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Cosine Change of Variable for Symm's Integral Equation on Open Arcs

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By applying a cosine change of variable, the numerical solution of Symm's integral equation on open arcs can be obtained in an effective way so that the influence of the singularities at the endpoints is eliminated. Both Galerkin and collocation methods using a piecewise constant approximating space are analysed. The convergence of the approximate solutions and the superconvergence of the single layer potential as well as some numerical results are presented.

1. Introduction

WHEN THE contour Γ is an open arc, the solution $g(x)$ of Symm's integral equation

$$-\frac{1}{\pi}\int_{\Gamma} g(y)\log|x-y|\,dy = f(x) \quad \text{for } x\in\Gamma \tag{1}$$

is usually singular at the endpoints (see [16]), where dy denotes the arc length element at the point y. These singularities degrade the rates of convergence when numerical methods such as Galerkin and collocation methods are applied (see [17] and the numerical results at the end of this paper), and so one introduces modifications in order to restore the optimal rate of convergence. One possible modification is the augmented method for which the approximating spaces (test and trial) are augmented by appropriate singular functions which mimic the behaviour of the exact solution at the endpoints of the open contour Γ. This modification applied to the Galerkin method with piecewise polynomial test and trial functions has been analysed by Stephan and Wendland in [13]. Another modification is to grade the mesh in a suitable way near the endpoints. Its application to the Galerkin method with piecewise constant test and trial functions has been analysed by Yan and Sloan [17], and its application to the collocation method with piecewise linear trial functions has been done by Costabel, Ervin, & Stephan [4]. In this paper, however, we study the method of cosine change of variable for the numerical solution of equation (1).

The crucial idea is based on the works of Yan and Sloan [16] and Yan [14]. Instead of applying numerical methods directly to equation (1), we reformulate equation (1) as follows. While the open contour Γ is parametrized by a smooth function $v:[-1,1]\to\Gamma\subset\mathbb{R}^2$, with $|v'(s)|\geq\varepsilon>0$, we introduce a new parametrization function $a:[0,\pi]\to\Gamma$ by $a(t)=v(\cos t)$, so that Γ is now considered to be parametrized by $t = \cos^{-1} s$. This leads to

$$-\frac{1}{\pi}\int_0^{\pi} w(\tau)\log|a(t)-a(\tau)|\,\mathrm{d}\tau = f_a(t) \quad \text{for } t\in[0,\pi],$$

with $w(t) = |v'(\cos t)|\,|\sin t|\,g(a(t))$ and $f_a(t) = f(a(t))$, which are even and 2π-periodic because $a(t)$ is so. Written as an operator equation this becomes

$$Kw = f_a. \tag{2}$$

In contrast to (1) this is an equation for only the nonsingular part $w(t)$ of the original solution

$$g(a(t)) = w(t)\,|\sin t|^{-1}\,|v'(\cos t)|^{-1}.$$

Because w is well behaved, numerical methods applied to equation (2) rather than to equation (1) can be expected to have better convergence properties. We call this approach (i.e. reformulating equation (1) as equation (2)) the method of cosine change of variable.

Equation (2) has been investigated completely by employing Fourier analysis in [16], so it is a simple matter to analyse the Galerkin method when the approximating space consists of trigonometric polynomials. This numerical method and its discrete version (with the double and single integrals appearing in the resulting stiffness matrix substituted by numerical integrations) have been completely analysed by Atkinson & Sloan in [3]. For the case of Γ a straight line, Stephan & Sloan have recently analysed the collocation method for (1) with the trial space being Chebyshev polynomials [12]. This method is equivalent to the collocation method for (2) with the trial space being trigonometric polynomials.

Other possible methods are Galerkin and collocation methods with smooth spline approximating functions, which are our concerns in this paper. They can be analysed by employing and modifying those results for the smooth, closed contour cases [6, 1, 2, 10, 9]. However, in order to maintain simplicity, we only consider a special case in this paper. We apply and examine both Galerkin and collocation methods, but only the piecewise constant versions, because of their easy implementation, and because the expected $O(h^3)$ optimal order of convergence for the single layer potential is accurate enough from a practical point of view.

Let $\sigma_j = hj$, $\sigma_{j+\frac{1}{2}} = \sigma_j + \frac{1}{2}h$ $(j = 0,\dots,n-1)$, and $\sigma_n = \pi$, with $h = \pi/n$. Setting $\Pi_+ = \{\sigma_j : j = 0,\dots,n\}$, let $S^h(\Pi_+)$ be the space of piecewise constant functions on the interval $[0,\pi]$ with break point set Π_+. Suppose that P_+^h is the corresponding orthogonal projection from $L_2[0,\pi]$ to $S^h(\Pi_+)$, and Q_+^h is the midpoint interpolation projection from $C[0,\pi]$ to $S^h(\Pi_+)$, defined by

$$P_+^h v(t) = \sum_{j=0}^{n-1} h^{-1}\int_{\sigma_j}^{\sigma_{j+1}} v(\tau)\,\mathrm{d}\tau X_j(t), \qquad Q_+^h v(t) = \sum_{j=0}^{n-1} v(\sigma_{j+\frac{1}{2}})X_j(t),$$

where

$$X_j(t) = \begin{cases} 1 & \text{if } t\in(\sigma_j,\sigma_{j+1}),\\ 0 & \text{otherwise.}\end{cases}$$

The Galerkin method approximates w by $w_{h+}^g \in S^h(\Pi_+)$, such that

$$P_+^h K w_{h+}^g = P_+^h f_a, \tag{3}$$

and the collocation method approximates w by $w^c_{h_+} \in S^h(\Pi_+)$, such that

$$Q^h_+ K w^c_{h_+} = Q^h_+ f_a. \tag{4}$$

Setting $\Pi_- = \{-\sigma_j : j = 1,\dots, n\}$, $\Pi = \Pi_+ \cup \Pi_-$, let $S^h(\Pi)$ be the space of piecewise constant functions on the interval $[-\pi, \pi]$ with break point set Π, and let $S^h_e(\Pi) = \{v \in S^h(\Pi): v(x) = v(-x) \text{ for } x \in [-\pi, \pi]\}$. Thus, $S^h_e(\Pi)$ is the even extension of $S^h(\Pi_+)$ to the interval $[-\pi, \pi]$. Moreover, as extensions of the operators P^h_+ and Q^h_+ onto $S^h(\Pi)$, the orthogonal projection P^h from $L_2[-\pi, \pi]$ to $S^h(\Pi)$, and the midpoint interpolation projection Q^h from $C[-\pi, \pi]$ to $S^h(\Pi)$ are given respectively by

$$P^h w(t) = P^h_+ w(t) + P^h_+ w_-(-t), \qquad Q^h w(t) = Q^h_+ w(t) + Q^h_+ w_-(-t),$$

where $w_-(\tau) = w(-\tau)$.

Noting that f_a, $Kw^g_{h_+}$, and $Kw^c_{h_+}$ are even functions, (3) and (4) are respectively equivalent to

$$P^h K w^g_h = P^h f_a, \tag{3*}$$

$$Q^h K w^c_h = Q^h f_a, \tag{4*}$$

where w^g_h and $w^c_h \in S^h_e(\Pi)$ are the even extensions of $w^g_{h_+}$ and $w^c_{h_+}$ to the interval $[-\pi, \pi]$. Thus, we may consider (3*), (4*) in the following instead of (3), (4).

In order to give a more precise expression we give some notation here. Each 2π-periodic function v has a Fourier expansion

$$v(s) = \frac{1}{\sqrt{(2\pi)}} \sum_{m \in \mathbb{Z}} \hat{v}(m) e^{ims},$$

where the Fourier coefficients are given by

$$\hat{v}(m) = \frac{1}{\sqrt{(2\pi)}} \int_{-\pi}^{\pi} v(s) e^{-ims}\, ds.$$

For $r \in \mathbb{R}$ define the inner product

$$(w, v)_r = \hat{w}(0)\overline{\hat{v}(0)} + \sum_{m \in \mathbb{Z}\setminus\{0\}} |m|^{2r}\, \hat{w}(m)\overline{\hat{v}(m)}.$$

Then when $r = 0$, we have $(w, v)_0 = (w, v)$, where

$$(w, v) = \int_{-\pi}^{\pi} w(s)\overline{v(s)}\, ds.$$

The Sobolev space $H^r(2\pi)$ consists of all 2π-periodic real functions v for which the norm $|v|_r = \sqrt{(v, v)_r}$ is finite. The Sobolev space $H^r_e(2\pi)$ is the subspace of $H^r(2\pi)$, which consists of those even functions in $H^r(2\pi)$.

The operator K has its simplest form when Γ is a straight segment, say of length $4e^{-1}$. In this case the operator A_e, defined by

$$A_e w(t) = \int_0^{\pi} w(\tau) \Lambda_e(t, \tau)\, d\tau,$$

with

$$\Lambda_e(t, \tau) = -(1/\pi) \log |2e^{-1}(\cos t - \cos \tau)|,$$

is produced, which is isometric from $H_c^r(2\pi)$ to $H_c^{r+1}(2\pi)$. The operator $K: H_c^r(2\pi) \to H_c^{r+1}(2\pi)$ for other smooth open contours is shown in [16] and [14] to be a compact perturbation of A_c, with an exact decomposition

$$K = A_c(I + M),$$

where M is compact on $H_c^r(2\pi)$ for an arbitrary real number r.

Under this decomposition, equation (2) is equivalent to a Fredholm equation of the second kind on the space $H_c^r(2\pi)$

$$(I + M)w = \bar{f}_a, \tag{5}$$

with $\bar{f}_a = A_c^{-1} f_a$. Further, when the transfinite diameter (or logarithmic capacity) C_Γ is not one, the operator $I + M$ is invertible on $H_c^r(2\pi)$ for $r \in \mathbb{R}$.

The operator A_c has a close relation to the isometric operator from $H^r(2\pi)$ to $H^{r+1}(2\pi)$, say A, defined by

$$Aw(s) = \int_{-\pi}^{\pi} w(\sigma)\Lambda(s - \sigma)\, d\sigma,$$

with

$$\Lambda(s - \sigma) = -(1/\pi) \log |2e^{-\frac{1}{2}} \sin \tfrac{1}{2}(s - \sigma)|.$$

In fact, as shown in [16],

$$A_c w = Aw \quad \text{for } w \in H_c^0(2\pi). \tag{6}$$

By using these notations and properties we shall show that both w_h^g and w_h^c converge to w in certain norms, and have superconvergence properties

$$|(w_h^g - w, v)| \leq O(h^3)\, |w|_1\, |v|_2 \quad \text{for } w \in H_c^1(2\pi) \text{ and } v \in H_c^2(2\pi),$$
$$|(w_h^c - w, v)| \leq O(h^3)\, |w|_2\, |v|_2 \quad \text{for } w, v \in H_c^2(2\pi).$$

It is clear from these results that through the cosine change of variable and the application of the numerical methods to equation (2), the optimal rate of convergence $O(h^3)$ is restored.

2. Error analysis of the Galerkin method

Equation (1) comes from an exterior potential problem with a Dirichlet boundary condition given on the open arc Γ (see for example [14]). This potential theory problem with an arbitrary geometric configuration can always be rescaled to give rise to a new boundary integral equation of the first kind with $C_\Gamma < 1$. For this reason, we assume in the following that $C_\Gamma < 1$ without loss of generality. Under this assumption we are able to show that K is positive-definite, and hence introduce an energy norm similar to the one adopted in [11] and [17]. This approach enables us to perform a simple error analysis for the Galerkin method.

THEOREM 1 *Assume that* $0 < C_\Gamma < 1$. *Let* $w \in H_c^0(2\pi)$, then

$$(Kw, w) \geq 0$$

with equality only if $w = 0$.

Proof.

$$\begin{aligned}(Kw, w) &= -\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{0}^{\pi} w(t)w(\tau)\log|a(t)-a(\tau)|\,d\tau\,dt\\ &= -\frac{2}{\pi}\int_{0}^{\pi}\int_{0}^{\pi} w(t)w(\tau)\log|a(t)-a(\tau)|\,d\tau\,dt\\ &= -\frac{2}{\pi}\int_{\Gamma}\int_{\Gamma} g(x)g(y)\log|x-y|\,dy\,dx,\end{aligned}$$

where

$$g(x) = \frac{w\big(a^{-1}(x)\big)\,|(v^{-1})'(x)|}{\sin[a^{-1}(x)]}$$

(see [16], [14]). For $1<p<2$

$$\begin{aligned}\int_{\Gamma}|g(y)|^{p}\,dy &= \int_{0}^{\pi}\left|\frac{w(\tau)}{\sin(\tau)}(v^{-1})'\big(a(\tau)\big)\right|^{p}|v'(\cos\tau)|\sin\tau\,d\tau\\ &\leq c\int_{0}^{\pi} w^{p}(\tau)\sin^{1-p}\tau\,d\tau\\ &\leq c\left(\int_{0}^{\pi} w^{2}(\tau)\,d\tau\right)^{p/2}\left(\int_{0}^{\pi}\sin^{2(1-p)/(2-p)}\tau\,d\tau\right)^{(2-p)/2}.\end{aligned}$$

Here $\int_0^{\pi}\sin^{2(1-p)/(2-p)}\tau\,d\tau$ is finite when $1<p<\frac{4}{3}$, so that, for $1<p<\frac{4}{3}$,

$$\|g\|_{L_p(\Gamma)} \leq c\,|w|_0.$$

Let $\sigma(x)=\int_{x_1}^{x} g(y)\,dy$, where $\int_{x_1}^{x}$ denotes the integration along Γ from x_1 to x with x_1 an endpoint of Γ. Noting that $d\sigma(x)=g(x)\,dx$, we then have

$$(Kw, w) = (2/\pi)I(\sigma).$$

Since $g\in L_p(\Gamma)$ $(1<p<\frac{4}{3})$, $I(|\sigma|)<+\infty$. Hence the result under consideration follows from Theorem 1 in [16] or Theorem 2.1.1 in [14]. This completes the proof. □

Remark. A corollary of this theorem is that the Galerkin equation (3*) is uniquely solvable for any positive integer n.

For the analysis of convergence, we introduce an energy inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle w, v\rangle = (Kw, v),$$

with its corresponding norm given by

$$\|w\| = \langle w, w\rangle^{\frac{1}{2}} = (Kw, w)^{\frac{1}{2}}.$$

The analysis of convergence of the Galerkin approximation is now straightforward.

THEOREM 2 *Under the condition of Theorem* 1,

$$\|w - w_h^g\| \leq \min_{v_h \in \mathcal{S}_e^h(\Pi)} \|w - v_h\|,$$

and further

$$\|w - w_h^g\| \leq ch^{\frac{3}{2}} |w|_1 \quad \textit{for } w \in \mathrm{H}_e^1(2\pi).$$

Proof. Since (3*) is equivalent to

$$\big(K(w_h^g - w), v_h\big) = 0 \quad \text{for } v_h \in \mathcal{S}_e^h(\Pi),$$

or correspondingly

$$\langle w_h^g - w, v_h \rangle = 0 \quad \text{for } v_h \in \mathcal{S}_e^h(\Pi),$$

the function w_h^g is the orthogonal projection (with respect to the energy inner product) of w onto the space $\mathcal{S}_e^h(\Pi)$, and therefore

$$\|w_h^g - w\| \leq \min_{v_h \in \mathcal{S}_e^h(\Pi)} \|w - v_h\|.$$

For the second part of this theorem we note that

$$\min_{v_h \in \mathcal{S}_e^h(\Pi)} \|w - v_h\| \leq \|w - P^h w\|,$$

because $P^h w \in \mathcal{S}_e^h(\Pi)$. Now

$$\begin{aligned}\|w - P^h w\|^2 &= \big(K(I - P^h)w, (I - P^h)w\big) = \big((I - P^h)K(I - P^h)w, (I - P^h)w\big)\\ &\leq |(I - P^h)K|_0 \, |(I - P^h)w|_0^2 \leq ch^3 |w|_1^2.\end{aligned}$$

Then the second part of this theorem follows. □

3. Superconvergence of the Galerkin method and its applications

It has already been shown in [11], [17] and Chapter 3 of [14] that, when the Galerkin method with piecewise constant approximating functions on uniform meshes is applied to equation (1) on closed smooth contours, the optimal order of the convergence for the single-layer potential in the interior and exterior regions is $O(h^3)$. However, for the case of open arcs, the $O(h^3)$ order of convergence has been degraded to $O(h \log h^{-1})$. By employing the technique of mesh grading around the endpoints of Γ the optimal rate of convergence can be restored. In this section we show that the method of the cosine change of variable will also lead to the Galerkin method having the $O(h^3)$ order of convergence for the single layer potential.

THEOREM 3 *Under the condition of Theorem* 1,

$$|(w - w_h^g, v)| \leq O(h^3) \, |w|_1 \, |v|_2$$

for $w \in \mathrm{H}_e^1(2\pi)$ *and* $v \in \mathrm{H}_e^2(2\pi)$.

Proof. Let $v_1 = K^{-1}v$, then $v_1 \in \mathrm{H}_e^1(2\pi)$, and, by noting the symmetry of the operator K on $H_e^0(2\pi)$, we have

$$(w - w_h^g, v) = (w - w_h^g, Kv_1) = \big(K(w - w_h^g), v_1\big) = \big(K(w - w_h^g), (I - P^h)v_1\big).$$

It then follows that

$$\begin{aligned}|(w - w_h^g, v)| &\leq \|w - w_h^g\| \, \|(I - P^h)v_1\| \leq ch^3 |w|_1 |v_1|_1 \\ &\leq ch^3 |w|_1 |v|_2;\end{aligned}$$

this completes the proof. □

As an application of this superconvergence result, we consider an approximation

$$u_h^g(z) = -\frac{1}{\pi}\int_0^{\pi} w_h^g(\tau)\log|z - a(\tau)|\,d\tau = -\frac{1}{2\pi}\int_{-\pi}^{\pi} w_h^g(\tau)\log|z - a(\tau)|\,d\tau,$$

of the single layer potential

$$\begin{aligned}u(z) = -\frac{1}{\pi}\int_{\Gamma} g(y)\log|z - y|\,dy &= -\frac{1}{\pi}\int_0^{\pi} w(\tau)\log|z - a(\tau)|\,d\tau \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} w(\tau)\log|z - a(\tau)|\,d\tau\end{aligned}$$

for $z \notin \Gamma$. It is clear that

$$u_h^g(z) - u(z) = (g_z, w_h^g - w),$$

where

$$g_z(t) = -(1/2\pi)\log|z - a(t)| \in H_e^2(2\pi).$$

Thus, from Theorem 3, we have, for $z \notin \Gamma$, that

$$|u_h^g(z) - u(z)| \leq ch^3 |u|_1.$$

Remark. If the Galerkin method is considered in general cases, e.g. $\mathcal{S}_e^h(\Pi)$ being a space of even piecewise polynomials or even smooth splines, then convergence and superconvergence results are also attainable. The proofs are similar to those in both the last and the previous sections.

4. Error analysis of the collocation method

In the works of Yan [15] and Graham & Yan [5], an analysis of the collocation method for the case of polygonal boundaries Γ, even with a sharp angle, has been given. The crucial fact exploited is that a polygonal boundary can always be seen as a small perturbation of a unit circle. But an open arc, a degenerate case of a polygon, can never be simply seen as a small perturbation of the unit circle. That is why their theory no longer applies to the case of open arcs, although some numerical experiments have been attempted (see at the end of this paper). However, by using the method of cosine change of variable, a complete error analysis can be given for the collocation method applied to equation (2).

In fact, we employ here a similar approach to that adopted for the collocation method applied on a closed contour in [15], i.e. we work with an equivalent projection method for a second kind Fredholm equation on the space $H^0_e(2\pi)$. Since $H^0_e(2\pi)$ is a subspace of the space $H^0(2\pi)$, the analysis can be simplified by employing directly some related results for the smooth closed-contour case. This is a basic idea in the analysis of the collocation method in this paper. The positive definite property of K will not be used, so it is not necessary to have the assumption $0<C_\Gamma<1$. We only assume that $C_\Gamma \neq 1$ so as to ensure the invertibility of the operator K.

In the following we shall employ the 'interpolating' projection operator B^h introduced in [15], defined by

$$B^h = (Q^h A_h)^{-1} Q^h A.$$

B^h is a bounded projection operator onto $\mathcal{S}^h(\Pi)$ with a property:

$$|(I - B^h)|_0 \leq c\,|(I - P^h)w|_0.$$

To use the operator B^h limited on the space $H^0_e(2\pi)$, two propositions on operator A are given:

PROPOSITION 1 *Let* $v \in H^0(2\pi)$.
(i) *If* v *is an even function,* Av *is even*;
(ii) *If* v *is an odd function,* Av *is odd.*

Proof. The first part follows directly from (6). Thus we need only to prove the second part. If v is odd, then

$$\begin{aligned} Av(s) &= \int_{-\pi}^{\pi} \Lambda(s-\sigma)v(\sigma)\,d\sigma \\ &= \int_0^{\pi} \Lambda(s-\sigma)v(\sigma)\,d\sigma + \int_{-\pi}^{0} \Lambda(s-\sigma)v(\sigma)\,d\sigma \\ &= \int_0^{\pi} \Lambda(s-\sigma)v(\sigma)\,d\sigma + \int_{\pi}^{0} \Lambda(s+\sigma)v(\sigma)\,d\sigma \\ &= \int_0^{\pi} \Lambda(s-\sigma)v(\sigma)\,d\sigma - \int_0^{\pi} \Lambda(-s-\sigma)v(\sigma)\,d\sigma. \end{aligned}$$

It follows that $Av(-s) = -Av(s)$, which completes the proof. □

PROPOSITION 2 *If* $v \in H^0_e(2\pi)$, *then* $B^h v \in H^0_e(2\pi)$.

Proof. Let

$$v_1(s) = \tfrac{1}{2}[B^h v(s) + B^h v(-s)], \qquad v_2(s) = \tfrac{1}{2}[B^h v(s) - B^h v(-s)].$$

Since $Q^h Av = Q^h AB^h v = Q^h Av_1 + Q^h Av_2$, we have $Q^h Av - Q^h Av_1 = Q^h Av_2$, with functions v and v_1 being even, and with v_2 being odd. It follows from Proposition 1 that $Q^h Av - Q^h Av_1$ is even and $Q^h Av_2$ is odd. Thus $Q^h Av_2 = 0$. Since $v_2 \in \mathcal{S}^h(\Pi)$, it follows from the invertibility of the operator $Q^h A$ on $\mathcal{S}^h(\Pi)$ (see [15]) that $v_2 = 0$. Hence $B^h v(s) = B^h v(-s)$, as required. □

Having established these propositions, we are now able to give the convergence of the collocation method as a theorem.

THEOREM 4 *Assume $C_\Gamma \neq 1$. When n is sufficiently large, the collocation equation (4) is uniquely solvable in the approximating space $S_e^h(\Pi)$ for any continuous function f_a and, provided the exact solution $w \in H_e^0(2\pi)$,*

$$|w_h^c - w|_0 \leq c\,|(I - P^h)w|_0.$$

Proof. Since $K = A_e(I + M)$, the equation $Q^hKw_h^c = Q^hf_a$ can be written as

$$Q^hA_ew_h^c + Q^hA_eMw_h^c = Q^hA_e\bar{f}_a, \tag{7}$$

where $\bar{f}_a = A_e^{-1}f_a$. Because w_h^c, Mw_h^c, and $\bar{f}_a$ are even, equation (7) can be rewritten as

$$Q^hAw_h^c + Q^hAMw_h^c = Q^hA\bar{f}_a$$

or equivalently as

$$(I + B^hM)w_h^c = B^h\bar{f}_a.$$

Now, Proposition 2 ensures that the operator $I + B^hM$ maps $H_e^0(2\pi)$ into $H_e^0(2\pi)$, and the compactness of the operator M implies that $(I - B^h)M \to 0$ uniformly on $H_e^0(2\pi)$ as $h \to 0$. The operator $I + M$ is invertible on $H_e^0(2\pi)$ because $C_\Gamma \neq 1$, so $I + B^hM = I + M - (I - B^h)M$ is invertible on $H_e^0(2\pi)$ when n is sufficiently large. The first part of the theorem then follows immediately. Moreover, $(I + B^hM)^{-1}$ is uniformly bounded. By noting that

$$(I + M)w = \bar{f}_a,$$

we have

$$w + B^hMw = B^h\bar{f}_a + (I - B^h)w.$$

Thus

$$(I + B^hM)(w - w_h^c) = (I - B^h)w,$$

and then

$$|w_h^c - w|_0 = |(I + B^hM)^{-1}(I - B^h)w| \leq c\,|(I - B^h)w|_0 \leq c\,|(I - P^h)w|_0.$$

This completes the proof of the second part of the theorem. □

5. Superconvergence of the collocation method and its application

An important property of the interpolation projection operator B^h is its superconvergence. As its application we are able to show the superconvergence of the collocation method in our consideration.

But, to establish B^h superconvergent, a rigorous analysis is needed. This is arranged as two lemmas and the following theorem. Let $\Lambda_h = \{p \in \mathbb{Z}: -n < p \leq n\}$. We define an operator T^h by

$$T^hu = \frac{1}{\sqrt{(2\pi)}} \sum_{p \in \Lambda_h} \hat{u}(p)e^{ips},$$

so that T^hu is a finite truncation of the Fourier series of u, then the following is a well known property of T^h (see for example [8] and [7]).

LEMMA 1 *If* $w \in H^2(2\pi)$,

$$|(I-T^h)w|_0 \leq ch^2|w|_2 \quad \textit{and} \quad |(I-T^h)w|_1 \leq ch\,|w|_2.$$

The next property of T^h related to B^h is not quite so straightforward.

LEMMA 2 *If* $w \in H^1(2\pi)$,

$$|T^h(I-B^h)T^h w|_{-2} \leq ch^3\,|w|_1.$$

In order to prove this lemma we need to present another property of the operator B^h, which is that for $p,q \in \Lambda_h$,

$$(B^h e^{ip\cdot}, e^{iq\cdot})_0 = 2\pi\beta_p\,|\underline{p}|^{-1}\lambda_p^{-1}\bar{\alpha}_q\delta_{p,q}.$$

Here $\beta_p = (-1)^p e^{ip\sigma_1}$, $\alpha_q = 2(qh)^{-1}\sin\tfrac{1}{2}qh(-1)^q e^{iq\sigma_1}$,

$$\underline{p} = \begin{cases} p & \text{for } p \neq 0, \\ 1 & \text{for } p = 0, \end{cases}$$

$$\lambda_p = \begin{cases} 1 & \text{for } p = 0, \\ \dfrac{2n}{\pi}\sin\dfrac{\pi|p|}{2n}\displaystyle\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{(2nk+|p|)^2} + \frac{1}{(2nk+2n-|p|)^2}\right) & \text{for } p \neq 0. \end{cases}$$

λ_p is the eigenvalue of the operator Q^hA on space $\mathcal{S}^h(\Pi)$, and $Q^hAQ^he^{ipt} = \lambda_p Q^h e^{ipt}$ (see [15]).

Proof. It is clear that

$$|T^h(I-B^h)T^hw|_{-2}^2 = \sum_{q\in\Lambda_h}|\underline{q}|^{-4}\,|((I-B^h)T^hw, e^{iq\cdot})_0|^2.$$

By using the property of B^h above, we find that

$$|((I-B^h)T^hw, e^{iq\cdot})_0| = \surd(2\pi)|\hat{w}(q)|\,|1-\beta_q\overline{\alpha_q}\,|\underline{q}|^{-1}\lambda_q^{-1}|$$

and, through a simple calculation, we have

$$|1-\beta_q\overline{\alpha_q}\,|\underline{q}|^{-1}\lambda_q^{-1}| = \frac{1}{n^2}\frac{|\sum_{k=1}^{\infty}(-1)^{k-1}\{[1/(k-q/n)^2]-[1/(k+q/n)^2]\}|}{|\sum_{k=0}^{\infty}(-1)^k\{[1/(kn+q)^2]+[1/(kn+n-q)^2]\}|}.$$

Since

$$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{(kn+q)^2}+\frac{1}{(kn+n-q)^2}\right) \geq \frac{1}{q^2}+\frac{1}{(n-q)^2}-\frac{1}{(n+q)^2}-\frac{1}{(2n-q)^2} \geq \frac{1}{2q^2},$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{(k-q/n)^2}-\frac{1}{(k+q/n)^2}\right) \leq \frac{1}{(1-q/n)^2}-\frac{1}{(1+q/n)^2} \leq \frac{64}{9}\frac{q}{n},$$

$$|1-\beta_q\overline{\alpha_q}\,|\underline{q}|^{-1}\lambda_q^{-1}| \leq c(q/n)^3,$$

it follows that

$$|T^h(I-B^h)T^hw|_{-2}^2 \leq c\frac{1}{n^6}\sum_{q\in\Lambda_h}|\hat{w}(q)|^2\,|\underline{q}|^2 \leq c\frac{1}{n^6}|w|_1^2,$$

as claimed.

We are now ready to state the theorem on the superconvergence of B^h.

THEOREM 5 *If* $w \in \mathrm{H}^2(2\pi)$, $v \in \mathrm{H}^2(2\pi)$, *then*

$$|((I - B^h)w, v)| \leq ch^3(|w|_2\,|v|_1 + |w|_1\,|v|_2).$$

Proof.

$$|((I - B^h)w, v)|$$
$$\leq |((I - B^h)(I - T^h)w, v)| + |((I - B^h)T^h w, T^h v)| + |((I - B^h)T^h w, (I - T^h)v)|.$$

The third part of the right-hand side is bounded by

$$|(I - B^h)T^h w|_0\,|(I - T^h)v|_0 \leq ch^3\,|T^h w|_1\,|v|_2 \leq ch^3\,|w|_1\,|v|_2.$$

The second is bounded by $|(T^h(I - B^h)T^h w, v)|$, which is bounded from Lemma 2 by $ch^3\,|w|_1\,|v|_2$. For the estimate of the first part, writing $v = Av_1$,

$$\begin{aligned}|((I - B^h)(I - T^h)w, v)| &= |((I - B^h)(I - T^h)w, Av_1)| \\ &= |((I - Q^h)A(I - B^h)(I - T^h)w, v_1)| \\ &\leq |(I - Q^h)A|_0\,|(I - B^h)(I - T^h)w|_0\,|v_1|_0 \\ &\leq ch^3\,|w|_2\,|v|_1.\end{aligned}$$

Thus the theorem follows. □

Remark. One consequence of this theorem is that $|((B_h - I)w, v)| \leq ch^3\,|w|_2\,|v|_2$, a superconvergence result which was first proved in the work of Saranen [9]. However, the present approach is much simpler.

By using this property we are now able to show the superconvergence of the collocation solution w_h^c stated in the following theorem. This theorem also shows that by applying the collocation method to equation (2) instead of (1), we restore the optimal rate of convergence in the negative norm $|\bullet|_{-2}$.

THEOREM 6 *Assume* $C_\Gamma \neq 1$. *If* $w \in \mathrm{H}_c^2(2\pi)$, *then*

$$|w_h^c - w|_{-2} \leq ch^3\,|w|_2.$$

Proof. Note that

$$Q^h K w_h^c = Q^h K w, \qquad Q^h A_c w_h^c = Q^h(Kw - A_c M w_h^c),$$
$$Q^h A w_h^c = Q^h A((I + M)w - M w_h^c), \qquad w_h^c = B^h((I + M)w - M w_h^c).$$

By Theorem 5,

$$|w_h^c - [(I + M)w - M w_h^c]|_{-2} \leq ch^3\,|(I + M)w - M w_h^c|_2,$$

and since M is bounded on $\mathrm{H}_c^2(2\pi)$, and is bounded from $\mathrm{H}_c^0(2\pi)$ into $\mathrm{H}_c^2(2\pi)$, it follows that $|(I + M)w - M w_h^c|_2 \leq c(|w|_2 + |w_h^c|_0)$. From Theorem 4, we see that $|w_h^c|_0 \leq c\,|w|_0 \leq c\,|w|_2$, so

$$|w_h^c - [(I + M)w - M w_h^c]|_{-2} \leq ch^3\,|w|_2,$$
$$|(I + M)(w_h^c - w)|_{-2} \leq ch^3\,|w|^2.$$

From the discussions around (5), $(I + M)^{-1}$ is bounded on $\mathrm{H}_c^{-2}(2\pi)$, therefore

$$\begin{aligned}|w_h^c - w|_{-2} &= |(I + M)^{-1}(I + M)(w_h^c - w)|_{-2} \\ &\leq c|(I + M)(w_h^c - w)|_{-2} \leq ch^3\,|w|_2. \quad \square\end{aligned}$$

As an application of this superconvergence result, we consider an approximation

$$u_h^c(z) = -\frac{1}{\pi}\int_0^{\pi} w_h^c(\tau)\log|z-a(\tau)|\,d\tau = -\frac{1}{2\pi}\int_{-\pi}^{\pi} w_h^c(\tau)\log|z-a(\tau)|\,d\tau$$

of the single layer potential

$$u(z) = -\frac{1}{\pi}\int_{\Gamma} g(y)\log|z-y|\,dy = -\frac{1}{\pi}\int_0^{\pi} w(\tau)\log|z-a(\tau)|\,d\tau$$

for $z \notin \Gamma$. It is clear that

$$u_h^c(z) - u(z) = (g_z, w_h^c - w),$$

and, since $g_z(t) \in \mathrm{H}_e^2(2\pi)$ for $z \notin \Gamma$, we see from Theorem 6 that

$$|u_h^c(z) - u(z)| \leq ch^3\,|w|_2 \quad \text{for } z \notin \Gamma.$$

Remark. In the collocation method for equation (2), one can also employ the trial space consisting of those smooth splines of a given degree which are even functions. The analysis is similar to that carried out above, applying known results for closed contours to the case of an open contour. The results on convergence and superconvergence for closed contours have been well established in the work of Arnold, Wendland, and Sarenen [1, 2, 9, 10]. Thus, convergence and superconvergence of collocation methods with smooth splines applied to equation (2) can be obtained without any difficulty.

6. Numerical results

In the following we present numerical experiments for solving equations (1) and (2) by the collocation procedure. We compare the direct collocation for equation (1) with the collocation for the version of cosine change of variable, i.e. for equation (2). The approximating potential is calculated by applying Simpson's rule on each subinterval.

In Tables 1, 2, and 3, we consider the equation

$$-\int_{\Gamma} g(y)\log|x-y|\,dy = 1 \quad \text{on } \Gamma = [-1, 1], \tag{8}$$

and test the midpoint collocation for this equation on the space of piecewise constant functions, where the mesh grading technique is employed. The mesh consists of the points

$$-1 + (2i/n)^q \quad (i = 1, \ldots, \tfrac{1}{2}n)$$

plus their reflections in the line $x = 0$. The grading parameter q is chosen as $q = 1$ (uniform mesh), 2, and 4. The matrix elements are calculated analytically. The single layer potential $u_h(z)$ for $z \notin \Gamma$ is calculated by

$$u(z) = \int_{\Gamma} g_h(y)\log|z-y|\,dy,$$

TABLE 1
$q = 1$

$n = \frac{2}{h}$	C_Γ^*	γ_h	$u_h(0, 1)$	γ_h	$u_h(1\cdot2, 0)$	γ_h
2	0·41929		−0·15183		−0·03250	
4	0·45790		−0·20654		0·01370	
8	0·47853	0·9039	−0·23764	0·8149	0·05090	0·3125
16	0·48917	0·9553	−0·25421	0·9091	0·07469	0·6447
32	0·49456	0·9806	−0·26277	0·9505	0·08801	0·8367
64	0·49728	0·9911	−0·26714	0·9741	0·09499	0·9326
128	0·49864	0·9957	−0·26934	0·9867	0·09854	0·9747
256	0·49932	0·9979	−0·27044	0·9932	0·10083	0·9909

TABLE 2
$q = 2$

$n = \frac{2}{h}$	C_Γ^*	γ_h	$u_h(0, 1)$	γ_h	$u_h(1\cdot2, 0)$	γ_h
2	0·41929		−0·15183		−0·03250	
4	0·47768		−0·23545		0·05064	
8	0·49412	1·8278	−0·26190	1·6608	0·08718	1·1861
16	0·49849	1·9124	−0·26908	1·8816	0·09818	1·7312
32	0·49962	1·9558	−0·27093	1·9558	0·10112	1·9077
64	0·49990	1·9782	−0·27140	1·9828	0·10187	1·9668
128	0·49997	1·9892	−0·27151	1·9927	0·10205	1·9869
256	0·49999	1·9946	−0·27154	1·9967	0·10210	1·9973

TABLE 3
$q = 4$

$n = \frac{2}{h}$	C_Γ^*	γ_h	$u_h(0, 1)$	γ_h	$u_h(1\cdot2, 0)$	γ_h
2	0·41929		−0·15183		−0·03250	
4	0·48427		−0·24319		0·07631	
8	0·49753	2·2923	−0·26702	1·9387	0·09746	2·3632
16	0·49965	2·6464	−0·27093	2·6085	0·10141	2·4221
32	0·49995	2·8126	−0·27147	2·8431	0·10202	2·6839
64	0·49999	2·8966	−0·27154	2·9018	0·10211	2·8443
128	0·50000	2·9450	−0·27155	2·9466	0·10212	2·9235
256	0·50000	2·9704	−0·27155	2·9738	0·10212	2·9625

where g_h is the collocation solution. The approximating exponent γ_h is calculated by, for example,

$$\gamma_h = \log_2 [(u_{4h} - u_{2h})/(u_{2h} - u_h)].$$

It is clear that the optimal rate of convergence $O(h^3)$ can be restored when the grading parameter q is chosen to be at least 4.

Instead of considering direct collocation for equation (1), in Table 4 we apply collocation to the cosine change of variable version of equation (8)

$$-\int_0^{\pi} w(\tau)\log|\cos t-\cos\tau|\,d\tau=1\quad \text{for } t\in[0,\pi].$$

The matrix elements are calculated by a careful Fourier expansion. The calculated capacity and potential are obtained precisely, ignoring errors from the calculation of the matrix elements and Simpson's rule for the potential. This result is much better than those in Tables 1, 2, and 3. Since the operator K maps a constant to a constant, the precisely calculated results obtained in Table 4 are

TABLE 4

$n=\frac{\pi}{h}$	$\int w_h$	$u_h(0, 1)$
2	1·442695	−0·271553
4	1·442695	−0·271553
8	1·442695	−0·271553
16	1·442695	−0·271553
32	1·442695	−0·271553
64	1·442695	−0·271553
128	1·442695	−0·271553
256	1·442695	−0·271553

TABLE 5

$n=\frac{\pi}{h}$	$\int w_h$	$u_h(0, 0{\cdot}5)$	γ_h
2	1·570796	2·593941	
4	1·570796	4·362843	
8	1·570796	4·490656	3·7907
16	1·570796	4·502151	3·4750
32	1·570796	4·503573	3·0148
64	1·570796	4·503746	3·0407
128	1·570796	4·503768	2·9911

TABLE 6

$n=\frac{\pi}{h}$	$\int w_h$	γ_h	$u_h(0, 0{\cdot}5)$	γ_h
2	2·221441		3·668386	
4	2·052344		4·743006	
8	2·012909	2·1003	4·808458	4·0372
16	2·003216	2·0245	4·815582	3·1997
32	2·000803	2·0061	4·816591	2·8191
64	2·000201	2·0015	4·816746	2·7064
128	2·000050	2·0004	4·816773	2·4921
256	2·000013	2·0001	4·816779	2·3281

not surprising. By understanding this, the equation considered in Table 4 seems trivial. Hence we consider in Table 5 a different equation

$$-\frac{1}{\pi}\int_0^\pi w(\tau)\log|2e^{-1}(\cos t-\cos\tau)|\,d\tau=\sin^2 t \quad \text{for } t\in[0,\pi].$$

Not being a constant, the right-hand side of this equation is a smooth function. As we expected in Theorem 6, the optimal rate of convergence $O(h^3)$ is obtained. In Table 6, we try considering $|\sin t|$ as the right-hand side of the above equation: $|\sin t|$ is only a function of $H_e^{1+\varepsilon}(2\pi)$ $(0<\varepsilon<\frac{1}{2})$. It is clear that for this case the optimal rate of convergence $O(h^3)$ is not reached.

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