Numer. Math. 57, 379–391 (1990)

Numerische Mathematik

On the Convergence of the Multigrid Method for a Hypersingular Integral Equation of the First Kind

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Received March 10, 1989/August 30,1989

Summary. We present a multigrid method to solve linear systems arising from Galerkin schemes for a hypersingular boundary integral equation governing three dimensional Neumann problems for the Laplacian. Our algorithm uses damped Jacobi iteration, Gauss-Seidel iteration or SOR as pre- and post-smoothers. If the integral equation holds on a closed, Lipschitz surface we prove convergence of V- and W-cycles with any number of smoothing steps. If the integral equation holds on an open, Lipschitz surface (covering crack problems) we show convergence of the W-cycle. Numerical experiments are given which underline the theoretical results.

Subject Classifications: AMS(MOS); 65F10, 65N30, 45L10; CR: G1.9.

1 Introduction

This paper shows that the convergence theory of multigrid methods [5, 6] also holds for linear systems arising from Galerkin schemes for a hypersingular integral equation given by the operator of the normal derivative of the double layer potential. This operator is a strongly elliptic pseudodifferential operator of order plus one.

In order to introduce the integral operator we consider the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma = \partial\Omega$$

$$u(x) = O\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \tag{1.1}$$

for Ω being either the exterior of a bounded domain in $\mathbf{R}^3$ with a Lipschitz boundary $\partial\Omega$ or $\Omega = \mathbb{R}^3 \setminus \bar{\Gamma}$ where $\Gamma = \partial\Omega$ is a smooth open surface with a Lip-

schitz boundary. In (1.1) $\frac{\partial u}{\partial n}$ denotes the normal derivative on $\partial\Omega=\Gamma$. In [1] this problem was converted into a first kind boundary integral equation with the operators D and K' of the normal derivative of the double and single layer potential, respectively,

$$Dv=(1+K')g \quad \text{on } \partial\Omega=\Gamma \tag{1.2a}$$

or

$$Dv=g \quad \text{on } \Gamma=\partial\Omega \tag{1.2b}$$

The first equation holds in case $\Gamma=\partial\Omega$ is closed whereas the second equation corresponds to a crack problem with crack $\Gamma=\partial\Omega$ being a surface piece. The hypersingular integral operator D on Γ is given by

$$\begin{aligned} Dv(x) &= -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\frac{1}{|x-y|}v(y)\,ds_y \\ &= -\frac{1}{\pi}\,\text{f.p.}\int_\Gamma \frac{1}{|x-y|^3}v(y)\,ds_y \end{aligned}$$

Here f.p. denotes a finite part integral. The operator D is continuous from $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$. Furthermore it is strongly elliptic ([1]), i.e., there exists a constant $\gamma>0$ with

$$\forall v\in\tilde{H}^{1/2}(\Gamma) \quad \langle Dv, v\rangle_{H^{-1/2}(\Gamma)\times\tilde{H}^{1/2}(\Gamma)} \geqq \gamma\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{1.3}$$

Here the spaces $\tilde{H}^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$ are defined as follows: If Γ is a closed surface with $\Gamma=\partial\Omega$, then $\tilde{H}^{1/2}(\Gamma)=H^{1/2}(\Gamma)$ with the trace space

$$H^{1/2}(\Gamma)=\{u|_\Gamma : u\in H^1_{\text{loc}}(\Omega)\}.$$

If Γ is an open surface piece, then one can extend Γ by an additional surface piece Γ_1 such that $\bar{\Gamma}\cup\bar{\Gamma}_1$ is a closed surface. Then

$$\tilde{H}^{1/2}(\Gamma)=\{v|_\Gamma : v\in H^{1/2}(\bar{\Gamma}\cup\bar{\Gamma}_1), \quad v|_{\Gamma_1}=0\}.$$

The space $H^{-1/2}(\Gamma)$ is defined to be the dual space of $\tilde{H}^{1/2}(\Gamma)$.

As shown in [1, 7, 8] the integral Eqs. (1.2a), (1.2b) for $g\in H^{-1/2}(\Gamma)$ satisfying $\int_\Gamma g\,ds=0$ have a unique solution $v\in\tilde{H}^{1/2}(\Gamma)$ which is the trace on Γ of the solution $u\in H^1_{\text{loc}}(\Omega)$ of the original b.v.p. Note in the case of a crack Γ, v corresponds to the jump $[u]=u_+-u_-$ of the traces of u on the upper (lower) side of Γ.

After choosing a finite dimensional space of boundary elements $S_h\subset\tilde{H}^{1/2}(\Gamma)$ a *Galerkin* scheme for Eq. (1.2a), (1.2b) reads: Find a solution $u_h\in S_h$ of

$$\langle Du_h, w\rangle_{L^2(\Gamma)}=\langle F, w\rangle_{L^2(\Gamma)} \quad \forall w\in S_h \tag{1.4}$$

(with $F=(1+K')g$ or $F=g$).

Due to the strong ellipticity of D, the linear system (1.4) has a unique solution u_h which converges quasioptimally towards the solution of (1.2a) or (1.2b) in the energy norm $\|\cdot\|_{\tilde{H}^{1/2}(\Gamma)}$. As a simple choice for S_h we can take the space of continuous, piecewise linear polynomials on triangles (which also vanish on the boundary $\partial\Gamma$ of Γ in the case of a crack Γ).

In the sequel we are concerned with solving efficiently the linear system (1.4). Let $N := \dim S_h$, then (1.4) is a $N \times N$ linear system. A Gauss solver requires CN^3 operations. We may also use an iterative Jacobi or SOR method to solve (1.4) with an accuracy of the order of the Galerkin error. Then we need $C'N^{5/2}$ operations, since the convergence factor ρ of one iteration step behaves like $\rho \approx 1 - CN^{-1/2}$ (cf. Sect. 4). The aim of the multigrid method is to give an iteration procedure which has a convergence factor $\rho \leqq q < 1$ per iteration, with a constant q independent of N. Yet one iteration requires just CN^2 operations if the maximal number of levels is used (as one Jacobi iteration does). Hence the system (1.4) can be solved with an accuracy of the order of the Galerkin error using a total number $C'N^2$ of operations.

In Sect. 2, we describe the multigrid algorithm to be analyzed.

In Sect. 3 we show that the convergence factor of one multigrid step is uniformly bounded by a constant smaller than one (Theorem 1). We apply the convergence theory of multigrid methods by Mandel which was originally used for strongly elliptic differential operators [5, 6]. The convergence proofs in [5, 6] are based on a so-called *approximation assumption* which depends on a regularity property of the original problem. Using a regularity result for the boundary integral eq. (1.4) and by applying the Aubin-Nitsche trick to the Galerkin error of our hypersingular integral equation we show that this approximation assumption holds for boundary element discretizations when continuous, piecewise linear polynomials on uniform meshes are used as test and trial functions in the Galerkin scheme.

For Γ being a closed surface, the multigrid method with 2 levels for the integral eq. (1.2a) was recently analyzed in [4]. Our analysis also applies to the case of Γ being an open Lipschitz surface in $\mathbb{R}^3$ or an open curve in $\mathbb{R}^2$. Crack problems in linear elasticity with Neumann conditions corresponding to (1.1) are also included in our analysis since they, too, lead to strongly elliptic pseudodifferential operators (see [2]). Our result shows that the convergence factor of one multigrid step with an arbitrary number of levels and of smoothing steps is uniformly bounded by a constant smaller than 1. In the case of a crack a W-cycle is required, in the case of a closed surface this result holds also for the V-cycle.

In Sect. 4 we illustrate our theoretical results by some numerical experiments for a square crack in $\mathbb{R}^3$ using multigrid methods with the maximal number of levels.

2 The Multigrid Algorithm

In order to describe the multigrid method for solving (1.2a), (1.2b) we introduce a sequence of nested finite dimensional spaces of boundary elements $S_{h_0} \subset S_{h_1} \subset \ldots \subset S_{h_m}$. We use the following iterative scheme to solve approximately the Galerkin eq. (1.4) on the finest grid S_{h_m}.

First we choose a relaxation method, for example damped Jacobi relaxation, to solve the Galerkin eq. (1.4) on $S_{h_j}, j>0$. Note that the Galerkin scheme (1.4) for S_{h_j} is equivalent to the linear system

$$\tilde{A}_{h_j}\tilde{u}_{h_j}=\tilde{f}_{h_j}, \quad \tilde{A}_{h_j}:=(\langle D\phi_{k'}^{h_j},\phi_l^{h_j}\rangle)_{k,l=1}^{N_j}, \quad \tilde{f}_{h_j}:=(\langle f,\phi_l^{h_j}\rangle)_{l=1}^{N_j} \tag{2.1}$$

where $\tilde{u}_{h_j}$ is the coefficient vector of u_{h_j} with respect to a basis $\{\phi_l^{h_j}\}_{l=1}^{N_j}$ of S_{h_j}. Then the damped Jacobi iterate $v_{h_j}=F_j(w_{h_j},f)$ for $w_{h_j}\in S_{h_j}$ and $f\in H^{-1/2}(\Gamma)$ is given by

$$\tilde{v}_{h_j}=\tilde{w}_{h_j}-\omega_j\tilde{R}_j(\tilde{A}_{h_j}\tilde{w}_{h_j}-\tilde{f}_{h_j}) \tag{2.2}$$

with the relaxation parameter $\omega_j\in\mathbb{R}$ where $\tilde{f}_{h_j}$ is defined as above and with

$$\tilde{R}_j=\operatorname{diag}((\tilde{A}_{h_j})_{kk}^{-1})_{k=1}^{N_j}=\operatorname{diag}(\langle D\phi_k^{h_j},\phi_k^{h_j}\rangle^{-1})_{k=1}^{N_j}$$

In the following $F_j^\nu(w_{h_j},f)$ denotes the result of ν relaxation steps F_j.

Now, one step

$$v_{h_j}=M_j(w_{h_j},f) \tag{2.3}$$

of the multigrid algorithm is defined for $w_{h_j}\in S_{h_j}$, $f\in H^{-1/2}(\Gamma)$ recursively as follows: Let $\mu=1$, or 2 and $\nu_1,\nu_2\in\mathbb{N}\cup\{0\}$.

(i) for $j=0$: $M_0(w_{h_0},f):=v_{h_0}$ where v_{h_0} is the exact solution of the Galerkin eq. (1.4)

$$\langle Dv_{h_0},s_{h_0}\rangle=\langle f,s_{h_0}\rangle \qquad \forall s_{h_0}\in S_{h_0} \tag{2.4}$$

(ii) for $j>0$: $M_j(w_{h_j},f):=v_{h_j}^{(3)}$ where

$$v_{h_j}^{(1)}:=F_j^{\nu_1}(w_{h_j},f) \tag{2.5a}$$

$$v_{h_j}^{(2)}:=v_{h_j}^{(1)}+M_{j-1}^{\mu}(0,f-Dv_{h_j}^{(1)}) \tag{2.5b}$$

$$v_{h_j}^{(3)}:=F_j^{\nu_2}(v_{h_j}^{(2)},f) \tag{2.5c}$$

Note, $M_j^\mu(\cdot,g)$ denotes μ times application of $M_j(\cdot,g)$. The steps (2.5a) and (2.5c) are called presmoothing and postsmoothing, respectively. The iteration step (2.5b), the so-called correction step, is motivated as follows: One looks for a correction $v_{h_{j-1}}\in S_{h_{j-1}}$ for $v_{h_j}^{(1)}$ such that $v_{h_j}^{(1)}+v_{h_{j-1}}$ satisfies the Galerkin equation

$$\langle D(v_{h_j}^{(1)}+v_{h_{j-1}}),s_{h_{j-1}}\rangle=\langle f,s_{h_{j-1}}\rangle \qquad \forall s_{h_{j-1}}\in S_{h_{j-1}}$$

which is equivalent to

$$\langle Dv_{h_{j-1}},s_{h_{j-1}}\rangle=\langle f-Dv_{h_j}^{(1)},s_{h_{j-1}}\rangle \tag{2.6}$$

Its solution $v_{h_{j-1}}\in S_{h_{j-1}}$ is approximated by $z_{h_{j-1}}:=M_{j-1}^\mu(0,f-Dv_{h_j}^{(1)})$, which results from application of μ multigrid steps on level $j-1$ to problem (2.6) starting with initial guess 0. In order to determine $z_{h_{j-1}}$ it is only necessary to know the residual $\langle f-Dv_{h_j}^{(1)},s_{h_j}\rangle$ $\forall s_{h_j}\in S_{h_j}$. This yields the right side in (2.6) by simple algebraic manipulations, since $S_{h_{j-1}}\subset S_{h_j}$.

We remark that the above algorithm for $\mu=1$ is called V-cycle and for $\mu=2$ it is called W-cycle.

With $u_{h_j}=\sum_{k=1}^{N_j} \tilde{u}_{h_j}^k \phi_k^{h_j}$ we obtain (by passing from the coefficient vectors $\tilde{v}_{h_j}$ to the functions $v_{h_j}\in S_{h_j}$, etc.) for (2.2)

$$v_{h_j}=w_{h_j}-\omega_j \sum_{k=1}^{N_j} \langle D w_{h_j}-f, \phi_k^{h_j}\rangle \phi_k^{h_j} \tag{2.7}$$

Next, we define the mapping $D_{h_j}:=\Pi_{S_{h_j}}^{L^2} D\colon S_{h_j}\to S_{h_j}$ with the L^2-projection $\Pi_{S_{h_j}}^{L^2}\colon H^{-1/2}(\Gamma)\to S_{h_j}$ given by

$$\langle \Pi_{S_{h_j}}^{L^2} g, s_{h_j}\rangle=\langle g, s_{h_j}\rangle \qquad \forall s_{h_j}\in S_{h_j} \tag{2.8}$$

Hence we can rewrite (2.7) as

$$v_{h_j}=w_{h_j}-\omega_j R_j P_j(D_{h_j} w_{h_j}-f_{h_j}), \tag{2.9}$$

where $f_{h_j}:=\Pi_{S_{h_j}}^{L^2} f$. Here P_j is the mapping from S_{h_j} into S_{h_j} defined by

$$P_j s_{h_j}=\sum_{k=1}^{N_j} \langle s_{h_j}, \phi_k^{h_j}\rangle \phi_k^{h_j} \tag{2.10}$$

with a matrix representation $\tilde{P}$ with respect to the basis $\{\phi_k^{h_j}\}_{k=1}^{N_j}$ given by

$$\tilde{P}_j=(\langle \phi_k^{h_j}, \phi_l^{h_j}\rangle)_{k,l=1}^{N_j}$$

Here $R_j\colon S_{h_j}\to S_{h_j}$ is the mapping given by the matrix $\tilde{R}_j$. We remark that the relaxation step (2.9) is of the form (6.4) in [6], i.e.,

$$F_j(w_{h_j}, f_{h_j}):=w_{h_j}-\omega B_j^{-1}(D_{h_j} w_{h_j}-f_{h_j}) \tag{2.11}$$

with

$$B_j^{-1}=R_j P_j. \tag{2.12}$$

Therefore we can apply the convergence analysis of [6] to our multigrid algorithm. The analysis in [6] is based on two properties of the algorithm: the "smoothing assumption" and the "approximation assumption". By the above choice (2.11) of the relaxation scheme the smoothing assumption is automatically satisfied due to Theorem 6.1 in [6].

The approximation assumption is determined by the choice of boundary elements S_{h_j} and the operator D. For the formulation of the approximation assumption we define following [6] a scale of norms on S_{h_j}

$$|v_{h_j}|_\nu^2:=\langle B_j(B_j^{-1} D_{h_j})^\nu v_{h_j}, v_{h_j}\rangle, \qquad \nu\geqq 0.$$

Note

$$|v_{h_j}|_1^2=\langle D v_{h_j}, v_{h_j}\rangle\simeq \|v_{h_j}\|_{\tilde{H}^{1/2}(\Gamma)}^2 \quad \text{for } v_{h_j}\in S_{h_j}. \tag{2.13}$$

The "approximation assumption" is satisfied if

(A.1) $\rho(B_j^{-1} D_{h_j}) \leqq 1$

(A.2) *For* $j=1, \ldots, m$ *let* $v_{h_j} \in S_{h_j}$, $f := D v_{h_j}$ *and* $w_{h_{j-1}} \in S_{h_{j-1}}$ *be the Galerkin solution to* $Dw=f$, *i.e.,*

$$\langle D w_{h_{j-1}}, s_{h_{j-1}} \rangle = \langle f, s_{h_{j-1}} \rangle \qquad \forall s_{h_{j-1}} \in S_{h_{j-1}}.$$

Note, $w_{h_{j-1}}$ *is the Galerkin projection of* v_{h_j} *on* $S_{h_{j-1}}$. *Then we require fore some* α *with* $0 < \alpha \leqq 1$ *and constant* C

$$|v_{h_j} - w_{h_{j-1}}|_{1-\alpha} \leqq C |v_{h_j} - w_{h_{j-1}}|_1 \tag{2.14}$$

Note that instead of (A.1) above $\rho(B_j^{-1} D_{h_j}) \leqq C$, for a constant C, suffices as follows by scaling ω_j in (2.9).

3 Proof of Assumptions (A.1) and (A.2)

In order to prove assumption (A.1) we need the following property of the basis functions of S_{h_j}: *For any* $f_{h_j} \in S_{h_j}$ *there exist constants* $\eta_1, \eta_2 > 0$, *independent of* h, *such that*

$$\eta_1 h^d \sum_{k=1}^{N_j} (\vec{f}_{h_j}^k)^2 \leqq \| f_{h_j} \|_{L^2(\Gamma)}^2 \leqq \eta_2 h^d \sum_{k=1}^{N_j} (\vec{f}_{h_j}^k)^2 \tag{3.1}$$

with

$$f_{h_j} = \sum_{k=1}^{N_j} \vec{f}_{h_j}^k \phi_k^{h_j}.$$

Here d *denotes the dimension of* Γ ($d=2$ *in our case*).

For boundary elements on a uniform grid and basis functions having smallest support and maximum value one the assumption (3.1) is generally satisfied (see [3]).

The relation (3.1) can be rewritten as

$$\eta_1 h^2 \vec{f}_{h_j}^T \vec{f}_{h_j} \leqq \vec{f}_{h_j}^T \tilde{P}_j \vec{f}_{h_j} \leqq \eta_2 h^2 \vec{f}_{h_j}^T \vec{f}_{h_j}$$

yielding

$$\eta_1 h^2 \leqq \lambda_{\min}(\tilde{P}_j), \qquad \lambda_{\max}(\tilde{P}_j) \leqq \eta_2 h^2$$

and hence, since P_j is self adjoint,

$$\| P_j \|_{L^2(\Gamma) \to L^2(\Gamma)} \leqq \eta_2 h^2, \tag{3.2}$$

and

$$\| P_j^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \leqq \eta_1^{-1} h^{-2} \tag{3.3}$$

Suppose the grid to be quasiuniform such that the basis functions $\{\phi_k^{h_j}\}$ of S_{h_j} are obtained by an affine map via shape functions on the reference element. Then we have

Lemma 1. *There exists $C>0$, independent of h, such that*

$$\|R_j\|_{L^2(\Gamma)\to L^2(\Gamma)} \leqq C h^{-1}, \qquad \|R_j^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leqq C h$$

Proof. For simplicity we consider first the case of a mapping $x \to \tilde{x} = x_0 + h^{-1} x$ with fixed $x_0 \in \mathbb{R}^2$ and $C \in \mathbb{R}$. Then we have by homogeneity

$$\langle D\phi_k, \phi_k\rangle = \iint \frac{\phi_k(x)\,\phi_k(y)}{|x-y|^3}\,dx\,dy = \iint \frac{\phi_k(h(\tilde{x}-x_0))\,\phi_k(h(\tilde{y}-y_0))}{h^3|\tilde{x}-\tilde{y}|^3}\,h^2\,d\tilde{x}\,h^2\,d\tilde{y} = Ch$$

The argument applies to the general case of an affine map $x \to \tilde{x} = x_0 + Tx$ with $\|T\| \leqq Ch^{-1}$, $\|T^{-1}\| \leqq Ch$ with C independent of h. Furthermore we observe

$$\|R_j\|_{L^2(\Gamma)\to L^2(\Gamma)} \leqq \max_k (\langle D\phi_k, \phi_k\rangle^{-1})$$

and

$$\|R_j^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leqq \max_k (\langle D\phi_k, \phi_k\rangle)$$

which yields the assertion. □

Lemma 2. *There exists a constant C, independent of h_j such that*

$$\|D_{h_j}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leqq C h_j^{-1}.$$

Proof. We have with $D_{h_j} = \Pi^{L^2}_{S_{h_j}} D|_{S_{h_j}}$ and $D\colon \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ continuous

$$\begin{aligned}\|D_{h_j} u_{h_j}\|^2_{L^2} = \langle D_{h_j} u_{h_j}, D_{h_j} u_{h_j}\rangle = \langle D u_{h_j}, D_{h_j} u_{h_j}\rangle &\leqq \|D u_{h_j}\|_{H^{-1/2}(\Gamma)}\,\|D_{h_j} u_{h_j}\|_{\tilde{H}^{1/2}(\Gamma)} \\ &\leqq C\,\|u_{h_j}\|_{\tilde{H}^{1/2}(\Gamma)}\,\|D_{h_j} u_{h_j}\|_{\tilde{H}^{1/2}(\Gamma)} \leqq C h_j^{-1/2}\,\|u_{h_j}\|_{L^2(\Gamma)}\, h_j^{-1/2}\,\|D_{h_j} u_{h_j}\|_{L^2(\Gamma)}\end{aligned}$$

In the last estimate we have used the inverse assumption which holds for a quasiuniform mesh. □

Lemma 3. *There exists a constant C independent of h_j such that*

$$\rho(B_j^{-1} D_{h_j}) \leqq C. \tag{3.4}$$

Proof. With $B_j^{-1} D_{h_j} = R_j P_j D_{h_j}$ we have

$$\begin{aligned}\rho(B_j^{-1} D_{h_j}) &\leqq \|R_j P_j D_{h_j}\|_{L^2(\Gamma)\to L^2(\Gamma)} \\ &\leqq \|R_j\|_{L^2(\Gamma)\to L^2(\Gamma)}\,\|P_j\|_{L^2(\Gamma)\to L^2(\Gamma)}\,\|D_{h_j}\|_{L^2(\Gamma)\to L^2(\Gamma)}\end{aligned}$$

Hence the estimates (3.2), (3.3) together with Lemma 1 and 2 yield (3.4). □

Now we formulate assumption (A.2) using standard Sobolev norms. First we observe

$$|v_{h_j}|_0^2 = \langle B_j v_{h_j}, v_{h_j}\rangle \leqq \|B_j\|_{L^2(\Gamma)\to L^2(\Gamma)}\,\|v_{h_j}\|^2_{L^2(\Gamma)} \tag{3.5}$$

hence, with $B_j = P_j^{-1} R_j^{-1}$, and (3.3) and Lemma 1

$$\begin{aligned}\|B_j\|_{L^2(\Gamma)\to L^2(\Gamma)} &\leqq \|P_j^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \|R_j^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \\ &\leqq \eta_1^{-1} h^{-2} Ch \leqq \tilde{C} h^{-1}.\end{aligned}$$

From (2.13) we have

$$|v_{h_j}|_1^2 \leqq C \|v_{h_j}\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{3.6}$$

Hence interpolation between (3.5) and (3.6) yields

$$|v_{h_j}|_{1-\alpha} \leqq C h^{-\alpha/2} \|v_{h_j}\|_{\tilde{H}^{\frac{1-\alpha}{2}}(\Gamma)}$$

On the other hand (2.13) gives

$$|v_{h_j} - w_{h_{j-1}}|_1 \geqq \tilde{C} \|v_{h_j} - w_{h_{j-1}}\|_{\tilde{H}^{1/2}(\Gamma)} \tag{3.7}$$

Hence the desired approximation property (2.14) is satisfied if there holds

$$\|v_{h_j} - w_{h_{j-1}}\|_{\tilde{H}^{\frac{1-\alpha}{2}}(\Gamma)} \leqq C h^{\alpha/2} \|v_{h_j} - w_{h_{j-1}}\|_{\tilde{H}^{1/2}(\Gamma)}. \tag{3.8}$$

This is a regularity assumption on the solution of the integral equation as can be seen by application of the Aubin-Nitsche trick. We know that the solution u of $Du = w \in H^{-1/2+s}(\Gamma)$ belongs to $\tilde{H}^{1/2+s}(\Gamma)$ with $s = \frac{1}{2}$ if Γ is a closed, Lipschitz surface (this following from [1]) and with $s = \frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$ if Γ is a smooth, open surface with a smooth boundary [8] or even a Lipschitz boundary [7].

Lemma 4. *The inequality* (3.8) *holds with* C, *independent of* h, *and with* $\alpha = 1$ *for a closed, Lipschitz surface* Γ *and* $\alpha = 1 - 2\varepsilon$, *for arbitrary* $\varepsilon > 0$, *for an open Lipschitz surface* Γ.

Proof. We apply the Aubin-Nitsche trick to the Galerkin error $v_{h_j} - w_{h_{j-1}}$. We set $\tilde{u} := D^{-1}\tilde{w}$ and let $\tilde{u}_h$ be the Galerkin solution to $\tilde{u}$. Then taking the sup over $\tilde{w} \in \tilde{H}^{\eta}(\Gamma)$ with $\|\tilde{w}\|_{\tilde{H}^{\eta}(\Gamma)} = 1$ where $\eta = 0$ if Γ is closed and $\eta = -\varepsilon$ if Γ is open, we have

$$\begin{aligned}\|v_{h_j} - w_{h_{j-1}}\|_{\tilde{H}^{-\eta}(\Gamma)} &= \sup \langle v_{h_j} - w_{h_{j-1}}, \tilde{w} \rangle = \sup \langle v_{h_j} - w_{h_{j-1}}, D\tilde{u} \rangle \\ &= \sup \langle D(v_{h_j} - w_{h_{j-1}}), \tilde{u} - \tilde{u}_h \rangle \\ &\leqq \sup \|D(v_{h_j} - w_{h_{j-1}})\|_{H^{-1/2}(\Gamma)} \|\tilde{u} - \tilde{u}_h\|_{\tilde{H}^{1/2}(\Gamma)} \\ &\leqq C \sup \|v_{h_j} - w_{h_{j-1}}\|_{\tilde{H}^{1/2}(\Gamma)} C h^{1/2+\eta} \|\tilde{u}\|_{\tilde{H}^{1+\eta}(\Gamma)}\end{aligned}$$

This yields (3.8) after using $\|\tilde{u}\|_{H^{1+\eta}(\Gamma)} \leqq C \|\tilde{w}\|_{\tilde{H}^{\eta}(\Gamma)}$. □

Now, we are in the position to apply the results in [6] which guarantee the convergence of our multigrid algorithm. First, we define the convergence factor ρ of one multigrid step with respect to the energy norm (2.13) by

$$\rho_j := \sup_{w_{h_j} \in S_{h_j}} \frac{|M_j(w_{h_j}, 0)|_1}{|w_{h_j}|_1} \tag{3.9}$$

Theorem 1. *Let $\nu_1+\nu_2>0$. If Γ is a closed, Lipschitz surface, let $\mu=1$ or 2 whereas if Γ is an open, Lipschitz surface, let $\mu=2$. Then the convergence factor ρ of one multigrid step (2.3) satisfies $\rho_j\leqq C_0<1$ with C_0 independent of h_j.*

Proof. Due to Lemmas 1–4 the assumptions of Theorem 3.1 in [6] are satisfied. Hence Theorem 1 follows as a consequence of this theorem. □

Furthermore, due to Remark 5.3 in [6] one obtains

$$\rho=O(\nu_1^{-\alpha/2}\,\nu_2^{-\alpha/2})$$

if one applies ν_1 pre-smoothing and ν_2 post-smoothing steps. Finally, we remark that the above analysis also applies to Gauss-Seidel and SOR due to Remark 6.6 and Theorems 6.7, 6.8 in [6].

4 Numerical Results

We consider $Du=f$ on $\Gamma=[-1,1]^2$ with S_{h_j} being the space of piecewise linear, continuous functions on a criss-cross grid (with maximal side-length h_j) and $v_{h_j}\in S_{h_j}$ satisfies $v_{h_j}=0$ on $\partial\Gamma$. This is the less regular case, considered above, where Theorem 1 guarantees the convergence of the W-cycle. Our experiments show convergence of the multigrid algorithm with uniformly bounded convergence factors even for the V-cycle. The numerical experiments were performed on an IBM PS/2, Model 80 at Georgia Tech. The numerical convergence factors are obtained as follows: We start with an arbitrary element $w_{h_j}^{(0)}\in S_{h_j}$ and perform the iterations

$$w_{h_j}^{(k)}:=M_j(w_{h_j}^{(k-1)},0),\qquad k=1,2,\ldots \tag{4.1}$$

Then we define

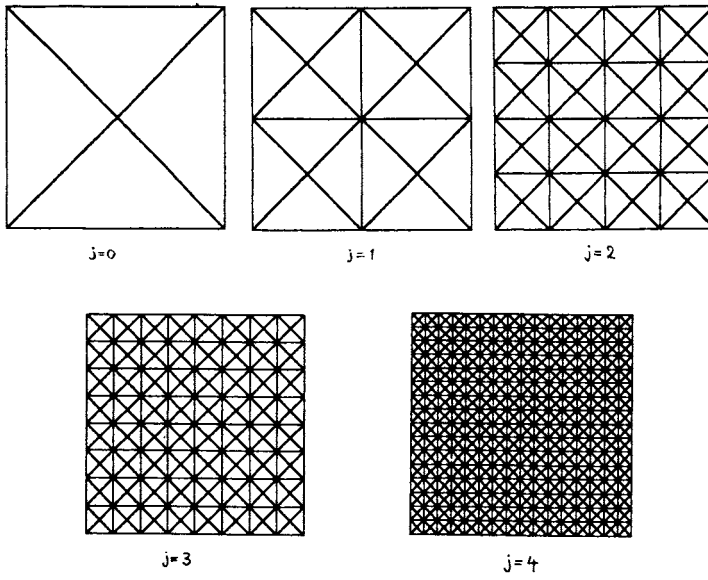
$$\rho_j^{(k)}:=\frac{|w_{h_j}^{(k)}|_1}{|w_{h_j}^{(k-1)}|_1}. \tag{4.2}$$

Thus we have with the coefficient vectors $\tilde{w}_{h_j}^{(k)}$ of $w_{h_j}^{(k)}$

$$\begin{aligned}\rho_j^{(k)}&=\frac{\langle D w_{h_j}^{(k)}, w_{h_j}^{(k)}\rangle^{1/2}}{\langle D w_{h_j}^{(k-1)}, w_{h_j}^{(k-1)}\rangle^{1/2}}\\&=\frac{\langle \tilde{A}_{h_j}\tilde{w}_{h_j}^{(k)}, \tilde{w}_{h_j}^{(k)}\rangle^{1/2}}{\langle \tilde{A}_{h_j}\tilde{w}_{h_j}^{(k-1)}, \tilde{w}_{h_j}^{(k-1)}\rangle^{1/2}}.\end{aligned} \tag{4.3}$$

Note, by definition we have $\rho_j^{(k)}\leqq\rho_j$ with ρ_j as in (3.9).

The following lemma shows that $\rho_j^{(k)}$ approaches the convergence factor ρ_j for $k\to\infty$. Here we also consider SOR as a relaxation scheme. Reversed SOR denotes a SOR iteration where the components are treated in reversed order.



One symmetric SOR (SSOR) step consists of one SOR step and one reversed SOR step.

Lemma 5. *Let $M_j(\cdot, f)$ denote one multigrid step as in* (2.3)–(2.5) *with* $\nu_1=\nu_2$. *Assume that either Jacobi or* SSOR *is used for pre- and post-smoothing, or that* SOR *is used as pre-smoother and reversed* SOR *as post-smoother. Let* $w_{h_j}^{(0)}\in S_{h_j}\setminus T$ *where* T *is a proper subspace of* S_{h_j} *and define the sequence* $w_{h_j}^{(k)}$, $k\in\mathbb{N}$ *by* (4.1). *Then there holds for* $\rho_j^{(k)}$ *given by* (4.2)

$$\lim_{k\to\infty}\rho_j^{(k)}=\rho_j \tag{4.4}$$

with the convergence factor ρ_j *of a multigrid step defined by* (3.9).

Proof. By induction we obtain that the multigrid iteration M_j in (2.4)–(2.5c) has the matrix representation

$$\tilde{M}_j(\tilde{w},\tilde{f})=G\tilde{w}+H\tilde{f},\qquad \tilde{w},\tilde{f}\in\mathbb{R}^{N_j},$$

where the matrix H is symmetric. This allows us to reduce (4.1)–(4.3) to the standard power method for obtaining the largest eigenvalue of a symmetric matrix, yielding (4.4). □

In the following tables we list the values of $\rho_j^{(50)}$ obtained after 50 iterations. In our numerical experiments we have used 5 different meshes, one being a refinement of the others (compare Fig. 1) with meshsizes $h_j=2^{1-j}, j=0,\ldots,4$. The boundary element spaces S_{h_j} are the spaces of piecewise linear, continuous

Table 1

	$j=1$ $(h=1)$	$j=2$ $(h=\frac{1}{2})$	$j=3$ $(h=\frac{1}{4})$	$j=4$ $(h=\frac{1}{8})$	$j=\infty$ $(h=0)$
Jacobi	0.1943	0.4283	0.6810	0.8366	1.0
MG($j+1$ lev.)	0.0218	0.0518	0.0641	0.08359	0.11
MG(2 lev.)	0.0218	0.0512	0.0615	0.08357	0.11

functions on those meshes which vanish on $\partial\Gamma$, the boundary of Γ. Note that S_{h_0} consits of just one element.

Table 1 shows for $\omega=1$ the numerical convergence factors for the Jacobi iteration and for the multigrid method with Jacobi and a maximal number of levels and $\nu_1=\nu_2=1$. The last column shows the limits for $j\to\infty$ (i.e., $h\to 0$) which were obtained by extrapolation assuming $\log\rho_j=\log\rho_\infty+Ch_j$.

First, we note that the number of operations for one multigrid step is proportional to the number of operations of one Jacobi iteration with a proportionality constant independent of j. The convergence factors in Table 1 for the Jacobi iteration show the behavior $\log\rho_j^{\text{Jacobi}}\simeq -Ch_j$. Thus with Jacobi iteration one needs h_j^{-5} operations to solve the linear system for the Galerkin scheme up to an error which is of the same order as the theoretical Galerkin error. The numerical results in Table 1 show that the "numerical" convergence factors for the multigrid method satisfy $\rho_j<C_0<1$ with C_0 independent of h_j (compare Theorem 1). Therefore with the multigrid method one needs only h_j^{-4} operations to solve the linear system for the Galerkin scheme up to an error which is of the same order as the theoretical Galerkin error. Note, the number of the entries of the Galerkin matrix $\tilde{A}_{h_j}$ is also Ch_j^{-4} with a constant C independent of h_j.

Furthermore Table 1 gives the numerical convergence factors for the 2 level-multigrid scheme where on level $j-1$ a Gauss solver is used. In this case the convergence factors are almost identical with those of the multigrid scheme with $j+1$ levels. Note that in case of the multigrid scheme with $j+1$ levels one solves the linear system exactly only on level 0 where the system consists only of one equation.

Table 2 gives the convergence factors of one SSOR iteration, taking the optimal relaxation parameter ω^* for each j. Assuming a behavior $\log\rho_j=-Ch_j^\eta$, the last row of Table 2 gives numerical values for η obtained from ρ_j and ρ_{j-1}. Apparently one has $\log\rho_j=O(h_j)$ asymptotically as for the Jacobi iteration.

Our numerical experiments showed that the Jacobi iteration converges for $0.1\leqq\omega\leqq 1.5$. Therefore we also performed the multigrid scheme with Jacobi

Table 2

	$j=1$	$j=2$	$j=3$	$j=4$
Optimal ω^*	1.0	1.1	1.25	1.4
ρ of SSOR with ω^*	0.0217	0.0618	0.204	0.431
η		0.46	0.81	0.92

Table 3

ω	$j=1$ 2 lev.	$j=2$ 3 lev.	$j=3$ 4 lev.	$j=4$ 5 lev.	$j=\infty$
0.1	0.8303	0.84	0.841	0.845	0.85
0.3	0.5389	0.57	0.586		0.60
0.5	0.3113	0.356	0.376		0.40
0.8	0.0897	0.132	0.149		0.17
1.0	0.0218	0.0518	0.0641	0.0836	0.11
1.1	0.0800	0.144	0.136		0.16
1.5	0.5614	0.778	0.746		0.82

Table 4

ω	$j=1$ 2 lev.	$j=2$ 3 lev.	$j=3$ 4 lev.	$j=4$ 5 lev.	$j=\infty$
0.1	0.6883	0.710	0.72	0.72	0.72
0.3	0.2852	0.32048	0.3352		0.35
0.5	0.09063	0.1168	0.1281		0.14
0.8	0.005953	0.01423	0.02124		0.032
1.0	0.0003472	0.002767	0.009399	0.01742	0.032
1.1	0.001282	0.002276	0.008921	0.01934	0.042
1.5	0.09840	0.1152	0.1195		0.12
1.9	0.7	0.7	0.712		0.72

for those values of ω. In Table 3 we list corresponding numerical convergence factors with $\nu_1=\nu_2=1$ and $j=1, 2, 3, 4$. The table shows that all convergence factors are bounded by a constant smaller than one uniformly for all j. Furthermore Table 3 shows that for all j the optimal convergence factor is obtained for $\omega=1$.

Due to [6] our convergence result of Theorem 1 applies also to the multigrid scheme with SOR as smoother. Therefore we list in Table 4 the numerical convergence factors for the multigrid scheme with the symmetric SOR-relaxation (SSOR). The last columns in Tables 3 and 4 were obtained by extrapolation as in Table 1.

Acknowledgement. The authors would like to thank F.V. Postell for assistance with the computations. The second author was supported by the National Science Foundation under Grant DMS-8704463.

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