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The application of integral equation methods to the numerical solution of some exterior boundary-value problems

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The application of integral equation methods to exterior boundary-value problems for Laplace's equation and for the Helmholtz (or reduced wave) equation is discussed. In the latter case the straightforward formulation in terms of a single integral equation may give rise to difficulties of non-uniqueness; it is shown that uniqueness can be restored by deriving a second integral equation and suitably combining it with the first. Finally, an outline is given of methods for transforming the integral operators with strongly singular kernels which occur in the second equation.

1. Introduction

The importance of integral equation methods in the solution, both theoretical and practical, of certain types of boundary-value problems is universally recognized. The essential feature and main advantage of such methods is that they often allow the problem to be reduced from one involving the whole of the domain of interest to one involving only its boundary, so that the dimension of the problem is reduced by one. From a practical viewpoint the gain may not always be very spectacular where interior problems are concerned, but for exterior problems, where the region of interest is of infinite extent, an integral equation formulation may be virtually indispensable.

What is perhaps less well appreciated (though the question has received a good deal of attention, particularly in the acoustical literature, just recently) is that the very process of reduction from the exterior domain to the boundary may give rise to difficulties of non-uniqueness which are not inherent in the original problem. Our main purpose here will be to demonstrate this phenomenon as it affects the exterior problem for the Helmholtz (or reduced wave) equation and to propose a practical method for overcoming the difficulty (§§ 3 and 4). This compares favourably with earlier methods (some of which we briefly review at the conclusion of § 3), and has been successfully applied to a number of two-dimensional problems in acoustic diffraction and fluid dynamics.

First, however, with the object of establishing basic concepts and notation we briefly discuss the application of integral equation methods to the possibly more familiar problems of potential theory.

2. Integral equations in potential theory

We here consider the solution of Laplace's equation

$$\nabla^2 u = 0, \tag{1}$$

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subject to prescribed boundary conditions, in an exterior domain E with boundary B; for simplicity we suppose that B is smooth and encloses a simply connected domain D. We denote by $G(\boldsymbol{p}, \boldsymbol{q})$ the free-space Green function for equation (1), i.e. the potential at a point $\boldsymbol{p}$ due to a unit point source at $\boldsymbol{q}$. In three-dimensional space we have

$$G(\boldsymbol{p}, \boldsymbol{q}) = 1/(4\pi R) \quad (R = |\boldsymbol{p}-\boldsymbol{q}|), \tag{2}$$

while in two dimensions (though here the physical concept is less well defined) an analogous rôle is played by the function

$$G(\boldsymbol{p}, \boldsymbol{q}) = -(1/2\pi) \ln R. \tag{3}$$

Frequently one seeks to express the unknown exterior potential $u(\boldsymbol{p})$ (or a closely related function) as the potential arising from some particular form of source distribution $\sigma(\boldsymbol{q})$ on the boundary; for example as a single-layer potential

$$u(\boldsymbol{p}) = \int_B \sigma(\boldsymbol{q})\, G(\boldsymbol{p}, \boldsymbol{q})\, \mathrm{d}S_q \equiv L[\sigma], \tag{4}$$

or as a double-layer potential

$$u(\boldsymbol{p}) = \int_B \sigma(\boldsymbol{q}) \frac{\partial}{\partial n_q} G(\boldsymbol{p}, \boldsymbol{q})\, \mathrm{d}S_q \equiv M[\sigma], \tag{5}$$

where $\partial/\partial n_q$ denotes differentiation along the outward normal at $\boldsymbol{q}$. In the latter case $\sigma(\boldsymbol{q})$ represents a distribution of dipoles or doublets on B. Evidently once the boundary function $\sigma(\boldsymbol{q})$ has been determined the representations (4) and (5) define $u(\boldsymbol{p})$ explicitly at all points of E.

When formulating a boundary-value problem as an integral equation it is necessary to have regard to the discontinuity properties, or jump relations, for the layer potentials. We recall that, under suitable conditions on $\sigma(\boldsymbol{q})$ and the boundary B, as $\boldsymbol{p}$ passes through B the single-layer potential $L[\sigma]$ and the normal derivative of $M[\sigma]$ remain continuous, while $(\partial/\partial n_p)\, L[\sigma]$ and $M[\sigma]$ are discontinuous, their interior and exterior limiting values and the value on the boundary itself being related as follows:

$$\left(\frac{\partial L[\sigma]}{\partial n}\right)_B = \left(\frac{\partial L[\sigma]}{\partial n}\right)_{\text{int}} - \tfrac{1}{2}\sigma = \left(\frac{\partial L[\sigma]}{\partial n}\right)_{\text{ext}} + \tfrac{1}{2}\sigma, \tag{6}$$

$$(M[\sigma])_B = (M[\sigma])_{\text{int}} + \tfrac{1}{2}\sigma = (M[\sigma])_{\text{ext}} - \tfrac{1}{2}\sigma. \tag{7}$$

Let us consider the application of the single-layer potential representation to the solution of the Dirichlet and Neumann problems in three dimensions. We require a function $u(\boldsymbol{p})$ which is twice continuously differentiable in E and satisfies

(i) $\nabla^2 u = 0 \quad (\boldsymbol{p} \in E)$,

(ii) (*a*) $u = g(\boldsymbol{p}) \quad (\boldsymbol{p} \in B)$ (Dirichlet),

or (*b*) $\partial u/\partial n = g(\boldsymbol{p}) \quad (\boldsymbol{p} \in B)$ (Neumann),

(iii) $u(\boldsymbol{p}) \to 0$ as $|\boldsymbol{p}| \to \infty$.

It is known that in either case, provided g is continuous, the solution exists and is unique (Kellogg 1929, pp. 311–314).

Applying conditions (ii a) and (ii b) respectively, we obtain

$$g = L[\sigma] \quad (\boldsymbol{p} \in B), \tag{8}$$

and

$$g = \partial L[\sigma]/\partial n_p - \tfrac{1}{2}\sigma \quad (\boldsymbol{p} \in B), \tag{9}$$

which are integral equations of the first and second kind respectively for $\sigma(\boldsymbol{q})$. Both equations possess unique solutions and are well adapted to numerical solution; the difficulties sometimes associated with equations of the first kind do not arise with equation (8) to any marked degree because of the presence of the singularity of the kernel at $\boldsymbol{p} = \boldsymbol{q}$.

Clearly, problems involving more general boundary conditions may be dealt with in a similar way.

It should be observed that the preceding discussion relates only to three-dimensional problems. A general review of the more complicated two-dimensional problem is beyond the scope of this paper but for completeness we note the following modifications:

(i) The appropriate representation is now

$$u(\boldsymbol{p}) = C + L[\sigma],$$

where C is a constant to be determined.

(ii) A bounded solution exists if and only if the condition

$$\int_B (\partial u/\partial n)\, \mathrm{d}S_q = 0$$

is satisfied. In the Neumann case, provided that the prescribed values of $\partial u/\partial n$ satisfy this condition, there exists a unique solution vanishing at infinity. In other cases the fulfilment of the condition can be secured by appropriate choice of C, and there is a unique solution bounded at infinity. (Petrovskii (1967) contains a lucid exposition of the two-dimensional existence theory.)

(iii) There exist particular contours for which the boundary integral equation derived from (8) fails to possess a unique solution. In such cases, however, the difficulty is readily overcome by replacing $L[\sigma]$ by $L[\sigma] + \text{const.} \times \int \sigma\, \mathrm{d}S_q$. (See Jaswon (1963) for a discussion of this phenomenon; a companion paper by Symm (1963) describes some applications of integral equation methods to two-dimensional problems.)

Use of Green's formula

With the exception of the two-dimensional Dirichlet problem, double-layer potentials appear to be less well suited than single-layer potentials to the solution of exterior problems for Laplace's equation. However, a useful alternative is the mixed representation provided by Green's formula:

$$\int_B \left\{ u(\boldsymbol{q}) \frac{\partial}{\partial n_q} G(\boldsymbol{p}, \boldsymbol{q}) - G(\boldsymbol{p}, \boldsymbol{q}) \frac{\partial}{\partial n_q} u(\boldsymbol{q}) \right\} \mathrm{d}S_q = \left\{ \begin{array}{ll} u(\boldsymbol{p}) & (\boldsymbol{p} \in E), \\ \tfrac{1}{2} u(\boldsymbol{p}) & (\boldsymbol{p} \in B), \\ 0 & (\boldsymbol{p} \in D). \end{array} \right\} \tag{10}$$

In terms of our operator notation the boundary integral equation has the form

$$\tfrac{1}{2}u + L[\partial u/\partial n] - M[u] = 0 \quad (\boldsymbol{p} \in B). \tag{11}$$

If, for example, u is given in B we obtain an equation of the same form as equation (8), while if $\partial u/\partial n$ is given the operator in the integral equation is simply the transpose of the operator in equation (9). In general, the boundary integral equations which arise with the use of Green's formula are very similar to those which result from the use of layer representations.

The Green's formula approach has the slight disadvantage that the evaluation of $u(\boldsymbol{p})$ at interior points requires the calculation of two potentials instead of one. However, this disadvantage is offset by the fact that the representation (10) remains valid in circumstances when one or other of the layer representations may fail. This consideration is of some relevance to the solution of the Helmholtz equation which we now consider.

3. The Helmholtz equation

Let $u_{\mathrm{i}}(\boldsymbol{p})$ be a given incident wave and $u_{\mathrm{s}}(\boldsymbol{p})$ the scattered wave produced by its reflexion from B; both are required to satisfy the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$, $\boldsymbol{p} \in E$. Then the total wave is given by

$$u = u_{\mathrm{i}} + u_{\mathrm{s}}.$$

For simplicity we suppose that the medium is non-dissipative, so that k is real ($k \neq 0$), and consider only the case of total reflexion (the case of a rigid scatterer in acoustics); the analysis may be readily extended to allow for complex values of k and more general boundary conditions. The total wavefunction is required to satisfy the following conditions:

(i) u is twice continuously differentiable in E and

$$\nabla^2 u + k^2 u = 0 \quad (\boldsymbol{p} \in E); \tag{12}$$

(ii)
$$\partial u/\partial n = 0 \quad (\boldsymbol{p} \in B). \tag{13}$$

In addition the scattered wave must satisfy the following radiation condition:

(iii) If r denotes distance from a fixed origin, then as $r \to \infty$

$$\partial u_{\mathrm{s}}/\partial r - \mathrm{i}ku_{\mathrm{s}} = \begin{cases} o(r^{-\frac{1}{2}}) \text{ in two dimensions,} \\ o(r^{-1}) \text{ in three dimensions.} \end{cases} \tag{14}$$

The radiation condition ensures that u_{s} is purely an outgoing wave (a time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ being assumed). The existence and uniqueness of the solution of the above problem were demonstrated by Leis (1958).

Green's functions for the Helmholtz equation, analogous to the expressions (2) and (3), are given by

$$G_k(\boldsymbol{p}, \boldsymbol{q}) = \begin{cases} \mathrm{e}^{\mathrm{i}kR}/(4\pi R) \text{ in three dimensions,} \\ \tfrac{1}{4}\mathrm{i}H_0^{(1)}(kR) \text{ in two dimensions,} \end{cases} \tag{15}$$

where $H_0^{(1)}(kR)$ is the Hankel function of the first kind. With the substitution of G_k for G the scattered wave u_s satisfies Green's formula as given by equation (10). However, in order to take advantage of the homogeneity of the boundary condition (13) it is more convenient to work with the total wave u. It is readily verified that since u_i is free from singularities in D, u satisfies the modified Green's formula

$$u_i(\boldsymbol{p})+\int_B\left\{u(\boldsymbol{q})\frac{\partial}{\partial n_q}G_k(\boldsymbol{p},\boldsymbol{q})-G_k(\boldsymbol{p},\boldsymbol{q})\frac{\partial}{\partial n_q}u(\boldsymbol{q})\right\}\mathrm{d}S_q=\begin{cases}u(\boldsymbol{p}) & (\boldsymbol{p}\in E),\\ \tfrac{1}{2}u(\boldsymbol{p}) & (\boldsymbol{p}\in B),\\ 0 & (\boldsymbol{p}\in D).\end{cases} \tag{16}$$

By analogy with (4) and (5), we may define single- and double-layer Helmholtz potentials by

$$L_k[\sigma]=\int_B\sigma(\boldsymbol{q})\,G_k(\boldsymbol{p},\boldsymbol{q})\,\mathrm{d}S_q,\quad M_k[\sigma]=\int_B\sigma(\boldsymbol{q})\frac{\partial}{\partial n_q}G_k(\boldsymbol{p},\boldsymbol{q})\,\mathrm{d}S_q. \tag{17}$$

In terms of this notation the boundary equation in (16) becomes

$$\tfrac{1}{2}u+L_k\left[\frac{\partial u}{\partial n}\right]-M_k[u]=u_i\quad(\boldsymbol{p}\in B), \tag{18}$$

which, when the boundary condition $\partial u/\partial n = 0$ is applied, reduces simply to

$$\tfrac{1}{2}u-M_k[u]=u_i\quad(\boldsymbol{p}\in B). \tag{19}$$

Although the boundary values of the sought function $u(\boldsymbol{p})$ certainly satisfy this integral equation, it may happen that the solution is not unique. Indeed there exists an infinite set of values of k for which the equation has a multiplicity of solutions, and these values are found to coincide with the 'resonant' wavenumbers (or eigenvalues) for a related *interior* problem. It is of interest to investigate this connexion. First let us note that non-uniqueness of the solution of equation (19) arises if and only if the corresponding homogeneous equation

$$\tfrac{1}{2}\phi-M_k[\phi]=0 \tag{20}$$

has a non-trivial solution. In this event, by a well-known theorem in the theory of Fredholm integral equations, the transposed equation

$$\tfrac{1}{2}\psi-\partial L_k[\psi]/\partial n_p=0 \tag{21}$$

will also possess a non-trivial solution, and conversely. Now consider the interior problem for which $\nabla^2 v+k^2 v=0$ in D and $v=0$ on B. It is readily seen that the boundary values $\partial v/\partial n$ satisfy equation (21). In general this interior problem has only the solution $v\equiv 0$, but if k is one of an infinite set, K_1 say, of discrete resonant wavenumbers (eigenvalues) for this problem, there exists a non-trivial solution $v(\boldsymbol{p})$ such that $\partial v/\partial n \not\equiv 0$. Hence equation (21) possesses a non-trivial solution, and the solution of equation (19) is accordingly non-unique.

In the two-dimensional case of a circle of radius a, for example, the set K_1 consists of values of k such that $J_n(ka)=0$ for some integer n, the corresponding eigenfunctions being $J_n(kr)\cos n\theta$ and $J_n(kr)\sin n\theta$. (The value $k=0$ must clearly be excluded.) These values are sparsely distributed when k is small, but as k increases they become progressively more dense. It is noteworthy that from a computational view-point ill-conditioning is likely to occur whenever k lies in the immediate vicinity of a critical wavenumber; the difficulty is consequently more severe when k is large.

It is again emphasized that the uniqueness of the original exterior problem is not in question and that the connexion with the interior domain has no physical significance. The difficulty of non-uniqueness springs entirely from our formulation of the problem in terms of a boundary integral equation. It is consequently of interest to examine the nature of the family of solutions of this equation which arises when $k \in K_1$. If we take any member, $u^*(\boldsymbol{q})$ say, of the family and construct the function

$$u(\boldsymbol{p}) = u_i(\boldsymbol{p}) + M_k[u^*] \quad (\boldsymbol{p} \in E),$$

then $u(\boldsymbol{p})$ is indeed a solution of the Helmholtz equation satisfying the appropriate condition at infinity. However, it is found that its exterior normal derivative satisfies only the condition $L_k[\partial u/\partial n] = 0$, and we cannot in this instance (i.e. for $k \in K_1$) conclude that $\partial u/\partial n = 0$; only one particular choice of $u^*(\boldsymbol{q})$ will in fact yield a $u(\boldsymbol{p})$ with this property. Essentially, therefore, the problem is one of enforcing the boundary condition $\partial u/\partial n = 0$.

Now it is possible to derive a further integral equation for the unknown boundary values of u by applying the operation $\partial/\partial n_p$ to the first equation in (16) and subsequently placing $\boldsymbol{p}$ on the boundary. Setting $\partial u/\partial n = 0$ in this equation we obtain

$$\frac{\partial}{\partial n_p} M_k[u] = -\frac{\partial u_i}{\partial n_p}. \tag{22}$$

This equation, taken by itself, suffers from the same kind of defect as equation (19); it possesses a multiplicity of solutions whenever $k \in K_2$, the set of resonant wavenumbers for the interior Neumann problem.† It is even possible for the two equations to fail simultaneously. In the case of the circle of radius a, for example, the set K_2 consists of those values of k for which $J_n'(ka) = 0$ for some integer n, and clearly the subset for which $J_0'(ka) = -J_1(ka) = 0$ belongs to both K_1 and K_2. Nevertheless, it transpires that the two equations have always only one solution in common; in the case of the double failure this is explained by the fact that the corresponding eigenfunctions are necessarily different.

There remains the question of how to combine the equations (19) and (22). If we simply write down a set of approximating equations for each of the integral equations, we shall obtain an over-determined system. Since the two systems are not inconsistent this approach is perfectly feasible. However, it is more economical to proceed as follows. Adding a multiple of equation (22) to equation (19) we obtain

$$\tfrac{1}{2}u - M_k[u] + \alpha \frac{\partial}{\partial n_p} M_k[u] = u_i - \alpha \frac{\partial u_i}{\partial n_p} \quad (\boldsymbol{p} \in B), \tag{23}$$

† We note that $k = 0$ (corresponding to Laplace's equation) is always a member of K_2.

where α is a constant. We now demonstrate that the uniqueness of the solution of this combined integral equation can be ensured by suitable choice of α.

Uniqueness of the combined integral equation

The uniqueness of the solution of (23) will follow if we can show that the corresponding homogeneous equation

$$\tfrac{1}{2}\phi - M_k[\phi] + \alpha\,\partial M_k[\phi]/\partial n_p = 0 \quad (\boldsymbol{p} \in B), \tag{24}$$

has only the trivial solution $\phi = 0$. Consider the double-layer Helmholtz potential function

$$v(\boldsymbol{p}) = M_k[\phi] \quad (\boldsymbol{p} \in D+B+E).$$

Equation (24) expresses the fact that the interior boundary values of $v(\boldsymbol{p})$ and $\partial v/\partial n$ are related by

$$v_{\text{int}} - \alpha(\partial v/\partial n)_{\text{int}} = 0. \tag{25}$$

Applying Green's second identity to v and its complex conjugate $\bar{v}$ we obtain

$$\int_B \left(v\frac{\partial \bar{v}}{\partial n_q} - \bar{v}\frac{\partial v}{\partial n_q} \right) \mathrm{d}S_q = 2\mathrm{i}\,\mathrm{Im}\,\alpha \int_B \left| \frac{\partial v}{\partial n} \right|^2 \mathrm{d}S_q = 0.$$

Provided that $\mathrm{Im}\,\alpha \neq 0$ it follows that $(\partial v/\partial n)$ vanishes on B and equation (25) yields also $v_{\text{int}} = 0$. From the jump relations and the uniqueness of the solution of the exterior problem (whence $v \equiv 0$) we conclude that $\phi = 0$. Hence the solution of equation (23) is unique provided only that we choose α such that $\mathrm{Im}\,\alpha \neq 0$.

Related work

As we remarked earlier, a similar type of failure occurs at critical wavenumbers if a solution to the boundary-value problem is sought in the form of a surface-layer potential. The failure is then in fact more catastrophic inasmuch as no solution for the source density $\sigma(\boldsymbol{q})$ exists, in general, at a critical wavenumber. The phenomenon was apparently first encountered in attempts to prove the existence of the solution for various exterior problems (Kupradze 1965†; Weyl 1952; Müller 1952; Leis 1958). In the case of the Dirichlet problem, for example, it is found necessary to supplement the initially assumed double-layer representation at a critical k by the addition of a single-layer potential term. However, the abrupt transition in the mode of representation renders this approach quite unsuitable as a basis for numerical work.‡

It was shown independently by Panich (1965) and by Brakhage & Werner (1965) that for the Dirichlet problem the difficulty can be avoided by employing (independently of k) a suitable combination of single- and double-layer potentials of the same density function σ:

$$u_{\mathrm{s}} = L_k[\sigma] + \alpha M_k[\sigma] \quad (\boldsymbol{p} \in E), \tag{26}$$

where if k is real α must be chosen to be strictly complex ($\mathrm{Im}\,\alpha \neq 0$). This method was subsequently applied to the numerical solution of the two-dimensional Dirichlet problem by Greenspan & Werner (1966).

† The author's work dating from 1934 is herein recapitulated.

‡ It is also of interest to note that, for the same reason, most of the existence proofs fail to exhibit the analyticity of the solution with respect to k.

The Neumann (and mixed) boundary-value problem presents considerably greater difficulty because of the occurrence of the singular operator $\partial M_k/\partial n_p$. Panich (1965) proposed a method in which the term $M_k[\sigma]$ in (26) is replaced by $M_k[L_0[\sigma]]$ and showed that there results an integral equation for σ whose kernel is only weakly singular. Kussmaul (1969) employs the representation (26) directly and deals with singular operator by a regularization technique which, however, involves the solution of two auxiliary boundary-value problems; we propose a rather simpler regularization procedure in § 4.

A method of a somewhat different character has been described by Schenck (1968) who uses, in addition to Green's boundary formula, the corresponding interior formula applied at a selected set of interior points. This leads to an overdetermined (though consistent) system of approximating linear algebraic equations which may be solved by a least-squares procedure. The effectiveness of the method depends on the suitable choice of the supplementary points, which must be such that when k assumes a critical value every linear combination of the relevant eigenfunctions is non-zero (and not excessively small) at one or more of the points.

In special cases it may be possible to employ the interior equation alone. It is shown by Copley (1968), for example, that for problems with axial symmetry only points on the axis need be chosen. In general, however, the use of the boundary equation is necessary for a satisfactory determination of the boundary values of the solution. A valuable commentary on this and other methods for the solution of problems in acoustic radiation and diffraction is to be found in the paper of Schenck already cited.

4. Treatment of singular kernels

For the numerical solution of integral equations it is customary to approximate the integrals by suitable quadrature formulae, and then to satisfy the approximating equations at a selected set of nodal points in the range of integration. Solution of the resulting set of linear equations gives approximate values of the unknown function at the selected points. However, in the case of the boundary integral equations discussed above this procedure is complicated by the presence of singularities of the kernels, and some preliminary analysis may be necessary.

In equation (23), for example, the kernel of M_k is non-singular on a smooth boundary and therefore fairly readily approximated numerically, but the kernel of $\partial M_k[\phi]/\partial n_p$ is highly singular, for

$$\frac{\partial M_k[\phi]}{\partial n_p} = \frac{\partial}{\partial n_p}\int_B \frac{\partial G_k(\boldsymbol{p}, \boldsymbol{q})}{\partial n_q}\phi(\boldsymbol{q})\,\mathrm{d}S_q \equiv N_k[\phi], \quad \text{say,}$$

in which the kernel has the property that as $R \to 0$

$$\frac{\partial^2 G_k}{\partial n_p\,\partial n_q} = \begin{cases} O(R^{-2}) \text{ in two dimensions,} \\ O(R^{-3}) \text{ in three dimensions.} \end{cases}$$

Thus, differentiating through the integral gives rise to a non-integrable kernel, although it can be shown that $N_k[\phi]$, under suitable conditions on ϕ, is well-behaved.

Clearly, before attempting to represent N_k numerically it must be transformed so as to reduce the strength of the singularity, and we discuss two ways in which this may be achieved.

The first method consists in directly transforming the expression into one involving tangential rather than normal derivatives. By different methods Maue (1949) and later Mitzner (1966) derived the important result

$$N_k[\phi] = \int_B \{(\boldsymbol{n}_q \times \nabla_q \phi) . (\boldsymbol{n}_p \times \nabla_p G_k) + k^2 \boldsymbol{n}_p . \boldsymbol{n}_q G_k \phi(\boldsymbol{q})\} \, \mathrm{d}S_q \quad (\boldsymbol{p} \in B), \tag{27}$$

where $\boldsymbol{n}_p$, $\boldsymbol{n}_q$ are unit outward normal vectors at $\boldsymbol{p}$ and $\boldsymbol{q}$ and ∇_p, ∇_q are the gradient operators at these points. In two dimensions this reduces to

$$N_k[\phi] = \int_B \left\{ \frac{\partial \phi}{\partial t_q} \frac{\partial G_k}{\partial t_p} + k^2 \boldsymbol{n}_p . \boldsymbol{n}_q G_k \phi(\boldsymbol{q}) \right\} \mathrm{d}s_q, \tag{28}$$

in which t_p, t_q are the positive tangent directions at $\boldsymbol{p}$ and $\boldsymbol{q}$. Both terms under the integral are still singular ($\partial G_k / \partial t_p$ has a Cauchy singularity and G_k a logarithmic singularity at $\boldsymbol{p} = \boldsymbol{q}$); nevertheless it is now possible to construct numerical approximations to the operator, details of which will be reported elsewhere. In this connexion we may mention also the work of Atkinson (1966), of which a summary is given in Kussmaul (1969).

In three dimensions the use of the expression (27) for $N_k[\phi]$ is considerably more complicated, and in this case it may therefore be more convenient to employ a method of regularization, in which the non-integrable singularity is removed by premultiplication by a suitable auxiliary operator. By applying Green's theorem to the functions $M_k[\phi]$ and G_k in the interior, D, and making use of the jump properties of the various potentials, it can be shown that

$$L_k N_k[\phi] = (M_k^2 - \tfrac{1}{4} I)[\phi] \quad (\boldsymbol{p} \in B), \tag{29}$$

where I is the identity operator. Then, writing the composite integral equation (23) in the form

$$\tfrac{1}{2} u - M_k[u] + \alpha\{(N_k - N_0) + N_0\}[u] = u_{\mathrm{i}} - \alpha \, \partial u_{\mathrm{i}} / \partial n_p \quad (\boldsymbol{p} \in B),$$

we can premultiply by L_0 and use the operator relation (29) with $k = 0$ to transform the term $L_0 N_0$; thus

$$L_0\{\tfrac{1}{2} I - M_k + \alpha(N_k - N_0)\}[u] + \alpha(M_0^2 - \tfrac{1}{4} I)[u] = L_0[u_{\mathrm{i}} - \alpha \, \partial u_{\mathrm{i}} / \partial n_p]. \tag{30}$$

In the three-dimensional case the solution remains uniquely determined because L_0 always has a bounded inverse—a consequence of the uniqueness of the solution to the interior Dirichlet problem for Laplace's equation. In two dimensions, however, it may happen that the homogeneous equation $L_0[\phi] = 0$ possesses a non-trivial solution for certain boundary shapes and although, as noted in § 2, this situation can easily be dealt with by modifying L_0, it is probably simpler to use relation (28) for these problems.

The kernels of the operators L_0 and $(N_k - N_0)$ still contain weak singularities, of order R^{-1} as $R \to 0$ in three dimensions, but again numerical approximations can be constructed.

The composite integral equation formulation for the exterior Neumann problem presented above, together with the expression (28) for $N_k[\phi]$, has been found useful in the solution of a variety of two-dimensional problems. The value of the coupling constant, α, is not critical, and quite small values of $\mathrm{Im}\,(\alpha)$ have proved sufficient to ensure freedom from ill-conditioning.

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References (Burton & Miller)

Atkinson, K. E. 1966 Thesis, University of Wisconsin.

Brakhage, H. & Werner, P. 1965 Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung. *Arch. Math.* **16**, 325–329.

Copley, L. G. 1968 Fundamental results concerning integral representations in acoustic radiation. *J. Acoust. Soc. Am.* **44**, 28–32.

Greenspan, D. & Werner, P. 1966 A numerical method for the exterior Dirichlet problem for the reduced wave equation. *Arch. Ration. Mech. Anal.* **23**, 288–316.

Jaswon, M. A. 1963 Integral equation methods in potential theory. I. *Proc. Roy. Soc. Lond.* A **275**, 23–32.

Kellogg, O. D. 1929 *Foundations of potential theory.* New York: Frederick Ungar Publishing Company.

Kupradze, V. D. 1965 *Potential methods in the theory of elasticity.* Jerusalem: Israel program for scientific translations.

Kussmaul, R. 1969 Ein numerisches Verfahren zur Lösung des Neumannschen Aussenraumproblems für die Helmholtzsche Schwingungsgleichung. *Computing* **4**, 246–273.

Leis, R. 1958 Über das Neumannsche Randwertproblem für die Helmholtzsche Schwingungsgleichung. *Arch. Ration. Mech. Anal.* **2**, 101–113.

Maue, A.-W. 1949 Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung. *Z. Phys.* **126**, 601–618.

Mitzner, K. M. 1966 Acoustic scattering from an interface between media of greatly different density. *J. Math. Phys.* **7**, 2053–2060.

Müller, C. 1952 Zur Methode der Strahlungskapazität von H. Weyl. *Math. Z.* **56**, 80–83.

Panich, O. I. 1965 On the question of the solvability of the exterior boundary problem for the wave equation and Maxwell's equation. (In Russian.) *Uspekhi Mat. Nauk.* **20**, no. 1, 221–226.

Petrovskii, I. G. 1967 *Partial differential equations.* London: Iliffe Books Ltd.

Schenck, H. A. 1968 Improved integral formulation for acoustic radiation problems. *J. Acoust. Soc. Am.* **44**, 41–58.

Symm, G. T. 1963 Integral equation methods in potential theory. II. *Proc. Roy. Soc. Lond.* A **275**, 33–46.

Weyl, H. 1952 Kapazität von Strahlungsfeldern. *Math. Z.* **55**, 187–198.