SIAM J. MATH. ANAL.
Vol. 44, No. 4, pp. 2666–2694

EXPLICIT VARIATIONAL FORMS FOR THE INVERSES OF INTEGRAL LOGARITHMIC OPERATORS OVER AN INTERVAL*

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Abstract. We introduce explicit and exact variational formulations for the weakly singular and hypersingular operators over an open interval as well as for their corresponding inverses. Contrary to the case of a closed curve, these operators no longer map fractional Sobolev spaces in a dual fashion but degenerate into different subspaces depending on their extensibility by zero. We show that an average and jump decomposition leads to precise coercivity results and characterize the mismatch occurring between associated functional spaces. Through this setting, we naturally define Calderón-type identities with their potential use as preconditioners. Moreover, we provide an interesting relation between the logarithmic operators and one-dimensional Laplace Dirichlet and Neumann problems. This work is a detailed and extended version of the article "Variational Forms for the Inverses of Integral Logarithmic Operators over an Interval" by Jerez-Hanckes and Nédélec [*C.R. Acad. Sci. Paris Ser.* I, 349 (2011), pp. 547–552].

Key words. open surface problems, integral logarithmic equations, boundary integral equations, Laplace equation, Calderón projectors

AMS subject classifications. 45P05, 65N38, 31A10, 46E35

DOI. 10.1137/100806771

1. Introduction.

We study the ubiquitous logarithmic singular operators arising when solving screen [34, 22, 31], crack, or interface problems [4, 27, 28, 35], with piecewise constant coefficients in $\mathbb{R}^2$ via boundary integral equations. In general, solutions over a domain $\mathcal{O} \subset \mathbb{R}^2$, with a nonempty interior and boundary $\partial\mathcal{O}$, can be constructed in terms of boundary data using the single and double layer potentials [33, 20], defined over $\mathbb{R}^2 \setminus \partial\mathcal{O}$ as

$$(\Psi_{\mathrm{SL}}\varphi)(\mathbf{x}) := \int_{\partial\mathcal{O}} \log \frac{1}{\|\mathbf{x}-\mathbf{x}'\|}\varphi(\mathbf{x}')\mathrm{d}\mathbf{x}' , \tag{1.1}$$

$$(\Psi_{\mathrm{DL}}\alpha)(\mathbf{x}) := \int_{\partial\mathcal{O}} \partial_n \log \frac{1}{\|\mathbf{x}-\mathbf{x}'\|}\alpha(\mathbf{x}')\mathrm{d}\mathbf{x}' , \tag{1.2}$$

respectively, and where the normal derivative $\partial_n = \mathbf{n}\cdot\nabla_{\mathbf{x}'}$, with $\mathbf{n}$ being the unit normal vector pointing outwards for closed boundaries. After taking Dirichlet and/or Neumann traces of these potentials and imposing boundary conditions, one needs to solve a Fredholm integral equation of either first or second kind. When the boundary is closed, one obtains so-called Calderón identities which hold even for Lipschitz boundaries with their beneficial properties as preconditioners [12]. Also, Dirichlet and Neumann trace spaces are dual to each other, e.g., $H^{1/2}(\partial\mathcal{O})$ and $H^{-1/2}(\partial\mathcal{O})$ defined below.

*Received by the editors August 27, 2010; accepted for publication (in revised form) April 4, 2012; published electronically August 9, 2012. This work was partially funded by the ETH Zurich Postdoctoral Fellowship.
http://www.siam.org/journals/sima/44-4/80677.html

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The situation changes drastically when considering open boundaries, i.e., when $\partial(\partial\mathcal{O})$ is not empty. Indeed, Calderón identities break down due to the disappearance of the double layer boundary operator (and its adjoint), and the mapping properties of the boundary operators degenerate. Instead of working on standard Sobolev trace spaces $H^{\pm 1/2}(\partial\mathcal{O})$, one must consider the subspaces commonly denoted as either $\widetilde{H}^{\pm 1/2}(\partial\mathcal{O})$ or $H_{00}^{\pm 1/2}(\partial\mathcal{O})$, obtained for the positive sign by extending by zero or, in the negative case, by duality with $H^{1/2}(\partial\mathcal{O})$, and which are endowed with finer topologies [10, 18]. Thus, most existing works tackle the arising integral equations separately, i.e., equations involving the hypersingular or weakly singular operators. For example, several collocation methods have been proposed [32, 7, 11] for which the use of Chebyshev expansions is key to determining convergence rates in weighted Sobolev spaces. The reader will later observe that some of our proofs are reminiscent of the tools employed in these cited works, but our aim differs greatly in that we focus predominantly on variational formulations and preconditioners.

Currently, neither the hypersingular nor weakly singular operators have explicit inverses [33, 30]. Moreover, the hypersingular operator cannot be interpreted as an integral in the classical sense, and one must regularize it or use a variational approach to solve it [9, 3, 30]. Numerically ill-conditioned, the resulting integral equation can be preconditioned using the standard weakly singular operator [21], but the conditioning number still grows logarithmically with mesh size. On the other hand, solutions for the weakly singular operator present strong singularities at the endpoints of the interval. More precisely, they behave as $1/\sqrt{d}$, where d is the distance within the interval to the endpoints [5, 22, 16]. Consequently, solving this type of first-kind Fredholm equation has received considerable attention in the past as the vast literature proves; see [6, 1, 17, 8, 29, 13, 15], to name a few. Unfortunately, classical Galerkin schemes with both uniform and nonuniform meshes yield ill-conditioned matrices for which preconditioning via standard Calderón projectors performs poorly.

In this work, we address the aforementioned issues systematically and holistically by decomposing solutions over the plane into average and jump parts. By doing so, we obtain the following:

(i) exact characterizations of occurring functional spaces;
(ii) precise mapping properties of the weakly singular and hypersingular operators;
(iii) explicit and exact variational formulations for the operators as well as for their corresponding inverses.

Although we carry out the analysis only for the Laplace equation over the unbounded domain surrounding an interval with Dirichlet and Neumann conditions (cf. section 2), extensions can be immediately carried out as perturbations. Main results are condensed in section 3, and proofs are given in section 4. These last ones are based on the previous observations together with extensions and combinations of many results formerly derived in Hölder spaces [25], weighted L^2-spaces [23], and Chebyshev polynomials [19]. Finally, and as a by-product of our investigations, we give an interesting connection between these integral operators and one-dimensional Laplace Dirichlet and Neumann problems.

2. Preliminaries.

2.1. Geometry. Without loss of generality, introduce the canonic splitting of the $\mathbb{R}^2$ into two half-planes $\pi_\pm := \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 \gtrless 0\right\}$, with interface Γ given by the line $x_2 = 0$. Let the interval $I := (-1, 1)$ divide the interface into the open disjoint segments $\Gamma_c := I \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_c$. Henceforth, the problem domain is denoted by $\Omega := \mathbb{R}^2 \setminus \bar{\Gamma}_c$. Extension to smooth or Jordan arcs in $\mathbb{R}^2$ can be treated as compact perturbations and is not considered in the following.

2.2. Notation. Let $\mathcal{O} \subseteq \mathbb{R}^d$, with $d = 1, 2$, be open. We denote by $\mathscr{C}^k(\mathcal{O})$ the space of k-times differentiable continuous functions over $\mathcal{O}$ with $k \in \mathbb{N}_0$. Its subspace of compactly supported functions is $\mathscr{C}_0^k(\mathcal{O})$, and for infinitely differentiable functions we write $\mathscr{D}(\mathcal{O}) \equiv \mathscr{C}_0^\infty(D)$. The space of distributions or linear functionals over $\mathscr{D}(\mathcal{O})$ is $\mathscr{D}'(\mathcal{O})$. Also, let $L^p(\mathcal{O})$ be the standard class of functions with bounded L^p-norm over $\mathcal{O}$. By $\mathbb{S}'(\mathcal{O})$ we denote the Schwartz space of tempered distributions [2, Chapter 9].

Duality products are denoted by angular brackets, $\langle \cdot\,, \cdot \rangle$, with subscripts accounting for the duality pairing. Inner products are denoted by parentheses, $(\cdot\,, \cdot)$, with integration domains specified by subscripts. Furthermore, operators are denoted in mild calligraphic style and complex conjugates by overline. The adjoint of an operator will be specified by an asterisk.

2.3. Standard Sobolev spaces. For $s \in \mathbb{R}$, $H^s(\mathcal{O})$ denote standard Sobolev spaces [20, Chapter 3]. Let $s \geq 0$; we say that a distribution belongs to the local Sobolev space $H^s_{\text{loc}}(\mathcal{O})$ if its restriction to every compact set $K \Subset \mathbb{R}^d$ lies in $H^s(K)$. If $s > 0$ and $\mathcal{O}$ is Lipschitz, $\widetilde{H}^s(\mathcal{O})$ denotes the space of functions whose extension by zero over a closed domain $\widetilde{O}$ belongs to $H^s(\widetilde{O})$. We identify

$$\widetilde{H}^{-1/2}(\mathcal{O}) \;\equiv\; \left(H^{1/2}(\mathcal{O})\right)' \qquad \text{and} \qquad H^{-1/2}(\mathcal{O}) \;\equiv\; \left(\widetilde{H}^{1/2}(\mathcal{O})\right)', \tag{2.1}$$

and if $\mathcal{O} = \widetilde{O}$, then $\widetilde{H}^{\pm 1/2}(\mathcal{O}) \equiv H^{\pm 1/2}(\mathcal{O})$.

2.4. Traces. Define restrictions over the half-planes

$$u^\pm \;:=\; u|_{\pi_\pm}.$$

We introduce the *trace operators* $\gamma^\pm : \mathscr{D}(\overline{\pi}_\pm) \to \mathscr{D}(\Gamma)$ as

$$\gamma^\pm u \;:=\; \lim_{\epsilon \to 0^\pm} u(x_1, \epsilon) \;=\; \gamma^\pm u^\pm. \tag{2.2}$$

If $s > 1/2$, the operators $\gamma^\pm$ have unique extensions to bounded linear operators $H^s_{\text{loc}}(\pi_\pm) \to H^{s-1/2}_{\text{loc}}(\Gamma)$ [20, Chapter 3]. Furthermore, one can define the trace over a subdomain Γ_b in the following way.

THEOREM 2.1. *We denote by $\gamma_b^\pm$ the trace operator*

$$\begin{array}{rcl} \gamma_b^\pm \;:\; \mathscr{D}(\overline{\pi}_\pm) & \longrightarrow & \mathscr{D}(\Gamma_b), \\ u^\pm & \longmapsto & \gamma_b^\pm u^\pm \;=\; \gamma^\pm u^\pm|_{\Gamma_b}. \end{array} \tag{2.3}$$

If $s > 1/2$, a unique extension to a bounded linear operator $\gamma_b^\pm : H^s_{loc}(\pi_\pm) \to H^{s-1/2}(\Gamma_b)$ can be obtained by density of $\mathscr{D}(\overline{\pi}_\pm)$ in $H^s(\pi_\pm)$.

Let $[\gamma] := \gamma^+ - \gamma^-$ represent the jump operator across Γ, and similarly for γ_b. As Γ is nonorientable, we set $\mathbf{n}$ pointing along the positive x_2-axis, i.e., $\mathbf{n} = \hat{\mathbf{x}}_2$.

2.5. Weighted Sobolev spaces. Since the problem domain Ω is unbounded (cf. section 2.1), one usually works in either local Sobolev spaces or in weighted ones such as

$$W^{1,-1}(\Omega) \;=\; \left\{ u \in \mathscr{D}'(\Omega) \;:\; \frac{u}{(1+r^2)^{1/2}\log(2+r^2)} \;\in\; L^2(\Omega),\; \nabla u \in L^2(\Omega) \right\}, \tag{2.4}$$

which coincides with the standard $H^1_{\mathrm{loc}}(\Omega)$ for a bounded part of Ω and avoids specifying behaviors at infinity [26]. Furthermore, these weighted spaces are Hilbert, whereas local Sobolev spaces are only of Fréchet type. We also define the subspace:

$$W_0^{1,-1}(\Omega) \;=\; \left\{ u \in W^{1,-1}(\Omega) \;:\; \gamma_c^{\pm} u \;=\; 0 \right\} \,. \tag{2.5}$$

Lemma 2.2 (see [26, section 2.5.4]). *Define the seminorm*

$$|u|^2_{1,-1,\Omega} \;:=\; \int_{\Omega} |\nabla u(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \,. \tag{2.6}$$

Then there exists $c > 0$ such that

$$\|u\|_{W_0^{1,-1}(\Omega)} \;\le\; c\,|u|_{1,-1,\Omega} \quad \forall\, u \in W_0^{1,-1}(\Omega). \tag{2.7}$$

Moreover, the seminorm constitutes a norm on the space $W^{1,-1}(\Omega)/\mathbb{C}$. Specifically, there exists $c > 0$ such that

$$\inf_{p\in\mathbb{C}} \|u \,-\, p\|_{W^{1,-1}(\Omega)} \;\le\; c\,|u|_{1,-1,\Omega} \quad \forall\, u \in W^{1,-1}(\Omega) \,. \tag{2.8}$$

Now, traces on Γ for elements in $W^{1,-1}(\Omega)$ lie in the usual $H^{1/2}_{\mathrm{loc}}(\Gamma)$, and their restriction to a bounded Γ_c generates the subspace $H^{1/2}(\Gamma_c)$. Lastly, let us introduce the space $\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$ as the subspace of $\widetilde{H}^{-1/2}(\Gamma_c)$-distributions with zero mean value, i.e.,

$$\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c) \;:=\; \left\{ \varphi \in \widetilde{H}^{-1/2}(\Gamma_c) \;:\; \langle \varphi \,,\, 1 \rangle_{H^{1/2}(\Gamma_c)} \;=\; 0 \right\}, \tag{2.9}$$

which is related to the compatibility condition for Neumann problems.

2.6. Dirichlet problems. Instead of directly considering the standard Laplace problems, we start by tackling a slightly different Laplace problem with two different Dirichlet conditions $g^{\pm}$ from above and below on Γ_c. These boundary data lie in the Hilbert space:

$$\mathbb{X} := \left\{ \mathbf{g} = (g^+, g^-) \in H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c) : g^+ - g^- \in \widetilde{H}^{1/2}(\Gamma_c) \right\} \tag{2.10}$$

with norm

$$\|\mathbf{g}\|^2_{\mathbb{X}} := \left\|g^+\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^-\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^+ - g^-\right\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \,.$$

Equivalently, we define the Hilbert space for Neumann data:

$$\mathbb{Y} := \left\{ \boldsymbol{\varphi} = (\varphi^+, \varphi^-) \in H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c) : \varphi^+ - \varphi^- \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c) \right\} \tag{2.11}$$

with similar norm. The Dirichlet problem we consider is as follows.

Problem 2.3. *For $\mathbf{g} \in \mathbb{X}$, find $u \in W^{1,-1}(\Omega)$ such that*

$$\begin{cases} -\Delta\, u \;=\; 0, & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} u \;=\; \mathbf{g}, & \mathbf{x} \in \Gamma_c. \end{cases} \tag{2.12}$$

2.6.1. Uniqueness of solutions. Any function u in $W^{1,-1}(\Omega)$ can be split into its restrictions on $\pi_\pm$:

$$u^\pm := u|_{\pi_\pm} \in W^{1,-1}(\pi_\pm), \tag{2.13}$$

with traces $\gamma^\pm u^\pm \in H^{1/2}_{\mathrm{loc}}(\Gamma)$ well defined. By definition, if u is a solution of Problem 2.3, then $\gamma_c^\pm u^\pm = g^\pm$ with $[\gamma_c u] \in \widetilde{H}^{1/2}(\Gamma_c)$. Furthermore, due to the regularity of the solution in the interior of Ω, the transmission conditions

$$[\gamma_f u] = 0 \qquad \text{and} \qquad [\gamma_f \partial_n u] = 0 \tag{2.14}$$

hold. By the extension theorem [20, Theorem 3.18], there exists a continuous operator $\mathcal{E}_\Gamma^+ : H^{1/2}(\Gamma_c) \to H^{1/2}(\Gamma)$ extending g^+ over Γ satisfying

$$\mathcal{E}_\Gamma^+ g^+ \in H^{1/2}(\Gamma), \qquad \operatorname{supp}\left(\mathcal{E}_\Gamma^+ g^+\right) \Subset \Gamma, \qquad \text{and} \qquad \left(\mathcal{E}_\Gamma^+ g^+\right)\big|_{\Gamma_c} = g^+. \tag{2.15}$$

Furthermore, $[g] \in \widetilde{H}^{1/2}(\Gamma_c)$, so that its extension by zero, denoted by $\widetilde{[g]}$, lies in $H^{1/2}(\Gamma)$ and one can set $\mathcal{E}_\Gamma^- g^- := \mathcal{E}_\Gamma^+ g^+ - \widetilde{[g]}$, which is also continuous. Now, $\mathcal{E}_\Gamma^+ g^+$ and $\mathcal{E}_\Gamma^- g^-$ also admit liftings with compact support in the upper and lower half-planes, respectively, provided by the continuous operators $\mathcal{R}^\pm : H^{1/2}(\Gamma) \to W^{1,-1}(\pi_\pm)$. Define $v^\pm \in W^{1,-1}(\pi_\pm)$ through the operator composition

$$v^\pm := \left(\mathcal{R}^\pm \circ \mathcal{E}_\Gamma^\pm\right) g^\pm \tag{2.16}$$

having compact support, i.e., $\operatorname{supp}(v^\pm) \Subset \bar{\pi}_\pm$. Introduce

$$v := v^\pm \quad \text{if } \mathbf{x} \in \pi_\pm, \tag{2.17}$$

so that by (2.14), $v \in W^{1,-1}(\Omega)$. This allows the definition of an operator $\mathcal{A} : \mathbb{X} \to W^{1,-1}(\Omega)$ such that $v := \mathcal{A}\,\mathbf{g}$, and for which it holds

$$\|\mathcal{A}\,\mathbf{g}\|_{W^{1,-1}(\Omega)} \le C_{\mathcal{A}}\,\|\mathbf{g}\|_{\mathbb{X}} \tag{2.18}$$

by continuity of all the composing operators. Continuity of the trace operators provides the following result.

LEMMA 2.4. *If $u \in W^{1,-1}(\Omega)$ is such that $\gamma_c^\pm u = g^\pm$ with $(g^+, g^-) \in \mathbb{X}$, there exists a real positive constant C_X such that*

$$\|\mathbf{g}\|_{\mathbb{X}} \le C_X\,\|u\|_{W^{1,-1}(\Omega)}\,. \tag{2.19}$$

Since by construction $\gamma_c^\pm v^\pm = g^\pm$, it holds that $\gamma_c^\pm(u - v) = 0$, and we can rewrite Problem 2.3 with a homogeneous Dirichlet condition.

PROBLEM 2.5. *Let $v := \mathcal{A}\,\mathbf{g}$ and $f := \Delta\, v \in \left(W_0^{1,-1}(\Omega)\right)'$. We look for w in $W_0^{1,-1}(\Omega)$ such that*

$$\begin{cases} -\Delta\, w = f, & \mathbf{x} \in \Omega, \\ \gamma_c^\pm\, w = 0, & \mathbf{x} \in \Gamma_c. \end{cases} \tag{2.20}$$

PROPOSITION 2.6. *Problem* 2.5 *has a unique solution in* $W_0^{1,-1}(\Omega)$.

Proof. From the distributional sense of (2.20), we first observe

$$-\left\langle \Delta\, w\,,\, w^t \right\rangle_{W_0^{1,-1}(\Omega)} = \left\langle f\,,\, w^t \right\rangle_{W_0^{1,-1}(\Omega)} \quad \forall\, w^t \in W_0^{1,-1}(\Omega). \tag{2.21}$$

Now, let B_R be the open ball of radius $R > 0$ centered at zero with boundary ∂B_R and such that $\mathrm{supp}(f) \Subset B_R$. Let $\Gamma_R := \Gamma \cap B_R$ and $B_R^{\pm} := \pi_{\pm} \cap B_R$ be the upper and lower semicircles with boundaries $\partial B_R^{\pm} = \Gamma_R \cup (\partial B_R \cap \pi_{\pm})$. For every $w^t \in W_0^{1,-1}(\Omega)$, it holds that

$$-\left\langle \Delta w\,,\, w^t \right\rangle_{W_0^{1,-1}(B_R^{\pm})} \;=\; \left(\nabla w\,,\, \nabla w^t\right)_{B_R^{\pm}} - \left\langle \gamma_{\partial B_R^{\pm}} \partial_n w\,,\, \gamma_{\partial B_R^{\pm}} w^t \right\rangle_{H^{1/2}(\partial B_R^{\pm})}, \tag{2.22}$$

and addition of both semicircle contributions yields

$$\begin{aligned} -\left\langle \Delta\, w\,,\, w^t \right\rangle_{W_0^{1,-1}(B_R \cap \Omega)} \;=\; & \left(\nabla w\,,\, \nabla w^t\right)_{B_R \cap \Omega} \;-\; \left\langle \gamma_{\partial B_R} \partial_n w\,,\, \gamma_{\partial B_R} w^t \right\rangle_{H^{1/2}(\partial B_R)} \\ & \sum_{\pm} \mp \left\langle \gamma_R^{\pm} \partial_n w\,,\, \gamma_R^{\pm} w^t \right\rangle_{H^{1/2}(\Gamma_R)}. \end{aligned} \tag{2.23}$$

By definition of $W^{1,-1}(\Omega)$, when R tends to infinity, the second term on the right-hand side vanishes. The remaining boundary term over Γ_R extends now over Γ wherein the splitting into Γ_c and Γ_f holds. Since $\gamma_c^{\pm} w^t = 0$ and $\gamma_f^{\pm} w^t = \gamma_f w^t$, the duality products over Γ_c cancel out and yield

$$-\left\langle \gamma^{+} \partial_n w\,,\, \gamma^{+} w^t \right\rangle_{H^{1/2}(\Gamma)} + \left\langle \gamma^{-} \partial_n w\,,\, \gamma^{-} w^t \right\rangle_{H^{1/2}(\Gamma)} \;=\; -\left\langle [\gamma_f \partial_n w]\,,\, \gamma_f w^t \right\rangle_{H^{1/2}(\Gamma_f)}. \tag{2.24}$$

By the transmission conditions (2.14), the above contribution disappears to obtain

$$\Phi_D(w, w^t) \;:=\; \left(\nabla w\,,\, \nabla w^t\right)_{\Omega} \;=\; \left\langle f\,,\, w^t \right\rangle_{W_0^{1,-1}(\Omega)} \qquad \forall\, w^t \;\in\; W_0^{1,-1}(\Omega). \tag{2.25}$$

The associated bilinear form is continuous and coercive on $W_0^{1,-1}(\Omega)$. Indeed,

$$\Phi_D(w, w) = (\nabla w\,,\, \nabla w)_{\Omega} \;=\; |w|^2_{1,-1,\Omega} \;\geq\; c^{-2}\, \|w\|^2_{W_0^{1,-1}(\Omega)} \tag{2.26}$$

by Lemma 2.2. Thus, by the Lax–Milgram theorem, we have uniqueness of w since f belongs to the dual space of $W_0^{1,-1}(\Omega)$. $\square$

With this, one can easily prove the following result.

PROPOSITION 2.7. *If* $\mathbf{g} \in \mathbb{X}$*, then Problem* 2.3 *has a unique solution in* $W^{1,-1}(\Omega)$*.*

2.6.2. Average and jump decomposition. The solution to Problem 2.3 can be split as follows. To any function u in $W^{1,-1}(\Omega)$, one associates restrictions $u^{\pm}$ on $\pi_{\pm}$ belonging to $W^{1,-1}(\pi_{\pm})$. Denote by $\breve{u}^{\pm} \in W^{1,-1}(\mathbb{R}^d)$ the mirror reflection of $u^{\pm}$ over $\pi_{\mp}$. Average and jump solutions defined over $\mathbb{R}^2$ are written as

$$\begin{cases} u_{\mathrm{avg}} \;:=\; \frac{\breve{u}^{+}+\breve{u}^{-}}{2}, \\ u_{\mathrm{jmp}} \;:=\; \frac{\breve{u}^{+}-\breve{u}^{-}}{2}, \end{cases} \qquad \text{associated to the data} \qquad \begin{cases} g_{\mathrm{avg}} \;:=\; \frac{g^{+}+g^{-}}{2}, \\ g_{\mathrm{jmp}} \;:=\; \frac{g^{+}-g^{-}}{2}. \end{cases} \tag{2.27}$$

Normal traces can also be similarly decomposed. Due to the set orientation of the normal, they take the form

$$\begin{cases} \gamma_c \partial_n u_{\mathrm{avg}} \;:=\; \frac{1}{2} \hat{\mathbf{x}}_2 \cdot \nabla(\breve{u}^{+} - \breve{u}^{-}), \\ \gamma_c \partial_n u_{\mathrm{jmp}} \;:=\; \frac{1}{2} \hat{\mathbf{x}}_2 \cdot \nabla(\breve{u}^{+} + \breve{u}^{-}), \end{cases} \qquad \text{associated to the values} \qquad \begin{cases} u_{\mathrm{avg}}, \\ u_{\mathrm{jmp}}, \end{cases} \tag{2.28}$$

and we have the associated Green's formula (as $(\nabla u_{\mathrm{avg}}\,,\, \nabla v_{\mathrm{jmp}})_{\Omega} = 0$):

$$(\nabla u\,,\, \nabla v)_{\Omega} = \langle \gamma_c \partial_n u_{\mathrm{avg}}\,,\, \gamma_c v_{\mathrm{avg}} \rangle_{H^{1/2}(\Gamma_c)} + \langle \gamma_c \partial_n u_{\mathrm{jmp}}\,,\, \gamma_c v_{\mathrm{jmp}} \rangle_{\widetilde{H}^{1/2}(\Gamma_c)} \tag{2.29}$$

for all $v \in W^{1,-1}(\mathbb{R}^2)$ split into average and jump parts.

PROPOSITION 2.8. *The solution of the Dirichlet Problem* 2.3 *is such that its Neumann trace at* Γ_c *belongs to the space* $\mathbb{Y}$. *There exists a unique Dirichlet-to-Neumann (DtN) map* $\boldsymbol{\mathcal{D}} : \mathbb{X} \to \mathbb{Y}$ *satisfying*

$$\langle \boldsymbol{\mathcal{D}}\,\mathbf{g}\,,\,\mathbf{g}\rangle_{\mathbb{X}} \;\geq\; C\,\|\mathbf{g}\|^2_{\mathbb{X}} \tag{2.30}$$

for $\mathbf{g}$ *in* $\mathbb{X}$, *and where the vector duality product is given by*

$$\langle \boldsymbol{\mathcal{D}}\,\mathbf{g}\,,\,\mathbf{g}\rangle_{\mathbb{X}} = \langle \boldsymbol{\mathcal{D}}\,\mathbf{g}_{avg}\,,\,\mathbf{g}_{avg}\rangle_{H^{1/2}(\Gamma_c)} + \langle \boldsymbol{\mathcal{D}}\,\mathbf{g}_{jmp}\,,\,\mathbf{g}_{jmp}\rangle_{\widetilde{H}^{1/2}(\Gamma_c)}\,. \tag{2.31}$$

Proof. By Proposition 2.7, a unique continuous application $\mathcal{T}_D$ exists such that

$$\begin{array}{rccl} \mathcal{T}_D \;: & \mathbb{X} & \longrightarrow & W^{1,-1}(\Omega), \\ & \mathbf{g} & \longmapsto & u = \mathcal{T}_D\mathbf{g}. \end{array} \tag{2.32}$$

Due to the trace theorem, Theorem 2.1, one can construct a continuous operator,

$$\boldsymbol{\mathcal{D}} \;:=\; \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} \circ \partial_n \circ \mathcal{T}_D \;:\; \mathbb{X} \longrightarrow H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c),$$

belonging to $\mathbb{Y}$ since $\gamma_c^+\partial_n u - \gamma_c^-\partial_n u \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$. Parity decomposition follows by taking duality with v split into average and jump parts using formula (2.29). ∎

COROLLARY 2.9. *For* $g^\pm =: g \in H^{1/2}(\Gamma_c) \setminus \mathbb{C}$, *the corresponding solution of Problem* 2.3 *in* Ω *is symmetric with respect to* Γ. *Moreover, there exists a unique DtN operator* $\mathcal{D}_s : H^{1/2}(\Gamma_c) \setminus \mathbb{C} \to \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$ *satisfying*

$$\langle \mathcal{D}_s\, g\,,\,g\rangle_{H^{1/2}(\Gamma_c)} \;\geq\; C_s\,\|g\|^2_{H^{1/2}(\Gamma_c)}\,. \tag{2.33}$$

Proof. Since $\mathbf{g} = (g, g)$, the difference $g^+ - g^- \equiv 0$ lies trivially in $\widetilde{H}^{1/2}(\Gamma_c)$ and $\mathbf{g} \in \mathbb{X}$. Thus, Proposition 2.8 holds, but now the norm is

$$\|\mathbf{g}\|_{\mathbb{X}} = 2\,\|g\|_{H^{1/2}(\Gamma_c)}\,,$$

and the duality product becomes

$$\sum_{\pm} \left\langle \gamma_c^\pm \partial_n \mathcal{T}_D\mathbf{g}\,,\,g^\pm\right\rangle_{\Gamma_c} \;=\; 2\,\langle [\gamma_c\partial_n\mathcal{T}_D\mathbf{g}]\;,\,g\rangle_{\Gamma_c}\,, \tag{2.34}$$

where $\mathcal{T}_D$ is given in (2.32) and factors of two cancel out. We obtain the desired inequality by defining $\mathcal{D}_s := [\gamma_c\partial_n\mathcal{T}_D\mathcal{I}_{2\times 2}]$, where $\mathcal{I}_{n\times n}$ is the identity matrix of dimension n. ∎

COROLLARY 2.10. *For* $g^\pm = \pm g \in \widetilde{H}^{1/2}(\Gamma_c)$, *the associated solution of Problem* 2.3 *is antisymmetric with respect to* Γ *and there exists a unique DtN operator* $\mathcal{D}_{as}$: $\widetilde{H}^{1/2}(\Gamma_c) \to H^{-1/2}(\Gamma_c)$. *Moreover, the energy inequality holds:*

$$\langle \mathcal{D}_{as}\, g\,,\,g\rangle_{\widetilde{H}^{1/2}(\Gamma_c)} \;\geq\; C_{as}\,\|g\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}\,. \tag{2.35}$$

Proof. Define $\mathbf{g} := (g, -g)$. The difference $g^+ - g^-$ lies trivially in $\widetilde{H}^{1/2}(\Gamma_c)$ and $\mathbf{g} \in \mathbb{X}$. Thus, Proposition 2.8 holds with

$$\|\mathbf{g}\|_{\mathbb{X}} = 2\,\|g\|_{\widetilde{H}^{1/2}(\Gamma_c)}\,,$$

with duality product

$$\sum_{\pm} \left\langle \gamma_c^\pm \partial_n \mathcal{T}_D\mathbf{g}\,,\,g^\pm\right\rangle_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; 2\,\langle \gamma_c\partial_n\mathcal{T}_D\mathbf{g}\,,\,g\rangle_{\widetilde{H}^{1/2}(\Gamma_c)}\,, \tag{2.36}$$

so that the factors cancel and we obtain the desired inequality. ∎

2.7. Neumann problems. As in the Dirichlet case, we now define the general problem.

PROBLEM 2.11. *Find* $u \in W^{1,-1}(\mathbb{R}^2)$ *such that*

$$\begin{cases} -\Delta u = 0, & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_c^+ \partial_n u \\ \gamma_c^- \partial_n u \end{pmatrix} = \boldsymbol{\varphi}, & \mathbf{x} \in \Gamma_c, \end{cases} \tag{2.37}$$

where $\boldsymbol{\varphi}$ *belongs to the space* $\mathbb{Y}$*.*

PROPOSITION 2.12. *The Neumann Problem* 2.11 *has a unique solution in the space* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\boldsymbol{\varphi} \in \mathbb{Y}$*.*

Proof of Proposition 2.6. For $\boldsymbol{\varphi} = (\varphi^+, \varphi^-)$ and u satisfying (2.37), we have the following variational formulation:

$$\Phi_N(u,v) := (\nabla u\,,\, \nabla v)_{\mathbb{R}^2} = \sum_{\pm} \pm \left\langle \varphi^\pm\,,\, \gamma^\pm v \right\rangle_{H^{1/2}(\Gamma_c)} \quad \forall\, v \in W^{1,-1}(\mathbb{R}^2). \tag{2.38}$$

Clearly, the bilinear form Φ_N is coercive and continuous. On the right-hand side, the dual form is well defined only if $\varphi^+ - \varphi^- \in \widetilde{H}^{1/2}(\Gamma_c)$, since $\gamma_c^\pm v = \gamma_c v \in H^{1/2}(\Gamma_c)$. Moreover, if v is equal to a constant, the bilinear form is zero and thus $\boldsymbol{\varphi}$ must satisfy the compatibility condition

$$\left\langle \varphi^+ - \varphi^-\,,\, 1 \right\rangle_{\Gamma_c} = 0\,. \tag{2.39}$$

Consequently, if $\boldsymbol{\varphi}$ belongs to $\mathbb{Y}$, by the Lax–Milgram theorem, the problem has a unique solution in $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$. ∎

PROPOSITION 2.13. *The solution of the Neumann Problem* 2.11 *is such that its Dirichlet trace at* Γ_c *belongs to the space* $\mathbb{X}$*. There exists a unique Neumann-to-Dirichlet (NtD) map* $\mathcal{N} : \mathbb{Y} \to \mathbb{X}$ *satisfying*

$$\langle \mathcal{N}\boldsymbol{\varphi}\,,\, \boldsymbol{\varphi} \rangle_{\mathbb{Y}} \ \geq\ C\, \|\boldsymbol{\varphi}\|^2_{\mathbb{Y}} \tag{2.40}$$

for $\boldsymbol{\varphi}$ *in* $\mathbb{Y}$*, and where the vector duality product is given by*

$$\langle \mathcal{N}\boldsymbol{\varphi}\,,\, \boldsymbol{\varphi} \rangle_{\mathbb{Y}} = \left\langle \mathcal{N}\boldsymbol{\varphi}_{avg}\,,\, \boldsymbol{\varphi}_{avg} \right\rangle_{\widetilde{H}^{-1/2}(\Gamma_c)} + \left\langle \mathcal{N}\boldsymbol{\varphi}_{jmp}\,,\, \boldsymbol{\varphi}_{jmp} \right\rangle_{H^{-1/2}(\Gamma_c)}\,. \tag{2.41}$$

Proof. By Proposition 2.12, a unique continuous application $\mathcal{T}_N$ exists such that

$$\begin{array}{rccl} \mathcal{T}_N\ : & \mathbb{Y} & \longrightarrow & W^{1,-1}(\Omega), \\ & \boldsymbol{\varphi} & \longmapsto & u = \mathcal{T}_N \boldsymbol{\varphi}. \end{array} \tag{2.42}$$

Due to the trace theorem (Theorem 2.1), one can construct a continuous operator,

$$\mathcal{N} := \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} \circ \mathcal{T}_D\ :\ \mathbb{Y} \longrightarrow H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c),$$

belonging to $\mathbb{X}$ since $\gamma_c^+ u - \gamma_c^- u \in \widetilde{H}^{1/2}(\Gamma_c)$. Parity decomposition follows by taking the duality pairing with v split into average and jump parts using formula (2.29). ∎

Symmetric (antisymmetric) Neumann problems can be stated as follows.

PROBLEM 2.14. *Find* $u_s, u_{as} \in W^{1,-1}(\mathbb{R}^2)$ *such that*

$$\begin{cases} -\Delta u_s = 0, & \mathbf{x} \in \Omega, \\ [\gamma_c \partial_n u_s] = \varphi, & \mathbf{x} \in \Gamma_c, \end{cases} \quad \textit{and} \quad \begin{cases} -\Delta u_{as} = 0, & \mathbf{x} \in \Omega, \\ \gamma_c^\pm \partial_n u_{as} = \phi, & \mathbf{x} \in \Gamma_c, \end{cases} \tag{2.43}$$

for data φ *in the space* $\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$ *and* ϕ *in* $H^{-1/2}(\Gamma_c)$.

COROLLARY 2.15. *The symmetric Neumann Problem* 2.14 *has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\varphi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$. *Thus, there exists a unique continuous and invertible NtD, denoted* $\mathcal{N}_s : \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c) \to H^{1/2}(\Gamma_c)/\mathbb{C}$. *Moreover, the energy inequality holds:*

$$\langle \mathcal{N}_s \varphi \, , \, \varphi \rangle_{\Gamma_c} \;\geq\; C \, \|\varphi\|^2_{\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)} . \tag{2.44}$$

The inverse of this application is the operator $\mathcal{D}_s$ *defined in Corollary* 2.9.

Proof. The proof is the same as for Proposition 2.12 using

$$\Phi_N(u,v) \;=\; (\nabla u \, , \, \nabla v)_{\mathbb{R}^2} \;=\; \langle \varphi \, , \, \gamma_c v \rangle_{\Gamma_c} \qquad \forall\, v \in W^{1,-1}(\mathbb{R}^2), \tag{2.45}$$

and replacing $\varphi^+ - \varphi^-$ with φ. ◻

COROLLARY 2.16. *The antisymmetric Neumann Problem* 2.14 *has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\phi \in H^{-1/2}(\Gamma_c)$. *Hence, there exists a unique continuous and invertible* $\mathcal{N}_{as} : H^{-1/2}(\Gamma_c) \to \widetilde{H}^{1/2}(\Gamma_c)$ *satisfying*

$$\langle \mathcal{N}_{as} \, \phi \, , \, \phi \rangle_{\Gamma_c} \;\geq\; C \, \|\phi\|^2_{H^{-1/2}(\Gamma_c)} . \tag{2.46}$$

The inverse of this application is the operator $\mathcal{D}_{as}$ *defined in Corollary* 2.10.

Proof. The proof follows that of Proposition 2.13. Operator $\mathcal{T}_N$ becomes

$$\begin{aligned} \mathcal{T}_N \; : \; H^{-1/2}(\Gamma_c) \quad &\longrightarrow \quad W^{1,-1}(\mathbb{R}^2)/\mathbb{C}, \\ \varphi \quad &\longmapsto \quad u \,=\, \mathcal{T}_N \varphi, \end{aligned}$$

and one constructs an operator $\mathcal{N}_{as} = [\gamma_c \circ \mathcal{T}_N]$ with range in $\widetilde{H}^{1/2}(\Gamma_c)$. Thus,

$$\langle \mathcal{N}_{as} \, \varphi \, , \, \varphi \rangle_{\Gamma_c} \;=\; (\nabla u \, , \, \nabla u)_{\mathbb{R}^2} \;=\; |u|^2_{1,-1,\mathbb{R}^2} \;\geq\; C_1 \, \|\gamma_c u\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \tag{2.47}$$

by continuity of the lifting operator. This proves the invertibility of $\mathcal{N}_{as}$. Moreover, since $\mathcal{N}_{as}$ is also continuous, it holds that

$$\|\varphi\|_{H^{-1/2}(\Gamma_c)} \;=\; \left\|\mathcal{N}^{-1}_{as} \, \gamma_c u\right\|_{H^{-1/2}(\Gamma_c)} \;\leq\; C_2 \, \|\gamma_c u\|_{\widetilde{H}^{1/2}(\Gamma_c)} , \tag{2.48}$$

which combined with the previous inequality yields the desired result. ◻

3. Main results. We now present the main results of this work: explicit variational forms or regularizations for the weakly singular and hypersingular operators over an interval and their inverses as well as associated Calderón-type identities. In fact, we will show that there exist two equivalent forms for the inverse of the weakly singular operator and two equivalent representations for the hypersingular operator. Moreover, we study the mapping properties of the underlying operators and derive useful identities for numerical applications. Proofs are given in the following section.

Introduce the following integral logarithmic operators for $x \in I$:

$$\mathcal{L}_1 \, \varphi(y) \;:=\; \int_I \log \frac{1}{|x-y|} \varphi(x) \, dx, \tag{3.1}$$

$$\mathcal{L}_2 \, \varphi(y) \;:=\; \int_I \log \frac{M(x,y)}{|x-y|} \varphi(x) \, dx, \tag{3.2}$$

where the first one is the standard weakly singular operator and where in the second

$$M(x,y) := \frac{1}{2}\left((y-x)^2 + (w(x)+w(y))^2\right), \tag{3.3}$$

with $w(x) := \sqrt{1-x^2}$ for $x \in I$. Lastly, introduce the subspace $H_*^{1/2}(\Gamma_c)$ of functions $g \in H^{1/2}(\Gamma_c)$ satisfying

$$\left\langle g\,,\, w^{-1}\right\rangle_{\Gamma_c} \;=\; 0. \tag{3.4}$$

3.1. Symmetric problem and weakly singular operator. The symmetric Dirichlet and Neumann solutions are given via the simple layer potential (1.1) with $\partial\mathcal{O}$ replaced by Γ_c. For the Neumann version, one just simply introduces the data in the potential, whereas for the Dirichlet problem one needs to solve the following: find φ such that

$$\mathcal{L}_1\,\varphi(x) \;=\; g(x), \qquad x \in I. \tag{3.5}$$

PROPOSITION 3.1. *The symmetric variational formulation of the integral equation* (3.5) *in the Hilbert space* $\widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c)$ *is*

$$\left\langle \mathcal{L}_1\,\varphi\,,\,\varphi^t\right\rangle_{\Gamma_c} \;=\; \left\langle g\,,\,\varphi^t\right\rangle_{\Gamma_c} \qquad \forall\,\varphi^t \in \widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c), \tag{3.6}$$

which is coercive, i.e.,

$$\langle \mathcal{L}_1\,\varphi\,,\,\varphi\rangle_{\Gamma_c} \geq C\,\|\varphi\|^2_{\widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c)} \qquad \forall\,\varphi \in \widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c). \tag{3.7}$$

The associated operator, $\mathcal{N}_s$ (cf. Corollary 2.15*), is a bijection between $\widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c)$ and $H_*^{1/2}(\Gamma_c)$. The inverse operator is associated to $\mathcal{D}_s$ (cf. Corollary* 2.9*), which is symmetric and coercive in $H_*^{1/2}(\Gamma_c)$. It admits two variational formulations:*

$$\frac{1}{\pi^2}\left\langle \mathcal{L}_2\,g'\,,\,\left(g^t\right)'\right\rangle_{\Gamma_c} \;=\; \left\langle \varphi\,,\,g^t\right\rangle_{\Gamma_c} \qquad \forall\,g^t \in H_*^{1/2}(\Gamma_c) \tag{3.8}$$

and

$$\frac{1}{2\pi^2}\int_{I\times I}\frac{d^2}{dxdy}\left(\log\frac{M(x,y)}{|x-y|}\right)[g(x)-g(y)]\left[g^t(x)-g^t(y)\right]dydx = \left\langle\varphi\,,\,g^t\right\rangle_{\Gamma_c} \tag{3.9}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$. These formulations in turn yield two expressions for the $H_*^{1/2}(\Gamma_c)$-norm:

$$\frac{1}{\pi^2}\,\langle \mathcal{L}_2\,g'\,,\,g'\rangle_{\Gamma_c} \;\geq\; C\,\|g\|^2_{H_*^{1/2}(\Gamma_c)} \qquad \forall\,g \in H_*^{1/2}(\Gamma_c) \tag{3.10}$$

and

$$\int_{I\times I}\frac{1-xy}{w(x)w(y)}\,\frac{(g(x)-g(y))^2}{(x-y)^2}dydx \;\geq\; C\,\|g\|^2_{H_*^{1/2}(\Gamma_c)} \qquad \forall\,g \in H_*^{1/2}(\Gamma_c). \tag{3.11}$$

Remark 3.2. Although Dirichlet Problem 2.3 admits a unique solution for all $g^{\pm} = g$ in $H^{1/2}(\Gamma_c)$, the solution to a constant data, e.g., $\gamma_c^{\pm}u = 1$, is such that $\varphi = 0$. Thus, the integral representation (3.5) cannot describe this constant solution and the exact image of $\mathcal{N}_s$ for $\widetilde{H}_{\langle 0\rangle}^{-1/2}(\Gamma_c)$ is the subspace $H_*^{1/2}(\Gamma_c)$.

3.2. Antisymmetric problem and hypersingular operator. For the Dirichlet case, the solution is retrieved using the double layer potential (1.2). However, for the Neumann version, one must first solve the hypersingular integral equation for α (the jump of the Dirichlet trace):

$$\varphi(x) \;=\; \fint_I \frac{1}{|x-y|^2}\alpha(y)dy \qquad \text{for} \quad x \in I, \tag{3.12}$$

where the dashed integral is understood as either a finite part integral for sufficiently regular α or in a weak sense for functions in Sobolev spaces.

PROPOSITION 3.3. *A symmetric variational formulation for* (3.12) *in the Hilbert space* $\widetilde{H}^{1/2}(\Gamma_c)$ *is given by*

$$\left\langle \mathcal{L}_1\, \alpha' \,,\, (\alpha^t)' \right\rangle_{\Gamma_c} \;=\; \left\langle \varphi \,,\, \alpha^t \right\rangle_{\Gamma_c} \qquad \forall\, \alpha^t \in \widetilde{H}^{1/2}(\Gamma_c). \tag{3.13}$$

Moreover, this bilinear form is coercive, i.e.,

$$\langle \mathcal{L}_1\, \alpha' \,,\, \alpha' \rangle_{\Gamma_c} \;\geq\; C\, \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \qquad \forall\, \alpha \in \widetilde{H}^{1/2}(\Gamma_c). \tag{3.14}$$

The associated operator, $\mathcal{D}_{as}$ *(Corollary* 2.10*), is a bijection from the space* $\widetilde{H}^{1/2}(\Gamma_c)$ *to* $H^{-1/2}(\Gamma_c)$*. This operator admits an alternative variational formulation:*

$$\frac{1}{2}\int_{I\times I} \frac{(\alpha(x)-\alpha(y))\,(\alpha^t(x)-\alpha^t(y))}{|x-y|^2}dxdy + 2\int_I \frac{\alpha(x)\alpha^t(x)}{1-x^2}dx \;=\; \left\langle \varphi \,,\, \alpha^t \right\rangle_{\Gamma_c} \tag{3.15}$$

for all $\alpha^t \in \widetilde{H}^{1/2}(\Gamma_c)$*, defining a norm on* $\widetilde{H}^{1/2}(\Gamma_c)$*:*

$$\int_{I\times I} \frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy + 4\int_I \frac{\alpha(x)^2}{1-x^2}dx \geq C\, \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \quad \forall\, \alpha \in \widetilde{H}^{1/2}(\Gamma_c). \tag{3.16}$$

The inverse operator is associated to the operator $\mathcal{N}_{as}^{-1} = \mathcal{D}_{as}$ *and is a bijection of* $H^{-1/2}(\Gamma_c)$ *onto* $\widetilde{H}^{1/2}(\Gamma_c)$*, symmetric and coercive. It admits the variational formulation*

$$\frac{1}{\pi^2}\left\langle \mathcal{L}_2\, \varphi \,,\, \varphi^t \right\rangle_{\Gamma_c} \;=\; \left\langle \alpha \,,\, \varphi^t \right\rangle_{\Gamma_c} \qquad \forall\, \varphi \in H^{-1/2}(\Gamma_c) \tag{3.17}$$

and thus provides a norm on the space $H^{-1/2}(\Gamma_c)$*,*

$$\langle \mathcal{L}_2\, \varphi \,,\, \varphi \rangle \;\geq\; C\, \|\varphi\|^2_{H^{-1/2}(\Gamma_c)} \qquad \forall\, \varphi \in H^{-1/2}(\Gamma_c). \tag{3.18}$$

The reader will certainly recognize the variational form (3.15) when divided by π (see [9, 21]). The above proposition emphasizes the connections between the standard hypersingular operator and $\mathcal{L}_2$.

PROPOSITION 3.4. $\widetilde{H}^{1/2}(\Gamma_c)$ *is exactly the space of functions* $g \in H^{1/2}(\Gamma_c)$ *such that* $w^{-1}g$ *is in* $L^2(\Gamma_c)$*. The space* $\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$ *is exactly the image of the distributional derivative of functions in* $\widetilde{H}^{1/2}(\Gamma_c)$*.*

3.3. Calderón-type identities. In the above results, two derivation operators have appeared: one denoted by D, whose domain lies on $\widetilde{H}^{1/2}(\Gamma_c)$, and another, $-D^*$, acting on $H_*^{1/2}(\Gamma_c)$. Since $\widetilde{H}^{1/2}(\Gamma_c)$ can be extended by zero to be a subspace of $H^{1/2}(\mathbb{R}) \subset \mathbb{S}'(\mathbb{R})$ the first derivation operator is defined distributionally. The second one is taken in a classical sense.

PROPOSITION 3.5. *The derivation operator D is continuous and surjective from the space $\widetilde{H}^{1/2}(\Gamma_c)$ onto $\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)$, while the derivation operator D^* is continuous and surjective from the space $H_*^{1/2}(\Gamma_c)$ onto the space $H^{-1/2}(\Gamma_c)$. Moreover, the operator $-D^*$ is the adjoint of D with respect to the duality product in $L^2(\Gamma_c)$.*

Finally, one can prove some properties linking these derivation operators D and D^* and the logarithmic operators previously introduced just by considering the variational forms of Propositions 3.1 and 3.3.

PROPOSITION 3.6. *The following identities hold:*

$$-\mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 \circ D \;=\; \mathcal{I}_{\widetilde{H}^{1/2}(\Gamma_c)}, \qquad -\mathcal{L}_1 \circ D \circ \mathcal{L}_2 \circ D^* \;=\; \mathcal{I}_{H_*^{1/2}(\Gamma_c)}, \tag{3.19a}$$

$$-D \circ \mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 \;=\; \mathcal{I}_{\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma_c)}, \qquad -D^* \circ \mathcal{L}_2 \circ D \circ \mathcal{L}_1 \;=\; \mathcal{I}_{H^{-1/2}(\Gamma_c)}. \tag{3.19b}$$

Remark 3.7. Clearly, $-D^* \mathcal{L}_1 D$ corresponds to the integration-by-parts formulation of the hypersingular operator (see (3.13)).

3.4. Links with one-dimensional Laplace problems. Let $H_0^1(\Gamma_c)$ denote the subspace of $H^1(\Gamma_c)$-functions vanishing at the endpoints. We now introduce the two classical one-dimensional Neumann and Dirichlet boundary value problems on the segment Γ_c.

PROBLEM 3.8 (one-dimensional Dirichlet problem). *Find $u \in H_0^1(\Gamma_c)$ such that*

$$\begin{cases} -u''(x) \;=\; \alpha(x), & x \in I, \\ u(\pm 1) \;=\; 0. \end{cases} \tag{3.20}$$

PROBLEM 3.9 (one-dimensional Neumann problem). *Find $u \in H^1(\Gamma_c)$ such that*

$$\begin{cases} -u''(x) \;=\; \varphi(x), & x \in I, \\ u'(\pm 1) \;=\; 0. \end{cases} \tag{3.21}$$

The Neumann problem admits a coercive variational formulation in the subspace of $H^1_{\langle 0\rangle}(\Gamma_c)$ defined as

$$H^1_{\langle 0\rangle}(\Gamma_c) \;:=\; \{u \in H^1(\Gamma_c) \;:\; \langle u\,,\,1\rangle_{H^1(\Gamma_c)} \;=\; 0\}, \tag{3.22}$$

and one seeks $u \in H^1_{\langle 0\rangle}(\Gamma_c)$ such that

$$(u'\,,\,v')_{\Gamma_c} \;=\; \langle \varphi\,,\,v\rangle_{H^1(\Gamma_c)} \qquad \forall\, v \in H^1_{\langle 0\rangle}(\Gamma_c). \tag{3.23}$$

Similarly, the Dirichlet problem has a coercive variational formulation in $H_0^1(\Gamma_c)$: find $u \in H_0^1(\Gamma_c)$ such that

$$(u'\,,\,v')_{\Gamma_c} \;=\; \langle \varphi\,,\,v\rangle_{H^1(\Gamma_c)} \qquad \forall\, v \in H_0^1(\Gamma_c). \tag{3.24}$$

The inverse of the Neumann operator is given by the integral operator

$$\mathcal{G}_N\, u(y) \;:=\; \int_I G_N(x,y) u(x)\, dx, \tag{3.25}$$

with kernel

$$G_N(x,y) := \begin{cases} -\frac{1}{4}\left\{(y+1)^2+(x-1)^2-\frac{4}{3}\right\} & \text{for} \quad y \le x \le 1, \\ -\frac{1}{4}\left\{(y-1)^2+(x+1)^2-\frac{4}{3}\right\} & \text{for} \quad -1 \le x \le y. \end{cases} \tag{3.26}$$

Correspondingly, the inverse of the Dirichlet operator is associated to the kernel

$$G_D(x,y) := \begin{cases} \frac{1}{2}(1+y)(1-x) & \text{for} \quad y \le x \le 1 \\ \frac{1}{2}(1-y)(1+x) & \text{for} \quad -1 \le x \le y, \end{cases} \tag{3.27}$$

with integral operator

$$\mathcal{G}_D\, u(y) := \int_I G_D(x,y)u(x)\,dx. \tag{3.28}$$

Proposition 3.10. *The operators $\mathcal{L}_1$, $\mathcal{L}_2$, D, D^*, $\mathcal{G}_D$, $\mathcal{G}_N$ are linked by the identities*

$$\frac{1}{\pi^2}\mathcal{L}_2 \circ \mathcal{L}_2 = (-D \circ D^*)^{-1} = \mathcal{G}_N,$$
$$\frac{1}{\pi^2}\mathcal{L}_1 \circ \mathcal{L}_1 = (-D^* \circ D)^{-1} = \mathcal{G}_D\,.$$

Remark 3.11. In other words, $\mathcal{L}_2$ is the square root of the inverse of the one-dimensional Neumann problem, while $\mathcal{L}_1$ is the square root of the inverse of the one-dimensional Dirichlet problem.

3.5. Examples for the functional spaces presented. As a by-product of these investigations, we state explicit examples of functions lying in the aforementioned Sobolev spaces. These are useful to grasp the main differences. Let us introduce the following functions:

$$V_\beta(x) := \log^\beta w^2(x) \qquad \text{and} \qquad W_\beta(x) := w^{-2}(x)V_\beta(x) \quad \text{for } x \in \Gamma_c, \tag{3.29}$$

dependent on the real parameter β.

Proposition 3.12. *The function V_β is in the space $H^{1/2}(\Gamma_c)$ if $\beta < 1/2$ and in the space $\widetilde{H}^{1/2}(\Gamma_c)$ if $\beta < -1/2$. The function W_β is in the space $H^{-1/2}(\Gamma_c)$ if $\beta < -1/2$ and in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ if $\beta < -3/2$.*

Proof. We study the functions at one endpoint, e.g., at $\mathbf{x}_0 = (-1,0)$. Introduce local polar coordinates $(r,\theta) \in \mathbb{R}^2$ with $r \in [0,\infty)$ and $\theta \in [-\pi/2, 3\pi/2)$ centered at $\mathbf{x}_0$ and directly take traces. Thus, when $\theta = 0$, the coordinate r along the segment is equivalent to $w^2(x)$. The function V_β is locally associated to the trace of the function $\log^\beta r$ which is in $H^1(\Omega)$ for $\beta < 1/2$ implying $V_\beta \in H^{1/2}(\Gamma_c)$. For the extension by zero, we use the function

$$Z_\beta(r,\theta) := \begin{cases} \log^\beta(r) & \text{for} \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ \sin\theta \log^\beta(r) & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases} \tag{3.30}$$

whose trace is zero for $x < -1$ and thus belongs to $\widetilde{H}^{1/2}(\Gamma_c)$. This function is in $H^1(\Omega)$ for $\beta < -1/2$.

The results for the function W_β are a direct consequence of the properties of the operators D and D^*. ∎

4. Proofs for the main results.

4.1. Analytical tools. We summarize all the results required in the proof of the above propositions. More details are provided in [14] and the references therein.

4.1.1. Hölder spaces. Denote by $\mathcal{C}^{0,\alpha}(I)$ the class of real (or complex) functions that satisfy the *Hölder condition* for every $\tau, \tau' \in I$:

$$|\varphi(\tau) - \varphi(\tau')| \leq M_\alpha |\tau - \tau'|^\alpha, \qquad M_\alpha > 0,$$

for $\alpha \in]0,1[$. $\mathcal{C}_0^{0,\alpha}(I)$ is the space of $\mathcal{C}^{0,\alpha}$ functions extended by zero at the endpoints. The set $\mathcal{C}^{0,\alpha}(I)$ is a Banach space with the norm

$$\|\varphi\|_{\mathcal{C}^{0,\alpha}(I)} = \|\varphi\|_{L^\infty(I)} + \|\varphi\|_{\alpha,I}$$

and

$$\|\varphi\|_{\alpha,I} = \sup_{\tau,\tau' \in I} \frac{|\varphi(\tau) - \varphi(\tau')|}{|\tau - \tau'|^\alpha}.$$

LEMMA 4.1 (see [31]). *Let K be compact and $0 < \alpha < \beta \leq 1$. Then the embeddings*

$$\mathcal{C}^{0,\beta}(K) \subset \mathcal{C}^{0,\alpha}(K) \subset \mathcal{C}(K)$$

are compact.

DEFINITION 4.2. *Denote by $\mathcal{H}_\mu(I)$ the set of functions that can be represented as*

$$\varphi(t) = w^{-1}(t)\tilde{\varphi}(t), \tag{4.1}$$

where $\tilde{\varphi}(t) \in \mathcal{C}^{0,\mu}(I)$ and $w(t) = \sqrt{1-t^2}$, with norm

$$\|\varphi\|_{\mathcal{H}_\mu(I)} = \|w\varphi\|_{\mathcal{C}^{0,\mu}(I)}.$$

LEMMA 4.3 (see [25, 24]). *Let $f \in \mathcal{H}_\mu(I)$ with $\mu < 1/2$. The solution $\varphi \in \mathcal{H}_\mu(I)$ of*

$$⨍_I \frac{\varphi(t)}{t-x}\,dt = f(x) \qquad \forall\, x \in I \tag{4.2}$$

is given by

$$\varphi(x) = -\left[\frac{1}{\pi^2} ⨍_I \frac{w(\tau)}{w(x)} \frac{f(\tau)\,d\tau}{\tau - x} + \frac{A}{w(x)}\right] \qquad \forall\, x \in I, \tag{4.3}$$

where A is a real constant.

4.1.2. Weighted L^2-spaces and Chebyshev polynomials. The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of first and second kinds, respectively, are polynomials of degree n, defined in $x \in I$ as [19]

$$T_n(x) = \cos n\theta \qquad \text{and} \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{4.4}$$

with $x = \cos\theta$. These satisfy the recurrence relations

$$P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x), \quad n = 2, 3, \ldots, \tag{4.5}$$

together with initial conditions $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$, and $U_1(x) = 2x$. Furthermore, it holds that

$$U_n(x) \;-\; U_{n-2}(x) \;=\; 2T_n(x)\;, \tag{4.6}$$

$$T_n'(x) \;=\; n\,U_{n-1}(x)\;, \tag{4.7}$$

$$(wU_{n-1})' \;=-\; nw^{-1}T_n \tag{4.8}$$

for $n \in \mathbb{N}$, with w defined as before. Moreover, the T_n are orthogonal with respect to w^{-1}:

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,w^{-1}(x)\,dx \;=\; \begin{cases} 0, & n \neq m\;,\\ \pi/2, & n = m \neq 0\;,\\ \pi, & n = m = 0\;. \end{cases} \tag{4.9}$$

For the second-kind Chebyshev polynomials U_n, it holds that

$$\int_{-1}^{1} U_n(x)\,U_m(x)\,w(x)\,dx \;=\; \begin{cases} 0, & n \neq m\;,\\ \pi/2, & n = m \neq 0. \end{cases} \tag{4.10}$$

We also have

$$\int_{-1}^{1} w^{-1}(x)U_n(x)dx \;=\; \begin{cases} \pi, & n = 2p, \quad p \geq 0,\\ 0, & n = 2p+1,\; p \geq 0, \end{cases} \tag{4.11a}$$

$$\int_{-1}^{1} \frac{xU_n(x)}{w(x)}dx \;=\; \begin{cases} 0, & n = 2p, \quad p \geq 0,\\ \pi, & n = 2p+1,\; p \geq 0. \end{cases} \tag{4.11b}$$

Based on the above, one can define weighted function spaces and norms,

$$L^2_{1/w} \;:=\; \left\{ u \text{ measurable } : \; \|f\|^2_{1/w} \;:=\; \int_{-1}^{1} |f(x)|^2\; w^{-1}(x)\,dx \;<\; \infty \right\},$$

$$L^2_{w} \;:=\; \left\{ u \text{ measurable } : \; \|f\|^2_{w} \;:=\; \int_{-1}^{1} |f(x)|^2\; w(x)\,dx \;<\; \infty \right\},$$

and the associated space

$$W \;=\; \left\{ u \text{ measurable } : \; u \in L^2_{1/w}\;,\; u' \in L^2_w \right\} \tag{4.12}$$

with evident graph norm.

THEOREM 4.4. *For a given $x \in I$, the logarithmic kernel admits the expansion on Chebyshev polynomials as a function in $L^2_{1/w}$:*

$$\log\frac{1}{|x-y|} \;=\; \log 2 \;+\; \sum_{n=1}^{\infty} \frac{2}{n} T_n(x)T_n(y) \qquad \forall\, y \in I. \tag{4.13}$$

For all $(x, y) \in I \times I$, with $x \neq y$, its derivatives have the following expressions:

$$\frac{1}{x-y} \;=\; \sum_{n=1}^{\infty} 2U_{n-1}(x)T_n(y), \tag{4.14}$$

$$\frac{1}{x-y} \;=-\; \sum_{n=1}^{\infty} 2T_n(x)U_{n-1}(y), \tag{4.15}$$

$$\frac{d^2}{dxdy}\log\frac{1}{|x-y|} \;=\; \frac{1}{|x-y|^2} \;=\; \sum_{n=1}^{\infty} 2nU_{n-1}(x)U_{n-1}(y). \tag{4.16}$$

We also have for fixed $x \in I$ the following equality in $L^2_{1/w}$:

$$\log \frac{M(x,y)}{|x-y|} \;=\; \sum_{n=1}^{\infty} \frac{2w(x)w(y)}{n} U_{n-1}(x)U_{n-1}(y) \qquad \forall\, y \in I, y \neq x, \tag{4.17}$$

whose derivatives are

$$\frac{d}{dx} \log \frac{M(x,y)}{|x-y|} \;=\; -\frac{w(y)}{w(x)} \frac{1}{x-y}, \tag{4.18}$$

$$\frac{d}{dy} \log \frac{M(x,y)}{|x-y|} \;=\; \frac{w(x)}{w(y)} \frac{1}{x-y}, \tag{4.19}$$

$$\frac{d^2}{dxdy} \log \frac{M(x,y)}{|x-y|} \;=\; \frac{1-xy}{w(x)w(y)} \frac{1}{(x-y)^2}, \tag{4.20}$$

$$\frac{d^2}{dxdy} \log \frac{M(x,y)}{|x-y|} \;=\; \sum_{n=1}^{\infty} 2n \frac{T_n(x)T_n(y)}{w(x)w(y)}. \tag{4.21}$$

Proof. See Appendix A for the proofs of (4.13) and (4.17). These expansions are valid in the space $L^2_{1/w}$, which can be easily seen using Chebyshev orthogonality relations and their known bounds. The expressions for their derivatives are quite formal as these series diverge when $x - y$ is small. $\square$

The following proposition links Hölder and weighted L^2-spaces.

PROPOSITION 4.5 (see [14]). *Any function $u \in \mathcal{H}_\mu(\Gamma_c)$ can be written as a series of weighted first-kind Chebyshev polynomials. Moreover, $\mathcal{H}_\mu(\Gamma_c) \subset L^2_w$.*

4.2. Proof of Proposition 3.1. We obtain the coercivity of $\mathcal{L}_1$ via the variational formulation (3.7) and the coercivity of the aforementioned Neumann problem. To characterize the image of the operator, multiply (4.13) by w^{-1} and use orthogonality properties of T_n, and first observe that

$$\mathcal{L}_1\, w^{-1}(\tau) \;=\; \pi \log 2 \qquad \forall\, \tau \,\in\, I. \tag{4.22}$$

We then multiply this expression by $\varphi(\tau) \in \widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma_c)$ and integrate on τ over Γ_c (3.5). Since φ has a zero mean value, we obtain condition (3.4). The remaining statement concerning the logarithmic operator $\mathcal{L}_1$ follows from Corollary 2.9.

In order to obtain an expression for the inverse operator $\mathcal{L}_1^{-1}$, we use again (4.13). This inverse operator is in fact the restriction of $\mathcal{D}_s$ to the subspace of $H^{1/2}(\Gamma_c)$ whose functions satisfy $[\gamma_c g] = 0$ and condition (3.4). Starting from (3.5), we obtain by derivation

$$\fint_I \frac{\varphi(t)}{x\,-\,t}\, dt \;=\; g'(x) \qquad \forall\, x \,\in\, I. \tag{4.23}$$

Using the inverse of this operator (cf. Lemma 4.3), it holds that

$$\varphi(x) \;=\; -\frac{1}{\pi^2} \fint_I \frac{w(\tau)}{w(x)} \frac{g'(\tau)\, d\tau}{\tau - x} \;+\; \frac{A}{w(x)} \qquad \forall\, x \,\in\, I, \tag{4.24}$$

with

$$A \;=\; \frac{1}{\pi \log 2} \left(g(x) \,+\, \frac{1}{\pi^2} \int_I \log \frac{1}{|x\,-\,t|} \fint_I \frac{w(\tau)}{w(t)} \frac{g'(\tau)\, d\tau}{\tau - t}\, dt \right), \tag{4.25}$$

which is a real constant. One can also write

$$\varphi(x) \;=\; \frac{1}{\pi^2 w(x)}\frac{d}{dx}\int_I \log\frac{1}{|x-\tau|}w(\tau)\,g'(\tau)\,d\tau + \frac{A}{w(x)} \qquad \forall\, x\in\, I. \tag{4.26}$$

We now multiply (4.26) by a test function g^t in the space $H_*^{1/2}(\Gamma_c)$ and integrate. From (3.4), it holds that

$$\int_I \varphi(x)g^t(x)dx \;=\; \frac{1}{\pi^2}\int_I \frac{g^t(x)}{w(x)}\fint_I \frac{w(\tau)}{x-\tau}g'(\tau)\,d\tau\,dx \qquad \forall\, g^t\in H_*^{1/2}(\Gamma_c). \tag{4.27}$$

Integration by parts leads to a first expression of the variational formulation:

$$\frac{1}{\pi^2}\int_{I\times I}\log\frac{1}{|x-\tau|}\left(\frac{g^t(x)}{w(x)}\right)'(w(\tau)\,g'(\tau))\,d\tau\,dx \;=\; -\int_I \varphi(x)g^t(x)dx \tag{4.28}$$

for all $g^t\in H_*^{1/2}(\Gamma_c)$. Unfortunately, this formulation is not symmetric. Since we know that the result is a symmetric bilinear form, we can add its adjoint to obtain a symmetric formulation. But this is not satisfactory. To obtain a different expression, we expand both g and g^t on the Chebyshev basis:

$$g_n \;=\; \frac{2}{\pi}\int_I \frac{g(x)T_n(x)}{w(x)}dx \qquad \forall\, n\;\in\;\mathbb{N}_0, \tag{4.29}$$

and thus $g_0 \;=\; 0$ by definition of $H_*^{1/2}(\Gamma_c)$. Hence,

$$g(x) \;=\; \sum_{n=1}^{\infty} g_n T_n(x)\,, \qquad x\;\in\; I, \tag{4.30}$$

and an equivalent expansion holds for $g^t(x)$. With this, the bilinear form in (4.27) becomes

$$\begin{aligned}&\frac{1}{\pi^2}\int_{I\times I}\frac{1}{x-\tau}\frac{g^t(x)}{w(x)}w(\tau)\,g'(\tau)\,d\tau\,dx\\ &= \frac{1}{\pi^2}\int_{I\times I}\frac{1}{x-\tau}\left(\sum_{n=1}^{\infty} g_n^t\frac{T_n(x)}{w(x)}\right)\left(w(\tau)\,\frac{d}{d\tau}\sum_{m=1}^{\infty} g_m T_m(\tau)\right)d\tau\,dx.\end{aligned} \tag{4.31}$$

Using (4.14) and (4.31), it takes the form

$$\begin{aligned}&\frac{1}{\pi^2}\int_{I\times I}\frac{1}{x-\tau}\frac{g^t(x)}{w(x)}w(\tau)\,g'(\tau)\,d\tau\,dx\\ &= \frac{1}{\pi^2}\int_{I\times I}\sum_{p=1}^{\infty}2T_p(x)U_{p-1}(\tau)\sum_{n=1}^{\infty}g_n^t\frac{T_n(x)}{w(x)}w(\tau)\sum_{m=1}^{\infty}mg_mU_{m-1}(\tau)\,d\tau dx.\end{aligned} \tag{4.32}$$

By identities (4.9) and (4.10), the only nonzero contributions occur when $n=p$ and $m=p$, and so

$$\begin{aligned}&\frac{1}{\pi^2}\int_{I\times I}\frac{1}{x-\tau}\left(\frac{g^t(x)}{w(x)}\right)(w(\tau)\,g'(\tau))\;d\tau\,dx\\ &= \frac{2}{\pi^2}\int_{I\times I}\left(\sum_{p=1}^{\infty}p\,g_p\,g_p^t\frac{T_p^2(x)}{w(x)}w(\tau)\,U_{p-1}^2(\tau)\right)d\tau\,dx\\ &= \frac{1}{2}\sum_{p=1}^{\infty}p\,g_p\,g_p^t.\end{aligned} \tag{4.33}$$

We seek an expression using only the derivatives of g and g^t (cf. (4.7)):

$$g'(x) \;=\; \sum_{n=1}^{\infty} n\, g_n\, U_{n-1}(x)\,, \qquad x \,\in\, I, \tag{4.34a}$$

$$\left(g^t(x)\right)' \;=\; \sum_{n=1}^{\infty} n\, g_n^t\, U_{n-1}(x)\,, \qquad x \,\in\, I. \tag{4.34b}$$

Employing the U_n-orthogonality relation, i.e., (4.10) or (4.8), we obtain, for all $n \in \mathbb{N}$,

$$g_n \;=\; \frac{2}{n\pi}\int_I U_{n-1}(x)w(x)g'(x)\,dx, \tag{4.35}$$

$$g_n^t \;=\; \frac{2}{n\pi}\int_I U_{n-1}(x)w(x)\left(g^t(x)\right)'\,dx, \tag{4.36}$$

and thus, starting from (4.33) together with (4.35) and (4.36), we have

$$\frac{1}{2}\sum_{n=1}^{\infty} n g_n g_n^t = \frac{1}{\pi^2}\int_{I\times I}\sum_{n=1}^{\infty}\frac{2w(x)w(y)}{n}U_{n-1}(x)U_{n-1}(y)g'(x)\left(g^t(y)\right)'\,dy\,dx. \tag{4.37}$$

Due to (4.17), the variational formulation for the inverse operator is

$$\frac{1}{\pi^2}\int_{I\times I}\log\frac{M(x,y)}{|x-y|}g'(x)\left(g^t(y)\right)'\,dy\,dx \;=\; \int_I \varphi(x)g^t(x)dx \tag{4.38}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$, giving the stated result by density arguments.

Lastly, one obtains the variational formulation (3.9) by first noticing that the finite part of the associated kernel is such that

$$⨍_I \frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|}\,dx = \;0 \qquad \forall\, y \,\in\, I. \tag{4.39}$$

This identity is obtained by integrating expression (4.21) over I and using the orthogonality of the basis T_n (4.9). From (4.39), we also have

$$g(y)g^t(y)⨍_I \frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|}\,dx = \;0. \tag{4.40}$$

Finally, we express the kernel and functions in the bilinear form (3.9) using their expansions on Chebyshev polynomials T_n. In this long expression, all terms related to the products $gg^t(x)$ vanish, and we recover the expression (4.37) with a factor of two.

Remark 4.6. The different expansions of the integral kernels in terms of Chebyshev polynomials given by (4.13), (4.14), (4.17), (4.18), and (4.21) are absolutely essential in our proof. By doing so, one takes exactly into account all the finite parts which appear due to the nonintegrable kernels.

4.3. Proof of Proposition 3.3. Using the variational formulation and coercivity of Neumann Problem 2.14, one proves the coercivity of the hypersingular operator. The inverse operator is the restriction of $\mathcal{N}_{as}$ (cf. Corollary 2.16) to $\widetilde{H}^{1/2}(\Gamma_c)$ and is also coercive in this space (Corollary 2.10).

Starting from (3.12), integration by parts yields

$$\varphi(x) \;=\; \fint_I \frac{1}{x-y}\alpha'(y)dy \quad \text{for} \quad x \in I. \tag{4.41}$$

This equation can also be written as

$$\varphi(x) \;=\; -\frac{d}{dx}(\mathcal{L}_1\,\alpha')(x) \quad \text{for} \quad x \in I. \tag{4.42}$$

Multiplying by a test function α^t and integrating by parts, we obtain the variational formulation (3.13):

$$\big\langle \mathcal{L}_1\,\alpha'\,,\,(\alpha^t)'\big\rangle \;=\; \big\langle \varphi\,,\,\alpha^t\big\rangle \qquad \forall\,\alpha^t \in\, \widetilde{H}^{1/2}(\Gamma_c). \tag{4.43}$$

We can expand the functions $w^{-1}\alpha$ and $w^{-1}\alpha^t$ on the Chebyshev polynomials U_n. All of these functions are zero at the endpoints of I, thus belonging to $\widetilde{H}^{1/2}(\Gamma_c)$, and consequently the summation starts at $n=0$. We have, by density arguments,

$$\alpha(x) \;=\; \sum_{n=0}^{\infty} \alpha_n w(x) U_n(x)\,, \qquad x \,\in\, I, \tag{4.44a}$$

$$\text{with} \qquad \alpha_n \;=\; \frac{2}{\pi}\int_I \alpha(x)U_n(x)dx\,, \qquad n \,\in\, \mathbb{N}, \tag{4.44b}$$

and equivalently for a test function α^t. Thus, the quadratic form associated to the integral kernel in (4.41) is formally

$$\fint_{I\times I} \frac{1}{|x-y|^2}\alpha(x)\alpha(y)dxdy \;=\; \frac{\pi^2}{2}\sum_{n=0}^{\infty}(n+1)\,\alpha_n^2, \tag{4.45}$$

but the finite part appearing in the first hand of the equality is not clearly defined. Using (4.8), we express the derivatives of the functions α and α^t on functions $w^{-1}T_n$:

$$\frac{d}{dx}\alpha(x) \;=\; -\sum_{n=1}^{\infty} n\alpha_{n-1}\frac{T_n(x)}{w(x)}\,, \qquad x \,\in\, I, \tag{4.46a}$$

$$\frac{d}{dy}\alpha^t(y) \;=\; -\sum_{n=1}^{\infty} n\alpha_{n-1}^t\frac{T_n(y)}{w(y)}\,, \qquad y \,\in\, I. \tag{4.46b}$$

Finally, using the orthogonality of U_n and expression (4.13), we recover the bilinear form (3.15) as

$$\big\langle \mathcal{L}_1\,\alpha'\,,\,(\alpha^t)'\big\rangle \;=\; \frac{\pi^2}{2}\sum_{n=0}^{\infty}(n+1)\,\alpha_n\alpha_n^t. \tag{4.47}$$

In order to obtain the variational formulation (3.15), we first remark that the finite part of the associated kernel is such that

$$\fint_I \frac{1}{x-y}dx \;=\; -\log\frac{|1-y|}{|1+y|} \qquad \forall\, y \,\in\, I, \tag{4.48}$$

and thus derivation in y yields

$$\fint_I \frac{1}{|x-y|^2}dx \;=\; \frac{d}{dy}\fint_I \frac{1}{x-y}dx = \frac{2}{1-y^2} \qquad \forall\, y \,\in\, I. \tag{4.49}$$

From this identity we also have

$$\alpha(y)\alpha^t(y) \fint_I \frac{1}{|x-y|^2}\,dx \;=\; 2\frac{\alpha(y)\alpha^t(y)}{1-y^2} \qquad \forall\, y \in I. \tag{4.50}$$

One can express the kernel and the function in the bilinear form (3.15) using their expansions on functions wU_n. Terms related to products $\alpha\alpha^t$ are known from identity (4.50) and, consequently, we recover the expression (4.45) with a factor of two.

An expression for the inverse of the double layer potential is retrieved by using the inverse of the operator in (4.41) given by Lemma 4.3. This yields

$$\alpha'(x) \;=\; -\frac{1}{\pi^2}\fint_I \frac{w(\tau)}{w(x)}\frac{\varphi(\tau)\,d\tau}{\tau - x} \;+\; \frac{A}{w(x)} \qquad \forall\, x \in I. \tag{4.51}$$

As the function α' has a zero coefficient on T_0, the coefficient A is zero. Thus, (4.51) becomes

$$\alpha'(x) \;=\; -\frac{1}{\pi^2}\fint_I \frac{w(\tau)}{w(x)}\frac{\varphi(\tau)\,d\tau}{\tau - x} \qquad \forall\, x \in I. \tag{4.52}$$

We expand φ and φ^t on Chebyshev polynomials U_n:

$$\varphi(x) \;=\; \sum_{n=0}^{\infty} \varphi_n U_n(x) \qquad \forall\, x \in I, \tag{4.53a}$$

$$\text{with} \qquad \varphi_n \;=\; \frac{2}{\pi}\int_I \varphi(x)w(x)U_n(x)dx \qquad \forall\, n \in \mathbb{N}. \tag{4.53b}$$

The function φ^t admits a primitive:

$$\beta^t(y) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n}\varphi^t_{n-1} T_n(y)\,, \qquad y \in I. \tag{4.54}$$

Multiplication of (4.52) by a test function β^t and integration by parts on the left-hand side yields

$$\int_I \alpha(x)\varphi^t(x)dx \;=\; \frac{1}{\pi^2}\int_I \frac{\beta^t(x)}{w(x)} \fint_I \frac{w(\tau)}{x-\tau}\varphi(\tau)\,d\tau dx \tag{4.55}$$

for all $\varphi^t \in H^{-1/2}(\Gamma_c)$. The bilinear form in (4.55) is

$$\begin{aligned} &\frac{1}{\pi^2}\int_{I\times I} \frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)} w(\tau)\,\varphi(\tau)\,d\tau dx \\ &= \frac{1}{\pi^2}\int_{I\times I} \frac{1}{x-\tau}\left[\sum_{n=1}^{\infty}\frac{1}{n}\varphi^t_{n-1}\frac{T_n(x)}{w(x)}\right]\left[w(\tau)\sum_{m=0}^{\infty}\varphi_m U_m(\tau)\right] d\tau\, dx. \end{aligned} \tag{4.56}$$

Using (4.14), (4.56) takes the form

$$\begin{aligned} &\frac{1}{\pi^2}\int_{I\times I} \frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)} w(\tau)\,\varphi(\tau)\,d\tau dx \\ &= \frac{1}{\pi^2}\int_{I\times I} \sum_{p=1}^{\infty} 2T_p(x)U_{p-1}(\tau) \sum_{n=1}^{\infty}\frac{1}{n}\varphi^t_{n-1}\frac{T_n(x)}{w(x)} w(\tau) \sum_{m=0}^{\infty}\varphi_m U_m(\tau) d\tau dx. \end{aligned} \tag{4.57}$$

From the identities (4.9) and (4.10), the only nonzero contributions occur when $n = p$ and $m = p-1$, and so

$$
\begin{aligned}
&\frac{1}{\pi^2}\int_{I\times I}\frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)}w(\tau)\,\varphi(\tau)\,d\tau\,dx\\
&= \frac{2}{\pi^2}\int_{I\times I}\left[\sum_{p=1}^{\infty}\frac{1}{p}\varphi_{p-1}\,\varphi^t_{p-1}\frac{T_p^2(x)}{w(x)}w(\tau)\,U^2_{p-1}(\tau)\right]d\tau\,dx\\
&= \frac{1}{2}\sum_{p=0}^{\infty}\frac{1}{p+1}\,\varphi_p\,\varphi^t_p\,.
\end{aligned}
\tag{4.58}
$$

By expanding φ and φ^t according to (4.53) and (4.53b), it holds that

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\,\varphi_n\,\varphi^t_n\\
&= \frac{1}{\pi^2}\int_{I\times I}\left[\sum_{n=1}^{\infty}\frac{2w(x)w(y)}{n}U_{n-1}(x)U_{n-1}(y)\right]\varphi(x)\varphi^t(y)dydx.
\end{aligned}
\tag{4.59}
$$

Hence, due to (4.17), the variational formulation for the inverse operator is

$$
\frac{1}{\pi^2}\int_{I\times I}\log\frac{M(x,y)}{|x-y|}\varphi(x)\varphi^t(y)\,dy\,dx \;=\; \int_I \alpha(x)\varphi^t(y)dx \qquad \forall\,\varphi^t\in H^{-1/2}(\Gamma_c),
\tag{4.60}
$$

thus giving the stated result by density arguments.

4.4. Proof of Proposition 3.4. The space $\widetilde{H}^{1/2}(\Gamma_c)$ lies in $H^{1/2}(\Gamma_c)$ but not in $H^{1/2}_*(\Gamma_c)$ as not every function in $\widetilde{H}^{1/2}(\Gamma_c)$ satisfies (3.4). However, we only need to prove the result on the subspace of $\widetilde{H}^{1/2}(\Gamma_c)$ which complies with (3.4), whose codimension is one. Accordingly, we consider the norms on the spaces $\widetilde{H}^{1/2}(\Gamma_c)$ and $H^{1/2}_*(\Gamma_c)$ given by (3.16) and (3.11), respectively, which are

$$
\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \int_{I\times I}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dx\;dy\;+4\int_I\frac{\alpha^2(x)}{w^2(x)}dx,
\tag{4.61a}
$$

$$
\|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} \;=\; \int_{I\times I}\frac{1-xy}{w(x)w(y)}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy.
\tag{4.61b}
$$

For $x, y \in I$, it holds that

$$
1 \;\le\; \frac{1-xy}{w(x)w(y)} \;\le\; \frac{2}{w(x)w(y)}.
$$

The difference between the two squared norms, (4.61a), (4.61b), is given by

$$
\begin{aligned}
\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} &= \int_{I\times I}\left\{1-\frac{1-xy}{w(x)w(y)}\right\}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy\\
&\quad +4\int_I\frac{\alpha^2(x)}{w^2(x)}dx,
\end{aligned}
\tag{4.62}
$$

or, equivalently,

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} = -\int_{I\times I} \frac{(\alpha(x)-\alpha(y))^2}{w(x)w(y)(1-xy+w(x)w(y))}dxdy + 4\int_I \frac{\alpha^2(x)}{w^2(x)}dx. \tag{4.63}$$

As the first term on the right-hand side is negative, we have

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \le \|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} + 4\left\|\frac{\alpha}{w}\right\|^2_{L^2(\Gamma_c)}. \tag{4.64}$$

We introduce the change of variable $x = \cos(\theta), y = \cos(\varphi)$, and write $\hat{\alpha}(\theta) := \alpha(\cos\theta)$. With this,

$$\int_{I\times I} \frac{(\alpha(x)-\alpha(y))^2}{w(x)w(y)(1-xy+w(x)w(y))}dxdy = \int_0^\pi\int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi. \tag{4.65}$$

This last quantity can be decomposed as

$$\int_0^\pi\int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi = \int_0^\pi\int_0^\pi \frac{\hat{\alpha}^2(\theta)-\hat{\alpha}(\theta)\hat{\alpha}(\varphi)}{\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi. \tag{4.66}$$

On the other hand,

$$\int_0^\pi \frac{1}{\sin^2(\frac{\theta+\varphi}{2})}d\varphi \;=\; \frac{4}{\sin(\theta)}, \tag{4.67}$$

and so (4.66) becomes

$$\int_0^\pi\int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi = 4\int_I \frac{\alpha^2(x)}{w^2(x)}dx - \int_0^\pi\int_0^\pi \frac{\hat{\alpha}(\theta)\hat{\alpha}(\varphi)}{\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi. \tag{4.68}$$

Thus the difference of the squares of these two norms is also

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} = \int_I\int_I \frac{\alpha(x)\alpha(y)}{w(x)w(y)(1-xy+w(x)w(y))}dxdy. \tag{4.69}$$

Using the previous estimates and Cauchy–Schwarz inequality, we have the bounds

$$\int_0^\pi\int_0^\pi \frac{\hat{\alpha}(\theta)\hat{\alpha}(\varphi)}{\sin^2(\frac{\theta+\varphi}{2})}d\theta\, d\varphi \;\le\; \int_0^\pi\int_0^\pi \frac{\hat{\alpha}^2(\theta)}{\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi, \tag{4.70}$$

or

$$\int_{I\times I} \frac{\alpha(x)\alpha(y)}{w(x)w(y)(1-xy+w(x)w(y))}dx \;\ge\; -2\int_I \frac{\alpha^2(x)}{w^2(x)}dx. \tag{4.71}$$

Hence, we have proven that

$$\|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} + 2\left\|\frac{\alpha}{w}\right\|^2_{L^2(\Gamma_c)} \;\le\; \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}, \tag{4.72}$$

from which we obtain also

$$\|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} \leq \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}\,. \tag{4.73}$$

If we consider the norms on $\widetilde{H}^{1/2}(\Gamma_c)$ and $H^{1/2}_*(\Gamma_c)$ given, respectively, by (3.14) and (3.10), the difference of the squares of these two norms (with one multiplied by π^2) is given by

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H^{1/2}_*(\Gamma_c)} = \int_{I\times I} \log M(x,y)\alpha'(x)\alpha(y)'\,dy\,dx. \tag{4.74}$$

Using two integrations by parts, we recover the expression (4.69) as we have

$$\frac{d^2}{dxdy}\log M(x,y) = \frac{1}{w(x)w(y)(1-xy+w(x)w(y))}. \tag{4.75}$$

The result concerning the dual spaces is just a direct consequence of the duality.

4.5. Proof of Proposition 3.5. The proof is based on Chebyshev expansions to write down functions in different spaces and term-by-term derivation to conclude. These expansions are similar to those given by Stephan and coworkers [32, 7, 11]. However, in our case we conclude by density and convergence results previously used. Specifically, one can do the following:

1. Expand a function in $\widetilde{H}^{1/2}(\Gamma_c)$ on functions $w(x)U_n(x)$ as

$$\alpha(x) \;=\; \sum_{n=0}^{\infty} \alpha_n w(x) U_n(x)\,, \qquad x\;\in\; I. \tag{4.76}$$

Then it results from Proposition 3.3 that the expression

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty}(n+1)\,\alpha_n^2 \tag{4.77}$$

defines a norm in the space $\widetilde{H}^{1/2}(\Gamma_c)$.

2. Expand a function in the space $H^{1/2}(\Gamma_c)$ on polynomials $T_n(x)$ as

$$g(x) \;=\; \sum_{n=0}^{\infty} g_n T_n(x)\,, \qquad x\;\in\; I, \tag{4.78}$$

and, from Proposition 3.1,

$$\|g\|^2_{H^{1/2}(\Gamma_c)} \;=\; g_0^2 + \sum_{n=1}^{\infty} n\, g_n^2 \tag{4.79}$$

is a norm for the space $H^{1/2}(\Gamma_c)$. The subspace $H^{1/2}_*(\Gamma_c)$ is such that $g_0 = 0$.

3. Expand a function in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ on $w^{-1}T_n$ as

$$\varphi(x) \;=\; \sum_{n=0}^{\infty} \varphi_n \frac{T_n(x)}{w(x)}\,, \qquad x\;\in\; I. \tag{4.80}$$

Then, from Proposition 3.1,

$$\|\varphi\|^2_{\widetilde{H}^{-1/2}(\Gamma_c)} \;=\; \varphi_0^2 \;+ \sum_{n=1}^{\infty}\frac{1}{n}\,\varphi_n^2 \tag{4.81}$$

is a norm in the space $\widetilde{H}^{-1/2}(\Gamma_c)$.

4. Expand a function in the space $H^{-1/2}(\Gamma_c)$ on $U_n(x)$ as

$$\varphi(x) \;=\; \sum_{n=0}^{\infty} \varphi_n U_n(x)\,, \qquad x \,\in\, I. \tag{4.82}$$

Then it results from Proposition 3.3 that

$$\|\varphi\|^2_{H^{-1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty} \frac{1}{n+1}\,\varphi_n^2 \tag{4.83}$$

is a norm in the space $H^{-1/2}(\Gamma_c)$.

5. Choose a function in the space $\widetilde{H}^{1/2}(\Gamma_c)$ given by (4.76), whose norm squared is

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty} (n+1)\,\alpha_n^2. \tag{4.84}$$

Its derivative is given by

$$\alpha'(x) \;=\; -\sum_{n=1}^{\infty} n\,\alpha_{n-1}\frac{T_n(x)}{w(x)}\,, \qquad x \,\in\, I, \tag{4.85}$$

and its norm squared in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ is thus

$$\|\alpha'\|^2_{\widetilde{H}^{-1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty} (n+1)\,\alpha_n^2. \tag{4.86}$$

This proves the continuity of this operator. The surjectivity is clear on the expression of the derivative.

6. Choose now a function in the space $H^{1/2}_*(\Gamma_c)$ given by (4.78), whose norm squared is

$$\|g\|^2_{H^{1/2}(\Gamma_c)} \;=\; \sum_{n=1}^{\infty} n\,g_n^2. \tag{4.87}$$

Its derivative is given by

$$g'(x) \;=\; \sum_{n=1}^{\infty} n\,g_n\,U_{n-1}(x)\,, \qquad x \,\in\, I, \tag{4.88}$$

and its norm squared in the space $H^{-1/2}(\Gamma_c)$ is thus

$$\|g'\|^2_{H^{-1/2}(\Gamma_c)} \;=\; \sum_{n=1}^{\infty} n\,g_n^2. \tag{4.89}$$

This proves the continuity of this operator. The surjectivity is clear based on the expression of the derivative.

4.6. Proof of Proposition 3.10. Expand a function in $H^1_*(\Gamma_c)$ on $T_n(x)$ as

$$g(x) \;=\; \sum_{n=1}^{\infty} g_n T_n(x)\,, \qquad x \,\in\, I. \tag{4.90}$$

The norm of this function in $L^2(\Gamma_c)$ is

$$\|g\|^2_{L^2(\Gamma_c)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_n g_m B_n^m, \quad \text{where} \quad B_n^m = \int_I T_n\, T_m dx. \tag{4.91}$$

Its derivative, using D^*, is given by

$$D^* g \;=\; \frac{dg}{dx}(x) \;=\; \sum_{n=0}^{\infty} n g_n U_{n-1}(x)\,, \qquad x \,\in\, I. \tag{4.92}$$

This function lies in $H^1_*(\Gamma_c)$ if its derivative is in $L^2(\Gamma_c)$ and if it satisfies (3.22). The norm of its derivative in $L^2(\Gamma_c)$ is given by

$$\|g\|^2_{H^1_*(\Gamma_c)} = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} nm g_n g_m A_n^m, \quad \text{where} \quad A_n^m = \int_I U_{n-1}\, U_{m-1} dx. \tag{4.93}$$

The bilinear form defined in (4.93) is exactly the one associated to the operator DD^*, which is also Neumann Problem 3.23, whose inverse is the operator $\mathcal{G}_N$. We now use the expression for $\mathcal{L}_2^{-1}$, given by (cf. Proposition 3.6)

$$\mathcal{L}_2^{-1} = -D^* \circ \mathcal{L}_1 \circ D. \tag{4.94}$$

Expand a function in $\widetilde{H}^{1/2}(\Gamma_c)$ on $w(x)U_{n-1}(x)$ as follows:

$$\alpha(x) \;=\; \sum_{n=1}^{\infty} \alpha_n w(x) U_{n-1}(x)\,, \qquad x \,\in\, I. \tag{4.95}$$

Using the above property of the expansion we have

$$(\mathcal{L}_2)^{-1}\alpha(x) \;=\; -\pi \sum_{n=1}^{\infty} n\alpha_n U_{n-1}(x)\,, \qquad x \,\in\, I. \tag{4.96}$$

Since $\mathcal{L}_2^{-1}$ is self-adjoint in $L^2(\Gamma_c)$, its square is associated to the bilinear form given by the squared norm in $L^2(\Gamma_c)$ of the above expression. This form is the same as the one associated to (4.93) except for the multiplicative factor π^2. This proves the first result of the proposition.

The function α is equal to zero at $x = \pm 1$ if it holds that

$$\sum_{p=1}^{\infty} 2p\alpha_{2p} = 0 \qquad \text{and} \qquad \sum_{p=0}^{\infty} (2p+1)\alpha_{2p+1} = 0 \tag{4.97}$$

and if α belongs to $H^1_0(\Gamma_c)$. As we have (4.97), its derivative, using D, can be written as

$$\frac{d\alpha}{dx}(x) = -\sum_{p=1}^{\infty} \left(2p\, \alpha_{2p} \frac{(T_{2p} - T_0)}{w}(x) + (2p-1)\alpha_{2p-1} \frac{(T_{2p-1} - T_1)}{w}(x) \right). \tag{4.98}$$

As the functions $w^{-1}(T_{2p} - T_0)$ and $w^{-1}(T_{2p-1} - T_1)$ are in $L^2(\Gamma_c)$, we can define the coefficients C_{2q}^{2p} and C_{2q-1}^{2p-1} as

$$C_{2q}^{2p} = \int_I \frac{(T_{2p} - T_0)(T_{2p} - T_0)}{w^2} dx, \tag{4.99a}$$

$$C_{2q-1}^{2p-1} = \int_I \frac{(T_{2p-1} - T_1)(T_{2q-1} - T_1)}{w^2} dx. \tag{4.99b}$$

Then, using the parity (or imparity) of these functions, the squared norm in $L^2(\Gamma_c)$ of the derivative of α is given by

$$\|\alpha\|^2_{H_0^1(\Gamma_c)} = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} 2p2q\alpha_{2p}\alpha_{2q}C_{2q}^{2p} + (2p-1)(2q-1)\alpha_{2p-1}\alpha_{2q-1}C_{2q-1}^{2p-1}. \tag{4.100}$$

This bilinear form is thus the one associated to Dirichlet Problem 3.8. The operator $\mathcal{L}_1^{-1}$ has the expression (cf. Proposition 3.6)

$$(\mathcal{L}_1)^{-1} = -D \circ \mathcal{L}_2 \circ D^*. \tag{4.101}$$

A function g given by

$$g(x) \;=\; \sum_{n=0}^{\infty} g_n T_n(x)\,, \qquad x \in I, \tag{4.102}$$

is in $H_0^1(\Gamma_c)$ when the bilinear form defined in (4.93) is bounded and is zero-valued at $x = \pm 1$, i.e.,

$$\sum_{p=1}^{\infty} g_{2p} = 0 \qquad \text{and} \qquad \sum_{p=0}^{\infty} g_{2p+1} = 0. \tag{4.103}$$

Furthermore, we assume that g satisfies

$$\sum_{p=0}^{\infty} 2p g_{2p} = 0, \qquad \sum_{p=0}^{\infty} (2p+1) g_{2p+1} = 0. \tag{4.104}$$

Using identities (4.11a), (4.11b), this is also equivalent to the two identities

$$\int_{-1}^{1} \frac{dg}{dx}\frac{1}{w(x)}\,dx \;=0, \qquad \int_{-1}^{1} \frac{dg}{dx}\frac{x}{w(x)}\,dx \;=0. \tag{4.105}$$

or, equivalently,

$$\int_{-1}^{1} g(x)\arcsin(x)\,dx \;=0, \qquad \int_{-1}^{1} g(x)w(x)\,dx \;=0. \tag{4.106}$$

Using (4.101) and properties of the Chebyshev polynomials, we have

$$(\mathcal{L}_1)^{-1}g(x) \;=\; -\pi\sum_{n=0}^{\infty} n g_n \frac{T_n}{w}(x)\,, \qquad x \in I. \tag{4.107}$$

Using identities (4.11), this last expression can be also written as (4.100) when g_n is replaced by α_n, which satisfies (4.11) and proves the result.

Remark 4.7. The eigenfunctions of the Dirichlet and the Neumann one-dimensional Laplace problems are function sets $\{\sin(n\pi x)\}_{n\geq 1}$ and $\{\cos(n\pi x)\}_{n\geq 1}$, respectively, and in each case the associated eigenvalue is $n\pi$. Thus, using the identities (3.10), we obtain the eigenfunctions of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathcal{L}_1 \sin(n\pi x) = n^{-1}\sin(n\pi x), \qquad n \geq 1, \tag{4.108a}$$

$$\mathcal{L}_2 \cos(n\pi x) = n^{-1}\cos(n\pi x), \qquad n \geq 1, \tag{4.108b}$$

respectively. The first identity can be proved directly using a Fourier transform and a property of the Hilbert transform. Identity (4.108b) can be proved using the first one and (4.94).

5. Conclusions. We have systematically derived precise variational forms and characterizations for norms, domains, and ranges for the weakly and hypersingular operators arising from the Laplace equation in two dimensions with a bounded cut in two-dimensional space. In particular, we observe that the derivation operator is key to understanding the differences in the associated functional spaces, and we provide examples for these. Moreover, we provide Calderón-type identities which will be used as preconditioners in an upcoming work. Additionally, we found a natural though surprising link between the logarithmic integral operators and one-dimensional Laplace problems.

Appendix A. Series expansions. We give here a short proof of the expansion formulas (4.13) and (4.17). Let $x = \cos\theta_1$ and $y = \cos\theta_2$ and associate complex numbers $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ whose moduli are equal to one. Introduce the complex logarithm function and write

$$\log(z_2 - z_1) = \log(|z_2 - z_1|) + \frac{i}{2}(\pi + \theta_1 + \theta_2), \tag{A.1a}$$

$$\log(z_2 - \overline{z}_1) = \log(|z_2 - \overline{z}_1|) + \frac{i}{2}(\pi + \theta_2 - \theta_1). \tag{A.1b}$$

Using the identity $\log(z_2 - z_1) = \log(z_2) + \log(1 - \overline{z}_2 z_1)$, we obtain the expansions

$$\log(z_2 - z_1) = i\theta_2 + \sum_{n=1}^{\infty} \frac{1}{n} e^{-in\theta_2} e^{in\theta_1}, \tag{A.2a}$$

$$\log(z_2 - \overline{z}_1) = i\theta_2 + \sum_{n=1}^{\infty} \frac{1}{n} e^{-in\theta_2} e^{-in\theta_1}, \tag{A.2b}$$

so that

$$\log\{(z_2 - z_1)(z_2 - \overline{z}_1)\} = 2i\theta_2 + \sum_{n=1}^{\infty} \frac{2}{n}\{\cos n\theta_1 \cos n\theta_2 - i \sin n\theta_2 \cos n\theta_1\}, \tag{A.3a}$$

$$\log\left\{\frac{z_2 - z_1}{z_2 - \overline{z}_1}\right\} = \sum_{n=1}^{\infty} \frac{2}{n}\{\sin(n\theta_1)\sin(n\theta_2) - i\cos(n\theta_2)\sin(n\theta_1)\}. \tag{A.3b}$$

Now using Pythagoras' theorem (or some computations in the complex plane), we obtain that

$$M(x, y) = |z_2 - \overline{z}_1|^2 = (x - y)^2 + w^2(x) + w^2(y), \tag{A.4a}$$

$$|z_2 - z_1|^2 |z_2 - \overline{z}_1|^2 = 4(x - y)^2. \tag{A.4b}$$

Using these two identities we obtain the expressions (4.13) and (4.17) and moreover

$$\theta_2 = \sum_{n=1}^{\infty} \frac{2}{n} w(y) U_n(y) T_n(x), \tag{A.5a}$$

$$\theta_1 = \sum_{n=1}^{\infty} \frac{2}{n} w(x) U_n(x) T_n(y). \tag{A.5b}$$

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