Inegral equations in 2D

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The single-layer at frequency k has the following expression:

$$S_k\phi(x) = \int_\Gamma G_k(y-x)\phi(y)d\sigma(y)$$

with

$$G_k(x) = \begin{cases} \frac{-1}{2\pi}\ln|x| & \text{if } k=0 \\ \frac{i}{4}H_0^{(1)}(k|x|) & \text{otherwise} \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of order 0 defined by

$$H_0^{(1)}(r) = J_0(r) + iY_0(r)\,.$$

The double layer is defined by

$$D_k\phi(x) = \int_\Gamma n(y)\cdot\nabla_y G_k(y-x)\phi(y)d\sigma(y)\,.$$

The hypersingular operator admits the representation:

$$\begin{aligned}\langle N_k\mu,\nu\rangle = \int_{\Gamma\times\Gamma} & G_k(x-y)\mu'(x)\nu'(y) \\ & - k^2 G_k(x,y)\mu(x)\nu(y)n(x)\cdot n(y)d\sigma(x)d\sigma(y)\,.\end{aligned} \tag{1}$$

where μ' is the tangential derivative of μ.

For the regularization of S_k, our aim is to write the semi analytic integration of S_k as

$$\int_\Gamma G_k(y-x)\phi(y)dy = \tilde{\int}_\Gamma G_k(y_x)\phi(y)dy + \left(\int - \tilde{\int}\right) G_k(y-x)\phi(y)dy\,.$$

If we note $R = \tilde{\int}_\Gamma - \int_\Gamma$ the regularization operator, we thus have

$$\begin{aligned}\int_\Gamma G_k(y-x)\phi(y)dy = & \tilde{\int}_\Gamma G_k(y_x)\phi(y)dy + R\left[\left(G_k(y-x) - G_0(y-x)\right)\phi(y)\right] \\ & + R\left(G_0(y-x)\phi(y)\right)\end{aligned}$$

Here, we use the fact that $G_k - G_0$ is a smooth function (it is C^1) so that

$$R\left[\left(G_k(y-x) - G_0(y-x)\right)\phi(y)\right] \approx 0\,.$$

We must thus ensure that the arbitrary values C_k and C_0 assigned respectively to the elementary kernels $r \mapsto H_0^{(1)}(kr)$ and $r \mapsto \log(r)$ implemented in Gypsilab are such that

$$G_k(0) - G_0(0) = \lim_{r\to 0} G_k(r) - G_0(r)\,.$$

that is

$$\frac{i}{4}C_k - \frac{1}{2\pi}C_0 = \lim_{r\to 0} G_k(r) - G_0(r)\,. \tag{2}$$

For C_0, we make the arbitrary choice

$$C_0 := 0.$$

We must now choose the value of C_k accordingly. To evaluate the limit of the right hand side, we can write [1, Eq. 10.8.2]

$$Y_0(r) = \frac{2}{\pi}\left(\ln\frac{r}{2} + \gamma\right) J_0(r) + r^2 F(r)$$

where γ is the Euler constant. This gives

$$\lim_{r\to 0} G_k - G_0 = -\frac{1}{2\pi}\left(\ln\frac{k}{2} + \gamma\right) + \frac{i}{4}\,.$$

This means we have to set

$$C_k = 1 + \frac{2i}{\pi}\left(\ln\frac{k}{2} + \gamma\right)\,.$$

References

[1] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders. *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.16 of 2017-09-18.