Square root preconditioners for the Helmholtz integral equation

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Abstract

We apply pseudo-differential operators theory to the first-kind integral equations on open curves, allowing us to analyze two new preconditioners and study the convergence orders of a Galerkin method on weighted L^2 spaces.

Introduction

1 Analytical setting

The Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$$

for $x \in [-1, 1]$ []. They satisfy the ordinary differential equations

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0 \quad (1)$$

$$(1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0 \quad (2)$$

Let ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$ and let ∂_x the derivation operator. Then (2) and (2) can be rewritten under the form

$$-(\omega\partial_x)^2 T_n = n^2 T_n\,, \quad (3)$$

$$-(\partial_x\omega)^2 U_n = (n+1)^2 U_n\,. \quad (4)$$

Notice that by $(\partial_x\omega)f$ we mean $(\omega f)'$.

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1.1 Spaces T^s and U^s

1.1.1 Definitions

Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} f^2(x)\sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_{\omega}$ the inner products in $L^2_{\frac{1}{\omega}}$ and L^2_{ω} respectively. The Chebyshev polynomials satisfy

$$\langle T_n, T_m \rangle_{\frac{1}{\omega}} = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } m = n = 0 \\ \pi/2 \text{ otherwise} \end{cases} \tag{5}$$

and

$$\langle U_n, U_m \rangle_{\omega} = \begin{cases} 0 \text{ if } n \neq m \\ \pi/2 \text{ otherwise} \end{cases} \tag{6}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{7}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by

$$\hat{u}_n := \begin{cases} \dfrac{2}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x)T_n(x)}{\sqrt{1-x^2}} dx & \text{if } n \neq 0\,, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^{1} \frac{u(x)}{\sqrt{1-x^2}} dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^{1} \frac{u^2(x)}{\sqrt{1-x^2}} dx = \frac{\pi \hat{u}_0^2}{2} + \pi \sum_{n=1}^{+\infty} \hat{u}_n^2\,.$$

When u is furthermore a smooth function, the series (7) converges uniformly to u. Similarly, any function $v \in L^2_{\omega}$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^{1} v(x) U_n(x) \sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^{1} v^2(x)\sqrt{1-x^2}dx = \frac{\pi}{2}\sum_{n=0}^{+\infty} \check{v}_n^2\,.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we may define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \,\middle|\, \sum_{n=0}^{+\infty} (1+n^2)^s \hat{u}_n^2 < +\infty \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u, v \rangle_{T^s} = \frac{\pi}{2}\hat{u}_0\hat{v}_0 + \pi \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty} n^{2s} |\hat{u}_n|^2\,.$$

We denote by T^∞ the Fréchet space $T^\infty := \cap_{s\in\mathbb{R}} T^s$, and by $T^{-\infty}$ the set of continuous linear forms on T^∞. For $l \in T^{-\infty}$, we note $\hat{l}_n = l(T_n)$, so that for $u \in T^\infty$,

$$l(u) = \frac{\pi}{2}\hat{l}_0\hat{u}_0 + \pi \sum_{n=1}^{+\infty} \hat{l}_n \hat{u}_n\,.$$

We choose to identify the dual of $L^2_{\frac{1}{\omega}}$ to itself using the previous bilinear form. With this identification, any element of T^s with $s \geq 0$ can also be seen as an element of $T^{-\infty}$. Furthermore, the space T^{-s} can be defined for all $s \geq 0$ as

$$T^{-s} = \left\{ u \in T^{-\infty} \,\middle|\, \sum_{n=0}^{+\infty} (1+n^2)^{-s} \hat{u}_n^2 < \infty \right\}.$$

Using the former identification T^{-s} becomes the dual of T^s. For $s < t$, the inclusion $T^s \subset T^t$ is compact.

Remark 1. *The spaces T^n correspond, up to a variable change, to the spaces H^n_e defined in [1, 3, 16, 17] among other works, that is, the restriction of the usual Sobolev space H^n to even periodic functions, as stated in Lemma 7.*

In a similar fashion, we define the following spaces:

Definition 2. *For all $s \geq 0$, we set*

$$U^s = \left\{ u \in L^2_\omega \,\middle|\, \sum_{n=0}^{+\infty} (1+n^2)^s \check{u}_n^2 \right\}.$$

We extend as before the definition to negative indices by setting U^{-s} to be the dual of U^s for $s \geq 0$, this time with respect to the duality $\langle \cdot, \cdot \rangle_\omega$.

1.1.2 Basic properties

Obviously, for any real s, if $u \in T^s$ the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x)$$

converges to u in T^s. The same assertion holds for $u \in U^s$ when T_n is replaced by U_n. Therefore

Lemma 1. *$C^\infty([-1,1])$ is dense in T^s and U^s for all $s \in \mathbb{R}$.*

The polynomials T_n and U_n are connected by the following formulas:

$$\forall n \geq 2, \quad T_n(x) = \frac{1}{2}\left(U_n - U_{n-2}\right), \tag{8}$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}. \tag{9}$$

We deduce the following inclusions:

Lemma 2. *For all real s, $T^s \subset U^s$ and for all $s > 1/2$, $U^s \subset T^{s-1}$.*

Before starting the proof, we introduce the Cesàro operator C defined on $l^2(\mathbb{N}^*)$ by

$$(Cu)_n = \frac{1}{n}\sum_{k=1}^{n} u_k\,.$$

As is well-known, this is a linear continuous operator on $l^2(\mathbb{N}^*)$. Its adjoint

$$(C^* u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k}\,,$$

is therefore also continuous on $l^2(\mathbb{N}^*)$. In other words, for all $u \in l^2(\mathbb{N})$,

$$\sum_{n=1}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{u_k}{k}\right)^2 \leq C\sum_{k=1}^{+\infty} u_k^2\,.$$

Proof. The first property is immediate from (8). When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging, allowing to identify u to a function in $T^{-\infty}$, with, in view of (9),

$$\hat{u}_0 = 2\sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2\sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since $u \in U^s$, the sequence $\left((1+n^2)^{s/2}\,|\check{u}|\right)_{n\geq 1}$ is in $l^2(\mathbb{N}^*)$. Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the sequence $r_n := \left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}|\check{u}_k|\right)_{n\geq 0}$ is in $l^2(\mathbb{N})$. But

$$\begin{aligned}
\|u\|^2_{T^{s-1}} &= \sum_{n=0}^{+\infty}(1+n^2)^{s-1}|\hat{u}_n|^2 \\
&\leq 4\sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left(\sum_{k=n}^{+\infty}|\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}|\check{u}_k|\right)^2. \\
&= 4\|r_n\|^2_{l^2}.
\end{aligned}$$

□

One immediate consequence is that $T^\infty = U^\infty$. Moreover, we have the following result:

Lemma 3.

$$T^\infty = C^\infty([-1,1]).$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (3), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}}\int_{-1}^{1}\frac{(\omega\partial_x)^{2k}u(x)T_n(x)}{\omega(x)}dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, the function $(\omega\partial_x)^{2k}u$ is C^∞, and since $\|T_n\|_\infty = 1$, the integral is bounded independently of n. Thus, the coefficients $\hat{u}_n$ have a fast decay, proving that $C^\infty([-1,1]) \subset T^\infty$.

For the converse inclusion, if $u \in T^\infty$, the series

$$u(x) = \sum_{n=0}\hat{u}_nT_n(x)$$

is normally converging since $\|T_n\|_\infty = 1$, so u is a continuous function. This proves $T^\infty \subset C^0([-1,1])$. It suffices to show that $u' \in T^\infty$ and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty}nu_nU_{n-1}(x).$$

Therefore, u' is in $U^\infty = T^\infty$. This proves the result. □

Lemma 4. *For $s \leq \frac{1}{2}$, the functions of U^s cannot be identified to functions in $T^{-\infty}$.*

Proof. Assume by contradiction that the functions of $U^{\frac{1}{2}}$ can be identified to elements of $T^{-\infty}$. Then, there must exist a map I continuous from $U^{\frac{1}{2}}$ to $T^{-\infty}$ with the property

$$\forall u \in U^\infty, \quad Iu = u.$$

Now, let us consider for example the function u defined by $\check{u}_n = \frac{1}{n\ln(n)}$. Note that u is in $U^{1/2}$. Let $u_N = \sum_{n=0}^{N} \check{u}_n U_n$. Then (u_N) is a sequence of elements of U^∞ converging to u in $U^{1/2}$. By continuity of I, and since $Iu_N = u_N$, the sequence $(\langle u_N, T_0\rangle_{\frac{1}{\omega}})_{N\in\mathbb{N}}$ must converge with limit $\langle Iu, T_0\rangle$. This is not the case since

$$\langle u_N, T_0\rangle_{\frac{1}{\omega}} = \sum_{n=0}^{N} \check{u}_n \langle U_n, T_0\rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2k\ln(2k)}$$

is diverging. The result for $s \leq \frac{1}{2}$ comes from the fact that $U^{\frac{1}{2}} \subset U^s$ for $s \leq \frac{1}{2}$. □

Two natural derivation operators arise in our context, that give another link between T^s and U^s. They are given by the identities

$$\partial_x T_n = nU_{n-1}\,, \tag{10}$$

$$-\omega\partial_x\omega U_n = (n+1)T_{n+1}\,. \tag{11}$$

The first one is obtained for example from the trigonometric definition of T_n. This combined with $-(\omega\partial_x)^2 T_n = n^2 T_n$ gives the second identity.

Definition 3. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s defined as*

$$\forall v \in U^\infty, \quad \langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}}\,.$$

In a similar fashion, the operator $\omega\partial_x\omega$ can be extended into a continuous map from U^{s+1} to T^s defined as

$$\forall v \in T^\infty, \quad \langle \omega\partial_x\omega u, v\rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v\rangle_\omega\,.$$

Proof. Using the identities (10) and (11), one can check that the formulas indeed extend the usual definition of the two operators for smooth functions. We now show that the map ∂_x extended this way is continuous from T^{s+1} to U^s. The definition

$$\forall v \in U^\infty, \langle \partial_x u, v\rangle := -\langle u, \omega\partial_x\omega v\rangle$$

gives a sense to $\partial_x u$ for all u in $T^{-\infty}$, as a duality $T^{-\infty} \times T^\infty$ product, because if $v \in U^\infty (= C^\infty)$, then $\omega\partial_x\omega v = (1-x^2)v' - xv$ also lies in $C^\infty (= T^\infty)$. It remains to check the announced continuity. Letting $w = \partial_x u$, we have, by definition, for all n

$$\check{w}_n = \langle w, U_n\rangle_\omega = -\langle u, \omega\partial_x\omega U_n\rangle_{\frac{1}{\omega}} = n\,\langle u, T_{n+1}\rangle_{\frac{1}{\omega}} = n\hat{u}_{n+1}$$

Obviously, this implies the announced continuity with

$$\|w\|_{U^s} \leq \|u\|_{T^{s+1}}\,.$$

The properties of $\omega\partial_x\omega$ on T^s are established in a similar way. □

The operator ∂_x is not continuous from T^s to T^{s-1}. However, the following result holds:

Corollary 1. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$. On the other hand, $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s and from U^{s+1} to U^s for all $s \in \mathbb{R}$.*

Proof. For the continuity of ∂_x from T^{s+2} to T^s, we se the continuity of ∂_x from T^{s+2} to U^{s+1} and then of the identity from U^{s+1} to T^s. For the continuity of ∂_x from U^{s+2} to U^s, we use the same arguments in reverse order.

On the other hand, we have, for $n \geq 2$,

$$\omega\partial_x\omega T_n = \omega\partial_x\omega \frac{U_n - U_{n-2}}{2} = \frac{(n+1)T_{n+1} - (n-1)T_{n-1}}{2}\,.$$

Therefore $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s. Finally, $\omega\partial_x\omega$ is continuous from U^{s+1} to T^s and the inclusion $T^s \subset U^s$ is continuous for all s. □

Lemma 5. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and there exists a constant C such that for all $x \in [-1,1]$,*

$$|u(x)| \leq C \left\|u\right\|_{T^{1/2+\varepsilon}}\,.$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$|u(x)| \leq C \left\|u\right\|_{U^{3/2+\varepsilon}}\,.$$

Proof. Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\left\|T_n\right\|_{L^\infty} = 1$. Cauchy-Schwarz's inequality then yields

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}}\left\|u\right\|_{T^{\frac{1}{2}+\varepsilon}}\,.$$

For the second statement, we use the inclusion $U^s \subset T^{s-1}$ valid for $s > 1/2$, as established in Lemma 2. □

1.1.3 Characterization of T^n and U^n.

In this section, we provide a characterization of the spaces T^s and U^s in terms of L^2 norms of the derivatives. In what follows, $H^s_p(0,T)$ denotes the space of functions in $H^s(\mathbb{T}_T)$, where $\mathbb{T}_T$ is the torus $\mathbb{R}/(T\mathbb{Z})$, $H^s_e(0,T)$ (resp. $H^s_o(0,T)$) is the set of even (resp. odd) functions if $H^s_p(0,T)$.

Lemma 6. *The operator $\omega\partial_x$ has a continuous extension from T^1 to T^0. Similarly, the operator $\partial_x\omega$ has a continuous extension from U^1 to U^0.*

Proof. Obviously, the operator ω maps $L^2_\omega = U^0$ to $L^2_{\frac{1}{\omega}} = T^0$. This is in fact a bijective isometry with inverse $\frac{1}{\omega}$. Since ∂_x is continuous from T^1 to U^0, we have the announced continuity of $\omega\partial_x$. For the second part, we write

$$\partial_x\omega = \frac{1}{\omega}\left(\omega\partial_x\omega\right).$$

Where $\omega\partial_x\omega$ is continuous from U^1 to T^0, and the multiplication by $\frac{1}{\omega}$ is continuous from T^0 to U^0. □

We can now state the main result of this paragraph. For a function u defined on $[-1,1]$, we denote by Cu the function defined on $\mathbb{T}_{2\pi}$ by

$$Cu(\theta) = u(\cos(\theta))$$

and by Su the function defined as

$$Su(\theta) := \sin(\theta)Cu(\theta).$$

Lemma 7. *A function u belongs to the space T^n if and only if $u = \tilde{u} \circ \arccos$ for some function $\tilde{u} \in H_e^n(0, 2\pi)$. In this case, $Cu = \tilde{u}$ and*

$$\|u\|_{T_n} \sim \|Cu\|_{H^n} \text{ and } |u|_{T^n} \sim |Cu|_{H^n}.$$

Similarly, u belongs to the space U^n if and only if $u = \frac{1}{\sqrt{1-x^2}}\tilde{u} \circ \arccos$ for some function $\tilde{u} \in H_o^n(0, 2\pi)$. In this case, $Su = \tilde{u}$ and

$$\|u\|_{U_n} \sim |Su|_{H^n}.$$

Moreover, if $u \in T^n$, then $(\omega\partial_x)^n u$ is in $L^2_{\frac{1}{\omega}}$ and

$$|u|^2_{T^n} \sim \int_{-1}^{1} \frac{((\omega\partial_x)^n u)^2}{\omega}.$$

Similarly, if $u \in U^n$, then $(\partial_x\omega)^n u \in L^2_\omega$ and

$$\|u\|_{U_n} = \int_{-1}^{1} \omega((\partial_x\omega)^n u)^2.$$

Proof. The first two equivalences stem from the fact that

$$\hat{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Cu(\theta)\cos(n\theta), \quad \check{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Su(\theta)\sin((n+1)\theta)d\theta,$$

which can be verified by using the change of variables $x = \cos\theta$ in the definitions of $\hat{u}_n$ and $\check{u}_n$. Now, let us show that if $u \in T^n$, then $(\omega\partial_x)^n$ is in $L^2_{\frac{1}{\omega}}$. The operator $(\omega\partial_x)^2$ is continuous from T^s to T^{s-2} for all real s which implies the result if n is even. If n is odd, say $n = 2k+1$, we write $(\omega\partial_x)((\omega\partial_x)^2)^k$, and conclude using Lemma 6. The same kind of proof also shows that if $u \in U^n$, $(\partial_x\omega)^n u \in L^2_\omega$. The rest of the proof can be performed by computing the quantities for functions in $C^\infty([-1,1])$, performing integrations by parts and concluding with the density of T^∞ in T^s and U^s. $\square$

1.2 Periodic pseudo-differential operators

On the family of spaces $H_p^s(0,T)$, a class of periodic pseudo-differential operators (PPDO) has been introduced in [15], with symbolic calculus. A PPDO on $H_p^s(0,T)$ of order p is an operator of the form

$$A : u \in H_p^s(0,T) \mapsto \sum_{n\in\mathbb{Z}} \sigma_A(t,n)\hat{u}_n e^{\frac{2in\pi t}{T}}.$$

for a "prolongated symbol" $\sigma_A \in C^\infty(\mathbb{T}_T \times \mathbb{R})$ satisfying

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \quad \left|\partial_t^j \partial_\xi^k \sigma_A(t,\xi)\right| \leq C_{j,k}(1+|\xi|)^{p-k}\,. \tag{12}$$

Here, $\hat{u}_n = \frac{1}{T}\int_0^T u(t)e^{-i\frac{2n\pi t}{T}}dt$ are the usual Fourier coefficients of u and

$$\partial_t := \frac{T}{2i\pi}\frac{\partial}{\partial t}, \quad \partial_\xi := \frac{T}{2i\pi}\frac{\partial}{\partial \xi},$$

with for $j \geq 1$, $\partial_t^{j+1} = \partial_t \partial_t^j$, and $\partial_\xi^{j+1} = \partial_\xi \partial_\xi^j$. The class of symbols that satisfy (12) is denoted by Σ^α, and $\Sigma^{-\infty} := \cup_{\alpha \in \mathbb{Z}} \Sigma^\alpha$. The operator corresponding to a symbol σ is denoted by $Op(\sigma)$ and the set of PPDO of order α is denoted by $Op(\Sigma^\alpha)$.

The prolongated symbol is not unique but is determined uniquely at the integer values of ξ by

$$\sigma_A(t,n) = e^{-\frac{2in\pi t}{T}} A(e^{\frac{2in\pi t}{T}})\,. \tag{13}$$

as shown in [15], justifying the terminology of "prolongated symbol". $A \in Op(\Sigma^\alpha)$ if and only if those values satisfy

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \left|\partial_t^j \Delta_n^k \sigma_A(t,n)\right| \leq C_{j,k}(1+|n|)^{\alpha-k}\,,$$

where $\Delta_n\phi(t,n) = \phi(t,n+1) - \phi(t,n)$ and for $k \geq 1$, $\Delta^{k+1}\phi = \Delta(\Delta^k\phi)$. That is, if the symbol defined in (13) satisfies this condition, then there exists a prolongated symbol satisfying (12). Because of this, we will abusively write $\sigma \in \Sigma^p$ for a function $\sigma(t,n)$ that can be prolongated to a function $\tilde{\sigma}(t,\xi) \in \Sigma^p$. An operator in $Op(\Sigma^\alpha)$ maps continuously $H_p^s(0,T)$ to $H_p^{s+\alpha}(0,T)$ for all $s \in \mathbb{R}$. The composition of two operators in $Op(\Sigma^\alpha)$ and $Op(\Sigma^\beta)$ gives rise to an operator in $Op(\Sigma^{\alpha+\beta})$. If two symbols a and b in $\Sigma^{-\infty}$ satisfy $a - b \in \Sigma^\alpha$, we write $a = b + \Sigma^\alpha$.

Definition 4. *Let $a \in \Sigma^{-\infty}$. If there exists a sequence of reals $(p_j)_{j\in\mathbb{N}}$ such that $p_j < p_{j+1}$ and a sequence of symbols $a_j \in \Sigma^{p_j}$ such that for all N,*

$$a = \sum_{i=0}^{N} a_i + \Sigma^{p_j+1}\,,$$

we then write

$$a = \sum_{i=0}^{+\infty} a_j\,.$$

This is called an asymptotic expansion of the symbol a.

The composition of two periodic pseudi-differential operators A and B with symbols σ_A and σ_B has a symbol σ_C with the following asymptotic expansion

$$\sigma_C(t,\xi) = \sum_{j=0}^{+\infty} \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \sigma_A(t,\xi)\partial_t^j \sigma_B(t,\xi)\,. \tag{14}$$

We will use the following result, see [15]:

Theorem 1. *Consider an integral operator K of the form*

$$K : u \mapsto \frac{1}{T} \int_0^T a(t,s)\kappa(t-s)u(s)ds.$$

where a is T-periodic and C^∞ in both arguments and κ is a T-periodic distribution. Assume that the Fourier coefficients $\hat{\kappa}(n)$ of κ can be prolonged to a function $\hat{\kappa}(\xi)$ on $\mathbb{R}$ such that

$$\forall k \in \mathbb{N}, \exists C_k > 0 : \quad \left|\partial_\xi^k \hat{\kappa}(\xi)\right| \le C_k (1+|\xi|)^{\alpha - k}.$$

for some α. Then K is in $Op(\Sigma^\alpha)$ with a symbol satisfying the asymptotic expansion

$$\sigma_K(t,\xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \hat{\kappa}(\xi) \partial_s^j a(t,s)_{|s=t}.$$

In particular, taking $\kappa = 1$, we see that for all functions $a \in C^\infty(\mathbb{T}_T^2)$

$$Ku = \frac{1}{T} \int_0^T a(t,s)u(s)ds$$

is in $Op\left(\Sigma^{-\infty}\right)$.

1.3 Pseudo-differential operators on T^s.

Lemma 8. *Let A a PPDO that stabilizes the set of smooth real even functions. Then A coincides on this set with the operator B with symbol given by*

$$\sigma_B(n,\theta) = \frac{\sigma_A(n,\theta) + \sigma_A(-n,-\theta)}{2}.$$

Moreover, σ_B admits the following decomposition:

$$\sigma_B(n,\theta) = a_1(n,\cos\theta) + i\sin(\theta) a_2(n,\cos\theta)$$

with

$$\begin{aligned} a_1(n,x) &= \frac{\sigma_B(n,\arccos(x)) + \sigma_B(-n,\arccos(x))}{2} \\ a_2(n,x) &= \frac{\sigma_B(n,\arccos(x)) - \sigma_B(-n,\arccos(x))}{2i\sqrt{1-x^2}} \end{aligned}$$

and a_1 and a_2 are real and C^∞.

Proof. Since A stabilizes the set of real even functions, for all even function u we have

$$Au(\theta) = Bu(\theta) := \frac{Au(\theta) + Au(-\theta)}{2}.$$

We have

$$Bu(\theta) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \sigma_A(n,\theta) \hat{u}_n e^{in\theta} + \frac{1}{2} \sum_{n\in\mathbb{Z}} \sigma_A(n,-\theta) \hat{u}_n e^{-in\theta},$$

so the symbol of B is $\sigma_B(n,\theta) = \frac{\sigma_A(n,\theta)+\sigma_A(-n,-\theta)}{2}$. In particular, it satisfies the following symmetry:

$$\sigma_B(-n,-\theta) = \sigma_B(n,\theta)\,.$$

We write $\sigma_B(n,\theta) = f_B(n,\theta) + g_B(n,\theta)$ where $f_B(n,\theta) = \frac{\sigma_B(n,\theta)+\sigma_B(-n,\theta)}{2}$ and $g_B(n,\theta) = \frac{\sigma_B(n,\theta)-\sigma_B(-n,\theta)}{2}$. Notice that f_B (resp. g_B) is even (resp. odd) in both θ and n. For a real even function u we have $\hat{u}_n = \hat{u}_{-n}$, thus

$$Bu(\theta) = f_B(0,\theta)\hat{u}_0 + 2\sum_{n=1}^{+\infty} f_B(n,\theta)\hat{u}_n\cos(n\theta) + 2i\sum_{n=1}^{+\infty} g_B(n,\theta)\hat{u}_n\sin(n\theta)\,.$$

Since Bu must be real and since $\hat{u}_n$ is real, we find that f_B is a real and g_B is imaginary. The functions a_1 and a_2 defined in the lemma satisfy

$$a_1(n,x) = f_B(n,\arccos,x), \quad a_2(n,x) = \frac{g_B(n,\arccos(x))}{i\sqrt{1-x^2}}\,.$$

so they are both real. By Lemma 7, they lie in T^∞ since f_B (resp. g_B) is a smooth even (resp. odd) function and $f_B(n,\theta) = a_1(n,\cos\theta)$, while $g_B(n,\theta) = i\sin\theta a_2(n,\cos\theta)$. Thus

$$\sigma_B(n,\theta) = a_1(n,\cos\theta) + i\sin\theta a_2(n,\cos\theta)\,.$$

$\square$

We use this result to transport the notion of periodic pseudo-differential operators by the change of variable $x = \cos\theta$. Let A an operator on $T^{-\infty}$, assume that there exists a couple of smooth functions a_1 and a_2 in $C^\infty([-1,1]\times\mathbb{N})$ such that for all $n\in\mathbb{N}$,

$$AT_n = a_1(x,n)T_n - \omega^2 a_2(x,n)U_{n-1}\,.$$

Such a (non-unique) couple of functions is called a pair of symbols of A. For $n\in\mathbb{Z}$ and $\theta\in[0,2\pi]$, define the symbol $\tilde{\sigma}(a_1,a_2)$ by

$$\tilde{\sigma}(a_1,a_2)(\theta,n) = a_1(\cos\theta,|n|) + i\sin\theta\,\mathrm{sign}(n)a_2(\cos\theta,|n|)\,.$$

We say that $(a_1,a_2)\in S^p$ if $\tilde{\sigma}(a_1,a_2)\in\Sigma^p$, and $S^{-\infty} := \cup_{p\in\mathbb{Z}} S^p$. The operator defined by a pair of symbols (a_1,a_2) is denoted by $Op(a_1,a_2)$ and the set of pseudo-differential operatos of order p in $T^{-\infty}$ by $Op(S^p)$. Then, we have the following properties of $Op(S^p)$:

Lemma 9. *If $A\in Op(S^p)$, then letting $\tilde{A} := Op(\tilde{\sigma}_A)$, for $u\in T^s$, we have*

$$C(Au) = \tilde{A}(Cu)\,,$$

where we recall that for any function v in T^s, $Cv(\theta) = v(\cos(\theta))$. Reciprocally, if A is a linear operator that maps T^∞ to itself such that $CA = \tilde{A}C$ where $\tilde{A}$ is a PPDO of order p, then A is in $Op(S^p)$ and if $\sigma_{\tilde{A}}$ is the symbol of $\tilde{A}$, the $A = Op(a_1,a_2)$ with the functions a_1 and a_2 of Lemma 8.

Proof. For the direct result, by linearity, it suffices to show the equality for $u = T_n$ for some $n \in \mathbb{N}$. In this case, we have $Cu(\theta) = T_n(\cos(\theta)) = \cos(n\theta)$.

$$\tilde{A}\,(Cu)\,(\theta) = \frac{\tilde{A}e^{in\theta} + \tilde{A}e^{-in\theta}}{2}$$

which gives, by definition of $\tilde{A}$ and using the determination of the symbol (13),

$$\tilde{A}\,(Cu)\,(\theta) = \tilde{\sigma}_A(\theta, n)e^{in\theta} + \tilde{\sigma}_A(\theta, -n)e^{-in\theta}\,.$$

By definition of $\tilde{\sigma}_A$, this gives

$$\tilde{A}\,(Cu)\,(\theta) = a_1(\cos\theta, n)\cos(n\theta) - \sin\theta a_2(\cos\theta, n)\sin(n\theta).$$

Using the identities $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n\theta) = \sin\theta U_{n-1}(\cos\theta)$ we obtain

$$\tilde{A}\,(Cu)\,(\theta) = a_1(\cos\theta, n)T_n(\cos\theta) - (1 - \cos^2\theta)a_2(\cos\theta, n)U_{n-1}(\cos\theta)\,,$$

as claimed. For the converse result, assume that $CA = \tilde{A}C$ where $\tilde{A}$ is a PPDO of order p. Then $\tilde{A}$ stabilizes the set of smooth real even functions since $CAu(\theta) = Au(\cos\theta)$ is real and even and A maps smooth functions to smooth functions. Let B, a_1 and a_2 be defined as in Lemma 8 for the operator $\tilde{A}$. Using the same calculations as before, we find

$$A(T_n)(\cos\theta) = a_1(n, \cos\theta)T_n(\cos\theta) - \omega^2 a_2(n, \cos\theta)U_{n-1}(\cos\theta)\,,$$

that is $A = Op(a_1, a_2)$. Since $\sigma_B \in \Sigma^p$, we get $(a_1, a_2) \in S^p$ as claimed. $\square$

Definition 5. *Let A and B in $Op(S^{-\infty})$. If $A - B \in Op(S^\alpha)$, we write*

$$A = B + T_\alpha\,.$$

Furthermore, if A and B are such that $A - B$ is continuous from U^s to $U^{s-\alpha}$ for all $s \in \mathbb{R}$, we write

$$A = B + U_\alpha\,.$$

2 Application to preconditioning for the Helmholtz scattering problem

In this section, we apply the analytical tools introduced in the previous section to the study of the Helmholtz scattering problems. The two main results are **??** and **??**. We start by introducing the notations.

2.1 The scattering problem for an open curve

Let Γ be a smooth non-intersecting open curve in $\mathbb{R}^2$, with a smooth unit normal vector $n_\Gamma : \Gamma \to \mathbb{R}^2$ and let $k \geq 0$ the wave number. We seek a solution to the two problems

$$-\Delta u_i - k^2 u_i = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma, \quad i = 1, 2 \tag{15}$$

with the following additional conditions

- Dirichlet or Neumann boundary conditions, respectively

$$u_1 = u_D, \text{ and } \frac{\partial u_2}{\partial n} = u_N \text{ on } \Gamma \tag{16}$$

  where $\frac{\partial u}{\partial n} = n_\Gamma \cdot \nabla u$.

- Suitable decay at infinity, given for $k > 0$ by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{17}$$

  with $r = |x|$ for $x \in \mathbb{R}^2$.

When $k = 0$, the radiation condition must be replaced by an appropriate decay of u and ∇u at infinity, see for example [13, 14], or [9, Chap. 7] Vérifier le chapitre et la page. Existence and uniqueness results are available for those problems, but the solutions fail to be regular even with smooth data u_D and u_N. More precisely, let $\lambda = \left[\frac{\partial u_1}{\partial n}\right]_\Gamma$ and $\mu = [u_2]_\Gamma$ where $[\cdot]_\Gamma$ refers to the jump of a quantity across Γ, we have the following result.

Theorem 2. *(see e.g. [10,13,14]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (15,16,17) both possess a unique solution $u_i \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ, λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

while μ satisfies

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$.

For the definition of Sobolev spaces on smooth open curves, we follow [9] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma}) \right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential We define the single-layer potential by

$$\mathcal{S}_k \lambda(x) = \int_\Gamma G_k(x - y)\lambda(y) d\sigma(y) \tag{18}$$

where G_k is the Green's function

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi} \ln |z|, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{19}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here H_0 is the Hankel function of the first kind. For $k > 0$, the solution u_1 to the Dirichlet problem admits the representation

$$u_1 = \mathcal{S}_k \lambda \tag{20}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the jump of the normal derivative of u_1 across Γ and is the unique solution to

$$S_k \lambda = u_D \,. \tag{21}$$

Here, $S_k := \gamma \mathcal{S}_k$ where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. When $k = 0$, the computation of u_1 also involves the resolution of (21) but some subtleties arise in the representation of u_1 by (20). On this topic, see [13, Theorem 1.4].

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k \mu(x) = \int_\Gamma n_\Gamma(y) \cdot \nabla G_k(x-y) \mu(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \Gamma$$

for any smooth function μ defined on Γ. The normal derivative of $\mathcal{D}_k \mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \frac{\partial}{\partial_n} \mathcal{D}_k$. This operator admits the following representation for $x \in \Gamma$

$$N_k \mu(x) = \lim_{\varepsilon \to 0^+} \frac{\partial}{\partial \varepsilon} \int_\Gamma n_\Gamma(y) \cdot \nabla G(x + \varepsilon n_\Gamma(x) - y) \mu(y) d\sigma(y). \tag{22}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned} \langle N_k \mu, \nu \rangle_{L^2(\Gamma)} = \int_{\Gamma \times \Gamma} & G_k(x-y) \partial_\tau \mu(x) \partial_\tau \nu(y) \\ & - k^2 G_k(x-y) \mu(x) \nu(y) n_\Gamma(x) \cdot n_\Gamma(y) d\sigma(x) d\sigma(y) \,. \end{aligned} \tag{23}$$

where ∂_τ denotes the tangential derivative on Γ defined for $u : \Gamma \to \mathbb{R}$ as follows. Let $r : [a, b] \to \Gamma$ a smooth parametrization such that $|r'(x)| \neq 0$ for $x \in [a, b]$. Then $\partial_\tau u := \frac{1}{|r'(x)|} \frac{d}{dx} u(r(x))$. It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u_2 to the Neumann problem can be written as

$$u_2 = \mathcal{D}_k \mu \tag{24}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ is the jump of u_2 across Γ and is the unique solution to

$$N_k \mu = u_N \,. \tag{25}$$

Weighted layer potentials. Theorem 2 implies that even if u_D and u_N are smooth, the solutions λ and μ to the corresponding integral equations have singularities. As a remedy, we consider weighted versions of those integral operators. Let ω_Γ the operator $u \mapsto \omega_\Gamma(x) u(x)$ such that $\omega_\Gamma(r(x)) := \frac{|\Gamma|}{2} \omega(x)$ where $|\Gamma|$ is the length of Γ, $\omega(x) = \sqrt{1 - x^2}$ as in the previous section, and

$r:[-1,1]\to\Gamma$ is a smooth parametrization of Γ with constant speed $|r'| = \frac{|\Gamma|}{2}$. Let

$$R : L^2(\Gamma) \longrightarrow L^2(-1,1)$$

defined by

$$Ru(x) = u(r(x))$$

Also let $n = Rn_\Gamma$. We have

$$\omega_\Gamma = \frac{|\Gamma|}{2} R^{-1}\omega R\,, \quad \frac{1}{\omega_\Gamma} = \frac{2}{|\Gamma|} R^{-1}\frac{1}{\omega} R\,, \quad \partial_\tau = \frac{2}{|\Gamma|} R^{-1}\partial_x R\,.$$

For $u, v \in L^2(\Gamma)$,

$$\langle u, v\rangle_{L^2(\Gamma)} = \frac{|\Gamma|^2}{4} \langle Ru, Rv\rangle_{L^2(-1,1)}\ .$$

Definition 6. *The weighted layer potentials S_{k,ω_Γ} and N_{k,ω_Γ} are defined as follows:*

$$S_{k,\omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}, \quad N_{k,\omega_\Gamma} := N_k \omega_\Gamma\,.$$

Solving the integral equations (21) and (25), is equivalent to solving

$$\begin{aligned} S_{k,\omega_\Gamma}\alpha &= u_D \\ N_{k,\omega_\Gamma}\beta &= u_N \end{aligned}$$

and letting $\lambda = \frac{\alpha}{\omega_\Gamma}$, $\mu = \omega_\Gamma u_N$. Those weighted integral operators appear in many related works such as [3, 6, 7]. We also define the rescaled parametric representations $S_{k,\omega}$ and $N_{k,\omega}$ from $[-1,1]$ to $\mathbb{C}$ by $S_{k,\omega} := \frac{2}{|\Gamma|} R S_{k,\omega_\Gamma} R^{-1}$ and $N_{k,\omega} := \frac{|\Gamma|}{2} R N_{k,\omega_\Gamma} R^{-1}$.

Lemma 10. *The operator $N_{k,\omega}$ satisfies, for all $\beta, \beta' \in C^\infty([-1,1])$*

$$\begin{aligned} \langle N_{k,\omega}\beta, \beta'\rangle_\omega &= \langle S_{k,\omega}(\omega\partial_x\omega)\beta, (\omega\partial_x\omega)\beta'\rangle_{\frac{1}{\omega}} \\ &\quad - k^2\frac{|\Gamma|^2}{2}\int_{-1}^{1} G_k(r(x)-r(y))\omega(x)\beta(x)\omega(y)\beta'(y) n(x)\cdot n(y) dx dy \end{aligned}$$

Proof. By definition, we have

$$\begin{aligned} \langle N_{k,\omega}\beta, \beta'\rangle_\omega &= \langle N_{k,\omega}\beta, \omega\beta'\rangle_{L^2(-1,1)} \\ &= \left(\frac{2}{|\Gamma|}\right)^2 \left\langle R^{-1}N_{k,\omega}\beta, R^{-1}\omega\beta'\right\rangle_{L^2(\Gamma)} \\ &= \left(\frac{2}{|\Gamma|}\right)^2 \left\langle N_k\omega_\Gamma R^{-1}\beta, \omega_\Gamma R^{-1}\beta'\right\rangle_{L^2(\Gamma)} \end{aligned}$$

which gives, using the identity (23),

$$\langle N_{k,\omega}\beta, \beta'\rangle_\omega = \left(\frac{2}{|\Gamma|}\right)^2 \left(I_1 - k^2 I_2\right)\,,$$

where

$$I_1 = \left\langle S_k\partial_\tau\omega_\Gamma R^{-1}\beta, \partial_\tau\omega_\Gamma R^{-1}\beta'\right\rangle_{L^2(\Gamma)}$$

and

$$I_2 = \int_{\Gamma\times\Gamma} G_k(x-y)\omega_\Gamma(x)\beta(r^{-1}(x))\omega_\Gamma(y)\beta'(r^{-1}(y))n_\Gamma(x)\cdot n_\Gamma(y)dxdy\,.$$

Using the parametrization r of Γ, we can rewrite

$$I_2 = \left(\frac{|\Gamma|}{2}\right)^4 \int_{-1}^{1} G_k(r(x)-r(y))\omega(x)\beta(x)\omega(y)\beta(y)n(x)\cdot n(y)dxdy$$

For I_1, we write

$$I_1 = \frac{|\Gamma|^2}{4}\left\langle RS_k\partial_\tau\omega_\Gamma R^{-1}\beta, R\partial_\tau\omega_\Gamma R^{-1}\beta'\right\rangle_{L^2(-1,1)}$$

And we have

$$\begin{aligned} RS_k\partial_\tau\omega_\Gamma R^{-1} &= RS_k\frac{1}{\omega_\Gamma}R^{-1}R\omega_\Gamma R^{-1}R\partial_\tau R^{-1}R\omega_\Gamma R^{-1} \\ &= S_{k,\omega}\omega\partial_x\omega \end{aligned}$$

similarly, $R\partial\tau\omega_\Gamma R^{-1} = \partial_x\omega$. Thus,

$$I_1 = \frac{|\Gamma|^2}{2}\left\langle S_{k,\omega}(\omega\partial_x\omega)\beta, \omega\partial_x\omega\beta'\right\rangle_{\frac{1}{\omega}}\,,$$

and the identity is proved. □

2.2 Operators S_ω and N_ω on the flat segment

In this section, the wavenumber is equal to 0 and the curve Γ is the flat segment $(-1,1)\times 0$. The parametrization r is the constant function equal to 1, $\partial_\tau = \partial_x$ and $\omega_\Gamma = \omega$. In this simple context, S_ω and N_ω have elementary properties that allow us to characterize T^s and U^s for $s = \pm\frac{1}{2}$.

Single layer potential The operator S_ω takes the form

$$S_\omega\alpha(x) = \int_{-1}^{1}\frac{\ln|x-y|\,\alpha(y)}{\sqrt{1-y^2}}dy\,.$$

There holds

$$S_\omega T_n = \sigma_n T_n \tag{26}$$

where

$$\sigma_n = \begin{cases} \dfrac{\ln(2)}{2} & \text{if } n=0 \\ \dfrac{1}{2n} & \text{otherwise.} \end{cases}.$$

In particular S_ω is in the class $Op(S_T^{-1})$. As a consequence, S_ω maps T^∞ to itself, so the image of a smooth function is a smooth function. This gives the possibility to derive an explicit inverse of S_ω as the square root of a local operator, as a prelude to the main theorem Theorem 3. Recall that

$$-(\omega\partial_x)^2T_n = n^2T_n$$

the operator $-(\omega\partial_x)^2$ is thus in $Op(S_T^2)$ and

$$-(\omega\partial_x)^2 S_\omega^2 = \frac{I_d}{4} + T_\infty. \tag{27}$$

This shows that $\sqrt{-(\omega\partial_x)^2}$ and S_ω are inverse operators modulo smoothing operators and that $\sqrt{-(\omega\partial_x)^2}$ can thus be used as an efficient preconditioner for S_ω. We now proceed to show the following characterization of $T^{-1/2}$ and $T^{1/2}$, which is a reformulation of a result obtained in [6] Ou bien [5], vérifier et citer le thm.

Lemma 11. *We have* $T^{-1/2} = \omega\tilde{H}^{-1/2}(-1,1)$ *and for all* $u \in \tilde{H}^{-1/2}(-1,1)$,

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}\,.$$

Moreover, $T^{1/2} = H^{1/2}(-1,1)$ *and*

$$\|u\|_{H^{1/2}} = \|u\|_{T^{1/2}}$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the (unweighted) single-layer operator S is positive and bounded from below on $\tilde{H}^{-1/2}(-1,1)$, (see [9] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1,1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle Su, u\rangle}.$$

On the other hand, the explicit expression (26) imply that if $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_\omega\alpha, \alpha\rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since $\alpha = \omega u$, $\langle S_\omega\alpha, \alpha\rangle_{\frac{1}{\omega}} = \langle Su, u\rangle$. This proves the first result. For the second result, we know that, $(H^{1/2}(-1,1))' = \tilde{H}^{-1/2}(-1,1)$ (taking the dual with respect to the usual L^2 duality, [8] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v\neq 0} \frac{\langle u, v\rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}}\,.$$

According to the previous result, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v\rangle = \langle u, \alpha\rangle_\omega$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha\neq 0} \frac{\langle u, \alpha\rangle_\omega}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle\cdot,\cdot\rangle_\omega$.

□

Hypersingular operator For $k = 0$ and when $\Gamma = (-1,1)\times\{0\}$, the identity (2.1) becomes

$$\langle N_\omega\beta, \beta'\rangle_\omega = \langle S_\omega(\omega\partial_x\omega)\beta, (\omega\partial_x\omega)\beta'\rangle_{\frac{1}{\omega}}$$

Noticing that $(\omega\partial_x\omega)U_n = -(n+1)T_{n+1}$, we have for all $n \neq m$

$$\langle N_\omega U_n, U_m\rangle_\omega = 0\,.$$

Therefore, we have

$$N_\omega U_n = \nu_n U_n$$

with $\nu_n \|U_n\|_\omega^2 = (n+1)^2\sigma_{n+1}\|T_{n+1}\|_{\frac{1}{\omega}}^2$, that is, $\nu_n = \frac{(n+1)}{2}$. Thus N_ω maps U^s to U^{s-1} for all $s \in \mathbb{R}$. In particular, N_ω maps smooth functions to smooth functions. Here again, we can express the inverse of N_ω in the form of the square root of a local operator. Recall that

$$-(\partial_x\omega)^2 U_n = (n+1)^2 U_n\,,$$

thus,

$$N_\omega = \frac{1}{2}\sqrt{-(\partial_x\omega)^2}\,. \tag{28}$$

As before, we obtain a characterization of U^s for $s = \pm\frac{1}{2}$ from the previous formula:

Lemma 12. *TWe have $U^{1/2} = \frac{1}{\omega}\tilde{H}^{1/2}(-1,1)$ and for all $u \in \tilde{H}^{1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{1/2}} \sim \left\|\frac{u}{\omega}\right\|_{U^{1/2}}\,.$$

Moreover, $U^{-1/2} = H^{1/2}(-1,1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{U^{1/2}}\,.$$

Proof. It suffices to remark that

$$\|u\|_{\tilde{H}^{1/2}} \sim \sqrt{\left\langle N\omega\frac{u}{\omega}, \omega\frac{u}{\omega}\right\rangle} = \sqrt{\left\langle N_\omega\frac{u}{\omega}, \frac{u}{\omega}\right\rangle_\omega} \sim \left\|\frac{u}{\omega}\right\|_{U^{1/2}}\,.$$

The second equality follows from the same calculations that were done in Lemma 11, as well as the norm equivalence. ☐

In what follows, we show that (27) and (28) can be generalized to non-zero wavenumber k and arbitrary smooth and non-intersecting open curve Γ.

2.3 Non-flat arc and non-zero frequency

We first focus on the weighted single-layer operator problem with non-zero frequency, and establish the following result, announced in [2].

Theorem 3. *$S_{k,\omega}$ is in S^1 and*

$$\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right] S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + T_4.$$

Remark 2. *The previous result also implies that*

$$-(\omega_\Gamma\partial_\tau)^2 S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + R$$

where R is in $Op(S_T^2(\Gamma))$. This is also a compact perturbation of the identity. Nevertheless, we have $R = k^2\omega_\Gamma^2 S_{k,\omega_\Gamma}^2 + T_4$ and therefore, the term $k^2\omega^2 S_{k,\omega}^2$ is the leading first order correction accounting for the wavenumber. The inclusion of this term in the preconditioner leads to a drastic reduction of the number of GMRES iterations in numerical applications, as demonstrated in [2].

Proof. Our perturbation analysis hinges on the following property of the Hankel function:

$$H_0(z) = \frac{-1}{2\pi}\ln|z|J_0(z) + F_1(z^2) \tag{29}$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. One has

$$\left(S_{k,\omega}u\right)(x) = \int_{-1}^{1} H_0(k|r(x)-r(y)|)\frac{u(y)}{\omega(y)}dy\,.$$

Using the change of variables $x = \cos\theta$, $y = \cos\theta'$, we get

$$S_{k,\omega}u(\cos\theta) = \int_0^\pi H_0(k\,|r(\cos\theta)-r(\cos\theta')|)u(\cos(\theta))d\theta\,.$$

We can rewrite this using (29)

$$\begin{aligned} S_{k,\omega}u(\cos\theta) =& \frac{-1}{2\pi}\int_0^\pi \ln|\cos\theta-\cos\theta'|\,J_0(k\,|r(\cos\theta)-r(\cos\theta')|)Cu(\theta)d\theta \\ &+\int_0^\pi F_k(\cos\theta,\cos\theta')Cu(\theta)d\theta' \end{aligned}$$

where

$$F_2(x,y) = \ln\frac{|r(x)-r(y)|}{|x-y|} + F_1(k^2(x-y)^2)$$

is a C^∞ function. By parity, the second integral is equal to

$$\frac{1}{2}\int_{-\pi}^{\pi} F_2(\cos\theta,\cos\theta')Cu(\theta)d\theta.$$

By Theorem 1, this is of the form RCu where R is in $Op(\Sigma^{-\infty})$. For the first integral, we make the following classical manipulations. We first write $\cos\theta-\cos\theta' = -2\sin\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}$. Thus $\ln|\cos\theta-\cos\theta'| = \ln\left|\sqrt{2}\sin\frac{\theta+\theta'}{2}\right| + \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right|$. Using the change of variables $\theta\to-\theta$, we get

$$C\left(S_{k,\omega}u\right)(\theta) = \frac{-1}{2\pi}\int_{-\pi}^{\pi}\ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right|J_0(k\,|\cos\theta-\cos\theta'|)Cu(\theta')d\theta' + RCu\,.$$

Let $\kappa := t\mapsto -\frac{1}{2\pi}\ln\left|\sqrt{2}\sin\frac{t}{2}\right|$. It is well-known that $\hat{\kappa}(n) = \frac{1}{2n}$ for $n\neq 0$. We may prolonge this by $\kappa(\xi) = \frac{1}{2\xi}$ away from $\xi = 0$. Let $a(t,s) = J_0(k\,|r(\cos t)-r(\cos s)|)$, which is a smooth function. Applying Theorem 1, the operator

$$\tilde{S}_k u \mapsto \int_{-\pi}^{\pi}\kappa(t-s)a(t,s)u(s)ds$$

is in $Op(\Sigma^{-1})$ with its symbol satisfying, for $\xi>0$,

$$\sigma_{\tilde{S}_k}(t,\xi) = \frac{1}{2\xi}+\frac{\sin^2 t}{4\xi^3}+3i\frac{k^2\sin t\cos t}{4\xi^4}+\frac{4k^2(4\sin^2 t-3\cos^2 t)+3k^4\sin^4 t}{16\xi^5}+\Sigma^6\,.$$

Since $CS_{k,\omega} = \tilde{S}_kC$, we have $CS_{k,\omega}^2 = \tilde{S}_kCS_{k,\omega} = \tilde{S}_k^2C$. Applying symbolic calculus, the symbol of σ_2 of $\tilde{S}_k^2$ satisfies

$$\sigma_2 = \frac{1}{4\xi^2}+k^2\frac{\sin^2 t}{4\xi^4}+ik^2\frac{\sin t\cos t}{\xi^5}+k^2\frac{13\sin^2 t-11\cos^2 t}{8\xi^6}+k^4\frac{\sin^4(t)}{4\xi^6}+\Sigma^7\,.$$

We can now notice that for $u \in T^s$, we have

$$C\left[-(\omega\partial_x)^2 - k^2\omega^2\right]u = \left[-\frac{\partial^2}{\partial\theta^2} - k^2\sin^2\theta\right]Cu.$$

Thus

$$C\left[-(\omega\partial_x)^2 - k^2\omega^2\right]S_{k,\omega}^2 = \left[-\frac{\partial^2}{\partial\theta^2} - k^2\sin^2\theta\right]\tilde{S}_k^2 C$$

Of course, $\frac{-\partial^2}{\partial\theta^2} - k^2\sin^2$ is a PPDO with symbol $\sigma_\Delta(\theta,\xi) = \xi^2 - k^2\sin^2(\theta)$. We apply again symbolic calculus to find the symbol σ_3 of $\left[-\frac{\partial^2}{\partial\theta^2} - k^2\sin^2\theta\right]\tilde{S}_k^2$. We find

$$\sigma_3 = \frac{1}{4} + \frac{k^2}{8\xi^4} + \Sigma^6\,.$$

Using maple, we find that the order 5 and 7 terms are null and

$$\sigma_3 = \frac{1}{4} + \frac{k^2}{8\xi^4} + \frac{2k^4\sin^2(\theta) + k^2}{8\xi^6} + \Sigma^8\,.$$

$\square$

2.4 Neumann problem

Similarly, if we define $N_{k,\omega} := N_k\omega$, we have

Theorem 4.

$$N_{k,\omega}^2 = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] + U_2.$$

This result suggests $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]^{-1/2}$ as a candidate preconditioner for $N_{k,\omega}$.

2.5 Non-flat arc

In the more general case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, the results of the previous sections can be extended using again compact perturbations arguments. Essentially, in the decomposition Equation 29, x and y must be replaced by $r(x)$ and $r(y)$, where the function r is a smooth, constant-speed parametrisation of Γ defined on $[-1,1]$ and satisfying $|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4}|x-y|^2 + |x-y|^4 G(x,y)$ where $|\Gamma|$ is the length of Γ and G is a C^∞ function on $[-1,1]$. Let $\omega_\Gamma(x) = |\Gamma|\,\omega(x)$, ∂_τ the tangential derivative on Γ and $S_{k,\omega_\Gamma} := S_k\frac{1}{\omega_\Gamma}$.

Theorem 5. *One has $S_{k,\omega_\Gamma} \in S^{-1}$ and*

$$\left(-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right)S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + S^4$$

Similarly,

$$N_{k,\omega_\Gamma}^2 = -(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2 + U_2.$$

where U_2 maps U^s to U^{s+2} for all $s \in \mathbb{R}$.

3 Galerkin analysis

In this section, we describe and analyze the Galerkin scheme used to solve the integral equations in this work. To keep matters simple, we focus on equations (**??**) and (**??**) on the flat strip. The results extend to the general case using standard arguments in the theory of boundary element methods. Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [12, Chap. 4,] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [11] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [3], and some results exist for piecewise linear functions in the colocation setting [4].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 6 and Theorem 7.

In what follows, we introduce a discretization of the segment $[-1,1]$ as $-1 = x_0 < x_1 < \cdots < x_N = 1$, and let $\theta_i := \arccos(x_i)$. We define the parameter h of the discretization as

$$h := \min_{i=0\cdots N-1} |\theta_{i+1} - \theta_i| \,.$$

In practice, one should use a mesh for which $|\theta_i - \theta_{i+1}|$ is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the $i-th$ interval is approximately $(ih)^2$. In comparison, in the h-BEM method with $p = 1$ polynomial order, this would only lead to a convergence rate in $O(h)$ (cf. [11, Theorem 1.3]).

3.0.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution λ of (**??**). To achieve it, we use a variational formulation of (**??**) to compute an approximation α_h of α, and set $\lambda_h = \frac{\alpha_h}{\omega}$. Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at x_i. Let α_h the unique solution in V_h to

$$\langle S_\omega \alpha_h, \alpha_h' \rangle_{\frac{1}{\omega}} = - \langle u_D, \alpha_h' \rangle_{\frac{1}{\omega}} \,, \quad \forall \alpha_h' \in V_h.$$

We shall prove the following result:

Theorem 6. *If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C h^{s+1/2} \|u_D\|_{T^{s+1}} \,.$$

In particular, when u_D is smooth, it belongs to T^∞ so the rate of convergence is $h^{5/2}$. We start by proving an equivalent of Céa's lemma:

Lemma 13. *There exists a constant C such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

Proof. In view of the properties of S_ω stated in **??**, we have the equivalent norm

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \left\langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \right\rangle.$$

Since $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = -\langle u_D, \alpha'_h \rangle$ for all $\alpha'_h \in V_h$, we deduce

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq \left\langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \right\rangle, \quad \forall \alpha'_h \in V_N.$$

By duality

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of S_ω from $T^{-1/2}$ to $T^{1/2}$. □

From this we can derive the rate of convergence for α_h to the true solution α. We use the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h, which satisfies the following properties:

Lemma 14. *For any function u,*

$$\|(\mathrm{I} - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C \|u\|_{L^2_{\frac{1}{\omega}}},$$

$$\|(\mathrm{I} - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C h^2 \|u\|_{T_2}.$$

The proof requires the following well-known result:

Lemma 15. *Let $\tilde{u}$ in the Sobolev space $H^2(\theta_1, \theta_2)$,such that $\tilde{u}(\theta_1) = \tilde{u}(\theta_2) = 0$. Then there exists a constant C independent of θ_1 and θ_2 such that*

$$\int_{\theta_1}^{\theta_2} \tilde{u}(\theta)^2 \leq C(\theta_1 - \theta_2)^4 \int_{\theta_1}^{\theta_2} \tilde{u}''(\theta)^2 d\theta$$

Proof. The first inequality is obvious since $\mathbb{P}_h$ is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the $L^2_{\frac{1}{\omega}}$ norm,

$$\|I - \mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I - I_h u\|_{L^2_{\frac{1}{\omega}}}, \tag{30}$$

where $I_h u$ is the piecewise affine (continuous) function that matches the values of u at the breakpoints x_i. By Lemma 7, on each interval $[x_i, x_{i+1}]$, the function $\tilde{u}(\theta) := u(\cos(\theta))$ is in the Sobolev space $H^2(\theta_i, \theta_{i+1})$ so we can apply Lemma 15:

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I_h u})^2 \leq (\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I_h u})''^2.$$

This gives

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} \leq 2h^4 \left(\int_{\theta_i}^{\theta_{i+1}} \tilde{u}''^2 + \int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 \right). \tag{31}$$

Before continuing, we need to establish the following result

Lemma 16. *There holds*

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 \leq C \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega}$$

Proof. The expression of $I_h u$ is given by

$$\tilde{I_h u}(\theta) = u(x_i) + \frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)}(\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 = \left(\frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)}\right)^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left(\int_{x_i}^{x_{i+1}} u'(t)dt\right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \leq \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. □

We can now conclude the proof of Lemma 14. Summing all inequalities (31) for $i = 0, \cdots N+1$, we get

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}}^2 \leq Ch^4 \left(\|u\|_{T^2}^2 + \|u'\|_{T_0}^2\right).$$

By Corollary 1, the operator ∂_x is continuous from T^2 to T^0 which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T^2}.$$

Thanks to (30), this concludes the proof. □

We obtain the following corollary by interpolation:

Corollary 2. *The operator* $\mathrm{I} - \mathbb{P}_N$ *is continuous from* $L^2_{\frac{1}{\omega}}$ *to* T^s *for* $0 \leq s \leq 2$ *with*

$$\|(\mathrm{I} - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s}.$$

We can now prove Theorem 6:

Proof. First, using Lemma 11, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}}.$$

Moreover, if u_D is in T^{s+1}, then $\alpha = S_\omega^{-1} u_D$ is in T^s and $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$. By the analog of Céa's lemma, Lemma 13, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq C h^{s+1/2} \|\alpha\|_{T^s}.$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

Using Cauchy-Schwarz's inequality and Corollary 2 ($s = \frac{1}{2}$),

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s}.$$

□

3.0.2 Neumann problem

We now turn to the numerical resolution of (**??**). We use a variational form for equation (**??**), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce W_h the space of continuous piecewise affine functions with breakpoints at x_i, and we denote by β_h the unique solution in W_h to the variational equation:

$$\langle N_\omega \beta_h, \beta_h' \rangle_\omega = \langle u_N, \beta_h' \rangle_\omega, \quad \forall \beta_h' \in W_h. \tag{32}$$

Then, $\mu_h = \omega \beta_h$ is the proposed approximation for μ. We shall prove the following:

Theorem 7. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C h^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Like before, we start with an analog of Céa's lemma:

Lemma 17. *There exists a constant C such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta_h' \in W_h} \|\beta - \beta_h'\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator I_h:

Lemma 18. *There holds*

$$\|u - I_h u\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_h u\|_{U^1} \leq Ch \|u\|_{U^2}$$

Proof. We only show the first estimation, the method of proof for the second being similar. Using again Lemma 15 on each segment $[x_i, x_{i+1}]$, one can write

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} \omega(u - I_h u)^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_h u)''^2 \\
&\leq Ch^4 \left(2\int_{\theta_i}^{\theta_{i+1}} Vu''^2 + 2\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \right)
\end{aligned}$$

where we recall that for any function u, Vu is defined as

$$Vu(\theta) = \sin(\theta) u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

Lemma 19.

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(\|u\|_{U_2}^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

Proof. Using the expression of I_h, one can write

$$\begin{aligned}
\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \Bigg(& |u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \\
& + \left(\frac{u(x_{i+1}) - u(x_i)}{\cos\theta_{i+1} - \cos\theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2) \Bigg)
\end{aligned} \tag{33}$$

We can estimate the first term, thanks to Lemma 5:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned}
(u(x_{i+1}) - u(x_i))^2 &= \left(\int_{x_i}^{x_{i+1}} \partial_x u \right)^2 \\
&\leq \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\
&= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2.
\end{aligned}$$

to conclude, it remains to observe that the quantity

$$\frac{|(\theta_{i+1} - \theta_i)| \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2)}{(\cos(\theta_i) - \cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2(\theta_i)$ □

We now plug the estimate Lemma 19 in (33), and sum over i:

$$\|u - I_h u\|_{L^2_\omega}^2 \leq Ch^4(\|u\|_{U^2}^2 + \|u'\|_{L^2_\omega}^2).$$

This implies the claim once we use the continuity of ∂_x from U^2 to U^0, cf. Corollary 1. ☐

We can now prove Theorem 7

Proof. Let us denote by Π_h the Galerkin projection operator defined by $\beta \mapsto \beta_h$. Since it is an orthogonal projection on W_h with respect to the scalar product $(\beta, \beta') := \langle N_\omega \beta, \beta' \rangle$, it is continuous from $U^{1/2}$ to itself, so we have for any u in $U^{1/2}$.

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C \|u\|_{U^{1/2}}.$$

We are now going to show the estimate

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2}.$$

By the analog of Céa's lemma Lemma 17, one has

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq \|(I - I_h)u\|_{U^{1/2}}.$$

By interpolation, this norm satisfies

$$\|(I - I_h)u\|_{U^{1/2}} \leq C\sqrt{\|(I - I_h)u\|_{U^0}}\sqrt{\|(I - I_h)u\|_{U^1}},$$

which yields, applying Lemma 18,

$$\|(I - I_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2}.$$

By interpolation, for all $s \in [1/2, 2]$, we get

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{s-1/2} \|u\|_{U^s}.$$

In view of **??**, we have $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$. In addition, since N_ω is a continuous bijection from U^{s+1} to U^s for all s, there holds

$$\|\beta\|_{U^s} = \left\|N_\omega^{-1} u_N\right\|_{U^s} = \|u_N\|_{U^{s-1}}.$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-1/2} \|\beta\|_{U^s} \leq Ch^{s-1/2} \|u_N\|_{U^{s-1}}.$$

☐

4 Conclusion

5 Proof of Theorem 4

From equation (23), we can deduce the following formula for the weighted operator:

$$N_{k,\omega} = -\partial_x S_{k,\omega} \omega \partial_x \omega - k^2 S_{k,\omega} \omega^2 \tag{34}$$

If we define $L_n := -\partial_x O_{n+2}\omega\partial_x\omega$, then using the mapping properties of ∂_x and $\omega\partial_x\omega$ given by Definition 3, and since, by **??**, O_{n+2} is of order $n+2$ in the scale T^s, we deduce that L_n is of order n in the scale U^s. The expansion obtained for the weighted single-layer operator in **??** yields the following expansion for $N_{k,\omega}$.

Lemma 20.

$$N_{k,\omega} = N_\omega + k^2\left(-\frac{L_1}{4} - S_\omega\omega^2\right) + U_3$$

As a consequence, $N_{k,\omega}$ is an operator of order -1 in the scale U^s. Using equation (34), we have the following expression:

$$N_{k,\omega}^2 = N_\omega^2 - k^2\left(\frac{L_1N_\omega + N_\omega L_1}{4} + N_\omega S_\omega\omega^2 + S_\omega\omega^2 N_\omega\right) + U_2.$$

We have proved in By definition, $L_1 = -\partial_x O_3\omega\partial_x\omega$, while $N_\omega = -\partial_x S_\omega\omega\partial_x\omega$, thus

$$L_1N_\omega = \partial_x(O_3(\omega\partial_x)^2 S_\omega)\omega\partial_x\omega.$$

Moreover,

$$N_\omega L_1 = \partial_x(S_\omega(\omega\partial_x)^2 O_3)\omega\partial_x\omega.$$

Adding these two inequalities and using **??**, we get

$$\frac{L_1N_\omega + N_\omega L_1}{4} = \partial_x(S_\omega\omega^2 S_\omega)\omega\partial_x\omega + U_2.$$

Here again, we use the formula $\partial_x S_\omega\omega^2 = S_\omega\omega\partial_x\omega$, which yields

$$\frac{L_1N_\omega + N_\omega L_1}{4} = S_\omega\omega\partial_x\omega\partial_x S_\omega\omega^2 = \left(-\frac{I_d}{4} + T_\infty\right)\omega^2.$$

Since ω^2 is continuous from U^s to T^s by **??** and using the injections $T^s \subset U^s$, any operator of the form $R\omega^2$ is smoothing in the scale U^s as soon as R is smoothing in the scale T^s. Therefore,

$$\frac{L_1N_\omega + N_\omega L_1}{4} = -\frac{\omega^2}{4} + U_\infty.$$

Moreover, we have

$$\begin{aligned} S_\omega\omega^2 N_\omega &= -S_\omega\omega^2\partial_x S_\omega\omega\partial_x\omega \\ &= -S_\omega\omega^2\partial_x^2 S_\omega\omega^2 \end{aligned}$$

using again **??**. Since $\omega^2\partial_x^2 = (\omega\partial_x)^2 + x\partial_x$, we get

$$S_\omega\omega^2 N_\omega = \frac{\omega^2}{4} - S_\omega x\partial_x S_\omega\omega^2 + U_\infty$$

Futhermore,

$$N_\omega S_\omega\omega^2 = -\partial_x S_\omega\omega\partial_x\omega S_\omega\omega^2.$$

We use $\omega\partial_x\omega = \omega^2\partial_x - x$:

$$\begin{aligned} N_\omega S_\omega \omega^2 &= -\partial_x S_\omega \omega^2 \partial_x S_\omega \omega^2 + \partial_x S_\omega x S_\omega \omega^2 \\ &= \frac{\omega^2}{4} + \partial_x S_\omega x S_\omega \omega^2 \end{aligned}$$

Thus,

$$S_\omega \omega^2 N_\omega + N_\omega S_\omega \omega^2 = \frac{\omega^2}{2} + \left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2\right) + U_\infty.$$

We are done if we prove that the operator in parenthesis is of order 2 in the scale U^s. For this, we may compute the action of each one of them on U_n. Using the various identities at our disposal, we obtain on the one hand for $n \geq 2$

$$\partial_x S_\omega x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8(n+2)} + \frac{U_n + U_{n-2}}{8n(n+2)}.$$

and on the other hand for $n > 0$

$$S_\omega x \partial_x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8n}.$$

After substracting, this gives the rather surprising identity identity for $n \geq 2$

$$\left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2\right) U_n = \frac{U_n}{4n(n+2)}$$

which of course proves our claim.

6 Suggestion de découpage

J'y ai un tout petit peu réfléchi :

- Les analyses pseudo-diffs des espaces T^s, bien qu'intéressantes, sont trop longues et ne se justifient pas vraiment dans le simple but de faire une méthode numérique.
- La méthode de Galerkine est bien analysée et nouvelle (à ma connaissance) mais n'est pas vraiment essentielle pour le message.

Je pense qu'on pourrait envisager 3 articles. Un très concis sur la méthode numérique en elle-même. Utiliser le minimum d'info pour k=0, donner les inverses exacts, prouver la commutation des opérateurs pour k non nul, puis balancer les préconditionneurs, et mettre les figures.

Un article un peu à part sur la méthode de Galerkine, et tous les aspects numériques (bcp moins d'impact)

Un article (peut-être juste sur arxiv ?) sur les espaces T^s et U^s, qui donne toutes les justifications théoriques. (une sorte de version étendue de cet article.)

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