Arch. Rational Mech. Anal. 112 (1990) 363–390. © Springer-Verlag 1990

A Hypersingular Boundary Integral Method for Two-Dimensional Screen and Crack Problems

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Communicated by D. OWEN

Abstract

Here we analyze hypersingular integral equations on a curved open smooth arc in $\mathbb{R}^2$ that model either curved cracks in an elastic medium or the scattering of acoustic and elastic waves at a hard screen. By using the Mellin transformation we obtain sharp regularity results for the solution of these equations in Sobolev spaces in the form of singular expansions. In particular we show that the expansions do not contain logarithmic singularities.

Introduction

We analyze hypersingular integral equations of the first kind on open, bounded, nonintersecting curves $\Gamma \subset \mathbb{R}^2$. We derive the integral equations by using the "direct method" for

1) the exterior Neumann problem of the Helmholtz equation,
2) the exterior traction problem of the Navier equations.

These equations model the following problems:

1) the scattering of time-harmonic acoustic waves at a hard scatterer Γ,
2) the stationary elastic displacement field in an isotropic homogeneous unbounded medium with a curved crack Γ or the scattering of elastic-time harmonic waves at the crack Γ (as in the case of scattered earthquake waves, for example).

On the screen we consider the Neumann condition and on the crack the traction condition. The corresponding Dirichlet or displacement problems yield Fredholm boundary integral equations of the first kind and will be presented in the paper [27].

For the derivation of the hypersingular integral equations we only require that the desired fields have locally finite energy, *i.e.*, that the fields belong to

$H^1_{\mathrm{loc}}(\Omega_\Gamma)$ for $\Omega_\Gamma = \mathbb{R}^2 \setminus \overline{\Gamma}$. Alternatively, we could impose corresponding edge conditions for the fields u or conditions for the trace of u and its normal derivative such as

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} u \frac{\partial u}{\partial n}\, ds = 0$$

where Γ_ε denotes any piece of Γ of length ε. This last condition for the trace of u on Γ will be a consequence of the assumption that $u \in H^1_{\mathrm{loc}}(\Omega_\Gamma)$. Therefore, the integral equations are equivalent to the variational formulations of the original boundary value problems. We prove existence, uniqueness and regularity of solutions of the integral equations in appropriate Sobolev spaces.

The existence of solutions follows from the strong ellipticity of the hypersingular integral operator, a pseudodifferential operator of order 1 on Γ that provides a Gårding inequality in the energy space $\tilde{H}^{1/2}(\Gamma)$. Therefore, this integral operator is a Fredholm operator of index zero, and hence uniqueness of the solution of the original boundary value problem implies existence of the solution of our integral equation. The main result of our investigation is an explicit description of the solutions of the integral equations near the screen and crack tips. We show sharp regularity results in augmented Sobolev spaces by providing local singular Lehmann expansions and regularity of the remainders. In particular we find the *same* singular behavior near a crack tip of the solutions in the stationary and in the time-harmonic problems. More explicitly, the form of the singularities is $\sqrt{\varrho}$, independent of wave numbers, where ϱ denotes the distance from a crack tip. In addition, our analysis yields $\varrho^{3/2}$ for the next term in the expansion, so that the curvature of Γ does *not* produce logarithmic contributions. We obtain these sharp regularity results by applying the Mellin transform to the hypersingular integral equations and by using the fact that the equations are Wiener-Hopf equations that involve a pseudodifferential operator of order 1. We generalize the purely variational formulation of the screen problem by DURAND [18] and clarify the local behavior of the elastic field near the crack tips. This behavior is basic for the computational method presented in [34].

Our Mellin calculus was developed in references [10], [12], [15], and was applied to the regularity analysis for various boundary integral equations in references [12], [13], [14], [16], [26], [27], [40].

The local expansions about the crack tips provide, with the help of the Green representation formula, a corresponding description of the fields near Γ. The hypersingular integral equations are well suited to numerical computations with the Galerkin method. Augmentation by means of the explicitly known singular functions obtained here will both prevent pollution effects and improve the accuracy of the computations. It also leads to the optimal grading of meshes (see references [3], [34], [41]).

The variational formulations of the hypersingular equations have already been studied by several authors and in various applications (*cf.*, for example, references [4], [5], [6], [9], [20], [21], [22], [23], [47], [48]). In several of these contributions, the variational formulations have also been used for numerical analysis and computations with Galerkin's method.

Here, however, we do not present the asymptotic error analysis for corresponding augmented spline Galerkin methods, which can be obtained along the lines of the analysis in references [31] and [40]; the details, however, are yet to be carried out.

We mark the end of a proof by $\square$.

1. The Neumann Screen Problem

The Neumann screen problem describes the acoustic scattering of a plane wave at a hard obstacle Γ. The latter is given by an oriented open arc, which is a finite piece of a smooth curve in $\mathbb{R}^2$. The orientation defines the normal vector $\boldsymbol{n}$ pointing to the side Γ_2. The opposite side of Γ will be denoted by Γ_1. The scattering problem leads to the problem of finding the velocity field u satisfying

$$(\Delta + k^2)\, u = 0 \quad \text{in } \Omega_\Gamma = \mathbb{R}^2 \setminus \bar{\Gamma}, \tag{1.1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} = g_1, \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = g_2$$

where $g_1 = g_2 = -\dfrac{\partial u_{in}}{\partial n} \in \mathrm{H}^{-1/2}(\Gamma)$ is given by the incident field u_{in}. In the slightly more general mathematical model we may also choose arbitrary $g_1, g_2 \in \mathrm{H}^{-1/2}(\Gamma)$ satisfying $g = g_1 - g_2 \in \tilde{\mathrm{H}}^{-1/2}(\Gamma)$. In addition, we require the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}) \quad \text{and} \quad u = O(r^{-1/2}) \tag{1.2}$$

as $r = |x| \to \infty$ and $\operatorname{Im} k \geqq 0$.

As in references [37], [39], [40] we define variational solutions of (1.1), (1.2) by using the local Sobolev spaces. (See [46, p. 51 ff.] and closely related presentations in references [18], [21], [23], and [47]). We call u a *variational* solution of (1.1), (1.2) if u has the following properties:

(i) $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$, *i.e.*, $u \in \mathrm{H}^1(\Omega_\Gamma \cap B_R)$ for every $R > 0$ where $B_R = \{x \mid |x| \leqq R\}$.

(ii) u satisfies

$$\int_{\Omega_\Gamma} (\nabla u \cdot \nabla \phi - k^2 u \phi)\, dx - \int_\Gamma (g_1 \phi_1 - g_2 \phi_2)\, ds = 0 \tag{1.3}$$

for all test functions $\phi \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ having compact support in $\mathbb{R}^2$. ϕ_j denotes the limits of ϕ on Γ from the sides, Γ_j $i = 1, 2$, respectively.

(iii) u satisfies the radiation condition (1.2).

Note that (1.3) with $\phi \in C_0^\infty(\Omega_\Gamma)$ already implies that u is a solution of the Helmholtz equation in (1.1) and hence that $u \in C^\infty(\Omega_\Gamma)$. Therefore, (1.2) can be required to hold pointwise. On the other hand, note that u and ϕ may both possess different limits u_j and ϕ_j on Γ_j for $j = 1$ or 2. For brevity, let us introduce the space of distributional solutions,

$$\mathscr{L}_\Gamma := \{u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma) \mid \Delta u + k^2 u = 0 \text{ in } \Omega_\Gamma,\ u \text{ satisfies (1.2)}\}. \tag{1.4}$$

Lemma 1.1. *For* $\operatorname{Im} k \geqq 0$, $k \neq 0$ *and* $g_1 = g_2 = 0$ *the Neumann screen problem* (1.1), (1.2) *has at most the trivial solution in* $\mathscr{L}_\Gamma$.

Proof. For the proof and for further investigation we extend Γ to an arbitrary smooth simple curve ∂G_1 enclosing a bounded domain G_1 (see Figure 1). Let $\boldsymbol{n}$ denote the exterior normal vector of ∂G_1 and $\partial/\partial n$ the corresponding normal derivative. For any function v given in Ω_Γ let

$$[v] = v_1 - v_2$$

where v_1 denotes the limit of v from G_1 and v_2 the limit from $G_2 := (\mathbb{R}^2 \setminus \overline{G}_1) \cap B_R$ and where $R > 0$ has been chosen sufficiently large (see Figure 1).

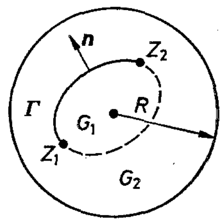

Fig. 1

Now, let $u \in \mathscr{L}_\Gamma$ with $g_j = 0$. Then $u|_{G_1} \in \mathrm{H}^1(G_1)$, $u|_{G_2} \in \mathrm{H}^1(G_2)$ for every $R > 0$ and u satisfies the following transmission problem:

$$u = u_1 \text{ in } G_1 \quad \text{with} \quad \Delta u_1 + k^2 u_1 = 0 \text{ in } G_1;$$

$$u = u_2 \text{ in } \mathbb{R}^2 \setminus \overline{G}_1 \quad \text{with} \quad \Delta u_2 + k^2 u_2 = 0 \text{ in } \mathbb{R}^2 \setminus \overline{G}_1 \tag{1.5}$$

and u_2 satisfies the radiation conditions (1.2);

$$u_1 = u_2 \text{ on } \partial G_1 \setminus \overline{\Gamma} \quad \text{and} \quad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial G_1 \setminus \overline{\Gamma}.$$

From problem (1.5), from conditions $g_1 = g_2 = 0$ on Γ, and from the inclusion $\Delta u_j = -k^2 u_j \in L_2(G_j)$, $j = 1, 2$, we obtain

$$\int\limits_{|x|=R} \frac{\partial u_2}{\partial n} \bar{u}_2 \, dx = \int\limits_{G_1} (|\nabla u|^2 - k^2 |u|^2) \, dx + \int\limits_{G_2} (|\nabla u|^2 - k^2 |u|^2) \, dx \tag{1.6}$$

by applying Green's formula in G_1 and G_2. For $\operatorname{Re} k > 0$ and $\operatorname{Im} k > 0$ we take the imaginary part of (1.6) to obtain

$$\operatorname{Im} \int\limits_{|x|=R} \frac{\partial u_2}{\partial n} \bar{u}_2 \, ds = -2 \operatorname{Im} k \cdot \operatorname{Re} k \int\limits_{G_1 \cup G_2} |u|^2 \, dx.$$

The radiation condition (1.2) implies that the left-hand side tends to zero for $R \to \infty$. Hence we find that $u \equiv 0$.

For $\operatorname{Re} k = 0$ and $\operatorname{Im} k > 0$ we find that

$$\int_{|x|=R} \frac{\partial u_2}{\partial n} \bar{u}_2 \, ds = \int_{G_1 \cup G_2} (|\nabla u|^2 + (\operatorname{Im} k)^2 |u|^2) \, dx .$$

Again the radiation condition (1.2) yields that the left-hand side vanishes for $R \to \infty$, which implies that $u \equiv 0$.

For $\operatorname{Im} k = 0$ and $\operatorname{Re} k > 0$ we insert into

$$I := \int_{|x|=R} \frac{\partial u_2}{\partial n} \bar{u}_2 \, ds$$

the condition (1.2) to obtain

$$k \int_{|x|=R} |u_2|^2 \, ds = \operatorname{Im} I + o(1) = o(1)$$

for $R \to \infty$, since the imaginary part of the right-hand side of (1.6) is zero for real k. By Rellich's theorem [46, Theorem 4.2] this gives $u_2 \equiv 0$. Using (1.5) and the unique continuation property for the Helmholtz equation across $\partial G_1 \setminus \bar{\Gamma}$ into G_1 we therefore find that $u_1 \equiv 0$. $\square$

For the formulation and analysis of properties of the boundary integral equation on Γ we shall need the following property of the traces of variational solutions.

Lemma 1.2. *Let* $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ *be a variational solution of the Neumann screen problem* (1.1), (1.2) *and let* $u_1 = u|_{G_1}$, $u_2 = u|_{G_2}$. *Then*

$$[u]|_\Gamma = (u_1 - u_2)|_\Gamma \in \tilde{\mathrm{H}}^{1/2}(\Gamma) \tag{1.7}$$
$$= \{w \in \mathrm{H}^{1/2}(\partial G_1) \text{ with } \operatorname{supp} w \subset \bar{\Gamma} \text{ equipped with the } \mathrm{H}^{1/2}(\partial G_1)\text{-norm}\}.$$

Proof. The relations

$$u_1|_{\partial G_1} \in \mathrm{H}^{1/2}(\partial G_1) \quad \text{and} \quad u_2|_{\partial G_2} \in \mathrm{H}^{1/2}(\partial G_1)$$

are simple consequences of the trace theorem applied to u_1 in G_1 and u_2 in G_2 and hence

$$[u]|_{\partial G_1} \in \mathrm{H}^{1/2}(\partial G_1).$$

In addition we see from (1.5) that

$$\operatorname{supp}\, [u]|_{\partial G_1} \subset \bar{\Gamma}. \quad \square$$

Next, we develop a solution procedure for (1.1), (1.2) by making use of integral equations that allow us to obtain the explicit singular behavior of the field u at the endpoint of the arc Γ.

We begin with Green's second identity applied to a given variational solution $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ of (1.1), (1.2) with G_1 and G_2 as in Fig. 1. Green's formula provides a representation of u by potentials of double and single layers. For any fixed $z \in G_1$ we have

$$\begin{aligned} u(z) &= -\int_{\partial G_1} \Phi(z,\zeta)\frac{\partial u_1}{\partial n}(\zeta)\,ds_\zeta + \int_{\partial G_1}\left(\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\right)u_1(\zeta)\,ds_\zeta, \\ 0 &= -\int_{\partial G_2}\left(\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\right)u_2(\zeta)\,ds_\zeta + \int_{\partial G_2}\Phi(z,\zeta)\frac{\partial u_2}{\partial n}(\zeta)\,ds_\zeta \end{aligned} \tag{1.8}$$

where

$$\Phi(z,\zeta) = -\frac{i}{4}\mathrm{H}_0^{(1)}(k\,|z-\zeta|) \tag{1.9}$$

is the fundamental solution of the Helmholtz equation.

Taking the normal derivative for $z \in \partial G_1$ in both equations of (1.8), using the jump relations of boundary potentials, and summing, we obtain

$$\begin{aligned} \left.\frac{\partial u_1}{\partial n} + \frac{\partial u_2}{\partial n}\right|_{\partial G_1} &= -2\int_\Gamma (g_1 - g_2)\frac{\partial}{\partial n_z}\Phi\,ds_\zeta + 2\frac{\partial}{\partial n_z}\int_\Gamma\left(\frac{\partial}{\partial n_\zeta}\Phi\right)[u]\,ds_\zeta \\ &\quad + \int_{|\zeta|=R}\frac{\partial}{\partial n_z}\Phi\frac{\partial u_2}{\partial n}\,ds_\zeta - \frac{\partial}{\partial n_z}\int_{|\zeta|=R}\left(\frac{\partial}{\partial n_\zeta}\Phi\right)u_2\,ds_\zeta. \end{aligned}$$

Thus, specializing this equation to Γ, letting $R\to\infty$, and using the radiation condition (1.2), which holds for u_2 as well as for Φ, we obtain

$$D[u](z) := -2\frac{\partial}{\partial n_z}\int_\Gamma [u](\zeta)\left(\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\right)ds_\zeta \tag{1.10}$$

$$= -g_1(z) - g_2(z) - 2\int_\Gamma (g_1(\zeta) - g_2(\zeta))\frac{\partial}{\partial n_z}\Phi(z,\zeta)\,ds_\zeta \quad \text{for } z\in\Gamma.$$

Now, let us suppose that $[u]\in\tilde{\mathrm{H}}^{1/2}(\Gamma)$ satisfies (1.10). Then, for $z\in\Omega_\Gamma = \mathbb{R}^2\setminus\bar{\Gamma}$, the two integrals in the right-hand side of the relation

$$u(z) = \int_\Gamma [u](\zeta)\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\,ds_\zeta - \int_\Gamma \Phi(z,\zeta)\,(g_1(\zeta) - g_2(\zeta))\,ds_\zeta \tag{1.11}$$

define potentials $K[u]$ and $V(g_1 - g_2)$, respectively, since for fixed $z\notin\bar{\Gamma}$, we have $\Phi(z,\zeta)\in\mathrm{H}^{1/2}(\Gamma) = (\tilde{\mathrm{H}}^{-1/2}(\Gamma))'$ and $\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\in\mathrm{H}^{-1/2}(\Gamma) = (\tilde{\mathrm{H}}^{1/2}(\Gamma))'$.

The mapping properties of the single-layer potential operator V and the double-layer potential operator K can be obtained from the well known expansion of the Hankel function appearing in the formula (1.9) for the fundamental solution:

$$\begin{aligned} -\frac{i}{4}\mathrm{H}_0^{(1)}(k\,|z-\zeta|) &= \sum_{j=0}^{\infty} c_j\,|k(z-\zeta)|^{2j} + \frac{1}{2\pi}\log|k(z-\zeta)| \\ &\quad\times\left\{1 + \sum_{j=1}^{\infty} d_j\,|k(z-\zeta)|^{2j}\right\}, \end{aligned} \tag{1.12}$$

which converges for $|k(z-\zeta)| < R_0$ for some $R_0 > 0$. Accordingly, we have an *asymptotic* expansion of V,

$$\begin{aligned} Vg(z) = &\int_{\partial G_1} g\, ds \left(c_0 + \frac{1}{2\pi}\log|k|\right) + \sum_{j=1}^{\infty} c_j k^{2j} \int_{\partial G_1} |z-\zeta|^{2j} g(\zeta)\, ds_\zeta \\ &+ \sum_{j=1}^{\infty} d_j k^{2j} \frac{1}{2\pi} \int_{\partial G_1} \log|k(z-\zeta)| \cdot |z-\zeta|^{2j} g(\zeta)\, ds_\zeta \\ &+ \frac{1}{2\pi}\int_{\partial G_1} g(\zeta)\log|z-\zeta|\, ds_\zeta. \end{aligned} \tag{1.13}$$

Every operator in the expansion (1.13) can be regarded as an operator of potential type defined on ∂G_1, provided that g is extended by zero to a function in $\mathrm{H}^{-1/2}(\partial G_1)$ and $z \in G_l$, $l = 1, 2$. By introducing local coordinates which locally transform ∂G_1 (which is of class C^∞) onto a straight line and by following Eskin [19, p. 106ff.], we easily verify that the operators in (1.31) admit two-dimensional symbols of the form const $(k)\,|\xi' + \xi_2|^{-2-2j}$, $j = 0, 1, \ldots$ that possess analytic extensions

$$\text{const } (k)\,|\xi' + (\xi_2 \pm i\tau)|^{-2-2j}$$

for $\xi' \neq 0$, $(\xi', \xi_2) \in \mathbb{R}^2$ and $\tau > 0$ with respect to $(\xi_2 \pm i\tau)$ and that satisfy the hypotheses of Lemma 8.1 in [19]. Hence the operators in (1.13) define continuous mappings from $\mathrm{H}^s(\partial G_1)$ into $\mathrm{H}^{s+2j+3/2}(G_1)$ and $\mathrm{H}^{s+2j+3/2}(G_2)$ for every $s \in \mathbb{R}$ and $j = 0, 1, \ldots$. In particular, with the above extension of g from Γ to ∂G_1 and with $s = -\frac{1}{2}$ we find that

$$V(g_1 - g_2)(\cdot) = \int_\Gamma \Phi(\cdot,\zeta)\,(g_1(\zeta) - g_2(\zeta))\, ds_\zeta \in \mathrm{H}^1(G_1)$$

and $V(g_1 - g_2) \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{G}_1)$.

For the double-layer operator

$$Kv(z) = \int_{\partial G_1} v(\zeta)\frac{\partial}{\partial n_\zeta}\Phi(z,\zeta)\, ds_\zeta$$

we obtain the asymptotic expansion

$$\begin{aligned} Kv(z) = &\sum_{j=1}^{\infty}\left(2jc_j + \frac{d_j}{2\pi}\right)k^{2j}\int_{\partial G_1} |\zeta - z|^{2(j-1)}\, \boldsymbol{n}(\zeta)\cdot(\zeta - z)\, v(\zeta)\, ds_\zeta \\ &+ \sum_{j=1}^{\infty} 2j\, d_j k^{2j}\frac{1}{2\pi}\int_{\partial G_1}\log|k(\zeta - z)|\,|\zeta - z|^{2(j-1)}\, \boldsymbol{n}(\zeta)\cdot(\zeta - z)\, v(\zeta)\, ds_\zeta \\ &+ \frac{1}{2\pi}\int_{\partial G_1}\frac{\boldsymbol{n}(\zeta)\cdot(\zeta - z)}{|\zeta - z|^2}\, v(\zeta)\, ds_\zeta. \end{aligned} \tag{1.14}$$

Every operator in the expansion (1.14) also defines on ∂G_1 an operator of potential type in the sense of Eskin [19]. The corresponding two-dimensional symbols are of the form

$$\text{const } (k)\,\xi_2\,|\xi' + \xi_2|^{-2-2j}$$

and possess analytic extensions

$$\text{const}\,(k)\,(\xi_2 \pm i\tau)\,|\xi' + (\xi_2 \pm i\tau)|^{-2-2j}$$

with respect to $(\xi_2 \pm i\tau)$ for $\xi' \neq 0$, $(\xi', \xi_2) \in \mathbb{R}^2$ and $\tau > 0$. These symbols also satisfy the hypotheses of Lemma 8.1 in [19] with $\alpha' = \alpha = -2j - 1$, $j = 0, 1, \ldots,$ so that the operators in (1.14) define continuous mappings from $\mathrm{H}^s(\partial G_1)$ into $\mathrm{H}^{s+2j+1/2}(G_1)$ and $\mathrm{H}^{s+2j+1/2}(G_2)$, respectively, for every $s \in \mathbb{R}$ and $j = 0, 1, 2, \ldots$ Thus ,with the extension of $[u]$ by zero from Γ to ∂G_1 and with $s = \frac{1}{2}$ we see that $K[u](z)$ in (1.11) belongs to $\mathrm{H}^1(G_1)$ and $\mathrm{H}^1(G_2)$ for $z \in G_1$ and $z \in G_1$, respectively.

All the above considerations are valid for extensions of Γ to arbitrary smooth simple curves ∂G_1. Hence, for $(g_1 - g_2) \in \tilde{\mathrm{H}}^{-1/2}(\Gamma)$ and $[u] \in \tilde{\mathrm{H}}^{1/2}(\Gamma)$ we find $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ for u given by (1.11). Moreover, for $z \notin \Gamma$ we may differentiate under the integral sign in (1.11) to obtain

$$(\Delta + k^2)\,u = 0 \quad \text{in } \Omega_\Gamma.$$

Since Φ satisfies the radiation conditions (1.2) for $\zeta \in \Gamma$ and for $|z| \to \infty$, it follows that u also satisfies (1.2).

In summary, we find equivalence of the variational solution of the boundary value problem (1.1), (1.2) and the boundary integral equation (1.10) for $[u]$ in the following sense.

Theorem 1.3. *Let* $g_1, g_2 \in \mathrm{H}^{-1/2}(\Gamma)$ *be given with* $g_1 - g_2 \in \tilde{\mathrm{H}}^{-1/2}(\Gamma)$. *If* $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ *is the solution of the Neumann screen problem* (1.1), (1.2), *then the jump* $[u]|_\Gamma \in \tilde{\mathrm{H}}^{1/2}(\Gamma)$ *satisfies the integral equation* (1.10) *for* $z \in \Gamma$. *If the jump* $[u]|_\Gamma \in \tilde{\mathrm{H}}^{1/2}(\Gamma)$ *satisfies* (1.10), *then* u *defined by* (1.11) *satisfies the Neumann screen problem* (1.1), (1.2) *in* $\mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$.

Note that for the derivation of the boundary integral equation (1.10) we only assumed that the solution has *locally bounded energy*: We need only the regularity of the variational solution, *i.e.*, that $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$, and *not any additional* regularity assumptions. Thus, the solution of the scattering-screen problem can be reduced to the resolution of equation (1.10) with respect to $[u]$. The expansion (1.12) also leads to an asymptotic expansion of the operator D in (1.10):

$$\begin{aligned} Dw(z) = {} & \sum_{j=2}^{\infty} 4(j-1)\left(2jc_j + (2j+1)\frac{d_j}{2\pi}\right) k^{2j} \int_{\partial G_1} |\zeta - z|^{2(j-2)}\, \boldsymbol{n}(\zeta) \cdot (\zeta - z) \\ & \times \boldsymbol{n}(z) \cdot (z - \zeta)\, w(\zeta)\, ds_\zeta \\ & - \sum_{j=1}^{\infty} 2\left(2jc_j + \frac{d_j}{2\pi}\right) k^{2j} \int_{\partial G_1} |\zeta - z|^{2(j-1)}\, \boldsymbol{n}(z) \cdot \boldsymbol{n}(\zeta)\, w(\zeta)\, ds_\zeta \\ & + \sum_{j=2}^{\infty} 8j(j-1)\, d_j \frac{1}{2\pi} k^{2j} \int_{\partial G_1} (\log |k(z - \zeta)|)\, |\zeta - z|^{2(j-2)} \\ & \times \boldsymbol{n}(\zeta) \cdot (z - \zeta)\, \boldsymbol{n}(z) \cdot (\zeta - z)\, w(\zeta)\, ds_\zeta \end{aligned} \tag{1.15}$$

$$- \sum_{j=1}^{\infty} 4j\, d_j k^{2j} \frac{1}{2\pi} \int_{\partial G_1} (\log |k(z-z)|)\, |\zeta - z|^{2(j-1)}\, \boldsymbol{n}(\zeta)\cdot\boldsymbol{n}(z)\, w(\zeta)\, ds$$

$$- \frac{1}{\pi} \int_{\partial G_1} \frac{\boldsymbol{n}(\zeta)\cdot\boldsymbol{n}(z)}{|\zeta - z|^2} w(\zeta)\, ds_\zeta - \frac{2}{\pi} \int_{\partial G_1} \frac{\boldsymbol{n}(z)\cdot(z-\zeta)\,\boldsymbol{n}(\zeta)\cdot(\zeta - z)}{|\zeta - z|^4}$$

$$\times\, w(\zeta)\, ds_\zeta, \qquad z \in \partial G_1.$$

Let $Z(s)$ be the C^∞ parametric representation of ∂G_1 with respect to the arc length s. Further let $\Gamma: \zeta = Z(s)$ for $0 < s < L_0$ and $\partial G_1: \zeta = Z(s)$ for $0 < s \leq L$. For $\zeta = Z(s)$ and $z = Z(t)$ we use Taylor's formula to convert the operator D in (1.15) to the form

$$(Dw)\,(Z(t)) = -\frac{1}{\pi} \text{finite part} \int_{s=0}^{L} \frac{w(s)}{(s-t)^2}\, ds + \text{p.v.} \int_{s=0}^{L} \frac{D_1(t,s)}{s-t} w(s)\, ds \tag{1.16}$$

$$+\, k^2 \int_{s=0}^{L} \log |k(t-s)|\, D_2(t,s,k)\, w(s)\, ds + \int_{s=0}^{L} D_3(t,s,k)\, w(s)\, ds, \quad t \in [0, L].$$

Here the first integral on the right-hand side is defined by a Hadamard finite-part integral, the second by a Cauchy principal-value integral. The kernel functions D_1, D_2, and D_3 are infinitely differentiable in *all their arguments*. We also denote the corresponding operators by D_1, D_2, and D_3. For the operator D_0 defined by the Hadamard integral, a Fourier transform provides the symbol (see references [1], [37], [38], [44])

$$\sigma(D_0, \xi') = |\xi'|. \tag{1.17}$$

Hence, D_0 is a pseudodifferential operator of order 1 on ∂G_1; see reference [42].

Remark. For $\psi \in \tilde{H}^{1/2}(\mathbb{R}^+)$ having compact support, the operator D_0 admits the following equivalent representations, which can be obtained by integration by parts (see also reference [9]),

$$\begin{aligned} D_0\psi(t) &= -\frac{1}{\pi} \text{ finite part} \int_{s=0}^{\infty} \frac{\psi(s)}{(s-1)^2}\, ds = \frac{1}{\pi} \text{p.v.} \int_{s=0}^{\infty} \frac{d\psi}{ds} \frac{ds}{s-t} \\ &= -\frac{1}{\pi} \frac{d}{dt} \text{p.v.} \int_{s=0}^{\infty} \frac{\psi(s)\, ds}{s-t} = \frac{1}{\pi} \frac{d^2}{dt^2} \int_{s=0}^{\infty} \psi(s) \log|s-t|\, ds. \end{aligned} \tag{1.18}$$

Moreover, D_1, D_2, and D_3 are pseudodifferential operators of orders 0, -1 and $-\infty$, respectively. Therefore, the operator D on ∂G_1 given by (1.16) can be written as

$$D = D_0 + C_0 \tag{1.19}$$

where C_0 is a pseudodifferential operator of order zero on ∂G_1. Since $\sigma(D_0, \xi')$ in (1.17) is positive for $|\xi'| = 1$ and homogeneous of degree 1, it follows that D_0 satisfies a *Gårding inequality* (see [43, Theorem IV 3.2]): *There exists a positive* *constant* γ *and a compact operator* C_1 *such that*

$$\operatorname{Re}\langle (D_0 + C_1)\,\psi, \psi\rangle_{L^2(\partial G_1)} \geqq \gamma\, \|\psi\|^2_{H^{1/2}(\partial G_1)} \tag{1.20}$$

for all $\psi \in H^{1/2}(\partial G_1)$.)

Now, we are in the position to establish the bijectivity of the integral operator D in (1.10) and its coerciveness in the form of a Gårding inequality in the energy space $\tilde{H}^{1/2}(\Gamma)$ given by (1.7).

Theorem 1.4. *For* $k \neq 0$, $\operatorname{Im} k \geqq 0$, *the integral equation* (1.10) *defines a continuous bijective mapping* $D: \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. *Furthermore, a Gårding inequality holds in* $\tilde{H}^{1/2}(\Gamma)$, i.e., *there exists a constant* $\gamma > 0$ *and a compact mapping* $C: \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *such that*

$$\operatorname{Re}\langle (D + C)\,\psi, \psi\rangle_{L^2(\Gamma)} \geqq \gamma\, \|\psi\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{1.21}$$

for every $\psi \in \tilde{H}^{1/2}(\Gamma)$.

Proof. Extend $\tilde{\psi} \in H^{1/2}(\Gamma)$ to ψ^* by zero to ∂G_1. Then (1.19) and (1.20) imply that

$$\operatorname{Re}\langle (D + C_1 - C_0)\,\psi^*, \psi^*\rangle_{L^2(\partial G_1)} \geqq \gamma\, \|\psi^*\|^2_{H^{1/2}(\partial G_1)} = \gamma\, \|\psi\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

for $\psi^* \in H^{1/2}(\partial G_1)$. Since C_1 is compact (see inequality (1.20)) and since C_0 is a pseudodifferential operator of order zero (see equation (1.19)), it follows that $C_1 - C_0$ is a compact mapping from $\tilde{H}^{1/2}(\Gamma)$ into $H_j^{-1/2}(\Gamma)$. This implies (1.21). Since D is a pseudodifferential operator of order 1 (see reference [42]), the mapping $D: \tilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\partial G_1) \to H^{-1/2}(\partial G_1) \to H^{-1/2}(\Gamma)$ is continuous. This implies with (1.21) that D is a Fredholm operator with in dex zero. Therefore the classical Fredholm alternative is valid for D and the injectivity, *i.e.*, Lemma 1.1, Lemma 1.2 and Theorem 1.3, yield the surjectivity which completes the proof. □

Remarks. (i) The unique solvability of the hypersingular equation (1.10) in $\tilde{H}^{1/2}(\Gamma)$ is in full agreement with DURAND's result in reference [18]. (ii) Note that the Gårding inequality (1.21) provides the convergence of conforming boundary element methods as in referencs [35], [36], [21], [22], [23].

Our main concern in this paper is the local regularity and singular behavior of u near the screen tips Z_1, Z_2 of Γ. It is well known from G. Fichera's fundamental work (see the article [45] for extensive references) that at the screen tips Z_i, $i = 1, 2$ (see Fig. 1) the variational solution u of (1.1), (1.2) has in general unbounded gradients even for given C^∞-data. Since any improvement of constructive methods — like the approximation by finite elements and boundary elements — requires higher regularity of the exact solution u, one seeks a decomposition of the solution into special singular terms concentrated near Z_i, and a regular remainder. In the following we derive such a decomposition for the solution of the integral equation (1.10). A detailed analysis of equation (1.10) based on the Mellin transform as developed in references [10]–[17] gives us the appropriate function spaces together with the exact form of the singularities at the screen tips.

The following analysis rests on the interpretation of equation (1.10) as an equation on the half line $\mathbb{R}_+ = [0, \infty)$ instead of Γ. Then we use the results on $\mathbb{R}_+$ that there have been obtained by means of Mellin transforms and Wiener-Hopf techniques as in references [10], [14], [15]. Let us begin with the analysis of the principal part D_0.

Lemma 1.5. *Let* $G \in \mathrm{H}^\eta(\mathbb{R}_+)$ *with* $\eta \geqq -1/2$ *and let* $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ *be a fixed cut-off functions with* $\chi \equiv 1$ *in a neighborhood of* 0. *Then the equation*

$$D_0\psi(t) := -\frac{1}{\pi}\ \text{finite part} \int_0^\infty \psi(s)\frac{ds}{(t-s)^2} = G(t) \tag{1.22}$$

for $t \in (0, \infty)$ *has exactly one solution* $\psi \in \tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)$. *This solution has the form*

$$\psi = s^{1/2}\left\{\sum_{j=0}^{[\eta-\frac{1}{2}]} \mathscr{L}_j(G)\, s^j\chi(s)\right\} + \psi_0(s) \tag{1.23}$$

where $\psi_0 \in \tilde{\mathrm{H}}^{\eta+1}_{\mathrm{loc}}(\mathbb{R}_+) = \mathring{\mathrm{W}}_0^{\eta+1}(\mathbb{R}_+)$ *and where* l_j *are bounded linear functionals on* $\mathrm{H}^\eta(\mathbb{R}_+)$. *The mapping* $P_0\colon G \mapsto \psi_0$ *defines a bounded linear mapping*

$$P_0\colon \mathrm{H}^\eta(\mathbb{R}_+) \to \tilde{\mathrm{H}}^{\eta+1}_{\mathrm{loc}}(\mathbb{R}_+).$$

Proof. In order to apply the Mellin transform to equation (1.22) we use (1.18) to obtain

$$D_0\psi(t) = \frac{1}{\pi}\frac{d^2}{dt^2}\int_{s=0}^\infty \psi(s)\log\left|1-\frac{t}{s}\right|ds + \frac{1}{\pi}\frac{d^2}{dt^2}\int_0^\infty \psi(s)\log|s|\,ds = G(t) \tag{1.24}$$

for ψ with compact support. For $\psi \in \tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)$, the extension ψ^* by zero to $\mathbb{R}_-$ belongs to $\mathrm{H}^{1/2}(\mathbb{R})$ and

$$\left(t \mapsto \int_{\mathbb{R}} \psi^*(s)\log|s-t|\,ds\right) \in \mathrm{H}^{3/2}(\mathbb{R}) \subset \mathrm{C}^0.$$

Therefore, the second integral in (1.24) exists and its derivatives vanish. Integrating (1.24) twice we obtain

$$V_0\psi(t) = -\frac{1}{\pi}\int_0^\infty \psi(s)\log\left|1-\frac{t}{s}\right|ds = H(t) := -\int_{s=0}^{t}\int_{\tau=0}^{s} G(\tau)\,d\tau\,ds + c_0 + c_1 t \tag{1.25}$$

where c_0, c_1 are constants of integration. These are chosen so that the right-hand side of (1.25) has compact support when G has compact support, *i.e.*,

$$c_1 = \int_{\mathbb{R}_+} G\,d\tau, \qquad c_0 = \int_{\mathrm{supp}G}\left\{\int_{\tau=0}^{s} G(\tau)\,d\tau - c_1\right\}ds. \tag{1.26}$$

For further information on V_0 in (1.25) and its properties see also references [12], [16, (2.8)ff.], and [26], [27], [33]. The Mellin transform is defined by

$$\hat{v}(\lambda) = \int_0^\infty x^{i\lambda-1} v(x)\, dx \tag{1.27}$$

and its inversion is given by

$$v(x) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda=\text{const}} x^{-i\lambda} \hat{v}(\lambda)\, d\lambda. \tag{1.28}$$

For $\psi \in \tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)$ and $H \in \mathrm{H}^{3/2}(\mathbb{R}_+)$, equation (1.26) is valid. Its Mellin transform for $\operatorname{Im}\lambda < 1$ is

$$\widehat{V_0\psi}(\lambda) = \frac{\cosh \pi\lambda}{\lambda \sinh \pi\lambda} \hat{\psi}(\lambda - i) = \hat{H}(\lambda), \qquad \operatorname{Im}\lambda < 1.$$

Since $G \in \mathrm{H}^\eta$, the right-hand side of (1.25) belongs to $\mathrm{H}^{\eta+2}(\mathbb{R}_+)$. For the proposed form of the solution (1.23) we want to recover ψ by using the inverse Mellin transform applied to the *analytic continuation* of $\hat{\psi}$ given by

$$\hat{\psi}(\lambda - i) = \hat{H}(\lambda) \frac{\lambda \sinh \pi\lambda}{\cosh \pi\lambda} \tag{1.29}$$

for $\operatorname{Im}\lambda = \eta + \frac{3}{2} > 1$. Hence, we need to perturb the path of integration from $\frac{1}{2} < \operatorname{Im}\lambda = \text{const} < 1$ to $\operatorname{Im}\lambda = \eta + 0$. In doing this we need to pick up the residues of $\hat{\psi}$ given by the right-hand side of (1.29). Since $H \in \mathrm{H}^{\eta+2}(\mathbb{R}_+)$ has compact support, the Mellin image $\hat{\mathrm{H}}(\lambda)$ is a meromorphic function having simple poles at $\lambda = ik$, $k \in \mathbb{N}_0$.

But there the factor $\sinh \pi\lambda$ in (1.29) also has its zeros. Hence, $\hat{\psi}(\mu)$ has simple poles only at $\mu = i(k - \frac{1}{2})$, $k \in \mathbb{N}$, in the strip $-\frac{1}{2} < \operatorname{Im}\mu < \eta + \frac{1}{2}$. Moreover, $\hat{\psi}$ decays appropriately at infinity due to the behavior of $\hat{H}(\lambda)$. (This follows from the arguments given in reference [16, Proofs of Lemma 4.2 and of Theorem 4.3].) Therefore, application of the residue theorem to (1.29) and the inverse Mellin transform (1.28) yields the explicit solution formula

$$\begin{aligned}\psi(s) = &-s^{1/2} \left[\sum_{j=0}^{[\eta-\frac{1}{2}]} s^j \hat{H}(i(j+\tfrac{1}{2}))\, (j+\tfrac{1}{2})\, (-1)^j\right] \chi(s) \\ &- s^{1/2} \left[\sum_{j=0}^{[\eta-\frac{1}{2}]} s^j \hat{H}(i(j+\tfrac{1}{2}))\, (j+\tfrac{1}{2})\, (-1)^j\right] (1-\chi(s)) \\ &+ \int_{\operatorname{Im}\mu=\eta+\frac{1}{2}} s^{-i\mu} \hat{H}(\mu+i) \cdot \frac{(\mu+i)\sinh \pi\mu}{\cosh \pi\mu}\, d\mu.\end{aligned} \tag{1.30}$$

The first sum in (1.30) explicitly defines the singularity terms in (1.23). The last two expressions in (1.30) together define ψ_0 and the operator P_0 explicitly and provide the desired mapping properties. □

In order to extend Theorem 1.4 by describing the mapping properties of equation (1.10) in Sobolev spaces $\tilde{H}^{1/2+\sigma}(\Gamma)$ and $H^{-1/2+\sigma}(\Gamma)$, we augment the usual Sobolev spaces by the singularity functions that appear in Lemma 1.5.

Definition. *Let* ϱ_i *denote the Euclidean distance between* $z \in \Gamma$ *and the endpoint* Z_i *of* Γ. χ_i *is a* C^∞*-cut-off function on* Γ *with* $0 \leqq \chi_i \leqq 1$ *and* $\chi_i = 1$ *near to* Z_i, $\chi_i = 0$ *at the opposite end,* $i = 1, 2$. *Let* $\psi = \sum_{i=1}^{2} \alpha_i \varrho_i^{1/2} \chi_i + \psi_0$ *with* $\alpha_i \in \mathbb{R}$ *and* $\psi_0 \in \tilde{H}^\tau(\Gamma)$, $\tau < 2$, *let*

$$\|\psi\|_{\mathscr{X}^\tau(\Gamma)} := \begin{cases} \sum_{i=1}^{2} |\alpha_i| + \|\psi_0\|_{\tilde{H}^\tau(\Gamma)} & \text{for } 1 \leqq \tau < 2, \\ \|\psi\|_{\tilde{H}^\tau(\Gamma)} & \text{for } \tau < 1, \end{cases} \tag{1.31}$$

and let $\mathscr{X}^\tau$ *denote the space* $\mathbb{R}^2 \times \tilde{H}^\tau(\Gamma)$ *or* $\tilde{H}^\tau(\Gamma)$. *For functions* ψ *described as above with*

$$\psi_0 = \sum_{i=1}^{2} \beta_i \varrho_i^{3/2} \chi_i + \psi_1, \quad \beta_i \in \mathbb{R} \quad \text{and} \quad \psi_1 \in \tilde{H}^\tau(\Gamma), \quad \tau < 3,$$

let

$$\|\psi\|_{\mathscr{X}^\tau(\Gamma)} = \begin{cases} \sum_{i=1}^{2} |\alpha_i| + |\beta_i| + \|\psi_1\|_{\tilde{H}^\tau(\Gamma)} & \text{for } 2 \leqq \tau < 3, \\ \|\psi\|_{\mathscr{X}^\tau} & \text{for } \tau < 2, \end{cases} \tag{1.32}$$

where $\mathscr{X}^\tau$ *denotes the space* $\mathbb{R}^4 \times \tilde{H}^\tau(\Gamma)$ *for* $2 \leqq \tau < 3$.

Lemma 1.6. *For* $|\sigma| < \frac{1}{2}$ *let*
(i) $g_j \in H^{1/2+\sigma}(\Gamma)$ *or*
(ii) $g_j \in H^{3/2+\sigma}(\Gamma)$, $j = 1, 2$,
be given. The solution $[u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ *of* (1.10) *then has the form*

(i) $[u]|_\Gamma = \sum_{i=1}^{2} \alpha_i \varrho_i^{1/2} \chi_i + v_0$ *with* $v_0 \in \tilde{H}^{3/2+\sigma}(\Gamma)$, $\alpha_i \in \mathbb{R}$

or

(ii) $[u]|_\Gamma = \sum_{i=1}^{2} (\alpha_i \varrho_i^{1/2} + \beta_i \varrho_i^{3/2}) \chi_i + v_1$ *with* $v_1 \in \tilde{H}^{5/2+\sigma}(\Gamma)$, $\alpha_i, \beta_i \in \mathbb{R}$,

respectively.

Proof. For the proof we rewrite equation (1.10) with (1.16), (1.19) as

$$D[u] = D_0[u] + C_0[u] = G,$$

where G denotes the right-hand side in (1.10) and $C_0 = D_1 + D_2 + D_3$ as in equation (1.16). Let χ_1 be a C^∞ cut-off function that is identically 1 near the tip

Z_1 and identically zero near Z_2 and let χ_1^* be a similar cut-off function having support in the region where χ_1 is 1. Then $\chi_1^*\chi_1 = \chi_1^*$ and we easily obtain the formula

$$D_0(\chi_1[u]) = (1 - \chi_1^*)\, D_0(\chi_1[u]) + \chi_1^* D_0(\chi_1 - 1)\,[u] - \chi_1^* C_0[u] + \chi_1^* G. \tag{1.33}$$

Case (i): For $g_j \in \mathrm{H}^{1/2+\sigma}(\Gamma)$ we also have $G \in \mathrm{H}^{1/2+\sigma}$. Furthermore, with $[u] \in \tilde{\mathrm{H}}^{1/2}(\Gamma)$ we find that $C_0[u] \in \mathrm{H}^{1/2}(\partial G_1)$. Since $\chi_1^*(\chi_1 - 1) \equiv 0$, the operator $\chi_1^* D_0(\chi_1 - 1)$ is C^∞-smoothing; hence $\chi_1^* D_0(\chi_1 - 1)\,[u] \in C^\infty$. Finally, note that $[u]$ satisfies (1.10) not only on Γ but by extension also on ∂G_1 where D is a classical pseudodifferential operator of order 1. As such, D has pseudolocal properties and for $g \in \mathrm{H}^{1/2+\sigma}(\Gamma)$ we get $[u] \in \mathrm{H}^{3/2+\sigma}(\Gamma_0)$ for every $\Gamma_0 = \overline{\Gamma}_0 \subset\subset \Gamma$ which is a compact subarc of Γ. For the further analysis we want to apply Lemma 1.5. Therefore we now consider the functions depending on the real variable s and extend them to the half axis corresponding to the respective crack tip. Since $(1 - \chi_1^*)$ $\chi_1 \neq 0$ in the interior of a compact subarc Γ_0, the term $(1 - \chi_1^*)\, D_0\chi_1[u]$ is in $\mathrm{H}^{1/2+\sigma}(\mathbb{R}_+)$. Moreover, this term decays for $t \to \infty$ quadratically and, therefore, lies in the range of D_0, as is the case for the other four terms in (1.33). Therefore, Lemma 1.5 can be applied to (1.33) implying proposition (i) in Lemma 1.6, first with $v_0 \in \tilde{\mathrm{H}}^{3/2}(\Gamma)$ since $C_0[u] \in \mathrm{H}^{1/2}$. Now, $\sqrt{s}\,\chi_1 \in \mathrm{H}^{1/2+\sigma}$ and, hence, we obtain $C_0[u] \in \mathrm{H}^{1/2+\sigma}$. Therefore, the whole right-hand side in (1.33) is in $\mathrm{H}^{1/2+\sigma}$ and another application of Lemma 1.5 to (1.33) yields the proposed form (i) of $\chi|[u]$. Using the same arguments at the other tip Z_2, we complete the verification of (i).

Case (ii): If $g_j \in \mathrm{H}^{3/2+\sigma}(\Gamma)$, we obtain with the above arguments the form (i) for $[u]$, and in addition, that all the terms on the right-hand side of (1.33) except $C_0[u]$ are in $\mathrm{H}^{3/2+\sigma}$. Since $v_0 \in \tilde{\mathrm{H}}^{3/2+\sigma}$ from (i), we have $C_0 v_0 \in \mathrm{H}^{3/2+\sigma}$. For the square-root term in (i) we see from the representation (1.16) that $D_2\sqrt{s}\,\chi_1$, $D_3\sqrt{s}\,\chi_2$ are in $\mathrm{H}^{3/2+\sigma}$. For the remaining term we write

$$\int\limits_{s=0}^{L} \frac{D_1(t,s)}{s-t}\sqrt{s}\;\chi_1(s)\,ds = D_1(t,t)\int\limits_{s=0}^{L}\frac{\sqrt{s}\,\chi_1}{s-t}\,ds + \int\limits_{s=0}^{L}\frac{\chi_1\sqrt{s}\,D(_1(s,t) - D_1(t,t))}{s-t}\,ds.$$

where the last term is in C^∞ because its kernel is. For the first term we apply the Mellin transform obtaining

$$\widehat{\int\limits_{s=0}^{\infty}\frac{\sqrt{s}\,\chi_1(s)}{s-t}\,ds}(\lambda) = +i\,\frac{\cosh \pi\lambda}{\sinh \pi\lambda}\,\frac{\hat{\chi}_1(\lambda)}{(\lambda - i/2)} \quad \text{for } \operatorname{Im}\lambda < \tfrac{1}{2}.$$

Here, the right-hand side has simple poles only at $\lambda = \mathrm{i}k$, $k \in \mathbb{Z}$ and decays exponentially for $|\operatorname{Re}\lambda| \to \infty$ and for $\operatorname{Im}\lambda = \text{const.}$, because $\hat{\chi}_1(\lambda)$ is the Mellin transform of the C_0^∞-function χ_1. Therefore,

$$\int\limits_{s=0}^{L}\frac{\sqrt{s}\,\chi_1}{s-t}\,ds \in C^\infty. \tag{1.34}$$

Finally, for $g_j \in H^{3/2+\sigma}(\Gamma)$ we have from (1.10) that $G \in H^{3/2+\sigma}(\Gamma)$. So, the complete right hand side of (1.33) is in $H^{3/2+\sigma}$ and in the range of D_0 on $\mathbb{R}_+$. Hence, Lemma 1.5 and (1.23) provide for $[u]$ the proposed form (ii). $\square$

As a consequence of Lemma 1.5 and the proof of Lemma 1.6 we obtain

Corollary 1.7. *For* $-\frac{1}{2} \leqq \eta < 0$, *and* $g_j \in H^{\eta}(\Gamma)$, $j = 1, 2$, *the solution* $[u]|_\Gamma \in \tilde{H}^{1/2}(\Gamma)$ *of* (1.10) *belongs to* $\tilde{H}^{\eta+1}(\Gamma)$.

Remark. The arguments in the proof of Lemma 1.5 and the use of the Mellin transform with $\operatorname{Im} \lambda = \eta + 3/2$, but with $-1 < \eta \leqq -\frac{1}{2}$, imply that Corollary 1.7 also holds for $-1 < \eta \leqq -\frac{1}{2}$. Moreover, $D_\Gamma \colon \tilde{H}^{\eta+1}(\Gamma) \to H^{\eta}(\Gamma)$ is bijective for $-1 < \eta < 0$.

Now we are in the position to extend the bijectivity result in Theorem 1.4 to the augmented higher-order spaces defined in (1.31), (1.32).

Theorem 1.8. *For each* $\sigma \in (-\frac{1}{2}, \frac{1}{2})$ *the operators in* (1.10),

(i) $D_\Gamma \colon \mathscr{Z}^{3/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma)$,

(ii) $D_\Gamma \colon \mathscr{Z}^{5/2+\sigma}(\Gamma) \to H^{3/2+\sigma}(\Gamma)$, *respectively, with*

$$\{\alpha_1, \alpha_2, \psi_0\} \mapsto D_\Gamma \left(\sum_{i=1}^{2} \alpha_i \varrho_i^{1/2} \chi_i + \psi_0 \right) = 2g,$$

$$\{\alpha_1, \alpha_2, \beta_1, \beta_2, \psi\} \mapsto D_\Gamma \left(\sum_{i=1}^{2} (\alpha_i \varrho_i^{1/2} + \beta_i \varrho_i^{3/2}) \chi_i + \psi_1 \right) = 2g$$

are bijective and continuous and provide the a-priori estimates

$$\begin{aligned} \|\psi\|_{\mathscr{Z}^{3/2+\sigma}(\Gamma)} &\leqq c\, \|g\|_{H^{1/2+\sigma}(\Gamma)}, \\ \|\psi\|_{\mathscr{Z}^{5/2+\sigma}(\Gamma)} &\leqq c\, \|g\|_{H^{3/2+\sigma}(\Gamma)}. \end{aligned} \tag{1.35}$$

Remark. Theorem 1.8 shows clearly that, even for $g \in C^\infty$, the solution $[u]_\Gamma$ generally possesses an unbounded gradient at the tips, *i.e.*, $\alpha_i \neq 0$.

Proof. *Continuity*: *Case* (i). For

$$\psi = \sum_{i=1}^{2} \alpha_i \varrho_i^{1/2} \chi_i + \psi_0 \in \mathscr{Z}^{3/2+\sigma}$$

we find that $\psi_0 \in \tilde{H}^{3/2+\sigma}(\Gamma)$ and that ψ_0 can be extended by zero to a function ψ_0^* in $H^{3/2+\sigma}(\partial G_1)$. Now $D_\Gamma \psi_0 = D_{\partial G_1} \psi_0^*$ belongs to $H^{1/2+\sigma}(\partial G_1)$, which implies the continuity of $D_\Gamma \psi_0|_\Gamma$. The continuity for the two-dimensional additional terms follows from (1.16) and (1.18):

$$D_\Gamma(\varrho_1^{1/2}\chi_1)(t) = -\frac{1}{\pi}\frac{d}{dt}\,\text{p.v.}\int\limits_{s=0}^{\infty}\frac{\sqrt{s}\,\chi_1(s)}{s-t}\,ds + D_1(t,t)\int\limits_{s=0}^{\infty}\frac{\sqrt{s}\,\chi_1(s)}{s-t}\,ds$$

$$+\int\limits_{s=0}^{\infty}\sqrt{s}\,\chi_1(s)\,\frac{D_1(t,s)-D_1(t,t)}{s-t}\,ds$$

$$+k^2\int\limits_{s=0}^{\infty}\sqrt{s}\,\chi_1(s)\log|k(s-t)|\,D_2(t,s,k)\,ds$$

$$+\int\limits_{s=0}^{\infty}\sqrt{s}\,\chi_1(s)\,D_3(t,s,k)\,ds.$$

From (1.34), we know that the first two terms on the right-hand side are of class C^∞, whereas the remaining kernels are sufficiently regular to provide the property $D_\Gamma(\varrho_1^{1/2}\chi_1)\in H^{3/2+\sigma'}(\Gamma)$ for every $\sigma'<\frac{1}{2}$. Of course, all the above mappings are linear and continuous.

Case (ii). For $\psi = \sum\limits_{i=1}^{2}(\alpha_i\varrho_i^{1/2}+\beta_i\varrho_i^{3/2})\,\chi_i+\psi_1\in\mathscr{L}^{5/2+\sigma}$, we proceed in the same manner as in Case (i), now with images in $H^{3/2+\sigma}$. The property $D_\Gamma(\varrho_1^{1/2}\chi_1)\in H^{3/2+\sigma'}$ has already been shown whereas $D_\Gamma(\varrho_1^{1/2}\chi_1)\in H^{3/2+\sigma'}$ with any $\sigma'<\frac{1}{2}$ follows in exactly the same manner by replacing χ_1 by $(\varrho_1\chi_1)$.

Injectivity: *Cases* (i) *and* (ii). Consider the homogeneous equation $D_\Gamma\left(\sum\limits_{i=1}^{2}\alpha_i\varrho_i^{1/2}\chi_i+v_0\right)=0$ where $v_0=\sum\limits_{i=1}^{2}\beta_i\varrho_i^{3/2}\chi_i+\psi_1$ in Case (ii). Then by Theorem 1.4 we obtain

$$\sum_{i=1}^{2}\alpha_i\varrho_i^{1/2}\chi_i+v_0=0 \quad \text{in } \tilde{H}^{1/2}(\Gamma) \tag{1.36}$$

since D_Γ is bijective from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$. But then we have that (1.36) also holds in $\tilde{H}^{3/2+\sigma}(\Gamma)$, which can only hold if $\alpha_i=0$ (since $\varrho_i^{1/2}\chi_i\notin\tilde{H}^{3/2+\sigma}$) and $v_0=0$, too. In case (ii) we see that $v_0=0$ in $\tilde{H}^{3/2+\sigma}$ implies $v_0=0$ in $\tilde{H}^{5/2+\sigma}$, which yields $\beta_i=0$ and $\psi_1=0$.

Surjectivity: *Cases* (i) *and* (ii). In both cases (i) and (ii) with g_j given, $j=1,2$ first determine $[u]\in\tilde{H}^{1/2}(\Gamma)$ by solving equation (1.10) using Theorem 1.4. Then with Lemma 1.6 we find the desired form of $[u]$ containing the constants α_i, β_i, $i=1,2$ in both cases (i) and (ii).

Since $\mathscr{L}^{3/2+\sigma}$, $H^{1/2+\sigma}(\Gamma)$ and $\mathscr{L}^{5/2+\sigma}$, $H^{3/2+\sigma}(\Gamma)$ are all Banach spaces and since D_Γ defines continuous bijective linear mappings, Banach's closed graph theorem implies continuity of the inverses, *i.e.*, the desired a-priori estimates (1.35). □

2. The Stationary Neumann Crack Problem in Elasticity

With the same notation as in Section 1 (see Fig. 1) let us state the *Neumann crack problem* for a homogeneous, isotropic, elastic material with given tractions at the crack line Γ. We write $\mathbb{H}^{-1/2}(\Gamma)$ for $(H^{-1/2}(\Gamma))^2$ and $\mathbb{H}^1_\Gamma$ for $(H^1_\Gamma)^2$, *etc. For given* $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathbb{H}^{-1/2}(\Gamma)$ *satisfying* $\boldsymbol{\psi} := \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 \in \tilde{\mathbb{H}}^{-1/2}(\Gamma) := (\tilde{H}^{-1/2}(\Gamma))^2$ *and* $\int_\Gamma \boldsymbol{\psi}\, ds = \mathbf{0}$ *find the displacement field* $\boldsymbol{u} \in \mathbb{H}^1_\Gamma$ *satisfying*

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} = \mathbf{0} \quad \text{in } \Omega_\Gamma = \mathbb{R}^2 \setminus \Gamma, \tag{2.1}$$

$$T(\boldsymbol{u})|_{\Gamma_1} = \boldsymbol{\psi}_1, \quad T(\boldsymbol{u})|_{\Gamma_2} = \boldsymbol{\psi}_2, \tag{2.2}$$

where T denotes the traction operator

$$T(\boldsymbol{u}) := \lambda (\operatorname{div} \boldsymbol{u})\, \boldsymbol{n} + 2\mu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + \mu \boldsymbol{n} \wedge \operatorname{curl} \boldsymbol{u} \qquad \text{on } \Gamma_j, \quad j = 1,2,$$

and where $\mu > 0$, $\lambda > -\mu$ are the given Lamé constants. In addition we need a condition of decay at infinity, for which we follow the choice made in references [27] and [28]: We require that there is a rigid motion r_u such that $\boldsymbol{u} - \boldsymbol{r}_u$ is regular at ∞, *i.e.*, such that

$$\frac{\partial^\alpha}{\partial x_i^\alpha} (\boldsymbol{u} - \boldsymbol{r}_u) = O(|\boldsymbol{x}|^{-1-\alpha}) \tag{2.3}$$

with $\alpha = 0, 1$ and $i = 1, 2$ as $|\boldsymbol{x}| \to \infty$. The motion $\boldsymbol{r}_u$ can be written in the form

$$\boldsymbol{r}_u(\boldsymbol{x}) = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 (x_1 \boldsymbol{e}_1 - x_1 \boldsymbol{e}_2), \tag{2.4}$$

where $\boldsymbol{e}_j$ are the unit vectors of $\mathbb{R}^2$ in the x_j directions and where ω_j are suitable constants. As noted in references [25] and [28], the *equilibrium conditions*

$$\int_\Gamma (\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2)\, ds = \int_\Gamma \boldsymbol{\psi}\, ds = 0 \tag{2.5}$$

are necessary for the existence of a displacement field $\boldsymbol{u}$ satisfying (2.1), (2.2); the conditions of decay and additional conditions can be imposed that yield a unique solution. Here we impose a standard condition for uniqueness by *giving the values* of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$, *i.e.*, we specify the rigid motion belonging to $\boldsymbol{u}$.

We first give a variational formulation of (2.1)–(2.3) providing a solution $\boldsymbol{u}$ of finite energy, *i.e.*, $\boldsymbol{u} \in \mathbb{H}^1_\Gamma$ where $\mathbb{H}^1_\Gamma$ is the completion of all C^∞-functions $f(x)$ of the form

$$f(x) = f_0(x) + r_f(x)$$

with respect to the norm

$$\|f\|_{1,\Gamma} := \left\{ \int_{\Omega_\Gamma} \mathscr{E}(f, f)\, dx + \int_\Gamma |f|^2\, ds \right\}^{1/2}$$

where f_0 is regular at infinity and r_f is some rigid motion associated with f (see (2.4)). Note that f can be *discontinuous* across Γ. Here, as in reference [30], we put

$$\mathscr{E}(f,\phi) := (\lambda+\mu)\operatorname{div} f \operatorname{div}\phi + \frac{\mu}{2}\sum_{j\neq k}^{2}\left(\frac{\partial f_j}{\partial x_k}+\frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial \phi_j}{\partial x_k}+\frac{\partial \phi_k}{\partial x_j}\right)$$
$$+\frac{\mu}{2}\sum_{j,k=1}^{2}\left(\frac{\partial f_j}{\partial x_k}-\frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial \phi_j}{\partial x_k}-\frac{\partial \phi_k}{\partial x_j}\right) \tag{2.6}$$

and the variational formulation of (2.1)–(2.3) then reads:
Find $u \in \mathbb{H}^1_\Gamma$ *such that for all* $v \in \mathbb{H}^1_\Gamma$

$$\int_{\Omega_\Gamma} \mathscr{E}(u,v)\,dx = \int_\Gamma \{\psi_1\cdot(v_1 - r) - \psi_2\cdot(v_2 - r)\}\,ds \tag{2.7}$$

where r_v *denotes the rigid motion associated with* v.

For the above formulation we may consider the following two different motivations.

1. For given $\psi_j \in \mathbb{H}^{-1/2}(\Gamma)$, $j = 1, 2$, find two fields u_1 in G_1 and u_2 in G_2 with

$$u_j \in \mathbb{H}^1_{\mathrm{loc}}(G_j) \quad \text{and} \quad T(u_j)_{\Gamma_j} = \psi_j.$$

The existence of these fields u_j is due to the extended trace theorem (see reference [32, p. 188]). Since $\psi := \psi_1 - \psi_2 \in \tilde{\mathbb{H}}^{-1/2}(\Gamma)$, the fields can even be chosen with

$$T(u_2) = T(u_1) \quad \text{on } \partial G_1 \setminus \Gamma.$$

Then the Green formula for $w_j := u - u_j$ in G_j gives

$$\int_{G_j} \mathscr{E}(w_j, v)\,dx = -(-1)_j\left\{\int_\Gamma \psi_j\cdot v\,ds + \int_{\partial G_1\setminus\Gamma} T(u)\cdot v\,ds\right\} - \int_{G_j}\mathscr{E}(u_j,v)\,dx$$
$$+(-1)^j\left\{\int_\Gamma r\cdot\psi_2\,ds + \int_{\partial G_1\setminus\Gamma} r\cdot T(u)\,ds\right\}$$

for $j = 1, 2$, where r belongs to v (see Theorem 2.2 in reference [28]). Now we add these two equations, use the well known integrability condition

$$\int_{\partial G_1} r\cdot T(u)\,ds = 0$$

for rigid motions r and interior domains and use

$$[u]|_{\partial G_1\setminus\Gamma} = 0 \quad \text{and} \quad [T(u)]|_{\partial G_1\setminus\Gamma} = 0,$$

which follows from the membership of u in $\mathbb{H}^1_\Gamma$, to obtain (2.7).

2. Approximate Ω_Γ by a suitable sequence of smoothly bounded exterior domains $\Omega_{\Gamma_\varepsilon}$ as indicated in Figure 2. Then consider the sequence of elastic fields u_ε with $T(u_\varepsilon)_{\Gamma_\varepsilon} = \psi_\varepsilon$ where ψ_ε is given by ψ_1 and ψ_2 on the upper and lower parts of Γ_ε, respectively. An extension of DURAND's arguments [18] to the elastic fields

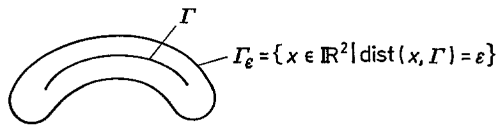

Fig. 2

shows that the variational solution $\boldsymbol{u}$ of (2.7) is the limit of the finite-energy solutions in $\Omega_{\Gamma_\varepsilon}$ defined in reference [28].

It should be noted that our approach concerns only the case of a homogeneous body. If the crack appears along a curve separating different materials, then the present equations no longer apply (see reference [8]). Moreover, the linear functional on the right-hand side in (2.7) is continuous on $\mathbb{H}^1_\Gamma$, because $(\boldsymbol{v} - \boldsymbol{r})|_\Gamma$ is in $\mathbb{H}^{1/2}(\Gamma) := (\mathrm{H}^{1/2}(\Gamma))^2$ and $\boldsymbol{\psi}$ is in $(\mathbb{H}^{1/2}(\Gamma))' = \tilde{\mathbb{H}}^{-1/2}(\Gamma)$. In addition, Korn's inequality is valid for the interior domain G_1 as well as for the exterior domain G_2, thereby guaranteeing coerciveness of the bilinear form of the left-hand side of (2.7) in the form of a Gårding inequality:

$$\int_{\Omega_\Gamma} \mathscr{E}(\boldsymbol{u}, \boldsymbol{u})\, dx \geqq \|\boldsymbol{u}\|^2_{1,\Gamma} - \int_\Gamma |\boldsymbol{u}|^2\, ds, \qquad \boldsymbol{u} \in \mathbb{H}^1_\Gamma. \tag{2.8}$$

Since we also have uniqueness for the solution of (2.7) for $\boldsymbol{r}_u$ determined from $\boldsymbol{w}$, the well known Lax-Milgram theorem (see reference [24]) implies:

Theorem 2.1. *For given $\boldsymbol{\psi}_j \in \mathbb{H}^{-1/2}(\Gamma)$, $j = 1, 2$ with $\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 \in \tilde{\mathbb{H}}^{-1/2}(\Gamma)$ and given $\boldsymbol{r}_u$ there exists exactly one variational solution $\boldsymbol{u} \in \mathbb{H}^1_\Gamma$ to the crack problem* (2.1), (2.2) *or to* (2.7).

In order to derive the Betti representation for the above variational solution we proceed in exactly the same manner as in the first section. Now the fundamental solution of (2.1) is given by the matrix

$$\boldsymbol{M}(x, y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \log|x - y|\, \boldsymbol{I} - \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^T}{|x - y|^2} \right\}. \tag{2.9}$$

Here T denotes the transposed tensor and $\boldsymbol{I}$ is the identity matrix. Correspondingly, the boundary-stress matrix is given by

$$\boldsymbol{M}_1(x, y) = (T_y \boldsymbol{M}(x, y))^T = \frac{\mu}{2\pi(\lambda + 2\mu)} \tag{2.10}$$
$$\times \left\{ \boldsymbol{I} + \frac{2(\lambda + \mu)}{\mu|x - y|^2} (x - y)(x - y)^T \frac{\partial}{\partial n_y} + \frac{\mu}{\lambda + 2\mu} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial s_y} \right\} \log|x - y|.$$

Applying the Betti formula to G_1 and to G_2 and summing the result we obtain

$$\boldsymbol{u}(x) = \int_\Gamma \boldsymbol{M}(x, y)\, \boldsymbol{\psi}(y)\, ds_y - \int_\Gamma \boldsymbol{M}_1(x, y)\, [\boldsymbol{u}]\, ds_y + (\omega_1 + \omega_3 x_2, \omega_2 - \omega_3 x_1)^T$$
$$\text{for } x \in \Omega_\Gamma. \tag{2.11}$$

From (2.11) and the jump relations of boundary potentials we find the Somigliana identity

$$\begin{aligned}\tfrac{1}{2}(\boldsymbol{u}_1(x)+\boldsymbol{u}_2(x)) = \int_{\Gamma\setminus\{x\}} \{\boldsymbol{M}(x,y)\,\boldsymbol{\psi}(y) - \boldsymbol{M}_1(x,y)\,[\boldsymbol{u}]\}\,ds_y \\ +(\omega_1+\omega_3 x_2, \omega_2-\omega_3 x_1)^T \quad \text{for } x\in\Gamma.\end{aligned} \tag{2.12}$$

on the crack. Clearly (2.12) cannot be used for finding $[\boldsymbol{u}]|_\Gamma$, the unknown quantity for (2.11). We therefore apply the traction operator on Γ_j, $j=1,2$, to the Betti formula (2.11) and use the decay condition (2.3) to obtain

$$\tfrac{1}{2}(\boldsymbol{\psi}_1(x)+\boldsymbol{\psi}_2(x)) = -T_x \int_{\Gamma\setminus\{x\}} \boldsymbol{M}_1(x,y)\,[\boldsymbol{u}](y)\,ds_y + \int_{\Gamma\setminus\{x\}} (T_x\boldsymbol{M}(x,y))\,\boldsymbol{\psi}(y)\,ds_y \quad \text{for } x\in\Gamma. \tag{2.13}$$

In this equation, the unknown function is $[\boldsymbol{u}]$ and therefore (2.13) is suitable for its determination, *i.e.*,

$$\begin{aligned}\mathcal{D}_\Gamma[\boldsymbol{u}](x) &:= -T_x \int_{\Gamma\setminus\{x\}} (T_y\boldsymbol{M}(x,y))^T\,[\boldsymbol{u}](y)\,ds_y \\ &= \tfrac{1}{2}(\boldsymbol{\psi}_1(x)+\boldsymbol{\psi}_2(x)) - \int_{\Gamma\setminus\{x\}} (T_x\boldsymbol{M}(x,y))\,\boldsymbol{\psi}(y)\,ds_y.\end{aligned} \tag{2.14}$$

Theorem 2.2. *Let* $\boldsymbol{\psi}_j \in \mathbb{H}^{-1/2}(\Gamma)$ *with* $\boldsymbol{\psi}_1-\boldsymbol{\psi}_2\in\tilde{\mathbb{H}}^{-1/2}(\Gamma)$ *and* $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)^T$ *be given. Then* $\boldsymbol{u}\in\mathbb{H}^1_\Gamma$ *is the unique solution of the Neumann crack problem* (2.1)–(2.4) *if and only if the jump of the displacement* $[\boldsymbol{u}]\in\tilde{\mathbb{H}}^{1/2}(\Gamma) := (\tilde{\mathrm{H}}^{1/2}(\Gamma))^2$ *is a solution of the hypersingular integral equation of the first kind* (2.14).

Remark. The reduction to (2.14) on Γ and the analysis of $\mathcal{D}_\Gamma$ as a mapping from $\tilde{\mathbb{H}}^{1/2}(\Gamma)$ into $\mathbb{H}^{-1/2}(\Gamma)$ has also been performed in references [4], [5], [6], and [36].

Proof. Corresponding to the derivation of (2.14), any solution $\boldsymbol{u}\in\mathbb{H}^1_\Gamma$ of (2.1)–(2.4) together with its traces $[\boldsymbol{u}]|_\Gamma\in\tilde{\mathbb{H}}^{1/2}(\Gamma)$ provides a solution of (2.14).

Conversely, if $[\boldsymbol{u}]\in\tilde{\mathbb{H}}^{1/2}(\Gamma)$ is a solution of (2.14), then the displacement field $\boldsymbol{u}(x)$ defined by (2.11) satisfies the boundary conditions (2.2) since (2.14) is satisfied. The differential equations (2.1) and the radiation condition (2.3) are satisfied by the right-hand sides of (2.11) since $\boldsymbol{M}(x,y)$ is the fundamental solution satisfying (2.3) with $\boldsymbol{r}_M=\boldsymbol{0}$. □

Next we have the following uniqueness and existence result.

Theorem 2.3. *The integral equation* (2.14) *defines a continuous, bijective mapping from* $\tilde{\mathbb{H}}^{1/2}(\Gamma)$ *onto* $\mathbb{H}^{-1/2}(\Gamma)$. *Furthermore, a Gårding inequality in* $\tilde{\mathbb{H}}^{1/2}(\Gamma)$ *holds, i.e., there exist a constant* $\gamma>0$ *and a continuous map* $\mathcal{C}\colon\tilde{\mathbb{H}}^{1/2}(\Gamma)\to\mathbb{H}^{-1/2+\varepsilon}(\Gamma)$ *for some* $\varepsilon>0$ *such that*

$$\langle(\mathcal{D}_\Gamma+\mathcal{C})\boldsymbol{\psi},\boldsymbol{\psi}\rangle_{\mathcal{L}^2(\Gamma)} \geqq \gamma\,\|\boldsymbol{\psi}\|^2_{\tilde{\mathbb{H}}^{1/2}(\Gamma)} \tag{2.15}$$

for every $\boldsymbol{\psi}\in\mathbb{H}^\Gamma_2(\Gamma)$.

Proof. As is shown in the Appendix (A.4), the operator $\mathscr{D}_\Gamma$ assumes the special form

$$\mathscr{D}_\Gamma[\boldsymbol{u}](t) = \frac{\mu}{2(1-\nu)}(I + t\chi_1(t)\,C_1 + (L-t)\,\chi_2(t)\,C_2)\,D_0[\boldsymbol{u}](t)$$
$$+ D_1(t)\,\text{p.v.}\int_0^L \frac{[\boldsymbol{u}]\,ds}{s-t} + \int_0^L K(s,t)\,[\boldsymbol{u}](s)\,ds \tag{2.16}$$

for $0 \leqq t \leqq L$ where I denotes the identity, $D_1(t)$ and $K(s,t)$ are smooth matrix valued functions, C_1, C_2 are constant matrices and χ_j are the cut-off functions of Chapter 1 having sufficiently small supports near the tips. Hence, with

$$C(t) := t\chi_1(t)\,C_1 + (L-t)\,\chi_2(t)\,C_2,$$

we have

$$|C(t)| \leqq \delta < 1$$

where δ can be chosen in connection with the supports of χ_j. As in the proof of Theorem 1.4 we find that

$$\langle(\mathscr{D}_\Gamma + \mathscr{C})\,\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_{\mathscr{L}^2(\Gamma)} \geqq \gamma_0\,\|\boldsymbol{\psi}\|^2_{\tilde{\mathbb{H}}^{1/2}(\Gamma)} - c\,\delta\,\|\boldsymbol{\psi}\|^2_{\tilde{\mathbb{H}}^{1/2}(\Gamma)}$$

for any test function $\boldsymbol{\psi} \in (C_0^\infty(\Gamma))^2$ where $\mathscr{C}$ is given by the last two terms of (2.16). With $c\,\delta < \gamma_0$ we have the desired estimate (2.15). The mapping and continuity properties in Theorem 2.3 follow in exactly the same way as in the proof of Theorem 1.4. □

Theorem 2.4. *For* $|\sigma| < \frac{1}{2}$ *let*

(i) $\boldsymbol{\psi} \in \mathbb{H}^{1/2+\sigma}(\Gamma)$ *with* (2.5) *or*

(ii) $\boldsymbol{\psi} \in \mathbb{H}^{3/2+\sigma}(\Gamma)$ *with* (2.5)

be given, respectively. Then the solution $[\boldsymbol{u}] \in \tilde{\mathbb{H}}^{1/2}(\Gamma)$ *has the form*

$$[\boldsymbol{u}]|_\Gamma = \sum_{i=1}^{2} \alpha_i \varrho_i^{1/2}\chi_i + \boldsymbol{v} \quad \text{with} \quad \boldsymbol{v} \in \tilde{\mathbb{H}}^{3/2+\sigma}(\Gamma), \quad \alpha_i \in \mathbb{R}^2,$$

or

$$[\boldsymbol{u}]|_\Gamma = \sum_{i=1}^{2} (\alpha_i \varrho_i^{1/2} + \beta_i \varrho_i^{3/2})\,\chi_i + \boldsymbol{v}_1 \quad \text{with} \quad \boldsymbol{v}_1 \in \tilde{\mathbb{H}}^{5/2+\sigma}(\Gamma), \quad \beta_i \in \mathbb{R}^2$$

respectively. For fixed $\sigma \in (-\frac{1}{2}, \frac{1}{2})$, *the operators in* (2.14)

(i) $\mathscr{D}_\Gamma\colon (\mathscr{L}^{3/2+\sigma}(\Gamma))^2 \to (\mathbb{H}^{1/2+\sigma}(\Gamma))^2$,

(ii) $\mathscr{D}_\Gamma\colon (\mathscr{L}^{5/2+\sigma}(\Gamma))^2 \to (\mathbb{H}^{3/2+\sigma}(\Gamma))^2$,

are bijective and continuous and provide the a-priori estimates corresponding to those in (1.35), *with* $\mathscr{D}_\Gamma$ *in place of* D_Γ.

Proof. Equation (2.16) implies that $\mathscr{D}_\Gamma$ is a pseudodifferential operator of order 1 having the principal symbol

$$\sigma_0(\mathscr{D}_\Gamma, t, \xi) = \frac{\mu}{2(1-\nu)}(1 + C(t))\begin{pmatrix}|\xi| & 0\\ 0 & |\xi|\end{pmatrix}. \tag{2.17}$$

Hence, the multiplication of the principal part by $(1 + C(t))^{-1}$ causes the system (2.14) to decouple. Now the proofs of Lemmata 1.6 and 1.7 carry over to this case without difficulty. □

Corollary 2.5. *For* $|\sigma| < \frac{1}{2}$, *the mapping*

$$\mathscr{D}_\Gamma\colon (\tilde{\mathrm{H}}^{1/2+\sigma}(\Gamma))^2 \to (\mathrm{H}^{-1/2+\sigma}(\Gamma))^2$$

is continuous and bijective.

Proof. The proposition follows from Corollary 1.7 and the corresponding remark applied to each component of (2.16) by virtue of the arguments in the proof of Theorem 2.4. □

3. Elastic Wave Scattering at a Crack

The scattering of an elastic wave at a crack can be modeled by an exterior boundary value problem for the amplitude field,

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0 \quad \text{in } \Omega_\Gamma, \tag{3.1}$$

$$T(\boldsymbol{u})|_{\Gamma_1} = \boldsymbol{\psi}_1, \qquad T(\boldsymbol{u})|_{\Gamma_2} = \boldsymbol{\psi}_2 \tag{3.2}$$

with $\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 \in \tilde{\mathbb{H}}^{-1/2}(\Gamma)$, subject to the radiation conditions

$$\frac{\partial \boldsymbol{u}_j}{\partial r} - ik_j \boldsymbol{u}_j = o(r^{-1/2}) \quad \text{and} \quad \boldsymbol{u}_j = O(r^{-1/2}) \tag{3.3}$$

as $r = |x| \to \infty$, with $j = p$ or s; here, $\boldsymbol{u}_p = -\frac{1}{k_p^2} \operatorname{grad} \operatorname{div} \boldsymbol{u}$ is the potential field and $\boldsymbol{u}_s = \boldsymbol{u} - \boldsymbol{u}_p$ is the solonoidal field associated with the wave numbers $k_p, k_s > 0$,

$$k_p^2 = \omega^2/\mu, \quad k_s^2 = \omega^2/(\lambda + 2\mu).$$

As in the case of the acoustic field in Chapter 1 we call $\boldsymbol{u}$ a *variational solution* if

(i) $\boldsymbol{u} \in \mathbb{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$, *i.e.*, $\boldsymbol{u} \in (H^1(\Omega_\Gamma \cap B_R))^2$, for every $R > 0$,

(ii) $$\int_{\Omega_\Gamma} (\mathscr{E}(\boldsymbol{u}, \boldsymbol{v}) - \omega^2 \boldsymbol{u} \cdot \boldsymbol{v})\, dx - \int_\Gamma (\boldsymbol{\psi}_1 \cdot \boldsymbol{v}_1 - \boldsymbol{\psi}_2 \cdot \boldsymbol{v}_2)\, ds = 0 \tag{3.4}$$

for all test functions $\boldsymbol{v} \in \mathbb{H}^1_{\mathrm{loc}}(\Omega_\Gamma)$ having compact support in $\mathbb{R}^2$,

(iii) $\boldsymbol{u}$ satisfies the radiation conditions (3.3).

In the following we extend the results for the scalar Helmholtz equation in Section 1 to the boundary value problem (3.1)–(3.3) for elastic vibrations (which also is studied in the article [9]). We outline only those parts of the proofs that are peculiar to elastic vibrations.

We begin with the uniqueness for the variational solution corresponding to Lemma 1.1. By using the decomposition $\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_s$, and $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \boldsymbol{0}$, we can reduce (3.1)–(3.3) to the exterior modified homogeneous Neumann problems for vector Helmholtz equations for $\boldsymbol{u}_p$ and $\boldsymbol{u}_s$ with the corresponding wave numbers k_p and k_s. The corresponding boundary conditions become on both sides of Γ

$$T(\boldsymbol{u}_p)|_{\Gamma_j} = 2\mu \frac{\partial \boldsymbol{u}_p}{\partial \boldsymbol{n}} + \lambda \boldsymbol{n} \wedge \operatorname{div} \boldsymbol{u}_p = \boldsymbol{0},$$

$$T(\boldsymbol{u}_s)|_{\Gamma_j} = 2\mu \frac{\partial \boldsymbol{u}_s}{\partial \boldsymbol{n}} + \mu \boldsymbol{n} \wedge \operatorname{curl} \boldsymbol{u}_s = \boldsymbol{0},$$

(see reference [30, pp. 41 ff.). As in the proof of Lemma 1.1, we can show that $\boldsymbol{u}_p = \boldsymbol{0}$ and $\boldsymbol{u}_s = \boldsymbol{0}$ in Ω_Γ, which yield uniqueness of (3.1)–(3.3) in the space $\mathscr{L}_\Gamma = \{u \in \mathbb{H}^1_{\text{loc}}(\Omega_\Gamma)$ satisfying (3.1) and (3.3)$\}$.

Furthermore, Lemma 1.2 remains valid, *i.e.*, $[u]|_\Gamma \in \tilde{\mathbb{H}}^{1/2}(\Gamma) := (\tilde{H}^{1/2}(\Gamma))^2$, for the variational solution $\boldsymbol{u}$, since its proof applies to both components of $\boldsymbol{u}$ directly.

Here we convert the boundary value problem into an equivalent hypersingular integral equation of the first kind by making use of the fundamental solution

$$\boldsymbol{M}_\omega(x, y) = -\frac{i}{4\mu} H_0^{(1)}(k_2 |x-y|) - \frac{i}{4\omega^2} \times \left(\left(\frac{\partial^2}{\partial y_j \, \partial y_k}\{H_0^{(1)}(k_2 |x-y|) - H_0^{(1)}(k_1 |x-y|)\}\right)\right) \quad j, k = 1, 2. \tag{3.5}$$

The hypersingular equation for $[\boldsymbol{u}]$ becomes

$$\mathscr{D}^\omega_\Gamma[\boldsymbol{u}](x) := -T_x \int_{\Gamma\setminus\{x\}} (T_y \boldsymbol{M}_\omega(x, y))^T [\boldsymbol{u}](y) \, ds_y = \boldsymbol{f}(x) \tag{3.6}$$

where

$$\boldsymbol{f}(x) = \tfrac{1}{2}(\boldsymbol{\psi}_1(x) + \boldsymbol{\psi}_2(x)) - \int_{\Gamma\setminus\{x\}} (T_x \boldsymbol{M}_\omega(x, y)) \, \boldsymbol{\psi}(y) \, ds_y.$$

This is equation (2.14) with $\boldsymbol{M}_\omega$ instead of $\boldsymbol{M}$.

Because the fundamental solution $\boldsymbol{M}_\omega$ can be expanded into pseudohomogeneous kernels via the expansion (1.12) of the Hankel function (see reference [2]), we can proceed as in the proof of Theorem 1.3 to obtain the equivalence of the variational solution $\boldsymbol{u} \in \mathbb{H}^1_{\text{loc}}(\Omega_\Gamma)$ of (3.1)–(3.3) with the solution $[\boldsymbol{u}] \in \tilde{\mathbb{H}}^{1/2}(\Gamma)$ of the integral equation (3.6). Thus, Theorem 1.3 remains valid word for word.

The solvability of the hypersingular equation (3.6), the regularity of its solution and the corresponding singular expansions about the crack tips again can be established. Inspection of the fundamental solution shows that

$$T_y \boldsymbol{M}_\omega = T_y \boldsymbol{M} + K(x, y; \omega)$$

where K is the kernel of an integral operator defining a pseudodifferential operator of order -2 and $K = O(\omega^2 |\log \omega|)$ for $\omega \to 0$ (see reference [28]). Consequently,

$$\mathscr{D}_\Gamma^\omega = \mathscr{D}_\Gamma + \mathscr{B}_\omega \tag{3.7}$$

where $\mathscr{B}_\omega$ is a pseudodifferential operator of order -1 and $|||\mathscr{B}_\omega||| = O(\omega^2 |\log \omega|)$ as $\omega \to 0$ with respect to the operator norms for the spaces of linear mappings

$$L(\tilde{\mathbb{H}}^{1/2+\sigma}(\Gamma), \mathbb{H}^{3/2+\sigma}(\Gamma)), \quad |\sigma| < \tfrac{1}{2}.$$

Therefore $\mathscr{D}_\Gamma^\omega$ is a perturbation of $\mathscr{D}_\Gamma$ by a compact operator $\mathscr{B}_\omega$ and thus the Gårding inequality (2.15) remains valid for $\mathscr{D}_\Gamma^\omega$. Hence $\mathscr{D}_\Gamma^\omega$: $\tilde{\mathbb{H}}^{1/2}(\Gamma) \to \mathbb{H}^{-1/2}(\Gamma)$ is also bijective due to the uniqueness of (3.6) obtained above. In view of Corollary 2.5 and the form of (3.7), the bijectivity extends to $\mathscr{D}_\Gamma^\omega$: $\tilde{\mathbb{H}}^{1/2+\sigma}(\Gamma) \to \mathbb{H}^{-1/2+\sigma}(\Gamma)$ for $|\sigma| < \frac{1}{2}$.

Next, we note that the solution of the equations (3.6) has the same regularity as for the case $\omega = 0$ given in Theorem 2.4. This can be seen from (3.7) as follows. Since $\mathscr{B}_\omega(\varrho_i^{1/2}\chi_i) \in \mathbb{H}^{2-\varepsilon}(\Gamma)$ and $\mathscr{B}_\omega(\varrho_i^{3/2}\chi_i) \in \mathbb{H}^{3-\varepsilon}(\Gamma)$ for every $\varepsilon > 0$, we obtain

$$\mathscr{B}_\omega: \mathscr{X}^{3/2+\sigma}(\Gamma) \to \mathbb{H}^{1/2+\sigma}(\Gamma) \quad \text{is compact, } \mathscr{D}_\Gamma^{-1}\mathscr{B}_\omega \in \mathscr{X}^{3/2+\sigma}(\Gamma);$$

$$\mathscr{B}_\omega: \mathscr{X}^{5/2+\sigma}(\Gamma) \to \mathbb{H}^{3/2+\sigma}(\Gamma) \quad \text{is compact, } \mathscr{D}_\Gamma^{-1}\mathscr{B}_\omega \in \mathscr{X}^{5/2+\sigma}(\Gamma).$$

Hence

$$\mathscr{D}_\Gamma^\omega = \mathscr{D}_\Gamma(I + \mathscr{D}_\Gamma^{-1}\mathscr{B}_\omega)$$

is the composition of the isomorphism $\mathscr{D}_\Gamma$ and a Riesz-Schauder operator. Hence, the uniqueness of equations (3.6) implies by Nikolski's theorem that $\mathscr{D}_\Gamma^\omega$ is also an isomorphism from $(\mathscr{X}^{3/2+\sigma}(\Gamma))^2$ onto $\mathbb{H}^{1/2+\sigma}(\Gamma)$ and from $(\mathscr{X}^{5/2+\sigma}(\Gamma))^2$ onto $\mathbb{H}^{3/2+\sigma}(\Gamma)$, respectively. Therefore, Theorem 2.4 also remains valid word-for-word with $\omega > 0$. In particular, we find for $\omega \neq 0$ the *same* form of the singular expansions for $[\boldsymbol{u}]$ as in the stationary case $\omega = 0$.

4. Appendix: The Hypersingular Kernel in Elasticity

The non-integrable kernel of the operator $\mathscr{D}_\Gamma$ in (2.14) can be found explicitly in [7, p. 187ff. and p. 191]. With

$$\boldsymbol{n} = \boldsymbol{n}(y) = \begin{pmatrix} \dot{Z}_2(s) \\ -\dot{Z}_1(s) \end{pmatrix}, \quad \boldsymbol{m} = \boldsymbol{n}(x) = \begin{pmatrix} \dot{Z}_2(t) \\ -\dot{Z}_1(t) \end{pmatrix}, \quad r = |x - y|,$$

$$\frac{\partial}{\partial \boldsymbol{n}}(\log r) = \frac{1}{r}\frac{\partial r}{\partial \boldsymbol{n}} = \frac{(y-x)}{r^2} \cdot \boldsymbol{n}(y); \quad \nu = (\lambda + \mu)/2\mu$$

we find the components of $\mathscr{D} = ((D_{jk}))$ to be

$$D_{jj}(y, x) = \frac{-\mu}{2\pi(1-\nu)}\left\{\frac{n(y)\cdot n(x)}{r^2} + \frac{2\nu}{r^2}\left(n_j(y)\,n_j(x) - n(y)\cdot n(x)\,(y_{j+1} - x_{j+1})^2/r^2\right)\right\}$$
$$-\frac{\mu}{\pi(1-\nu)}\left\{(x_j - y_j)\left(\left(\frac{1}{r}\frac{\partial r}{\partial n(y)}\right) n_j(x) + (1-\nu)\left(\frac{1}{r}\frac{\partial r}{\partial n(x)}\right) n_j(y)\right)\Big/ r^2\right.$$
$$\left. - 8\left(\frac{1}{r}\frac{\partial r}{\partial n(x)}\right)\left(\frac{1}{r}\frac{\partial r}{\partial n(y)}\right)(x_j - y_j)^2/r^2\right\}, \quad \text{for } j = 1, 2,; \tag{A.1}$$

$$D_{12}(y, x) = D_{21}(x, y)$$
$$= \frac{-\mu\nu}{\pi(1-\nu)}\{n_1(x)\,n_2(y) + n(x)\cdot n(y)\,(y_1 - x_1)\,(y_2 - x_2)/r^2\}/r^2$$
$$-\frac{\mu}{2\pi(1-\nu)}\left\{(1-2\nu)\,(n_1(y)\,n_2(x) - n_1(x)\,n_2(y))\right.$$
$$+ 2(y_2 - x_2)\left((1-2\nu)\,n_1(x)\left(\frac{1}{r}\frac{\partial r}{\partial n(y)}\right) + \nu n_1(y)\left(\frac{1}{r}\frac{\partial r}{\partial n(x)}\right)\right)$$
$$\left. + 2(y_1 - x_1)\left((1-2\nu)\,n_2(y)\left(\frac{1}{r}\frac{\partial r}{\partial n(x)}\right) + \nu n_2(x)\left(\frac{1}{r}\frac{\partial r}{\partial n(y)}\right)\right)\right\}\Big/ r^2$$
$$+ \frac{4\mu}{\pi(1-\nu)}\left\{\left(\frac{1}{r}\frac{\partial r}{\partial n(y)}\right)\left(\frac{1}{r}\frac{\partial r}{\partial n(x)}\right)(x_1 - y_1)(x_2 - y_2)\right\}\Big/ r^2. \tag{A.2}$$

We insert Taylor expansions for $x = Z(t)$ and $y = Z(s)$ into (A.1) and (A.2) to obtain the following *explicit* representation for $\mathscr{D}$ near $t = 0$ corresponding to the crack tip Z_1,

$$\mathscr{D}_\Gamma[u] = \frac{\mu}{2(1-\nu)}(I + tC)\,D_0[u] + D_1(t)\,\text{p.v.}\int_0^L \frac{[u]\,ds}{s - t} + \int_0^L K(s, t)\,[u]\,(s)\,ds. \tag{A.3}$$

Here I denotes the identity, C is a constant matrix and $D_1(t)$ and $K(s, t)$ are smooth matrix-valued functions. At the second crack tip Z_2 we find the corresponding behavior with t in the first term replaced by $L - t$. Hence (A.3) can also be written as

$$\mathscr{D}_\Gamma[u)]\,(t) = \frac{\mu}{2(1-\nu)}(I + \chi_1(t)\,tC_1 + \chi_2(t)\,(L - t)\,C_2)\,D_0[u] + D_1(t)\;\text{p.v.}\int_0^L \frac{[u]\,ds}{s - t} + \int_0^L K(s, t)\,[u]\,(s)\,ds. \tag{A.4}$$

Note that C_1, C_2 depend only on the curvatures of Γ at Z_1, Z_2, respectively, but not on the cut-off functions χ_1, χ_2.

Acknowledgements. The work of STEPHAN was supported by the NSF grants DMS-8501797 and DMS-8603954. The work of WENDLAND was supported by the "Stiftung Volkswagenwerk." We thank D. OWEN for many helpful suggestions.

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(Received January 16, 1990)