using (8.41), (8.42), and (8.43) we obtain that

$$\begin{aligned} \frac{\arcsin s - s}{s^3}\left[K_0 - \frac{\pi^2}{6}\right] - \frac{K_1}{s^3} + \frac{1}{6} &\geq 0.2214\left(3.7516 - \frac{\pi^2}{6}\right) \\ &\quad - 0.5835 + \frac{1}{6} \geq 0.0495 \geq 0 \,. \end{aligned} \tag{8.44}$$

This finishes the proof of (8.30).

Case 4: Assume $x \in (0, 1/2]$ and $s \in [\sqrt{3}/2, 1]$.

This time $\theta = \arcsin\theta \in [\pi/3, \pi/2]$. Since $\theta \leq \pi/2$, we have

$$m \geq \pi - \frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) \geq 2.8561 \,. \tag{8.45}$$

Therefore,

$$K_0 \geq \frac{2.8561^2}{2} - \frac{2.8561^4}{96} \geq 3.3855 \,. \tag{8.46}$$

Since $xs \leq 1/2$, an upper bound on K_1 is given by

$$K_1 \leq s^3 \frac{\left(1 - \left(\frac{1}{2}\right)^2\right)^{-\frac{1}{2}} - 1}{\left(\frac{1}{2}\right)^2} \leq 0.6189 s^3 \,. \tag{8.47}$$

Again, based on the monotonicity of the function $\dfrac{\arcsin s - s}{s^3}$ we obtain

$$\frac{\arcsin s - s}{s^3} \geq \frac{\arcsin\left(\sqrt{3}/2\right) - \sqrt{3}/2}{\left(\sqrt{3}/2\right)^3} \geq 0.2789 \,. \tag{8.48}$$

Finally, using (8.45), (8.47), and (8.48) we arrive at

$$\begin{aligned} \frac{\arcsin s - s}{s^3}\left[K_0 - \frac{\pi^2}{6}\right] - \frac{K_1}{s^3} + \frac{1}{6} &\geq 0.2789\left(3.3855 - \frac{\pi^2}{6}\right) \\ &\quad - 0.6189 + \frac{1}{6} \geq 0.0332 \geq 0 \,. \end{aligned} \tag{8.49}$$

This finishes the proof of (8.30) in Case 4 and hence concludes the proof of Proposition 2. □

Acknowledgments

I am greatly indebted to Carlos Kenig for suggesting this theme and for his constant support and encouragement. Also I would like to thank Mikhail Safonov, Walter Littman and Max Jodeit for their kind interest in my work and many stimulating discussions.

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Received December 14, 2000

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