Double Layer Potentials for Domains with Corners and Edges

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Introduction. Classically, the Dirichlet problem for Laplace's equation in a bounded domain $D \subset R^n$, with smooth boundary, B, and continuous data may be solved by the use of the double layer potential [2]:

$$u(x) = \int_B \frac{(x - Q)\cdot N_Q}{|x - Q|^n} f(Q)\, dQ.$$

This technique reduces the problem of existence to that of solving an integral equation on the boundary. The same approach applies when the data is not required to be continuous, but to belong to some L^p class [9]. The double layer potential approach also works for smooth or L^p data for the heat equation in the cylindrical domain $D \times (0, T)$, [3].

The L^p situation is different when the boundary is not smooth. For example, the double layer potential in the first quadrant (for the Laplace operator) leads to an integral equation not solvable for $p = 3/2$ but solvable for all other $p > 1$. Our examples indicate that the double layer potential always works for p large enough but not for all p. For the continuous case or for $p = \infty$, see J. Kral [5] and G. Miranda [7]. Corners and edges in the boundary determine the "bad" p's (those for which the double layer potential is not efficacious). In particular, an edge causes an interval of bad p's while isolated vertices yield isolated bad p's (see Section 4).

One of the main concerns of this paper is the efficacy (or lack of it) of the double layer potential method for solving the Dirichlet problem with L^p boundary data. In the example of the first quadrant (Section 1) the double layer potential method fails for $p = 3/2$ in the case of the Laplacian. Nevertheless, the Poisson kernel for the quadrant leads to a solution even for data in $L^{3/2}$. On the other hand, for $1 < p < 3/2$, the Dirichlet problem in the exterior of the first quadrant is solvable by means of the double layer potential but not by means of the Poisson kernel. This is discussed in Section 2. Later sections include material on bounded domains with corners, on the analogous problems with the heat operator in place of the Laplacian, and on some 3-dimensional cases.

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In Section 3, we consider a 3-dimensional cone and show that the boundary integral equation is uniquely solvable for all $p > 1$, except those in a nonempty discrete set. In Section 4, we show that a 3-dimensional wedge leads to an interval of "bad" p's (Cf. Theorem (4.3)). Section 5 treats the case of a bounded plane domain with a corner. Section 6 is the analogue of Section 4, with the Laplacian replaced by the heat operator.

The simplicity of the domains in our examples leads to boundary operators which can be interpreted as convolutions on certain Lie groups. The cases in which the groups are non-commutative correspond to intervals of "bad" p's.

We refer the reader to the extensive bibliography of Avantaggiati and Troisi [1].

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1. The method of potentials for a plane sector. We begin with the L^p-Dirichlet problem for the first quadrant in the plane. The results stated here are applications of [4]. For this reason details are omitted.

Suppose f_1, f_2 are in $L^p(0, \infty)$, $1 < p \leqq \infty$. For $x > 0$, $y > 0$, set

$$u(x, y) = \frac{1}{\pi} \int_0^\infty \frac{y}{(x - t)^2 + y^2} f_1(t)\, dt + \frac{1}{\pi} \int_0^\infty \frac{x}{(y - s)^2 + x^2} f_2(s)\, ds. \tag{1.1}$$

The function u is the double layer potential of the density which is f_1 on the positive x-axis and f_2 on the positive y-axis. Thus u is harmonic inside the first quadrant. For $r > 0$, let B_r denote the "parallel boundary",

$$(r, r) + B$$

where B denotes the boundary of the first quadrant. One can view the restriction u_r of u to B_r as a function on B. Then the norm of u_r in $L^p(B)$ is bounded in r. Moreover, as r tends to 0, u_r converges a.e. and in $L^p(B)$ for finite p. The limit function is

$$f_1(x) + \frac{1}{\pi} \int_0^\infty \frac{x}{x^2 + s^2} f_2(s)\, ds \equiv f_1(x) + Kf_2(x)$$

on the positive x-axis and $Kf_1(y) + f_2(y)$ on the positive y-axis.

When a function $g \in L^p(B)$ is given as Dirichlet data we can think of it as a pair, $(g_1(x), g_2(y))$, of functions in $L^p(0, \infty)$. This leads to two equations in two unknowns in $L^p(0, \infty)$:

$$f_1 + Kf_2 = g_1\,, \qquad Kf_1 + f_2 = g_2\,. \tag{1.2}$$

The system has a solution for every pair g_1, g_2 if and only if $I - K^2$ maps $L^p(0, \infty)$ onto itself. It is shown in [4, (3.3)] that $I + K$ is invertible for all $p > 1$ and that $I - K$ is invertible for all $p > 1$ except $p = 3/2$. In the latter case $I - K$ is not onto. (See [4, remark following (2.3)].) Since the adjoint of $I - K$ is $1 - 1$ (see [4]) on $L^{3/2}(0, \infty)$, we conclude that the range of $I - K$ is not closed. This in turn implies the range of $I - K^2$ is not closed and from this we can say that the range of the matrix

$$\begin{pmatrix} I & K \\ K & I \end{pmatrix}$$

is not closed in $L^{3/2}(0, \infty) \times L^{3/2}(0, \infty)$.

Concerning the question of uniqueness in the L^p Dirichlet problem we find that $u \equiv 0$ if (1) u is harmonic in the first quadrant, (2) u_r is bounded in $L^p(B)$ for $r > 0$ and, (3) $u_r \to 0$ in $L^p(B)$ as $r \to 0$. (See Section 2.)

The foregoing can be summarized as follows:

Theorem (1.2). *If $p \neq 3/2$ and $1 < p < \infty$, the method of double layer potentials yields the unique solution of the following L^p-Dirichlet problem:*

(i) $\Delta u = 0$ *in the interior of the first quadrant,*

$$\text{(ii)} \qquad ||u_r||_{L^p(B)} = \left\{\int_0^\infty |u(r, y + r)|^p \, dy + \int_0^\infty |u(x + r, r)|^p \, dx\right\}^{1/p}$$

is bounded in $r > 0$,

(iii) $u_r \to g$ *in $L^p(B)$ as $r \to 0$ where $g \in L^p(B)$ is the given data.*

If $p = \infty$, (iii) is replaced by: $u_r \to g$ a.e. where $g \in L^\infty(B)$.

If $p = 3/2$, the method of double layer potentials fails to give existence for some data $g \in L^{3/2}(B)$.

We want to emphasize what Theorem (1.2) says, namely, the use of the double layer potential may fail to yield a solution of the Dirichlet problem when the data belongs to $L^{3/2}(B)$. Nevertheless, there is a solution constructed using the Poisson kernel *for the first quadrant.* In fact,

$$u(x, y) = \frac{1}{\pi}\int_0^\infty \left(\frac{y}{(x - t)^2 + y^2} - \frac{y}{(x + t)^2 + y^2}\right) g_1(t) \, dt$$
$$+ \frac{1}{\pi}\int_0^\infty \left(\frac{x}{(y - s)^2 + x^2} - \frac{x}{(y + s)^2 + x^2}\right) g_2(s) \, ds.$$

In this case the Poisson kernel for the region is well-behaved, that is, the boundary integral against any L^p density $p \geqq 1$ is well-defined. The situation changes dramatically in the case of the exterior problem.

We pass now from the quadrant to the sector

$$Q_\theta = \{(x, y) : x = r \cos \alpha, \, y = r \sin \alpha, \, r > 0, \, 0 < \alpha < \theta\}.$$

When $0 < \theta < \pi$, the method of double layer potentials behaves as it does for the quadrant. However, the operator, K, in the boundary system (1.1) is replaced by the operator K_θ restricting the Poisson kernel to the ray, $(r \cos \theta, r \sin \theta)$, $0 < r < \infty$. The formula is

$$K_\theta(f)(u) = \frac{\sin \theta}{\pi}\int_0^\infty \frac{u}{u^2 + v^2 - 2uv \cos \theta} f(v) \, dv, \qquad u > 0.$$

We can extend Theorem (1.2) to the case $0 < \theta < \pi$ as follows:

Theorem (1.3). *If $p \neq 2 - \theta/\pi$, and $1 < p < \infty$, the method of double layer potentials yields the unique solution of the following L^p-Dirichlet problem:*

(i) $\Delta u = 0$ *in* Q_θ .

(ii) $$||u_r||_{L^p(B_\theta)} \equiv \left\{ \int_0^\infty |u(x + r\cos\tfrac{1}{2}\theta, r\sin\tfrac{1}{2}\theta)|^p\,dx + \int_{-\infty}^0 |u(x\cos\theta + r\cos\tfrac{1}{2}\theta, -x\sin\theta + r\sin\tfrac{1}{2}\theta)|^p\,dx \right\}^{1/p}$$

is bounded for $r > 0$.

(iii) $u_r \to g$ *in* $L^p(B_\theta)$, *where* $g \in L^p(B_\theta)$ *is the given data.*

If $p = \infty$, (iii) is replaced by: $u_r \to g$ a.e. where $g \in L^\infty(B_\theta)$. If $p = 2 - \theta/\pi$ the method of double layer potentials fails to give existence for some $g \in L^{2-(\theta/\pi)}(B_\theta)$.

To prove (1.3) using [4] one needs the Mellin transform, $\hat{K}_\theta(\zeta)$, of the kernel defining K_θ . This is given by

$$\hat{k}_\theta(\zeta) = \frac{\sin(\pi - \theta)\zeta}{\sin\pi\zeta}, \qquad 0 < \operatorname{Re}\zeta < 1.$$

As in the case $\theta = \pi/2$, $I - K_\theta$ has dense range as an operator on $L^{2-(\theta/\pi)}(0, \infty)$, but is not onto. This in turn implies that the boundary operator

$$\begin{pmatrix} I & K_\theta \\ K_\theta & I \end{pmatrix}$$

has dense range but is not onto.

2. The exterior problem. existence and uniqueness. Let us consider the exterior of the first quadrant. Here the double layer potential with density f, given by $f_1(x)$ on the positive x-axis and by $f_2(y)$ on the positive y-axis, is

$$u(x, y) = \frac{1}{\pi}\int_0^\infty \frac{-y}{(x - t)^2 + y^2} f_1(t)\,dt + \frac{1}{\pi}\int_0^\infty \frac{-x}{(y - s)^2 + x^2} f_2(s)\,ds$$

where x and y are not both positive. In this case when we go to the boundary we obtain the following limits:

$$\begin{aligned} &f_1(x) - Kf_2(x), \text{ on the positive } x\text{-axis,} \\ (2.1)\qquad & \\ &-Kf_1(y) + f_2(y), \text{ on the positive } y\text{-axis,} \end{aligned}$$

where K is as in Section 1. The discussion following (1.1) proves the existence part of the following theorem:

Theorem (2.2). *If $p \neq 3/2$ and $1 < p < \infty$, there exists a unique solution, u, of the L^p-Dirichlet problem:*

(i) $\Delta u = 0$ *in the exterior of the first quadrant,*

(ii) $$||u_r||_{L^p(B)} \equiv \left\{\int_0^\infty |u(-r, y - r)|^p \, dy + \int_0^\infty |u(x - r, -r)|^p \, dx\right\}^{1/p}$$

is bounded for $r > 0$.

(iii) $u_r \to g$ *in* $L^p(B)$ *as* $r \to 0$, *where* $g \in L^p(B)$.

If $p = \infty$, (iii) is replaced by $u_r \to g$ a.e., $g \in L^\infty(B)$.

If $p = 3/2$, the method of double layer potentials fails to give existence for some $g \in L^{3/2}(B)$.

Before proving uniqueness we would like to remark on the non-efficacy of the method of Poisson integrals in the exterior angle case.

In Section 1, the Poisson kernel of the first quadrant gave the existence of a solution for $p = 3/2$. The conformal map $z \to z^{2/3}$ which carries the exterior of the first quadrant onto the lower half-plane can be used to compute explicitly the Poisson kernel of the exterior region. To do this, we compose the Green's function for the lower half-plane with the conformal map, and then compute the normal derivative at a point on the boundary, B. For example, if the boundary point is on the positive x-axis, we apply

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(= \frac{1}{r}\frac{\partial}{\partial\theta} \text{ when } \theta = 0\right)$$

to the Green's function

$$G(u, v; x, y) = \frac{1}{2\pi}\left[\log|z^{2/3} - w^{2/3}| - \log|z^{2/3} - \overline{w^{2/3}}|\right]$$

where $z = x + iy$ and $w = u + iv$, and obtain

$$\frac{\partial}{\partial y} G|_{(u,v;x,0)} = \text{Const.} \frac{\operatorname{Im} \overline{w^{2/3}}}{|w^{2/3} - x^{2/3}|^2} \cdot x^{-1/3}.$$

This shows that you cannot form the boundary potential of this Poisson kenrel against every function in $L^p(B)$ for $p \leqq 3/2$.

We now prove uniqueness by means of a representation lemma.

Lemma (2.3). *Let* $1 < p \leqq \infty$, $p \neq 3/2$. *Suppose*

(i) $\Delta u = 0$ *in the exterior of the first quadrant,*

(ii) $||u_r||_{L^p(B)}$ *is bounded for* $r > 0$,

(iii) $u_r \to g$ *as* $r \to 0$, *in* $L^p(B)$ *for* $p < \infty$ *or* a.e., *when* $p = \infty$.

Then

$$u(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{y}{(x - t)^2 + y^2} T_1(g)(t) \, dt - \frac{1}{\pi}\int_0^\infty \frac{x}{(y - s)^2 + x^2} T_2(g)(s) \, ds \tag{2.4}$$

where

$$\begin{pmatrix} T_1 g \\ T_2 g \end{pmatrix} = \begin{pmatrix} I & -K \\ -K & I \end{pmatrix}^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and

$$g_1(x) = g(x, 0), \qquad g_2(y) = g(0, y).$$

Proof. Let

$$u_k(x, y) = u\left(x - \frac{1}{k}, y - \frac{1}{k}\right).$$

Then by the mean-value property u_k is bounded and continuous up to the boundary. A conformal mapping to the unit disc, D, shows uniqueness in the case $p = \infty$ since the transformed functions are in the Hardy Class $H^\infty(D)$. Since we know an existence result, namely (2.2),

$$(2.5) \qquad u_k(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{y}{(x - t)^2 + y^2} f_{k1}(t)\, dt - \frac{1}{\pi}\int_0^\infty \frac{x}{(y - s)^2 + x^2} f_{k2}(s)\, ds$$

where $f_k = (f_{k1}, f_{k2}) \in L^\infty(0, \infty) \times L^\infty(0, \infty)$.

$$f_k = \begin{pmatrix} I & -K \\ -K & I \end{pmatrix}^{-1} \begin{pmatrix} u_{k1} \\ u_{k2} \end{pmatrix},$$

$$u_{k1}(t) = u\left(t - \frac{1}{k}, -\frac{1}{k}\right) = u_k(t, 0),$$

and

$$u_{k2}(s) = u\left(-\frac{1}{k}, s - \frac{1}{k}\right) = u_k(0, s).$$

In fact

$$f_{k1} = (I - K^2)^{-1}(u_{k1} + Ku_{k2}) \equiv (I - K)^{-1}h_{k1}$$
$$f_{k2} = (I - K^2)^{-1}(Ku_{k1} + u_{k2}) \equiv (I - K)^{-1}h_{k2}.$$

Here the inverses are inverses of operators acting on L^∞. It is shown in [4] that if $p > 3/2$, $I - K$, as an operator on L^p, has an inverse which coincides with its inverse on L^∞ (i.e., they coincide on the intersection). The same holds for $I + K$, for all $p > 1$.

In the case $3/2 < p < \infty$, u_{k1}, u_{k2} converge in $L^p(0, \infty)$ to g_1 and g_2 respectively. Hence $h_{ki} = (I + K)^{-1}(u_{ki} + Ku_{kj})(i \neq j)$ converges (in L^p) to $(I + K)^{-1}(g_i + Kg_j)$. ($g_1 = g(x, 0)$ and $g_2 = g(0, y)$.) The lemma for $p > 3/2$ follows on letting $k \to \infty$ in (2.5).

The argument when $1 < p < 3/2$ is similar, but is made more complicated because $(I - K)^{-1}$ for $1 < p < 3/2$ does not coincide with $(I - K)^{-1}$ (acting on L^∞) on $L^p \cap L^\infty$. Let us denote by T_p, $(I - K)^{-1}$ on L^p. In [4] we show that when $f \in L^p \cap L^\infty$,

$$T_p f(x) = T_\infty f(x) + c(f)\cdot x^{-2/3},$$

where

$$c(f) = A\int_0^\infty f(v)v^{-1/3}\, dv,$$

and $A \neq 0$. This implies

$$f_{ki} = T_p h_{ki} + c(h_{ki})x^{-2/3}.$$

Hence (2.5) may be written

$$(2.6) \qquad u_k(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{y}{(x-t)^2 + y^2} T_p h_{k1}(t)\, dt$$
$$- \frac{1}{\pi}\int_0^\infty \frac{x}{(y-s)^2 + x^2} T_p h_{k2}(s)\, ds$$
$$- c(h_{k1})\frac{1}{\pi}\int_0^\infty \frac{y}{(x-t)^2 + y^2} t^{-2/3}\, dt$$
$$- c(h_{k2})\frac{1}{\pi}\int_0^\infty \frac{x}{(y-s)^2 + x^2} s^{-2/3}\, dt.$$

The last two terms have the form

$$w(x, y) = c_1 G(x, y) + c_2 G(y, x),$$

where

$$G(x, y) = \frac{1}{\pi}\int_0^\infty \frac{y}{(x-t)^2 + y^2} t^{-2/3}\, dt.$$

For $\lambda > 0$, $w(\lambda x, \lambda y) = \lambda^{-2/3} w(x, y)$. In order that the $L^p(B)$-norm of $w(x - r, y - r)$ be bounded for all $r > 0$ we must have $w \equiv 0$. Now we can let $k \to \infty$ and obtain the representation (2.4).

Remark (2.7). In the definition of $w(x, y)$ the constants can be chosen so that $w(x - r, y - r)$ has $L^p(B)$-norm bounded for small r; indeed we can get

$$w(r \cos \theta, r \sin \theta) = r^{-2/3} \sin (2\theta/3) \qquad (r > 0, -3/2\pi < \theta < 0).$$

This argument may be used to show that if conditions (i) and (iii) of Lemma (2.3) hold with $g = 0$ and u bounded at infinity, then $u(x, y) = \operatorname{Im} (x + iy)^{-2/3}$.

3. The method of potentials for a 3-dimensional cone. In this section we study the method of double layer potentials in the simplest 3-dimensional domain with a simple corner, namely a cone.

We consider the right circular cone, $\{(x, y, z) : z^2 > x^2 + y^2, z > 0\}$. Let B denote the boundary of the cone. We will write a point $P \in B$ in terms of its cylindrical coordinates, $P = (r \cos \theta, r \sin \theta, z)$, $r > 0$, $0 \leqq \theta < 2\pi$. The inner normal to B at P is given by $N_P = (-\cos \theta, -\sin \theta, 1)/\sqrt{2}$. We express points X_0 inside the cone and off its axis in cylindrical coordinates and in the form $P_0 + sN_{P_0}$:

$$X_0 = (r_0 \cos \theta_0, r_0 \sin \theta_0, z_0)$$
$$= \left(r^0 \cos \theta_0 - \frac{s}{\sqrt{2}} \cos \theta_0, r^0 \sin \theta_0 - \frac{s}{\sqrt{2}} \sin \theta_0, r^0 + \frac{s}{\sqrt{2}}\right).$$

Here

$$r^0 = \tfrac{1}{2}(z_0 + r_0), \qquad s = \frac{\sqrt{2}}{2}(z_0 - r_0), \quad \text{so} \quad r_0 = r^0 - (s/\sqrt{2}).$$

The double layer potential of a density $f(r, \theta)$ on B is

$$u(X_0) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{z_0 - r_0 \cos(\theta_0 - \theta)}{|X_0 - P|^3} f(r, \theta) r \, dr \, d\theta.$$

Put $t = s/\sqrt{2}$ and for $t < r_0$ set $u_t(r_0, \theta_0) = u(P_0 + sN_{P_0}) = u((r_0 - t)\cos\theta_0, (r_0 - t)\sin\theta_0, r_0 + t)$. When $t \geqq r_0$ we set $u_t = 0$. As in Section 1, $\|u_t\|_{L^p(B)}$ $(1 < p \leqq \infty)$ is bounded for $t > 0$. Moreover, u_t converges a.e. (and in $L^p(B)$ for finite p) to $(I + K)f$ where

$$Kf(r_0, \theta_0) = \frac{1}{2^{5/2}\pi} \int_0^\infty \int_0^{2\pi} \frac{r_0(1 + \cos(\theta_0 - \theta))}{[r_0^2 + r^2 - r_0 r(1 + \cos(\theta_0 - \theta))]^{3/2}} f(r, \theta) r \, dr \, d\theta$$

(in this formula, r_0 is used for r^0 in the previous one).

We observe that multiplication by $r_0^{2/p}$ transforms Kf into a convolution on the commutative group $R^+ \times T$ where T is the circle group; the Haar measure is $(dr/r)d\theta$. The harmonic analysis used in [4], namely Wiener's Theorem [6], shows that $I + K$ is invertible on $L^p(B)$ if and only if

$$\hat{k}(\zeta, m) \equiv \frac{1}{2^{5/2}\pi} \int_0^\infty r^\zeta \int_0^{2\pi} \frac{r(1 - \cos\theta)\cos m\theta}{[1 + r^2 - r(1 + \cos\theta)]^{3/2}} \, d\theta \, \frac{dr}{r} \neq 1$$

for any integer m or any $\zeta = 2/p + i\xi$, ξ real.

We can show that the set N of values of $p > 1$ for which $\hat{k}(2/p + i\xi, m) = 1$ for some ξ and m is non-empty, contained in $(1, 2)$, and discrete. That N is not empty is seen by observing that $\hat{k}(2/p, 0) \to \infty$ as $p \downarrow 1$. Also $|\hat{k}(\zeta, m)| \leqq 1/\sqrt{2}$ when $\operatorname{Re}(\zeta) \leqq 1$. The sequence of functions $\hat{k}(\zeta, m)$, holomorphic in $0 < \operatorname{Re}(\zeta) < 2$, converges to 0 uniformly in compact substrips. An application of Hurwicz's theorems yields the discreteness of N.

4. The method of potentials for a 3-dimensional wedge. In this section we study the method of double layer potentials for solving the L^p-Dirichlet problem for the domain

$$W = \{(x, y, z) : x \in (-\infty, \infty), y > 0, z > 0\}.$$

Again we denote by B the boundary of W. The double layer potential with density $f = (f_1, f_2)$ on B is given by

$$u(x_0, y_0, z_0) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{y_0}{[(x - x_0)^2 + (z - z_0)^2 + y_0^2]^{3/2}} f_1(x, z) \, dx \, dz$$

$$+ \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{z_0}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}} f_2(x, y) \, dx \, dy.$$

Proceeding as in the case of the first quadrant (Section 1) we define parallel boundaries

$$B_r = B + r(0, 1, 1) = \{(x, y + r, r) : y > 0\} \cup \{(x, r, z + r) : z > 0\}$$

and we let u_r denote the restriction of u to B_r. As in Section 1 we view u_r as a function on B. As such $||u_r||_{L^p(B)} (1 < p \leqq \infty)$ is bounded for $r > 0$. Moreover, u_r converges a.e. (and in $L^p(B)$ for finite p) to

$$f_1 + Kf_2 \quad \text{on} \quad \{(x, y, 0) : y > 0\} \equiv B^1$$

$$Kf_1 + f_2 \quad \text{on} \quad \{(x, 0, z) : z > 0\} \equiv B^2$$

where now K is given by

$$Kf(x, u) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{u}{[(x - t)^2 + u^2 + v^2]^{3/2}} f(t, v)\, dt\, dv. \tag{4.1}$$

***Remark* (4.2).** If we multiply $Kf(x, u)$ by $u^{1/p}$ and make appropriate changes of variables we can write

$$u^{1/p} Kf(x, u) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{s^{1+(1/p)}}{(t^2 + s^2 + 1)^{3/2}} \left(\frac{u}{s}\right)^{1/p} f\left(x - \frac{u}{s} t, \frac{u}{s}\right) dt\, \frac{ds}{s}.$$

This is seen to be convolution $f \to f * k_p$ with a non-negative kernel k_p on the group V of 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}, \qquad x \in R, \qquad y > 0,$$

with right-invariant Haar measure $dx(dv/v)$. The group is neither Abelian nor unimodular and does not support the harmonic analysis used in [4].

However, since multiplication by $u^{1/p}$ is an isometry from $L^p(R \times (0, \infty))$ with Lebesgue measure onto $L^p(V)$ we can use convolution to find the norm of the operator K as an operator on L^p; it is just the L^1 norm of the non-negative kernel k_p and is given by the expression

$$\frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \frac{s^{1+(1/p)}}{(t^2 + s^2 + 1)^{3/2}}\, dt\, \frac{ds}{s} = \tfrac{1}{2} \sec \frac{\pi}{2p}.$$

Clearly, the norm of K is <1 if $p > 3/2$, equals 1 when $p = 3/2$, is >1 for $1 < p < 3/2$ and tends to infinity as $p \to 1$.

***Theorem* (4.3).** *The system*

$$f_1 + Kf_2 = g_1,$$

$$Kf_1 + f_2 = g_2,$$

where K is given by (4.1), *has a unique solution* $(f_1, f_2) = f \in L^p(B)$ *for any data* $(g_1, g_2) = g \in L^p(B)$ *for* $p > 3/2$.

When $1 < p < 3/2$, the space of solutions to the homogeneous problem has infinite dimension.

Proof. The invertibility of the matrix

$$M = \begin{pmatrix} I & K \\ K & I \end{pmatrix}$$

on $L^p(\mathbf{R} \times (0, \infty)) \times L^p(\mathbf{R} \times (0, \infty))$ is equivalent to the invertibility of $(I - K^2)$ on $L^p(\mathbf{R} \times (0, \infty))$. Hence, by Remark (4.2), $(I - K^2)$ is invertible for $p > 3/2$, and the first part of the theorem follows.

For the second part we observe that the null space of M consists of pairs $(g, -Kg)$ where g belongs to the null space of $I - K^2$. It is enough ,therefore, to show that the null space of $I - K$ has infinite dimension. This is the purpose of the following lemmas. These lemmas describe the null space of $I - K$ in terms of the kernels of operators $I - K_z$ on $L^p(0, \infty)$ obtained by taking the partial Fourier transform in the x-variable. More precisely, if $(I - K)f = 0$ for some non-zero $f \in L^p(R \times (0, \infty))$, then the partial Fourier transform

$$\left(\mathfrak{F}_x f(\xi, u) = \int_{-\infty}^{\infty} f(x, u)e^{ix\xi}\, dx\right)$$

yields

$$0 = \mathfrak{F}_x[(I - K)f](\xi, u) = [(I - K_{\xi^2/4})(\mathfrak{F}_x f)](u) \tag{4.4}$$

where

$$K_z f(u) = \frac{1}{\pi} \int_0^{\infty} \frac{u}{u^2 + v^2} H_1(z(u^2 + v^2))f(v)\, dv, \tag{4.5}$$

and

$$H_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-zs} e^{-1/s} \frac{ds}{s^{1+\nu}} \tag{4.6}$$

where Re $(z) \geqq 0$, $\nu > 0$, and Γ denotes the gamma function. An integration by parts gives

$$H_\nu(z) = \frac{1}{z} (\nu + 1)\nu[H_{\nu+2}(z) - H_{\nu+1}(z)]. \tag{4.7}$$

Remark. $H_\nu(z)$ can be expressed in terms of modified Bessel functions; see [10, p. 183]. We will need values of ν, z other than 1 and $\xi^2/4$ later. Note that K_0 is the operator K used in Section 1.

We note in passing our use of the fact that $(\mathfrak{F}_x f)(\xi, u)$ is in L^p as a function of $u > 0$ for a.e. ξ. This is easily checked using the Hausdorff-Young Theorem since $p < 2$.

Lemma (4.8). *If* $1 < p < 3/2$, Re $(z) \geqq 0$, $z \neq 0$, *then* $I - K_z$ *has a one-dimensional null space spanned by*

$$m_z(u) = \lim_{n \to \infty} K_z^n(v^{-2/3})(u).$$

Proof. First, assume $(I - K_z)(f) = 0$ with $f \in L^p(0, \infty)$. Then $(I - K_0)f = (K_z - K_0)f$; we will show presently that $(K_z - K_0)f \in L^r \cap L^p$ for all $r \geqq p$. Let us choose a finite $r > 3/2$. Denote by T_p, T_r the inverses of $I - K_0$ as operators on L^p, L^r respectively. Then

$$\begin{aligned} f &= T_p(I - K_0)f = T_p(K_z - K_0)f \\ &= (T_p - T_r)(K_z - K_0)f + T_r(K_z - K_0)f. \end{aligned}$$

It is shown in [4, Example (3.3)] that on $L^p \cap L^r$,

$$(T_p - T_r)g = Au^{-2/3} \int_0^\infty g(v)v^{-1/3}\,dv$$

where A is a non-zero constant. Therefore

$$f(u) = cu^{-2/3} + T_r(K_z - K_0)f \equiv cu^{-2/3} + g$$

where c depends on f, and $g \in L^r$. Then

$$f = K_z^n(f) = cK_z^n(v^{-2/3}) + K_z^n(g).$$

Since $|K_z(g)| \leqq K_0(|g|)$, the norm of K_z as an operator on L^r is <1 $(r > 3/2)$. Hence $K_z^n(g) \to 0$ a.e., so m_z spans the null space. It remains to show that $(K_z - K_0)f \in L^r$, $r > p$.

Now,

$$|(K_z - K_0)f(u)| \leqq 2K_0(|f|)(u) \leqq cu^{-1/p}$$

by Hölder's inequality, an estimate useful for $u \geqq 1$. When $0 < u < 1$ we break the interval of integration into two parts to get

$$(K_z - K_0)f(u) = \frac{1}{\pi}\int_0^1 \frac{u}{u^2 + v^2}\,(H_1(z(u^2 + v^2)) - H_1(0))f(v)\,dv$$
$$+ c\left(\int_1^\infty \frac{u}{u^2 + v^2}\,|f(v)|\,dv\right).$$

Again by Hölder's inequality the second term is bounded in u, $0 < u < 1$. For the first term we use the inequality (for $|z| \leqq 2$, Re $(z) \geqq 0$)

$$\begin{aligned} |H_1(z) - H_1(0)| &\leqq \int_0^\infty |e^{-zs} - 1|\,\frac{e^{-1/s}}{s^2}\,ds \leqq |z| \int_0^{2/|z|} \frac{e^{-1/s}}{s}\,ds + 2\int_{2/|z|}^\infty \frac{ds}{s^2} \\ &= 0\left(|z| \log \frac{4}{|z|}\right). \end{aligned}$$

Thus if $|z| \leqq 1$ and Re $(z) \geqq 0$ then $|z(u^2 + v^2)| \leqq 2$; and we obtain the estimate

$$0\left(u \log \frac{2}{u}\right) \quad \text{for} \quad |(K_z - K_0)f(u)| \quad \text{for} \quad 0 < u < 1.$$

To complete the proof of the lemma we need to show that $m_z(u) \not\equiv 0$ and that $m_z \in L^p$. For convenience we let

$$h(u) = K_z(v^{-2/3}) = \frac{1}{\pi}\int_0^\infty \frac{u}{u^2+v^2}\int_0^\infty e^{-z(u^2+v^2)s}e^{-1/s}\frac{ds}{s^2}\frac{dv}{v^{2/3}}.$$

Change s into $s/(u^2+v^2)$ and then interchange the order of integration; this leads to

$$h = u^{-2/3}H_{5/6}(zu^2) = u^{-2/3} + u^{-2/3}(H_{5/6}(zu^2) - 1).$$

For $0 < \nu < 1$, $|H_\nu(z) - 1| = 0(|z|^\nu)$, and is bounded. Hence $h = u^{-2/3} + g$, where $g \in L^r$ for $r > 3/2$ only. Then

$$K_z(h) = h + K_z(g).$$

By induction,

$$K_z^{N+1}(v^{-2/3}) = K_z^N h = h + \sum_{m=1}^{N} K_z^m g.$$

Since $g \in L^r$, $r > 3/2$, the series converges pointwise a.e. to a function in L^r. Thus

$$m_z = h + \sum_{m=1}^{\infty} K_z^m g.$$

But $h = u^{-2/3}H_{5/6}(zu^2) \sim u^{-2/3}$ as $u \to 0$, which implies $h \notin L^r$, $r > 3/2$, and therefore, that $m_z(u) \not\equiv 0$.

To show that $m_z \in L^p$, we observe by (4.7) that $h \in L^p(z \neq 0)$; it is sufficient to show that the series is also in L^p. To this end we shall show that $||K_z g||_p \leqq c\,||g||_r$, $r \geqq p$. Since $K_z g \in L^r$, $K_z g$ is in L^p on (0, 1) and

$$\left(\int_0^1 |K_z g|^p\,du\right)^{1/p} \leqq ||K_z g||_r \leqq c_r\,||g||_r\,.$$

For $u > 1$ we obtain from (4.7) that

$$|K_z g(u)| \leqq \frac{c}{|z|}\int_0^\infty \frac{u}{(u^2+v^2)^2}\,|g(v)|\,dv$$

$$\leqq \frac{c}{|z|\,u^{1+1/r}}\,||g||_r\,.$$

This completes the proof of Lemma (4.8).

The preceding lemma gives a formal technique for obtaining nontrivial solutions of $(I - K)f = 0$ in the case $1 < p < 3/2$. Start with a function $\varphi(x)$ and form

$$f(x, u) = \mathfrak{F}_\xi^{-1}(\mathfrak{F}_x\varphi(\xi)m_{\xi^2/4}(u))(x). \tag{4.9}$$

Then

$$\mathfrak{F}_x((I - K)f)(\xi, u) = \mathfrak{F}_x\varphi(\xi)(I - K_{\xi^2/4})m_{\xi^2/4}(u) = 0.$$

This argument becomes rigorous when we show $f \in L^p(\mathbf{R} \times (0, \infty))$, using suitable functions φ. In the following lemma, $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions [8].

Lemma (4.10). *If $\varphi \in \mathcal{S}$ and $\int \varphi(x)\, dx = 0$ the function f defined by* (4.9) *belongs to $L^p(\mathbf{R} \times (0, \infty))$, $1 < p < 3/2$.*

Proof. For $u > 0$, fixed, $m_{\xi^2}{}^n(u) \equiv (K_{\xi^2}{}^n(v^{-2/3}))(u)$ converges pointwise and boundedly in ξ to $m_{\xi^2}(u)$. In fact, $0 \leqq m_{\xi^2}{}^n(u) \leqq u^{-2/3}$. It will be enough to show that $f_n(x, u) = (m_{\xi^2/4}{}^n(u)\hat{\varphi}(\xi))^{\check{}}$ has $L^p(\mathbf{R} \times (0, \infty))$-norm bounded independent of n (we use $\hat{\varphi}$ for $\mathcal{F}\varphi$ and $\check{\varphi}$ for $\mathcal{F}^{-1}\varphi$).

We need the following formula for $m_z{}^n(u)$, proved by induction:

$$(4.11) \qquad m_z{}^n(u) = u^{-2/3}(2\pi)^{-n+1} \int_0^\infty \cdots \int_0^\infty H_{5/6}(zu^2\psi(s)) \frac{ds_1 \cdots ds_{n-1}}{\prod_{m=1}^{n-1} (1 + s_m)s_m{}^{5/6}}$$

where $\psi(s) = \psi(s_1, \cdots, s_{n-1}) = (1 + s_{n-1} + s_{n-1}s_{n-2} + \cdots + s_{n-1} \cdots s_1)$. A proof of this formula will be indicated at the end of the argument. If $H_{5/6}(zu^2\psi(s))$ is replaced by 1, the result is $u^{-2/3}$. We can then use Minkowski's inequality and the fact that $\psi(s) > 1$ to reduce the problem of estimating the norm of f_n to that of showing that for $\lambda > 0$,

$$f_\lambda(x, u) \equiv \left[u^{-2/3}H_{5/6}\left(\frac{\xi^2}{4}\, u^2\lambda\right)\hat{\varphi}(\xi)\right]^{\check{}}$$

has $L^p(\mathbf{R} \times (0, \infty))$-norm bounded by a constant times a *negative* power of λ.

When we pass the Fourier transform inside the integral we obtain

$$f_\lambda(x, u) = c \int_0^\infty \int \varphi(y)u^{-2/3} \frac{e^{-(x-y)^2/u^2\lambda s}}{(u^2\lambda s)^{1/2}}\, e^{-1/s}\, dy\, \frac{ds}{s^{1+5/6}}.$$

An interchange in the order of integration leads to

$$f_\lambda(x, u) = c \int \varphi(y) \frac{\lambda^{5/6}u}{((x - y)^2 + u^2\lambda)^{4/3}}\, dy.$$

First, consider the case $|x| \leqq 1$. By Minkowski's inequality,

$$\begin{aligned} \|f_\lambda(x, \cdot)\|_{L^p(0,\infty)} &\leqq c\lambda^{5/6} \int |\varphi(y)| \left\{\int_0^\infty \frac{u^p}{[(x - y)^2 + u^2\lambda]^{4p/3}}\, du\right\}^{1/p} dy \\ &\leqq c\lambda^{1/3-1/2p} \int \frac{|\varphi(y)|}{|x - y|^{5/3-1/p}}\, dy. \end{aligned}$$

Since $p < 3/2$ the exponent of λ is negative and the integral is a bounded function of x because $\varphi \in L^1 \cap L^\infty$.

Next, suppose $|x| > 1$. Since $\int \varphi = 0$

$$f_\lambda(x, u) = \lambda^{5/6} \int \varphi(y)\left\{\frac{u}{[(x-y)^2 + u^2\lambda]^{4/3}} - \frac{u}{[x^2 + u^2\lambda]^{4/3}}\right\} dy$$

$$= \lambda^{5/6} \int_{|y|<|x|/2} + \lambda^{5/6} \int_{|y| \geqq |x|/2} \equiv g(x, u) + h(x, u).$$

We treat each term defining h separately and again by Minkowski's inequality,

$$||h(x, \cdot)||_{L^p(0,\infty)} \leqq c\lambda^{1/3-1/2p} \int_{|y|<|x|/2} \frac{|\varphi(y)|}{|x-y|^{5/3-1/p}}\, dy$$

$$+ c\lambda^{1/3-1/2p} \frac{1}{|x|^{5/3-1/p}} \int_{|y|>|x|/2} |\varphi(y)|\, dy.$$

Because $|\varphi(y)| \leqq c_k(1 + |y|^2)^{-2k}$ for any k, the first term is dominated by a constant times

$$\lambda^{1/3-1/2p}\, |x|^{-2k} \int_{|y|>|x|/2} \frac{(1 + |y|^2)^{-k}}{|x-y|^{5/3-1/p}}\, dy \leqq c\lambda^{1/3-1/2p}\, |x|^{-2k}.$$

Again, the second term is in L^p for large x since $|\varphi(y)| \leqq C_k(1 + |y|^2)^{-2k}$ for any k.

Finally, we estimate g using the mean value theorem and the relations $\frac{1}{2}|x| \leqq |x - y| \leqq 2|x|$:

$$|g(x, u)| \leqq c\lambda^{5/6} \int_{|y|<|x|/2} |\varphi(y)| \frac{u\, |y|\, |x|}{(|x|^2 + u^2\lambda)^{4/3+1}}\, dy.$$

Hence

$$||g(x, \cdot)||_{L^p(0,\infty)} \leqq c\lambda^{1/3-1/2p}\left(\int_{|y|<|x|/2} |\varphi(y)|\, |y|\, dy\right) |x|^{1/p-8/3}$$

$$\leqq c\lambda^{1/3-1/2p}\, |x|^{1/p-8/3}.$$

It follows that

$$||f_\lambda||_{L^p(\mathbf{R}\times(0,\infty))} \leqq c_p(\varphi)\lambda^{1/3-1/2p}.$$

This completes the proof of Lemma 4.10.

We indicate the steps of the induction proof of (4.11). Assuming (4.11) for n, we have

$$m_z^{n+1}(u) = \frac{1}{\pi} \int_0^\infty \frac{u}{u^2 + v^2} H_1(z(u^2 + v^2)) m_z^n(v)\, dv. \tag{4.12}$$

We write

$$H_1(z(u^2 + v^2)) = \int_0^\infty e^{-z(u^2+v^2)r} e^{-1/r} \frac{dr}{r^2},$$

and in (4.11) with u replaced by v we write

$$H_{5/6}(zv^2\psi(s)) = \frac{1}{\Gamma(5/6)} \int_0^\infty e^{-zv^2\psi(s)s_n} e^{-1/s_n} \frac{ds_n}{s_n^{1+5/6}}.$$

When these expressions are substituted into (4.11) and (4.12), we get an integral in the variables $v > 0$, $r > 0$, $s_n > 0$ and $s \in \mathbf{R}_+^{n+1}$. Successively, we replace r by $r/(u^2 + v^2)$, s_n by s_n/v^2, integrate in v, replace s_n by rs_n, and finally r by u^2r. The resulting expression is integrated in r to obtain $H_{5/6}(zu^2\psi(s_n, s_{n-1}, \cdots, s_1))$ and thus (4.11) with $n + 1$ in place of n.

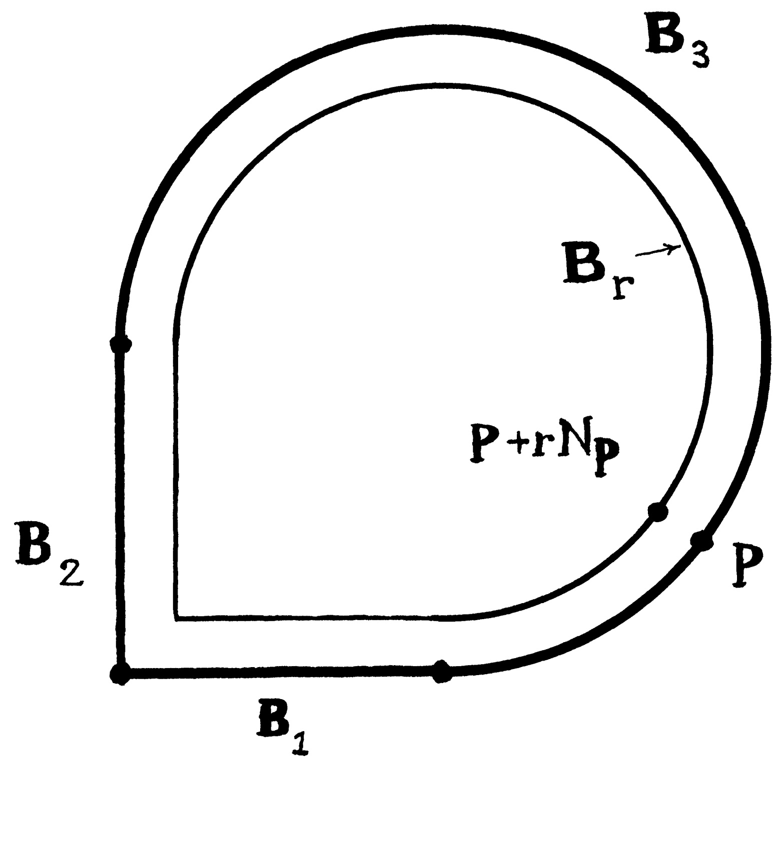
5. The method of potentials for a bounded plane domain. In this section we shall examine the method of double-layer potentials for solving the L^p-Dirichlet problem in the domain bounded by the three curves

$$\gamma_1(x) = (x, 0), \quad 0 \leqq x \leqq 1, \quad \gamma_3(\theta) = (1 + \cos\theta, 1 + \sin\theta), \quad -\pi/2 \leqq \theta \leqq \pi,$$

and

$$\gamma_2(y) = (0, 1 - y), \qquad 0 \leqq y \leqq 1.$$

As shown in the figure we denote by $B = B_1 \cup B_2 \cup B_3$ the boundary of the domain and by B_r the set of points of the form $P + rN_P$ where $P \in B$ are $|P| > r$. Define $u_r(P) = u(P + rN_P)$ when $P + rN_P \in B_r$ and zero otherwise.



As before, we form the double-layer potential u of a density f on the boundary and then find the limit of u_r as $r \to 0$. If we write $f = f_1 + f_2 + f_3$ with $f_i = f|_{B_i}$ we obtain

$$u|_{B_1} = f_1 + K_1 f_2 + c_{13} f_3$$
$$u|_{B_2} = K_1 f_1 + f_2 + c_{23} f_3$$
$$u|_{B_3} = c_{31} f_1 + c_{32} f_2 + (I + c_{33}) f_3$$

where

$$K_1 f(u) = \frac{1}{\pi} \int_0^1 \frac{u}{u^2 + v^2} f(v)\, dv, \qquad 0 < u < 1,$$

and the operators c_{ij} are compact. They are compact because away from the corner the boundary is locally represented by the graphs of C^1 functions having Lipschitz continuous first derivatives, and hence, when i or j is three,

$$\chi_{B_i}(P) \int_{B_j} \frac{\langle P - Q, N_Q \rangle}{|P - Q|^2} f_j(Q)\, dQ$$

can be bounded locally by convolution with an L^1-kernel. Thus we are led to study the Fredholm alternative or its absence for the matrix

$$\begin{bmatrix} I & K_1 & 0 \\ K_1 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 & c_{13} \\ 0 & 0 & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

of operators. This in turn is equivalent to the Fredholm properties for the matrix

$$\begin{bmatrix} I & K_1 \\ K_1 & I \end{bmatrix} \tag{5.1}$$

Theorem (5.2). *As an operator on* $L^p(0, 1) \times L^p(0, 1)$ *the matrix* (5.1)
(*i*) *is invertible for* $p > 3/2$,
(*ii*) *is not Fredholm for* $p = 3/2$,
(*iii*) *is Fredholm for* $1 < p < 3/2$.

Proof. To prove (i) we note that K_1 as an operator on $L^p(0, 1)$ has norm < 1 for $p > 3/2$.

For (ii) we shall show that (5.1) has dense range but is not onto. Since $L^p(0, 1) \subset L^{3/2}(0, 1)$ for $p > 3/2$, (i) implies the density of the range. Let us suppose that (5.1) is indeed onto. Then $I - K_1^2$ is onto. Let $K_R : L^p(0, R) \equiv L_R^p \to L_R^p$ be given by

$$K_R f(u) = \frac{1}{\pi} \int_0^R \frac{u}{u^2 + v^2} f(v)\, dv, \qquad 0 < R \leqq \infty.$$

The operator K_∞ is the operator K of Section 1. We know that $(I - K^2)$ is not onto (Section 1). Now $K_R = M_{1/R} K_1 M_R$ when $M_s f(u) = f(su)$. Thus if I_R denotes the identity on $L_R^{3/2}$, we have that $I_R - K_R^2 = M_{1/R}(I_1 - K_1^2) M_R$ is onto. We shall deduce that $I - K^2$ is onto. For $g \in L^{3/2}(0, \infty)$ let $g_R = \chi_{(0,R)} g$. Then there exists $f^R \in L_R^{3/2}$ such that

$$(I_R - K_R^2) f^R = g_R .$$

Because the range is closed the f^R can be chosen in such a way that

$$\|f^R\|_{3/2} \leqq c\, \|g_R\|_{3/2} \leqq c\, \|g\|_{L^{3/2}}$$

with c independent of R. If we extend f^R to $(0, \infty)$ by letting $f^R(u) = 0$ for $u > R$ we can get a weakly convergent sequence of f^R's tending weakly to $f \in L^{3/2}(0, \infty)$. We claim that for such f, we would have $(I - K^2) f = g$. For if $h \in L^3$,

$$\int_0^\infty (I - K^2) fh\, du = \lim_{R\to\infty} \int_0^\infty f^R (I - K^2)^* h$$
$$= \lim_{R\to\infty} \int_0^\infty g_R h = \int_0^\infty gh.$$

To prove (iii) we note that, as before, $I - K_1^2$ has dense range in $L^p(0, 1)$, $1 < p < 3/2$. Clearly for $f \in L^p(0, 1)$ (and extended by zero)

$$\|f\|_{L^p(0,1)} \leqq c\, \|(I - K^2) f\|_{L^p(0,\infty)} \tag{5.3}$$
$$\leqq c\, \|(I - K_1^2) f\|_{L^p(0,1)} + c\, \|K^2 f\|_{L^p(1,\infty)} .$$

Now when

$$u > 1, \qquad |K^2 f(u)| \leqq c \frac{\log 2u}{u} \|f\|_p$$

and if $f_n \to f$ weakly then $K^2 f_n \to K^2 f$ pointwise. Let $A = (I - K_1^2)$. We show next that A has closed range. If not, there exist f_n's $\in L^p(0, 1)$ such that dist $(f_n, \operatorname{Ker} A) = 1$ and $\|Af_n\|_{L^p(0,1)} \to 0$. We may suppose that the f_n converge weakly to some $f \in \operatorname{Ker} A$. But (5.3) and Fatou's lemma give $\|f_n - f\|_p \to 0$. This contradicts dist $(f_n, \operatorname{Ker} A) = 1$. We have actually shown that $I - K_1^2$ is onto.

Finally, we show that $I - K_1^2$ has finite dimensional null space by showing that the identity map on the null space is compact. This follows from (5.3).

6. The heat equation in a wedge. We consider an L^p-Dirichlet problem for the heat equation in $\Omega \equiv W \times (0, T)$ where W is the 3-dimensional wedge considered in Section 4. The problem involves data specified on the lateral boundary $B \times (0, T)$ where $B = \partial W$. More precisely, we seek u defined in Ω such that

(i) $\dfrac{\partial u}{\partial t} - \Delta u = 0$ in Ω,

(ii) $u(x, 0) = 0, \qquad x \in W,$

(iii) with $u_r(P, t) \equiv u(P + rN_P, t)$, $P \in B$, $\sup_{r>0} \|u_r\|_{L^p(B\times(0,T))} < \infty$,

(iv) $\lim_{r\downarrow 0} u_r(P, t) = g(P, t) \in L^p(B \times (0, T))$

a.e. and in $L^p(B \times (0, T))$ if $p < \infty$.

As before, we attempt to solve the problem using the double layer potential u of a density f on the lateral boundary $B \times (0, T)$, which we again write as (f_1, f_2). That is, we define

$$u(x_0, y_0, z_0, t) = \frac{1}{(4\pi)^{3/2}} \int_0^t \int_0^\infty \int \frac{z_0}{(t-s)^{5/2}} \cdot \exp\left\{-\frac{(x_0 - x)^2 + (y_0 - y)^2 + z_0^2}{4(t-s)}\right\} f_1(x, y, s)\, dx\, dy\, ds$$
$$+ \frac{1}{(4\pi)^{3/2}} \int_0^t \int_0^\infty \int \frac{y_0}{(t-s)^{5/2}} \cdot \exp\left\{-\frac{(x_0 - x)^2 + (z_0 - z)^2 + y_0^2}{4(t-s)}\right\} f_2(x, z, s)\, dx\, dz\, ds.$$

When we pass to $B \times (0, T)$ we are led to the system of integral equations

$$f_1 + Kf_2 = g_1$$
$$Kf_1 + f_2 = g_2, \tag{6.1}$$

where g_1, g_2 are the parts of g on the two flat parts of $B \times (0, T)$. The operator K is now defined by

$$Kf(x, u, t) = \int_0^t \int_0^\infty \int K(x - y, u, v, t - s) f(y, v, s)\, dy\, dv\, ds, \tag{6.2}$$

where

$$K(x, u, v, t) = \frac{1}{(4\pi)^{3/2}} \frac{u}{t^{5/2}} \exp\left\{-\frac{x^2 + u^2 + v^2}{4t}\right\}.$$

Remark (6.3). As in Section 4, we can express $u^{1/p}Kf(x, u, t)$ as a convolution operator over the group of 3×3 matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ x & u & 0 \\ t & 0 & u^2 \end{pmatrix} \qquad (x, t \in \mathbf{R}, u > 0)$$

with right-invariant Haar measure $dx(du/u)\, dt$. Hence the L^p operator norm of K is $\frac{1}{2} \sec \pi/2p$.

The following theorem is the analog of Theorem (4.3).

Theorem (6.4). *The system* (6.1), *with K given by* (6.2), *has a unique solution*

$f = (f_1, f_2) \in L^p(B \times (0, T))$ *for any data* $g = (g_1, g_2) \in L^p(B \times (0, T))$, *provided* $p > 3/2$.

When $1 < p < 3/2$ the null space of the system (6.1) has infinite dimension.

Proof. The first statement has the same proof as the corresponding one in (4.3).

Let $1 < p < 3/2$. We construct functions in the null space of $I - K$ by using the partial Fourier transform in the variables x, t, applied to functions supported in $t > 0$. Let us denote this by $\hat{f}$: that is,

$$\hat{f}(\xi, u, \tau) = \int_0^\infty \int e^{ix\xi + it\tau} f(x, u, t)\, dx\, dt.$$

Formally, if $(I - K)f = 0$ then

$$[(I - K)f]^\wedge(\xi, u, \tau)$$
$$= \hat{f}(\xi, u, \tau) - \frac{1}{\pi} \int_0^\infty \frac{u}{u^2 + v^2} H_1(\tfrac{1}{4}(\xi^2 - i\tau)(u^2 + v^2))\hat{f}(\xi, v, \tau)\, dv = 0.$$

For fixed ξ, τ, $\hat{f}(\xi, v, \tau)$ is in the kernel of $I - K_z$, $z = \frac{1}{4}(\xi^2 - i\tau)$ with K_z the operator of (4.5). The following lemma is the analog of (4.10) and completes the proof of (6.4):

Lemma (6.4). *If* $\varphi(x, y) \in L^p(\mathbf{R} \times (0, T))$ *then*

$$f(x, u, t) \equiv (m_{1/4(\xi^2 - i\tau)}(u)\varphi(\xi, \tau))^\vee$$

is in $L^p(\mathbf{R} \times R^+ \times (0, T))$ *and* $(I - K)f = 0$. *Here* $m_z(u)$ *is the function defined in Lemma* (4.8).

Proof. As in Lemma (4.10) it suffices to show that for $\lambda > 0$,

$$f_\lambda(x, u, t) \equiv [u^{-2/3} H_{5/6}((\xi^2 - i\tau)u^2\lambda)\hat{\varphi}(\xi, \tau)]^\vee$$

has $L^p(\mathbf{R} \times \mathbf{R}^+ \times (0, T))$ norm bounded by a constant times a negative power of λ. In this case we have that

$$f_\lambda(x, u, t) = \text{const.}\, u^{-2/3} \int_0^t \int (\lambda u^2)^{5/6} e^{-\lambda u^2/s} e^{-y^2/4s} s^{-1-(4/3)} \varphi(x - y, t - s)\, dy\, ds.$$

Then,

$$||f_\lambda(x, \cdot, t)||_{L^p(0,\infty)}$$
$$\leqq c\lambda^{5/6} \int_0^t \int \left(\frac{s}{\lambda}\right)^{(1/2)+(1/2)p} e^{-y^2/4s} s^{-1-(4/3)}\, |\varphi(x - y, t - s)|\, dy\, ds.$$

Hence

$$||f_\lambda||_{L^p(R \times R^+ \times (0,T))}$$
$$\leqq c\lambda^{(1/3)-(1/2p)}\, ||\varphi||_{L^p(R \times (0,T))} \int_0^T \int e^{-y^2/4s} s^{(-1/2)-(4/3)+(1/2)p}\, dy\, ds$$
$$\leqq c_T \lambda^{(1/3)-(1/2p)}\, ||\varphi||_{L^p(R \times (0,T))} \quad \text{if} \quad 1 < _Q < \tfrac{3}{2}.$$

Remark (6.5). The results of this section hold when the 3-dimensional wedge W is replaced by the 2-dimensional quadrant Q. The operator to consider on the boundary is simply

$$Kf(u, t) = \int_0^t \int_0^\infty \left\{ \int K(y, u, v, t - s)\, dy \right\} f(v, s)\, dv\, ds,$$

where $K(y, u, v, t)$ is the kernel of (6.2). The operator $K_{1/4(\xi^2 - i\tau)}$ is replaced by $K_{1/4(-i\tau)}$ in the proof of (6.4).

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