Preconditionning the Helmholtz problem on curved arcs in dimension 2

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Abstract

We introduce new efficient preconditioners for the Helmholtz weighted single layer potential and hypersingular on open curves. Those preconditioners are in the form of square roots of local differential operators.

Introduction

The problem of preconditionning the linear systems coming from the discretization of first kind integral equations has received considerable attention since two decades roughly. Among the possible strategies are the so-called pseudo-differential preconditionner, whose analysis uses tools from pseudo-differential calculus [**?**,**?**,**?**,**?**]. Ai-je cité les bons articles ?. Roughly speaking, if the original problem is written in the abstract way

$$\mathcal{L}u = f\,, \tag{1}$$

the strategy consists in findind a suitable operator $\mathcal{K}$ such that, when left mulitplying the (1) by $\mathcal{K}$, one needs to solve

$$\mathcal{K}\mathcal{L}u = \mathcal{K}f\,. \tag{2}$$

Now if $\mathcal{K}\mathcal{L}$ is a compact perturbation of the identity, the condition number of the discretized underlying system is independant of the chosen size of the mesh, leading to optimal convergence rate of the numerical methods used to solve the system.

Several strategies, depending on the problem to solve have been studied in the literature [] Tu voulais mettre qqch ici ? to propose such operators $\mathcal{K}$, that often turn out to be very effective in practice, when numerical applications are considered. However, all the preceding results and theories are limited to smooth domains and very little is known when the integral equation is posed on domains with corners (in 2D), wedges or conical points (in 3D). One of the reason might be the fact that pseudo-differential calculus is difficult to generalize on such manifolds, and more problematic, the underlying operators are difficult to analyze on such domains.

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Nevertheless, a program similar to those given on smooth manifold was proposed a few years ago [?, ?, ?] autres à rajouter ? who tackled the problem of preconditionning the first or second layer potential for Laplace or Helmholtz equation in dimension 2 or 3 but for very particular domains: a straight and then curved segment in 2D and a unit disc in 3D. Je ne suis pas vraiment d'accord, ils disent à chaque fois que par perturbation compacte on peut traiter le cas non-flat.

Here, we build preconditioners for the single-layer potential and the hypersingular operator, using ideas inspired by pseudo-differential theory. These preconditioners are in the form of the square root of local operators, in close analogy to [?], and lead to a drastic reduction of the number of the iterations in the GMRES method [?]. Since our aim is to generalize our ideas to more complex geometries, both in 2D and 3D we do not use the high order spectral methods available for this problem [?], but rather introduce the simplest possible Galerkin method with piecewise polynomial functions. The only difference as compared to usual boundary element methods is the use of a weighted L^2 space as the underlying Hilbert space. This translates into numerical quadratures with a weight on the open arc, together with a slightly refined mesh towards the edges of the arc. As a result, our numerical method involves only minor modifications with respect to standard BEM, and thus, can be implemented easily. We show that the orders of convergence obtained are the same as those on smooth closed curves for piecewise polynomials.

The paper is organized as follows: in the first section, we give an overview of the resolution of the Laplace and Helmholtz scattering problems by means of layer potentials on open arcs. We then define two family of spaces, T^s and U^s in the second section, that will be the basis of our pseudo-differential theory. We introduce a class of operators for which we are able to estimate the order. In the third section, we apply these new tools to the Laplace equation in the case of the unit segment, and give explicit inverses of the weighted layer potentials in terms of square roots of local operators. We then generalize our results to non-zero frequency and non-flat arc in the fourth section. The last section is dedicated to expose the numerical method.

Peaufiner l'intro

Throughout this article, C refers to a constant which specific value can change from line to line.

1 The scattering problem outside an open curve

Let Γ be a smooth non-intersecting open curve, and let $k \geq 0$ the wave number. We seek a solution of the problem

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \tag{3}$$

when one considers furthermore

- Dirichlet or Neumann boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \tag{4}$$

  or

$$\frac{\partial u}{\partial n} = u_N \text{ on } \Gamma \tag{5}$$

respectively.

- Suitable decay at infinity, given by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{6}$$

with $r = |x|$ for $x \in \mathbb{R}^2$.

In the preceding equation n stands for a smooth unit normal vector to Γ.

Existence and uniqueness of solutions to the previous problems are guaranteed by the following theorem.

Theorem 1. *(see e.g. [?, ?, ?]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (3,4,6) and (3,5,6) both possess a unique solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

Near an edge of the arc, μ is locally given by

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$.

For the definition of Sobolev spaces on smooth open curves, we follow [?] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \mid U \in H^s(\tilde{\Gamma}) \right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \mid \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential   We define the single-layer potential by

$$\mathcal{S}_k \lambda(x) = \int_\Gamma G_k(x - y)\lambda(y)d\sigma(y) \tag{7}$$

where G_k is the Green's function defined by

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi}\ln|z|, & \text{if } k = 0, \\[2mm] G_k(z) = \dfrac{i}{4}H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{8}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here H_0 is the Hankel function of the first kind. The solution u of the Dirichlet problem can be computed from (3,4,6) as

$$u = \mathcal{S}_k \lambda \tag{9}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the unique solution to

$$S_k \lambda = u_D. \tag{10}$$

Here, $S_k := \gamma \mathcal{S}_k$ where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k\mu(x) = \int_\Gamma n(y) \cdot \nabla G_k(x-y)\mu(y)d\sigma(y).$$

for any smooth function μ defined on γ. The normal derivative of $\mathcal{D}_k\mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \partial_n\mathcal{D}_k$. This operator admits the representation for $x \in \Gamma$

$$N_k\mu = \lim_{\varepsilon\to 0^+} \int_\Gamma n(y) \cdot \nabla G(x+\varepsilon n(x)-y)\mu(y)d\sigma(y). \tag{11}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\langle N_k\mu,\nu\rangle = \int_{\Gamma\times\Gamma} G_k(x-y)\mu'(x)\nu'(y)-k^2 G_k(x,y)\mu(x)\nu(y)n(x)\cdot n(y)d\sigma(x)d\sigma(y). \tag{12}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u to the Neumann problem (3,5,6) can be written as

$$u = \mathcal{D}_k\mu \tag{13}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ solves

$$N_k\mu = u_N\,. \tag{14}$$

2 Analytical setting

In this section, we introduce a new analytical setting that will allow us to analyze the integral equations in a unified and simple framework. We introduce the Chebyshev polynomials of first and second kinds, respectively given by

$$T_n(x) = \cos(n\arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}\,.$$

Let ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$. Moreover, let ∂_x the derivation operator. The Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0 \text{ and } (1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0$$

which can be rewritten under the form

$$
\begin{aligned}
(\omega\partial_x)^2 T_n &= -n^2 T_n\,, &\tag{15}\\
(\partial_x\omega)^2 U_n &= -(n+1)^2 U_n\,. &\tag{16}
\end{aligned}
$$

(Notice that by $(\partial_x\omega)f$ we mean $(\omega f)'$.) As we shall see, the preceding equations are crucial in our analysis.

2.1 Spaces T^s and U^s

Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \,\middle|\, \int_{-1}^1 \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L^2_\omega := \left\{ u \in L^1_{\text{loc}}(-1,1) \,\middle|\, \int_{-1}^1 f^2(x)\sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot, \cdot \rangle_\omega$ the inner products in $L^2_{\frac{1}{\omega}}$ and L^2_ω respectively. The Chebyshev polynomials satisfy

$$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } m = n = 0 \\ \pi/2 \text{ otherwise} \end{cases} \tag{17}$$

and

$$\int_{-1}^1 U_n(x)U_m(x)\sqrt{1-x^2} dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi/2 \text{ otherwise} \end{cases} \tag{18}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{19}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by

$$\hat{u}_n := \begin{cases} \dfrac{2}{\pi} \displaystyle\int_{-1}^1 \dfrac{u(x)T_n(x)}{\sqrt{1-x^2}} dx & \text{if } n \neq 0, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^1 \dfrac{u(x)}{\sqrt{1-x^2}} dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^1 \frac{u^2(x)}{\sqrt{1-x^2}} dx = \frac{\pi \hat{u}_0^2}{2} + \pi \sum_{n=1}^{+\infty} \hat{u}_n^2.$$

When u is furthermore a smooth function, the series (19) converges uniformly to u. Similarly, any function $v \in L^2_\omega$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^1 v(x)U_n(x)\sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^{1} v^2(x)\sqrt{1-x^2}dx = \frac{\pi}{2}\sum_{n=0}^{+\infty}\check{v}_n^2\,.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we may define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^s\hat{u}_n^2 < +\infty \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u,v\rangle_{T^s} = \frac{\pi}{2}\hat{u}_0\hat{v}_0 + \pi\sum_{n=1}^{+\infty}(1+n^2)^s\hat{u}_n\hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty}n^{2s}\left|\hat{u}_n\right|^2.$$

We denote by T^∞ the Fréchet space $T^\infty := \cap_{s\in\mathbb{R}}T^s$, and by $T^{-\infty}$ the set of continuous linear forms on T^∞. For $l \in T^{-\infty}$, we note $\hat{l}_n = l(T_n)$, so that for $u \in T^\infty$,

$$l(u) = \frac{\pi}{2}\hat{l}_0\hat{u}_0 + \pi\sum_{n=1}^{+\infty}\hat{l}_n\hat{u}_n\,.$$

We choose to identify the dual of $L^2_{\frac{1}{\omega}}$ to itself using the previous bilinear form. With this identification, any element of T^s with $s \geq 0$ can also be seen as an element of $T^{-\infty}$. Furthermore, the space T^{-s} can be defined for all $s \geq 0$ as

$$T^{-s} = \left\{ u \in T^{-\infty} \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^{-s}\hat{u}_n^2 < \infty \right\}.$$

Using the former identification T^{-s} becomes the dual of T^s. For $s < t$, the inclusion $T^s \subset T^t$ is compact.

Remark 1. *The spaces T^n correspond, up to a variable change, to the spaces H^n_e defined in [?, ?, ?, ?] among other works, that is, the restriction of the usual Sobolev space H^n to even periodic functions, as stated in Lemma 9.*

In a similar fashion, we define the following spaces:

Definition 2. *For all $s \geq 0$, we set*

$$U^s = \left\{ u \in L^2_\omega \;\middle|\; \sum_{n=0}^{+\infty}(1+n^2)^s\check{u}_n^2 \right\}.$$

We extend as before the definition to negative indices by setting U^{-s} to be the dual of U^s for $s \geq 0$, this time with respect to the duality $\langle\cdot,\cdot\rangle_\omega$.

Obviously, for any real s, if $u \in T^s$ the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x)$$

converges to u in T^s. The same assertion holds for $u \in U^s$ when T_n is replaced by U_n. Therefore

Lemma 1. $C^\infty([-1,1])$ *is dense in T^s and U^s for all $s \in \mathbb{R}$.*

The polynomials T_n and U_n are connected by the following formulas:

$$\forall n \geq 2, \quad T_n(x) = \frac{1}{2}\left(U_n - U_{n-2}\right), \tag{20}$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}. \tag{21}$$

Lemma 2. *For all real s, $T^s \subset U^s$ and for all $s > 1/2$, $U^s \subset T^{s-1}$.*

Proof. The first property is immediate from (20). When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging, allowing to identify u to a function in $T^{-\infty}$, with, in view of (21),

$$\hat{u}_0 = \sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2\sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since $u \in U^s$, $(1+n^2)^{s/2} |\check{u}|$ is in l^2 and by continuity of the adjoint of the Cesàro operator in l^2, the sequence $r_n := \left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}} |\check{u}_k|\right)_n$ is in l^2. But

$$
\begin{aligned}
\|u\|_{T^{s-1}}^2 &= \sum_{n=0}^{+\infty}(1+n^2)^{s-1} |\hat{u}_n|^2 \\
&\leq 4\sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left(\sum_{k=n}^{+\infty} |\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}} |\check{u}_k|\right)^2 . \\
&= 4\|r_n\|_{l^2}^2 .
\end{aligned}
$$

$\square$

One immediate consequence is that $T^\infty = U^\infty$. Moreover, we have the following result:

Lemma 3.
$$T^\infty = C^\infty([-1,1]).$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (15), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^1 \frac{(\omega\partial_x)^{2k}u(x)T_n(x)}{\omega(x)}dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, this proves that $C^\infty([-1,1]) \subset T^\infty$.

To prove the converse inclusion, first notice that, by normal convergence of the series, $T^\infty \subset C^0([-1,1])$. Now, let $u \in T^\infty$, it suffices to show that $u' \in T^\infty$ and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty} nu_n U_{n-1}(x).$$

Therefore, u' is in $U^\infty = T^\infty$. $\qquad\qquad\square$

Lemma 4. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and there exists a constant C such that for all $x \in [-1,1]$,*

$$|u(x)| \leq C \, \|u\|_{T^{1/2+\varepsilon}} \,.$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$|u(x)| \leq C \, \|u\|_{U^{3/2+\varepsilon}} \,.$$

Proof. We write

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\|T_n\|_{L^\infty} = 1$. Cauchy-Schwarz's inequality then yields

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \, \|u\|_{T^{\frac{1}{2}+\varepsilon}} \,.$$

For the second statement, we use the inclusion $U^s \subset T^{s-1}$ valid for $s > 1/2$, as established in Lemma 2. $\qquad\qquad\square$

Lemma 5. *We have $L_\omega^2 = \frac{1}{\omega}L_{\frac{1}{\omega}}^2$ and the pointwise multiplication by $\frac{1}{\omega}$ defines an bijective isometry from $L_{\frac{1}{\omega}}^2$ to L_ω^2 with inverse ω.*

Lemma 6. *The multiplication by ω^2 defines a continuous operator from U^s to T^s.*

Proof. This fact is a consequence of the identity $\omega^2 U_n = \frac{T_{n+2}-T_n}{2}$. $\qquad\square$

Lemma 7. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s as*

$$\langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}} \,.$$

For all real s, the operator $\omega\partial_x\omega$ can be extended into a continuous operator from $U^{s+1} \to T^s$ as

$$\langle \omega\partial_x\omega u, v\rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v\rangle_\omega \,.$$

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Proof. First, note that if $v \in T^\infty$, then $\omega \partial_x \omega v = (1-x^2)v' - xv \in T^\infty$. The definition of ∂_x is thus correct. The claimed continuity is an immediate consequence of

$$\omega \partial_x \omega U_n = -(n+1)T_{n+1}.$$

The second assertion comes from $T_n' = nU_{n-1}$. $\qquad \square$

Contrary to intuition, the operator ∂_x isn't continuous from T^s to T^{s-1}. However, the following result holds:

Corollary 1. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$.*

Proof. For the first case we use ∂_x is continuous from T^{s+2} to U^{s+1} and then the identity is continuous from U^{s+1} to T^s. For the second, we use the same arguments in the reverse order. $\qquad \square$

Lemma 8. *The operator $\omega \partial_x$ defined on T^∞ as*

$$\omega \partial_x u = \omega u'$$

has a continuous extension from T^1 to T^0. Similarly, the operator $\partial_x \omega$ has a continuous extension from U^1 to U^0.

Proof. For the first part, we use the fact that ∂_x is continuous from T^1 to U^0, and, by Lemma 5, ω is continuous from U^0 to T^0. For the second part, we use the fact that $\omega \partial_x \omega$ is continuous from U^1 to T^0, and from the same lemma, the multiplication by $\frac{1}{\omega}$ is continuous from T^0 to U^0. $\qquad \square$

We can now provide a characterization of the spaces T^s and U^s in terms of L^2 norms of the derivatives. For a function u defined on $[a, b]$, we denote by $\tilde{u}$ the function defined on $[\theta_a, \theta_b] := [\arccos(a), \arccos(b)]$ as

$$\tilde{u}(\theta) = u(\cos(\theta))$$

and by Vu the function defined as

$$Vu(\theta) := \sin(\theta)\tilde{u}(\theta).$$

Lemma 9. *A function u belongs to the space T^n if and only if $u = \tilde{u} \circ \arccos$ for some even function $\tilde{u}$ satisfying $\tilde{u} \in H^n(-\pi, \pi)$. In this case,*

$$\|u\|_{T_n} \sim \|\tilde{u}\|_{H^n}, |u|_{T^n} \sim |\tilde{u}|_{H^n}.$$

Similarly, u belongs to $\in U^n$ if and only if $u = \frac{1}{\sqrt{1-x^2}} Vu \circ \arccos$ for some odd function Vu in $H^n(-\pi, \pi)$. In this case,

$$\|u\|_{U_n} \sim |Vu|_{H^n}.$$

Moreover, if $u \in T^n$, then $(\omega \partial_x)^n u$ is in $L^2_{\frac{1}{\omega}}$ and

$$|u|_{T^n}^2 \sim \int_{-1}^1 \frac{((\omega \partial_x)^n u)^2}{\omega}.$$

If $u \in U^n$, then $(\partial_x \omega)^n u \in L^2_\omega$ and

$$\|u\|_{U_n} = \int_{-1}^1 \omega((\partial_x \omega)^n u)^2.$$

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Proof. The first two equivalences stem from the fact that

$$\hat{u}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{u}(\theta) \cos(n\theta), \quad \check{u}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} Vu(\theta) \sin((n+1)\theta) d\theta,$$

which can be verified by using the change of variables $x = \cos\theta$ in the definitions of $\hat{u}_n$ and $\check{u}_n$. Now, let us show that if $u \in T^n$, then $(\omega\partial_x)^n$ is in $L^2_{\frac{1}{\omega}}$. The operator $(\omega\partial_x)^2$ is continuous from T^s to T^{s-2} for all real s which implies the result if n is even. If n is odd, say $n = 2k + 1$, we write $(\omega\partial_x)((\omega\partial_x)^2)^k$, and conclude using Lemma 8. The same kind of proof also shows that if $u \in U^n$, $(\partial_x\omega)^n u \in L^2_\omega$. The rest of the proof can be performed by computing the quantities for functions in $C^\infty([-1,1])$, performing integrations by part and concluding with the density of T^∞ in T^s and U^s. □

2.2 Classes of operators on T^s

We now proceed to define a class of operators on T^s that is stable under composition and enjoys some properties similar to pseudo-differential operators. However, we shall not look for asymptotic expansions of symbols, to keep matters as simple as possible. The class we define may thus be quite far from a proper definition of pseudo-differential operators. The main result we need is Lemma 12. To ease the computations, we define $T_n = T_{|n|}$ for $n \in \mathbb{Z}$.

Definition 3. *Let $p \in \mathbb{R}$. If $A : T^\infty \to T^{-\infty}$ can be extended into a continuous operator from T^s to T^{s+p} for any $s \in \mathbb{R}$, we shall say that it is of order p in the scale T^s. When an operator is of order p for all $p \in \mathbb{N}$, we call it a smoothing operator.*

An operator $A : U^\infty \to U^{-\infty}$ which maps continuously U^s to U^{s+p} for all real s is said to be of order p in the scale U^s. When the family (U^s or T^s) is clear from the context, we simply say that A is of order p.

Definition 4. *Let A and B to operators of order s in the scale T^s. When the operator $A - B$ is of order $p \in (0, +\infty]$, we shall write*

$$A = B + T_p.$$

When the scale is U^s instead of T^s, we write

$$A = B + U_p.$$

Definition 5. *For any real p, an operator A belongs to the class S^p if there exists a function $a : \mathbb{N}^2 \to \mathbb{R}$ satisfying:*

(i) $\forall k \in \mathbb{Z}, \quad AT_k = \sum_{i=-\infty}^{+\infty} a(k,i)T_{k-i}$

(ii) $\forall \alpha \in \mathbb{N}, \forall i, k \in \mathbb{Z}, \quad |\Delta_k^\alpha a(k,i)| \leq C_{\alpha,i}(1+k^2)^{p-\alpha}$

(iii) *There exists $N \in \mathbb{N}$ such that*

$$|i| \geq N \implies \forall k \in \mathbb{Z}, \quad a(k,i) = 0.$$

The function a is called the symbol of A and is denoted by σ_A.

Remark 2. *The condition (i) does not define uniquely the symbol of A, since $T_{-n} = T_n$. We will not tackle the question whether conditions (ii) and (iii) ensure uniqueness. The notation σ_A thus refers to one possible symbol of A. The symbols can be viewed as infinite matrix with a finite band of diagonals. Condition (ii) further requires that the coefficients on each diagonal be smooth functions of k.*

Lemma 10. *If $A \in S^p$, then A is of order p in the scale T^s.*

Proof. Since the sum in the condition (i) is finite, by linearity, showing that the operator A is of order p amounts to proving that the operator A_i defined by

$$\forall k \in \mathbb{Z}, \quad A_i T_k = a(k, i) T_{k-i}$$

is of order p. We treat the case $i > 0$, the opposite case being analogous. Let $u \in T^s$ for some s, there holds

$$A_i u = \sum_{k=0}^{+\infty} a(k+i, i) \hat{u}_{k+i} T_k + \sum_{k=0}^{i} a(i-k, i) \hat{u}_{i-k} T_k.$$

Let Vu and Ru respectively the two terms of the rhs. Obviously, R is a smoothing operator. Now, for all $k \in \mathbb{N}$ let

$$\hat{v}_k := a(i+k, i) \hat{u}_{i+k}.$$

Applying Peetre's inequality, one has

$$(1+k^2)^{n+s} |\hat{v}_k|^2 \leq C \left(1+i^2\right)^{|n+s|} \left(1+(i+k)^2\right)^{n+s} |a(k+i, i)|^2 |\hat{u}_{k+i}|^2.$$

Condition (ii) with $\alpha = 0$ yields

$$|a(k+i, i)|^2 \leq C \left(1 + (k+i)\right)^{-2n} \leq 2C \left(1 + (k+i)^2\right)^{-n}.$$

Therefore, $\|Vu\|_{T^{s+n}} \leq C(1+i)^{|n+s|} \|u\|_{T^s}$ which shows that A is of order p. $\square$

Lemma 11. *If $A \in S^p$, $B \in S^q$, then AB and BA are in S^{p+q},*

Proof. The composition of two operators $AT_k = \sum_{i=-\infty}^{+\infty} a(k, i) T_{k-i}$ and $BT_k = \sum_{i=-\infty}^{+\infty} b(k, i) T_{k-i}$ gives rise to an operator $C = \sum_{i=-\infty}^{+\infty} c(k, i) T_{k-i}$ for which a symbol is given by

$$c(k, i) = \sum_{j=-\infty}^{+\infty} a(k-j, i-j) b(k, j).$$

This formula is obtained writing the expression of ABT_n using their symbol and using the identity $T_i T_j = T_{i+j} + T_{i-j}$. The number of non-zero terms in this decomposition is obviously finite, so we must simply check the requirement (ii). It is enough to show that for any $j \in \mathbb{Z}$, the function $c_{i,j}(k) := a(k-j, i-j) b(k, j)$ satisfies

$$\forall \alpha \in \mathbb{N}, \forall i, k, \in \mathbb{Z}, |\Delta_k^\alpha c_{i,j}| \leq C_{i,j,k} (1+k^2)^{p+q-\alpha}.$$

To prove this, one can check by induction that for any two functions $c, d : \mathbb{N} \to \mathbb{R}$, $\Delta^\alpha(cd)(k)$ is a linear combination of terms of the form $\Delta^\beta c(k_1) \Delta^{\alpha-\beta} d(k_2)$, where k_1 and k_2 lie in the interval $[k, k-\alpha]$. The conclusion follows after applying asumption (ii) for A and B and applying Peetre's inequality to the terms $(1 + (k_1)^2)^{p+\beta}(1 + (k_2)^2)^{q+\alpha-\beta}$. and apply asumption (ii) for A and B. $\square$

Lemma 12. *If $A \in S^p$ and $B \in S^q$, the $AB - BA$ is in S^{p+q-1}.*

Proof. A symbol of $AB - BA$ is given by

$$\sigma_{AB-BA}(k,i) = \sum_{j=-\infty}^{+\infty} a(k-j, i-j)b(k,j) - \sum_{j=-\infty}^{+\infty} b(k-j, i-j)a(k,j).$$

In the second sum, we change the index to $j' = i - j$, yielding

$$
\begin{aligned}
\sigma_{AB-BA}(k,i) &= \sum_{j=-\infty}^{+\infty} a(k-j, i-j)b(k,j) - \sum_{j=-\infty}^{+\infty} b(k-i+j', j')a(k, i-j'). \\
&= \sum_{j=-\infty}^{+\infty} \left[a(k-j, i-j) - a(k, i-j) \right] b(k,j) \\
&\quad - \sum_{j=-\infty}^{+\infty} a(k, i-j) \left[b(k-i+j') - b(k,j) \right].
\end{aligned}
$$

We show that asumption (ii) holds for one of the terms $\left[a(k-j, i-j) - a(k, i-j) \right] b(k,j)$ when j is positive. The other cases are treated analogously. We can write

$$
\begin{aligned}
a(k-j, i-j) - a(k, i-j) &= \sum_{l=0}^{j-1} a(k-j+l, i-j) - a(k-j+l+1, i-j) \\
&= -\sum_{l=0}^{j-1} \Delta a(k-j+l+1, i-j).
\end{aligned}
$$

The result then follows from the same considerations as in the proof of the previous result. $\square$

Lemma 13. *If P is a polynomial, the multiplication by P defines an operator of S^0.*

Proof. For all n, we have $xT_n = \frac{T_{n+1} + T_{n-1}}{2}$, thus x is in the class S^0. By Lemma 11, the same is true for x^n for any n, and by linearity, for any polynomial. $\square$

The next two lemmas enlarge the class of operators for which we can assess the order. Lemma 15 allows us to treat a rather general class of operators and will allow us to evaluate the order of the remainders in Taylor expansions of the kernels.

Lemma 14. *If ψ is a C^∞ function on $[-1, 1]$, then the operator*

$$u(t) \mapsto \psi(t)u(t)$$

is of order 0, and for any $s \in \mathbb{R}$,

$$\|\psi u\|_{T^s} \leq C 2^{|s|/2} \|u\|_{T^s} \|\psi\|_{T^{|s|+1}}.$$

where C is independent of ψ and s.

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Proof. Let $u \in T^s$, we rewrite u as

$$u = \sum_{n=-\infty}^{+\infty} u'_n T_n$$

where for $n < 0$ we define $T_n = T_{|n|}$, and with

$$u'_n = \begin{cases} u_0 & \text{if } n = 0 \\ \frac{u_{|n|}}{2} & \text{otherwise.} \end{cases}$$

We apply the same idea to ψ, and using $T_m T_n = T_{m+n} + T_{m-n}$,

$$\psi u = \sum_{m,n} u'_n \psi'_m (T_{m+n} + T_{m-n}) = \sum_m \left(\sum_n u'_n (\psi'_{n+m} + \psi'_{n-m}) \right) T_m$$

that is, by parity

$$\psi u = 2 \sum_{m,n} u'_m \psi'_{m-n} T_n$$

Using Peetre's inequality, we have

$$(1 + n^2)^{s/2} |(\psi u)_n| \le 2^{|s|/2+1} \sum_m (1 + m^2)^{s/2} |u'_m| (1 + |n - m|^2)^{|s/2|} \left|\psi'_{n-m}\right|$$

and by Young's inequality with $r = 2, p = 2, q = 1$,

$$\|\psi u\|_s^2 \le 2^{|s|+2} \|u\|_s^2 \sum_{m=-\infty}^{+\infty} (1 + m^2)^{|s|/2} |\psi'_m|$$

The last sum is finite because $\psi \in T^\infty$ and

$$\sum_{m=-\infty}^{\infty} (1 + m^2)^{|s|/2} |\psi'_m| \le \left(\sum_{m=-\infty}^{+\infty} \frac{1}{1 + m^2} \right) \sum_{m=-\infty}^{+\infty} (1 + m^2)^{|s|+1} |\psi'|_m^2.$$

$\square$

Lemma 15. *Let $g(x, y)$ the kernel of an integral operator of order $\alpha \in \mathbb{R}$ in the spaces T^s, that is,*

$$G : u \mapsto \int_{-1}^1 \frac{g(x,y)u(y)}{\omega(y)} dy$$

is continuous from T^s to $T^{s+\alpha}$ for all s. Let $R(x, y)$ a C^∞ function. Then the operator

$$K : \int_{-1}^1 \frac{g(x,y)R(x,y)u(y)}{\omega(y)} dy$$

is of order α.

Proof. Since R is in C^∞, one can show that R admits the following expression:

$$R(x, y) = \sum_{m,n} r_{m,n} T_m(x) T_n(y) \tag{22}$$

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Moreover, the regularity of R ensures $r_{m,n}$ satisfies for all $s, t \in \mathbb{R}$,

$$\sum_{m,n} (1+m^2)^s (1+n^2)^t \, |r_{m,n}|^2 < +\infty.$$

To prove this property, one can for example apply the operator $(\omega \partial_x)^2$ repeatedly in the two variables. The resulting function is C^∞, and in particular, square integrable on $[0,1] \times [0,1]$. We then write the Parseval's identity and the result follows. We can thus write

$$Ku = \sum_{m,n} r_{m,n} T_m G T_n u$$

where for each m, n, the operator $T_m G T_n$ is defined by

$$T_m G T_n u(x) = T_m(x) \int_{-1}^1 \frac{G(x,y) T_n(y) u(y)}{\omega(y)} dy.$$

This operator is in $L(T^s, T^{s+\alpha})$ by the previous lemma, with

$$\|T_m G T_n\|_{T^s \to T^{s+\alpha}} \le \|G\|_{T^s \to T^{s+\alpha}} \, 2^{|s|+|s+\alpha|} (1+n^2)^{|s|+1} (1+m^2)^{|s+\alpha|+1}.$$

thus, the series in (22) is normally convergent in $L(T^s, T^{s+\alpha})$, which proves the claim. $\qquad\square$

In particular, when $g(x,y) \equiv 1$, G is a smoothing operator thus

Corollary 2. *Let* $F \in C^\infty([-1,1]^2)$. *Then*

$$u \mapsto \int_{-1}^1 \frac{F(x,y) u(y)}{\omega(y)} dy$$

is a smoothing operator.

3   Laplace equation on a flat segment

In this section, we restrict our attention to the case where Γ is the open segment $(-1,1) \times \{0\}$, and $k = 0$. We study the properties of the equations

$$S\lambda = -u_D$$

and

$$N\mu = u_N$$

and show their invertibility in a range of Sobolev-like spaces. This problem has been considered thoroughly, both in terms of analytical and numerical properties in the literature (see for instance [**?, ?**]), and it turns out that the Chebyshev polynomials of first and second kind play a very important role. However, we go further compared to the literature by by constructing a functional framework close to Sobolev spaces, based on Chebyshev polynomials that allows us to give a complete framework for the existence and uniqueness of the solutions to the preceding equations, as well as new preconditioners.

3.1 Single layer equation

In this section we focus on the equation $S\lambda = g$, that is we seek $\lambda \in \tilde{H}^{-1/2}$ such that

$$-\frac{1}{2\pi} \int_{-1}^{1} \log |x - y| \lambda(y) = -g(x), \quad \forall x \in (-1, 1). \tag{23}$$

This equation is sometimes called "Symm's integral equation" and its resolution has received a lot of attention in the 1990's. Numerical methods, using both collocation and Galerkin have been presented and analyzed [?, ?, ?, ?, ?].

Our analysis lies on the following formula. For a proof, see for example [?] Theorem 9.2. Note that this is also the main ingredient in several connected works, such as [?] and [?].

Lemma 16.

$$-\frac{1}{2\pi} \int_{-1}^{1} \frac{\ln |x - y|}{\sqrt{1 - y^2}} T_n(y) dy = s_n T_n(x)$$

where

$$s_n = \begin{cases} \dfrac{\ln(2)}{2} & \text{if } n = 0 \\ \dfrac{1}{2n} & \text{otherwise.} \end{cases}$$

Using the decomposition of g and of the logarithmic kernel on the basis T_n, we see that the solution λ to equation (23) admits the following expansion

$$\lambda(x) = \frac{1}{\sqrt{1 - x^2}} \sum_{n=0}^{+\infty} \frac{\hat{g}_n}{s_n} T_n(x). \tag{24}$$

We deduce the following well-known fact:

Corollary 3. *If the data g is in $C^\infty([-1, 1])$, the solution λ to the equation*

$$S\lambda = g$$

is of the form

$$\lambda = \frac{\alpha}{\sqrt{1 - x^2}}$$

with $\alpha \in C^\infty([-1, 1])$.

Proof. Let $\alpha = \sqrt{1 - x^2}\lambda$ where λ is the solution of $S\lambda = g$ with $g \in C^\infty([-1, 1])$. Lemma 3 implies that $g \in T^\infty$, and by equation (24),

$$\hat{\alpha}_n = \frac{\hat{g}_n}{s_n},$$

so α also belongs to $T^\infty = C^\infty([-1, 1])$. $\qquad\square$

We follow [?] by noticing that the behavior in $\frac{1}{\sqrt{1-x^2}}$ is consistent with the expected singularity near the edges and introduce the weighted single layer operator as the operator that appeared in Proposition 16.

Definition 6. *(See [?]) Let S_ω be the weighted single layer operator defined by*

$$S_\omega : \alpha \in C^\infty([-1,1]) \longrightarrow -\frac{1}{2\pi} \int_{-1}^1 \frac{\ln|x-y|}{\omega(y)} \alpha(y) dy$$

We also recall that the operator $(\omega \partial_x)^2$ is defined by

$$(\omega \partial_x)^2 : \alpha \in C^\infty([-1,1]) \longrightarrow (1-x^2)\alpha''(x) - x\alpha'(x).$$

The action of these operators on T^∞ is easy to analyze using (15) and Lemma 16. By density of T^∞ in T^s for all s, we get:

Proposition 1. *The operator S_ω is a self-adjoint, positive definite operator, and defines a continuous bijection from T^s to T^{s+1} for all real s. In particular, S_ω is of order 1 and is compact in T^s. Similarly, for any $s \in \mathbb{R}$, the operator $-(\omega \partial_x)^2$ is positive, self-adjoint, and of order -2.*

Proof. It suffices to remark that if $u = \sum_{n=0}^\infty \hat{u}_n T_n \in T^s$, then

$$S_\omega u = \frac{\ln(2)}{2} \hat{u}_0 T_0 + \sum_{n=1}^\infty \frac{\hat{u}_n}{2n} T_n$$

while

$$-(\omega \partial_x)^2 u = \sum_{n=0}^\infty n^2 \hat{u}_n T_n \,.$$

$\square$

Remark 3. *Of course, S_ω is in the class S^1 defined in subsection 2.2*

To obtain the solution of (23), we can thus solve

$$S_\omega \alpha = -u_D, \tag{25}$$

and let $\lambda = \frac{\alpha}{\omega}$. The next lemmas make clear the connection between the spaces T^s with $s = \pm\frac{1}{2}$ and the usual Sobolev spaces. Equivalent results are proved in [?].

Lemma 17. *We have $\tilde{H}^{-1/2}(-1,1) = \frac{1}{\omega} T^{-1/2}$ and for all $u \in \tilde{H}^{-1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}} \,.$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the operator S is positive and bounded from below on $\tilde{H}^{-1/2}(-1,1)$, (see [?] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1,1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle Su, u \rangle}.$$

On the other hand, it is obvious that if $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_\omega \alpha, \alpha \rangle_\omega}.$$

It remains to notice that for $\alpha = \omega u$, the two duality products coincide. $\square$

Lemma 18. *Similarly,*

$$H^{1/2}(-1,1) = T^{1/2}$$

and for $u \in H^{1/2}(-1,1)$ we have

$$\|u\|_{H^{1/2}} \sim \|u\|_{T^{1/2}} \; .$$

Proof. We know that $(H^{1/2})'(-1,1) = \tilde{H}^{-1/2}(-1,1)$ (cf [**?**] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v \neq 0} \frac{\langle u, v \rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}} \; .$$

According to the previous lemma, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v \rangle = \langle u, \alpha \rangle_{\omega}$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha \neq 0} \frac{\langle u, \alpha \rangle_{\omega}}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle \cdot, \cdot \rangle_{\omega}$. $\qquad\square$

We are now in a position to find an expression for the inverse of S_{ω}. An explicit inverse has already appeared in the literature. In particular, in [**?**, **?**], explicit variational forms for this inverse operator are derived rigorously. (A similar method is also employed in the recent paper [**?**] in $\mathbb{R}^3$ for the case of the unit disk.) We just state here the following formal decomposition:

$$\frac{d^2}{dxdy} \log \frac{M(x,y)}{|x-y|^2} = \frac{-1+xy}{2|x-y|^2} = \sum_{n=1}^{+\infty} n T_n(x) T_n(y)$$

with $M(x,y) = \frac{1}{2} \left((y-x)^2 + (\omega(x) + \omega(y))^2 \right)$.

However, using the preceding analysis, we have an alternative way of defining this exact inverse, which leads to an expression in the form of the square root of a local operator. To state the next result, we define the operator π_0 as the $L^2_{1/\omega}$ orthogonal projector on T_0. Namely

$$\pi_0 \alpha(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\alpha(y)}{\omega(y)} dy \; .$$

The preceding definition can be extended to $u \in T^s$ for any $s \in \mathbb{R}$ by setting $\pi_0 u$ as the solution of

$$\begin{cases} \langle \alpha - \pi_0 \alpha, T_0 \rangle_{\frac{1}{\omega}} = 0, \\ \pi_0 \alpha \in \text{Span}(T_0), \end{cases}$$

since $T_0 \in T^{\infty}$. Of course, π_0 is continuous from T^s to T^{∞} for any s.

Theorem 2. *There holds*

$$S_{\omega}^2 = \frac{1}{4} \left(-(\omega \partial_x)^2 + \frac{1}{\ln(2)^2} \pi_0 \right)^{-1} \; . \tag{26}$$

Proof. The Chebyshev polynomials (T_n) are a common Hilbert basis of eigenvectors for the three operators S_ω, $-(\omega\partial_x)^2$ and π_0, so it is sufficient to compute $4S_\omega^2\left(-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0\right)T_n$ and check that this is indeed equal to T_n. One has using the explicit eigenvalues for S_ω. This is easily verified using the following expressions

$$S_\omega T_n = \frac{1}{2n}T_n\,,\ \pi_0 T_n = 0 \text{ and } -(\omega\partial_x)^2 T_n = n^2 T_n \text{ if } n \neq 0\,,$$

while

$$S_\omega T_0 = \frac{\ln(2)}{2}T_0\,,\ \pi_0 T_0 = T_0 \text{ and } -(\omega\partial_x)^2 T_0 = 0 \text{ otherwise.}$$

$\square$

From the preceding formula, we can extract the explicit inverse of S_ω in terms of the square root of the inner operator.

Corollary 4. *The inverse of S_ω can be equivalently expressed as*

$$S_\omega^{-1} = 2\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}\,. \tag{27}$$

The last result shows that $\sqrt{-(\omega\partial_x)^2}$ and S_ω can be thought as inverse operators (modulo constant terms) and that, at the very least, $\sqrt{-(\omega\partial_x)^2}$ can be used as a preconditioner for S_ω. Indeed, $2\sqrt{-(\omega\partial_x)^2}S_\omega$ is a compact perturbation of identity. This makes a clear link with the approximation of the Dirichlet to Neumann map proposed in [?] in terms of a square root operator. The link will be even clearer when Helmholtz equation will be considered. In Table 1, we compare the number of iterations for the numerical resolution of Equation (25) by the method detailed in ?? without preconditioner, and with a preconditioner given by $M^{-1}[B]M^{-1}$ where M is the mass matrix and $[B]$ is the Galerkin matrix of the operator $\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$. The right hand side in (25) is chosen as $u_D(x) = (x^2 + 0.001)^{-1/2}, x \in (-1,1)$.

N	with Prec.		without Prec.	
	n_{it}	t(s)	n_{it}	t(s)
50	7	0.11	35	0.21
200	7	0.14	53	0.55
800	7	0.29	76	2.0
3200	7	0.95	107	9.5

Table 1: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.
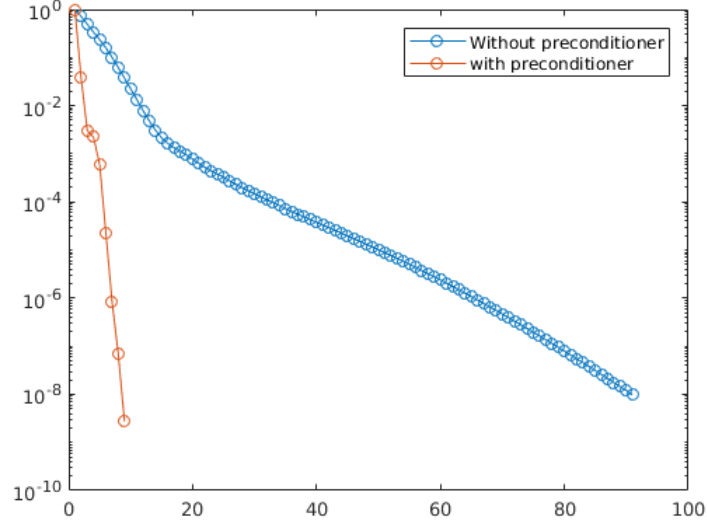
Figure 1: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N = 1600$.

3.2 Hypersingular equation

We now turn our attention to the equation

$$N\mu = g \tag{28}$$

Similarly to the previous section and following again the idea of [?], we consider a rescaled version of the hypersingular operator $N_\omega := N\omega$ defined by

$$N_\omega \mu = \lim_{\varepsilon \to 0} \int_{-1}^{1} n(y) \cdot \nabla G(x + \varepsilon n(x) - y)\sqrt{1 - y^2} dy$$

We can get the solution to equation (28) by solving

$$N_\omega \beta = u_N, \tag{29}$$

and letting $\mu = \omega\beta$. We now show that N_ω can also be analyzed in our functional framework, using this time the spaces U^s.

Lemma 19. *For any β, β', one has*

$$\langle N_\omega \beta, \beta' \rangle_\omega = \langle S_\omega \omega \partial_x \omega \beta, \omega \partial_x \omega \beta' \rangle_{\frac{1}{\omega}} .$$

Proof. It is sufficient to show this formula for β and β' in U^∞ by density. Indeed, for such β, β', both sides of the identity define continuous bilinear forms on T^∞. We use the well-known integration by part formula

$$\langle Nu, v \rangle = \langle S\partial_x u, \partial_x v \rangle ,$$

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valid when u and v vanish at the extremities of the segment (see for example [?]). For a smooth β, we thus have

$$\langle N(\omega\beta), (\omega\beta')\rangle = \langle S\partial_x(\omega\beta), \partial_x(\omega\beta')\rangle$$

which obviously implies the announced identity. $\qquad\square$

Proposition 2. N_ω *is a positive definite, self-adjoint operator continuous from* U^s *to* U^{s-1} *for all real s. For all $n \in \mathbb{N}$, we have*

$$N_\omega U_n = \frac{n+1}{2}U_n.$$

Moreover, $-(\partial_x\omega)^2$ is also positive definite of order 2.

Proof. From identity $T'_{n+1} = (n+1)U_n$ and Equation (15) we obtain

$$\omega\partial_x\omega U_n = -(n+1)T_{n+1}.$$

Therefore, by Lemma 19

$$
\begin{aligned}
\langle N_\omega U_m, U_n\rangle_\omega &= (n+1)(m+1)\langle S_\omega T_{m+1}, T_{n+1}\rangle_{\frac{1}{\omega}} \\
&= \delta_{m=n}\frac{n+1}{2}.
\end{aligned}
$$

The fact that $-(\partial_x\omega)^2$ is self-adjoint positive definite of order 2 is a consequence of Equation (16). $\qquad\square$

Like before, we have the following link between $U^{-1/2}$, $U^{1/2}$ and the usual Sobolev spaces.

Lemma 20. *The following identities hold,*

$$\omega U^{1/2} = \tilde{H}^{1/2}(-1, 1),$$

$$U^{-1/2} = H^{-1/2}(-1, 1),$$

with

$$\|\omega u\|_{\tilde{H}^{1/2}} \sim \|u\|_{U^{1/2}}, \quad \|u\|_{U^{-1/2}} \sim \|u\|_{H^{-1/2}}.$$

Proof. It suffices to remark that

$$\|\omega u\|_{\tilde{H}^{1/2}} \sim \sqrt{\langle N\omega u, \omega u\rangle} = \sqrt{\langle N_\omega u, u\rangle_\omega} \sim \|u\|_{U^{1/2}}.$$

The second equality follows from the same calculations that were done in Lemma 18,, as well as the norm equivalence. $\qquad\square$

As an application of this result, one can also derive the formal expansions as in [?]

$$\frac{1}{(x-y)^2} = \sum_{n=0}^{+\infty} 2(n+1)U_n(x)U_n(y),$$

that lead, by applying for $(\partial_x \omega)^{-2}$ on both sides, to the following explicit kernel for the inverse of N_ω:

$$\ln\left(\frac{(y-x)^2 + (\omega(x)+\omega(y))^2}{2|x-y|}\right) = \sum_{n=0}^{+\infty} \frac{2U_n(x)U_n(y)}{n+1}.$$

Here instead, we give a simple expression of the inverse of N_ω as the inverse square root of a local operator:

Theorem 3. *There holds*

$$N_\omega^2 = -\frac{1}{4}(\partial_x \omega)^2.$$

The inverse of N_ω is therefore

$$N_\omega^{-1} = 2\sqrt{-(\partial_x \omega)^{-2}}. \tag{30}$$

In Table 2, we compare the number of iterations for the numerical resolution of Equation (29) by the method detailed in **??** without preconditioner, and with a preconditioner given by $M^{-1}[B]M^{-1}$ where M is the mass matrix and $[B]$ is the Galerkin matrix of the operator $\sqrt{-(\partial_x \omega)^{-2}}$. The right hand side in (29) is chosen as $u_N(x) = (x^2 + 0.001)^{1/2}, x \in (-1, 1)$.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
50	4	0.05	50	0.05
200	3	0.05	200	0.25
800	3	0.06	799	3.7
3200	3	0.6	3007	630

Table 2: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.
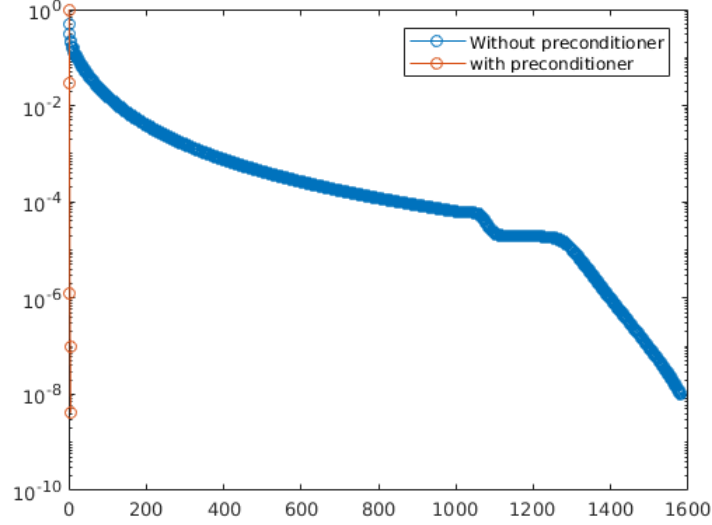
Figure 2: Number of iteration in the resolution of the hypersingular integral equation with a mesh of size $N = 1600$. The importance of preconditioning in this case is more obvious than in the case of the single-layer equation.

Remark 4. *In [?], the main theorem is equivalent to stating that $N_\omega S_\omega$ and $S_\omega N_\omega$ are bicontinuous operators in T^s, with a spectrum concentrated around $\frac{1}{4}$, which can be exploited for preconditioning purposes. It is shown that N_ω is continuous from T^s to T^{s-1} for all $s > 1$ (in fact, this remains true for $s > \frac{1}{2}$). The two main arguments involved in the proof are the explicit expression of $N_\omega T_n$ and the continuity of the adjoint of the Cesaro operator in $l^2(\mathbb{N})$. The same arguments can be used to prove that S_ω is also a bicontinuous operator from U^s to U^{s+1} for all $s > 1/2$.*

4   Helmholtz equation

In this section, we introduce preconditioners for the integral equations on $\Gamma = (-1, 1)$ for the Helmholtz equation. Recall the definition of the single layer and hypersingular operators, S_k and N_k, given in (7) and (11), and the integral equations for the Dirichlet and Neumann problems, (10) and (14). As before, let $S_{k,\omega} := S_k \frac{1}{\omega}$ and $N_{k,\omega} := N_k \omega$. We begin by establishing the following result:

Theorem 4. *The following commutations hold:*

$$S_{k,\omega} \left[-(\omega \partial_x)^2 - k^2 \omega^2 \right] = \left[-(\omega \partial_x)^2 - k^2 \omega^2 \right] S_{k,\omega},$$

$$N_{k,\omega} \left[-(\partial_x \omega)^2 - k^2 \omega^2 \right] = \left[-(\partial_x \omega)^2 - k^2 \omega^2 \right] N_{k,\omega}.$$

Proof. We start with the first commutation. Since $(\omega \partial_x)^2$ is self adjoint and

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symmetric, we have

$$S_{k,\omega}(\omega\partial_x)^2 = \int_{-1}^{1} \frac{(\omega_y\partial_y)^2 \left[G_k(x-y)\right] u(y)}{\omega(y)},$$

where we use the notation ω_y and ∂_y to emphasize the dependence in the variable y. Thus,

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = \int_{-1}^{1} \frac{D_k(x,y)u(y)}{\omega(y)},$$

where $D_k(x,y) := \left[(\omega_y\partial_y)^2 - (\omega_x\partial_x)^2\right]\left[G_k(x-y)\right]$. One has

$$D_k(x,y) = G_k''(x-y)(\omega_y^2 - \omega_x^2) + G_k'(x-y)(y+x).$$

Since G_k is a solution of the Helmholtz equation, we have for all $(x \neq y) \in \mathbb{R}^2$

$$G_k'(x-y) = (y-x)(G_k''(x-y) + k^2 G(x-y)),$$

thus

$$D_k(x,y) = G_k''(x-y)\left(\omega_y^2 - \omega_x^2 + y^2 - x^2\right) + k^2(y^2 - x^2)G_k(x-y).$$

A careful analysis shows that no Dirac mass appears in the previous formula, that is, the previous formula is an equality of two functions in $T^{-\infty}$. Note that $y^2 - x^2 = \omega_x^2 - \omega_y^2$ so the first term vanishes and we find

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = k^2 \left(\omega^2 S_{k,\omega} - S_{k,\omega}\omega^2\right).$$

The proof of the second commutation is postponed to Appendix A. $\qquad\square$

This theorem implies that the operators $S_{k,\omega}$ and $N_{k,\omega}$ share the same eigenvectors as, respectively, $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$ and $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]$. We can look for eigenfunctions of the operator $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$, to find a diagonal basis for $S_{k,\omega}$. They are the solutions to the differential equation

$$(1-x^2)y'' - xy' - k^2\omega^2 y = \lambda y.$$

Once we set $x = \cos\theta$, $\tilde{y}(\theta) = y(x)$, $q = \frac{k^2}{4}$, $a = \lambda + 2q$, $\tilde{y}$ is a solution of the standard Mathieu equation

$$\tilde{y}'' + (a - 2q\cos(2\theta))\tilde{y} = 0. \qquad (31)$$

There are a discrete set of values $a_{2n}(q)$ for which this equation possesses an even and 2π periodic function. The corresponding solution is known as the Mathieu cosine, and usually denoted by ce_n. Here, we use the notation ce_n^k to emphasize the dependency in the parameter $k = \sqrt{2q}$ of those functions. The normalization is taken as

$$\int_0^{2\pi} \mathrm{ce}_n^k(\theta)^2 d\theta = \pi.$$

They satisfy

$$\int_{-\pi}^{\pi} \mathrm{ce}_n^k(\theta)\mathrm{ce}_m^k(\theta) = \pi\delta_{m,n}.$$

Any even 2π periodic function in $L^2(-\pi,\pi)$ can be expanded along the functions ce_n, with the coefficients obtained by orthonormal projection. Letting

$$T_n^k := \mathrm{ce}_n^k(\arccos(x)),$$

in analogy to the zero-frequency case, we have

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] T_n^k = \lambda_{n,k}^2 T_n^k.$$

For large n, using the general results from the theory of Hill's equations (see [?, eq. 28.29.21]) we have the following asymptotic formula for $\lambda_{n,k}$:

$$\lambda_{n,k}^2 = n^2 - \frac{k^4}{16n^2} + o\left(n^{-2}\right).$$

The first commutation established in Theorem 4 implies that the Matthieu cosines are also the eigenfunctions of the single-layer operator. An equivalent statement is given in [?, Thm 4.2], if we allow the degenerate case $\mu = 0$. A similar analysis can be applied to the hypersingular operator. The eigenfunctions of $\left[(\partial_x\omega)^2 - k^2\omega^2\right]$ are given by

$$U_n^k := \frac{\mathrm{se}_n^k(\arccos(x))}{\omega(x)}$$

where se_n^k are the so-called Matthieu sines, which also satisfy the Matthieu differential equation (31), but with the condition that they must be odd 2π periodic functions. Unfortunately, the lack of knowledge about the eigenvalues of $S_{k,\omega}$ and $N_{k,\omega}$ prevents us from applying a similar analysis as that performed in the first part of this work. Instead, we will perform a perturbation analysis, much like [?].

4.1 Dirichlet problem

In this section, we focus on the Dirichlet problem with non-zero frequency, and the corresponding integral equation

$$S_k\lambda = u_D \tag{32}$$

Here again, we introduce the weighted operator $S_{k,\omega} := S_k \frac{1}{\omega}$, i.e.

$$S_{k,\omega}\alpha : x \mapsto \int_{-1}^1 \frac{H_0(k|x-y|)\alpha(y)}{\omega(y)}\,dy.$$

If we let $\lambda = \dfrac{\alpha}{\omega}$, then equation (32) is equivalent to

$$S_{k,\omega}\alpha = u_D.$$

The Hankel function can be written as

$$H_0(z) = \frac{-1}{2\pi}\ln|z|J_0(z) + F_1(z^2)$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. Using the power series definition of J_0,

$$
\begin{aligned}
\frac{i}{4} H_0(k\,|x-y|) = {} & \frac{-1}{2\pi} \ln|x-y| \\
& + \frac{1}{2\pi} \frac{k^2}{4} (x-y)^2 \ln|x-y| \\
& + (x-y)^4 \ln|x-y| F_2(x,y) + F_3(x,y)
\end{aligned}
\tag{33}
$$

where F_2 and F_3 are C^∞. Let us introduce the operators O_n defined for $n \geq 1$ as

$$
O_n : \alpha \mapsto -\frac{1}{2\pi} \int_{-1}^{1} (x-y)^{n-1} \ln|x-y| \frac{\alpha(y)}{\omega(y)}.
$$

recall the definition of the class S^n introduced in subsection 2.2. We have the following result:

Lemma 21. *For every n, O_n is in the class S^n.*

Proof. This can be shown by a simple induction. O_1 is just S_ω, which is indeed in S^1. Let $n \geq 2$, and assume $O_{n-1} \in S^{n-1}$. We have

$$
O_n = x O_{n-1} - O_{n-1} x.
$$

As shown in Lemma 13, the multiplication by x defines an operator of S^0. By assumption, O_{n-1} is in S^{n-1}, thus Lemma 12 implies that $O_n \in S^n$, which concludes the proof. $\qquad\square$

Using the notation introduced in Definition 4, we have the following result:

Lemma 22. *The operator $S_{k,\omega}$ admits the following expansion*

$$
S_{k,\omega} = S_\omega - \frac{k^2}{4} O_3 + T_5.
$$

Proof. From equation (33), it suffices to show that the operator

$$
R_5 : \alpha \mapsto \int_{-1}^{1} (x-y)^4 \ln|x-y| F_2(x,y) \frac{\alpha(y)}{\omega(y)}
$$

is of order 5. Since O_5 is of order 5, this is true in view of Lemma 15. $\qquad\square$

In particular, the operator $S_{k,\omega}$ is well defined on $T^{-\infty}$, and is of order 1. Moreover, we see that $S_{k,\omega}$ is a compact perturbation of S_ω in T^s. Therefore, the product $\sqrt{(-\omega\partial_x)^2} S_{k,\omega}$ is a compact perturbation of identity. In fact, using Equation 33, we can see that $S_{k,\omega}(-\omega\partial_x)^2 S_{k,\omega} = \frac{I_d}{4} + K$ where K is a compact operator of order 2. However, even if the condition number of $\sqrt{-(\omega\partial_x)^2} S_{k,\omega}$ is independent of the mesh size, numerical experiments show that it increases with k. It is thus desirable to include in the preconditioner a correction for the wave number, which is the object of the next result. We first give an identity which will be useful also in the treatment of the hypersingular equation.

Lemma 23. *There holds*

$$S_\omega(\omega\partial_x)^2 O_3 + O_3(\omega\partial_x)^2 S_\omega = 4S_\omega\omega^2 S_\omega + T_4.$$

Proof. One has

$$(\omega\partial_x)^2\left((x-y)^2\ln|x-y|\right) = 2\omega^2(x)\ln|x-y| - 2x(x-y)\ln|x-y| + P(x,y)$$

where P is a polynomial in x and y. Dividing by $-2\pi\omega(y)$ and integrating on both sides with respect to y, we get

$$(\omega\partial_x)^2 O_3 = 2\omega^2 S_\omega - 2x O_2 + R$$

where R is a smoothing operator in view of Corollary 2. Equalizing the adjoint operators (with respect to the T^0 scalar product), we get

$$O_3(\omega\partial_x)^2 = 2S_\omega\omega^2 + 2O_2 x + R^*$$

where R^* is also a smoothing operator (note that O_1 is self-adjoint while $O_2^* = -O_2$). This gives

$$S_\omega(\omega\partial_x)^2 O_3 + O_3(\omega\partial_x)^2 S_\omega = 4S_\omega\omega^2 S_\omega - 2S_\omega x O_2 + 2O_2 x S_\omega + R_\infty$$

where R_∞ is again a smoothing operator. Now, notice that, by definition of O_3,

$$O_2 x S_\omega - S_\omega x O_2 = (O_2 x S_\omega - S_\omega O_2 x) - S_\omega O_3.$$

Since $O^3 \in S^3$ and $S_\omega \in S^1$, Lemma 11 implies that the term $S_\omega O_3$ is in S^4. Moreover, $O_2 x \in S^2$ and $S_\omega \in S^1$, so the commutation in parenthesis is also in S^4 by Lemma 12, which implies the announced result. $\square$

Theorem 5. *There holds*

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega}^2 = \frac{I_d}{4} + T_4.$$

Proof. Using the expansion of Lemma 22, we can write

$$
\begin{aligned}
-S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} &= -S_\omega(\omega\partial_x)^2 S_\omega \\
&\quad + \frac{k^2}{4}\left(S_\omega(\omega\partial_x)^2 O_3 + O_3(\omega\partial_x)^2 S_\omega\right) + T_4
\end{aligned}
$$

By Theorem 2, the first term is $\frac{I_d}{4} + T_\infty$ and by Lemma 23 the second term is $k^2\omega^2 + T_4$ Finally, using Lemma 22, on can check that

$$S_\omega\omega^2 S_\omega = S_{k,\omega}\omega^2 S_{k,\omega} + T_4$$

We have thus proved

$$-S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} = \frac{I_d}{4} + k^2 S_{k,\omega}\omega^2 S_{k,\omega} + T_4.$$

If we substract the term $k^2 S_{k,\omega}\omega^2 S_{k,\omega}$ of each side, and use the first commutation proved in Theorem 4, we finally get

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega}^2 = \frac{I_d}{4} + T_4,$$

and the result is proved. $\square$

Recall that $\lambda_{n,k}^2$ are the eigenvalues of $-(\omega\partial_x) - k^2\omega^2$. Let $s_{n,k}$ the eigenvalues of $S_{k,\omega}$ on the basis of Mathieu cosines, that is

$$S_{k,\omega}T_n^k = s_{n,k}T_n^k.$$

The previous theorem has the following consequence:

Corollary 5. *One has*

$$s_{n,k}\lambda_{n,k} = \frac{1}{4} + r_{n,k}$$

where $r_{n,k}$ satisfies

$$\sum_{n=0}^{+\infty}(1+n^2)^4\,|r_{n,k}|^2 < +\infty$$

The results of this section prompt us to use $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$ as a preconditioner for $S_{k,\omega}$. Problème d'inversibilité possible pour certaines valeurs de k. Je n'arrive pas à l'écarter.

4.2 Neumann problem

Similarly, if we define $N_{k,\omega} := N_k\omega$, we have

Theorem 6.
$$N_{k,\omega}^2 = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] + U_2.$$

This result suggests $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]^{-1/2}$ as a candidate preconditioner for $N_{k,\omega}$. Problème d'inversibilité idem. The proof is reported to Appendix B.

4.3 Non-flat arc

In the more general case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, the results of the previous sections can be extended using again compact perturbations arguments. Essentially, in the decomposition Equation 33, x and y must be replaced by $r(x)$ and $r(y)$, where the function r is a smooth, constant-speed parametrisation of Γ defined on $[-1, 1]$ and satisfying $|r(x) - r(y)|^2 = \frac{|\Gamma|}{4}|x-y|^2 + |x-y|^4\,G(x,y)$ where $|\Gamma|$ is the length of Γ and G is a C^∞ function on $[-1, 1]$. Letting $\omega_\Gamma(x) = |\Gamma|\,\omega(x)$, ∂_τ the tangential derivative on Γ and $S_{k,\omega_\Gamma} := S_k\frac{1}{\omega_\Gamma}$, we have:

Theorem 7.
$$S_{k,\omega_\Gamma}\left(-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right)S_{k,\omega_\Gamma} = \frac{I_d}{4} + T_4$$

Theorem 8.
$$N_{k,\omega_\Gamma}^2 = -(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2 + U_2.$$

where R_2 is of order 2 in the scale U^s.

The proofs of those facts are omitted.

4.4 Numerical results

In this section, we show some numerical results to illustrate the efficiency of our approach. Numerical tests are run with a high-level programming language, on a 16GB RAM laptop, Spec de l'ordis à terminer. Numerical methods are detailed in **??**. In each case, to fully resolve the frequency, the number of segments in the discretization is set to $N \approx 10k$, where $k = \frac{\pi}{\lambda}$ is the wavenumber. In the GMRES iteration, we require a relative residual below $1e - 8$.

Flat segment, Dirichlet boundary condition. In Table 3 we report the number of GMRES iterations for the numerical resolution of Equation (25) on the segment $\Gamma = (-1, 1)$, when the linear system is preconditioned by the operator $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$, as compared to the case where no preconditioner is used. In Figure 3, we plot the relative residual history in the GMRES method with and without preconditioner for a problem with $L = 800\lambda$. The Dirichlet data is set to $u_D(x) = e^{ikx}$.

	with Prec.		without Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	8	0.1	73	0.28
200	10	1.3	116	17
800	15	34	148	300

Table 3: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.
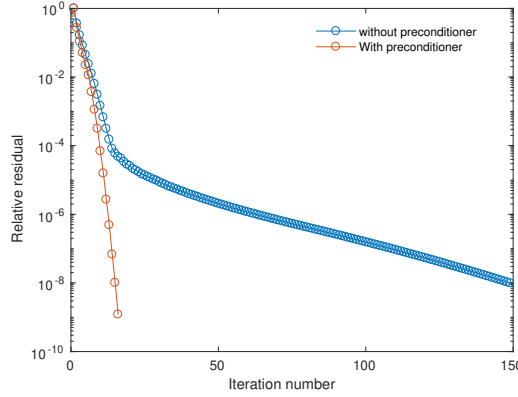

Figure 3: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N \approx 3700$, $L = 800\lambda$.

Flat segment, Neumann boundary condition. We run the same numerical comparisons, this time with the precondioning operator $\left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{-1/2}$.

	with Prec.		without Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	8	0.08	785	9.4
200	10	3.6	-	> 2min
800	17	73	-	-

Table 4: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.
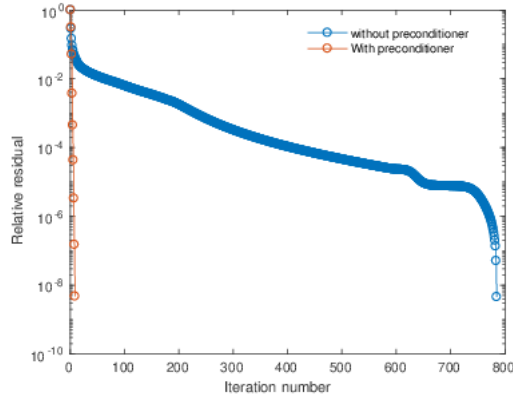

Figure 4: Number of iteration in the resolution of Hypersingular integral equation with a mesh of size $N \approx 800$, $L = 50\lambda$.

Non-flat arc. Here, we also report numerical results when the curve is an portion of spiral (see Figure 5), for both boundary conditions. This shows that the preconditioning strategy is also efficient in presence of non-zero curvature.

	Dirichlet		Neumann	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	23	0.6	785	9.4
200	27	9	-	> 2min
800	40	35	-	-

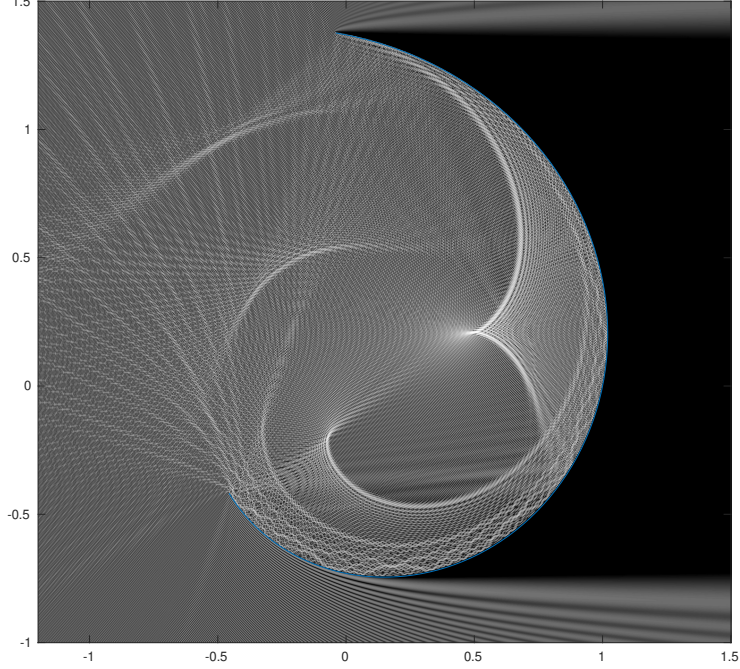Table 5: Number of iterations of the preconditioned system

Figure 5: Sample diffraction pattern (Dirichlet boundary conditions) with left to right incidence for an arc of spiral of size $L = 800\lambda$. After the resolution of the integral equation, the image is obtained in a few seconds.

Comparison with the generalized Calderon relations. As shown in [**?**], S_ω and N_ω can be used efficiently as mutual preconditioners. This alternative method is also very efficient in our numerical setting (here, we use simple piecewise affine functions, whereas in [**?**], spectral discretization with trigonometric polynomials is used.) We report in Table 6 the number of iterations and computing times for the Neumann problem with an angle of incidence $\frac{\pi}{4}$ on the flat segment. The number iterations are comparable for the two methods. However, the matrix-vector product time in our method is significantly slower as it only involves sparse operators. This leads to a faster resolution of the linear system.

	Calderon Prec.		Square root Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	14	0.1	8	0.1
200	15	7.5	11	3.6
800	16	130	15	70

Table 6: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.

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5 Numerical methods

5.1 Galerkine setting

In this section, we describe and analyze the Galerkin scheme used to solve the integral equations in this work. To keep matters simple, we focus on equations (23) and (28) on the flat strip. The results extend to the general case using standard arguments in the theory of boundary element methods. Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [**?**, Chap. 4,] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [**?**] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [**?**], and some results exist for piecewise linear functions in the colocation setting [**?**].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 9 and Theorem 10.

In what follows, we introduce a discretization of the segment $[-1, 1]$ as $-1 = x_0 < x_1 < \cdots < x_N = 1$, and let $\theta_i := \arccos(x_i)$. We define the parameter h of the discretization as

$$h := \min_{i=0\cdots N-1} |\theta_{i+1} - \theta_i| .$$

In practice, one should use a mesh for which $|\theta_i - \theta_{i+1}|$ is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the $i - th$ interval is approximately $(ih)^2$. In comparison, in the h-BEM method with $p = 1$ polynomial order, this would only lead to a convergence rate in $O(h)$ (cf. [**?**, Theorem 1.3]).

5.1.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution λ of (23). To achieve it, we use a variational formulation of (25) to compute an approximation α_h of α, and set $\lambda_h = \frac{\alpha_h}{\omega}$. Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at x_i. Let α_h the unique solution in V_h to

$$\langle S_\omega \alpha_h, \alpha_h' \rangle_{\frac{1}{\omega}} = - \langle u_D, \alpha_h' \rangle_{\frac{1}{\omega}}, \quad \forall \alpha_h' \in V_h.$$

We shall prove the following result:

Theorem 9. *If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C h^{s+1/2} \|u_D\|_{T^{s+1}} .$$

In particular, when u_D is smooth, it belongs to T^∞ so the rate of convergence is $h^{5/2}$. We start by proving an equivalent of Céa's lemma:

Lemma 24. *There exists a constant C such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

Proof. In view of the properties of S_ω stated in Lemma 16, we have the equivalent norm

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \rangle .$$

Since $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = - \langle u_D, \alpha'_h \rangle$ for all $\alpha'_h \in V_h$, we deduce

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \rangle , \quad \forall \alpha'_h \in V_N.$$

By duality

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of S_ω from $T^{-1/2}$ to $T^{1/2}$. $\qquad \square$

From this we can derive the rate of convergence for α_h to the true solution α. We use the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h, which satisfies the following properties:

Lemma 25. *For any function u,*

$$\|(I - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C \|u\|_{L^2_{\frac{1}{\omega}}} ,$$

$$\|(I - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T_2} .$$

The proof requires the following well-known result:

Lemma 26. *Let $\tilde{u}$ in the Sobolev space $H^2(\theta_1, \theta_2)$, such that $\tilde{u}(\theta_1) = \tilde{u}(\theta_2) = 0$. Then there exists a constant C independent of θ_1 and θ_2 such that*

$$\int_{\theta_1}^{\theta_2} \tilde{u}(\theta)^2 \leq C(\theta_1 - \theta_2)^4 \int_{\theta_1}^{\theta_2} \tilde{u}''(\theta)^2 d\theta$$

Proof. The first inequality is obvious since $\mathbb{P}_h$ is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the $L^2_{\frac{1}{\omega}}$ norm,

$$\|I - \mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I - I_h u\|_{L^2_{\frac{1}{\omega}}} , \tag{34}$$

where $I_h u$ is the piecewise affine (continuous) function that matches the values of u at the breakpoints x_i. By Lemma 9, on each interval $[x_i, x_{i+1}]$, the function $\tilde{u}(\theta) := u(\cos(\theta))$ is in the Sobolev space $H^2(\theta_i, \theta_{i+1})$ so we can apply Lemma 26:

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I}_h u)^2 \leq (\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I}_h u)''^2 .$$

This gives

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} \le 2h^4 \left(\int_{\theta_i}^{\theta_{i+1}} \tilde{u}''^2 + \int_{\theta_i}^{\theta_{i+1}} \tilde{I_h} u''^2 \right). \tag{35}$$

Before continuing, we need to establish the following result

Lemma 27. *There holds*

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h} u''^2 \le C \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega}$$

Proof. The expression of $I_h u$ is given by

$$\tilde{I_h} u(\theta) = u(x_i) + \frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} (\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h} u''^2 = \left(\frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left(\int_{x_i}^{x_{i+1}} u'(t) dt \right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \le \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. $\qquad \square$

We can now conclude the proof of Lemma 25. Summing all inequalities (35) for $i = 0, \cdots N + 1$, we get

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}}^2 \le C h^4 \left(\|u\|_{T^2}^2 + \|u'\|_{T_0}^2 \right).$$

By Corollary 1, the operator ∂_x is continuous from T^2 to T^0 which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \le C h^2 \|u\|_{T^2}.$$

Thanks to (34), this concludes the proof. $\qquad \square$

We obtain the following corollary by interpolation:

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Corollary 6. *The operator* $\mathrm{I} - \mathbb{P}_N$ *is continuous from* $L^2_{\frac{1}{\omega}}$ *to* T^s *for* $0 \leq s \leq 2$ *with*

$$\|(\mathrm{I} - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s} .$$

We can now prove Theorem 9:

Proof. First, using Lemma 17, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}} .$$

Moreover, if u_D is in T^{s+1}, then $\alpha = S_\omega^{-1} u_D$ is in T^s and $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$. By the analog of Céa's lemma, Lemma 24, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s} .$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}} .$$

Using Cauchy-Schwarz's inequality and Corollary 6 ($s = \frac{1}{2}$),

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s} .$$

$\square$

5.1.2 Neumann problem

We now turn to the numerical resolution of (28). We use a variational form for equation (29), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce W_h the space of continuous piecewise affine functions with breakpoints at x_i, and we denote by β_h the unique solution in W_h to the variational equation:

$$\langle N_\omega \beta_h, \beta_h' \rangle_\omega = \langle u_N, \beta_h' \rangle_\omega, \quad \forall \beta_h' \in W_h. \tag{36}$$

Then, $\mu_h = \omega \beta_h$ is the proposed approximation for μ. We shall prove the following:

Theorem 10. *If* $u_N \in U^{s-1}$, *for some* $\frac{1}{2} \leq s \leq 2$, *there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}} .$$

Like before, we start with an analog of Céa's lemma:

Lemma 28. *There exists a constant C such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta'_h \in W_h} \|\beta - \beta'_h\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator I_h:

Lemma 29. *There holds*

$$\|u - I_h u\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_h u\|_{U^1} \leq Ch \|u\|_{U^2}$$

Proof. We only show the first estimation, the method of proof for the second being similar. Using again Lemma 26 on each segment $[x_i, x_{i+1}]$, one can write

$$
\begin{aligned}
\int_{x_i}^{x_{i+1}} \omega(u - I_h u)^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_h u)''^2 \\
&\leq Ch^4 \left(2 \int_{\theta_i}^{\theta_{i+1}} Vu''^2 + 2 \int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \right)
\end{aligned}
$$

where we recall that for any function u, Vu is defined as

$$Vu(\theta) = \sin(\theta)u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

Lemma 30.

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(\|u\|^2_{U_2} \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

Proof. Using the expression of I_h, one can write

$$
\begin{aligned}
\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \Bigg(&|u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \\
&+ \left(\frac{u(x_{i+1}) - u(x_i)}{\cos\theta_{i+1} - \cos\theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2) \Bigg) \quad (37)
\end{aligned}
$$

We can estimate the first term, thanks to Lemma 4:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$
\begin{aligned}
(u(x_{i+1}) - u(x_i))^2 &= \left(\int_{x_i}^{x_{i+1}} \partial_x u \right)^2 \\
&\leq \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\
&= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2.
\end{aligned}
$$

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to conclude, it remains to observe that the quantity

$$\frac{|(\theta_{i+1} - \theta_i)| \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2)}{(\cos(\theta_i) - \cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2(\theta_i)$ $\qquad\square$

We now plug the estimate Lemma 30 in (37), and sum over i:

$$\|u - I_h u\|_{L_\omega^2}^2 \leq C h^4 (\|u\|_{U^2}^2 + \|u'\|_{L_\omega^2}^2).$$

This implies the claim once we use the continuity of ∂_x from U^2 to U^0, cf. Corollary 1. $\qquad\square$

We can now prove Theorem 10

Proof. Let us denote by Π_h the Galerkin projection operator defined by $\beta \mapsto \beta_h$. Since it is an orthogonal projection on W_h with respect to the scalar product $(\beta, \beta') := \langle N_\omega \beta, \beta' \rangle$, it is continuous from $U^{1/2}$ to itself, so we have for any u in $U^{1/2}$.

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C \|u\|_{U^{1/2}}.$$

We are now going to show the estimate

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C h^{3/2} \|u\|_{U^2}.$$

By the analog of Céa's lemma Lemma 28, one has $\|(I - \Pi_h)u\|_{U^{1/2}} \leq \|(I - I_h)u\|_{U^{1/2}}$. By interpolation, this norm satisfies

$$\|(I - I_h)u\|_{U^{1/2}} \leq C \sqrt{\|(I - I_h)u\|_{U^0}} \sqrt{\|(I - I_h)u\|_{U^1}},$$

which yields, applying Lemma 29,

$$\|(I - I_h)u\|_{U^{1/2}} \leq C h^{3/2} \|u\|_{U^2}.$$

By interpolation, for all $s \in [1/2, 2]$, we get

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C h^{s-1/2} \|u\|_{U^s}.$$

In view of Lemma 20, we have $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$. Moreover, since N_ω is a continuous bijection from U^{s+1} to U^s for all s, there holds

$$\|\beta\|_{U^s} = \|N_\omega^{-1} u_N\|_{U^s} = \|u_N\|_{U^{s-1}}.$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq C h^{s-1/2} \|\beta\|_{U^s} \leq C h^{s-1/2} \|u_N\|_{U^{s-1}}.$$

$\qquad\square$

5.1.3 Numerical quadratures and compression method

Weakly singular kernels. To compute the elements of the matrix of the operators which kernel have a weak singularity, we use Gaussian quadratures for the weights $\frac{1}{\omega}$ and ω, with 3 points on each segment. As a result, we obtain a vector (x_k) of $3N$ points in $\mathbb{R}^2$ and a vector (w_k) of $3N$ quadrature weights. If p stands for the weight ($1/\omega$ or ω), then quantities of the form $\int_\Gamma \int_\Gamma p(x)p(y)\phi_i(x)G_k(x,y)\phi_j(y)$, are approximated by

$$\int_\Gamma \int_\Gamma p(x)p(y)\phi_i(x)G_k(x,y)\phi_j(y) \approx \sum_{k=1}^N w_k\phi_i(x_k)\sum_{l=1}^N w_l G(x_k,x_l)\phi_j(x_l). \quad (38)$$

Since the kernel G_k is singular, we must correct the previous formula when x_k and x_l are close from each other. This is done by the following method. For each segment I_n in the mesh, we find the subset $(x'_k)$of the points x_k that are closer than a fixed threshold from I_n. For those points, we compute more accurately the integral

$$\int_{I_n} p(y)G_k(x_k,y)\phi_j(y),$$

by any method at our disposal (variable change, substraction of the singularity...). The second integral is then approximated using the previous quadrature. Note that in this framework, the addition of the wait p requires only minor modification over a standard BEM code. Basically, one only needs to compute numerical quadratures for the weight p, for which open source routines are available, and implement a method to compute the close integrals with the weight p. Note that using the variable change $x = \cos(\theta)$, the weight disappears and the only singularity left is that of G_k.

Hypersingular operator. Although the weighted hypersingular $N_{k,\omega}$ operator has a non-integrable kernel, the formulawe have the formula

$$\langle N_{k,\omega}\beta, \beta' \rangle_\omega = \langle S_{k,\omega}(\omega\partial_x\omega)u, (\omega\partial_x\omega)v\rangle_{\frac{1}{\omega}} - k^2\langle S_\omega\omega^2 u, \omega^2 v\rangle_{\frac{1}{\omega}}.$$

This only involves weakly singular kernels and is thus treated by the previous method.

Compression method The rhs of (38) is the scalar product of a vector with a discrete convolution. To accelerate the computation and to avoid assembling the full matrix $(G_k(x_k,x_l))$, we use Citer EBD alors que pas encore accepté ?.

5.2 Preconditioning the linear systems

For an operator A, let us denote by $[A]_p$ the Galerkine matrix of the operator for the weight $p(x)$. When the operator BA is a compact perturbation of the identity (either in T^s or U^s) then, following [**?**], we precondition the linear system $[A]_p x = y$ by the matrix $[I]_p^{-1}[B]_p[I]_p^{-1}$. If B is the inverse of a local operator C, then it is easier to compute $[C]_p$. This time, the good choice for the preconditioner is $[C]_p^{-1}$. When testing the pair $S_{k,\omega}, N_{k,\omega}$ as mutual preconditioner, the operator $S_{k,\omega}N_{k,\omega}$ is discretized as $[I_d]_{\frac{1}{\omega}}^{-1}[S_{k,\omega}]_{\frac{1}{\omega}}[I_d]_\omega^{-1}[N_{k,\omega}]_\omega$, yielding very satisfactory results.

The preconditioners introduced in this work are in the form of square roots of local operators. More precisely, we introduced two preconditioners P_1 and P_2 with

$$
\begin{array}{rcl}
P_1(k) & = & \left(-(\omega\partial_x)^2 - k^2\omega^2\right)^{1/2} \\
P_2(k) & = & \left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{-1/2}
\end{array}
$$

For the second equation, we rewrite

$$
P_2(k) = \left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{-1} \left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{1/2},
$$

which brings us back to computing the square root of a sparse matrix. When the frequency is 0, we use the method exposed in [**?**]. When the frequency is non-zero, the previous method fails since the spectrum of the matrix contains negative values. In [**?**], a method involving a Padé approximation of the square root, with a rotated branch cut, is used to compute the matrix of an operator of the form $\sqrt{X - k^2 I_d}$ where X is a positive definite operator. This method gives excellent results in our context when using $X = -(\partial\omega_x)^2 + k^2 \left(I_d - \omega^2\right)$.

6   Conclu

Résumé de ce qu'on a fait, du lien qu'on a fait. Ouverture sur les singularités de type coin puis 3D. Beaucoup plus compliqué car pas de relations analytiques qui nous aident. Expliquer la beauté de l'approche numérique avec un poids. On propose le préconditioneur avec un test numérique ?

Possible analyse pseudo-diff ? En reparler ? Lien avec Antoine et Darbas.

A   Commutation of N_ω and $(\partial_x\omega)^2 + k^2\omega^2$

Lemma 31. *For any ϕ, there holds*

$$
\partial_x S_\omega \omega^2 \phi = S_\omega \omega \partial_x \omega \phi.
$$

Proof. In our context, the largest space encountered is $U^{-\infty}$ (As a consequence of Lemma 4) so we shall show that this identity holds in that space. For any s, by Lemma 6, ω^2 is a continuous operator from U^s to T^s, S_ω is continuous from T^s to T^{s+1} and ∂_x is continuous from T^{s+1} to U^s by Lemma 7. Therefore, the left operator is continuous from U^s to U^s. Similarly, the right operator is continuous from U^s to U^s for all s. If we fix $\phi \in U^s$ for some s, Lemma 1 ensures that there exists a sequence ϕ_N in U^∞ converging to ϕ in U^s. It is thus sufficient to check the identity for $\phi = U_n$. For $n \geq 1$,

$$
\begin{array}{rcl}
\partial_x S_\omega \omega^2 U_n & = & \partial_x S_\omega \left(\dfrac{T_n - T_{n+2}}{2}\right) \qquad\qquad (39) \\[2ex]
& = & \partial_x \left(\dfrac{T_n}{4n} - \dfrac{T_{n+2}}{4(n+2)}\right) \qquad (40) \\[2ex]
& = & \dfrac{U_{n-1} - U_{n+1}}{4} \qquad\qquad\qquad (41) \\[2ex]
& = & -\dfrac{T_{n+1}}{2}. \qquad\qquad\qquad\quad (42)
\end{array}
$$

One can check that the result also holds for $n = 0$. On the other hand, for all n,

$$S_\omega \omega \partial_x \omega U_n = -(n+1)S_\omega T_{n+1} \tag{43}$$

$$= -\frac{1}{2}T_{n+1} \tag{44}$$

which proves the result. $\qquad \square$

We now turn to the proof of the theorem. To ease the computations, we take some notations: let $\Delta_\omega := (\omega \partial_x)^2$, $\Delta_\omega^T := (\partial_x \omega)^2$, $N_\omega := N_{k,\omega}$, and $S_\omega := S_{k,\omega}$. Using, equation (12) we can write

$$N_\omega = -\partial_x S_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2.$$

To show that N_ω and $\Delta_\omega^T + k^2 \omega^2$ commute, we compute their commutator C and show that it is null. We have

$$
\begin{aligned}
C & := N_\omega \Delta_\omega^T - \Delta_\omega^T N_\omega + k^2 N_\omega \omega^2 - k^2 \omega^2 N_\omega \\
& = -\partial_x S_\omega \Delta_\omega \omega \partial_x \omega - k^2 S_\omega \omega^2 \Delta_\omega^T \\
& \quad +\partial_x \Delta_\omega S_\omega \omega \partial_x \omega + k^2 \Delta_\omega^T S_\omega \omega^2 \\
& \quad -k^2 \partial_x S_\omega \omega \partial_x \omega^3 - k^4 S_\omega \omega^4 \\
& \quad +k^2 \omega^2 \partial_x S_\omega \omega \partial_x \omega + k^4 \omega^2 S_\omega \omega^2
\end{aligned}
$$

where each term in the r.h.s. of the first equality gives rise to a line in the second. We rearrange the terms as follows:

$$
\begin{aligned}
C & = \partial_x (\Delta_\omega S_\omega - S_\omega \Delta_\omega)\omega \partial_x \omega - k^2 \partial_x S_\omega \omega \partial_x \omega^3 + k^2 \omega^2 \partial_x S_\omega \omega \partial_x \omega \\
& \quad +k^4 (\omega^2 S_\omega - S_\omega \omega^2)\omega^2 \\
& \quad +k^2 (\Delta_\omega^T S_\omega \omega^2 - S_\omega \omega^2 \Delta_\omega^T)
\end{aligned}
$$

For the first term, we inject the commutation shown in Theorem 4. For the last line, we use the following identities:

$$\Delta_\omega^T = \Delta_\omega - 2x\partial_x - 1$$

$$\omega^2 \Delta_\omega^T = \Delta_\omega \omega^2 + \omega^2 + 2\omega x \partial_x \omega$$

Let $D = \frac{C}{k^2}$,

$$
\begin{aligned}
D & = \partial_x S_\omega \omega (\omega^2 \partial_x - \partial_x \omega^2)\omega + (\omega^2 \partial_x - \partial_x \omega^2)S_\omega \omega \partial_x \omega \\
& \quad +k^2 (\omega^2 S_\omega - S_\omega \omega^2)\omega^2 \\
& \quad +(\Delta_\omega - 2x\partial_x - 1)S_\omega \omega^2 - S_\omega (\Delta_\omega \omega^2 + \omega^2 + 2\omega x \partial_x \omega)
\end{aligned}
$$

We use $\omega^2 \partial_x - \partial_x \omega^2 = 2x$, and the relation $\partial_x S\omega^2 = S_\omega \omega \partial_x \omega$ to get

$$
\begin{aligned}
D & = 2S_\omega \omega \partial_x x\omega + 2xS_\omega \omega \partial_x \omega \\
& \quad + \left(k^2(\omega^2 S_\omega - S_\omega \omega^2) + \Delta_\omega S_\omega - S_\omega \Delta_\omega\right)\omega^2 \\
& \quad -2S_\omega \omega^2 - 2xS_\omega \omega \partial_x \omega - 2S_\omega \omega x \partial_x \omega.
\end{aligned}
$$

Using again the commutation shown in Theorem 4, we are left with

$$D = 2S_\omega \omega (\partial_x x - x\partial_x)\omega - 2S_\omega \omega^2$$

This is null since $\partial_x x - x\partial_x = 1$.

B Proof of Theorem 6

From equation (12), we can deduce the following formula for the weighted operator:

$$N_{k,\omega} = -\partial_x S_{k,\omega} \omega \partial_x \omega - k^2 S_{k,\omega} \omega^2 \tag{45}$$

If we define $L_n := -\partial_x O_{n+2} \omega \partial_x \omega$, then using the mapping properties of ∂_x and $\omega \partial_x \omega$ given by Lemma 7, and since, by Lemma 21, O_{n+2} is of order $n + 2$ in the scale T^s, we deduce that L_n is of order n in the scale U^s. The expansion obtained for the weighted single-layer operator in Lemma 22 yields the following expansion for $N_{k,\omega}$.

Lemma 32.

$$N_{k,\omega} = N_\omega + k^2 \left(-\frac{L_1}{4} - S_\omega \omega^2 \right) + U_3$$

As a consequence, $N_{k,\omega}$ is an operator of order -1 in the scale U^s. Using equation (45), we have the following expression:

$$N_{k,\omega}^2 = N_\omega^2 - k^2 \left(\frac{L_1 N_\omega + N_\omega L_1}{4} + N_\omega S_\omega \omega^2 + S_\omega \omega^2 N_\omega \right) + U_2.$$

We have proved in By definition, $L_1 = -\partial_x O_3 \omega \partial_x \omega$, while $N_\omega = -\partial_x S_\omega \omega \partial_x \omega$, thus

$$L_1 N_\omega = \partial_x (O_3 (\omega \partial_x)^2 S_\omega) \omega \partial_x \omega.$$

Moreover,

$$N_\omega L_1 = \partial_x (S_\omega (\omega \partial_x)^2 O_3) \omega \partial_x \omega.$$

Adding these two inequalities and using Lemma 23, we get

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = \partial_x (S_\omega \omega^2 S_\omega) \omega \partial_x \omega + U_2.$$

Here again, we use the formula $\partial_x S_\omega \omega^2 = S_\omega \omega \partial_x \omega$, which yields

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = S_\omega \omega \partial_x \omega \partial_x S_\omega \omega^2 = \left(-\frac{I_d}{4} + T_\infty \right) \omega^2.$$

Since ω^2 is continuous from U^s to T^s, and using the injections $T^s \subset U^s$, any operator of the form $R\omega^2$ is smoothing in the scale U^s as soon as R is smoothing in the scale T^s. Therefore,

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = -\frac{\omega^2}{4} + U_\infty.$$

Moreover, we have

$$\begin{aligned} S_\omega \omega^2 N_\omega &= -S_\omega \omega^2 \partial_x S_\omega \omega \partial_x \omega \\ &= -S_\omega \omega^2 \partial_x^2 S_\omega \omega^2 \end{aligned}$$

using again Lemma 31. Since $\omega^2 \partial_x^2 = (\omega \partial_x)^2 + x\partial_x$, we get

$$S_\omega \omega^2 N_\omega = \frac{\omega^2}{4} - S_\omega x \partial_x S_\omega \omega^2 + U_\infty$$

Futhermore,

$$N_\omega S_\omega \omega^2 \quad = \quad -\partial_x S_\omega \omega \partial_x \omega S_\omega \omega^2.$$

We use $\omega \partial_x \omega = \omega^2 \partial_x - x$:

$$
\begin{aligned}
N_\omega S_\omega \omega^2 \quad &= \quad -\partial_x S_\omega \omega^2 \partial_x S_\omega \omega^2 + \partial_x S_\omega x S_\omega \omega^2 \\
&= \quad \frac{\omega^2}{4} + \partial_x S_\omega x S_\omega \omega^2
\end{aligned}
$$

Thus,

$$S_\omega \omega^2 N_\omega + N_\omega S_\omega \omega^2 \quad = \quad \frac{\omega^2}{2} + \left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2 \right) + U_\infty.$$

We are done if we prove that the operator in parenthesis is of order 2 in the scale U^s. For this, we may compute the action of each one of them on U_n. Using the various identities at our disposal, we obtain on the on hand for $n \geq 2$

$$\partial_x S_\omega x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8(n+2)} + \frac{U_n + U_{n-2}}{8n(n+2)}.$$

and on the other hand

$$S_\omega x \partial_x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8n}.$$

After substracting, this gives the rather surprising identity identity

$$\left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2 \right) U_n = \frac{U_n}{4n(n+2)}$$

which of course proves our claim.

C   Suggestion de découpage

J'y ai un tout petit peu réfléchi :

- Les analyses pseudo-diffs des espaces T^s, bien qu'intéressantes, sont trop longues et ne se justifient pas vraiment dans le simple but de faire une méthode numérique.

- La méthode de Galerkine est bien analysée et nouvelle (à ma connaissance) mais n'est pas vraiment essentielle pour le message.

Je pense qu'on pourrait envisager 3 articles. Un très concis sur la méthode numérique en elle-même. Utiliser le minimum d'info pour k=0, donner les inverses exacts, prouver la commutation des opérateurs pour k non nul, puis balancer les préconditionneurs, et mettre les figures.

Un article un peu à part sur la méthode de Galerkine, et tous les aspects numériques (bcp moins d'impact)

Un article (peut-être juste sur arxiv ?) sur les espaces T^s et U^s, qui donne toutes les justifications théoriques. (une sorte de version étendue de cet article.)

References