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On the Multi-Level Splitting of Finite Element Spaces

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Summary. In this paper we analyze the condition number of the stiffness matrices arising in the discretization of selfadjoint and positive definite plane elliptic boundary value problems of second order by finite element methods when using hierarchical bases of the finite element spaces instead of the usual nodal bases. We show that the condition number of such a stiffness matrix behaves like $O((\log\kappa)^2)$ where κ is the condition number of the stiffness matrix with respect to a nodal basis. In the case of a triangulation with uniform mesh size h this means that the stiffness matrix with respect to a hierarchical basis of the finite element space has a condition number behaving like $O\left(\left(\log\frac{1}{h}\right)^2\right)$ instead of $O\left(\left(\frac{1}{h}\right)^2\right)$ for a nodal basis. The proofs of our theorems do not need any regularity properties of neither the continuous problem nor its discretization. Especially we do not need the quasiuniformity of the employed triangulations. As the representation of a finite element function with respect to a hierarchical basis can be converted very easily and quickly to its representation with respect to a nodal basis, our results mean that the method of conjugate gradients needs only $O(\log n)$ steps and $O(n\log n)$ computer operations to reduce the energy norm of the error by a given factor if one uses hierarchical bases or related preconditioning procedures. Here n denotes the dimension of the finite element space and of the discrete linear problem to be solved.

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1. Introduction

Today multigrid or multi-level methods are thought to be the most efficient methods to solve the large systems of linear equations arising in connection with the approximate solution of linear elliptic boundary value problems by finite element or finite difference methods. Multigrid methods are iterative methods. Their convergence properties are studied in the classical papers of Bank and Dupont [4] and Hackbusch [7, 8]. A convergence proof giving some more information for the positive definite and symmetric case can be found, for example, in [14]. It can be shown that, for properly chosen families of grids, the amount of work necessary to reduce the error by a constant factor is proportional to the number of unknowns. "Properly chosen" is an essential because the multigrid philosophy requires that the difference between the exact approximations on two successive grids is fast oscillating in the sense of the local grid width. For appropriate grids this property can be proved via the elliptic regularity theory; see [4, 7, 8] and [13]. But what happens for the complicated families of grids adapted to the behaviour of real life problems is very difficult to predict.

In this paper we propose a new method which is nearly of the same optimal computational complexity as the conventional multigrid methods but which is free of many of its restrictions. Especially its speed of convergence does not depend on the regularity properties of the boundary value problem to be solved.

The main idea is to replace the usual nodal bases of the finite element spaces by hierarchical bases. Such hierarchical bases originate in a very natural way if one starts with a coarse initial triangulation of the domain under consideration and refines it several times to get the final triangulation. We prove that in the two-dimensional case the condition number of the stiffness matrix of a given selfadjoint and positive definite second order elliptic boundary value problem with respect to such hierarchical bases increases only like $O(j^2)$ for a growing number j of refinement levels. This has to be compared with the exponential growth of the condition numbers known from nodal bases.

To derive the result we introduce in Sects. 2 and 3 mesh-dependent norms on the finite element spaces which are essentially the Euclidean norms of the coefficient vectors with respect to hierarchical bases. We prove that on the given finite element spaces these norms essentially behave like the usual Sobolev-norms associated with second order elliptic boundary value problems and therefore like the corresponding energy norms. Section 2 deals with uniformly refined families of grids whereas in Sect. 3 arbitrary families of nested triangulations are considered.

At first sight one gets rather dense and complicated discretization matrices when using hierarchical bases, and one could think that the well developed mechanism of finite element computations has to be abandoned. In Sect. 4 we show how one can overcome these difficulties. The trick is not to assemble the discretization matrix but to use a factorization of it into the usual nodal basis stiffness matrix and some other very simple and sparse matrices depending only on the triangulations. Thus it is possible to compute the product of the hierarchical basis stiffness matrix with a given vector with nearly the same amount of work as one needs for computing the product of the nodal basis matrix with a vector.

These facts lead to the desired result: using hierarchical bases and the method of conjugate gradients, the number of floating point operations necessary to reduce the energy norm of the error of an initial approximation by a factor $\varepsilon \in (0,1)$ is bounded by

$$Cn\log(n)\left|\log\left(\frac{\varepsilon}{2}\right)\right|, \tag{1.1}$$

where n is the dimension of the finite element space. The constant C depends neither on ε or n nor on the regularity of the problem to be solved. (1.1) is a very favourable operation count because it says that our method is of nearly optimal computational complexity. The number of operations necessary to reach a given accuracy grows nearly proportionally with the number of unknowns. So far such nearly optimal or optimal operation counts for general classes of problems were only known for multigrid methods.

In Sect. 5 we explore whether the asymptotic behaviour of our estimates can be improved. Our answer is a definite no. By a very simple example on a uniformly refined grid we show that the norm estimates of Sects. 2 and 3 are asymptotically optimal. Using this fact we prove that the condition number of the bilinear form defining the boundary value problem with respect to any other bilinear form decoupling the different refinement levels grows at least quadratically in the number of these levels.

Our results are dimension dependent. The one-dimensional case has been treated by Zienkiewicz, Kelly, Gago and Babuška [16]. These authors prove that the condition numbers with respect to appropriately scaled hierarchical bases are bounded independently of the number of refinement levels. This case is much simpler than the two-dimensional case because one can easily show that hierarchical bases of spaces of piecewise linear functions are orthogonal with respect to the canonical bilinear form

$$D(u,v)=\int_a^b u'(x)v'(x)dx. \tag{1.2}$$

In the three-dimensional case the asymptotical reduction of the condition numbers is unfortunately less spectacular. One can see by our Lemma 5.2 and a simple homogeneity argument that the condition number with respect to an optimally scaled hierarchical basis grows at least as $O(2^j)$ with the number j of refinement levels in the case of a quasiuniform family of grids. We conjecture that this is also an upper bound. This would mean that one gets the same reduction of the condition number as when using incomplete factorizations or a symmetric successive overrelaxation technique; see [1].

A hierarchical basis corresponds to a multi-level splitting of the finite element space. Iterative methods, with are based on a two-level splitting and whose speed of convergence is bounded independent of the regularity, have been developed by Bank and Dupont in [5], a paper which influenced many workers on multigrid methods. Axelsson and Gustafsson [2] use two-level splittings of finite element spaces as preconditioners for conjugate gradient type algorithms. If the employed grids are not too fine, such iterative methods can compete with true multigrid methods but asymptotically the amount of work grows much faster than the number of unknowns because one has to solve in each iteration step a linear system with the coarse level discretization matrix as coefficient matrix. The mathematical basis of both papers is a strengthened Cauchy-Schwarz inequality, a technique of proof which is also used by Braess in his convergence analysis [6] of a standard multigrid method. In our situation with many refinement levels we cannot make use of this technique, the constants would explode exponentially with the number of refinement levels.

2. The Grid Dependent Norms: the Case of Uniformly Refined Families of Triangulations

We think that one of the main advantages of our method is its robustness and the fact that its speed of convergence does not depend on the regularity properties of the considered boundary value problem or on a regular refinement. Nevertheless we begin in this section by stating the basic results for regularly refined triangulations. In the next section we show how these results can be generalized to other families of nested triangulations.

By a triangulation $\mathscr{T}$ of a polygonal planar region Ω we mean a set of triangles such that the union of these triangles is $\bar{\Omega}$ and such that the intersection of two triangles of $\mathscr{T}$ either consists of a common side or a common vertex of both triangles or is empty. Here we start with a coarse initial triangulation $\mathscr{T}_0$ of the polygonal domain Ω under consideration. Beginning with this triangulation we construct a nested family $\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2, \ldots$ of triangulations of Ω. In this section $\mathscr{T}_{k+1}$ is obtained from $\mathscr{T}_k$ by subdividing any triangle of $\mathscr{T}_k$ into four congruent subtriangles as shown in Fig. 1.

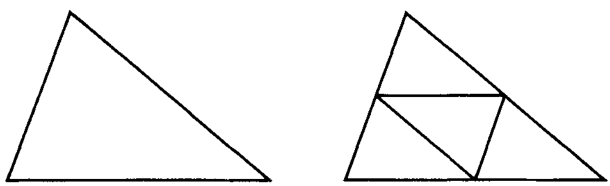

Fig. 1

Let $\mathscr{N}_k$ be the set of the vertices of the triangles of $\mathscr{T}_k$, and let $\mathscr{S}_k$ be the space of all functions being continuous on $\bar{\Omega}$ and linear on the triangles of $\mathscr{T}_k$. We call the points in $\mathscr{N}_k$ nodes and the functions in $\mathscr{S}_k$ finite element functions of level k. Obviously we have $\mathscr{N}_k \subseteq \mathscr{N}_{k+1}$ and $\mathscr{S}_k \subseteq \mathscr{S}_{k+1}$, and a function $u \in \mathscr{S}_k$ is determined by its values in the nodes $x \in \mathscr{N}_k$.

For a given continuous function u we denote by $I_k u$ the function of $\mathscr{S}_k$ interpolating u at the nodes of $\mathscr{N}_k$: We have

$$I_k u \in \mathscr{S}_k; \quad (I_k u)(x) = u(x), \quad x \in \mathscr{N}_k. \tag{2.1}$$

For $j \geqq k$ and $u \in \mathscr{S}_k$ u is reproduced by the interpolation operator I_j. Therefore any function $u \in \mathscr{S}_j$ has the representation

$$u = I_0 u + \sum_{k=1}^{j} (I_k u - I_{k-1} u). \tag{2.2}$$

This is a self-evident but nevertheless very interesting formulas: (2.2) is a decomposition of u into fast oscillating functions corresponding to the different refinement levels; $I_0 u$ is a function of the finite element space corresponding to the initial triangulation, whereas $I_k u - I_{k-1} u \in \mathscr{S}_k$ vanishes at all nodal points of level $k-1$.

Let $\mathscr{V}_k$, $k = 1, 2, \ldots$, be the subspace of $\mathscr{S}_k$ consisting of all finite element functions vanishing in the nodes of level $k-1$. $\mathscr{V}_k$ is the range of $I_k - I_{k-1}$, and (2.2) means that $\mathscr{S}_j$ is the direct sum of $\mathscr{V}_0 := \mathscr{S}_0$ and $\mathscr{V}_1, \ldots, \mathscr{V}_j$. Using the spaces $\mathscr{V}_k$ we give a recursive definition of the hierarchical basis of $\mathscr{S}_j$. We start with the nodal basis of $\mathscr{S}_0$, that means the basis consisting of functions which take the value 1 at an associated nodal point of level 0 and the value 0 at all other nodal points of this level. A hierarchical basis of $\mathscr{S}_k$, $k \geqq 1$, consists of the old hierarchical basis functions of level $k-1$ and the functions forming a nodal basis of $\mathscr{V}_k$. This construction is illustrated in Fig. 2 for a simple one dimensional example.

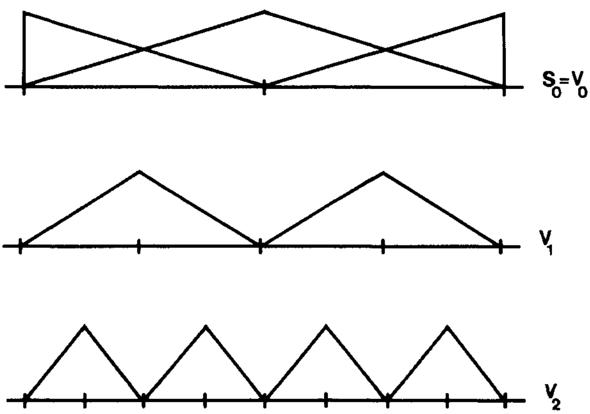

Fig. 2

With the multi-level splitting (2.2) of the finite element space $\mathscr{S}_j$ we associate a mesh dependent seminorm defined by

$$|u|^2 = \sum_{k=1}^{j} \sum_{x \in \mathscr{N}_k \setminus \mathscr{N}_{k-1}} |(I_k u - I_{k-1} u)(x)|^2, \quad u \in \mathscr{S}_j. \tag{2.3}$$

This seminorm has a very simple representation when the function u is represented with respect to the hierarchical basis of $\mathscr{S}_j$: it is the Euclidean length of the vector of its coefficients with the exception of those corresponding to the initial level.

To state the main results of this section we need the L_2-norm

$$\|u\|_{0,2;\Omega}=(\int_\Omega |u(x)|^2 dx)^{1/2} \tag{2.4}$$

on $L_2(\Omega)$, and the seminorm

$$|u|_{1,2;\Omega}=(\|D_1 u\|_{0,2;\Omega}^2+\|D_2 u\|_{0,2;\Omega}^2)^{1/2} \tag{2.5}$$

and the norm

$$\|u\|_{1,2;\Omega}=(\|u\|_{0,2;\Omega}^2+|u|_{1,2;\Omega}^2)^{1/2} \tag{2.6}$$

on the Hilbert-space $W^{1,2}(\Omega)$. Note that the spaces $\mathcal{S}_k$ are subspaces of $W^{1,2}(\Omega)$. In addition we make use of the maximum norm

$$\|u\|_{0,\infty;\Omega}=\max_{x\in\Omega}|u(x)| \tag{2.7}$$

on $C(\bar{\Omega})$. We have:

Theorem 2.1. *There are positive constants K_1 and K_2 with*

$$\frac{K_1}{(j+1)^2}\{|I_0 u|_{1,2;\Omega}^2+|u|^2\}\leqq|u|_{1,2;\Omega}^2\leqq K_2\{|I_0 u|_{1,2;\Omega}^2+|u|^2\}$$

for all functions $u\in\mathcal{S}_j$. These constants depend only on a lower bound for the interior angles of the triangles under consideration. Especially they are independent of the number j of refinement levels and of the size and shape of the domain Ω.

Theorem 2.2. *There are positive constants K_1^* and K_2^* with*

$$\frac{K_1^*}{(j+1)^2}\{\|I_0 u\|_{1,2;\Omega}^2+|u|^2\}\leqq\|u\|_{1,2;\Omega}^2\leqq K_2^*\{\|I_0 u\|_{1,2;\Omega}^2+|u|^2\}$$

for all functions $u\in\mathcal{S}_j$. These constants depend only on the diameter of the domain Ω and on a lower bound for the interior angles of the triangles under consideration. Especially they are independent of the number j of refinement levels and of the shape of the domain Ω.

Note that j is of order $|\log h|$ where h is the characteristic gridsize on level j. As the finite element space corresponding to the initial triangulation is fixed independent of the number of refinement levels, these theorems mean that the Euclidean norm of the coefficient vector of a function $u\in\mathcal{S}_j$ with respect to the hierarchical basis of this space essentially behaves like its Sobolev-norm associated with second order elliptic boundary value problems.

The rest of this section is devoted to the proofs of the two theorems. We subdivide these proofs in a series of lemmas each of which is of its own interest. Our starting point for the proof of the lower estimates in both theorems is the following result:

Lemma 2.1. *Let T be a given triangle of diameter H which is subdivided in an arbitrary manner into further triangles of diameters greater than or equal to h. Let u be a function which is continuous on the triangle T and linear on all these small triangles. Then one has*

$$\|u\|_{0,\infty;T} \leqq C\left(\log\frac{H}{h}+\frac{1}{4}\right)^{1/2}\|u\|_{1,2;T}$$

where the constant C only depends on H and on a lower bound for the interior angles of the small triangles.

Similar results can be found in Wendland [12] (Lemma 8.3.3), Oganesjan and Ruhovec [10 (p. 74–77)] or Thomée [11 (p. 77)]. As Lemma 2.1 is of crucial importance for our theory we recall a proof given in [15] which is quite elementary and gives a rather detailed insight in the structure and the size of the constant.

We derive the estimate of Lemma 2.1 from the estimate

$$\frac{1}{\pi\sigma^2}\int_{|x-x_0|\leqq\sigma}|u(x)|\,dx \leqq \frac{1}{\sqrt{2\pi}}\left(\log\frac{R}{\sigma}+\frac{1}{4}\right)^{1/2}|u|_{1,2;K} \tag{2.8}$$

where K denotes the circle of radius $R \geqq \sigma > 0$ with center x_0 and u is an arbitrary function of $W_0^{1,2}(K)$. This estimate is proved in [15]; see also the appendix of this paper. It is sharp, this means that for every R and σ a function u exists for which the inequality becomes an equality. Ultimately the constant on the right hand side of (2.8) is responsible for the growth of the condition numbers of finite element discretization matrices when the number j of refinement levels increases. For the triangulations considered in this section R/σ is of order 2^j. Note that the constant on the right hand side of (2.8) tends only very slowly to infinity for a growing value R/σ; for $R/\sigma \leqq 2^{225}$ it is smaller than 5.

To apply (2.8) to the piecewise linear functions defined on our triangle T, which is assumed for the moment to be the reference triangle

$$T = \{(x, y) \mid x, y \geqq 0, x+y \leqq 1\}, \tag{2.9}$$

we need an extension operator E extending these functions, for example, to functions $Eu \in W_0^{1,2}(S)$, where S denotes the square

$$S = \{(x, y) \mid |x|+|y| \leqq 2\}. \tag{2.10}$$

Of course, one wants to have the estimate

$$|Eu|_{1,2;S} \leqq C_0\|u\|_{1,2;T}. \tag{2.11}$$

Such an extension operator can be constructed in a very elementary way. At first u is extended by continued reflection to a function v defined on S as indicated in Fig. 3.

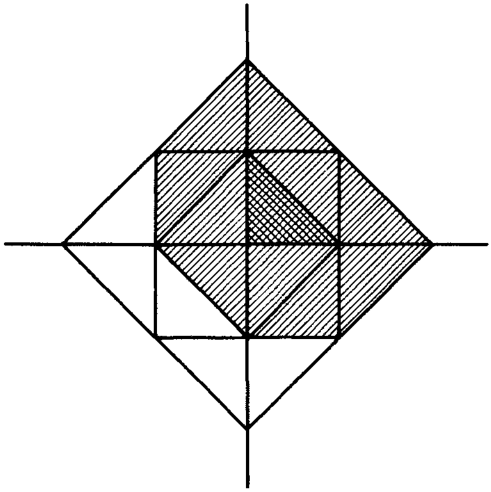

Fig. 3

Let ω be the function which is linear on the triangles in Fig. 3 and which takes the value 1 at the vertices of the triangle T and the value 0 at all other vertices of these triangles. Set $Eu = \omega v$. Via this extension operator we get from (2.8) the estimate

$$\frac{1}{\pi\sigma^2} \int_{\substack{|x-x_0|\leq\sigma \\ x\in T}} |u(x)|\,dx \leq C_1 \left(\log\frac{H}{\sigma}+\frac{1}{4}\right)^{1/2} \|u\|_{1,2;T} \tag{2.12}$$

with a harmless constant C_1. $H=\sqrt{2}$ is the diameter of the reference triangle T; $x_0 \in T$ and $\sigma \in (0, H]$ are arbitrary.

Now we can make use of the properties of piecewise linear functions. For every function u, which is linear on a given triangle Δ of diameter h, the well-known inverse inequality

$$\|u\|_{0,\infty;\Delta} \leq C_2 \frac{1}{h^2} \int_\Delta |u(x)|\,dx \tag{2.13}$$

holds where the constant C_2 depends only on the smallest interior angle of Δ but not on the size of this triangle. From (2.12) and (2.13) the estimate of Lemma 2.1 follows for the reference triangle T.

The next step, towards the proof of the lower estimates given in the two theorems, is to give estimates for the norms of the interpolation operators (2.1). Let

$$(Iu)(x_1, x_2) = x_1 u(1,0) + x_2 u(0,1) + (1 - x_1 - x_2) u(0,0) \tag{2.14}$$

be the linear function interpolating a given function u in the vertices of the reference triangle (2.9). Then one has

$$|Iu|_{1,2;T} \leq 2\|u\|_{0,\infty;T} \tag{2.15}$$

and therefore for the functions described in Lemma 2.1

$$|Iu|_{1,2;T} \leqq C_3 \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} \|u\|_{1,2;T}. \tag{2.16}$$

Next we want to replace the norm (2.6) on the right hand side of (2.16) by the seminorm (2.5). For this purpose we use Poincaré's inequality

$$\|u\|_{1,2;T}^2 \leqq \frac{1}{2\pi^2} \|u\|_{1,2;T}^2 + 2 \left|\int_T u(x)\,dx\right|^2 \tag{2.17}$$

which can be proved by extending u by reflection to an even function defined on the square $[-1,1]^2$ and by expanding this function in a Fourier-series. Using (2.16) and (2.17) we get

$$|Iu|_{1,2;T} \leqq C_4 \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} |u|_{1,2;T} + C_5 \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} \left|\int_T u(x)\,dx\right|. \tag{2.18}$$

The second term on the right hand side of (2.18) can be deleted: Consider the function

$$v(x) = u(x) - \alpha$$

with vanishing mean value on T. We have by (2.18) because $Iv = Iu - \alpha$

$$|Iu|_{1,2;T} = |Iv|_{1,2;T} \leqq C_4 \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} |v|_{1,2;T} = C_4 \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} |u|_{1,2;T}. \tag{2.19}$$

A deformation of T into another triangle of the same diameter does not affect the nature of this estimate; also in this case the constant only depends on a lower bound for the interior angles of the small triangles. Furthermore the seminorm (2.5) is invariant under changes of the size of T. Therefore we can summarize our considerations in the following lemma:

Lemma 2.2. *Let T be a triangle of diameter H which is subdivided in an arbitrary manner into further triangles of diameters greater than or equal to h. Let u be a function, which is continuous on the triangle T and linear on all these small triangles, and let Iu be the linear function which interpolates u in the vertices of T. Then one has*

$$|Iu|_{1,2;T} \leqq C \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} |u|_{1,2;T}$$

with a constant C depending only on a lower bound for the interior angles of the small triangles but not on the size of T or these triangles.

This lemma is the key to the proof of the left hand side estimate in Theorem 2.1. For the proof of Theorem 2.2 we need an additional estimate for the L_2-norm of Iu:

Lemma 2.3. *Under the assumptions of Lemma 2.2 one has*

$$\|Iu\|_{0,2;T} \leqq C \left(\log \frac{H}{h} + \frac{1}{4}\right)^{1/2} (\|u\|_{0,2;T}^2 + H^2 |u|_{1,2;T}^2)^{1/2}.$$

Again the constant C depends only on a lower bound for the interior angles of the small triangles.

Proof. We use the scaled triangle

$$\hat{T}=\left\{\frac{1}{H}x\,|\,x\in T\right\}$$

of diameter 1. For a given function u defined on T let the function $\hat{u}$ be defined on $\hat{T}$ by

$$\hat{u}(x)=u(Hx).$$

If u has the assumed properties we have

$$\|Iu\|_{0,2;T}\leqq\frac{H}{2}\|\hat{u}\|_{0,\infty;\hat{T}}.$$

Lemma 2.1 gives

$$\|\hat{u}\|_{0,\infty;\hat{T}}=C\left(\log\frac{H}{h}+\frac{1}{4}\right)^{1/2}\|\hat{u}\|_{1,2;\hat{T}}$$

As we have

$$\|\hat{u}\|_{1,2;\hat{T}}^2=H^{-2}\|u\|_{0,2;T}^2+|u|_{1,2;T}^2$$

the desired estimate follows. □

After these preparations we can return to the family of triangulations $\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2,\ldots$ and the associated finite element spaces $\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2,\ldots$ introduced above. Then we have, for example, by Lemma 2.2 for every function $u\in\mathcal{S}_j$ and every triangle $T\in\mathcal{T}_k$, $k\leqq j$,

$$|I_k u|_{1,2;T}\leqq C(\log 2^{j-k}+\tfrac{1}{4})^{1/2}|u|_{1,2;T} \tag{2.20}$$

and therefore also

$$\begin{aligned}|I_k u|_{1,2;\Omega}&\leqq C(\log 2^{j-k}+\tfrac{1}{4})^{1/2}|u|_{1,2;\Omega}\\&\leqq C(j-k+1)^{1/2}|u|_{1,2;\Omega}.\end{aligned} \tag{2.21}$$

To finish the proof of the lower estimate in Theorem 2.1 we still need another observation:

Lemma 2.4. *There exist positive constants C_1 and C_2 with*

$$C_1|u|^2\leqq\sum_{k=1}^{j}|I_k u-I_{k-1}u|_{1,2;\Omega}^2\leqq C_2|u|^2$$

for all functions $u\in\mathcal{S}_j$. These constants depend only on a lower bound for the interior angles of the triangles under consideration; they are independent of j.

Proof. Let T be a triangle of $\mathcal{T}_{k-1}$ which is subdivided into four triangles of $\mathcal{T}_k$ as shown in Fig. 1. As every function $v\in\mathcal{V}_k$ is piecewise linear on these four triangles and vanishes in the vertices of T, there are positive constants C_1 and

C_2 depending only on a lower bound for the interior angles of T with

$$2C_1|v|_{1,2;T}^2 \leqq \sum_{x\in T\cap\mathcal{N}_k\setminus\mathcal{N}_{k-1}} |v(x)|^2 \leqq C_2|v|_{1,2;T}^2$$

for all functions $v\in\mathscr{V}_k$. Every node $x\in\mathcal{N}_k\setminus\mathcal{N}_{k-1}$ is contained in at most two triangles of level $k-1$. Therefore a summation of this inequality over all triangles $T\in\mathscr{T}_{k-1}$ gives

$$C_1|v|_{1,2;\Omega}^2 \leqq \sum_{x\in\mathcal{N}_k\setminus\mathcal{N}_{k-1}} |v(x)|^2 \leqq C_2|v|_{1,2;\Omega}^2.$$

This is the proposition for $v\in\mathscr{V}_k$. By construction of the seminorm (2.3) the lemma follows. □

Using Lemma 2.4 and (2.21) we get the first part of Theorem 2.1:

Lemma 2.5. *There exists a positive constant C with*

$$|I_0u|_{1,2;\Omega}^2 + |u|^2 \leqq C(j+1)^2|u|_{1,2;\Omega}^2$$

for all functions $u\in\mathscr{S}_j$. This constant depends only on a lower bound for the interior angles of the triangles under consideration. Especially it is independent of the number j of refinement levels and of the size and shape of the domain Ω.

Proof. By (2.21) we have

$$\begin{aligned}
&|I_0u|_{1,2;\Omega}^2 + \sum_{k=1}^{j} |I_ku - I_{k-1}u|_{1,2;\Omega}^2 \\
&\quad\leqq |I_0u|_{1,2;\Omega}^2 + 2\sum_{k=1}^{j} (|I_ku|_{1,2;\Omega}^2 + |I_{k-1}u|_{1,2;\Omega}^2) \\
&\quad\leqq 4\sum_{k=0}^{j} |I_ku|_{1,2;\Omega}^2 \\
&\quad\leqq 4\sum_{k=0}^{j} C^2(j-k+1)|u|_{1,2;\Omega}^2 \\
&\quad= 2C^2(j+1)(j+2)|u|_{1,2;\Omega}^2.
\end{aligned}$$

By Lemma 2.4 the proposition follows. □

By Lemma 2.3 we can state

Lemma 2.6. *For all functions $u\in\mathscr{S}_j$ one has*

$$\|I_0u\|_{1,2;\Omega}^2 \leqq C(j+1)\{\|u\|_{0,2;\Omega}^2 + H^2|u|_{1,2;\Omega}^2\}$$

where the constant C depends only on a lower bound for the interior angles of the triangles under consideration and H is the maximal diameter of the tr-angles in the initial triangulation.

Lemma 2.5 and 2.6 imply

$$\|I_0u\|_{1,2;\Omega}^2 + |u|^2 \leqq C\left(1+\frac{H^2}{j+1}\right)(j+1)^2\|u\|_{1,2;\Omega}^2 \tag{2.22}$$

and therefore the left hand side estimate in Theorem 2.2.

Slightly weakened versions of the right hand side estimates in Theorem 2.1 and 2.2 can be proved very easily. Using the splitting (2.2), the triangle inequality and the Cauchy-Schwarz inequality for sums we find, for every function $u \in \mathscr{S}_j$,

$$|u|_{1,2;\Omega}^2 \leqq (j+1)\left\{|I_0 u|_{1,2;\Omega}^2 + \sum_{k=1}^{j} |I_k u - I_{k-1} u|_{1,2;\Omega}^2\right\}. \tag{2.23}$$

By Lemma 2.4

$$|u|_{1,2;\Omega}^2 \leqq K(j+1)\{|I_0 u|_{1,2;\Omega}^2 + |u|^2\} \tag{2.24}$$

follows for all functions $u \in \mathscr{S}_j$. In a similar way one can prove the estimate

$$\|u\|_{0,2;\Omega}^2 \leqq (j+1)\{\|I_0 u\|_{0,2;\Omega}^2 + H^2 |u|^2\} \tag{2.25}$$

for all functions $u \in \mathscr{S}_j$. Again H is the maximal diameter of the triangles in the initial triangulation. These considerations show that the desired upper estimates in Theorem 2.1 and 2.2 cannot depend very sensitively on the number j of refinement levels. But to prove that the constants are bounded independently of j requires a more detailed analysis.

The most essential part in this analysis is an orthogonality property of the spaces $\mathscr{V}_k$:

Lemma 2.7. *There is a constant C depending only on a lower bound for the interior angles of the triangles under consideration with*

$$D(u, v) \leqq C \left(\frac{1}{\sqrt{2}}\right)^{|k-l|} |u|_{1,2;\Omega} |v|_{1,2;\Omega}$$

for all functions $u \in \mathscr{V}_k$, $v \in \mathscr{V}_l$. The bilinear form D is defined by

$$D(u, v) = \sum_{i=1}^{2} \int_\Omega D_i u D_i v \, dx$$

and induces the seminorm (2.5).

Proof. We assume $l > k$. We fix a triangle T of the triangulation $\mathscr{T}_k$ and decompose $v \in \mathscr{V}_l$ as the sum $v = v_0 + v_1$ of two functions v_0 and v_1 of $\mathscr{V}_l$ where v_1 agrees with v at all nodes $x \in \mathscr{N}_l$ in the interior of T and v_0 with v at all nodes $x \in \mathscr{N}_l$ on the boundary ∂T of T. As we have $v_1 = 0$ on the boundary of T and u is linear on T we get by partial integration

$$\sum_{i=1}^{2} \int_T D_i u D_i v_1 \, dx = 0$$

or, more appropriately for us,

$$\sum_{i=1}^{2} \int_T D_i u D_i v \, dx = \sum_{i=1}^{2} \int_T D_i u D_i v_0 \, dx.$$

Define Γ as the boundary strip of T consisting of all those triangles of $\mathcal{T}_l$ which are subsets of T and met the boundary of T. As v_0 is identically zero outside Γ the last equality is reduced to

$$\sum_{i=1}^{2} \int_T D_i u D_i v \, dx = \sum_{i=1}^{2} \int_\Gamma D_i u D_i v_0 \, dx.$$

Therefore we get by the Cauchy-Schwarz inequality

$$\sum_{i=1}^{2} \int_T D_i u D_i v \, dx \leqq |u|_{1,2;\Gamma} |v_0|_{1,2;\Gamma}.$$

We analyze the two factors on the right hand side of this inequality separately. Using the same arguments as in the proof of Lemma 2.4 we get with the same constants

$$\begin{aligned} |v_0|_{1,2;T}^2 &\leqq \frac{1}{C_1} \sum_{x \in T \cap \mathcal{N}_l \setminus \mathcal{N}_{l-1}} |v_0(x)|^2 = \frac{1}{C_1} \sum_{x \in \partial T \cap \mathcal{N}_l \setminus \mathcal{N}_{l-1}} |v(x)|^2 \\ &\leqq \frac{1}{C_1} \sum_{x \in T \cap \mathcal{N}_l \setminus \mathcal{N}_{l-1}} |v(x)|^2 \leqq \frac{C_2}{C_1} |v|_{1,2;T}^2. \end{aligned}$$

On the other hand the derivatives of u are constant on T. This gives

$$|u|_{1,2;\Gamma}^2 = \frac{\operatorname{meas}(\Gamma)}{\operatorname{meas}(T)} |u|_{1,2;T}^2.$$

Because of our refinement structure we have

$$\frac{\operatorname{meas}(\Gamma)}{\operatorname{meas}(T)} = 1 - \left(1 - 2\left(\frac{1}{2}\right)^{l-k}\right)^2 \leqq 4 \left(\frac{1}{2}\right)^{l-k}.$$

These considerations result in the estimate

$$\sum_{i=1}^{2} \int_T D_i u D_i v \, dx \leqq C \left(\frac{1}{\sqrt{2}}\right)^{l-k} |u|_{1,2;T} |v|_{1,2;T}$$

with a constant C depending only on a lower bound for the interior angles of the triangle T. The summation of this inequality over all triangles $T \in \mathcal{T}_k$ and the application of the Cauchy-Schwarz inequality to the sum on the right hand side give the proposition. □

Now we are able to complete the proof of Theorem 2.1:

Lemma 2.8. *There is a constant K depending only on a lower bound for the interior angles of the triangles under consideration with*

$$|u|_{1,2;\Omega}^2 \leqq K \{ |I_0 u|_{1,2;\Omega}^2 + |u|^2 \}$$

for all functions $u \in \mathcal{S}_j$.

Proof. With $v_0 = I_0 u$ and $v_k = I_k u - I_{k-1} u$ for $k=1,\dots,j$ we have by Lemma 2.7

$$|u|_{1,2;\Omega}^2 = \left|\sum_{k=0}^{j} v_k\right|_{1,2;\Omega}^2 = \sum_{k,l=0}^{j} D(v_k, v_l) \leqq C \sum_{k,l=0}^{j} \left(\frac{1}{\sqrt{2}}\right)^{|k-l|} |v_k|_{1,2;\Omega} |v_k|_{1,2;\Omega}.$$

If we define the entries a_{kl} of the symmetric matrix A of dimension $j+1$ by

$$a_{kl} = \left(\frac{1}{\sqrt{2}}\right)^{|k-l|}$$

and the components of the vector η of the same dimension by

$$\eta_k = |v_k|_{1,2;\Omega}$$

this inequality is equivalent to

$$|u|_{1,2;\Omega}^2 \leqq C(\eta, A\eta)$$

where the brackets on the right hand side denote the Euclidean inner product. If λ is the largest eigenvalue of A,

$$|u|_{1,2;\Omega}^2 \leqq C\lambda(\eta,\eta) = C\lambda \sum_{k=0}^{j} |v_k|_{1,2;\Omega}^2$$

follows. The largest eigenvalue of A is bounded by the largest row sum of A, and the largest row sum of A by

$$1 + 2\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k = \frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Using Lemma 2.4 the proposition follows. □

The next lemma also completes the proof of Theorem 2.2:

Lemma 2.9. *For all functions $u \in \mathcal{S}_j$ one has*

$$\|u\|_{0,2;\Omega}^2 \leqq 8\{\|I_0 u\|_{0,2;\Omega}^2 + H^2 |u|^2\}$$

where H is the maximal diameter of the triangles in the initial triangulation.

Proof. As in the proof of Lemma 2.8 we set $v_0 = I_0 u$ and $v_k = I_k u - I_{k-1} u$ for $k = 1,\dots,j$. We have

$$\|u\|_{0,2;\Omega} = \left\|\sum_{k=0}^{j} v_k\right\|_{0,2;\Omega} \leqq \|v_0\|_{0,2;\Omega} + \sum_{k=1}^{j} \|v_k\|_{0,2;\Omega}.$$

By the linearity of the function $v_k \in \mathcal{V}_k$ on the triangles of $\mathcal{T}_k$ we get

$$\begin{aligned}\|v_k\|_{0,2;\Omega}^2 &= \sum_{T \in \mathcal{T}_{k-1}} \|v_k\|_{0,2;T}^2 \\ &\leqq \sum_{T \in \mathcal{T}_{k-1}} \operatorname{meas}(T) \sum_{x \in T \cap \mathcal{N}_k \setminus \mathcal{N}_{k-1}} |v_k(x)|^2.\end{aligned}$$

Because of the given refinement structure the areas of all triangles $T \in \mathcal{T}_{k-1}$ are bounded by

$$\operatorname{meas}(T) \leqq \left(\frac{1}{4}\right)^{k-1} \frac{H^2}{2}.$$

As every node $x \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$ is contained in at most two triangles $T \in \mathcal{T}_{k-1}$

$$\|v_k\|_{0,2;\Omega}^2 \leqq (\tfrac{1}{4})^{k-1} H^2 |v_k|^2$$

follows. Inserting this estimate above we get

$$\|u\|_{0,2;\Omega} \leqq \|v_0\|_{0,2;\Omega} + \sum_{k=1}^{j} \left(\frac{1}{2}\right)^k 2H|v_k|.$$

Using the Cauchy-Schwarz inequality the desired result

$$\|u\|_{0,2;\Omega}^2 \leqq 2\left\{\|v_0\|_{0,2;\Omega}^2 + 4H^2 \sum_{k=1}^{j} |v_k|^2\right\}$$

follows. □

3. Nonuniformly Refined Families of Triangulations

In the last section we have introduced discrete norms and seminorms on finite element spaces resulting from a uniform refinement of a given initial triangulation. We have shown how these norms and seminorms can be estimated by the corresponding Sobolev-norms and seminorms associated with second order elliptic boundary value problems and vice versa. That these estimates hold not only on the given finite element spaces but also on all their linear subspaces, is self-evident but nevertheless of great practical importance. For example, one can consider finite element spaces consisting of functions for which the higher level parts are different from zero only in certain subregions. Such spaces can be adapted perfectly to the behaviour of the solution of a given elliptic boundary value problem, but nevertheless they are covered by the theory of the preceding section.

A disadvantage of this straightforward approach is that one has to distinguish free and nonfree nodes, a fact which complicates the algorithmic realisation considerably. Therefore our favourite is a refinement scheme developed by R. Bank and used in his finite element package PLTMG [3]. Bank's program builds up families of finite element spaces which behave, on the one hand, like subspaces of the finite element spaces originating from a uniform refinement of a given initial triangulation and which can be imbedded into these spaces but, on the other hand, are based on conformal triangulations of the given domain. They can easily be adapted to any function to be approximated and are totally sufficient for all practical purposes. The incorporation of this approach into our theory will be described in a subsequent paper.

To demonstrate the flexibility of our theory and its inherent robustness, we discuss here its extension to more complicated families of nested triangulations which are based on very general refinement rules. The resulting finite element spaces are not necessarily related to subspaces of the finite element spaces obtained by a uniform refinement.

As in the last section we start with a given coarse initial triangulation $\mathcal{T}_0$ of the plane polygonal domain Ω under consideration and construct a family of nested triangulations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of Ω. A triangle of $\mathcal{T}_{k+1}$ is either a triangle of the triangulation $\mathcal{T}_k$ to be refined or is generated by subdividing a triangle of $\mathcal{T}_k$ into four congruent subtriangles as shown in Fig. 4a or into two triangles as shown in Fig. 4b.

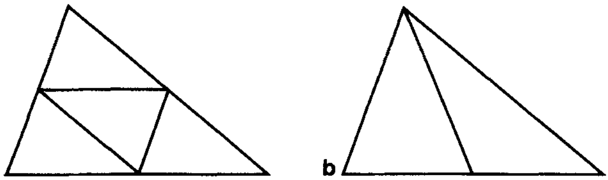

Fig. 4a, b

The two triangles of Fig. 4b are obtained by connecting a given vertex of the original triangle with the midpoint of the opposite side.

The refinement according to Fig. 4b is a potentially dangerous process because the minimal interior angles can decrease rapidly. Therefore one has to take care that the interior angles of the triangles of all levels are bounded away from zero by a constant $\theta > 0$. We claim that this is so.

Using these rules one can construct highly nonuniform triangulations. Consider as an example a square with its initial triangulation $\mathcal{T}_0$ shown in Fig. 5a.

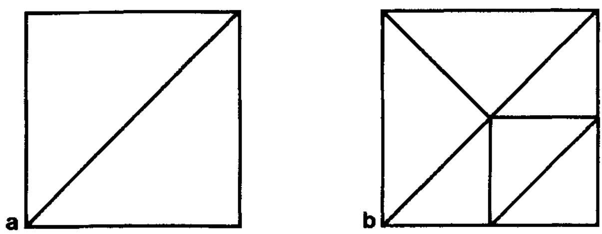

Fig. 5a, b

The first refinement step giving $\mathcal{T}_1$ is shown in Fig. 5b, whereas in the second step giving $\mathcal{T}_2$ all triangles are regularly refined as shown in Fig. 4a. In the next step only the two triangles bottom right in the square are refined as shown in Fig. 5b. The next step is a regular refinement step and so on. The triangulation $\mathcal{T}_7$ consisting of 468 triangles with 266 nodes is shown in Fig. 6. In this example the minimal interior angle never decreases, and the ratio between the diameters of the largest and the smallest triangle of level k behaves like $2^{[(k+1)/2]}$.

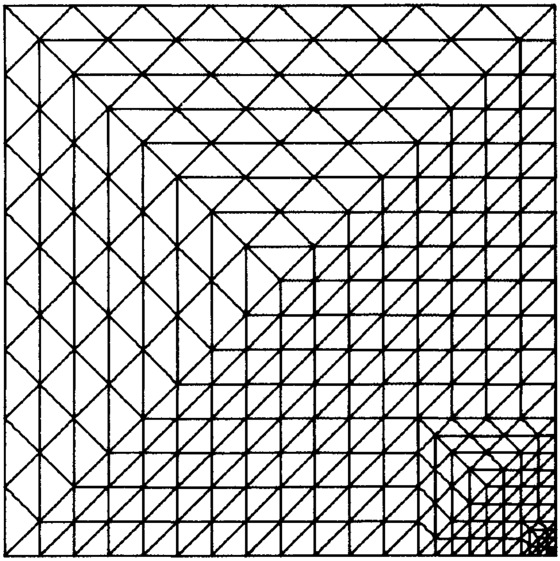

Fig. 6

As in the last section we denote by $\mathcal{N}_k$ the nodes of level k which are the vertices of the triangles in the triangulation $\mathcal{T}_k$. $\mathcal{S}_k$ is the space of the piecewise linear finite element functions corresponding to the triangulation $\mathcal{T}_k$. As before we introduce the finite element interpolation operators I_k; $I_k u$ is the finite element function of level k interpolating u at the nodes of this level. Corresponding to (2.3) we define a seminorm on $\mathcal{S}_j$ by

$$|u|^2 = \sum_{k=1}^{j} \sum_{x \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} |(I_k u - I_{k-1} u)(x)|^2, \qquad u \in \mathcal{S}_j. \tag{3.1}$$

Again $|u|$ is the Euclidean norm of the vector of coefficients of u with respect to the hierarchical basis of $\mathcal{S}_j$ with exception of the coefficients corresponding to the initial level. Our aim is to generalize Theorem 2.1 and 2.2 to the more general refinement structures considered here.

First, we observe that every triangle $T \in \mathcal{T}_k$ is a union of triangles $\mathcal{T}_j$, $j \geqq k$, with diameters h satisfying

$$h \geqq \left(\frac{\sin\theta}{2}\right)^{j-k} \operatorname{diam}(T) \tag{3.2}$$

where $\theta > 0$ is the lower bound for the interior angles introduced above. The sides of T have at least the length $H \sin\theta$, where H denotes the diameter of T, and therefore the triangles obtained by a refinement of T according to Fig. 4a or Fig. 4b have diameters greater than or equal to $H\sin(\theta)/2$. This shows (3.2) for $j = k+1$; the rest follows by induction. Now we can apply Lemma 2.2 to the triangles $T\in\mathscr{T}_k$, $k\leqq j$, and get for all functions $u\in\mathscr{S}_j$

$$|I_k u|_{1,2;T}\leqq C\left(\log\left(\left(\frac{2}{\sin\theta}\right)^{j-k}\right)+\frac{1}{2}\right)^{1/2}|u|_{1,2;T} \tag{3.3}$$

and therefore

$$|I_k u|_{1,2;\Omega}\leqq C(j-k+1)^{1/2}|u|_{1,2;\Omega} \tag{3.4}$$

with constants C depending only on the bound θ. Using an obvious generalization of Lemma 2.4 this results in the estimate

$$|I_0 u|_{1,2;\Omega}^2+|u|^2\leqq C(j+1)^2|u|_{1,2;\Omega}^2 \tag{3.5}$$

for the functions $u\in\mathscr{S}_j$ with a constant C depending only on θ. (3.5) corresponds to Lemma 2.5 and the left hand side estimate in Theorem 2.1. With (3.2) and Lemma 2.3 one gets for $u\in\mathscr{S}_j$

$$\|I_0 u\|_{0,2;\Omega}^2\leqq C(j+1)\{\|u\|_{0,2;\Omega}^2+H^2|u|_{1,2;\Omega}^2\} \tag{3.6}$$

where C depends only on the lower bound θ for the interior angles and H is the maximal diameter of a triangle in the initial triangulation. (3.6) corresponds to Lemma 2.6; with (3.5) and (3.6) the lower estimate in Theorem 2.2 has been generalized.

Using the decomposition

$$u=I_0 u+\sum_{k=1}^{j}(I_k u-I_{k-1}u) \tag{3.7}$$

of the functions $u\in\mathscr{S}_j$ and the same arguments as in the last section one can prove the upper estimates

$$|u|_{1,2;\Omega}^2\leqq K(j+1)\{|I_0 u|_{1,2;\Omega}^2+|u|^2\} \tag{3.8}$$

and

$$\|u\|_{0,2;\Omega}^2\leqq(j+1)\{\|I_0 u\|_{0,2;\Omega}^2+H^2|u|^2\} \tag{3.9}$$

for the functions $u\in\mathscr{S}_j$, where K depends only on θ and H is, as before, an upper bound for the diameters of the triangles in the initial triangulation. On the other hand the proofs of the asymptotically sharper estimates

$$|u|_{1,2;\Omega}^2\leqq K\{|I_0 u|_{1,2;\Omega}^2+|u|^2\}, \tag{3.10}$$

$$\|u\|_{0,2;\Omega}^2\leqq K\{\|I_0 u\|_{0,2;\Omega}^2+H^2|u|^2\} \tag{3.11}$$

for the functions $u\in\mathscr{S}_j$ of Lemma 2.8 and 2.9 depend on an additional property concerning the speed of refinement which cannot be guaranteed for all families of triangulations constructed according to the refinement rules given above.

Therefore we assume that there are positive constants α and $q<1$ such that for all triangles T of any level k and all levels $j\geqq k$ the triangles of level j, the union of which is T, have diameters h satisfying

$$h\leqq\alpha q^{j-k}\operatorname{diam}(T). \tag{3.12}$$

If one refines the triangles uniformly as in the last section, one has $\alpha=1$ and $q=1/2$, whereas in the example given above (3.12) holds because of

$$\left(\frac{1}{2}\right)^{\left[\frac{j-k}{2}\right]} \leqq \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{j-k}$$

with $\alpha=\sqrt{2}$ and $q=1/\sqrt{2}$. Our assumption can be deduced from very weak approximation properties of the finite element spaces and therefore it should be fulfilled in all practical applications.

Based on the assumption (3.12) an orthonality property corresponding to Lemma 2.7, which results in (3.10), and the estimate (3.11) can be proved. These are the last steps in the generalization of Theorem 2.1 and 2.2 to the situation considered here.

4. The Solution of Positive Definite and Symmetric Boundary Value Problems Using Hierarchical Bases

As in the preceding sections let $\bar{\Omega}$ be a polygonal but not necessarily convex region in the plane. By $H(\Omega)$ we denote either the Hilbert-space $W^{1,2}(\Omega)$ or a linear subspace of $W^{1,2}(\Omega)$ consisting of all functions in $W^{1,2}(\Omega)$ vanishing on a given boundary piece Γ of Ω in the sense of the trace operator. $H(\Omega)$ is the solution space of the elliptic boundary value problem to be solved. As usual in a finite element context this boundary value problem is given by a bilinear form B on $H(\Omega)$. Here we assume that the bilinear form B is positive definite and symmetric. B induces the energy norm

$$\|u\| = (B(u,u))^{1/2} \tag{4.1}$$

on $H(\Omega)$, and we assume that there are positive constants M and δ with

$$\delta |u|_{1,2;\Omega}^2 \leqq \|u\|^2 \leqq M |u|_{1,2;\Omega}^2 \tag{4.2}$$

for all functions $u \in H(\Omega)$, or with

$$\delta \|u\|_{1,2;\Omega}^2 \leqq \|u\|^2 \leqq M \|u\|_{1,2;\Omega}^2 \tag{4.3}$$

for all these functions. Our aim is to find a function $u \in H(\Omega)$ satisfying

$$B(u,v) = f^*(v), \qquad v \in H(\Omega), \tag{4.4}$$

where f^* is a given bounded linear functional on $H(\Omega)$.

A simple example is the boundary value problem

$$-\Delta u = f \quad \text{on } \Omega, \qquad u=0 \quad \text{on } \partial\Omega \tag{4.5}$$

for the Laplace-equation. Here the solution space $H(\Omega)$ is $W_0^{1,2}(\Omega)$, and the bilinear form B is given by

$$B(u,v) = \sum_{i=1}^{2} \int_{\Omega} D_i u D_i v \, dx. \tag{4.6}$$

(4.2) holds with $\delta=1$ and $M=1$. Another example is the boundary value problem

$$-\Delta u+u=f \quad \text{on } \Omega, \qquad \frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega. \tag{4.7}$$

In this case the solution space $H(\Omega)$ is the space $W^{1,2}(\Omega)$ itself, B is given by

$$B(u,v)=\int_{\Omega}\left(\sum_{i=1}^{2} D_i u D_i v+uv\right)dx, \tag{4.8}$$

and (4.3) holds with the same constants $\delta=1$ and $M=1$.

We want to solve the boundary value problem (4.4) by the finite element method using a family of finite element spaces as in Sects. 2 or 3. For this purpose we assume that the boundary piece Γ in the definition of $H(\Omega)$ is the union of certain sides of triangles of the initial triangulation $\mathcal{T}_0$. This implies that for every level k $\mathcal{S}_k \cap H(\Omega)$ consists of the finite element functions in $\mathcal{S}_k$ vanishing at the level k nodes contained in Γ. For $u\in\mathcal{S}_j\cap H(\Omega)$ one has $I_k u\in\mathcal{S}_k\cap H(\Omega)$, and the hierarchical basis functions of $\mathcal{S}_j$ vanishing on Γ form a basis of the space $\mathcal{S}_j\cap H(\Omega)$, its hierarchical basis. We formulate the discretized boundary value problem with respect to this hierarchical basis.

Using the discrete seminorm (2.3), (3.1) and the energy norm (4.1) we define the discrete norm

$$|||u|||^2=\|I_0 u\|^2+|u|^2, \qquad u\in\mathcal{S}_j\cap H(\Omega). \tag{4.9}$$

One of the main results of this paper is the following theorem which is an immediate consequence of Theorem 2.1 or Theorem 2.2 and the generalization of these two theorems to the nonuniformly refined families of grids considered in Sect. 3.

Theorem 4.1. *There exist positive constants K_1 and K_2, which are independent of the number j of refinement levels, with*

$$\frac{K_1}{(j+1)^2}\,|||u|||^2 \leqq \|u\|^2 \leqq K_2\,|||u|||^2$$

for all functions $u\in\mathcal{S}_j\cap H(\Omega)$. K_1 depends only on the constants M and δ in (4.2) *and* (4.3) *respectively, on a lower bound for the interior angles of the triangles in the final triangulation and possibly on the diameter of the domain Ω. For a nonuniformly refined family of nested triangulations as described in Sect. 3 K_2 depends in addition on the constants α and q from* (3.12).

In particular the constants in Theorem 4.1 are independent of the regularity of the boundary value problem to be solved, and the theorem holds for very general classes of triangulations which do not need to be uniform.

The norm (4.9) is induced by an inner product. By A_0 we denote the Grammian matrix given by this inner product and the hierarchical basis of $\mathcal{S}_j\cap H(\Omega)$. Essentially A_0 is the nodal basis discretization matrix of the boundary value problem with respect to the finite element space $\mathcal{S}_0\cap H(\Omega)$. Let A be the discretization matrix of the boundary value problem with respect to the hierarchical basis of $\mathscr{S}_j \cap H(\Omega)$. With this notation Theorem 4.1 says that for all coefficient vectors x one has

$$\frac{K_1}{(j+1)^2}(x, A_0 x) \leqq (x, Ax) \leqq K_2(x, A_0 x) \tag{4.10}$$

where the brackets denote the Euclidean inner product. Due to the usually comparatively small dimension of $\mathscr{S}_0 \cap H(\Omega)$ it is cheap to compute the Cholesky-decomposition LL^T of A_0, and (4.10) is equivalent to the fact that the spectral condition number of the matrix

$$L^{-1} A L^{-T} \tag{4.11}$$

is bounded by

$$\kappa(L^{-1} A L^{-T}) \leqq \frac{K_2}{K_1}(j+1)^2. \tag{4.12}$$

The two factors L^{-1} and L^{-T} are not very essential, their main task is to break the influence of the geometry of the initial triangulation. As the low-dimensional initial space $\mathscr{S}_0 \cap H(\Omega)$ is fixed independent of the number j of refinement levels, the spectral condition number of the hierarchical basis discretization matrix A itself grows only like $O(j^2)$. This is a surprising result because it is well known that the spectral condition number of the usual nodal basis discretization matrix $\hat{A}$ grows exponentially with the number j of refinement levels.

The nice mathematical properties of the hierarchical basis discretization matrix A seem to be compensated by its rather complicated structure and its many non-zero entries, at least compared with the nodal basis discretization matrix $\hat{A}$. Nevertheless it is possible to develop cheap and effective algorithms for computing the product Ax of A with any coefficient vector x. This is sufficient for many purposes, for example for the application of conjugate gradient type methods. Let S be the matrix which transforms the representations of the finite element functions of $\mathscr{S}_j \cap H(\Omega)$ with respect to the hierarchical basis of this space into their representations with respect to the usual nodal basis. Then one has for all coefficient vectors x and y

$$(x, Ay) = (Sx, \hat{A}Sy) = (x, S^T \hat{A} S y) \tag{4.13}$$

and therefore the representation

$$A = S^T \hat{A} S \tag{4.14}$$

of the hierarchical basis discretization matrix A. Our idea is to make use of this factorization for computing Ax, so that we have to provide fast and stable algorithms for computing Sx and $S^T x$.

The computation of the product of the matrix S with a vector means evaluating a finite element function $u \in \mathscr{S}_j \cap H(\Omega)$, which is given by its coefficients with respect to the hierarchical basis of this space, at the nodal points $x \in \mathscr{N}_j \backslash \Gamma$. This can be done recursively beginning with the nodes of level 0. At these nodes $x \in \mathscr{N}_0 \backslash \Gamma$ the values of u are given by the corresponding hierarchical basis coefficients because the higher level basis functions vanish there. If the values at the nodes $x \in \mathcal{N}_{k-1} \setminus \Gamma$ are already known, one first evaluates the function $I_{k-1}u$ at the nodes $x \in \mathcal{N}_k \setminus (\mathcal{N}_{k-1} \cap \Gamma)$. Due to the construction of our triangulations this is a simple interpolation process. Then one adds the values of $I_k u - I_{k-1} u$ at these nodes, which are stored in the hierarchical basis coefficient vector, to these interpolated values giving the values of u also at the nodes $x \in \mathcal{N}_k \setminus (\mathcal{N}_{k-1} \cap \Gamma)$. After this step has been completed, the values of u at the nodes $x \in \mathcal{N}_k \setminus \Gamma$ are known and one can continue with the next refinement level.

This algorithm can be implemented very easily on a computer. Assume that $x_{I1(i)}, x_{I2(i)} \in \mathcal{N}_{k-1}$ are the neighbours of the node $x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$ as shown in Fig. 7.

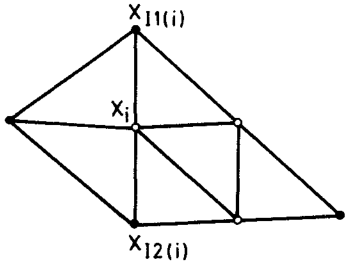

Fig. 7

x_i is the midpoint of a side of a triangle of level $k-1$ with endpoints $x_{I1(i)}$ and $x_{I2(i)}$. Denote by $\mathcal{M}_k$ the set of the indizes i of the nodes $x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$. Assume that the indices $I1(i)$ and $I2(i)$ are stored in the arrays $I1$ and $I2$ and that at the beginning of the computation the entry $X(i)$ of the array X contains the hierarchical basis coefficient of a given function u corresponding to the node with index i. Then the following algorithm, which is described in a simple pseudo-programming language, returns the value of the function u in the gridpoint with index i in the entry $X(i)$ of the array X:

for $k=1$ **to** j (4.15)
  for $i \in \mathcal{M}_k$
    $X(i) = X(i) + (X(I1(i)) + X(I2(i)))/2$
  next i
next k.

The outer loop is a loop on the refinement levels whereas the inner loop performs the operations which are necessary to get the values in the new nodes of level k, if the values in the nodes of level $k-1$ are already known and stored in the corresponding entries of the array X. Note that no additional storage for the new coefficient vector is needed and that the whole algorithm requires less than $2n$ additions and n divisions by 2 where n is the dimension of the finite element space.

Mathematically the given algorithm for computing Sx corresponds to a factorization

$$S = S_j S_{j-1} \dots S_1 \tag{4.16}$$

of the matrix S. The sparse matrix S_k describes the evaluation of the values at the new nodes of level k, if the values at the nodes of level $k-1$ are already known. S_k is the identity matrix with exception of the entries

$$S_k|_{i,I1(i)}=\tfrac{1}{2}, \quad S_k|_{i,I2(i)}=\tfrac{1}{2} \tag{4.17}$$

in the i-th row and columns $I1(i)$ and $I2(i)$ for $i\in\mathcal{M}_k$. The interior loop in the algorithm (4.15) means multiplication by the matrix S_k. (4.16) leads to the representation

$$S^T=S_1^T\ldots S_{j-1}^T S_j^T \tag{4.18}$$

of S^T and means that the computation of S^Tx is as cheap and as simple as the computation of Sx.

Using the notation introduced above S^Tx is computed by the following algorithm:

for $k=j$ **down to** 1 (4.19)
  for $i\in\mathcal{M}_k$
    $X(I1(i))=X(I1(i))+X(i)/2$
    $X(I2(i))=X(I2(i))+X(i)/2$
  next i
next k.

For different purposes the computation of $S^{-1}x$ and $S^{-T}x$ may also be necessary. Using the factorization

$$S^{-1}=S_1^{-1}\ldots S_{j-1}^{-1}S_j^{-1} \tag{4.20}$$

one gets the algorithm

for $k=j$ **down to** 1 (4.21)
  for $i\in\mathcal{M}_k$
    $X(i)=X(i)-(X(I1(i))+X(I2(i)))/2$
  next i
next k

for computing $S^{-1}x$, and using

$$S^{-T}=S_j^{-T}S_{j-1}^{-T}\ldots S_1^{-T} \tag{4.22}$$

the algorithm

for $k=1$ **to** j (4.23)
  for $i\in\mathcal{M}_k$
    $X(I1(i))=X(I1(i))-X(i)/2$
    $X(I2(i))=X(I2(i))-X(i)/2$
  next i
next k

for computing $S^{-T}x$.

We conclude that the computation of the product Ax of the hierarchical basis discretization matrix with a given vector x is not much more expensive than the computation of the product $\hat{A}x$ of the corresponding modal basis discretization matrix $\hat{A}$ with x, if we make use of the factorization (4.14) and the algorithms described above. For linear finite elements the additional amount of work consists of less than $4n$ additions and $2n$ divisions by 2 where n is the dimension of the finite element space, and, as additional storage, two integer arrays of a dimension less than n, containing the necessary information on the refinement process, are needed. Now we have all the tools at hand for constructing a fast algorithm for the solution of the discretized boundary value problem.

We want to compute the solution vector x of the linear system

$$\hat{A}x=\hat{b}. \tag{4.24}$$

As above, $\hat{A}$ is the nodal basis discretization matrix, $\hat{b}$ is the right hand side corresponding to the linear functional f^* in (4.4) and the nodal basis, and the vector x represents the values of the finite element solution of (4.4) in the nodal points. Instead of attacking the original system (4.24) we solve the transformed system

$$L^{-1}AL^{-T}y=L^{-1}S^T\hat{b} \tag{4.25}$$

or the slightly simplified system

$$Ay=S^T\hat{b} \tag{4.26}$$

by the conjugate gradient method; $A=S^T\hat{A}S$ is the hierarchical basis discretization matrix. The nodal basis solution vector $x=SL^{-T}y$ and $x=Sy$, respectively, can be obtained very easily by another application of the matrix S.

The speed of convergence of the conjugate gradient method is closely connected with the spectral condition number κ of the linear system to be solved. As shown in [1], k steps of the conjugate gradient method reduce the energy norm of the error, which is induced by the coefficient matrix, at least by the factor

$$\frac{2}{\left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^k+\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k} \tag{4.27}$$

if $\kappa>1$; if one has $\kappa=1$, the coefficient matrix is the identity matrix and the exact solution is obtained in the first step. Therefore one needs at most

$$k\geqq\frac{1}{2}\sqrt{\kappa}\left|\log\left(\frac{\varepsilon}{2}\right)\right| \tag{4.28}$$

steps to reduce the energy norm of the initial error by the factor $\varepsilon\in(0,1)$. In our application the condition number of the coefficient matrix grows quadratically with the number j of refinement levels. So we get the upper bound

$$C_1 j \left|\log\left(\frac{\varepsilon}{2}\right)\right| \tag{4.29}$$

for the number of conjugate gradient steps necessary to reduce the energy norm (4.1) of a given initial error by the factor ε. Besides some vector operations one step of the conjugate gradient method requires one multiplication of a vector with the coefficient matrix of the linear system. Therefore in our application the amount of work necessary to perform one conjugate gradient step is proportional to the dimension n of the linear system: we get an upper bound

$$C_2 n j \left|\log\left(\frac{\varepsilon}{2}\right)\right| \tag{4.30}$$

for the number of computer operations necessary to reduce the energy norm of the initial error by a factor ε. In the case of a uniform refinement, as in Sect. 2, the number of unknowns is nearly quadrupled in every refinement step, and also in the case of the nonuniform triangulations considered in Sect. 3 one can show by (3.12) that the number n of the unknowns grows exponentially with the number j of the refinement levels. Therefore one has

$$j \leqq C_3 \log(n), \tag{4.31}$$

and we can present the upper bound

$$C n \log(n) \left|\log\left(\frac{\varepsilon}{2}\right)\right| \tag{4.32}$$

for the number of computer operations necessary to reduce the energy norm (4.1) of a given initial error by the factor $\varepsilon \in (0,1)$. (4.32) states that our algorithm has a nearly optimal computational complexity, and this complexity is independent of any regularity properties of the boundary value problem or the quasiuniformity of the family of triangulations.

5. On the Optimality of the Estimates

We take the triangle T with the vertices $(0,0)$, $(0,1)$ and $(1,1)$ as the domain Ω in Sect. 2. By j regular refinement steps we subdivide the triangle into 4^j congruent subtriangles giving the triangulation $\mathcal{T}_j$. Let $u \in \mathcal{S}_j$ be the piecewise linear function defined by

$$u(x,y) = \begin{cases} (2-2^k x)k + (2^k x - 1)(k-1), & 2^{-k} \leqq x \leqq 2^{-(k-1)} \\ j, & 0 \leqq x \leqq 2^{-j} \end{cases} \tag{5.1}$$

where $k = 1, 2, \ldots, j$. An easy computation gives

$$|u|^2_{1,2;T} = \tfrac{3}{2} j. \tag{5.2}$$

On the other hand we have

$$|u|^2 \geqq \sum_{k=1}^{j} \{|(I_k u - I_{k-1} u)(2^{-k}, 0)|^2 + |(I_k u - I_{k-1} u)(2^{-k}, 2^{-k})|^2\}$$
$$= \frac{(j-2)(j-1)(2j+1)}{12} + \frac{1}{2}. \tag{5.3}$$

Similar simple examples show:

Theorem 5.1. *Theorem* 2.1 *and* 2.2 *are sharp in the sense that there exists a positive constant c with*

$$\Psi(j) \geqq c j^2, \quad j = 1, 2, 3, \ldots,$$

for all functions Ψ *with*

$$|I_0 u|_{1,2;\Omega}^2 + |u|^2 \leqq \Psi(j) |u|_{1,2;\Omega}^2, \quad u \in \mathcal{S}_j,$$

or even with

$$|u|^2 \leqq \Psi(j) \|u\|_{1,2;\Omega}^2, \quad u \in \mathcal{S}_j.$$

Therefore the condition numbers of finite element discretization matrices of plane second order elliptic boundary value problems with respect to hierarchical bases cannot be expected to grow slower than $O(j^2)$ with the number j of refinement levels. More generally we now prove that the relative condition number of the bilinear form B defining the boundary value problem with respect to any other bilinear form decoupling the different refinement levels grows at least as $O(j^2)$.

Let the symmetric bilinear form B on the solution space $H(\Omega)$ satisfy (4.2) or (4.3), and let $\tilde{B}$ be any other positive definite and symmetric bilinear form on the discrete solution space $\mathcal{S}_j \cap H(\Omega)$ satisfying

$$\tilde{B}(v_k, v_l) = 0 \tag{5.4}$$

for all functions $v_k \in \mathcal{V}_k \cap H(\Omega)$, $v_l \in \mathcal{V}_l \cap H(\Omega)$ with $k \neq l$ and $0 \leqq k, l \leqq j$. Let $\tilde{\sigma}_1(j)$ be the largest and $\tilde{\sigma}_2(j)$ be the smallest real number with

$$\tilde{\sigma}_1(j) \tilde{B}(u,u) \leqq B(u,u) \leqq \tilde{\sigma}_2(j) \tilde{B}(u,u) \tag{5.5}$$

for all functions $u \in \mathcal{S}_j \cap H(\Omega)$. Then one has

Theorem 5.2. *The relative condition number* $\tilde{\kappa}(j)$ *of the bilinear form* B *with respect to the bilinear form* $\tilde{B}$*, which is defined by*

$$\tilde{\kappa}(j) = \frac{\tilde{\sigma}_2(j)}{\tilde{\sigma}_1(j)},$$

grows at least as $O(j^2)$ *with the number* j *of refinement levels.*

We prove Theorem 5.2 by comparing B and $\tilde{B}$ with a standardized bilinear form $\hat{B}$ on $\mathcal{S}_j \cap H(\Omega)$ which is defined by

$$\hat{B}(u,v) = B(I_0 u, I_0 v) + \sum_{k=1}^{j} B(I_k u - I_{k-1} u, I_k v - I_{k-1} v). \tag{5.6}$$

Let $\sigma_1(j)$ be the largest and $\sigma_2(j)$ be the smallest real number with

$$\sigma_1(j)\hat{B}(u,u) \leqq B(u,u) \leqq \sigma_2(j)\hat{B}(u,u) \tag{5.7}$$

for all functions $u \in \mathscr{S}_j \cap H(\Omega)$. Using Lemma 2.4 and Theorem 4.1 one can see that the relative condition number

$$\kappa(j) = \frac{\sigma_2(j)}{\sigma_1(j)} \tag{5.8}$$

does not grow faster than $O(j^2)$; there are positive constants K_1 and K_2 independent of j with

$$\frac{K_1}{(j+1)^2} \leqq \sigma_1(j), \qquad \sigma_2(j) \leqq K_2. \tag{5.9}$$

On the other hand, using Lemma 2.4 and a similar example as in the proof of Theorem 5.1 one gets

Lemma 5.1. *There exists a positive constant K with*

$$\sigma_1(j) \leqq \frac{K}{j^2}$$

for $j = 1, 2, 3, \ldots$.

This proves Theorem 5.2 because one has the following purely algebraic lemma which is a simple generalization of a well-known result concerning the optimal scaling of positive definite and symmetric matrices:

Lemma 5.2. *The relative condition number $\tilde{\kappa}(j)$ of the bilinear form B with respect to the bilinear form $\tilde{B}$ is bounded from below by*

$$\tilde{\kappa}(j) \geqq \frac{1}{\sigma_1(j)}.$$

Proof. For all functions $v, w \in \mathscr{S}_j \cap H(\Omega)$ with $B(v,v) \neq 0$, $B(w,w) \neq 0$ one has

$$\tilde{\sigma}_1(j) \leqq \frac{B(w,w)}{\tilde{B}(w,w)}, \qquad \frac{B(v,v)}{\tilde{B}(v,v)} \leqq \tilde{\sigma}_2(j)$$

and therefore the lower estimate

$$\tilde{\kappa}(j) \geqq \frac{\tilde{B}(w,w)}{\tilde{B}(v,v)} \frac{B(v,v)}{B(w,w)}$$

for the relative condition number $\tilde{\kappa}(j)$. So we can prove the proposition by constructing functions $v^*, w^* \in \mathscr{S}_j \cap H(\Omega)$ with

$$B(v^*, v^*) = 1, \tag{5.10}$$

$$B(w^*, w^*) = \sigma_1(j) \tag{5.11}$$

and

$$\tilde{B}(v^*, v^*) \leqq \tilde{B}(w^*, w^*). \tag{5.12}$$

For this purpose let $\mathscr{K}$ be the compact set of all functions $v \in \mathscr{S}_j \cap H(\Omega)$ with $\hat{B}(v,v)=1$. Let $w^* \in \mathscr{K}$ be a function with

$$B(w^*, w^*) \leqq B(w,w), \qquad w \in \mathscr{K}.$$

This implies (5.11). Let $v_k^* \in \mathscr{V}_k \cap \mathscr{K}$, $k=0,1,\ldots,j$, be functions with

$$\tilde{B}(v_k^*, v_k^*) \leqq \tilde{B}(v_k, v_k), \qquad v_k \in \mathscr{V}_k \cap \mathscr{K}.$$

Define $v^* \in \mathscr{K}$ as the function v_k^* minimizing $\tilde{B}(v_k^*, v_k^*)$. Then (5.10) holds by the definition of $\hat{B}$. As the spaces $\mathscr{V}_k \cap H(\Omega)$ are pairwise orthogonal with respect to the bilinear forms $\tilde{B}$ and $\hat{B}$ one gets

$$\tilde{B}(v^*, v^*) \leqq \tilde{B}(v,v), \qquad v \in \mathscr{K}.$$

This implies (5.12) and proves the lemma. □

6. Some Numerical Results

Beginning with the triangulation $\mathscr{T}_0$ shown in Fig. 8a we subdivide the unit square $\bar{\Omega}$ by j regular refinement steps as described in Sect. 2 into $2 \cdot 4^j$ finite elements giving the triangulation $\mathscr{T}_j$; for $j=2$ the result is shown in Fig. 8b.

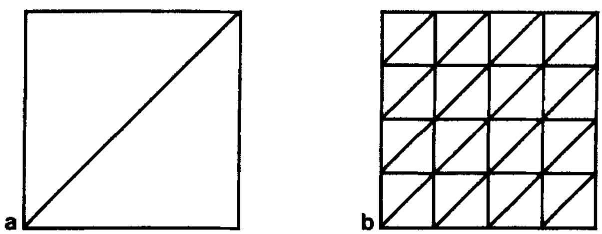

Fig. 8a, b

Our solution space $H(\Omega)$ is the space $W_0^{1,2}(\Omega)$. This means that the corresponding finite element space $\mathscr{S}_j \cap H(\Omega)$ consists of the functions of $\mathscr{S}_j$ vanishing on the boundary of the unit square. The bilinear form defining the boundary value problem is given by

$$B(u,v) = \sum_{i=1}^{2} \int_{\Omega} D_i u D_i v \, dx. \tag{6.1}$$

If one uses the standard nodal basis this situation results in a scaled five-point star discretization of the negative Laplacian.

In our case the nodes corresponding to the initial triangulation are points of the boundary. Therefore the functions in $\mathscr{S}_0 \cap H(\Omega)$ are identically zero and the matrix A_0 in (4.10) is the identity matrix. To check the theorems of the preceding sections and to get some feeling for the size of the constants we need

Table 1. The smallest and the largest eigenvalues of the hierarchical basis discretization matrices

j	Smallest eigenvalue	Largest eigenvalue	Gridpoints
3	0.820418	8.6914	9×9
4	0.534491	10.4364	17×17
5	0.378325	12.0480	33×33
6	0.282068	13.2975	65×65
7	0.218364	14.2770	129×129
8	0.173996	15.0530	257×257
9	0.141860	15.6741	513×513

Table 2. The condition numbers of the hierarchical basis discretization matrices

j	3	4	5	6	7	8	9
$\kappa(j)$	10.59	19.53	31.85	47.14	65.38	86.51	110.49

the smallest and the largest eigenvalues of the hierarchical basis discretization matrices.

Some of these eigenvalues have been computed numerically and are listed in Table 1.

One can see that the small eigenvalues decrease only very slowly, a nice confirmation of our theory. The large eigenvalues, which are bounded uniformly in j, increase unfortunately as fast as they are allowed the proof of Lemma 2.8. They and not the small eigenvalues are the dominating factors for the size of the spectral condition numbers $\kappa(j)$ shown in Table 2.

The values given by Table 1 and 2 contain not only information about our model problem but also about problems which use certain subspaces of our finite element spaces for the unit square as discrete solution spaces. As examples take the Dirichlet problems for the Laplace equation on the L-shaped domain of Fig. 9a and the slit domain of Fig. 9b.

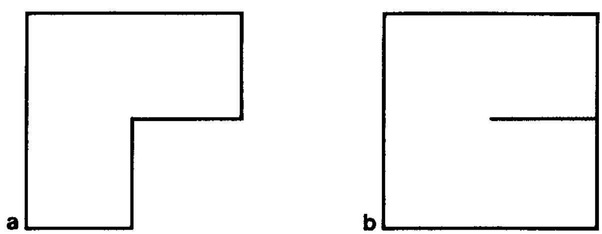

Fig. 9a, b

Because of their lack of regularity such problems are generally thought to be difficult for multi-level iterative methods. But if one uses as discrete solution spaces finite element spaces with the same uniform gridsizes as for the unit square, the hierarchical bases of these spaces are subsets of the hierarchical bases corresponding to the unit square. Therefore the smallest and the largest eigenvalue of the discretization matrices corresponding to these hierarchical bases are bounded from below and above, respectively, by the smallest and the largest eigenvalue for the unit square. This means that the condition numbers of the discretization matrices decrease although the problems become more difficult.

By formula (4.27) the number of steps, which are needed by the conjugate gradient method to reduce the energy norm of an initial error by the factor ε when using hierarchical bases, is at most the smallest integer greater than or equal to

$$c(j)\left|\log\left(\frac{\varepsilon}{2}\right)\right| \tag{6.2}$$

where the constant $c(j)$ is given by

$$\frac{1}{c(j)}=\log\left(\frac{\sqrt{\kappa(j)}+1}{\sqrt{\kappa(j)}-1}\right); \tag{6.3}$$

this is a slightly sharper estimate than (4.28). The computed eigenvalues from Table 1 lead to the values of $c(j)$ given in Table 3.

Table 3. The constants $c(j)$ given by (6.3)

j	3	4	5	6	7	8	9
$c(j)$	1.575	2.171	2.792	3.409	4.022	4.633	5.240

The linear polynomial fitting these computed values of $c(j)$ optimally in the sense of least squares is

$$c(j)\approx 0.6124j-0.2688. \tag{6.4}$$

The deviation of the values (6.4) from the computed values of $c(j)$ of Table 3 is very small as can be seen in Fig. 10. The upper bound for the number of conjugate gradient steps needed to reduce the energy norm of the error by a given factor grows linearly with the number j of refinement levels as one can expect by formula (4.29).

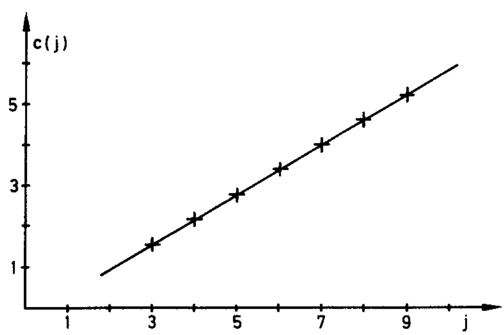

Fig. 10

As a final example let us consider the same boundary value problem as above but the nonuniform triangulation given by Fig. 6. The smallest triangle in this triangulation is of the same size as for a uniform grid with gridsize $h = 1/128$. The minimal eigenvalue of the hierarchical basis discretization matrix is 0.599... and the maximal eigenvalue 10.22... resulting in the relatively small condition number 17.05. The conjugate gradient method seems to take additional advantage of the eigenvalue structure. The convergence history for a point source in the only interior gridpoint of level 7 with $u=0$ as initial approximation of the finite element solution is shown in Fig. 11. R is the Euclidean norm of the actual residual, which is by our theory a good measure for the energy norm of the error, and $R\emptyset$ is the Euclidean norm of the initial residual.

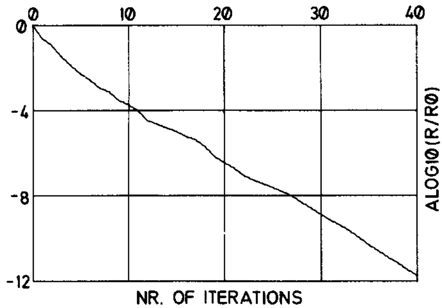

Fig. 11

Appendix

In Sect. 2 we derived upper bounds for the norms of certain finite element interpolation operators. These estimates are of crucial importance for our theory and depend mainly on the estimate (2.8). An elementary proof of this estimate is given in [15], but for the completeness of our exposition we add this proof here.

For convenience let $K(0,r)$ be the circle of radius $r>0$ with the origin as center.

Lemma 1. *Let u: $K(0,R)\to\mathbb{R}$ be a continuously differentiable function vanishing on the boundary of the circle $K(0,R)$. Let f: $\mathbb{R}_+\to\mathbb{R}_+$ be continuous and F: $\mathbb{R}_+\to\mathbb{R}$ be defined by*

$$F(r)=\int_0^r t f(t)\,dt.$$

Then we have

$$\left|\int_{K(0,R)} f(\sqrt{x^2+y^2})\,u(x,y)\,d(x,y)\right| \leqq \sqrt{2\pi}\left(\int_0^R \frac{F^2(r)}{r}\,dr\right)^{1/2} |u|_{1,2;K(0,R)}.$$

Proof. Using polar-coordinates we get

$$\int_{K(0,R)} f(\sqrt{x^2+y^2})\,u(x,y)\,d(x,y)=\int_0^R r f(r)\int_0^{2\pi} u(r\cos\varphi, r\sin\varphi)\,d\varphi\,dr.$$

Under the given assumptions the exterior integral on the right hand side of this equality can be transformed by partial integration giving

$$\begin{aligned}\int_{K(0,R)} & f(\sqrt{x^2+y^2})u(x,y)\,d(x,y)\\ &= -\int_0^R F(r)\int_0^{2\pi}(D_1u)(r\cos\varphi, r\sin\varphi)\cos\varphi\,d\varphi\,dr\\ &\quad -\int_0^R F(r)\int_0^{2\pi}(D_2u)(r\cos\varphi, r\sin\varphi)\sin\varphi\,d\varphi\,dr.\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the interior integrals on the right hand side

$$\begin{aligned}\Big|\int_{K(0,R)} & f(\sqrt{x^2+y^2})u(x,y)\,d(x,y)\Big|\\ &\leqq \sqrt{\pi}\int_0^R |F(r)|\left(\int_0^{2\pi}(D_1u)^2(r\cos\varphi, r\sin\varphi)\,d\varphi\right)^{1/2} dr\\ &\quad +\sqrt{\pi}\int_0^R |F(r)|\left(\int_0^{2\pi}(D_2u)^2(r\cos\varphi, r\sin\varphi)\,d\varphi\right)^{1/2} dr\end{aligned}$$

follows. If we apply the Cauchy-Schwarz inequality to the exterior integrals we get

$$\begin{aligned}\Big|\int_{K(0,R)} & f(\sqrt{x^2+y^2})u(x,y)\,d(x,y)\Big|\\ &\leqq \sqrt{\pi}\left(\int_0^R \frac{F^2(r)}{r}\,dr\right)^{1/2}\left(\int_0^R r\int_0^{2\pi}(D_1u)^2(r\cos\varphi, r\sin\varphi)\,d\varphi\,dr\right)^{1/2}\\ &\quad +\sqrt{\pi}\left(\int_0^R \frac{F^2(r)}{r}\,dr\right)^{1/2}\left(\int_0^R r\int_0^{2\pi}(D_2u)^2(r\cos\varphi, r\sin\varphi)\,d\varphi\,dr\right)^{1/2}.\end{aligned}$$

If we use the elementary inequality

$$a+b\leqq\sqrt{2}\sqrt{a^2+b^2}$$

and transform back to Cartesian coordinates we get the proposition. □

The next step is to improve the estimate of Lemma 1 for the weighted mean value of the function u to an estimate for the weighted mean value of the absolute value of u.

Lemma 2. *Under the assumptions of Lemma 1 the following estimate holds:*

$$\int_{K(0,R)} f(\sqrt{x^2+y^2})|u(x,y)|\,d(x,y)\leqq\sqrt{2\pi}\left(\int_0^R \frac{F^2(r)}{r}\,dr\right)^{1/2}|u|_{1,2;K(0,R)}.$$

Proof. As main tool we use the continuously differentiable functions

$$B_n(t)=(t^2+n^{-2})^{1/2}-n^{-1},\qquad n\in\mathbb{N},$$

which are defined for all real t. Because

$$|B_n(t)-|t||\leqq 1/n$$

and using the triangle inequality we get

$$\int_{K(0,R)} f(\sqrt{x^2+y^2})|u(x,y)|d(x,y)$$
$$\leqq \int_{K(0,R)} f(\sqrt{x^2+y^2})B_n(u(x,y))d(x,y)+\frac{1}{n}\int_{K(0,R)} |f(\sqrt{x^2+y^2})|d(x,y),$$

for all n. The functions $B_n(u(x,y))$ are continuously differentiable and vanish because $B_n(0)=0$ on the boundary of the circle $K(0,R)$. Therefore we can apply Lemma 1 to the first integral on the right hand side. We get

$$\int_{K(0,R)} f(\sqrt{x^2+y^2})B_n(u(x,y))d(x,y)\leqq\sqrt{2\pi}\left(\int_0^R \frac{F^2(r)}{r}\,dr\right)^{1/2}|B_n\circ u|_{1,2;K(0,R)}$$

with

$$|B_n\circ u|^2_{1,2;K(0,R)}=\int_{K(0,R)} |B_n'(u(x,y))|^2((D_1u)^2(x,y)+(D_2u)^2(x,y))d(x,y).$$

Using $|B_n'(t)|\leqq 1$ we have the assertion. □

Approximating the discontinuous weight function

$$f(r)=\begin{cases}\dfrac{1}{\pi\sigma^2}, & r\leqq\sigma\\ 0 & r>\sigma\end{cases}$$

by continuous, for example piecewise linear functions, and using Lemma 2 we get the following estimate.

Lemma 3. *Let $u\colon K(0,R)\to\mathbb{R}$ be a continuously differentiable function vanishing on the boundary of the circle $K(0,R)$. Then we have for $0<\sigma\leqq R$*

$$\frac{1}{\pi\sigma^2}\int_{K(0,\sigma)} |u(x,y)|d(x,y)\leqq\frac{1}{\sqrt{2\pi}}\left(\log\frac{R}{\sigma}+\frac{1}{4}\right)^{1/2}|u|_{1,2;K(0,R)}.$$

As every function from $W_0^{1,2}(K(0,R))$ can be approximated arbitrarily well by functions u as described in Lemma 3 the estimate (2.1) follows from this lemma.

The estimate of Lemma 3 is sharp. This is shown by the example of the continuously differentiable functions $u_{R,\sigma}\colon K(0,R)\to\mathbb{R}$, $0<\sigma\leqq R$, which are defined by

$$u_{R,\sigma}(x,y)=\phi_{R,\sigma}(\sqrt{x^2+y^2})$$

$$\phi_{R,\sigma}(r)=\begin{cases}-\dfrac{1}{\sigma^2}\dfrac{r^2}{2}+\dfrac{1}{2}+\log\dfrac{R}{\sigma}, & 0\leqq r\leqq\sigma\\ \log\dfrac{R}{r}, & \sigma\leqq r\leqq R.\end{cases}$$

For these functions the left and the right hand side in the estimate of Lemma 3 coincide.

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