Quadrature for Bessel functions

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In all this document, r is a positive real number $N \geq 1$ is an integer, φ a real number and J_0 denotes the Bessel function of first kind. We assume in addition that $r < N$. Under this condition, we shall prove the following estimation :

Proposition 0.1.

$$\left| J_0(r) - \frac{1}{N} \sum_{j=0}^{N-1} e^{ir \sin\left(\frac{2j\pi}{N} - \varphi\right)} \right| \leq C_N \left(\frac{er}{N} \right)^N$$

Where $C_N \leq 3$ *and* $C_N \underset{N \to +\infty}{\longrightarrow} 2$

In order to prove this proposition, we first prove a result on Fourier series

Lemma 0.1. *For any* $\mathcal{C}^2$ *function* f *defined on* $\mathbb{R}$ *and complex-valued, that is* $2\pi-$*periodic, one has*

$$\frac{1}{2\pi} \int_0^{2\pi} f - \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{2j\pi}{N} \right) = - \sum_{k \in \mathbb{Z}^*} c_{kN}(f)$$

Where $c_n(f)$ *denotes the Fourier coefficient of* f *defined as*

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

Proof. Since f is $\mathcal{C}^2$, it is equal to its Fourier Series, which converges normally :

$$\forall x \in \mathbb{R}, f(x) = \sum_{k \in \mathbb{Z}} c_k(f) e^{ikx}$$

Using this expression, we obtain

$$\frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{2j\pi}{N} \right) = \sum_{k \in \mathbb{Z}^*} c_k(f) \left(\frac{1}{N} \sum_{j=0}^{N-1} e^{ik\frac{2j\pi}{N}} \right)$$

Now observe that if $k \notin N\mathbb{Z}$,

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{ik\frac{2j\pi}{N}} = 0$$

and if $k \in N\mathbb{Z}$ then

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{ik\frac{2j\pi}{N}} = 1$$

Therefore

$$\int_0^{2\pi} f(x)dx - \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{2j\pi}{N} \right) = c_0(f) - \sum_{k \in N\mathbb{Z}} c_k(f) = - \sum_{k \in \mathbb{Z}^*} c_{kN}(f)$$

$\square$

Let us now prove the proposition :

Proof. The result is based on the fact that

$$J_0(r) = \int_0^{2\pi} e^{ir\sin(x)}dx = \int_0^{2\pi} e^{ir\sin(x-\varphi)}dx$$

Let $f : x \mapsto e^{ir\sin(x-\varphi)}$. Let us recall the integral representation of the Bessel function of the first kind and of order k where k is a relative integer :

$$J_k(r) = \int_0^{2\pi} e^{ir\sin(x)}e^{-ikx}dx = e^{-ik\varphi}\int_0^{2\pi} e^{ir\sin(x-\varphi)}e^{-ikx}dx$$

Thus, one has $c_k(f) = e^{ik\varphi}J_k(r)$. The former Lemma therefore writes

$$J_0(r) - \frac{1}{N}\sum_{j=0}^{N-1} e^{ir\sin\left(\frac{2j\pi}{N}-\varphi\right)} = -\sum_{k\in\mathbb{Z}^*} e^{iNk\varphi}J_{Nk}(r)$$

We shall now use the following estimation for J_k : $\forall R > 1$

$$|J_k(r)| \leq R^{-|k|}e^{rR}$$

Since $N > r$, we have $N|k| > r$ for all $k \in Z^*$. We can choose $R = \frac{N|k|}{r} > 1$, implying that

$$|J_{Nk}(r)| \leq \left(\frac{er}{N|k|}\right)^{N|k|}$$

Applying this estimate we obtain :

$$\left|J_0(r) - \frac{1}{N}\sum_{j=0}^{N-1} e^{ir\sin\left(\frac{2j\pi}{N}-\varphi\right)}\right| \leq \sum_{k\in\mathbb{Z}^*}\left(\frac{er}{N|k|}\right)^{N|k|}$$

Therefore,

$$\left|J_0(r) - \frac{1}{N}\sum_{j=0}^{N-1} e^{ir\sin\left(\frac{2j\pi}{N}-\varphi\right)}\right| \leq 2\left(\frac{er}{N}\right)^N \sum_{k\in\mathbb{N}^*}\left(\frac{1}{k}\right)^{Nk}$$

Let γ_N be defined as

$$\gamma_N = \sum_{k\in\mathbb{N}^*}\left(\frac{1}{k}\right)^{Nk}$$

Observe that

$$0 \leq \gamma_N - 1 \leq \sum_{k\geq 2}\frac{1}{2^{kN}} = \frac{1}{2^{2N} - 2^N}$$

showing that $\gamma_N \leq \frac{3}{2}$ and $\gamma_N \underset{N\to+\infty}{\longrightarrow} 1$. The result is finally proved by setting $C_N = 2\gamma_N$

□