MULTILEVEL ADDITIVE SCHWARZ METHODS

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Abstract. We consider the solution of the algebraic system of equations which result from the discretization of elliptic equations. A class of multilevel algorithms are studied using the additive Schwarz framework. We establish that, in the general case, the condition number of the iterative operator grows at most linearly with the number of levels. The bound is independent of the mesh sizes and the number of levels under the H^2-regularity assumption of the equation. This is an improvement on Dryja and Widlund's result on a multilevel additive Schwarz algorithm, as well as Bramble, Pasciak and Xu's result on the BPX algorithm.

1. Introduction. Multilevel methods, such as multigrid methods, are considered the most efficient methods to solve the large systems of linear equations arising from the finite element or finite difference discretization of elliptic PDES; cf. Hackbusch [8], McCormick [9] and the references therein. Recently, with the increasing interest in parallel computation, several new multilevel methods have been developed and analyzed, e.g., Yserentant's hierarchical basis method [12], the hierarchical basis multigrid method of Bank, Dupont and Yserentant [1], the parallel multilevel preconditioners developed in Bramble, Pasciak and Xu [4] and Xu [11], and the multilevel additive Schwarz methods of Dryja and Widlund [7].

We consider second order, self-adjoint, uniformly elliptic differential equations, on a two or three-dimensional polygonal domain, approximated by using continuous, piecewise linear finite elements. We use multilevel additive Schwarz (MAS) algorithms to solve the resulting linear system. When such algorithms are used, an equivalent equation is solved by an iterative method such as the conjugate gradient method. In each iteration, a number of independent problems corresponding to subdomains are solved, while the size of all the subproblems can be very small. Dryja and Widlund have shown that the condition number of a multilevel additive Schwarz operator grows at most quadratically with the number of levels; cf. [7]. Similar results for the BPX algorithm were established in Bramble, Pasciak and Xu [4] and Xu [11]. In this paper, we improve the results for a class of multilevel methods by showing that the condition number of the MAS operator grows at most linearly with the number of levels in general, and is bounded by a constant independent of mesh sizes and the number of levels under the H^2-regularity assumption of the elliptic operator. Such regularity assumption can be verified for elliptic problems with smooth coefficients in a convex domain. We note that similar results are already known for multigrid methods.

The rest of the paper is organized as follows. In section 2, we describe a class of multilevel additive Schwarz algorithms. In section 3, we establish a bound for the condition number of the iterative operator of the algorithm. In section 4, we describe a variant of the algorithm. We construct a very special decomposition of the space, and show that it can also be regarded as a multilevel diagonal scaling (MDS) algorithm. In the case of constant coeffi-

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cients and uniform triangulation, the MDS algorithm is identical, up to a constant multiple, to the BPX algorithm of Bramble, Pasciak and Xu [4]. As a consequence, we also obtain an improved result on the BPX algorithm. In section 5, we report on some numerical results of the multilevel additive Schwarz method for the model problem. In these experiments, we only concern with the convergence properties of the algorithms. For implementations of the algorithms on a parallel computer, see Bjørstad, Moe and Skogen [2] and Bjørstad and Skogen [3]. They implemented multilevel additive Schwarz algorithms on a massively parallel, SIMD machine (MasPar MP-1). Approximate solvers for the subproblems are also discussed.

2. Description of the Multilevel Additive Schwarz Methods. We describe the method and carry out the analysis for Poisson's equation, although the algorithm and analysis carry over to more general second order problems without any difficulty. In particular, we can obtain a good upper bound as long as the coefficients do not change very much inside individual substructure. We consider the model problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{in} \quad \partial\Omega. \end{array} \right. \tag{1}$$

Here Ω is a bounded polygonal region in R^2 or R^3. The variational form is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \; dx \quad \text{ and } \quad f(v) = \int_\Omega fv \; dx.$$

We define a sequence of nested triangulations $\{\mathcal{T}^l\}_{l=1}^L$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau_i^1\}_{i=1}^{N_1}$, where τ_i^1 represents an individual triangle. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}_{i=1}^{N_l}$ are defined by dividing triangles in the triangulation $\mathcal{T}^{l-1}$ into several triangles. We assume that all the triangulations are shape regular. Let $h_i^l = \text{diameter}(\tau_i^l), h_l = \max_i h_i^l$ and $h = h_L$. We also assume that $h_{l_1}/h_{l_2} \leq Cr^{2(l_1-l_2)}$ with $r < 1$.

Let $V^l, l = 1, \cdots, L$, be the space of continuous piecewise linear element associated with the triangulation $\mathcal{T}^l$. The finite element solution, $u_h = P_{V^h} u \in V^h = V^L$, satisfies

$$a(u_h, \phi_h) = f(\phi_h), \quad \forall \phi_h \in V^h = V^L. \tag{3}$$

We assume that there are $L-1$ sets of overlapping subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, 3, \cdots, L$. On each level, we have an overlapping decomposition

$$\Omega = \cup_{i=1}^{N_l} \hat{\Omega}_i^l.$$

We assume that the sets $\{\hat{\Omega}_i^l\}$ satisfy

Assumption 2.1. *The decomposition* $\Omega = \cup_{i=1}^{N_l} \hat{\Omega}_i^l$ *satisfies*

- *$\partial\hat{\Omega}_i^l$ aligns with the boundaries of level l triangles, i.e. $\hat{\Omega}_i^l$ is the union of level l triangles. Diameter($\hat{\Omega}_i^l$) $= O(h_{l-1})$.*
- *On each level, the subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form a* finite covering *of Ω, with a covering constant N_c, i.e. we can color $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$, using at most N_c colors in such a way that subdomains of the same color are disjoint.*
- *On each level, associated with $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$, there exists a* partition of unity *$\{\theta_i^l\}$ satisfying*

$$\sum_i \theta_i^l = 1, \text{ with } \theta_i^l \in H_0^1(\hat{\Omega}_i^l) \cap C^0(\hat{\Omega}_i^l), 0 \le \theta_i^l \le 1 \text{ and } |\nabla\theta_i^l| \le C/h_{l-1}.$$

The first property is very natural; it simply says that the restriction of the triangulation $\mathcal{T}^l$ to subdomain $\hat{\Omega}_i^l$ defines a triangulation for $\hat{\Omega}_i^l$ and the finite element problem on $\hat{\Omega}_i^l$ is well defined. The second condition is used to establish the upper bound of the spectrum of the additive Schwarz operator. The last condition is used for the lower bound of the spectrum.

One way of constructing subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}, l = 2, \cdots, L$, with the above properties is described in Dryja and Widlund [5,6]. Each triangle $\tau_i^{l-1}, i = 1, \cdots, N_l, l = 2, \cdots, L$, is extended to a larger region $\hat{\tau}_i^{l-1}$ so that $ch_i^{l-1} \le \text{dist}(\partial\hat{\tau}_i^{l-1}, \partial\tau_i^{l-1}) \le Ch_i^{l-1}$, aligning $\partial\hat{\tau}_i^{l-1}$ with the boundaries of level l triangles. We cut off the part of $\hat{\tau}_i^{l-1}$ that is outside Ω. We use $\hat{\tau}_i^{l-1}$ as the subdomains $\hat{\Omega}_i^l$. Another way of constructing $\{\hat{\Omega}_i^l\}$ is given in section 4.

Let $N_1 = 1, V_1^1 = V^1$ and $V_i^l = V^l \cap H_0^1(\hat{\Omega}_i^l)$ for $i = 1, \cdots, N_l, l = 2, \cdots, L$. The finite element space $V^h = V^L$ is represented by

$$V^L = \sum_{l=1}^{L} V^l = \sum_{l=1}^{L}\sum_{i=1}^{N_l} V_i^l. \tag{4}$$

Let $P_i^l : V^h \to V_i^l$, be the projection defined by

$$a(P_i^l u, \phi) = a(u, \phi), \quad \forall\phi \in V_i^l.$$

The L-level additive Schwarz operator P is defined by

$$P = \sum_{l=0}^{L}\sum_{i=1}^{N_l} P_i^l. \tag{5}$$

Instead of solving the original finite element equation (3), we solve the following equivalent equation:

Algorithm 2.1. *Find $u_h \in V^L$ by solving iteratively the equation*

$$Pu_h = g_h$$

with $g_h = \sum_l \sum_i g_i^l$. Here the g_i^l are the solutions for the following finite element problems

$$a(g_i^l, \phi) = a(P_{V_i^l} u, \phi) = f(\phi), \quad \forall\phi_h \in V_i^l. \tag{6}$$

To find u_h, we first find the right hand side g_h, by solving (6), and we then use the conjugate gradient method to solve the system. In each iteration, we need to compute $P_i^l v_h$ for a given $v_h \in V^h$ by solving the equation

$$a(P_{V_i^l} v_h, \phi_h) = a(v_h, \phi_h) \quad \forall \phi_h \in V_i^l.$$

This is a finite element equation on $\hat{\Omega}_i^l$ with mesh size h_l, and $\dim(V_i^l) \approx c(h_{l-1}/h_l)^2$. Thus the size of all such problems can be very small.

3. Condition Number Estimate. When using the conjugate gradient method to solve a linear system, the crucial issue is the condition number of the iteration operator. Dryja and Widlund [7] established the following estimates for the spectrum of P:

$$C_1 L^{-1} a(u_h, u_h) \le a(Pu_h, u_h) \le C_2 L\, a(u_h, u_h) \ \forall u_h \in V^h. \tag{7}$$

Thus $\kappa(P) \le C_2 C_1^{-1} L^2$, i.e. the condition number of P grows at most quadratically with the number of levels. In this section, we improve the upper bound in (7) by eliminating the dependence on L. Using H^2-regularity, which holds for convex regions, we can also eliminate the dependence on L in the lower bound.

THEOREM 3.1. *For the multilevel additive Schwarz operator P defined above, we have*

$$C_1 L^{-1} a(u_h, u_h) \le a(Pu_h, u_h) \le C_2\, a(u_h, u_h) \ \forall u_h \in V^h.$$

If the equation has H^2-regularity, then the lower bound can also be improved, and we have

$$C_1 a(u_h, u_h) \le a(Pu_h, u_h) \le C_2\, a(u_h, u_h) \ \forall u_h \in V^h.$$

All the constants are independent of $\{h_l\}$ and L.

To prove the theorem, we use two lemmas. The first lemma, known as Lions' lemma, is very important in estimating the minimum eigenvalue of P; see Dryja and Widlund [5,6,7] and Nepomnyaschikh [10] for different variants of this result.

LEMMA 3.1. *Let V be a Hilbert space, V_i be subspaces of V, and $V = \sum V_i$. Let P_{V_i} be the projections from V onto V_i, and $P = \sum_i P_{V_i}$. Then*

$$\lambda_{\min}^{-1}(P) = \lambda_{\max}(P^{-1}) = \max_u \frac{a(P^{-1}u, u)}{a(u,u)} = \max_u \min_{\sum u_i = u} \frac{\sum_i a(u_i, u_i)}{a(u,u)}.$$

Proof. We note that

$$a(P^{-1}u, u) = \sum_i a(P^{-1}u, u_i) = \sum_i a(P_i P^{-1}u, u_i) \le a(P^{-1}u, u)^{1/2} (\sum_i a(u_i, u_i))^{1/2}.$$

Thus

$$a(P^{-1}u, u) = \min_{\sum u_i = u} \sum_i a(u_i, u_i), \tag{8}$$

and that the minimum is achieved for $u_i = P_i P^{-1} u$. The lemma follows from (8). ∎

Let $P^l : V^h \to V^l$, be the orthogonal projection. Then, for $u_h \in V^h$ and $1 \leq l \leq L$, we have

$$(9) \qquad P^l u_h = \sum_{i=1}^{l} u^i, \text{ with } u^i = (P^i - P^{i-1}) u_h, \quad u^1 = P^1 u_h.$$

In particular, $u_h = P^L u_h = \sum_{l=1}^{L} u^l$. Using the fact that $V^k \subset V^l$, for $k \leq l$, we obtain $P^k P^l = P^l P^k = P^k$. Thus $P^l - P^{l-1}$ is also a projection, and

$$(P^l - P^{l-1})(P^k - P^{k-1}) = \delta_{lk}(P^l - P^{l-1}).$$

Therefore $u_h = \sum_{i=0}^{L} u^i$ is an $a(\cdot,\cdot)$-orthogonal decomposition of u_h, and

$$(10) \qquad a(u_h, u_h) = \sum_{l=1}^{L} a(u^l, u^l)$$

Since, on each level, the subdomains $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form a finite covering of Ω with a covering constant N_c, we can color the $\hat{\Omega}_i^l$ using only N_c colors, in such a way that all subdomains of the same color are disjoint. On each level, we can group the subdomains $\hat{\Omega}_i^l$ by color, obtaining N_c sets of subregions. Let $\Lambda_s, s = 1, \cdots, N_c$, be the sets of indices of these sets. Since the subdomains $\hat{\Omega}_i^l$ of the same color are disjoint, subspaces V_i^l of the same color are mutually orthogonal and $P_{i_1}^l P_{i_2}^l = 0$ for i_1, i_2 in the same Λ_s. This in turn implies that $P_{\Lambda_s}^l = \sum_{i \in \Lambda_s} P_i^l$ is a projection from V^h onto $V_{\Lambda_s}^l = \sum_{i \in \Lambda_s} V_i^l$. In particular, $(P_{\Lambda_s}^l)^2 = P_{\Lambda_s}^l$. We can therefore write

$$\sum_{i=1}^{N_l} P_i^l = \sum_{s=1}^{N_c} \sum_{i \in \Lambda_s} P_i^l \equiv \sum_{s=1}^{N_c} P_{\Lambda_s}^l,$$

where the $P_{\Lambda_s}^l$ are projections.

The next lemma, similar to the strengthened Cauchy inequality, plays an important role in obtaining an upper bound for $\lambda_{\max}(P)$.

Lemma 3.2. *For $k \leq l$, and $u^k \in V^k$, we have*

$$a(P_{\Lambda_s}^l u^k, u^k)^{1/2} \leq r_{k,l}\, a(u^k, u^k)^{1/2}$$

with $r_{k,k} \leq 1, r_{k,k+1} \leq 1$ and $r_{k,l} \leq C r^{l-1-k}$, for $k < l - 1$.

Proof. Since $P_{\Lambda_s}^l$ is a projection, the conclusions for $l = k$ and $l = k+1$ are trivial. For $l - 1 > k$, we decompose Λ_s into two disjoint sets Λ_I and Λ_B; $i \in \Lambda_I$ if the subdomain $\hat{\Omega}_i^l$ lies in the interior of a triangle τ^k, while $i \in \Lambda_B$ if $\hat{\Omega}_i^l$ intersects $\partial \tau_{i'}^k$ for some i'. We write $P_{\Lambda_s} = P_{\Lambda_I} + P_{\Lambda_B}$. We note that u^k is linear in each triangle τ_j^k and therefore harmonic in τ_j^k. For each $i \in \Lambda_I$, $P_i^l u^k \in H_0^1(\hat{\Omega}_i^l) \subset H_0^1(\tau_{i'}^k)$ for some i', and therefore, $a(P_{\Lambda_I}^l u^k, u^k) = 0$. Thus,

$$a(P_{\Lambda_s}^l u^k, u^k) = a(P_{\Lambda_B}^l u^k, u^k)$$

Let $S = \text{supp}\{P_{\Lambda_B} u^k\} = \cup_{i\in\Lambda_B} \hat{\Omega}_i^l$. Then

$$a(P_{\Lambda_B}^l u^k, u^k) = a_S(P_{\Lambda_B}^l u^k, u^k) \le a_S(u^k, u^k)$$

Since u^k is linear in τ_j^k, ∇u^k is constant in τ_j^k. Therefore

$$a_{S\cap\tau_j^k}(u^k, u^k) = \frac{\text{mes}(S\cap\tau_j^k)}{\text{mes}(\tau_j^k)} a_{\tau_j^k}(u^k, u^k) \le C\frac{h_{l-1}}{h_k} a_{\tau_j^k}(u^k, u^k) = Cr_{k,l}^2 a_{\tau_j^k}(u^k, u^k).$$

Summing over j, we obtain

$$a_S(u^k, u^k) \le Cr_{k,l}^2 a(u^k, u^k).$$

Therefore

$$a(P_{\Lambda_s}^l u^k, u^k) = a(P_{\Lambda_B}^l u^k, u^k) \le a_S(u^k, u^k) \le Cr_{k,l}^2 a(u^k, u^k).$$

∎

Proof of Theorem 3.1. We first establish the upper bound. Since $V_{\Lambda_s}^l = \sum_{i\in\Lambda_s} V_i^l \subset V^l$, we have $P_{\Lambda_s}^l P^l = P_{\Lambda_s}^l$. Thus,

$$a(P_{\Lambda_s}^l u_h, u_h) = a(P_{\Lambda_s}^l u_h, P_{\Lambda_s}^l u_h) = a(P_{\Lambda_s}^l P^l u_h, P_{\Lambda_s}^l P^l u_h).$$

Substituting (9) into the above equation, we obtain

$$\begin{aligned} a(P_{\Lambda_s^l} u_h, u_h) &= \sum_{k=1}^{l}\sum_{j=1}^{l} a(P_{\Lambda_s}^l u^k, P_{\Lambda_s}^l u^j) \le \sum_{k,j=1}^{l} |P_{\Lambda_s}^l u^k|_a |P_{\Lambda_s}^l u^j|_a \\ &= (\sum_{k=1}^{l} |P_{\Lambda_s}^l u^k|)^2 \le C(\sum_{k=1}^{l} r_{k,l}|u^k|_a)^2 \\ &\le C(\sum_{k=1}^{l} r_{k,l})(\sum_{k=1}^{l} r_{k,l} a(u^k, u^k)) \le C\frac{2}{1-r}\{\sum_{k=1}^{l} r_{k,l} a(u^k, u^k)\}. \end{aligned}$$

Summing over all colors ($1 \le s \le N_c$), we get

$$\sum_{i=1}^{N_l} a(P_i^l u_h, u_h) = \sum_{s=1}^{N_c} a(P_{\Lambda_s}^l u_h, u_h) \le CN_c\frac{2}{1-r}(\sum_{k=1}^{l} r_{k,l} a(u^k, u^k)).$$

Summing over l and changing the order of the summation for k and l, we get

$$\begin{aligned} \sum_{l=1}^{L}\sum_{i=1}^{N_l} a(P_i^l u_h, u_h) &\le CN_c\frac{2}{1-r}(\sum_{k=1}^{L}\sum_{l=k}^{L} r_{k,l} a(u^k, u^k)) \\ &\le CN_c\frac{2}{1-r}\{\sum_{k=1}^{L} a(u^k, u^k)(\sum_{l=k}^{L} r_{k,l})\} \\ &\le CN_c(\frac{2}{1-r})^2 \sum_{k=1}^{L} a(u^k, u^k) \\ &= CN_c(\frac{2}{1-r})^2 a(u_h, u_h). \end{aligned}$$

In the last step, we have used the orthogonality property (10) of the u^k. This concludes the proof of the upper bound of $P = \sum_l \sum_i P_i^l$.

We now establish the lower bound. We note that, for the general case, it is given in Dryja and Widlund [7]. When the problem is H^2-regular, we can use Nitsche's trick to show that the $a(\cdot,\cdot)$-projection P^l satisfies the approximation property

$$\|P^l u - u\|_{L^2(\Omega)} \leq C h_l |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)\,. \tag{11}$$

We use the orthogonal decomposition

$$u_h = P^L u_h = \sum_{l=1}^{L} u^l \equiv P^1 u_h + (P^2 - P^1) u_h + \cdots + (P^L - P^{L-1}) u_h.$$

Since $u^l = (P^l - P^{l-1}) u_h = (P^l - P^{l-1})^2 u_h = (I - P^{l-1}) u^l$, we get, using (11),

$$\|u^l\|_{L^2(\Omega)} \leq C\, h_{l-1} |u^l|_{H^1(\Omega)}. \tag{12}$$

We further decompose u^l as

$$u^l = \sum_{i=1}^{N_l} u_i^l\,, \text{ with } u_i^l \equiv \Pi^{h_l}(\theta_i^l u^l) \in V_i^l.$$

Here Π^{h_l} is the interpolation operator from V^h to V^{h_l} and $\{\theta_i^l\}$ a partition of unity as in assumption 2.1. It can be shown that

$$\begin{aligned}
|u_i^l|^2_{H^1(\hat{\Omega}_i^l)} &= |\Pi^{h_l}(\theta_i u^l)|^2_{H^1(\hat{\Omega}_i^l)} \\
&\leq C(|\theta_i|^2_{L^\infty(\Omega)} |u^l|^2_{H^1(\hat{\Omega}_i^l)} + |\theta_i|^2_{W^{1,\infty}(\Omega)} \|u_i^l\|^2_{L^2(\hat{\Omega}_i^l)}) \\
&\leq C(|u^l|^2_{H^1(\hat{\Omega}_i^l)} + (1/h_{l-1}^2) \|u_i^l\|^2_{L^2(\hat{\Omega}_i^l)}).
\end{aligned}$$

Summing the above inequality over i, using the finite covering property of $\{\hat{\Omega}_i^l\}$ and inequality (12), we obtain

$$\begin{aligned}
\sum_i |u_i^l|^2_{H^1(\Omega)} &= \sum_i |u_i^l|^2_{H^1(\hat{\Omega}_i^l)} \leq C \sum_i \{|u^l|^2_{H^1(\hat{\Omega}_i^l)} + 1/h_{l-1}^2 \|u^l\|^2_{L^2(\hat{\Omega}_i^l)}\} \\
&\leq C\{|u^l|^2_{H^1(\Omega)} + 1/h_{l-1}^2 \|u^l\|^2_{L^2(\Omega)}\} \leq C |u^l|^2_{H^1(\Omega)}.
\end{aligned}$$

Summing over l, $1 \leq l \leq L$, and using the orthogonality of u^l, we get

$$\sum_{l=1}^{L} \sum_i |u_i^l|^2_{H^1(\Omega)} \leq C |u_h|^2_{H^1(\Omega)}.$$

The lower bound for P now follows from Lions' lemma. ∎

REMARK 3.1. Although the proof of the upper bound is given for the model problem, it is easy to see that it works for any uniform elliptic operator. Since we can confine our study to one substructure at a time, we also see that the upper bound is independent of jumps in the coefficients between the substructures.

4. A Multilevel Diagonal Scaling. We begin this section by constructing a special decomposition of the domain Ω. We then show that this decomposition, and the corresponding decomposition of the finite element subspaces, satisfies Assumption 2.1. We then demonstrate that the algorithm is a multilevel diagonal scaling (MDS), a natural generalization of diagonal scaling. For problems with constant coefficients and with uniform triangulations, the multilevel diagonal scaling algorithm is identical, up to a constant multiple, to the BPX algorithm of Bramble, Pasciak and Xu [4]. In the general case, BPX with diagonal scaling results in MDS algorithm.

Let $\{\mathcal{T}^l\}_{l=1}^L$ be a nested sequence of triangulations, with $\mathcal{T}^{l+1}$ obtained from $\mathcal{T}^l$ by dividing the triangles (rectangles) of $\mathcal{T}^l$ into four triangles (rectangles). In three dimension, we make a similar construction. We consider the piecewise linear and bilinear elements or trilinear elements, respectively. As in the previous section, the finite element space associated with $\mathcal{T}^l$ is denoted by V^l, and $V^h = V^L$. Let ϕ_i^l be a nodal basis function of V^l, and associate with each ϕ_i^l a subdomain $\hat{\Omega}_i^l = \mathrm{supp}\{\phi_i^l\}$. We choose $V_i^l = \mathrm{span}\{\phi_i^l\} = V^l \cap H_0^1(\hat{\Omega}_i^l)$ and obtain the decomposition

$$V^L = \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l$$

and the projections $P_{V_i^l} : V^L \xrightarrow{H^1} V_i^l$. Using $P = \sum_{l=1}^L \sum_i P_{V_i^l}$, we define an additive Schwarz algorithm

Algorithm 4.1 (MDS). *Find the finite element solution $u_h \in V^h$ by solving iteratively the equation*

$$Pu_h = g_h$$

with an appropriate right hand side g_h.

We define the degree of a vertex x_i as the number of edges directly connected to x_i, and the degree of a triangulation $\mathcal{T}^h$ as the maximum of the degrees of its vertices. It is easy to see that the overlapping subdomains $\{\hat{\Omega}_i^l\}$ satisfy Assumption 2.1. In particular, we see that on each level, $\{\hat{\Omega}_i^l\}_{i=1}^{N_l}$ form an finite covering of Ω with a covering constant less than or equal to degree($\mathcal{T}^l$)+1. We also see that on each level, $\{\hat{\Omega}_i^l\}$ provides relatively generous overlap. The nodal basis $\{\phi_i^l\}$ can be chosen as a partition of unity.

The optimal convergence properties of the algorithm follow from theorem 3.1, and as a consequence, we obtain an improved estimate for the BPX algorithm; see further discussion below.

To see that the above algorithm is in fact a generalization of the diagonal scaling method, we first consider a matrix representation of two simple algorithms.

Algorithm 1. In the two level additive Schwarz algorithm, the matrix form of the projections P_{V_i} is given by

$$P_{V_i} = \begin{pmatrix} K_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} K_h = \tilde{K}_i^+ K_h$$

after a permutation. Here K_i is the stiffness matrix associated with the subspace V_i. Let $\Pi_H^h : V^H \to V^h$ be the standard interpolation operator and Π_h^H be its adjoint. In matrix form, the two level additive Schwarz algorithm can then be written as a preconditioned system

$$B_h^{-1} K_h x = B_h^{-1} b,$$

where

$$B_h^{-1} = \Pi_H^h K_H^{-1} \Pi_h^H + \sum_{i=1}^{N} \tilde{K}_i^{+}.$$

Algorithm 2. With the special choice of the subregions $\hat{\Omega}_i = \mathrm{supp}\{\phi_i\}$, and $V_i = \mathrm{span}\{\phi_i\}$, the additive Schwarz algorithm corresponds to

$$D^{-1} K_h \vec{u} = D^{-1} b,$$

where $D = \mathrm{diag}(K_h)$. This gives us the Jacobi conjugate gradient method.

In the multilevel case, let $\Pi_{l_1}^{l_2} : V^{l_1} \to V^{l_2}, (l_1 \geq l_2)$ be the standard interpolation (prolongation) operator, and let $\Pi_{l_2}^{l_1} : V^{l_2} \to V^{l_1} = (\Pi_{l_1}^{l_2})^t, (l_1 \geq l_2)$ be a local averaging operator, which is the adjoint operator of $\Pi_{l_1}^{l_2}$. Algorithm 4.1 can then be written as: Find the solution of $K_L x = b$ by solving the preconditioned system

$$B_L^{-1} K_L x = B_L^{-1} b,$$

where

$$B_L^{-1} = \Pi_1^L K_1^{-1} \Pi_L^1 + \Pi_2^L D_2^{-1} \Pi_L^2 + \cdots + \Pi_{L-1}^L D_{L-1}^{-1} \Pi_L^{L-1} + D_L^{-1}.$$

Here K_l is the stiffness matrix associated with V^l and $D_l = \mathrm{diag}(K_l)$. We note that K_1^{-1} can be replaced by any good preconditioner B_1 of K_1.

If we replace the matrices D_l by identity matrices, we obtain the BPX algorithm. However, since the diagonal elements contain information on the shapes of the triangle and the coefficients of the problems, we expect that the multilevel diagonal preconditioner will work better in practice for non-model problems, since it more closely reflects the properties of the problem.

The method described in this section is similar to the hierarchical basis method [12]. The work in each iteration is about $\frac{4}{3}$ of that of the hierarchical basis method. However, the condition number is much better than for the hierarchical basis method, and the method also works well in higher dimensions, at least for problems with smooth coefficients. A detailed comparison of the BPX algorithm and the hierarchical basis method is given in Yserentant [13].

total no. of elements	no. of subdomains of a lower level $N \times N$	ovlp ratio	no. of levels L	cond no. $\kappa(P)$	no. of iter. for $\epsilon = 10^{-6}$
8^2	2×2	1/2	3	7.2	11
16^2	2×2	1/2	4	9.3	17
32^2	2×2	1/2	5	10.7	20
64^2	2×2	1/2	6	11.7	21
9^2	3×3	1/3	2	4.6	9
27^2	3×3	1/3	3	7.1	16
81^2	3×3	1/3	4	8.4	19
243^2	3×3	1/3	5	9.5	21
27^2	3×3	1/3	2	4.8	8
81^2	3×3	1/3	2	4.7	7
16^2	4×4	1/4	2	5.1	13
64^2	4×4	1/4	3	7.3	17
256^2	4×4	1/4	4	8.4	20
64^2	4×4	1/4	2	5.3	8
25^2	5×5	1/5	2	5.7	14
125^2	5×5	1/5	3	7.6	17

Table 1
Multilevel Additive Schwarz Scheme, Using Bilinear Element

5. Numerical Experiments. In this section, we report on some the numerical experiments with the multilevel additive Schwarz methods. These experiments were carried out for Poisson's equation on a unit square with homogeneous Dirichlet boundary conditions

$$\left\{ \begin{array}{rcll} -\Delta u & = & f(x) & \text{in} \quad \Omega, \\ u & = & 0 & \text{in} \quad \partial\Omega. \end{array} \right. \tag{13}$$

We divide the domain Ω into $N \times N$ square elements $\tau_{ij}^1, i, j = 1, \cdots, N$, and obtain a triangulation $\mathcal{T}^1 = \{\tau_{ij}^1\}$. We then divide each $\tau_{i,j}^1$ into $N \times N$ squares to obtain the triangulation $\mathcal{T}^2 = \{\tau_{ij}^2\}$, etc. The length of an edge of τ_{ij}^l is denoted by H_l and $H_l = (1/N)^l$. For $l = 2, \cdots, L$, we extend τ_{ij}^{l-1} to a larger square $\hat{\tau}_{ij}^{l-1}$. The overlap ratio

$$\text{overlap ratio} = \frac{\text{dist}\{\partial\hat{\tau}_{ij}^{l-1}, \partial\tau_{ij}^{l-1}\}}{H_{l-1}}$$

measures the width of $\hat{\tau}_{ij}^{l-1} \setminus \tau_{ij}^{l-1}$ in terms of H_{l-1}, the side of the square τ_{ij}^{l-1}. We use Ω as our domain for $l = 1$, and $\Omega_{ij}^l = \hat{\tau}_{ij}^{l-1}$ as our subdomains for $l = 2, \cdots, N$.

In these experiments, we take $N = 2, 3, 4$ or 5, and $\hat{\tau}_{ij}^{l-1} \setminus \tau_{ij}^{l-1}$ is one element wide(H_l), i.e. the overlap ratio is $1/N$. Therefore, we only need to solve very small linear systems of order 9, 16 25 and 36, respectively. We use the conjugate gradient method to solve the system $Pu_h = g_h$ iteratively. The last column of the table gives the number of iterations required to decrease the l_2 norm of the residual by a factor of $\epsilon = 10^{-6}$.

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