Mathematical Methods in the Applied Sciences, Vol. 18, 267–293 (1995)
MOS subject classification: 35 J 05, 35 P 25, 35 R 30, 45 A 05, 65 R 20, 65 R 30, 65 T 05

Inverse Scattering from an Open Arc

Rainer Kress

Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Lotzestr.16-18, 37083 Göttingen, Germany

Communicated by B. Brosowski

A Newton method is presented for the approximate solution of the inverse problem to determine the shape of a sound-soft or perfectly conducting arc from a knowledge of the far-field pattern for the scattering of time-harmonic plane waves. Fréchet differentiability with respect to the boundary is shown for the far-field operator, which for a fixed incident wave maps the boundary arc onto the far-field pattern of the scattered wave. For the sake of completeness, the first part of the paper gives a short outline on the corresponding direct problem via an integral equation method including the numerical solution.

1. Introduction

In the recent monograph by Colton and the present author [7] an up to date survey is attempted on the inverse Dirichlet problem for the Helmholtz equation. Since this presentation is confined to closed boundary curves in $\mathbb{R}^2$ and closed boundary surfaces in $\mathbb{R}^3$, we eventually felt the need to extend the corresponding analysis as far as possible to the case of open boundaries. For the sake of simplicity, in a first step we confine ourselves to the considerably simpler two-dimensional case of the inverse Dirichlet problem for an open arc in $\mathbb{R}^2$.

Studying an inverse problem always requires a solid knowledge of the corresponding direct problem. Therefore, and also in order to introduce notations, the first part of this paper will present the basic results on the existence of the solution of the Dirichlet problem for the Helmholtz equation in the exterior of an arc including its numerical approximation by an integral equation approach. Then the second part of the paper is devoted to the inverse Dirichlet problem.

The plan of the paper is as follows. In section 2 we will introduce the precise notion of the Dirichlet problem and the direct scattering problem for an open arc and prove uniqueness and existence of a classical solution via a single-layer potential approach. Of course, we will not obtain any new results in this section as compared with previous papers on this subject (cf. [10] or the appendix in the Russian translation [6] of [5] and the references therein). However, by using the cosine transformation which has been introduced by Yan and Sloan [26] for the corresponding single-layer integral equation for an open arc in the potential theoretic case of Laplace's equation, our presentation is very concise. In section 3 we will introduce the far-field pattern for

0170–4214/95/040267–27 *Received 20 November 1992*

the scattering of plane waves from a sound-soft (or perfectly conducting) arc and, in particular, we will prove the reciprocity relation. We will then use this reciprocity to extend results on the completeness of the far-field patterns for all incident plane waves due to Colton and Kirsch [4] from the case of a closed boundary contour to the case of an arc. In section 4 we will present a quadrature method for the numerical solution of the integral equation of section 2 including its error analysis. This method consists in the combination of the cosine substitution mentioned above and the application of a quadrature method which has been developed by Chapko and Kress [3] for the single-layer approach to the Dirichlet problem in the case of a closed boundary curve. A numerical example exhibits the fast convergence of this method for analytic arcs. This numerical method will also be needed in the second part of the paper for the implementation of a Newton method for the inverse problem.

In section 5 we begin our study of the inverse problem which consists of determining the shape of a sound-soft (or perfectly conducting) arc from a knowledge of the far-field pattern for the scattering of one or several incident plane waves. We begin with extending a classical uniqueness result due to Schiffer from the case of scattering from on obstacle with a closed boundary curve to the case of an arc. However, since the original approach by Schiffer cannot be extended to the case of an arc, we will use a different technique which was originated by Isakov [12] and Kirsch and Kress [16] for closed boundary curves. We note that as opposed to the direct Dirichlet problem the inverse problem is improperly posed and non-linear. As far as numerical methods for the approximate solution for the case of a close boundary curve is concerned, in the monograph [7] special emphasize is given towards iterative methods which circumvent the necessity of solving a direct Dirichlet problem at each iteration step and which separate the inverse problem into a linear ill-posed part and a non-linear well-posed part like the two methods proposed by Colton and Monk [8] and Kirsch and Kress [15]. However, these methods cannot be extended to the case of scattering from an arc since they require a non-empty interior of the scatterer. Therefore, the rest of the paper is concerned with the investigation of a Newton method for the solution of the inverse problem, which, of course, is a method requiring the solution of the direct problem in each iteration step. Newton methods for inverse scattering from a closed contour in two dimensions have been investigated and numerically tested by Roger [23], Kirsch [14], Murch *et al.* [20], Tobocman [24] and Wang and Chen [25].

For the theoretical foundation of this Newton method, in section 6 we will establish Fréchet differentiability with respect to the boundary arc for the far-field operator which for a fixed incident plane wave maps the boundary onto the far-field pattern of the scattered wave. Following a recent approach by Potthast [21] for the case of closed boundary curves or surfaces, we base our analysis on establishing Fréchet differentiability for the boundary integral operators used in the existence analysis of section 2. Finally, in section 7 we describe the Newton method including its numerical implementation and provide a numerical example.

2. The direct scattering problem

Assume that $\Gamma \subset \mathbb{R}^2$ is an arc of class C^3, i.e.

$$\Gamma = \{z(s)\colon s \in [-1, 1]\},$$

where $z:[-1,1] \to \mathbb{R}^2$ is an injective and three times continuously differentiable function. The mathematical treatment of the scattering of time-harmonic acoustic or electromagnetic plane waves from thin infinitely long cylindrical obstacles leads to the following *direct scattering problem* for the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma, \tag{2.1}$$

with wave number $k > 0$ and Dirichlet boundary condition

$$u = 0 \quad \text{on } \Gamma. \tag{2.2}$$

In acoustics the Dirichlet boundary condition (2.2) corresponds to scattering from a sound-soft obstacle, whereas in electromagnetics it models scattering from a perfect conductor with the electromagnetic field E-polarized.

The total field $u \in C^2(\mathbb{R}^2 \backslash \Gamma) \cap C(\mathbb{R}^2)$ is decomposed $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ into the given incident field $u^{\mathrm{i}}(x) = \mathrm{e}^{ikx \cdot d}$, $|d| = 1$, and the unknown scattered field u^{s} which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - iku^{\mathrm{s}}\right) = 0, \quad r = |x|, \tag{2.3}$$

uniformly in all directions $\hat{x} = x/|x|$. After renaming the unknown function, this direct scattering problem is a special case of the following *exterior Dirichlet problem* for an arc: Given a function $f \in C(\Gamma)$, find a solution $u \in C^2(\mathbb{R}^2 \backslash \Gamma) \cap C(\mathbb{R}^2)$ to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma,$$

which satisfies the boundary condition

$$u = f \quad \text{on } \Gamma$$

and the Sommerfeld radiation condition. Note that we do not explicitly assume any edge condition for the behaviour of the solution at the two end points $z_1 := z(1)$ and $z_{-1} := z(-1)$ of Γ.

We briefly sketch uniqueness, existence and well-posedness for this boundary value problem.

Theorem 2.1. *The exterior Dirichlet problem for an open arc has at most one solution.*

The proof relies on Rellich's lemma and the application of Green's theorem and is essentially the same as in the case of a closed boundary curve (cf. [7]). However, some technical problems arise from the fact that u is only assumed to be continuous up to the boundary, whereas the application of Green's theorem requires continuous differentiability up to the boundary. In addition, in the application of Green's theorem the end points of Γ need special consideration. These difficulties can be overcome using an approximation idea due to Heinz (see [9]) and convergence theorems for Lebesgue integration to prove the following lemma.

Lemma 2.2. *Let $u \in C^2(\mathbb{R}^2 \backslash \Gamma) \cap C(\mathbb{R}^2)$ be a solution to the Helmholtz equation in $\mathbb{R}^2 \backslash \Gamma$ which satisfies the homogeneous boundary condition $u = 0$ on Γ. Then for sufficiently large R we have that* grad $u \in L^2(B_R)$ *where B_R denotes the open disk of radius R centred at the origin and*

$$\int_{|x| \leqslant R} |\operatorname{grad} u|^2 \, \mathrm{d}x - k^2 \int_{|x| \leqslant R} |u|^2 \, \mathrm{d}x = \int_{|x| = R} u \frac{\partial \bar{u}}{\partial \nu} \, \mathrm{d}s. \tag{2.4}$$

Proof. We omit the technical details since the proof is essentially the same as for a closed boundary curve which can be found in[7]. The basic idea is to choose an odd function $\psi \in C^1(\mathbb{R})$ such that $\psi(t) = 0$ for $0 \leqslant t \leqslant 1$, $\psi(t) = t$ for $t \geqslant 2$ and $\psi'(t) \geqslant 0$ for all t, and set $u_n := [\psi(n \operatorname{Re} u) + i\psi(n \operatorname{Im} u)]/n$. We then have uniform convergence $u_n \to u$, $n \to \infty$. Since $u = 0$ on the boundary Γ, the functions u_n vanish in a neighbourhood of Γ and we can apply Green's theorem to obtain

$$\int_{|x| \leqslant R} \operatorname{grad} u_n \cdot \operatorname{grad} \bar{u}\, dx = k^2 \int_{|x| \leqslant R} u_n \bar{u}\, dx + \int_{|x| = R} u_n \frac{\partial \bar{u}}{\partial \nu}\, ds.$$

From this the statement of the lemma can be deduced by passing to the limit $n \to \infty$ with the aid of Fatou's lemma and Lebesgue's dominated convergence theorem. □

We will establish existence by seeking the solution in the form of a single-layer potential

$$u(x) = \int_\Gamma \Phi(x, y)\, \varphi(y)\, ds(y), \quad x \in \mathbb{R}^2 \backslash \Gamma, \tag{2.5}$$

with the fundamental solution to the Helmholtz equation in two dimensions given by

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y,$$

in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. The unknown density φ is assumed to be of the form

$$\varphi(x) = \frac{\tilde{\varphi}(x)}{\sqrt{|x - z_1||x - z_{-1}|}}, \quad x \in \Gamma \backslash \{z_1\} \cup \{z_{-1}\}, \tag{2.6}$$

where $\tilde{\varphi} \in C(\Gamma)$. Clearly, the single-layer potential (2.5) satisfies the Helmholtz equation in $\mathbb{R}^2 \backslash \Gamma$ and the Sommerfeld radiation condition. In addition, since densities φ of the form (2.6) belong to $L^p(\Gamma)$ for $1 < p < 2$, the single-layer potential (2.5) is continuous throughout $\mathbb{R}^2$ (cf. [11, p. 276]). From the potential theoretic jump relations (cf. [5]) it follows that the potential u given by (2.5) solves the exterior Dirichlet problem provided the density φ is a solution to the integral equation

$$\int_\Gamma \Phi(x, y)\, \varphi(y)\, ds(y) = f(x), \quad x \in \Gamma. \tag{2.7}$$

We will prove existence of a solution to this integral equation under appropriate additional assumptions on the regularity of φ and f. This will be done by using the cosine transformation which has been introduced by Yan and Sloan [26] and successfully applied for the numerical solution of the integral equation corresponding to (2.7) in the potential theoretic case $k = 0$ (see also [1]). We first note that from the uniqueness Theorem 2.2 and the potential theoretic jump relations, we can readily deduce that the integral equation (2.7) has at most one solution of the form (2.6), i.e. we have uniqueness for (2.7).

We substitute $s = \cos t$, $t \in [0, \pi]$, into $\Gamma = \{z(s)\colon s \in [-1, 1]\}$ and transform (2.7) into the parametric form

$$\frac{1}{2\pi} \int_0^\pi H(t, \tau)\, \psi(\tau)\, d\tau = g(t), \quad t \in [0, \pi]. \tag{2.8}$$

Here we have set

$$\psi(t) := |\sin(t)|\,|z'(\cos t)|\,\varphi(z(\cos t)), \qquad g(t) := -2f(z(\cos t)) \tag{2.9}$$

for $0 \leqslant t \leqslant \pi$ and the kernel is given by

$$H(t,\tau) := \frac{\pi}{\mathrm{i}} H_0^{(1)}(k\,|z(\cos t) - z(\cos \tau)|), \quad t \neq \tau. \tag{2.10}$$

For the further analysis it is convenient to transform the integral equation into an equation over the interval $[0, 2\pi]$ rather than $[0, \pi]$. In view of (2.9) and (2.10) it is obvious that finding a solution $\psi \in C[0, \pi]$ of the integral equation (2.8) is equivalent to finding an even 2π-periodic solution $\psi \in C(\mathbb{R})$ of

$$\frac{1}{4\pi}\int_0^{2\pi} H(t,\tau)\psi(\tau)\,\mathrm{d}\tau = g(t), \quad t \in [0, 2\pi]. \tag{2.11}$$

For notational convenience, we bring the relations (2.9) in operator form. After defining bijective operators Y, Z between functions on Γ and even 2π-periodic functions on $\mathbb{R}$ by setting

$$(Y\varphi)(t) := |\sin(t)|\,|z'(\cos t)|\,\varphi(z(\cos t)), \qquad (Zf)(t) := f(z(\cos t)) \tag{2.12}$$

for $t \in \mathbb{R}$, we have the equivalence that φ is a solution of the form (2.6) with $\tilde{\varphi} \in C(\Gamma)$ to the integral equation (2.7) if and only if $\psi = Y\varphi \in C(\mathbb{R})$ is an even 2π-periodic solution of (2.11) where $g = -2Zf$.

We recall the definition $H_0^{(1)} = J_0 + \mathrm{i}N_0$ of the Hankel function, and use the power series

$$J_0(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{w}{2}\right)^{2n} \tag{2.13}$$

for the Bessel function J_0 of order zero and

$$N_0(w) = \frac{2}{\pi}\left(\ln\frac{w}{2} + C\right)J_0(w) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{n}\frac{1}{m}\right\}\frac{(-1)^{n+1}}{(n!)^2}\left(\frac{w}{2}\right)^{2n} \tag{2.14}$$

for the Neumann function N_0 of order zero with Euler's constant $C = 0.57721\ldots$. From the expansion (2.13) and (2.14) we see that the kernel K can be written in the form

$$H(t,\tau) = \{1 + H_1(t,\tau)\}\ln\left(\frac{4}{e^2}[\cos t - \cos\tau]^2\right) + H_2(t,\tau), \tag{2.15}$$

where

$$H_1(t,\tau) := J_0(k\,|z(\cos t) - z(\cos\tau)|) - 1$$

is clearly three times continuously differentiable and

$$H_2(t,\tau) := H(t,\tau) - \{1 + H_1(t,\tau)\}\ln\left(\frac{4}{e^2}[\cos t - \cos\tau]^2\right), \quad t \neq \tau,$$

can be extended as a twice continuously differentiable function on $\mathbb{R}\times\mathbb{R}$. The latter follows from the fact that

$$(s,\sigma) \mapsto \frac{|z(s) - z(\sigma)|^2}{|s - \sigma|^2}, \quad s \neq \sigma,$$

can be extended as a twice continuously differentiable function on $[-1, 1] \times [-1, 1]$ which is positive for all $s, \sigma \in [-1, 1]$ since z is injective. In particular, using the expansions (2.13) and (2.14), we can deduce that

$$H_1(t, t) = \frac{\partial}{\partial t} H_1(t, t) = 0 \tag{2.16}$$

and

$$H_2(t, t) = \frac{\pi}{i} + 2C + 2 \ln \left\{ \frac{ke}{4} |z'(\cos t)| \right\}.$$

From the identity

$$\ln\left(\frac{4}{e^2}[\cos t - \cos \tau]^2\right) = \ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + \ln\left(\frac{4}{e}\sin^2\frac{t+\tau}{2}\right)$$

and the splitting (2.15), we see that the integral equation (2.11) can be equivalently written as

$$\frac{1}{2\pi}\int_0^{2\pi}\left\{[1 + H_1(t,\tau)]\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + \frac{1}{2}H_2(t,\tau)\right\}\psi(\tau)\,d\tau = g(t), \quad t \in [0, 2\pi]. \tag{2.17}$$

We now rewrite this final form of our integral equation in operator notation as

$$K\psi = g, \tag{2.18}$$

where the operator K is split into $K = L + A + B$ with the integral operators

$$(L\psi)(t) := \frac{1}{2\pi}\int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)\psi(\tau)\,d\tau, \quad t \in [0, 2\pi],$$

$$(A\psi)(t) := \frac{1}{2\pi}\int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)H_1(t, \tau)\psi(\tau)\,d\tau, \quad t \in [0, 2\pi],$$

$$(B\psi)(t) := \frac{1}{4\pi}\int_0^{2\pi} H_2(t, \tau)\psi(\tau)\,d\tau, \quad t \in [0, 2\pi].$$

For $0 < \alpha < 1$, by $C^{0,\alpha}$ and $C^{1,\alpha}$ we denote the normed spaces of Hölder continuous and Hölder continuously differentiable functions, respectively, equipped with the usual Hölder norms (cf. [5, 7]). In particular, by $C^{0,\alpha}_{2\pi}(\mathbb{R})$ and $C^{1,\alpha}_{2\pi}(\mathbb{R})$, we denote the subspaces of 2π-periodic functions from $C^{0,\alpha}(\mathbb{R})$ and $C^{1,\alpha}(\mathbb{R})$, and by $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ and $C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ we denote the subspaces of even functions from $C^{0,\alpha}_{2\pi}(\mathbb{R})$ and $C^{1,\alpha}_{2\pi}(\mathbb{R})$, respectively. We will show that the operator $K: C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ is bounded and has a bounded inverse.

Theorem 2.3. *The operator $L: C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ is bounded and has a bounded inverse. The operators $A, B: C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ are compact.*

Proof. The operator L represents the parametric form of the logarithmic single-layer potential for the Laplace equation for a circle of radius $e^{-1/2}$. From Theorem 7.29 in [17] it can be deduced that $L: C^{0,\alpha}_{2\pi}(\mathbb{R}) \to C^{1,\alpha}_{2\pi}(\mathbb{R})$ is bounded and has a bounded inverse. Since for $f_m(t) = e^{imt}$ we have (cf. [17, p. 177])

$$Lf_m = c_m f_m, \quad m = 0, \pm 1, \pm 2, \ldots,$$

where

$$c_m = -\frac{1}{\max(1, |m|)}, \tag{2.19}$$

it can be easily seen that L and L^{-1} map even functions onto even functions. Therefore, the assertion of the theorem on L follows.

In view of (2.16), the second derivative of $A\psi$ can be seen to have a weakly singular kernel and therefore A maps $C[0, 2\pi]$ boundedly into $C^2[0, 2\pi]$. The same is true for B since the second derivative of $B\psi$ has a continuous kernel. Now compactness of A and B from $C^{0,\alpha}[0, 2\pi]$ into $C^{1,\alpha}[0, 2\pi]$ follows from the compact imbedding of $C^{0,\alpha}[0, 2\pi]$ into $C[0, 2\pi]$ (cf. [5, 17]).

Theorem 2.4. *For each $g \in C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ the integral equation* (2.17) *has a unique solution $\psi \in C^{0,\alpha}_{2\pi,e}(\mathbb{R})$.*

Proof. This follows from the Riesz–Fredholm theory (cf. [17]) applied to the equation of the second kind.

$$\psi + L^{-1}(A + B)\psi = L^{-1}g,$$

which is equivalent to (2.18). The operator $L^{-1}(A + B)$ is compact from $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ into $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ by Theorem 2.3, and the homogeneous equation (2.18) has only the trivial solution in $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ as already mentioned above. Therefore the Riesz–Fredholm theory ensures that the inverse $(L + A + B)^{-1}$ of $L + A + B$ exists and is bounded from $C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ into $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$.

Theorem 2.5. *For each $f \in C^{1,\alpha}(\Gamma)$ the Dirichlet problem for the exterior of an arc has a unique solution. This solution depends continuously on the boundary data, i.e. small deviations of the boundary values in the $C^{1,\alpha}$ norm ensure small deviations in the solution with respect to uniform convergence of the solution and all its derivatives on closed subsets of $\mathbb{R}^2 \backslash \Gamma$.*

Proof. From (2.12) we observe that $f \in C^{1,\alpha}(\Gamma)$ implies that $g = -2Zf \in C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ and $\|g\|_{C^{1,\alpha}_{2\pi,e}(\mathbb{R})} \leqslant C\|f\|_{C^{1,\alpha}(\Gamma)}$ for some constant C. We use the cosine substitution to transform the single-layer potential u with density φ into

$$u(x) = (S\psi)(x), \quad x \in \mathbb{R}^2 \backslash \Gamma, \tag{2.20}$$

where the integral operator $S: C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C(\mathbb{R}^2 \backslash \Gamma)$ is defined by

$$(S\psi)(x) := \int_0^\pi \Phi(x, z(\cos\tau))\, \psi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \backslash \Gamma. \tag{2.21}$$

Now the assertion of the theorem follows from the boundedness of the inverse operator K^{-1} established in the proof of Theorem 2.4 and the observation that the fundamental solution Φ and all its derivatives are uniformly bounded on $D \times \Gamma$ for each set $D \subset \mathbb{R}^2$ with $\bar{D} \cap \Gamma = \emptyset$. □

For densities $\psi \in C^{0,\alpha}_{2\pi,e}(\mathbb{R})$, from the jump relations for single-layer potentials (cf. [5]) it can be seen that the first derivatives of u given by (2.20) can be extended as a continuous function from $\mathbb{R}^2\backslash\Gamma$ into all of $\mathbb{R}^2\backslash\{z_1\}\cup\{z_{-1}\}$ (with different limiting values on both sides of Γ). In particular, if we denote by ν a normal vector of Γ (there are two possible orientations), then

$$\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h\to+0} \nu(x)\cdot \operatorname{grad} u(x \pm h\nu(x))$$

exists for all $x\in\Gamma\backslash\{z_1\}\cup\{z_{-1}\}$ and

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = -\varphi \quad \text{on } \Gamma\backslash\{z_1\}\cup\{z_{-1}\}. \tag{2.22}$$

For the exterior scattering problem we now have the following representation for the scattered field through the so-called secondary sources on the boundary which in physics is known as Huygen's principle (see Theorem 3.12 in [7]).

Theorem 2.6. *For the scattering of an entire field u^{i} from a sound-soft (or perfectly conducting) arc Γ we have*

$$u(x) = u^{\mathrm{i}}(x) - \int_\Gamma \left\{\frac{\partial u_+}{\partial \nu}(y) - \frac{\partial u_-}{\partial \nu}(y)\right\}\Phi(x,y)\,\mathrm{d}s(y), \quad x\in\mathbb{R}^2\backslash\Gamma.$$

Proof. This follows from the single-layer approach

$$u(x) = u^{\mathrm{i}}(x) + u^{\mathrm{s}}(x) = u^{\mathrm{i}}(x) + \int_\Gamma \Phi(x,y)\,\varphi(y)\,\mathrm{d}s(y), \quad x\in\mathbb{R}^2\backslash\Gamma, \tag{2.23}$$

and the jump relation (2.22), i.e.

$$-\varphi = \frac{\partial u^{\mathrm{s}}_+}{\partial \nu} - \frac{\partial u^{\mathrm{s}}_-}{\partial \nu} = \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} \quad \text{on } \Gamma\backslash\{z_1\}\cup\{z_{-1}\}. \tag{2.24}$$

3. The far-field pattern

Any solution u^{s} to the Helmholtz equation satisfying the Sommerfeld radiation condition has an asymptotic behaviour of the form

$$u^{\mathrm{s}}(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x|\to\infty, \tag{3.1}$$

uniformly in all directions $\hat{x} := x/|x|$. The function u_∞, defined on the unit circle Ω in $\mathbb{R}^2$, is known as the *far-field pattern* of u^{s}. We note (cf. [7, p. 66]) that with the aid of Green's representation formula the far-field pattern may be expressed in the form

$$u_\infty(\hat{x}) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_{\Omega_R}\left\{u(y)\frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\right\}\mathrm{d}s(y) \tag{3.2}$$

for $|\hat{x}| = x/|x|$, where Ω_R denotes a sufficiently large circle of radius R centred at the origin with outward unit normal ν.

From Theorem 2.6 and the asymptotics for the Hankel function $H_0^{(1)}$, we readily see that the far-field pattern for the scattering of an entire field u^{i} from a sound-soft (or perfectly conducting) arc Γ is given by

$$u_\infty(\hat{x}) = -\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{8\pi k}}\int_\Gamma\left\{\frac{\partial u_+}{\partial \nu}(y) - \frac{\partial u_-}{\partial \nu}(y)\right\}\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(y), \quad \hat{x}\in\Omega. \tag{3.3}$$

In the sequel, for an incident plane wave $u^i(x) = u^i(x; d) = e^{ikx\cdot d}$ we will indicate the dependence of the scattered field, of the total field and of the far-field pattern on the incident direction d by writing, respectively, $u^s(x; d)$, $u(x; d)$ and $u_\infty(\hat{x}; d)$. With the help of the representation (3.3) we can prove the following reciprocity relation.

Theorem 3.1. *The far-field pattern for the scattering of a sound-soft (or perfectly conducting) arc satisfies the reciprocity relation*

$$u_\infty(\hat{x}; d) = u_\infty(-d; -\hat{x}), \quad \hat{x}, d \in \Omega. \tag{3.4}$$

Proof. In order to avoid technical difficulties due to the two end points of the arc in the application of Green's theorem, we proceed differently from the proof of the corresponding Theorem 3.13 in [7] for the case of a closed boundary curve. For $d \in \Omega$ we define

$$g_d(t) := e^{ikd\cdot z(\cos t)}, \quad 0 \leqslant t \leqslant \pi,$$

and denote by φ_d the density of the single-layer potential (2.23). Then $\psi_d = Y\varphi_d$ is the unique solution of $K\psi_d = 2g_d$. Now, using the representation (3.3), the jump relation (2.24) and the symmetry $H(t, \tau) = H(\tau, t)$ of the kernel of the integral operator K, we obtain

$$\begin{aligned}
\frac{\sqrt{8\pi k}}{e^{i\pi/4}} u_\infty(\hat{x}; d) &= \int_\Gamma \varphi_d(y) e^{-ik\hat{x}\cdot y}\, ds(y) = \int_0^\pi \psi_d g_{-\hat{x}}\, dt \\
&= 2\int_0^\pi K^{-1} g_d g_{-\hat{x}}\, dt = 2\int_0^\pi g_d K^{-1} g_{-\hat{x}}\, dt = \int_0^\pi g_d \psi_{-\hat{x}}\, dt \\
&= \int_\Gamma e^{ikd\cdot y} \varphi_{-\hat{x}}(y)\, ds(y) = \frac{\sqrt{8\pi k}}{e^{i\pi/4}} u_\infty(-d; -\hat{x})
\end{aligned}$$

and the relation (3.4) is proven. □

The following result on the question whether the far-field patterns for a fixed sound-soft (or perfectly conducting) arc Γ and all incident plane waves are complete in $L^2(\Omega)$ corresponds to the results by Colton and Kirsch [4] for the case of a closed boundary curve. For the notion of Herglotz wave functions we refer to [7].

Theorem 3.2. *Let (d_n) be a sequence of unit vectors that is dense on Ω and define the set $\mathscr{F}$ of far-field patterns by*

$$\mathscr{F} := \{u_\infty(\cdot; d_n) \colon n = 1, 2, \ldots\}.$$

Then $\mathscr{F}$ is complete in $L^2(\Omega)$ if and only if there does not exist a Herglotz wave function which vanishes on Γ.

Proof. We extend the proof of Theorem 3.17 in [7] to the present situation. By Theorem 2.5 the far-field pattern u_∞ depends continuously on d. Therefore, in view of the reciprocity relation (3.4), the completeness condition

$$\int_\Omega u_\infty(\hat{x}; d_n) h(\hat{x})\, ds(\hat{x}) = 0, \quad n = 1, 2, \ldots,$$

for a function $h \in L^2(\Omega)$ is equivalent to the condition

$$\int_\Omega u_\infty(\hat{x}; d) g(d)\, ds(d) = 0, \quad \hat{x} \in \Omega, \tag{3.5}$$

for $g \in L^2(\Omega)$ with $g(d) = h(-d)$. However, by superposition,

$$v_\infty(\hat{x}) := \int_\Omega u_\infty(\hat{x}; d) g(d)\, ds(d), \quad \hat{x} \in \Omega,$$

describes the far-field pattern for the scattering of the Herglotz wave function

$$v^i(x) := \int_\Omega e^{ikx\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^2,$$

with kernel g from the arc Γ. Now, condition (3.5), i.e. $v_\infty = 0$ on Ω is equivalent to $v^s = 0$ in $\mathbb{R}^2$ for the corresponding scattered wave since the far-field pattern uniquely determines the scattered field (see Theorem 2.13 in [7]). By the boundary condition $v^i + v^s = 0$ on Γ for the total field this is equivalent to $v^i = 0$ on Γ. This completes the proof by the observation that a Herglotz wave function with kernel g vanishes in all of $\mathbb{R}^2$ if and only if the kernel $g = 0$ (see Theorem 3.15 in [7]). $\square$

In polar co-ordinates $x = (r, \theta)$ the cylindrical wave functions

$$u_n(x) = J_n(kr)\cos n\theta, \quad v_n(x) = J_n(kr)\sin n\theta, \qquad n = 0, 1, \dots,$$

where J_n denotes the Bessel function of order n provide examples of Herglotz wave functions. Therefore, by Theorem 3.2, for any straight line segment Γ the far-field patterns are not complete. The same is true for circular arcs with radius R such that kR is a zero of one of the Bessel functions J_n.

4. On the numerical solution

We wish to apply a quadrature method which has been developed by Chapko and Kress [3] for the single-layer approach to solve the Dirichlet problem in the case of a closed boundary curve. A similar analysis was carried out by Bürger [2]. First we recall the integral equation (2.17) and slightly rewrite it in the form

$$\frac{1}{2\pi}\int_0^{2\pi} K(t, \tau)\psi(\tau)\, d\tau = g(t), \quad t \in [0, 2\pi], \tag{4.1}$$

where

$$K(t, \tau) = \left\{1 + \sin^2\frac{t-\tau}{2} K_1(t, \tau)\right\}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + K_2(t, \tau), \tag{4.2}$$

with

$$K_1(t, \tau) := \frac{H_1(t, \tau)}{\sin^2(t-\tau)/2}, \quad t \neq \tau$$

and

$$K_2(t, \tau) := \frac{1}{2} H_2(t, \tau).$$

From (2.13) we see that K_1 is twice continuously differentiable with diagonal term

$$K_1(t, t) = -k^2 \sin^2 t\, |z'(\cos t)|^2.$$

We also rewrite our integral operators into

$$(A\psi)(t) = \frac{1}{2\pi}\int_0^\pi \sin^2\frac{t-\tau}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)K_1(t,\tau)\psi(\tau)\,d\tau, \quad t\in[0,2\pi],$$

and

$$(B\psi)(t) = \frac{1}{2\pi}\int_0^\pi K_2(t,\tau)\psi(\tau)\,d\tau, \quad t\in[0,2\pi].$$

The incorporation of the extra factor in the kernel of A is necessary in order to exploit the vanishing diagonal terms (2.16) in setting up the numerical quadrature and in the corresponding error analysis.

Our quadrature method is based on trigonometric interpolation with $n\in\mathbb{N}$ equidistant nodal points

$$t_j^{(n)} := \frac{j\pi}{n}, \quad j = 0, \dots, 2n-1.$$

We will use the following interpolatory quadrature rules:

$$\frac{1}{2\pi}\int_0^{2\pi}\psi(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t)\psi(t_j^{(n)}), \tag{4.3}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\psi(\tau)\sin^2\frac{t-\tau}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{j=0}^{2n-1} F_j^{(n)}(t)\psi(t_j^{(n)}), \tag{4.4}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\psi(\tau)\,d\tau \approx \frac{1}{2n}\sum_{j=0}^{2n-1}\psi(t_j^{(n)}), \tag{4.5}$$

with the weights

$$R_j^{(n)}(t) = \frac{1}{2n}\left\{c_0 + 2\sum_{m=1}^{n-1} c_m\cos m(t-t_j^{(n)}) + c_n\cos n(t-t_j^{(n)})\right\}$$

and

$$F_j^{(n)}(t) = \frac{1}{2n}\left\{\gamma_0 + 2\sum_{m=1}^{n-1} \gamma_m\cos m(t-t_j^{(n)}) + \gamma_n\cos n(t-t_j^{(n)})\right\},$$

where the c_m are given by (2.19) and

$$\gamma_m := \tfrac{1}{4}(2c_m - c_{m+1} - c_{m-1}).$$

These quadratures are obtained by replacing ψ by its trigonometric interpolation polynomial with respect to the nodal points $t_j^{(n)}, j = 0, \dots, 2n-1$, and then integrating exactly (cf. [3]).

We apply the quadrature rules (4.3)–(4.5) to the integral equation (4.1) according to the splitting (4.2) of the kernel K and obtain the approximating equation

$$\sum_{j=0}^{2n-1}\tilde{\psi}_n(t_j^{(n)})\left\{R_j^{(n)}(t) + F_j^{(n)}(t)K_1(t,t_j^{(n)}) + \frac{1}{2n}K_2(t,t_j^{(n)})\right\} = g(t), \quad t\in[0,2\pi]. \tag{4.6}$$

We seek a solution $\tilde{\psi}_n$ *of* (4.6) in the space $T_{n,e}$ of even trigonometric polynomials of degree less than or equal to n. The interpolation problem with respect to the $(n+1)$-dimensional space $T_{n,e}$ and the nodal points $t_j^{(n)}$, $j = 0, \dots, n$, is uniquely solvable. We denote by $P_{n,e}$ the corresponding interpolation operator.

Using the fact that $LP_{n,e}\psi = L\psi$ for $\psi \in T_{n,e}$, we can rewrite (4.6) in operator notation as

$$L\tilde{\psi}_n + A_n\tilde{\psi}_n + B_n\tilde{\psi}_n = g, \tag{4.7}$$

with the numerical quadrature operators

$$(A_n\psi)(t) := \sum_{j=0}^{2n-1} F_j^{(n)}(t) K_1(t, t_j^{(n)})\psi(t_j^{(n)}), \quad t \in [0, 2\pi],$$

and

$$(B_n\psi)(t) := \frac{1}{2n}\sum_{j=0}^{2n-1} K_2(t, t_j^{(n)})\psi(t_j^{(n)}), \quad t \in [0, 2\pi].$$

To obtain an approximating equation which can be reduced to solving a finite-dimensional linear system, we use collocation with the interpolation operator $P_{n,e}$ for (4.7). Thus, our approximation scheme finally consists in solving

$$P_{n,e}L\psi_n + P_{n,e}A_n\psi_n + P_{n,e}B_n\psi_n = P_{n,e}g \tag{4.8}$$

for $\psi_n \in T_{n,e}$. Clearly, this is equivalent to the linear system

$$\sum_{j=0}^{2n-1} \psi_n(t_j^{(n)})\left\{R_{|k-j|}^{(n)} + F_{|k-j|}^{(n)} K_1(t_k^{(n)}, t_j^{(n)}) + \frac{1}{2n} K_2(t_k^{(n)}, t_j^{(n)})\right\} = g(t_k^{(n)}), \quad k = 0, \dots, n, \tag{4.9}$$

which we have to solve for the nodal values $\psi_n(t_k^{(n)})$ of $\psi_n \in T_{n,e}$ and where

$$R_j^{(n)} := R_j^{(n)}(0) = \frac{1}{2n}\left\{c_0 + 2\sum_{m=1}^{n-1} c_m \cos\frac{mj\pi}{n} + (-1)^j c_n\right\}$$

and

$$F_j^{(n)} : = F_j^{(n)}(0) = \frac{1}{2n}\left\{\gamma_0 + 2\sum_{m=1}^{n-1} \gamma_m \cos\frac{mj\pi}{n} + (-1)^j \gamma_n\right\}.$$

Note that (4.9) is an $(n+1)\times(n+1)$-system since the nodal values for $\psi_n \in T_{n,e}$ have the symmetry property $\psi_n(t_j^{(n)}) = \psi_n(t_{2n-k}^{(n)})$ for $k = 0, \dots, n$.

For the details of an error analysis for this quadrature method based on Anselone's concept of collectivity compact operators (cf. [17]), we refer to Chapko and Kress [3]. We just state the following main result.

Theorem 4.1. *Assume that the kernels H_1 and H_2 are even 2π-periodic functions of both variables and twice continuously differentiable and that $0 < \alpha < 1$. Then both operator sequences $(P_{n,e}A_n)$ and $(P_{n,e}B_n)$ are collectively compact from $C_{2\pi,e}^{0,\alpha}(\mathbb{R})$ into $C_{2\pi,e}^{1,\alpha}(\mathbb{R})$ and converge pointwise to the operators A and B, respectively. Provided the original equation (4.1) has a unique even solution ψ, then for sufficiently large n the approximating equation (4.9) has a unique solution ψ_n and we have an error estimate*

$$\|\psi_n - \psi\|_{0,\alpha} \leqslant C(\|P_{n,e}g - g\|_{1,\alpha} + \|P_{n,e}(A_n + B_n)\psi - (A + B)\psi\|_{1,\alpha})$$

for some constant $C = C(\alpha)$.

Proof. The analysis of [3] can be easily extended to the present case where we consider only even functions. The restriction $0 < \alpha < \frac{1}{2}$ made in [3] can be avoided by replacing Lemma 4.1 of [3] by the stronger error estimate

$$\| P_{n,e} f - f \|_{1,\alpha} \leqslant C \frac{\ln n}{n^{1-\alpha}} \| f \|_{C^2},$$

which is valid for all $f \in C^{1,\alpha}_{2\pi,e}(\mathbb{R})$, all $0 < \alpha < 1$ and some constant C depending only on α and which can be derived from Jackson's theorem on best approximations (cf. [22, p. 40]). □

The error estimate of Theorem 4.1 shows that the accuracy of the approximation depends on how well $P_{n,e}(A_n + B_n)\psi$ approximates $(A + B)\psi$ for the exact solution ψ. In particular, if the kernels K_1 and K_2 and the exact solution ψ are analytic, then from the error estimates for the trigonometric interpolation of periodic analytic functions (cf. [17, p. 160]) it can be derived that the error $\|\psi_n - \psi\|_{0,\alpha}$ decreases at least exponentially with increasing n. Proceeding as in Problem 12.4 of [17] it can be shown that for analytic kernels K_1 and K_2 and analytic right-hand sides g the solution ψ indeed is analytic. In particular, this means that for the scattering from analytic arcs the above numerical method shows exponential convergence.

For a numerical example, we consider the scattering of a plane wave $u^i(x) = e^{ikd \cdot x}$ by the sound-soft (or perfectly conducting) arc Γ with the parametric representation

$$z(s) = \left(2 \sin \frac{s}{2}, \sin s\right), \quad \frac{\pi}{4} \leqslant s \leqslant \frac{7\pi}{4}, \tag{4.10}$$

which is illustrated in Fig. 1. From the asymptotics for the Hankel function $H_0^{(1)}$, we see that the far-field pattern of the single-layer potential (2.5) is given by

$$u_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_\Gamma e^{-ik\hat{x}\cdot y} \varphi(y) \, ds(y), \quad \hat{x} \in \Omega,$$

(cf. (3.3)) or in parametrized form by

$$u_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_0^\pi e^{-ik\hat{x}\cdot z(\cos\tau)} \psi(\tau) \, d\tau, \quad \hat{x} \in \Omega. \tag{4.11}$$

After solving the integral equation (2.7) numerically by the method of this section, the integral (4.11) is evaluated by the trapezoidal rule (4.5). Table 1 gives some approximate values for the far-field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$. Note that the fast convergence is clearly exhibited.

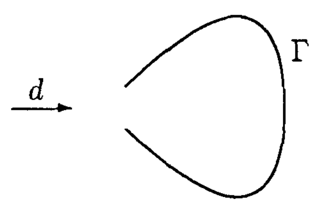

Fig. 1. Boundary curve (4.10)

Table 1. Numerical results for the boundary curve (4.10)

	n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
$k = 1$	8	− 1.16849335	0.35035875	0.19929045	− 0.92767817
	16	− 1.16852614	0.35050704	0.19959601	− 0.92780409
	32	− 1.16852614	0.35050704	0.19959601	− 0.92780408
$k = 5$	16	− 1.75628107	1.12249635	− 0.66287723	0.44511201
	32	− 1.75620002	1.12234933	− 0.66277407	0.44528313
	64	− 1.75620002	1.12234933	− 0.66277407	0.44528313
$k = 10$	32	− 2.22397222	1.65514924	− 0.51843579	− 0.88672310
	64	− 2.22395069	1.65514284	− 0.51825644	− 0.88663460
	128	− 2.22395069	1.65514284	− 0.51825644	− 0.88663460

5. The inverse problem

Up until now we have been considering the direct scattering problem, i.e. given the shape of the sound-soft (or perfectly conducting) arc Γ to determine the scattered field and, in particular, its far-field pattern. In this section we shall study the *inverse problem* to determine the shape of the arc Γ from a knowledge of the far-field pattern for the scattering of one or several incident plane waves. The first question to ask about this inverse scattering problem is uniqueness, i.e. under what conditions is an arc Γ uniquely determined by a knowledge of its far-field patterns for incident plane waves. The following theorem extends the classical uniqueness result by Schiffer (see [19]) from the case of scattering from an obstacle with a closed boundary curve to the case of an arc.

Theorem 5.1. *Assume that Γ_1 and Γ_2 are two sound-soft (or perfectly conducting) arcs such that the far-field patterns coincide for all incident directions and one fixed wave number. Then $\Gamma_1 = \Gamma_2$.*

Schiffer's approach to proving the corresponding uniqueness result for closed boundary curves uses eigenspaces for the negative Laplacian and cannot be extended to the present situation. Therefore, we will use a different technique which was developed by Isakov [12] and Kirsch and Kress [16]. For this we will need the following lemma on the approximation of arbitrary solutions to the Helmholtz equation by plane waves. Its proof can be found in [16].

Lemma 5.2. *Let $D \subset \mathbb{R}^2$ be a simply connected bounded domain with C^2 boundary ∂D and let $u \in C^2(D) \cap C^1(\bar{D})$ be a solution to the Helmholtz equation. Then there exists a sequence (v_n) in the span of plane waves*

$$V := \operatorname{span}\{u^{\mathrm{i}}(\cdot\,; d) \colon d \in \Omega\},$$

such that

$$v_n \to u, \quad \operatorname{grad} v_n \to \operatorname{grad} u, \quad n \to \infty, \tag{5.1}$$

uniformly on compact subsets of D.

Proof of Theorem 5.1. Since the far-field pattern uniquely determines the scattered field (see Theorem 2.13 in [7]), the assumption of the theorem implies that the scattered waves for both arcs coincide:

$$u_1^s(\cdot\,;d) = u_2^s(\cdot\,;d) \quad \text{in } G \tag{5.2}$$

for all $d \in \Omega$ where $G := \mathbb{R}^2 \backslash (\Gamma_1 \cup \Gamma_2)$. Let $x_0 \in G$ be arbitrary and consider the two Dirichlet problems

$$\Delta w_j^s + k^2 w_j^s = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma_j, \quad j = 1, 2, \tag{5.3}$$

with the boundary condition

$$w_j^s + \Phi(\cdot, x_0) = 0 \quad \text{on } \Gamma_j, \quad j = 1, 2, \tag{5.4}$$

subject to the Sommerfeld radiation condition. We will show that

$$w_1^s = w_2^s \quad \text{in } G. \tag{5.5}$$

To this end, we choose a simply connected bounded domain D with C^2 boundary ∂D such that $(\Gamma_1 \cup \Gamma_2) \subset D$ and $x_0 \notin \bar{D}$. Then, by Lemma 5.2, there exists a sequence (v_n) of linear combinations of plane waves such that

$$\| v_n - \Phi(\cdot, x_0) \|_{C^{1,\alpha}(\Gamma_j)} \to 0, \quad n \to \infty, \; j = 1, 2. \tag{5.6}$$

Since the v_n are linear combinations of plane waves, as a consequence of (5.2), the corresponding scattered waves $v_{n,1}^s$ and $v_{n,2}^s$ for the arcs Γ_1 and Γ_2 coincide in G. Hence, for $v_n^s = v_{n,1}^s = v_{n,2}^s$ we have

$$v_n^s + v_n = 0 \quad \text{or } \Gamma_j, \quad j = 1, 2. \tag{5.7}$$

Now the well-posedness for the Dirichlet problem (Theorem 2.5), the boundary conditions (5.4) and (5.7) and the convergence (5.6) imply that

$$v_n^s \to w_j^s, \quad n \to \infty,$$

uniformly on compact subsets of G for $j = 1, 2$; whence (5.5) follows.

Now assume that $\Gamma_1 \neq \Gamma_2$. Then, without loss of generality, there exists $x^* \in \Gamma_1$ such that $x^* \notin \Gamma_2$. We can choose a sequence (x_n) from G with $x_n \to x^*$, $n \to \infty$, such that

$$\sup_{x \in \Gamma_2,\, n \in \mathbb{N}} |x - x_n| > 0. \tag{5.8}$$

Consider the solutions $w_{n,j}^s$ to the exterior Dirichlet problems (5.3), (5.4) with x_0 replaced by x_n. By (5.5) we have $w_{n,1}^s = w_{n,2}^s$ in G. Considering $w_n^s = w_{n,2}^s$ as the scattered wave corresponding to the arc Γ_2, from (5.8) we observe that the Dirichlet data

$$w_n^s = -\Phi(\cdot, x_n) \quad \text{on } \Gamma_2$$

are uniformly bounded with respect to the $C^{1,\alpha}$ norm on Γ_2. Therefore, by Theorem 2.5 we have that the w_n^s are uniformly bounded with respect to the maximum norm on closed subsets of $\mathbb{R}^2 \backslash \Gamma_2$. In particular, this implies that

$$|w_n^s(x^*)| \leqslant C$$

for all n and some positive constant C. On the other hand, considering $w_n^s = w_{n,1}^s$ as the scattered wave corresponding to the arc Γ_1, from the boundary condition we have

$$|w_n^s(x^*)| = |\Phi(x^*, x_n)| = \tfrac{1}{4} |H_0^{(1)}(k\,|x^* - x_n|)| \to \infty, \quad n \to \infty.$$

This is a contradiction. Therefore, $\Gamma_1 = \Gamma_2$. □

Turning to the question of the existence of a solution to the inverse scattering problem, we first note that this is the wrong question to ask since in contrast to the direct problem the inverse problem is improperly posed. In particular, from (3.3) we see that the far-field pattern depends analytically on the observation direction. Hence, small perturbations of the far-field pattern in any norm which is suitable to describe measurement errors, e.g. the L^2 norm, lead to a function which lies outside the class of far-field patterns. Therefore, in general for measured far-field patterns a solution to the inverse problem does not exist, and if it exists the solution does not depend continuously on the data, i.e. small changes in the measured data can lead to large errors in the reconstruction. Hence, the proper question to ask is how can the inverse problem be stabilized and approximate solutions found to the stabilized problem.

For the approximate solution of the inverse scattering problem for closed boundary contours, one can distinguish between two different approaches. In one group of methods the inverse obstacle problem is reformulated as a nonlinear optimization problem in an *output least squares* sense. These methods require the solution of the direct scattering problem for different domains at each step of the iteration procedure used to arrive at a solution. In a second group of methods the need of solving a direct scattering problem at each iteration step is avoided. This is achieved by separating the inverse obstacle problem into a linear ill-posed part for the reconstruction of the scattered wave from the far-field pattern and a non-linear well-posed part for finding the location of the boundary of the scatterer from the boundary condition for the total field. However, unfortunately these latter methods heavily rely on the fact that the unknown scattering object has an open interior. For example, in the method suggested by Colton and Monk [8], one needs to put the origin into the interior of the unknown obstacle, and in the method proposed by Kirsch and Kress [15], one has to put an auxiliary curve into the interior. Therefore, unfortunately, there are no immediate counterparts available for these approaches in the case of scattering from an arc. Consequently, in the sequel, we will study more closely a Newton method as one possibility form the first group mentioned above.

6. Fréchet derivatives

The solution to the direct scattering problem with a fixed incident wave u^i defines an operator

$$F: \Gamma \mapsto u_\infty,$$

which maps the boundary Γ onto the far-field pattern u_∞ of the scattered wave u^s. In terms of this operator, given a (measured) far-field pattern u_∞, the inverse problem consists in solving the non-linear and ill-posed equation

$$F(\Gamma) = u_\infty \tag{6.1}$$

for the unknown arc Γ. In order to use Newton's iteration method for the approximate solution of (6.1), it is necessary to establish differentiability of the operator F with respect to Γ.

Using a Hilbert space approach for weak solutions to the direct scattering problem, Kress and Zinn [18] have shown that the operator corresponding to F in the case of closed boundary curves is Fréchet differentiable. The explicit form of the derivative has been established by Kirsch [13]. More recently, Potthast [21] was able to obtain the same results based on the classical boundary integral equation approach to the direct scattering problem. In the sequel, we will follow Potthast and establish Fréchet differentiability of the operator F for the case of an arc. To this end, we first recall the definition of the operator $K = L + A + B$ from section 2 and show its Fréchet differentiability.

Theorem 6.1. *The mapping*

$$z \mapsto K$$

is Fréchet differentiable from $C^3[-1, 1]$ into the space $\mathscr{L}(C^{0,\alpha}_{2\pi,e}(\mathbb{R}), C^{1,\alpha}_{2\pi,e}(\mathbb{R}))$ of bounded linear operators from $C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ into $C^{1,\alpha}_{2\pi,e}(\mathbb{R})$. The derivative is given by $h \to K'_z(\cdot\,; h)$ where $K'_z(\cdot\,; h): C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ denotes the integral operator

$$K'_z(\psi; h)(t) := \frac{1}{2\pi} \int_0^\pi H'(t, \tau; z, h)\psi(\tau)\,d\tau, \quad t \in [0, 2\pi], \tag{6.2}$$

with the kernel

$$H'(t, \tau; z, h) := -4\pi \operatorname{grad}_x \Phi(z(\cos t), z(\cos \tau)) \cdot [h(\cos t) - h(\cos \tau)], \quad t \neq \tau.$$

Proof. Let $z: [-1, 1] \to \mathbb{R}^2$ be a fixed but arbitrary injective function of class C^3. Then for all $h: [-1, 1] \to \mathbb{R}^2$ of class C^3 with sufficiently small $\|h\|_{C^3[-1,1]}$ the function $z + h$ is injective and describes an arc of class C^3.

Define the analytic function $\Psi: \mathbb{R}^2 \backslash \{0\} \to \mathbb{C}$ by

$$\Psi(x) := \frac{\pi}{i} H_0^{(1)}(k|x|).$$

Provided $0 \notin \{x + \lambda\xi : \lambda \in [0, 1]\}$, by Taylor's formula we have that

$$\Psi(x + \xi) - \Psi(x) - \operatorname{grad} \Psi(x) \cdot \xi = \int_0^1 (1 - \lambda)\Psi''(x + \lambda\xi)\xi \cdot \xi\,d\lambda, \tag{6.3}$$

where Ψ'' denotes the Hessian of Ψ. Recalling the kernel

$$H(t, \tau; z) = H(t, \tau) = \Psi(z(\cos t) - z(\cos \tau)), \quad t \neq \tau,$$

of the operator K from (2.10) and defining

$$H'(t, \tau; z, h) := \operatorname{grad} \Psi(z(\cos t) - z(\cos \tau)) \cdot [h(\cos t) - h(\cos \tau)], \quad t \neq \tau,$$

and

$$\begin{aligned} H''(t, \tau; z, h) := \Psi''(z(\cos t) - z(\cos \tau))[h(\cos t) - h(\cos \tau)] \\ \cdot [h(\cos t) - h(\cos \tau)], \quad t \neq \tau, \end{aligned}$$

Taylor's formula (6.3) tells us that

$$H_1(t, \tau; z + h) - H_1(t, \tau; z) - H'(t, \tau; z, h) = \int_0^1 (1 - \lambda)H''(t, \tau; z + \lambda h, h)\,d\lambda \tag{6.4}$$

for $t \neq \tau$ and for all h with sufficiently small $\|h\|_{C^3[-1,1]}$. Writing

$$H'(t, \tau; z, h) = \frac{k H_0^{(1)\prime}(k|z(\cos t) - z(\cos \tau)|)}{|z(\cos t) - z(\cos \tau)|} [z(\cos t) - z(\cos \tau)]$$
$$\cdot [h(\cos t) - h(\cos \tau)],$$

we can see that H' can be extended as a continuous function on $[0, \pi] \times [0, \pi]$ such that it has a continuous first derivative and a weakly singular (with a logarithmic singularity) second derivative with respect to t. For establishing this, obviously, in view of the power series (2.13) and (2.14), we only need to investigate the logarithmic singular part

$$\Psi_1(x) := J_0(k|x|) \ln |x|^2$$

of Ψ. Since

$$\operatorname{grad} \Psi_1(x) = \frac{1}{|x|^2} \Psi_2(x) x,$$

where

$$\Psi_2(x) := 2 J_0(k|x|) + k|x| J_0'(k|x|) \ln |x|^2,$$

the corresponding part H_1' of H' is given by

$$H_1'(t, \tau; z, h) = M(t, \tau; z) N(t, \tau; z, h),$$

where

$$M(t, \tau; z) := \Psi_2(|z(\cos t) - z(\cos \tau)|), \quad t \neq \tau,$$

and

$$N(t, \tau; z, h) := \frac{[z(\cos t) - z(\cos \tau)] \cdot [h(\cos t) - h(\cos \tau)]}{|z(\cos t) - z(\cos \tau)|^2}, \quad t \neq \tau.$$

The function M is continuously differentiable with respect to t on all of $[0, \pi] \times [0, \pi]$ and its second derivative has a logarithmic singularity since the first and second derivatives of Ψ_2 have these properties. In view of the fact that z is injective, i.e. $z'(s) \neq 0$ for all $s \in [-1, 1]$, using Taylor's formula to write

$$\frac{[z(s) - z(\sigma)] \cdot [h(s) - h(\sigma)]}{|z(s) - z(\sigma)|^2} = \frac{\displaystyle\int_0^1 \int_0^1 z'(s + \lambda(\sigma - s)) \cdot h'(s + \mu(\sigma - s)) \, \mathrm{d}\lambda \, \mathrm{d}\mu}{\displaystyle\int_0^1 \int_0^1 z'(s + \lambda(\sigma - s)) \cdot z'(s + \mu(\sigma - s)) \, \mathrm{d}\lambda \, \mathrm{d}\mu}$$

it can be seen that N can be extended as a twice continuously differentiable function on $[0, \pi] \times [0, \pi]$. Summarizing our analysis and using the fact that H' is even and 2π-periodic with respect to t and τ, we now have established that the integral operator with kernel $H'(\cdot, \cdot; z, h)$, i.e. the operator $K_z'(\cdot; h)$ given by (6.2) is a bounded operator from $C_{2\pi,e}(\mathbb{R})$ into $C_{2\pi,e}^2(\mathbb{R})$.

Elementary differentiation shows that

$$\psi''(x)\xi \cdot \xi = \frac{1}{|x|^4} \chi_1(x) [x \cdot \xi]^2 + \frac{1}{|x|^2} \chi_2(x) |\xi|^2,$$

where

$$\chi_1(x) := \frac{\pi}{i}[k^2|x|^2 H_0^{(1)''}(k|x|) - k|x| H_0^{(1)'}(k|x|)]$$

and

$$\chi_2(x) := \frac{\pi}{i} k|x| H_0^{(1)'}(k|x|).$$

Inserting this into (6.4) we can write

$$H''(t, \tau; z, h) = M_1(t, \tau; z) N_1(t, \tau; z, h) + M_2(t, \tau; z) N_2(t, \tau; z, h),$$

where

$$M_j(t, \tau; z) := \chi_j(|z(\cos t) - z(\cos \tau)|), \quad t \neq \tau,$$

for $j = 1, 2$,

$$N_1(t, \tau; z, h) := \left[\frac{[z(\cos t) - z(\cos \tau)] \cdot [h(\cos t) - h(\cos \tau)]}{|z(\cos t) - z(\cos \tau)|^2}\right]^2, \quad t \neq \tau,$$

and

$$N_2(t, \tau; z, h) := \frac{[h(\cos t) - z(\cos \tau)] \cdot [h(\cos t) - h(\cos \tau)]}{|z(\cos t) - z(\cos \tau)|^2}, \quad t \neq \tau.$$

Now proceeding as above, it can be shown that M_1 and M_2 are continuously differentiable with respect to t on all of $[0, \pi] \times [0, \pi]$ and that their second derivatives with respect to t have a logarithmic singularity at $t = \tau$. Similarly, using Taylor's formula, it can be seen that N_1 and N_2 can be extended as twice continuously differentiable functions on $[0, \pi] \times [0, \pi]$ and, in addition, that we have estimates of the form

$$\left\| \frac{\partial^m}{\partial t^m} N_j(\cdot, \cdot; z, h) \right\|_\infty \leqslant C \| h \|_{C^3[-1, 1]} \tag{6.5}$$

for $j = 1, 2$, $m = 0, 1, 2$, some constant C independent of h and all sufficiently small $h \in C^3[-1, 1]$. If we now use these estimates on the right-hand side of Taylor's formula (6.4) and its first and second derivative with respect to t and integrate we obtain that

$$\| K_{z+h}\psi - K_z\psi - K_z'(\psi; h) \|_{C^2[0, \pi]} \leqslant M \| h \|^2_{C^3[-1, 1]} \| \psi \|_{C[0, \pi]}$$

for all sufficiently small $h \in C^3[-1, 1]$ and some constant M, that is, the linear mapping $h \to K_z'(\cdot; h)$ is the Fréchet derivative of $z \mapsto K_z$ from $C^3[-1, 1]$ into $\mathscr{L}(C_{2\pi, e}(\mathbb{R}), C^2_{2\pi, e}(\mathbb{R}))$ at the point z. Note that the term on the left-hand side corresponding to the logarithmic operator L cancels due to taking the difference $K_{z+h} - K_z$.

Finally, the proof is finished by the remark that the norm of an operator in $\mathscr{L}(C_{2\pi, e}(\mathbb{R}), C^2_{2\pi, e}(\mathbb{R}))$ trivially is an upper bound for the norm of this operator considered as an element of $\mathscr{L}(C^{0,\alpha}_{2\pi, e}(\mathbb{R}), C^{1,\alpha}_{2\pi, e}(\mathbb{R}))$. □

In order to establish Fréchet differentiability of F we also need to find the derivative of the single-layer potential (2.5) with respect to Γ, that is, with respect to z. To this end we recall the definition (2.21) of the single-layer operator S and the definition (2.12) of the operator Z and note that obviously

$$K = -2ZS \quad \text{on } \Gamma. \tag{6.6}$$

We proceed with establishing Fréchet differentiability of S.

Theorem 6.2. *Let $D \subset \mathbb{R}^2$ be a domain such that $\bar{D} \cup \Gamma = \emptyset$. Then the mapping*

$$z \mapsto S$$

is Fréchet differentiable from $C^3[-1, 1]$ into $\mathscr{L}(C^{0,\alpha}_{2\pi,e}(\mathbb{R}), C^2(D))$. The derivative is given by $h \to S'_z(\cdot, h)$ where $S'_z(\cdot, h): C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^2(D)$ denotes the integral operator

$$S'_z(\psi; h)(x) := -\int_0^\pi \operatorname{grad}_x \Phi(x, z(\cos\tau)) \cdot h(\cos\tau)\psi(\tau)\, d\tau, \quad x \in D. \tag{6.7}$$

Proof. Since $\bar{D} \cup \Gamma = \emptyset$ the kernel of the integral operator S is analytic with respect to z and three times continuously differentiable with respect to t and τ. Therefore, we can proceed similarly as in the proof of the preceding theorem and use Taylor's formula with respect to z in order to establish that (6.7) describes the Fréchet derivative of $z \mapsto S$. □

From (6.7) it is obvious that

$$v(x) := S'_z(\psi; h)(x), \quad x \in \mathbb{R}^2 \backslash \Gamma,$$

defines a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition. We are now interested in the behaviour of v at the boundary Γ and its boundary values. To this end, by (6.7) we interpret v as a derivative of a single-layer potential. For densities $\psi \in C^{0,\alpha}_{2\pi,e}(\mathbb{R})$ from the jump relations for the gradient of a single-layer potential (see [5]) we then have that v can be extended locally Hölder continuously from $\mathbb{R}^2 \backslash \Gamma$ into $\mathbb{R}^2 \backslash \{z_1\} \cup \{z_{-1}\}$ with limiting values

$$v_\pm = -\int_0^\pi \operatorname{grad} \Phi(\cdot, z(\cos\tau)) \cdot h(\cos\tau)\psi(\tau)\, d\tau \pm \tfrac{1}{2} \nu \cdot Z^{-1} \tilde{h} \varphi$$

on $\Gamma \backslash \{z_1\} \cup \{z_{-1}\}$ where $\tilde{h}$ is given by $\tilde{h}(t) = h(\cos t)$, $t \in \mathbb{R}$. Comparing this with the form (6.2) of the derivative K'_z and again employing the jump relations, we obviously can write

$$v_\pm = -Z^{-1}\tilde{h} \cdot \operatorname{grad} u_\pm - \tfrac{1}{2} Z^{-1} K'_z(\psi; h) \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}. \tag{6.8}$$

We are now ready to establish the main result of this section.

Theorem 6.3. *The far-field operator $F: C^3[-1, 1] \to L^2(\Omega)$ is Fréchet differentiable. The derivative is given by $F'_z h = v_\infty$ where v_∞ denotes the far-field pattern of the solution $v \in C^2(\mathbb{R}^2 \backslash \Gamma)$ to the Helmholtz equation*

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^2 \backslash \Gamma \tag{6.9}$$

satisfying the Sommerfeld radiation condition and the boundary condition

$$v_\pm = -\nu \cdot Z^{-1} \tilde{h} \frac{\partial u_\pm}{\partial \nu} \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}, \tag{6.10}$$

such that $v - v_0$ is continuous in $\mathbb{R}^2$ where

$$v_0(x) := \int_\Gamma \operatorname{grad}_x \Phi(x, \cdot) \cdot Z^{-1} \tilde{h} \left\{ \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} \right\} ds, \quad x \in \mathbb{R}^2 \backslash \Gamma. \tag{6.11}$$

Here u denotes the solution to the scattering problem (2.1)–(2.3) *and, as before, $\tilde{h}$ is given by $\tilde{h}(t) = h(\cos t)$, $t \in \mathbb{R}$.*

Proof. By the analysis on the direct scattering problem of section 2, we can represent the scattered wave as a single-layer potential with density φ, that is,

$$u^s = S\psi, \tag{6.12}$$

where $\psi = Y\varphi$ is given as the solution of the integral equation $K\psi = 2Ru^i$, that is,

$$\psi = 2K^{-1}Ru^i.$$

Here the restriction operator $R: C^2(\mathbb{R}^2) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ is defined by

$$Ru_i := Zu^i|_\Gamma.$$

Clearly, the mapping $z \mapsto R_z$ is Fréchet differentiable with the derivative given by

$$R'_z(u^i; h) = (Z \operatorname{grad} u^i|_\Gamma) \cdot \tilde{h}. \tag{6.13}$$

Since $K: C^{0,\alpha}_{2\pi,e}(\mathbb{R}) \to C^{1,\alpha}_{2\pi,e}(\mathbb{R})$ is invertible and $z \mapsto K$ is Fréchet differentiable, the mapping $z \mapsto K^{-1}$ is also Fréchet differentiable with the derivative given by $h \mapsto -K^{-1}K'_zK^{-1}$ (see [7, p. 120]). Hence, from

$$u^s = 2SK^{-1}Ru^i,$$

using the chain rule and Theorems 6.1 and 6.2, we see that for any domain $D \subset \mathbb{R}^2$ with $\bar{D} \cup \Gamma = \emptyset$ the mapping $z \to u^s$ from $C^3[-1, 1]$ into $C^2(D)$ is Fréchet differentiable with the derivative given through $h \mapsto v$ where

$$v = 2(SK^{-1}R)'_z(u^i; h) = v_1 + v_2 + v_3,$$

$$v_1 := 2S'_z(K^{-1}Ru^i; h),$$

$$v_2 := -2SK^{-1}K'_z(K^{-1}Ru^i; h),$$

$$v_3 := 2SK^{-1}R'_z(u^i; h).$$

Clearly, v solves the Helmholtz equation and fulfills the Sommerfeld radiation condition. Recall that by the jump relation (2.24) we have that

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = -\varphi \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}. \tag{6.14}$$

Now by (6.7) it follows that $v_1 = v_0$ with v_0 given by (6.11). By (6.8), the boundary values of v_1 are

$$v_{1,\pm} = -Z^{-1}\tilde{h} \cdot \operatorname{grad} u^s_\pm - \tfrac{1}{2}Z^{-1}K'_z(\psi; h) \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}.$$

From (6.6) and (6.13) we see that

$$v_2 = \tfrac{1}{2}Z^{-1}K'_z(\psi; h) \quad \text{on } \Gamma$$

and

$$v_3 = -Z^{-1}R'_z(u^i; h) = -Z^{-1}\tilde{h} \cdot \operatorname{grad} u^i \quad \text{on } \Gamma.$$

Since $u = 0$ on Γ we have that

$$\operatorname{grad} u_\pm = \frac{\partial u_\pm}{\partial \nu} \nu \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}.$$

Therefore, adding up the boundary conditions for v_1, v_2 and v_3 now verifies the boundary condition (6.10). The assertion on the smoothness of $v - v_0$ follows from the fact that the single-layer potentials v_2 and v_3 with densities of the form (2.6) are continuous throughout $\mathbb{R}^2$.

The Fréchet differentiability of the mapping $z \to u^s$ which we now have established means that

$$\| u^s_{z+h} - u^s_h - v \|_{C^2(D)} \leqslant \gamma \| h \|^2_{C^3[-1,1]}$$

for all sufficiently small h and some constant γ independent of h. From the general far-field representation (3.2), we now can conclude that

$$\| u_{\infty, z+h} - u_{\infty, z} - v_\infty \|_{L^2(\Omega)} \leqslant C \| h \|^2_{C^3[-1,1]} \tag{6.15}$$

for all sufficiently small h and some constant C which finishes our proof. □

We note the parametrized form

$$v_0(x) = \frac{ik}{4} \int_0^\pi \frac{H_1^{(1)}(k|x - z(\cos\tau)|)}{|x - z(\cos\tau)|} [x - z(\cos\tau)] \cdot h(\cos\tau)\psi(\tau)\, d\tau, \quad x \in \mathbb{R}^2 \backslash \Gamma,$$

of (6.11) where we have used that $H_0^{(1)\prime} = - H_1^{(1)}$ with the Hankel function $H_1^{(1)}$ of order one and of the first kind. From the asymptotics for the Hankel function $H_1^{(1)}$, we find that the corresponding far-field pattern is given by

$$v_{0,\infty}(\hat{x}) = \frac{\sqrt{k}\, e^{-i\pi/4}}{\sqrt{8\pi}} \int_0^\pi e^{-ik\hat{x}\cdot z(\cos\tau)} \hat{x} \cdot h(\cos\tau)\psi(\tau)\, d\tau, \quad \hat{x} \in \Omega. \tag{6.16}$$

For the actual numerical computation of the Fréchet derivative, i.e. the solution of the boundary value problem (6.9)–(6.11), we solve for $w := v - v_0$, which is continuous in all of $\mathbb{R}^2$. Writing the boundary condition (6.10) as

$$v_\pm = - Z^{-1} \tilde{h} \cdot \operatorname{grad} u_\pm \quad \text{on } \Gamma \backslash \{z_1\} \cup \{z_{-1}\}$$

and using the single-layer representation (6.12) for u^s and the relation (6.14), we find that the boundary values of w are given by

$$\begin{aligned} w(x) = &\int_\Gamma \operatorname{grad}_x \Phi(x, y) \cdot [(Z^{-1}\tilde{h})(y) - (Z^{-1}\tilde{h})(x)]\, \varphi(y)\, ds(y) \\ &- (Z^{-1}\tilde{h})(x) \cdot \operatorname{grad} u^i(x) \end{aligned}$$

for $x \in \Gamma$, or in parametrized form

$$w(z(\cos t)) = \int_0^\pi M(t, \tau)\psi(\tau)\, d\tau - h(\cos t) \cdot \operatorname{grad} u^i(z(\cos t)), \quad t \in [0, \pi], \tag{6.17}$$

with the kernel given by

$$\begin{aligned} M(t, \tau) := &\frac{ik}{4} \frac{H_1^{(1)}(k|z(\cos t) - z(\cos\tau)|)}{|z(\cos t) - z(\cos\tau)|} [z(\cos t) - z(\cos\tau)] \\ &\cdot [h(\cos t) - h(\cos\tau)] \end{aligned}$$

for $t \neq \tau$. Using the power series (2.13) and (2.14), it can be shown that the kernel

M can be written in the form

$$M(t, \tau) = M_1(t, \tau)\ln\left(\frac{4}{e^2}[\cos t - \cos \tau]^2\right) + M_2(t, \tau),$$

where

$$M_1(t, \tau) := \frac{-k}{4\pi}\,\frac{J_1(k|z(\cos t) - z(\cos \tau)|)}{|z(\cos t) - z(\cos \tau)|}[z(\cos t) - z(\cos \tau)]$$
$$\cdot [h(\cos t) - h(\cos \tau)]$$

is three times continuously differentiable with

$$M_1(t,t) = \frac{\partial}{\partial t} M_1(t, t) = 0$$

and where

$$M_2(t, \tau) := M(t, \tau) - M_1(t, \tau)\ln\left(\frac{4}{e^2}[\cos t - \cos \tau]^2\right), \quad t \neq \tau,$$

can be extended as a twice continuously differentiable function on $[0, \pi] \times [0, \pi]$ with

$$M_2(t, t) = \frac{z'(\cos t) \cdot h'(\cos t)}{2\pi|z'(\cos t)|^2}.$$

For numerical evaluations the integral in (6.17) can be approximated using the quadrature rules (4.4) and (4.5) after the integral has been transformed into an integral over $[0, 2\pi]$ analogous to (2.17). For the solution of the exterior Dirichlet problem the numerical method described in section 4 can be applied. Note, that since the kernel M has the same structure as the kernel H of the operator K, by the corresponding analog of Theorem 2.3 the boundary data given through (6.17) belong to $C^{1,\alpha}_{2\pi,e}(\mathbb{R})$.

7. The Newton method

We now proceed with describing the application of Newton's method to the solution of (6.1), that is, the solution of

$$F_z = u_\infty,$$

where we write $F(\Gamma) = F_z$ for a parametrization z of Γ. We replace (7.1) by the linearized equation

$$F_z'h + F_z = u_\infty, \tag{7.2}$$

which we have to solve for h in order to improve an approximate boundary given by the function z into the new approximation given by $\tilde{z} = z + h$. In the usual fashion, Newton's method consists in iterating this procedure. The following theorem settles the question of uniqueness for the linear equation (7.2).

Theorem 7.1. *The nullspace of the linear operator F_z' is given by*

$$N(F_z') = \{h\colon \nu(z(\cos t)) \cdot h(\cos t) = 0,\ t \in \mathbb{R}\}. \tag{7.3}$$

Proof. Assume that $F_z'h = 0$. Then the solution v to the Dirichlet problem (6.9)–(6.11) has vanishing far-field pattern. Hence, by Rellich's lemma (cf. [5, 7]) we have $v = 0$ in $\mathbb{R}^2 \backslash \Gamma$ and consequently $v_{\pm} = 0$ on Γ. By the form of the boundary condition (6.10) this in turn implies

$$v(x) \cdot (Z^{-1}\tilde{h})(x) = 0, \quad x \in \Gamma \backslash \{z_1\} \cup \{z_{-1}\}, \tag{7.4}$$

since the normal derivative $\partial u/\partial v$ cannot vanish on open subsets of Γ as a consequence of Holmgren's uniqueness theorem (cf. [5]) and the boundary condition $u = 0$ on Γ. Substituting $x = z(\cos t)$, $t \in \mathbb{R}$, now completes the proof. □

The form (7.3) of the nullspace reflects the fact that the far-field pattern, of course, remains the same when the parametrization of the arc Γ is changed. This inherent non-uniqueness has to be taken care of by choosing appropriate classes of parametrizations. One such possibility which we will pursue a little further is to consider only arcs which are graphs of functions defined on a fixed interval, that is, we assume the function $z:[-1, 1] \to \mathbb{R}^2$ to be of the form

$$z(s) = (s, y(s)), \quad s \in [-1, 1], \tag{7.5}$$

where $y:[-1, 1] \to \mathbb{R}$ is of class C^3. Consequently, the difference h between two such arcs has to be of the form $h(s) = (0, q(s))$, $s \in [-1, 1]$. Then (7.4) clearly is equivalent to $q(s) = 0$, $s \in (-1, 1)$, that is, in the class of arcs of the form (7.5) the Fréchet derivative F_z' is injective.

Of course, for practical computations q, that is, $h = (0, q)$ is taken from a finite-dimensional subspace U_p and equation (7.2) is approximately solved by projecting it on a finite-dimensional subspace of $L^2(\Omega)$. For example, in accordance with our numerical method for the solution of the integral equation, we may choose U_p as polynomials of degree less than or equal to p with the Chebyshev polynomials $T_0, \ldots, T_p$ as basis, and take as projection the collocation at m equidistant points $\hat{x}_1, \ldots, \hat{x}_m \in \Omega$. Then writing

$$q = \sum_{j=0}^{p} a_j T_j, \tag{7.6}$$

we have to solve the linear system

$$\sum_{j=0}^{p} a_j (F_z' h_j)(\hat{x}_i) = u_\infty(\hat{x}_i) - F_z(\hat{x}_i), \quad i = 1, \ldots, m, \tag{7.7}$$

for the coefficients a_j where $h_j = (0, T_j)$. In order to compute the coefficients and the right-hand side of the linear system (7.7), in each iteration step of the Newton method not only the direct problem for the arc Γ given by the function z has to be solved for the evaluation of the right-hand side $F_z(\hat{x}_i)$. In addition, to compute the matrix entries $(F_z' h_j)(\hat{x}_i)$, we need to solve $p + 1$ additional direct problems for the same boundary arc Γ and different boundary values given by (6.10) for $h = (0, T_j)$, $j = 0, \ldots, p$. If we use our quadrature methods described in section 4, then the approximating linear system (4.9) has to be solved for $p + 2$ different right-hand sides. This can be efficiently done by a suitable LU-decomposition of the linear system.

Before we conclude the paper with a numerical example, we summarize the description of one step of the Newton method as follows:

1. For the arc given by the function z, that is, by the function y set up the linear system

(4.9) and solve for the right-hand side given through the incident plane wave. Compute the far-field pattern $F_z(\hat{x}_i)$ via (4.11) for $i = 1, \dots, m$.

2. Compute the boundary values (6.17) and solve (4.9) for the corresponding right-hand sides for $h = (0, T_j)$, $j = 0, \dots, p$. Compute the far-field pattern $w_\infty(\hat{x}_i)$ via (4.11) and the far-field patterns $v_{0,\infty}(\hat{x}_i)$ from (6.16) to obtain $(F_z' h_j)(\hat{x}_i) = w_\infty(\hat{x}_i) + v_{0,\infty}(\hat{x}_i)$ for $i = 1, \dots, m$.
3. Solve the linear system (7.7) for the coefficients $a_0, \dots, a_p$ of $h = (0, q)$ and update $\tilde{z} = z + h$, that is, $\tilde{y} = y + q$.

Since the linearized equation (7.2) inherits the ill-posedness from equation (6.1), for its solution some stabilization has to be incorporated, for example, by a Tikhonov regularization (cf. [7, 17]). This regularization will also take care of overdetermination in (7.7) when $2m > p + 1$. And it also allows the use of more than one incident field by simultaneously solving (7.7) for different incident directions d.

In the following numerical example, we chose an arc of the form (7.5) with the function y given by

$$y(s) = 0.5 \cos\frac{\pi s}{2} + 0.2 \sin\frac{\pi s}{2} - 0.1 \cos\frac{3\pi s}{2}, \quad s \in [-1, 1]. \tag{7.8}$$

Note that (7.8) is not a polynomial, that is, it does not belong to the class of functions (7.6) in which we seek our approximation. For the solution of the forward problem generating the synthetic data u_∞, we used the method of section 4. In order not to commit an *inverse crime* (see [7]), we chose different numbers of quadrature points for the solution of the forward and of the inverse problem. The following results are all obtained with $n = 64$ quadrature points for the forward problem and $n = 32$ quadrature points for the inverse problem. In addition, in order to illustrate the stability of the inverse algorithm, we also tested it with random errors added to the synthetic far-field pattern. We used $m = 64$ equidistant points for the far-field evaluations in (7.7). We chose the regularization parameter α for the Tikonov regularization of (7.7) by trial and error. We note that keeping the degree p in (7.6) small creates an additional regularizing effect. In all our numerical experiments we found that we ran into serious stability problems when we chose $p \geqslant 7$.

As initial guess for the Newton iteration in our example we chose $q = 0$ for $p = 2$. Then we recursively used the result for degree $p \geqslant 2$ as starting values for degree $p + 1$ with the additional coefficient $a_{p+1} = 0$. The iterations were stopped when the difference of the Tikonov cost functional in two consecutive steps was less than a tolerance 0.001. For each additional degree of freedom about two or three Newton steps were sufficient for this accuracy.

Table 2 gives the numerical results for the wave number $k = 3$ and different incident directions d. The parameter ε describes the noise level added to the exact far-field data. For comparison, the last row gives the coefficients of the best L^2 approximation to y with respect to the Chebyshev polynomials.

In addition, we also illustrate our results by two figures to indicate the influence of the wave number k, the degree p and the noise level ε on the reconstruction. The incident direction is $d = (1,0)$. In Figs. 2 and 3 the thin line represents the original arc and the thick line its reconstruction by our inverse algorithm. The parameters corresponding to the four curves (a)–(d) are as follows: $p = 5$ for (a) and (b) and $p = 6$ for (c) and (d); $\varepsilon = 0$ and $\alpha = 0$ for (a) and (c) and $\varepsilon = 0.1$ and $\alpha = 0.01$ for (b) and (d).

Table 2. Numerical results for the arc (7.8)

u^i	ε	α	a_0	a_1	a_2	a_3	a_4	a_5
$d = (1, 0)$	0	0	0.28	0.24	– 0.21	– 0.03	– 0.06	– 0.02
	0.1	0.01	0.27	0.24	– 0.22	– 0.05	– 0.08	– 0.02
$d = (0, 1)$	0	0	0.26	0.23	– 0.21	– 0.00	– 0.03	– 0.02
	0.1	0.01	0.26	0.22	– 0.21	0.01	– 0.02	0.01
$d = (-1, 0)$	0	0	0.27	0.23	– 0.21	– 0.03	– 0.05	0.02
	0.1	0.01	0.27	0.23	– 0.23	– 0.02	– 0.03	0.00
$d = (0, -1)$	0	0	0.27	0.23	– 0.22	– 0.03	– 0.06	– 0.00
	0.1	0.01	0.27	0.23	– 0.22	– 0.03	– 0.04	0.01
			0.26	0.23	– 0.22	– 0.03	– 0.06	0.00

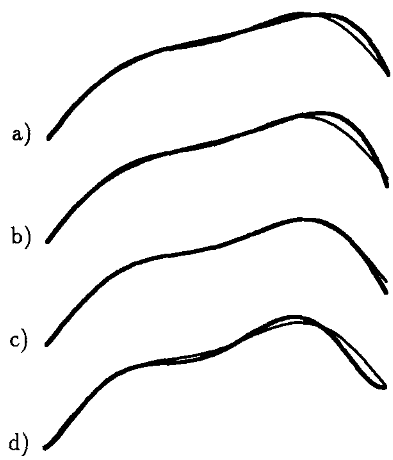

Fig. 2. Reconstructions for $k = 3$

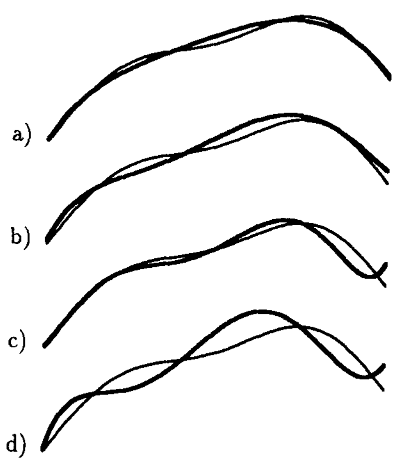

Fig. 3. Reconstructions for $k = 1$

References

1. Atkinson, K. E. and Sloan, I. H., 'The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs', *Math. Comp*, **56**, 119–139 (1991).
2. Bürger, J., 'Die numerische Behandlung von Integralgleichungen erster Art bei Dirichletschen Randwertproblemen zur Helmholtzgleichung', Diplomarbeit, Göttingen, 1994.
3. Chapko, R. and Kress, R., 'On a quadrature method for a logarithmic integral equation of the first kind', in: *World Scientific Series in Applicable Analysis — Vol. 2. Contributions in Numerical Mathematics* (Agarwal, ed.), pp. 127–140, World Scientific, Singapore, 1993.
4. Colton, D. and Kirsch, A., 'Dense sets and far field patterns in acoustic wave propagation', *SIAM J. Math. Anal.*, **15**, 996–1006 (1984).
5. Colton, D. and Kress, R., *Integral Equation Methods in Scattering Theory*, Wiley-Interscience, New York, 1983.
6. Colton, D. and Kress, R., *Integral Equation Methods in Scattering Theory*, Mir, Moscow, 1987 (Russian translation of [5]).
7. Colton, D. and Kress, R., *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, Berlin, 1992.
8. Colton, D. and Monk, P., 'A novel method for solving the inverse scattering problem for time-harmonic acoustic waves in the resonance region', *SIAM J. Appl. Math.*, **45**, 1039–1053 (1985).
9. Gieseke, B., 'Zum Dirichletschen Prinzip für selbstadjungierte elliptische Differentialoperatoren', *Math. Z.*, **68**, 54–62 (1964).
10. Hayashi, Y., 'The Dirichlet problem for the two-dimensional Helmholtz equation for an open boundary', *J. Math. Anal. Appl.*, **44**, 489–530 (1973).
11. Henrici, P., *Applied and Computational Complex Analysis*, Vol 3, Wiley-Interscience, New York, 1986.
12. Isakov, V., 'On uniqueness in the inverse transmission scattering problem', *Comm. Part. Diff. Eqs.*, **15**, 1565–1587 (1990).
13. Kirsch, A., 'The domain derivative and two applications in inverse scattering theory', *Inverse Problems*, **9**, 81–96 (1993).
14. Kirsch, A., 'Numerical algorithms in inverse scattering theory', in: *Ordinary and Partial Differential Equations* (Jarvis and Sleeman, eds), Pitman Research Notes in Mathematics **289**, pp. 93–111, Longman, London, 1993.
15. Kirsch, A. and Kress, R., 'On an integral equation of the first kind in inverse acoustic scattering', in: *Inverse Problems* (Cannon and Hornung, eds), pp. 93–102, ISNM **77**, 1986.
16. Kirsch, A. and Kress, R., 'Uniqueness in inverse obstacle scattering', *Inverse Problems*, **9**, 285–299 (1993).
17. Kress, R., *Linear Integral Equations*. Springer, Berlin, Heidelberg, New York, 1989.
18. Kress, R. and Zinn, A., 'On the numerical solution of the three dimensional inverse obstacle scattering problem', *J. Comp. Appl. Math.*, **42**, 49–61 (1992).
19. Lax, P. D. and Pillips, R. S., *Scattering Theory*, Academic Press, New York, 1967.
20. Murch, R. D., Tan, D. G. H. and Wall, D. J. N., Newton–Kantorovich method applied to two-dimensional inverse scattering for an exterior Helmholtz problem. *Inverse Problems*, **4**, 1117–1128 (1988).
21. Potthast, R., Fréchet differentiability of boundary integral operators in inverse acoustic scattering, *Inverse Problems*, **10**, 431–477 (1994).
22. Prössdorf, S. and Silbermann, B., *Projektionsverfahren und die näherungsweise Lösung singulärer Gleichungen*, Teubner, Leipzig, 1977.
23. Roger, A., 'Newton Kontorovich algorithm applied to an electromagnetic inverse problem', *IEEE Trans. Ant. Prop.*, **AP-29**, 232–238 (1981).
24. Tobocman, W., 'Inverse acoustic wave scattering in two dimensions for impenetrable targets', *Inverse Problems*, **5**, 1131–1144 (1989).
25. Wang, S. L. and Chen, Y. M., 'An efficient numerical method for exterior and interior inverse problems of Helmholtz equation', *Wave Motion*, **13**, 387–399 (1991).
26. Yan, Y. and Sloan, I. H., 'On integral equations of the first-kind with logarithmic kernels', *J. Integral Equations Appl.*, **1**, 549–579 (1988).