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§1. Introduction

1. In this paper we investigate the order of approximation of functions of the Sobolev-Slobodeckiĭ classes $W_p^\alpha(Q^m)$ (Q^m is the m-dimensional unit cube) by piecewise-polynomial functions. We consider uniform approximation and approximation in the metric of the spaces $L_{q,\sigma} = L_q(Q^m;\ \sigma)$, where $q \geq 1$ and σ is a finite Borel measure. For the approximation device we use functions which become polynomials of some fixed degree $l = l(\alpha)$ on each of the cubes in a suitable partitioning of Q^m into cubes. The partition according to which we construct the approximating piecewise-polynomial function is not fixed in advance; only the number of cubes in it is restricted. The partition itself is chosen (according to the formulation of the problem) as a function either of the function being approximated or of the measure σ. Naturally such a choice makes possible a better rate of approximation. Indeed, as one of the results of this article shows, the rate of approximation obtained in the uniform metric for $p\alpha > m$ is the same as that for functions which have smoothness of order α in the classical sense.

The problem investigated in this paper arose first in connection with the development of the theory of double Stieltjes operator integrals [1]–[4]. Estimates of singular numbers of integral operators acting from the space $L_{2,\sigma}$ into another space $L_{2,\tau}$ are of considerable value for this theory. The specific problem is that we need estimates not depending (in the usual sense) on the measures σ and τ. The method of approximation presented here was in fact developed with the aim of investigating integral operators.

This method of approximation found another application in the theory of double Stieltjes operator integrals in so-called "interpolation in smoothness" (see [4]). Here the basic approximation results can be used directly, without the corresponding theorems on integral operators. We note that new results in the multidimensional problem of multipliers in l_p spaces (see [3], [4]) are a consequence of "interpolation in smoothness".

It has also become clear that the proposed untraditional tool for approximation is also useful in other problems. First of all, with it we can find the exact order of ϵ-entropy of the unit sphere [1]) $SW_p^\alpha(Q^m)$, $p\alpha > m$, as a compactum in $C(Q^m)$.

It is known [5] that the ϵ-entropy of the sphere $SC^\alpha(Q^m)$ as a compactum in $C(Q^m)$ has order $\epsilon^{-m\alpha-1}$. In this case "strong" and "weak" norms are similar in nature, so that we can use relatively simple approximation methods for estimating entropy. Sometimes we are concerned with estimating the

1) The unit sphere of the Banach space X is denoted by SX.

entropy of the set $SW_p^\alpha(Q^m)$ in the metric of the space $C(Q^m)$. The norms in $W_p^\alpha(Q^m)$ and in $C(Q^m)$ are different in nature, and the norm in $C(Q^m)$ is essentially more "restrictive" than the norm in $L_p(Q^m)$. As a result the classical linear approximation methods do not lead to an exact result in this case. The approximation method proposed in this paper makes it possible to find the precise order of ϵ-entropy, which again turns out to be $\epsilon^{-m\alpha^{-1}}$. 1)

All the results obtained in this article are, in the final analysis, based on one special theorem on set functions. The proof of this theorem rests on a detailed investigation of a certain concrete algorithm for partitions of the cube Q^m. When applied to approximation theory, this algorithm generates a tool for approximations which is closely linked with the peculiarities in the formulation of the problem, which makes possible a good rate of approximations.

As has already been noted, in this paper we study the classes W_p^α. In the proofs, however, we use only a few properties of these spaces: imbedding theorems in the spaces C and L_q, and also the property of homogeneity of the principal term of the norm with respect to similarity transformations of the region. As a result of this all the results also hold for other functional classes having the same properties. In particular, this refers to the spaces H_p^α of S. M. Nikol'skiĭ and B_p^α of O. V. Besov (see e.g. [6]).

In the case $m = 1$ the theorems on approximation carry over to classes of functions of bounded β-variation. These results are of definite interest for estimating singular numbers of integral operators.

Besides the results relating to the theory of approximation, estimates of ϵ-entropy are also presented in this paper. Applications of the basic results of this article to estimating singular numbers of integral operators are treated in [7].

We give a brief summary of our work. In §2 we prove the basic theorem on partition functions. In §3 we derive theorems on approximation of functions of the class W_p^α. In §4 we obtain analogous results for functions of bounded β-variation. In §5 we give estimates of the ϵ-entropy of the set $SW_p^\alpha(Q^m)$ in $C(Q^m)$ for $p\alpha > m$ and in $L_q(Q^m)$ for $p\alpha \le m$, $q < mp(m - p\alpha)^{-1}$ (§5 can be read immediately after §3).

Results for the case $p\alpha > m$ (though more important) were published without proof in [8]

2. We introduce some concepts and notation used below.

Let R^m be the m-dimensional Euclidean space of points (vectors) $x = (x_1, \cdots, x_m)$, $|x|$ the length of the vector x. If $\kappa = (\kappa_1, \cdots, \kappa_m)$ is a multi-index (all κ_i are integers, $\kappa \ge 0$), then $x^\kappa = \prod_{i=1}^m x_i^{\kappa_i}$, $|\kappa| = \sum_{i=1}^m \kappa_i$. Let D^κ denote the differentiation operator:

$$D^{\varkappa} = \frac{\partial^{|\varkappa|}}{\partial x_1^{\varkappa_1} \dots \partial x_m^{\varkappa_m}}.$$

Let $\Delta \subset R^m$ be a cube with edges parallel to the coordinate axes, $p \ge 1$, $\alpha > 0$, $[\alpha]$ the integral part of α, $\theta = \alpha - [\alpha]$. We introduce the spaces 2) $L_p(\Delta)$, $L_\infty(\Delta)$, $C(\Delta)$, $C^\alpha(\Delta)$ in the usual manner. We use $W_p^\alpha(\Delta)$ to denote the Sobolev-Slobodeckiĭ space (see e.g. [6]). The norm in $W_p^\alpha(\Delta)$ is defined by

1) V. M. Tihomirov kindly communicated to us another proof of this result which is suitable for $m = 1$ and integral α.

2) If the cube Δ is not closed and $\bar{\Delta}$ is its closure, then we use $C(\Delta)$ and $C^\alpha(\Delta)$ for the spaces $C(\bar{\Delta})$ and $C^\alpha(\bar{\Delta})$.

$$\|u\|_{W_p^\alpha(\Delta)} = \|u\|_{L_p(\Delta)} + \|u\|_{L_p^\alpha(\Delta)}, \tag{1.1}$$

where for integral α

$$\|u\|^p_{L_p^\alpha(\Delta)} = \sum_{|\varkappa|=\alpha} \int_\Delta |D^\varkappa u|^p \, dx, \tag{1.2}$$

and for nonintegral α

$$\|u\|^p_{L_p^\alpha(\Delta)} = \sum_{|\varkappa|=[\alpha]} \int_\Delta \int_\Delta \frac{|D^\varkappa u(x) - D^\varkappa u(y)|^p}{|x-y|^{p\theta+m}} \, dx \, dy. \tag{1.3}$$

Below Q^m is the halfopen unit cube $0 \leq x_i < 1$ $(i = 1, \cdots, m)$ of the space R^m; if $\Delta = Q^m$ the symbol Δ will be omitted in the notation for spaces and their corresponding norms.

The seminorm $\|\cdot\|_{L_p^\alpha(\Delta)}$ has the property of homogeneity with respect to similarity transformations of the cube Δ. Indeed, let $\Delta = x_0 + hQ^m$, suppose $u \in W_p^\alpha$ and v is the function defined for $x \in \Delta$ by $v(x) = u(h^{-1}(x - x_0))$. Then $v \in W_p^\alpha(\Delta)$ and

$$\|v\|_{L_p^\alpha(\Delta)} = h^{mp^{-1}-\alpha} \|u\|_{L_p^\alpha}. \tag{1.4}$$

In the one-dimensional case, besides the classes $W_p^\alpha(\Delta)$ we shall also consider classes of functions of bounded β-variation. The function u given on the (possibly infinite) interval $\Delta \subset R_1$ *belongs to the class $V_\beta(\Delta)$ of functions of bounded β-variation* $(\beta \geq 1)$ *if the quantity*

$$\|u\|^\beta_{\mathring{V}_\beta(\Delta)} = \sup \sum_{k=1}^n |u(x_k) - u(x_{k-1})|^\beta$$

is finite; here the least upper bound is taken relative to all possible finite sets of points $x_0 < x_1 < \cdots < x_n$ from the interval Δ. The class $V_\beta(\Delta)$ is a Banach space relative to the norm

$$\|u\|_{V_\beta(\Delta)} = \|u\|_{\mathring{V}_\beta(\Delta)} + \sup_{x \in \Delta} |u(x)|. \tag{1.5}$$

Every function $u \in V_\beta(\Delta)$ has limits from the left and from the right at each point of the interval Δ. To simplify the exposition we normalize the functions of bounded β-variation, making them continuous from the right, and restrict consideration to classes $V_\beta(\Delta)$ for intervals Δ which are halfopen from the right. (In particular, if the left end of the interval Δ is infinite, then we include the improper end $x = -\infty$ in the interval Δ, using as the value $u(-\infty)$ the corresponding limit.) We note another obvious property of the classes V_β: their invariance with respect to monotonic replacements of the independent variable.

We recall the definition of ϵ-entropy and the n-diameter of a compact set in a normed linear space (see [5], [9]). Let X be a Banach space and suppose the set $K \subset X$ is compact. Let $\mathcal{N}_\epsilon(K; X)$ be the minimal number of elements of the ϵ-net of the set K in the metric of the space X; the *ϵ-entropy of the set K in X* is the quantity

$$\mathcal{H}_\varepsilon(K; X) = \log_2 \mathcal{N}_\varepsilon(K; X).$$

The number

$$d_n(K; X) = \inf \sup_{x \in K} \min_{y \in L_n} \|x - y\|$$

is called the *n-diameter of the set* K in X; the inf is taken relative to all possible n-dimensional hyperplanes $L_n \subset X$.

If Y is a Banach space compactly imbedded in X and $K = SY$, then we write $\mathcal{H}_\epsilon(Y; X)$ instead of $\mathcal{H}_\epsilon(SY; X)$ and $d_n(Y; X)$ instead of $d_n(SY; X)$.

We use the following notation for the constants encountered in the various estimates. Constants whose values are not essential are denoted by the letter c with no subscript, and essential constants are denoted by C with subscripts. Also we write $f \asymp g$ if $f \leq cg$ and $g \leq cf$.

§2. Theorems on partition functions

1. Let Ξ be a partition of the cube Q^m into a finite number of halfopen m-dimensional cubes Δ_k; let $|\Xi|$ be the number of cubes in the partition Ξ. We write $\Xi = \{\Delta_k\}$ $(k = 1, \cdots, |\Xi|)$ and $\Delta_k \in \Xi$. A partition Ξ' obtained from Ξ by dividing certain cubes $\Delta_k \in \Xi$ into 2^m different cubes is called an *elementary extension of the partition* Ξ. Below a basic role is played by the class $\mathfrak{R}$ of all partitions which can be obtained from the trivial partition $\Xi_0 (|\Xi_0| = 1)$ by a finite number of elementary extensions. The symbol Ξ_0 will always denote the trivial partition.

Let J be a nonnegative function of halfopen cubes $\Delta \subset Q^m$, semiadditive from below in the following sense: if the cube $\Delta \subset Q^m$ is decomposed into a finite number of disjoint cubes Δ_j, then $\Sigma_j J(\Delta_j) \leq J(\Delta)$. Let $|\Delta|$ be the Euclidean volume of the cube Δ, $a > 0$ some number. Set

$$g_a(J; \Delta) = |\Delta|^a J(\Delta) \quad (\Delta \subset Q^m)$$

and consider the following function of partitions Ξ of the cube Q^m:

$$G_a(J; \Xi) = \max_{\Delta \in \Xi} g_a(J; \Delta). \tag{2.1}$$

The basic goal of this section is to estimate the quantity $\min_{|\Xi| \leq n} G_a(J; \Xi)$, depending on n. To obtain such an estimate we construct a special sequence of partitions. Roughly speaking, this sequence is constructed by the method of "successive division." Indeed, the first step is to divide the cube Q^m into 2^m different cubes. Then we again partition that cube Δ for which the quantity $g_a(J; \Delta)$ is maximal into 2^m cubes. This process is continued and we obtain a sequence of partitions for which we can give a good estimate of the rate of decrease of the quantity (2.1). The considerations presented below follow basically from this procedure. There is a difference, however, in that we allow simultaneous division of several cubes of one partition.

Thus, let J be a given function semiadditive from below and with it associate a sequence of partitions $\{\Xi_i\}_0^\infty$ which is constructed as follows. We start with the trivial partition Ξ_0. Suppose the partition Ξ_i has already been constructed and let $\Delta_j \in \Xi_i$ $(j = 1, \cdots, s_i)$ be those cubes of the partition Ξ_i for which

$$g_a(J; \Delta_j) \geqslant 2^{-ma} G_a(J; \Xi_i) \quad (j = 1, \ldots, s_i). \tag{2.2}$$

Then as Ξ_{i+1} we take the elementary extension of the partition Ξ_i obtained by dividing these cubes. Thus s_i is the number of cubes in the partition Ξ_i which were divided in passing to Ξ_{i+1}. It is clear that $\Xi_i \in \mathfrak{R}$ $(i = 0, 1, \cdots)$.

The sequence of partitions Ξ_i obtained by this construction is denoted as follows:

$$\{\Xi_i\}_0^\infty = T_a(J).$$

For the quantities characterizing the sequence $T_a(J)$, we use the notation

$$n_i = n_i(J; a) = |\Xi_i| \quad (\Xi_i \in T_a(J)), \tag{2.3}$$

$$\delta_i = \delta_i(J; a) = G_a(J; \Xi_i) = \max_{\Delta \in \Xi_i} |\Delta|^a J(\Delta) \qquad (\Xi_i \in T_a(J)). \tag{2.4}$$

It is clear that $n_0 = 1$ and

$$n_{i+1} \leqslant 2^m n_i \quad (i = 0, 1, \ldots). \tag{2.5}$$

The basic result of this section is

Theorem 2.1. *For every function J semiadditive from below and for each natural number n there is a partition $\Xi \in \mathfrak{R}$ of the cube Q^m such that $|\Xi| \leq n$ and*

$$G_a(J; \Xi) \leqslant C_1 n^{-(a+1)} J(Q^m), \tag{2.6}$$

where the constant $C_1 = C_1(a, m)$ does not depend on the function J.

The validity of Theorem 2.1 stems from the following assertion.

Theorem 2.1′. *For every function J semiadditive from below the quantities $n_i(J; a)$ and $\delta_i(J; a)$ are related by the inequality*

$$\delta_i \leqslant C_2 n_i^{-(a+1)} \quad (i = 0, 1, \ldots), \tag{2.7}$$

where the constant $C_2 = C_2(a, m)$ does not depend on J.

We note that according to (2.5) the constant C_1 in (2.6) can be taken equal to $2^{m(a+1)}C_2$. The proof of Theorem 2.1′ is preceded by two simple lemmas.

Lemma 2.1. *Suppose the cube $\Delta \subset Q^m$ is divided into 2^m different cubes Δ_j $(j = 1, \cdots, 2^m)$. Then*

$$\max_j g_a(J; \Delta_j) \leqslant 2^{-ma} g_a(J; \Delta).$$

Lemma 2.2. *Let s be a natural number and let $x_j > 0$, $y_j > 0$ $(j = 1, \cdots, s)$ be numbers satisfying the relationships*

$$\sum_{j=1}^{s} x_j \leqslant 1, \quad \sum_{j=1}^{s} y_j \leqslant 1, \quad x_j y_j^a \geqslant b \qquad (j = 1, \ldots, s).$$

for some $a > 0$, $b > 0$. Then $b \leq s^{-(a+1)}$.

Lemma 2.1 is obvious and Lemma 2.2 is proved with the help of elementary work with extrema.

Proof of Theorem 2.1′. Without loss of generality we assume $J(Q^m) \leq 1$. We investigate certain properties of the sequences (2.3) and (2.4). It follows from Lemma 2.1 and the inequality (2.2) that

$$\delta_{i+1} \leqslant 2^{-ma}\delta_i \qquad (i = 0, 1, \ldots). \tag{2.8}$$

Another inequality for the quantities δ_i follows from Lemma 2.2. Namely, setting

$x_j = J(\Delta_j)$, $y_j = |\Delta_j|$ $(j = 1, \cdots, s_i)$ and taking account of (2.2), we find that the conditions of the lemma are satisfied for $b = 2^{-ma}\delta_i$. Hence

$$\delta_i \leqslant 2^{ma} s_i^{-(a+1)} \qquad (i = 0, 1, \ldots). \tag{2.9}$$

We note the obvious relationships

$$n_0 = 1,\ s_i \leqslant n_i,\ n_{i+1} - n_i = (2^m - 1) s_i,$$
$$n_i \leqslant 2^m \sum_{j=0}^{i-1} s_j \qquad (i = 1, 2, \ldots). \tag{2.10}$$

Let $k \geq i \geq 0$; from (2.8) and (2.9) we obtain that

$$\delta_k \leqslant 2^{-(k-i-1)ma} s_i^{-(a+1)}.$$

Hence for every i $(0 \leq i \leq k)$

$$s_i \leqslant 2^{-(k-i-1)ma(a+1)^{-1}} \delta_k^{-(a+1)^{-1}} \tag{2.11}$$

Further, for $k \geq 1$, taking account of (2.10) and (2.11), we find that

$$n_k \leqslant 2^m \delta_k^{-(a+1)^{-1}} \sum_{i=0}^{k-1} 2^{-(k-i-1)ma(a+1)^{-1}}$$

$$= 2^m \delta_k^{-(a+1)^{-1}} \sum_{j=0}^{k-1} 2^{-jma(a+1)^{-1}} < 2^m [1 - 2^{-ma(a+1)^{-1}}]^{-1} \delta_k^{-(a+1)^{-1}}.$$

Thus for $k \geq 1$

$$\delta_k \leqslant C_2 n_k^{-(a+1)}, \tag{2.12}$$

where the constant

$$C_2 = 2^{m(a+1)} [1 - 2^{-ma(a+1)^{-1}}]^{-(a+1)}$$

does not depend on J. It is also obvious that (2.12) holds for $k = 0$ too. Thus Theorem 2.1′ has been proved.

Remark 2.1. In the class of all functions semiadditive from below the order of the estimate (2.6) is exact. It is attained, for example, for the function $J(\Delta) = |\Delta|$. However, it is clear that there also exist functions J for which the estimate (2.6) can be significantly improved; in particular this is so when J is a point load type of function.

2. The considerations in this subsection are of an "entropic" nature and are used only in §5. We let $\mathfrak{J}$ denote the set of all functions J semiadditive from below which satisfy the condition $J(Q^m) \leq 1$. We combine the functions $J \in \mathfrak{J}$ which are close in a certain sense into a class and estimate the number of such classes.

Together with the sequences (2.3), (2.4) we also consider the sequence of numbers

$$\tilde{\delta}_i = \tilde{\delta}_i(J; a) = C_2 \min_{0 \leqslant j \leqslant i} [2^{-am(i-j)} n_j^{-(a+1)}] \qquad (i = 0, 1, \ldots). \tag{2.13}$$

It follows from (2.8) and (2.7) that

$$\delta_i \leqslant \tilde{\delta}_i \qquad (i = 0, 1, \ldots). \tag{2.14}$$

It is also clear that

$$\tilde{\delta}_i \leqslant C_2 n_i^{-(a+1)} \qquad (i = 0, 1, \ldots). \tag{2.15}$$

Thus the sequence (2.13) majorizes (2.4) and satisfies an inequality analogous to (2.7). Together with this the sequence $\{\tilde{\delta}_i\}$ behaves more regularly than the sequence $\{\delta_i\}$: the following inequalities hold for it:

$$2^{-(a+1)m} \tilde{\delta}_i \leqslant \tilde{\delta}_{i+1} \leqslant 2^{-am} \tilde{\delta}_i. \tag{2.16}$$

Indeed, [1]

$$\tilde{\delta}_{i+1} = C_2 \min_{0 \leqslant j \leqslant i+1} [2^{-am(i-j+1)} n_j^{-(a+1)}] = \min [2^{-am} \tilde{\delta}_i;\ C_2 n_{i+1}^{-(a+1)}].$$

The right inequality in (2.16) now follows immediately; it remains to refer to (2.5) and (2.15) to derive the left inequality.

Now let η be a fixed number $(0 < \eta \leq C_2)$. Let $T_a^\eta(J)$ denote the interval $\{\Xi_i\}_0^k$ of the sequence $T_a(J)$, where the number k is determined by the conditions

$$\tilde{\delta}_k < \eta \leqslant \tilde{\delta}_{k-1}. \tag{2.17}$$

We shall assume that the functions $J, J' \in \mathfrak{J}$ belong to the same class if and only if

$$T_a^\eta(J) = T_a^\eta(J').$$

The number of classes into which the set $\mathfrak{J}$ can be separated here is denoted by $N(a; \eta)$.

Lemma 2.3. *The estimate*

$$\log_2 N(a; \eta) \leqslant C_3 \eta^{-(a+1)^{-1}}, \qquad C_3 = C_3(a, m). \tag{2.18}$$

holds for all values of η, $0 < \eta \leq C_2$.

Proof. Let $\{\Xi_i\}_0^k$ be a finite sequence of partitions such that $\{\Xi_i\}_0^k = T_a^\eta(J)$ for at least one function $J \in \mathfrak{J}$. Then according to (2.5), (2.15) and (2.17) we obtain

$$n_k \leqslant 2^m n_{k-1} \leqslant 2^m (C_2 \tilde{\delta}_{k-1}^{-1})^{(a+1)^{-1}} \leqslant 2^m (C_2 \eta^{-1})^{(a+1)^{-1}}. \tag{2.19}$$

First of all we estimate the number of different sequences $n_i = |\Xi_i|$ $(i = 1, \cdots, k)$ whose last terms satisfy (2.19). We use n^* to denote the integral part of the number $2^m (C_2 \eta^{-1})^{(a+1)^{-1}}$. Writing n^* in the form

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1) We note that an estimate of δ_{i+1} in terms of δ_i from below is, generally speaking, impossible. Therefore we introduced the numbers $\tilde{\delta}_i$.

$$n^* = 1 + \sum_{i=1}^{k} (n_i - n_{i-1}) + (n^* - n_k),$$

we see that the number of such sequences does not exceed the number of representations of the number $n^* - 1$ in the form of a sum of positive integral terms; here representations which differ in the order of terms are considered different. The number of such representations is equal to 2^{n^*-2} (see for example [10], part 1, problem 21).

Let $\{n_i\}_0^k$ $(n_k \le n^*)$ be a fixed sequence of the form under consideration. We estimate the number of all possible sequences of partitions $\{\Xi_i\}_0^k = T_a^\eta(J)$ for which $|\Xi_i| = n_i$. For this we note that if the partition Ξ_i $(i = 0, 1, \cdots, k-1)$ is already fixed, then the partition Ξ_{i+1} is uniquely determined by which $s_i = (n_{i+1} - n_i)(2^m - 1)^{-1}$ of the cubes of the partition Ξ_i (from the overall number of cubes n_i) are decomposed in passing to Ξ_{i+1}. The number of possible variants here is equal to $\binom{n_i}{s_i} < 2^{n_i}$. Hence the number of all sequences of partitions of the form under consideration with a fixed sequence of numbers $\{n_i\}_0^k$ is less than

$$2^{n_0 + n_1 + \cdots + n_{k-1}}.$$

We note that from the definition (2.13) of the number $\tilde\delta_{k-1}$ we obtain the inequalities

$$n_i \leqslant (C_2 \tilde\delta_{k-1}^{-1})^{(a+1)^{-1}} 2^{-(k-1-i)ma(a+1)^{-1}} \quad (i = 0, 1, \ldots, k-1).$$

Hence we find that

$$\sum_{i=0}^{k-1} n_i \leqslant (C_2 \tilde\delta_{k-1}^{-1})^{(a+1)^{-1}} \sum_{i=0}^{k-1} 2^{-ima(a+1)^{-1}} < C_3' \tilde\delta_{k-1}^{-(a+1)^{-1}}, \tag{2.20}$$

where $C_3' = C_2^{(a+1)^{-1}} [1 - 2^{-ma(a+1)^{-1}}]^{-1}$. Combining the estimates and taking account of (2.17), we find that

$$\log_2 N(a; \eta) \leqslant n^* - 2 + C_3' \tilde\delta_{k-1}^{-(a+1)^{-1}} < [C_3' + 2^m C_2^{(a+1)^{-1}}] \eta^{-(a+1)^{-1}}.$$

Thus inequality (2.18) is obtained for $C_3 = C_3' + 2^m C_2^{(a+1)^{-1}}$. The lemma is proved.

3. We turn to a discussion of Theorem 2.1. The condition $a > 0$ in it is essential; in fact if, for example, J is a point load type of function, then the estimate (2.6) is not true for $a = 0$. In the one-dimensional case, nevertheless, a modification of Theorem 2.1, valid also for $a = 0$, is possible. This modification will be needed in §4 in studying functions of bounded β-variation. Here, however, we cannot even restrict ourselves to partitions of the class $\mathfrak{R}$. As a result of this the considerations of subsection 2 lose their meaning.

On the other hand, failure of the condition $\Xi \in \mathfrak{R}$ leads to an improvement of the constant in (2.6): an analogous estimate holds for $C_1 = 1$ and is attained for the function $J(\Delta) = |\Delta|$. We note that for $m = 1$ we cannot obtain the estimate (2.6) with $C_1 = 1$ even by passing to a wider class of partitions.

Thus, let $m = 1$ and $Q^1 = [0, 1)$. We write $J[x', x'')$ instead of $J([x', x''))$ for every interval $[x', x'') \subset Q^1$. In view of the condition of semiadditivity, the function $\psi(t) = J[t, x'')$ does not increase on (x', x''), is bounded, and consequently has a finite limit as $t \to x' + 0$ which we denote by $\tilde{J}[x', x'')$. Obviously $\tilde{J}[x', x'') \leq J[x', x'')$.

Theorem 2.2. *Suppose the nonnegative function* J, *semiadditive from below, of halfopen intervals* $\Delta \subset Q^1$ *is continuous from the left:*

$$J[x', t) \to J[x', x'') \quad as \quad t \to x'' - 0.$$

For every such function and arbitrary $a \geq 0$ *for each natural number* n *there is a partition* Ξ *of the interval* Q^1 *such that* $|\Xi| \leq n$ *and*

$$G_a(\tilde{J};\ \Xi) \leqslant n^{-(a+1)} J\,(Q^1). \tag{2.21}$$

The proof is by induction, assuming $J[0, 1) = 1$. For $n = 1$ the inequality (2.21) is obvious. Suppose the assertion of the theorem is true for some $n \geq 1$; we shall show that then it also holds for $n + 1$. First of all we note that if we take $[0, x_0)$ for the basic interval, then (2.21) becomes

$$\check{G}_a(\tilde{J}; \Xi) \leqslant n^{-(a+1)} x_0^a J\,[0, x_0).$$

We introduce the notation $\phi(x) = J[0, x)$ and consider the function

$$\varphi(x) - \left(\frac{n}{n+1}\right)^{a+1} x^{-a}.$$

Since this function is continuous from the left, does not increase and changes sign in the interval $(0, 1)$, there is a point $x_0 \in (0, 1)$ such that

$$\varphi(x_0) \leqslant \left(\frac{n}{n+1}\right)^{a+1} x_0^{-a} \leqslant \varphi(x_0 + 0).$$

In accord with the induction hypothesis, we can divide the interval $[0, x_0)$ into halfopen intervals $\Delta_1, \cdots, \Delta_k$ $(k \leq n)$ so that

$$\max_{i=1,\ldots,k} g_a(\tilde{J};\ \Delta_i) \leqslant n^{-(a+1)} x_0^a \varphi(x_0) \leqslant (n+1)^{-(a+1)}. \tag{2.22}$$

Further, from the inequality

$$\phi(x) + J[x, 1) \leq 1,$$

passing to the limit as $x \to x_0 + 0$, we find that

$$\varphi(x_0 + 0) + \tilde{J}[x_0, 1) \leqslant 1.$$

Hence

$$\tilde{J}[x_0, 1) \leqslant 1 - \varphi(x_0 + 0) \leqslant 1 - \left(\frac{n}{n+1}\right)^{a+1} x_0^{-a}.$$

It is an elementary matter to verify that for $x_0 \in (0, 1)$ the right side of the last inequality does not exceed $(n + 1)^{-(a+1)} (1 - x_0)^{-a}$, and consequently

$$(1 - x_0)^a \tilde{J}[x_0, 1) \leqslant (n+1)^{-(a+1)}. \tag{2.23}$$

The inequalities (2.22) and (2.23) show that the partition of the interval $[0, 1)$ into intervals $\Delta_1, \ldots, \Delta_k, [x_0, 1)$ is the desired one. Thus the induction has been verified and the theorem proved.

We note that under the conditions of the theorem it is possible to relax the requirement that the function J be continuous from the left. An inequality of the form (2.21) remains valid but with a factor $c > 1$ on the right-hand side.

The basic difference between the inequalities (2.21) and (2.6) is that the function J is replaced by $\tilde{J}$ on the left-hand side. As an example of a point load type function shows, for $a = 0$ this is essential.

§3. Theorems on approximation of functions of the classes W_p^α

In this section we investigate the rate of approximation of functions of the class W_p^α by piecewise polynomial functions. The degree l of the approximating polynomials is fixed, with $l = \alpha - 1$ for integral α and $l = [\alpha]$ for nonintegral α. Below we use the notation $\omega = \alpha m^{-1}$ and, when $p\omega \le 1$, $q^* = p(1 - p\omega)^{-1}$. Here q^* is the so-called limit exponent in the theorem of imbedding of the space W_p^α into the space L_q. As usual, we set $q^* = \infty$ when $p\omega = 1$.

1. Let $\Delta \subset Q^m$ be a cube. With every function $u \in W_p^\alpha(\Delta)$ we associate a polynomial r of degree l satisfying the conditions

$$\int_\Delta x^\varkappa r(x)\,dx = \int_\Delta x^\varkappa u(x)\,dx \quad (|\varkappa| \leqslant l). \tag{3.1}$$

Conditions (3.1) obviously determine r uniquely. Set $r = P_\Delta u$; thus P_Δ is a linear projection operator mapping the space $W_p^\alpha(\Delta)$ onto the finite-dimensional space of polynomials of degree l in m variables.[1)] The dimensionality of this space is denoted by $\nu = \nu(m, l)$.

We note the following simple assertions.

Lemma 3.1. *When $p\omega > 1$ for every function $u \in W_p^\alpha(\Delta)$ the following inequality is satisfied:*

$$\| u - P_\Delta u \|_{C(\Delta)} \leqslant C_4 |\Delta|^{\omega - p^{-1}} \| u \|_{L_p^\alpha(\Delta)}, \tag{3.2}$$

where the constant $C_4 = C_4(p, \alpha, m)$ does not depend on Δ.

Lemma 3.2. *When $p\omega \le 1$ and $q < q^*$ for every function $u \in W_p^\alpha(\Delta)$ the following inequality is satisfied:*

$$\| u - P_\Delta u \|_{L_q(\Delta)} \leqslant C_5 |\Delta|^{q^{-1} - q^{*-1}} \| u \|_{L_p^\alpha(\Delta)}, \tag{3.3}$$

where the constant $C_5 = C_5(p, q, \alpha, m)$ does not depend on Δ.

For the proof of both assertions we first consider the case in which $\Delta = Q^m$. We introduce a new norming in the space W_p^α:

$$|||u|||_{W_p^\alpha} = \sum_{|\varkappa| \leqslant l} \left| \int_{Q^m} x^\varkappa u(x)\,dx \right| + \| u \|_{L_p^\alpha}.$$

---

1)In all future constructions the operator P_Δ can be replaced by any other projection operator onto the same space.

Equivalence of the norms $|||\cdot|||_{W_p^\alpha}$ and $\|\cdot\|_{W_p^\alpha}$ follows from considerations of S. L. Sobolev [11] for integral α and can be proved quite analogously for nonintegral α. It follows from conditions (3.1) that

$$|||u - P_{Q^m}u|||_{W_p^\alpha} = \|u\|_{L_p^\alpha}.$$

The theorem of imbedding of the space W_p^α into the space C (for $p\omega > 1$) and into L_q (for $p\omega \le 1$) shows that the inequality (3.2) or, respectively, (3.3), is satisfied in the cube $\Delta = Q^m$. To pass to an arbitrary cube Δ we need to implement the similarity transformation and for this use the property of homogeneity (1.4) of the seminorm $\|\cdot\|_{L_p^\alpha(\Delta)}$. Thus Lemmas 3.1 and 3.2 are proved.

2. As in §2, let Ξ be a partition of the (halfopen) cube Q^m into halfopen cubes. We use $\mathcal{P}(\Xi; l)$ to denote the linear set of all functions whose restriction to each of the cubes $\Delta \in \Xi$ is a polynomial of degree l. We introduce the projection operator P_Ξ defined as follows: $v = P_\Xi u$ is the function of the class $\mathcal{P}(\Xi; l)$ coinciding with the polynomial $P_\Delta u$ on each cube $\Delta \in \Xi$.

Theorem 3.1. *For every function $u \in W_p^\alpha$ $(p\omega > 1)$ and for every natural n there is a partition $\Xi \in \mathfrak{R}$ of the cube Q^m such that $|\Xi| \le n$ and*

$$\|u - P_\Xi u\|_{L_\infty} \leqslant C_6 n^{-\omega} \|u\|_{L_p^\alpha}, \qquad C_6 = C_6(p, \alpha, m). \tag{3.4}$$

Proof. Let Ξ be an arbitrary partition of Q^m into cubes and $v = P_\Xi u$. Then, according to (3.2),

$$\sup_{x\in Q^m} |u(x) - v(x)| \leqslant C_4 [\max_{\Delta\in\Xi} |\Delta|^{p\omega-1} \|u\|^p_{L_p^\alpha(\Delta)}]^{p^{-1}}. \tag{3.5}$$

We consider the following function $J_u(\Delta)$ of cubes $\Delta \subset Q^m$:

$$J_u(\Delta) = \|u\|^p_{L_p^\alpha(\Delta)}. \tag{3.6}$$

The function J_u is semiadditive from below and additive [1]) for integral α.

In the square brackets on the right-hand side of (3.5) we have the partition function $G_{p\omega-1}(J_u; \Xi)$ constructed from the function (3.6). By Theorem 2.1 there exists a partition $\Xi \in \mathfrak{R}$ of the cube Q^m for which $|\Xi| \le n$ and

$$G_{p\omega-1}(J_u; \Xi) \leqslant C_1 n^{-p\omega} J_u(Q^m).$$

The last inequality together with (3.5) leads to the estimate (3.14) with constant $C_6 = C_4 C_1^{p-1}$. The theorem is proved.

Theorem 3.2. *Let $p\omega \le 1$, $q < q^*$. For every function $u \in W_p^\alpha$ and every natural number n there is a partition $\Xi \in \mathfrak{R}$ of the cube Q^m such that $|\Xi| \le n$ and*

$$\|u - P_\Xi u\|_{L_q} \leqslant C_7 n^{-\omega} \|u\|_{L_p^\alpha}, \quad C_7 = C_7(p, q, \alpha, m). \tag{3.7}$$

Proof. Consider the partition function $G_a(J_u; \Xi)$ with $a = p(q^{-1} - q^{*-1})$. According to Theorem 2.1 there is a partition $\Xi \in \mathfrak{R}$ for which $|\Xi| \le n$ and

1) The notation J_u does not express the dependence of the function (3.6) on α and p. However, this will not lead to any ambiguities in what follows.

$$G_a(J_u;\ \Xi) = \max_{\Delta\in\Xi} |\Delta|^{p(q^{-1}-q_*^{-1})} \|u\|^p_{L_p^\alpha(\Delta)} \leqslant C_1 n^{-p(\omega+q^{-1})} \|u\|^p_{L_p^\alpha}.$$

Taking Lemma 3.2 into account for this partition Ξ and the function $v = P_\Xi u$, we find that

$$\|u-v\|^q_{L_q} = \sum_{\Delta\in\Xi} \|u-v\|^q_{L_q(\Delta)} \leqslant |\Xi| C_5^q [G_a(J_u;\ \Xi)]^{qp^{-1}}$$

$$\leqslant |\Xi| C_5^q C_1^{qp^{-1}} n^{-(q\omega+1)} \|u\|^q_{L_p^\alpha} \leqslant [C_7 n^{-\omega} \|u\|_{L_p^\alpha}]^q,$$

where $C_7 = C_1^{p-1} C_5$. The theorem is proved.

3. Now let σ be a finite Borel measure defined on subsets of Q^m. We consider approximation of functions from W_p^α in the metric of $L_{q,\sigma} = L_q(Q^m;\ \sigma)$, $q \geq 1$. In contrast to Theorems 3.1 and 3.2, we are here concerned with the rate of approximation which can be attained by choosing partitions not depending on the function u (but depending, generally speaking, on the measure σ).

Let $\mathfrak{M}$ be the set of all finite Borel measures on Q^m satisfying the condition $\sigma(Q^m) \leq 1$. By $\mathfrak{M}_\lambda$ $(1 \leq \lambda < \infty)$ we denote the set of all absolutely continuous measures σ on Q^m whose density $d\sigma/dx$ belongs to the space L_λ and satisfies the condition

$$\int_{Q^m} \left(\frac{d\sigma}{dx}\right)^\lambda dx \leqslant 1. \tag{3.8}$$

Theorem 3.3. *Let $p\omega > 1$. For every Borel measure $\sigma \in \mathfrak{M}$ and every natural n there exists a partition $\Xi \in \mathfrak{R}$ of the cube Q^m such that $|\Xi| \leq n$, and for every function $u \in W_p^\alpha$ the following inequality is satisfied:*

$$\|u - P_\Xi u\|_{L_{q,\sigma}} \leqslant C_8 n^{-\gamma} \|u\|_{L_p^\alpha}, \quad C_8 = C_8(p,\ q,\ \alpha,\ m), \tag{3.9}$$

where $\gamma = \omega$ when $p \geq q$ and $\gamma = \omega - p^{-1} + q^{-1}$ when $p < q$. The constant C_8 does not depend on σ.

Proof. It suffices to give the reasoning for the case $q \geq p$ since the validity of the assertion of the theorem for $q < p$ follows in an obvious way from the assertion for $q = p$.

Thus let Ξ be an arbitrary partition of Q^m into (halfopen) cubes. For every function $u \in W_p^\alpha$ and the function $v = P_\Xi u$ by Lemma 3.1 we have

$$\|u-v\|^q_{L_{q,\sigma}} \leqslant \sum_{\Delta\in\Xi} \sup_{x\in\Xi} |u-v|^q \sigma(\Delta) \leqslant C_4^q \sum_{\Delta\in\Xi} |\Delta|^{(\omega-p^{-1})q} \|u\|^q_{L_p^\alpha(\Delta)} \sigma(\Delta).$$

When $p \leq q$ we have

$$\sum_{\Delta\in\Xi} \|u\|^q_{L_p^\alpha(\Delta)} \leqslant \left[\sum_{\Delta\in\Xi} \|u\|^p_{L_p^\alpha(\Delta)}\right]^{qp^{-1}} \leqslant \|u\|^q_{L_p^\alpha}. \tag{3.10}$$

Consequently

$$\|u-v\|^q_{L_{q,\sigma}} \leqslant C_4^q \|u\|^q_{L_p^\alpha} \cdot \max_{\Delta\in\Xi} \{|\Delta|^{(\omega-p^{-1})q} \sigma(\Delta)\}.$$

Now we apply Theorem 2.1 to the function $G_a(\sigma; \Xi)$ with $a = (\omega - p^{-1})q$, which leads to (3.9). The theorem is proved.

The following theorem extends our result to the case $p\omega \leq 1$. Naturally here inequality (3.9) may not be true for arbitrary measures. It remains valid, however, for absolutely continuous measures whose density is summable when raised to a sufficiently high degree.

Theorem 3.4. *Let* $p\omega \leq 1$ *and*

$$\lambda^{-1} + qq^{*-1} < 1. \tag{3.11}$$

Then for every measure $\sigma \in \mathfrak{M}_\lambda$ *and every natural* n *there exists a partition* $\Xi \in \mathfrak{R}$ *of the cube* Q^m *such that* $|\Xi| \leq n$ *and for every function* $u \in W_p^{\alpha}$ *the inequality* (3.9) *is satisfied with some constant* $C_8 = C'_8(p, q, \lambda, \alpha, m)$ *not depending on* σ.

Proof. For every cube $\Delta \subset Q^m$ and for $v = P_\Delta u$ we find, by Lemma 3.2, that

$$\int_\Delta |u-v|^q d\sigma = \int_\Delta |u-v|^q \frac{d\sigma}{dx} dx \leqslant \left\{\int_\Delta |u-v|^{q(1-\lambda^{-1})^{-1}} dx\right\}^{1-\lambda^{-1}} \left\{\int_\Delta \left(\frac{d\sigma}{dx}\right)^\lambda dx\right\}^{\lambda^{-1}}$$

$$\leqslant c\,|\Delta|^{1-\lambda^{-1}-qq^{*-1}} \|u\|^q_{L_p^\alpha(\Delta)} \left\{\int_\Delta \left(\frac{d\sigma}{dx}\right)^\lambda dx\right\}^{\lambda^{-1}}.$$

Hence for every partition Ξ of Q^m into cubes, for an arbitrary function $u \in W_p^\alpha$ and for the function $v = P_\Xi u$ we obtain

$$\|u-v\|^q_{L_{q,\sigma}} \leqslant c \sum_{\Delta\in\Xi} |\Delta|^{1-\lambda^{-1}-qq^{*-1}} \|u\|^q_{L_p^\alpha(\Delta)} \left\{\int_\Delta \left(\frac{d\sigma}{dx}\right)^\lambda dx\right\}^{\lambda^{-1}}. \tag{3.12}$$

Now we assume [1] that $q \leq p$. We note that for $q \leq p$

$$\sum_{\Delta\in\Xi} \|u\|^q_{L_p^\alpha(\Delta)} \leqslant |\Xi|^{1-qp^{-1}} \left\{\sum_{\Delta\in\Xi} \|u\|^p_{L_p^\alpha(\Delta)}\right\}^{qp^{-1}} \leqslant |\Xi|^{1-qp^{-1}} \|u\|^q_{L_p^\alpha}. \tag{3.13}$$

Together with (3.12) this leads to the inequality

$$\|u-v\|^q_{L_{q,\sigma}} \leqslant c\,|\Xi|^{1-qp^{-1}} \|u\|^q_{L_p^\alpha} \max_{\Delta\in\Xi} \left\{|\Delta|^{\lambda-1-\lambda qq^{*-1}} \int_\Delta \left(\frac{d\sigma}{dx}\right)^\lambda dx\right\}^{\lambda^{-1}} \tag{3.14}$$

The choice of partition Ξ is made according to Theorem 2.1, applied to the function

$$J(\Delta) = \int_\Delta \left(\frac{d\sigma}{dx}\right)^\lambda dx$$

with $a = \lambda - 1 - \lambda qq^{*-1}$. Then inequality (3.14) leads directly to (3.9) if we take into account (3.8) and the fact that $|\Xi| \leq n$.

The case $q > p$ is considered analogously. The inequality (3.10) is to be used in place of (3.13). The theorem is proved.

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1) Here we cannot reduce the case $q < p$ to the case $q = p$ because, by (3.11), this would lead to a tighter restriction on λ.

In concluding this section we make several remarks.

Remark 3.1. The assertions of Theorems 3.3 and 3.4 are well known for the case in which σ is Lebesgue measure. For Ξ we can then obviously choose partitions into equal cubes. Theorems 3.3 and 3.4 show that analogous results also hold (with an appropriate choice of partitions) in a considerably more general case.

Remark 3.2. If under the conditions of Theorems 3.3 and 3.4 we allow independence of the partition Ξ from the function u, then we can replace (3.9) by the inequality

$$\| u - P_\Xi u \|_{L_{q,\sigma}} \leqslant cn^{-\omega} \| u \|_{L_p^\alpha}. \tag{3.15}$$

Under the conditions of Theorem 3.3 this follows directly from (3.4); under the conditions of Theorem 3.4 it can be derived easily from (3.7). A comparison of (3.15) and (3.9) shows that passing to a method of partitions which does not depend on the function u yields a worse result only for $q > p$.

Remark 3.3. Theorems 3.1–3.4 remain valid if the classes W_p^α are replaced by the classes H_p^α or B_p^α. Here the determination of the projection operators P_Δ is somewhat changed. This is connected with the fact that for the approximation in each cube $\Delta \in \Xi$ we use polynomials on which the principal term of the norm of the corresponding space is equal to zero. Thus, for the classes B_p^α, using the usual definition of norm (see [6]) for integral α, O. V. Besov considers polynomials of degree α in each of the variables $x_1, \cdots, x_m$.

§4. On approximation of functions of the classes V_β by piecewise-constant functions

Theorems on approximation of functions of the classes V_β ($\beta \geq 1$) (Theorems 4.1–4.3) are proved by the same procedure as the results of §3, on the basis of Theorem 2.2. We note that the first two theorems of this section (Theorems 4.1, 4.2) are easy to prove directly without using theorems on partition functions. [1]

We restrict consideration to functions $u \in V_\beta$ which are normalized to continuity from the right, and we assume the basic interval (denoted by X) to be halfopen.

1. With every function $u \in V_\beta(X)$ we associate a function I_u of halfopen intervals $\Delta \subset X$ by the rule

$$I_u(\Delta) = \| u \|^\beta_{\overset{\circ}{V}_\beta(\Delta)}. \tag{4.1}$$

This function plays the same role in the investigation of the classes V_β as the function (3.6) for the classes W_p^α. The function (4.1) is obviously semiadditive from below. In view of the assumed normalization of the functions $u \in V_\beta$, the following assertion is also true.

Lemma 4.1. *Suppose the function $u \in V_\beta(X)$ is continuous from the right and I_u is the function of the halfopen intervals $\Delta \subset X$ defined by the formula* (4.1). *Then the function I_u is continuous from the left, and the function $\tilde{I}_u$ generated by it according to the definition in* §2.3 *coincides with I_u.*

Proof. Let $\Delta = [x', x'') \subset X$. For a given $\epsilon > 0$ there is a system of points $x' \leq x_0 < x_1 < \cdots$ $\cdots < x_n < x''$ for which

$$\sum_{k=1}^{n} | u(x_k) - u(x_{k-1}) |^\beta > I_u(\Delta) - \varepsilon. \tag{4.2}$$

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1) Ju. A. Brudnyĭ kindly brought this to our attention.

In view of the continuity from the right of the function u, we can assume that $x_0 > x'$ in (4.2). For every $x \in (x_m, x'')$ it follows from (4.2) that

$$I_u(\Delta) - \varepsilon < \|u\|^{\beta}_{\overset{\circ}{V}_{\beta}([x_0, x_n])} \leqslant I_u[x', x) \leqslant I_u(\Delta).$$

Analogously for $x \in (x', x_0]$ we find that $I_u(\Delta) - \epsilon < \|u\|^{\beta}_{\overset{\circ}{V}_{\beta}([x_0, x_n])} \leq I_u[x, x'') \leq I_u(\Delta)$. Both assertions of the lemma follow from these inequalities.

2. Let $\Delta = [x', x'') \subset X$ be an interval. The role of inequality (3.2) for the functions $u \in V_\beta(X)$ is played by the obvious inequality

$$\sup_{x \in \Delta} |u(x) - u(x')| \leqslant \|u\|_{\overset{\circ}{V}_{\beta}(\Delta)}. \tag{4.3}$$

Let Ξ be a partition of the basic interval X into intervals $\Delta_i = [x_{i-1}, x_i)$ $(i = 1, \cdots, |\Xi|)$. Let P_Ξ denote the operator which associates with the function $u \in V_\beta(X)$ the piecewise-constant function $v \in \mathcal{P}(\Xi;\ 0)$ assuming the constant value equal to $u(x_{i-1})$ in each interval $\Delta_i \in \Xi$.

Theorem 4.1. *Let $X = [b', b'')$ be a finite or infinite interval. For every function $u \in V_\beta(X)$ continuous from the right and every natural number n there is a partition Ξ of the interval X into halfopen intervals such that $|\Xi| \leq n$ and*

$$\|u - P_\Xi u\|_{L_\infty(X)} \leqslant n^{-\beta^{-1}} \|u\|_{\overset{\circ}{V}_{\beta}(X)}.$$

Proof. As a result of the invariance of the classes V_β with respect to monotonic replacement of the independent variable, we can assume that $X = Q^1 = [0, 1)$. Let Ξ be a partition of the interval Q^1 into halfopen intervals. From inequality (4.3) for every function $u \in V_\beta$ and the function $v = P_\Xi u$ we obtain that

$$\sup_{x \in X} |u(x) - v(x)| \leqslant [\max_{\Delta \in \Xi} I_u(\Delta)]^{\beta^{-1}} = [G_0(I_u;\ \Xi)]^{\beta^{-1}}. \tag{4.4}$$

Lemma 4.1 shows that Theorem 2.2 applies to the function I_u. Indeed, there exists a partition Ξ of the interval Q^1 into halfopen intervals such that $|\Xi| \leq n$ and

$$G_0(\widetilde{I}_u;\ \Xi) = G_0(I_u;\ \Xi) \leqslant n^{-1} I_u(Q^1).$$

The last inequality together with (4.4) proves the theorem.

Now let σ be a Borel measure defined on subsets of the interval X and satisfying the condition $\sigma(X) \leq 1$. If the measure σ is considered on halfopen intervals $\Delta = [x', x'') \subset X$, then in view of the complete additivity of the measure, the function $\sigma(\Delta) = \sigma[x', x'')$ is continuous from the left. It is also clear that the corresponding function $\widetilde{\sigma}(\Delta)$ coincides with the measure of the interval (x', x''), i.e. $\widetilde{\sigma}[x', x'') = \sigma(x', x'')$.

In the following theorem we estimate the rate of approximation of functions of the class $V_\beta(X)$ by piecewise-constant functions in the metric of the space $L_{q,\sigma} = L_q(X;\ \sigma)$, $q \geq 1$.

Theorem 4.2. *Let $X = [b', b'')$ be a finite or infinite interval. For every Borel measure σ $(\sigma(X) \leq 1)$ and every natural number n there exists a partition Ξ of the interval X into halfopen intervals such that $|\Xi| \leq n$ and for every function $u \in V_\beta(X)$ continuous from the right we have*

$$\| u - P_\Xi u \|_{L_{q,\sigma}} \leqslant n^{-\vartheta} \| u \|_{\overset{\circ}{V}_{\beta}(X)}, \quad \vartheta = \min(\beta^{-1}, q^{-1}).$$

Proof. As in the proof of the preceding theorem, it suffices to consider the case $X = Q^1$. Let Ξ be some partition of the interval Q^1. For every function $u \in V_\beta$ and the function $v = P_\Xi u$ from (4.3) we find that

$$\| u - v \|^q_{L_{q,\sigma}} \leqslant \sum_{\Delta \in \Xi} \sup_{x \in \Delta} | u(x) - v(x) |^q \tilde{\sigma}(\Delta) \leqslant \sum_{\Delta \in \Xi} \| u \|^q_{\overset{\circ}{V}_{\beta}(\Delta)} \tilde{\sigma}(\Delta). \tag{4.5}$$

(We note that in (4.5) it was possible to replace $\sigma(\Delta)$ by $\tilde{\sigma}(\Delta)$ because the function $u - v$ vanishes at the left end of every interval $\Delta \in \Xi$.) From (4.5) we find that when $q \leq \beta$

$$\| u - v \|^q_{L_{q,\sigma}} \leqslant [\max_{\Delta \in \Xi} \tilde{\sigma}(\Delta)] \, | \Xi |^{1 - q\beta^{-1}} \left[\sum_{\Delta \in \Xi} \| u \|^\beta_{\overset{\circ}{V}_{\beta}(\Delta)} \right]^{q\beta^{-1}}$$

$$\leqslant [\max_{\Delta \in \Xi} \tilde{\sigma}(\Delta)] \, | \Xi |^{1 - q\beta^{-1}} \| u \|^q_{\overset{\circ}{V}_{\beta}(\Delta)}$$

and when $q > \beta$

$$\| u - v \|^q_{L_{q,\sigma}} \leqslant [\max_{\Delta \in \Xi} \tilde{\sigma}(\Delta)] \left[\sum_{\Delta \in \Xi} \| u \|^\beta_{\overset{\circ}{V}_{\beta}(\Delta)} \right]^{q\beta^{-1}} \leqslant [\max_{\Delta \in \Xi} \tilde{\sigma}(\Delta)] \| u \|^q_{\overset{\circ}{V}_{\beta}(\Delta)}.$$

Now we have only to note that by Theorem 2.2 applied (with $a = 0$) to the function $J = \sigma$, there is a partition Ξ ($|\Xi| \leq n$) such that

$$\max_{\Delta \in \Xi} \tilde{\sigma}(\Delta) \leqslant n^{-1}.$$

The theorem is proved.

3. For the functions $u \in V_\beta(X) \cap C^\mu(X)$ the assertion of Theorem 4.2 can be somewhat strengthened in the case where the interval X is finite. To simplify the statement we assume that $X = Q^1$. As a result of the obvious imbedding $C^\mu \subset V_\beta$ for $\beta = \max(1, \mu^{-1})$, it makes sense to consider the given problem only under the condition $0 < \mu < \beta^{-1}$.

Theorem 4.3. *Suppose $X = Q^1$ and the exponents μ, β and q satisfy the condition $1 \leq \beta < q$, $0 < \mu < \beta^{-1}$. For every Borel measure σ ($\sigma(Q^1) \leq 1$) and every natural number n there is a partition Ξ of the interval Q^1 into halfopen intervals so that $|\Xi| \leq n$ and for every function $u \in V_\beta \cap C^\mu$ we have*

$$\| u - P_\Xi u \|_{L_{q,\sigma}} \leqslant n^{-\rho} L^{1 - \beta q^{-1}} \| u \|^{\beta q^{-1}}_{\overset{\circ}{V}_{\beta}}, \quad \rho = \mu(1 - \beta q^{-1}) + q^{-1},$$

where L is the Hölder constant of the function u.

Proof. Let Ξ be a partition of the interval Q^1 into intervals $\Delta_k = [x_{k-1}, x_k)$, $0 = x_0 < x_1 < \cdots$ $\cdots < x_{|\Xi|} = 1$. We have

$$\| u - P_\Xi u \|^q_{L_{q,\sigma}} \leqslant \sum_{k=1}^{|\Xi|} \sup_{x \in \Delta_k} | u(x) - u(x_{k-1}) |^q \tilde{\sigma}(\Delta_k)$$

$$\leqslant \sum_{k=1}^{|\Xi|} \sup_{x \in \Delta_k} | u(x) - u(x_{k-1}) |^\beta L^{q-\beta} | \Delta_k |^{\mu(q-\beta)} \tilde{\sigma}(\Delta_k) \leqslant$$

$$\leqslant L^{q-\beta}\sum_{k=1}^{|\Xi|}\|u\|^{\beta}_{\overset{\circ}{V}_{\beta}(\Delta_k)}|\Delta_k|^{\mu(q-\beta)}\widetilde{\sigma}(\Delta_k)\leqslant L^{q-\beta}[\max_k|\Delta_k|^{\mu(q-\beta)}\widetilde{\sigma}(\Delta_k)]\cdot\|u\|^{\beta}_{\overset{\circ}{V}_{\beta}}.$$

Now it remains to choose the partition Ξ according to Theorem 2.2, which is to be applied to the function $J = \sigma$ with $a = \mu(q - \beta)$. The theorem is proved.

§5. Estimates of ϵ-entropy and n-diameters

1. The class of functions $\mathcal{P}(\Xi;\, l)$ is a linear set of dimensionality $|\Xi|\cdot\nu$. As a result of this, Theorems 3.3 and 3.4 can be interpreted in terms of n-diameters. Indeed, we have the following assertion.

Theorem 5.1. *Under the conditions of Theorem 3.3 or 3.4 the n-diameters d_n of the set SW^{α}_p in the metric of the space $L_{q,\sigma}$ satisfy the inequality*

$$d_n(W^{\alpha}_p;\, L_{q,\sigma})\leqslant cn^{-\gamma}, \tag{5.1}$$

where the exponent γ is the same as in (3.9) and the constant c does not depend on σ.

We also note that in Theorems 3.3 and 3.4 a linear approximation operator (the operator P_{Ξ}) is constructed for which (5.1) is realized. Thus (5.1) is actually valid even for *linear* n-diameters (see [9]) of SW^{α}_p in the metric of $L_{q,\sigma}$.

All of the above also holds for the classes H^{α}_p and B^{α}_p.

The approximation method used for the proof of Theorems 3.1 and 3.2 is different: there the partition Ξ depends on the function being approximated, and consequently the class of functions used for approximation is nonlinear. As a consequence of this, Theorems 3.1 and 3.2 are useless in the problem of estimating n-diameters of the set SW^{α}_p in the spaces C or L_q. However, an analysis of the method of proof of these two theorems allows us to estimate another metric characteristic of the set SW^{α}_p — its ϵ-entropy.

2. The basic result of this section is

Theorem 5.2. *For the ϵ-entropy of the set SW^{α}_p in the metric of L_q we have (for sufficiently small $\epsilon > 0$) the estimate*

$$\mathcal{H}_{\varepsilon}(W^{\alpha}_p;\, L_q)\leqslant c\varepsilon^{-\omega^{-1}}. \tag{5.2}$$

Here $1 \le q \le \infty$ for $p\omega > 1$ and $1 \le q < q^*$ for $p\omega \le 1$.

First of all we explain the general plan of the proof. The set SW^{α}_p is first divided into classes, with each class comprising those functions whose approximation with a given accuracy requires, according to Theorem 3.1 or 3.2, the same sequence of partitions. The number of classes is estimated on the basis of Lemma 2.3, after which we want to estimate the ϵ-entropy of each class. Suppose a sequence of partitions $\{\Xi_i\}_0^k$ corresponds to a certain class. A crude method for calculating ϵ-entropies (approximation of the function u by the function $P_{\Xi}u$ and estimation of the ϵ-entropy of the unit sphere of the finite-dimensional space $\mathcal{P}(\Xi;\, l)$) leads to an excessive estimate. This method does not take into account that the polynomials $P_{\Delta'}u$ and $P_{\Delta''}u$ for neighboring cubes $\Delta', \Delta'' \in \Xi_k$ cannot differ from each other very strongly. We will take into account the closeness of such polynomials $P_{\Delta'}u$ and $P_{\Delta''}u$ as follows. We consider all piecewise-polynomial approximations $P_{\Xi_i}u$ $(i=0, 1, \cdots, k)$, and in passing from the number i to the number $i+1$ we make use of the fact that for cubes

$\Delta', \Delta'' \in \Xi_{i+1}$ contained in the same cube $\Delta \in \Xi_i$, both polynomials $P_{\Delta'}u$ and $P_{\Delta''}u$ differ little from $P_\Delta u$.

3. In the proof of Theorem 5.2 we require preliminary estimates in a special metric related to a fixed partition. Let Ξ be some partition of the cube Q^m. For the function $u \in L_q$, along with the usual norm of the space L_q, we consider the norm

$$\|u\|_{q,\Xi} = \max_{\Delta\in\Xi} \|u\|_{L_q(\Delta)}. \tag{5.3}$$

The space of functions generated by the norm (5.3) is denoted by $L_{q,\Xi}$. We note the obvious inequalities

$$\|u\|_{q,\Xi} \leqslant \|u\|_{L_q} \leqslant |\Xi|^{q^{-1}} \|u\|_{q,\Xi}, \tag{5.4}$$

which become equalities when $q = \infty$. If Ξ' is an extension of Ξ, then

$$\|u\|_{q,\Xi'} \leqslant \|u\|_{q,\Xi}.$$

We establish two auxiliary assertions.

Lemma 5.1. *Let Ξ be a partition of the cube Q^m and $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(\Xi; l, M)$ be the set of functions $v \in \mathcal{P}(\Xi; l)$ satisfying the condition*

$$\|v\|_{q,\Xi} \leqslant M. \tag{5.5}$$

Then for every $\epsilon \le M$ we have

$$\mathcal{N}_\varepsilon(\tilde{\mathcal{P}}; L_{q,\Xi}) \leqslant C_9^{|\Xi|}(M\varepsilon^{-1})^{\nu|\Xi|}, \tag{5.6}$$

where the constant $C_9 = C_9(m, l, q)$ does not depend on Ξ.

Proof. Let $\mathcal{R} = \mathcal{R}(\Delta; l, M)$ be the set of polynomials r of degree l of m variables satisfying the condition

$$\|r\|_{L_q(\Delta)} \leqslant M. \tag{5.7}$$

For every $\epsilon \le M$ we have

$$\mathcal{N}_\varepsilon(\mathcal{R}; L_q(\Delta)) \leqslant C_9(M\varepsilon^{-1})^\nu. \tag{5.8}$$

Indeed, for the case $\Delta = Q^m$ inequality (5.8) follows from the fact that the norm $\|\cdot\|_{L_q(\Delta)}$ on the finite-dimensional (ν-dimensional) space of polynomials is equivalent to the Euclidean norm on the space of coefficients. To pass to an arbitrary cube it suffices to make a transformation of the independent variables, which does not affect the size of the constant C_9.

Condition (5.5) obviously implies condition (5.7) for polynomials r obtained by restricting the functions $v \in \tilde{\mathcal{P}}$ to some cube $\Delta \in \Xi$. Here estimate (5.6) is easily obtained from (5.8). Indeed, the required ϵ-net in the set $\tilde{\mathcal{P}}$ can be formed by means of all possible "pastings together" of elements of ϵ-nets constructed in the sets $\mathcal{R}(\Delta; l, M)$ for all cubes $\Delta \in \Xi$. The lemma is proved.

Assume that $\{\Xi_i\}_0^k \subset \mathfrak{R}$ is a sequence of partitions of the cube Q^m where each partition Ξ_i is an extension of the preceding partition Ξ_{i-1}. As usual, we write $|\Xi_i| = n_i$. Let ζ_i be numbers satisfying the conditions

$$b\zeta_i \leqslant \zeta_{i+1} \leqslant \zeta_i \quad (b>0;\ i=0,\ 1,\ \ldots,\ k-1), \tag{5.9}$$

and let $\hat{\mathcal{P}}_i \subset \mathcal{P}(\Xi_i;\ l)$ $(i = 0, 1, \cdots, k)$ be certain sets of piecewise-polynomial functions, where $\hat{\mathcal{P}}_i$ is the $2\zeta_i$-net for the set $\hat{\mathcal{P}}_{i+1}$ in the metric of $L_{q,\Xi_{i+1}}$ $(i = 0, 1, \cdots, k-1)$.

Lemma 5.2. *Under the above assumptions we have*

$$\mathcal{N}_{\zeta_i}(\hat{\mathcal{P}}_i;\ L_{q,\Xi_i}) \leqslant C_{10}^{n_1+\ldots+n_i}\mathcal{N}_{\zeta_0}(\hat{\mathcal{P}}_0;\ L_q) \quad (i=1,\ \ldots,\ k). \tag{5.10}$$

The constant $C_{10} = C_{10}(m, l, q, b)$ does not depend on the sequence of partitions being considered or on the numbers ζ_i.

Proof. For every element $v \in \hat{\mathcal{P}}_{i+1}$ there is an element $v' \in \hat{\mathcal{P}}_i$ such that

$$\| v - v' \|_{q,\Xi_{i+1}} \leqslant 2\zeta_i.$$

Let V_i be the ζ_i-net of minimal cardinality constructed for $\hat{\mathcal{P}}_i$ relative to the metric of L_{q,Ξ_i}. Then for a suitable element $\tilde{v} \in V_i$ we obtain

$$\| v - \tilde{v} \|_{q,\Xi_{i+1}} \leqslant \| v - v' \|_{q,\Xi_{i+1}} + \| v' - \tilde{v} \|_{q,\Xi_i} \leqslant 3\zeta_i.$$

On the strength of Lemma 5.1 we can construct a ζ_{i+1}-net Z_{i+1}, in the set $\tilde{\mathcal{P}}(\Xi_{i+1};\ l,\ 3\zeta_i)$ whose cardinality does not exceed the quantity

$$C_9^{n_{i+1}}(3\zeta_i\zeta_{i+1}^{-1})^{\nu n_{i+1}} \leqslant C_{10}^{n_{i+1}} \quad (C_{10} = C_9 3^\nu b^{-\nu}).$$

Since $v - \tilde{v} \in \tilde{\mathcal{P}}(\Xi_{i+1};\ l,\ 3\zeta_i)$, for some element $z \in Z_{i+1}$ we have

$$\| v - \tilde{v} - z \|_{q,\Xi_{i+1}} \leqslant \zeta_{i+1}.$$

All possible elements of the form $\tilde{v} + z$, where $\tilde{v} \in V_i$, $z \in Z_{i+1}$, form a ζ_{i+1}-net for the set $\hat{\mathcal{P}}_{i+1}$ relative to the metric of the space $L_{q,\Xi_{i+1}}$. The cardinality of this net is estimated, obviously, by the quantity

$$C_{10}^{n_{i+1}}\mathcal{N}_{\zeta_i}(\hat{\mathcal{P}}_i;\ L_{q,\Xi_i}).$$

Thus we have obtained the estimate

$$\mathcal{N}_{\zeta_{i+1}}(\hat{\mathcal{P}}_{i+1};\ L_{q,\Xi_{i+1}}) \leqslant C_{10}^{n_{i+1}}\mathcal{N}_{\zeta_i}(\hat{\mathcal{P}}_i;\ L_{q,\Xi_i}),$$

from which (5.10) follows. The lemma is proved.

4. We proceed to the proof of Theorem 5.2. We use the notation $T_a^\eta(J)$, (2.3), (2.4), (2.13) introduced in §2, and relate it to the partition function $J = J_u$ defined by the formula (3.6). In the case $p\omega > 1$ it obviously suffices to prove the theorem for $q = \infty$.

Let η be a fixed number, $0 < \eta \leq C_2$. Set $a = p\omega - 1$ for $p\omega > 1$ and $a = p(q^{-1} - q^{*-1})$ for $p\omega \leq 1$, $q < q^*$. Divide the set SW_p^α into classes, associating two functions $u_1, u_2 \in SW_p^\alpha$ with the same class if and only if $T_a^{\eta}(J_{u_1}) = T_a^{\eta}(J_{u_2})$. The number of distinct classes obviously does not exceed the number $N(a, \eta)$ introduced in §2.2. On the strength of Lemma 2.3, the latter can be estimated as follows:

$$\log_2 N(a, \eta) \leqslant C_3 \eta^{-(a+1)^{-1}} \quad (0 < \eta \leqslant C_2). \tag{5.11}$$

Now we estimate the entropy of each of the classes. We use $\hat{S}$ to denote some class. With all functions $u \in \hat{S}$ we associate the same sequence of partitions $\{\Xi_i\}_0^k = T_a^{\eta}(J_u)$ and also the same sequence of numbers $\tilde{\delta}_i = \tilde{\delta}_i(J_u; a)$ $(i = 0, 1, \cdots, k)$. The number k is determined by (2.17).

Using the notation (2.4), (3.6) and (5.3), we observe that we can write the assertions of Lemmas 3.1 (the case $q = \infty$) and 3.2 (the case $q < q^*$) in the form

$$\| u - P_{\Xi_i} u \|_{q, \Xi_i} \leqslant C_{11} \delta_i^{p^{-1}} \quad (i = 0, 1, \ldots, k). \tag{5.12}$$

Here $C_{11} = C_4$ for $q = \infty$ and $C_{11} = C_5$ for $q < q^*$. Taking (2.14) into account, we obtain the relationships

$$\| u - P_{\Xi_i} u \|_{q, \Xi_i} \leqslant \zeta_i \quad (\zeta_i = C_{11} \tilde{\delta}_i^{p^{-1}};\ i = 0, 1, \ldots, k) \tag{5.13}$$

which are somewhat cruder but will be more convenient below. We set $\hat{\mathcal{P}}_i = P_{\Xi_i} \hat{S}$ and note that the sets $\hat{\mathcal{P}}_i$ and the numbers ζ_i satisfy the conditions of Lemma 5.2. Indeed, from (5.13) we find that

$$\begin{aligned} \| P_{\Xi_i} u - P_{\Xi_{i+1}} u \|_{q, \Xi_{i+1}} &\leqslant \| u - P_{\Xi_{i+1}} u \|_{q, \Xi_{i+1}} + \| u - P_{\Xi_i} u \|_{q, \Xi_i} \\ &\leqslant \zeta_{i+1} + \zeta_i \leqslant 2\zeta i. \end{aligned}$$

The inequalities (5.9) are obviously satisfied in view of (2.16).

We estimate the quantity $\mathcal{N}_{\zeta_0}(\hat{\mathcal{P}}_0; L_q)$. Since $P_{\Xi_0} = P_{Q^m}$ and $\tilde{\delta}_0 = C_2$, from (5.13) and the imbedding theorem we obtain

$$\| P_{Q^m} u \|_{L_q} \leqslant C_{11} C_2^{p^{-1}} + \| u \|_{L_q} \leqslant C_{11} C_2^{p^{-1}} + C_{11} \| u \|_{W_p^{\alpha}} \leqslant C_{11} (C_2^{p^{-1}} + 1).$$

It follows from (5.8) that

$$\mathcal{N}_{\zeta_0}(\hat{\mathcal{P}}_0; L_q) \leqslant C_9 (1 + C_2^{-p^{-1}})^{\nu} \equiv C_{12}. \tag{5.14}$$

We use the result of Lemma 5.2. From (5.13), (5.10) and (5.14) we find that

$$\mathcal{N}_{2\zeta_k}(\hat{S}; L_{q, \Xi_k}) \leqslant \mathcal{N}_{\zeta_k}(\hat{\mathcal{P}}_k; L_{q, \Xi_k}) \leqslant C_{12} C_{10}^{n_1 + \cdots + n_k}. \tag{5.15}$$

According to (2.20), (2.16) and (2.17),

$$n_1 + \ldots + n_k \leqslant C_3' \tilde{\delta}_k^{-(a+1)^{-1}} \leqslant 2^m C_3' \tilde{\delta}_{k-1}^{-(a+1)^{-1}} \leqslant 2^m C_3' \eta^{-(a+1)^{-1}}.$$

Hence and from (5.15) we obtain an estimate of the form

$$\mathcal{H}_{2\zeta_k}(\hat{S};\ L_{q,\Xi_k}) \leqslant C_{13}\eta^{-(a+1)^{-1}}. \tag{5.16}$$

To obtain the final result we have to pass to estimating the entropy in the original metric, [1] i.e. in the metric of L_q. Setting $\epsilon_k = 2\zeta_k n_k^{q-1}$ and comparing (5.16) and (5.4), we obtain

$$\mathcal{H}_{\varepsilon_k}(\hat{S}; L_q) \leqslant C_{13}\eta^{-(a+1)^{-1}}.$$

Further, it follows from (2.19) and (2.17) that

$$\varepsilon_k \leqslant c\tilde{\delta}_k^{p^{-1}}\eta^{-q^{-1}(a+1)^{-1}} \leqslant c\eta^{p^{-1}-q^{-1}(a+1)^{-1}}.$$

Thus

$$\mathcal{H}_{\varepsilon}(\hat{S};\ L_q) \leqslant C_{13}\eta^{-(a+1)^{-1}}, \tag{5.17}$$

where

$$\varepsilon = c\eta^{p^{-1}-q^{-1}(a+1)^{-1}}.$$

Finally, combining the inequalities (5.17) and (5.11), we arrive at the estimate

$$\mathcal{H}_{\varepsilon}(SW_p^{\alpha};\ L_q) \leqslant c\varepsilon^{-[p^{-1}(a+1)-q^{-1}]^{-1}}.$$

It is easy to see that the last inequality coincides with the estimate (5.2). Indeed, the relationship $p^{-1}(a+1) - q^{-1} = \omega$ holds for both $q = \infty$, $a = p\omega - 1$ and for $q < q^*$, $a = p(q^{-1} - q^{*-1})$, $q^* = p(1 - p\omega)^{-1}$. The theorem is proved.

We note that we can establish the estimates

$$\mathcal{H}_{\varepsilon}(H_p^{\alpha}; L_q) \leqslant c\varepsilon^{-\omega^{-1}}, \quad \mathcal{H}_{\varepsilon}(B_p^{\alpha}; L_q) \leqslant c\varepsilon^{-\omega^{-1}}$$

in exactly the same way.

Corollary. *When* $p\omega > 1$ *the relationship* $\mathcal{H}_{\epsilon}(W_p^{\alpha};\ C) \asymp \epsilon^{-\omega^{-1}}$ *holds.*

Indeed, the estimate from above obviously coincides with the estimate (5.2) for $q = \infty$. The estimate from below for integral α follows from the inclusion $C^{\alpha} \subset W_p^{\alpha}$ and the inequality obtained in [5]:

$$\mathcal{H}_{\varepsilon}(C^{\alpha}; C) \geqslant c\varepsilon^{-\omega^{-1}}. \tag{5.18}$$

For nonintegral α the class C^{α} is not in W_p^{α}. However, in this case we can also obtain the required estimate from below of $\mathcal{H}_{\epsilon}(W_p^{\alpha};\ C)$ with the help of the system of functions which was used in [5] to obtain (5.18).

5. In conclusion we make some remarks about our estimates for ϵ-entropy and n-diameters. For simplicity we restrict consideration to the case of the sphere $SW_2^1(Q^1)$ considered in the metric of $C(Q^1)$.

The inclusions

$$SC^1 \subset SW_2^1 \subset SC^{1/2}$$

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1) For $q = \infty$ the metrics of L_q and $L_{q,\Xi}$ coincide; it is more convenient for our exposition, however, not to isolate the case $q = \infty$.

imply the inequalities

$$\mathcal{H}_\varepsilon(C^1; C) \leqslant \mathcal{H}_\varepsilon(W_2^1; C) \leqslant \mathcal{H}_\varepsilon(C^{1/2}; C),$$
$$d_n(C^1; C) \leqslant d_n(W_2^1; C) \leqslant d_n(C^{1/2}; C).$$

Above we saw that the quantity $\mathcal{H}_\epsilon(W_2^1; C)$ has the same order of magnitude as $\mathcal{H}_\epsilon(C^1; C)$:

$$\mathcal{H}_\varepsilon(W_2^1; C) \asymp \mathcal{H}_\varepsilon(C^1; C) \asymp \varepsilon^{-1}.$$

As for the n-diameters, the precise order of $d_n(W_2^1; C)$ is unknown. The most natural assumption is

$$d_n(W_2^1; C) \asymp d_n(C^{1/2}; C) \asymp n^{-1/2}. \tag{5.19}$$

This would mean that from the point of view of approximation by linear sets in the metric of the space C the sphere SW_2^1 is not better than the widest set $SC^{1/2}$. At the same time, from the point of view of ϵ-entropy, the sphere SW_2^1 is constructed essentially like the narrowest set SC^1. We note that for several other metric characteristics—n-diameters in the sense of I. M. Gel'fand (see [12])—relationships of the form (5.19) are indeed valid.

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