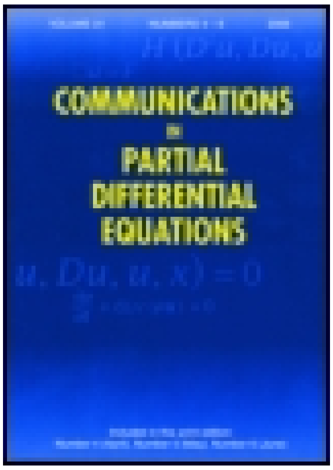

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PSEUDODIFFERENTIAL OPERATORS OF MELLIN TYPE

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INTRODUCTION

This article studies several algebras of pseudo-differential operators (pdo's) on the half line R^+. The operators of order 0 include the Hilbert transform on $L^p(R^+)$

$$Hf(t) = \text{p.v.}\ \frac{1}{\pi}\int_0^\infty \frac{f(s)}{t-s}\,ds$$

and the point of view is motivated by the representation of H as a Mellin multiplier,

$$Hf(t) = \frac{-1}{2\pi i}\int_{\operatorname{Re} z = 1/p} t^{-z}\cot \pi z\, \tilde{f}(z)\,dz$$

where $f(z) = \int_0^\infty t^{z-1} f(t)\,dt$, $f \in C_0^\infty(R^+)$, is the Mellin transform of f. Since $\left(-t\frac{d}{dt}f\right)^\sim(z) = z\tilde{f}(z)$, we consider operators of the form

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(1) $$a\left(t,-t\frac{d}{dt}\right)f(t)=\frac{1}{2\pi i}\int_{\mathrm{Re}\,z=1/p} t^{-z}a(t,z)\tilde{f}(z)dz$$

where the symbol $a(t,z)$ satisfies suitable smoothness properties in t and holomorphy and growth properties in z. The spectral and continuity properties of the operators (1) on various L^p and Sobolev spaces depend on the domain of holomorphy of the functions $a(0,z)$ and $a(\infty,z)$.

In Chapter 1 we develop a C^∞ theory for the operators (1) which is motivated by the usual theory of pdo's on R^n. The basic operator is

$$\partial = -t\frac{d}{dt}$$

and we study the action of (1) on functions in $C^\infty(R^+)$ which satisfy Holder type continuity conditions at $t = 0$ and $t = \infty$. To include the Hilbert transform, H, we allow the symbols $a(\cdot,z)$ to have a sequence of poles $\{z_k\}$, $\mathrm{Re}\,z_k \to -\infty$; this makes the proof of the symbolic calculus theorems for compositions and adjoints more delicate than for the corresponding theorems on R^n. We develop algebras ample enough to include H as well as many Hardy kernel operators of the form

$$Kf(t) = \int_c^\infty k\left(\frac{t}{s}\right)f(s)\frac{ds}{s} = \frac{1}{2\pi i}\int_{\mathrm{Re}\,z=1/p} t^{-z}\tilde{k}(z)\tilde{f}(z)dz$$

and multiplication by cutoff functions $\chi(t) \in C_o^\infty(R)$, $\chi(t) = 1$ near $t = 0$.

In Chapter 2 we study the continuity properties of pdo's on various weighted and unweighted L^p and Sobolev spaces on the half line R^+.

A <u>principal symbol</u> for a class of pdo's on $L^p(R^+)$ is introduced in Chapter 3. The notion of <u>ellipticity on $L^p(R^+)$</u> of a pdo (i.e., the existence of a parametrix modulo compact operators) depends on p as in [LP1]. We obtain that a pdo is elliptic on $L^p(R^+)$ iff its principal symbol never vanishes. An index theorem for pdo's on $L^2(R^+)$ was proved by Cordes and Herman [CH]. We give an index theorem for an elliptic system on $L^p(R^+)$ which generalizes [CH] and [LP1].

In Chapter 4 we define an algebra of pdo's of order 0 on $L^p(I)$, where I is a finite interval. We work modulo compact operators on $L^p(I)$ to define a principal symbol and prove that a pdo is a Fredholm operator on $L^p(I)$ iff its principal symbol never vanishes. For an elliptic system of pdo's on $L^p(I)$, we prove an index theorem which relates the index of the system to the topological degree of the principal symbol. In this case the principal symbol may be identified with a map from the unit circle S^1 into $GL(N,C)$; of course the degree and hence the index depend on the L^p space under consideration.

In Chapter 5, we apply the theory of pdo's to the study of single and double layer potentials for

Laplace's equation in a plane polygon. We study the Fredholm and index properties of the integral equations which arise for Dirichlet, Neumann, and oblique derivative and mixed problems.

Various classes of pdo's on R^+, halfspaces, and manifolds with boundary have been considered by many authors. Boutet de Monvel [B] studied pdo's with the transmission property. Melrose [M] defined a class of operators on a half-space via oscillatory integrals. Nourrigat [N1] considered symbols $a(\cdot,z)$ holomorphic in a left half plane and gave applications to Fuchs type equations in [N2]. Shamir [S1,S2] observed how the spectrum of a pdo on R^+ depends on the L^p space on which it acts. Cordes and Herman [CH] and Cordes [C] used Banach algebra techniques to introduce the notion of a principal symbol. Eskin [E] used the Mellin transform representation of the Hilbert transform and Hardy kernels to study mixed boundary problems. Plamenevskii [P1], using the Mellin transform, assigned a pdo on the sphere S^{n-1} (actually a meromorphic function of a complex parameter λ) to an operator on R^n. He then reduced the study of an algebra of operators on R^n to the investigation of an algebra of meromorphic operator-valued functions on S^{n-1}. The authors [LP1] developed an algebra which is properly included in the algebra $Op - \Sigma_{1/p}$ studied in Chapter 3.

For the index properties of boundary value problems in polynomial domains, see the extensive work and bibliographies of Grisvard [G], Kondratiev [K], and Avantaggiati [Av1]. The L^p theory of boundary value problems was studied by Merigot [Mer]. Double layer potentials in sectors and bounded domains were considered by Fabes, Jodeit, and Lewis [JFL]. Applications of the present theory to the biharmonic operator in a polygon have been given by Diomeda and Lisena [DL].

1. The C^∞ theory of pdo's on R^+.

We shall denote the open half line $(0,\infty)$ as R^+ and $[0,\infty)$ as $\overline{R^+}$. The basic differential operator used will be

$$\partial = -t\frac{d}{dt}$$

and ∂_t will denote $-t\frac{\partial}{\partial t}$. If a is a real number, the integer $(a]$ will be defined as

$$(a] = \text{greatest integer} < a.$$

The following function spaces will be used for the C^∞ theory of pdo's on R^+.

DEFINITION 1.1. *Let* $-\infty < a < b < \infty$. *By* $\mathcal{F}_{a,b}$ *we denote the class of functions* $f \in C^\infty(R^+)$ *such that*:

(1) *For* $j = 0,\ldots,(-a]$, *there are scalars* f_{jo}

<u>such that for every</u> k <u>and every</u> $\delta > 0$,

$$(1.1)\quad \partial^k\Big(f(t) - \sum_{j=0}^{(-a]} \frac{1}{j!} f_{j0} t^j\Big) = O(t^{-a-\delta}),\ \ t \to 0^+,$$

(2) <u>For</u> $j = 0,\ldots,(b]$, <u>there are scalars</u> $f_{j\infty}$ <u>such that for every</u> k <u>and every</u> $\delta > 0$,

$$(1.2)\quad \partial^k\Big(f(t) - \sum_{j=0}^{(b]} \frac{1}{j!} f_{j\infty} t^{-j}\Big) = O(t^{-b+\delta}),\ \ t \to \infty.$$

$\mathcal{F}_{a,b}$ <u>is a Frechet space with the seminorms</u>

$$|f_{j0}|,\ j = 0,\ldots,(-a];\ |f_{j\infty}|,\ j = 0,\ldots,(b];$$

$$(1.3)\quad \sup_{0<t<2} \left| t^{a+\delta}\partial^k\Big(f(t) - \sum_{j=0}^{(-a]} \frac{1}{j!} f_{j0} t^j\Big)\right|;$$

$$\sup_{1<t<\infty} \left| t^{b-\delta}\partial^k\Big(f(t) - \sum_{j=0}^{(b]} \frac{1}{j!} f_{j\infty} t^{-j}\Big)\right| .$$

We shall abbreviate (1.1) as $f(t) \sim \Sigma \frac{1}{j!} f_{j0} t^j$, $t \to 0^+$, and shall write (1.2) as $f(t) \sim \Sigma \frac{1}{j!} f_{j\infty} t^{-j}$, $t \to \infty$. If $b > 0$, we denote by $\acute{\mathcal{F}}_{a,b}$ the space of functions $f \in \mathcal{F}_{a,b}$ such that $f \sim 0$, $t \to \infty$; i.e., $\partial^k f(t) = O(t^{-b+\delta})$, $t \to \infty$, for every $\delta > 0$. Similarly $\mathring{\mathcal{F}}_{a,b}$ consists of these functions $f \in \mathcal{F}_{a,b}$ such that $f(t) \sim 0$, $t \to 0^+$. We define

$$\mathring{\acute{\mathcal{F}}}_{a,b} = \mathring{\mathcal{F}}_{a,b} \cap \acute{\mathcal{F}}_{a,b}.$$

The space $\mathcal{J}''_{a,b}$ is stable under multiplication by $\log t$, and $C_0^\infty(R^+)$ is dense in $\mathcal{J}''_{a,b}$. We use the convention that $\mathcal{J}_{-\infty,b} = \bigcap_{a<b} \mathcal{J}_{a,b}$ and note that $\mathcal{J}_{a,b} = \bigcap_{\delta>0} \mathcal{J}_{a+\delta,b-\delta}$. If we define

$$\check{f}(t) = f\left(\frac{1}{t}\right), \tag{1.4}$$

then $f \in \mathcal{J}'_{a,b}$ iff $\check{f} \in \mathcal{J}'_{-b,-a}$.

The spaces $\mathcal{M}'_{a,b}$ defined in Definition 1.2 will be the Mellin transforms of function in $\mathcal{J}'_{a,b}$. By $S_{a,b}$ we denote the strip

$$S_{a,b} = \{z \in C : a < \operatorname{Re} z < b\} \tag{1.5}$$

and let Z_- be the set of nonpositive integers.

<u>DEFINITION 1.2.</u> <u>Let</u> $0 < b$. <u>The space</u> $\mathcal{M}'_{a,b}$ <u>consists of functions</u> $g(z)$ <u>defined and meromorphic on the strip</u> $S_{a,b}$ <u>such that</u>

1. g <u>has poles only at</u> $z = -j \in Z_- \cap S_{a,b}$ <u>which are at most simple; denote the residue at</u> $z = -j$ <u>by</u> $(1/j!)g_j$.
2. $g(z)$ <u>is rapidly decreasing as</u> $|\operatorname{Im} z| \to \infty$ <u>in</u> $S_{a,b}$; <u>i.e., on every strip with base</u> $(a',b') \subset\subset (a,b)$

$$|g(z)| = O(|\operatorname{Im} z|^{-N}) \tag{1.6}$$

for every N *as* $|\mathrm{Im}\, z| \to \infty$, $z \in S_{a',b'}$.

The following Lemma follows from the arguments of Avantaggati [Av2], Nourrigat [N2], or [LP2].

LEMMA 1.1. *Let* $0 < b$. *For* $f \in \mathcal{F}'_{a,b}$ *and* $\max(a,0) < \mathrm{Re}\, z < b$, *define*

$$\tilde{f}(z) = \int_0^\infty t^{z-1} f(t)\,dt. \tag{1.7}$$

Then $f(z)$ *may be extended as a meromorphic function in* $\mathcal{M}'_{a,b}$ *with residues at* $z = -j \in Z_- \cap S_{a,b}$ *given by* $(1/j!)f_{jo}$.

Conversely, given $g(z) \in \mathcal{M}'_{a,b}$, *for any* γ *such that* $\max(a,0) < \gamma < b$, *define*

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, z = \gamma} t^{-z} g(z)\,dz. \tag{1.8}$$

Then $f \in \mathcal{F}'_{a,b}$, $f \sim \sum_{j=o}^{(-a]} \frac{g_j}{j!} t^j$, $t \to 0^+$, *and* $\tilde{f}(z) = g(z)$, *where* $\tilde{f}$ *is defined by* (1.7).

DEFINITION 1.3. *For* $-\infty \leq c < d \leq +\infty$ *and* m *real the class* $\mathcal{O}^m_{c,d}$ *consists of those functions* $a(z)$ *holomorphic in the strip* $S_{c,d}$ *such that for every* $(c',d') \subset\subset (c,d)$, *for every* k

(1.9) $$\frac{d^k}{dz^k} a(z) = O(|\mathrm{Im}\, z|^{m-k})$$

as $|\mathrm{Im}\, z| \to \infty$, $z \in S_{c',d'}$.

$\Theta^m_{c,d}$ is a Frechet space with the seminorms

(1.10) $$\sup_{z \in S_{c',d'}} \left|(1+|z|)^{k-m} \frac{d^k}{dz^k} a(z)\right|, \quad (c',d') \subset\subset (c,d).$$

For $-\infty \leq c < d \leq +\infty$ and $-\infty \leq a < b \leq +\infty$, $\mathfrak{F}_{a,b}(\Theta^m_{c,d})$ denotes the Frechet space $\mathfrak{F}_{a,b}(R^+;\Theta^m_{c,d})$ of functions $a(t,z) \in \mathfrak{F}_{a,b}$ as functions of t with values in $\Theta^m_{c,d}$.

We shall write $\Theta^{-\infty}_{c,d} = \bigcap_m \Theta^m_{c,d}$.

DEFINITION 1.4. *For $-\infty \leq c < d \leq \infty$ and $m \in R$, the symbol class $\Sigma^m_{c,d}$ is defined by*

(1.11) $$\Sigma^m_{c,d} = \bigcap_{(c',d') \subset\subset (c,d)} \mathfrak{F}_{c-c',d-d'}(\Theta^m_{c',d'}).$$

Also

(1.12) $$\mathring{\Sigma}^m_{c,d} = \bigcap_{(c',d') \subset\subset (c,d)} \mathring{\mathfrak{F}}_{c-c',d-d'}(\Theta^m_{c',d'}).$$

REMARKS.

(1) If $\chi(t)$ is a *cutoff function* (i.e., $\chi(t) \in C_0^\infty(R)$, $\chi(t) = 1$ near $t = 0$), $\alpha > 0$, and $\mathrm{Re}\, z_0 = c$, then for any $a(z) \in \Theta^{-\infty}_{c,d}$, the function

$$(1.13)\qquad a(t,z) = \chi(t)(\alpha t)^{z-z_0}a(z)$$

is in $\Sigma_{c,d}^{c'-\infty}$.

2. If $a(t,z) \in \Sigma_{c,d}^{m}$ and $c < c' < \operatorname{Re} z < d$, we have the asymptotic expansion

$$(1.14)\qquad a(t,z) \sim \sum_{j=0}^{(c'-c]} \frac{a_{jo}(z)}{j!} t^{j},\ t \to 0^{+},$$

where $a_{jo}(z) \in \mathcal{O}_{c+j,d}^{m}$.

We are now ready to define a class of pseudo-differential operators.

<u>DEFINITION 1.5.</u> <u>Let</u> $-\infty \leq c < d \leq +\infty$ <u>and suppose that</u>

$$(1.15)\qquad J = (c,d) \cap (0,1) \neq \emptyset.$$

<u>If</u> $a(t,z) \in \Sigma_{c,d}^{m}$ <u>define the operator</u> $A = a(t,\partial) \in \mathrm{Op}-\Sigma_{c,d}^{m}$ <u>as follows:</u>

<u>Choose</u> $\gamma \in J$ <u>and for</u> $f \in \mathcal{J}'_{c,d}$, <u>let</u>

$$(1.16)\qquad Af(t) = a(t,\partial)f(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\gamma} t^{-z}a(t,z)\tilde{f}(z)dz.$$

The definition (1.16) is independent of $\gamma \in J$.

<u>THEOREM 1.1.</u> <u>If</u> $a(t,z) \in \Sigma_{c,d}^{m}$, <u>then</u>

$$a(t,\partial) : \mathfrak{F}'_{c,d} \to \mathfrak{F}'_{c,d}$$

is continuous; moreover, if

$$(1.17) \qquad f(t) \sim \sum_{j=o}^{[-c]} \frac{1}{j!} f_{jo} t^j, \quad t \to 0^+$$

then

$$(1.18) \quad Af(t) \sim \sum_{k=o}^{[-c]} \frac{1}{k!} \left\{ \sum_{j=o}^{k} \binom{k}{j} a_{k-j,o}(-j) f_{jo} \right\} t^k, \quad t \to 0^+,$$

where $a_{k-j,o}(z)$ is given by (1.14). Hence

$$(1.19) \qquad a(t,\partial) : \mathfrak{F}''_{c,d} \to \mathfrak{F}''_{c,d}$$

is continuous.

PROOF: Let $f \in \mathfrak{F}'_{c,d}$ satisfy (1.17). For all sufficiently small $\delta > 0$, we shift the contour of integration in (1.16) to $\operatorname{Re} z = d-\delta$ and rewrite (1.16) as

$$(1.20) \qquad Af(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = d-\delta} t^{-z} a(t,z) \tilde{f}(z) dz.$$

This shows that $Af(t)$ is in $C^\infty(R^+)$. Since for $\operatorname{Re} z = d-\delta$, $\partial_t^k(a(t,z) - a(\infty,z)) = O(t^{-\delta+\epsilon}(1+|z|)^m)$, $t \to \infty$, we readily obtain from (1.20) that $\partial^k f(t) = O(t^{-d+\delta})$, $t \to \infty$.

To study the behavior of (1.16) near $t = 0$, we choose any small $\delta > 0$ such that $c + \delta$ is not a nonnegative integer. In (1.16) shift the contour of integration to $\operatorname{Re} z = c+\delta$, and obtain that

$$Af(t) = \sum_{j=0}^{(-c]} \operatorname{Res}\{t^{-z}a(t,z)\tilde{f}(z);\ z = -j\} + I(t),$$

where $I(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} z = c+\delta} t^{-z}a(t,z)\tilde{f}(z)dz$. It is easy to show that $\partial^k I(t) = O(t^{-c+\delta})$, $t \to 0^+$. We now study the residue

$$(1.22) \qquad R_{-j} = \operatorname{Res}\{t^{-z}a(t,z)\tilde{f}(z);\ z = -j\}.$$

Choose a small $\epsilon > 0$ such that $(-c-\epsilon] = (-c]$. Since $a(t,z) \in \mathfrak{F}_{c+j+\epsilon,d+j-\epsilon}(\mathcal{O}^m_{-j-\epsilon,-j+\epsilon})$, then for $|z+j| \leq \epsilon/2$,

$$\partial_t^\ell\Big\{a(t,z) - \sum_{k=0}^{(-c-j]} \frac{1}{k!}\, a_{ko}(z)t^k\Big\} = O(t^{-c-k-2\epsilon}),$$

with $a_{ko}(z) \in \mathcal{O}^m_{-j-\epsilon,-j+\epsilon}$.

Using the representation

$$R_{-j} = \frac{1}{2\pi i}\int_{|z+j| = \epsilon/2} t^{-z}a(t,z)\tilde{f}(z)dz$$

for (1.22) we have that $\partial_t^\ell R_{-j}$ is

$$\partial_t^\ell\Big\{\sum_{k=0}^{(-c-j]} \frac{1}{k!}\, a_{ko}(-j)\, \frac{1}{j!}\, f_{jo}t^{k+j}\Big\} + O(t^{-c-j-2\epsilon}).$$

This proves that $Af \in \mathfrak{F}'_{c,d}$ and formulas (1.18) and (1.19). q.e.d.

We now study the composition of two operators.

THEOREM 1.2. *Let* $a(t,z) \in \Sigma^{m}_{c,d}$ *and* $b(t,z) \in \Sigma^{m'}_{c,d}$. *Then there is a symbol* $c(t,z) \in \Sigma^{m+m'}_{c,d}$ *such that*

1. *For every* N

$$(1.23)\quad c(t,z) - \sum_{k=0}^{N} \frac{1}{k!}\frac{\partial^k}{\partial z^k} b(t,z)(\partial_t^k a)(t,z) \in \Sigma^{m+m'-N-1}_{c,d}\ .$$

2. $b(t,\partial) \circ a(t,\partial) = c(t,\partial)$ (*as operators on* $\mathcal{J}'_{c,d}$).

PROOF: Fixing a cutoff function $\chi(t)$, we split the symbol of $a(t,z)$ as

$$(1.24)\qquad a(t,z) = \chi(t)a(t,z) + (1-\chi(t))a(t,z)$$
$$\equiv a_0(t,z) + a_\infty(t,z).$$

Consider the contribution of $b(t,\partial) \circ a_\infty(t,\partial)$. Since $a_\infty(t,z) \in \mathring{\Sigma}^{m}_{c,d}$, for $\operatorname{Re} z - d < \operatorname{Re} w < 0$ we define

$$(1.25)\qquad \mathfrak{m}a_\infty(w,z) = \int_0^\infty t^{w-1}a_\infty(t,z)dt.$$

Since $\mathfrak{m}a_\infty(w,z)$ is the Mellin transform of the function $t \to a_\infty\left(\frac{1}{t}, z\right)$ evaluated at the point $(-w)$, and $t \to a_\infty\left(\frac{1}{t}, z\right)$ is a function in $\mathcal{J}'_{\operatorname{Re} z - d, \operatorname{Re} z - c}$, the function $\mathfrak{m}a_\infty(w,z)$ is a meromorphic function of w in $S_z \equiv S_{c - \operatorname{Re} z, \operatorname{Re} z}$, with simple poles at $\{0,1,2,\dots\} \cap S_z$ with residues given by $-a_{j\infty}(z)/j!$; $j = 0,\dots,(d-\operatorname{Re} z]$; and we have the inversion formula

$$(1.26)\qquad a_\infty(t,z) = \frac{1}{2\pi i}\int_{\operatorname{Re} w = \gamma'} t^{-w}\mathfrak{m}a_\infty(w,z)dw,$$

$\operatorname{Re} z - d < \gamma' < 0.$

Fix $f \in \mathfrak{F}'_{c,d}$ and let $g(t) = a_\infty(t,\partial)f(t)$. If $\max(0,c) < \gamma < d$ and $c < \operatorname{Re} w = \gamma' < \gamma$, then

$$\tilde{g}(w) = \int_0^\infty t^{w-1}\left\{\frac{1}{2\pi i}\int_{\operatorname{Re} z=\gamma} t^{-z} a_\infty(t,z)\tilde{f}(z)\,dz\right\}dw$$

(1.27)

$$= \frac{1}{2\pi i}\int_{\operatorname{Re} z=\gamma} \mathfrak{m}a_\infty(w-z,z)\tilde{f}(z)\,dz.$$

The use of Fubini's Theorem in (1.27) is justified since $c - \operatorname{Re} z < \operatorname{Re} w - \operatorname{Re} z < 0$. Since $g(t) \in \mathfrak{F}_{c,d}$ $\tilde{g}(w)$ has no pole in $S_{c,d}$; hence for <u>any</u> γ', $c < \gamma' < d$, we have

$$(1.28)\quad b(t,\partial)g(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} w=\gamma'} t^{-w} b(t,w)\tilde{g}(w)\,dw.$$

Using (1.27) and Peetre's inequality, $(1+|w-z|)^{-N} \leq C(1+|w|)^{-N}(1+|z|)^{N}$, we apply Fubini's Theorem to obtain

$$(1.29)\quad b(t,\partial)\circ a_\infty(t,\partial)f(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} z=\gamma} t^{-z} c_\infty(t,z)\tilde{f}(z)\,dz,$$

where

$$(1.30)\quad c_\infty(t,z) = \frac{1}{2\pi i}\int_{\operatorname{Re} w=\gamma'} t^{z-w} b(t,w)\mathfrak{m}a_\infty(w-z,z)\,dw.$$

In (1.30) choose any $\gamma' \in (c, \operatorname{Re} z)$. We make the change of variables $w-z = v$, $\operatorname{Re} v = \gamma_1 \in (c-\operatorname{Re} z, 0)$, and apply Taylor's formula with integral remainder and the inversion formula (1.26), so that

$$(1.31)\quad c_\infty(t,z) = \sum_{k=o}^{N} \frac{1}{k!} \frac{\partial^k}{\partial z^k} b(t,z)(\partial_t^k a_\infty)(t,z) + R_\infty^N(t,z),$$

with

$$(1.32)\quad R_\infty^N(t,z)$$

$$= \frac{1}{N!}\int_o^1 (1-s)^N \frac{1}{2\pi i}\int_{\mathrm{Re}\, v=\gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t,z+sv)v^N \mathfrak{m} a_\infty(v,z)\,dv\,ds$$

Fix s, $0 \leq s \leq 1$, and let

$$(1.33)\quad c_s(t,z) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, v=\gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t,z+sv)v^N \mathfrak{m} a_\infty(v,z)\,dv$$

where $\gamma_1 \in (c-\mathrm{Re}\, z, 0)$. The function $c_s(t,z)$, originally defined by (1.33) for $\mathrm{Re}\, z > 0$ may be prolonged as a holomorphic function of z in the entire strip $S_{c,d}$ by using formula (1.33) with small γ_1; shifting the contour of integration to $\mathrm{Re}\, v = c - \mathrm{Re}\, z + \delta$ or $d - \mathrm{Re}\, z - \delta$ and taking into account the poles $v^N \mathfrak{m} a_\infty(v,a)$ at $v = 1,\ldots,(d-\mathrm{Re}\, z]$, we may show that

$$c_s(t,z) \in \mathring{\mathfrak{F}}_{c-c'}(\mathfrak{S}_{c',d'}^{m+m'-N-1})$$

if $(c',d') \subset\subset (c,d)$. Hence $R_\infty^N(t,z) \in \Sigma_{c,d}^{m+m'-N-1}$.

A similar, but simpler argument shows that if $a_o(t,z)$ is defined as in (1.24) then

(1.34) $$b(t,\partial) \circ a_o(t,\partial)f(t) = c_o(t,\partial)f(t),$$

where for $\max(0,c) < \operatorname{Re} z < d$

(1.35) $$c_o(t,z) = \sum_{k=o}^{N} \frac{1}{k!} \frac{\partial^k}{\partial z^k} b(t,z)(\partial_t^k a_o)(t,z) + R_o^N(t,z).$$

and

(1.36) $$R_o^N(t,z)$$

$$= \frac{1}{N!} \int_o^1 (1-s)^N \frac{1}{2\pi i} \int_{\operatorname{Re} v = \gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t,z+sv) v^N \tilde{a}_o(v,z)\,dv\,ds,$$

where $\gamma_1 \in (0, d-\operatorname{Re} z)$ and $\tilde{a}_o(v,z) = \int_o^\infty t^{v-1} a_o(t,z)\,dt$.
Again it follows that $R_o^N(t,z) \in \Sigma_{c,d}^{m+m'-N-1}$.

q.e.d.

We consider the transpose and the adjoint of an operator $A = a(t,\partial) \in Op - \Sigma_{c,d}^m$. The transpose of the operator ${}^tA$, and the adjoint of the operator, A^*, are defined by the relations:

For $f \in \mathcal{F}_{c,d}$, $\varphi \in C_o^\infty(R^+)$,

(1.37) $$\int_o^\infty Af(t)\varphi(t)\,dt = \int_o^\infty f(t)({}^tA\varphi)(t)\,dt,$$

$$\int_o^\infty Af(t)\overline{\varphi(t)}\,dt = \int_o^\infty f(t)\overline{A^*\varphi(t)}\,dt.$$

THEOREM 1.3. *Suppose that* $(c,d) \cap (0,1) \neq \emptyset$, *and that* $a(t,z) \in Op - \Sigma_{c,d}^m$. *Then there are symbols* ${}^ta(t,z)$ *and* $a^*(t,z) \in \Sigma_{1-d,1-c}^m$, *such that*

$$(1.38)\quad {}^{t}a(t,z) - \sum_{k=o}^{N} \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial w}\right)^k \left(\partial_t^k a\right)(t,w)\Big|_{w=1-z} \in \Sigma^{m-N-1}_{1-d,1-c}$$

$$(1.39)\quad a^*(t,z) - \sum_{k=o}^{N} \frac{(-1)^k}{k!}\left\{\left(\frac{\partial}{\partial w}\right)^k \left(\partial_t^k a\right)(t,w)\Big|_{w=1-\bar{z}}\right\}^* \in \Sigma^{m-N-1}_{1-d,1-c}$$

<u>and</u> <u>if</u> $A = a(t,\partial) \in \mathrm{Op} - \Sigma^m_{c,d}$, <u>then</u>

$$(1.40)\quad {}^{t}A = {}^{t}a(t,\partial) \in \mathrm{Op} - \Sigma^m_{1-d,1-c},$$

$$(1.41)\quad A^* = a^*(t,\partial) \in \mathrm{Op} - \Sigma^m_{1-d,1-c}.$$

We sketch the proof of Theorem 5.2. Split $a(t,z)$ as in (1.24). If $m \leq -2$ and $\varphi \in C_o^\infty(R^+)$ show that with $\max(0,c) < \gamma < 1$,

$$(1.42)\quad {}^{t}A_o\varphi(s) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, w = 1-\gamma} s^{-w} \int_o^\infty t^{w-1} a_o(t,1-w)\varphi(t)\,dt\,dw.$$

The proof then proceeds in the spirit of the proof of Theorem 5.1. See also Theorem 4 of [LP2]. In the case $m > -2$, choose $\lambda_o > d$ and write $a(t,z) = a_1(t,z)(z-\lambda_o)^{m+2}$ with $a_1(t,z) \in \Sigma^{-2}_{c,d}$. Then $a(t,\partial) = a_1(t,\partial)(\partial-\lambda_o)^{m+2}$. The transpose of the differential operator $(\partial-\lambda_o)^{m+2}$ may be calculated directly and ${}^{t}a(t,\partial)$ is handled by the first part of the proof. We leave the details to the reader.

2. <u>L^p estimates for pseudodifferential operators on R^+.</u>

Let $1 < p < \infty$. We shall prove estimates for the operators considered in Chapter 1 on $L^p(R^+)$ and on various weighted and Sobolev spaces on R^+. Let

$$\|f\|_p = \|f;L^p(R^+)\| = \left(\int_0^\infty |f(t)|^p dt\right)^{1/p}.$$

<u>THEOREM 2.1.</u> <u>Suppose that</u> $1 < p < \infty$ <u>and that</u> $c < 1/p < d$. <u>If</u> $a(t,z) \in \Sigma^0_{c,d}$ <u>and</u>

$$(2.1) \qquad Af(t) = a(t,\partial)f(t)$$

$$= \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z} a(t,z)\tilde{f}(t)dt,$$

<u>for</u> $f \in \mathcal{F}'_{c,d}$, <u>then there is a constant</u> $C = C(a,p)$ <u>such that</u>

$$(2.2) \qquad \|Af\|_p \le C\|f\|_p \ .$$

<u>PROOF</u>: For $f \in \mathcal{F}'_{c,d}$ define

$$(2.3) \quad T_p f(x) = F_p(x) = e^{-(1/p)x} f(e^{-x}) \in \mathcal{S}(R).$$

Then $\|F_p;L^p(R)\| = \|f;L^p(R^+)\|$ and

$$T_p Af(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi} a(e^{-x};1/p + i\xi)\mathcal{F}(F_p)(\xi)d\xi.$$

The symbol $b_p(x,\xi) = a(e^{-x},1/p + i\xi)$ satisfies

$$\frac{\partial^{j+k}}{\partial x^j\, \partial\xi^k} b_p(x,\xi) = O((1 + |\xi|)^{-k})$$

and hence $T_p A T_p^{-1}$ is a classical Calderon-Zygmund operator of class $S^0_{1,0}$, and hence

$$(2.4) \qquad \|T_p A T_p^{-1} F_p ; L^p(R)\| \le C_p \|F_p ; L^p(R)\| .$$

Cf. Meyer [M] or Coifman and Meyer [CM]. Inequality (2.3) now follows from (2.4).

q.e.d.

Before studying the continuity of operator $a(t,\partial) \in Op - \Sigma^m_{c,d}$ on Sobolev spaces we remark that if $a(t,z) \in \Sigma^m_{c,d}$, then $a(t,z-1) \in \Sigma^m_{c+1,d+1}$ and that if $c+1<d$, $(\partial/\partial t)a(t,z) \in \Sigma^m_{c+1,d}$.

For $k \in Z_+ = \{0,1,2,\ldots\}$, we let $W^{p,k}$ be the usual Sobolev space on R^+:

$$W^{p,k} = \{f \in L^p(R^+); f^{(j)} \in L^p(R^+),\ j = 0,\ldots,k\};$$

$$W^{p,k} = \{f \in W^{p,k}; f^{(j)}(0) = 0,\ j = 0,\ldots,k-1\}.$$

If $k \in Z_- = \{0,-1,-2,\ldots\}$, then $W^{p,k}$ is the dual of $W^{q,-k}$, $1/p + 1/q = 1$, under the bilinear form

$$\langle f,\varphi\rangle = \int_0^\infty f(t)\varphi(t),\ f \in C^\infty(\overline{R^+}),\ \varphi \in C_0^\infty(R^+).$$

<u>THEOREM 2.2.</u> <u>Suppose that</u> $a(t,z) \in \Sigma^0_{c,d}$ <u>and that</u> $A = a(t,\partial)$ <u>is defined by (2.1)</u>. <u>Then</u>

1. <u>If</u> $k \ge 0$ <u>and</u> $c+k < 1/p < d$,

$$(2.5) \qquad a(t,\partial): W^{p,k} \to W^{p,k}$$

<u>is continuous</u>; <u>moreover</u>,

(2.6) $$a(t,\partial):\mathring{W}^{p,k} \to \mathring{W}^{p,k}$$

<u>is continuous</u>.

2. <u>If</u> $j \geq 0$ <u>and</u> $c < 1/p < d-j$, <u>then</u>

(2.7) $$a(t,\partial):W^{p,-j} \to W^{p,-j}$$

<u>is continuous</u>.

<u>PROOF</u>: If $c+1 < 1/p < d$ and $f \in \mathcal{F}'_{c,d}$, then

$$\frac{d}{dt}\, a(t,\partial)f(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z-1}a(t,z)(-z\tilde{f}(z))dz$$

(2.8)

$$= \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}(\partial/\partial t)a(t,z)\tilde{f}(z)dz.$$

The first term on the r.h.s. of (2.8) is

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p+1} t^{-z}a(t,z-1)[f']^{\sim}(t)dt$$

$$= \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}a(t,z-1)[f']^{\sim}(z)dz.$$ The

second term on the r.h.s. of (2.8) is $(\partial/\partial t)a(t,\partial)f$. Hence by Theorem 2.1,

(2.9) $$\left\|\frac{d}{dt}\, a(t,\partial)f\right\|_p \leq C\|f'\|_p + C\|f\|_p \; .$$

The continuity of (2.5) now follows by induction on k; (2.6) follows since $a(t,\partial):\mathring{\mathcal{F}}'_{c,d} \to \mathring{\mathcal{F}}'_{c,d}$.

To prove 2., fix $f \in C_0^\infty(R)$ and $\varphi \in C_0^\infty(R^+)$. By Theorem 1.3,

$$\langle Af,\varphi\rangle = \langle f, {}^tA\varphi\rangle,$$

with ${}^tA = {}^ta(t,\partial)$, where ${}^ta(t,z) \in \Sigma^0_{1-d,1-c}$. If $1/p + 1/q = 1$ and $1-d+j < 1/q < 1-c$, i.e., $c < 1/p < d-j$, then ${}^tA: W^{q,j} \to W^{q,j}$ is continuous by part 1 of the Theorem. Part 2 follows.

q.e.d.

To study operator $a(t,\partial) \in Op - \Sigma^m_{c,d}$ with $m \neq 0$, we introduce weighted Sobolev spaces following the ideas and notation of Nourrigat [N2].

DEFINITION 2.1. For $\ell \in Z_+$ define

(2.10) $$W^{p,k}_\ell = \{f \in W^{p,k}; \partial^j f \in W^{p,k},\ j = 0,\dots,\ell\},$$

and

(2.11) $$\mathring{W}^{p,k}_\ell = \{f \in W^{p,k}; \partial^j f \in \mathring{W}^{p,k},\ j = 0,\dots,\ell\}.$$

To define $W^{p,k}_\ell$ with $\ell \in Z_-$, we introduce an invertible pdo $\Lambda = \lambda(\partial)$ of order 1. Let d_0 be a large positive number and define

(2.11) $$\lambda(z) = z - d_0 \in \Sigma^{-1}_{-\infty,\infty},$$

$$\lambda^{-1}(z) = (z-d_o)^{-1} \in \Sigma^{-1}_{-\infty,d_o}$$

and let $\Lambda = \lambda(\partial)$, $\Lambda^{-1} = \lambda^{-1}(\partial)$. If $f \in \mathscr{S}'_{c,d}$, $d \leq d_o$

$$\Lambda^{-1}f(t) = \int_o^t \left(\frac{t}{s}\right)^{-d_o} f(s)\,\frac{ds}{s}\,. \tag{2.12}$$

and

$$\Lambda\Lambda^{-1}f = \Lambda^{-1}\Lambda f = f. \tag{2.13}$$

LEMMA 2.1. If $\ell \in Z_+$, then $f \in W^{p,k}_\ell$ iff $\Lambda^\ell \in W^{p,k}$; moreover, $f \in W^{p,k}_\ell$ iff $\Lambda^\ell f \in W^{p,k}$. Also

$$\|f;W^{p,k}_\ell\| \approx \|\Lambda^\ell f;W^{p,k}\|.$$

PROOF: If $f \in W^{p,k}_\ell$, then $\|\Lambda^\ell f;W^{p,k}_\ell\| \leq C \sum_{j=o}^{\ell} \|\partial^j f;W^{p,k}\| = C\|f:W^{p,k}_\ell\|$.

If $f \in C^\infty_o(R)$, then for $j = 0,\ldots,\ell$, $\partial^j f = a_j(\partial)\Lambda^\ell f$ where $a_j(z) = z^j(z-d_o)^{-\ell} \in \Sigma^{j-\ell}_{-\infty,d_o}$. By Theorem 2.2, $\|\partial^j f;W^{p,k}\| \leq C\|\Lambda^\ell f:W^{p,k}\|$.

q.e.d.

DEFINITION 2.2. For $\ell \in Z_-$, a distribution $f \in \mathscr{S}'(\mathbb{R}^+)$ is in $W^{p,k}_\ell$ (respectively $\mathring{W}^{p,k}_\ell$) iff $f = \Lambda^{-\ell}g$, $g \in W^{p,k}$ (respectively $\mathring{W}^{p,k}$). The norm on $W^{p,k}_\ell$ is

(2.15) $$\|f;W^{p,k}_{\ell}\| = \|\Lambda^{\ell}f;W^{p,k}\|.$$

THEOREM 2.3. *Let* $m \in Z$ *and* $d \leq d_o$. *Let* $a(t,z) \in \Sigma^m_{c,d}$ *and define* $A = a(t,\partial)$. *Then*

1. *If* $k \in Z_+$ *and* $c + k \leq 1/p < d$, *then for all* $\ell \in Z$,

(2.16)
$$A : W^{p,k}_{\ell} \to W^{p,k}_{\ell-m}$$
$$A : \mathring{W}^{p,k}_{\ell} \to \mathring{W}^{p,k}_{\ell-m}$$

is continuous.

2. *If* $j \geq 0$ *and* $c < 1/p < d-j$, *then*

(2.17) $$A : W^{p,-j}_{\ell} \to W^{p,-j}_{\ell}$$

is continuous.

PROOF: If $f = \Lambda^{-\ell}g$, $g \in W^{p,k}$, then $\Lambda^{\ell-m}a(t,\partial)f = \Lambda^{\ell-m}a(t,\partial)\Lambda^{-\ell}g$ and the operator

$$b(t,\partial) = \Lambda^{\ell-m}a(t,\partial)\Lambda^{-\ell}$$

is in $Op - \Sigma^0_{c,d}$ by the Symbolic Calculus Theorem 1.2. Hence by Theorem 2.2, $\|Af;W^{p,k}_{\ell-m}\| \approx \|\Lambda^{\ell-m}Af;W^{p,k}\|$ $\leq C\|b(t,\partial)g;W^{p,k}\| \leq C\|g;W^{p,k}\| \approx \|f;W^{p,k}_{\ell}\|$.

q.e.d.

3. An algebra of pseudodifferential operators on $L^p(R^+)$.

Let $a(t,\partial) \in Op - \Sigma^{o}_{c,d}$ with $c < 1/p < d$. We wish to give conditions on the symbol $a(t,z)$ which determine whether the operator $a(t,\partial)$ is a Fredholm operator on $L^p(R^+)$ and to link the index of a Fredholm operator to topological properties of the symbol. It may happen that the operator $a(t,\partial)$ is Fredholm on $L^p(R^+)$ for some choices of $1/p \in (c,d)$ and not for others; the index of the operator (on $L^p(R^+)$) will change as we cross a non-Fredholm value of $1/p \in (c,d)$.

Before defining the notion of principal symbol, we introduce the functions in $\Theta^{o}_{o,1}$;

$$
\begin{aligned}
& h(z) = -\cot \pi z, \\
(3.1)\qquad & \theta(z) = (1 - e^{2\pi iz})^{-1}, \\
& 1 - \theta(z) = -e^{2\pi iz}(1 - e^{2\pi iz})^{-1}.
\end{aligned}
$$

Note that $\theta(1/p + i\infty) = 1$, $\theta(1/p - i\infty) = 0$, and that for $j \geq 1$, $(d/dz)^j\theta(z) \in \Theta^{-\infty}_{o,1}$. The corresponding operators in $Op - \Sigma^{o}_{o,1}$ have the representations:

$$
\begin{aligned}
& Hf(t) = h(\partial)f(t) = p.v.\ \frac{1}{\pi}\int_o^\infty \frac{f(s)}{t-s}\,ds, \\
(3.2)\qquad & \theta(\partial) = \frac{1}{2}(I - iH), \\
& I - \theta(\partial) = \frac{1}{2}(I + iH);
\end{aligned}
$$

H is the Hilbert transform on $L^p(R^+)$: Note that if $\mathcal{H}$ is the Hilbert transform on $L^p(R), (1/2)(I \pm i\mathcal{H})$ form a basis for singular integral operators on R.

DEFINITION 3.1. A symbol $a(t,z)$ is in the class $\Sigma_{1/p}$ iff for some c,d with $0 \leq c < 1/p < d \leq 1$,

1. $a(t,z) \in \Sigma^0_{c,d}$,

2. There are functions $a_+(t), a_-(t) \in \mathcal{F}_{c-d,d-c}$ such that

$$(3.3)\qquad a(t,z) - a_+(t)\theta(z) - a_-(t)(1-\theta(z)) \in \Sigma^{-1}_{c,d} .$$

If $a(t,z) \in \Sigma_{1/p}$, the functions $a_+(t), a_-(t)$ in (3.3) are uniquely determined by the relations

$$(3.4)\qquad a_+(t) = a(t, 1/p + i\infty),\ a_-(t) = a(t, 1/p - i\infty).$$

The function $a(t,z)$ is a continuous function defined on the compact rectangle:

$$(3.5)\qquad R_{1/p} = \{(t,z): 0 \leq t \leq \infty,\ z = 1/p + i\xi, -\infty \leq \xi \leq \infty\}.$$

DEFINITION 3.2. If $a(t,z) \in \Sigma_{1/p}$ and

$$(3.6)\qquad \begin{aligned} Af(t) &= a(t,\partial)f(t) \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z} a(t,z)\tilde{f}(z)\,dz, \end{aligned}$$

<u>for</u> $f \in C_o^\infty(R)$, <u>then the principal symbol</u>, $\sigma(A)$, <u>of the operator</u> A <u>is the function</u> $a(t,z)$ <u>restricted to the boundary of the compact rectangle</u> $R_{1/p}$.

REMARKS

1. $\sigma(A) = 0$ iff $a(t,z) \in \Sigma^{\alpha-1}_{c,d}$ for some c,d.

2. If for some $\delta > 0$ we are given functions $a_+(t)$, $a_-(t) \in \mathcal{F}_{-\delta,\delta}$, and $a_o(z)$, $a_\infty(z)$ such that the functions

$$
(3.7)\quad
\begin{aligned}
b_o(z) &= a_o(z) - a_+(0)\theta(z) - a_-(0)(1-\theta(z)),\\
b_\infty(z) &= a_\infty(z) - a_+(\infty)\theta(z) - a_-(\infty)(1-\theta(z)),
\end{aligned}
$$

are in $\Theta^{-1}_{1/p-\delta,1/p+\delta}$, we select a cutoff function $\chi(t) \in C_o^\infty(R)$, $\chi(t) = 1$ near $t = 0$, and define

$$
(3.8)\quad a(t,z) = \chi(t)b_o(z) + (1-\chi(t))b_\infty(z) + a_+(t)\theta(z) + a_-(t)(1-\theta(z)).
$$

Then $a(t,z) \in \Sigma_{1/p}$ and

$$
(3.9)\quad
\begin{cases}
a(0,z) = a_o(z), \quad a(\infty,z) = a_\infty(z),\\
a(t,1/p+i\infty) = a_+(t), \quad a(t,1/p-i\infty) = a_-(t).
\end{cases}
$$

3. That the principal symbol, $\sigma(A)$, of an operator $A = a(t,\partial)$ is uniquely determined by the operator A is a consequence of the following observations. Fix

$f \in C_0^\infty(R^+)$. Then for $\lambda \in R$,

$$(3.10) \qquad t^{i\lambda}A(t^{-i\lambda}f)(t) =$$

$$= \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}a(t,z+i\lambda)\tilde{f}(z)dz.$$

As $\lambda \to \pm\infty$, $t^{i\lambda}A(t^{-i\lambda}f)$ converges in $L^p(R^+)$ to $a_\pm(t)f(t)$. Next, for $\lambda > 0$, let $T_\lambda f(t) = \lambda^{-1/p}f(t/\lambda)$. Then $\|T_\lambda f\|_p = \|f\|_p$ and

$$(3.11) \quad (T_{1/\lambda}AT_\lambda)f(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}a(\lambda t,z)\tilde{f}(z)dz.$$

Hence in $L^p(R^+)$,

$$\lim_{\epsilon\to 0}(T_{1/\epsilon}AT_\epsilon f)(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}a(0,z)\tilde{f}(z)dz,$$

(3.12)

$$\lim_{\lambda\to\infty}(T_{1/\lambda}AT_\lambda f)(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, z = 1/p} t^{-z}a(\infty,z)\tilde{f}(z)dz.$$

The following result is fundamental in studying the Fredholm properties on $L^p(R^+)$ operators defined by (3.6).

__THEOREM 3.1.__ __If__ $a(t,z) \in \Sigma_{1/p}$, __then__

$$A = a(t,\partial) : L^p \to L^p$$

__is a compact operator iff the principal symbol__, $\sigma(A)$, __is identically__ 0; __i.e.__, $a(t,z) \in \tilde{\Sigma}^{-1}_{c,d}$ __for some__ c,d, $c < 1/p < d$.

PROOF: We first show that if $a(t,z) \in \Sigma^{-1}_{c,d}$ then $a(t,\partial)$ is compact on $L^p(\mathbb{R}^+)$. If $a(t,z) \in \Sigma^{-1}_{c,d}$, then $f \to -t\frac{\partial}{\partial t}Af$ is bounded on $L^p(\mathbb{R}^+)$ by Theorems 1.2 and 2.1. Hence if $a(t,z) = 0$ for t outside $[\epsilon,T]$, $0 < \epsilon < T < \infty$, A maps L^p into $W^{p,1}(\epsilon,T)$, which is compactly embedded in L^p. If $a(t,z) \in \mathring{\Sigma}^{-1}_{c,d}$, and $\epsilon > 0$, let $\chi(t)$ be a cutoff function, and define

$$a_\epsilon(t,z) = (1 - \chi(t/\epsilon))\chi(\epsilon t)a(t,z);$$

$a_\epsilon(t,z)$ converges to $a(t,z)$ in $\mathring{\Sigma}^{-1}_{c,d}$ as $\epsilon \to 0^+$. The corresponding operators $a_\epsilon(t,\partial)$ are compact on L^p and converge in operator norm to $a(t,\partial)$.

Conversely, suppose $A = a(t,\partial) : L^p \to L^p$ is compact. We use the constructions of Remark 3. following Definition 3.2. Fix $f \in C_0^\infty(R^+)$. Then as $\lambda \to \pm\infty$, $t^{-i\lambda}f(t)$ converges to 0 weakly in L^p, by the Riemann-Lebesgue Lemma, so that $t^{i\lambda}A(t^{-i\lambda}f)$ converges to 0 strongly in L^p. Hence $a_+(t) = a_-(t) \equiv 0$. If $T_\lambda f(t) = \lambda^{-1/p}f(t/\lambda)$, as $\lambda \to 0^+$ or $\lambda \to \infty$, $T_\lambda f$ converges to 0 weakly in L^p, and $A(T_\lambda f)$ converges to 0 in L^p. Each of the operators in (3.11) is 0 so that $a(0,z) = a(\infty,z) \equiv 0$.

q.e.d.

DEFINITION 3.4. Let $a(t,z) \in \Sigma_{1/p}$ and let $a(t,\partial) \in Op - \Sigma_{1/p}$ be defined by (3.6). Then A *is said to be elliptic on* $L^p(R^+)$ *iff its principal symbol*, $\sigma(A)$, *satisfies*

$$\sigma(A)(t,z) \neq 0, \ (t,z) \in \partial R_{1/p} \ . \tag{3.13}$$

If $a(t,z)$ is the symbol of an elliptic operator on $L^p(R^+)$, then for $0 \leq t \leq \infty$,

$$\begin{aligned} a_+(t) &= a(t,1/p + i\infty) \neq 0, \\ a_-(t) &= a(t,1/p - i\infty) \neq 0, \end{aligned} \tag{3.14}$$

which are conditions independent of p, and reflect the fact that $a(t,\partial)$, considered as a classical pdo of order 0 on the noncompact manifold R^+ is elliptic. If $a(t,z) \in \Sigma_{1/p}$, the conditions

$$a(0,z) \neq 0, \ a(\infty,z) \neq 0, \ \mathrm{Re}\, z = 1/p \tag{3.15}$$

are then satisfied for all p such that $1/p$ lies outside a discrete set in (c,d).

We now relate "ellipticity on $L^p(R^+)$" to the Fredholm properties of the operator $A = a(t,\partial)$.

THEOREM 3.2. *Let* $A = a(t,\partial) \in Op - \Sigma_{1/p}$. *The following conditions are equivalent:*

1. $a(t,\partial)$ *is elliptic on* $L^p(R^+)$.
2. *For some* c,d, $0 \leq c < 1/p < d \leq 1$, *there is a symbol* $b(t,z) \in \Sigma^0_{c,d}$ *such that*

$a(t,\partial)b(t,\partial) - I,\ b(t,\partial)a(t,\partial) - I \in Op - \Sigma_{c,d}^{-1}$.

3. __There is a bounded operator__ B __on__ $L^p(R^+)$ __such that__ $AB - I$ __and__ $BA - I$ __are compact operators on__ $L^p(R^+)$.

4. __For__ $f \in C_0^\infty(R^+)$, __there is an apriori estimate__

$$\|f\|_p \leq C\|Af\|_p + \|Kf\|_p, \tag{3.16}$$

__where__ K __is a compact operator on__ $L^p(R^+)$.

PROOF: 1 implies 2: Choose $c < 1/p < d$ such that $a(0,z)$ and $a(\infty,z)$ have no zeroes in the strip $S_{c,d}$. Let $b(t,z) \in \Sigma_{c,d}^{0}$ be such that for $c < 1/q < d$, $b(t,z)a(t,z) = a(t,z)b(t,z) \equiv 1$ on the boundary of $R_{1/q}$; $b(t,\partial)$ could be constructed as in Remark 2 following Definition 3.2. Condition 2 now follows from Theorem 1.2.

2 implies 3: This follows from Theorem 3.1.

3 implies 4: This is obvious.

4 implies 1: We use the constructions of Remark 3 following Definition 3.2. Fix $f \in C_0^\infty(R^+)$. Then if K is compact on L^p, $\|t^{i\lambda}K(t^{-i\lambda}f)\|_p \to 0$ as $\lambda \to \pm\infty$. Inequality (3.16) implies that

$$\|f\|_p \leq C\|t^{i\lambda}A(t^{-i\lambda}f)\|_p + \|t^{i\lambda}K(t^{-i\lambda}f)\|_p$$

and letting $\lambda \to \pm\infty$, we obtain the apriori inequality

$$(3.17) \qquad \|f\|_p \leq C\|a_\pm(t)f\|_p$$

which implies that $|a_\pm(t)| \geq \frac{1}{C}$. If $T_\lambda f(t) = \lambda^{-1/p}f(t/\lambda)$, then

$$(3.18) \qquad \|f\|_p \leq C\|T_{1/\lambda}A(T_\lambda f)\|_p + \|T_{1/\lambda}KT_\lambda f\|_p.$$

Letting $\lambda \to 0$ or $\lambda \to \infty$ in (3.18) and applying (3.12) we obtain

$$(3.19) \qquad \|f\|_p \leq C\|a(0,\partial)f\|_p, \ \|f\|_p \leq C\|a(\infty,\partial)f\|_p,$$

for $f \in C_0^\infty(R^+)$. Inequalities (3.19) imply that the Mellin multiplier operators $a(0,\partial)$ and $a(\infty,\partial)$ have closed range and hence the symbols $a(0,z)$ and $a(\infty,z)$ have no zeroes on $\operatorname{Re} z = 1/p$.

q.e.d.

For an NxN system of operators in $Op - \Sigma_{1/p}$, the notion of ellipticity on $L^p(\equiv [L^p(R^+)]^N)$ is that

$$(3.20) \qquad \det[\sigma(A_{ij})] \neq 0, \ (t,z) \in \partial R_{1/p},$$

where $[\sigma(A_{ij})]$ is the matrix of principal symbols. A theorem analogous to Theorem 3.2 may be proved for such a system.

Since $\partial R_{1/p}$ is homeomorphic to the unit circle S^1, if A is an NxN system of operators which is elliptic on L^p, then $[\sigma(A_{ij})]$, considered as a map from $\partial R_{1/p}$ to $GL(N,C)$, has a <u>topological degree</u>, $n_{1/p}$, defined as

$$(3.21)\quad \begin{cases} n_{1/p} = n_{1/p}(\sigma(A)) = \text{winding number } \det[\sigma(A_{ij})] \\ \quad = \frac{1}{2\pi}\,\Delta_{\partial R_{1/p}} \operatorname{Arg}\det[\sigma(A_{ij})], \end{cases}$$

where the change in argument is calculated as $\partial R_{1/p}$ is traversed in the <u>clockwise direction</u>.

We emphasize that $n_{1/p}$ depends on $1/p$ and that if p is changed so that the line $\operatorname{Re} z = 1/p$ crosses a zero of $\det[\sigma(A_{ij})](0,z)$ or $\det[\sigma(A_{ij})](\infty,z)$, the jump in $n_{1/p}$ may be calculated by the Argument Principle.

The <u>analytical index</u> of an $N\times N$ system $A = [a_{ij}(t,\partial)]$ of operators in $Op - \Sigma_{1/p}$ which is elliptic on L^p is defined as

$$(3.22)\qquad \operatorname{ind}_p(A) = \dim\ker A - \dim\ker A^*,$$

where in (3.22) we have considered A as an operator on $[L^p(R^+)]^N$ and $A^* = [a^*_{ji}(t,\partial)]$ as an operator on $[L^q(R^+)]^N$, $1/p + 1/q = 1$.

The index theorem we shall develop in Theorem 3.3 is that

$$(3.23)\qquad \operatorname{ind}_p(A) = n_{1/p}(\sigma(A)).$$

The first step in the proof of (3.23) is the construction of a canonical elliptic system A for which

$$\operatorname{ind}_p(A) = 1 = n_{1/p}(\sigma(A)). \tag{3.24}$$

For this see the example of Cordes and Herman [CH,C], or consider the following example.

For $f \in L^p(R)$ define the (column) vector function $f = (f_1, f_2) \in [L^p(R^+)]^2$ by

$$f_1(t) = f(t),\ f_2(t) = f(-t). \tag{3.25}$$

The Hilbert transform on $L^p(R)$

$$\mathcal{H}f(x) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy, \tag{3.26}$$

may be represented as the 2×2 system of operators in $Op - \Sigma_{1/p}$:

$$\begin{aligned} (\mathcal{H}f)_1(t) &= Hf_1(t) + Sf_2(t) \\ (\mathcal{H}f)_2(t) &= -Sf_1(t) - Hf_2(t), \end{aligned} \tag{3.27}$$

when $H = h(\partial)$ is a Hilbert transform on $L^p(R^+)$ defined in (3.1-2) and

$$Sf(t) = s(\partial)f(t) = \frac{1}{\pi}\int_0^{\infty} \frac{f(s)}{t+s}\,ds \tag{3.28}$$

is the Stieltjes transform on $L^p(R^+)$ with symbol $s(z) = \csc \pi z \in \mathfrak{S}^{-\infty}_{0,1}$. For $f \in L^p(R)$, define

$$(3.29)\quad \begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x-z}\,dx, \quad \operatorname{Im} z \neq 0, \\ f^{\pm}(x) &= \lim_{\epsilon\to 0^+} f(x\pm i\epsilon) = \frac{1}{2}(I \pm i\mathcal{H})f(x). \end{aligned}$$

For $x \in R$ let $w = w(x) = \frac{x-i}{x+i}$; then $f(x) \to M_w f(x) = w(x)f(x)$ has the system representation

$$(3.30)\quad (M_w f)_1(t) = w(t)f_1(t), \quad (M_w f)_2(t) = \overline{w}(t)f_2(t).$$

It is well known that if on $L^p(\mathbb{R})$ we define $\mathcal{C}f(x) = w(x)f^-(x) + f^+(x)$, $\mathcal{C}$ has kernel spanned by $f(x) = \frac{1}{w(x)} - 1$ and cokernel 0. When $\mathcal{C}$ is written as a 2×2 system of pseudodifferential operators we obtain the matrices of principal symbols

$$\sigma(\mathcal{C})\left(t, \tfrac{1}{p} + i\infty\right) = \begin{bmatrix} w(t) & 0 \\ 0 & 1 \end{bmatrix}$$

$$(3.31)\quad \sigma(\mathcal{C})(0,z) = \begin{bmatrix} -ih & is \\ -is & ih \end{bmatrix}, \qquad \sigma(\mathcal{C})(\infty,z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma(\mathcal{C})\left(t, \tfrac{1}{p} - i\infty\right) = \begin{bmatrix} 1 & 0 \\ 0 & \overline{w}(t) \end{bmatrix}.$$

Using (3.31) we have that $\operatorname{ind}_p(\mathcal{C}) = 1 = n_{1/p}(\sigma(\mathcal{C}))$.

The second step in the proof of (3.23) is, given a system A of operators which is elliptic on L^p and such that $n_{1/p}(\sigma(A)) = 0$, to construct a homotopy through elliptic operators to the identity operator.

Suppose that A_0 and A_1 are systems which are elliptic on L^p and there is a homotopy between the matrices of principal symbols

$$\sigma(A_s) : \partial R_{1/p} \to GL(N,C), \quad 0 \leq s \leq 1 \tag{3.32}$$

such that for each s, $\sigma(A_s)$ is a matrix of principal symbols, and

1. If $A_{s\pm}(t) = \sigma(A_s)(t.1/p \pm i\infty)$, then

$$\lim_{s\to s'} \sup_{0\leq t\leq\infty} \|A_{s\pm}(t) - A_{s'\pm}(t)\| = 0. \tag{3.33}$$

2. $\displaystyle\lim_{s\to s'} \sup_{\operatorname{Re} z = 1/p} \|\sigma(A_s)(0,z) - \sigma(A_{s'})(0,z)\| = 0,$

$$\lim_{s\to s'} \sup_{\operatorname{Re} z = 1/p} \left\|z \frac{d}{dz}\left\{\sigma(A_s)(0,z) - \sigma(A_{s'})(0,z)\right\}\right\| = 0. \tag{3.34}$$

3. $\displaystyle\lim_{s\to s'} \sup_{\operatorname{Re} z = 1/p} \|\sigma(A_s)(\infty,z) - \sigma(A_{s'})(\infty,z)\| = 0,$

$$\lim_{s\to s'} \sup_{\operatorname{Re} z = 1/p} \left\|z \frac{d}{dz}\left\{\sigma(A_s)(\infty,z) - \sigma(A_{s'})(\infty,z)\right\}\right\| = 0. \tag{3.35}$$

A homotopy (3.32) satisfying (3.33-35) will be called a <u>homotopy through elliptic symbols</u>; for each s we use the construction of (3.8) for a matrix of operators, A_s, with the given symbols $\sigma(A_s)$; by the 1-

dimensional Marcinkiewicz Theorem we have that $s \to A_s$ is continuous from $[0,1]$ into the bounded operators on $[L^p(R^+)]^N$. Hence $\text{ind}_p(A_0) = \text{ind}_p(A_1)$.

We now outline the proof of the Index Theorem. As a first step consider the scalar case.

LEMMA 3.1. *Let* $\sigma(A)$ *be the symbol of an operator* $A = a(t,\partial) \in Op - \Sigma_{1/p}$ *which is elliptic on* $L^p(R^+)$ *and suppose that*

$$n_{1/p} = 0$$

where $n_{1/p} = n_{1/p}(\sigma(A))$ *is defined by (3.21) with* $N = 1$. *Then*

$$\text{ind}_p(A) = 0.$$

PROOF: The required homotopy through elliptic symbols is given by

$$\sigma(A_s)(t,z) = e^{s\log\sigma(A)}, \quad 0 \leq s \leq 1, \ (t,z) \in \partial R_{1/p}.$$

q.e.d.

We now introduce a nonzero scalar function $\varphi \in \mathcal{F}_{-\delta,\delta}$ such that $\varphi(0) = \varphi(\infty) = 1$ and $n(\varphi) = \frac{1}{2\pi}\Delta \underset{0\leq t\leq\infty}{\text{Arg}}\, \varphi = 1$. For $k \in Z$ define the operator

(3.36) $$A_k = a_k(t,\partial) = \varphi(t)^k\theta(\partial) + (I - \theta(\partial)).$$

Then A_k is elliptic on L^p for $1 < p < \infty$, and $n_{1/p}(\sigma(A_k)) = n(\varphi^k) = k$. Define

$$J = \operatorname{ind}_p(A_1). \tag{3.37}$$

Then

$$\operatorname{ind}_p(A_k) = k \operatorname{ind}_p(A_1) = Jk. \tag{3.38}$$

(We will show in Theorem 3.3 that in fact, $J = 1$.)

The next Lemma is evident.

<u>LEMMA 3.2. If</u> $A = [\delta_{ij} a_{ii}(t,\partial)]$ <u>is a diagonal system of operators which is elliptic on</u> L^p. <u>Then</u>

$$\operatorname{ind}_p(A) = J\, n_{1/p}(\sigma(A)), \tag{3.39}$$

<u>where</u> J <u>is defined as in (3.37)</u>.

We next calculate the index of a system whose symbols is the identity matrix on 3 sides of $\partial R_{1/p}$.

<u>LEMMA 3.3. Let</u> $\sigma(A) = [\sigma(A_{ij})]$ <u>be an</u> $N \times N$ <u>matrix of principal symbols a system</u> A <u>which is elliptic on</u> L^p. <u>Suppose moreover that</u>:

$$\begin{aligned} &\sigma(A)(t, 1/p + i\infty) = \sigma(A)(t, 1/p - i\infty) \equiv I, \\ &\sigma(A)(\infty, z) = I. \end{aligned} \tag{3.40}$$

<u>Then</u>

$$(3.41) \qquad \operatorname{ind}_p(A) = J\, n_{1/p}(\sigma(A)).$$

PROOF: Multiplying A by a diagonal system, B, of operators satisfying the hypotheses of Lemma 3.2, condition (3.40), and such that $n_{1/p}(\sigma(B)) = -n_{1/p}(\sigma(A))$, we may assume that $n_{1/p}(\sigma(A)) = 0$.

From condition (3.40) we may choose c,d, $0 \leq c < 1/p < d \leq 1$ such that $a_{ij}(0,z) = \delta_{ij} + g_{ij}(z)$, where $g_{ij} \in \mathfrak{G}^{-1}_{c,d}$ and $[a_{ij}(0,z)]$ has nonvanishing determinant in $S_{c,d}$.

Given $\epsilon > 0$, shrinking (c,d) if necessary, by Runge's Theorem there are rational functions $g_{ij\epsilon} \in \mathfrak{G}^{-1}_{c,d}$ such that

$$\sup_{S_{c,d}} |g_{ij} - g_{ij\epsilon}| < \epsilon, \ \sup_{S_{c,d}} |z \tfrac{d}{dz}\{g_{ij} - g_{ij\epsilon}\}| < \epsilon.$$

Let $a_{ij\epsilon}(0,z) = \delta_{ij} + g_{ij\epsilon}(z)$ and let A_ϵ be a system of operators with principal symbol, $\sigma(A_\epsilon)$ which satisfies (3.40) and such that $\sigma(A_\epsilon)(0,z) = [a_{ij\epsilon}(0,z)]$. If ϵ is small, then $n_{1/p}(\sigma(A_\epsilon)) = 0$ and $\operatorname{ind}_p A_\epsilon = \operatorname{ind}_p A$. We have reduced the problem to the case that $\sigma(A_\epsilon)(0,z)$ consists of rational functions.

Make a fractional linear transformation, $z \to w$, which maps the half plane $\{\operatorname{Re} z < 1/p\}$ onto the interior of the unit circle $\{|w| < 1\}$ and such that as z traverses $\operatorname{Re} z = 1/p + i\xi$, $-\infty \leq \xi \leq \infty$, w traverses $\{|w| = 1\}$ in the clockwise direction. Define $G(w) = [a_{ij\epsilon}(0,z)]$. Then $G : S^1 \to GL(N,C)$ is a rational function. Now (Cf., e.g., [V, pp. 45-49]) there are matrices of rational functions, $G_0(w)$, $G_\infty(w)$ holomorphic in $\{|w| \leq 1\}$ and $\{|w| \geq 1\}$ respectively such that

$$G(w) = G_0(w)\, D\, G_\infty(w) \tag{3.42}$$

where

$$D = D(n_1,\ldots,n_N) = [\delta_{ij} w^{n_i}] \tag{3.43}$$

is a diagonal matrix and $n_1 + \ldots + n_N = 0 = \frac{1}{2\pi} \Delta \underset{|w|=1}{\operatorname{Arg}} \det G(w)$.

For $0 \leq s \leq 1$, define

$$G_s(w) = G_0^{-1}(s) G_0(sw) D\, G_\infty(w/s) G_\infty^{-1}(1/s). \tag{3.44}$$

Formula (3.44) gives a homotopy through elliptic symbols to a diagonal matrix of operators which satisfies the hypotheses of Lemma 3.2.

<u>q.e.d.</u>

It is now straightforward to prove the formula

$$\operatorname{ind}_p(A) = J\, n_{1/p}(\sigma(A)) \tag{3.45}$$

in the case that $\sigma(A)$ is the principal symbol of an elliptic operator whose principal symbol is the identity matrix I at the corners of $\partial R_{1/p}$; i.e.,

$$\sigma(A)(0,1/p \pm i\infty) = \sigma(A)(\infty,1/p \pm i\infty) = I. \tag{3.46}$$

To handle the general case we use the following Lemma whose proof we leave to the reader.

LEMMA 3.4. *Given 4 matrices A_{0+}, A_{0-}, $A_{\infty+}$, $A_{\infty-}$ in $GL(N,C)$, there is a symbol, $\sigma(A)$, of an elliptic system on L^p such that*

$$\begin{aligned} \sigma(A)\left(0, \frac{1}{p} \pm i\infty\right) &= A_{0,\pm} \\ \sigma(A)\left(0, \frac{1}{p} \pm i\infty\right) &= A_{\infty,\pm} \end{aligned} \tag{3.47}$$

and such that

$$\mathrm{ind}_p(A) = J\, n_{1/p}(\sigma(A)). \tag{3.48}$$

We are now ready to prove (3.23).

THEOREM 3.3. (The Index Theorem). *If A is a system of operators in $\mathrm{Op}\, \Sigma_{1/p}$ which is elliptic on L^p, then*

$$\mathrm{ind}_p(A) = n_{1/p}(\sigma(A)). \tag{3.23}$$

PROOF: By Lemmas 3.1 to 3.4 we have that $\mathrm{ind}_p(A) = J\, n_{1/p}(\sigma(A))$. The example studied in (3.25) to (3.31) shows that $J = 1$. q.e.d.

4. Pseudodifferential operators on a finite interval.

We shall develop an algebra of pdo's on a finite interval $I = [0,\pi]$. For $f \in L^p(I)$, define

$$Tf(t) = f(\pi-t).$$

Then $T^* = T$ and $TT = I$. If A is a bounded operator on $L^p(I)$, the essential requirement for A to be in our class of pdo's is that, when acting on functions supported in $[0,\pi)$, both A and TAT, modulo compact operators, are in $Op - \Sigma_{1/p}$ on R^+.

DEFINITION 4.1. *A bounded operator A on $L^p(I)$ is a pdo of class $Op - \Sigma_{1/p}(I)$ iff*

1. *If $\varphi,\psi \in C_o^\infty(R)$ have disjoint supports then the map*

$$(4.1) \qquad f \to (\varphi A\psi)f \equiv \varphi\chi_{[0,\pi]}A(\chi_{[0,\pi]}\psi f)$$

is a compact operator on $L^p(R^+)$.

2. *If $\varphi,\psi \in C_o^\infty([0,\pi))$, there is an operator $A_{\varphi\psi} \in Op - \Sigma_{1/p}$ and a compact operator $K_{\varphi\psi}$ on $L^p(I)$ such that*

$$(4.2) \qquad \varphi A\psi = A_{\varphi\psi} + K_{\varphi\psi}.$$

3. *The operator*

$$(4.3) \qquad f \to TATf = A_T f$$

<u>with</u> $Tf(t) = f(\pi - t)$ <u>satisfies</u> <u>conditions</u> 1 <u>and</u> 2.

An example of an operator of class $Op - \Sigma_{1/p}(I)$ is the finite Hilbert transform

$$(4.4) \qquad Hf(t) = \text{p.v.}\ \frac{1}{\pi}\int_0^\pi \frac{f(s)}{t-s}\,ds, \ (THT = -H),$$

and the finite Stieltjes transform

$$(4.5) \qquad Sf(t) = \text{p.v.}\ \frac{1}{\pi}\int_0^\pi \frac{f(s)}{t+s}\,ds.$$

For $\varphi,\psi \in C_0^\infty([0,\pi))$, $\varphi TS T\psi$ is a compact operator on $L^p(I)$.

To define the principal symbol of a pdo of class $Op - \Sigma_{1/p}(I)$, we first note that if $\varphi,\psi \in C_0^\infty([0,\pi))$ and $a_{O\varphi\psi}(t,\partial) = A_{O\varphi\psi} \in Op - \Sigma_{1/p}$ is defined so that

$$(4.6) \qquad \varphi A\psi = A_{O\varphi\psi} + K_{O\varphi\psi}$$

as in (4.2), then there is a well defined function $a_0(t,z)$ defined on

$$(4.7) \qquad \Lambda_{1/p} = \{(t,z) : (t,z) \in \partial R_{1/p},\ 0 \le t < \pi\}$$

such that

$$(4.8) \qquad \sigma(a_{O\varphi\psi}(t,\partial)) = \varphi(t)a_0(t,z)\psi(t)$$

for $(t,z) \in \Lambda_{1/p}$. The function $a_0(t,z)$ is defined by choosing $\varphi,\psi \in C_0^\infty([0,\pi))$ which are $= 1$ in a neighborhood of t and letting

$$(4.9) \qquad a_0(t,z) = \sigma(a_{0\varphi\psi})(t,z).$$

The formula (4.9) uniquely determines $a_0(t,z)$ on $\Lambda_{1/p}$; if φ_1 and ψ_1 are also $= 1$ near t then $\varphi_1\varphi A\psi\psi_1 = \varphi_1\psi_1 a_{0\varphi\psi}(t,\partial) + K = \varphi\psi a_{0\varphi_1\psi_1} + K_1$ where K and K_1 are compact operators on $L^p(R^+)$, and by Theorem 3.1, $\varphi_1\psi_1 a_{0\varphi\psi}$ and $\varphi\psi a_{0\varphi_1\psi_1}$ have the same principal symbols.

The operator $A_T = TAT$ has a well defined principal symbol on $\Lambda_{1/p}$ which we denote by $a_\pi(t,z)$.

To glue together the symbols $a_0(t,z)$ and $a_\pi(t,z)$ we will show that for $0 < t < \pi$, the following compatibility conditions are satisfied:

$$(4.10) \qquad \begin{aligned} a_0(t,1/p + i\infty) &= a_\pi(\pi - t,1/p - i\infty), \\ a_0(t,1/p - i\infty) &= a_\pi(\pi - t,1/p + i\infty). \end{aligned}$$

To prove (4.10) we introduce the operators

$$(4.11) \qquad \begin{aligned} \Theta &= \tfrac{1}{2}(I - iH) \\ I - \Theta &= \tfrac{1}{2}(I + iH) = T\Theta T \end{aligned}$$

with H the finite Hilbert transform given by (4.4). Fix $\varphi \in C_0^\infty((0,\pi))$ such that $T\varphi(t) = \varphi(t)$ and $\varphi(t) = 1$ on $(\delta,\pi-\delta)$. As operators on $L^p(R^+)$,

$$(4.12) \qquad \varphi A\varphi = \varphi\{a_0(t,1/p + i\infty)\theta(\partial) + a_0(t,1/p - i\infty)\}\varphi + K_0,$$

and

$$(4.13) \qquad \varphi TAT\varphi = \varphi\{a_\pi(t,1/p + i\infty)\theta(\partial) + a_\pi(t,1/p - i\infty)\}\varphi + K_\pi,$$

with K_0 and K_π compact on $L^p(R^+)$ and $\theta(\partial)$ defined by (3.2). Modulo compact operators, on $L^p(I)$, we have the representation:

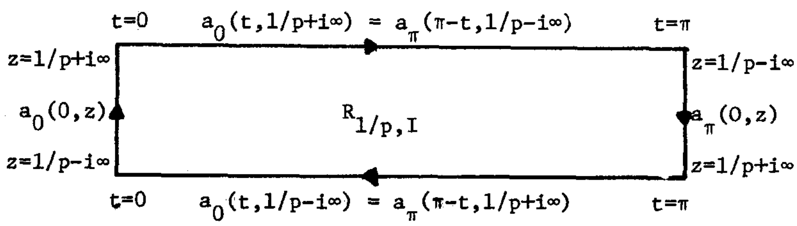
$$\varphi A\varphi = \varphi(a_0(t,1/p + i\infty)\Theta + a_0(t,1/p - i\infty)(I - \Theta))$$

and since $T\Theta T = I - \Theta$,

$$(4.14) \qquad T\varphi A\varphi T = \varphi\{a_0(\pi - t,1/p + i\infty)(I - \Theta) + a_0(\pi - t,1/p - i\infty)\Theta\}\varphi.$$

Comparing (4.14) to (4.13) we obtain (4.10).

If A is a pdo of class $Op - \Sigma_{1/p}(I)$, the <u>principal symbol</u> of A, $\sigma(A)$, is defined as the pair of functions $a_0(t,z)$, $a_\pi(t,z)$ defined on $\Lambda_{1/p}$ which satisfy the compatibility conditions (4.10). The natural setting for the <u>principal</u> symbol as a continuous function defined on the boundary of a compact rectangle $R_{1/p,I}$ in (4.15):



It can be shown that a pdo of class $Op - \Sigma_{1/p}(I)$ is compact iff its principal symbol is $\equiv 0$.

The principal symbol of an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ is defined as the matrix of principal symbols.

DEFINITION 4.2. *An $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ is elliptic on $L^p \equiv [L^p(I)]^N$ iff the determinant of the matrix of principal symbols does not vanish on $\partial R_{1/p,I}$.*

THEOREM 4.1. *Let $A = [A_{ij}]$ be an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$. The following are equivalent:*

1. *A is elliptic on L^p.*
2. *There is an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ such that*

$$AB - I \quad \text{and} \quad BA - I$$

are compact operators on $[L^p(I)]^N$.

3. *There is a bounded operator B on $[L^p(I)]^N$ such that $BA - I$ is a compact operator on $[L^p(I)]^N$..*
4. *For $\vec{f} = (f_1,\dots,f_N)$ and N-tuple of functions in $C_0^\infty((0,1))$, there is an a priori estimate*

$$(4.16)\qquad \|f\,;L^p(I)\| \le c\|Af:L^p(I)\| + \|Kf:L^p(I)\|$$

<u>where</u> K <u>is</u> <u>a</u> <u>compact</u> <u>operator</u> <u>on</u> $[L^p(I)]^N$.

<u>PROOF</u>: That 1. implies 2. implies 3. implies 4. is obvious. If (4.16) holds, let $f\in[C_0^\infty((2\delta,\pi-2\delta))]^N$, and let $\varphi\in C_0^\infty([0,\pi))$ be such that $\varphi=1$ on $[0,\pi-\delta)$. Then $Af=\varphi A_0\varphi f+K$, where K is a compact operator on $L^p(R^+)$. Using the techniques in the proof of Theorem 3.2, we obtain the apriori estimates

$$(4.17)\qquad \|f\,;L^p(R^+)\| \le c\|A_0(t;1/p\pm i\infty)f;L^p(R^+)\|,$$

$$(4.18)\qquad \|f\,;L^p(R^+)\| \le c\|A_0(0,\partial)f;L^p(R^+)\|.$$

Inequalities (4.17) and (4.18) are valid for all $\vec{f}\in[C_0^\infty((\delta,\pi-2\delta))]^N$, with c independent of δ. Since $A_0(0,\partial)$ commutes with dilations, (4.18) is valid for all $\vec{f}\in[C_0^\infty(R^+)]^N$ and therefore $\det[a_{ij0}(0,z)]\neq 0$, $z=1/p+i\xi$, $-\infty\le\xi\le\infty$. The continuity of $\sigma(A)(t,z)$ and (4.17) show that $[a_{ij0}(t,z)]$ is nonsingular on $\Lambda_{1/p}$.

<u>q.e.d.</u>

If A is a system of pdo's of class $Op-\Sigma_{1/p}(I)$ which is elliptic on L^p we define

$$(4.19)\qquad n_{1/p}(\sigma(A)) = \frac{1}{2\pi}\,\Delta_{\partial R_{1/p,I}}\operatorname{Arg}\ \det(\sigma(A));$$

the change in argument being taken as $\partial R_{1/p,I}$ in (4.15) is traversed in the clockwise direction. The analytical index of the system A is defined as

$$\text{(4.20)} \qquad \operatorname{ind}_p(A) = \dim\ker A - \dim\ker A^*,$$

where A is considered as an operator on $[L^p(I)]^N$ and A^* is considered as an operator on $[L^q(I)]^N$, $1/p + 1/q = 1$.

Before proving the Index Theorem 4.2, i.e., that

$$\text{(4.21)} \qquad \operatorname{ind}_p(A) = n_{1/p}(\sigma(A)),$$

we consider a canonical system A for which we show directly that

$$\text{(4.22)} \qquad \operatorname{ind}_p(A) = 1 = n_{1/p}(\sigma(A)).$$

Let $S^1 = \{w \in C ; |w| = 1\}$ be the unit circle and for $f(w) = f(e^{i\varphi}) \in C^\infty(S^1)$ define for $z = e^{i\theta}$

$$\text{(4.23)} \qquad \begin{aligned} Bf(z) &= \text{p.v.}\ \frac{1}{\pi i}\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})d\varphi - \text{p.v.}\ \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(e^{i\varphi}) \cot\left(\frac{\theta - \varphi}{2}\right)d\varphi . \end{aligned}$$

Note that $f \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})d\varphi$ is a compact operator on $L^p(-\pi,\pi)$. Given $f \in L^p(S^1)$ we define the column vector function $\vec{f} = (f_1,f_2) \in [L^p(I)]^2$ by

$$\text{(4.24)}\qquad \begin{aligned} f_1(t) &= f(e^{it}) \qquad 0 \le t \le \pi \\ f_2(t) &= f(e^{-it}) \qquad 0 \le t \le \pi . \end{aligned}$$

We consider the operator

$$\text{(4.25)}\quad B_{11}f(t) = \text{p.v.}\ \frac{1}{\pi}\int_0^{\pi} f(s)\ \frac{1}{2}\cot\left(\frac{t-s}{2}\right)ds,\ t \in [0,\pi].$$

If $\varphi \in C_0^\infty([0,\pi))$, $\varphi B_{11}\varphi f = \varphi H\varphi f + Kf$, where K is a compact operator on $L^p(0,\pi)$ and H is the finite Hilbert transform defined by (4.4). Now

$$TB_{11}T = -B_{11}$$

so that B_{11} differs from H by a compact operator.

Now consider

$$\text{(4.26)}\qquad B_{12}f(t) = \frac{1}{\pi}\int_0^{\pi} f(s)\ \frac{1}{2}\cot\left(\frac{t+s}{2}\right)ds,\ t \in [0,\pi].$$

The singularity of the kernel $\frac{1}{2}\cot\left(\frac{t+s}{2}\right)$ occurs when $t+s$ is near 0 or near 2π. If $\varphi \in C_0^\infty([0,\pi))$,

$$\text{(4.27)}\qquad \begin{aligned} \varphi B_{12}\varphi f(t) &= \frac{1}{\pi}\int_0^{\pi} \frac{\varphi(t)\varphi(s)}{t+s}\ f(s)ds + Kf(t), \\ &= (\varphi S\varphi)f + Kf, \end{aligned}$$

where K is compact on $L^p(I)$ and S is the finite Stieltjes transform defined by (4.5). Also

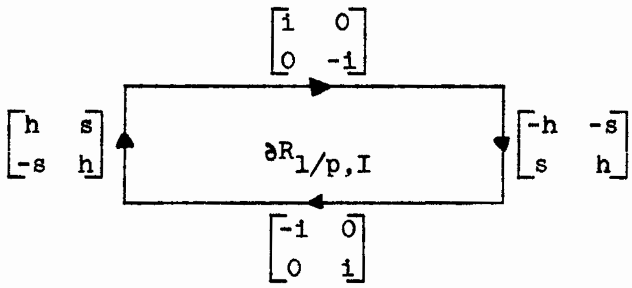
$$\text{(4.28)}\qquad TB_{12}T = -B_{12}.$$

The operator B defined in (4.23) on $L^p(S^1)$, using the correspondence (4.24) may be represented, modulo a compact operator, as a system on $L^p(I) \times L^p(I)$:

$$(4.29) \qquad \begin{aligned} (Bf)_1(t) &= B_{11}f_1(t) + B_{12}f_2(t) \\ (Bf)_2(t) &= -B_{12}f_1(t) - B_{11}f_2(t), \end{aligned}$$

which is a 2×2 system of pdo's of class $Op - \Sigma_{1/p}(I)$ whose principal symbols are given in (4.30). (In (4.30) $h(z) = -\cot \pi z$, $s(z) = \csc \pi z$.)

(4.30)



Given $f(e^{i\theta}) \in C^\infty(S^1)$, define for $z = e^{i\theta}$, let

$f^+(z) = \lim\limits_{r\uparrow 1} \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - rz}\, d\zeta$, and $f^-(z) = \lim\limits_{r\downarrow 1} \oint \frac{f(\zeta)}{\zeta - rz}\, d\zeta$.

Modulo a compact operator,

$$f^\pm(e^{i\theta}) = \frac{1}{2}\{f(e^{i\theta}) \pm iBf(e^{i\theta})\}.$$

Finally, note that the multiplication operator $f(w) \to wf(w)$ has the representation as a system: $f_1 \to e^{it}f_1$; $f_2 \to e^{-it}f_2$.

The operator

$$(4.31) \qquad Af(w) = wf^-(w) + f^+$$

has a one dimensional kernel spanned by $1/w - 1$ and cokernel $\{0\}$. As a system of pdo's of class Op - $\Sigma_{1/p}(I)$, a direct calculation shows that $n_{1/p}(\sigma(A)) = 1$ so that

$$\text{ind}_p(A) = 1 = n_{1/p}(\sigma(A)). \tag{4.32}$$

We are now ready to prove the Index Theorem.

<u>THEOREM 4.2.</u> <u>If</u> A <u>is an</u> N×N <u>matrix of pdo's of class</u> $\Sigma_{1/p}(I)$ <u>which is elliptic on</u> L^p, <u>then</u>

$$\text{ind}_p(A) = n_{1/p}(\sigma(A)).$$

<u>PROOF</u>: We may follow the proofs of Lemmas 3.1-3.4 with only a small change in notation to prove the same special cases. Hence there is an integer J_I such that

$$\text{ind}_p(A) = J_I n_{1/p}(\sigma(A)).$$

By the example considered above, $J_I = 1$.

<u>q.e.d.</u>

5. <u>Single and double layer potentials in polygonal domains.</u>

As an application of pdo's of class $\Sigma_{1/p}(I)$ we use single and double layer potentials to study the

Dirichlet, Neumann, and Oblique Derivative Problems for Laplace's equation in a polygon.

Let Ω be a simply connected polygon in R^2 with N successive vertices labeled as $P_1, P_3, \ldots, P_{2N-1} = P_1$ as $\partial\Omega$ is traversed in the counter clockwise direction. Label the interior angles at P_{2k-1} as θ_{2k-1}, $k = 1,\ldots,N$. At the midpoint of $\overline{P_{2k-1}P_{2k+1}}$ we introduce a false vertex P_{2k} with interior angle $\theta_{2k} = \pi$ and parametrize the new half sides with $t \in [0,1]$ so that $t = 0$ at P_{2k-1} or P_{2k+1} and $t = 1$ at P_{2k}; i.e.:

For $t \in [0,1]$, define

$$
\text{(5.1)} \qquad
\begin{aligned}
P_t &= P_{t,2k-1} = tP_{2k} + (1-t)P_{2k-1} \in \overline{P_{2k-1}P_{2k}}, \\
P_t &= P_{t,2k-2} = tP_{2k-2} + (1-t)P_{2k-2} \in \overline{P_{2k-1}P_{2k-2}}
\end{aligned}
$$

If ℓ_1 is the length of $\overline{P_iP_{i+1}}$, the arclength $d\sigma$ is given by

$$
\text{(5.2)} \qquad d\sigma = (-1)^{i+1}\ell_i dt.
$$

For $\phi \in L^p(\partial\Omega)$ define the double layer potential

$$
\text{(5.3)} \qquad u(X) = \frac{1}{\pi}\int_{\partial\Omega} \frac{\langle X-Q, n_Q\rangle}{|X-Q|^2}\phi(Q)d\sigma_Q,
$$

where n_Q is the interior unit normal to a point $Q \in \partial\Omega$.

For $P \in \partial\Omega$, let

$$(5.4)\quad \begin{aligned} u^+(P) &= \lim_{X\to P,\, X\in\Omega} u(X) = \phi(P) + K\phi(P), \\ u^-(P) &= \lim_{X\to P,\, X\in R^2-\Omega} u(X) = -\phi(P) + K\phi(P), \end{aligned}$$

where

$$(5.5)\quad K\phi(P) = \frac{1}{\pi}\int_{\partial\Omega} \frac{\langle P-Q, n_Q\rangle}{|P-Q|^2}\,\phi(Q)\,d\sigma Q,$$

and the limits in (5.4) are taken nontangentially in $L^p(\partial\Omega)$ as $X \to P \in \partial\Omega$. Using the correspondences given by (5.1), consider ϕ and $K\phi$ as functions in $[L^p(0,1)]^{2N}$, where $\phi_i(t) = \phi(P_t)$; $P_t \in \overline{P_iP_{i+1}}$. Then K may be represented as a $2N \times 2N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$, $I = [0,1]$. We obtain the following operators for $K = [K_{ij}]$:

$$K_{ii} = 0,\ K_{2k-1,2k} = K_{2k,2k-1} = 0,$$

K_{ij} is compact if $\overline{P_iP_{i+1}}$ and $\overline{P_jP_{j+1}}$ do not touch,

$$(5.6)\quad \begin{aligned} K_{2k-2,2k-1}\varphi(t) &= \int_0^1 k_\theta(\ell_{2k-2}t/\ell_{2k-1}s)\varphi(s)\,\frac{ds}{s}, \\ K_{2k-1,2k-2}\varphi(t) &= \int_0^1 k_\theta(\ell_{2k-1}t/\ell_{2k-2}s)\varphi(s)\,\frac{ds}{s}, \end{aligned}$$

with $\theta = \theta_{2k-1}$

where

$$(5.7)\quad k_\theta(t) = \frac{1}{\pi}\,\frac{t\sin\theta}{t^2 + 1 - 2t\cos\theta}.$$

The Mellin transform of the kernel $k_\theta(t)$ is given by

(5.8) $$\tilde{k}_\theta(z) = \frac{\sin(\pi-\theta)z}{\sin \pi z} \in \Theta^{-\infty}_{0,1}.$$

The matrix of principal symbols of the operator $[\delta_{ij}I + K_{ij}]$ is the identity on the top, bottom, and r.h.s. of $\partial R_{1/p,I}$; at $t = 0$ its determinant can be calculated by using an even number of row and column transpositions to obtain a matrix with 2x2 blocks on the diagonal; the vertex P_{2k-1} contributes

(5.8) $$\begin{bmatrix} 1 & b^z\tilde{k}_\theta(z) \\ b^{-z}\tilde{k}_\theta(z) & 1 \end{bmatrix}$$

where $b = \ell_{2k-2}/\ell_{2k-1}$ and $\theta = \theta_{2k-1}$. Hence

$$\det(\sigma(I+K))(0,z) = \prod_{j=1}^{N}\left(1 - \left[\tilde{k}_{\theta_{2j-1}}(z)\right]^2\right).$$

The zeroes of $1 - [\tilde{k}_\theta(z)]^2$ occur at $z = n\pi/\theta$ or $z = n\pi/(2\pi-\theta)$. If $\theta \neq \pi$ we define p_θ by

(5.9) $$1/p\theta = \begin{cases} \pi/(2\pi-\theta), & 0 < \theta < \pi, \\ \pi/\theta, & \pi < \theta < 2\pi, \end{cases}$$

so that $z_\theta = 1/p_\theta$ is the simple root of $1 - k_\theta^2 = 0$ satisfying $0 < \operatorname{Re} z < 1$; in fact $1 < p_\theta < 2$. By the Argument Principle

(5.10) $$n_{1/p}(\theta) = \frac{1}{2\pi}\, \Delta \underset{\substack{z=1/p+i\xi \\ -\infty\le\xi\le+\infty}}{\operatorname{Arg}} (1 - [k_\theta]^2)$$

$$= \begin{cases} 0, 0 < 1/p < 1/p_\theta \\ 1, 1/p_\theta < 1/p < 1 \end{cases} .$$

From the results of Chapter 4, the system

$$(5.11) \qquad (I+K)\phi = \psi$$

is Fredholm on $L^p(\partial\Omega)$ iff $p \neq p_{\theta_1}, \ldots, p_{\theta_{2N-1}}$; in case (5.11) is Fredholm on $L^p(\partial\Omega)$ we have

$$(5.12) \qquad \text{ind}_p(I+K) = \sum_{p<p_{\theta_{2k-1}}} 1 \quad .$$

In particular, since $p_\theta < 2$, for $p \geq 2$, $\text{ind}_p(I+K) = 0$.

To study the kernel and cokernel of (5.11) we introduce the single layer potential

$$(5.13) \qquad v(X) = \frac{1}{2\pi} \int_{\partial\Omega} |X-Q|^2 \psi(Q) d\sigma_Q .$$

Then

$$(5.14) \qquad \begin{aligned} \frac{\partial v}{\partial n^+}(P) &= \lim_{X\to P, X\in Q} \frac{\partial v}{\partial n_P}(X) = (I-K^*)\psi \\ \frac{\partial v}{\partial n^-}(P) &= \lim_{X\to P, X\in R^2-\Omega} \frac{\partial v}{\partial n_P}(X) = -(I+K^*)\psi , \end{aligned}$$

where

$$(5.15) \qquad K^*\psi(P) = \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle Q-P, n_P\rangle}{|P-Q|^2} \psi(Q) d\sigma_Q .$$

Let M be the (compact) mean value operator

$$M\phi = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} \phi(Q)\,d\sigma_Q .$$

From, e.g., Petrovsky [Pe, §34], we recall the following facts for these operators on $L^p(\partial\Omega)$:

$\ker(I-M) = \{\text{constant functions}\}$; $(I+K)M = 2M$; and $\ker(I+K^*)(I-M) = \{\text{constant functions}\}$.

Define

$$p_o = \max p_{\theta_{2k-1}} \tag{5.16}$$

with p_θ given by (5.9). Note that $p_o < 2$.

<u>LEMMA 5.1.</u> <u>For</u> $p > p_o$ <u>the kernel of</u> $(I-M)(I+K)$ <u>consists of constant functions</u>.

<u>PROOF</u>: Constant functions are in the kernel of $(I-M)(I+K)$ and $\mathrm{ind}_p(I-M)(I+K) = \mathrm{ind}_p(I+K) = 0$. Moreover, $(I+K^*)(I-M)$ has kernel of dimension 1.

<u>q.e.d.</u>

<u>LEMMA 5.2.</u> <u>For</u> $p > p_o$,

$$\ker(I+K) = \{0\}. \tag{5.17}$$

<u>PROOF</u>: If $(I+K)\phi = 0$ then $\phi = $ constant by Lemma 5.1. But then $(I+K)\phi = 2\phi = 0$.

<u>q.e.d.</u>

LEMMA 5.3. *For* $1 < q < \infty$

$$\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}.$$

PROOF: If $1 < q \leq 2$ and $1/p + 1/q = 1$, then $p > p_o$. Hence $\text{ind}_q(I+K^*) = 0$ and $\ker(I+K) \cap L^p(\partial\Omega) = \{0\}$. Hence $\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}$. If $2 \leq q$, $\ker(I+K^*) \cap L^q(\partial\Omega) \subset \ker(I+K^*) \cap L^2(\partial\Omega) = \{0\}$.

q.e.d.

THEOREM 5.1. *If* K *is defined by* (5.5) *then the integral equation*

$$(I+K)\phi = \psi \tag{5.11}$$

considered on $L^p(\partial\Omega)$ *has the following properties*:

1. (5.11) *is Fredholm on* $L^p(\partial\Omega)$ *iff*

$$p \neq p_{\theta_1}, \ldots, p_{\theta_{2N-1}}, \tag{5.16}$$

2. *For* $1 < q < \infty$, $\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}$.

3. *For every* p *satisfying* (5.16),

$$\dim\ker(I+K) \cap L^p(\partial\Omega) = \begin{cases} 0, & p \geq 2 \\ \text{no. of vertices with angle } \theta \text{ such that } p < p_\theta \end{cases} \tag{5.17}$$

(p_θ *given by* (5.9)).

The proof of Theorem 5.1 follows from the preceding discussion.

We next seek to resolve the Neumann problem $\Delta v = 0$ in Ω, $\frac{\partial v}{\partial n^+}(P) = g(P) \in L^q(\partial\Omega)$ in the form of a single layer potential given by (5.13); using (5.14) we study the integral equation

$$(5.18) \qquad (I-K^*)\phi = g$$

on $L^q(\partial\Omega)$. The operator $I-K$ is a Fredholm operator on $L^p(\partial\Omega)$ iff $p \neq p_{\theta_1},\ldots,p_{\theta_{2N-1}}$ and for such values of p has the same index on $L^p(\partial\Omega)$ as the operator $I+K$. We show that the kernel of $(I-K^*)$ consists of constants.

We show that the kernel of $I-K^*$ has dimension 1. Let q be near 1 and $S_q = \ker(I-K^*) \cap L^q(\partial\Omega)$. Now $S_q \neq \{0\}$ since $\{\text{constants}\} \subset \ker(I-K)$ and $\text{ind}_q(I-K) = 0$. It is easy to show that if $\phi \in S_q$ and $M\phi = 0$, then $\phi = 0$. Hence there is a $\phi_1 \in S_q$ such that $M\phi_1 = 1$. Then if $\phi \in S_q$, $\phi = (M\phi)\phi_1$.

For (5.18) we have the following theorem.

<u>THEOREM 5.2.</u> <u>The integral equation</u>

$$(5.18) \qquad (I-K^*)\phi = g$$

<u>on</u> $L^q(\partial\Omega)$ <u>has the following properties</u>.

1. <u>For</u> $1 < q < \infty$,

(5.19) $$\dim \ker(I-K^*) \cap L^q(\partial\Omega) = 1.$$

2. <u>(5.18) is Fredholm on</u> $L^q(\partial\Omega)$ <u>iff</u>

(5.20) $$q \neq q_{\theta_1},\ldots,q_{\theta_{2N-1}};\ (1/q_{\theta_i} + 1/p_{\theta_i} = 1).$$

(p_θ <u>given by (5.9)</u>).

3. <u>If</u> q <u>satisfies (5.20)</u>,

$$\mathrm{ind}_q(I-K^*) = \begin{cases} 0, & q \leq 2, \\ -(\text{no. of vertices with angle } \theta \text{ for which } q > q_\theta). \end{cases}$$

<u>COROLLARY 5.1. The integral equation</u>

(5.21) $$(I-K)\phi = \psi$$

<u>on</u> $L^p(\partial\Omega)$ <u>has the following properties</u>.

1. <u>If</u> $p > p_o$, $\ker(I-K) \cap L^p(\partial\Omega) = \{\text{constant functions}\}$.

2. <u>For</u> $p \neq p_{\theta_1},\ldots,p_{\theta_{2N-1}}$

$$\dim \ker(I-K) \cap L^p(\partial\Omega) = 1 + (\text{no. of vertices with angle } \theta \text{ for which } p < p_\theta).$$

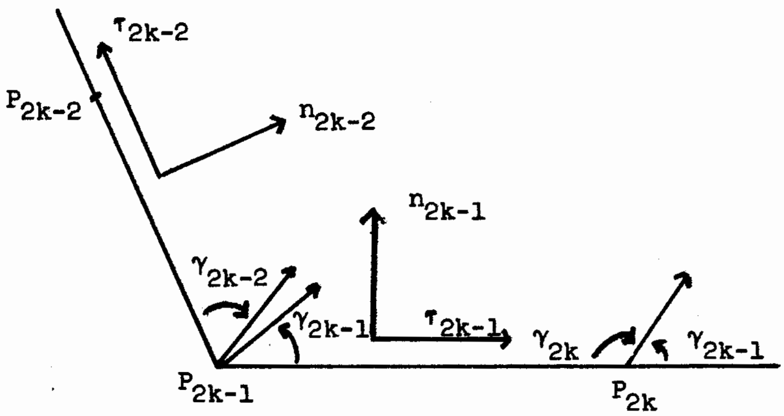
Finally we apply the pdo theory to study the Oblique Derivative Problem or a Mixed Boundary Value

Problem in a polygon. We use the same notation as before, and use single layer potentials to study:

$$(5.22)\qquad \begin{aligned} &\Delta v = 0 \quad \text{in} \quad \Omega, \\ &\frac{\partial v}{\partial \nu} = g(P) \in L^p(\partial\Omega), \end{aligned}$$

where on each segment $\overline{P_{2k-1}P_{2k+1}}$, $\nu = \nu(P)$ is a smooth nonvanishing vector field. Dirichlet data can be handled by taking ν tangent to $\partial\Omega$; jump discontinuities in ν may be considered by introducing a new vertex with angle $\theta = \pi$ at the jump.

For simplicity we assume that each segment $\overline{P_iP_{i+1}}$ has unit length. On each segment $\overline{P_iP_{i+1}}$ we introduce unit tangent vectors τ_i in the direction of increasing t as in (5.23):



Normalize so that $|\nu(P)| = 1$ and define smooth angles $\gamma_i = \gamma_i(t)$ so that on $\overline{P_iP_{i+1}}$;

$$\nu(P_t) = \cos\gamma_i(t)\tau_i + \sin\gamma_i(t)n_i .$$

The continuity of $\nu(P)$ at the even vertices may be expressed as:

$$(5.24) \qquad \gamma_{2k-1}(1) + \gamma_{2k}(1) = \pi \pmod{2\pi}.$$

Consider the operators

$$(5.25) \qquad A^\tau\psi(P) = \lim_{X\to P, X\in\Omega} \frac{\partial v}{\partial\tau_P}(X)$$

$$= \text{p.v.}\ \frac{1}{\pi}\int_{\partial\Omega} \frac{\langle P-Q, \tau_P\rangle}{|P-Q|^2}\psi(Q)d\sigma_Q ,$$

$$(5.26) \quad A^n\psi(P) = (I-K^*)\psi(P) = \lim_{\epsilon\to 0+} \frac{\partial v}{\partial n_P}(P + \epsilon n_P),$$

with v the single layer potential given by (5.13). To represent the operators A^τ and A^n as systems of pdo's of class $Op - \Sigma_{1/p}(I)$, we introduce the Hardy kernels (and their Mellin transforms):

$$k_{\tau\theta}(t) = \frac{1}{\pi}\,\frac{t - \cos\theta}{t^2 + 1 - 2t\cos\theta}$$

$$\tilde{k}_{\tau\theta}(z) = -\,\frac{\cos((\pi-\theta)z + \theta)}{\sin\pi z}$$

$$(5.27) \qquad k_{n\theta}(t) = \frac{1}{\pi}\,\frac{-\sin\theta}{t^2 + 1 - 2t\cos\theta}$$

$$\tilde{k}_{n\theta}(z) = -\,\frac{\sin((\pi-\theta)z + \theta)}{\sin\pi z}$$

$$s(t) = k_{\tau\pi}(t) = \frac{1}{\pi}\,\frac{1}{t+1}$$

$$\tilde{s}(z) = \frac{1}{\sin \pi z}$$

and the corresponding Hardy kernel operations:

$$K\psi(t) = \int_0^1 k\left(\frac{t}{s}\right) f(s)\,\frac{ds}{s} \tag{5.28}$$

which, as operators in $Op - \Sigma_{1/p}(I)$ have principal symbols, $\sigma(K)$, given by

$$\tilde{k}(z) \quad \begin{array}{c} 0 \\ \boxed{\quad R_{1/p,I} \quad} \; 0 \\ 0 \end{array} \quad . \tag{5.29}$$

For the operator $A^\tau = [A^\tau_{ij}]$, we obtain

1. $A^\tau_{ii}\psi_i(t) = H\psi_i(t) = \text{p.v.}\ \frac{1}{\pi}\int_0^1 \frac{\psi_i(s)}{t-s}\,ds.$

2. $A^\tau_{2k-1,2k}\psi_{2k}(t) = \frac{1}{\pi}\int_1^2 \frac{1}{t-s}\,\psi_{2k}(2-s)ds$

$$= -TST\psi_{2k}(t).$$

3. $A^\tau_{2k,2k-1}\psi_{2k-1}(t) = \frac{1}{\pi}\int_0^1 \frac{-1}{2-t-s}\,\psi_{2k-1}(s)ds$

$$= -TST\psi_{2k-1}(t).$$

4. $A^\tau_{2k-2,2k-1} = A^\tau_{2k-1,2k-2} = K_{\tau\theta},\ \theta = \theta_{2k-1}.$

5. A^τ_{ij} is compact if $\overline{P_iP_{i+1}}$ and $\overline{P_jP_{j+1}}$ do not touch.

For the operator $A^n = [A^n_{ij}] = (I-K^*)$ we have

6. $A^n_{ii} = I$.

7. $A^n_{2k-1,2k} = A^n_{2k,2k-1} = 0$.

8. $A^n_{2k-2,2k-1} = A^n_{2k-1,2k-2} = K_{n\theta}$.

9. A^n_{ij} is compact if $\overline{P_iP_{i+1}}$ and $\overline{P_jP_{j+1}}$ do not touch.

The system which arises to study (5.22) is now

$$(\nu\cdot\tau_p)A^\tau\psi(P) + (\nu\cdot n_p)A^n\psi(P) = g(P) \tag{5.30}$$

on $L^p(\partial\Omega)$.

We now calculate the determinant of the matrices of principal symbols of (5.40) defined on $\partial R_{1/p,I}$.

On the r.h.s. of $\partial R_{1/p,I}$; i.e., when $t = 1$, we make an even number of row and column transpositions to obtain N 2×2 blocks along the diagonal; each even numbered vertex P_{2k} contributes with $\gamma = \gamma_{2k-1}(1) = \pi - \gamma_{2k}(1) \pmod{2\pi}$:

$$\begin{bmatrix} -\cos\gamma\cot\pi z + \sin\gamma & \cos\gamma\csc\pi z \\ -\cos\gamma\csc\pi z & \cos\gamma\cot\pi z + \sin\gamma \end{bmatrix} \tag{5.31}$$

which has determinant $\equiv 1$ for $0 < \operatorname{Re} z < 1$. Hence on the r.h.s. of $\partial R_{1/p,I}$:

$$\det[\sigma(A)] = \prod_{j=1}^{N} 1 = 1.$$

To calculate the determinant of the matrix of principal symbols on $\partial R_{1/p,I}$, with $0 \le t < 1$, we are again reduced to studying 2x2 blocks which arise from considering the operator near a vertex P_{2k-1}. Calling $i = 2k-1$, $\theta = \theta_i$, $\gamma_{i-1} = \gamma_{2k-2}(t)$, $\gamma_i = \gamma_{2k-1}(t)$ the matrix which comes from P_i is given by

$$(5.32)\quad \frac{-1}{\sin\pi z}\begin{bmatrix} \cos(\pi z+\gamma_{i-1}) & \cos((\pi-\theta)z+\theta-\gamma_{i-1}) \\ \cos((\pi-\theta)z+\theta-\gamma_i) & \cos(\pi z+\gamma_i) \end{bmatrix}$$

which has determinant

$$(5.33)\quad \frac{1}{2}\,\frac{\cos(2\pi z + 2\gamma) - \cos(2(\pi-\theta)z + 2\theta + 2\gamma)}{\sin^2 \pi z}$$

$2\gamma = \gamma_{i-1} + \gamma_i$.

The function in (5.33) may be expressed as

$$(5.34)\quad F_{\theta 2\gamma}(z) = \frac{\sin((2\pi-\theta)z + \theta + 2\gamma)\sin(\theta(1-z))}{\sin^2 \pi z}\ .$$

$F_{\theta 2\gamma}$ satisfies

$$(5.35)\quad F_{\theta 2\gamma}(1/p + i\infty) = -e^{-i2\gamma},\ F_{\theta 2\gamma}(1/p - i\infty) = -e^{-i2\gamma}$$

and has zeroes when

$$(5.36)\quad 1 - z = \frac{n\pi}{\theta}\ ,\ 1 - z = \frac{n\pi + 2\gamma}{2\pi - \theta}\ .$$

Hence

$$\det[\sigma(A)](t,1/p+i\infty) = (-1)^N \prod_{j=1}^{N} e^{-i(\gamma_{2k-2}(t)+\gamma_{2k-1}(t))}$$

(5.37)

$$\det[\sigma(A)](t,1/p-i\infty) = (-1)^N \prod_{j=1}^{N} e^{+i(\gamma_{2k-2}(t)+\gamma_{2k-1}(t))}.$$

Note that $\det[\sigma(A)](1,1/p\pm i\infty) = 1$ by the continuity condition (5.24). We conclude that (5.40) is Fredholm on $L^p(\partial\Omega)$ iff $1/q = 1 - 1/p$ satisfies

$$\frac{1}{q} \neq \frac{n\pi}{\theta}, \quad \frac{1}{q} \neq \frac{n\pi + 2\gamma}{2\pi - \theta}$$

(5.38)

$$\theta = \theta_{2k-1},\ 2\gamma = \gamma_{2k-2}(0) + \gamma_{2k-1}(0),\ k = 1,\ldots,N.$$

If (5.38) is satisfied, the index of (5.30) on $L^p(\partial\Omega)$ can be readily calculated using the Index Theorem 4.2 and (5.37).

For an Index Theorem for the Oblique Derivative Problem in the case of a domain with smooth boundary, see Hormander [H]. We also refer the reader to the work of Costabel [Cos] who developes an algebra of singular integral operators on curves with corners which includes the double layer potential and proves an Index Theorem. His approach also allows weights such as t^β at the vertices of the polygon.

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