IMA Journal of Numerical Analysis (2000) **20**, 601–619

On the numerical solution of a hypersingular integral equation for elastic scattering from a planar crack

ROMAN CHAPKO
Department of Applied Mathematics and Computer Science, Lviv University, 290000 Lviv, Ukraine

RAINER KRESS
Institut für Numerische und Angewandte Mathematik, Universität Göttingen, 37083 Göttingen, Germany

AND

LARS MÖNCH
Softlab GmbH, Zamdorfer Str. 120, 81677 München, Germany

[Received 30 November 1998 and in revised form 22 November 1999]

In this paper we describe a fully discrete quadrature method for the numerical solution of a hypersingular integral equation of the first kind for the scattering of time-harmonic elastic waves by a cavity crack. We establish convergence of the method and prove error estimates in a Hölder space setting. Numerical examples illustrate the convergence results.

1. Introduction

Direct and inverse elastic scattering problems are of significant interest in different fields of mathematical physics and applied sciences with applications, for example, in fracture mechanics and in nondestructive testing. For a two-dimensional crack, using integral equation techniques, two of the present authors have investigated the solution of the direct and inverse scattering problems for time-harmonic elastic waves with a Dirichlet boundary condition (see Kress, 1996) and for time-harmonic acoustic waves with a Dirichlet (see Kress, 1995a) and a Neumann boundary condition (see Mönch, 1996, 1997). In this paper we initiate the extension of these methods to the case of elastic waves with a Neumann boundary condition. We describe the numerical solution of the hypersingular integral equation arising from an elastic double-layer potential approach for the direct scattering problem by a fully discrete method and investigate its convergence and error analysis in a Hölder space setting. The corresponding inverse problem will be the subject of a forthcoming paper.

Consider the scattering of time-harmonic elastic waves by a planar crack described by an arc $\Gamma \subset \mathbb{R}^2$ of class C^∞, i.e., $\Gamma = \{\gamma(t) : t \in [-1, 1]\}$ where $\gamma : [-1, 1] \to \mathbb{R}^2$ is injective and infinitely differentiable. By θ we denote the unit tangent vector to Γ given by $\theta(\gamma(t)) = |\gamma'(t)|^{-1}\gamma'(t)$ and by $\nu = Q\theta$ the unit normal vector, where Q denotes the unitary matrix

$$Q := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We denote the two end points of the arc by $x^*_{\pm 1}$.

Assume that the exterior of the crack is filled with an isotropic and homogeneous elastic medium with Lamé constants μ and λ satisfying $\lambda > -\mu$ and $\mu > 0$. Then the displacement field u of the scattered wave is described by a solution of the Navier equation

$$\mu \Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u + \omega^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \tag{1.1}$$

where $\omega > 0$ is the frequency. The total displacement field is given by the superposition $u + u^i$ of the scattered field u and the incident field u^i which we assume to be an entire solution of the Navier equation. For a cavity crack the total displacement has to satisfy the Neumann boundary condition

$$T(u + u^i) = 0 \quad \text{on } \Gamma \setminus \{x_1^* \cup x_{-1}^*\} \tag{1.2}$$

with the traction operator T given by

$$T v := \lambda \operatorname{div} v \, \nu + 2\mu \, (\nu \cdot \operatorname{grad}) \, v + \mu \operatorname{div}(Qv) \, Q\nu. \tag{1.3}$$

In addition, to ensure uniqueness, the scattered field u is required to satisfy the Kupradze radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u_p}{\partial r} - i k_p u_p \right) = 0, \quad \lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u_s}{\partial r} - i k_s u_s \right) = 0, \quad r = |x|, \tag{1.4}$$

uniformly in all directions. Here, the longitudinal wave u_p and the transversal wave are defined by

$$u_p := -\frac{1}{k_p^2} \operatorname{grad} \operatorname{div} u, \quad u_s := u - u_p$$

and the associated wave numbers are given by

$$k_p^2 := \frac{\omega^2}{\lambda + 2\mu}, \quad k_s^2 := \frac{\omega^2}{\mu},$$

respectively. For regularity, we require $u \in C^2(\mathbb{R}^2 \setminus \Gamma)$ such that u is bounded and the boundary condition (1.2) is satisfied in the sense of locally uniform convergence for all $x \in \Gamma \setminus \{x_1^* \cup x_{-1}^*\}$. (Throughout the paper the function spaces such as $C^2(\mathbb{R}^2 \setminus \Gamma)$ and $C(\Gamma)$ have to be understood as vector valued, i.e., $u : \mathbb{R}^2 \setminus \Gamma \to \mathbb{C}^2$ and $\varphi : \Gamma \to \mathbb{C}^2$.) Under these assumptions we have the following uniqueness result due to Wickham (1981) (see also Lewis & Wickham, 1992).

THEOREM 1 The direct scattering problem (1.1), (1.2), (1.4) has at most one solution.

Proof. Wickham considered only the case of a straight crack. However, it is possible to extend Wickham's analysis to the case of arbitrarily shaped cracks. The proof makes use of Betti's integral theorem and the Kupradze radiation condition. □

Wickham also obtained an existence result for a straight crack by using a modified Green's function approach that leads to an integral equation of the second kind. A weak solution approach in a Sobolev space setting via integral equations of the first kind was described by Wendland & Stephan (1990). For a treatment of the Neumann boundary value problem for elastic waves by an integral equation of the second kind in the case of closed boundary curves we refer to Ahner & Hsiao (1976). For further related work on acoustic and elastic scattering from a crack in $\mathbb{R}^2$ we refer to Hsiao *et al.* (1991), Krishnasamy *et al.* (1990), Martin (1981), Martin & Rizzo (1989), and Sáez *et al.* (1995).

In our paper we reduce the boundary value problem (1.1), (1.2), (1.4) to an integral equation of the first kind by seeking the solution in the form of an elastic double-layer potential. We decompose this integral equation into a hypersingular part given by the limiting static case and the remaining part given by the difference between the time-harmonic and static case which contains only a logarithmic singularity. For the static hypersingular integral operator we make use of a Maue-type transformation. Then, by using the cosine transformation introduced by Multhopp (1938) and Yan & Sloan (1988) for related potential theoretic problems, we reduce the integral equation for a crack to an integral equation for a closed boundary curve making the existence analysis very concise.

For the numerical solution of the hypersingular integral equation and its error analysis we adopt the approach to the corresponding integral equations for the Neumann problem in the scalar case of the Helmholtz equation as described in detail by Kress (1995b) and Mönch (1996). It consists of a collocation method using trigonometric interpolation at an equidistant grid. Then the collocation method is made fully discrete by using product quadratures for the singular integrals which again are based on trigonometric interpolation. The error analysis of Kress (1995b) and Mönch (1996) carries over to the present case.

2. Static double-layer potential

In the static case $\omega = 0$, the fundamental solution to the Navier equation is given by

$$\Phi_0(x, y) := \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \Psi(x, y) I + \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)} J(x - y),$$

where

$$\Psi(x, y) := \ln \frac{1}{|x - y|}, \quad x \neq y,$$

I is the identity matrix, and the matrix J is defined by

$$J(w) := \frac{w\, w^\top}{|w|^2}$$

in terms of a dyadic product of $w \in \mathbb{R}^2 \setminus \{0\}$ and its transpose $w^\top$. Then the static double-layer potential with vector density φ on Γ is defined by

$$(V_0\varphi)(x) := \int_\Gamma [T_y \Phi_0(x, y)]^\top \varphi(y)\, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Gamma.$$

As is well known (see Chen & Zhou, 1992; Maz'ya & Nikol'skii, 1991), $V_0\varphi$ solves the static Navier equation. For the evaluation of its traction on the boundary, we will derive a Maue-type formula as follows.

For a function f and a matrix G we note the product rule

$$T(fG) = T(fI)\,G + f\,TG, \tag{2.1}$$

where for a matrix, the traction operator has to be understood as acting on the columns of the matrix. When $f : (0,\infty) \to \mathbb{C}$ is continuously differentiable, a lengthy, but straightforward calculation shows that

$$T_x\{f(|x-y|)I\} = \frac{f'(|x-y|)}{|x-y|}U_1(x,y), \tag{2.2}$$

where

$$U_1(x,y) := \lambda\nu(x)(x-y)^\top + \mu(x-y)\nu(x)^\top + \mu\nu(x)^\top(x-y)I. \tag{2.3}$$

By further calculations, using

$$T(xx^\top) = 3\lambda\,\nu(x)x^\top + 2\mu\,\nu(x)x^\top + \mu\,x\nu(x)^\top + \mu\,\nu(x)^\top x\,I,$$

it can be seen that

$$T_x\{J(x-y)\} = \frac{1}{|x-y|^2}\,U_2(x,y), \tag{2.4}$$

where

$$\begin{aligned} U_2(x,y) := {}& (\lambda+2\mu)\,\nu(x)\,(x-y)^\top + \mu\,(x-y)\,\nu(x)^\top \\ & + \mu\,\nu(x)^\top\,(x-y)[I - 4J(x-y)]. \end{aligned} \tag{2.5}$$

By employing (2.2) and (2.4) and using

$$\nu(y)\,(y-x)^\top - (y-x)\,\nu(y)^\top \;= \theta(y)^\top(x-y)Q$$

it can be shown that (see also Chen & Zhou, 1992)

$$T_y\Phi_0(x,y) := \frac{1}{2\pi(\lambda+2\mu)}\left\{[\mu I + 2(\lambda+\mu)J(x-y)]\frac{\partial\Psi(x,y)}{\partial\nu(y)} + \mu Q\,\frac{\partial\Psi(x,y)}{\partial\theta(y)}\right\}. \tag{2.6}$$

Now we are in a position to state the following theorem.

THEOREM 2 Assume that $\varphi \in C(\Gamma)$ vanishes $\varphi(x_1^*) = \varphi(x_{-1}^*) = 0$ at the end points of Γ and is differentiable on $\Gamma \setminus \{x_1^* \cup x_{-1}^*\}$ such that the derivative is locally uniformly Hölder continuous on $\Gamma\setminus\{x_1^*\cup x_{-1}^*\}$ and integrable over Γ. Then for the static double-layer potential we have that

$$(TV_0\varphi)(x) = \frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\int_\Gamma \frac{\partial}{\partial\theta(x)}\,\{\Psi(x,y)I + J(x-y)\}\,\frac{\partial\varphi(y)}{\partial\theta(y)}\,\mathrm{d}s(y) \tag{2.7}$$

for $x \in \Gamma \setminus \{x_1^* \cup x_{-1}^*\}$.

Proof. Using the Laplace equation for Ψ it can be seen that

$$\frac{\partial}{\partial x_1}\frac{\partial \Psi(x,y)}{\partial \nu(y)} = -\frac{\partial}{\partial x_2}\frac{\partial \Psi(x,y)}{\partial \theta(y)}, \quad \frac{\partial}{\partial x_2}\frac{\partial \Psi(x,y)}{\partial \nu(y)} = \frac{\partial}{\partial x_1}\frac{\partial \Psi(x,y)}{\partial \theta(y)}.$$

Lengthy, but straightforward calculations show that

$$\frac{\partial J(x-y)}{\partial \theta(x)} = \frac{\partial \Psi(x,y)}{\partial \nu(x)}[I - 2J(x-y)]Q, \tag{2.8}$$

and

$$\begin{aligned}
2\frac{\partial}{\partial x_1}\left\{J(x-y)\frac{\partial \Psi(x,y)}{\partial \nu(y)}\right\} &= -\frac{\partial}{\partial x_2}\left\{\frac{\partial J(x-y)}{\partial \theta(y)} + \frac{\partial \Psi(x,y)}{\partial \nu(y)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right.\\
&\qquad\left. + 2\frac{\partial \Psi(x,y)}{\partial \theta(y)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\},\\
2\frac{\partial}{\partial x_2}\left\{J(x-y)\frac{\partial \Psi(x,y)}{\partial \nu(y)}\right\} &= \frac{\partial}{\partial x_1}\left\{\frac{\partial J(x-y)}{\partial \theta(y)} - \frac{\partial \Psi(x,y)}{\partial \nu(y)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right.\\
&\qquad\left. + 2\frac{\partial \Psi(x,y)}{\partial \theta(y)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}.
\end{aligned}$$

These relations can now be employed to deduce from (2.6), via the definition (1.3), that

$$(TV_0\varphi)(z) = -\frac{\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\int_\Gamma \frac{\partial^2}{\partial\theta(x)\partial\theta(y)}\{\Psi(z,y)I + J(z-y)\}\,\varphi(y)\,\mathrm{d}s(y) \tag{2.9}$$

for $x \in \Gamma\setminus\{x_1^* \cup x_{-1}^*\}$ and $z = x + h\nu(x)$ with sufficiently small h. Now, first performing an integration by parts in (2.9) (and using $\varphi(x_{\pm 1}^*) = 0$) and then applying the jump relations for the gradient of acoustic and elastic single-layer potentials (see Chen & Zhou, 1992; Maz'ya & Nikol'skii, 1991) yields the assertion. □

We note that obviously (2.7) is also valid for the static double-layer potential on a closed contour. An analogous Maue-type expression was given by Nedelec (1982) (for the corresponding bilinear form) in a weak solution context.

In order to analyse the principal part of the integral operator on the right-hand side of (2.7) we use the parametric representation for Γ and perform a further partial integration as follows. From (2.8) we deduce that

$$\begin{aligned}
\frac{\partial^2 J(x-y)}{\partial\theta(y)\partial\theta(x)} &= \frac{\partial^2 \Psi(x,y)}{\partial\theta(y)\partial\nu(x)}[I - 2J(x-y)]Q\\
&\quad + 2\frac{\partial \Psi(x,y)}{\partial \nu(x)}\frac{\partial \Psi(x,y)}{\partial \nu(y)}[I - 2J(x-y)].
\end{aligned}$$

Hence, substituting

$$x = \gamma(t), \quad y = \gamma(\tau),$$

and performing a partial integration, in view of (2.7) we obtain

$$(TV_0\varphi)(\gamma(t)) = \frac{1}{\pi\,|\gamma'(t)|}\int_{-1}^{1}\left\{\frac{c_0\psi'(\tau)}{\tau - t} + K_0(t,\tau)\psi(\tau)\right\}\mathrm{d}\tau, \quad -1 < t < 1, \tag{2.10}$$

for $\psi(t) := \varphi(\gamma(t))$ and

$$c_0 := \frac{\mu(\lambda + \mu)}{\lambda + 2\mu}.$$

The kernel K_0 has the form

$$K_0(t,\tau) = c_0 \left\{ L_1(t,\tau) I + L_2(t,\tau)[I - 2\widetilde{J}(t,\tau)]Q + L_3(t,\tau)[I - 2\widetilde{J}(t,\tau)] \right\}, \quad (2.11)$$

where we have set

$$\widetilde{J}(t,\tau) := J(\gamma(t) - \gamma(\tau))$$

and L_1, L_2, and L_3 are given by

$$L_1(t,\tau) := \frac{\partial}{\partial \tau} \left\{ \frac{1}{\tau - t} - \frac{[\gamma(\tau) - \gamma(t)]^\top \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2} \right\}$$

$$= \frac{-1}{(\tau - t)^2} - \frac{[\gamma'(\tau)]^\top \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2} + 2\, \frac{[\gamma(\tau) - \gamma(t)]^\top \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2}\, \frac{[\gamma(\tau) - \gamma(t)]^\top \gamma'(\tau)}{|\gamma(t) - \gamma(\tau)|^2},$$

$$L_2(t,\tau) := 2\, \frac{[\gamma(\tau) - \gamma(t)]^\top Q \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2}\, \frac{[\gamma(\tau) - \gamma(t)]^\top \gamma'(\tau)}{|\gamma(t) - \gamma(\tau)|^2} - \frac{[\gamma'(\tau)]^\top Q \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2},$$

$$L_3(t,\tau) := 2\, \frac{[\gamma(\tau) - \gamma(t)]^\top Q \gamma'(t)}{|\gamma(t) - \gamma(\tau)|^2}\, \frac{[\gamma(\tau) - \gamma(t)]^\top Q \gamma'(\tau)}{|\gamma(t) - \gamma(\tau)|^2}.$$

Using Taylor's formula in integral form, the matrix $\widetilde{J}$ and the functions L_1, L_2, and L_3 can be seen to be infinitely differentiable on $[-1, 1] \times [-1, 1]$ with the diagonal values

$$\widetilde{J}(t,t) = \frac{\gamma'(t)\gamma'(t)^\top}{|\gamma'(t)|^2}$$

and

$$L_1(t,t) = -\frac{[\gamma'(t)^\top \gamma''(t)]^2}{2|\gamma'(t)|^4} + \frac{\gamma'''(t)^\top \gamma'(t)}{6|\gamma'(t)|^2} + \frac{|\gamma''(t)|^2}{4|\gamma'(t)|^2},$$

$$L_2(t,t) = \frac{\gamma'(t)^\top \gamma''(t) \gamma''(t)^\top Q \gamma'(t)}{2|\gamma'(t)|^4} - \frac{\gamma'''(t)^\top Q \gamma'(t)}{6|\gamma'(t)|^2},$$

$$L_3(t,t) = -\frac{[\gamma''(t)^\top Q \gamma'(t)]^2}{2|\gamma'(t)|^4}.$$

Hence, the kernel K_0 is infinitely differentiable.

3. Dynamic double-layer potential

The fundamental solution for the Navier equation is given by

$$\Phi(x,y) := \frac{i}{4\mu} H_0^{(1)}(k_s|x - y|) I + \frac{i}{4\omega^2} \operatorname{grad}_x \operatorname{grad}_x^\top [H_0^{(1)}(k_s|x - y|) - H_0^{(1)}(k_p|x - y|)]$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. Using Bessel's differential equation, straightforward computations yield a representation of the form

$$\Phi(x, y) = \widetilde{\Phi}_1(|x-y|)I + \widetilde{\Phi}_2(|x-y|)J(x-y),$$

where for $v > 0$ the functions $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ are given by

$$\widetilde{\Phi}_1(v) := \frac{i}{4\mu} H_0^{(1)}(k_s v) - \frac{i}{4\omega^2 v}[k_s H_1^{(1)}(k_s v) - k_p H_1^{(1)}(k_p v)],$$
$$\widetilde{\Phi}_2(v) := \frac{i}{4\omega^2}[\frac{2k_s}{v} H_1^{(1)}(k_s v) - k_s^2 H_0^{(1)}(k_s v) - \frac{2k_p}{v} H_1^{(1)}(k_p v) + k_p^2 H_0^{(1)}(k_p v)]$$

with the Hankel function of order one $H_1^{(1)} = -H_0^{(1)\prime}$.

The dynamic double-layer potential with vector density φ on Γ defined by

$$(V\varphi)(x) := \int_\Gamma [T_y \Phi(x, y)]^\top \varphi(y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \Gamma, \tag{3.1}$$

solves the Navier equation (1.1) and satisfies the Kupradze radiation condition (1.4). To determine the traction of the double-layer potential it suffices to consider the difference $\Phi - \Phi_0$ of the dynamic and the static fundamental solutions. For this we write

$$\Phi(x, y) - \Phi_0(x, y) = \Phi_1(|x-y|)I + \Phi_2(|x-y|)J(x-y), \tag{3.2}$$

where

$$\Phi_1(v) := \widetilde{\Phi}_1(v) - \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \ln\frac{1}{v},$$
$$\Phi_2(v) := \widetilde{\Phi}_2(v) - \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}.$$

With the aid of

$$T(xa^\top) = 2(\lambda + \mu)\nu(x)a^\top,$$
$$T(ax^\top) = T(a^\top x I) = \lambda\nu(x)a^\top + \mu a\nu(x)^\top + \mu\nu(x)^\top a I$$

for constant vectors a, from (2.1), (2.3), and (2.5) we find that

$$\begin{aligned} T_x[U_1(y, x)^\top] = &- 2\lambda(\lambda + 2\mu)\,\nu(x)\,\nu(y)^\top \\ &- 2\mu^2[\nu(y)\,\nu(x)^\top \\ &+ \nu(x)^\top \nu(y) I] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} T_x[U_2(y, x)^\top] = &- 4\mu\nu(y)^\top(y-x)\frac{U_2(x, y)}{|x-y|^2} \\ &- 2(\lambda + \mu)(\lambda + 2\mu)\nu(x)\nu(y)^\top \\ &- 2\mu[\lambda\nu(x)\nu(y)^\top + \mu\nu(y)\nu(x)^\top \\ &+ \mu\nu(x)^\top\nu(y)I][I - 2J(x-y)]. \end{aligned} \tag{3.4}$$

Finally, from the product

$$J(x-y)U_1(y,x)^\top = \lambda\,(y-x)\,\nu(y)^\top + 2\mu\,\nu(y)^\top(y-x)\,J(x-y),$$

making use of (2.1) and (2.4) we derive

$$\begin{aligned} T_x[J(x-y)U_1(y,x)^\top] = {} & 2\mu\nu(y)^\top(y-x)\frac{U_2(x,y)}{|x-y|^2} \\ & - 2\lambda(\lambda+2\mu)\nu(x)\nu(y)^\top \\ & - 2\mu[\lambda\nu(x)\nu(y)^\top + \mu\nu(y)\nu(x)^\top \\ & + \mu\nu(x)^\top\nu(y)I]J(x-y). \end{aligned} \tag{3.5}$$

Now, in view of (2.2) and (2.4), from (3.2) we obtain

$$\begin{aligned} [T_y\{\Phi(x,y) - \Phi_0(x,y)\}]^\top = {} & \frac{\Phi_1'(|x-y|)}{|x-y|}U_1(y,x)^\top \\ & + \frac{\Phi_2'(|x-y|)}{|x-y|}J(x-y)U_1(y,x)^\top \\ & + \frac{\Phi_2(|x-y|)}{|x-y|^2}U_2(y,x)^\top. \end{aligned}$$

From this, applying the traction operator again, we deduce that

$$T_x[T_y\{\Phi(x,y) - \Phi_0(x,y)\}]^\top = M(x,y), \tag{3.6}$$

where

$$M(x,y) := \sum_{j=1}^{2}\sum_{k=0}^{2} \Phi_j^{(k)}(|x-y|)W_j^{(k)}(x,y), \quad x \neq y,$$

with the functions $\Phi_j^{(k)}$ and the matrices $W_j^{(k)}$ given by

$$\Phi_j^{(0)}(v) := \frac{1}{v^2}\Phi_j(v), \quad \Phi_j^{(1)}(v) := \frac{1}{v}\Phi_j'(v), \quad \Phi_j^{(2)}(v) := \Phi_j''(v), \quad j = 1, 2, \tag{3.7}$$

and

$$\begin{aligned} W_1^{(2)}(x,y) &:= \frac{1}{|x-y|^2}U_1(x,y)U_1(y,x)^\top, \\ W_1^{(1)}(x,y) &:= T_x[U_1(y,x)^\top] - W_1^{(2)}(x,y), \\ W_1^{(0)}(x,y) &:= 0, \\ W_2^{(2)}(x,y) &:= \frac{1}{|x-y|^2}U_1(x,y)J(x-y)U_1(y,x)^\top, \\ W_2^{(1)}(x,y) &:= T_x[J(x-y)U_1(y,x)^\top] - W_2^{(2)}(x,y) + \frac{1}{|x-y|^2}U_1(x,y)U_2(y,x)^\top, \\ W_2^{(0)}(x,y) &:= T_x[U_2(y,x)^\top] - \frac{2}{|x-y|^2}U_1(x,y)U_2(y,x)^\top. \end{aligned}$$

We note that these matrices can be computed via (2.3), (2.5), (3.3), (3.4), and (3.5) and that they are infinitely differentiable on $\Gamma \times \Gamma$.

Using $H_0^{(1)\prime} = -H_1^{(1)}$ and Bessel's differential equation for $H_0^{(1)}$, from (3.7) we derive that

$$\begin{aligned}
\Phi_1^{(1)}(v) = &- \frac{ik_s^3}{4\omega^2 v} H_1^{(1)}(k_s v) - \frac{i}{4\omega^2}\left\{\frac{1}{v^2}[k_s^2 H_0^{(1)}(k_s v) - k_p^2 H_0^{(1)}(k_p v)]\right.\\
&\left. - \frac{2}{v^3}[k_s H_1^{(1)}(k_s v) - k_p H_1^{(1)}(k_p v)]\right\} + \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)v^2},\\
\Phi_2^{(1)}(v) = &\frac{i}{4\omega^2 v}\left\{k_s^3 H_1^{(1)}(k_s v) - k_p^3 H_1^{(1)}(k_p v) + \frac{2}{v}[k_s^2 H_0^{(1)}(k_s v) - k_p^2 H_0^{(1)}(k_p v)]\right.\\
&\left. - \frac{4}{v^2}[k_s H_1^{(1)}(k_s v) - k_p H_1^{(1)}(k_p v)]\right\},\\
\Phi_1^{(2)}(v) = &- \frac{ik_s^4}{4\omega^2} H_0^{(1)}(k_s v) + \frac{i}{4\omega^2}\left\{\frac{1}{v}[2k_s^3 H_1^{(1)}(k_s v) - k_p^3 H_1^{(1)}(k_p v)]\right.\\
&+ \frac{3}{v^2}[k_s^2 H_0^{(1)}(k_s v) - k_p^2 H_0^{(1)}(k_p v)]\\
&\left. - \frac{6}{v^3}[k_s H_1^{(1)}(k_s v) - k_p H_1^{(1)}(k_p v)]\right\} - \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)v^2},\\
\Phi_2^{(2)}(v) = &\frac{i}{4\omega^2}\left\{k_s^4 H_0^{(1)}(k_s v) - k_p^4 H_0^{(1)}(k_p v) - \frac{3}{v}[k_s^3 H_1^{(1)}(k_s v) - k_p^3 H_1^{(1)}(k_p v)]\right.\\
&\left. - \frac{6}{v^2}[k_s^2 H_0^{(1)}(k_s v) - k_p^2 H_0^{(1)}(k_p v)] + \frac{12}{v^3}[k_s H_1^{(1)}(k_s v) - k_p H_1^{(1)}(k_p v)]\right\}.
\end{aligned}$$

To take care of the logarithmic singularity of the Hankel functions we decompose

$$\Phi_j^{(k)}(v) = \frac{1}{\pi}\Psi_j^{(k)}(v)\ln v + \chi_j^{(k)}(v),$$

where

$$\begin{aligned}
\Psi_1^{(0)}(v) := &- \frac{k_s^2}{2\omega^2 v^2} J_0(k_s v) + \frac{1}{2\omega^2 v^3}[k_s J_1(k_s v) - k_p J_1(k_p v)] + \frac{\lambda + 3\mu}{4\mu(\lambda + 2\mu)v^2},\\
\Psi_2^{(0)}(v) := &\frac{1}{2\omega^2 v^2}\left\{k_s^2 J_0(k_s v) - k_p^2 J_0(k_p v) - \frac{2}{v}[k_s J_1(k_s v) - k_p J_1(k_p v)]\right\},\\
\Psi_1^{(1)}(v) := &\frac{k_s^3}{2\omega^2 v} J_1(k_s v) + \frac{1}{2\omega^2}\left\{\frac{1}{v^2}[k_s^2 J_0(k_s v) - k_p^2 J_0(k_p v)]\right.\\
&\left. - \frac{2}{v^3}[k_s J_1(k_s v) - k_p J_1(k_p v)]\right\},
\end{aligned}$$

$$\begin{aligned}
\Psi_2^{(1)}(v) :=& -\frac{1}{2\omega^2 v}\Bigg\{k_s^3 J_1(k_s v) - k_p^3 J_1(k_p v) + \frac{2}{v}[k_s^2 J_0(k_s v) - k_p^2 J_0(k_p v)] \\
&- \frac{4}{v^2}[k_s J_1(k_s v) - k_p J_1(k_p v)]\Bigg\}, \\
\Psi_1^{(2)}(v) :=& \frac{k_s^4}{2\omega^2} J_0(k_s v) - \frac{1}{2\omega^2}\Bigg\{\frac{1}{v}[2k_s^3 J_1(k_s v) - k_p^3 J_1(k_p v)] \\
&+ \frac{3}{v^2}[k_s^2 J_0(k_s v) - k_p^2 J_0(k_p v)] - \frac{6}{v^3}[k_s J_1(k_s v) - k_p J_1(k_p v)]\Bigg\}, \\
\Psi_2^{(2)}(v) :=& -\frac{1}{2\omega^2}\Bigg\{k_s^4 J_0(k_s v) - k_p^4 J_0(k_p v) - \frac{3}{v}[k_s^3 J_1(k_s v) - k_p^3 J_1(k_p v)] \\
&- \frac{6}{v^2}[k_s^2 J_0(k_s v) - k_p^2 J_0(k_p v)] + \frac{12}{v^3}[k_s J_1(k_s v) - k_p J_1(k_p v)]\Bigg\}
\end{aligned}$$

in terms of the Bessel functions J_0 and J_1. Then, using the power series expansions for the Bessel and Hankel functions, it can be seen that with the exception of $\chi_1^{(0)}$ the functions $\Psi_j^{(k)}$ and $\chi_j^{(k)}$ are analytic on $\mathbb{R}$ with

$$\begin{aligned}
&\Psi_1^{(0)}(0) = \frac{1}{32\omega^2}[3k_s^4 + k_p^4], \quad \Psi_2^{(0)}(0) = \frac{1}{16\omega^2}[k_p^4 - k_s^4], \\
&\Psi_j^{(k)}(0) = 2\,\Psi_j^{(0)}(0), \quad j, k = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
\chi_j^{(k)}(0) &= \frac{2k-1}{\pi}\,\Psi_j^{(0)}(0) + 2\alpha_j, \quad j, k = 1, 2, \\
\alpha_1 &:= \frac{1}{32\pi\omega^2}\left\{3k_s^4 \ln\frac{k_s}{2} + k_p^4 \ln\frac{k_p}{2} - \frac{11}{4}k_s^4 - \frac{5}{4}k_p^4 + \left(C_e - \frac{i\pi}{2}\right)(3k_s^4 + k_p^4)\right\}, \\
\alpha_2 &:= \chi_2^{(0)}(0) = -\frac{1}{16\pi\omega^2}\left\{k_s^4 \ln\frac{k_s}{2} - k_p^4 \ln\frac{k_p}{2} + \left(C_e - \frac{3}{4} - \frac{i\pi}{2}\right)(k_s^4 - k_p^4)\right\}.
\end{aligned}$$

Here, $C_e = 0{\cdot}57721\ldots$ denotes Euler's constant. Note that the functions $\Psi_1^{(0)}$ and $\chi_1^{(0)}$ do not enter into (3.6), since $W_1^{(0)} = 0$.

We can now decompose

$$M(x, y) = \frac{1}{\pi}\, M_1(x, y) \ln|x - y| + M_2(x, y),$$

where

$$M_1(x, y) := \sum_{j=1}^{2}\sum_{k=0}^{2} \Psi_j^{(k)}(|x - y|) W_j^{(k)}(x, y)$$

and

$$M_2(x, y) := M(x, y) - \frac{1}{\pi}\, M_1(x, y) \ln|x - y|.$$

Since M only has a logarithmic singularity, for the traction operator we can interchange differentiation and integration to obtain

$$(T[V - V_0]\varphi)(x) = \int_\Gamma \left\{ \frac{1}{\pi} M_1(x, y) \ln|x - y| + M_2(x, y) \right\} \varphi(y) \, ds(y)$$

for $x \in \Gamma \setminus \{x_1^* \cup x_{-1}^*\}$. From this, substituting $x = \gamma(t)$, $y = \gamma(\tau)$, and incorporating (2.10) we finally obtain that

$$(TV\varphi)(\gamma(t)) = \frac{1}{\pi\,|\gamma'(t)|} \int_{-1}^{1} \left\{ \frac{c_0\,\psi'(\tau)}{\tau - t} + [K_1(t,\tau) \ln 2|t - \tau| + K_2(t,\tau)]\,\psi(\tau) \right\} d\tau \tag{3.8}$$

for $-1 < t < 1$. As in 2.10 we have set $\psi(t) := \varphi(\gamma(t))$ and the kernels are given by

$$\begin{aligned} K_1(t,\tau) &:= |\gamma'(t)|\,|\gamma'(\tau)|\, M_1(\gamma(t), \gamma(\tau)) \\ K_2(t,\tau) &:= |\gamma'(t)|\,|\gamma'(\tau)|\, \{\pi\, M(\gamma(t), \gamma(\tau)) - K_1(t,\tau) \ln 2|t - \tau|\} + K_0(t,\tau) \end{aligned}$$

with K_0 as defined in (2.11). Note that, due to the occurrence of the normal vectors $\nu(x)$ and $\nu(y)$, all the matrices $W_j^{(k)}(\gamma(t), \gamma(\tau))$ contain a common factor $[|\gamma'(t)|\,|\gamma'(\tau)|]^{-1}$. The kernels K_1 and K_2 again can be shown to be infinitely differentiable.

4. The hypersingular integral equation

For a continuous vector density φ satisfying $\varphi(x_1^*) = \varphi(x_{-1}^*) = 0$, the double-layer potential (3.1) is bounded on $\mathbb{R}^2$. This follows from the regularity properties for the elastic double-layer potential (see Chen & Zhou, 1992) by considering the arc Γ as a subset of a closed contour and extending the density φ by zero outside of Γ. Therefore, from our above analysis we can conclude the following theorem.

THEOREM 3 The elastic double-layer potential (3.1) solves the scattering problem (1.1), (1.2), (1.4) provided that the density φ satisfies the assumptions of Theorem 2 and the integral equation

$$T_x \int_\Gamma [T_y \Phi(x, y)]^\top \varphi(y) \, ds(y) = -(Tu^i)(x), \quad x \in \Gamma \setminus \{x_1^* \cup x_{-1}^*\}. \tag{4.1}$$

From the jump relations for the elastic double-layer potential and the uniqueness Theorem 1 we can deduce that under the above regularity assumptions on φ the integral equation (4.1) has at most one solution.

To establish existence of a solution, in view of (3.8), we consider the parametrized form

$$\frac{1}{\pi} \int_{-1}^{1} \left\{ \frac{c_0\,\psi'(\tau)}{\tau - t} + [K_1(t,\tau) \ln 2|t - \tau| + K_2(t,\tau)]\,\psi(\tau) \right\} d\tau = g(t), \quad -1 < t < 1, \tag{4.2}$$

where we have set $g(t) := -|\gamma'(t)|\,(Tu^i)(\gamma(t))$. Substituting $t = \cos s$ and $\tau = \cos\sigma$ and multiplying by $\sin s$, we equivalently transform 4.2 into

$$\frac{1}{\pi} \int_0^\pi \left\{ \frac{c_0 \sin s}{\cos s - \cos\sigma}\, \chi'(\sigma) + [H_1(s,\sigma) \ln 2|\cos s - \cos\sigma| + H_2(s,\sigma)]\,\chi(\sigma) \right\} d\sigma = f(s) \tag{4.3}$$

for $0 < s < \pi$. Here, we have set $\chi(s) := \psi(\cos s)$ and $f(s) := \sin s\ g(\cos s)$ and the kernels are given by

$$H_j(s,\sigma) := \sin s \sin\sigma\, K_j(\cos s, \cos\sigma), \quad j = 1, 2.$$

Since we want to apply simple error estimates for trigonometric interpolation, rather than working in the original interval $[0, \pi]$ as proposed by Capobianco *et al.* (1997) for a related equation, for the further investigation and also for the numerical solution of the hypersingular integral equation (4.3) we extend it onto $[0, 2\pi]$ by extending both the solution χ and the right-hand side f as odd and 2π-periodic functions. The odd extension of χ makes sense, since we need to have $\chi(0) = \chi(\pi) = 0$. We note that this extension does not require any additional regularity assumptions on χ and f and that as indicated later, for the numerical solution we can restrict the resulting linear system for the approximate solution to the nodal values on the original interval $[0, \pi]$.

By using the identities

$$\frac{\sin s}{\cos s - \cos\sigma} = \frac{1}{2}\cot\frac{\sigma - s}{2} - \frac{1}{2}\cot\frac{\sigma + s}{2}$$

and

$$\ln 4(\cos s - \cos\sigma)^2 = \ln 4\sin^2\frac{s-\sigma}{2} + \ln 4\sin^2\frac{s+\sigma}{2}$$

we obtain the integral equation

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{c_0\cot\frac{\sigma - s}{2}\ \chi'(\sigma) + \left[H_1(s,\sigma)\ln 4\sin^2\frac{s-\sigma}{2} + H_2(s,\sigma)\right]\chi(\sigma)\right\}\mathrm{d}\sigma = f(s) \tag{4.4}$$

for $0 \leqslant s \leqslant 2\pi$ which is equivalent to (4.3). Note that the uniqueness of a solution to the boundary integral equation (4.1) implies the uniqueness of an odd solution to the parametrized integral equation (4.4). We also note that for an odd $\chi \in C^{1,\alpha}[0, 2\pi]$ the corresponding density φ satisfies the regularity assumptions of Theorem 2. Due to

$$\psi'(\cos s) = -\frac{\chi'(s)}{\sin s},$$

the derivative develops a square root singularity at the end points of Γ.

We rewrite (4.4) in the operator form

$$U\chi + A\chi = f$$

with the integral operators defined by

$$(U\chi)(s) := \frac{c_0}{2\pi}\int_0^{2\pi}\cot\frac{\sigma - s}{2}\chi'(\sigma)\,\mathrm{d}\sigma,$$

$$(A\chi)(s) := \frac{1}{2\pi}\int_0^{2\pi}\left\{H_1(s,\sigma)\ln 4\sin^2\frac{s-\sigma}{2} + H_2(s,\sigma)\right\}\chi(\sigma)\,\mathrm{d}\sigma$$

for $s \in [0, 2\pi]$. For $m \in \mathbb{N}\cup\{0\}$ and $0 < \alpha < 1$, by $C^{m,\alpha}_{odd}[0, 2\pi]$ we denote the space of m-times uniformly Hölder continuously differentiable and odd 2π-periodic vector functions furnished with the usual Hölder norm.

LEMMA 4 For $m \in \mathbb{N}$, the operator U is bounded from $C^{m,\alpha}_{odd}[0, 2\pi]$ into $C^{m-1,\alpha}_{odd}[0, 2\pi]$ and has a bounded inverse.

Proof. This follows from the standard mapping properties of the operator with Hilbert kernel (see Kress, 1999) and the observation that U maps odd functions into odd functions. $\square$

Now we can assure unique solvability of the hypersingular integral equation (4.4).

LEMMA 5 For $m \in \mathbb{N}$ and $f \in C^{m-1,\alpha}_{odd}[0, 2\pi]$, the hypersingular integral equation (4.4) has exactly one solution $\chi \in C^{m,\alpha}_{odd}[0, 2\pi]$.

Proof. Since the kernels H_1 and H_2 are infinitely differentiable and odd with respect to both variables, the integral operator A is bounded from $C^{m,\alpha}_{odd}[0, 2\pi]$ into $C^{m+1,\alpha}_{odd}[0, 2\pi]$ and therefore compact from $C^{m,\alpha}_{odd}[0, 2\pi]$ into $C^{m-1,\alpha}_{odd}[0, 2\pi]$ (see Lemma 4.1 in Kress, 1995b). Hence, by the Riesz theory applied to the equivalent equation of the second kind

$$\chi + U^{-1}A\chi = U^{-1}f \tag{4.5}$$

and the uniqueness for the integral equation mentioned above, we can assert the existence of a unique solution of (4.4). $\square$

We summarize the above analysis, which can also be carried out in a Sobolev space setting, in the following theorem.

THEOREM 6 The elastic scattering problem (1.1), (1.2), (1.4) for a cavity crack is uniquely solvable.

5. Numerical solution

In this section we will develop a numerical solution method for the hypersingular integral equation (4.4) by combining a collocation and a quadrature method as suggested and analysed by Kress (1995b) and Mönch (1996, 1997) for the case of the corresponding hypersingular equations in acoustic and electromagnetic scattering. We begin by describing the appropriate quadrature rules. For this we consider trigonometric interpolation with $2n$ equidistant nodal points

$$s_j^{(n)} := \frac{j\pi}{n}, \quad j = 0, \dots, 2n-1,$$

and with respect to the $2n$-dimensional space T_n of trigonometric polynomials of the form

$$T_n := \left\{ v(t) = \sum_{m=0}^{n} a_m \cos mt + \sum_{m=1}^{n-1} b_m \sin mt, \quad a_m, b_m \in \mathbb{C}^2 \right\}.$$

Denoting by $P_n : C[0, 2\pi] \to T_n$ the corresponding interpolation operator, we use the following interpolatory quadrature rules

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\sigma - s}{2}\chi'(\sigma)\,\mathrm{d}\sigma &\approx \sum_{k=0}^{2n-1} T_k^{(n)}(s)\chi(s_k^{(n)}),\\
\frac{1}{2\pi}\int_0^{2\pi} H_1(s,\sigma)\ln 4\sin^2\frac{\sigma - s}{2}\,\chi(\sigma)\,\mathrm{d}\sigma &\approx \sum_{k=0}^{2n-1} R_k^{(n)}(s)H_1(s, s_k^{(n)})\chi(s_k^{(n)}),\\
\frac{1}{2\pi}\int_0^{2\pi} H_2(s,\sigma)\chi(\sigma)\,\mathrm{d}\sigma &\approx \frac{1}{2n}\sum_{k=0}^{2n-1} H_2(s, s_k^{(n)})\chi(s_k^{(n)})
\end{aligned} \tag{5.1}$$

for $s \in [0, 2\pi]$. The weight functions $T_k^{(n)}$ and $R_k^{(n)}$ are given by

$$\begin{aligned}
T_k^{(n)}(s) &= \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\sigma - s}{2}\,L_k^{(n)\prime}(\sigma)\,d\sigma\\
&= -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(s - s_k^{(n)}) - \frac{1}{2}\cos n(s - s_k^{(n)}),\\
R_k^{(n)}(s) &= \frac{1}{2\pi}\int_0^{2\pi} \ln 4\sin^2\frac{\sigma - s}{2}\,L_k^{(n)}(\sigma)\,\mathrm{d}\sigma\\
&= -\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(s - s_k^{(n)}) - \frac{1}{2n^2}\cos n(s - s_k^{(n)})
\end{aligned}$$

in terms of the Lagrange factors $L_k^{(n)}$ for the trigonometric interpolation. For a derivation of these expressions for the weights we refer to Kress (1995b, 1999) and Mönch (1996). Now we introduce the quadrature operator A_n by

$$(A_n\chi)(s) := \sum_{k=0}^{2n-1}\left\{R_k^{(n)}(s)H_1(s, s_k^{(n)}) + \frac{1}{2n}H_2(s, s_k^{(n)})\right\}\chi(s_k^{(n)})$$

for $s \in [0, 2\pi]$. By first approximating the integral operator A by the quadrature operator A_n and then collocating the resulting equation with the aid of the trigonometric interpolation operator P_n we arrive at the approximating equation

$$U\chi_n + P_nA_n\chi_n = P_nf \tag{5.2}$$

which we need to solve for an odd $\chi_n \in T_n$. In the derivation of (5.2) we used the fact that $P_nU\chi_n = U\chi_n$ for odd $\chi_n \in T_n$. Clearly, (5.2) is equivalent to the linear system

$$\sum_{k=0}^{2n-1}\left\{T_{|j-k|}^{(n)} + R_{|j-k|}^{(n)}H_1(s_j^{(n)}, s_k^{(n)}) + \frac{1}{2n}H_2(s_j^{(n)}, s_k^{(n)})\right\}\chi_n(s_k^{(n)}) = f(s_j^{(n)}) \tag{5.3}$$

for $j = 0, 1, \dots, 2n-1$, which we have to solve for the nodal values $\chi_n(s_k^{(n)})$ of the odd $\chi_n \in T_n$, and where

$$T_j^{(n)} := T_j^{(n)}(0) = \begin{cases} \frac{1}{2n}\sin^{-2}\frac{j\pi}{2n}, & j \text{ odd}, \\ 0, & j \text{ even}, j \neq 0, \\ -\dfrac{n}{2}, & j = 0, \end{cases}$$

$$R_j^{(n)} := R_j^{(n)}(0) = -\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} - \frac{(-1)^j}{2n^2}.$$

In the derivation of (5.3) we have used the fact that the first quadrature rule in (5.1) is exact for trigonometric polynomials in T_n. Since χ_n is odd, the $2n \times 2n$ system (5.3) for the nodal values $\chi_n(s_k^{(n)})$, $k = 0, 1, \dots, 2n-1$, reduces to an $(n-1)\times(n-1)$ system for the nodal values $\chi_n(s_k^{(n)})$, $k = 1, \dots, n-1$.

The error and convergence analysis is based on the estimate

$$\|P_n\chi - \chi\|_{m,\alpha} \leqslant c\frac{\ln n}{n^{\ell-m+\beta-\alpha}}\|\chi\|_{\ell,\beta}, \tag{5.4}$$

for the trigonometric interpolation which is valid for $0 \leqslant m \leqslant \ell$, $0 < \alpha \leqslant \beta < 1$, and some constant c depending only on m, ℓ, α and β (see Prössdorf & Silbermann, 1991, p 78). It can be used to show that

$$\|(P_nA_n - A)\chi\|_{m-1,\alpha} \leqslant \tilde{c}\frac{\ln n}{n^{\ell-m+1}}\|\chi\|_{\ell,\alpha} \tag{5.5}$$

for $1 \leqslant m \leqslant \ell$, $0 < \alpha < 1$ and some constant $\tilde{c}$. The proof of (5.5) given in Kress (1995b) for the special case $m = \ell = 1$ can be extended to the general case.

THEOREM 7 For $f \in C_{odd}^{\ell-1,\beta}[0, 2\pi]$ and sufficiently large n the approximate equation (5.2) has a unique odd solution $\chi_n \in T_n$ and, for the exact solution χ of (4.4) we have the error estimate

$$\|\chi - \chi_n\|_{m,\alpha} \leqslant C\frac{\ln n}{n^{\ell-m+\beta-\alpha}}\|\chi\|_{\ell,\beta} \tag{5.6}$$

for $0 \leqslant m \leqslant \ell$, $0 < \alpha \leqslant \beta < 1$ and some constant C depending only on α, β, m, ℓ.

Proof. For $\ell = m$ the estimate (5.5) implies norm convergence

$$\|P_nA_n - A\|_{C_{odd}^{m,\alpha}[0,2\pi]\to C_{odd}^{m-1,\alpha}[0,2\pi]} \to 0, \quad n \to \infty.$$

Therefore, from the Neumann series, we can conclude that for sufficiently large n the operators $U + P_nA_n : C_{odd}^{m,\alpha}[0, 2\pi] \to C_{odd}^{m-1,\alpha}[0, 2\pi]$ are invertible and the inverse operators are uniformly bounded. Then the error estimate (5.6) follows by writing

$$\chi_n - \chi = (U + P_nA_n)^{-1}\{(P_nf - f) + (A - P_nA_n)\chi\}$$

and then using the uniform boundedness of the inverse operators, the estimates (5.4) and (5.5), and the fact that $f = (U + A)\chi$. □

We remark that for analytic arcs Γ we can improve the above error estimate to the form

$$\|\chi - \chi_n\|_{m,\alpha} \leqslant C\mathrm{e}^{-qn}$$

for some constant $q > 0$ (see Kress, 1995b; Mönch, 1996), i.e., the error between the exact and the approximate solution decreases at least exponentially. In addition, following Kress (1999), Section 13.4, or Kress & Sloan (1993), the error analysis can also be carried out in a Sobolev space setting.

6. Numerical examples

For a numerical example we consider the scattering of an elastic plane wave u^i by an arc as illustrated in Fig. 1 and described by the parametrization

$$\gamma(t) = \left(2\sin\frac{3\pi}{8}\left(\frac{4}{3}+t\right), -\sin\frac{3\pi}{4}\left(\frac{4}{3}+t+\frac{2}{3\pi}\right)\right), \quad t \in [-1, 1].$$

The incident wave is either a longitudinal plane wave $u_p^i(x; d) = d\,\mathrm{e}^{ik_p d^\top x}$ or a transversal plane wave $u_s^i(x; d) = -Qd\,\mathrm{e}^{ik_s d^\top x}$ with a unit vector d describing the direction of propagation and the polarization.

From Betti's integral representation formula it can by deduced that any solution $u = u_p + u_s$ to the Navier equations satisfying the Kupradze radiation condition has an asymptotic behaviour of the form

$$u_p(x) = \frac{\mathrm{e}^{ik_p|x|}}{\sqrt{|x|}}\left\{u_{p,\infty}(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad u_s(x) = \frac{\mathrm{e}^{ik_s|x|}}{\sqrt{|x|}}\left\{u_{s,\infty}(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$ uniformly in all directions $\hat{x} := x/|x|$. The vector fields $u_{p,\infty}$ and $u_{s,\infty}$ defined on the unit circle Ω are called the longitudinal far field pattern and transversal far field pattern, respectively.

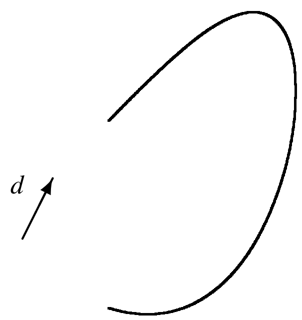

FIG. 1. Arc for numerical example.

TABLE 1
Numerical results for the longitudinal far field

ω	n	$\mathrm{Re}\, u^1_{p,\infty}$	$\mathrm{Re}\, u^2_{p,\infty}$	$\mathrm{Im}\, u^1_{p,\infty}$	$\mathrm{Im}\, u^2_{p,\infty}$
2	8	−0·178990618149	−0·310020844712	0·205418434812	0·355795165905
	16	−0·187241556029	−0·324311888330	0·218100370126	0·377760922208
	32	−0·187271864387	−0·324364383947	0·218228326722	0·377982549533
	64	−0·187271864878	−0·324364384796	0·218228328043	0·377982551820
	128	−0·187271864878	−0·324364384797	0·218228328043	0·377982551821
4	8	−0·167483165002	−0·290089351195	0·255252828938	0·442110868497
	16	−0·164256483650	−0·284500575154	0·267688744928	0·463650506830
	32	−0·164266040504	−0·284517128111	0·267583944391	0·463468986974
	64	−0·164266040470	−0·284517128052	0·267583944234	0·463468986703
	128	−0·164266040470	−0·284517128052	0·267583944234	0·463468986703

From the asymptotics for the Hankel function $H_1^{(1)}$ we obtain that the far field patterns of the double-layer potential (3.1) are given by

$$u_{p,\infty}(\hat{x}) = \frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\lambda + 2\mu}\sqrt{\frac{k_p}{8\pi}} \int_\Gamma J(\hat{x})F(\hat{x}, y)\varphi(y)\mathrm{e}^{-ik_p\hat{x}^\top y}\,\mathrm{d}s(y), \quad \hat{x} \in \Omega,$$

and

$$u_{s,\infty}(\hat{x}) = \frac{\mathrm{e}^{-i\frac{\pi}{4}}}{\mu}\sqrt{\frac{k_s}{8\pi}} \int_\Gamma [I - J(\hat{x})]F(\hat{x}, y)\varphi(y)\mathrm{e}^{-ik_s\hat{x}^\top y}\,\mathrm{d}s(y), \quad \hat{x} \in \Omega,$$

in terms of the matrix

$$F(\hat{x}, y) = \lambda\hat{x}\nu(y)^\top + \mu\nu(y)\hat{x}^\top + \mu\nu(y)^\top\hat{x}I.$$

To evaluate these integrals for the far field expressions we use the trapezoidal rule. The vector fields $u_{p,\infty}$ and $u_{s,\infty}$ are complex valued, i.e., we have to compute 8 scalar real functions. Tables 1 and 2 show some numerical values for the far field patterns $u_{p,\infty}(d)$ and $u_{s,\infty}(d)$ for the incident direction $d = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})^\top$ as indicated in Fig. 1. The incident wave is chosen as a longitudinal wave, and the Lamé constants are $\mu = 1{\cdot}5$ and $\lambda = 3$. Note that the exponential convergence is clearly exhibited.

Since there is no explicit solution available for elastic scattering by an arc, to check the validity of the algorithm and our implementation we confirmed that the corresponding scheme for a closed boundary curve gives the correct numerical solution for scattering of point sources located in the interior of the scatterer.

Acknowledgements

This research was supported by the Deutsche Forschungsgemeinschaft. The authors would like to express their thanks to Professor George Hsiao (University of Delaware) for drawing their attention to the paper of Nedelec (1982). This work was initiated whilst Lars Mönch was employed by the University of Göttingen and completed whilst Roman Chapko was visiting the University of Göttingen.

TABLE 2
Numerical results for the transversal far field

ω	n	$\mathrm{Re}\, u^1_{s,\infty}$	$\mathrm{Re}\, u^2_{s,\infty}$	$\mathrm{Im}\, u^1_{s,\infty}$	$\mathrm{Im}\, u^2_{s,\infty}$
2	8	0·090156723969	−0·052052008853	−0·338915978584	0·195673231468
	16	0·129713730669	−0·074890257319	−0·336533518548	0·194297717525
	32	0·129871972595	−0·074981618338	−0·336861491886	0·194487073020
	64	0·129871972547	−0·074981618310	−0·336861494894	0·194487074757
	128	0·129871972547	−0·074981618310	−0·336861494895	0·194487074757
4	8	−0·286462295247	0·165389083274	0·229061779628	−0·132248880129
	16	0·082253709361	−0·047489201241	−0·082921716645	0·047874875427
	32	0·082597432696	−0·047687650002	−0·081597332868	0·047110242096
	64	0·082597434631	−0·047687651118	−0·081597333231	0·047110242306
	128	0·082597434631	−0·047687651118	−0·081597333231	0·047110242306

REFERENCES

AHNER, J. F. & HSIAO, G. C. 1976 On the two-dimensional exterior boundary value problem in elasticity. *SIAM J. Appl. Math.* **31**, 677–685.

CAPOBIANCO, M. R., CRISCUOLO, G., & JUNGHANNS, P. 1997 A fast algorithm for Prandtl's integro-differential equation. *J. Comput. Appl. Math.* **77**, 103–128.

CHEN, G. & ZHOU, J. 1992 *Boundary Element Methods*. San Diego: Academic.

HSIAO, G. C., STEPHAN, E. P., & WENDLAND, W. L. 1991 On the Dirichlet problem in elasticity for a domain exterior to an arc. *J. Comput. Appl. Math.* **34**, 1–19.

KRESS, R. 1995a Inverse scattering from an open arc. *Math. Methods Appl. Sci.* **18**, 267–293.

KRESS, R. 1995b On the numerical solution of a hypersingular integral equation in scattering theory. *J. Comput. Appl. Math.* **61**, 345–360.

KRESS, R. 1996 Inverse elastic scattering from a crack. *Inv. Problems* **12**, 667–684.

KRESS, R. 1999 *Linear Integral Equations* 2nd edn. New York: Springer.

KRESS, R. & SLOAN, I. H. 1993 On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation. *Numer. Math.* **66**, 193–214.

KRISHNASAMY, G., SCHMERR, L. W., RUDOLPHI, T. J., & RIZZO, F. J. 1990 Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering. *J. Appl. Mech.* **57**, 404–414.

LEWIS, P. A. & WICKHAM, G. R. 1992 The diffraction of SH waves by an arbitrary shaped crack in two dimensions. *Phil. Trans. R. Soc.* A **340**, 503–529.

MARTIN, P. A. 1981 Diffraction of elastic waves by a penny-shaped crack. *Proc. R. Soc. London* A **378**, 263–285.

MARTIN, P. A. & RIZZO, F. J. 1989 On boundary integral equations for crack problems. *Proc. R. Soc. London* A **421**, 341–355.

MAZ'YA, V. G. & NIKOL'SKII, S. M. 1991 *Analysis IV*. Berlin: Springer.

MÖNCH, L. 1996 On the numerical solution of the direct scattering problem for a sound-hard open arc. *J. Comput. Appl. Math.* **71**, 343–356.

MÖNCH, L. 1997 On the inverse acoustic scattering problem by an open arc: the sound-hard case. *Inv. Problems* **13**, 1379–1392.

MULTHOPP, H. 1938 Die Berechnung der Auftriebsverteilung von Tragflügeln. *Luftfahrt-Forschung* **4**, 153–169.

NEDELEC, J. C. 1982 Integral equations with nonintegrable kernels. *Integral Equations Operator Theory* **5**, 562–573.

PRÖSSDORF, S. & SILBERMANN, B. 1991 *Numerical Analysis for Integral and Related Operator Equations*. Berlin: Akademie Basel: Birkhäuser.

SÁEZ, A., GALLEGO, R., & DOMÍNGUEZ, J. 1995 Hypersingular quarter-point boundary elements for crack problems. *Int. J. Numer. Methods Eng.* **38**, 1681–1701.

WENDLAND, W. L. & STEPHAN, E. P. 1990 A hypersingular boundary integral method for two-dimensional screen and crack problems. *Arch. Ration. Mech. Anal.* **112**, 363–390.

WICKHAM, G. R. 1981 The diffraction of stress waves by a plane finite crack in two dimensions: uniqueness and existence. *Proc. R. Soc. London* A **378**, 241–261.

YAN, Y. & SLOAN, I. H. 1988 On integral equations of the first kind with logarithmic kernels. *J. Integral Equations Appl.* **1**, 549–579.