A Preconditioning Strategy for Boundary Element Galerkin Methods

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The Dirichlet and Neumann problems for the Laplacian are reformulated in the usual way as boundary integral equations of the first kind with symmetric kernels. These integral equations are solved using Galerkin's method with piecewise-constant and piecewise-linear boundary elements, respectively. In both cases, the stiffness matrix is symmetric and positive-definite, and has a condition number of order N, the number of degrees of freedom. By contrast, the condition number of the product of the two stiffness matrices is bounded independently of N. Hence, we can use the Neumann stiffness matrix to precondition the Dirichlet stiffness matrix, and vice versa. © 1997 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq **13:** 283–301, 1997

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I. INTRODUCTION

Consider the Dirichlet and Neumann problems for the Laplace equation on a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^2$, with boundary $\Gamma = \partial\Omega$. Via the standard fundamental solution, we reformulate these problems as boundary integral equations of the first kind with symmetric kernels. The Dirichlet problem leads to an integral operator S with a logarithmically singular kernel, whereas the Neumann problem leads to a hypersingular operator R. The integral equations $Su = f$ and $Ru = f$ are solved using Galerkin's method, the first with piecewise constants and the second with piecewise linears. The same uniform mesh with N subintervals is used in both cases, and we put $h = 1/N$. Working with the usual nodal bases, we denote the stiffness matrices of S and R by S_h and R_h, respectively, modifying the usual definitions of S and R so as to ensure that S_h and R_h are strictly positive definite. The main result of this article is that R_h is a good preconditioner for S_h, and vice versa. More precisely, we are able to prove that, if Γ is smooth, then the condition numbers of the products $R_h S_h$ and $S_h R_h$ are bounded uniformly in h.

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Since S_h and R_h are dense, $N \times N$ matrices, direct solution of the Galerkin equations via a Cholesky factorization costs $O(N^3)$ operations. It is well known that the condition numbers of S_h and R_h are both $O(h^{-1})$, so we can solve the Galerkin equations using at most $O(h^{-1/2})$ conjugate gradient iterations [1]. With our choice of preconditioner, the number of iterations becomes independent of h, but the cost of each iteration is roughly doubled. Assuming that each matrix-vector multiplication costs $O(N^2)$ operations, we see that the preconditioner reduces the overall cost of the iterative solver from $O(N^{5/2})$ to $O(N^2)$ operations.

A fast multipole [2] or panel clustering [3] method reduces the cost of a matrix-vector multiplication to $O(N)$ operations (ignoring factors of $\log N$). Moreover, these fast methods have a setup cost that is $O(N)$, as against $O(N^2)$ if every entry of the stiffness matrix is calculated. Hence, by combining our preconditioner with one of these fast matrix-vector multiplication methods, we arrive at a solution technique with (essentially) linear complexity, i.e., one requiring only $O(N)$ operations.

It seems likely that our preconditioning strategy can be adapted to problems in three dimensions, as well as to other elliptic equations and to higher-order boundary elements. A practical virtue of the method is that it is simple to implement with existing software components.

Several authors have investigated preconditioners for various types of boundary element methods. Vavasis [4] reports numerical results for collocation methods applied to problems in three dimensions with mixed boundary conditions. Multigrid preconditioners have been studied by von Petersdorff and Stephan [5], [6] and by Bramble, Leyk, and Pasciak [7]. Tran and Stephan [8] consider additive Schwarz preconditioners. However, none of these approaches is closely related to the method studied here. We point out that if one reformulates the Dirichlet or Neumann problem as a Fredholm integral equation of the *second* kind, then the condition number of the Galerkin equations remains bounded as $N \to \infty$, but the stiffness matrix is not symmetric; see Hackbusch [9, §4.5].

The article is organized as follows. Section II describes the boundary integral formulations, and serves to introduce our notation and to motivate our preconditioning strategy. Section III is a short note explaining how the integral equations are transferred from Γ to the unit interval via a parametric representation, in such a way that an important relation between S and R is preserved. In Section IV, we set up a framework for studying the conditioning of the Galerkin stiffness matrices. These matrices turn out to be circulant if Γ is a circle, and this special case is treated in Section V. Next, in Section VI, we allow Γ to be an arbitrary smooth, closed curve, and use a perturbation argument to obtain our main result, Theorem 6.1. Sections VII and VIII discuss some numerical experiments, the results of which confirm that our preconditioning strategy works in practice, even under less restrictive assumptions on Γ and the mesh. We obtain good results when Γ is only *piecewise* smooth, and also when the mesh is graded at the corners. In the latter case, however, the entries of S_h must be scaled in a certain way.

II. BOUNDARY INTEGRAL EQUATIONS

We recall some standard facts from [10], [11], and [12] about boundary integral formulations of the Laplace equation on a Lipschitz domain,

$$\Delta u = 0 \quad \text{on } \Omega,$$

with either a Dirichlet boundary condition,

$$u = g \quad \text{on } \Gamma,$$

or else a Neumann boundary condition

$$\partial_\nu u = g \quad \text{on } \Gamma,$$

where $\partial_\nu = \partial/\partial\nu$ denotes differentiation in the direction of the outward unit normal ν.

The weak solutions of the Dirichlet and Neumann problems belong to the Sobolev space $H^1(\Omega)$. Any function $u \in H^1(\Omega)$ has a trace $\gamma u = u|_\Gamma \in H^{1/2}(\Gamma)$, and if $\Delta u = 0$ on Ω, then via the first Green identity we can define a weak normal derivative $\partial_\nu u \in H^{-1/2}(\Gamma)$. Thus, for the Dirichlet problem it is natural to assume that $g \in H^{1/2}(\Gamma)$, whereas for the Neumann problem we assume $g \in H^{-1/2}(\Gamma)$ and

$$\int_\Gamma g d\sigma = 0, \tag{2.1}$$

where $d\sigma$ is the element of arc length. The condition (2.1) is necessary for the existence of a solution to the Neumann problem, because if $u \in H^1(\Omega)$ is harmonic on Ω, then $\int_\Gamma \partial_\nu u d\sigma = 0$. Of course, the solution of the Neumann problem is unique only up to an arbitrary constant term.

The single-layer potential $\mathcal{K}_1$ and the double-layer potential $\mathcal{K}_2$ are defined by

$$\mathcal{K}_1 v(\mathbf{x}) = \frac{1}{2\pi}\int_\Gamma v(\mathbf{y}) \log \frac{\alpha}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}} \quad \text{and} \quad \mathcal{K}_2 v(\mathbf{x}) = \frac{1}{2\pi}\int_\Gamma v(\mathbf{y}) \frac{\partial}{\partial \nu_y} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}},$$

for $\mathbf{x} \in \Omega$, where $\alpha > 0$ is an arbitrary parameter. We define the boundary operators

$$Sv(\mathbf{x}) = \frac{1}{2\pi}\int_\Gamma v(\mathbf{y}) \log \frac{\alpha}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}},$$

$$Tv(\mathbf{x}) = \frac{1}{2\pi}\int_\Gamma v(\mathbf{y}) \frac{\partial}{\partial \nu_{\mathbf{y}}} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}},$$

$$T^* v(\mathbf{x}) = \frac{1}{2\pi}\frac{\partial}{\partial \nu_{\mathbf{x}}}\int_\Gamma v(\mathbf{y}) \log \frac{1}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}},$$

$$R_1 v(\mathbf{x}) = -\frac{1}{2\pi}\frac{\partial}{\partial \nu_{\mathbf{x}}}\int_\Gamma v(\mathbf{y}) \frac{\partial}{\partial \nu_{\mathbf{y}}} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} d\sigma_{\mathbf{y}}, \tag{2.2}$$

for $\mathbf{x} \in \Gamma$. It is possible to express R_1 in terms of S, as follows. Orient the curve Γ so that Ω lies on the left, and let τ be the forward unit tangent vector, so that the ordered pair (ν, τ) forms a right-handed basis for $\mathbb{R}^2$. It follows that

$$-\frac{\partial}{\partial \nu_{\mathbf{x}}}\frac{\partial}{\partial \nu_{\mathbf{y}}} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{\partial}{\partial \tau_{\mathbf{x}}}\frac{\partial}{\partial \tau_{\mathbf{y}}} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \quad \text{for } \mathbf{x}, \mathbf{y} \in \Gamma \text{ with } \mathbf{x} \neq \mathbf{y}, \tag{2.3}$$

and, after an integration by parts,

$$R_1 v = -\partial_\tau S \partial_\tau v. \tag{2.4}$$

(If Γ is smooth, then $R_1 v(\mathbf{x})$ can also be written as the Hadamard finite part integral with the symmetric kernel (2.3); see [13].) The operator T^* is the formal transpose of T with respect to the bilinear pairing

$$\langle v, w \rangle_\Gamma = \int_\Gamma v(\mathbf{x}) w(\mathbf{x}) d\sigma_{\mathbf{x}},$$

and we have

$$\langle Sv, w\rangle_\Gamma = \langle v, Sw\rangle_\Gamma, \qquad \langle R_1 v, w\rangle_\Gamma = \langle v, R_1 w\rangle_\Gamma, \quad \text{and} \quad \langle Tv, w\rangle_\Gamma = \langle v, T^* w\rangle_\Gamma,$$

if v and w are, say, restrictions to Γ of functions in $C^\infty(\mathbb{R}^2)$.

The following mappings are bounded for $-1/2 < s < 1/2$:

$$\mathcal{K}_1\colon H^{s-1/2}(\Gamma) \to H^{s+1}(\Omega), \qquad \mathcal{K}_2\colon H^{s+1/2}(\Gamma) \to H^{s+1}(\Omega),$$

$$S\colon H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \qquad R_1\colon H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma),$$

$$T^*\colon H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \qquad T\colon H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma).$$

If the boundary Γ is smooth, then S and R_1 are pseudo-differential operators of order -1 and $+1$, respectively, and the mappings above are bounded for all $s \in \mathbb{R}$; in fact, T and T^* are smoothing operators in this case.

The boundary values and normal derivatives of the single- and double-layer potentials are given by

$$\gamma\mathcal{K}_1 v = Sv \quad \text{and} \quad \partial_\nu \mathcal{K}_1 v = \tfrac{1}{2} v + T^* v, \quad \text{if } v \in H^{-1/2}(\Gamma),$$

and

$$\gamma\mathcal{K}_2 v = -\tfrac{1}{2} v + Tv \quad \text{and} \quad \partial_\nu \mathcal{K}_2 v = -R_1 v, \quad \text{if } v \in H^{1/2}(\Gamma).$$

If $u \in H^1(\Omega)$ satisfies $\Delta u = 0$ on Ω, then we have the representation formula

$$u = \mathcal{K}_1 \partial_\nu u - \mathcal{K}_2 \gamma u \quad \text{on } \Omega, \tag{2.5}$$

allowing us to compute u from a knowledge of both γu and $\partial_\nu u$. Taking the trace of both sides of (2.5), one finds that

$$S\partial_\nu u = \tfrac{1}{2}\gamma u + T\gamma u \quad \text{on } \Gamma, \tag{2.6}$$

and, taking the normal derivative of both sides, one finds that

$$R_1 \gamma u = \tfrac{1}{2}\partial_\nu u - T^* \partial_\nu u \quad \text{on } \Gamma. \tag{2.7}$$

Thus, for the Dirichlet problem, we put $f = \frac{1}{2} g + Tg$, and try to solve

$$Sv = f \quad \text{on } \Gamma. \tag{2.8}$$

If $v \in H^{-1/2}(\Gamma)$ satisfies this equation, then the function $u = \mathcal{K}_1 v - \mathcal{K}_2 g \in H^1(\Omega)$ is a weak solution of the Dirichlet problem, and $\partial_\nu u = v$. Conversely, if $u \in H^1(\Omega)$ is a weak solution of the Dirichlet problem, then the function $v = \partial_\nu u \in H^{1/2}(\Gamma)$ satisfies $Sv = f$.

To deal with the Neumann problem, we put $f = \frac{1}{2} g - T^* g$, and try to solve

$$R_1 v = f \quad \text{on } \Gamma. \tag{2.9}$$

If $v \in H^{1/2}(\Gamma)$ satisfies this equation, then the function $u = \mathcal{K}_1 g - \mathcal{K}_2 v \in H^1(\Omega)$ is a weak solution of the Neumann problem, and $\gamma u = v$. Conversely, if $u \in H^1(\Omega)$ is a weak solution of the Neumann problem, then the function $v = \gamma u \in H^{1/2}(\Gamma)$ satisfies $R_1 v = f$.

The operators S and R_1 inherit from Δ the property of being strongly elliptic. In fact, if the parameter α satisfies

$$\alpha > \text{ logarithmic capacity of } \Gamma,$$

then

$$\langle Sv, v\rangle_\Gamma \geq c\|v\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } v \in H^{-1/2}(\Gamma);$$

for an explanation of the logarithmic capacity, see Hille [14]. The hypersingular operator R_1 annihilates constants, but, for any $\beta > 0$, the modified operator

$$Rv = R_1 v + \beta\langle v, 1\rangle_\Gamma$$

satisfies

$$\langle Rv, v\rangle_\Gamma \geq c\|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Hence, the operators $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $R : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ have bounded inverses.

For the Neumann problem, we claim that the solution $v \in H^{1/2}(\Gamma)$ of the modified boundary integral equation

$$Rv = f \quad \text{on } \Gamma \tag{2.10}$$

is also a solution of the original Eq. (2.9). Indeed, taking $u = 1$ in (2.6), we see that $T1 = -\frac{1}{2}$, so by (2.1) the right-hand side $f = \frac{1}{2}g - T^*g$ satisfies

$$\langle f, 1\rangle_\Gamma = \langle g, 1\rangle_\Gamma = 0.$$

Thus, (2.10) implies $\beta\langle v, 1\rangle_\Gamma = Rv - R_1 v = f - R_1 v$ and so, by integrating this equation over Γ, we have $\langle v, 1\rangle_\Gamma = 0$ and, hence, $R_1 v = Rv = f$, as claimed.

To motivate our preconditioning scheme, consider the operators RS and SR. By taking $u = \mathcal{K}_1 v$ in (2.7), and $u = \mathcal{K}_2 v$ in (2.6), one finds that

$$R_1 S v = \left(\tfrac{1}{2} - T^*\right)\left(\tfrac{1}{2} + T^*\right) v \quad \text{and} \quad SR_1 v = \left(\tfrac{1}{2} + T\right)\left(\tfrac{1}{2} - T\right) v.$$

(Notice that each formula follows from the other by duality.) It is known from [12] that $\frac{1}{2} \pm T : L_2(\Gamma) \to L_2(\Gamma)$ is a Fredholm operator with zero index, so the operators

$$RS : L_2(\Gamma) \to L_2(\Gamma) \quad \text{and} \quad SR : L_2(\Gamma) \to L_2(\Gamma)$$

are invertible. If Γ is smooth, then T and T^* actually have smooth kernels, so $S^{-1} - 4R$ and $R^{-1} - 4S$ are compact operators on $L_2(\Gamma)$. In this sense, $4R$ is an approximate inverse for S, so it is reasonable to hope that the stiffness matrix R_h will be a good preconditioner for S_h, and vice versa.

III. INTEGRAL EQUATIONS ON THE UNIT INTERVAL

Assume now that the boundary Γ is smooth, and take a 1-periodic, regular parametric representation $\gamma : \mathbb{R} \to \Gamma$. In other words, γ is a C^∞ function from $\mathbb{R}$ onto Γ, the restriction of γ to the half-open interval [0, 1) is one-one, and

$$\gamma(x + 1) = \gamma(x) \quad \text{and} \quad |\gamma'(x)| \neq 0 \quad \text{for } x \in \mathbb{R}.$$

With any function $v : \Gamma \to \mathbb{R}$ we associate two 1-periodic functions, $v_\gamma : \mathbb{R} \to \mathbb{R}$ and $\tilde{v}_\gamma : \mathbb{R} \to \mathbb{R}$, defined by

$$v_\gamma(x) = v[\gamma(x)] \quad \text{and} \quad \tilde{v}_\gamma(x) = v_\gamma(x)\frac{|\gamma'(x)|}{2\pi} \quad \text{for } x \in \mathbb{R}.$$

We then define the linear operators S_γ and R_γ by

$$S_\gamma \tilde{v}_\gamma = (Sv)_\gamma \quad \text{and} \quad \langle R_\gamma v_\gamma, w_\gamma \rangle = \langle Rv, w \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L_2(0,1)$. In this way, the weakly singular boundary integral Eq. (2.8) is recast as an integral equation on the unit interval,

$$S_\gamma \tilde{v}_\gamma = f_\gamma,$$

and likewise the (modified) hypersingular boundary integral Eq. (2.10) is recast as

$$R_\gamma v_\gamma = 2\pi \tilde{f}_\gamma.$$

With the above definitions, we have the following identity, in which $v'_\gamma = dv_\gamma/dx$.

Lemma 3.1. *For $v, w \in H^{1/2}(\Gamma)$,*

$$\langle R_\gamma v_\gamma, w_\gamma \rangle = 4\pi^2 \beta \langle \tilde{v}_\gamma, 1 \rangle \langle \tilde{w}_\gamma, 1 \rangle + \frac{1}{2\pi} \langle S_\gamma v'_\gamma, w'_\gamma \rangle.$$

Proof. The operators R_γ and R_1 are related by

$$\langle R_\gamma v_\gamma, w_\gamma \rangle = \langle Rv, w \rangle_\Gamma = \beta \langle v, 1 \rangle_\Gamma \langle w, 1 \rangle_\Gamma + \langle R_1 v, w \rangle_\Gamma = \beta \langle 2\pi \tilde{v}_\gamma, 1 \rangle \langle 2\pi \tilde{w}_\gamma, 1 \rangle + \langle R_1 v, w \rangle_\Gamma,$$

and, recalling (2.4) and integrating by parts,

$$\langle R_1 v, w \rangle_\Gamma = -\langle \partial_\tau S \partial_\tau v, w \rangle_\Gamma = \langle S \partial_\tau v, \partial_\tau w \rangle_\Gamma = \langle S_\gamma (\widetilde{\partial_\tau v})_\gamma, 2\pi (\widetilde{\partial_\tau w})_\gamma \rangle.$$

Finally,

$$2\pi (\widetilde{\partial_\tau v})_\gamma(x) = |\gamma'(x)| \tau[\gamma(x)] \cdot \text{grad } v[\gamma(x)] = \gamma'(x) \cdot \text{grad } v[\gamma(x)] = v'_\gamma(x),$$

and the result follows. ∎

From now on, we shall work only with the integral equations on the unit interval, and will, therefore, write $S = S_\gamma$ and $R = R_\gamma$.

IV. GALERKIN'S METHOD

We choose an integer $N > 1$, put $h = 1/N$ and define a uniform mesh in the unit interval,

$$x_j = jh \quad \text{for } 0 \le j \le N.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ \frac{1}{2} & \text{if } t = 0 \text{ or } 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} 1 + t & \text{if } -1 < t < 0, \\ 1 - t & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and define the 1-periodic functions

$$\phi_j(x) = \sum_{n=-\infty}^{\infty} \phi\left(\frac{x - x_{j-1}}{h} + n \right) \quad \text{and} \quad \psi_j(x) = \sum_{n=-\infty}^{\infty} \psi\left(\frac{x - x_j}{h} + n \right) \quad \text{for } 1 \le j \le N.$$

Applying Galerkin's method to the integral equation $Sv = f$, we approximate the solution v using a piecewise constant function,

$$v_h(x) = \sum_{k=1}^{N} V_k \phi_k(x),$$

satisfying

$$\langle \phi_j, Sv_h \rangle = \langle \phi_j, f \rangle \quad \text{for } 1 \le j \le N.$$

Finding v_h amounts to solving the $N \times N$ linear system

$$\sum_{k=1}^{N} \langle \phi_j, S\phi_k \rangle V_k = \langle \phi_j, f \rangle \quad \text{for } 1 \le j \le N. \tag{4.1}$$

Similarly, we approximate the solution of $Rv = f$ by a piecewise linear function v_h satisfying $\langle \psi_j, Rv_h \rangle = \langle \psi_j, f \rangle$ for $1 \le j \le N$, i.e., we solve the $N \times N$ linear system

$$\sum_{k=1}^{N} \langle \psi_j, R\psi_k \rangle V_k = \langle \psi_j, f \rangle \quad \text{for } 1 \le j \le N. \tag{4.2}$$

For $s \in \mathbb{R}$, let H^s denote the usual 1-periodic Sobolev space of order s. We put $H_h^0 = \mathbb{R}^N$, and equip this space with the normalized ℓ_2 inner product,

$$\langle V, W \rangle_h = h \sum_{j=1}^{N} V_j W_j \quad \text{for } V = (V_1, \ldots, V_N) \quad \text{and} \quad W = (W_1, \ldots, W_N).$$

We define the prolongation operators

$$p_h : H_h^0 \to H^0 \quad \text{and} \quad q_h : H_h^0 \to H^1$$

by

$$p_h V(x) = \sum_{j=1}^{N} V_j \phi_j(x) \quad \text{and} \quad q_h V(x) = \sum_{j=1}^{N} V_j \psi_j(x),$$

and also the transposed operators

$$p_h^* : H^0 \to H_h^0 \quad \text{and} \quad q_h^* : H^{-1} \to H_h^0,$$

given by

$$(p_h^* w)_j = \frac{1}{h} \langle \phi_j, w \rangle \quad \text{and} \quad (q_h^* w)_j = \frac{1}{h} \langle \psi_j, w \rangle \quad \text{for } 1 \le j \le N,$$

so that

$$\langle p_h V, w \rangle = \langle V, p_h^* w \rangle_h \quad \text{and} \quad \langle q_h V, w \rangle = \langle V, q_h^* w \rangle_h.$$

The linear systems (4.1) and (4.2) can then be written as

$$S_h V = p_h^* f \quad \text{and} \quad R_h V = q_h^* f,$$

where

$$S_h = p_h^* S p_h : H_h^0 \to H_h^0 \quad \text{and} \quad R_h = q_h^* R q_h : H_h^0 \to H_h^0.$$

Viewed as matrices, the jk-entries of S_h and R_h are $h^{-1}\langle \phi_j, S\phi_k \rangle$ and $h^{-1}\langle \psi_j, R\psi_k \rangle$, respectively.

V. SPECIAL CASE

In this section, we treat the special case when Γ is the unit circle. Via the obvious parametric representation $\gamma(x) = (\cos 2\pi x, \sin 2\pi x)$, the integral operators take the form

$$Sv(x) = \int_0^1 v(y) \log \frac{\alpha}{|2 \sin \pi(x-y)|}\, dy,$$

$$Rv(x) = 4\pi^2 \beta \int_0^1 v(y)\, dy - \frac{1}{2\pi}(Sv')'(x),$$

where $v'(x) = dv/dx$; see Section III. The kernel of S has the Fourier cosine expansion

$$\log \frac{\alpha}{|2 \sin \pi x|} = \log \alpha + \sum_{m=1}^{\infty} \frac{1}{m} \cos 2\pi m x,$$

and, therefore, if we write the complex Fourier series of v as

$$v(x) = \sum_{m=-\infty}^{\infty} \hat{v}(m) e^{i2\pi m x} \quad \text{where} \quad \hat{v}(m) = \int_0^1 e^{-i2\pi m x} v(x)\, dx,$$

then

$$Sv(x) = \sum_{m=-\infty}^{\infty} \sigma_S(m) \hat{v}(m) e^{i2\pi m x} \quad \text{and} \quad Rv(x) = \sum_{m=-\infty}^{\infty} \sigma_R(m) \hat{v}(m) e^{i2\pi m x},$$

where the symbols of S and R are given by

$$\sigma_S(m) = \begin{cases} \log \alpha & \text{if } m = 0, \\ \frac{1}{2|m|} & \text{if } m \neq 0, \end{cases} \quad \text{and} \quad \sigma_R(m) = \begin{cases} 4\pi^2 \beta & \text{if } m = 0, \\ |m|\pi & \text{if } m \neq 0. \end{cases} \tag{5.1}$$

We choose $\alpha > 1$ and $\beta > 0$ to ensure that S and R are positive-definite.

In the definitions of x_j, ϕ_j, and ψ_j, it is convenient to allow j to be *any* integer, so that

$$x_{j+N} = x_j + 1, \quad \phi_{j+N} = \phi_j, \quad \text{and} \quad \psi_{j+N} = \psi_j.$$

The operators S and R are translation-invariant, because they have 1-periodic, convolution kernels, so

$$h^{-1} \langle \phi_j, S\phi_k \rangle = a_{j-k} \quad \text{and} \quad h^{-1} \langle \psi_j, R\psi_k \rangle = b_{j-k},$$

where

$$a_j = h^{-1} \langle \phi_j, S\phi_0 \rangle \quad \text{and} \quad b_j = h^{-1} \langle \psi_j, R\psi_0 \rangle \quad \text{for } j \in \mathbb{Z}.$$

Obviously, $a_{j+N} = a_j$ and $b_{j+N} = b_j$, and, therefore, the stiffness matrices are circulant. Hence,

$$S_h W_k = \lambda_k W_k \quad \text{and} \quad R_h W_k = \mu_k W_k \quad \text{for } -N/2 \leq k < N/2, \tag{5.2}$$

where the components of the k-th eigenvector W_k are

$$(W_k)_j = e^{i2\pi k x_j} \quad \text{for } 1 \leq j \leq N,$$

and the corresponding eigenvalues are

$$\lambda_k = \sum_{j=1}^{N} e^{-i2\pi k x_j} a_j \quad \text{and} \quad \mu_k = \sum_{j=1}^{N} e^{-i2\pi k x_j} b_j.$$

We point out that the eigenfunctions of S and R are just the complex monomials $w_k(x) = e^{i2\pi kx}$, and the symbols give the corresponding eigenvalues:

$$Sw_k(x) = \sigma_S(k)w_k(x) \quad \text{and} \quad Rw_k(x) = \sigma_R(k)w_k(x) \quad \text{for } k \in \mathbb{Z}.$$

Thus, one expects λ_k and μ_k to converge to $\sigma_S(k)$ and $\sigma_R(k)$, respectively, if $h \to 0$ holding k fixed.

Lemma 5.1. *If Γ is the unit circle, then the eigenvalues of the stiffness matrices S_h and R_h are given by*

$$\lambda_k = \begin{cases} \sigma_S(0) & \text{if } k = 0, \\ \sigma_S(k)F_2(kh) & \text{if } k \neq 0, \end{cases} \quad \textit{and} \quad \mu_k = \begin{cases} \sigma_R(0) & \text{if } k = 0, \\ \sigma_R(k)F_4(kh) & \text{if } k \neq 0, \end{cases}$$

where

$$-\frac{N}{2} \leq k < \frac{N}{2} \quad \textit{and} \quad F_r(y) = \left(\frac{\sin \pi y}{\pi y}\right)^r \left(1 + \sum_{m \neq 0} \frac{|y|^3}{|m+y|^3}\right).$$

Proof. The Fourier *transform* of the (non-periodic) function ϕ is

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{-i2\pi\xi t}\phi(t)\,dt = e^{-i\pi\xi}\frac{\sin \pi\xi}{\pi\xi},$$

and so the Fourier *coefficients* of ϕ_j are

$$\hat{\phi}_j(m) = he^{-i2\pi m x_{j-1}}\hat{\phi}(mh) \quad \text{for } m \in \mathbb{Z}.$$

Since $\widehat{Sv}(m) = \sigma_S(m)\hat{v}(m)$, Parseval's formula gives

$$a_j = h^{-1}\sum_{m=-\infty}^{\infty} \overline{\hat{\phi}_j(m)}\widehat{S\phi_0}(m) = h\sum_{m=-\infty}^{\infty} \sigma_S(m)e^{i2\pi m x_j}|\hat{\phi}(mh)|^2,$$

and this series is absolutely convergent, so

$$\lambda_k = \sum_{m=-\infty}^{\infty} \sigma_S(m)|\hat{\phi}(mh)|^2 h\sum_{j=1}^{N} e^{i2\pi(m-k)x_j}.$$

The inner sum satisfies

$$h\sum_{j=1}^{N} e^{i2\pi(m-k)x_j} = \begin{cases} 1 & \text{if } m \equiv k \bmod N, \\ 0 & \text{otherwise,} \end{cases}$$

implying that

$$\lambda_k = \sum_{m \equiv k \bmod N} \sigma_S(m)|\hat{\phi}(mh)|^2.$$

If $m = k + lN$ and $k \neq 0$, then

$$\sigma_S(m)|\hat{\phi}(mh)|^2 = \sigma_S(k)\left(\frac{\sin \pi kh}{\pi kh}\right)^2 \frac{|kh|^3}{|l+kh|^3},$$

and the formula for λ_k follows. A similar argument shows that

$$\hat{\psi}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2, \quad \hat{\psi}_j(m) = he^{-i2\pi m x_j}\hat{\psi}(mh), \quad \mu_k = \sum_{m \equiv k \bmod N} \sigma_R(m)|\hat{\psi}(mh)|^2,$$

and

$$\sigma_R(m)|\hat{\psi}(mh)|^2 = \sigma_R(k)\left(\frac{\sin \pi kh}{\pi kh}\right)^4 \frac{|kh|^3}{|l+kh|^3},$$

completing the proof. ∎

We can now prove the main result for this section.

Theorem 5.1. *If Γ is the unit circle, then $R_hS_h = S_hR_h$ and*

$$c\|V\|_{H_h^0} \le \|R_hS_hV\|_{H_h^0} \le C\|V\|_{H_h^0} \quad \textit{for all } V \in H_h^0,$$

where c and C are positive constants, independent of h.

Proof. We see from (5.2) that

$$R_hS_hW_k = \mu_k\lambda_kW_k = S_hR_hW_k \quad \text{for } -N/2 \le k < N/2,$$

so $R_hS_h = S_hR_h$ is symmetric and positive definite, and it suffices to show that

$$c \le \mu_k\lambda_k \le C \quad \text{for } -N/2 \le k < N/2.$$

In fact, by (5.1) and Lemma 5.1,

$$\mu_k\lambda_k = \begin{cases} 4\pi^2\beta\log\alpha & \text{if } k = 0, \\ (\pi/2)F_2(kh)F_4(kh) & \text{if } k \ne 0, \end{cases}$$

and it is easily verified that

$$(\pi/2)^{-r} \le F_r(y) \le 2(1 + 3^{-3} + 5^{-3} + 7^{-3} + \cdots) \quad \text{for } -\tfrac{1}{2} \le y \le \tfrac{1}{2} \text{ and } r \ge 0,$$

so we can choose c and C independent of h. ∎

For any invertible linear map $A : H_h^0 \to H_h^0$, let

$$\kappa(A) = \|A\|\,\|A^{-1}\|$$

denote the condition number of A induced by the norm of H_h^0. (If A is viewed as a matrix, then $\kappa(A)$ is just the usual ℓ_2 condition number.) Theorem 5.1 shows that $\kappa(R_hS_h) = \kappa(S_hR_h)$ remains bounded as $h \to 0$. However, for S_h and R_h separately, we have

$$ch\|V\|_{H_h^0} \le \|S_hV\|_{H_h^0} \le \|V\|_{H_h^0}$$

and

$$c\|V\|_{H_h^0} \le \|R_hV\|_{H_h^0} \le Ch^{-1}\|V\|_{H_h^0}$$

for all $V \in H_h^0$, implying that

$$\kappa(S_h) = O(h^{-1}) \quad \text{and} \quad \kappa(R_h) = O(h^{-1}),$$

as stated in the Introduction.

VI. PERTURBATION ARGUMENT

In this section, we consider the case when Γ is an arbitrary smooth and closed curve. The boundary integral operators S and R can now be represented as

$$S = S_0 + K_S \quad \text{and} \quad R = R_0 + K_R,$$

where S_0 and R_0 are the integral operators for the case when Γ is the unit circle, and K_S and K_R are smoothing operators, i.e., K_S and K_R have C^∞ 1-periodic kernels; see [15]. We define

$$A = RS, \quad r_h = q_h p_h^*, \quad A^h = R r_h S, \quad A_h = q_h^* A^h p_h,$$

with p_h, q_h, p_h^* and q_h^* as in Section IV. The operator A_h is the discrete form of A, i.e.,

$$A_h = R_h S_h.$$

We can also represent A, A^h, and A_h as

$$\begin{array}{llllll} A &= A_0 + K & \text{with } A_0 &= R_0 S_0 & \text{and } K &= K_R S_0 + R K_S, \\ A^h &= A_0^h + K^h & \text{with } A_0^h &= R_0 r_h S_0 & \text{and } K^h &= K_R r_h S_0 + R r_h K_S, \\ A_h &= A_{0,h} + K_h & \text{with } A_{0,h} &= q_h^* A_0^h p_h & \text{and } K_h &= q_h^* K^h p_h. \end{array}$$

Our analysis uses four technical lemmas.

Lemma 6.1. *The operators $q_h^* : H^0 \to H_h^0$ and $r_h : H^1 \to H^1$ satisfy the uniform bounds*

$$\|q_h^* v\|_{H_h^0} \le 2\|v\|_{H^0} \quad \textit{for } v \in H^0,$$

and

$$\|r_h v\|_{H^1} \le 4\|v\|_{H^1} \quad \textit{for } v \in H^1.$$

Proof. Let $v \in H^0$. From the definitions of q_h^* and the norm in H_h^0, we have

$$\|q_h^* v\|_{H_h^0}^2 = h \sum_{j=1}^N |(q_h^* v)_j|^2 = \frac{1}{h} \sum_{j=1}^N \left(\int_{x_{j-1}}^{x_{j+1}} v(x) \psi_j(x)\, dx \right)^2.$$

A quick calculation shows that $\|\psi_j\|_{H^0}^2 = 2h/3$, so, by the Cauchy–Schwarz inequality,

$$\|q_h^* v\|_{H_h^0}^2 \le \frac{2}{3} \sum_{j=1}^N \int_{x_{j-1}}^{x_{j+1}} |v(x)|^2\, dx = \frac{4}{3} \|v\|_{H^0}^2 \le (2\|v\|_{H^0})^2,$$

proving the bound for q_h^*.

Now let $v \in H^1$, and denote the interval (x_{j-1}, x_j) by I_j. We have

$$(r_h v)(x) = (q_h p_h^* v)(x) = \sum_{j=1}^N (p_h^* v)_j \psi_j(x) = \frac{1}{h} \sum_{j=1}^N \left(\int_{I_j} v(y)\, dy \right) \psi_j(x), \tag{6.1}$$

so if $x \in I_k$ then, by the Cauchy–Schwarz inequality,

$$|(r_h v)(x)|^2 \le \frac{2}{h^2} \sum_{j=k-1}^{k} |\psi_j(x)|^2 \left(\int_{I_j} v(y)\, dy \right)^2 \le \frac{2}{h} \sum_{j=1}^N |\psi_j(x)|^2 \int_{I_j} |v(y)|^2\, dy,$$

and, thus,

$$\|r_h v\|_{H^0}^2 \le \frac{2}{h}\sum_{j=1}^{N}\|\psi_j\|_{H^0}^2\int_{I_j}|v(y)|^2\,dy = \frac{4}{3}\|v\|_{H^0}^2.$$

To estimate the derivative $(r_h v)'$, we note that, if $x \in I_k$, then $\sum_{j=k-1}^{k}\psi_j'(x) = 0$ and so

$$\begin{aligned}|(r_h v)'(x)|^2 &= \left|\frac{1}{h}\sum_{j=k-1}^{k}\psi_j'(x)\int_{I_j}[v(y)-v(x)]\,dy\right|^2\\ &\le \frac{2}{h^2}\sum_{j=k-1}^{k}|\psi_j'(x)|^2\left(\int_{I_j}[v(y)-v(x)]\,dy\right)^2.\end{aligned}$$

Next, for $x \in I_k$ and $j = k-1$ or k,

$$\begin{aligned}\left(\int_{I_j}[v(y)-v(x)]\,dy\right)^2 &\le h\int_{I_j}|v(y)-v(x)|^2\,dy = h\int_{I_j}\left|\int_x^y v'(t)\,dt\right|^2\,dy\\ &\le 2h^2\int_{I_j}\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt\,dy = 2h^3\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt,\end{aligned}$$

giving

$$\begin{aligned}\int_{I_k}|(r_h v)'(x)|^2\,dx &\le 4h\left(\sum_{j=k-1}^{k}\int_{I_k}|\psi_j'(x)|^2\,dx\right)\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt\\ &= 8\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt.\end{aligned}$$

Summing over k, we deduce that $\|(r_h v)'\|_{H^0}^2 \le 16\|v'\|_{H^0}^2 = (4\|v'\|_{H^0})^2$. ∎

Lemma 6.2. *The operators $r_h = q_h p_h^*$ and $r_h^* = p_h q_h^*$ have the approximation properties*

$$\|v - r_h v\|_{H^0} \le 4h\|v'\|_{H^0} \quad \textit{for } v \in H^1,$$

and

$$\|v - r_h^* v\|_{H^{-1}} \le 4h\|v\|_{H^0} \quad \textit{for } v \in H^0.$$

Proof. Let $v \in H^1$. If $x \in I_k$, then $\sum_{j=k-1}^{k}\psi_j(x) = 1$ and so, by (6.1),

$$v(x) - r_h v(x) = \frac{1}{h}\sum_{j=k-1}^{k}\psi_j(x)\int_{I_j}[v(x)-v(y)]\,dy.$$

Repeating the arguments of the second half of the proof of Lemma 6.1 (but with ψ_j in place of ψ_j'), we see that

$$\begin{aligned}\int_{I_k}|v(x)-r_h v(x)|^2\,dx &\le 4h\left(\sum_{j=k-1}^{k}\int_{I_k}|\psi_j(x)|^2\,dx\right)\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt\\ &= \frac{8h^2}{3}\int_{I_{k-1}\cup I_k}|v'(t)|^2\,dt.\end{aligned}$$

After summing over k, we obtain $\|v - r_h v\|_{H^0}^2 \le (16h^2/3)\|v'\|_{H^0}^2 \le (4h\|v'\|_{H^0})^2$, as required. The estimate for $\|v - r_h^* v\|_{H^{-1}}$ follows at once by duality. ∎

Lemma 6.3. *The operator A^h converges to A in $\mathcal{L}(H^0, H^{-1})$; in fact,*

$$|\langle (A - A^h)v, w\rangle| \le Ch\|v\|_{H^0}\|w\|_{H^1} \quad \textit{for } v \in H^0 \textit{ and } w \in H^1.$$

Proof. Since $A^* = SR$ and $(A^h)^* = Sr_h^* R$,

$$|\langle (A - A^h)v, w\rangle| = |\langle v, (SR - Sr_h^* R)w\rangle| \le \|v\|_{H^0}\|S(Rv - r_h^* Rw)\|_{H^0},$$

and because $S : H^{-1} \to H^0$ and $R : H^1 \to H^0$ are bounded, it follows from Lemma 6.2 that

$$\|S(Rv - r_h^* Rw)\|_{H^0} \le C\|Rw - r_h^* Rw\|_{H^{-1}} \le Ch\|Rw\|_{H^0} \le Ch\|w\|_{H^1},$$

yielding the desired estimate. ∎

We shall use the symbol $\rightharpoonup$ to indicate weak convergence.

Lemma 6.4. *For any sequence $\{v_h\}$ in H^0,*

$$v_h \rightharpoonup 0 \textit{ in } H^0 \quad \textit{implies} \quad (K - K^h)v_h \to 0 \textit{ in } H^0.$$

Proof. The triangle inequality gives

$$\|(K - K^h)v_h\|_{H^0} \le \|K_R(S_0 v_h - r_h S_0 v_h)\|_{H^0} + \|R(K_S v_h - r_h K_S v_h)\|_{H^0}.$$

The operators $K_R : H^0 \to H^0$ and $S_0 : H^0 \to H^1$ are bounded, so, by Lemma 6.2,

$$\|K_R(S_0 v_h - r_h S_0 v_h)\|_{H^0} \le Ch\|S_0 v_h\|_{H^1} \le Ch\|v_h\|_{H^0},$$

and $R : H^1 \to H^0$ is bounded, so, by Lemma 6.1,

$$\|R(K_S v_h - r_h K_S v_h)\|_{H^0} \le C\|K_S v_h - r_h K_S v_h\|_{H^1} \le C\|K_S v_h\|_{H^1}.$$

If $v_h \rightharpoonup 0$ in H^0, then $\|v_h\|_{H^0} \le C$ by the uniform boundedness principle, and $K_S v_h \to 0$ in H^1, because $K_S : H^0 \to H^1$ is compact. Hence, $\|(K - K^h)v_h\|_{H^0} \to 0$. ∎

We are now ready to prove the main result of the article.

Theorem 6.1. *If Γ is a smooth and closed curve, then*

$$c\|V\|_{H_h^0} \le \|R_h S_h V\|_{H_h^0} \le C\|V\|_{H_h^0} \quad \textit{for all } V \in H_h^0,$$

where c and C are positive constants, independent of h.

Proof. To prove the left-hand inequality, suppose for a contradiction that there is a sequence $V_h \in H_h^0$ satisfying

$$\|A_h V_h\|_{H_h^0} \to 0 \text{ as } h \to 0 \quad \text{but} \quad \|V_h\|_{H_h^0} = 1 \text{ for all } h.$$

Since $\|p_h V_h\|_{H^0} = \|V_h\|_{H_h^0} = 1$, the weak compactness of the unit sphere in H^0 implies the existence of $v \in H^0$ and a subsequence of $p_h V_h$, which we again denote by $p_h V_h$, such that

$$p_h V_h \rightharpoonup v \text{ in } H^0. \tag{6.2}$$

We claim that

$$A p_h V_h \rightharpoonup 0 \text{ in } H^0. \tag{6.3}$$

Let $w \in H^1$ and write $\langle Ap_hV_h, w\rangle = I + II + III$, where

$$I = \langle Ap_hV_h, w - r_hw\rangle, \qquad II = \langle (A - A^h)p_hV_h, r_hw\rangle, \qquad III = \langle A^hp_hV_h, r_hw\rangle.$$

The operator $A : H^0 \to H^0$ is bounded, so, by Lemma 6.2,

$$|I| \le \|Ap_hV_h\|_{H^0}\|w - r_hw\|_{H^0} \le Ch\|p_hV_h\|_{H^0}\|w\|_{H^1} = Ch\|w\|_{H^1}.$$

By Lemmas 6.1 and 6.3,

$$|II| \le Ch\|p_hV_h\|_{H^0}\|r_hw\|_{H^1} \le Ch\|w\|_{H^1},$$

and because $r_h^* = p_hq_h^*$,

$$|III| = |\langle p_hq_h^*A^hp_hV_h, w\rangle| = |\langle p_hA_hV_h, w\rangle| \le \|p_hA_hV_h\|_{H^0}\|w\|_{H^0} = \|A_hV_h\|_{H_h^0}\|w\|_{H^0}.$$

Thus,

$$\lim_{h\to 0}\langle Ap_hV_h, w\rangle = 0 \quad \text{for each } w \in H^1.$$

Since $\|Ap_hV_h\|_{H^0} \le C$ and H^1 is dense in H^0, we deduce that $\langle Ap_hV_h, w\rangle \to 0$ for each $w \in H^0$, or, in other words, (6.3) holds.

The operator $A : H^0 \to H^0$ is bounded, so (6.2) implies that $Ap_hV_h \rightharpoonup Av$ in H^0. Thus, $Av = 0$, so $v = 0$, and (6.2) reduces to

$$p_hV_h \rightharpoonup 0 \text{ in } H^0.$$

Applying Lemma 6.4 and using the compactness of $K : H^0 \to H^0$, we see that

$$K^hp_hV_h = (K^h - K)p_hV_h + Kp_hV_h \to 0 \quad \text{in } H^0.$$

By Lemma 6.1,

$$\|K_hV_h\|_{H_h^0} = \|q_h^*K^hp_hV_h\|_{H_h^0} \le 2\|K^hp_hV_h\|_{H^0},$$

so both $\|A_hV_h\|_{H_h^0}$ and $\|K_hV_h\|_{H_h^0}$ tend to zero, implying that

$$\|A_{0,h}V_h\|_{H_h^0} = \|A_hV_h - K_hV_h\|_{H_h^0} \to 0 \quad \text{as } h \to 0,$$

which contradicts Theorem 5.1 because $\|V_h\|_{H_h^0} = 1$.

The right inequality is proved directly, using successively Lemma 6.1, the boundedness of $R : H^1 \to H^0$, Lemma 6.1 (again), and the boundedness of $S : H^0 \to H^1$:

$$\begin{aligned}\|R_hS_hV\|_{H_h^0} = \|q_h^*Rr_hSp_hV\|_{H_h^0} &\le 2\|Rr_hSp_hV\|_{H^0} \le C\|r_hSp_hV\|_{H^1}\\ &\le C\|Sp_hV\|_{H^1} \le C\|p_hV\|_{H^0} = C\|V\|_{H_h^0}.\end{aligned}$$

The theorem is now proved. ∎

If the conjugate gradient (CG) method is used to solve the linear system $S_hV = F$, then the iterates $V_0, V_1, V_2, \cdots$ satisfy the error bound

$$\|V_j - V\|_{S_h} \le 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^j \|V_0 - V\|_{S_h} \quad \text{for } j \ge 0, \tag{6.4}$$

where $\|W\|_{S_h} = (W^TS_hW)^{1/2}$ is the norm induced by S_h, and $\kappa = \kappa(S_h)$ is the ℓ_2 condition number of S_h; see [1]. Using instead the *preconditioned* conjugate gradient (PCG) method, with R_h as the preconditioning matrix, the error bound (6.4) is again valid, but now with $\kappa = \kappa(R_h^{1/2}S_hR_h^{1/2}) \le \kappa(R_hS_h)$. Hence, Theorem 6.1 shows that our preconditioning strategy leads to an error reduction factor that is bounded away from 1 as $h \to 0$. Furthermore, we can interchange the roles of S_h and R_h, because $\kappa(R_hS_h) = \kappa(S_hR_h)$.

VII. NUMERICAL EXPERIMENTS

In our numerical experiments we considered the Dirichlet problem for two choices of the domain and boundary data. The first domain has a smooth boundary, as required by our theory, but the second has corners. Recall that our definitions of S and R involve the parameters α and β, and that α must be chosen greater than the logarithmic capacity of Γ.

Problem 1. Let Γ be the ellipse with semi-axes a_1 and a_2, and let the Dirichlet boundary data be $g(\mathbf{x}) = \sqrt{|x_1 + x_2^2|}$. The logarithmic capacity of Γ in this case is $(a_1 + a_2)/2$. In our numerical calculations we used the parametric representation $\mathbf{x} = (a_1 \cos 2\pi x, a_2 \sin 2\pi x)$, and put $a_1 = 4, a_2 = 2, \alpha = 3.5$, and $\beta = 0.01$.

Problem 2. Let Γ be the L-shaped polygon having vertices (0, 0), (0, 1), $(-1, 1)$, $(-1, -1)$, (1, -1), and (1, 0), and define $g(z)$ to be the imaginary part of $z^{2/3}$, where $-7\pi/4 < \arg z \leq \pi/4$. Because the logarithmic capacity of a curve is not greater than its Euclidean diameter, it suffices to take $\alpha > 2\sqrt{2}$, and, in our numerical calculations, we chose $\alpha = 5$ and $\beta = 0.25$.

We solved these two problems using the piecewise-constant Galerkin method described in Section III. The linear systems were solved via the conjugate gradient method, with and without preconditioning. The iterations were stopped when the ℓ_2 norm of the residual was less than 10^{-8}, and all calculations were performed in double precision, IEEE arithmetic. We computed accurately the ℓ_2 condition numbers $\kappa(S_h), \kappa(R_h)$ and $\kappa(R_h S_h)$. In the case of the symmetric and positive definite matrices S_h and R_h, the condition numbers were found simply by computing the ratio of the largest to the smallest eigenvalue. However, $R_h S_h$ is generally nonsymmetric, so to find $\kappa(R_h S_h)$ we computed the ratio of the largest to the smallest *singular* value; see [1].

Table I shows our results for Problem 1. As expected, the condition numbers of S_h and R_h are $O(h^{-1})$, but the condition number of the product $R_h S_h$ is bounded independently of h. However, the mesh has to be quite fine before the cost of using the preconditioner is justified by the reduction in the number of iterations. It is worth mentioning that we experimented with various choices of the parameter β before selecting $\beta = 0.01$. With other choices of β it was possible to obtain significantly larger values for $\kappa(R_h)$ and $\kappa(R_h S_h)$ on the coarser grids, although the numbers of PCG iterations were essentially unchanged. We conjecture that this behavior occurs because the spectrum of $R_h^{1/2} S_h R_h^{1/2}$ consists of two parts: a single eigenvalue $\lambda_0(\beta)$, and the remaining eigenvalues that lie in an interval $[\tilde{\lambda}_{\min}, \tilde{\lambda}_{\max}]$, where $\tilde{\lambda}_{\max}$ and $\tilde{\lambda}_{\min}$, the maximum and minimum eigenvalues *excluding* $\lambda_0(\beta)$, are insensitive to the choice of β. (For the unit circle, our conjecture is proved by Lemma 5.1.) As far as the convergence of the PCG method is concerned, the condition number is effectively $\tilde{\lambda}_{\max}/\tilde{\lambda}_{\min}$, and, with $\beta = 0.01$, one has $\lambda_0(\beta) \in [\tilde{\lambda}_{\min}, \tilde{\lambda}_{\max}]$ on all the grids considered.

TABLE I. Results for Problem 1 using R_h as preconditioner.

				Iterations	
N	$\kappa(S_h)$	$\kappa(R_h)$	$\kappa(R_h S_h)$	CG	PCG
4	3.65e+00	2.61e+00	3.77e+00	1	4
8	6.03e+00	2.55e+00	3.52e+00	3	8
16	1.24e+01	4.48e+00	3.36e+00	7	10
32	2.49e+01	8.72e+00	3.48e+00	14	12
64	5.00e+01	1.74e+01	3.53e+00	20	11
128	1.00e+02	3.47e+01	3.54e+00	27	11
256	2.00e+02	6.93e+01	3.54e+00	35	10

TABLE II. Results for Problem 1 using $\tilde{R}_h$ as preconditioner.

				Iterations	
N	$\kappa(S_h)$	$\kappa(\tilde{R}_h)$	$\kappa(\tilde{R}_h S_h)$	CG	PCG
4	3.65e+00	1.61e+00	4.64e+00	1	2
8	6.03e+00	1.74e+00	4.45e+00	3	8
16	1.24e+01	3.23e+00	4.83e+00	7	11
32	2.49e+01	6.01e+00	4.99e+00	14	12
64	5.00e+01	1.17e+01	5.10e+00	20	12
128	1.00e+02	2.30e+01	5.16e+00	27	11
256	2.00e+02	4.57e+01	5.19e+00	35	10

We also investigated what happens if the matrix $R_h = [a_{jk}]$ is replaced by $\tilde{R}_h = [\tilde{a}_{jk}]$, where

$$\tilde{a}_{jk} = \begin{cases} a_{jk} & \text{if } \min_{l\in\mathbb{Z}} |j-k+lN| \le B_h, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\tilde{R}_h$ has bandwidth B_h. Empirically we found, as one would expect, that the larger the value of B_h, the fewer PCG iterations were needed, although, of course, the cost of each iteration increases with B_h. Table II shows the results obtained with $B_h = N/8$, which seemed to be an efficient choice in terms of CPU time. We also observed that $\kappa(\tilde{R}_h S_h)$ was not bounded uniformly in h, if the bandwidth B_h was held constant.

Table III gives our results for Problem 2. It seems that the presence of corners does not reduce the effectiveness of R_h as a preconditioner for S_h, encouraging us to believe that the theory could be extended to cover domains with nonsmooth boundaries.

VIII. MESH GRADING

If Γ is a polygon, then it is usually desirable to grade the mesh to compensate for corner singularities in the the solution of the integral equation; see Yan and Sloan [16], and von Petersdorff [17]. Unfortunately, such mesh grading increases the condition number of S_h, and destroys the effectiveness of our preconditioning strategy. In this section, we shall present empirical evidence suggesting that this problem can be overcome by scaling the entries of the matrix S_h in a certain way. A similar kind of scaling has been used by Bank and Scott [18] to improve the conditioning of finite element stiffness matrices.

TABLE III. Results for Problem 2 with a uniform mesh.

				Iterations	
N	$\kappa(S_h)$	$\kappa(R_h)$	$\kappa(R_h S_h)$	CG	PCG
8	1.77e+01	3.84e+00	3.99e+01	4	9
16	3.29e+01	4.29e+00	3.40e+01	8	14
32	6.30e+01	8.19e+00	3.14e+01	16	16
64	1.23e+02	1.63e+01	3.04e+01	25	15
128	2.45e+02	3.25e+01	3.03e+01	31	15
256	4.90e+02	6.51e+01	3.03e+01	43	14

TABLE IV. Mesh grading $q = 2$ with the original, unscaled stiffness matrix S_h.

				Iterations	
N	$\kappa(S_h)$	$\kappa(R_h)$	$\kappa(R_h S_h)$	CG	PCG
12	4.86e+01	4.82e+00	7.65e+01	6	13
24	3.74e+02	8.34e+00	5.12e+02	15	29
48	3.14e+03	1.66e+01	3.73e+03	39	66
96	2.55e+04	3.35e+01	2.30e+04	106	157
192	2.04e+05	6.71e+01	1.35e+05	285	338

We define a graded mesh as follows. Denote the vertices of Γ by $\mathbf{v}_1, \ldots, \mathbf{v}_M$, following a counterclockwise orientation, and put $\mathbf{v}_0 = \mathbf{v}_M$. We associate a mesh grading exponent $q_m \geq 1$ with the vertex $\mathbf{v}_m$ (taking $q_0 = q_M$), and then divide the edge $[\mathbf{v}_{m-1}, \mathbf{v}_m]$ into an even number N_m of subintervals by letting

$$\mathbf{x}_{mj} = \begin{cases} \mathbf{v}_{m-1} + \frac{1}{2}(jh_m)^{q_{m-1}}(\mathbf{v}_m - \mathbf{v}_{m-1}) & \text{for } j = 0, \ldots, N_m/2, \\ \mathbf{v}_m - \frac{1}{2}(2 - jh_m)^{q_m}(\mathbf{v}_m - \mathbf{v}_{m-1}) & \text{for } j = N_m/2, \ldots, N_m, \end{cases}$$

where $h_m = 2/N_m$. In this way, the whole polygon Γ is divided into $N = \sum_{m=1}^{M} N_m$ boundary elements.

We used such a mesh to solve Problem 2 of Section VII. In this case, the L-shaped polygon has $M = 6$ sides, and each corner is a right angle. For simplicity, we used $N/6$ subintervals on each of the six sides, with the same mesh grading exponent q at each vertex. We found experimentally that $\kappa(S_h) = O(h^{1-2q})$ and $\kappa(R_h) = O(h^{-1})$. Table IV shows the results when $q = 2$. The condition numbers of S_h and R_h seem to be $O(h^{-3})$ and $O(h^{-1})$, respectively, and our preconditioning strategy fails completely.

The remedy is to work with the scaled basis $\{h_j^{-1}\phi_j : j = 1, \ldots, N\}$ for the trial space of piecewise constant functions. Thus, we write

$$v_h(x) = \sum_{k=1}^{N} \bar{V}_k h_k^{-1} \phi_k(x),$$

and the new unknowns $\bar{V}_1, \ldots, \bar{V}_N$ must then satisfy

$$\sum_{k=1}^{N} \frac{\langle \phi_j, S\phi_k \rangle}{h_j h_k} \bar{V}_k = h_j^{-1} \langle \phi_j, f \rangle \quad \text{for } 1 \leq j \leq N;$$

TABLE V. Mesh grading $q = 2$ with the scaled stiffness matrix $\bar{S}_h$.

				Iterations	
N	$\kappa(\bar{S}_h)$	$\kappa(R_h)$	$\kappa(R_h \bar{S}_h)$	CG	PCG
12	2.58e+01	4.82e+00	3.58e+01	6	13
24	4.75e+01	8.34e+00	3.25e+01	14	17
48	9.31e+01	1.66e+01	3.26e+01	22	18
96	1.85e+02	3.35e+01	3.26e+01	30	17
192	3.70e+02	6.71e+01	3.26e+01	40	17

TABLE VI. Mesh grading $q = 3$ with the scaled stiffness matrix $\bar{S}_h$.

				Iterations	
N	$\kappa(\bar{S}_h)$	$\kappa(R_h)$	$\kappa(R_h\bar{S}_h)$	CG	PCG
12	2.58e+01	4.82e+00	3.58e+01	6	13
24	4.77e+01	1.07e+01	3.86e+01	14	18
48	9.38e+01	2.19e+01	3.64e+01	24	19
96	1.89e+02	4.49e+01	3.65e+01	34	18
192	3.79e+02	9.06e+01	3.67e+01	48	17

cf. (4.1). In this way, we arrive at the scaled stiffness matrix $\bar{S}_h = [h_j^{-1}h_k^{-1}a_{jk}]$, where $a_{jk} = \langle\phi_j, S\phi_k\rangle$, the jk-entry of the original, unscaled matrix S_h. The results shown in Table V indicate that $\kappa(\bar{S}_h) = O(h^{-1})$ and $\kappa(R_h\bar{S}_h) = O(1)$, and in Table VI we obtain similar results with the stronger mesh grading $q = 3$. Notice that for both mesh gradings the numbers of CG and PCG iterations are comparable to those appearing in Table III for the uniform mesh.

NOTE ADDED IN PROOF

The recent thesis of Steinbach [19, Kapitel 4] analyses in a different way a preconditioner similar to ours.

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