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The *h-p* boundary element method for solving 2- and 3-dimensional problems

E.P. Stephan

Institut für Angewandte Mathematik, Universität Hannover, Hannover, Germany

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Abstract

In this survey paper we report on recent developments of the *hp*-version of the boundary element method. We consider weakly singular and hypersingular integral equations on polygons and polyhedral surfaces. We show that the Galerkin solutions (computed with the *hp*-version on geometric meshes) converge exponentially fast towards the exact solutions of the integral equations. The implementation of the *hp*-version of the boundary element method is discussed and adaptive algorithms are given. Model problems for the coupling of finite elements and boundary elements are also considered. Numerical results are presented which underline the theoretical results.

1. Introduction

The *h-p* version was first analyzed theoretically for the finite element method (FEM) by Babuška and others (for an overview see [2]). Meanwhile, the technique of the *h-p* version was applied to the boundary element method (BEM) for solving Dirichlet and Neumann problems for the Laplacian in plane polygonal domains [3, 4, 9], for corresponding mixed boundary value problems [13] and transmission problems [11]. Recently, three-dimensional boundary value problems have been treated with the *h-p* version of the BEM [16, 19].

All boundary value/transmission problems have in common that they can be reduced to strongly elliptic boundary integral equations (pseudodifferential equations) on the boundary/interface. The basic idea of the convergence proofs is the observation that for strongly elliptic pseudodifferential equations one obtains quasioptimal convergence in the energy norm for any Galerkin scheme with conforming elements [32]. This result is used to analyze the *h-p* version which turns out to be the most suitable numerical approach to achieve accuracy and reliable solutions: the *h-p* version with a geometric mesh refinement yields exponentially fast convergence. This is not restricted to two-dimensional problems. As recently shown, the *h-p* version of the BEM can also be applied to three-dimensional boundary value problems [16, 19] and transmission problems [12], it is a very robust method since a geometric mesh refinement towards edges and corners of the domain together with an appropriate *p*-distribution suffices to obtain exponentially fast converging numerical solutions.

Sometimes, one uses a combination of finite elements and boundary elements, e.g. in transmission problems, where the interior problem is described by a (possibly non-linear) differential operator with variable coefficients, and the exterior problem is given via a linear differential operator with constant coefficients. In this case the symmetric coupling method of FEM and BEM can also be performed with an *h-p* version, and again exponentially fast convergence can be obtained [14, 15].

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2. Boundary integral equations

We consider integral equation methods for solving Dirichlet and Neumann boundary value problems for the Laplacian in polygonial domains $\Omega \subset \mathbb{R}^2$ or polyhedral domains $\Omega \subset \mathbb{R}^3$.

Dirichlet: For given $f \in H^{1/2}(\Gamma)$ on the boundary Γ of Ω find $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \quad \text{in } \Omega\,, \qquad u = f \quad \text{on } \Gamma\,. \tag{1}$$

Neumann: For given $g \in H^{-1/2}(\Gamma)$ on $\Gamma = \partial\Omega$ with $\int_\Gamma g \, \mathrm{d}s = 0$ find $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \quad \text{in } \Omega\,, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma\,. \tag{2}$$

Here, Ω is a polygon or polyhedron with piecewise C^∞ boundary Γ, cusps are excluded. The normal vector always points into the exterior of Ω. Here, the Sobolev spaces $H^s(\Gamma)$ for $s > 0$ are defined to be the restriction of $H^{s+1/2}(\mathbb{R}^n)$ to Γ, $n = 2$ or 3;

$$H^s(\Gamma) := \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^3)\}$$

for $s = 0$ to be $L^2(\Gamma)$ and for $s < 0$ by duality (see [17]). It is well known [5] that problems (1) and (2) can be reduced to integral equations of the first kind on the boundary Γ for the Cauchy data $v = u|_\Gamma$ and $\psi = \partial u / \partial n|_\Gamma$. Namely, we have, respectively

$$V\psi = (1 + K)f \quad \text{on } \Gamma \tag{3}$$

$$Wv = (1 - K')g \quad \text{on } \Gamma \tag{4}$$

with the integral operators

$$Vw(x) := 2\int_\Gamma G(x, y)w(y)\, \mathrm{d}s_y\,, \qquad Kw(x) := 2\int_\Gamma \frac{\partial}{\partial n_y} G(x, y)w(y)\, \mathrm{d}s_y\,,$$

$$K'w(x) := 2\int_\Gamma \frac{\partial}{\partial n_x} G(x, y)w(y)\, \mathrm{d}s_y\,, \qquad Ww(x) := 2\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y} G(x, y)w(y)\, \mathrm{d}s_y\,,$$

with the fundamental solution of the Laplacian

$$G(x, y) := \begin{cases} -\dfrac{1}{2\pi} \ln |x - y|\,, & n = 2\,, \\ \dfrac{1}{4\pi} \dfrac{1}{|x - y|}\,, & n = 3\,. \end{cases} \tag{5}$$

Whereas for $n = 3$, Ω can be completely arbitrary, let us assume that for the case of a two-dimensional domain Ω its boundary Γ has conformal radius less than one; this can always be achieved by an appropriate scaling of the domain. Then there exist unique solutions $\psi \in H^{-1/2}(\Gamma)$ of (3) and $v \in H^{1/2}(\Gamma)$ of (4), where the solution of (4) is only unique up to constants. Eq. (3) is equivalent to problem (1); i.e. let $u \in H^1(\Omega)$ solve (1), then $\psi = \partial u/\partial n|_\Gamma$ solves (3), conversely, let $\psi \in H^{-1/2}(\Gamma)$ solve (3) then u defined by

$$u(x) = \int_\Gamma \left\{ G(x, y)\psi(y) - \frac{\partial}{\partial n_y} G(x, y)v(y) \right\} \mathrm{d}s_y\,, \quad x \in \Omega \tag{6}$$

with $v = f$ solves (1). On the other hand, (4) is equivalent to (2) in the sense that let $u \in H^1(\Omega)$ solve (2), then $v = u|_\Gamma$ solves (4), conversely, let $v \in H^{1/2}(\Gamma)$ solve (4), then u defined by (6) with $\psi = g$

solves (2). Of course, boundary integral equations can also be used to solve boundary value problems in unbounded regions. For brevity, we consider here only screen problems in $\mathbb{R}^3$, i.e. boundary value problems exterior to an open surface Γ (for corresponding two-dimensional problems see [25]).

Dirichlet screen: For given $f \in H^{1/2}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Gamma})$ such that

$$\Delta u = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{\Gamma}, \qquad u = f \quad \text{on } \Gamma, \qquad u = O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \tag{7}$$

Neumann screen: For given $g \in H^{-1/2}(\Gamma)$ find $u \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Gamma})$ such that

$$\Delta u = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{\Gamma}, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma, \qquad u = O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \tag{8}$$

As shown in [29], these screen problems can be converted into the integral equations

$$V\psi = 2f \quad \text{on } \Gamma \tag{9}$$

$$Wv = 2g \quad \text{on } \Gamma \tag{10}$$

with the jumps $\psi := [\partial u/\partial n]|_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$ and $v := [u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ across Γ. Again, there holds existence and uniqueness of solutions $\psi \in \tilde{H}^{-1/2}(\Gamma)$ and $v \in \tilde{H}^{1/2}(\Gamma)$ of (9) and (10). Here, the Sobolev space for an open surface piece Γ is introduced with the help of a closed surface $\tilde{\Gamma}$, where $\Gamma \subset \tilde{\Gamma}$, namely $\tilde{H}^s(\Gamma) = \{u \in H^s(\tilde{\Gamma}) : \operatorname{supp} u \subset \bar{\Gamma}\}$. Note that $\tilde{H}^{1/2}(\Gamma) = H^{1/2}_{\infty}(\Gamma)$ (for $H^{1/2}_{\infty}(\Gamma)$ see [17]) and $\tilde{H}^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))'$ by duality.

In electrostatics ψ describes the charge density of the surface with given potential f

Clearly, the application of boundary integral equations is not restricted to the above benchmark problems, e.g. they can be applied to the Lamé system of linear elasticity in a similar manner when the corresponding fundamental solution is taken instead of (5).

The boundary integral equations (3) and (4) (or (9) and (10)) can be solved approximately by the Galerkin method using conforming subspaces $\{X_N\}$ of $H^{-1/2}(\Gamma)$ (or $\tilde{H}^{-1/2}(\Gamma)$) and $\{Y_N\}$ of $H^{1/2}(\Gamma)$ (or $\tilde{H}^{1/2}(\Gamma)$), respectively. In the *boundary element method* one uses finite element spaces as such subspaces. The majority of results and boundary element software are based on the classical h-version. Recently, the p-version [31] and the h-p version were developed [30] and here we report on the progress of the latter version.

Since the operators of the single layer potential V and the normal derivative of the double layer potential W satisfy a Gårding's inequality in $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively, in case of a closed manifold Γ (and in $\tilde{H}^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$, respectively, if Γ is open), any conforming Galerkin scheme converges quasioptimally [32], i.e. we have with a constant C independent of N

$$\|\psi - \psi_N\|_X \leq C\|\psi - \phi\|_X \tag{11}$$

and

$$\|v - v_N\|_Y \leq C\|v - w\|_Y \tag{12}$$

for any $\phi \in X_N \subset H^{-1/2}(\Gamma) =: X$ and $w \in Y_N \subset H^{1/2}(\Gamma) =: Y$. Here, ψ solves (3) and v solves (4) and ψ_N, v_N are the solutions of the corresponding Galerkin equations with $\langle \cdot, \cdot \rangle$ denoting the L^2-inner product on Γ:

$$\langle V\psi_N, \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in X_N \tag{13}$$

and

$$\langle Wv_N, w \rangle = \langle g, w \rangle \quad \forall w \in Y_N . \tag{14}$$

For the h-version optimal convergence results for the above Galerkin schemes together with mesh grading towards the edges of the polyhedron Γ have been given in [24]. For the p-version applied to the weakly singular integral equation (3) and the hypersingular equation (4), see [28] and [26], respectively. The key of the error analysis for both h- and p-version are sharp regularity results for the solutions of (3) and (4); these results are derived in [23, 24] and are based on the work in [7].

In order to obtain these results we analyse the original boundary value problems (1) and (2). Since the solutions of (3) and (4) are the corresponding unknown Cauchy data, we get their regularity results by taking traces of the solutions of the boundary value problems.

Let us shortly review this approach by considering the three-dimensional Dirichlet problem which was analysed in [23]; for the Neumann problem see [24].

Let us consider the case of a polyhedron Ω with $\Gamma = \partial\Omega = \cup_{j=1}^{J} \bar{\Gamma}^j$ in $\mathbb{R}^3$ with plane faces Γ^j. We describe Dirichlet data $f \in H^s(\Gamma)$ where $H^s(\Gamma)$ is defined above. Then, the Neumann data ψ of the solution of (1) has regularity H^{s-1} away from the edges and corners. Near an edge with opening ω there are singularities of the form

$$c(y)\rho^{m\nu+2p-1}\,.$$

Here $\nu = \pi/\omega$, $m > 0$ and $p \geq 0$ are integers and ρ denotes the distance to the edge while the stress intensity factor $c(y)$ is a function defined on the edge. Near the corners we get additional corner singularities of the form

$$r^{\lambda_k - 1} w_k(\zeta)\,,\quad \zeta \in \Gamma_0\,,$$

where r denotes the distance to the vertex and w_k is a function on the spherical polygon $\Gamma_0 = \Gamma \cap S_2$. S_2 is a sphere centered in the vertex, $\zeta = (\theta, \varphi)$ are polar coordinates on S_2. The exponent λ_k and the function w_k are obtained as follows: Consider the eigenvalue problem for the Laplace–Beltrami operator $\Delta_{\theta,\varphi}$ on Γ_0, let μ_k be the kth eigenvalue with corresponding eigenfunction v_k:

$$\Delta_{\theta,\varphi} v_k(\theta, \varphi) = \mu_k v_k(\theta, \varphi)$$
$$v_k(\theta, \varphi)|_{\Gamma_0} = 0\,.$$

Then

$$\lambda_k = -\frac{1}{2} + \sqrt{\mu_k + \frac{1}{4}} \qquad \text{and} \qquad w_k := \frac{\partial}{\partial n} v_k(\theta, \varphi)\bigg|_{\Gamma_0}\,.$$

With this notation there holds the following decomposition result for the solution of (3) into edge and corner singularities where the edge terms are of tensor product form.

THEOREM 1. Let f be smooth on Γ^j, $j = 1, \ldots, J$, $\Gamma = \cup_{j=1}^{J} \bar{\Gamma}^j$. Then, the solution $\psi = \partial u/\partial n$ on Γ of (3) has the decomposition on Γ^j

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma^j} = \psi_0 + \sum_{\text{corners}} \chi_c r^{\lambda_k - 1} w_k(\theta) + \sum_{\text{edges}} \chi_e c_j(y) \rho^{\nu_j - 1} \tag{15}$$

where the sums of edge and corner singularity terms are finite. Here, $\chi_c(\chi_e)$ are suitable C^∞ cut-off-functions near corners (edges) and ψ_0 consists of a global remainder term belonging to some $H^s(\Gamma^j)$ plus higher-order edge and corner singularity terms. c_j is a function on the jth edge belonging to specific weighted Sobolev spaces and $\nu_j := \pi/\omega_j$ where ω_j is the opening angle at the jth edge. (For a detailed formulation of the theorem see [23]).

REMARK. Since $\rho^{\nu_j} \in H^{1/2+\nu_j-\epsilon}(\Gamma)$ and $r^{\lambda_k} \in H^{1+\lambda_k-\epsilon}(\Gamma)$ the regularity of the solution ψ of (3) for sufficiently smooth f is

$$\psi|_{\Gamma^j} \in H^{-1/2+\alpha}(\Gamma^j) \tag{16}$$

with

$$\alpha = \min\{\nu_j, \lambda_k + 1/2\}\,. \tag{17}$$

In case of screen problems, the polyhedron degenerates to a surface piece Γ and Ω becomes the region exterior to the screen Γ. The regularity theory in [23] shows that even for a smooth right-hand side f, the solution ψ of the integral equation (9) becomes unbounded near the edges of Γ, it exhibits so-called edge and corner singularities. From [23] we cite the regularity result for the square plate $\Gamma = (0, 1) \times (0, 1)$. Therefore, a corresponding regularity analysis leads to the situation in Example 1.

THEOREM 2. Let $f \in H^2(\Gamma)$. Then, the unique solution $\psi \in \tilde{H}^{-1/2}(\Gamma)$ of (9) has near the origin a decomposition into edge and corner singularities of the form

$$\psi = \psi_0 + \chi_c(r) a_1 r^{\lambda-1} w(\theta) + \chi_e(\theta) e_1(x) y^{-1/2} + \chi_e(\pi/2 - \theta) e_2(y) x^{-1/2}$$

with $\psi_0 \in H^{1-\epsilon}(\Gamma)$, $a_1 \in \mathbb{R}$, $w \in H^{1-\epsilon}[0, \pi/2]$, $e_1(x) = b_1 x^{\lambda-1/2} + c_1$, $e_2(y) = b_2 y^{\lambda-1/2} + c_2$, $c_i \in H_0^1(\mathbb{R}^+)$, $b_i \in \mathbb{R}$, $i = 1, 2$. A corresponding decomposition holds at the other vertices of Γ.

Here, (r, θ) are plane polar coordinates concentrated at the origin. χ_c and χ_e are C^∞ cut-off functions with $\chi_c \equiv 1$ for $r < 1/4$, $\chi_e \equiv 1$ for $\theta < \pi/4$. We have $\lambda \approx 0.2966$ (see [21]) and $\nu_j = \frac{1}{2}$ since $\omega_j = 2\pi$.

3. Standard boundary element methods

3.1. The h-method with graded mesh

For simplicity we only consider the case where all the faces meeting at the vertex are convex. Then each of these sectors can be mapped linearly on the quadrant $\mathbb{R}^1_+ \times \mathbb{R}^1_+ \subset \mathbb{R}^2$. On the square $[0, 1]^2$ we introduce a graded mesh given by the lines $x_1 = (ih)^\beta$, $x_2 = (jh)^\beta$, $h = 1/N$, $i, j = 0, \ldots, N$. For the construction of a graded mesh on a general polyhedral domain see [22].

Let $X_N = S^\beta_{0,h}$ be the space of piecewise constant functions on such a graded mesh. We can compensate the effect of the singularities by an appropriately graded mesh and get the convergence rate $h^{3/2}$:

THEOREM 3 [24]. Let h be sufficiently small and α as in (17). Then we have

$$\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \le C \begin{cases} h^{\alpha\beta-\epsilon} & \text{if } \beta < \dfrac{3}{2\alpha} \\ h^{3/2} & \text{if } \beta > \dfrac{3}{2\alpha} \end{cases}$$

where the constant $C = C(\beta)$ is independent of h.

To illustrate this result we consider the screen $\Gamma = (0, 1)^2$ (cf. Theorem 2). Let Ω be the exterior of the square. Then, $\nu_j = 1/2$, $\lambda_1 = 0.297$ thus, $\alpha = 1/2$ and Theorem 3 we have the convergence rate

$$\mathcal{O}(h^{\beta/2-\epsilon}) \quad \text{if } \beta < 3 \quad \text{and} \quad \mathcal{O}(h^{3/2}) \quad \text{if } \beta > 3\,.$$

For brevity we have considered here only the Dirichlet problem; for the Neumann problem see [24].

3.2. The h-p version with quasiuniform mesh

Construct a quasiuniform mesh with width h and define the space $X_N = S_{p,h}(\Gamma)$ of discontinuous piecewise polynomials of degree p on this mesh. Then again, the rate of convergence of the Galerkin solution is determined by the regularity of the solution, i.e. the parameter α in (17).

THEOREM 4 [19, 28]. Let p be sufficiently large and h be small enough. Then, the Galerkin equations (13) are uniquely solvable in $S_{p,h}(\Gamma)$. Let $\psi \in H^{-1/2}(\Gamma)$ be the solution of the integral equation (3) with sufficiently smooth right-hand side f, and let $\psi_{p,h} \in X_N := S_{p,h}(\Gamma)$ be the Galerkin solution of (13), then we have with α in (17) and arbitrary $\epsilon > 0$

$$\|\psi - \psi_{p,h}\|_{H^{-1/2}(\Gamma)} \leq C h^{\alpha-\epsilon} p^{-2\alpha+2\epsilon} . \tag{18}$$

REMARK. If $N = \dim X_N$ denotes the degrees of freedom, then Theorem 4 gives for the h-method the rate $\mathcal{O}(N^{-\alpha})$ and for the p-method the rate $\mathcal{O}(N^{-2\alpha})$. Corresponding results are given in [8] for the finite element method.

In the screen problem (9) with $\Gamma = (0, 1)^2$ we have $\alpha = \min\{\frac{1}{2}, 0.297 + \frac{1}{2}\} = \frac{1}{2}$, hence Theorem 4 yields the convergence rate $\mathcal{O}(h^{1/2-\epsilon})$ for the h-method and $\mathcal{O}(p^{-1+2\epsilon})$ for the p-method. Corresponding results can be derived for the Neumann problem, too. For the two-dimensional problems see Stephan and Suri [30].

4. The *h-p* version of the BEM for 2D problems

4.1. Exponential convergence

Let us consider in more detail the problems (1) and (2) when Ω is a bounded, plane domain with polygonal boundary $\Gamma = \cup_{j=1}^{J} \bar{\Gamma}^j$, Γ^j being open straight line segments. The interior angle at the vertex t_j is denoted by ω_j. It is well known that the solutions of (1) and (2) have special singular forms at the corners of Ω [10]. (Note for 2D problems $\alpha = \min \nu_j$ in (17) with $\nu_j = \pi/\omega_j$.) When these problems are converted into the boundary integral equations (3) and (4), then the solutions have corresponding corner singularities which reduce the convergence of the Galerkin schemes (13) and (14) when the h-version or the p-version are performed (cf. Theorems 3 and 4). Nevertheless the h-p version with geometrically spaced mesh (near the vertices of Ω) converges exponentially fast. To describe the h-p version of the boundary element method let us first introduce a geometric mesh Γ_σ^n on the boundary Γ of Ω.

As in [9, 13], we refine geometrically the mesh towards each vertex and introduce on this geometric mesh Γ_σ^n the spaces $S^{n-1}(\Gamma_\sigma^n)$ of boundary elements as follows:

$$S^{n-1}(\Gamma_\sigma^n) := \{\psi | \psi|_{\Gamma_\pm^{j,k}} \in P_{k-1}(\Gamma_\pm^{j,k}), k = 1, \ldots, n+1\}$$

where $P_k(\Gamma_\pm^{j,k})$ denotes the space of polynomials of degree $\leq k$ on the subinterval $\Gamma_\pm^{j,k}$. The geometric mesh Γ_σ^n is obtained by bisecting each side Γ^j with length d_j into two pieces Γ_-^j (containing the vertex t_{j-1}) and Γ_+^j (containing the vertex t_j). Then, $\mathrm{dist}(t_{j-1}, \Gamma_+^{j,k}) = d_j/2\,\sigma^{n-k+1}$ for $k \leq n+1$ where $\sigma \in (0, 1)$ and n is an integer. Then, with the choice $X_N = S^{n-1}(\Gamma_\sigma^n)$, where $N = \dim S^{n-1}(\Gamma_\sigma^n)$, we obtain the exponential convergence of the Galerkin method for the interior problem (1) as shown in [3, 4].

THEOREM 5. Provided the given data f in (1) is piecewise analytic, then there holds the estimate

$$\|\psi - \psi_N\|_{H^{-1/2}(\Gamma)} \leq C\, \mathrm{e}^{-b\sqrt{N}} \tag{19}$$

between the Galerkin solution $\psi_N \in S^{n-1}(\Gamma_\sigma^n)$ of (13) and the exact solution $\psi = \partial u/\partial n|_\Gamma$ of (3) where the positive constants C and $b > 0$ depend on the mesh parameter σ but not on $N = \dim S^{n-1}(\Gamma_\sigma^n)$.

Next, we consider the *interior Neumann problem (2)* and since W is only positive semidefinite, we consider as in [14] the system

$$Wv = (1 - K')g - a \quad \text{on } \Gamma\,, \qquad \int_\Gamma v \,\mathrm{d}s = 0 \tag{20}$$

which has a unique solution $(v, a) \in H^{1/2}(\Gamma) \times \mathbb{C}$. The corresponding Galerkin scheme reads: Determine $(v_N, a_N) \in Y_N \times \mathbb{C}$ with

$$\begin{aligned} \langle W v_N, w\rangle + \langle a_N, w\rangle &= \langle (I - K')g, w\rangle \\ \langle v_N, 1\rangle &= 0 \end{aligned} \tag{21}$$

for all $w \in Y_N \subset H^{1/2}(\Gamma)$. For the h-p version of (21) we take continuous, piecewise polynomials on the geometric mesh Γ^n_σ:

$$\mathring{S}^n(\Gamma^n_\sigma) = S^n(\Gamma^n_\sigma) \cap C^0(\Gamma) \subset H^{1/2}(\Gamma) \tag{22}$$

where

$$S^n(\Gamma^n_\sigma) := \{ v \,|\, v|_{\Gamma^{n,k}_\sigma} \in P_k(\Gamma^{n,k}_\sigma),\ k = 1, \ldots, n+1 \}\,.$$

Then, with the choice $Y_N = \mathring{S}^n(\Gamma^n_\sigma)$ we obtain the exponential convergence of the Galerkin method (21) for the interior Neumann problem (2) (compare Babuška et al. [3, 4]).

THEOREM 6. Provided the given data g in (2) is piecewise analytic, there holds the estimate

$$\|v - v_N\|_{H^{1/2}(\Gamma)} + |a - a_N| \le C\, \mathrm{e}^{-b\sqrt{N}} \tag{23}$$

between the Galerkin solution $(v_N, a_N) \in \mathring{S}^n(\Gamma^n_\sigma) \times \mathbb{C}$ of (21) and the exact solution $(v, a) \in H^{1/2}(\Gamma) \times \mathbb{C}$ of (20). Again, the positive constants C, b depend on σ but not on $N = \dim \mathring{S}^n(\Gamma^n_\sigma)$.

4.2. Implementation

For $X_N \subset H^{-1/2}(\Gamma)$ the basis functions can be discontinuous at the grid points of the geometric mesh $\Delta := \Gamma^n_\sigma$. We can use linearly transformed monomials. For an element $K \subset \Gamma^n_\sigma$ with polynomial degree p_K, the endpoints (a_1, a_2), (b_1, b_2) and the transformation

$$T_K := \begin{cases} [-1, 1] & \to \quad K \\ \quad x & \mapsto \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[(b_1 - a_1)x + a_1 + b_1] \\ \frac{1}{2}[(b_2 - a_2)x + a_2 + b_2] \end{pmatrix} \end{cases}$$

we take the functions

$$Q^K_i : \begin{cases} K & \to \quad \mathbb{R} \\ \begin{pmatrix} \xi \\ \eta \end{pmatrix} & \mapsto \quad \left(T_K^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right)^i \quad (i = 0, \ldots, p_K)\,. \end{cases}$$

A function $f \in X_N$ with mesh $\Gamma^n_\sigma = \{K_i : i = 1, \ldots, m(\Delta)\}$ and polynomial degree vector $\boldsymbol{p} = (p_1, \ldots, p_{m(\Delta)})$ has the representation

$$f : \begin{cases} \displaystyle\bigcup_{i=1}^{m(\Delta)} K_i & \to \quad \mathbb{R} \\ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in K_i & \mapsto \quad \displaystyle\sum_{j=0}^{p_i} a_j^{K_i} Q_j^{K_i} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{cases}$$

where $a_j^{K_i} \in \mathbb{R}$, $j = 0, \ldots, p_i$, $i = 1, \ldots, m(\Delta)$.

For $Y_N \subset H^{1/2}(\Gamma)$ the basis functions must be continuous on Γ. We construct a basis using the affine images of antiderivatives of Legendre polynomials, then these basis functions vanish at the grid points of Γ''_σ. The missing degrees of freedom at those points are delivered by the standard piecewise linear roof functions. The standard basis functions on the reference interval $[-1, 1]$ are defined by

$$\psi_i(x) = \frac{1 - (-1)^i x}{2} \quad (i = 0, 1)$$

$$\psi_i(x) = \sqrt{\frac{2i-1}{2}} \int_{-1}^{x} l_{i-1}(t)\,\mathrm{d}t \quad (i = 2, 3, \ldots)$$

with the Legendre polynomials l_i. There holds $\psi_i(\pm 1) = 0$ $(i \geq 2)$ and

$$\psi_i = \frac{1}{\sqrt{2(2i-1)}} (l_i - l_{i-2}) \quad (i \geq 2)\,.$$

Hence, for $f \in Y_N$ there holds

$$f: \begin{cases} \bigcup_{i=1}^{m(\Delta)} K_i & \to \quad \mathbb{R} \\ \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in K_i & \mapsto \quad \sum_{j=0}^{p_i} a_j^{K_i} L_j^{K_i} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{cases}$$

with $a_0^{K_i} = a_1^{K_{i-1}}$ $(i = 1, 2, \ldots, m(\Delta)$, $K_0 = K_{m(\Delta)})$ by continuity $(a_j^{K_i} \in \mathbb{R}$, $j = 0, \ldots, p_i$, $i = 1, \ldots, m(\Delta))$. Here, the $L_j^{K_i}$'s are the transformed basis functions

$$L_i^{K}: \begin{cases} K & \to \quad \mathbb{R} \\ \begin{pmatrix} \xi \\ \eta \end{pmatrix} & \mapsto \quad \psi_i\left(T_K^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) \end{cases} \quad (i = 0, \ldots, p_k)\,.$$

In our computations there appear four integral operators, the single layer potential V, the hypersingular operator W, the double layer potential K and its adjoint K'. All test and trial functions in the Galerkin schemes can be represented as combinations of transformed monomials $e_i^k = Q_k^{K_i}$ (degree k, subinterval $K_i \subset \Gamma''_\sigma$). Hence, it suffices to consider those functions.

The *single layer potential* terms

$$\langle Ve_j^l, e_i^k \rangle = -\frac{1}{\pi} \int_{\Gamma_i} e_i^k(\eta) \int_{\Gamma_j} e_j^l(\xi) \ln|\xi - \eta|\,\mathrm{d}s_\xi\,\mathrm{d}s_\eta$$

are computed as follows: The outer integral by 32 point Gaussian quadrature, the inner integral analytically making use of the expression

$$\int x^l \log|x - z|\,\mathrm{d}x = \frac{1}{l+1} [x^{l+1} - z^{l+1}] \log|x - z| - \frac{1}{l+1} \sum_{k=1}^{l+1} \frac{x^{l-k+2} z^{k-1}}{l-k+2}\,.$$

For details see [9, 25].

For the terms involving the *hypersingular operator* there holds due to integration by parts

$$\langle We_j^l, e_i^k \rangle = \langle V\dot{e}_j^l, \dot{e}_i^k \rangle = c\langle Ve_j^{l-1}, e_i^{k-1} \rangle$$

with constants $c = c(K_j, K_i)$. Here, $\dot{e}_j^l$ denotes the derivative with respect to the arc length of e_j^l. Note in the terms with the hypersingular operator the test and trial functions are continuous throughout Γ.

For the terms with the double layer potential operator we observe the relation

$$\langle K e_j^l, e_i^k \rangle = \langle e_j^l, K' e_i^k \rangle .$$

Again, the outer integral is computed by Gaussian quadrature whereas the inner integral is computed analytically (compare Ervin et al. [9]). The linear systems (13) and (21) for the various versions are solved directly.

4.3. Numerical results

EXAMPLE 1. (Dirichlet Problem) $u = \mathrm{Im}\, z^{2/3}$ and Ω be the L-shaped region with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, $(1,0)$.

In Table 1, N denotes the number of unknowns of the Galerkin approximation ψ_N from (13). n is the number of subintervals on the edges joining at the reentrant corner of Ω. e_E denotes the relative error in the energy norm

$$e_E = \frac{\|\psi - \psi_N\|_E}{\|\psi\|_E} \quad \text{with } \|\psi - \psi_N\|_E := \langle V(\psi - \psi_N), (\psi - \psi_N) \rangle^{1/2} .$$

EXAMPLE 2. (Neumann Problem) $u = \mathrm{Im}\, z^{1/7}$ and Ω as in Example 1. $e_0 = \|u - u_N\|_{L^2(\Gamma)} / \|u\|_{L^2(\Gamma)}$ where u_N solves (21).

Note: In Tables 1 and 2 the experimental convergence rates α_N agree with the theoretical convergence rates α^* for the h and p versions, whereas the error of the h-p version decreases exponentially underlining (19) and (23).

Table 1
Relative error in energy norm for h-, p- and h-p version of (3)

h-version, $p = 0$, $\alpha^* = 0.66$			p-version, 8 elem., $\alpha^* = 1.33$			h-p version, $\sigma = 0.25$			
N	e_E	α_N	N	e_E	α_N	n	N	e_E	α_N
8	0.1880		8	0.1880		1	8	0.1880	
16	0.1126	0.74	16	0.0874	1.11	2	18	0.0699	1.22
32	0.0710	0.67	24	0.0543	1.17	3	30	0.0242	2.08
64	0.0447	0.67	32	0.0375	1.29	4	44	0.0100	2.31
128	0.0282	0.66	40	0.0273	1.43				

Table 2
Relative error in L^2-norm for h-, p- and h-p version of (20)

h-version, $p = 0$, $\alpha^* = 0.64$			p-version, 8 elem.		h-p version, $\sigma = 0.15$			
N	e_0	α_N	e_0	α_N	n	N	e_0	α_N
8	0.0241		0.0241		1	8	0.02411	
16	0.0149	0.70	0.0136	0.83	2	18	0.00692	1.54
24	0.0115	0.64	0.0094	0.90	3	30	0.00217	2.27
32	0.0095	0.65	0.0071	0.98	4	44	0.00067	3.07
40	0.0083	0.64	0.0056	1.03	5	60	0.00023	3.45
48	0.0074	0.64	0.0047	1.07				

5. The *h-p* version of the BEM for 3D problems

5.1. Exponential convergence

Let Γ be the boundary of a simply connected bounded polyhedron or a screen Ω in $\mathbb{R}^3$. We first introduce appropriate countably normed spaces $B^l_\beta(\Gamma)$, $(0<\beta<1)$. Let F be a face of Ω having the corner points $e_1,\dots,e_m$. For each i, f_i denotes the edge of F connecting e_i with e_{i+1}, where the periodicity convention $e_{m+1}=e_1$, $f_{m+1}=f_1$ is adopted. Consider a covering of F by neighbourhoods U_i $(i=1,\dots,m)$ of e_i not containing e_j, $j\neq i$, and introduce polar coordinates (r_i,θ_i) of origin e_i in U_i such that $r_i=\operatorname{dist}(x,e_i)$ and the edges f_{i-1} and f_i are given by $\theta_i=\omega_i$ and $\theta_i=0$, respectively. For $0<\beta<1$, let

$$B^l_\beta(F)=\{u\,|\,u\in H^{l-1}(F)\colon \exists C,d>0 \text{ independent of } k \text{ such that}$$

$$\|r_i^{\beta+\alpha_\theta-l}(\theta_i(\omega_i-\theta_i))^{(\beta+\alpha_\theta-l)_+}\cdot(\partial/\partial r_i)^{\alpha_r}(\partial/\partial\theta_i)^{\alpha_\theta}u\|_{L^2(U_i)}$$

$$\leqslant Cd^{\alpha_r+\alpha_\theta-l}(\alpha_r+\alpha_\theta-l)!\qquad \alpha_r+\alpha_\theta=k=l,l+1,\dots,i=1,\dots,m\}\,,$$

$$B^l_\beta(\Gamma)=\{u\,|\,u\in H^{l-1}(\Gamma),\,u\in B^l_\beta(F)\text{ for all faces }F\text{ of }\Gamma\}\tag{24}$$

with $(a)_+:=\max(a,0)$. If we would like to emphasize the dependence on the constants C,d we will write $B^l_\beta(F)=B^l_{\beta,C,d}(F)$, etc.

Now we define the geometric mesh on the faces of the polyhedron. We assume that the face F is a triangle. There is no loss of generality because every polygonal domain can be decomposed into triangles. We divide this triangle into three parallelograms and three triangles where each parallelogram lies in a corner of the face F and each triangle lies at the edge of F apart of the corners. By linear transformations φ_i we can transform the parallelograms on to the reference square $Q=[0,1]^2$ such that the vertices of the face F are transformed to $(0,0)$. The triangles can be transformed by a linear transformation $\tilde\varphi_i$ on to the reference triangle $\tilde Q=\{(x,y)\in Q\,|\,y\leqslant x\}$ such that the corner point of the triangle in the interior of the face F is transformed to $(1,1)$ of the reference triangle. By the following definition the geometric mesh and appropriate spline spaces are defined on the reference element Q. Analogously, the geometric mesh can be defined on the reference triangle $\tilde Q$ (see Fig. 1). Via the transformations φ_i^{-1}, $\tilde\varphi_i^{-1}$ the geometric mesh Γ^n_a is also defined on the faces of the polyhedron. The approximation on the reference square is the more interesting case because it handles the corner-edge singularities.

Now let us introduce the spaces of spline-functions on geometric meshes. For a given $0<\sigma<1$ we use the partition I^n_σ of $I=[0,1]$ into n subintervals (x_{j-1},x_j), $j=1,\dots,n$, where

$$x_0=0\,,\qquad x_j=\sigma^{n-j}\,,\quad j=1,\dots,n\,.\tag{25}$$

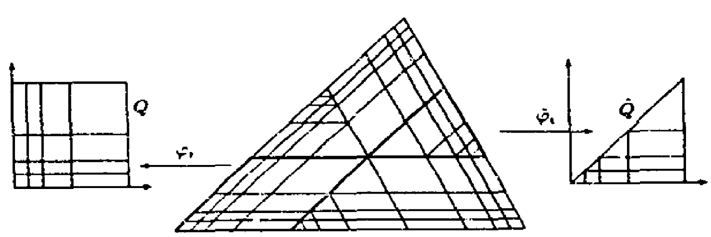


Fig. 1. Geometric mesh for $\sigma=1/2$ and $n=5$.

With I^n_σ we associate a degree-vector $p = (p_0, \ldots, p_{n-1})$ and define $S^{p,l}(I^n_\sigma)$ as the vectorspace of all piecewise polynomials v on I having degree p_j on (x_j, x_{j+1}), $j = 0, \ldots, n-1$, i.e. $v|_{(x_j, x_{j+1})} \in P_{p_j}((x_j, x_{j+1}))$. For a given $0 < \sigma < 1$ we use the partition Q^n_σ of $Q = [0, 1]^2$ into n^2 rectangles R_{kl}

$$R_{kl} = [x_{k-1}, x_k] \times [x_{l-1}, x_l], \quad (k, l = 1, \ldots, n), \qquad Q = \bigcup_{k,l=1}^{n} R_{kl}. \tag{26}$$

With Q^n_σ we associate a degree vector $p = (p_0, \ldots, p_{n-1})$ and define $S^{p,l}(Q^n_\sigma)$ as the vectorspace of all piecewise polynomials v on Q having degree p_{k-1} in x and p_{l-1} in y on $[x_{k-1}, x_k] \times [x_{l-1}, x_l]$, $k, l = 1, \ldots, n$, i.e. $v|_{[x_{k-1}, x_k] \times [x_{l-1}, x_l]} \in P_{p_{k-1}, p_{l-1}}(R_{kl})$. For the differences $h_k = x_k - x_{k-1}$ we have

$$h_k = x_k - x_{k-1} = x_{k-1}\left(\frac{1}{\sigma} - 1\right) \leq x\left(\frac{1}{\sigma} - 1\right) = x\gamma, \quad \forall x \in [x_{k-1}, x_k], \quad 2 \leq k \leq n \tag{27}$$

with $\gamma = 1/\sigma - 1$. The index $l \in \{0, 1\}$ in $S^{p,l}(I^n_\sigma)$ (resp. $S^{p,l}(Q^n_\sigma)$) determines the regularity of the piecewise polynomials, i.e. discontinuity in case of $l = 0$ and continuity in case of $l = 1$.

Then we have by construction:

$$S^{p,l}(Q^n_\sigma) \supset S^{p,l}(I^n_\sigma) \times S^{p,l}(I^n_\sigma) \tag{28}$$

and via the transformations φ_i, $\tilde{\varphi}_i$ the corresponding boundary element spaces $S^{p,l}(\Gamma^n_\sigma)$ are obtained on the geometric mesh Γ^n_σ.

For the countably normed spaces there holds the following approximation results.

THEOREM 7 [20, 18].

(*i*) *Let $u(x, y) \in B^1_\beta(Q)$ with $0 < \beta < 1$, $\beta \neq 1/2$. Then there is a spline function $u_N \in S^{p,0}(Q^n_\sigma)$ and constants $C_1, b_1 > 0$ independent of N, but dependent on σ, μ, β such that*

$$\|u(x, y) - u_N(x, y)\|_{L^2(Q)} \leq C_1 \, e^{-b_1 N^{1/4}}$$

with $p = (p_0, \ldots, p_n)$, $p_k = [\mu k]$ for a $\mu > 0$ and $N = \dim S^{p,0}(Q^n_\sigma)$.

(*ii*) *Let $v(x, y) \in B^2_\beta(Q) \cap C^0(Q)$ with $0 < \beta < 1$, $\beta \neq 1/2$. Then there is a spline function $v_N \in S^{p,1}(Q^n_\sigma)$ and constants $C_2, b_2 > 0$ independent of N, but dependent on σ, μ, β such that*

$$\|v(x, y) - v_N(x, y)\|_{H^1(Q)} \leq C_1 \, e^{-b_2 N^{1/4}}$$

with $p = (p_0, \ldots, p_n)$, $p_0 = 1$, $p_k = \max(2, [\mu k] + 1)$ $(k \geq 1)$ for a $\mu > 0$ and $N = \dim S^{p,1}(Q^n_\sigma)$.

In the following we want to apply these results to the Galerkin solutions of the boundary integral equations (3) and (4).

First, we consider the screen problem (9) and take for simplicity $\Gamma = [0, 2]^2$ and divide into 4 pieces by symmetry (for the general case see [18, 19]).

Now we consider the Galerkin equations to (9) when $X_N = S^{p,0}(Q^n_\sigma)$, $N = \dim S^{p,0}(Q^n_\sigma)$. Estimate (11) causes us to look for a good approximation in X_N to obtain an upper bound for the Galerkin error. Due to the tensor product construction of $S^{p,0}(Q^n_\sigma)$ and the special decomposition of ψ in Theorem 2 the approximation properties of $S^{p,0}(Q^n_\sigma)$ can be studies term by term, what will be indicated in the following.

Thus, (due to Theorem 2) we need only to look at the behavior near the origin. Hence, let us consider the most singular part of the exact solution near the origin, namely the corner-edge singularity in $(0, 0)$ and the edge-singularity for the edge $\overline{(0, 0), (1, 0)}$. That means we have to approximate the function

$$\Psi_Q := r^{\lambda - 1}\theta^{1/2} + x^{\lambda - 1/2}y^{-1/2} \tag{29}$$

by a spline-function $\Psi_g \in S^{p,0}(Q^n_\sigma)$. The term $r^{\lambda-1}\theta^{1/2}$ has a singular behaviour for $r \to 0$ and $\theta \to 0$ and

$y^{-1/2}$ corresponds to the edge-singularity. The function $x^{\lambda-1/2}$ has the same behaviour as the stress-intensity distribution.

We use a similar decomposition of Ψ_g like (29) to approximate the terms separately,

$$\Psi_g := \phi_g(r, \theta) + a_g(x)\psi_g(y) . \tag{30}$$

Note that ϕ_g is a two-dimensional spline, while a_g and ψ_g are splines on a one-dimensional cut of the two-dimensional mesh parallel to the axis, i.e. $\phi_g \in S^{p,0}(Q^n_\sigma)$ and $a_g, \psi_g \in S^{p,0}(I^n_\sigma)$. From [18, 20] we know that

$$r^\alpha \theta^\nu \in B^1_\beta(Q) \tag{31}$$

for $\alpha > -\beta$, $\nu > 1/2 - \beta$, i.e. with $\alpha = \lambda - 1 \approx -0.7$ and $\nu = 1/2$ we have

$$r^{\lambda-1}\theta^{1/2} \in B^1_\beta(Q) \tag{32}$$

with $\beta > \max(1-\lambda, 0) = 1 - \lambda \approx 0.7$. Thus, due to Theorem 7 there exists a $\phi_g \in S^{p,0}(Q^n_\sigma)$ such that for a $\max(1/2, 1-\lambda) = 1 - \lambda < \beta < 1$

$$\|r^{\lambda-1}\theta^{1/2} - \phi_g\|_{L^2(Q)} \leq C_1 \, e^{-b_1 \sqrt[4]{N}}$$

The difference $y^{-1/2}x^{\lambda-1/2} - a_g(x)\psi_g(y)$ can be estimated in a similar way (see [16]).

For an piecewise analytic right-hand side Theorem 1 (see [23, 24]) shows that the exact solution can be decomposed near the origin in edge and corner-edge singularities analogously to (29) with an arbitrary smooth remainder term. Using (31) (see [18, 20]) it can be shown that the decomposition into finitely many edge and corner-edge singularities belongs to a $B^1_\beta(\Gamma)$ space and therefore can be approximated exponentially fast. The remainder term is arbitrarily smooth and can therefore be approximated algebraically with an order according to the smoothness.

Using expansions like (15) the analysis can also be extended to the Dirichlet and Neumann problem for the Laplacian in polyhedral domains (see [16, 18, 19]). This leads to the following theorems for the *h-p* version of the boundary element method.

THEOREM 8.

(i) *Let f in (1) be piecewise analytic and let ψ be given in (3) and ψ_N, its Galerkin approximation, be defined by (13) with $X_N = S^{p,0}(\Gamma^n_\sigma)$ on geometric meshes Γ^n_σ on Γ. Then there holds for any $\alpha > 0$*

$$\|\psi - \psi_N\|_{H^{-1/2}(\Gamma)} \leq C_1 \, e^{-b_1 N^{1/4}} + O(N^{-\alpha})$$

with constants C_1, b_1 independent of $N = \dim X_N$.

(ii) *Let g in (2) be piecewise analytic and let v be defined by (4) and v_N, its Galerkin approximation, be defined by (14) with $Y_N = S^{p,1}(\Gamma^n_\sigma)$ on geometric meshes Γ^n_σ on Γ. Then there holds for any $\alpha > 0$*

$$\|v - v_N\|_{H^{1/2}(\Gamma)} \leq C_2 \, e^{-b_2 N^{1/4}} + O(N^{-\alpha}) ,$$

with constants C_2, b_2 independent of $N = \dim Y_N$.

5.2. Implementation

Here, we consider only the case that the polyhedron Γ consists of surfaces which can be completely divided into rectangles (no triangles are needed). Since we use tensor products of Legendre polynomials as test and trial functions in the Galerkin schemes all arising integrals can be computed analytically.

Now the rectangles which are the supports of trial functions may have 3 different orientations to each other. Namely, (i) both rectangles are in the same plane, (ii) the rectangles are in parallel planes, (iii) the rectangles are perpendicular to each other. This leads to 2 different antiderivatives for the elements of the Galerkin matrix:

$$\text{(i)}\quad F_{klmn}(x_1, x_2, y_1, y_2; x_3 - y_3) := \int dx_1 \int dx_2 \int dy_1 \int dy_2 \frac{x_1^k x_2^l y_1^m y_2^n}{\sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}}$$

$$\text{(ii)}\quad F_{klmn}(x_1, x_2, y_2, y_3; x_3, y_1) := \int dx_1 \int dx_2 \int dy_2 \int dy_3 \frac{x_1^k x_2^l y_2^m y_3^n}{\sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}} .$$

In all cases the integrals are of convolution form $D_{kl}(x, y) = \int dx \int dy\, x^k y^l D(x - y)$, where $D(x - y)$ can be a general kernel, e.g. here we have $D(x - y) = 1/\sqrt{((x - y)^2 + a^2)}$. Hence, the number of integrations can be reduced.

$$\begin{aligned} D_{kl}(x, y) &= \int dx \int dy\, x^k y^l D(x - y) \\ &= \frac{1}{l+1} y^{l+1} \int dx\, x^k D(x - y) - \frac{1}{l+1} \int dy\, y^{l+1} \partial_y \int dx\, x^k D(x - y) \\ &= \frac{1}{l+1} \left\{ y^{l+1} \int dx\, x^k D(x - y) + x^k \int dy\, y^{l+1} D(x - y) - k D_{k-1,l+1}(x, y) \right\} . \end{aligned}$$

In case (i) we apply the reduction twice and obtain integrals of the form

$$G^*_{kl}(y_1, y_2; x_2, a) := \int dy_1 \int dy_2 \frac{y_1^k y_2^l}{\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + a^2}} .$$

In case (ii) we can only apply this reduction once and obtain

$$G^*_{klm}(y_1, y_2, y_3; x_1, x_2, x_3) := \int dy_1 \int dy_2 \int dy_3 \frac{y_1^k y_2^l y_3^m}{\sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}} .$$

With the antiderivative

$$g_l(y; x, a) := \int dy \frac{y^l}{\sqrt{(y - x)^2 + a^2}}$$

we can write G^*_{kl} and G^*_{klm} as

$$G^*_{kl}(y_1, y_2; x_1, x_2, a) = \int dy_2\, y_2^l g_k(y_1; x_1, \sqrt{(y_2 - x_2)^2 + a^2})$$

and

$$G^*_{klm}(y_1, y_2, y_3; x_1, x_2, y_3) = \int dy_3\, y_3^m G^*_{kl}(y_1, y_2; x_1, x_2, y_3 - x_3) .$$

Thus, with the recurrence relation

$$g_0(y; x, a) = \operatorname{arsinh} \frac{y - x}{|a|}$$

$$g_l(y; x, a) = \frac{1}{l} \{ y^{l-1} \sqrt{(y - x)^2 + a^2} + (2l - 1) x g_{l-1}(y; x, a) - (l - 1)(x^2 + a^2) g_{l-2}(y; x, a) \}$$

we get also recurrence relations for G^*_{kl} and G^*_{klm}.

5.3. Crack problems

Crack problems in linear elasticity lead to systems of integral equations of the first kind which can also be solved by the h-p version of Galerkin's method, e.g. the inclusion (Dirichlet) problem on a plane crack surface Γ leads for given $\bar{g}^0$ to

$$\int_\Gamma c_1 \begin{bmatrix} \frac{1}{|x-y|} & 0 & 0 \\ 0 & \frac{1}{|x-y|} + c_2 \frac{(x_2-y_2)^2}{|x-y|^3} & c_2 \frac{(x_2-y_2)(x_3-y_3)}{|x-y|^3} \\ 0 & c_2 \frac{(x_2-y_2)(x_3-y_3)}{|x-y|^3} & \frac{1}{|x-y|} + c_2 \frac{(x_3-y_3)^2}{|x-y|^3} \end{bmatrix} \begin{pmatrix} \psi_1(y) \\ \psi_2(y) \\ \psi_3(y) \end{pmatrix} \mathrm{d}s_y$$
$$= \begin{pmatrix} g_1^0(x) \\ g_2^0(x) \\ g_3^0(x) \end{pmatrix}, \quad x \in \Gamma \tag{34}$$

where the constants c_1, c_2 depend on the Lamé constants. A generalisation of the analysis in [16] will yield again exponentially fast convergence for the h-p version Galerkin scheme with geometric mesh as in Theorem 8. Note that the necessary explicit decomposition of the solution into edge and corner singularities can be found in [22].

5.4. Adapative h-p algorithm for screen problems

(a) For the weakly singular integral equation we proceed as follows: Starting with some Galerkin solution ψ_N one computes the local residual

$$R_i^2 := \int_{\Gamma_i} (f(x) - V\psi_N(x))^2 \, \mathrm{d}x .$$

For given $0 \le \theta \le 1$, if $R_i^2 \le \theta R_{\max}^2$ the Galerkin solution is already accurate enough. If $R_i^2 > \theta R_{\max}^2$ for some elements Γ_i we call this element critical. In the case of rectangles (triangles) we can also perform a direction control on the critical elements. In the case of rectangles temporarily we divide every critical element Γ_i uniformly into four elements (see Fig. 2) and take a closer look at the four local residues

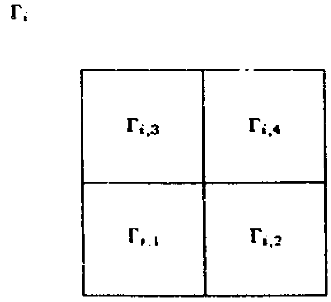


Fig. 2. Decomposition of an element to determine the refinement direction.

$$R_{i,k}^2 := \int_{\Gamma_{i,k}} (f(x) - V\Psi_N(x))^2 \, dx , \quad k = 1, \ldots, 4 . \tag{35}$$

Let $\mu > 0$ be a given number ($\mu = 1.5$ by default), called the direction parameter, and define the two indicators

$$\begin{aligned} ex_i &:= \max\left(\frac{R_{i,1}^2 + R_{i,3}^2}{R_{i,2}^2 + R_{i,4}^2}, \frac{R_{i,2}^2 + R_{i,4}^2}{R_{i,1}^2 + R_{i,3}^2}\right) , \\ ey_i &:= \max\left(\frac{R_{i,1}^2 + R_{i,2}^2}{R_{i,3}^2 + R_{i,4}^2}, \frac{R_{i,3}^2 + R_{i,4}^2}{R_{i,1}^2 + R_{i,2}^2}\right) . \end{aligned} \tag{36}$$

Then, perform a refinement in x-direction on Γ_i (x-refinement) if $ex_i \geq ey_i$ or $ex_i > \mu$ and a y-refinement on Γ_i if $ey_i \geq ex_i$ or $ey_i > \mu$.

The critical elements Γ_i have to be refined or the polynomial degrees have to be increased. This is decided as follows. Firstly, one computes a new Galerkin solution $\tilde{\psi}_N$ with the same mesh but with polynomial degree increased by one on the critical elements Γ_i. Now one compares the old with the new residual

$$\tilde{R}_i := \int_{\Gamma_i} (f(x) - V\tilde{\psi}_N(x))^2 \, dx .$$

Then, if $\tilde{R}_i^2 \leq \gamma R_i^2$ on a critical element one performs a p-refinement whereas if $\tilde{R}_i^2 > \gamma R_i^2$ on a critical element one performs a h-refinement, where γ is a preset parameter.

The h-refinement is done by interval halving when Γ_i is an interval and by subdividing into rectangles (triangles) when Γ_i is a rectangle (triangle).

If the rectangles are obtained by just dividing the critical element Γ_i into 4 subsquares we speak of an adaptive routine without xy-control (otherwise with xy-control). The p-refinement is done by increasing the polynomial degree by 1. Dividing the elements into more than 4 subsquares or increasing the polynomial degree by more than 1 usually gives a lower convergence rate because we introduce in this way more degrees of freedom away from the singularities of the solution than necessary. If we use a direction (or xy-) control, then we have to do the refinement in the indicated direction, i.e. appropriate halving of the rectangle (triangle) in the case of a h-refinement and increasing the polynomial degree by 1 in the case of a p-refinement.

(b) For the hypersingular integral equation we proceed as follows: Starting with some Galerkin solution v_N we compute an approximate value for the local residual

$$R_i^2 := \int_{\Gamma_i} (g(x) - Wv_N(x))^2 \, dx .$$

Since v vanishes on the edges of $\Gamma = [-1, 1]^2$ there holds with the surface Laplacian Δ_T for $x \in \Gamma$

$$Wv(x) := \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_x} \left(\frac{1}{|x-y|}\right) v(y) \, dy = -\Delta_T \int_\Gamma \frac{v(y)}{|x-y|} \, dy .$$

Therefore we approximate R_i^2 by $\tilde{\tilde{R}}_i^2 := \int_{\Gamma_i} (g(x) + \Delta_T^h V v_N(x))^2 \, dx$ using the 5 point difference star Δ_T^h. With this trick we avoid to compute directly the action of the hypersingular integral operator W. Now the adaptive algorithm can be performed as above (see [16]).

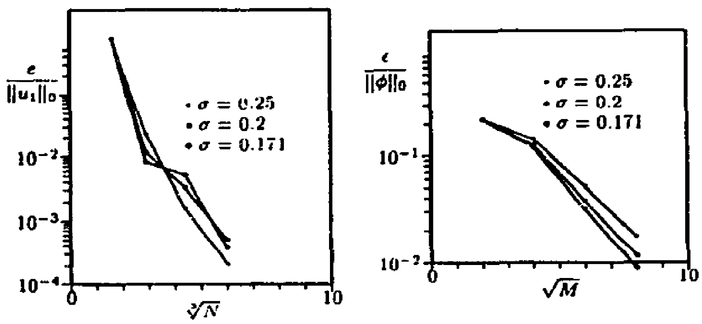


Fig. 3. L^2-errors $e = \|u_1 - u_N\|_{L^2(\Omega)}$, $\epsilon = \|\phi - \phi_N\|_{L^2(\Gamma)}$.

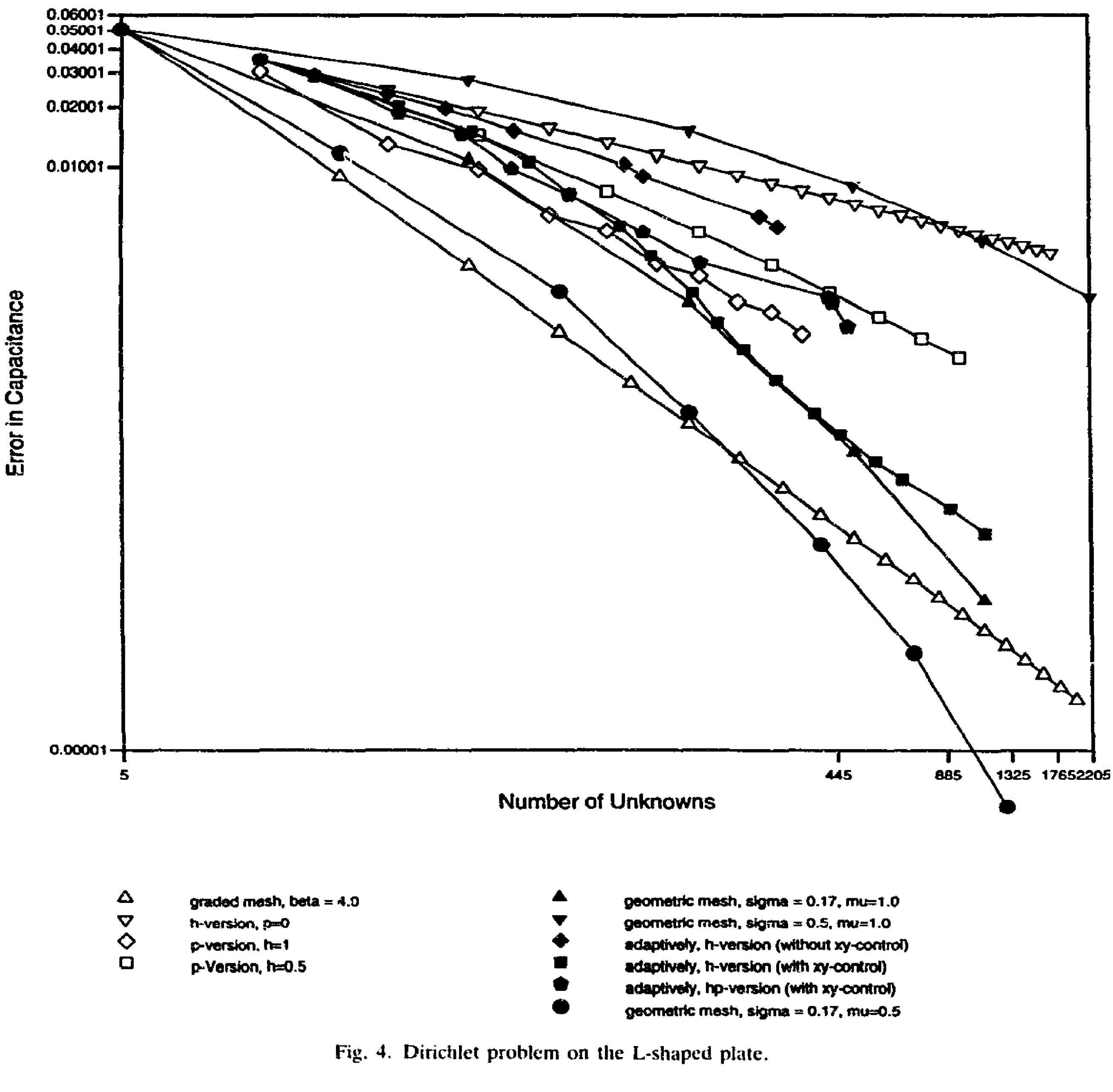


Fig. 4. Dirichlet problem on the L-shaped plate.

5.5. *Numerical results*

As described in Sections 3.1, 3.2, 5.1 and 5.2, we implement boundary element Galerkin schemes to obtain approximations ψ_N to the solution ψ of (9) and approximations v_N to the solution v of (10). We consider the screens given by the L-shaped plate in Fig. 6 and the square plate with sidelength 2. In all cases we choose the functions $f = g = 1$ on the right-hand side in (9) and (10). For the Dirichlet problem we look for the capacitance C of Γ which is given by $C = 1/4\pi \int_\Gamma \psi \, ds$ where the charge density ψ solves the integral equation (9). As approximation we compute $C_N = 1/4\pi \int_\Gamma \psi_N \, ds$ with the Galerkin approximation ψ_N of ψ. Due to $f = 1$ there holds with constants $c_1, c_2 > 0$

$$c_1 \|\psi - \psi_N\|^2_{\tilde{H}^{-1/2}(\Gamma)} \leq |C - C_N| \leq c_2 \|\psi - \psi_N\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{37}$$

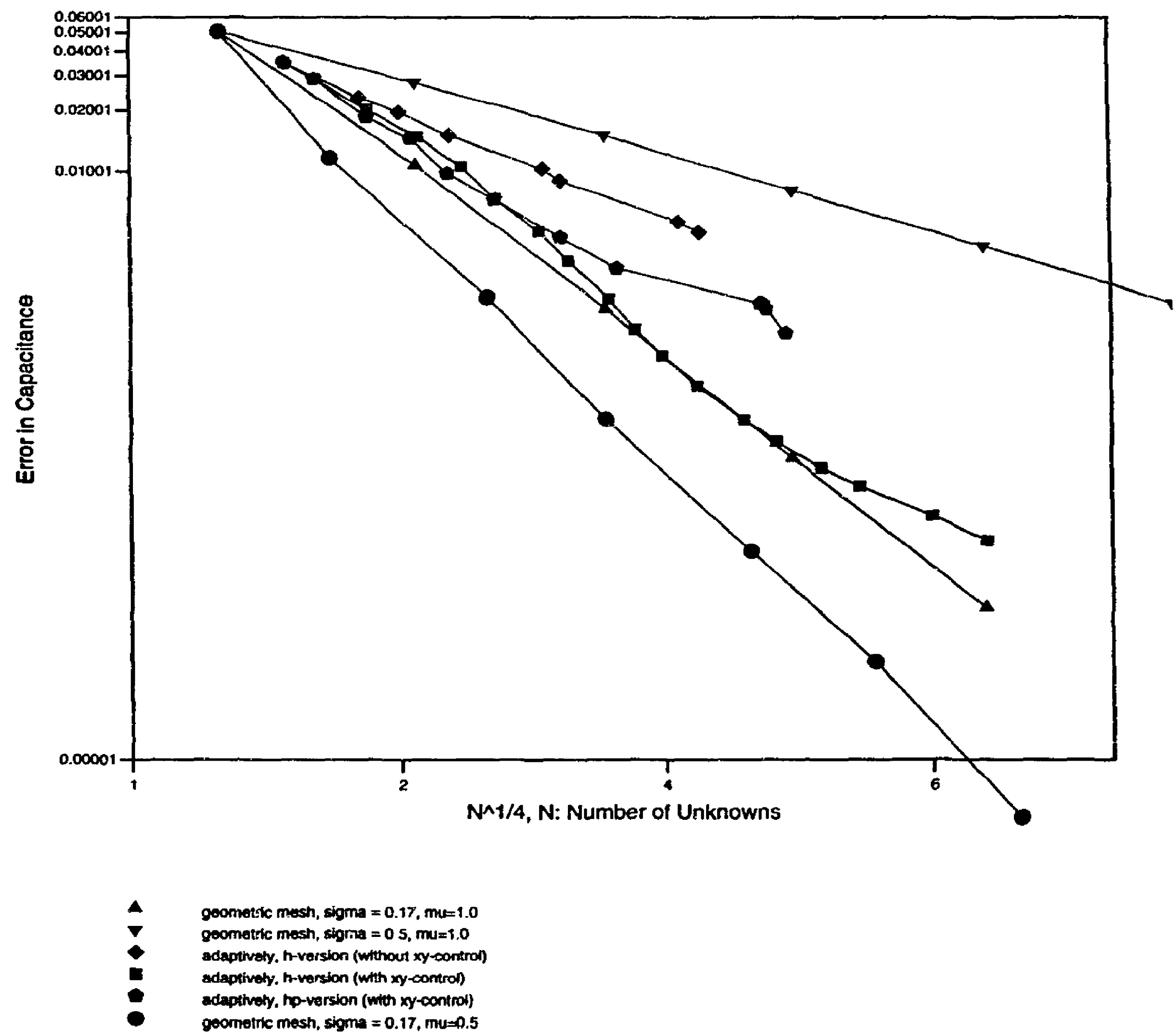


Fig. 5. Dirichlet problem on the L-shaped plate.

yielding with (18) the error estimates for the pure h- and p-versions

$$|C - C_N| \leq c_3 \begin{cases} h^{1-\epsilon} \\ p^{-2+2\epsilon} \end{cases} \tag{38}$$

with $c_3 > 0$ and arbitrary $\epsilon > 0$. For the h-p version with geometric mesh, Figs. 4 and 5 show the predicted exponentially fast convergence (cf. Theorem 8), whereas they show only algebraic convergence for the pure h- and p-versions as predicted by (38). In Fig. 6 we present typical mesh sequences and degree distributions for the adaptive hp-version with direction control. For the Neumann problem we consider the relative error in the energy norm $\|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} / \|v\|_{\tilde{H}^{1/2}(\Gamma)}$ for the solution v of (10) and its Galerkin approximation v_N. There hold the error estimates [24, 26]

$$\|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} \leq c_4 \begin{cases} h^{1/2-\epsilon} \\ p^{-1+2\epsilon} \end{cases}. \tag{39}$$

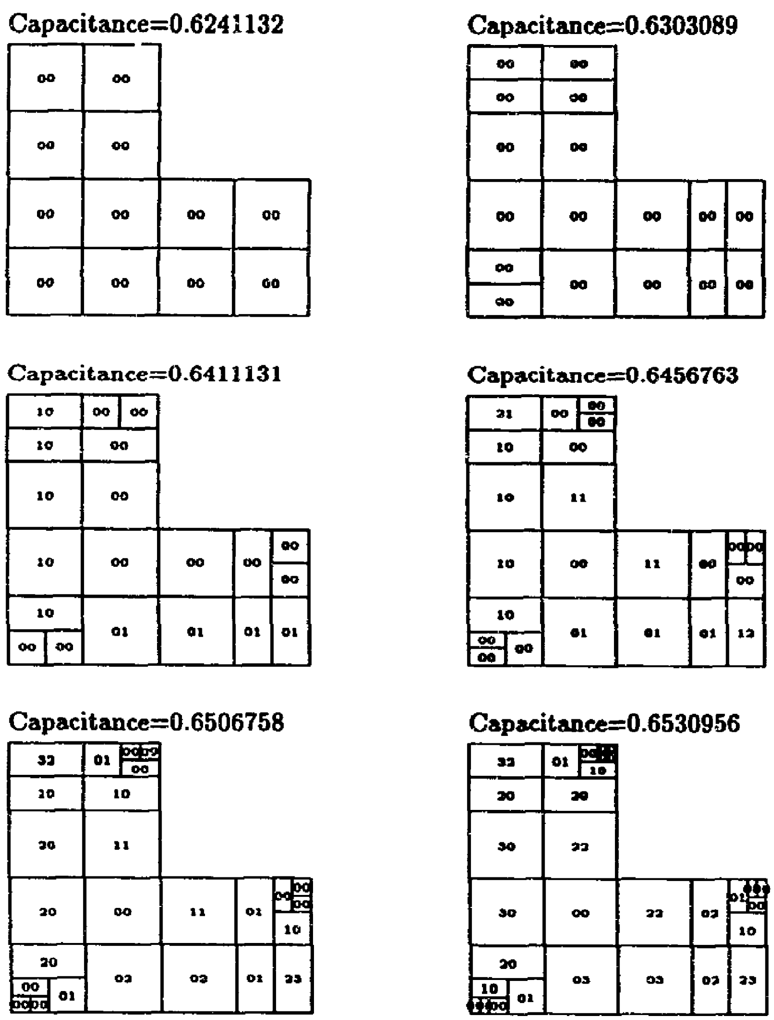


Fig. 6. The adaptive hp-version with direction control for the Dirichlet problem ($C = 0.659265$).

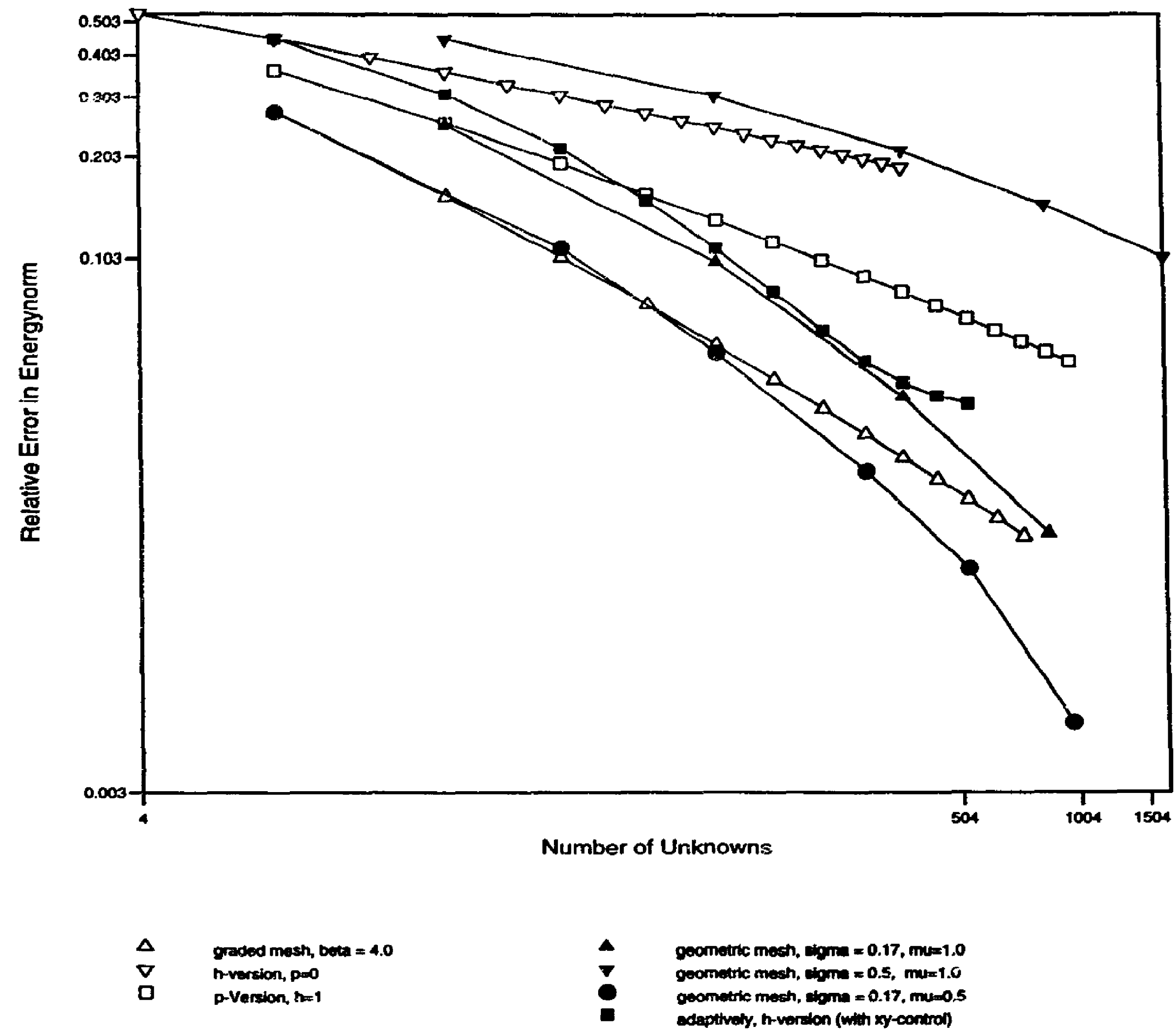


Fig. 7. The Neumann problem on the square plate.

For the hp-version with geometric mesh, Figs. 7 and 8 show the predicted exponentially fast convergence (cf. Theorem 8), whereas they show only algebraic convergence for the pure h- and p-versions as predicted by (39). We have also plotted in those figures the convergence rate for the graded mesh (cf. Theorem 3). Next, we present some numerical experiments for the weakly singular Eq. (3) on the L-block given in Fig. 10. Again, Fig. 9 shows only algebraic convergence for the pure h- and p-versions and exponentially fast convergence for the h-p version on the geometric mesh. Fig. 10 shows the mesh of the adaptive h-version with xy-control. Finally, Fig. 11 shows the numerical experiments for the system of integral equations (34) which corresponds to the inclusion problem for the Lamé system.

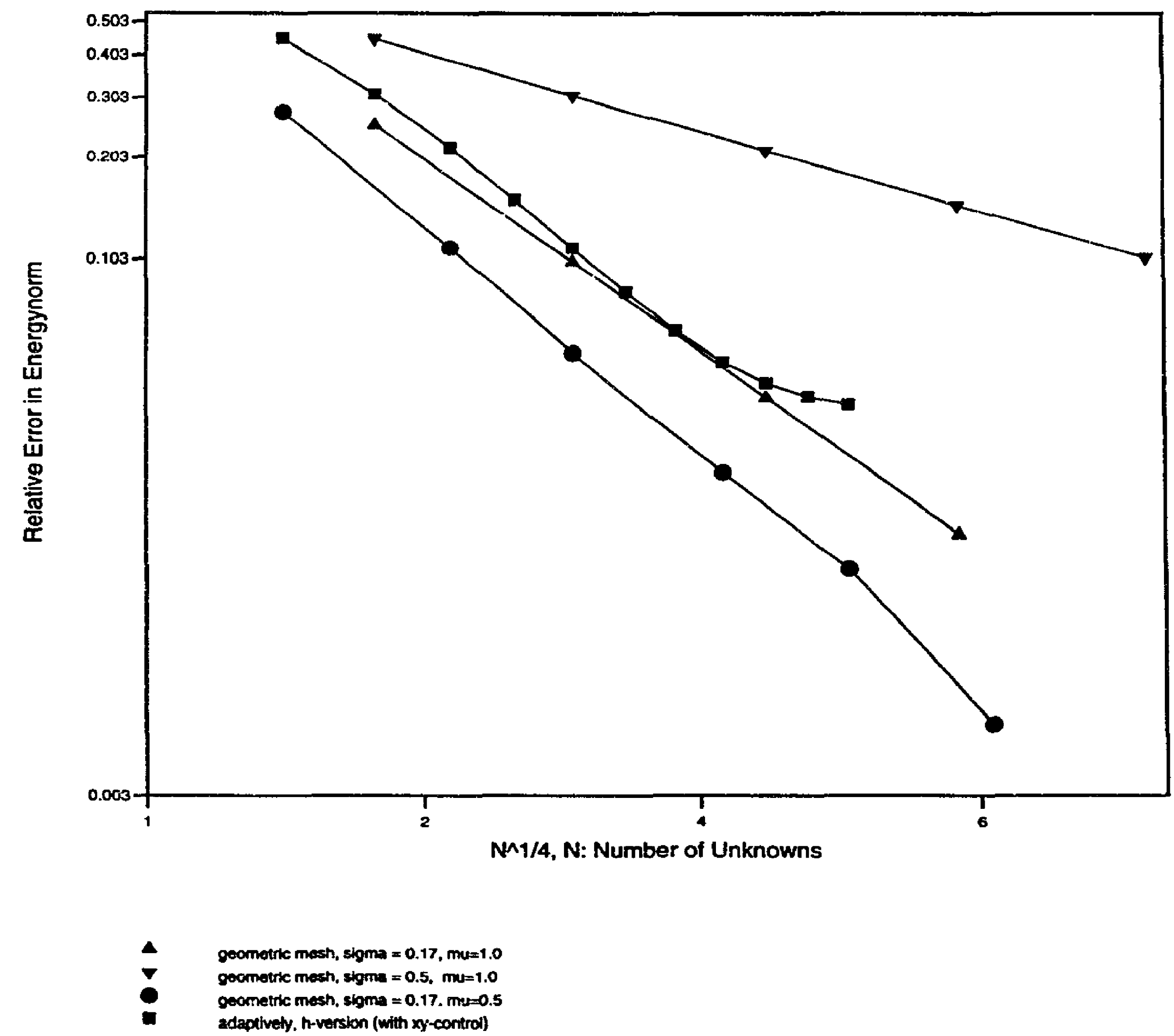


Fig. 8. The Neumann problem on the square plate.

To solve the linear systems (13) and (14) for the different versions of the boundary element method we used the conjugate gradient (CG) algorithm. In case of the p-version we investigate various kinds of preconditioning: (i) diagonal preconditioning, (ii) preconditioning by the diagonal plus the subblock of piecewise constant or piecewise linear functions in (a) one variable or (b) both variables. The numerical experiments show that preconditioning reduces significantly the number of iterations which are necessary to reach a given precision. In Fig. 12 the number of iterations are plotted versus the number of unknowns for the p-version to solve the Dirichlet problem on the square plate. For the h-p version we have always used CG with diagonal preconditioning.

For further details and further numerical results, see [16, 18, 19].

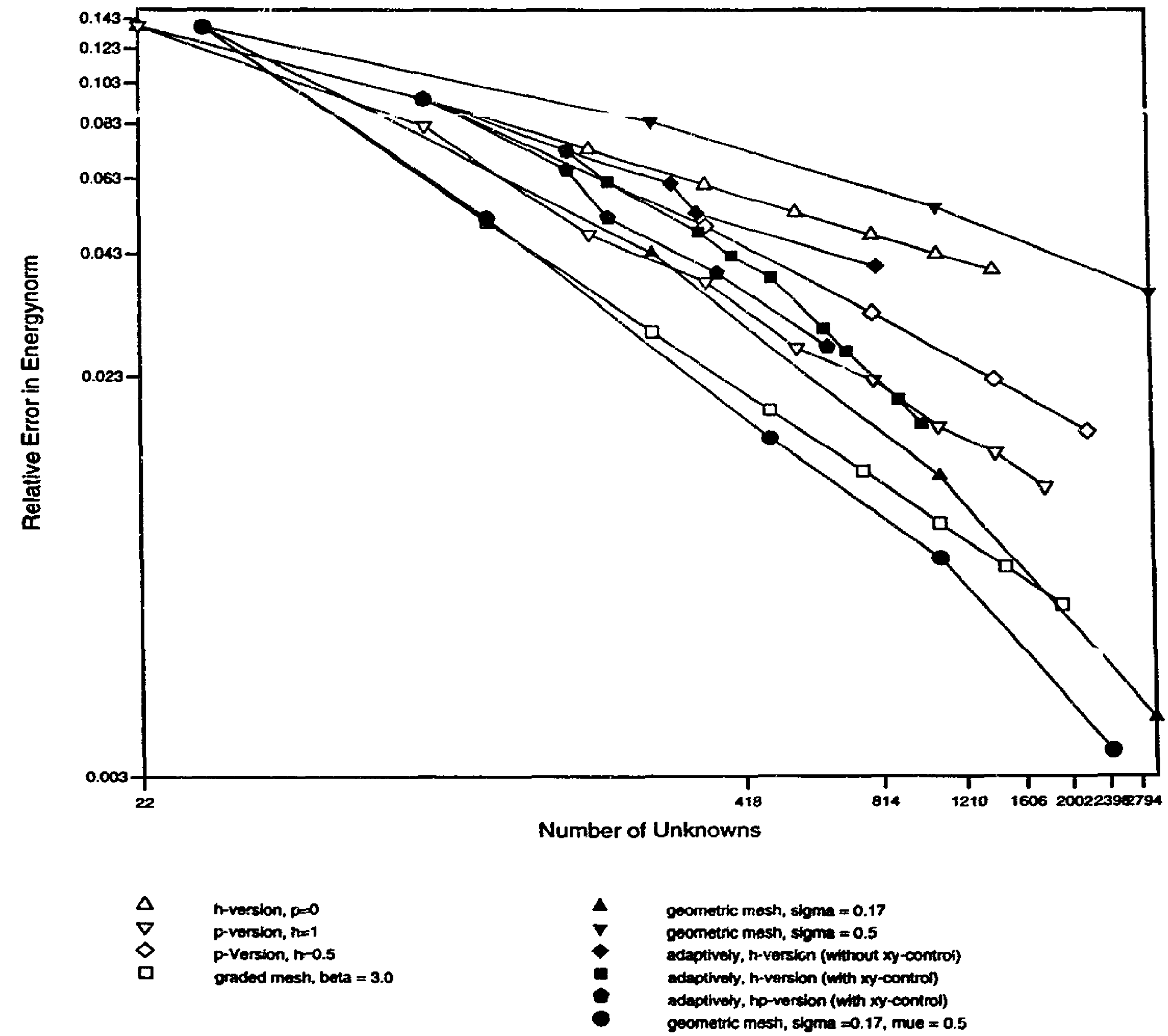


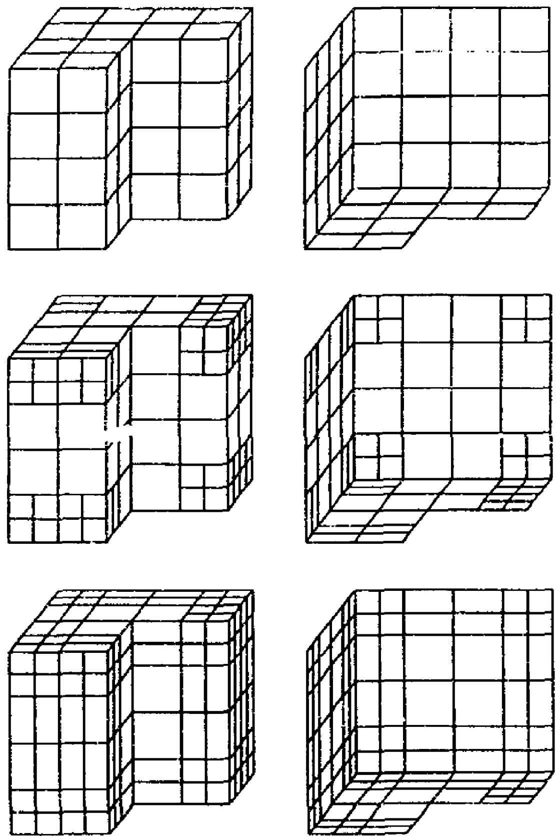
Fig. 9. Dirichlet problem on the L-block.

6. Coupling of FEM and BEM—*h-p* version

Here, we consider the *h-p* version of the FEM/BEM coupling for transmission problems in $\mathbb{R}^2$ (see [14]). For simplicity, we consider the interface problem

$$\begin{aligned}
&-\Delta u = f \quad \text{in } \Omega, \quad \Delta u = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{\Omega} \\
&\text{with given jumps } [u]|_\Gamma = u_0 \quad \text{and} \quad \left[\frac{\partial u}{\partial n}\right]\Big|_\Gamma = t_0 \quad \text{on } \Gamma = \partial\Omega \\
&\text{and growth condition } u(x) = \beta \log|x| + \mathcal{O}(1) \quad \text{as } |x| \to \infty
\end{aligned} \tag{40}$$

where $\beta \in \mathbb{C}$ is not known in advance.



38

Fig. 10. The adaptive h-version with direction control for the Dirichlet problem on the L-block.

It is well known that (40) has the following variational formulation:

Find $u \in H^1(\Omega)$, $\phi \in H^{-1/2}(\Gamma)$ such that for all $v \in H^1(\Omega)$, $\psi \in H^{-1/2}(\Gamma)$ there holds with $a(u, v) := \int_\Omega \nabla u \nabla v \, dx$

$$B\left(\binom{u}{\phi}, \binom{v}{\psi}\right) = L\left(\binom{v}{\psi}\right) \tag{41}$$

where

$$B\left(\binom{u}{\phi}, \binom{v}{\psi}\right) := 2a(u, v) + \langle (K' - 1)\phi + Wu|_\Gamma, v \rangle + \langle (K - 1)u|_\Gamma - V\phi, \psi \rangle$$

$$L\left(\binom{v}{\psi}\right) := \langle (K' + 1)t_0 + Wu_0, v \rangle + \langle (K - 1)u_0 - Vt_0, \psi \rangle .$$

For $X_N \subset H^1(\Omega)$ and $Y_M \subset H^{-1/2}(\Gamma)$ the Galerkin scheme reads: Find $u_N \in X_N$ and $\phi_M \in Y_M$ satisfying

$$B\left(\binom{u_N}{\phi_M}, \binom{w}{\psi}\right) = L\left(\binom{w}{\psi}\right) \quad \forall\, w \in X_N,\ \psi \in Y_M \tag{42}$$

Next, we consider the h-p version on a geometric mesh. For simplicity we consider only the case where Γ is an L-shaped polygon. Let Ω^n_σ be a geometric mesh on Ω, refined at the reentrant corner. Then, Ω^n_σ induces a geometric mesh Γ^n_σ on $\Gamma = \partial\Omega$. On the finite element mesh Ω^n_σ we approximate u by tensor products of antiderivatives of Legendre polynomials together with continuous piecewise linear functions, this gives the space $\hat{S}^{n,1}(\Omega^n_\sigma)$. On the boundary element mesh Γ^n_σ we approximate $\partial u / \partial n$ as

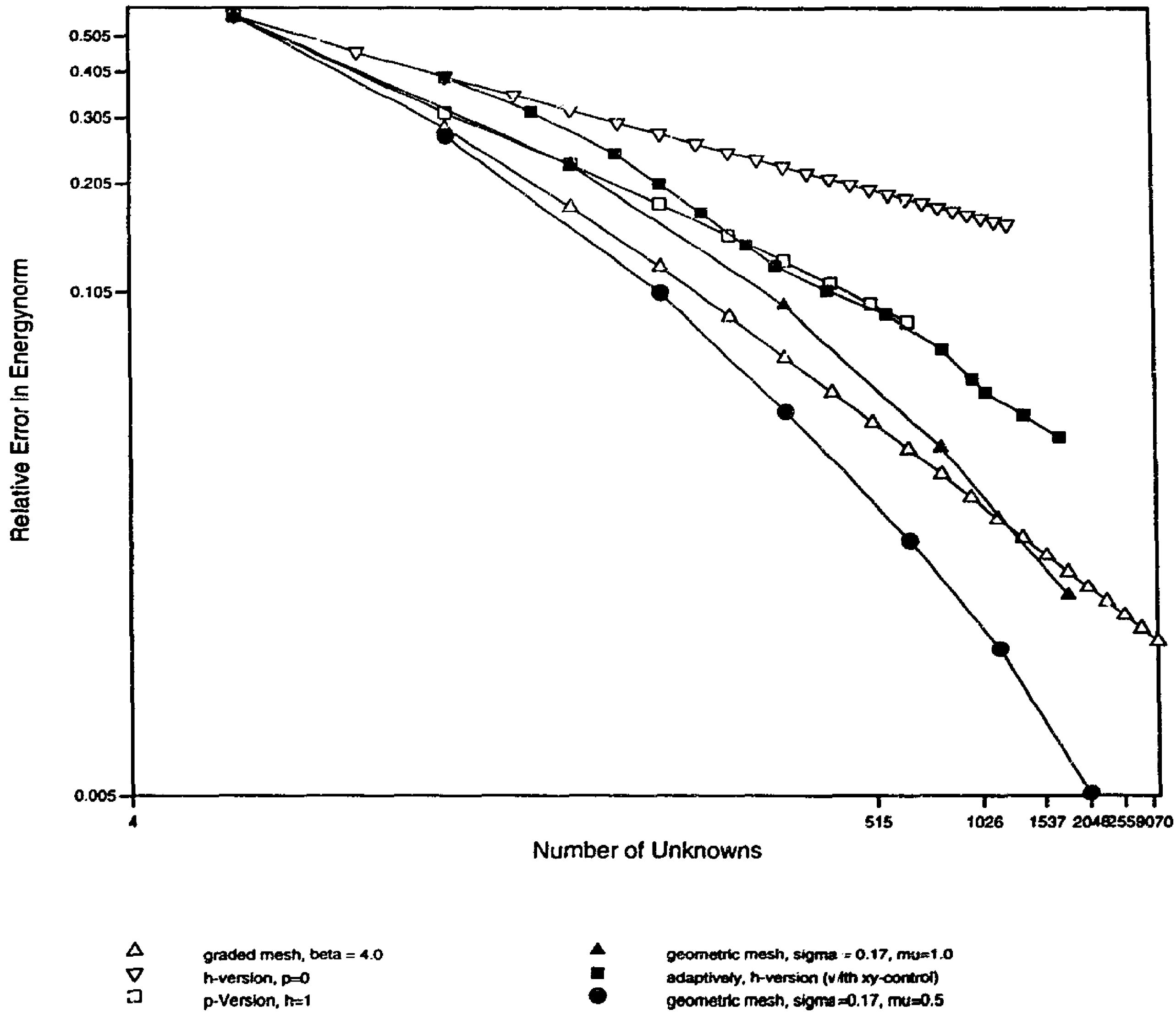


Fig. 11. Lame problem (inclusion) on the square plate.

in Sections 4.1 and 4.2 by elements in $S^{n-1}(\Gamma^n_\sigma)$. Then, the above symmetric coupling procedure converges exponentially (see [15]).

THEOREM 9. Let f, u_0, t_0 piecewise analytic, then there holds the estimate

$$\|u - u_N\|_{H^1(\Omega)} + \|\phi - \phi_M\|_{H^{-1/2}(\Gamma)} \leq C(e^{-b_1\sqrt[3]{N}} + e^{-b_2\sqrt[2]{M}})$$

between the Galerkin solution $u_N \in X_N = \mathring{S}^{n,1}(\Omega^n_\sigma)$, $\phi_M \in Y_M = S^{n-1}(\Gamma^n_\sigma)$ and the exact solution $u, \phi := \partial u/\partial n|_\Gamma$ of (41), where the positive constants C, b_1, b_2 are independent of $N = \dim X_N$ and $M = \dim Y_M$.

REMARK. For transmission problems in $\mathbb{R}^3$, Guo and Stephan [12] show exponential convergence for the *h-p* version of the FEM/BEM coupling,

$$\|u - u_N\|_{H^1(\Omega)} + \|\phi - \phi_M\|_{H^{-1/2}(\Gamma)} \leq C(e^{-b_1\sqrt[5]{N}} + e^{-b_2\sqrt[4]{M}}),$$

with corresponding finite element and boundary element spaces X_N and Y_M. The coupling approach can

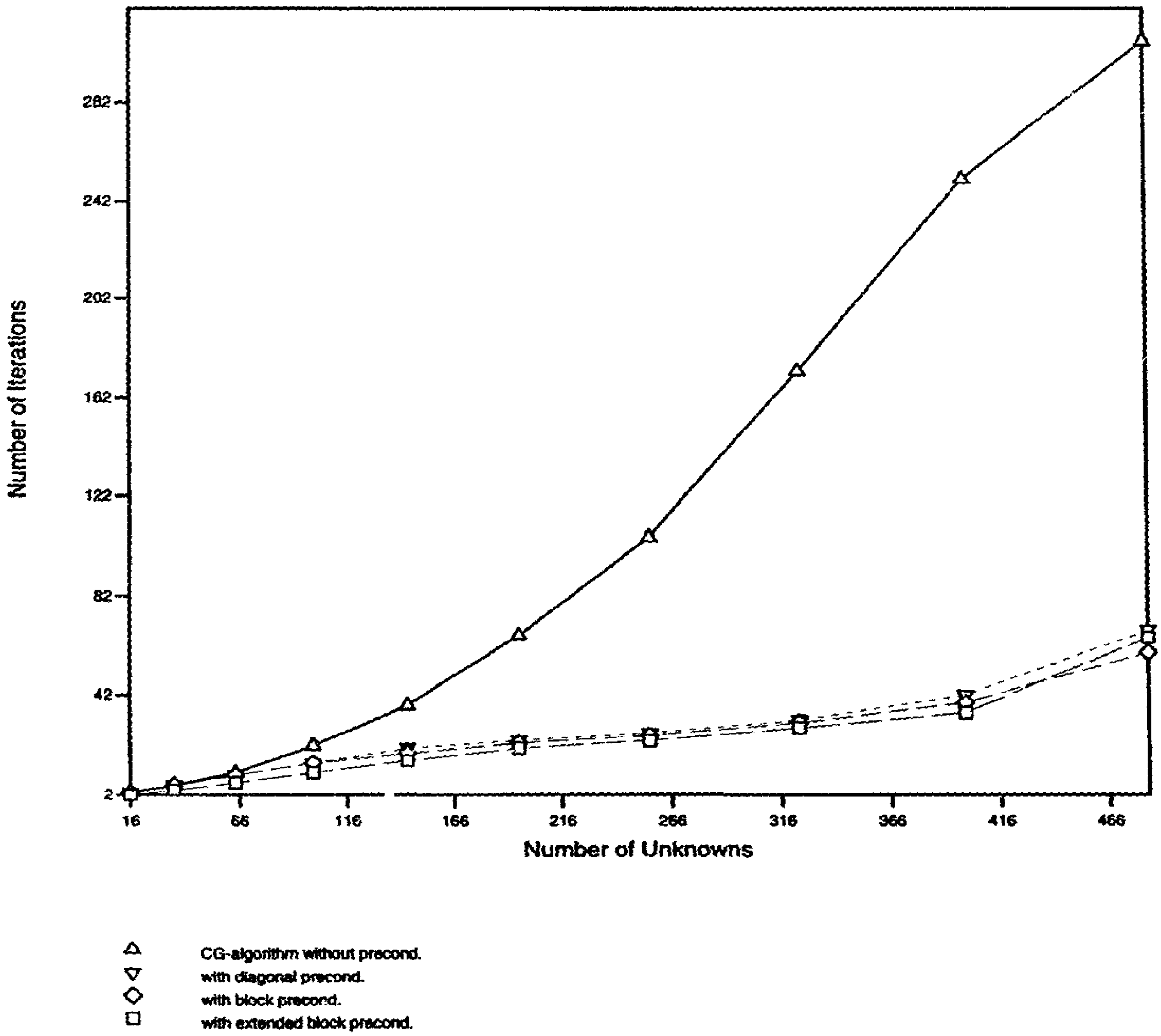


Fig. 12. Dirichlet problem on the square plate in R^3. Number of iterations for the CG-algorithm (p-version, 4 elements).

also be applied to interface problems with a non-linear differential operator in the bounded domain Ω (see [27]), e.g. the Hencky problem in non-linear elasticity. For brevity we consider here the problem

$$-\operatorname{div}(p|\nabla u_1|)\nabla u_1) + u_1 = f \quad \text{in } \Omega\,, \qquad -\Delta u_2 = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}$$

with

$$u_1 = u_2 + u_0\,, \qquad p(|\nabla u_1|)\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} + t_0 \quad \text{on } \Gamma\,, \qquad p(t) := 2 + (1+t)^{-1}$$

and $u_2 \sim \log|x|$ as $|x| \to \infty$.

This leads to (41) with

$$a(u, w) := \int_\Omega \left\{ \left(2 + \frac{1}{1+|\nabla u|}\right)\nabla u \nabla w + u \cdot w \right\} \mathrm{d}x$$

EXAMPLE. Let $\Omega = [-1, 1]^2$ and $u_1 = (2 - x_1^2 - x_2^2)^{4/3}$, $u_2 = \frac{1}{2}\log(x_1^2 + x_2^2)$, and the mesh Ω_σ^n be geometrically graded towards the corners of Ω. Then we obtain exponentially fast convergence in the energy norm for the Galerkin solution of (42). Fig. 3 shows L^2-errors for meshes with a different σ.

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