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52. Chipot: **Variational Inequalities and Flow in Porous Media**.
53. Majda: **Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables**.

(continued following index)

Rainer Kress

Linear Integral Equations



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To the Memory of My Parents

Preface

I fell in love with integral equations about twenty years ago when I was working on my thesis, and I am still attracted by their mathematical beauty. This book will try to stimulate the reader to share this love with me.

Having taught integral equations a number of times I felt a lack of a text which adequately combines theory, applications and numerical methods. Therefore, in this book I intend to cover each of these fields with the same weight. The first part provides the basic Riesz-Fredholm theory for equations of the second kind with compact operators in dual systems including all functional analytic concepts necessary for developing this theory. The second part then illustrates the classical applications of integral equation methods to boundary value problems for the Laplace and the heat equation as one of the main historical sources for the development of integral equations, and also introduces Cauchy type singular integral equations. The third part is devoted to describing the fundamental ideas for the numerical solution of integral equations. Finally, in a fourth part, ill-posed integral equations of the first kind and their regularization are studied in a Hilbert space setting.

In order to make the book accessible not only to mathematicians but also to physicists and engineers I have planned it as self-contained as possible by requiring only a solid foundation in differential and integral calculus and, for parts of the book, in complex function theory. Some background in functional analysis will be helpful, but the basic concepts of the theory of normed spaces will be briefly reviewed, and all functional analytic tools which are relevant in the study of integral equations will be developed in the book. Of course, I expect the reader to be willing to accept the functional analytic language for describing the theory and the numerical solution of integral equations. I hope that I succeeded in finding the adequate compromise between presenting integral equations in the proper modern framework and the danger of being too abstract.

An introduction to integral equations cannot present a complete picture of all classical aspects of the theory and of all recent developments. In this sense, this book intends to tell the reader what I think appropriate to teach students in a two-semester course on integral equations. I am willing to admit that the choice of a few of the topics might be biased by my own preferences and that some important subjects are omitted.

I am indebted to Dipl.-Math. Peter Hähner for carefully reading the book, for checking the solutions to the problems and for a number of suggestions for valuable improvements. Thanks also go to Frau Petra Trapp who spent some

time assisting me in the preparation of the L^AT_EX version of the text. And a particular note of thanks is given to my friend David Colton for reading over the book and helping me with the English language. Part of the book was written while I was on sabbatical leave at the Department of Mathematics at the University of Delaware. I gratefully acknowledge the hospitality.

Göttingen, April 1989

Rainer Kress

Contents

1. Normed Spaces	
1.1 Convergence and Continuity	2
1.2 Open and Closed Sets	4
1.3 Completeness	5
1.4 Compactness	5
1.5 Scalar Products	8
1.6 Best Approximation	10
2. Bounded and Compact Operators	
2.1 Bounded Operators	13
2.2 Integral Operators	14
2.3 Neumann Series	16
2.4 Compact Operators	17
3. The Riesz Theory	
3.1 Riesz Theory for Compact Operators	25
3.2 Spectral Theory for Compact Operators	31
3.3 Volterra Integral Equations	33
4. Dual Systems and Fredholm Theory	
4.1 Dual Systems Via Bilinear Forms	36
4.2 Dual Systems Via Sesquilinear Forms	37
4.3 Positive Dual Systems	39
4.4 The Fredholm Alternative	41
4.5 Boundary Value Problems	46
5. Regularization in Dual Systems	
5.1 Regularizers	50
5.2 Normal Solvability	51
5.3 Index	56
6. Potential Theory	
6.1 Harmonic Functions	58
6.2 Boundary Value Problems: Uniqueness	62
6.3 Surface Potentials	67
6.4 Boundary Value Problems: Existence	71
6.5 Supplements	74

7. Singular Integral Equations	
7.1 Hölder Continuity	82
7.2 The Cauchy Integral Operator	84
7.3 The Riemann Problem	91
7.4 Singular Integral Equations with Cauchy Kernel	94
7.5 Cauchy Integral and Logarithmic Potential	99
7.6 Supplements	105
8. Sobolev Spaces	
8.1 Fourier Expansion	108
8.2 The Sobolev Space $H^p[0, 2\pi]$	109
8.3 The Sobolev Space $H^p[\Gamma]$	117
8.4 Weak Solutions to Boundary Value Problems	124
9. The Heat Equation	
9.1 Initial Boundary Value Problem: Uniqueness	132
9.2 Heat Potentials	134
9.3 Initial Boundary Value Problem: Existence	139
10. Operator Approximations	
10.1 Approximations Based on Norm Convergence	142
10.2 Uniform Boundedness Principle	144
10.3 Collectively Compact Operators	146
10.4 Approximations Based on Pointwise Convergence	147
10.5 Successive Approximations	149
11. Degenerate Kernel Approximation	
11.1 Finite Dimensional Operators	154
11.2 Degenerate Kernels Via Interpolation	156
11.3 Degenerate Kernels Via Expansions	164
12. Quadrature Methods	
12.1 Numerical Integration	168
12.2 Nyström's Method	171
12.3 Nyström's Method for Weakly Singular Kernels	175
13. Projection Methods	
13.1 The Projection Method	184
13.2 The Collocation Method	189
13.3 The Galerkin Method	199
14. Iterative Solution and Stability	
14.1 The Method of Residual Correction	206
14.2 Multi-Grid Methods	210
14.3 Stability of Linear Systems	215

15. Equations of the First Kind	
15.1 Ill-Posed Problems	221
15.2 Regularization of Ill-Posed Problems	224
15.3 Compact Self Adjoint Operators	226
15.4 Singular Value Decomposition	232
15.5 Regularization Schemes	236
16. Tikhonov Regularization	
16.1 The Tikhonov Functional	243
16.2 Weak Convergence	244
16.3 Quasi-Solutions	245
16.4 Minimum Norm Solutions	250
16.5 Classical Tikhonov Regularization	253
17. Regularization by Discretization	
17.1 Projection Methods for Ill-Posed Equations	259
17.2 The Moment Method	263
17.3 Hilbert Spaces with Reproducing Kernel	265
17.4 Moment Collocation	267
18. Inverse Scattering Theory	
18.1 Ill-Posed Integral Equations in Potential Theory	270
18.2 An Inverse Acoustic Scattering Problem	277
18.3 Numerical Methods in Inverse Scattering	279
Bibliography	289
Index	297

1. Normed Spaces

The topic of this book is linear integral equations for which

$$\int_a^b K(x,y)\varphi(y)dy = f(x), \quad x \in [a,b],$$

and

$$\varphi(x) - \int_a^b K(x,y)\varphi(y)dy = f(x), \quad x \in [a,b],$$

are typical examples. In these equations the function φ is the unknown and the kernel K and the right hand side f are given functions. The above equations are denoted as *Fredholm integral equations* of the *first* and *second kind*. We will regard them as *operator equations*

$$A\varphi = f$$

and

$$\varphi - A\varphi = f$$

of the *first* and *second kind* in appropriate normed function spaces.

The symbol $A : X \rightarrow Y$ will mean a single valued mapping whose domain of definition is X and whose range is contained in Y , that is, for every $\varphi \in X$ the mapping A assigns a unique element $A\varphi \in Y$. The *range* $A(X)$ is the set $A(X) := \{A\varphi : \varphi \in X\}$ of all image elements. The terms *mapping*, *function* and *operator* will be used synonymously.

Existence and uniqueness of a solution to an operator equation can be equivalently expressed by the existence of the *inverse operator*. If for each $f \in A(X)$ there is only one element $\varphi \in X$ with $A\varphi = f$ then A is said to be *injective* and to have an inverse mapping $A^{-1} : A(X) \rightarrow X$ defined by $A^{-1}f := \varphi$. The inverse mapping has domain $A(X)$ and range X . It satisfies $A^{-1}A = I$ on X and $AA^{-1} = I$ on $A(X)$ where I denotes the identity operator mapping each element into itself. If $A(X) = Y$ then the mapping is said to be *surjective*. The mapping is called *bijective* if it is injective and surjective, that is, if the inverse mapping $A^{-1} : Y \rightarrow X$ exists.

In the first part of the book we will present the Riesz–Fredholm theory for compact operators which, in particular, answers the question of existence and uniqueness of solutions to integral equations of the second kind with sufficiently smooth kernel. In order to develop the theory we will assume that the reader is familiar with the elementary properties of linear spaces, normed spaces and bounded linear operators. For convenience and to introduce notations we briefly recall a few basics from the theory of normed spaces omitting most of the proofs. For a more detailed study we refer to any introduction to functional analysis.

1.1 Convergence and Continuity

Definition 1.1. Let X be a complex (or real) linear space (vector space). A function $\|\cdot\| : X \rightarrow \mathbb{R}$ with the properties

$$(N1) \quad \|\varphi\| \geq 0 \quad (\text{positivity})$$

$$(N2) \quad \|\varphi\| = 0 \quad \text{if and only if} \quad \varphi = 0 \quad (\text{definiteness})$$

$$(N3) \quad \|\alpha\varphi\| = |\alpha| \|\varphi\| \quad (\text{homogeneity})$$

$$(N4) \quad \|\varphi + \psi\| \leq \|\varphi\| + \|\psi\| \quad (\text{triangle inequality})$$

for all $\varphi, \psi \in X$ and all $\alpha \in \mathbb{C}$ (or \mathbb{R}) is called a norm on X . A linear space X equipped with a norm is called a normed space.

As a consequence of (N3) and (N4) we note the second triangle inequality

$$(N4') \quad \|\|\varphi\| - \|\psi\|\| \leq \|\varphi - \psi\|.$$

For two elements in a normed space $\|\varphi - \psi\|$ is called the *distance* between φ and ψ .

Definition 1.2. A sequence (φ_n) of elements in a normed space X converges to an element $\varphi \in X$ if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$\|\varphi_n - \varphi\| < \varepsilon$$

for all $n \geq N(\varepsilon)$, that is, if $\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\| = 0$. The element φ is called the limit of the sequence (φ_n) .

For a convergent sequence we will write $\lim_{n \rightarrow \infty} \varphi_n = \varphi$ or $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$. Note that by (N4) a sequence in a normed space cannot converge to two different elements. A sequence which does not converge is called *divergent*.

Definition 1.3. A function (operator) A mapping a subset U of a normed space X into a normed space Y is called continuous at $\varphi \in U$ if $\lim_{n \rightarrow \infty} A\varphi_n = A\varphi$ for every sequence (φ_n) from U with $\lim_{n \rightarrow \infty} \varphi_n = \varphi$. The function $A : U \rightarrow Y$ is called continuous if it is continuous for all $\varphi \in U$.

An equivalent definition is: a function $A : U \subset X \rightarrow Y$ is continuous at $\varphi \in U$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|A\varphi - A\psi\| < \varepsilon$ for all $\psi \in U$ with $\|\varphi - \psi\| < \delta$. Here, we have used the same symbol $\|\cdot\|$ for the norms on X and Y . The function A is called *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|A\varphi - A\psi\| < \varepsilon$ for all $\varphi, \psi \in U$ with $\|\varphi - \psi\| < \delta$.

Note that by (N4') the norm is a continuous function.

In our study of integral equations the basic example of a normed space will be the linear space $C[a, b]$ of continuous real or complex valued functions φ defined on an interval $[a, b] \subset \mathbb{R}$ furnished either with the *maximum norm*

$$\|\varphi\|_\infty := \max_{x \in [a, b]} |\varphi(x)|$$

or the *mean square norm*

$$\|\varphi\|_2 := \left(\int_a^b |\varphi(x)|^2 dx \right)^{1/2}.$$

Convergence of a sequence of continuous functions in the maximum norm is equivalent to uniform convergence and convergence in the mean square norm is called mean square convergence.

Definition 1.4. Two norms on a linear space X are called equivalent if each sequence converging with respect to the first norm also converges with respect to the second norm and vice versa.

Theorem 1.5. Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a linear space X are equivalent if and only if there exist positive numbers c and C such that

$$c\|\varphi\|_a \leq \|\varphi\|_b \leq C\|\varphi\|_a$$

for all $\varphi \in X$. The limits with respect to the two norms coincide.

Proof: Provided the conditions are satisfied, from $\|\varphi_n - \varphi\|_a \rightarrow 0$, $n \rightarrow \infty$, there follows $\|\varphi_n - \varphi\|_b \rightarrow 0$, $n \rightarrow \infty$, and vice versa.

Conversely, let the two norms be equivalent and assume there is no $C > 0$ such that $\|\varphi\|_b \leq C\|\varphi\|_a$ for all $\varphi \in X$. Then there exists a sequence (φ_n) with $\|\varphi_n\|_a = 1$ and $\|\varphi_n\|_b \geq n^2$. Now the sequence (ψ_n) with $\psi_n := \varphi_n/n$ converges to zero with respect to $\|\cdot\|_a$ whereas with respect to $\|\cdot\|_b$ it is divergent because of $\|\psi_n\|_b \geq n$. \square

In a finite dimensional linear space $X_m = \text{span}\{f_1, \dots, f_m\}$ of dimension m with basis f_1, \dots, f_m every element can be expressed in the form

$$\varphi = \sum_{k=1}^m \alpha_k f_k.$$

As is easily verified,

$$(1.1) \quad \|\varphi\|_\infty := \max_{k=1, \dots, m} |\alpha_k|$$

defines a norm on X_m . Let $\|\cdot\|$ denote any other norm on X_m . Then,

$$\|\varphi\| \leq C\|\varphi\|_\infty$$

for all $\varphi \in X_m$, where

$$C := \sum_{k=1}^m \|f_k\|.$$

Assume there is no $c > 0$ such that $c\|\varphi\|_\infty \leq \|\varphi\|$ for all $\varphi \in X_m$. Then there exists a sequence (φ_n) with $\|\varphi_n\| = 1$ such that $\|\varphi_n\|_\infty \geq n$. Consider the sequence (ψ_n) with $\psi_n := \varphi_n/\|\varphi_n\|_\infty$ and write

$$\psi_n = \sum_{k=1}^m \alpha_{kn} f_k.$$

Because of $\|\psi_n\|_\infty = 1$ each of the sequences (α_{kn}) , $k = 1, \dots, m$, is bounded in \mathbb{C} . Hence, by the Bolzano-Weierstrass theorem we can select convergent subsequences $\alpha_{k,n(j)} \rightarrow \alpha_k$, $j \rightarrow \infty$, for each $k = 1, \dots, m$. This now implies $\|\psi_{n(j)} - \psi\|_\infty \rightarrow 0$, $j \rightarrow \infty$, where

$$\psi := \sum_{k=1}^m \alpha_k f_k,$$

and also $\|\psi_{n(j)} - \psi\| \leq C\|\psi_{n(j)} - \psi\|_\infty \rightarrow 0$, $j \rightarrow \infty$. But on the other hand we have $\|\psi_n\| = 1/\|\varphi_n\|_\infty \rightarrow 0$, $n \rightarrow \infty$. Therefore, $\psi = 0$ and consequently $\|\psi_{n(j)}\|_\infty \rightarrow 0$, $j \rightarrow \infty$, which contradicts $\|\psi_n\|_\infty = 1$ for all n . Thus we have proven

Theorem 1.6. *On a finite dimensional linear space all norms are equivalent.*

1.2 Open and Closed Sets

For each element φ of a normed space X and each positive number r the set $B(\varphi; r) := \{\psi \in X : \|\psi - \varphi\| < r\}$ is called an *open ball* with center φ and radius r , the set $B[\varphi; r] := \{\psi \in X : \|\psi - \varphi\| \leq r\}$ is called a *closed ball*.

Definition 1.7. *A subset U of a normed space X is called open if for each element $\varphi \in U$ there exists $r > 0$ such that $B(\varphi; r) \subset U$.*

Obviously, open balls are open.

Definition 1.8. *A subset U of a normed space X is called closed if it contains all limits of convergent sequences of U .*

A subset U of a normed space X is closed if and only if its complement $X \setminus U$ is open. Obviously, closed balls are closed. In particular, using the norm (1.1), we see that finite dimensional subspaces of a normed space are closed.

Definition 1.9. *The closure \bar{U} of a subset U of a normed space X is the set of all limits of convergent sequences of U . A set U is called dense in another set V if $V \subset \bar{U}$, that is, if each element in V is the limit of a convergent sequence from U .*

A subset U is closed if and only if it coincides with its closure. By the Weierstrass approximation theorem (cf. Davis [2]) the linear subspace P of polynomials is dense in $C[a, b]$ with respect to the maximum norm and the mean square norm.

Definition 1.10. *A subset U of a normed space X is called bounded if there exists a positive number C such that $\|\varphi\| \leq C$ for all $\varphi \in U$.*

Convergent sequences are bounded.

1.3 Completeness

Definition 1.11. A sequence (φ_n) of elements in a normed space X is called a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$\|\varphi_n - \varphi_m\| < \varepsilon$$

for all $n, m \geq N(\varepsilon)$, that is, if $\lim_{n,m \rightarrow \infty} \|\varphi_n - \varphi_m\| = 0$.

Every convergent sequence is a Cauchy sequence whereas the converse in general is not true. This gives rise to the

Definition 1.12. A subset U of a normed space X is called complete if every Cauchy sequence of elements in U converges to an element in U . A normed space X is called a Banach space if it is complete.

Complete sets are closed, and closed subsets of a complete set are complete. Since the Cauchy criterion is sufficient for convergence of sequences of complex numbers, using the norm (1.1), we observe that finite dimensional normed spaces are Banach spaces. The Cauchy criterion is also sufficient for uniform convergence of a sequence of continuous functions towards a continuous limit function. Therefore, the space $C[a, b]$ is complete with respect to the maximum norm. As can be seen from elementary counterexamples $C[a, b]$ is not complete with respect to the mean square norm.

It is very important that every incomplete normed space can be completed in the sense of

Theorem 1.13. For each normed space X there exists a Banach space \tilde{X} such that X is isomorphic and isometric to a dense subspace of \tilde{X} , that is, there is a linear bijective mapping I from X into a dense subspace of \tilde{X} such that $\|I\varphi\|_{\tilde{X}} = \|\varphi\|_X$ for all $\varphi \in X$. The space \tilde{X} is uniquely determined up to isometric isomorphisms, that is, to any two such Banach spaces there exists a linear bijective mapping between the two spaces leaving the norms invariant.

For a proof of this concept of completion we refer to any introduction to functional analysis. As a matter of fact it will not play a central role in our presentation. The completion of $C[a, b]$ with respect to the mean square norm yields the complete space $L^2[a, b]$ of measurable and Lebesgue square integrable functions, or to be more precise, of equivalence classes of these functions which coincide almost everywhere.

1.4 Compactness

Definition 1.14. A subset U of a normed space X is called compact if every open covering of U contains a finite subcovering, that is, if for every family V_j , $j \in J$, of open sets with the property

$$U \subset \bigcup_{j \in J} V_j$$

there exists a finite subfamily $V_{j(k)}$, $j(k) \in J$, $k = 1, \dots, n$, with

$$U \subset \bigcup_{k=1}^n V_{j(k)}.$$

A subset U is called sequentially compact if every sequence of elements from U contains a subsequence converging to an element in U .

A subset U of a normed space is called totally bounded if for each $\varepsilon > 0$ there exists a finite number of elements $\varphi_1, \dots, \varphi_n$ in U such that

$$U \subset \bigcup_{j=1}^n B(\varphi_j; \varepsilon),$$

that is, each element $\varphi \in U$ has a distance less than ε from at least one of the elements $\varphi_1, \dots, \varphi_n$. We note that each sequentially compact set U is totally bounded. Otherwise there would exist a positive ε and a sequence (φ_n) in U with the property $\|\varphi_n - \varphi_m\| \geq \varepsilon$ for all $n \neq m$. This would imply that the sequence (φ_n) does not contain a convergent subsequence which contradicts the sequential compactness of U . For each totally bounded set U , letting $\varepsilon = 1/m$, $m = 1, 2, \dots$, and collecting the corresponding finite systems of elements, we obtain a sequence which is dense in U .

Theorem 1.15. *A subset of a normed space is compact if and only if it is sequentially compact.*

Proof: Let U be compact and assume it is not sequentially compact. Then there exists a sequence (φ_n) in U which does not contain a convergent subsequence with limit in U . Consequently, for each $\varphi \in U$ there exists an open ball $B(\varphi; r)$ with center φ and radius $r(\varphi)$ containing at most a finite number of the elements of the sequence (φ_n) . The set of these balls clearly is an open covering of U , and since U is compact it follows that U contains only a finite number of the elements of the sequence (φ_n) which is a contradiction.

Conversely, let U be sequentially compact and let V_j , $j \in J$, be an open covering of U . First, we show that there exists a positive number ε such that for every $\varphi \in U$ the ball $B(\varphi; \varepsilon)$ is contained in at least one of the sets V_j . Otherwise there would exist a sequence (φ_n) in U such that the ball $B(\varphi_n; 1/n)$ is not contained in one of the V_j . Since U is sequentially compact this sequence (φ_n) contains a convergent subsequence $(\varphi_{n(k)})$ with limit $\varphi \in U$. The element φ is contained in some V_j , and since V_j is open, using the triangle inequality, we see that $B(\varphi_{n(k)}; 1/n(k)) \subset V_j$ for sufficiently large k which is a contradiction. Now, since the sequentially compact set U is totally bounded there exists a finite number of elements $\varphi_1, \dots, \varphi_n$ in U such that the balls $B(\varphi_k; \varepsilon)$, $k = 1, \dots, n$, cover U . But for each of these balls then there exists a set $V_{j(k)}$, $j(k) \in J$, such that $B(\varphi_k; \varepsilon) \subset V_{j(k)}$. Hence, the finite family $V_{j(k)}$, $k = 1, \dots, n$, covers U . \square

In particular, from Theorem 1.15 we observe that compact sets are bounded, closed and complete.

Definition 1.16. A subset of a normed space is called relatively compact if its closure is compact.

As a consequence of Theorem 1.15, a set U is relatively compact if and only if each sequence of elements from U contains a convergent subsequence. Hence, as in the case of compact sets, each relatively compact set is totally bounded.

Theorem 1.17. Bounded and finite dimensional subsets of a normed space are relatively compact.

Proof: This follows from the Bolzano–Weierstrass theorem using the norm (1.1). \square

For the compactness in $C(G)$, the space of continuous real or complex valued functions defined on a compact set $G \subset \mathbb{R}^m$ furnished with the maximum norm

$$\|\varphi\|_\infty := \max_{x \in G} |\varphi(x)|,$$

we have

Theorem 1.18 (Arzelà–Ascoli). A set $U \subset C(G)$ is relatively compact if and only if it is bounded and equicontinuous, that is, there exists a constant C such that

$$|\varphi(x)| \leq C$$

for all $x \in G$ and all $\varphi \in U$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\varphi(x) - \varphi(y)| < \varepsilon$$

for all $x, y \in G$ with $|x - y| < \delta$ and all $\varphi \in U$. (Here and henceforth by $|x|$ we denote the Euclidean norm of a point $x \in \mathbb{R}^m$).

Proof: Let U be bounded and equicontinuous and let (φ_n) be a sequence in U . We choose a sequence (x_i) in G that is dense in G . Since the sequence $(\varphi_n(x_i))$ is bounded in \mathbb{C} for each x_i , by the standard diagonalization procedure we can choose a subsequence $(\varphi_{n(k)})$ such that $(\varphi_{n(k)}(x_i))$ converges in \mathbb{C} as $k \rightarrow \infty$ for each x_i . More precisely, since $(\varphi_n(x_1))$ is bounded, we can choose a subsequence $(\varphi_{n_1(k)})$ such that $(\varphi_{n_1(k)}(x_1))$ converges as $k \rightarrow \infty$. The sequence $(\varphi_{n_1(k)}(x_2))$ again is bounded and we can choose a subsequence $(\varphi_{n_2(k)})$ of $(\varphi_{n_1(k)})$ such that $(\varphi_{n_2(k)}(x_2))$ converges as $k \rightarrow \infty$. Repeating this process of selecting subsequences, we arrive at a double array $(\varphi_{n_i(k)})$ such that each row $(\varphi_{n_i(k)})$ is a subsequence of the previous row $(\varphi_{n_{i-1}(k)})$ and each sequence $(\varphi_{n_i(k)}(x_i))$ converges as $k \rightarrow \infty$. For the diagonal sequence $\varphi_{n(k)} := \varphi_{n_k(k)}$ we have $(\varphi_{n(k)}(x_i))$ converges as $k \rightarrow \infty$ for all x_i .

Because the set (x_i) is dense in G , given $\varepsilon > 0$, we can choose $i \in \mathbb{N}$ such that each point $x \in G$ has a distance less than δ from at least one element x_i of the set x_1, \dots, x_i . Next choose $N(\varepsilon) \in \mathbb{N}$ such that

$$|\varphi_{n(k)}(x_j) - \varphi_{n(l)}(x_j)| < \varepsilon$$

for all $k, l \geq N(\varepsilon)$ and all $j = 1, \dots, i$. From the equicontinuity we obtain

$$|\varphi_{n(k)}(x) - \varphi_{n(l)}(x)|$$

$$\leq |\varphi_{n(k)}(x) - \varphi_{n(k)}(x_j)| + |\varphi_{n(k)}(x_j) - \varphi_{n(l)}(x_j)| + |\varphi_{n(l)}(x_j) - \varphi_{n(l)}(x)| < 3\varepsilon$$

for all $k, l \geq N(\varepsilon)$ and all $x \in G$ which establishes the uniform convergence, that is, convergence in the maximum norm of the subsequence $(\varphi_{n(k)})$. Hence U is relatively compact.

Conversely, let U be relatively compact. Then U is totally bounded, i.e., given $\varepsilon > 0$ there exist functions $\varphi_1, \dots, \varphi_i \in U$ such that

$$\min_{j=1, \dots, i} \|\varphi - \varphi_j\|_\infty < \frac{\varepsilon}{3}$$

for all $\varphi \in U$. Since each of the $\varphi_1, \dots, \varphi_i$ is uniformly continuous on the compact set G , there exists $\delta > 0$ such that

$$|\varphi_j(x) - \varphi_j(y)| < \frac{\varepsilon}{3}$$

for all $x, y \in G$ with $|x - y| < \delta$ and all $j = 1, \dots, i$. Then for all $\varphi \in U$, choosing j_0 such that

$$\|\varphi - \varphi_{j_0}\|_\infty = \min_{j=1, \dots, i} \|\varphi - \varphi_j\|_\infty$$

we obtain

$$|\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi_{j_0}(x)| + |\varphi_{j_0}(x) - \varphi_{j_0}(y)| + |\varphi_{j_0}(y) - \varphi(y)| < \varepsilon$$

for all $x, y \in G$ with $|x - y| < \delta$. Therefore U is equicontinuous. Finally, U is bounded, since compact sets are bounded.

1.5 Scalar Products

Definition 1.19. Let X be a complex (or real) linear space. Then a function $(\cdot, \cdot) : X \rightarrow \mathbb{C}$ (or \mathbb{R}) with the properties

$$(H1) \quad (\varphi, \varphi) \geq 0 \quad (\text{positivity})$$

$$(H2) \quad (\varphi, \varphi) = 0 \quad \text{if and only if} \quad \varphi = 0 \quad (\text{definiteness})$$

$$(H3) \quad (\varphi, \psi) = \overline{(\psi, \varphi)} \quad (\text{symmetry})$$

$$(H4) \quad (\alpha\varphi + \beta\psi, \chi) = \alpha(\varphi, \chi) + \beta(\psi, \chi) \quad (\text{linearity})$$

for all $\varphi, \psi, \chi \in X$ and $\alpha, \beta \in \mathbb{C}$ (or \mathbb{R}) is called a scalar product on X .

By the bar we denote the complex conjugate. As a consequence of (H3) and (H4) we note that

$$(H4') \quad (\varphi, \alpha\psi + \beta\chi) = \bar{\alpha}(\varphi, \psi) + \bar{\beta}(\varphi, \chi) \quad (\text{antilinearity}).$$

Theorem 1.20. *For a scalar product there holds the Cauchy–Schwarz inequality*

$$|(\varphi, \psi)|^2 \leq (\varphi, \varphi)(\psi, \psi)$$

for all $\varphi, \psi \in X$ with equality if and only if φ and ψ are linearly dependent.

Proof: The inequality is trivial for $\varphi = 0$. For $\varphi \neq 0$ it follows from

$$\begin{aligned} (\alpha\varphi + \beta\psi, \alpha\varphi + \beta\psi) &= |\alpha|^2(\varphi, \varphi) + 2\operatorname{Re}\{\alpha\bar{\beta}(\varphi, \psi)\} + |\beta|^2(\psi, \psi) \\ &= (\varphi, \varphi)(\psi, \psi) - |(\varphi, \psi)|^2 \end{aligned}$$

where we have set $\alpha = -(\varphi, \varphi)^{-1/2}\overline{(\varphi, \psi)}$ and $\beta = (\varphi, \varphi)^{1/2}$. Since (\cdot, \cdot) is positive definite this expression is nonnegative and it is equal to zero if and only if $\alpha\varphi + \beta\psi = 0$. In the latter case φ and ψ are linearly dependent because $\beta \neq 0$.

Theorem 1.21. *A scalar product (\cdot, \cdot) on a linear space X defines a norm by*

$$\|\varphi\| := (\varphi, \varphi)^{1/2}$$

for all $\varphi \in X$. If X is complete with respect to this norm it is called a Hilbert space, otherwise it is called a pre–Hilbert space.

Proof: We leave it as an exercise to verify the norm axioms. The triangle inequality follows from the Cauchy–Schwarz inequality. \square

Note that $L^2[a, b]$ is a Hilbert space with the scalar product given by

$$(\varphi, \psi) := \int_a^b \varphi(x)\overline{\psi(x)}dx.$$

Definition 1.22. *Two elements φ and ψ of a pre–Hilbert space X are called orthogonal if*

$$(\varphi, \psi) = 0.$$

Two subsets U and V of X are called orthogonal if each pair of elements $\varphi \in U$ and $\psi \in V$ is orthogonal. For two orthogonal elements or subsets we write $\varphi \perp \psi$ and $U \perp V$, respectively. A subset U of X is called an orthogonal system if $(\varphi, \psi) = 0$ for all $\varphi, \psi \in U$ with $\varphi \neq \psi$. An orthogonal system U is called an orthonormal system if $\|\varphi\| = 1$ for all $\varphi \in U$. The set

$$U^\perp := \{\psi \in X : \psi \perp U\}$$

is called the orthogonal complement of the subset U .

1.6 Best Approximation

Definition 1.23. Let $U \subset X$ be a subset of a normed space X and let $\varphi \in X$. An element $v \in U$ is called a best approximation to φ with respect to U if

$$\|\varphi - v\| = \inf_{u \in U} \|\varphi - u\|,$$

that is, $v \in U$ has smallest distance from φ .

Theorem 1.24. Let U be a finite dimensional subspace of a normed space X . Then for every element of X there exists a best approximation with respect to U .

Proof: Let $\varphi \in X$ and choose a minimizing sequence (u_n) for φ , i.e., $u_n \in U$ satisfies

$$\|\varphi - u_n\| \rightarrow d := \inf_{u \in U} \|\varphi - u\|, \quad n \rightarrow \infty.$$

Because of $\|u_n\| \leq \|\varphi - u_n\| + \|\varphi\|$ the sequence (u_n) is bounded. Since U has finite dimension it is closed and by Theorem 1.17 the sequence (u_n) contains a convergent subsequence $(u_{n(k)})$ with limit $v \in U$. Then

$$\|\varphi - v\| = \lim_{k \rightarrow \infty} \|\varphi - u_{n(k)}\| = d.$$

Theorem 1.25. Let U be a linear subspace of a pre-Hilbert space X . An element v is a best approximation to $\varphi \in X$ with respect to U if and only if

$$(\varphi - v, u) = 0$$

for all $u \in U$, that is, if and only if $\varphi - v \perp U$. To each $\varphi \in X$ there exists at most one best approximation with respect to U .

Proof: This follows from the equality

$$\|(\varphi - v) + \alpha u\|^2 = \|\varphi - v\|^2 + 2\alpha \operatorname{Re}(\varphi - v, u) + \alpha^2 \|u\|^2$$

which is valid for all $v, u \in U$ and all $\alpha \in \mathbb{R}$.

Theorem 1.26. Let U be a complete linear subspace of a pre-Hilbert space X . Then to every element of X there exists a unique best approximation with respect to U .

Proof: Let $\varphi \in X$ and choose a sequence (u_n) with

$$(1.2) \quad \|\varphi - u_n\|^2 \leq d^2 + \frac{1}{n}, \quad n \in \mathbb{N},$$

where $d := \inf_{u \in U} \|\varphi - u\|$. Then

$$\begin{aligned} & \|(\varphi - u_n) + (\varphi - u_m)\|^2 + \|u_n - u_m\|^2 \\ &= 2\|\varphi - u_n\|^2 + 2\|\varphi - u_m\|^2 \leq 4d^2 + \frac{2}{n} + \frac{2}{m} \end{aligned}$$

for all $n, m \in \mathbb{N}$, and since $\frac{1}{2}(u_n + u_m) \in U$ it follows that

$$\|u_n - u_m\|^2 \leq 4d^2 + \frac{2}{n} + \frac{2}{m} - 4 \left\| \varphi - \frac{1}{2}(u_n + u_m) \right\|^2 \leq \frac{2}{n} + \frac{2}{m}.$$

Hence, (u_n) is a Cauchy sequence and since U is complete there exists an element $v \in U$ such that $u_n \rightarrow v$, $n \rightarrow \infty$. Passing to the limit $n \rightarrow \infty$ in (1.2) shows that v is a best approximation of φ with respect to U . \square

We wish to extend Theorem 1.25 to the case of convex subsets of a pre-Hilbert space. A subset U of a linear space X is called *convex* if for all $\varphi_1, \varphi_2 \in U$ and all $\lambda \in (0, 1)$ there holds $\lambda\varphi_1 + (1 - \lambda)\varphi_2 \in U$.

Theorem 1.27. *Let U be a convex subset of a pre-Hilbert space X . An element $v \in U$ is a best approximation to $\varphi \in X$ with respect to U if and only if*

$$\operatorname{Re}(\varphi - v, u - v) \leq 0$$

for all $u \in U$. To each $\varphi \in X$ there exists at most one best approximation with respect to U .

Proof: From the equality

$$\|\varphi - [(1 - \lambda)v + \lambda u]\|^2 = \|\varphi - v\|^2 - 2\lambda\operatorname{Re}(\varphi - v, u - v) + \lambda^2\|u - v\|^2$$

which is valid for all $u, v \in U$ and all $\lambda \in (0, 1)$. \square

It is left to the reader to carry Theorem 1.26 over from the linear case to the convex case.

Theorem 1.28. *Let $\{u_n : n \in \mathbb{N}\}$ be an orthonormal system in a pre-Hilbert space X . Then the following properties are equivalent:*

- a) $\operatorname{span}\{u_n : n \in \mathbb{N}\}$ is dense in X .
- b) Each $\varphi \in X$ can be expanded in a Fourier series

$$\varphi = \sum_{n=1}^{\infty} (\varphi, u_n) u_n.$$

- c) For each $\varphi \in X$ there holds Parseval's equality

$$\|\varphi\|^2 = \sum_{n=1}^{\infty} |(\varphi, u_n)|^2.$$

An orthonormal system with these properties is called complete.

Proof: What has to be understood by a series is explained in Problem 1.5.

a) \Rightarrow b): By Theorems 1.25 and 1.26, the partial sum

$$\varphi_n = \sum_{k=1}^n (\varphi, u_k) u_k$$

can be seen to be the best approximation to φ with respect to $\text{span}\{u_1, \dots, u_n\}$. Since $\text{span}\{u_n : n \in \mathbb{N}\}$ is dense in X , the sequence of the best approximations converges $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$.

b) \Rightarrow c): Take the scalar product of the Fourier series with φ .

c) \Rightarrow a): From

$$\left\| \varphi - \sum_{k=1}^n (\varphi, u_k) u_k \right\|^2 = \|\varphi\|^2 - \sum_{k=1}^n |(\varphi, u_k)|^2.$$

Problems

- 1.1. Show that finite dimensional subspaces of normed spaces are closed and complete.
- 1.2. A norm $\|\cdot\|_a$ on a linear space X is called *stronger* than a norm $\|\cdot\|_b$ if every sequence converging with respect to the norm $\|\cdot\|_a$ also converges with respect to the norm $\|\cdot\|_b$. The same fact is also expressed by saying that the norm $\|\cdot\|_b$ is *weaker* than the norm $\|\cdot\|_a$. Show that $\|\cdot\|_a$ is stronger than $\|\cdot\|_b$ if and only if there exists a positive number C such that $\|\varphi_b\| \leq C\|\varphi\|_a$ for all $\varphi \in X$. Show that on $C[a, b]$ the maximum norm is stronger than the mean square norm. Construct a counterexample to demonstrate that these two norms are not equivalent.
- 1.3. Show that a continuous real valued function on a compact subset of a normed space assumes its supremum and its infimum and that it is uniformly continuous.
- 1.4. Construct a counterexample to demonstrate that $C[a, b]$ is not complete with respect to the mean square norm.
- 1.5. Let (φ_n) be a sequence of elements of a normed space X . The *series*

$$\sum_{k=1}^{\infty} \varphi_k$$

is called convergent if the sequence (S_n) of partial sums

$$S_n := \sum_{k=1}^n \varphi_k$$

converges. The limit $S = \lim_{n \rightarrow \infty} S_n$ is called the *sum* of the series. Show that in a Banach space X the convergence of the series

$$\sum_{k=1}^{\infty} \|\varphi_k\| < \infty$$

is a sufficient condition for the convergence of the series

$$\sum_{k=1}^{\infty} \varphi_k$$

and that

$$\left\| \sum_{k=1}^{\infty} \varphi_k \right\| \leq \sum_{k=1}^{\infty} \|\varphi_k\|.$$

Hint: Show that (S_n) is a Cauchy sequence.

2. Bounded and Compact Operators

We briefly explain the basic properties of bounded linear operators and then introduce the concept of compact operators which is of fundamental importance in the study of integral equations.

2.1 Bounded Operators

Definition 2.1. An operator $A : X \rightarrow Y$ mapping a linear space X into a linear space Y is called linear if

$$A(\alpha\varphi + \beta\psi) = \alpha A\varphi + \beta A\psi$$

for all $\varphi, \psi \in X$ and all $\alpha, \beta \in \mathbb{C}$ (or \mathbb{R}).

Recall Definition 1.3 for the continuity of an operator $A : X \rightarrow Y$ mapping a normed space X into a normed space Y .

Theorem 2.2. A linear operator is continuous if it is continuous at one element.

Proof: Let $A : X \rightarrow Y$ be continuous at $\varphi_0 \in X$. Then for every $\varphi \in X$ and every sequence (φ_n) with $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, we have

$$A\varphi_n = A(\varphi_n - \varphi + \varphi) + A(\varphi - \varphi_0) \rightarrow A(\varphi_0) + A(\varphi - \varphi_0) = A(\varphi), \quad n \rightarrow \infty,$$

since $\varphi_n - \varphi + \varphi_0 \rightarrow \varphi_0$, $n \rightarrow \infty$.

Definition 2.3. A linear operator $A : X \rightarrow Y$ from a normed space X into a normed space Y is called bounded if there exists a positive number C such that

$$\|A\varphi\| \leq C\|\varphi\|$$

for all $\varphi \in X$. Each number C for which this inequality holds is called a bound for the operator A .

With the aid of the linearity of A it is easy to see that A is bounded if and only if

$$(2.1) \quad \|A\| := \sup_{\|\varphi\|=1} \|A\varphi\| = \sup_{\|\varphi\|\leq 1} \|A\varphi\| < \infty.$$

Hence a linear operator is bounded if and only if it maps bounded sets in X into bounded sets in Y . The number $\|A\|$ is the smallest bound for A and is

called the *norm* of A . Every linear combination of bounded linear operators again is a bounded linear operator, that is, the set $L(X, Y)$ of bounded linear operators from X into Y forms a linear space.

Theorem 2.4. *The linear space $L(X, Y)$ of bounded linear operators from a normed space X into a normed space Y is a normed space with the norm (2.1). If Y is a Banach space then $L(X, Y)$ also is a Banach space.*

The proof consists in carrying over the norm axioms and the completeness from Y onto $L(X, Y)$. For the second part, let (A_n) be a Cauchy sequence in $L(X, Y)$. Then for each $\varphi \in X$ the sequence $(A_n\varphi)$ is a Cauchy sequence in Y and converges since Y is complete. Then $A\varphi := \lim_{n \rightarrow \infty} A_n\varphi$ defines a bounded linear operator $A : X \rightarrow Y$ which is the limit of the sequence (A_n) , i.e., $\|A_n - A\| \rightarrow 0$, $n \rightarrow \infty$. \square

Note that for sequences of operators we have to distinguish between convergence with respect to the operator norm (2.1) and *pointwise convergence*. A sequence (A_n) of operators $A_n : X \rightarrow Y$ from a normed space X into a normed space Y is called pointwise convergent if for every $\varphi \in X$ the sequence $(A_n\varphi)$ converges in Y . Norm convergence implies pointwise convergence, but the converse is not true.

Theorem 2.5. *A linear operator is continuous if and only if it is bounded.*

Proof: Let $A : X \rightarrow Y$ be bounded and let (φ_n) be a sequence in X with $\varphi_n \rightarrow 0$, $n \rightarrow \infty$. Then from $\|A\varphi_n\| \leq C\|\varphi_n\|$ there follows $A\varphi_n \rightarrow 0$, $n \rightarrow \infty$. Thus, A is continuous at $\varphi = 0$ and because of Theorem 2.2 it is continuous everywhere in X .

Conversely, let A be continuous and assume there is no $C > 0$ such that $\|A\varphi\| \leq C\|\varphi\|$ for all $\varphi \in X$. Then there exists a sequence (φ_n) in X with $\|\varphi_n\| = 1$ and $\|A\varphi_n\| \geq n$. Consider the sequence $\psi_n := \varphi_n/\|A\varphi_n\|$. Then $\psi_n \rightarrow 0$, $n \rightarrow \infty$, and since A is continuous $A\psi_n \rightarrow A(0) = 0$, $n \rightarrow \infty$. This is a contradiction to $\|A\psi_n\| = 1$ for all n . Hence, A is bounded.

Theorem 2.6. *Let X, Y and Z be normed spaces and let $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ be bounded linear operators. Then the product $BA : X \rightarrow Z$, defined by $(BA)\varphi := B(A\varphi)$ for all $\varphi \in X$, is a bounded linear operator with $\|BA\| \leq \|A\| \|B\|$.*

Proof: This follows from $\|(BA)\varphi\| = \|B(A\varphi)\| \leq \|B\| \|A\| \|\varphi\|$.

2.2 Integral Operators

Now we want to introduce integral operators. We assume that the reader is familiar with the Riemann integral for real and complex valued functions in \mathbb{R}^m . A set $G \subset \mathbb{R}^m$ is called Jordan measurable if the characteristic function χ_G , defined by $\chi_G(x) = 1$ for $x \in G$ and $\chi_G(x) = 0$ for $x \notin G$, is Riemann integrable.

The Jordan measure $|G|$ is the integral of χ_G . For each Jordan measurable set G the closure \bar{G} and the boundary ∂G also are Jordan measurable with $|\bar{G}| = |G|$ and $|\partial G| = 0$. If G is compact and Jordan measurable, then each function $f \in C(G)$ is Riemann integrable. In addition, we assume that G is the closure of an open set or, equivalently, that G coincides with the closure of its interior. This ensures that for $f \in C(G)$ from $f(x) \geq 0$ for all $x \in G$ and $\int_G f(x)dx = 0$ there follows that $f(x) = 0$ for all $x \in G$.

Theorem 2.7. *Let $G \subset \mathbb{R}^m$ be a non-empty compact and Jordan measurable set which coincides with the closure of its interior. Let $K : G \times G \rightarrow \mathbb{C}$ be a continuous function. Then the linear operator $A : C(G) \rightarrow C(G)$ defined by*

$$(2.2) \quad (A\varphi)(x) := \int_G K(x, y)\varphi(y)dy, \quad x \in G,$$

is called an integral operator with continuous kernel K . It is a bounded operator with

$$\|A\|_\infty = \max_{x \in G} \int_G |K(x, y)|dy.$$

Proof: Clearly (2.2) defines a linear operator $A : C(G) \rightarrow C(G)$. For each $\varphi \in C(G)$ with $\|\varphi\|_\infty \leq 1$ we have

$$|(A\varphi)(x)| \leq \int_G |K(x, y)|dy$$

and thus

$$\|A\varphi\|_\infty = \max_{x \in G} |(A\varphi)(x)| \leq \max_{x \in G} \int_G |K(x, y)|dy.$$

Therefore,

$$\|A\|_\infty = \sup_{\|\varphi\|_\infty \leq 1} \|A\varphi\|_\infty \leq \max_{x \in G} \int_G |K(x, y)|dy.$$

Let $x_0 \in G$ be such that

$$\int_G |K(x_0, y)|dy = \max_{x \in G} \int_G |K(x, y)|dy.$$

For $\varepsilon > 0$ choose $\psi \in C(G)$ with

$$\psi(y) := \frac{\overline{K(x_0, y)}}{|K(x_0, y)| + \varepsilon}, \quad y \in G.$$

Then $\|\psi\|_\infty \leq 1$ and

$$\begin{aligned} \|A\psi\|_\infty &\geq |(A\psi)(x_0)| = \int_G \frac{|K(x_0, y)|^2}{|K(x_0, y)| + \varepsilon} dy \geq \int_G \frac{|K(x_0, y)|^2 - \varepsilon^2}{|K(x_0, y)| + \varepsilon} dy \\ &= \int_G |K(x_0, y)|dy - \varepsilon|G|. \end{aligned}$$

Hence

$$\|A\|_\infty = \sup_{\|\varphi\|_\infty \leq 1} \|A\varphi\|_\infty \geq \|A\psi\|_\infty \geq \int_G |K(x_0, y)|dy - \varepsilon|G|,$$

and, finally, since this holds for all $\varepsilon > 0$ we have

$$\|A\|_\infty \geq \int_G |K(x_0, y)|dy = \max_{x \in G} \int_G |K(x, y)|dy.$$

2.3 Neumann Series

For operator equations of the *second kind*

$$\varphi - A\varphi = f$$

existence and uniqueness of a solution can be established by the Neumann series provided A is a *contraction*, i.e., $\|A\| < 1$.

Theorem 2.8. *Let $A : X \rightarrow X$ be a bounded linear operator mapping a Banach space X into itself with $\|A\| < 1$ and let $I : X \rightarrow X$ denote the identity operator. Then $I - A$ has a bounded inverse operator on X which is given by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and which satisfies

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

The iterated operators A^n are defined by $A^0 := I$ and $A^n := AA^{n-1}$ for $n \in \mathbb{N}$.

Proof: By Theorem 2.6 there holds $\|A^n\| \leq \|A\|^n$. Since $\|A\| < 1$ we have absolute convergence

$$\sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}$$

in the Banach space $L(X, X)$ and therefore, by Problem 1.5, the Neumann series converges in the operator norm and defines a bounded linear operator

$$S := \sum_{k=0}^{\infty} A^k$$

with $\|S\| \leq (1 - \|A\|)^{-1}$. S is the inverse of $I - A$ as can be seen from

$$(I - A)S = (I - A) \lim_{n \rightarrow \infty} \sum_{k=0}^n A^k = \lim_{n \rightarrow \infty} (I - A^{n+1}) = I$$

and

$$S(I - A) = \lim_{n \rightarrow \infty} \sum_{k=0}^n A^k (I - A) = \lim_{n \rightarrow \infty} (I - A^{n+1}) = I$$

since $\|A^{n+1}\| \leq \|A\|^{n+1} \rightarrow 0$, $n \rightarrow \infty$. \square

Obviously, the partial sums

$$\varphi_n := \sum_{k=0}^n A^k f$$

of the Neumann series satisfy $\varphi_{n+1} = A\varphi_n + f$ for $n \geq 0$. Hence, the Neumann series is related to successive approximations by the following

Theorem 2.9. *Under the assumptions of Theorem 2.8 for all $f \in X$ the successive approximations*

$$(2.3) \quad \varphi_{n+1} := A\varphi_n + f, \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in X$ converge to the unique solution φ of $\varphi - A\varphi = f$.

Proof: By induction, it is readily seen that

$$\varphi_n = A^n \varphi_0 + \sum_{k=0}^{n-1} A^k f, \quad n = 1, 2, \dots,$$

whence

$$\lim_{n \rightarrow \infty} \varphi_n = \sum_{k=0}^{\infty} A^k f = (I - A)^{-1} f$$

follows. \square

We explicitly want to state this result for integral equations in

Corollary 2.10. *Let K be a continuous kernel satisfying*

$$\max_{x \in G} \int_G |K(x, y)| dy < 1.$$

Then for all $f \in C(G)$ the integral equation of the second kind

$$\varphi(x) - \int_G K(x, y)\varphi(y) dy = f(x), \quad x \in G,$$

has a unique solution $\varphi \in C(G)$. The successive approximations

$$\varphi_{n+1}(x) := \int_G K(x, y)\varphi_n(y) dy + f(x), \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in C(G)$ uniformly converge to this solution.

The method of successive approximations has two drawbacks. Firstly, the Neumann series ensures existence of solutions to integral equations of the second kind only for sufficiently small kernels, and secondly, in general, it cannot be summed in closed form. Later in the book we will have more to say about employing successive approximations to obtain approximate solutions (see Section 10.5).

2.4 Compact Operators

To provide the tools for establishing the existence of solutions to a wider class of integral equations we now turn to the introduction and investigation of compact operators.

Definition 2.11. *A linear operator $A : X \rightarrow Y$ from a normed space X into a normed space Y is called compact if it maps each bounded set in X into a relatively compact set in Y .*

Since by Definition 1.16 and Theorem 1.15 a subset U of a normed space Y is relatively compact if each sequence in U contains a subsequence converging in Y we have the following equivalent condition for an operator to be compact.

Theorem 2.12. *A linear operator $A : X \rightarrow Y$ is compact if and only if for each bounded sequence (φ_n) in X the sequence $(A\varphi_n)$ contains a convergent subsequence in Y .*

We proceed by establishing the basic properties of compact operators.

Theorem 2.13. *Compact linear operators are bounded.*

Proof: This is obvious since relatively compact sets are bounded (see Theorem 1.15).

Theorem 2.14. *Linear combinations of compact linear operators are compact.*

Proof: Let $A, B : X \rightarrow Y$ be compact linear operators and let $\alpha, \beta \in \mathbb{C}$. Then for each bounded sequence (φ_n) in X since A and B are compact we two times can select a subsequence to arrive at a subsequence $(\varphi_{n(k)})$ such that $(A\varphi_{n(k)})$ and $(B\varphi_{n(k)})$ converge. Hence $(\alpha A + \beta B)\varphi_{n(k)}$ converges and therefore $\alpha A + \beta B$ is compact.

Theorem 2.15. *Let X, Y, Z be normed spaces and let $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ be bounded linear operators. Then the product $BA : X \rightarrow Z$ is compact if one of the two operators A or B is compact.*

Proof: Let (φ_n) be a bounded sequence in X . If A is compact, then there exists a subsequence $(\varphi_{n(k)})$ such that $A\varphi_{n(k)} \rightarrow \psi \in Y$, $k \rightarrow \infty$. Since B is bounded and therefore continuous, we have $B(A\varphi_{n(k)}) \rightarrow B\psi \in Z$, $k \rightarrow \infty$. Hence BA is compact. If A is bounded and B is compact, the sequence $(A\varphi_n)$ is bounded in Y since bounded operators map bounded sets into bounded sets. Therefore, there exists a subsequence $(\varphi_{n(k)})$ such that $(BA)\varphi_{n(k)} = B(A\varphi_{n(k)}) \rightarrow \chi \in Z$, $k \rightarrow \infty$. Hence again BA is compact.

Theorem 2.16. *Let X be a normed space and Y be a Banach space. Let the sequence $A_n : X \rightarrow Y$ of compact linear operators be norm convergent to a linear operator $A : X \rightarrow Y$, that is, $\|A_n - A\| \rightarrow 0$, $n \rightarrow \infty$. Then A is compact.*

Proof: Let (φ_m) be a bounded sequence in X : $\|\varphi_m\| \leq C$ for all $m \in \mathbb{N}$. Since the A_n are compact, we can choose, by the standard diagonalization procedure (see the proof of Theorem 1.18), a subsequence $(\varphi_{m(k)})$ such that $(A_n\varphi_{m(k)})$ converges for every fixed n as $k \rightarrow \infty$. Given $\varepsilon > 0$, since $\|A_n - A\| \rightarrow 0$, $n \rightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $\|A_{n_0} - A\| < \varepsilon/3C$. Because $(A_{n_0}\varphi_{m(k)})$ converges there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\|A_{n_0}\varphi_{m(k)} - A_{n_0}\varphi_{m(l)}\| < \frac{\varepsilon}{3}$$

for all $k, l \geq N(\varepsilon)$. But then we have

$$\|A\varphi_{m(k)} - A\varphi_{m(l)}\|$$

$$\leq \|A\varphi_{m(k)} - A_{n_0}\varphi_{m(k)}\| + \|A_{n_0}\varphi_{m(k)} - A_{n_0}\varphi_{m(l)}\| + \|A_{n_0}\varphi_{m(l)} - A\varphi_{m(l)}\| < \varepsilon.$$

Thus $(A\varphi_{m(k)})$ is a Cauchy sequence and therefore convergent in the Banach space Y .

Theorem 2.17. *Let $A : X \rightarrow Y$ be a bounded linear operator with finite dimensional range $A(X)$. Then A is compact.*

Proof: Let $U \subset X$ be bounded. Then the bounded operator A maps U into the bounded set $A(U)$ contained in the finite dimensional space $A(X)$. By the Bolzano–Weierstrass Theorem 1.17 the set $A(U)$ is relatively compact. Therefore A is compact.

Lemma 2.18 (Riesz). *Let X be a normed space, $U \subset X$ a closed subspace, and $\alpha \in (0, 1)$. Then there exists an element $\psi \in X$ with $\|\psi\| = 1$ such that*

$$\|\psi - \varphi\| \geq \alpha$$

for all $\varphi \in U$.

Proof: Since $U \neq X$, there exists an element $f \in X$ with $f \notin U$ and because U is closed we have

$$\beta := \inf_{\varphi \in U} \|f - \varphi\| > 0.$$

We can choose $g \in U$ such that

$$\beta \leq \|f - g\| \leq \frac{\beta}{\alpha}.$$

Now we define

$$\psi := \frac{f - g}{\|f - g\|}.$$

Then $\|\psi\| = 1$ and for all $\varphi \in U$ we have

$$\|\psi - \varphi\| = \frac{1}{\|f - g\|} \|f - \{g + \|f - g\| \varphi\}\| \geq \frac{\beta}{\|f - g\|} \geq \alpha$$

since $g + \|f - g\| \varphi \in U$.

Theorem 2.19. *The identity operator $I : X \rightarrow X$ is compact if and only if X has finite dimension.*

Proof: Assume I is compact and X is not finite dimensional. Choose an arbitrary $\varphi_1 \in X$ with $\|\varphi_1\| = 1$. Then $U_1 := \text{span}\{\varphi_1\}$ is a finite dimensional and therefore closed subspace of X . By Lemma 2.18 there exists $\varphi_2 \in X$ with $\|\varphi_2\| = 1$ and $\|\varphi_2 - \varphi_1\| \geq 1/2$. Now consider $U_2 := \text{span}\{\varphi_1, \varphi_2\}$. Again by Lemma 2.18 there exists $\varphi_3 \in X$ with $\|\varphi_3\| = 1$ and $\|\varphi_3 - \varphi_1\| \geq 1/2$,

$\|\varphi_3 - \varphi_2\| \geq 1/2$. Repeating this procedure, we obtain a sequence (φ_n) with the properties $\|\varphi_n\| = 1$ and $\|\varphi_n - \varphi_m\| \geq 1/2$, $n \neq m$. This implies that the bounded sequence (φ_n) does not contain a convergent subsequence which contradicts the compactness of I . Therefore, if the identity operator is compact, X has finite dimension. The converse is an immediate consequence of Theorem 2.17. \square

This theorem, in particular, implies that the converse of Theorem 2.13 is false. It also justifies the distinction between operator equations of the first and second kind since obviously the operators A and $I - A$, where A is a compact operator, have different properties. Note, that by Theorems 2.15 and 2.19 the compact operator A cannot have a bounded inverse unless its range has finite dimension.

Theorem 2.20. *The integral operator with continuous kernel is a compact operator on $C(G)$.*

Proof: Let $U \subset C(G)$ be bounded: $\|\varphi\|_\infty \leq C$ for all $\varphi \in U$. Then

$$|(A\varphi)(x)| \leq C|G| \max_{x,y \in G} |K(x,y)|$$

for all $x \in G$ and all $\varphi \in U$, i.e., $A(U)$ is bounded. Since K is uniformly continuous on the compact set $G \times G$ for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|K(x,z) - K(y,z)| < \frac{\varepsilon}{|G|}$$

for all $x, y, z \in G$ with $|x - y| < \delta$. Then

$$|(A\varphi)(x) - (A\varphi)(y)| < \varepsilon$$

for all $x, y \in G$ with $|x - y| < \delta$ and all $\varphi \in U$, i.e., $A(U)$ is equicontinuous. Hence A is compact by the Arzelà–Ascoli Theorem 1.18. \square

We wish to mention that the compactness of the integral operator with continuous kernel also can be established through finite dimensional approximations using Theorems 2.16 and 2.17 in the Banach space $C(G)$. In this context note that the proofs of Theorems 2.16 and 1.18 are similar in structure. The finite dimensional operators can be obtained either by approximating the continuous kernel by polynomials through the Weierstrass approximation theorem or by approximating the integral through a finite sum (cf. Colton and Kress [1]).

Now we extend our investigation to integral operators with a *weakly singular kernel*, that is, the kernel K is defined and continuous for all $x, y \in G$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m]$ such that for all $x, y \in G$, $x \neq y$, we have

$$(2.4) \quad |K(x,y)| \leq M|x - y|^{\alpha-m}.$$

Theorem 2.21. *The integral operator with a weakly singular kernel is a compact operator on $C(G)$.*

Proof: The integral in (2.2) defining the operator A exists as an improper integral since

$$|K(x, y)\varphi(y)| \leq M\|\varphi\|_\infty|x - y|^{\alpha-m}$$

and

$$\int_G |x - y|^{\alpha-m} dy \leq \omega_m \int_0^d \rho^{\alpha-m} \rho^{m-1} d\rho = \frac{\omega_m}{\alpha} d^\alpha$$

where we have introduced polar coordinates with origin at x , d is the diameter of G and ω_m denotes the area of the unit sphere in \mathbb{R}^m .

Now we choose a piecewise linear continuous function $h : [0, \infty) \rightarrow \mathbb{R}$, with

$$h(t) := \begin{cases} 0, & 0 \leq t \leq 1/2, \\ 2t - 1, & 1/2 \leq t \leq 1, \\ 1, & 1 \leq t < \infty, \end{cases}$$

and define continuous kernels $K_n : G \times G \rightarrow \mathbb{C}$ by

$$K_n(x, y) := \begin{cases} h(n|x - y|)K(x, y), & x \neq y, \\ 0, & x = y. \end{cases}$$

The corresponding integral operators $A_n : C(G) \rightarrow C(G)$ are compact by Theorem 2.20. We have the estimate

$$\begin{aligned} |(A\varphi)(x) - (A_n\varphi)(x)| &= \left| \int_G \{K(x, y) - K_n(x, y)\}\varphi(y)dy \right| \\ &\leq \int_{G[x; 1/n]} |K(x, y)| \|\varphi\|_\infty dy \leq M\|\varphi\|_\infty \omega_m \int_0^{1/n} \rho^{\alpha-m} \rho^{m-1} d\rho \\ &= M\|\varphi\|_\infty \frac{\omega_m}{\alpha n^\alpha}, \quad x \in G, \end{aligned}$$

where $G[x; 1/n] := \{y \in G : |y - x| \leq 1/n\}$. From this we observe that $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, uniformly and therefore $A\varphi \in C(G)$. Furthermore it follows that

$$\|A - A_n\|_\infty \leq M \frac{\omega_m}{\alpha n^\alpha} \rightarrow 0, \quad n \rightarrow \infty,$$

and thus A is compact by Theorem 2.16. \square

Finally, we want to expand the analysis to integral operators defined on surfaces in \mathbb{R}^m . Having in mind applications to boundary value problems, we will confine our attention to those surfaces which are boundaries of smooth domains in \mathbb{R}^m . A bounded open domain $D \subset \mathbb{R}^m$ with boundary ∂D is said to be of *class C^k* , $k \in \mathbb{N}$, if the closure \bar{D} admits a finite open covering

$$\bar{D} \subset \bigcup_{q=1}^p V_q$$

such that for each of those V_q which intersects with the boundary ∂D there holds:

1. the intersection $V_q \cap \bar{D}$ can be mapped bijectively onto the half ball $H := \{x \in \mathbb{R}^m : |x| < 1, x_m \geq 0\}$ in \mathbb{R}^m ;
2. this mapping and its inverse are k times continuously differentiable;
3. the intersection $V_q \cap \partial D$ is mapped onto the circle $H \cap \{x \in \mathbb{R}^m : x_m = 0\}$.

In particular, this implies that the boundary can be represented locally through a *parametric representation*

$$x(u) = (x_1(u), \dots, x_m(u))$$

mapping an open parameter domain $U \subset \mathbb{R}^{m-1}$ bijectively onto a *surface patch* S of ∂D with the property that the vectors

$$\frac{\partial x}{\partial u_i}, \quad i = 1, \dots, m-1,$$

are linearly independent at each point x of S . Such a parametrisation we call a *regular parametric representation*. The whole boundary ∂D is obtained by matching together a finite number of such surface patches.

The vectors $\partial x / \partial u_i$, $i = 1, \dots, m-1$, span the tangent plane to the surface at the point x , the normal n is the vector perpendicular to the tangent plane and uniquely determined up to two opposite directions. The surface element at the point x is given through

$$ds = \sqrt{g} du_1 \cdots du_{m-1}$$

where g is the determinant of the positive definite matrix

$$g_{ij} := \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right), \quad i, j = 1, \dots, m-1.$$

Let ∂D be the boundary of a bounded open domain of class C^1 . In the Banach space $C(\partial D)$ of real or complex valued continuous functions defined on the surface ∂D equipped with the maximum norm $\|\varphi\|_\infty := \max_{x \in \partial D} |\varphi(x)|$ we consider the integral operator $A : C(\partial D) \rightarrow C(\partial D)$ defined by

$$(2.5) \quad (A\varphi)(x) := \int_{\partial D} K(x, y)\varphi(y)ds(y), \quad x \in \partial D,$$

where K is a continuous or weakly singular kernel. According to the dimension of the surface ∂D , a kernel K is said to be weakly singular, if K is defined and continuous for all $x, y \in \partial D$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m-1]$ such that

$$(2.6) \quad |K(x, y)| \leq M|x - y|^{\alpha-m+1}$$

for all $x, y \in \partial D$, $x \neq y$. Analogously to Theorems 2.20 and 2.21 we can prove the following

Theorem 2.22. *The integral operator with continuous or weakly singular kernel is a compact operator on $C(\partial D)$.*

Proof: For continuous kernels the proof of Theorem 2.20 essentially remains unaltered. For weakly singular kernels the only major difference in the proof as compared with the proof of Theorem 2.21 arises in the verification of the existence of the integral in (2.5). Since the surface ∂D is of class C^1 , the normal vector n is continuous on ∂D . Therefore, we can choose $R \in (0, 1]$ such that for the scalar product

$$(2.7) \quad (n(x), n(y)) \geq \frac{1}{2}$$

for all $x, y \in \partial D$ with $|x - y| \leq R$. Furthermore, we can assume that R is small enough such that the set $S[x; R] := \{y \in \partial D : |y - x| \leq R\}$ is connected for each $x \in \partial D$. Then the condition (2.7) implies that $S[x; R]$ can be bijectively projected into the tangent plane to ∂D at the point x . By using polar coordinates (ρ, ω) in the tangent plane with origin in x , we now can estimate

$$\begin{aligned} \left| \int_{S[x; R]} K(x, y) \varphi(y) ds(y) \right| &\leq M \|\varphi\|_\infty \int_{S[x; R]} |x - y|^{\alpha-m+1} ds(y) \\ &\leq 2M \|\varphi\|_\infty \omega_{m-1} \int_0^R \rho^{\alpha-m+1} \rho^{m-1} d\rho = 2M \|\varphi\|_\infty \omega_{m-1} \frac{R^\alpha}{\alpha}. \end{aligned}$$

Here we have used the facts that $|x - y| \geq \rho$, that the surface element

$$ds(y) = \frac{\rho^{m-1} d\rho d\omega}{(n(x), n(y))}$$

can be estimated with the aid of (2.7) by $ds(y) \leq 2\rho^{m-1} d\rho d\omega$, and that the projection of $S[x; R]$ into the tangent plane is contained in the interior of the sphere of radius R and center x . Furthermore,

$$\begin{aligned} \left| \int_{\partial D \setminus S[x; R]} K(x, y) \varphi(y) ds(y) \right| &\leq M \|\varphi\|_\infty \int_{\partial D \setminus S[x; R]} R^{\alpha-m+1} ds(y) \\ &\leq M \|\varphi\|_\infty R^{\alpha-m+1} |\partial D|. \end{aligned}$$

Hence, for all $x \in \partial D$ the integral (2.5) exists as an improper integral. For the compactness of A , we now can adopt the proof of Theorem 2.21.

Problems

- 2.1. Let $A : X \rightarrow Y$ be a bounded linear operator from a normed space X into a normed space Y and let \tilde{X} and \tilde{Y} be the completions of X and Y , respectively. Then there exists a uniquely determined bounded linear operator $\tilde{A} : \tilde{X} \rightarrow \tilde{Y}$ such that $\tilde{A}\varphi = A\varphi$ for all $\varphi \in X$. Furthermore there holds $\|\tilde{A}\| = \|A\|$. The operator A is called the *continuous extension* of A . (In the sense of Theorem 1.13 the space X is interpreted as a dense subspace of its completion \tilde{X})

Hint: For $\varphi \in \tilde{X}$ define $\tilde{A}\varphi = \lim_{n \rightarrow \infty} A\varphi_n$ where (φ_n) is a sequence of X with $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$.

- 2.2. Show that Theorem 2.9 remains valid for operators satisfying $\|A^k\| < 1$ for some $k \in \mathbb{N}$.
- 2.3. Write out the proofs for the compactness of the integral operator with continuous kernel in $C(G)$ by finite dimensional approximations as mentioned after the proof of Theorem 2.20.
- 2.4. Show that the result of Theorem 2.7 for the norm of the integral operator remains valid for weakly singular kernels.

Hint: Use the approximations from the proof of Theorem 2.21.

- 2.5. For the integral operator A with continuous kernel use the Cauchy–Schwarz inequality to establish that each set $U \subset C(G)$ which is bounded with respect to the mean square norm is mapped into a set $A(U) \subset C(G)$ which is bounded with respect to the maximum norm and equicontinuous. From this deduce that the integral operator with continuous kernel is compact with respect to the mean square norm. Use the same method and the approximations from the proof of Theorem 2.21 to extend this result to weakly singular kernels with $\alpha > m/2$.

3. The Riesz Theory

We now present the basic theory for an operator equation

$$\varphi - A\varphi = f$$

of the second kind with a compact linear operator $A : X \rightarrow X$ mapping a normed space X into itself. This theory was developed from Riesz [1] and originated by Fredholm's [1] work on integral equations of the second kind.

3.1 Riesz Theory for Compact Operators

We define

$$L := I - A$$

where I denotes the identity operator.

Theorem 3.1 (First Riesz Theorem). *The nullspace of the operator L*

$$N(L) := \{\varphi \in X : L\varphi = 0\}$$

is a finite dimensional subspace.

Proof: The nullspace of the bounded linear operator L is a closed subspace of X since for each sequence (φ_n) with $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, and $L\varphi_n = 0$ there holds $L\varphi = 0$. For all $\varphi \in N(L)$ we have $A\varphi = \varphi$; therefore the restriction of A to $N(L)$ coincides with the identity operator $A|_{N(L)} = I : N(L) \rightarrow N(L)$. A is compact on X and therefore also compact from $N(L)$ onto $N(L)$ since $N(L)$ is closed. Hence $N(L)$ is finite dimensional by Theorem 2.19.

Theorem 3.2 (Second Riesz Theorem). *The range of the operator L*

$$L(X) := \{L\varphi : \varphi \in X\}$$

is a closed linear subspace.

Proof: The range of the linear operator L is clearly a linear subspace. Let f be an element of the closure $\overline{L(X)}$. Then there exists a sequence (φ_n) in X such that $L\varphi_n \rightarrow f$, $n \rightarrow \infty$. By Theorem 1.24 to each φ_n we choose a best approximation χ_n with respect to $N(L)$, i.e.,

$$\|\varphi_n - \chi_n\| = \inf_{x \in N(L)} \|\varphi_n - x\|.$$

The sequence defined by

$$\tilde{\varphi}_n := \varphi_n - \chi_n$$

is bounded. We prove this indirectly, this is, we assume that it is not bounded. Then there exists a subsequence $(\tilde{\varphi}_{n(k)})$ such that $\|\tilde{\varphi}_{n(k)}\| \geq k$ for all $k \in \mathbb{N}$. Now we define

$$\psi_k := \frac{\tilde{\varphi}_{n(k)}}{\|\tilde{\varphi}_{n(k)}\|}$$

for all $k \in \mathbb{N}$. Since $\|\psi_k\| = 1$, there exists a subsequence $(\psi_{k(j)})$ such that

$$A\psi_{k(j)} \rightarrow \psi \in X, \quad j \rightarrow \infty.$$

Furthermore,

$$\|L\psi_k\| = \frac{\|L\tilde{\varphi}_{n(k)}\|}{\|\tilde{\varphi}_{n(k)}\|} \leq \frac{\|L\tilde{\varphi}_{n(k)}\|}{k} \rightarrow 0, \quad k \rightarrow \infty,$$

since the sequence $(L\varphi_n)$ is convergent and therefore bounded. Hence

$$L\psi_{k(j)} \rightarrow 0, \quad j \rightarrow \infty.$$

We now obtain

$$\psi_{k(j)} = L\psi_{k(j)} + A\psi_{k(j)} \rightarrow \psi, \quad j \rightarrow \infty,$$

and since L is bounded from the two previous equations we conclude that $L\psi = 0$. But then, since $\chi_{k(j)} + \|\tilde{\varphi}_{n(k)}\| \psi \in N(L)$ for all k , we find

$$\begin{aligned} \|\psi_k - \psi\| &= \frac{1}{\|\tilde{\varphi}_{n(k)}\|} \|\varphi_{n(k)} - \{\chi_{n(k)} + \|\tilde{\varphi}_{n(k)}\| \psi\}\| \\ &\geq \frac{1}{\|\tilde{\varphi}_{n(k)}\|} \inf_{\chi \in N(L)} \|\varphi_{n(k)} - \chi\| \geq \frac{1}{\|\tilde{\varphi}_{n(k)}\|} \|\varphi_{n(k)} - \chi_{n(k)}\| = 1 \end{aligned}$$

which contradicts the fact that $\psi_{k(j)} \rightarrow \psi$, $j \rightarrow \infty$.

Therefore the sequence $(\tilde{\varphi}_n)$ is bounded and we can select a subsequence $(\tilde{\varphi}_{n(k)})$ such that $(A\tilde{\varphi}_{n(k)})$ converges as $k \rightarrow \infty$. From the relation $\tilde{\varphi}_{n(k)} = L\tilde{\varphi}_{n(k)} + A\tilde{\varphi}_{n(k)}$ we observe that $(\tilde{\varphi}_{n(k)})$ converges $\tilde{\varphi}_{n(k)} \rightarrow \varphi \in X$, $k \rightarrow \infty$. But then $L\tilde{\varphi}_{n(k)} \rightarrow L\varphi \in X$, $k \rightarrow \infty$, and therefore $f = L\varphi \in L(X)$. Hence there holds $\overline{L(X)} = L(X)$. \square

For $n \geq 1$ the iterated operators L^n can be written in the form

$$L^n = (I - A)^n = I - A_n,$$

where

$$A_n = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} A^k$$

is compact by Theorems 2.14 and 2.15. Therefore by Theorem 3.1 the nullspaces $N(L^n)$ are finite dimensional and by Theorem 3.2 the ranges $L^n(X)$ are closed subspaces.

Theorem 3.3 (Third Riesz Theorem). *There exists a uniquely determined non-negative integer r , called the Riesz number of the operator A , such that*

$$\{0\} = N(L^0) \subsetneq N(L^1) \subsetneq \cdots \subsetneq N(L^r) = N(L^{r+1}) = \cdots,$$

$$X = L^0(X) \supsetneq L^1(X) \supsetneq \cdots \supsetneq L^r(X) = L^{r+1}(X) = \cdots.$$

Furthermore,

$$X = N(L^r) \oplus L^r(X).$$

Proof: Our proof consists of four steps.

1. Since for each φ with $L^n\varphi = 0$ it follows that $L^{n+1}\varphi = 0$, we trivially have

$$\{0\} = N(L^0) \subset N(L^1) \subset N(L^2) \subset \cdots.$$

Assume now that

$$\{0\} = N(L^0) \subsetneq N(L^1) \subsetneq N(L^2) \subsetneq \cdots.$$

Since the nullspaces $N(L^n)$ are finite dimensional by Theorem 3.1, it follows from the Riesz Lemma 2.18 that there exists $\varphi_n \in N(L^{n+1})$ such that $\|\varphi_n\| = 1$ and

$$\|\varphi_n - \varphi\| \geq \frac{1}{2}$$

for all $\varphi \in N(L^n)$. For $n > m$ we consider

$$A\varphi_n - A\varphi_m = \varphi_n - (\varphi_m + L\varphi_n - L\varphi_m).$$

Here $\varphi_m + L\varphi_n - L\varphi_m \in N(L^n)$ since

$$L^n(\varphi_m + L\varphi_n - L\varphi_m) = L^{n-m-1}L^{m+1}\varphi_m + L^{n+1}\varphi_n - L^{n-m}L^{m+1}\varphi_m = 0.$$

Hence

$$\|A\varphi_n - A\varphi_m\| \geq \frac{1}{2}$$

for $n > m$, and thus the sequence $(A\varphi_n)$ does not contain a convergent subsequence which is a contradiction to the compactness of A .

Therefore now we know that in the sequence $N(L^n)$ there are two successive nullspaces that are equal. Define

$$r := \min\{k : N(L^k) = N(L^{k+1})\}.$$

We now prove by induction that

$$N(L^r) = N(L^{r+1}) = N(L^{r+2}) = \cdots.$$

Assume that we already have $N(L^k) = N(L^{k+1})$ for some $k \geq r$. Then for each $\varphi \in N(L^{k+2})$ we have $L^{k+1}L\varphi = L^{k+2}\varphi = 0$. Thus $L\varphi \in N(L^{k+1}) = N(L^k)$. Hence $L^{k+1}\varphi = L^kL\varphi = 0$ and consequently $\varphi \in N(L^{k+1})$. Therefore we have $N(L^{k+2}) \subset N(L^{k+1})$.

We can summarize our results up to now by the formula

$$\{0\} = N(L^0) \subsetneq N(L^1) \subsetneq \cdots \subsetneq N(L^r) = N(L^{r+1}) = \cdots.$$

2. Since for each $\psi = L^{n+1}\varphi \in L^{n+1}(X)$ we can write $\psi = L^nL\varphi$, it trivially follows that

$$X = L^0(X) \supset L^1(X) \supset L^2(X) \supset \cdots.$$

Assume now that

$$X = L^0(X) \supsetneq L^1(X) \supsetneq L^2(X) \supsetneq \cdots.$$

Since the ranges $L^n(X)$ are closed subspaces by Theorem 3.2, it follows from the Riesz Lemma 2.18 that there exist $\psi_n \in L^n(X)$ such that $\|\psi_n\| = 1$ and

$$\|\psi_n - \psi\| \geq \frac{1}{2}$$

for all $\psi \in L^{n+1}(X)$. We write $\psi_n = L^n\varphi_n$ and for $m > n$ we consider

$$A\psi_n - A\psi_m = \psi_n - (\psi_m + L\psi_n - L\psi_m).$$

Here $\psi_m + L\psi_n - L\psi_m \in L^{n+1}(X)$ since

$$\psi_m + L\psi_n - L\psi_m = L^{n+1}(L^{m-n-1}\varphi_m + \varphi_n - L^{m-n}\varphi_m).$$

Hence

$$\|A\psi_n - A\psi_m\| \geq \frac{1}{2}$$

for $m > n$ and we can derive the same contradiction as above.

Therefore in the sequence $L^n(X)$ there are two successive ranges that are equal. Define

$$q := \min\{k : L^k(X) = L^{k+1}(X)\}.$$

We now prove by induction that

$$L^q(X) = L^{q+1}(X) = L^{q+2}(X) = \cdots.$$

Assume we already have that $L^k(X) = L^{k+1}(X)$ for some $k \geq q$. Then for each $\psi = L^{k+1}\varphi \in L^{k+1}(X)$ we can write $L^k\varphi = L^{k+1}\tilde{\varphi}$ with some $\tilde{\varphi} \in X$ since $L^k(X) = L^{k+1}(X)$. Hence $\psi = L^{k+2}\tilde{\varphi} \in L^{k+2}(X)$ and therefore we have $L^{k+1}(X) \subset L^{k+2}(X)$.

Again we can summarize our results by the formula

$$X = L^0(X) \supsetneq L^1(X) \supsetneq \cdots \supsetneq L^q(X) = L^{q+1}(X) = \cdots.$$

3. We now show that $r = q$. Assume that $r > q$ and let $\varphi \in N(L^r)$. Then since $L^{r-1}\varphi \in L^{r-1}(X) = L^r(X)$, we can write $L^{r-1}\varphi = L^r\tilde{\varphi}$ with some $\tilde{\varphi} \in X$. Since $L^{r+1}\tilde{\varphi} = L^r\varphi = 0$, we have $\tilde{\varphi} \in N(L^{r+1}) = N(L^r)$, that is, $L^{r-1}\varphi = L^r\tilde{\varphi} = 0$. Thus $\varphi \in N(L^{r-1})$ and hence $N(L^{r-1}) = N(L^r)$, which contradicts the definition of r .

On the other hand, assume that $r < q$ and let $\psi = L^{q-1}\varphi \in L^{q-1}(X)$. Since $L\psi = L^q\varphi \in L^q(X) = L^{q+1}(X)$ we can write $L\psi = L^{q+1}\tilde{\varphi}$ for some $\tilde{\varphi} \in X$. Therefore $L^q(\varphi - L\tilde{\varphi}) = L\psi - L^{q+1}\tilde{\varphi} = 0$ and since $N(L^{q-1}) = N(L^q)$, we conclude that $L^{q-1}(\varphi - L\tilde{\varphi}) = 0$, which implies $\psi = L^q\tilde{\varphi} \in L^q(X)$. Hence $L^{q-1}(X) = L^q(X)$, which contradicts the definition of q .

4. Let $\psi \in N(L^r) \cap L^r(X)$. Then $\psi = L^r\varphi$ for some $\varphi \in X$ and $L^r\psi = 0$. Hence $L^{2r}\varphi = 0$, which means $\varphi \in N(L^{2r}) = N(L^r)$. Hence $\psi = L^r\varphi = 0$. Let $\varphi \in X$ be arbitrary. Then $L^r\varphi \in L^r(X) = L^{2r}(X)$ and we can write $L^r\varphi = L^{2r}\tilde{\varphi}$ for some $\tilde{\varphi} \in X$. Define $\psi := L^r\tilde{\varphi} \in L^r(X)$ and $\chi := \varphi - \psi$. Then $L^r\chi = L^r\varphi - L^{2r}\tilde{\varphi} = 0$, which means $\chi \in N(L^r)$. Therefore the decomposition $\varphi = \chi + \psi$ proves the direct sum $X = N(L^r) \oplus L^r(X)$. \square

Now we derive the fundamental results of the Riesz theory by distinguishing the two cases $r = 0$ and $r > 0$.

Theorem 3.4. *Let X be a normed space, $A : X \rightarrow X$ a compact operator, and let $I - A$ be injective. Then the inverse operator $(I - A)^{-1} : X \rightarrow X$ exists and is bounded.*

Proof: By assumption, the operator L is injective, that is, $N(L) = \{0\}$. Therefore $r = 0$ and from Theorem 3.3 we conclude that $L(X) = X$, that is, the operator L is surjective. Hence the inverse operator $L^{-1} : X \rightarrow X$ exists. Assume L^{-1} is not bounded. Then there exists a sequence (f_n) with $\|f_n\| = 1$ such that the sequence $\varphi_n := L^{-1}f_n$ is not bounded. Define

$$g_n := \frac{f_n}{\|\varphi_n\|}, \quad \psi_n := \frac{\varphi_n}{\|\varphi_n\|}$$

for $n \in \mathbb{N}$. Then $g_n \rightarrow 0$, $n \rightarrow \infty$, and $\|\psi_n\| = 1$. Since A is compact, we can choose a subsequence $(\psi_{n(k)})$ such that $A\psi_{n(k)} \rightarrow \psi \in X$, $k \rightarrow \infty$. Then since

$$\psi_n - A\psi_n = g_n$$

we observe that $\psi_{n(k)} \rightarrow \psi$, $k \rightarrow \infty$, and therefore $\psi \in N(L)$. Hence $\psi = 0$, which contradicts $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$. \square

We can rewrite Theorem 3.4 in terms of the solvability of the operator equation of the second kind as follows.

Corollary 3.5. *Let X be a normed space and $A : X \rightarrow X$ a compact linear operator. If the homogeneous equation*

$$\varphi - A\varphi = 0$$

only has the trivial solution $\varphi = 0$, then for all $f \in X$ the inhomogeneous equation

$$\varphi - A\varphi = f$$

has a unique solution $\varphi \in X$ and this solution depends continuously on f .

Theorem 3.6. *Let X be a normed space and $A : X \rightarrow X$ a compact linear operator and assume $I - A$ is not injective. Then the nullspace $N(I - A)$ is finite dimensional and the range $(I - A)X \neq X$ is a proper closed subspace.*

Proof: By assumption, we have $N(L) \neq \{0\}$. This means $r > 0$ and from Theorem 3.3 we conclude that $L(X) \neq X$.

Corollary 3.7. *If the homogeneous equation*

$$\varphi - A\varphi = 0$$

has a nontrivial solution, then the inhomogeneous equation

$$\varphi - A\varphi = f$$

is either unsolvable or its general solution is of the form

$$\varphi = \tilde{\varphi} + \sum_{k=1}^m \alpha_k \varphi_k$$

where $\varphi_1, \dots, \varphi_m$ are linearly independent solutions of the homogeneous equation, $\alpha_1, \dots, \alpha_m$ are arbitrary complex numbers and $\tilde{\varphi}$ denotes a particular solution of the inhomogeneous equation.

It is left to the reader to formulate Theorems 3.4 and 3.6 for integral equations of the second kind with continuous or weakly singular kernel.

Corollary 3.8. *Theorems 3.4 and 3.6 and their Corollaries 3.5 and 3.7 remain valid when we replace $I - A$ by $S - A$, where S is a bounded linear operator that has a bounded inverse S^{-1} .*

Proof: This follows immediately from the fact that we can transform the equation

$$S\varphi - A\varphi = f$$

into the equivalent form

$$\varphi - S^{-1}A\varphi = S^{-1}f$$

where $S^{-1}A$ is compact by Theorem 2.15. □

The main importance of the results of the Riesz theory for compact operators lies in the fact that we can conclude existence from uniqueness as in the case of finite dimensional linear equations.

We conclude this section with the following theorem.

Theorem 3.9. *The projection operator $P : X \rightarrow N(L^r)$ defined by the decomposition*

$$X = N(L^r) \oplus L^r(X)$$

is compact. The operator $L - P = I - A - P$ is bijective.

Proof: The nullspace $N(L^r)$ is finite dimensional by Theorem 3.1. On $N(L^r)$ it is easily verified that

$$\|\psi\|_r := \inf_{x \in L^r(X)} \|\psi + x\|$$

defines a norm. In particular, we conclude from $\|\psi\|_r = 0$ that $\psi = 0$ since the range $L^r(X)$ is closed by Theorem 3.2. Since, by Theorem 1.5, on a finite dimensional linear space all norms are equivalent, there exists a positive number C such that $\|\psi\| \leq C\|\psi\|_r$, that is,

$$\|\psi\| \leq C \inf_{x \in L^r(X)} \|\psi + x\|$$

for all $\psi \in N(L^r)$. Then for all $\varphi \in X$ we have $P\varphi \in N(L^r)$ and therefore

$$\|P\varphi\| \leq C \inf_{x \in L^r(X)} \|P\varphi + x\| \leq C\|\varphi\|$$

since $\varphi - P\varphi \in L^r(X)$. Hence P is bounded and, since it has finite dimensional range $P(X) = N(L^r)$, by Theorem 2.17 it is compact.

It follows from Theorem 2.14 that the operator $A + P$ is compact. Let $\varphi \in N(L - P)$, that is,

$$L\varphi - P\varphi = 0.$$

Then $L^{r+1}\varphi = 0$ since $P\varphi \in N(L^r)$. Therefore $\varphi \in N(L^{r+1}) = N(L^r)$ and $P\varphi = \varphi$, which implies

$$L\varphi = \varphi.$$

From this we get by iteration that

$$\varphi = L^r\varphi = 0.$$

Therefore $N(L - P) = \{0\}$ and from Theorem 3.4 applied to the compact operator $A + P$, we conclude that $L - P$ is surjective.

3.2 Spectral Theory for Compact Operators

We continue by formulating the results of the Riesz theory in terms of *spectral analysis*.

Definition 3.10. *Let $A : X \rightarrow X$ be a bounded linear operator mapping a normed space X into itself. Then a complex number is called an eigenvalue of A if there exists an element $\varphi \in X$, $\varphi \neq 0$, such that $A\varphi = \lambda\varphi$. φ is called an eigenelement of A . A complex number λ is called a regular value of A if $(\lambda I - A)^{-1}$ exists and is bounded. The set of all regular values of A is called the resolvent set $\rho(A)$ and $R(\lambda; A) := (\lambda I - A)^{-1}$ is called the resolvent. The complement of $\rho(A)$ is called the spectrum $\sigma(A)$ and*

$$r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|$$

is called the spectral radius of A .

For the spectrum of a compact operator we have the following properties.

Theorem 3.11. *Let $A : X \rightarrow X$ be a compact linear operator in an infinite dimensional normed space X . Then $\lambda = 0$ belongs to the spectrum $\sigma(A)$ and $\sigma(A) \setminus \{0\}$ consists of at most a countable set of eigenvalues with no point of accumulation except, possibly, $\lambda = 0$.*

Proof: Suppose $\lambda = 0$ is a regular value of A , that is A^{-1} exists and is bounded. Then $I = A^{-1}A$ is compact by Theorem 2.15 and by Theorem 2.19 we obtain the contradiction that X is finite dimensional. Therefore $\lambda = 0$ belongs to the spectrum $\sigma(A)$.

For $\lambda \neq 0$ we can apply the Riesz theory to the operator $\lambda I - A$. Either $N(\lambda I - A) = \{0\}$ and $(\lambda I - A)^{-1}$ exists and is bounded by Theorem 3.4 or $N(\lambda I - A) \neq \{0\}$, which means λ is an eigenvalue. Thus each $\lambda \neq 0$ is either a regular value or an eigenvalue of A .

It remains to show that for each $R > 0$ there exists only a finite number of eigenvalues λ with $|\lambda| \geq R$. Assume, on the contrary, that we have a sequence (λ_n) of distinct eigenvalues satisfying $|\lambda_n| \geq R$. Choose eigenelements φ_n such that $A\varphi_n = \lambda_n\varphi_n$ and define finite dimensional subspaces

$$U_n := \text{span}\{\varphi_1, \dots, \varphi_n\}.$$

It is readily verified that eigenelements corresponding to distinct eigenvalues are linearly independent. Hence, we have $U_{n-1} \subsetneq U_n$ and, by the Riesz Lemma 2.18, we can choose a sequence (ψ_n) of elements $\psi_n \in U_n$ such that $\|\psi_n\| = 1$ and

$$\|\psi_n - \psi\| \geq \frac{1}{2}$$

for all $\psi \in U_{n-1}$. Writing

$$\psi_n = \sum_{k=1}^n \alpha_{nk} \varphi_k$$

we obtain

$$\lambda_n \psi_n - A\psi_n = \sum_{k=1}^{n-1} (\lambda_n - \lambda_k) \alpha_{nk} \varphi_k \in U_{n-1}.$$

Therefore, for $m < n$ we have

$$A\psi_n - A\psi_m = \lambda_n \psi_n - (\lambda_n \psi_n - A\psi_n + A\psi_m) = \lambda_n(\psi_n - \psi),$$

where $\psi := (\lambda_n \psi_n - A\psi_n + A\psi_m) / \lambda_n \in U_{n-1}$. Hence

$$\|A\psi_n - A\psi_m\| \geq \frac{|\lambda_n|}{2} \geq \frac{R}{2}$$

for $m < n$, and the sequence $(A\psi_n)$ does not contain a convergent subsequence which contradicts the compactness of A .

3.3 Volterra Integral Equations

Integral equations of the form

$$\int_a^x K(x, y)\varphi(y)dy = f(x), \quad x \in [a, b],$$

and

$$\varphi(x) - \int_a^x K(x, y)\varphi(y)dy = f(x), \quad x \in [a, b],$$

with variable limits of integration are called *Volterra integral equations* of the *first* and *second kind*, respectively. Equations of this type were first investigated by Volterra [1]. One can view Volterra equations as special cases of Fredholm equations with $K(x, y) = 0$ for $y > x$, but they have some special properties. In particular, Volterra integral equations of the second kind are always uniquely solvable.

Theorem 3.12. *The Volterra integral equation of the second kind*

$$\varphi(x) - \int_a^x K(x, y)\varphi(y)dy = f(x), \quad x \in [a, b],$$

with continuous kernel K for each right hand side $f \in C[a, b]$ has a unique solution $\varphi \in C[a, b]$.

Proof: We extend the kernel onto $[a, b] \times [a, b]$ by setting $K(x, y) := 0$ for $y > x$. Then K is continuous for $x \neq y$ and

$$|K(x, y)| \leq M := \max_{a \leq y \leq x \leq b} |K(x, y)|$$

for all $x \neq y$. Hence, K is weakly singular.

Now let $\varphi \in C[a, b]$ be a solution to the homogeneous equation

$$\varphi(x) - \int_a^x K(x, y)\varphi(y)dy = 0, \quad x \in [a, b].$$

By induction we show that

$$|\varphi(x)| \leq \|\varphi\|_\infty \frac{M^n(x-a)^n}{n!}, \quad x \in [a, b],$$

for $n = 0, 1, 2, \dots$. This certainly is true for $n = 0$. Assume the inequality is proven for some $n \geq 0$. Then

$$|\varphi(x)| \leq \left| \int_a^x K(x, y)\varphi(y)dy \right| \leq \|\varphi\|_\infty \frac{M^{n+1}(x-a)^{n+1}}{(n+1)!}.$$

Passing to the limit $n \rightarrow \infty$ yields $\varphi(x) = 0$ for all $x \in [a, b]$. Our statement now follows from Theorems 2.21 and 3.4. \square

In terms of spectral theory we can formulate the last result as follows: A Volterra integral operator with continuous kernel has no spectral values different from zero.

Despite the fact that, in general, integral equations of the first kind are more delicate than integral equations of the second kind, in some cases Volterra integral equations of the first kind can be treated by reducing them to equations of the second kind. Consider

$$(3.1) \quad \int_a^x K(x, y)\varphi(y)dy = f(x), \quad x \in [a, b],$$

and assume that the derivatives $\partial K/\partial x$ and f' exist and are continuous and that $K(x, x) \neq 0$ for all $x \in [a, b]$. Then differentiating with respect to x reduces (3.1) to

$$(3.2) \quad \varphi(x) + \int_a^x \frac{\frac{\partial}{\partial x} K(x, y)}{K(x, x)} \varphi(y)dy = \frac{f'(x)}{K(x, x)}, \quad x \in [a, b].$$

Equations (3.1) and (3.2) are equivalent if $f(a) = 0$. Provided that $\partial K/\partial y$ exists and is continuous and that again $K(x, x) \neq 0$ for all $x \in [a, b]$ then there is a second method to reduce the equation of the first kind to one of the second kind. In this case, setting

$$\psi(x) := \int_a^x \varphi(y)dy$$

and performing an integration by parts in (3.1) yields

$$(3.3) \quad \psi(x) - \int_a^x \frac{\frac{\partial}{\partial y} K(x, y)}{K(x, x)} \psi(y)dy = \frac{f(x)}{K(x, x)}, \quad x \in [a, b].$$

We leave it as an exercise to extend our short discussion of Volterra integral equations to the case of more than one independent variable.

Problems

- 3.1. Let $A : X \rightarrow Y$ be a compact linear operator from a normed space X into a normed space Y . The continuous extension $\tilde{A} : \tilde{X} \rightarrow \tilde{Y}$ of A is compact with $\tilde{A}(\tilde{X}) \subset Y$ (see Problem 2.1).
- 3.2. Let X be a linear space and let $A, B : X \rightarrow X$ be linear operators with $AB = BA$ and let AB have an inverse $(AB)^{-1} : X \rightarrow X$. Then A and B have inverse operators $A^{-1} = B(AB)^{-1}$ and $B^{-1} = A(AB)^{-1}$.
- 3.3. Prove Theorems 3.4 and 3.6 under the assumption that A^n is compact for some $n \geq 1$.

Hint: Use Problem 3.2 to prove that the spectrum $(\sigma(A))^n$ is contained in the spectrum $\sigma(A^n)$. Then use Theorem 3.11 to show that there exists an integer $m \geq n$ such that each of the operators

$$L_k := \exp \frac{2\pi ik}{m} I - A$$

for $k = 1, \dots, m-1$ has a bounded inverse. Then the equations $R(I - A)\varphi = Rf$ and $(I - A)\varphi = f$ where $R := \prod_{k=1}^{m-1} L_k$ are equivalent

- 3.4. Let X_i for $i = 1, \dots, n$ be normed spaces. Then the cartesian product $X := X_1 \times \dots \times X_n$ of n -tuples $\varphi := (\varphi_1, \dots, \varphi_n)$ is a normed space with the maximum norm

$$\|\varphi\|_\infty := \max_{i=1, \dots, n} \|\varphi_i\|.$$

Let $A_{ik} : X_k \rightarrow X_i$ for $i, k = 1, \dots, n$ be linear operators. Then the matrix operator $A : X \rightarrow X$ defined by

$$(A\varphi)_i := \sum_{k=1}^n A_{ik} \varphi_k$$

is bounded or compact if and only if each of its components $A_{ik} : X_k \rightarrow X_i$ is bounded or compact, respectively. Formulate Theorems 3.4 and 3.6 for systems of operator and integral equations of the second kind.

- 3.5. Show that the integral operator with continuous kernel

$$K(x, y) := \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \{ \cos(k+1)x \sin ky - \sin(k+1)x \cos ky \}$$

on the interval $[0, 2\pi]$ has no eigenvalues.

4. Dual Systems and Fredholm Theory

In the case of Theorem 3.6 in which the homogeneous equation has nontrivial solutions, the Riesz theory gives no answer to the question whether the nonhomogeneous equation for a given inhomogeneity is solvable or not. This question is settled by the Fredholm theory that we shall develop for compact operators which are adjoint in dual systems generated by nondegenerate bilinear or sesquilinear forms. We follow Jörgens [1] and Wendland [1], [3] but as opposed to their analysis we do not assume the bilinear or sesquilinear form to be bounded (see also Kress [7]). In a more general framework, our form of the Fredholm alternative is contained in Heuser [1]. However, our proof is more elementary and straightforward, in particular, we do not have to use the Hahn–Banach theorem, i.e., Zorn’s lemma (cf. Martensen [3]).

4.1 Dual Systems Via Bilinear Forms

Definition 4.1. Let X, Y be linear spaces. A mapping $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$ is called a *bilinear form* if

$$\begin{aligned}\langle \alpha_1\varphi_1 + \alpha_2\varphi_2, \psi \rangle &= \alpha_1\langle \varphi_1, \psi \rangle + \alpha_2\langle \varphi_2, \psi \rangle, \\ \langle \varphi, \beta_1\psi_1 + \beta_2\psi_2 \rangle &= \beta_1\langle \varphi, \psi_1 \rangle + \beta_2\langle \varphi, \psi_2 \rangle\end{aligned}$$

for all $\varphi_1, \varphi_2, \varphi \in X$, $\psi_1, \psi_2, \psi \in Y$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The bilinear form is called *nondegenerate* if for every $\varphi \in X$, $\varphi \neq 0$, there exists $\psi \in Y$ such that $\langle \varphi, \psi \rangle \neq 0$; and for every $\psi \in Y$, $\psi \neq 0$, there exists $\varphi \in X$ such that $\langle \varphi, \psi \rangle \neq 0$.

Definition 4.2. Two normed spaces X and Y equipped with a nondegenerate bilinear form are called a *dual system* and denoted by $\langle X, Y \rangle$.

Theorem 4.3. Let $G \subset \mathbb{R}^m$ be as in Theorem 2.7. Then $\langle C(G), C(G) \rangle$ is a dual system with the bilinear form

$$(4.1) \quad \langle \varphi, \psi \rangle := \int_G \varphi(x)\psi(x)dx, \quad \varphi, \psi \in C(G).$$

Proof: Obvious from Definition 4.2.

Definition 4.4. Let $\langle X_1, Y_1 \rangle_1$ and $\langle X_2, Y_2 \rangle_2$ be dual systems. Then two operators $A : X_1 \rightarrow X_2$, $B : Y_2 \rightarrow Y_1$ are called *adjoint* if for every $\varphi \in X_1$, $\psi \in Y_2$ there holds

$$\langle A\varphi, \psi \rangle_2 = \langle \varphi, B\psi \rangle_1.$$

Theorem 4.5. Let $\langle X_1, Y_1 \rangle_1$ and $\langle X_2, Y_2 \rangle_2$ be two dual systems. If an operator $A : X_1 \rightarrow X_2$ has an adjoint $B : Y_2 \rightarrow Y_1$, then B is uniquely determined and A and B are linear.

Proof: Suppose there existed two adjoints to A and denote these by B_1 and B_2 . Let $B := B_1 - B_2$. Then we have $\langle \varphi, B\psi \rangle_1 = \langle \varphi, B_1\psi \rangle_1 - \langle \varphi, B_2\psi \rangle_1 = \langle A\varphi, \psi \rangle_2 - \langle A\varphi, \psi \rangle_2 = 0$ for all $\varphi \in X_1$ and $\psi \in Y_2$. Hence, since $\langle \cdot, \cdot \rangle_1$ is nondegenerate we have $B\psi = 0$ for every $\psi \in Y_2$, that is, $B_1 = B_2$. To show that B is linear we simply observe that for every $\varphi \in X_1$ there holds

$$\langle \varphi, \beta_1 B\psi_1 + \beta_2 B\psi_2 \rangle_1 = \beta_1 \langle \varphi, B\psi_1 \rangle_1 + \beta_2 \langle \varphi, B\psi_2 \rangle_1$$

$$= \beta_1 \langle A\varphi, \psi_1 \rangle_2 + \beta_2 \langle A\varphi, \psi_2 \rangle_2 = \langle A\varphi, \beta_1\psi_1 + \beta_2\psi_2 \rangle_2 = \langle \varphi, B(\beta_1\psi_1 + \beta_2\psi_2) \rangle_1,$$

that is, $\beta_1 B\psi_1 + \beta_2 B\psi_2 = B(\beta_1\psi_1 + \beta_2\psi_2)$. In a similar manner, it is seen that A is linear.

Theorem 4.6. Let K be a continuous or a weakly singular kernel. Then in the dual system $\langle C(G), C(G) \rangle$ the (compact) integral operators defined by

$$(A\varphi)(x) := \int_G K(x, y)\varphi(y)dy, \quad x \in G,$$

$$(B\psi)(x) := \int_G K(y, x)\psi(y)dy, \quad x \in G,$$

are adjoint.

Proof: The theorem follows from

$$\begin{aligned} \langle A\varphi, \psi \rangle &= \int_G (A\varphi)(x)\psi(x)dx = \int_G \left(\int_G K(x, y)\varphi(y)dy \right) \psi(x)dx \\ &= \int_G \varphi(y) \left(\int_G K(x, y)\psi(x)dx \right) dy = \int_G \varphi(y)(B\psi)(y)dy = \langle \varphi, B\psi \rangle. \end{aligned}$$

In the case of a weakly singular kernel the interchanging of the order of integration is justified by the fact that $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, uniformly on G , where A_n is the integral operator with continuous kernel K_n introduced in the proof of Theorem 2.21.

4.2 Dual Systems Via Sesquilinear Forms

Definition 4.7. Let X, Y be linear spaces. A mapping $(\cdot, \cdot) : X \times Y \rightarrow \mathbb{C}$ is called a sesquilinear form if

$$(\alpha_1\varphi_1 + \alpha_2\varphi_2, \psi) = \alpha_1(\varphi_1, \psi) + \alpha_2(\varphi_2, \psi),$$

$$(\varphi, \beta_1\psi_1 + \beta_2\psi_2) = \bar{\beta}_1(\varphi, \psi_1) + \bar{\beta}_2(\varphi, \psi_2)$$

for all $\varphi_1, \varphi_2, \varphi \in X$, $\psi_1, \psi_2, \psi \in Y$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. Here, the bar indicates the complex conjugate.

We leave it as an exercise to formulate the Definition 4.4 and Theorem 4.5 in a dual system generated by a nondegenerate sesquilinear form.

Of course, there is a close relation between bilinear and sesquilinear forms. Assume there exists a mapping $* : Y \rightarrow Y$ such that $(\beta_1\psi_1 + \beta_2\psi_2)^* = \bar{\beta}_1\psi_1^* + \bar{\beta}_2\psi_2^*$ and $(\psi^*)^* = \psi$ for all $\psi_1, \psi_2, \psi \in Y$ and $\beta_1, \beta_2 \in \mathbb{C}$. Such a mapping is called an *involution* and provides a one-to-one correspondence between bilinear and sesquilinear forms by $\langle \varphi, \psi \rangle = \langle \varphi, \psi^* \rangle$. In the space $C(G)$ the natural involution is given by $\psi^*(x) := \psi(\bar{x})$ for all $x \in G$ and all $\psi \in C(G)$. Again we leave it as an exercise to formulate the Theorems 4.3 and 4.6 for the corresponding sesquilinear form

$$(4.2) \quad (\varphi, \psi) := \int_G \varphi(x)\overline{\psi(x)}dx, \quad \varphi, \psi \in C(G).$$

The operator $A : C[0, 1] \rightarrow C[0, 1]$ defined by $(A\varphi)(x) := \varphi(1)$ provides an example of a compact operator which does not have an adjoint operator with respect to the dual system $\langle C[0, 1], C[0, 1] \rangle$ or $\langle C[0, 1], C[0, 1] \rangle$. Assume $B : C[0, 1] \rightarrow C[0, 1]$ is an adjoint of A and choose a function $\psi \in C[0, 1]$ with $\int_0^1 \psi(x)dx = 1$. Then, by the Cauchy–Schwarz inequality, there follows

$$|\varphi(1)| = |\langle A\varphi, \psi \rangle| = |\langle \varphi, B\psi \rangle| \leq \|\varphi\|_2 \|B\psi\|_2$$

for all $\varphi \in C[0, 1]$. Considering the sequence (φ_n) with $\varphi_n(x) = x^n$ we arrive at a contradiction since the right hand side tends to zero if $n \rightarrow \infty$.

In the sequel, we will demonstrate that in Hilbert spaces for bounded linear operators the adjoint operators always exist. From Definition 1.19, we observe that each scalar product on a linear space X may be considered as a non-degenerate sesquilinear form which is *symmetric*, i.e., $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$ for all $\varphi, \psi \in X$, and *positive*, i.e., $\langle \varphi, \varphi \rangle > 0$ for all $\varphi \in X$ with $\varphi \neq 0$. Thus each pre–Hilbert space canonically is a dual system.

Theorem 4.8 (Riesz). *Let X be a Hilbert space. Then for each bounded linear function $F : X \rightarrow \mathbb{C}$ there exists a unique element $f \in X$ such that for all $\varphi \in X$ there holds*

$$F(\varphi) = \langle \varphi, f \rangle.$$

Proof: Uniqueness follows from the observation that, because of the positive definiteness of the scalar product, $f = 0$ is the only representer of the zero function $F = 0$. For $F \neq 0$ choose $w \in X$ with $F(w) \neq 0$. Since F is continuous the nullspace $N(F) = \{\varphi \in X : F(\varphi) = 0\}$ is a closed and consequently complete subspace of the Hilbert space X . By the approximation Theorems 1.25 and 1.26 there exists an element $v \in N(F)$ with $w - v \perp N(F)$. Then, setting $g := w - v$, since $F(g)\varphi - F(\varphi)g \in N(F)$ for all $\varphi \in X$ we have $(F(g)\varphi - F(\varphi)g, g) = 0$. Hence,

$$F(\varphi) = \left(\varphi, \frac{\overline{F(g)}g}{\|g\|^2} \right)$$

for all $\varphi \in X$ which completes the proof.

Theorem 4.9. *Let X and Y be Hilbert spaces, and let $A : X \rightarrow Y$ be a bounded linear operator. Then there exists a uniquely determined bounded linear operator $A^* : Y \rightarrow X$ with the property*

$$(A\varphi, \psi)_Y = (\varphi, A^*\psi)_X$$

for all $\varphi \in X$ and $\psi \in Y$, that is, A and A^* are adjoint with respect to the dual systems (X, X) and (Y, Y) generated by the scalar products on X and Y . The operator A^* is bounded with $\|A^*\| = \|A\|$.

Proof: For each $\psi \in Y$ the mapping $\varphi \mapsto (A\varphi, \psi)$ clearly defines a bounded linear function on X , since $|(A\varphi, \psi)| \leq \|A\| \|\varphi\| \|\psi\|$. By Theorem 4.8 we can write $(A\varphi, \psi) = (\varphi, f)$ with some $f \in X$. Therefore, setting $A^*\psi := f$ we define an operator $A^* : Y \rightarrow X$ which is an adjoint of A . By Theorem 4.5, the adjoint is uniquely determined and linear. Using the Cauchy–Schwarz inequality, we derive

$$\|A^*\psi\|^2 = (A^*\psi, A^*\psi) = (AA^*\psi, \psi) \leq \|A\| \|A^*\psi\| \|\psi\|$$

for all $\psi \in Y$. Hence, A^* is bounded with $\|A^*\| \leq \|A\|$. Conversely, since A is the adjoint of A^* we also have $\|A\| \leq \|A^*\|$. Hence $\|A^*\| = \|A\|$.

Theorem 4.10. *Let X and Y be Hilbert spaces and let $A : X \rightarrow Y$ be a compact linear operator. Then the adjoint operator $A^* : Y \rightarrow X$ is also compact.*

Proof: Let (ψ_n) be a bounded sequence in Y : $\|\psi_n\| \leq C$. By Theorem 4.9 the adjoint operator $A^* : Y \rightarrow X$ is bounded and consequently the operator $AA^* : Y \rightarrow Y$ is compact by Theorem 2.15. Hence there exists a subsequence $(\psi_{n(k)})$ such that $(AA^*\psi_{n(k)})$ converges in Y . But then from

$$\begin{aligned} \|A^*(\psi_{n(k)} - \psi_{n(j)})\|^2 &= (AA^*(\psi_{n(k)} - \psi_{n(j)}), \psi_{n(k)} - \psi_{n(j)}) \\ &\leq 2C\|AA^*(\psi_{n(k)} - \psi_{n(j)})\| \end{aligned}$$

we observe that $(A^*\psi_{n(k)})$ is a Cauchy sequence and therefore it converges in the Hilbert space X .

4.3 Positive Dual Systems

The following theorem is due to Lax [1] and provides a useful tool to extend results on the boundedness and compactness of linear operators from a given norm to a weaker scalar product norm.

Definition 4.11. *A dual system (X, X) generated by a scalar product (\cdot, \cdot) is called a positive dual system if the scalar product (\cdot, \cdot) is bounded with respect to the norm $\|\cdot\|$ on X , that is, if there exists a constant C such that for all $\varphi, \psi \in X$ there holds*

$$|(\varphi, \psi)| \leq C\|\varphi\| \|\psi\|.$$

Because of the boundedness of the sesquilinear form (\cdot, \cdot) the scalar product norm $\|\cdot\|_s$ induced by the positive dual system is weaker than the original norm $\|\cdot\|$. The Banach space $C(G)$ with the maximum norm and the scalar product (4.2) is an example for a positive dual system. Each pre-Hilbert space canonically is a positive dual system.

Theorem 4.12 (Lax). *Let (X, X) and (Y, Y) be positive dual systems and let $A : X \rightarrow Y$ and $B : Y \rightarrow X$ be bounded adjoint linear operators, that is, $(A\varphi, \psi) = (\varphi, B\psi)$ for all $\varphi \in X$ and $\psi \in Y$. Then A and B are bounded with respect to the scalar product norms induced by the positive dual systems with*

$$\|A\|_s^2 \leq \|A\| \|B\| \quad \text{and} \quad \|B\|_s^2 \leq \|B\| \|A\|.$$

Proof: It suffices to carry out the proof for A . Consider the bounded operator $M : X \rightarrow X$ given by $M := BA$. Then M is self adjoint, that is, $(M\varphi, \psi) = (\varphi, M\psi)$ for all $\varphi, \psi \in X$. Therefore, using the Cauchy-Schwarz inequality, we obtain

$$\|M^n\varphi\|_s^2 = (M^n\varphi, M^n\varphi) = (\varphi, M^{2n}\varphi) \leq \|M^{2n}\varphi\|_s$$

for all $\varphi \in X$ with $\|\varphi\|_s \leq 1$ and all $n \in \mathbb{N}$. From this, by induction, it follows that

$$\|M\varphi\|_s \leq \|M^{2^n}\varphi\|_s^{2^{-n}}.$$

Since $\|\cdot\|_s$ is weaker than $\|\cdot\|$, for all $\varphi \in X$ there holds $\|\varphi\|_s \leq C\|\varphi\|$ with some constant C . Hence,

$$\|M\varphi\|_s \leq [C\|M^{2^n}\varphi\|]^{2^{-n}} \leq [C\|\varphi\| \|M\|^{2^n}]^{2^{-n}} \leq [C\|\varphi\|]^{2^{-n}} \|M\|.$$

Passing to the limit $n \rightarrow \infty$ now yields

$$\|M\varphi\|_s \leq \|M\|$$

for all $\varphi \in X$ with $\|\varphi\|_s \leq 1$. Thus $\|M\|_s \leq \|M\| \leq \|B\| \|A\|$ by Theorem 2.6. Finally, for all $\varphi \in X$ with $\|\varphi\|_s \leq 1$, again using the Cauchy-Schwarz inequality, we have

$$\|A\varphi\|_s^2 = (A\varphi, A\varphi) = (\varphi, M\varphi) \leq \|M\varphi\|_s \leq \|M\| \leq \|B\| \|A\|.$$

From this the statement follows. \square

We note that the Lax Theorem 4.12 also can be used to derive compactness in the scalar product norm $\|\cdot\|_s$ in cases where the compactness in the original norm $\|\cdot\|$ is obtained via finite dimensional approximations using Theorem 2.16 in the completion \tilde{X} and \tilde{Y} of X and Y with respect to the scalar product norm (see Problem 4.5). In particular, using finite dimensional approximations for the compactness in $C(G)$ of the integral operator with continuous kernel and the approximations of Theorem 2.21 for the weakly singular kernel the compactness of these integral operators in the Hilbert space $L^2(G)$ follows from Theorem 4.12 without any further analysis.

4.4 The Fredholm Alternative

Now we proceed to develop the Fredholm theory which we will write out for a dual system generated by a bilinear form. We begin with the following technical

Lemma 4.13. *Let $\langle X, Y \rangle$ be a dual system. Then to every set of linearly independent elements $\varphi_1, \dots, \varphi_n \in X$ there exists a set $\psi_1, \dots, \psi_n \in Y$ such that*

$$\langle \varphi_i, \psi_k \rangle = \delta_{ik}, \quad i, k = 1, \dots, n.$$

The same statement holds with the roles of X and Y interchanged.

Proof: For one linearly independent element the lemma is true since $\langle \cdot, \cdot \rangle$ is nondegenerate. Assume the assertion of the lemma is proven for $n \geq 1$ linearly independent elements. Let $\varphi_1, \dots, \varphi_{n+1}$ be $n+1$ linearly independent elements. Then, by our induction assumption, for every $m = 1, \dots, n+1$, to the set $\varphi_1, \dots, \varphi_{m-1}, \varphi_{m+1}, \dots, \varphi_{n+1}$ of n elements in X there exists a set of n elements $\psi_1^{(m)}, \dots, \psi_{m-1}^{(m)}, \psi_{m+1}^{(m)}, \dots, \psi_{n+1}^{(m)}$ in Y such that

$$(4.3) \quad \langle \varphi_i, \psi_k^{(m)} \rangle = \delta_{ik}, \quad i, k = 1, \dots, n+1, \quad i, k \neq m.$$

Then there exists $\chi_m \in Y$ such that

$$\alpha_m := \langle \varphi_m, \chi_m - \sum_{\substack{k=1 \\ k \neq m}}^{n+1} \psi_k^{(m)} \langle \varphi_k, \chi_m \rangle \rangle = \langle \varphi_m - \sum_{\substack{k=1 \\ k \neq m}}^{n+1} \langle \varphi_m, \psi_k^{(m)} \rangle \varphi_k, \chi_m \rangle \neq 0$$

since otherwise

$$\varphi_m - \sum_{\substack{k=1 \\ k \neq m}}^{n+1} \langle \varphi_m, \psi_k^{(m)} \rangle \varphi_k = 0,$$

a contradiction to the linear independence of the $\varphi_1, \dots, \varphi_{n+1}$. Define

$$\psi_m := \frac{1}{\alpha_m} \left\{ \chi_m - \sum_{\substack{k=1 \\ k \neq m}}^{n+1} \psi_k^{(m)} \langle \varphi_k, \chi_m \rangle \right\}.$$

Then, obviously, $\langle \varphi_m, \psi_m \rangle = 1$ and for $i \neq m$,

$$\langle \varphi_i, \psi_m \rangle = \frac{1}{\alpha_m} \left\{ \langle \varphi_i, \chi_m \rangle - \sum_{\substack{k=1 \\ k \neq m}}^{n+1} \langle \varphi_i, \psi_k^{(m)} \rangle \langle \varphi_k, \chi_m \rangle \right\} = 0$$

because of (4.3). Hence we obtain $\psi_1, \dots, \psi_{n+1}$ such that

$$\langle \varphi_i, \psi_k \rangle = \delta_{ik}, \quad i, k = 1, \dots, n+1.$$

Theorem 4.14 (First Fredholm Theorem). *Let $\langle X, Y \rangle$ be a dual system, and $A : X \rightarrow X$, $B : Y \rightarrow Y$ be compact adjoint operators. Then the nullspaces of the operators $I - A$ and $I - B$ have the same finite dimension.*

Proof: By the first Riesz Theorem 3.1 we have

$$m := \dim N(I - A) < \infty, \quad n := \dim N(I - B) < \infty.$$

We assume $m < n$. Choose a basis $\varphi_1, \dots, \varphi_m$ for the nullspace $N(I - A)$ (if $m > 0$) and a basis ψ_1, \dots, ψ_n for the nullspace $N(I - B)$. By Lemma 4.13 there exist elements $a_1, \dots, a_m \in Y$ (if $m > 0$) and $b_1, \dots, b_n \in X$ such that

$$\langle \varphi_i, a_k \rangle = \delta_{ik}, \quad i, k = 1, \dots, m,$$

$$\langle b_i, \psi_k \rangle = \delta_{ik}, \quad i, k = 1, \dots, n.$$

Define a linear operator $T : X \rightarrow X$ with finite dimensional range by

$$T\varphi := \begin{cases} 0 & \text{if } m = 0, \\ \sum_{i=1}^m \langle \varphi, a_i \rangle b_i & \text{if } m > 0. \end{cases}$$

Let $\varphi \in N(I - A + T)$, that is, (if $m > 0$)

$$\varphi - A\varphi + \sum_{i=1}^m \langle \varphi, a_i \rangle b_i = 0.$$

It follows that

$$\langle \varphi, a_k \rangle = \langle \varphi, \psi_k - B\psi_k \rangle + \langle \varphi, a_k \rangle = \langle \varphi - A\varphi + \sum_{i=1}^m \langle \varphi, a_i \rangle b_i, \psi_k \rangle = 0$$

for $k = 1, \dots, m$. Therefore $\varphi - A\varphi = 0$ and hence we can write

$$\varphi = \sum_{i=1}^m \alpha_i \varphi_i.$$

Now from

$$\langle \varphi, a_k \rangle = \sum_{i=1}^m \alpha_i \langle \varphi_i, a_k \rangle = \alpha_k$$

we obtain $\alpha_k = 0$ for $k = 1, \dots, m$ and therefore $\varphi = 0$. Thus we have proved that $N(I - A + T) = \{0\}$, that is, $I - A + T$ is injective, which of course also is true for the case $m = 0$.

Now we show that $I - A + T$ is surjective. Let $P : X \rightarrow N(L^*)$ be the projection defined by the decomposition $X = N(L^*) \oplus L^*(X)$ and recall that the inverse operator $(I - A - P)^{-1} : X \rightarrow X$ exists by Theorem 3.9. The operator $T + P$ has a finite dimensional range $U := (T + P)(X)$. Define an operator $K : U \rightarrow U$ on the finite dimensional space U by

$$K := I + (T + P)(I - A - P)^{-1} = (I - A + T)(I - A - P)^{-1}.$$

Let $g \in N(K)$. Since $I - A + T$ is injective we have $(I - A - P)^{-1}g = 0$ and from this $g = 0$. Therefore the linear operator K on the finite dimensional space U is

injective and therefore surjective. Thus, given any $f \in X$, the inhomogeneous equation

$$Kg = (T + P)(I - A - P)^{-1}f$$

has a unique solution $g \in U$. Now we set

$$\varphi := (I - A - P)^{-1}(f - g)$$

and obtain

$$\begin{aligned} (I - A + T)\varphi &= f - g + (T + P)(I - A - P)^{-1}(f - g) \\ &= f - Kg + (T + P)(I - A - P)^{-1}f = f. \end{aligned}$$

Hence, $I - A + T$ is surjective.

Now, since $I - A + T$ is bijective, the inhomogeneous equation

$$\varphi - A\varphi + T\varphi = b_{m+1}$$

has a unique solution φ . We now arrive at the contradiction

$$\begin{aligned} 1 &= \langle b_{m+1}, \psi_{m+1} \rangle = \langle \varphi - A\varphi + T\varphi, \psi_{m+1} \rangle \\ &= \langle \varphi - A\varphi, \psi_{m+1} \rangle = \langle \varphi, \psi_{m+1} - B\psi_{m+1} \rangle = 0 \end{aligned}$$

since $\langle T\varphi, \psi_{m+1} \rangle = 0$. Therefore $m \geq n$ and a similar argument shows $n \geq m$. Hence $m = n$.

Theorem 4.15 (Second Fredholm Theorem). *The nonhomogeneous equation*

$$\varphi - A\varphi = f$$

is solvable if and only if the condition

$$\langle f, \psi \rangle = 0$$

is satisfied for all solutions of the homogeneous adjoint equation

$$\psi - B\psi = 0.$$

The same statement holds with the roles of A and B interchanged.

Proof: Necessity: Let φ be a solution of $\varphi - A\varphi = f$. Then for all solutions ψ of $\psi - B\psi = 0$, we obtain

$$\langle f, \psi \rangle = \langle \varphi - A\varphi, \psi \rangle = \langle \varphi, \psi - B\psi \rangle = 0.$$

Sufficiency: By the first Fredholm theorem we have

$$m = \dim N(I - A) = \dim N(I - B) < \infty.$$

In the case $m = 0$, the condition $\langle f, \psi \rangle = 0$ is satisfied for all $f \in X$ and by Corollary 3.5 the equation $\varphi - A\varphi = f$ is indeed solvable for all $f \in X$. In the

case $m > 0$, from the proof of the previous theorem, we know that $I - A + T$ is bijective. Hence there exists a unique solution φ of the equation

$$\varphi - A\varphi + T\varphi = f.$$

Then it follows that

$$\begin{aligned} \langle \varphi, a_k \rangle &= \langle \varphi, \psi_k - B\psi_k \rangle + \langle \varphi, a_k \rangle \\ &= \langle \varphi - A\varphi, \psi_k \rangle + \sum_{i=1}^m \langle \varphi, a_i \rangle \langle b_i, \psi_k \rangle = \langle \varphi - A\varphi + T\varphi, \psi_k \rangle = \langle f, \psi_k \rangle = 0 \end{aligned}$$

for $k = 1, \dots, m$ since we are assuming that the solvability condition of the theorem is satisfied. Hence $T\varphi = 0$ and thus φ also satisfies the original equation $\varphi - A\varphi = f$. \square

For convenience, we introduce the following notation (see also Definition 1.22).

Definition 4.16. Let $\langle X, Y \rangle$ be a dual system. Then the set

$$U^\perp := \{g \in Y : \langle \varphi, g \rangle = 0, \varphi \in U\}$$

is called the orthogonal complement of $U \subset X$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$. Analogously, the set

$$V^\perp := \{f \in X : \langle f, \psi \rangle = 0, \psi \in V\}$$

is called the orthogonal complement of $V \subset Y$.

We now summarize our results in

Theorem 4.17 (Fredholm Alternative). Let $\langle X, Y \rangle$ be a dual system and let $A : X \rightarrow X$, $B : Y \rightarrow Y$ be compact adjoint operators. Then either

$$N(I - A) = \{0\} \quad \text{and} \quad N(I - B) = \{0\}$$

and

$$(I - A)(X) = X \quad \text{and} \quad (I - B)(Y) = Y$$

or

$$\dim N(I - A) = \dim N(I - B) \in \mathbb{N}$$

and

$$(I - A)(X) = N(I - B)^\perp \quad \text{and} \quad (I - B)(Y) = N(I - A)^\perp.$$

Choosing the dual system introduced in Theorem 4.3 and the integral operators with continuous or weakly singular kernels considered in Theorem 4.6, our results include the classical Fredholm alternative for integral equations of the second kind first obtained by Fredholm [1] which we state as a corollary.

Corollary 4.18. *Let $G \subset \mathbb{R}^m$ be as in Theorem 4.3 and let K be a continuous or weakly singular kernel. Then either the homogeneous integral equations*

$$\varphi(x) - \int_G K(x, y)\varphi(y)dy = 0, \quad x \in G,$$

and

$$\psi(x) - \int_G K(y, x)\psi(y)dy = 0, \quad x \in G,$$

only have the trivial solutions $\varphi = 0$ and $\psi = 0$ and the inhomogeneous integral equations

$$\varphi(x) - \int_G K(x, y)\varphi(y)dy = f(x), \quad x \in G,$$

and

$$\psi(x) - \int_G K(y, x)\psi(y)dy = g(x), \quad x \in G,$$

for each right hand side $f \in C(G)$ and $g \in C(G)$ have a unique solution $\varphi \in C(G)$ and $\psi \in C(G)$ or the homogeneous integral equations have the same finite number of linearly independent solutions and the inhomogeneous integral equations are solvable if and only if the right hand sides satisfy

$$\int_G f(x)\psi(x)dx = 0$$

for all solutions ψ of the homogeneous adjoint equation or

$$\int_G \varphi(x)g(x)dx = 0$$

for all solutions φ of the homogeneous equation, respectively.

We wish to mention that the original proof by Fredholm (see Jörgens [1]) is shorter than our more general approach, but it is limited to the case of integral equations with continuous kernels.

Example 4.19. Consider the integral equation

$$(4.4) \quad \varphi(x) - \int_a^b e^{x-y}\varphi(y)dy = f(x), \quad x \in [a, b].$$

Obviously, a solution of (4.4) must be of the form

$$(4.5) \quad \varphi(x) = f(x) + ce^x$$

where c is constant. Inserting (4.5) into (4.4), we observe that φ solves the integral equation provided c satisfies

$$(4.6) \quad c\{1 - (b-a)\} = \int_a^b e^{-y}f(y)dy.$$

Now either $b-a \neq 1$ or $b-a = 1$. In the first case (4.6) has a unique solution leading to the unique solution

$$\varphi(x) = f(x) + \frac{\int_a^b e^{-y}f(y)dy}{1 - (b-a)} e^x$$

of the integral equation. In the second case (4.6) has a solution if and only if

$$(4.7) \quad \int_a^b e^{-y} f(y) dy = 0$$

and then any c satisfies (4.6). Note that $\psi(x) = e^{-x}$ is the solution of the homogeneous adjoint equation

$$\psi(x) - \int_a^b e^{y-x} \psi(y) dy = 0, \quad x \in [a, b],$$

and therefore (4.7) coincides with the solvability condition of the Fredholm alternative. \square

Our results also include the Schauder theory (cf. Schauder [1], Jörgens [1]) by taking $Y = X^*$, the *dual space* of X , which is defined as the normed space $X^* = L(X, \mathbb{C})$ of *bounded linear functionals*, that is, bounded linear operators $\psi : X \rightarrow \mathbb{C}$. A normed space X together with its dual space X^* forms a canonical dual system $\langle X, X^* \rangle$ with the bilinear form defined by $\langle \varphi, \psi \rangle := \psi(\varphi)$ for all elements $\varphi \in X$ and all functionals $\psi \in X^*$. It is a consequence of the Hahn–Banach theorem (cf. Jörgens [1]) that this bilinear form is nondegenerate. Our more general form of the Fredholm alternative seems to be more appropriate for the discussion of integral equations because of its symmetric structure. In particular, in our setting the adjoint of an integral equation again is an integral equation, whereas in the Schauder theory the adjoint equation is an equation in the dual space of bounded linear functionals. Hence, the Schauder theory does not include the classical results of Fredholm on integral equations with continuous or weakly singular kernels.

4.5 Boundary Value Problems

We conclude this chapter by giving some flavour of the use of the Riesz–Fredholm theory to boundary value problems by considering the ordinary differential equation of the second order

$$(4.8) \quad u'' + a_1 u' + a_2 u = v$$

on the interval $[0, 1]$ with coefficients $a_1, a_2, v \in C[0, 1]$ subject to the boundary conditions

$$(4.9) \quad u(0) = u_0, \quad u(1) = u_1.$$

The general idea in the application of integral equations in the treatment of boundary value problems is to equivalently transform the boundary value problem into an integral equation and then solve the integral equation.

Let u be twice continuously differentiable and set $\varphi := -u''$. Then, by partial integration, we find

$$u(x) = u(0) + u'(0)x - \int_0^x (x-y)\varphi(y) dy$$

and

$$u(x) = u(1) - u'(1)(1-x) + \int_x^1 (x-y)\varphi(y)dy$$

and

$$0 = \int_0^1 \varphi(y)dy + u'(1) - u'(0).$$

Multiplying the first equation by $(1-x)$, the second by x and the third by $(1-x)x$ and then adding we obtain

$$(4.10) \quad u(x) = u(0)(1-x) + u(1)x + \int_0^x (1-x)y\varphi(y)dy + \int_x^1 x(1-y)\varphi(y)dy.$$

Differentiating this equation yields

$$(4.11) \quad u'(x) = u(1) - u(0) - \int_0^x y\varphi(y)dy + \int_x^1 (1-y)\varphi(y)dy.$$

Now let u be a solution to the boundary value problem. Then, using the boundary condition $u(0) = u_0$ and $u(1) = u_1$ and the differential equation $\varphi = a_1u' + a_2u - v$, from (4.10) and (4.11) we deduce the integral equation

$$(4.12) \quad \varphi(x) - \int_0^1 K(x,y)\varphi(y)dy = f(x), \quad x \in [0,1],$$

with the kernel

$$K(x,y) := \begin{cases} y\{a_2(x)(1-x) - a_1(x)\}, & y < x, \\ (1-y)\{a_2(x)x + a_1(x)\}, & x < y, \end{cases}$$

and the right hand side

$$f(x) := (u_1 - u_0)a_1(x) + \{u_0(1-x) + u_1x\}a_2(x) - v(x), \quad x \in [0,1].$$

Conversely, let $\varphi \in C[0,1]$ be a solution to the integral equation and define a twice continuously differentiable function u by

$$u(x) := u_0(1-x) + u_1x + \int_0^x (1-x)y\varphi(y)dy + \int_x^1 x(1-y)\varphi(y)dy, \quad x \in [0,1].$$

Then $u(0) = u_0$ and $u(1) = u_1$ and by construction of the integral equation there holds $-u'' = \varphi = a_1u' + a_2u - v$. Therefore, the boundary value problem and the integral equation are equivalent.

Since the kernel K is continuous on $0 \leq y \leq x \leq 1$ and on $0 \leq x \leq y \leq 1$ it is weakly singular with $\alpha = 1$. Hence, the Fredholm alternative is valid for the integral equation (4.12). Either the inhomogeneous integral equation is uniquely solvable for each right hand side $f \in C[0,1]$ and therefore the boundary value problem itself also is uniquely solvable for all inhomogeneities $v \in C[0,1]$ and all boundary values u_0 and u_1 , or the homogeneous integral equation has a nontrivial solution $\varphi \in C[0,1]$. In the latter case,

$$w(x) := \int_0^x (1-x)y\varphi(y)dy + \int_x^1 (1-y)x\varphi(y)dy, \quad x \in [0,1],$$

is a nontrivial solution to the homogeneous boundary value problem

$$(4.13) \quad w'' + a_1 w' + a_2 w = 0, \quad w(0) = w(1) = 0.$$

In particular $w \neq 0$ because of $w'' = -\varphi$. For any nontrivial solution w to the homogeneous boundary value problem $\varphi = -w''$ cannot vanish identically and satisfies the homogeneous integral equation. Since the boundary value problem (4.13) admits at most one linearly independent solution the same is true for the homogeneous integral equation and its adjoint equation. We have to show that two nontrivial solutions to (4.13) are linearly dependent. Let w_1, w_2 be two solutions. Then $w'_1(0) \neq 0$ and $w'_2(0) \neq 0$ since otherwise $w_1 = 0$ and $w_2 = 0$ because of the Picard–Lindelöf uniqueness theorem for initial value problems. Therefore we have $w'_1(0) = \lambda w'_2(0)$ with some $\lambda \in \mathbb{C}$. Furthermore, $0 = w_1(0) = \lambda w_2(0) = 0$. Hence, again by the uniqueness for the initial value problem, there follows $w_1 = \lambda w_2$.

Let ψ be a solution to the homogeneous adjoint equation

$$\psi(x) - \int_0^1 K(y, x)\psi(y)dy = 0, \quad x \in [0, 1],$$

that is,

$$\psi(x) = \int_0^x (1-x)\{a_2(y)y + a_1(y)\}\psi(y)dy + \int_x^1 x\{a_2(y)(1-y) - a_1(y)\}\psi(y)dy.$$

Then $\psi(0) = \psi(1) = 0$ and for the derivative we find

$$\psi'(x) - a_1(x)\psi(x) = - \int_0^1 \{a_2(y)y + a_1(y)\}\psi(y)dy + \int_x^1 a_2(y)\psi(y)dy.$$

Hence,

$$(\psi' - a_1\psi)' + a_2\psi = 0.$$

Since this boundary value problem again admits at most one linearly independent solution, the homogeneous adjoint integral equation and the homogeneous adjoint boundary value problem are equivalent. By the Fredholm alternative, the inhomogeneous integral equation (4.12) and therefore the inhomogeneous boundary value problem (4.8) and (4.9) is solvable if and only if

$$\int_0^1 f(x)\psi(x)dx = 0.$$

Using the differential equation and the boundary condition for ψ , we transform

$$\begin{aligned} & \int_0^1 \{(u_1 - u_0)a_1(x) + [u_0(1-x) + u_1x]a_2(x)\}\psi(x)dx \\ &= (u_1 - u_0) \int_0^1 a_1(x)\psi(x)dx + \int_0^1 \{u_0(1-x) + u_1x\}\{a_1(x)\psi(x) - \psi'(x)\}'dx \\ &= u_0\psi'(0) - u_1\psi'(1). \end{aligned}$$

Hence, the conditions (4.14) and

$$\int_0^1 v(x)\psi(x)dx = u_0\psi'(0) - u_1\psi'(1)$$

are equivalent and we can summarize our results into the following form of the Fredholm alternative.

Theorem 4.20. Either the inhomogeneous boundary value problem

$$u'' + a_1 u' + a_2 u = v, \quad u(0) = u_0, \quad u(1) = u_1,$$

is uniquely solvable for all right hand sides v and boundary values u_0, u_1 or the homogeneous boundary value problem

$$w'' + a_1 w' + a_2 w = 0, \quad w(0) = w(1) = 0,$$

and the homogeneous adjoint boundary value problem

$$(\psi' - a_1 \psi)' + a_2 \psi = 0, \quad \psi(0) = \psi(1) = 0,$$

both have one linearly independent solution w and ψ , respectively. In the latter case, the inhomogeneous boundary value problem is solvable if and only if

$$\int_0^1 v(x) \psi(x) dx = u_0 \psi'(0) - u_1 \psi'(1).$$

Problems

- 4.1. Let $\langle \cdot, \cdot \rangle : C[a, b] \times C[a, b] \rightarrow \mathbb{R}$ be a degenerate bilinear form. Then there exists a function $f \in C[a, b]$ such that $\langle f, \psi \rangle = 0$ for all $\psi \in C[a, b]$. Since $f \neq 0$, without loss of generality we may assume that $f(a) = 1$. The compact operators $A, B : C[a, b] \rightarrow C[a, b]$ defined by $A\varphi := \varphi(a)f$ and $B\psi := 0$ are adjoint with respect to $\langle \cdot, \cdot \rangle$. By showing that $N(I - A) = \text{span}\{f\}$ and $N(I - B) = \{0\}$ demonstrate that for the validity of the Fredholm alternative the bilinear form necessarily must be nondegenerate.
- 4.2. Let X be the linear space of all functions $\varphi \in C(0, 1)$ for which positive numbers M and α (depending on φ) exist such that $|\varphi(x)| \leq Mx^{\alpha-1/2}$ for all $x \in (0, 1]$. Then X is a normed space with norm

$$\|\varphi\| := \sup_{x \in (0, 1]} \sqrt{x} |\varphi(x)|$$

and $\langle X, X \rangle$ is a dual system with the bilinear form

$$\langle \varphi, \psi \rangle := \int_0^1 \varphi(x) \psi(x) dx.$$

Show that the integral operators $A, B : X \rightarrow X$ with continuous kernel K defined as in Theorem 4.6 are compact and adjoint. By using the sequence (φ_n) in X given by the functions $\varphi_n(x) = x^{1/n-1/2}$ show that the bilinear form is not bounded.

- 4.3. Formulate and prove the Fredholm alternative for a pair of operators $S - A$ and $T - B$ where S and T each have a bounded inverse and A and B are compact.
- 4.4. Show that under the assumptions of Theorem 4.17 the operators A and B have Riesz number one if and only if for each pair of bases $\varphi_1, \dots, \varphi_m$ and ψ_1, \dots, ψ_m of the nullspaces $N(I - A)$ and $N(I - B)$ the matrix $\langle \varphi_i, \psi_k \rangle$, $i, k = 1, \dots, m$, is nonsingular.
- 4.5. Use the Lax Theorem 4.12 to show that the integral operator with weakly singular kernel considered in Theorem 2.21 is also a compact operator from $L^2(G) \rightarrow L^2(G)$ (see also Problem 2.5).

5. Regularization in Dual Systems

In this chapter we will consider equations which are singular in the sense that they are not of the second kind with a compact operator. We will demonstrate that it is still possible to obtain results on the solvability of singular equations provided they can be regularized, that is, they can be transformed into equations of the second kind with a compact operator.

5.1 Regularizers

The following definition will say more precisely what we mean by regularizing a bounded linear operator.

Definition 5.1. *Let X_1, X_2 be normed spaces and let $K : X_1 \rightarrow X_2$ be a bounded linear operator. A bounded linear operator $R_l : X_2 \rightarrow X_1$ is called a left regularizer of K if*

$$(5.1) \quad R_l K = I - A_l$$

where $A_l : X_1 \rightarrow X_1$ is compact; a bounded linear operator $R_r : X_2 \rightarrow X_1$ is called a right regularizer of K if

$$(5.2) \quad K R_r = I - A_r$$

where $A_r : X_2 \rightarrow X_2$ is compact. A bounded linear operator $R : X_2 \rightarrow X_1$ is called a regularizer of K if

$$(5.3) \quad R K = I - A_l \quad \text{and} \quad K R = I - A_r$$

where $A_l : X_1 \rightarrow X_1$ and $A_r : X_2 \rightarrow X_2$ are compact.

The difference between a left and a right regularizer is compact, since by multiplying (5.1) from the right by R_r and (5.2) from the left by R_l and then subtracting we obtain $R_r - R_l = R_l A_r - A_l R_r$ which is compact by Theorems 2.14 and 2.15. Again by Theorem 2.15 we observe that adding a compact operator to a regularizer preserves the regularizing property. Therefore, provided there exist a left and a right regularizer we always may assume that $R_l = R_r = R$.

Let us first consider regularizing from the left. Any solution to the original equation

$$(5.4) \quad K\varphi = f$$

also solves the regularized equation

$$(5.5) \quad \varphi - A_l \varphi = R_l f.$$

Therefore, by regularizing from the left we do not lose any solutions. Conversely, let φ be a solution of the regularized equation (5.5). Then

$$R_l(K\varphi - f) = 0,$$

and φ solves the original equation (5.4) provided $N(R_l) = \{0\}$. We call a left regularizer an *equivalent left regularizer* if the original and the regularized equation have the same solutions. Then, we can summarize in

Theorem 5.2. *A left regularizer is an equivalent left regularizer if and only if it is injective.*

Now let us treat regularizing from the right. Here we have to compare the original equation

$$(5.6) \quad K\varphi = f$$

and the regularized equation

$$(5.7) \quad \psi - A_r \psi = f.$$

Provided ψ solves the regularized equation (5.7) then $\varphi := R_r \psi$ is a solution to the original equation (5.6). Therefore, by regularizing from the right we do not create additional solutions. Conversely, to each solution φ of the original equation (5.6) there corresponds a solution ψ of the regularized equation (5.7) with $R_r \psi = \varphi$ provided $R_r(X_2) = X_1$. We call a right regularizer an *equivalent right regularizer* if it maps the solutions to the regularized equation bijectively onto the solutions of the original equation. Then, we can summarize in

Theorem 5.3. *A right regularizer is an equivalent right regularizer if and only if it is surjective.*

From Theorems 5.2 and 5.3 we conclude that in a situation where we can establish the existence of an injective left regularizer or a surjective right regularizer the results of the Riesz theory immediately carry over to the singular equation $K\varphi = f$. In particular, in both cases injectivity of the operator K implies surjectivity of K by Theorem 3.4. The transformation of a Volterra equation of the first kind into Volterra equations of the second kind as described in Section 3.3 may serve as a first example (see Problem 5.1).

5.2 Normal Solvability

In the remainder of this chapter we want to demonstrate that it is also possible to obtain solvability results by regularization which is not equivalent. Before we proceed, for convenience, we recall a technical lemma on finite dimensional linear systems.

Lemma 5.4. Let $A : \mathbb{C}^m \rightarrow \mathbb{C}^n$ and $A^* : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be adjoint matrices. The linear system

$$Ax = b$$

is solvable if and only if

$$(b, y) = 0$$

for all solutions y of $A^*y = 0$. Here, by (\cdot, \cdot) we denote the usual scalar product in Euclidean space.

Proof: Writing $A = (a_1, \dots, a_m)$ with column vectors $a_1, \dots, a_m \in \mathbb{C}^n$ we have

$$A(\mathbb{C}^m) = \text{span}\{a_1, \dots, a_m\}.$$

Now let $y \in A(\mathbb{C}^m)^\perp$, i.e., $(a_j, y) = 0$, $j = 1, \dots, m$. This can be expressed in the form $A^*y = 0$. Hence, $y \in N(A^*)$. Conversely, let $y \in N(A^*)$ and $b \in A(\mathbb{C}^m)$, that is, $b = Ax$ with some $x \in \mathbb{C}^m$. Then

$$(b, y) = (Ax, y) = (x, A^*y) = 0.$$

Therefore, $y \in A(\mathbb{C}^m)^\perp$. Since we now have established that

$$N(A^*) = A(\mathbb{C}^m)^\perp$$

the orthogonal decomposition

$$\mathbb{C}^n = A(\mathbb{C}^m) \oplus A(\mathbb{C}^m)^\perp = A(\mathbb{C}^m) \oplus N(A^*)$$

yields the assertion of the lemma.

Theorem 5.5. Let X_1, X_2 be normed spaces, let $K : X_1 \rightarrow X_2$ be a bounded linear operator and let $R : X_2 \rightarrow X_1$ be a regularizer of K . Then $\dim N(K) < \infty$ and $\dim N(R) < \infty$.

Proof: Let $K\varphi = 0$. Then $RK\varphi = 0$ and $N(K) \subset N(RK) = N(I - A_l)$. By the first Riesz Theorem 3.1, the nullspace of $I - A_l$ has finite dimension. Therefore $\dim N(K) \leq \dim N(I - A_l) < \infty$. Since K can be considered as a regularizer of R we also have $\dim N(R) < \infty$.

Theorem 5.6. Let $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ be dual systems and let the bounded operators $K : X_1 \rightarrow X_2$ and $K' : Y_2 \rightarrow Y_1$ be adjoint, that is,

$$\langle K\varphi, \psi \rangle_2 = \langle \varphi, K'\psi \rangle_1$$

for all $\varphi \in X_1$, $\psi \in X_2$. Assume there exist regularizers $R : X_2 \rightarrow X_1$ and $R' : Y_1 \rightarrow Y_2$ of K and K' which are adjoint. Then there holds

$$\dim N(K) < \infty \quad \text{and} \quad \dim N(K') < \infty$$

and

$$K(X_1) = N(K')^\perp \quad \text{and} \quad K'(Y_2) = N(K)^\perp.$$

Remark: Comparing this result with the Fredholm alternative Theorem 4.17 for compact operators we note that the only difference lies in the fact that the dimensions of the nullspaces in general are no longer the same. This will give rise to the introduction of the index of an operator in Section 5.3.

Proof: The first statement is contained in Theorem 5.5. For the second statement it suffices to carry out the proof for

$$(5.8) \quad K(X_1) = N(K')^\perp.$$

Let $f \in K(X_1)$, i.e., $f = K\varphi$ with some $\varphi \in X_1$. Then for all $\psi \in N(K')$ we have

$$\langle f, \psi \rangle_2 = \langle K\varphi, \psi \rangle_2 = \langle \varphi, K'\psi \rangle_1 = 0.$$

Hence, $f \in N(K')^\perp$ and $K(X_1) \subset N(K')^\perp$.

By assumption, we have

$$(5.9) \quad RK = I - A, \quad K'R' = I - A',$$

where $A : X_1 \rightarrow X_1$, $A' : Y_1 \rightarrow Y_1$ are compact and easily seen to be adjoint. Therefore, for $I - A$ and $I - A'$ the Fredholm alternative Theorem 4.17 is at our disposal.

Let $f \in N(K')^\perp$ and $\chi \in N(I - A') = N(K'R')$. Then

$$\langle Rf, \chi \rangle_1 = \langle f, R'\chi \rangle_2 = 0$$

since $R'\chi \in N(K')$. Thus, $Rf \in N(I - A')^\perp$ and, by Theorem 4.17, we conclude that the regularized equation

$$(5.10) \quad \varphi - A\varphi = Rf$$

is solvable. We now want to show that we can choose a solution φ of (5.10) in such a way that φ also satisfies the original equation

$$(5.11) \quad K\varphi = f.$$

For this we have to recall a few facts from Section 4.4 on the form of the solution of (5.10). But we first note that by Theorem 5.2 we only need to consider the case where R is not injective.

Let $m := \dim N(I - A) = \dim N(I - A')$ and choose bases $\varphi_1, \dots, \varphi_m$ for $N(I - A)$ and ψ_1, \dots, ψ_m for $N(I - A')$ (if $m > 0$). By Lemma 4.13 we select elements $a_1, \dots, a_m \in Y_1$ and $b_1, \dots, b_m \in X_1$ such that

$$\langle \varphi_i, a_k \rangle_1 = \langle b_i, \psi_k \rangle_1 = \delta_{ik}, \quad i, k = 1, \dots, m.$$

Now consider the adjoint operators $T : X_1 \rightarrow X_1$ and $T' : Y_1 \rightarrow Y_1$ defined by

$$T\varphi := \sum_{i=1}^m \langle \varphi, a_i \rangle_1 b_i, \quad \varphi \in X_1,$$

$$T'\psi := \sum_{i=1}^m \langle b_i, \psi \rangle_1 a_i, \quad \psi \in Y_1,$$

if $m > 0$ and $T\varphi = 0$ and $T'\psi = 0$ if $m = 0$. From the proof of Theorem 4.14 we know that $I - A + T$ and $I - A' + T'$ are bijective. The inverse operators $S := (I - A + T)^{-1}$ and $S' := (I - A' + T')^{-1}$ again are adjoint. From the proof of Theorem 4.15 we recall that

$$\varphi_0 := (I - A + T)^{-1}Rf = SRf$$

is a solution of (5.10) since $Rf \in N(I - A')^\perp$. Therefore, (if $m > 0$) we can write the general solution of (5.10) in the form

$$(5.12) \quad \varphi = SRf + \sum_{i=1}^m \alpha_i \varphi_i$$

with complex coefficients $\alpha_1, \dots, \alpha_m$. Obviously, (5.12) solves (5.11) if and only if the coefficients are chosen such that

$$(5.13) \quad \sum_{i=1}^m \alpha_i K\varphi_i = f - KSRf.$$

Because of $RK\varphi_i = (I - A)\varphi_i = 0$ and $R(f - KSRf) = Rf - RK\varphi_0 = Rf - (I - A)\varphi_0 = 0$ equation (5.13) is an equation in the finite dimensional nullspace of R . Let $n := \dim N(R)$ and choose a basis h_1, \dots, h_n of $N(R)$. By Lemma 4.13 select elements $c_1, \dots, c_n \in Y_2$ such that

$$\langle h_i, c_k \rangle_2 = \delta_{ik}, \quad i, k = 1, \dots, n.$$

Then (5.13) is equivalent to

$$\sum_{i=1}^m \alpha_i \langle K\varphi_i, c_k \rangle_2 = \langle f - KSRf, c_k \rangle_2, \quad k = 1, \dots, n,$$

which in turn is equivalent to

$$(5.14) \quad \sum_{i=1}^m \alpha_i \langle \varphi_i, K'c_k \rangle_1 = \langle f, c_k - R'S'K'c_k \rangle_2, \quad k = 1, \dots, n.$$

This is a linear system of n equations for m unknowns. By Lemma 5.4 the conditions which are necessary and sufficient for the solvability of (5.14) can be expressed by a finite number p of equations of the form

$$\sum_{k=1}^n \rho_{jk} \langle f, c_k - R'S'K'c_k \rangle_2 = 0, \quad j = 1, \dots, p,$$

or in short form

$$(5.15) \quad \langle f, g_j \rangle_2 = 0, \quad j = 1, \dots, p,$$

where

$$g_j := \sum_{k=1}^n \rho_{jk} (c_k - R'S'K'c_k).$$

Now the proof will be completed by showing that $g_j \in N(K')$, $j = 1, \dots, p$. For each given $\chi \in X_1$ the equation

$$K\varphi = K\chi$$

obviously possesses a solution, namely $\varphi = \chi$. Since, as just stated, the conditions (5.15) are necessary for the solvability of (5.11) in the case of the right hand side given by $f = K\chi$ we get

$$\langle K\chi, g_j \rangle_2 = 0, \quad j = 1, \dots, p,$$

that is,

$$\langle \chi, K'g_j \rangle_1 = 0, \quad j = 1, \dots, p.$$

This holds for every $\chi \in X_1$ and therefore, since $\langle \cdot, \cdot \rangle_1$ is nondegenerate, we have $K'g_j = 0$, $j = 1, \dots, p$.

Note, that the proof also covers the case $m = 0$ when $I - A$ and $I - A'$ are bijective. In this case there are $p = n$ conditions of the form (5.15) with $\rho_{jk} = \delta_{jk}$, $j, k = 1, \dots, n$. \square

As in Chapter 4 our analysis includes the special case of the canonical dual systems $\langle X_1, X_1^* \rangle$ and $\langle X_2, X_2^* \rangle$ with the dual spaces X_1^* and X_2^* of X_1 and X_2 . In this setting the solvability conditions of Theorem 5.6 usually are referred to as *normal solvability* of the operator K and in this case the results of Theorem 5.6 were first obtained by Atkinson [1]. For the reasons already mentioned in Chapter 4 our more general setting again seems to be more appropriate for the discussion of integral equations. This will become obvious from the various examples discussed in the following chapters which also will include the classical results by Noether [1] on singular integral equations with Hilbert type kernels. For convenience we reformulate Theorem 5.6 into the form of solvability conditions.

Corollary 5.7. *Under the assumptions of Theorem 5.6 each of the homogeneous equations*

$$K\varphi = 0 \quad \text{and} \quad K'\psi = 0$$

has at most a finite number of linearly independent solutions. The inhomogeneous equations

$$K\varphi = f \quad \text{and} \quad K'\psi = g$$

are solvable if and only if the conditions

$$\langle f, \psi \rangle_2 = 0 \quad \text{and} \quad \langle \varphi, g \rangle_1 = 0$$

are satisfied for all solutions of the homogeneous equations

$$K'\psi = 0 \quad \text{and} \quad K\varphi = 0,$$

respectively.

5.3 Index

We conclude this chapter by introducing the concept of the index of an operator.

Definition 5.8. Let $\langle X_1, Y_1 \rangle_1$ and $\langle X_2, Y_2 \rangle_2$ be two dual systems and assume $K : X_1 \rightarrow X_2$ and $K' : Y_2 \rightarrow Y_1$ to be bounded adjoint operators with finite dimensional nullspaces $N(K)$ and $N(K')$. Then the number

$$\text{ind } K := \dim N(K) - \dim N(K')$$

is called the index of the operator K .

Theorem 5.9. Under the assumptions of Theorem 5.6 there holds

$$\text{ind } K_1 K_2 = \text{ind } K_1 + \text{ind } K_2.$$

Proof: For the sake of brevity we confine ourselves to the case of a dual system $\langle X, Y \rangle$ and two operators $K_1, K_2 : X \rightarrow X$ which satisfy the assumptions of Theorem 5.6, i.e., together with their adjoint operators $K'_1, K'_2 : Y \rightarrow Y$ they possess regularizers.

Denote $m_j := \dim N(K_j)$, $m'_j := \dim N(K'_j)$, $j = 1, 2$, and choose bases of the nullspaces

$$\begin{aligned} N(K_j) &= \text{span}\{\varphi_{j,1}, \dots, \varphi_{j,m_j}\}, \\ N(K'_j) &= \text{span}\{\psi_{j,1}, \dots, \psi_{j,m'_j}\}, \end{aligned}$$

$j = 1, 2$. Let $\varphi \in N(K_1 K_2)$. Then $K_2 \varphi \in N(K_1)$, i.e.,

$$K_2 \varphi = \sum_{i=1}^{m_1} \alpha_i \varphi_{1,i}.$$

By Theorem 5.6 this equation is solvable if and only if

$$(5.16) \quad \sum_{i=1}^{m_1} \alpha_i \langle \varphi_{1,i}, \psi_{2,k} \rangle = 0, \quad k = 1, \dots, m'_2.$$

By r we denote the rank of the $m_1 \times m_2$ -matrix $\langle \varphi_{1,i}, \psi_{2,k} \rangle$. Then the solution space of (5.16) has dimension $m_1 - r$. Therefore

$$\dim N(K_1 K_2) = \dim N(K_2) + m_1 - r = m_2 + m_1 - r.$$

Similarly, let $\psi \in N(K'_2 K'_1)$. Then $K'_1 \psi \in N(K'_2)$, i.e.,

$$K'_1 \psi = \sum_{k=1}^{m'_2} \beta_k \psi_{2,k}.$$

This equation is solvable if and only if

$$(5.17) \quad \sum_{k=1}^{m'_2} \beta_k \langle \varphi_{1,i}, \psi_{2,k} \rangle = 0, \quad i = 1, \dots, m_1.$$

The solution space of (5.17) has dimension $m'_2 - r$. Therefore

$$\dim N(K'_1 K'_2) = \dim N(K'_1) + m'_2 - r = m'_1 + m'_2 - r.$$

Combining the two results yields

$$\text{ind}(K_1 K_2) = (m_1 - m'_1) + (m_2 - m'_2) = \text{ind} K_1 + \text{ind} K_2.$$

Corollary 5.10. *Under the assumptions of Theorem 5.6 the index is stable under compact perturbations, that is, it for compact operators C there holds*

$$\text{ind}(K + C) = \text{ind} K.$$

Proof: Let R regularize K . Then

$$\text{ind} R + \text{ind} K = \text{ind} RK = 0$$

since $RK = I - A$ where A is compact. Now by the first Fredholm Theorem 4.14, we have $\text{ind}(I - A) = 0$. For a compact operator C the operator R also regularizes $K + C$. Therefore

$$\text{ind}(K + C) = -\text{ind} R = \text{ind} K$$

and the proof is complete. \square

For the history of the development of the notion of the index of an operator we refer to Dieudonné [2].

Of course, this chapter can provide only a first glance into the theory of singular operators. For a detailed study – in the dual system $\langle X, X^* \rangle$ – we refer to the monograph by Michlin and Prössdorf [1].

Problems

- 5.1. Show that the transformations of the Volterra integral equation of the first kind (3.1) into the Volterra equations of the second kind (3.2) and (3.3) can be interpreted as regularizations from the left and from the right, respectively.

Hint: Use the space $C^1[a, b]$ of continuously differentiable functions furnished with the norm $\|\varphi\|_1 := \|\varphi\|_\infty + \|\varphi'\|_\infty$.

- 5.2. Use the setting of Problem 4.1 to show that for the validity of Theorem 5.6 the bilinear form necessarily must be nondegenerate.

- 5.3. Convince yourself where in the proof of Theorem 5.6 use is made of the fact that the operators K and K' possess regularizers from the left and from the right.

- 5.4. Let X_1, X_2 be Banach spaces, let $K : X_1 \rightarrow X_2$ be a bounded operator and let $R : X_2 \rightarrow X_1$ be a left (right) regularizer of K . Show that for all operators $C : X_1 \rightarrow X_2$ with $\|C\| < \|R\|$ the operator $A + C$ has a left (right) regularizer.

- 5.5. Use Problem 5.4. to show that in Banach spaces under the assumptions of Theorem 5.6 the index is stable under small perturbations, that is, there exists a number γ (depending on K and K') such that

$$\text{ind}(K + C) = \text{ind} K$$

for all operators C with adjoint C' satisfying $\|C\|, \|C'\| < \gamma$ (cf. Dieudonné [1], Atkinson [1]).

6. Potential Theory

The solution of boundary value problems for partial differential equations is one of the most important field of applications for integral equations. About a century ago the systematic development of the theory of integral equations was initiated by the treatment of boundary value problems and there has been an ongoing fruitful interaction between these two areas of applied mathematics. It is the aim of this chapter to introduce the main ideas of this field by studying the basic boundary value problems of potential theory. For the sake of simplicity we shall confine our presentation to the case of two and three space dimensions. The extension to more than three dimensions is straightforward. As we shall see, the treatment of the boundary integral equations for the potential theoretic boundary value problems delivers an instructive example for the application of the Fredholm alternative since its both cases occur in a natural way.

6.1 Harmonic Functions

We begin with a brief outline of the basic properties of harmonic functions going back to the early development of potential theory at the beginning of the 19th century with contributions by Dirichlet, Gauss, Green, Riemann and Weierstrass. For a more comprehensive study of potential theory we refer to Martensen [2] or Mikhlin [1].

Definition 6.1. *A twice continuously differentiable real valued function u , defined on a domain $D \subset \mathbb{R}^m$, $m = 2, 3$, is called harmonic if it satisfies Laplace's equation*

$$\Delta u = 0 \quad \text{in } D$$

where

$$\Delta u := \sum_{j=1}^m \frac{\partial^2 u}{\partial x_j^2}.$$

Harmonic functions describe time independent temperature distributions, potentials of electrostatic and magnetostatic fields and velocity potentials of incompressible irrotational fluid flows.

There is a close connection between harmonic functions in \mathbb{R}^2 and holomorphic functions in \mathbb{C} . From the Cauchy-Riemann equations we readily observe that both the real and imaginary part of a holomorphic function $f(z) = u(x_1, x_2) + iv(x_1, x_2)$, $z = x_1 + ix_2$, are harmonic functions.

Most of the basic properties of harmonic functions can be deduced from the fundamental solution which is introduced in the following theorem which

is obtained by straightforward differentiation. Recall that by $|x|$ we denote the Euclidean norm of a point $x \in \mathbb{R}^m$.

Theorem 6.2. *The function*

$$\Phi(x, y) := \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x - y|}, & m = 2, \\ \frac{1}{4\pi} \frac{1}{|x - y|}, & m = 3, \end{cases}$$

is called fundamental solution of Laplace's equation. For fixed $y \in \mathbb{R}^m$ it is harmonic in $\mathbb{R}^m \setminus \{y\}$.

By $C^k(D)$ for $k \in \mathbb{N}$ we denote the linear space of real or complex valued functions defined on the domain D which are k times continuously differentiable. By $C^k(\bar{D})$ we denote the subspace of all functions in $C^k(D)$ which together with all their derivatives up to order k can be extended continuously from D into \bar{D} . In this chapter, we mostly deal with real valued functions but with proper interpretation our results remain valid for complex valued functions. One of the basic tools in studying partial differential equations is provided by *Green's integral theorems*. From the end of Chapter 2 we recall what is meant by saying a bounded domain D belongs to class C^k , $k \in \mathbb{N}$. On occasion, we will express the property of a domain D to be of class C^k also by saying that its boundary ∂D is of class C^k . For the purpose of this introduction we confine ourselves to domains of class C^1 . Recall that by (\cdot, \cdot) we denote the usual scalar product in \mathbb{R}^m .

Theorem 6.3 (Green's Theorem). *Let D be a bounded domain of class C^1 and let n denote the unit normal vector to the boundary ∂D directed into the exterior of D . For $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$ there holds the first Green's theorem*

$$(6.1) \quad \int_D \{u \Delta v + (\operatorname{grad} u, \operatorname{grad} v)\} dx = \int_{\partial D} u \frac{\partial v}{\partial n} ds.$$

For $u, v \in C^2(\bar{D})$ there holds the second Green's theorem

$$(6.2) \quad \int_D (u \Delta v - v \Delta u) dx = \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds.$$

Proof: We apply Gauss' divergence theorem

$$\int_D \operatorname{div} A dx = \int_{\partial D} (n, A) ds$$

to the vector field $A \in C^1(\bar{D})$ defined by $A := u \operatorname{grad} v$ and use

$$\operatorname{div}(u \operatorname{grad} v) = (\operatorname{grad} u, \operatorname{grad} v) + u \operatorname{div} \operatorname{grad} v$$

to establish (6.1). To obtain (6.2) we interchange u and v and then subtract. \square

Note that our regularity assumptions on D are sufficient conditions for the validity of Gauss' and Green's theorems and can be weakened. In particular, the boundary can be allowed to have edges and corners. For a detailed study see for instance Mikhlin [1] or Müller [1].

Corollary 6.4. *Let $u \in C^2(\bar{D})$ be harmonic in D . Then*

$$(6.3) \quad \int_{\partial D} \frac{\partial u}{\partial n} ds = 0.$$

Proof: Choose $v = 1$.

Theorem 6.5 (Green's Formula). *Let D be as in Theorem 6.3 and let $u \in C^2(\bar{D})$ be harmonic in D . Then*

$$(6.4) \quad u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right\} ds(y), \quad x \in D.$$

Proof: We circumscribe the arbitrary fixed point $x \in D$ with a sphere $\Omega(x; r) := \{y \in \mathbb{R}^m : |x - y| = r\}$ and direct the unit normal n to $\Omega(x; r)$ into the interior of $\Omega(x; r)$. Now we apply the second Green's theorem (6.2) to the harmonic functions u and $\Phi(x, \cdot)$ in the domain $\{y \in D : |x - y| > r\}$ to obtain

$$\int_{\partial D + \Omega(x; r)} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y) = 0.$$

Since on $\Omega(x; r)$ we have

$$\text{grad}_y \Phi(x, y) = \frac{n(y)}{\omega_m r^{m-1}},$$

where $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, a straightforward calculation, using the mean value theorem and (6.3), shows that

$$\lim_{r \rightarrow 0} \int_{\Omega(x; r)} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y) = u(x),$$

whence (6.4) follows. \square

From the Green's formula we immediately conclude that harmonic functions are analytic functions of their independent variables.

Theorem 6.6. *Harmonic functions are analytic.*

Proof: We make use of the fact that each holomorphic function of several complex variables, that is, a function satisfying the Cauchy–Riemann equations with respect to each of the complex variables is also analytic, that is, it has a local power series expansion and vice versa (cf. Gilbert [1]). Our theorem now follows from the observation that the fundamental solution $\Phi(x, y)$ is an analytic function of the cartesian coordinates x_j , $j = 1, \dots, m$, of x and the fact

that the integrands in (6.4) are continuous with respect to y if x is contained in a compact subset of D . Therefore the Cauchy–Riemann equations for u can be verified by differentiating with respect to x under the integral. \square

It follows from Theorem 6.6 that a harmonic function which vanishes in an open subset of its domain of definition must vanish identically.

Theorem 6.7 (Mean Value Theorem). *Let u be a harmonic function in an open ball $B(x; R) = \{y \in \mathbb{R}^m : |y - x| < R\}$ and continuous in the closure $B[x; R]$. Then*

$$(6.5) \quad u(x) = \frac{m}{\omega_m R^m} \int_{|y-x|\leq R} u(y) dy = \frac{1}{\omega_m R^{m-1}} \int_{|y-x|=R} u(y) ds(y),$$

that is, the value of u at the center of the ball is equal to the integral mean values over both the ball and its surface ($\omega_2 = 2\pi$, $\omega_3 = 4\pi$).

Proof: For each $0 < \rho < R$ we have $u \in C^2(B[x; \rho])$ and can apply (6.3) and (6.4) with the result

$$(6.6) \quad u(x) = \frac{1}{\omega_m \rho^{m-1}} \int_{|y-x|=\rho} u(y) ds(y)$$

from which the second mean value formula follows by passing to the limit $\rho \rightarrow R$. Multiplying (6.6) by ρ^{m-1} and integrating with respect to ρ from 0 to R we obtain the first mean value formula.

Theorem 6.8 (Maximum-Minimum Principle). *A harmonic function on a domain cannot assume its maximum or its minimum unless it is constant.*

Proof: It suffices to carry out the proof for the maximum. Let u be a harmonic function in the domain D and assume that it attains its maximum value in D , that is, the set $D_M := \{x \in D : u(x) = M\}$ where $M := \sup_{x \in D} u(x)$ is not empty. Since u is continuous D_M is closed relative to D . Let x be any point in D_M and apply the mean value Theorem 6.7 to the harmonic function $M - u$ in a ball $B(x; R)$ with $B[x; R] \subset D$. Then

$$0 = M - u(x) = \frac{m}{\omega_m R^m} \int_{|y-x|\leq R} \{M - u(y)\} dy$$

so that $u = M$ in $B(x; R)$. Therefore D_M is open relative to D . Hence $D = D_M$ which means u is constant in D . Note that our proof does not use the analyticity of u .

Corollary 6.9. *Let D be a bounded domain and let u be harmonic in D and continuous in \bar{D} . Then u attains both its maximum and its minimum on the boundary.*

6.2 Boundary Value Problems: Uniqueness

The Green's formula (6.4) represents any harmonic function in terms of its boundary values and its normal derivative on the boundary, the so called *Cauchy data*. In the subsequent analysis we shall see that a harmonic function is already completely determined by either its boundary values or, up to a constant, its normal derivative alone.

Let $D_- \subset \mathbb{R}^m$ be a bounded domain of class C^2 . By $\Gamma := \partial D_-$ we denote its boundary and by $D_+ := \mathbb{R}^m \setminus \bar{D}_-$ its open complement. We assume the unit normal n to the boundary to be directed into the exterior D_+ . For the sake of simplicity for the rest of this chapter we assume that the boundary Γ is connected.

Interior Dirichlet Problem. *Find a function $u \in C^2(D_-) \cap C(\bar{D}_-)$ which is harmonic in D_- and satisfies the boundary condition*

$$u = f \quad \text{on } \Gamma$$

where f is a given continuous function.

Interior Neumann Problem. *Find a function $u \in C^2(D_-) \cap C(\bar{D}_-)$ which is harmonic in D_- and satisfies the boundary condition*

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma$$

in the sense

$$\lim_{h \rightarrow +0} (n(x), \operatorname{grad} u(x - hn(x))) = g(x), \quad x \in \Gamma,$$

of uniform convergence on Γ where g is a given continuous function.

Exterior Dirichlet Problem. *Find a function $u \in C^2(D_+) \cap C(\bar{D}_+)$ which is harmonic in D_+ and satisfies the boundary condition*

$$u = f \quad \text{on } \Gamma$$

where f is a given continuous function. For $|x| \rightarrow \infty$ it is required that

$$u(x) = O(1), \quad m = 2, \quad \text{and} \quad u(x) = o(1), \quad m = 3,$$

uniformly for all directions $x/|x|$.

Exterior Neumann Problem. *Find a function $u \in C^2(D_+) \cap C(\bar{D}_+)$ which is harmonic in D_+ and satisfies the boundary condition*

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma$$

in the sense of uniform convergence on Γ where g is a given continuous function. For $|x| \rightarrow \infty$ it is required that $u(x) = o(1)$ uniformly for all directions $x/|x|$.

Note that for the exterior problems we impose that $u(\infty) = 0$ with the exception of the Dirichlet problem in \mathbb{R}^2 where u is only required to be bounded.

These boundary value problems carry the names of Dirichlet, who made important contributions to potential theory, and Neumann, who gave the first rigorous existence proof (see Problem 6.5). From the numerous applications we mention:

- (1) determine the stationary temperature distribution in a heat conducting body from the temperature on the boundary or from the heat flux through the boundary;
- (2) find the potential of the electrostatic field in the exterior of a perfect conductor;
- (3) find the velocity potential of an incompressible irrotational flow around an obstacle.

Our aim is to establish that each of these potential theoretic boundary value problems has a unique solution depending continuously on the given boundary data, i.e., they are well-posed in the sense of Hadamard (see Section 15.1).

In our uniqueness proofs we need to apply the Green's Theorems 6.3 and 6.5. Since for solutions to the boundary value problems we do not assume differentiability up to the boundary we introduce the concept of *parallel surfaces*. These are described by

$$\Gamma_h := \{z = x + hn(x) : x \in \Gamma\}$$

where the positive or negative parameter h denotes the distance of Γ_h from the generating surface Γ . Because Γ is assumed to be of class C^2 , we observe that Γ_h is of class C^1 . Let $x(u) = (x_1(u), \dots, x_m(u))$, $u = (u_1, \dots, u_{m-1})$, be a regular parametric representation of a surface patch of Γ . Then straightforward differential geometric calculations show that the determinants

$$g(u) := \det \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right) \quad \text{and} \quad g(u; h) := \det \left(\frac{\partial z}{\partial u_i}, \frac{\partial z}{\partial u_j} \right)$$

are related by

$$g(u; h) = g(u)\{1 - 2hH(u) + h^2K(u)\}^2$$

where H and K denote the mean and Gaussian curvature of Γ . This verifies that the parallel surfaces are well defined provided the parameter h is restricted to be sufficiently small to ensure that the invariant $1 - 2hH + h^2K$ remains positive. In particular the surface elements ds on Γ and ds_h on Γ_h are related through

$$ds_h(z) = \{1 - 2hH + h^2K\}ds(x).$$

For a more detailed analysis of these parallel surfaces see Martensen [2].

Theorem 6.10. *Both the interior and the exterior Dirichlet problem have at most one solution.*

Proof: The difference $u := u_1 - u_2$ of two solutions to the Dirichlet problem is a harmonic function continuous up to the boundary satisfying the homogeneous boundary condition $u = 0$ on Γ . Then, from the maximum-minimum principle Corollary 6.9 we obtain $u = 0$ in D_- for the interior problem, and observing $u(x) = o(1)$, $|x| \rightarrow \infty$, we also obtain $u = 0$ in D_+ for the exterior problem in three dimensions.

For the uniqueness of the exterior Dirichlet problem in two dimensions and later for the uniqueness of the exterior Neumann problem we first have to investigate the behaviour of bounded harmonic functions at infinity. We begin with showing that the boundedness $|u(x)| \leq M$ for all $x \in D_+$ implies that

$$(6.7) \quad |\operatorname{grad} u(x)| \leq \frac{M^*}{|x|}$$

for all $|x| \geq R_0$ with some constants R_0 and M^* . Here, without loss of generality, we assume the origin to be contained in D_- . Choose R_0 large enough such that $y \in D_+$ for all $|y| \geq R_0/2$. Then for fixed x with $|x| \geq R_0$ we apply the mean value Theorem 6.7 and the Gauss integral theorem to obtain

$$\operatorname{grad} u(x) = \frac{m}{\omega_m R^m} \int_{|x-y| \leq R} \operatorname{grad} u(y) dy = \frac{m}{\omega_m R^m} \int_{|x-y|=R} n(y) u(y) ds(y),$$

where we choose the radius to be $R = |x|/2$. We can estimate

$$|\operatorname{grad} u(x)| \leq \frac{mM}{R} = \frac{2mM}{|x|}$$

which establishes the desired inequality (6.7).

For $x \in D_+$ now choose $R > |x|$ and denote by Ω_R the sphere of radius R with center at the origin and outward unit normal n . Furthermore choose a parallel surface $\Gamma_+ := \{x + hn(x) : x \in \Gamma\}$ with sufficiently small $h > 0$ separating x from the boundary Γ . Then by the Green's formula (6.4), applied in the domain between Γ_+ and Ω_R , we see that

$$\begin{aligned} u(x) &= \int_{\Gamma_+} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y) \\ &\quad + \int_{\Omega_R} \left\{ \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right\} ds(y). \end{aligned}$$

For the gradient of the second integral, making use of (6.7), we estimate

$$\begin{aligned} \left| \operatorname{grad} \int_{\Omega_R} \frac{\partial u}{\partial n}(y) \Phi(x, y) ds(y) \right| &\leq \frac{M^*}{\omega_m R} \int_{\Omega_R} \frac{1}{|x - y|^{m-1}} ds(y) \\ &\leq \frac{M^* R^{m-2}}{(R - |x|)^{m-1}} \rightarrow 0, \quad R \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} \left| \operatorname{grad} \int_{\Omega_R} u(y) \frac{\partial}{\partial n(y)} \Phi(x, y) ds(y) \right| &\leq \frac{2M(m+1)}{\omega_m} \int_{\Omega_R} \frac{1}{|x-y|^m} ds(y) \\ &\leq \frac{2M(m+1)R^{m-1}}{(R-|x|)^m} \rightarrow 0, \quad R \rightarrow \infty. \end{aligned}$$

Hence,

$$\operatorname{grad} u(x) = \operatorname{grad} \int_{\Gamma_+} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y)$$

and

$$u(x) = \int_{\Gamma_+} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y) + u_\infty$$

with some constant u_∞ . From this it is easily seen that

$$(6.8) \quad u(x) + \Phi(x, 0) \int_{\Gamma_+} \frac{\partial u}{\partial n} ds = u_\infty + O\left(\frac{1}{|x|^{m-1}}\right), \quad |x| \rightarrow \infty,$$

and

$$(6.9) \quad \operatorname{grad} u(x) + \operatorname{grad} \Phi(x, 0) \int_{\Gamma_+} \frac{\partial u}{\partial n} ds = O\left(\frac{1}{|x|^m}\right), \quad |x| \rightarrow \infty,$$

uniformly for all directions.

In two dimensions, since $2\pi\Phi(x, 0) = -\ln|x|$ is unbounded as $|x| \rightarrow \infty$, it follows that

$$(6.10) \quad \int_{\Gamma_+} \frac{\partial u}{\partial n} ds = 0$$

for each bounded harmonic function u in D_+ , whence

$$u(x) = u_\infty + O\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty.$$

By the second Green's theorem (6.2), applied to u and Φ in some annulus $R \leq |y| \leq R^*$, we see that

$$\begin{aligned} 0 &= \int_{|y|=R} \left\{ \frac{\partial u}{\partial n}(y) \ln \frac{1}{|y|} - u(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|y|} \right\} ds(y) \\ &\quad - \int_{|y|=R^*} \left\{ \frac{\partial u}{\partial n}(y) \ln \frac{1}{|y|} - u(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|y|} \right\} ds(y). \end{aligned}$$

From this, in view of (6.3) and (6.10), it follows that

$$\frac{1}{R} \int_{|y|=R} u(y) ds(y) = \frac{1}{R^*} \int_{|y|=R^*} u(y) ds(y),$$

and letting $R^* \rightarrow \infty$, we arrive at the mean value property at infinity

$$(6.11) \quad u_\infty = \frac{1}{2\pi R} \int_{|y|=R} u(y) ds(y).$$

Now we are able to return to uniqueness for the exterior Dirichlet problem in \mathbb{R}^2 . By the maximum-minimum principle Theorem 6.8 the supremum and the infimum of the bounded harmonic function u are either attained on the boundary or equal to u_∞ . When the maximum and minimum both are assumed on the boundary then from the homogeneous boundary condition we immediately have $u = 0$ in D_+ . When the supremum is equal to u_∞ then from $u(x) \leq u_\infty$ for all $x \in D_+$ and (6.11) we observe that $u = u_\infty$ in the exterior of some sphere. Now, we can apply the maximum principle to see that $u = u_\infty$ in all of D_+ and the homogeneous boundary condition finally implies $u = 0$ in D_+ . The case where the infimum is equal to u_∞ is settled by the same argument.

Theorem 6.11. *Two solutions of the interior Neumann problem can differ only by a constant. The exterior Neumann problem has at most one solution.*

Proof: The difference $u := u_1 - u_2$ of two solutions for the Neumann problem is a harmonic function continuous up to the boundary satisfying the homogeneous boundary condition $\partial u / \partial n = 0$ on Γ in the sense of uniform convergence. For the interior problem, suppose u is not constant in D_- . Then there exists some closed ball B contained in D_- such that

$$I := \int_B |\operatorname{grad} u|^2 dx > 0.$$

From the first Green's theorem (6.1), applied to the interior D_-^* of some parallel surface $\Gamma_- := \{x - hn(x) : x \in \Gamma\}$ with sufficiently small $h > 0$, we derive

$$I \leq \int_{D_-^*} |\operatorname{grad} u|^2 dx = \int_{\Gamma_-} u \frac{\partial u}{\partial n} ds.$$

Passing to the limit $h \rightarrow 0$, we now arrive at the contradiction $I \leq 0$.

For the exterior problem, from (6.3), (6.8) and (6.9) and the homogeneous boundary condition we first observe that

$$u(x) = O\left(\frac{1}{|x|^{m-1}}\right), \quad \operatorname{grad} u(x) = O\left(\frac{1}{|x|^m}\right), \quad |x| \rightarrow \infty,$$

uniformly for all directions. Now assume that $\operatorname{grad} u \neq 0$ in D_+ . Then, again, there exists some closed ball B contained in D_+ such that

$$I := \int_B |\operatorname{grad} u|^2 dx > 0.$$

From the first Green's theorem, applied to the domain $D_{+,R}^*$ between some parallel surface $\Gamma_+ := \{x + hn(x) : x \in \Gamma\}$ with sufficiently small $h > 0$ and some sufficiently large sphere Ω_R with radius R , there follows

$$I \leq \int_{D_{+,R}^*} |\operatorname{grad} u|^2 dx = \int_{\Omega_R} u \frac{\partial u}{\partial n} ds - \int_{\Gamma_+} u \frac{\partial u}{\partial n} ds.$$

Letting $R \rightarrow \infty$ and $h \rightarrow 0$, we arrive at the contradiction $I \leq 0$. Therefore, $u = \text{const}$ in D_+ and this constant must be zero since $u(\infty) = 0$. \square

From the proofs it is obvious that our uniqueness results remain valid under weaker regularity conditions on the boundary. Uniqueness for the Dirichlet problem via the maximum-minimum principle needs no regularity of the boundary, and uniqueness for the Neumann problem holds for those boundaries for which Green's integral theorem is valid. We have formulated the boundary value problems for C^2 -boundaries, since we shall establish existence of solutions under these conditions.

6.3 Surface Potentials

For fixed $y \in \mathbb{R}^m$ the fundamental solution $u = \Phi(\cdot, y)$ represents the potential of a unit *point source* located at the point y , i.e., $\text{grad}_x \Phi(x, y)$ gives the force-field of this point source acting at the point x . If we have two point sources with strength M and $-M$ at the points $y + he$ and y where e is a unit vector and $h > 0$, then for the corresponding potential by the mean value theorem we can write

$$v(x) = M\{\Phi(x, y + he) - \Phi(x, y)\} = Mh(e, \text{grad}_y \Phi(x, y + \delta he))$$

where $0 < \delta < 1$. Passing to the limit $h \rightarrow 0$ such that $Mh = N$ remains constant we obtain the potential

$$v(x) = N(e, \text{grad}_y \Phi(x, y))$$

of a *dipole* with dipole moment Ne located at the point y . For the treatment of the Dirichlet and Neumann problems we introduce surface potentials by distributing point sources and dipoles in a normal direction along the boundary Γ .

Definition 6.12. *Given a function $\varphi \in C(\Gamma)$ the functions*

$$(6.12) \quad u(x) := \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y), \quad x \in \mathbb{R}^m \setminus \Gamma,$$

and

$$(6.13) \quad v(x) := \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \mathbb{R}^m \setminus \Gamma,$$

are called single-layer and double-layer potential with density φ . In two dimensions, occasionally, for obvious reasons we will write logarithmic single-layer and logarithmic double-layer potential.

Since for points x not lying on the boundary Γ we can differentiate under the integral, we see that the single- and double-layer potentials represent harmonic functions in D_+ and D_- . For the investigation of the boundary value

problems we need the following properties of these potentials for points x on the boundary where the integrals become singular.

Theorem 6.13. *Let Γ be of class C^2 and $\varphi \in C(\Gamma)$. Then the single-layer potential u with density φ is continuous throughout \mathbb{R}^m . On the boundary there holds*

$$(6.14) \quad u(x) = \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y), \quad x \in \Gamma,$$

and

$$(6.15) \quad \frac{\partial u_{\pm}}{\partial n}(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y) \mp \frac{1}{2} \varphi(x), \quad x \in \Gamma,$$

where

$$\frac{\partial u_{\pm}}{\partial n}(x) := \lim_{h \rightarrow +0} (n(x), \operatorname{grad} u(x \pm hn(x)))$$

is to be understood in the sense of uniform convergence on Γ and where the integrals exist as improper integrals. The double-layer potential v with density φ can be continuously extended from D_+ to \bar{D}_+ and from D_- to \bar{D}_- with limiting values

$$(6.16) \quad v_{\pm}(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) \pm \frac{1}{2} \varphi(x), \quad x \in \Gamma,$$

where

$$v_{\pm}(x) := \lim_{h \rightarrow +0} v(x \pm hn(x))$$

and where the integrals exist as improper integrals. Furthermore

$$(6.17) \quad \lim_{h \rightarrow +0} \left\{ \frac{\partial v}{\partial n}(x + hn(x)) - \frac{\partial v}{\partial n}(x - hn(x)) \right\} = 0, \quad x \in \Gamma,$$

uniformly on Γ .

The proofs of these so called *jump relations* of potential theory are very elaborate and lengthy and therefore beyond the aim of this introduction. The reader interested in the details is referred to Colton and Kress [1], Leis [2] or Mikhlin [1]. We also refer to the following Section 7.5 where we will investigate the regularity properties of logarithmic single-layer and double-layer potentials with uniformly Hölder continuous densities. We illustrate the jump relations by the following

Example 6.14. *For the double-layer potential with constant density there holds*

$$(6.18) \quad 2 \int_{\Gamma} \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = \begin{cases} -2, & x \in D_-, \\ -1, & x \in \Gamma, \\ 0, & x \in D_+. \end{cases}$$

This follows for $x \in D_+$ from (6.3) applied to $\Phi(x, \cdot)$ and for $x \in D_-$ from (6.4) applied to $u = 1$ in D_- . Then the result for $x \in \Gamma$ is a consequence of the

jump relation (6.16). It also can be derived explicitly by excluding $x \in \Gamma$ from the integration by circumscribing it with a sphere. Let $H(x; r)$ be the part of the sphere with radius r and center at x lying within D_- and let $\Gamma(x; r)$ be the part of Γ lying in the exterior of this sphere. Let the normal to $H(x; r)$ be directed towards the center x . Then, by (6.3) applied to $\Phi(x, \cdot)$, we have

$$\int_{\Gamma(x; r) + H(x; r)} \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = 0$$

and from

$$\lim_{r \rightarrow 0} 2 \int_{H(x; r)} \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = \lim_{r \rightarrow 0} \frac{2}{\omega_m r^{m-1}} \int_{H(x; r)} ds(y) = 1$$

the result follows. \square

We also omit the proof of the following property of C^2 -surfaces which can be found in Colton and Kress [1] or in Martensen [2].

Theorem 6.15. *Let Γ be of class C^2 . Then there exists a positive constant L such that*

$$|(n(x), x - y)| \leq L|x - y|^2$$

for all $x, y \in \Gamma$.

This inequality expresses the fact that the vector $x - y$ for x close to y is almost orthogonal to the normal vector $n(x)$.

For our study of boundary integral equations for the Dirichlet and Neumann problem we introduce two integral operators $K, K' : C(\Gamma) \rightarrow C(\Gamma)$ by

$$(6.19) \quad (K\varphi)(x) := 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \Gamma,$$

and

$$(6.20) \quad (K'\psi)(x) := 2 \int_{\Gamma} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y), \quad x \in \Gamma.$$

Because of Theorem 6.15 we have the estimate

$$\omega_m \left| \frac{\partial}{\partial n(y)} \Phi(x, y) \right| = \frac{|(n(y), x - y)|}{|x - y|^m} \leq \frac{L}{|x - y|^{m-2}}, \quad x \neq y,$$

i.e., K and K' have weakly singular kernels and therefore are compact by Theorem 2.22. Note that in two dimensions for our C^2 -boundary the kernels of K and K' actually turn out to be continuous (see Problem 6.1). As seen by interchanging the order of integration, K and K' are adjoint with respect to the dual system $\langle C(\Gamma), C(\Gamma) \rangle$ defined by

$$(6.21) \quad \langle \varphi, \psi \rangle := \int_{\Gamma} \varphi \psi ds, \quad \varphi, \psi \in C(\Gamma).$$

Theorem 6.16. *The operators $I - K$ and $I - K'$ have trivial nullspaces*

$$N(I - K) = N(I - K') = \{0\}.$$

The nullspaces of the operators $I + K$ and $I + K'$ have dimension one

$$N(I + K) = \text{span}\{1\}, \quad N(I + K') = \text{span}\{\psi_0\}$$

with

$$\langle 1, \psi_0 \rangle = \int_{\Gamma} \psi_0 ds \neq 0,$$

that is, the Riesz number is one.

Proof: Let φ be a solution to the homogeneous equation $\varphi - K\varphi = 0$ and define a double-layer potential v by (6.13). Then by (6.16) we have $2v_- = K\varphi - \varphi = 0$ and from the uniqueness of the interior Dirichlet problem (Theorem 6.10) it follows that $v = 0$ in D_- . From (6.17) we see that $\partial v_+ / \partial n = 0$ on Γ and since $v(x) = o(1)$, $|x| \rightarrow \infty$, from the uniqueness for the exterior Neumann problem (Theorem 6.11) we find that $v = 0$ in D_+ . Hence, from (6.16) we deduce $\varphi = v_+ - v_- = 0$ on Γ . Thus $N(I - K) = \{0\}$ and, by the Fredholm alternative, there follows $N(I - K') = \{0\}$.

Now let φ be a solution to $\varphi + K\varphi = 0$ and again define v by (6.13). Then by (6.16) we have $2v_+ = K\varphi + \varphi = 0$ on Γ . Since $v(x) = o(1)$, $|x| \rightarrow \infty$, from the uniqueness of the exterior Dirichlet problem it follows that $v = 0$ in D_+ . From (6.17) we see that $\partial v_- / \partial n = 0$ on Γ and from the uniqueness for the interior Neumann problem we find that $v = \text{const}$ in D_- . Hence, from (6.16) we deduce $\varphi = \text{const}$ on Γ . Therefore, $N(I + K) \subset \text{span}\{1\}$, and since by (6.18) we have $1 + K1 = 0$ it follows that $N(I + K) = \text{span}\{1\}$.

By the Fredholm alternative $N(I + K')$ also has dimension one. Therefore $N(I + K') = \text{span}\{\psi_0\}$ with some function $\psi_0 \in C(\Gamma)$ which does not vanish identically. Assume that $\langle 1, \psi_0 \rangle = 0$ and define a single-layer potentials u_0 with density ψ_0 . Then by (6.14) and (6.15) we have $u_{0+} = u_{0-}$, $2\partial u_{0-} / \partial n = K'\psi_0 + \psi_0 = 0$ and

$$\frac{\partial u_{0+}}{\partial n} - \frac{\partial u_{0-}}{\partial n} = -\psi$$

in the sense of uniform convergence on Γ . From $\partial u_{0-} / \partial n = 0$ on Γ , by the uniqueness for the interior Neumann problem (Theorem 6.11), we conclude $u_0 = \text{const}$ in D_- . Assume that $\text{grad } u_0 \neq 0$ in D_+ and choose some closed ball B contained in D_+ with

$$I := \int_B |\text{grad } u_0|^2 dx > 0.$$

Then, by the Green's theorem (6.1), using $u_0 = \text{const}$ on Γ and $\langle 1, \psi_0 \rangle = 0$ we find

$$\begin{aligned} I &\leq \int_{\Omega_R} u_0 \frac{\partial u_0}{\partial n} ds - \int_{\Gamma} u_{0+} \frac{\partial u_{0+}}{\partial n} ds \\ &= \int_{\Omega_R} u_0 \frac{\partial u_0}{\partial n} ds - \int_{\Gamma} u_{0-} \frac{\partial u_{0-}}{\partial n} ds + \int_{\Gamma} u_0 \psi_0 ds = \int_{\Omega_R} u_0 \frac{\partial u_0}{\partial n} ds \end{aligned}$$

where Ω_R denotes a sphere with sufficiently large radius R . With the help of $\int_{\Gamma} \psi_0 ds = 0$ we see that

$$u_0(x) = O\left(\frac{1}{|x|^{m-1}}\right), \quad \text{grad } u_0(x) = O\left(\frac{1}{|x|^m}\right), \quad |x| \rightarrow \infty,$$

and therefore, passing to the limit $R \rightarrow \infty$, we arrive at the contradiction $I \leq 0$.

Hence, $u_0 = \text{const}$ in D_+ and because of $u_0(\infty) = 0$ this constant must be zero. But then we have $u_{0-} = 0$ on Γ whence $u_0 = 0$ in D_- follows by the uniqueness Theorem 6.10. Now, from the jump relation (6.15) we derive the contradiction $\psi_0 = 0$. Therefore, $\langle 1, \psi_0 \rangle \neq 0$. The statement on the Riesz number is a consequence of Problem 4.4.

6.4 Boundary Value Problems: Existence

Green's formula shows that each harmonic function can be represented as a combination of single- and double-layer potentials. For boundary value problems we try to find a solution in the form of one of these two potentials.

Theorem 6.17. *The double-layer potential*

$$(6.22) \quad u(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in D_-,$$

with continuous density φ is a solution of the interior Dirichlet problem provided φ is a solution of the integral equation

$$(6.23) \quad \varphi(x) - 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = -2f(x), \quad x \in \Gamma.$$

Proof: From Theorem 6.13.

Theorem 6.18. *The interior Dirichlet problem has a unique solution.*

Proof: The integral equation $\varphi - K\varphi = -2f$ of the interior Dirichlet problem is uniquely solvable by Theorem 3.4 since $N(I - K) = \{0\}$. \square

From Theorem 6.13, we see that in order to obtain an integral equation of the second kind for the Dirichlet problem it is crucial to seek the solution in the form of a double-layer potential rather than a single-layer potential which would lead to an integral equation of the first kind. Historically, this important observation goes back to Beer [1].

The double-layer potential approach (6.22) for the exterior Dirichlet problem leads to the integral equation $\varphi + K\varphi = 2f$ for the density φ . Since $N(I + K') = \text{span}\{\psi_0\}$, by the Fredholm alternative, this equation is solvable if and only if $\langle f, \psi_0 \rangle = 0$. Of course, for arbitrary boundary data f we cannot expect this condition to be satisfied. Therefore we modify our approach into the form of the following

Theorem 6.19. *The modified double-layer potential*

$$(6.24) \quad u(x) = \int_{\Gamma} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial n(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y), \quad x \in D_+,$$

with continuous density φ is a solution to the exterior Dirichlet problem provided φ is a solution of the integral equation

$$(6.25) \quad \varphi(x) + 2 \int_{\Gamma} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial n(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y) = 2f(x), \quad x \in \Gamma.$$

Here, we assume that the origin is contained in D_- .

Proof: From Theorem 6.13. Observe that u has the required behaviour for $|x| \rightarrow \infty$, namely $u(x) = O(1)$ if $m = 2$ and $u(x) = o(1)$ if $m = 3$.

Theorem 6.20. *The exterior Dirichlet problem has a unique solution.*

Proof: The integral operator $\tilde{K} : C(\Gamma) \rightarrow C(\Gamma)$ defined by

$$(\tilde{K}\varphi)(x) := 2 \int_{\Gamma} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial n(y)} + \frac{1}{|x|^{m-2}} \right\} ds(y), \quad x \in \Gamma,$$

is compact since the difference $\tilde{K} - K$ has continuous kernel. Let φ be a solution to the homogeneous equation $\varphi + \tilde{K}\varphi = 0$ and define u by (6.24). Then $2u = \tilde{K}\varphi + \varphi = 0$ on Γ and by the uniqueness of the exterior Dirichlet problem it follows that $u = 0$ in D_+ . Since

$$\frac{\partial \Phi(x, y)}{\partial n(y)} = O\left(\frac{1}{|x|^{m-1}}\right), \quad y \in \Gamma, \quad |x| \rightarrow \infty,$$

uniformly for all directions, we have the asymptotic behaviour

$$|x|^{m-2}u(x) = \int_{\Gamma} \varphi ds + O\left(\frac{1}{|x|}\right), \quad |x| \rightarrow \infty.$$

Since $u = 0$ in D_+ we deduce

$$(6.26) \quad \int_{\Gamma} \varphi ds = 0.$$

Therefore $\varphi + K\varphi = 0$, i.e., $\varphi \in N(I + K) = \text{span}\{1\}$ and from (6.26) we see that $\varphi = 0$. Now the existence of a unique solution to the integral equation (6.25) follows from the Riesz theory Theorem 3.4.

Theorem 6.21. *The single-layer potential*

$$(6.27) \quad u(x) = \int_{\Gamma} \psi(y) \Phi(x, y) ds(y), \quad x \in D_-,$$

with continuous density ψ is a solution of the interior Neumann problem provided ψ is a solution of the integral equation

$$(6.28) \quad \psi(x) + 2 \int_{\Gamma} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y) = 2g(x), \quad x \in \Gamma.$$

Proof: From Theorem 6.13.

Theorem 6.22. *The interior Neumann problem is solvable if and only if*

$$(6.29) \quad \int_{\Gamma} g ds = 0$$

is satisfied.

Proof: The necessity of condition (6.29) is a consequence of Green's theorem (6.3) applied to a solution u :

$$\int_{\Gamma} g ds = \lim_{h \rightarrow 0} \int_{\Gamma_-} g ds = \lim_{h \rightarrow 0} \int_{\Gamma_-} \frac{\partial u}{\partial n} ds = 0.$$

Here Γ_- denotes a parallel surface as in the proof of Theorem 6.11.

The sufficiency of condition (6.29) follows from the fact that by Theorem 6.16 it coincides with the solvability condition of Fredholm's alternative for the inhomogeneous integral equation $\psi + K'\psi = 2g$ of the interior Neumann problem.

Theorem 6.23. *The single-layer potential*

$$(6.30) \quad u(x) = \int_{\Gamma} \psi(y) \Phi(x, y) ds(y), \quad x \in D_+,$$

with continuous density ψ is a solution of the exterior Neumann problem provided ψ is a solution of the integral equation

$$(6.31) \quad \psi(x) - 2 \int_{\Gamma} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y) = -2g(x), \quad x \in \Gamma,$$

and, if $m = 2$, in addition satisfies

$$(6.32) \quad \int_{\Gamma} \psi ds = 0.$$

Proof: From Theorem 6.13. Observe that for $m = 2$ the additional condition (6.32) ensures that u has the required behaviour $u(x) = o(1)$, $|x| \rightarrow \infty$. This follows from

$$\ln \frac{1}{|x - y|} = \ln \frac{1}{|x|} + o(1), \quad y \in \Gamma, \quad |x| \rightarrow \infty,$$

uniformly for all directions $x/|x|$.

Theorem 6.24. *In \mathbb{R}^3 the exterior Neumann problem has a unique solution. In \mathbb{R}^2 the exterior Neumann problem is uniquely solvable if and only if*

$$(6.33) \quad \int_{\Gamma} g ds = 0$$

is satisfied.

Proof: By Theorem 6.16 and the Riesz theory Theorem 3.4 the inhomogeneous equation $\psi - K'\psi = -2g$ is uniquely solvable for each right hand side g . If $\langle 1, g \rangle = 0$, using the fact that $1 + K1 = 0$, we find

$$2\langle 1, \psi \rangle = \langle 1 - K1, \psi \rangle = \langle 1, \psi - K'\psi \rangle = -2\langle 1, g \rangle = 0.$$

Hence, the additional property (6.32) is satisfied in \mathbb{R}^2 . That condition (6.33) is necessary for the solvability in \mathbb{R}^2 follows from (6.10) by passing to the limit $h \rightarrow 0$. \square

We finally show that the solutions continuously depend on the given boundary data.

Theorem 6.25. *The solution to the Dirichlet and Neumann problem depend continuously in the maximum norm on the given data.*

Proof: For the Dirichlet problem the assertion follows from the maximum-minimum principle. In two dimensions, for the exterior problem, from the form (6.24) of the solution u we observe that we have to incorporate the value u_∞ at infinity through $\int_{\Gamma} \varphi ds$. But this integral continuously depends on the given boundary data since the inverse $(I + \tilde{K})^{-1}$ of $(I + \tilde{K})$ is bounded by Theorem 3.4.

For the Neumann problem we first observe that for single-layer potentials u with continuous density ψ for any closed ball B in \mathbb{R}^m we have an estimate of the form

$$\|u\|_{\infty, B} \leq \|w\|_{\infty, B} \|\psi\|_{\infty, \Gamma},$$

where the function

$$w(x) := \int_{\Gamma} |\Phi(x, y)| ds(y)$$

is continuous in \mathbb{R}^m by Theorem 6.13. Then for the exterior problem choose a sufficiently large ball B and the continuous dependence of the solution on the boundary data in B follows from the boundedness of the inverse $(I - K')^{-1}$ of $(I - K')$ (Theorem 3.4). In the remaining exterior of B continuity then follows through the maximum-minimum principle.

For the interior problem we can expect continuity only after making the solution unique by an additional condition, for instance by requiring that $\int_{\Gamma} u ds = 0$ for the solution u . From the equation $1 + K1 = 0$ it follows that $\langle 1, K'\psi \rangle = -\langle 1, \psi \rangle$. Therefore, K' maps $C_0(\Gamma) := \{\psi \in C(\Gamma) : \int_{\Gamma} \psi ds = 0\}$ into itself. By Theorem 6.16 the operator $I + K'$ has a trivial nullspace in $C_0(\Gamma)$. Hence, the inverse $(I + K')^{-1}$ is bounded from $C_0(\Gamma)$ onto $C_0(\Gamma)$. Finally, $u_0 := u - \int_{\Gamma} u ds / |\Gamma|$ yields a solution vanishing in the mean on the boundary.

6.5 Supplements

To conclude this chapter on the use of integral equations for boundary value problems in potential theory we briefly want to give a glimpse into a few further aspects.

Firstly, we wish to mention that in addition to transforming the boundary value problems to integral equations by seeking the solution in the form of surface potentials there is also a possibility to obtain integral equations based on the Green's formula Theorem 6.5. These equations turn out to be adjoint to those derived by the surface potential approach. For brevity, we shall confine our attention to the interior boundary value problems.

Note that Theorem 6.5 remains valid for harmonic functions which are continuous up to the boundary and possess normal derivatives in the sense of uniform convergence. This follows by first integrating over parallel surfaces to the boundary and then passing to the limit. From the formula (6.4), letting x tend to the boundary and using Theorem 6.13, we see that

$$(6.34) \quad u(x) + 2 \int_{\Gamma} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y) \Phi(x, y) \right\} ds(y) = 0$$

for all $x \in \Gamma$. Taking the normal derivative on the boundary and using again Theorem 6.13 shows that

$$(6.35) \quad \begin{aligned} \frac{\partial u}{\partial n}(x) &+ 2 \frac{\partial}{\partial n(x)} \int_{\Gamma} u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) \\ &- 2 \int_{\Gamma} \frac{\partial u}{\partial n}(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y) = 0 \end{aligned}$$

for all $x \in \Gamma$. Observe that the normal derivative of the double-layer potential exists since by assumption the normal derivative of u exists.

Now let u be the solution to the interior Dirichlet problem with boundary condition $u = f$ on Γ . Then from (6.35) we obtain the integral equation

$$(6.36) \quad \begin{aligned} \psi(x) &- 2 \int_{\Gamma} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y) \\ &= -2 \frac{\partial}{\partial n(x)} \int_{\Gamma} f(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \Gamma, \end{aligned}$$

of the second kind for the unknown normal derivative $\psi := \partial u / \partial n$ on Γ . This equation is obviously the adjoint equation of equation (6.23) obtained from the double-layer potential approach. By Theorem 6.16 and the Fredholm alternative, equation (6.36) is uniquely solvable. Having solved the integral equation for ψ , the solution of the boundary value problem is given through the Green's formula (6.4).

For a solution u to the interior Neumann problem, using $\partial u / \partial n = g$ on Γ and (6.34), we obtain the integral equation

$$(6.37) \quad \varphi(x) + 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = 2 \int_{\Gamma} g(y) \Phi(x, y) ds(y), \quad x \in \Gamma,$$

of the second kind for the unknown boundary values $\varphi := u$ on Γ . Since we have already established the existence of a solution to the Neumann problem, the existence of a solution to the integral equation (6.37) is immediate and we need

only to consider uniqueness. Because (6.37) is the adjoint of equation (6.28) obtained from the single-layer approach, by Theorem 6.16, we have uniqueness for equation (6.37) up to an additive constant.

The advantage of equations (6.36) and (6.37) based on Green's formula lies in the fact that their solution immediately yields the unknown Cauchy data without any further calculation. On the other hand, the inhomogeneous terms in these equations are more complicated and, for the Dirichlet problem, it has to be required that the solution has a normal derivative.

Despite the fact that the integral equation method provides an elegant approach to constructively prove existence of solutions for the boundary value problems of potential theory we do not want to disguise its major drawback: the relatively strong regularity assumption on the boundary to be of class C^2 . It is possible to slightly weaken the regularity and allow *Lyapunov boundaries* instead of C^2 -boundaries and still remain within the framework of compact operators. The boundary is said to satisfy a Lyapunov condition if at each point $x \in \Gamma$ the normal vector n to the surface exists and if there are positive constants L and α such that for the angle $\vartheta(x, y)$ between the normal vectors at x and y there holds $\vartheta(x, y) \leq M|x - y|^\alpha$ for all $x, y \in \Gamma$. For the treatment of the Dirichlet and Neumann problem for Lyapunov boundaries, which does not differ essentially from that for C^2 -boundaries, we refer to Mikhlin [1].

However, the situation changes considerably if the boundary is allowed to have edges and corners. This effects the form of the integral equations and the compactness of the integral operators as we will demonstrate by considering the interior Dirichlet problem in a two-dimensional domain with corners. We assume the boundary Γ piecewise twice differentiable, i.e., Γ consists of a finite number of arcs C_1, \dots, C_p which are all of class C^2 and which are connected at corners x_1, \dots, x_p . At the corners the normal vector is discontinuous. For simplicity, we restrict our analysis to boundaries which are straight lines in a neighborhood of each of the corners. In particular, this includes the case where Γ is a polygon. The interior angle at the corner x_i we denote by γ_i and assume that $0 < \gamma_i < 2\pi$, $i = 1, \dots, p$, i.e., we exclude cusps. For a boundary with corners, the continuity of the double-layer potential with continuous density stated in Theorem 6.13 remains valid, but at the corners the jump relations (6.16) have to be modified into the form

$$(6.38) \quad v_\pm(x_i) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x_i, y)}{\partial n(y)} ds(y) \pm \frac{1}{2} \delta_i^\pm \varphi(x_i), \quad i = 1, \dots, p,$$

where $\delta_i^+ = \gamma_i/\pi$ and $\delta_i^- = 2 - \gamma_i/\pi$. It is a matter of straightforward application of Green's theorem as in Example 6.14 to verify (6.38) for constant densities. For arbitrary continuous densities the result can be obtained from the C^2 -case of Theorem 6.13 by a superposition of two double-layer potentials on two C^2 -curves crossing at the corner with zero density on the parts of the two curves lying outside Γ .

Trying to find the solution to the interior Dirichlet problem in the form of a double-layer potential with continuous density φ as in Theorem 6.17 reduces

the boundary value problem to solving the integral equation $\varphi - \tilde{K}\varphi = -2f$, where the operator $\tilde{K} : C(\Gamma) \rightarrow C(\Gamma)$ is given by

$$(\tilde{K}\varphi)(x) := \begin{cases} (K\varphi)(x), & x \neq x_i, \quad i = 1, \dots, p, \\ (K\varphi)(x) + \left(\frac{\gamma_i}{\pi} - 1\right)\varphi(x_i), & x = x_i, \quad i = 1, \dots, p. \end{cases}$$

Note that by Problem 6.1 the kernel

$$k(x, y) := \frac{(n(y), x - y)}{\pi|x - y|^2}$$

of the integral operator K is continuous on $C_i \times C_i$, $i = 1, \dots, p$. For $m \in \mathbb{N}$ we use the continuous cutoff function h introduced in the proof of Theorem 2.21 to define the operators $K_m : C(\Gamma) \rightarrow C(\Gamma)$ by

$$(K_m\varphi)(x) := \int_{\Gamma} h(m|x - y|)k(x, y)\varphi(y)ds(y), \quad x \in \Gamma.$$

For each $m \in \mathbb{N}$ the operator K_m is compact since its kernel is continuous on $\Gamma \times C_i$, $i = 1, \dots, p$.

Now consider the operator $\tilde{K}_m := \tilde{K} - K_m$ and assume m large enough such that for each $x \in \Gamma$ the neighborhood $B[x; 1/m] = \{y \in \mathbb{R}^2 : |x - y| \leq 1/m\}$ intersects only either with one or – in the vicinity of the corners – two of the arcs C_i . By our assumption on the nature of the corners we can assume m large enough such that in the second case the intersection consists of two straight lines A and B . Let

$$M := \max_{i=1, \dots, p} \max_{x, y \in C_i} |k(x, y)|.$$

Then, by projection onto the tangent, for the first case we estimate

$$|(\tilde{K}_m\varphi)(x)| \leq M\|\varphi\|_{\infty} \int_{\Gamma \cap B[x; 1/m]} ds(y) \leq M\|\varphi\|_{\infty} \frac{4}{m}.$$

In the second case, we first note that for $x \notin A$, by Green's theorem (6.3) applied in the triangle T with basis A and corner x , we have

$$\int_A \left| \frac{\partial \Phi(x, y)}{\partial n(y)} \right| ds(y) = \left| \int_A \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) \right| = \frac{\alpha(x)}{2\pi}.$$

Here, $\alpha(x)$ denotes the angle in the triangle T at the corner x . Elementary triangle geometry shows that $\alpha(x) + \gamma_i \leq \pi$, where, without loss of generality, we have assumed $\gamma_i < \pi$. If $x \in A$ then $k(x, y) = 0$ for all $y \in A \setminus \{x\}$ and therefore

$$\int_A \left| \frac{\partial \Phi(x, y)}{\partial n(y)} \right| ds(y) = 0$$

in this case. Hence, for the corner x_i at the lines A and B it follows that

$$(\tilde{K}_m\varphi)(x_i) = \left(\frac{\gamma_i}{\pi} - 1\right)\varphi(x_i).$$

If x is not a corner, let say $x \in B$, then

$$|(\tilde{K}_m \varphi)(x)| \leq 2\|\varphi\|_\infty \left| \int_A \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) \right| \leq \frac{\alpha(x)}{\pi} \|\varphi\|_\infty \leq \left(1 - \frac{\gamma_i}{\pi}\right) \|\varphi\|_\infty.$$

Combining the three cases we observe that we can choose m large enough such that $\|\tilde{K}_m\|_\infty \leq q$ where

$$q := \max_{i=1,\dots,p} \left| 1 - \frac{\gamma_i}{\pi} \right| < 1.$$

Hence, we have a decomposition $I - \tilde{K} = I - \tilde{K}_m - K_m$, where $I - \tilde{K}_m$ has a bounded inverse by the Neumann series Theorem 2.8 and where K_m is compact. Now existence of a solution to the inhomogeneous equation $\varphi - \tilde{K}\varphi = -2f$ follows from $N(I - \tilde{K}) = \{0\}$ by Corollary 3.8. It is left to the reader to carry over the uniqueness proof from Theorem 6.16 to the case of a boundary with corners.

This idea of decomposing the integral operator into a compact operator and a bounded operator with norm less than one reflecting the behaviour at the corners goes back to Radon [1] and can be extended to the general two-dimensional case and also to three dimensions. For details we refer to Cryer [1], Kleinman and Wendland [1], Král [1], Leis [1], Ruland [1] and Wendland [2].

Finally, we wish to mention that the integral equations for the Dirichlet and Neumann problem can also be treated in the space $L^2(\Gamma)$ allowing boundary data in $L^2(\Gamma)$. This requires the boundary conditions to be understood in a weak sense which we want to illustrate by again considering the interior Dirichlet problem. We say that a harmonic function u in D_- assumes the boundary values $f \in L^2(\Gamma)$ in the L^2 -sense if

$$\lim_{h \rightarrow +0} \int_\Gamma |u(x - hn(x)) - f(x)|^2 ds(x) = 0.$$

To establish uniqueness under this weaker boundary condition, we choose parallel surfaces $\Gamma_h := \{x - hn(x) : x \in \Gamma\}$ to Γ with $h > 0$ sufficiently small. Now, write

$$\begin{aligned} J(h) &:= \int_{\Gamma_h} |u(z)|^2 ds_h(z) \\ &= \int_\Gamma \{1 + 2hH(x) + h^2K(x)\} |u(x - hn(x))|^2 ds(x), \quad h > 0, \end{aligned}$$

and differentiate to obtain

$$\frac{1}{2} \frac{dJ}{dh} = - \int_{\Gamma_h} u(z) \frac{\partial u}{\partial n}(z) ds_h(z) - \int_\Gamma \{H(x) + hK(x)\} |u(x - hn(x))|^2 ds(x).$$

Hence, using Green's theorem (6.1), we have

$$(6.39) \quad \frac{1}{2} \frac{dJ}{dh} = - \int_{D_h} |\operatorname{grad} u|^2 dx + \int_\Gamma \{H(x) + hK(x)\} |u(x - hn(x))|^2 ds(x),$$

where D_h denotes the interior of the parallel surface Γ_h . Assume that $\operatorname{grad} u \neq 0$ in D_- . Then there exists some closed ball B contained in D_- such that

$$I := \int_B |\operatorname{grad} u|^2 dx > 0.$$

Hence, using the homogeneous boundary condition in the L^2 -sense, from (6.39) we deduce that $dJ/dh \leq -I$ for all $0 < h \leq h_0$ with some sufficiently small $h_0 > 0$. Since J is continuous on $[0, h_0]$ and continuously differentiable on $(0, h_0]$ and since $J(0) = 0$ we see that $J(h) \leq -Ih$ for all $0 < h \leq h_0$. This is a contradiction to $J(h) \geq 0$ for all $h > 0$. Therefore u must be constant in D_- , and from $J(0) = 0$ we get $u = 0$ in D_- .

For the surface potential approach it is necessary to extend the jump relations of Theorem 6.13 from $C(\Gamma)$ onto $L^2(\Gamma)$. This can be achieved quite elegantly through the use of the Lax Theorem 4.12 as worked out by Kersten [1]. In particular, for the double-layer potential v with density $\varphi \in L^2(\Gamma)$ the jump relations (6.16) and (6.17) have to be replaced by

$$(6.40) \quad \lim_{h \rightarrow +0} \int_{\Gamma} |2v(x \pm hn(x)) - (K\varphi)(x) \mp \varphi(x)|^2 ds(x) = 0$$

and

$$(6.41) \quad \lim_{h \rightarrow +0} \int_{\Gamma} \left| \frac{\partial v}{\partial n}(x + hn(x)) - \frac{\partial v}{\partial n}(x - hn(x)) \right|^2 ds(x) = 0.$$

From this it follows that the double-layer potential with density $\varphi \in L^2(\Gamma)$ solves the Dirichlet problem with boundary values $f \in L^2(\Gamma)$ provided the density solves the integral equation $\varphi - K\varphi = -2f$ in the space $L^2(\Gamma)$. Now this integral equation can be treated analogously to Theorems 6.16 and 6.18.

For integral equation methods in vector potential theory, i.e., for boundary value problems for harmonic vector fields and the vector Laplace equation we refer to Kress [1,2,3], Martensen [2] and Werner [1]. For the treatment of boundary value problems for the Helmholtz and Maxwell equations see Colton and Kress [1].

We will come back to the potential theoretic boundary value problems in the following two chapters. In Section 7.5 we will solve the interior Dirichlet and Neumann problem in two dimensions through integral equations of the first kind in a Hölder space setting. And in Section 8.4 we will solve the integral equations of the second kind of this chapter in Sobolev spaces leading to weak solutions of the two-dimensional boundary value problems.

Problems

- 6.1. Use a regular parametric representation $x(t) = (x_1(t), x_2(t))$, $0 \leq t \leq 2\pi$, with counter-clockwise orientation for the boundary curve to transform the integral equation (6.23) of the interior two-dimensional Dirichlet problem into the form

$$\tilde{\varphi}(t) - \int_0^{2\pi} k(t, \tau) \tilde{\varphi}(\tau) d\tau = -2\tilde{f}(t), \quad 0 \leq t \leq 2\pi,$$

where $\tilde{\varphi}(t) := \varphi(\mathbf{x}(t))$, $\tilde{f}(t) := f(\mathbf{x}(t))$ and where the kernel is given by

$$k(t, \tau) = \begin{cases} \frac{1}{\pi} \frac{\dot{x}_2(\tau)[x_1(t) - x_1(\tau)] - \dot{x}_1(\tau)[x_2(t) - x_2(\tau)]}{[x_1(t) - x_1(\tau)]^2 + [x_2(t) - x_2(\tau)]^2}, & t \neq \tau, \\ \frac{1}{2\pi} \frac{\dot{x}_2(t)\dot{x}_1(t) - \dot{x}_1(t)\dot{x}_2(t)}{[\dot{x}_1(t)]^2 + [\dot{x}_2(t)]^2}, & t = \tau. \end{cases}$$

Show that this kernel is continuous provided Γ is of class C^2 .

- 6.2. Show that for an ellipse with parametric representation $x_1(t) = a \cos t$, $x_2(t) = b \sin t$ the kernel k of Problem 6.1 is given by

$$k(t, \tau) = -\frac{ab}{\pi} \frac{1}{a^2 + b^2 - (a^2 - b^2) \cos(t + \tau)}.$$

- 6.3. Extend Theorem 6.16 to domains D_- with nonconnected boundaries and, in particular, show that $\dim N(I - K) = p$, where p denotes the number of bounded components of D_+ . For the interior Dirichlet problem establish existence of a solution through a modification of the integral equation (6.23) analogous to (6.24) by adding a point source in each of the bounded components of D_+ .
- 6.4. Let $D_- \subset \mathbb{R}^2$ be of class C^2 and strictly convex in the sense that the curvature of the boundary Γ is strictly positive. Show that there exists a constant $0 < \delta < 1$ such that

$$\int_{\Gamma} \left| \frac{\partial \Phi(x_1, y)}{\partial n(y)} - \frac{\partial \Phi(x_2, y)}{\partial n(y)} \right| ds(y) \leq 1 - \delta$$

for all $x_1, x_2 \in \Gamma$.

Hint: Use Example 6.14, Problem 6.1 and the property that

$$\frac{\partial \Phi(x, y)}{\partial n(y)} = \frac{(n(y), x - y)}{2\pi|x - y|^2}$$

is negative on $\Gamma \times \Gamma$ to verify that for each Jordan measurable subset $\Gamma^* \subset \Gamma$ there holds

$$\left| \int_{\Gamma^*} \left\{ \frac{\partial \Phi(x_1, y)}{\partial n(y)} - \frac{\partial \Phi(x_2, y)}{\partial n(y)} \right\} ds(y) \right| \leq \frac{1}{2} - a|\Gamma|,$$

where

$$a := \min_{x, y \in \Gamma} \left| \frac{\partial \Phi(x, y)}{\partial n(y)} \right| > 0.$$

- 6.5. In 1870 Neumann [1] gave the first rigorous proof for the existence of a solution to the two-dimensional interior Dirichlet problem in a strictly convex domain of class C^2 . By completely elementary means he established that the successive approximations

$$\varphi_{m+1} := \frac{1}{2} \varphi_m + \frac{1}{2} K \varphi_m - f, \quad m = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in C(\Gamma)$ converge uniformly to the unique solution φ of $\varphi - K\varphi = -2f$. In functional analytic terms his proof amounted to showing that the operator L given by $L := (I + K)/2$ is a contraction with respect to the norm

$$\|\varphi\| := |\sup_{z \in \Gamma} \varphi(z) - \inf_{z \in \Gamma} \varphi(z)| + \alpha \sup_{z \in \Gamma} |\varphi(z)|$$

where $\alpha > 0$ is appropriately chosen. This norm is equivalent to the maximum norm. Derive the above results for yourself.

Hint: Use Problem 6.4 to show that $\|L\| \leq (2 - \delta + \alpha)/2$ by writing

$$(L\varphi)(x) = \int_{\Gamma} [\varphi(y) - \varphi(x)] \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y)$$

and

$$\begin{aligned} (L\varphi)(x_1) - (L\varphi)(x_2) &= \frac{1}{2} [\varphi(x_1) - \varphi(x)] - \frac{1}{2} [\varphi(x_2) - \varphi(x)] \\ &\quad + \int_{\Gamma} [\varphi(y) - \varphi(x)] \left\{ \frac{\partial \Phi(x_1, y)}{\partial n(y)} - \frac{\partial \Phi(x_2, y)}{\partial n(y)} \right\} ds(y) \end{aligned}$$

where $x \in \Gamma$ is chosen such that $\varphi(x) = \{\sup_{z \in \Gamma} \varphi(z) + \inf_{z \in \Gamma} \varphi(z)\}/2$ (cf. Kantorovich and Krylov [1]).

7. Singular Integral Equations

In this chapter we will consider one-dimensional singular integral equations involving Cauchy principal values which arise from boundary value problems for holomorphic functions. The investigations of these integral equations with Cauchy kernels by the Russian mathematicians Gakhov, Muskhelishvili, Vekua and others have had a great impact on the further development of the general theory of singular integral equations. For our introduction into integral equations they will provide an application of the general idea of regularizing singular operators as described in Chapter 5. We assume that the reader is acquainted with the basic theory of complex functions.

7.1 Hölder Continuity

For our investigation of singular integrals we first need to introduce the concept of Hölder continuity.

Definition 7.1. *A real or complex valued function φ defined on a set $G \subset \mathbb{R}^m$ is called uniformly Hölder continuous with Hölder exponent $0 < \alpha \leq 1$ if there exists a constant C such that*

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^\alpha$$

for all $x, y \in G$. By $C^{0,\alpha}(G)$ we denote the linear space of all functions defined on G which are bounded and uniformly Hölder continuous with exponent α . The space $C^{0,\alpha}(G)$ is called a Hölder space.

Note that each uniformly Hölder continuous function is uniformly continuous, whereas the converse is not true. We illustrate by two examples that uniformly Hölder continuous functions live between continuous and differentiable functions. The function $\varphi : [0, 1/2] \rightarrow \mathbb{R}$, given by $\varphi(x) = 1/\ln x$ for $x \in (0, 1/2]$ and $\varphi(0) = 0$, is uniformly continuous but not uniformly Hölder continuous. The function $\psi : [0, 1] \rightarrow \mathbb{R}$, given by $\psi(x) = \sqrt{x}$, is uniformly Hölder continuous with exponent $1/2$ but not continuously differentiable on $[0, 1]$. In general, by the mean value theorem, a continuously differentiable function on a convex set with bounded derivatives is uniformly Hölder continuous with exponent 1.

Theorem 7.2. *The Hölder space $C^{0,\alpha}(G)$ is a Banach space with the norm*

$$\|\varphi\|_\alpha := \sup_{x \in G} |\varphi(x)| + \sup_{\substack{x, y \in G \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}.$$

Proof: It is clear that

$$|\varphi|_\alpha := \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

defines a semi-norm on $C^{0,\alpha}(G)$, i.e., it satisfies all norm axioms with the exception of the definiteness (N2). Then $\|\cdot\|_\alpha = \|\cdot\|_\infty + |\cdot|_\alpha$ is a norm, since $\|\varphi\|_\infty := \sup_{x \in G} |\varphi(x)|$ defines a norm. Convergence in the supremum norm $\|\cdot\|_\infty$ is equivalent to uniform convergence on G . If G is compact the supremum norm and the maximum norm on $C(G)$ coincide.

It remains to be shown that $C^{0,\alpha}(G)$ is complete. Let (φ_n) denote a Cauchy sequence in $C^{0,\alpha}(G)$. Then obviously (φ_n) also is a Cauchy sequence with respect to the supremum norm and, by the sufficiency of the Cauchy criterion for uniform convergence, there exists a function $\varphi \in C(G)$ such that $\|\varphi_n - \varphi\|_\infty \rightarrow 0$, $n \rightarrow \infty$. Because (φ_n) is a Cauchy sequence in $C^{0,\alpha}(G)$, given $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|\varphi_n - \varphi_k|_\alpha < \varepsilon$ for all $n, k \geq N(\varepsilon)$, that is,

$$|\{\varphi_n(x) - \varphi_k(x)\} - \{\varphi_n(y) - \varphi_k(y)\}| < \varepsilon|x - y|^\alpha$$

for all $n, k \geq N(\varepsilon)$ and all $x, y \in G$. Since $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, uniformly on G , by letting $k \rightarrow \infty$ we have

$$|\{\varphi_n(x) - \varphi(x)\} - \{\varphi_n(y) - \varphi(y)\}| \leq \varepsilon|x - y|^\alpha$$

for all $n \geq N(\varepsilon)$ and all $x, y \in G$. From this we conclude that $\varphi \in C^{0,\alpha}(G)$ and $|\varphi_n - \varphi|_\alpha \leq \varepsilon$ for all $n \geq N(\varepsilon)$, which implies $\|\varphi_n - \varphi\|_\alpha \rightarrow 0$, $n \rightarrow \infty$. \square

Note that the product of two uniformly Hölder continuous functions φ and ψ is again uniformly Hölder continuous with $\|\varphi\psi\|_\alpha \leq \|\varphi\|_\alpha \|\psi\|_\alpha$.

By the following technical lemma we illustrate that Hölder continuity is a local property.

Lemma 7.3. *Let the function φ satisfy $|\varphi(x)| \leq M$ for all $x \in G$ and*

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^\alpha$$

for all $x, y \in G$ with $|x - y| \leq a$ with some constants a, C, M and $0 < \alpha \leq 1$. Then $\varphi \in C^{0,\alpha}(G)$ with

$$\|\varphi\|_\alpha \leq M + \max \left(C, \frac{2M}{a^\alpha} \right).$$

Proof: If $|x - y| > a$ then

$$|\varphi(x) - \varphi(y)| \leq 2M \leq 2M \left(\frac{|x - y|}{a} \right)^\alpha,$$

whence $|\varphi|_\alpha \leq \max(C, 2M/a^\alpha)$. \square

In particular, from Lemma 7.3, we see that for $\alpha < \beta$ clearly each function $\varphi \in C^{0,\beta}(G)$ is also contained in $C^{0,\alpha}(G)$. For this imbedding we have the following compactness property.

Theorem 7.4. *Let $0 < \alpha < \beta \leq 1$ and let G be compact. Then the imbedding operators*

$$I^\beta : C^{0,\beta}(G) \rightarrow C(G)$$

and

$$I^{\alpha,\beta} : C^{0,\beta}(G) \rightarrow C^{0,\alpha}(G)$$

are compact.

Proof: Let U be a bounded set in $C^{0,\beta}(G)$, that is, $\|\varphi\|_\beta \leq C$ for all $\varphi \in U$. Then obviously we have

$$|\varphi(x)| \leq C$$

for all $x \in G$ and

$$(7.1) \quad |\varphi(x) - \varphi(y)| \leq C|x - y|^\beta$$

for all $x, y \in G$ and all $\varphi \in U$, which implies that U is bounded and equicontinuous. Therefore, by the Arzelà–Ascoli Theorem 1.18, the set U is relatively compact in $C(G)$, which, in particular, means that the imbedding operator $I^\beta : C^{0,\beta}(G) \rightarrow C(G)$ is compact.

It remains to be verified that U is relatively compact in $C^{0,\alpha}(G)$. From (7.1) for all $\varphi, \psi \in U$ and for all $x, y \in G$ we have

$$\begin{aligned} & |\{\varphi(x) - \psi(x)\} - \{\varphi(y) - \psi(y)\}| \\ &= |\{\varphi(x) - \psi(x)\} - \{\varphi(y) - \psi(y)\}|^{\alpha/\beta} |\{\varphi(x) - \psi(x)\} - \{\varphi(y) - \psi(y)\}|^{1-\alpha/\beta} \\ &\leq (2C)^{\alpha/\beta} |x - y|^\alpha (2\|\varphi - \psi\|_\infty)^{1-\alpha/\beta} \end{aligned}$$

which implies that

$$|\varphi - \psi|_\alpha \leq (2C)^{\alpha/\beta} 2^{1-\alpha/\beta} \|\varphi - \psi\|_\infty^{1-\alpha/\beta}.$$

But from this we can conclude that each sequence taken from U and converging in $C(G)$ also converges in $C^{0,\alpha}(G)$.

7.2 The Cauchy Integral Operator

Let D_- be a bounded and simply connected domain in the complex plane and denote by $D_+ := \mathbb{C} \setminus \bar{D}_-$ its unbounded open complement. We denote the boundary by $\Gamma := \partial D$ and assume it to be of class C^2 . The normal vector n is directed into the exterior domain D_+ . For complex integration along the contour Γ we assume the direction of integration to be counterclockwise. We confine our analysis to this basic configuration and note that it can be extended, for instance, to multiply connected domains with less regular boundaries.

The *Cauchy integral*

$$f(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Gamma,$$

with density $\varphi \in C(\Gamma)$ defines a holomorphic function in D_+ and in D_- . Obviously, for z varying in an open domain not intersecting with Γ the integrand is continuous with respect to ζ and continuously differentiable with respect to z . Therefore, we can differentiate under the integral to verify the Cauchy–Riemann equations for f . Occasionally, we will call a function which is holomorphic in D_+ and in D_- *sectionally holomorphic*.

As in the case of the single- and double-layer potentials of Chapter 6 we are interested in the behaviour of the Cauchy integral for points on the boundary where the integral becomes singular. By integrating the equation $(\ln \ln x)' = 1/x \ln x$ between $0 \leq x \leq 1$ we observe that the Cauchy integral for points on the boundary, in general, will not exist if the density is merely continuous. Therefore we assume the density to be uniformly Hölder continuous. Note that the introduction of Hölder continuity through Definition 7.1 also covers functions on subsets of the complex plane by identifying \mathbb{R}^2 and \mathbb{C} .

For the remainder of this chapter we will always assume the Hölder exponent $\alpha \in (0, 1)$, despite the fact that part of our results remain valid for $\alpha = 1$.

Theorem 7.5. *For densities $\varphi \in C^{0,\alpha}(\Gamma)$ the Cauchy integral exists as a Cauchy principal value for all $z \in \Gamma$, that is, the limit*

$$\lim_{\rho \rightarrow 0} \int_{\Gamma \setminus \Gamma(z; \rho)} \frac{\varphi(\zeta)}{\zeta - z} d\zeta$$

exists where $\Gamma(z; \rho) := \{\zeta \in \Gamma : |\zeta - z| \leq \rho\}$.

Proof: Let $z \in \Gamma$ and let $H(z; \rho)$ be the part of the circle of radius ρ and center z lying within D_- . Then, by Cauchy's integral theorem,

$$\int_{\Gamma \setminus \Gamma(z; \rho)} \frac{d\zeta}{\zeta - z} = \int_{H(z; \rho)} \frac{d\zeta}{\zeta - z}.$$

Writing $\zeta = z + \rho e^{i\vartheta}$, $d\zeta = i\rho e^{i\vartheta} d\vartheta$, we find

$$\lim_{\rho \rightarrow 0} \int_{H(z; \rho)} \frac{d\zeta}{\zeta - z} = \lim_{\rho \rightarrow 0} \int_{H(z; \rho)} i d\vartheta = i\pi$$

since Γ is of class C^2 . Hence

$$(7.2) \quad \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\zeta - z} = \frac{1}{2}, \quad z \in \Gamma,$$

in the sense of a Cauchy principal value.

The normal vector n is continuous on Γ . Therefore, we can choose $R \in (0, 1]$ such that the scalar product satisfies

$$(7.3) \quad (n(z), n(\zeta)) \geq \frac{1}{2}$$

for all $z, \zeta \in \Gamma$ with $|z - \zeta| \leq R$. Furthermore, we can assume that R is small enough such that $\Gamma(z; R)$ is connected for each $z \in \Gamma$. Then the condition (7.3) implies that $\Gamma(z; R)$ can be bijectively projected into the tangent to Γ at the point z . The line elements $ds(\zeta)$ on Γ and $d\sigma$ on the tangent are related by

$$ds(\zeta) = \frac{d\sigma}{(n(z), n(\zeta))} \leq 2 d\sigma.$$

Hence,

$$\begin{aligned} \int_{\Gamma(z; R)} \left| \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} \right| ds &\leq 2|\varphi|_\alpha \int_{-R}^R |\zeta - z|^{\alpha-1} d\sigma \\ &\leq 4|\varphi|_\alpha \int_0^R \sigma^{\alpha-1} d\sigma = \frac{4R^\alpha}{\alpha} |\varphi|_\alpha \end{aligned}$$

and

$$\int_{\Gamma \setminus \Gamma(z; R)} \left| \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} \right| ds \leq |\varphi|_\alpha \int_{\Gamma \setminus \Gamma(z; R)} |\zeta - z|^{\alpha-1} ds \leq R^{\alpha-1} |\Gamma| |\varphi|_\alpha.$$

Therefore

$$\int_{\Gamma} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} dz$$

exists as an improper integral and by the decomposition

$$\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} d\zeta + \varphi(z) \int_{\Gamma} \frac{d\zeta}{\zeta - z}$$

the proof is completed. \square

After having established the existence of the Cauchy integral for points on the boundary, we now want to show that the function f defined by the Cauchy integral is uniformly Hölder continuous in \bar{D}_+ and in \bar{D}_- . For this we choose a number $h > 0$ small enough such that in the parallel strip

$$D_h := \{w + \eta hn(w) : w \in \Gamma, \eta \in [-1, 1]\}$$

each point z is uniquely representable through projection onto Γ in the form $z = w + \eta hn(w)$ with $w \in \Gamma$ and $\eta \in [-1, 1]$. Then we show that the function g defined on D_h by

$$g(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta) - \varphi(w)}{\zeta - z} d\zeta, \quad z \in D_h,$$

is uniformly Hölder continuous with

$$(7.4) \quad \|g\|_\alpha \leq C \|\varphi\|_\alpha$$

where C is some constant independent of φ .

We begin by noting that we can choose h sufficiently small such that

$$(7.5) \quad |w_1 - w_2| \leq 2|z_1 - z_2|$$

for each pair $z_1 = w_1 + \eta_1 h n(w_1)$, $z_2 = w_2 + \eta_2 h n(w_2)$. Since Γ is of class C^2 the normal is even continuously differentiable. Therefore, by the mean value theorem, there exists $M > 0$ such that $|n(w_1) - n(w_2)| \leq M|w_1 - w_2|$ for all $w_1, w_2 \in \Gamma$. Then for the minimal distance case $z_1^* = w_1 + hn(w_1)$, $z_2^* = w_2 + hn(w_2)$ – without loss of generality we may take the positive sign for the parameter h – we have $|w_1 - w_2| \leq |z_1^* - z_2^*| + hM|w_1 - w_2|$. Hence, if we restrict $h \leq 1/2M$ we indeed have $|w_1 - w_2| \leq 2|z_1^* - z_2^*| \leq 2|z_1 - z_2|$.

We can use (7.5) to estimate $|\varphi(\zeta) - \varphi(w)| \leq |\varphi|_\alpha |\zeta - w|^\alpha \leq 2^\alpha |\varphi|_\alpha |\zeta - z|^\alpha$ and then as in the proof of Theorem 7.5, by splitting the integral in two parts over $\Gamma(w_1; R)$ and $\Gamma \setminus \Gamma(w_1; R)$, we obtain

$$(7.6) \quad |g(z)| \leq C_0 |\varphi|_\alpha$$

for all $z \in D_h$ with some constant C_0 .

Now let $z_1, z_2 \in D_{h_0}$ with $0 < |z_1 - z_2| \leq R/4$ and set $\rho := 4|z_1 - z_2|$. We can estimate

$$(7.7) \quad \left| \int_{\Gamma(w_1; \rho)} \left\{ \frac{\varphi(\zeta) - \varphi(w_1)}{\zeta - z_1} - \frac{\varphi(\zeta) - \varphi(w_2)}{\zeta - z_2} \right\} d\zeta \right| \\ \leq 2^\alpha |\varphi|_\alpha \left\{ \int_{\Gamma(w_1; \rho)} |\zeta - z_1|^{\alpha-1} ds + \int_{\Gamma(w_2; 2\rho)} |\zeta - z_2|^{\alpha-1} ds \right\} \leq C_1 |z_1 - z_2|^\alpha |\varphi|_\alpha$$

with a constant C_1 depending on α . Here we have made use of the fact that $\Gamma(w_1; \rho) \subset \Gamma(w_2; 2\rho)$. For $|\zeta - w_1| \geq \rho = 4|z_1 - z_2|$ there holds

$$|\zeta - z_1| \leq |\zeta - z_2| + |z_2 - z_1| \leq |\zeta - z_2| + \frac{1}{4} |\zeta - w_1| \leq |\zeta - z_2| + \frac{1}{2} |\zeta - z_1|.$$

Hence, $|\zeta - z_1| \leq 2|\zeta - z_2|$ for $|\zeta - w_1| \geq \rho$. This inequality can now be used to obtain

$$(7.8) \quad \left| \int_{\Gamma \setminus \Gamma(w_1; \rho)} \{ \varphi(\zeta) - \varphi(w_2) \} \left\{ \frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2} \right\} d\zeta \right| \\ \leq |z_1 - z_2| |\varphi|_\alpha \int_{\Gamma \setminus \Gamma(w_1; \rho)} \frac{|\zeta - w_2|^\alpha}{|\zeta - z_1| |\zeta - z_2|} ds \\ \leq C'_2 |z_1 - z_2| |\varphi|_\alpha \left\{ \int_{\rho/2}^R \frac{d\sigma}{\sigma^{2-\alpha}} + \frac{|\Gamma|}{R^{2-\alpha}} \right\} \leq C_2 |z_1 - z_2|^\alpha |\varphi|_\alpha$$

with constants C_2, C'_2 depending on α and Γ . Note that for this estimate we need the restriction $\alpha < 1$, and that we have split the integral in two parts over $\Gamma(w_1; R) \setminus \Gamma(w_1; \rho)$ and $\Gamma \setminus \Gamma(w_1; R)$. Finally, since from the proof of Theorem 7.5 it is obvious that

$$\int_{\Gamma \setminus \Gamma(w_1; \rho)} \frac{d\zeta}{\zeta - z_1}$$

is bounded by 2π , we have the estimate

$$(7.9) \quad \left| \{ \varphi(w_1) - \varphi(w_2) \} \int_{\Gamma \setminus \Gamma(w_1; \rho)} \frac{d\zeta}{\zeta - z_1} \right| \leq C_3 |z_1 - z_2|^\alpha |\varphi|_\alpha$$

with a constant C_3 . Combining (7.7) to (7.9), we obtain

$$(7.10) \quad 2\pi |g(z_1) - g(z_2)| \leq (C_1 + C_2 + C_3)|z_1 - z_2|^\alpha |\varphi|_\alpha$$

for all $z_1, z_2 \in D_h$ with $|z_1 - z_2| \leq R/4$. Now the desired result (7.4) follows from (7.6) and (7.10) with the aid of Lemma 7.3.

From the decomposition

$$f(z) = g(z) + \varphi(w)F(z)$$

where, by Cauchy's integral theorem and (7.2),

$$F(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\zeta - z} = \begin{cases} 1, & z \in D_-, \\ 1/2, & z \in \Gamma, \\ 0, & z \in D_+, \end{cases}$$

we observe that $f \in C^{0,\alpha}(D_h \cap \bar{D}_+)$ and $f \in C^{0,\alpha}(D_h \cap \bar{D}_-)$ with boundary values

$$f_{\pm}(z) = g(z) + \varphi(z) \left(\frac{1}{2} \mp \frac{1}{2} \right) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta \mp \frac{1}{2} \varphi(z)$$

for $z \in \Gamma$. Hölder continuity of f in \bar{D}_+ and \bar{D}_- now follows by once again employing Lemma 7.3. We can summarize our results in

Theorem 7.6 (Sokhotski–Plemelj). *For densities $\varphi \in C^{0,\alpha}(\Gamma)$ the holomorphic function f defined by the Cauchy integral*

$$(7.11) \quad f(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Gamma,$$

can be uniformly Hölder continuously extended from D_+ into \bar{D}_+ and from D_- into \bar{D}_- with limiting values

$$(7.12) \quad f_{\pm}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta \mp \frac{1}{2} \varphi(z), \quad z \in \Gamma,$$

where $f_{\pm}(z) = \lim_{h \rightarrow +0} f(z \pm hn(z))$. Furthermore, we have the inequalities

$$\|f\|_{\alpha, \bar{D}_+} \leq C \|\varphi\|_\alpha, \quad \|f\|_{\alpha, \bar{D}_-} \leq C \|\varphi\|_\alpha$$

for some constant C depending on α and Γ .

The formula (7.12) was first derived by Sokhotski [1] in his doctor's thesis in 1873. A sufficiently rigorous proof was given by Plemelj [1] in 1908.

Corollary 7.7. *The Cauchy integral operator $A : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$, defined by*

$$(A\varphi)(z) := \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \Gamma,$$

is bounded. The kernel of the operator A is called a Cauchy kernel.

Proof: We write the Sokhotski–Plemelj formula (7.12) in the short form

$$(7.12') \quad f_{\pm} = \frac{1}{2} A\varphi \mp \frac{1}{2} \varphi$$

and obtain the boundedness of A from Theorem 7.6. \square

As a first application of the Sokhotski–Plemelj Theorem 7.6 we solve the problem of finding a sectionally holomorphic function with a given discontinuity along the contour Γ .

Theorem 7.8. *Let $\varphi \in C^{0,\alpha}(\Gamma)$. Then there exists a unique function f which is holomorphic in D_+ and D_- , which can be extended continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- satisfying the boundary condition*

$$f_- - f_+ = \varphi \quad \text{on } \Gamma,$$

and for which $f(z) \rightarrow 0$, $z \rightarrow \infty$, uniformly for all directions. This function is given by (7.11).

Proof: By Theorem 7.6 the function f , given by (7.11), has the required properties. To establish uniqueness, from $f_- - f_+ = 0$ on Γ , by Morera's theorem, we conclude that f is holomorphic everywhere in \mathbb{C} , that is, f is an entire function. Then from $f(z) \rightarrow 0$, $z \rightarrow \infty$, by Liouville's theorem it follows that $f = 0$ in \mathbb{C} . From this proof we see that the general sectionally holomorphic function f satisfying $f_- - f_+ = \varphi$ on Γ is obtained from (7.11) by adding an arbitrary entire function. \square

As a second application of Theorem 7.6 we state necessary and sufficient conditions for the existence of a holomorphic function in D_- or in D_+ with given boundary values.

Theorem 7.9. *For a given function $\varphi \in C^{0,\alpha}(\Gamma)$, there exists a function f holomorphic in D_- and continuous in \bar{D}_- with boundary values $f = \varphi$ on Γ if and only if φ is a solution of the homogeneous integral equation of the second kind*

$$\varphi - A\varphi = 0.$$

The solution is given through (7.11).

Proof: Let f be holomorphic with $f = \varphi$ on Γ . Then by Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in D_-,$$

and from Theorem 7.6 there follows $2\varphi = 2f_- = A\varphi + \varphi$ on Γ , therefore $\varphi - A\varphi = 0$.

Conversely, if φ is a solution of $\varphi - A\varphi = 0$, then again by Theorem 7.6 the function f , defined by (7.11), has boundary values $2f_- = A\varphi + \varphi = 2\varphi$ on Γ . \square

Obviously, for the corresponding exterior problem in D_+ with $f(\infty) = 0$ we have the homogeneous integral equation $\varphi + A\varphi = 0$. From these results we observe that the operator A provides an example for nullspaces $N(I - A)$ and $N(I + A)$ with infinite dimension, since there exist infinitely many holomorphic functions which are linearly independent. This, in particular, implies that A is not compact and it also means that $I - A$ and $I + A$ cannot be regularized.

We now can prove a property of the Cauchy integral operator A which is of central importance in our study of singular integral equations with Cauchy kernels since it will allow the construction of regularizers.

Theorem 7.10. *The Cauchy integral operator A satisfies*

$$A^2 = I.$$

Proof: For $\varphi \in C^{0,\alpha}(\Gamma)$ define f by (7.11). Then, by Theorem 7.9, we have $f_+ + Af_+ = 0$ and $f_- - Af_- = 0$. Hence, using (7.12), we derive

$$A^2\varphi = A(f_- + f_+) = f_- - f_+ = \varphi$$

and this is the desired result. \square

Note that Theorem 7.10 is a further indication that the Cauchy integral operator A cannot be compact.

Theorem 7.11. *The operators A and $-A$ are adjoint with respect to the dual system $\langle C^{0,\alpha}(\Gamma), C^{0,\alpha}(\Gamma) \rangle$ given through the nondegenerate bilinear form*

$$\langle \varphi, \psi \rangle := \int_{\Gamma} \varphi(z)\psi(z)dz, \quad \varphi, \psi \in C^{0,\alpha}(\Gamma).$$

Proof: It is left as an exercise to show that $\langle \cdot, \cdot \rangle$ is nondegenerate on $C^{0,\alpha}(\Gamma)$. For $\varphi, \psi \in C^{0,\alpha}(\Gamma)$ define

$$f(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad g(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\psi(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Gamma.$$

Then, by Theorem 7.6 and Cauchy's integral theorem, we find

$$\langle A\varphi, \psi \rangle + \langle \varphi, A\psi \rangle = \langle f_- + f_+, g_- - g_+ \rangle + \langle f_- - f_+, g_- + g_+ \rangle$$

$$= 2\langle f_-, g_- \rangle - 2\langle f_+, g_+ \rangle = -2 \int_{|z|=R} f(z)g(z)dz \rightarrow 0, \quad R \rightarrow \infty,$$

since $f(z), g(z) = O(1/|z|)$, $z \rightarrow \infty$. \square

Note that Theorem 7.11 justifies interchanging the order of integration in

$$\int_{\Gamma} \left(\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta \right) \psi(z)dz = \int_{\Gamma} \varphi(\zeta) \left(\int_{\Gamma} \frac{\psi(z)}{\zeta - z} dz \right) d\zeta.$$

Example 7.12. Let Γ be the unit circle. Then, substituting $z = e^{it}$, $\zeta = e^{i\tau}$, we find

$$\frac{d\zeta}{\zeta - z} = \frac{1}{2} \left(\cot \frac{\tau - t}{2} + i \right) d\tau,$$

i.e., the operator A can be expressed in the form

$$(A\varphi)(t) = \frac{1}{2\pi i} \int_0^{2\pi} \left\{ \cot \frac{\tau - t}{2} + i \right\} \varphi(\tau) d\tau, \quad t \in [0, 2\pi],$$

with the integral to be understood as a Cauchy principal value. Consider the integral equation of the first kind

$$(7.13) \quad \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \varphi(\tau) d\tau = \psi(t), \quad t \in [0, 2\pi],$$

in the space $C^{0,\alpha}[0, 2\pi]$. Since

$$\int_0^{2\pi} \cot \frac{t}{2} dt = 0,$$

by integrating (7.13) with respect to t and interchanging the order of integration (use Theorem 7.11 for $\psi(z) = 1/z$ and transform the integrals), we find that

$$\int_0^{2\pi} \psi(t) dt = 0$$

is a necessary condition for the solvability of (7.13). It is also sufficient, since by making use of $A^2 = I$ we see that the general solution of (7.13) then is given through

$$(7.14) \quad \varphi(t) = -\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - t}{2} \psi(\tau) d\tau + c, \quad t \in [0, 2\pi],$$

where c is an arbitrary constant. \square

The formulae (7.13) and (7.14) represent the *inversion formulae* of Hilbert. The kernel in (7.13) and (7.14) is called *Hilbert kernel* since singular equations with this kernel were first encountered in Hilbert [1]. Splitting the integral equation of Theorem 7.9 into its real and imaginary part, we readily find that the Hilbert inversion formulae relate the real and imaginary part of holomorphic functions in the unit disk.

7.3 The Riemann Problem

The following boundary value problem was first formulated by Riemann in his inaugural dissertation. Since a first attempt towards a solution was made by Hilbert [1] in 1904 through the use of integral equations, what we will denote as the Riemann problem in the literature sometimes also is called the Hilbert problem.

Riemann Problem. *Find a function f which is holomorphic in D_+ and in D_- , which can be extended uniformly Hölder continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- satisfying the boundary condition*

$$(7.15) \quad f_- = g f_+ + h \quad \text{on } \Gamma$$

and

$$(7.16) \quad f(z) \rightarrow 0, \quad z \rightarrow \infty,$$

uniformly for all directions. Here, g and h are given uniformly Hölder continuous functions on Γ .

Before we give a solution to this Riemann problem for the case where g vanishes nowhere on Γ , we first want to indicate its connection to integral equations by

Theorem 7.13. *Let $a, b, h \in C^{0,\alpha}(\Gamma)$. Then the Cauchy integral (7.11) maps solutions $\varphi \in C^{0,\alpha}(\Gamma)$ of the integral equation*

$$(7.17) \quad a\varphi + bA\varphi = h$$

linearly and bijectively into solutions f of the Riemann problem with the boundary condition

$$(7.18) \quad (a+b)f_- = (a-b)f_+ + h \quad \text{on } \Gamma.$$

Proof: By Theorems 7.6 and 7.8, the Cauchy integral maps $C^{0,\alpha}(\Gamma)$ linearly and bijectively into the linear space of sectionally holomorphic functions which vanish at infinity and can be extended uniformly Hölder continuously with Hölder exponent α from D_+ into \bar{D}_+ and from D_- into \bar{D}_- . From (7.12) and Theorem 7.9 we have the relation

$$a\varphi + bA\varphi = a(f_- - f_+) + b(Af_- - Af_+) = (a+b)f_- - (a-b)f_+$$

between the density φ and the function f defined through the Cauchy integral. This ends the proof.

Definition 7.14. *Let g be a complex valued and nowhere vanishing function defined on the contour Γ . The index of the function g is the integer $\text{ind } g$ given by the increment of its argument along the contour in a counterclockwise direction divided by 2π , that is,*

$$\text{ind } g = \frac{1}{2\pi} \int_{\Gamma} d\arg g = \frac{1}{2\pi} \left. \arg g \right|_{\Gamma}.$$

The index can also be expressed by the logarithm of g through

$$\text{ind } g = \frac{1}{2\pi i} \left. \ln g \right|_{\Gamma}.$$

Now, with the aid of the Sokhotski–Plemelj jump relations, we explicitly solve the homogeneous Riemann problem.

Theorem 7.15. *Let $g \in C^{0,\alpha}(\Gamma)$ be a nowhere vanishing function with $\text{ind } g = \kappa$. Then there exists a unique sectionally holomorphic function f_0 satisfying the homogeneous boundary condition*

$$f_{0-} = g f_{0+} \quad \text{on } \Gamma$$

and

$$\lim_{z \rightarrow \infty} z^\kappa f_0(z) = 1$$

uniformly for all directions. It is called canonical solution to the homogeneous Riemann problem, has the property that $f_0(z) \neq 0$ for all $z \in \mathbb{C}$ and is uniformly Hölder continuous up to the boundary.

Proof: Choose a point $a \in D_-$ and define $G \in C^{0,\alpha}(\Gamma)$ by

$$G(z) := (z - a)^{-\kappa} g(z), \quad z \in \Gamma.$$

Since G has index $\text{ind } G = 0$, we can reduce the homogeneous boundary condition $F_- = GF_+$ by taking logarithms to obtain $\ln F_- - \ln F_+ = \ln G$, where we may take any branch of the logarithm to arrive at a single valued function $\ln G \in C^{0,\alpha}(\Gamma)$. According to Theorem 7.8 we set

$$\psi(z) := \frac{1}{2\pi i} \int_\Gamma \frac{\ln G(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Gamma,$$

and $F(z) := e^{\psi(z)}$, $z \in \mathbb{C} \setminus \Gamma$. Then, from $\psi_- - \psi_+ = \ln G$ on Γ we derive $F_- = GF_+$ on Γ , and since $\psi(z) \rightarrow 0$, $z \rightarrow \infty$, there follows $F(z) \rightarrow 1$, $z \rightarrow \infty$. Now the function f_0 , defined by,

$$f_0(z) := \begin{cases} F(z), & z \in D_-, \\ (z - a)^{-\kappa} F(z), & z \in D_+, \end{cases}$$

has the required properties.

Let f be any other sectionally holomorphic function satisfying the homogeneous boundary condition $f_- = gf_+$ on Γ . Then the quotient $q := f/f_0$ is sectionally holomorphic (observe $f_0(z) \neq 0$ for all $z \in \mathbb{C}$) satisfying $q_- = q_+$ on Γ . Hence, by Morera's theorem, q is an entire function. From $z^\kappa f_0(z) \rightarrow 1$ and $z^\kappa f(z) \rightarrow 1$ for $z \rightarrow \infty$ it follows that $q(z) \rightarrow 1$ for $z \rightarrow \infty$, and Liouville's theorem implies $q = 1$ on \mathbb{C} . \square

From the proof we note that the general sectionally holomorphic function f satisfying the homogeneous boundary condition $f_- = gf_+$ on Γ is obtained from the canonical solution f_0 by multiplying it with an arbitrary entire function.

Theorem 7.16. *Under the assumptions of Theorem 7.15, the homogeneous Riemann problem admits $\max(\kappa, 0)$ linearly independent solutions.*

Proof: For p an entire function, the product $f := pf_0$ vanishes at infinity if and only if

$$\lim_{z \rightarrow \infty} z^{-\kappa} p(z) = 0.$$

Hence, p must be a polynomial of degree less than or equal to $\kappa - 1$ if $\kappa > 0$ and it must be zero if $\kappa \leq 0$. \square

By the procedure used in Theorem 7.15 it is also possible to treat the inhomogeneous Riemann problem. But here we will proceed differently and use Theorem 5.6 to obtain results on the existence of solutions to the inhomogeneous integral equation (7.17), which then by the equivalence stated in Theorem 7.13 imply results on the existence of solutions to the inhomogeneous Riemann problem (see Problem 7.1).

7.4 Singular Integral Equations with Cauchy Kernel

For $0 < \alpha, \beta \leq 1$ by $C^{0,\beta,\alpha}(\Gamma \times \Gamma)$ we denote the set of all functions k defined on $\Gamma \times \Gamma$ satisfying

$$(7.19) \quad |k(z_1, \zeta_1) - k(z_2, \zeta_2)| \leq M(|z_1 - z_2|^\beta + |\zeta_1 - \zeta_2|^\alpha)$$

for all points $z_1, z_2, \zeta_1, \zeta_2 \in \Gamma$ with some constant M depending on k . Let $0 < \alpha < \beta \leq 1$ and let $a \in C^{0,\alpha}(\Gamma)$ and $k \in C^{0,\beta,\alpha}(\Gamma \times \Gamma)$. Then the operator $K : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$, defined by

$$(K\varphi)(z) := a(z)\varphi(z) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(z, \zeta)}{\zeta - z} \varphi(\zeta) d\zeta, \quad z \in \Gamma,$$

is called a *singular integral operator* with a *Cauchy kernel*. And the operator $K^0 : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$, given by

$$(K^0\varphi)(z) := a(z)\varphi(z) + \frac{b(z)}{\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \Gamma,$$

where $b(z) := k(z, z)$, $z \in \Gamma$, is called the *dominant part* of the operator K . For the coefficients a and b of the dominant part we will assume that $a^2 - b^2$ vanishes nowhere on Γ . The dominant part can be written in the short form $K^0 = aI + bA$ and is bounded by Corollary 7.7. The splitting of the operator K into its dominant part K^0 and the remainder $K - K^0$ is justified by the following

Theorem 7.17. *The singular integral operator K is bounded and the difference between K and its dominant part K^0 is compact from $C^{0,\alpha}(\Gamma)$ into $C^{0,\alpha}(\Gamma)$.*

Proof: We choose $0 < \gamma < \alpha$ and show that the difference $H := K - K^0$ is bounded from $C^{0,\gamma}(\Gamma)$ into $C^{0,\alpha}(\Gamma)$. Then the assertion follows from Theorems 2.15 and 7.4.

We write

$$(H\varphi)(z) = \frac{1}{\pi i} \int_{\Gamma} h(z, \zeta) \varphi(\zeta) d\zeta, \quad z \in \Gamma,$$

with kernel

$$(7.20) \quad h(z, \zeta) := \frac{k(z, \zeta) - k(z, z)}{\zeta - z}, \quad z \neq \zeta.$$

Since $k \in C^{0,\beta,\alpha}(\Gamma \times \Gamma)$ there exists a constant M such that

$$(7.21) \quad |h(z, \zeta)| \leq M|z - \zeta|^{\alpha-1}.$$

Hence, the kernel h is weakly singular and therefore we have

$$(7.22) \quad \|H\varphi\|_\infty \leq C_0 \|\varphi\|_\infty$$

with some constant C_0 .

To establish Hölder continuity, we proceed analogously to the proof of Theorem 7.6. Let $z_1, z_2 \in \Gamma$ with $0 < |z_1 - z_2| \leq R/4$ and set $\rho := 4|z_1 - z_2|$. Then, using (7.21), we can estimate

$$(7.23) \quad \begin{aligned} & \left| \int_{\Gamma(z_1; \rho)} \{h(z_1, \zeta) - h(z_2, \zeta)\} \varphi(\zeta) d\zeta \right| \\ & \leq M \|\varphi\|_\infty \int_{\Gamma(z_1; \rho)} \{|z_1 - \zeta|^{\alpha-1} + |z_2 - \zeta|^{\alpha-1}\} ds \\ & \leq C'_1 \|\varphi\|_\infty \int_0^{2\rho} \sigma^{\alpha-1} d\sigma \leq C_1 \|\varphi\|_\infty |z_1 - z_2|^\alpha \end{aligned}$$

with constants C_1, C'_1 depending on M and α . Using (7.19), we can estimate

$$(7.24) \quad \begin{aligned} & \left| \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{k(z_1, \zeta) - k(z_2, \zeta)}{\zeta - z_1} \varphi(\zeta) d\zeta \right| \\ & \leq M \|\varphi\|_\infty |z_1 - z_2|^\beta \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{ds}{|\zeta - z_1|} \leq C'_2 \|\varphi\|_\infty |z_1 - z_2|^\beta \left\{ \int_{\rho/2}^R \frac{d\sigma}{\sigma} + \frac{|\Gamma|}{R} \right\} \\ & \leq C''_2 \|\varphi\|_\infty |z_1 - z_2|^\beta \left\{ \ln \frac{1}{\rho} + 1 \right\} \leq C_2 \|\varphi\|_\infty |z_1 - z_2|^\alpha \end{aligned}$$

with some constants C_2, C'_2, C''_2 depending on M, α, β and Γ . Since for $\zeta \in \Gamma$ with $|\zeta - z_1| \geq \rho$ there holds $|\zeta - z_1| \leq 2|\zeta - z_2|$ we derive the further estimate

$$(7.25) \quad \begin{aligned} & \left| \int_{\Gamma \setminus \Gamma(z_1; \rho)} \{k(z_2, \zeta) - k(z_2, z_2)\} \left\{ \frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2} \right\} \varphi(\zeta) d\zeta \right| \\ & \leq M \|\varphi\|_\infty |z_1 - z_2| \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{ds}{|\zeta - z_1||\zeta - z_2|^{1-\alpha}} \\ & \leq C'_3 \|\varphi\|_\infty |z_1 - z_2| \left\{ \int_{\rho/2}^R \sigma^{\alpha-2} d\sigma + \frac{|\Gamma|}{R^{2-\alpha}} \right\} \leq C_3 \|\varphi\|_\infty |z_1 - z_2|^\alpha \end{aligned}$$

with some constants C_3, C'_3 depending on M, α and Γ . Finally, we write

$$\int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{\varphi(\zeta)}{\zeta - z_1} d\zeta = \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{\varphi(\zeta) - \varphi(z_1)}{\zeta - z_1} d\zeta + \varphi(z_1) \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{d\zeta}{\zeta - z_1}$$

and note from the proof of Theorem 7.5 that the second integral on the right hand side is bounded by 2π . For the first integral we have

$$\left| \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{\varphi(\zeta) - \varphi(z_1)}{\zeta - z_1} d\zeta \right| \leq |\varphi|_\gamma \int_{\Gamma} \frac{ds}{|\zeta - z_1|^{1-\gamma}} \leq C'_4 |\varphi|_\gamma$$

with some constant C'_4 depending on Γ and γ . Hence, using again (7.19), we obtain

$$(7.26) \quad \left| \{k(z_1, z_1) - k(z_2, z_2)\} \int_{\Gamma \setminus \Gamma(z_1; \rho)} \frac{\varphi(\zeta)}{\zeta - z_1} d\zeta \right| \leq C_4 \|\varphi\|_\gamma |z_1 - z_2|^\alpha$$

with some constant C_4 depending on M, α and Γ . Combining (7.23) to (7.26), we obtain

$$(7.27) \quad \pi |(H\varphi)(z_1) - (H\varphi)(z_2)| \leq (C_1 + C_2 + C_3 + C_4) \|\varphi\|_\gamma |z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \Gamma$ with $|z_1 - z_2| \leq R/4$. Now the desired result

$$\|H\varphi\|_\alpha \leq C \|\varphi\|_\gamma$$

follows from (7.22) and (7.27) with the aid of Lemma 7.3.

Theorem 7.18. *The operators $K^0 = aI + bA$ and $K^{0'} = aI - Ab$ are adjoint with respect to the dual system $(C^{0,\alpha}(\Gamma), C^{0,\alpha}(\Gamma))$. They both have finite dimensional nullspaces and the index of the operator K^0 is given by*

$$\text{ind } K^0 = \text{ind } \frac{a - b}{a + b}.$$

Proof: That K^0 and $K^{0'}$ are adjoint is easily derived as a consequence of Theorem 7.11. From Theorems 7.13 and 7.16 we obtain

$$(7.28) \quad \dim N(K^0) = \max(\kappa, 0)$$

where

$$\kappa := \text{ind } \frac{a - b}{a + b}.$$

Analogous to Theorem 7.13, the homogeneous adjoint equation

$$(7.29) \quad a\psi - Ab\psi = 0$$

is equivalent to the Riemann problem with homogeneous boundary condition

$$(7.30) \quad (a - b)f_- = (a + b)f_+ \quad \text{on } \Gamma.$$

Let ψ solve the integral equation (7.29). Then $\varphi := b\psi$ solves $a\varphi - bA\varphi = 0$ and therefore, by Theorem 7.13, the sectionally holomorphic function f defined by (7.11) satisfies the boundary condition (7.30). Conversely, let f be a solution to the Riemann problem with boundary conditions (7.30). Then for the function $\psi := 2f_-/(a + b)$ we have $f_- + f_+ = a\psi$ and $f_- - f_+ = b\psi$. Hence, by Theorem 7.9, we find $a\psi - Ab\psi = f_- + f_+ - A(f_- - f_+) = 0$.

With the aid of $\text{ind } (a - b)/(a + b) = -\text{ind } (a + b)/(a - b)$, Theorem 7.16 now yields

$$(7.31) \quad \dim N(K^{0'}) = \max(-\kappa, 0)$$

and $\text{ind } K^0 = \dim N(K^0) - \dim N(K^{0'}) = \kappa$ follows by combining (7.28) and (7.31). \square

From now on, for symmetry, we will assume that $k \in C^{0,\beta,\beta}(\Gamma \times \Gamma)$ with $0 < \alpha < \beta \leq 1$. The singular integral operators with Cauchy kernel

$$(K\varphi)(z) := a(z)\varphi(z) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(z, \zeta)}{\zeta - z} \varphi(\zeta) d\zeta, \quad z \in \Gamma,$$

and

$$(K'\psi)(z) := a(z)\psi(z) - \frac{1}{\pi i} \int_{\Gamma} \frac{k(\zeta, z)}{\zeta - z} \psi(\zeta) d\zeta, \quad z \in \Gamma,$$

are adjoint. This follows by writing $K = K_0 + H$ and $K' = K^{0'} + H'$ where H and H' have weakly singular kernels $h(z, \zeta)$ and $h(\zeta, z)$, respectively, with h given by (7.20). For weakly singular kernels, as in Theorem 4.6, the order of integration may be interchanged.

Theorem 7.19. *The operator $(a^2 - b^2)^{-1}K'$ is a regularizer of K .*

Proof: First, we observe that for each $c \in C^{0,\alpha}(\Gamma)$ the difference $Ac - cA$ is compact from $C^{0,\alpha}(\Gamma)$ into $C^{0,\alpha}(\Gamma)$. This follows from Theorem 7.17, applied to the kernel $c(\zeta)$ which is contained in $C^{0,\beta,\alpha}(\Gamma \times \Gamma)$. Actually, the need for the compactness of the commutator $Ac - cA$ was our reason for allowing different exponents α and β in Theorem 7.17. Now, using $A^2 = I$, we derive

$$K^{0'}K^0 = (aI - Ab)(aI + bA) = (a^2 - b^2)I + M$$

where $M := abA - Aba - Ab^2A + b^2A^2$ is compact, i.e., $(a^2 - b^2)^{-1}K^{0'}$ is a left regularizer of K^0 . Analogously,

$$K^0K^{0'} = (aI + bA)(aI - Ab) = (a^2 - b^2)I + \tilde{M}$$

where $\tilde{M} := bAa - aAb$ is compact, i.e., $K^{0'}(a^2 - b^2)^{-1}$ is a right regularizer of K^0 . Again, the difference $(a^2 - b^2)^{-1}K^{0'} - K^0(a^2 - b^2)^{-1}$ is compact. Therefore, $(a^2 - b^2)^{-1}K^{0'}$ is a regularizer of K^0 . Now the assertion of the theorem follows from $K = K^0 + H$ and $K = K^{0'} + H'$ with compact H and H' . \square

We now are in the position to derive the classical Noether theorems from our general theory on regularization in dual systems of Chapter 5.

Theorem 7.20 (First Noether Theorem). *The singular integral operator with Cauchy kernel has a finite dimensional nullspace.*

Proof: Theorems 5.6 and 7.19.

Theorem 7.21 (Second Noether Theorem). *The index of the singular integral operator K with Cauchy kernel is given by*

$$\text{ind } K = \text{ind } \frac{a - b}{a + b}.$$

Proof: Theorems 7.18 and 7.19 and Corollary 5.10.

Theorem 7.22 (Third Noether Theorem). *The inhomogeneous singular integral equation*

$$K\varphi = h$$

is solvable if and only if

$$\langle h, \psi \rangle = 0$$

is satisfied for all $\psi \in N(K')$.

Proof: Theorems 5.6 and 7.19.

Theorem 7.23. *Let $0 < \alpha < \beta \leq 1$, let $a \in C^{0,\alpha}(\Gamma)$ and $k \in C^{0,\beta,\beta}(\Gamma \times \Gamma)$ and assume that $a^2 - b^2$ vanishes nowhere on Γ , where $b(z) := k(z, z)$, $z \in \Gamma$. Then, in the Hölder space $C^{0,\alpha}(\Gamma)$, the number of linearly independent solutions of the homogeneous singular integral equation*

$$a(z)\varphi(z) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(z, \zeta)}{\zeta - z} \varphi(\zeta) d\zeta = 0, \quad z \in \Gamma,$$

and of its adjoint equation

$$a(z)\psi(z) - \frac{1}{\pi i} \int_{\Gamma} \frac{k(\zeta, z)}{\zeta - z} \psi(\zeta) d\zeta = 0, \quad z \in \Gamma,$$

are both finite and their difference is given by the index

$$\frac{1}{2\pi i} \ln \left. \frac{a - b}{a + b} \right|_{\Gamma}.$$

The inhomogeneous singular integral equation

$$a(z)\varphi(z) + \frac{1}{\pi i} \int_{\Gamma} \frac{k(z, \zeta)}{\zeta - z} \varphi(\zeta) d\zeta = h(z), \quad z \in \Gamma,$$

is solvable if and only if

$$\int_{\Gamma} h(z)\psi(z) dz = 0$$

is satisfied for all solutions ψ of the homogeneous adjoint equation.

Proof: This is a reformulation of the three preceding theorems. \square

For equations of the first kind, i.e., equations with $a = 0$, we note the following

Corollary 7.24. *Singular integral operators of the first kind with Cauchy kernel have index zero. In particular, injective singular integral operators of the first kind are bijective with bounded inverse.*

Proof: The index zero follows from Theorem 7.21. The bounded inverse for an injective operator of the first kind is a consequence of the Riesz theory Theorem 3.4 together with the fact that A provides an equivalent regularizer. \square

With the aid of Example 7.12, choosing for Γ the unit circle, we obtain the following corollary, which contains the equations for which Noether [1] proved the theorems named after him.

Corollary 7.25. *Let a and k be real valued and 2π -periodic and assume that $a^2 + b^2$ is strictly positive, where $b(t) := k(t, t)$ for $t \in [0, 2\pi]$. Then for the singular integral equation with Hilbert kernel*

$$a(t)\varphi(t) - \frac{1}{2\pi} \int_0^{2\pi} k(t, \tau) \cot \frac{\tau - t}{2} \varphi(\tau) d\tau = h(t), \quad t \in [0, 2\pi],$$

and its adjoint equation

$$a(t)\psi(t) + \frac{1}{2\pi} \int_0^{2\pi} k(\tau, t) \cot \frac{\tau - t}{2} \psi(\tau) d\tau = 0, \quad t \in [0, 2\pi],$$

the three Noether theorems are valid, that is, the numbers of linearly independent solutions of the homogeneous equation and the homogeneous adjoint equation are both finite and their difference is given by the index

$$\frac{1}{\pi} \arctan \frac{b}{a} \Big|_0^{2\pi}.$$

The inhomogeneous equation is solvable if and only if

$$\int_0^{2\pi} h(t)\psi(t) dt = 0$$

is satisfied for all solutions ψ of the homogeneous adjoint equation.

Proof: After transforming them onto the unit circle Γ by setting $z = e^{it}$, $\zeta = e^{i\tau}$, the two equations read

$$\tilde{a}(z)\tilde{\varphi}(z) - \frac{1}{\pi} \int_{\Gamma} \frac{\tilde{k}(z, \zeta)}{\zeta - z} \tilde{\varphi}(\zeta) d\zeta = \tilde{h}(z), \quad z \in \Gamma,$$

and

$$\tilde{a}(z)\tilde{\psi}(z) + \frac{1}{\pi} \int_{\Gamma} \frac{\tilde{k}(\zeta, z)}{\zeta - z} \tilde{\psi}(\zeta) d\zeta = 0, \quad z \in \Gamma,$$

for $\tilde{\varphi}(e^{it}) := \varphi(t)$ and $\tilde{\psi}(e^{it}) := \psi(t)/e^{it}$ and where we have set $\tilde{a}(e^{it}) := a(t)$, $\tilde{h}(e^{it}) := h(t)$ and $\tilde{k}(e^{it}, e^{i\tau}) := (e^{it} + e^{i\tau})k(t, \tau)/2e^{i\tau}$. Hence, the corollary is established through transforming the Noether theorems for the pair of equations on the unit circle back to the equations on $[0, 2\pi]$. For the index we compute

$$\ln \frac{a + ib}{a - ib} \Big|_0^{2\pi} = 2 \ln(a + ib) \Big|_0^{2\pi} = 2i \arctan \frac{b}{a} \Big|_0^{2\pi}.$$

7.5 Cauchy Integral and Logarithmic Potential

We now want to utilize the close connection between holomorphic and two-dimensional harmonic functions to derive regularity results on the logarithmic single- and double-layer potential from the Sokhotski–Plemelj theorem.

In the Cauchy integral for fixed $z \in \mathbb{C}$ we substitute $\zeta - z = re^{i\vartheta}$ where $r = |\zeta - z|$ and $\vartheta = \arg(\zeta - z)$. Then

$$\frac{d\zeta}{\zeta - z} = d\ln(\zeta - z) = d(\ln r + i\vartheta) = \left(\frac{\partial \ln r}{\partial s} + i \frac{\partial \vartheta}{\partial s} \right) ds.$$

By the Cauchy–Riemann equations, applied to $\ln(\zeta - z) = \ln r + i\vartheta$, we have

$$\frac{\partial \vartheta}{\partial s} = \frac{\partial \ln r}{\partial n} = -\frac{\partial}{\partial n} \ln \frac{1}{r}.$$

Consequently,

$$(7.32) \quad \begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta &= -\frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta) \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial s(\zeta)} \ln \frac{1}{r} ds(\zeta). \end{aligned}$$

This formula indicates that for real valued densities the real part of the Cauchy integral coincides with the logarithmic double-layer potential

$$v(z) := \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma.$$

Hence, from Theorem 7.6 and the Sokhotski–Plemelj formula (7.12), we derive the following

Theorem 7.26. *The logarithmic double-layer potential v with Hölder continuous density φ can be extended uniformly Hölder continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- with limiting values*

$$(7.33) \quad v_{\pm}(z) = \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta) \pm \frac{1}{2} \varphi(z), \quad z \in \Gamma.$$

Complex integration and contour integration along Γ is related by

$$(7.34) \quad dz = e^{i\gamma(z)} ds(z)$$

for all $z \in \Gamma$ where $\gamma(z)$ denotes the angle between the tangent to Γ at the point z and the real axis. Since Γ is of class C^2 this angle depends continuously differentiable on z . Therefore, from Theorem 7.6 we immediately deduce that

$$(7.35) \quad f(z) := \frac{1}{2\pi} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma,$$

is holomorphic in D_+ and in D_- and that it can be extended uniformly Hölder continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- with limiting values

$$(7.36) \quad f_{\pm}(z) := \frac{1}{2\pi} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} ds(\zeta) \mp \frac{i}{2} e^{-i\gamma(z)} \varphi(z), \quad z \in \Gamma.$$

The gradient of the logarithmic single-layer potential

$$u(z) := \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \ln \frac{1}{r} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma,$$

is given by

$$\operatorname{grad} u(z) = \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \frac{\zeta - z}{|\zeta - z|^2} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma.$$

Note that we use the same symbol z for the complex number $z = x_1 + ix_2$ and the vector $z = (x_1, x_2)$. We observe that the gradient of the single-layer potential u is the complex conjugate of the holomorphic function f given by the Cauchy integral (7.35). Therefore, we can again use Theorem 7.6 to state

Theorem 7.27. *The first derivatives of the logarithmic single-layer potential u with Hölder continuous density can be extended uniformly Hölder continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- with limiting values*

$$(7.37) \quad \operatorname{grad} u_{\pm}(z) = \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \operatorname{grad} \ln \frac{1}{r} ds(\zeta) \mp \frac{1}{2} n(z) \varphi(z), \quad z \in \Gamma.$$

Here, we have used that the complex conjugate of $ie^{-i\gamma(z)}$ is equal to the normal $n(z)$ at $z \in \Gamma$. In particular, we see that the logarithmic single-layer integral operator S , defined by

$$(7.38) \quad (S\varphi)(z) := \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \ln \frac{1}{r} ds(\zeta), \quad z \in \Gamma,$$

is a bounded operator from $C^{0,\alpha}(\Gamma)$ into $C^{1,\alpha}(\Gamma)$. By $C^{1,\alpha}(\Gamma)$ we denote the normed space of all functions defined on Γ which have a uniformly Hölder continuous first derivative furnished with the norm

$$\|\varphi\|_{1,\alpha} := \|\varphi\|_{\infty} + \|\varphi'\|_{0,\alpha},$$

where the prime indicates differentiation with respect to the arclength.

Consider once again the logarithmic double-layer potential v and assume the density $\varphi \in C^{1,\alpha}(\Gamma)$. We write

$$\begin{aligned} \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} &= \left(n(\zeta), \operatorname{grad}_{\zeta} \ln \frac{1}{r} \right) = - \left(n(\zeta), \operatorname{grad}_z \ln \frac{1}{r} \right) \\ &= -t_2(\zeta) \frac{\partial}{\partial x_1} \ln \frac{1}{r} + t_1(\zeta) \frac{\partial}{\partial x_2} \ln \frac{1}{r}, \end{aligned}$$

where we have made use of the relation $n = (t_2, -t_1)$ between the tangent vector $t = (t_1, t_2)$ and the normal vector n to Γ . From this, using $\Delta \ln 1/r = 0$, we see that

$$\operatorname{grad}_z \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} = \left(\frac{\partial}{\partial x_2} (\operatorname{grad}_z \ln \frac{1}{r}, t(\zeta)), - \frac{\partial}{\partial x_1} (\operatorname{grad}_z \ln \frac{1}{r}, t(\zeta)) \right).$$

Then for the gradient of the double-layer potential we obtain

$$\operatorname{grad} v(z) = \left(\frac{\partial}{\partial x_2} w(z), -\frac{\partial}{\partial x_1} w(z) \right),$$

where

$$\begin{aligned} w(z) &= \frac{1}{2\pi} \int_{\Gamma} (\operatorname{grad}_z \ln \frac{1}{r}, t(\zeta)) \varphi(\zeta) ds(\zeta) \\ &= \frac{1}{2\pi} \int_{\Gamma} \frac{d\varphi}{ds}(\zeta) \ln \frac{1}{r} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma, \end{aligned}$$

is the single-layer potential with density $d\varphi/ds \in C^{0,\alpha}(\Gamma)$. Therefore, from Theorem 7.27 we derive

Theorem 7.28. *The first derivatives of the logarithmic double-layer potential v with density $\varphi \in C^{1,\alpha}(\Gamma)$ can be extended uniformly Hölder continuously from D_+ into \bar{D}_+ and from D_- into \bar{D}_- . The normal derivative is given by*

$$(7.39) \quad \frac{\partial v_{\pm}}{\partial n}(z) = \frac{1}{2\pi} \frac{d}{ds(z)} \int_{\Gamma} \frac{d\varphi}{ds}(\zeta) \ln \frac{1}{r} ds(\zeta), \quad z \in \Gamma,$$

and the tangential derivative by

$$(7.40) \quad \frac{\partial v_{\pm}}{\partial s}(z) = -\frac{1}{2\pi} \int_{\Gamma} \frac{d\varphi}{ds}(\zeta) \frac{\partial}{\partial n(z)} \ln \frac{1}{r} ds(\zeta) \pm \frac{1}{2} \frac{d\varphi}{ds}(z), \quad z \in \Gamma.$$

This implies that the operator T , defined by the normal derivative of the double-layer logarithmic potential

$$(7.41) \quad (T\varphi)(z) := \frac{1}{2\pi} \frac{\partial}{\partial n(z)} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta), \quad z \in \Gamma,$$

can be expressed through the single-layer integral operator S in the form

$$(7.42) \quad T\varphi = \frac{d}{ds} S \frac{d\varphi}{ds}.$$

Hence, T is a bounded operator from $C^{1,\alpha}(\Gamma)$ into $C^{0,\alpha}(\Gamma)$.

We wish to mention that these results on the regularity of single- and double-layer potentials with uniformly Hölder continuous densities can be extended to the three-dimensional case by techniques similar to those used in the proofs of Theorems 7.6 and 7.17 (see Colton and Kress [1]).

Now we want to demonstrate how the two-dimensional Dirichlet and Neumann problem for harmonic functions can be solved through the use of integral equations of the first kind. The single-layer potential

$$(7.43) \quad u(z) = \frac{1}{2\pi} \int_{\Gamma} \left\{ \psi(\zeta) - \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds \right\} \ln \frac{1}{r} ds(\zeta) + \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds, \quad z \in \mathbb{C},$$

solves the interior and the exterior Dirichlet problem with boundary condition $u = f$ on Γ provided the density $\psi \in C^{0,\alpha}(\Gamma)$ solves the equation

$$(7.44) \quad \frac{1}{2\pi} \int_{\Gamma} \left\{ \psi(\zeta) - \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds \right\} \ln \frac{1}{r} ds(\zeta) + \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds = f(z), \quad z \in \Gamma.$$

We abbreviate (7.44) into the form $S_0\psi = f$ where we have set

$$(7.45) \quad S_0\psi := S \left(\psi - \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds \right) + \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds.$$

Note that our modification of the logarithmic single-layer potential in (7.43) ensures the correct behaviour of u at infinity for the exterior problem. As we will see it is also crucial for the interior problem since it ensures uniqueness. Since S maps $C^{0,\alpha}(\Gamma)$ into $C^{1,\alpha}(\Gamma)$, the equation (7.44) can be solved only for inhomogeneities $f \in C^{1,\alpha}(\Gamma)$.

Theorem 7.29. *The modified single-layer operator $S_0 : C^{0,\alpha}(\Gamma) \rightarrow C^{1,\alpha}(\Gamma)$ is bijective with bounded inverse $S_0^{-1} : C^{1,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$.*

Proof: Let $f \in C^{1,\alpha}(\Gamma)$ and assume that $\psi \in C^{0,\alpha}(\Gamma)$ solves

$$(7.46) \quad S_0\psi = f$$

Then ψ also solves

$$(7.47) \quad \frac{d}{ds} S_0\psi + \int_{\Gamma} S_0\psi ds = \frac{df}{ds} + \int_{\Gamma} f ds.$$

The bounded operator $R : C^{1,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$, given by

$$Rf := \frac{df}{ds} + \int_{\Gamma} f ds,$$

is injective, since $Rf = 0$, by integration, implies $\int_{\Gamma} f ds = 0$ and $df/ds = 0$. Hence, the equations (7.46) and (7.47) are equivalent. By the jump relation (7.37), we may differentiate under the integral in (7.44) to see that equation (7.47) has the form of a singular integral equation with Cauchy kernel

$$(7.48) \quad \frac{1}{2\pi} \int_{\Gamma} \frac{k(z, \zeta)}{\zeta - z} \psi(\zeta) ds(\zeta) + (H\psi)(z) = \frac{df}{ds}(z) + \int_{\Gamma} f ds, \quad z \in \Gamma.$$

Here, $H : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$ is a compact operator, and the kernel k is given by

$$k(z, \zeta) = \frac{\partial}{\partial s(z)} \ln \frac{1}{r} = \frac{(t(z), \zeta - z)}{|\zeta - z|^2} (\zeta - z), \quad z \neq \zeta.$$

With the aid of the identity $\bar{a}b + a\bar{b} = 2(a, b)$ for complex numbers a and b , we rewrite the kernel into the form

$$k(z, \zeta) = \frac{1}{2} t(z) + \frac{1}{2} \overline{t(z)} \frac{(\zeta - z)^2}{|\zeta - z|^2}, \quad z \neq \zeta.$$

Since Γ is of class C^2 , the tangent vector t is continuously differentiable on Γ , and using a parametric representation, it can be seen that

$$\frac{(\zeta - z)^2}{|\zeta - z|^2}, \quad z \neq \zeta,$$

can be extended as a continuously differentiable function onto $\Gamma \times \Gamma$. This implies $k \in C^{0,1,1}(\Gamma \times \Gamma)$. Hence, we can apply Noether's theorems in the form of Problem 7.3. Since the integral equation is of the first kind its index is zero.

Let ψ be a solution to the homogeneous equation (7.47). Then ψ also satisfies the homogeneous form of (7.46) and therefore the single-layer potential u given by (7.43) solves the homogeneous interior and exterior Dirichlet problem. Hence, by the uniqueness Theorem 6.10, we have $u = 0$ in D_+ and in D_- . From the behaviour at infinity we deduce $\int_{\Gamma} \psi ds = 0$ and then the jump relations imply $\psi = 0$ on Γ . Now the assertion of the theorem follows from Corollary 7.24 and the boundedness of the differentiation operator R . \square

We now turn to the Neumann problem. The double-layer potential

$$(7.49) \quad u(z) = \frac{1}{2\pi} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta), \quad z \in \mathbb{C} \setminus \Gamma,$$

solves the interior and exterior Neumann problem with boundary condition $\partial u / \partial n = g$ on Γ provided the density $\varphi \in C^{1,\alpha}(\Gamma)$ solves the integral equation

$$(7.50) \quad \frac{1}{2\pi} \frac{\partial}{\partial n(z)} \int_{\Gamma} \varphi(\zeta) \frac{\partial}{\partial n(\zeta)} \ln \frac{1}{r} ds(\zeta) = g(z), \quad z \in \Gamma,$$

Theorem 7.30. *For each $g \in C^{0,\alpha}(\Gamma)$ satisfying the solvability condition*

$$(7.51) \quad \int_{\Gamma} g ds = 0$$

of the interior and exterior two-dimensional Neumann problem there exists a solution $\varphi \in C^{1,\alpha}(\Gamma)$ to the integral equation of the first kind (7.50). Two solutions can differ only by a constant.

Proof: The condition (7.51) ensures the existence of a single-valued function $G \in C^{1,\alpha}(\Gamma)$ with $dG/ds = g$. Now consider the equation $\tilde{S}\psi = G$. By Theorem 7.29 it is uniquely solvable. We set $\chi := \psi - \int_{\Gamma} \psi ds / |\Gamma|$. The condition $\int_{\Gamma} \chi ds = 0$ ensures the existence of a single-valued function $\varphi \in C^{1,\alpha}(\Gamma)$ such that $d\varphi/ds = \chi$. Then

$$T\varphi = \frac{d}{ds} S \frac{d\varphi}{ds} = \frac{d}{ds} S\chi = \frac{d}{ds} \tilde{S}\psi = \frac{dG}{ds} = g.$$

It is left as an exercise to the reader to show that the nullspace of T is given by $N(T) = \text{span}\{1\}$. \square

Our analysis follows Hsiao and MacCamy [1] who modified an earlier approach by Fichera [1] to make the integral equations uniquely solvable.

7.6 Supplements

We conclude this chapter, which was intended to serve as a first step into singular integral equations with Cauchy kernels, with a few references on the broad variety of extensions. First, we wish to mention that these integral equations can be treated in various other normed function spaces. In this context we make use of the Lax Theorem 4.12 to establish the boundedness of the Cauchy integral operator A in $L^2(\Gamma)$.

Theorem 7.31. *The Cauchy integral operator A is bounded from $L^2(\Gamma)$ into $L^2(\Gamma)$.*

Proof: We want to apply the Lax Theorem 4.12 in the positive dual system $(C^{0,\alpha}(\Gamma), C^{0,\alpha}(\Gamma))$ generated by the scalar product

$$(\varphi, \psi) := \int_{\Gamma} \varphi \bar{\psi} \, ds, \quad \varphi, \psi \in C^{0,\alpha}(\Gamma).$$

Therefore, we need the adjoint A^* of the Cauchy integral operator A with respect to this scalar product. Using Theorem 7.11 and the relation (7.34) between complex and contour integration, we see that

$$\int_{\Gamma} (A\varphi) \bar{\psi} \, ds = - \int_{\Gamma} \varphi \chi A(\overline{\psi \chi}) \, ds$$

where $\chi(z) := e^{i\gamma z}$, $z \in \Gamma$. Hence, the adjoint A^* is given by

$$A^* \psi = -\overline{\chi A(\overline{\psi \chi})},$$

and $A^* : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$ is bounded since $A : C^{0,\alpha}(\Gamma) \rightarrow C^{0,\alpha}(\Gamma)$ is bounded. Now the statement follows from Theorem 4.12 and the denseness of $C^{0,\alpha}(\Gamma)$ in $L^2(\Gamma)$. \square

In particular, as a consequence of Theorem 7.31, the property $A^2 = I$ of Theorem 7.10 remains valid in $L^2(\Gamma)$ and can be used to construct regularizers as in Theorem 7.19.

Next, we wish to draw attention to the extensions of the theory to open contours, for example to singular integral equations on the interval $[-1, 1]$. Similar to the case of closed contours treated in this chapter, these equations can be related to a Riemann problem for holomorphic functions in the complex plane \mathbb{C} cut along the segment $[-1, 1]$. In this case, at the endpoints of the interval there occur branchpoint singularities and the nature of these singularities plays a central role, in particular, it determines the index of the operators involved. The actual analysis requires a careful study of the behaviour of the Cauchy integral at the endpoints of the contour for densities which are singular at these endpoints and carrying out these details is beyond the aim of our introduction.

Finally, we mention the extensions to systems of one-dimensional singular integral equations with Cauchy kernels and the extensions of the boundary value problems from the case of the Cauchy–Riemann equations to more general elliptic systems in the framework of generalized holomorphic functions. For a

comprehensive study we refer to Gakhov [1], Gilbert and Buchanan [1], Meister [1], Michlin and Prössdorf [1], Muskhelishvili [1], Prössdorf [1], Vekua [1] and Wendland [4].

Problems

- 7.1. Let $g \in C^{0,\alpha}(\Gamma)$ be a nowhere vanishing function. Show that the inhomogeneous Riemann problem with boundary conditions

$$f_- = gf_+ + h \quad \text{on } \Gamma$$

is solvable if and only if the condition

$$\int_{\Gamma} h \tilde{f}_- dz = 0$$

is satisfied for all solutions \tilde{f} to the homogeneous adjoint Riemann problem with boundary condition

$$\tilde{f}_+ = g \tilde{f}_- \quad \text{on } \Gamma.$$

- 7.2. Use a regular parametric representation $x(t) = (x_1(t), x_2(t))$, $0 \leq t \leq 2\pi$, with counter-clockwise orientation for the boundary curve to transform the integral equation of the first kind (7.44) for the Dirichlet problem into the form

$$\frac{1}{4\pi} \int_0^{2\pi} \left\{ -\ln 4 \sin^2 \frac{t-\tau}{2} + p(t, \tau) \right\} \tilde{\chi}(\tau) d\tau + c = \tilde{f}(t), \quad 0 \leq t \leq 2\pi,$$

$$\int_0^{2\pi} \tilde{\chi}(\tau) d\tau = 0,$$

where $\tilde{\chi}(t) := \{[\dot{x}_1(t)]^2 + [\dot{x}_2(t)]^2\}^{1/2} \chi(x(t))$, $\chi := \psi - c$, $c := \frac{1}{|\Gamma|} \int_{\Gamma} \psi ds$, $\tilde{f}(t) := f(x(t))$, and where the kernel is given by

$$p(t, \tau) := \ln \frac{4 \sin^2 \frac{t-\tau}{2}}{[x_1(t) - x_1(\tau)]^2 + [x_2(t) - x_2(\tau)]^2}, \quad t \neq \tau.$$

Show that the kernel p is continuously differentiable provided Γ is of class C^2 . Transform the integral equation of the first kind (7.50) for the Neumann problem into the form

$$\frac{1}{4\pi} \int_0^{2\pi} \left\{ \cot \frac{\tau-t}{2} + q(t, \tau) \right\} \frac{d\tilde{\varphi}}{d\tau}(\tau) d\tau = \tilde{g}(t), \quad 0 \leq t \leq 2\pi,$$

where $\tilde{\varphi}(t) := \varphi(x(t))$, $g(t) := \{[\dot{x}_1(t)]^2 + [\dot{x}_2(t)]^2\}^{1/2} g(x(t))$, and where the kernel is given by

$$q(t, \tau) := 2 \frac{\dot{x}_1(t)[x_1(\tau) - x_1(t)] + \dot{x}_2(t)[x_2(\tau) - x_2(t)]}{[x_1(t) - x_1(\tau)]^2 + [x_2(t) - x_2(\tau)]^2} - \cot \frac{\tau-t}{2}, \quad t \neq \tau.$$

Show that the kernel q is continuous provided Γ is of class C^2 .

- 7.3. Formulate and prove Noether's theorems for the singular integral equation with Cauchy kernel where the complex integration is replaced by contour integration.

Hint: Use (7.34).

- 7.4. Evaluate the dominant parts of the singular integral operator occurring in (7.48) and its adjoint. Then use (7.32) together with Theorem 7.17 to deduce that the integral operators K and K' , given by (6.19) and (6.20) are compact from $C^{0,\alpha}(\Gamma)$ into $C^{0,\alpha}(\Gamma)$.

- 7.5. Show that for each real valued function $\varphi \in C^{0,\alpha}(\Gamma)$ there exists a function f which is holomorphic in D_- , which can be extended continuously from D_- into \bar{D}_- and which has real part $\operatorname{Re} f = \varphi$ on the boundary Γ . Show that two functions with this property can differ only by a purely imaginary constant.

Hint: Solve a Dirichlet problem for the real part of f and use (7.32) and Problem 7.4.

8. Sobolev Spaces

In this chapter we want to study the concept of weak solutions to boundary value problems for harmonic functions. We shall extend the classical theory of boundary integral equations as described in the two previous chapters from the spaces of continuous or Hölder continuous functions to appropriate Sobolev spaces. For the sake of brevity we will confine ourselves to interior boundary value problems in two dimensions.

We wish to mention that our introduction of Sobolev spaces of periodic functions and Sobolev spaces on a closed contour differs from the usual approach in two regards. Firstly, instead of using the Fourier transform we only need Fourier series, and secondly, the Sobolev spaces in the contour case are defined via a global rather than a local procedure. The motivation for our approach is the hope that this analysis might make the basic ideas on these Sobolev spaces more easily accessible. We also want to emphasize that we do not rely on Lebesgue integration as a necessary prerequisite. For our purpose it will be sufficient to understand the Hilbert space $L^2[0, 2\pi]$ as the completion of the space of 2π -periodic continuous functions with respect to the mean square norm.

8.1 Fourier Expansion

As basis of our presentation of Sobolev spaces we begin with a brief review on the classical Fourier series expansion. For a function $\varphi \in L^2[0, 2\pi]$ the series

$$(8.1) \quad \sum_{m=-\infty}^{\infty} a_m e^{imt}$$

where

$$a_m := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-imt} dt$$

is called the *Fourier series* of φ , its coefficients a_m are called the *Fourier coefficients* of φ . On $L^2[0, 2\pi]$, as usual, introduce the mean square norm by the scalar product

$$(\varphi, \psi) := \int_0^{2\pi} \varphi(t) \overline{\psi(t)} dt$$

Denote by f_m the trigonometric monomials

$$f_m(t) := e^{imt}$$

for $t \in \mathbb{R}$ and $m \in \mathbb{Z}$. Then, $\{f_m : m \in \mathbb{Z}\}$ is an orthogonal system. By the Weierstrass approximation theorem, the trigonometric polynomials are dense with respect to the maximum norm in the space $C[0, 2\pi]$ of 2π -periodic continuous functions, and $C[0, 2\pi]$ is dense in $L^2[0, 2\pi]$ in the mean square norm. Therefore, by Theorem 1.28, the orthogonal system is complete and the Fourier series (8.1) converges in the mean square norm. Because of the orthonormality factor $\|f_m\|_2^2 = 2\pi$, Parseval's equality assumes the form

$$(8.2) \quad \sum_{m=-\infty}^{\infty} |a_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(t)|^2 dt = \frac{1}{2\pi} \|\varphi\|_2^2.$$

Frequently, we will use the fact that the Fourier series of a continuously differentiable 2π -periodic function converges absolutely and uniformly. By partial integration we see that

$$2\pi a_m = \int_0^{2\pi} \varphi(t) e^{-imt} dt = \frac{1}{im} \int_0^{2\pi} \varphi'(t) e^{-imt} dt, \quad m \neq 0.$$

Therefore Parseval's equality, applied to φ' , yields

$$(8.3) \quad \sum_{m=-\infty}^{\infty} m^2 |a_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi'(t)|^2 dt.$$

Then, by the Cauchy–Schwarz inequality, it follows that

$$\left\{ \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} |a_m e^{imt}| \right\}^2 \leq \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m^2} \sum_{m=-\infty}^{\infty} m^2 |a_m|^2 = \frac{\pi}{6} \int_0^{2\pi} |\varphi'(\tau)|^2 d\tau$$

for all $t \in \mathbb{R}$, which establishes the absolute and uniform convergence of the Fourier series. Its limit must coincide with φ since we already have convergence of the Fourier series to φ in the mean square norm which is weaker than the maximum norm.

8.2 The Sobolev Space $H^p[0, 2\pi]$

We will now define subspaces $H^p[0, 2\pi]$ of $L^2[0, 2\pi]$ by requiring for their elements a certain decay of the Fourier coefficients a_m as $|m| \rightarrow \infty$.

Definition 8.1. Let $0 \leq p < \infty$. By $H^p[0, 2\pi]$ we denote the space of all functions $\varphi \in L^2[0, 2\pi]$ with the property

$$\sum_{m=-\infty}^{\infty} (1 + m^2)^p |a_m|^2 < \infty$$

for the Fourier coefficients a_m of φ . The space $H^p[0, 2\pi]$ is called a Sobolev space. Frequently we will abbreviate $H^p = H^p[0, 2\pi]$. Note that $H^0[0, 2\pi]$ coincides with $L^2[0, 2\pi]$.

Theorem 8.2. *The Sobolev Space $H^p[0, 2\pi]$ is a Hilbert space with the scalar product defined by*

$$(\varphi, \psi)_p := \sum_{m=-\infty}^{\infty} (1+m^2)^p a_m \bar{b}_m$$

for $\varphi, \psi \in H^p[0, 2\pi]$ with Fourier coefficients a_m and b_m , respectively. Note that the norm on $H^p[0, 2\pi]$ is given by

$$\|\varphi\|_p = \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^p |a_m|^2 \right\}^{1/2}.$$

The trigonometric polynomials are dense in $H^p[0, 2\pi]$.

Proof: We leave it as an exercise to verify that H^p is a linear space and that $(\cdot, \cdot)_p$ is a scalar product. That $(\cdot, \cdot)_p$ is well defined follows from the Cauchy-Schwarz inequality

$$\left| \sum_{m=-\infty}^{\infty} (1+m^2)^p a_m \bar{b}_m \right|^2 \leq \sum_{m=-\infty}^{\infty} (1+m^2)^p |a_m|^2 \sum_{m=-\infty}^{\infty} (1+m^2)^p |b_m|^2.$$

To prove that H^p is complete, let (φ_n) be a Cauchy sequence, i.e., given $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $\|\varphi_n - \varphi_k\|_p < \varepsilon$ for all $n, k \geq N(\varepsilon)$, or in terms of the Fourier coefficients $a_{n,m}$ of φ_n ,

$$\sum_{m=-\infty}^{\infty} (1+m^2)^p |a_{m,n} - a_{m,k}|^2 < \varepsilon^2$$

for all $n, k \geq N(\varepsilon)$. From this we observe that for each pair $M_1, M_2 \in \mathbb{N}$ there holds

$$(8.4) \quad \sum_{m=-M_1}^{M_2} (1+m^2)^p |a_{m,n} - a_{m,k}|^2 < \varepsilon^2$$

for all $n, k \geq N(\varepsilon)$. Therefore, since \mathbb{C} is complete, there exists a sequence (a_m) in \mathbb{C} such that $a_{m,n} \rightarrow a_m$, $n \rightarrow \infty$, for each $m \in \mathbb{Z}$. Passing to the limit $k \rightarrow \infty$ in (8.4) now yields

$$\sum_{m=-M_1}^{M_2} (1+m^2)^p |a_{m,n} - a_m|^2 \leq \varepsilon^2$$

for all $n \geq N(\varepsilon)$ and all $M_1, M_2 \in \mathbb{N}$. Hence

$$\sum_{m=-\infty}^{\infty} (1+m^2)^p |a_{m,n} - a_m|^2 \leq \varepsilon^2$$

for all $n \geq N(\varepsilon)$. From this we conclude that

$$\varphi := \sum_{m=-\infty}^{\infty} a_m f_m$$

defines a function $\varphi \in H^p$ with $\|\varphi - \varphi_n\|_p \rightarrow 0$, $n \rightarrow \infty$.

Let $\varphi \in H^p$ with Fourier coefficients a_m . Then for the partial sums φ_n of the Fourier series there holds

$$\|\varphi - \varphi_n\|_p^2 = \sum_{|m|=n+1}^{\infty} (1 + m^2)^p |a_m|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

since the series

$$\sum_{m=-\infty}^{\infty} (1 + m^2)^p |a_m|^2$$

converges. Therefore, the trigonometric polynomials are dense in H^p .

Theorem 8.3. *If $q > p$ then $H^q[0, 2\pi]$ is dense in $H^p[0, 2\pi]$ with compact imbedding from $H^q[0, 2\pi]$ into $H^p[0, 2\pi]$.*

Proof: From $(1 + m^2)^p \leq (1 + m^2)^q$ it follows that $H^q \subset H^p$ with bounded imbedding

$$(8.5) \quad \|\varphi\|_p \leq \|\varphi\|_q$$

for all $\varphi \in H^q$. Then the denseness of H^q in H^p is a consequence of the denseness of the trigonometric polynomials in H^p .

We denote the imbedding operator by $I : H^q \rightarrow H^p$. For $n \in \mathbb{N}$ we define finite dimensional operators $I_n : H^q \rightarrow H^p$ by

$$I_n \varphi := \sum_{m=-n}^n a_m f_m$$

for $\varphi \in H^q$ with Fourier coefficients a_m . Then

$$\begin{aligned} \|(I_n - I)\varphi\|_p^2 &= \sum_{|m|=n+1}^{\infty} (1 + m^2)^p |a_m|^2 \\ &\leq \frac{1}{(1 + n^2)^{q-p}} \sum_{|m|=n+1}^{\infty} (1 + m^2)^q |a_m|^2 \leq \frac{1}{(1 + n^2)^{q-p}} \|\varphi\|_q^2. \end{aligned}$$

Hence, compactness of the imbedding follows from Theorem 2.16. \square

For the introduction of the Sobolev spaces $H^p(\Gamma)$ for a closed contour Γ in \mathbb{R}^2 we shall need additional norms which are equivalent to $\|\cdot\|_p$. We will denote them by $\|\cdot\|_{p,[p]}$ where $[p]$ is the smallest integer less than or equal to p . For their definition we have to distinguish three cases. By $C^k[0, 2\pi]$ we will denote the space of k times continuously differentiable 2π -periodic functions from \mathbb{R} into \mathbb{C} .

Theorem 8.4. *For $k \in \mathbb{N}$ on $C^k[0, 2\pi] \subset H^k[0, 2\pi]$ the norm $\|\cdot\|_k$ is equivalent to*

$$(8.6) \quad \|\varphi\|_{k,k} := \left(\int_0^{2\pi} \{|\varphi(t)|^2 + |\varphi^{(k)}(t)|^2\} dt \right)^{1/2}.$$

Proof: Using

$$\int_0^{2\pi} \varphi^{(k)}(t) e^{-imt} dt = (im)^k \int_0^{2\pi} \varphi(t) e^{-imt} dt$$

and Parseval's equality (8.2) for the Fourier coefficients of φ and $\varphi^{(k)}$ yields

$$\|\varphi\|_{k,k}^2 = 2\pi \sum_{m=-\infty}^{\infty} (1 + m^{2k}) |a_m|^2.$$

Now the inequalities

$$(1 + m^{2k}) \leq (1 + m^2)^k \leq (2m^2)^k \leq 2^k (1 + m^{2k})$$

yield the equivalence of the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,k}$. \square

Theorem 8.4 shows that for integer values of k the Sobolev spaces $H^k[0, 2\pi]$ are homeomorphic to the classical Sobolev spaces given by the completion of the space of k times continuously differentiable functions with respect to the norm $\|\cdot\|_{k,k}$. Two normed spaces are called *homeomorphic* if there exists a bijective linear mapping from one space onto the other which together with its inverse is bounded.

Theorem 8.5. *For $0 < p < 1$ on $C^1[0, 2\pi]$ the norm $\|\cdot\|_p$ is equivalent to*

$$(8.7) \quad \|\varphi\|_{p,0} := \left\{ \int_0^{2\pi} |\varphi(t)|^2 dt + \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(t) - \varphi(\tau)|^2}{\left| \sin \frac{t-\tau}{2} \right|^{2p+1}} d\tau dt \right\}^{1/2}.$$

Proof: We first observe that the norm $\|\cdot\|_{p,0}$ corresponds to the scalar product given by

$$(\varphi, \psi) := \int_0^{2\pi} \varphi(t) \overline{\psi(t)} dt + \int_0^{2\pi} \int_0^{2\pi} \frac{\{\varphi(t) - \varphi(\tau)\} \{\bar{\psi}(t) - \bar{\psi}(\tau)\}}{\left| \sin \frac{t-\tau}{2} \right|^{2p+1}} d\tau dt$$

on $C^1[0, 2\pi]$. Note that the second term on the right is well defined since $C^1[0, 2\pi] \subset C^{0,1}[0, 2\pi]$. Straightforward integration shows that the trigonometric monomials f_m are orthogonal with respect to this scalar product, i.e.,

$$(f_m, f_k) = (2\pi + \gamma_m) \delta_{mk}$$

where

$$\gamma_m := 16\pi \int_0^\pi \frac{\sin^2 \frac{m}{2} t}{\sin^{2p+1} \frac{t}{2}} dt.$$

Using the inequality $2t/\pi < \sin t < t$ for $0 < t < \pi/2$, we can estimate

$$2^{2p+5}\pi \int_0^\pi \frac{\sin^2 \frac{m}{2} t}{t^{2p+1}} dt < \gamma_m < 16\pi^{2p+2} \int_0^\pi \frac{\sin^2 \frac{m}{2} t}{t^{2p+1}} dt.$$

For $m > 0$, we substitute $mt = \tau$ and use

$$0 < \int_0^\pi \frac{\sin^2 \frac{t}{2}}{t^{2p+1}} dt \leq \int_0^{m\pi} \frac{\sin^2 \frac{t}{2}}{t^{2p+1}} dt < \int_0^\infty \frac{\sin^2 \frac{t}{2}}{t^{2p+1}} dt < \infty$$

to obtain

$$(8.8) \quad c_0|m|^{2p} < \gamma_m < c_1|m|^{2p}$$

for all $m \neq 0$ with some constants c_0 and c_1 depending on p .

Using the estimate $|e^{imt} - e^{im\tau}| \leq |m(t - \tau)|$, from the Parseval equality (8.2) for $\varphi \in C^1[0, 2\pi]$ and the Cauchy–Schwarz inequality, we see that the series

$$\varphi(t) - \varphi(\tau) = \sum_{m=-\infty}^{\infty} a_m \{e^{imt} - e^{im\tau}\}$$

is absolutely and uniformly convergent for all $t, \tau \in [0, 2\pi]$. Therefore we can integrate termwise to derive

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(t) - \varphi(\tau)|^2}{\left| \sin \frac{t-\tau}{2} \right|^{2p+1}} d\tau dt = \sum_{m=-\infty}^{\infty} \gamma_m |a_m|^2.$$

Now the statement of the theorem follows from the inequalities (8.8) and

$$(1 + |m|^{2p}) \leq 2|m|^{2p} \leq 2(1 + m^2)^p \leq 2^{p+1}(1 + |m|^{2p})$$

together with Parseval's equality (8.2).

Corollary 8.6. *When $p = k+q$, where $k \in \mathbb{N}$ and $0 < q < 1$, then on $C^{k+1}[0, 2\pi]$ the norm $\|\cdot\|_p$ is equivalent to*

$$(8.9) \quad \|\varphi\|_{p,k} := \left(\|\varphi\|_0^2 + \|\varphi^{(k)}\|_{q,0}^2 \right)^{1/2}.$$

Proof: Combine the proofs of Theorems 8.4 and 8.5 (see Problem 8.1).

Corollary 8.7. *For a nonnegative integer k let $f \in C^k[0, 2\pi]$ and assume that $0 \leq p \leq k$. Then for all $\varphi \in H^p[0, 2\pi]$ the product $f\varphi$ belongs to $H^p[0, 2\pi]$ and*

$$\|f\varphi\|_p \leq C\|\varphi\|_p$$

with some constant C depending on f and p .

Proof: Use the equivalent norms of Theorems 8.4 and 8.5 and Corollary 8.6 (see Problem 8.1).

Definition 8.8. *For $0 \leq p < \infty$ by $H^{-p}[0, 2\pi]$ we denote the dual space of $H^p[0, 2\pi]$, that is, the space of bounded linear functionals on $H^p[0, 2\pi]$.*

The space H^{-p} is characterized by the following

Theorem 8.9. *For $F \in H^{-p}[0, 2\pi]$ there holds*

$$\|F\|_p = \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2 \right\}^{1/2}$$

where $c_m = F(f_m)$. Conversely, to each sequence (c_m) satisfying

$$\sum_{m=-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2 < \infty$$

there exists a bounded linear functional $F \in H^{-p}[0, 2\pi]$ with $F(f_m) = c_m$.

Proof: Assume the sequence (c_m) satisfies the required inequality and define a functional $F : H^p[0, 2\pi] \rightarrow \mathbb{C}$ by

$$F(\varphi) := \sum_{m=-\infty}^{\infty} a_m c_m$$

for $\varphi \in H^p[0, 2\pi]$ with Fourier coefficients a_m . Then from the Cauchy–Schwarz inequality we have

$$|F(\varphi)|^2 \leq \sum_{m=-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2 \sum_{m=-\infty}^{\infty} (1+m^2)^p |a_m|^2.$$

Hence, F is bounded with

$$\|F\|_p \leq \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2 \right\}^{1/2}.$$

On the other hand, for $n \in \mathbb{N}$ the function

$$\varphi_n := \sum_{m=-n}^n (1+m^2)^{-p} \bar{c}_m f_m$$

has norm

$$\|\varphi_n\|_p = \left\{ \sum_{m=-n}^n (1+m^2)^{-p} |c_m|^2 \right\}^{1/2}.$$

Therefore

$$\|F\|_p \geq \frac{|F(\varphi_n)|}{\|\varphi_n\|_p} = \left\{ \sum_{m=-n}^n (1+m^2)^{-p} |c_m|^2 \right\}^{1/2}.$$

Hence,

$$\left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^{-p} |c_m|^2 \right\}^{1/2} \leq \|F\|_p$$

which completes the proof.

Theorem 8.10. *For each function $g \in L^2[0, 2\pi]$ the duality pairing*

$$(8.10) \quad G(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \overline{g(t)} dt, \quad \varphi \in H^p[0, 2\pi],$$

canonically defines a linear functional $G \in H^{-p}[0, 2\pi]$. In this sense, $L^2[0, 2\pi]$ is a subspace of each dual space $H^{-p}[0, 2\pi]$, and the trigonometric polynomials are dense in $H^{-p}[0, 2\pi]$.

Proof: Denote by b_m the Fourier coefficients of g . Then, since $G(f_m) = \bar{b}_m$, by the second part of Theorem 8.9 we have $G \in H^{-p}$. Let $F \in H^{-p}$ with $F(f_m) = c_m$ and define a the sequence (F_n) in H^{-p} by

$$F_n(\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \overline{g_n(t)} dt,$$

where g_n is the trigonometric polynomial

$$g_n := \sum_{m=-n}^n \bar{c}_m f_m.$$

Then we have

$$\|F - F_n\|_p = \sum_{|m|=n+1}^{\infty} (1+m^2)^{-p} |c_m|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

analogous to Theorem 8.2. \square

Obviously, H^{-p} becomes a Hilbert space by appropriately extending the definition of the scalar product from Theorem 8.2 to negative p . For $p = 0$ the duality map described in Theorem 8.10 is bijective with $\|G\|_0 = \|g\|_0$. Therefore, we can identify H^{-0} and H^0 and obtain a *Hilbert scale* of Hilbert spaces H^p for all real p with compact imbedding from H^q into H^p for $q > p$. The Sobolev spaces H^p frequently are called *interpolation spaces* because of the interpolating properties indicated by the following two theorems.

Theorem 8.11. *Let $p < q$ and $r = \lambda p + (1-\lambda)q$ with $0 < \lambda < 1$. Then*

$$\|\varphi\|_r \leq \|\varphi\|_p^\lambda \|\varphi\|_q^{1-\lambda}$$

for all $\varphi \in H^q[0, 2\pi]$.

Proof: With the aid of Hölder's inequality we estimate

$$\begin{aligned} \|\varphi\|_r^2 &= \sum_{m=-\infty}^{\infty} \{(1+m^2)^p |a_m|^2\}^\lambda \{(1+m^2)^q |a_m|^2\}^{1-\lambda} \\ &\leq \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^p |a_m|^2 \right\}^\lambda \left\{ \sum_{m=-\infty}^{\infty} (1+m^2)^q |a_m|^2 \right\}^{1-\lambda} = \|\varphi\|_p^{2\lambda} \|\varphi\|_q^{2-2\lambda}. \end{aligned}$$

For real s consider the operator D^s transferring the trigonometric monomials $f_m(t) = e^{imt}$ into $D^s f_m := (1+m^2)^s f_m$. For each $p \in \mathbb{R}$ it clearly defines a bounded linear operator from H^{p+2s} onto H^p with

$$(8.11) \quad \|D^s \varphi\|_p = \|\varphi\|_{p+2s}$$

for all $\varphi \in H^{p+2s}$. Obviously, there holds $D^{s+t} = D^s D^t$ for all $s, t \in \mathbb{R}$. In particular, D^{-s} is the inverse of D^s . For $s = 1$ the operator $D = D^1$ corresponds to the second derivative $D\varphi = \varphi - \varphi''$ if φ is two times continuously differentiable. Therefore we may interprete D as generalization of this derivative for elements of the Sobolev spaces and the operators D^s as *intermediate derivatives*. With the aid of D^s we can transform the scalar products by

$$(8.12) \quad (\varphi, \psi)_{p+s} = (D^{s/2}\varphi, D^{s/2}\psi)_p$$

for all $\varphi, \psi \in H^{p+s}$ and

$$(8.13) \quad (\varphi, \psi)_{p+s} = (\varphi, D^s\psi)_p$$

for all $\varphi \in H^{p+s}$ and $\psi \in H^{p+2s}$. \square

Noting that $Df_m = (1 + m^2)f_m$, that is, the f_m are eigenfunctions of the operator D with eigenvalues $1 + m^2$, we indicate obvious possibilities for extending this procedure to introduce interpolation spaces generated by other operators.

Theorem 8.12. *For $p, q \in \mathbb{R}$ and $r > 0$ let A be a linear operator mapping H^p boundedly into H^q and H^{p+r} boundedly into H^{q+r} , that is,*

$$\|A\varphi\|_q \leq C_1 \|\varphi\|_p$$

for all $\varphi \in H^p$ and

$$\|A\varphi\|_{q+r} \leq C_2 \|\varphi\|_{p+r}$$

for all $\varphi \in H^{p+r}$ with some constants C_1 and C_2 . Then for each $0 < \lambda < 1$ there holds

$$\|A\varphi\|_{q+\lambda r} \leq C_1^{1-\lambda} C_2^\lambda \|\varphi\|_{p+\lambda r}$$

for all $\varphi \in H^{p+\lambda r}$, that is, A maps $H^{p+\lambda r}$ boundedly into $H^{q+\lambda r}$.

Proof: By $A^* : H^q \rightarrow H^p$ we denote the adjoint of $A : H^p \rightarrow H^q$, that is,

$$(8.14) \quad (A\varphi, \psi)_q = (\varphi, A^*\psi)_p$$

for all $\varphi \in H^p$ and $\psi \in H^q$. By Theorem 4.9 we have

$$\|A^*\psi\|_p \leq C_1 \|\psi\|_q$$

for all $\psi \in H^q$. From this, making use of (8.11), we observe that the operator $\tilde{A} := D^{-r/2} A^* D^{r/2}$ is bounded from H^{q+r} into H^{p+r} with

$$\|\tilde{A}\psi\|_{p+r} \leq C_1 \|\psi\|_{q+r}$$

for all $\psi \in H^{q+r}$.

For all $\varphi \in H^{p+r}$ and $\psi \in H^{q+r}$, with the aid of (8.13) and (8.14), we deduce

$$(A\varphi, \psi)_{q+r/2} = (A\varphi, D^{r/2}\psi)_q = (\varphi, A^* D^{r/2}\psi)_p = (\varphi, D^{-r/2} A^* D^{r/2}\psi)_{p+r/2}.$$

Therefore, the bounded operators $A : H^{p+r} \rightarrow H^{q+r}$ and $\tilde{A} : H^{q+r} \rightarrow H^{p+r}$ are adjoint with respect to the positive dual systems (H^{p+r}, H^{p+r}) and (H^{q+r}, H^{q+r}) generated by the $(p+r/2)$ - and $(q+r/2)$ -scalar products. Hence, applying the Lax Theorem 4.12 and using the denseness of H^{p+r} in $H^{p+r/2}$, we conclude that A is bounded from $H^{p+r/2}$ into $H^{q+r/2}$ with

$$\|A\varphi\|_{q+r/2} \leq C_1^{1/2} C_2^{1/2} \|\varphi\|_{p+r/2}$$

for all $\varphi \in H^{p+r/2}$.

Repeating this argument, we see that

$$\|A\varphi\|_{q+\lambda r} \leq C_1^{1-\lambda} C_2^\lambda \|\varphi\|_{p+\lambda r}$$

for all $\varphi \in H^{p+\lambda r}$ and all $0 < \lambda < 1$ which are rational numbers in the binary system. For arbitrary $0 < \lambda < 1$ we can choose a monotonic decreasing sequence (λ_n) of binary rationals such that $\lambda_n \rightarrow \lambda$, $n \rightarrow \infty$. Then, using (8.5), we have

$$\|A\varphi\|_{q+\lambda r} \leq \|A\varphi\|_{q+\lambda_n r} \leq C_1^{1-\lambda_n} C_2^{\lambda_n} \|\varphi\|_{p+\lambda_n r}$$

for all $\varphi \in H^{p+r}$ and $n \in \mathbb{N}$. Passing to the limit $n \rightarrow \infty$ we see that

$$\|A\varphi\|_{q+\lambda r} \leq C_1^{1-\lambda} C_2^\lambda \|\varphi\|_{p+\lambda r}$$

for all $\varphi \in H^{p+r}$, and since H^{p+r} is dense in $H^{p+\lambda r}$ also for all $\varphi \in H^{p+\lambda r}$. The idea to use the Lax theorem for proving this interpolation theorem is due to Kirsch [2].

8.3 The Sobolev Space $H^p(\Gamma)$

Let Γ be the boundary of a simply connected bounded domain $D \subset \mathbb{R}^2$ of class C^k , $k \in \mathbb{N}$. With the aid of a regular and k times continuously differentiable parametric representation

$$x(t) = (x_1(t), x_2(t)), \quad t \in [0, 2\pi],$$

for $0 \leq p \leq k$ we can define the Sobolev space $H^p(\Gamma)$ as the space of all functions $\varphi \in L^2(\Gamma)$ with the property that $\varphi \circ x \in H^p[0, 2\pi]$. By $\varphi \circ x$, as usual, we mean the function given by $(\varphi \circ x)(t) := \varphi(x(t))$, $t \in [0, 2\pi]$. The scalar product and norm on $H^p(\Gamma)$ are defined through the scalar product on $H^p[0, 2\pi]$ by

$$(\varphi, \psi)_{H^p(\Gamma)} := (\varphi \circ x, \psi \circ x)_{H^p[0, 2\pi]}.$$

Without loss of generality we have restricted the parameter domain to be the interval $[0, 2\pi]$. However, we must allow the possibility of different regular parametric representations for the contour Γ . Therefore, we have to convince ourselves that our definition is invariant under changes of the parametrisation.

Theorem 8.13. *Assume that $x(t) = (x_1(t), x_2(t))$ and $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$, $t \in [0, 2\pi]$, are two different regular parametric representations for the boundary*

Γ of a simply connected bounded domain $D \subset \mathbb{R}^2$ of class C^k , $k \in \mathbb{N}$. Then, for $0 \leq p \leq k$ the Sobolev spaces

$$H^p(\Gamma) := \{\varphi \in L^2(\Gamma) : \varphi \circ x \in H^p[0, 2\pi]\}$$

with scalar product

$$(\varphi, \psi)_{H^p(\Gamma)} := (\varphi \circ x, \psi \circ x)_{H^p[0, 2\pi]}$$

and

$$\tilde{H}^p(\Gamma) := \{\varphi \in L^2(\Gamma) : \varphi \circ \tilde{x} \in H^p[0, 2\pi]\}$$

with scalar product

$$(\varphi, \psi)_{\tilde{H}^p(\Gamma)} := (\varphi \circ \tilde{x}, \psi \circ \tilde{x})_{H^p[0, 2\pi]}$$

are homeomorphic.

Recall that a parametric representation is called regular if $x'(t) \neq 0$ for all $t \in [0, 2\pi]$. Without loss of generality we may assume that $x(0) = \tilde{x}(0)$. Then the statement is a consequence of the following lemma for the case $f = x^{-1} \circ \tilde{x}$.

Lemma 8.14. *Let f be a diffeomorphism of the interval $[0, 2\pi]$ onto itself of class $k \in \mathbb{N}$, that is, f maps the interval $[0, 2\pi]$ bijectively onto itself and together with its inverse belongs to $C^k[0, 2\pi]$, and let $0 \leq p \leq k$. Then for all $\varphi \in H^p[0, 2\pi]$ there holds $\varphi \circ f \in H^p[0, 2\pi]$ with*

$$\|\varphi \circ f\|_p \leq C \|\varphi\|_p$$

where C is some constant depending on f, k and p .

Proof: We make use of the equivalent norms of Theorems 8.4 and 8.5 and Corollary 8.6. If $p \in \mathbb{N}$, we use the norm given by (8.6). Because of Theorem 8.4 there exist positive constants c_1 and c_2 such that

$$c_1 \|\varphi\|_{j,j}^2 \leq \|\varphi\|_j^2 \leq c_2 \|\varphi\|_{j,j}^2$$

for all $\varphi \in C^p[0, 2\pi]$ and all $j = 1, \dots, p$. Then, using the chain rule and the Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned} \|\varphi \circ f\|_p^2 &\leq c_2 \int_0^{2\pi} \left\{ |\varphi(f(t))|^2 + \left| \frac{d^p}{dt^p} \varphi(f(t)) \right|^2 \right\} dt \\ &\leq c_2 M \sum_{j=0}^p \int_0^{2\pi} |\varphi^{(j)}(f(t))|^2 dt, \end{aligned}$$

where M is some constant depending on p and containing bounds on the derivatives of f up to order p . Substituting $\tau = f(t)$ on the right hand side and using (8.5), we get the further estimates

$$\begin{aligned} \|\varphi \circ f\|_p^2 &\leq c_2 M \gamma \sum_{j=0}^p \int_0^{2\pi} |\varphi^{(j)}(\tau)|^2 d\tau \leq c_2 M \gamma \sum_{j=1}^p \|\varphi\|_{j,j}^2 \\ &\leq \frac{c_2}{c_1} M \gamma \sum_{j=1}^p \|\varphi\|_j^2 \leq \frac{c_2}{c_1} M \gamma p \|\varphi\|_p^2. \end{aligned}$$

Here, we have set $\gamma := \|1/f'\|_\infty$. This completes the proof for $p \in \mathbb{N}$ since $C^p[0, 2\pi]$ is dense in $H^p[0, \pi]$.

If $0 < p < 1$, we use the norm given by (8.7). From the mean value theorem and the periodicity of f we have

$$|f(t) - f(\tau)| \leq \|f'\|_\infty h(|t - \tau|)$$

with the hat function

$$h(t) := \begin{cases} t, & 0 \leq t \leq \pi, \\ 2\pi - t, & \pi \leq t \leq 2\pi. \end{cases}$$

Then, estimating similarly as in the proof of Theorem 8.5, we derive

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(f(t)) - \varphi(f(\tau))|^2}{\left| \sin \frac{t - \tau}{2} \right|^{2p+1}} d\tau dt \\ & \leq \pi^{2p+1} \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(f(t)) - \varphi(f(\tau))|^2}{h(|t - \tau|)^{2p+1}} d\tau dt \\ & \leq [\pi \|f'\|_\infty]^{2p+1} \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(f(t)) - \varphi(f(\tau))|^2}{|f(t) - f(\tau)|^{2p+1}} d\tau dt \\ & = [\pi \|f'\|_\infty]^{2p+1} \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(t) - \varphi(\tau)|^2}{|t - \tau|^{2p+1}} f^{-1}'(t) f^{-1}'(\tau) d\tau dt \\ & \leq \gamma^2 \left[\frac{\pi}{2} \|f'\|_\infty \right]^{2p+1} \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(t) - \varphi(\tau)|^2}{\left| \sin \frac{t - \tau}{2} \right|^{2p+1}} d\tau dt. \end{aligned}$$

With this inequality the statement of the theorem now follows from the equivalence of the norms $\|\cdot\|_p$ and $\|\cdot\|_{p,0}$ on $C^1[0, 2\pi]$ and the denseness of $C^1[0, 2\pi]$ in $H^p[0, 2\pi]$.

Finally, the case of an arbitrary noninteger positive p is now settled by combining the two previous cases with the aid of the norm given by (8.9). \square

The classical Sobolev space $H^1(D)$ for a bounded domain $D \subset \mathbb{R}^2$ of class C^1 is defined as the completion of the space $C^1(\bar{D})$ of continuously differentiable functions with respect to the norm

$$\|u\|_{H^1(D)} := \left(\int_D \{|u(x)|^2 + |\text{grad } u(x)|^2\} dx \right)^{1/2}.$$

For a detailed description of this space, in particular, its equivalent definition through the concept of weak derivatives and its extension to the Sobolev spaces $H^k(D)$ of arbitrary order $k \in \mathbb{N}$ we refer to Adams [1], Gilbarg and Trudinger [1] or Treves [1]. Since each Cauchy sequence with respect to $\|\cdot\|_{H^1(D)}$ is also

a Cauchy sequence with respect to $\|\cdot\|_{L^2(D)}$ we may interprete $H^1(D)$ as a subspace of $L^2(D)$. Obviously the gradient can be extended from $C^1(\bar{D})$ as a bounded linear operator from $H^1(D)$ into $L^2(D)$ (see Problem 2.1).

We want to illustrate the connection between Sobolev spaces on the domain D and Sobolev spaces on its boundary Γ through the simplest case of the trace theorem. For functions defined on the closure \bar{D} their values on the boundary are clearly defined and the restriction of the function to the boundary Γ is called the *trace*, the operator mapping a function onto its trace is called the *trace operator*.

As a first step, we consider continuously differentiable functions u in the strip $\mathbb{R} \times [0, 1]$ which are 2π -periodic with respect to the first variable, that is, there holds $u(t + 2\pi, \cdot) = u(t, \cdot)$ for all $t \in \mathbb{R}$. By Q we denote the rectangle $Q := [0, 2\pi] \times [0, 1]$. For $0 \leq \eta \leq 1$ consider the Fourier coefficients

$$a_m(\eta) := \frac{1}{2\pi} \int_0^{2\pi} u(t, \eta) e^{-imt} dt.$$

By Parseval's equality we have

$$\sum_{m=-\infty}^{\infty} |a_m(\eta)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |u(t, \eta)|^2 dt, \quad 0 \leq \eta \leq 1.$$

Since the a_m and u are continuous, by Dini's theorem (see Problem 8.2), the series is uniformly convergent. Hence, we can integrate term by term to obtain

$$(8.15) \quad \sum_{m=-\infty}^{\infty} \int_0^1 |a_m(\eta)|^2 d\eta = \frac{1}{2\pi} \|u\|_{L^2(Q)}^2.$$

Similarly, from

$$a'_m(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial \eta}(t, \eta) e^{-imt} dt$$

and

$$ima_m(\eta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial t}(t, \eta) e^{-imt} dt$$

we see that

$$(8.16) \quad \sum_{m=-\infty}^{\infty} \int_0^1 |a'_m(\eta)|^2 d\eta = \frac{1}{2\pi} \left\| \frac{\partial u}{\partial \eta} \right\|_{L^2(Q)}^2$$

and

$$(8.17) \quad \sum_{m=-\infty}^{\infty} \int_0^1 m^2 |a_m(\eta)|^2 d\eta = \frac{1}{2\pi} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q)}^2.$$

In a second step we shall show that

$$(8.18) \quad \|u(\cdot, 0)\|_{H^{1/2}[0, 2\pi]}^2 \leq \frac{1}{\pi} \|u\|_{H^1(Q)}^2$$

for all u with $u(\cdot, 1) = 0$. The latter property implies that $a_m(1) = 0$ for all $m \in \mathbb{Z}$. In particular, using the Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned} \|u(\cdot, 0)\|_{H^{1/2}[0, 2\pi]}^2 &= \sum_{m=-\infty}^{\infty} (1+m^2)^{1/2} |a_m(0)|^2 \\ &= 2 \sum_{m=-\infty}^{\infty} (1+m^2)^{1/2} \operatorname{Re} \int_1^0 a'_m(\eta) \overline{a_m(\eta)} d\eta \\ &\leq 2 \sum_{m=-\infty}^{\infty} \left\{ \int_0^1 |a'_m(\eta)|^2 d\eta \right\}^{1/2} \left\{ (1+m^2) \int_0^1 |a_m(\eta)|^2 d\eta \right\}^{1/2} \\ &\leq 2 \left\{ \sum_{m=-\infty}^{\infty} \int_0^1 |a'_m(\eta)|^2 d\eta \right\}^{1/2} \left\{ \sum_{m=-\infty}^{\infty} (1+m^2) \int_0^1 |a_m(\eta)|^2 d\eta \right\}^{1/2} \\ &= \frac{1}{\pi} \left\| \frac{\partial u}{\partial \eta} \right\|_{L^2(Q)} \left\{ \|u\|_{L^2(Q)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q)}^2 \right\}^{1/2} \leq \frac{1}{\pi} \|u\|_{H^1(Q)}^2. \end{aligned}$$

We are now ready to establish

Theorem 8.15. *Let D be a simply connected bounded domain in \mathbb{R}^2 with boundary Γ of class C^2 . Then there exists a positive constant C such that*

$$\|u\|_{H^{1/2}(\Gamma)} \leq C \|u\|_{H^1(D)}$$

for all $u \in C^1(\bar{D})$.

Proof: As in the proof of Theorem 7.6, we choose a parallel strip $D_h := \{x + \eta hn(x) : x \in \Gamma, \eta \in [0, 1]\}$ with some $h > 0$ such that each $y \in D_h$ is uniquely representable through projection onto Γ in the form $y = x + \eta hn(x)$ with $x \in \Gamma$ and $\eta \in [0, 1]$. Here, for convenience and deviating from our general rule, we assume the normal n to be directed into D . By Γ_h we denote the interior boundary $\Gamma_h := \{y = x + hn(x) : x \in \Gamma\}$ of D_h . Then, through a regular parametric representation

$$x(t) = (x_1(t), x_2(t)), \quad 0 \leq t \leq 2\pi,$$

of the contour Γ we have a parametrisation

$$x(t, \eta) = x(t) + \eta hn(t), \quad 0 \leq t \leq 2\pi, \quad 0 \leq \eta \leq 1,$$

of the strip D_h . In particular, x provides a bijective mapping from Q onto D_h which together with its inverse is continuously differentiable. Thus, applying the chain rule, from (8.18) for all $u \in C^1(D_h)$ with $u = 0$ on Γ_h there follows

$$\|u\|_{H^{1/2}(\Gamma)} = \|u \circ x\|_{H^{1/2}[0, 2\pi]} \leq \frac{1}{\sqrt{\pi}} \|u \circ x\|_{H^1(Q)} \leq \tilde{C} \|u\|_{H^1(D_h)},$$

where \tilde{C} is some constant containing a bound on the first derivatives of the mapping x and its inverse.

Finally, to extend the estimate to arbitrary $u \in C^1(\bar{D})$, we choose a function $g \in C^1(\bar{D})$ given by $g(y) = 0$ for $y \notin D_h$ and $g(y) = f(\eta)$ for $y \in D_h$ with $y = x + \eta h n(x)$. Here, f is given by

$$f(\eta) = (1 - \eta)^2(1 + 3\eta)$$

and satisfies $f(0) = f'(0) = 1$ and $f(1) = f'(1) = 0$. The property $f'(0) = 1$ will not be used in this proof, but will be needed later in the proof of Theorem 8.17. Then, applying the product rule, for all $u \in C^1(\bar{D})$ we obtain

$$\|u\|_{H^{1/2}(\Gamma)} = \|gu\|_{H^{1/2}(\Gamma)} \leq \tilde{C}\|gu\|_{H^1(D)} \leq C\|u\|_{H^1(D)}$$

with some constant C containing a bound on g and its first derivatives.

Corollary 8.16. *The trace operator can be uniquely extended as a continuous operator from $H^1(D)$ into $H^{1/2}(\Gamma)$.*

Proof: Theorem 8.15 and Problem 2.1. □

In the following theorem, by $|\cdot|_{L^1(\Gamma)}$ we will denote the semi-norm

$$|u|_{L^1(\Gamma)} := \left| \int_{\Gamma} u ds \right|.$$

Theorem 8.17. *Let D be as in Theorem 8.15. Then there exists a positive constant C such that*

$$\|u\|_{L^2(D)}^2 \leq C \left\{ |u|_{L^1(\Gamma)}^2 + \|\operatorname{grad} u\|_{L^2(D)}^2 \right\}$$

and

$$\|u\|_{L^2(D)}^2 \leq C \left\{ \|u\|_{L^2(\Gamma)}^2 + \|\operatorname{grad} u\|_{L^2(D)}^2 \right\}$$

for all $u \in H^1(D)$.

Proof: Since $C^1(\bar{D})$ is dense in $H^1(D)$ it suffices to establish the inequalities for all $u \in C^1(\bar{D})$. In addition, obviously, we only need to verify the first inequality.

We use the notations introduced in the proof of Theorem 8.15 and consider a function u in the strip $\mathbb{R} \times [0, 1]$. From

$$\int_0^{2\pi} u(t, \eta) dt - \int_0^{2\pi} u(t, 0) dt = \int_0^{2\pi} \int_0^{\eta} \frac{\partial u}{\partial \xi}(t, \xi) d\xi dt,$$

using the Cauchy–Schwarz inequality, we derive

$$\left| \int_0^{2\pi} u(t, \eta) dt \right|^2 \leq 2 \left\{ \left| \int_0^{2\pi} u(t, 0) dt \right|^2 + 2\pi \left\| \frac{\partial u}{\partial \eta} \right\|_{L^2(Q)}^2 \right\}.$$

Integrating this inequality with respect to η we obtain

$$\int_0^1 \left| \int_0^{2\pi} u(t, \eta) dt \right|^2 d\eta \leq C_1 \left\{ \left| \int_0^{2\pi} u(t, 0) dt \right|^2 + \|\operatorname{grad} u\|_{L^2(Q)}^2 \right\}$$

with some constant C_1 . Combining this with (8.15) and (8.17), we see that

$$\begin{aligned}\|u\|_{L^2(Q)}^2 &\leq 2\pi \int_0^1 |a_0(\eta)|^2 d\eta + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q)}^2 \\ &\leq C_2 \left\{ \left| \int_0^{2\pi} u(t, 0) dt \right|^2 + \|\operatorname{grad} u\|_{L^2(Q)}^2 \right\}\end{aligned}$$

with some constant C_2 . Substituting back into the domain D as in the proof of Theorem 8.15, we see that there exists a constant C_3 such that

$$(8.19) \quad \|u\|_{L^2(D_h)}^2 \leq C_3 \left\{ |u|_{L^1(\Gamma)}^2 + \|\operatorname{grad} u\|_{L^2(D)}^2 \right\}.$$

For this transformation, without loss of generality we have assumed that the parameter t in the representation of the boundary Γ is given through the arclength multiplied by $2\pi/|\Gamma|$.

Now let $u \in C^1(\bar{D})$ satisfy $u = 0$ and $\operatorname{grad} u = 0$ on the boundary Γ . Then we can extend u to a continuously differentiable function on \mathbb{R}^2 by setting $u = 0$ in the exterior of D . Choose R large enough such that D is contained in the circle with radius R and center at the origin. Then from

$$u(x) = \int_{-R}^{x_1} \frac{\partial u}{\partial x_1} dx_1$$

with the aid of the Cauchy–Schwarz inequality we see that

$$|u(x)|^2 \leq 2R \int_{-R}^R \left| \frac{\partial u}{\partial x_1} \right|^2 dx_1.$$

Integrating this inequality and using $u = 0$ outside of D , yields

$$(8.20) \quad \|u\|_{L^2(D)}^2 \leq 4R^2 \|\operatorname{grad} u\|_{L^2(D)}^2$$

for all $u \in C^1(\bar{D})$ with $u = \operatorname{grad} u = 0$ on Γ .

Finally, for arbitrary $u \in C^1(\bar{D})$ we use the function g introduced in the proof of the previous theorem to decompose

$$u = gu + (1 - g)u$$

and then apply (8.19) to the first term and (8.20) to the second term on the right hand side. Observing that

$$\|\operatorname{grad} gu\|_{L^2(D)} \leq C_4 \left\{ \|\operatorname{grad} u\|_{L^2(D)} + \|u\|_{L^2(D_h)} \right\}$$

with some constant C_4 depending on g now ends the proof.

Corollary 8.18. *On $H^1(D)$ the norm $\|\cdot\|_{H^1(D)}$ is equivalent to each of the two norms*

$$\|u\| := \left(|u|_{L^1(\Gamma)}^2 + \|\operatorname{grad} u\|_{L^2(D)}^2 \right)^{1/2}$$

and

$$\|u\| := \left(\|u\|_{L^2(\Gamma)}^2 + \|\operatorname{grad} u\|_{L^2(D)}^2 \right)^{1/2}.$$

Proof: Theorems 8.3, 8.15 and 8.17.

8.4 Weak Solutions to Boundary Value Problems

We now want to demonstrate how the Sobolev spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ occur in a natural way through the solution of boundary value problems in a weak formulation. For a function $u \in H^1(D)$ the integral

$$\int_D |\operatorname{grad} u|^2 dx$$

is called the *Dirichlet integral*. For a harmonic function u the Dirichlet integral represents the energy of the potential u . Therefore, it is natural to attempt to develop an approach for the solution of boundary value problems for harmonic functions in which it is required that the solutions have finite energy, i.e., they belong to the Sobolev space $H^1(D)$. Since the functions contained in $H^1(D)$, in general, are not twice continuously differentiable in D and since they do not attain boundary values or normal derivatives on the boundary in the classical meaning, we have to extend the classical formulation of the boundary value problems by generalized versions. Our procedure will be quite typical in the sense that we use specific properties of classical solutions to formulate what we mean by a weak solution.

For each solution $u \in C^2(D) \cap C^1(\bar{D})$ to the Dirichlet problem $\Delta u = 0$ in D with $u = f$ on Γ by the first Green's Theorem 6.3 we have

$$\int_D (\operatorname{grad} u, \operatorname{grad} v) dx = 0$$

for all $v \in C^1(\bar{D})$ with $v = 0$ on Γ . Since the integral in this equation is well defined for all $u, v \in H^1(D)$ we can introduce the following weak formulation of the interior Dirichlet problem.

Weak Interior Dirichlet Problem. *Find a function $u \in H^1(D)$ satisfying*

$$(8.21) \quad \int_D (\operatorname{grad} u, \operatorname{grad} v) dx = 0$$

for all $v \in H^1(D)$ with $v = 0$ on Γ and

$$(8.22) \quad u = f \quad \text{on } \Gamma$$

where $f \in H^{1/2}(\Gamma)$ is a given function.

Note that boundary values for functions in $H^1(D)$ have to be understood in the sense of the trace operator according to Corollary 8.16.

For each solution $u \in C^2(D) \cap C^1(\bar{D})$ to the Neumann problem $\Delta u = 0$ in D with $\partial u / \partial n = g$ on Γ by the first Green's theorem we have

$$\int_D (\operatorname{grad} u, \operatorname{grad} v) dx = \int_{\Gamma} gv ds$$

for all $v \in C^1(\bar{D})$. Through the duality pairing (8.10) and the trace theorem the integral on the right is well defined for all $g \in H^{-1/2}(\Gamma)$ and $v \in H^1(D)$ by

$$\int_{\Gamma} gv ds := g(v).$$

Therefore we can introduce the following weak formulation of the interior Neumann problem.

Weak Interior Neumann Problem. *Find a function $u \in H^1(D)$ satisfying*

$$(8.23) \quad \int_D (\operatorname{grad} u, \operatorname{grad} v) dx = \int_{\Gamma} g v ds$$

for all $v \in H^1(D)$ where $g \in H^{-1/2}(\Gamma)$ is a given function.

Theorem 8.19. *The weak interior Dirichlet problem has at most one solution. Two solutions to the weak interior Neumann problem can differ only by a constant.*

Proof: The difference $u := u_1 - u_2$ between two solutions to the Dirichlet problem is a weak solution with homogeneous boundary condition $u = 0$ on Γ . Then, we may insert $v = u$ in (8.21) and obtain $\|\operatorname{grad} u\|_{L^2(D)} = 0$. Since $u = 0$ on Γ , from Corollary 8.18 we see that $u = 0$ in D . For two solutions of the Neumann problem the difference $u := u_1 - u_2 - c$ again satisfies $\|\operatorname{grad} u\|_{L^2(D)} = 0$ for all constants c . We choose c such that $\int_{\Gamma} u ds = 0$. Then from Corollary 8.18 we obtain $u = 0$ in D . \square

To establish the existence of weak solutions we will proceed analogously to the classical case and try to find the solution in the form of a logarithmic single- or double-layer potential with densities in $H^{-1/2}(\Gamma)$ or $H^{1/2}(\Gamma)$. Therefore, we first need to investigate properties of these potentials in our Sobolev space setting. This will be achieved through making use of corresponding properties in the spaces of uniformly Hölder continuous functions. For the remainder of this section by α we always will denote a Hölder exponent with $0 < \alpha < 1$.

Theorem 8.20. *The operators $K : H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ defined by the logarithmic double-layer potential*

$$(K\varphi)(x) := \frac{1}{\pi} \int_{\Gamma} \varphi(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|x-y|} ds(y), \quad x \in \Gamma,$$

and $K' : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ defined by the normal derivative of the logarithmic single-layer potential

$$(K'\psi)(x) := \frac{1}{\pi} \int_{\Gamma} \psi(y) \frac{\partial}{\partial n(x)} \ln \frac{1}{|x-y|} ds(y), \quad x \in \Gamma,$$

are compact and adjoint in the dual system $(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))_{L^2(\Gamma)}$, that is,

$$(8.24) \quad (K\varphi, \psi)_{L^2(\Gamma)} = (\varphi, K'\psi)_{L^2(\Gamma)}$$

for all $\varphi \in H^{1/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$. Here, again we use the notation

$$(f, g)_{L^2(\Gamma)} = \int_{\Gamma} f g ds := g(f)$$

for $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$.

Proof: We shall show that K is bounded from $H^{1/2}(\Gamma)$ into $H^1(\Gamma)$. Then the assertion on the compactness of K follows from Theorems 2.15 and 8.3.

For functions $\varphi \in C^{1,\alpha}(\Gamma)$ from Theorem 7.28 we know that the gradient of the double-layer potential v with density φ can be expressed in terms of the derivatives of the single-layer potential w with density $d\varphi/ds$. In particular, by (7.40) the tangential derivative is given by

$$\frac{\partial v_-}{\partial s}(x) = -\frac{1}{2} \frac{d\varphi}{ds}(x) - \frac{1}{2\pi} \int_{\Gamma} \frac{d\varphi}{ds}(y) \frac{\partial}{\partial n(x)} \ln \frac{1}{|x-y|} ds(y), \quad x \in \Gamma.$$

From this, with the aid of the jump relations of Theorem 7.26 for double-layer potentials with uniformly Hölder continuous densities, we conclude that

$$\frac{dK\varphi}{ds}(x) = -\frac{1}{\pi} \int_{\Gamma} \frac{d\varphi}{ds}(y) \frac{\partial}{\partial n(x)} \ln \frac{1}{|x-y|} ds(y), \quad x \in \Gamma.$$

Performing a partial integration now yields

$$\frac{dK\varphi}{ds}(x) = \frac{1}{\pi} \int_{\Gamma} \{\varphi(y) - \varphi(x)\} \frac{\partial}{\partial s(y)} \frac{\partial}{\partial n(x)} \ln \frac{1}{|x-y|} ds(y), \quad x \in \Gamma,$$

where the integral has to be understood in the sense of a Cauchy principal value. Finally, we use a parametric representation $x(t)$, $t \in [0, 2\pi]$, of Γ to transform the integral into

$$\frac{dK\varphi}{ds}(x(t)) = \int_0^{2\pi} \frac{k(t, \tau)}{\sin \frac{t-\tau}{2}} \{\varphi(x(\tau)) - \varphi(x(t))\} d\tau, \quad t \in [0, 2\pi],$$

where k is a bounded weakly singular kernel. Thus, by the Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned} \int_0^{2\pi} \left| \frac{dK\varphi}{ds}(x(t)) \right|^2 dt &\leq M \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(x(\tau)) - \varphi(x(t))|^2}{\sin^2 \frac{t-\tau}{2}} d\tau dt \\ &\leq M \|\varphi \circ x\|_{1/2,0}^2 \end{aligned}$$

with some constant M . Hence, by Theorem 8.5,

$$\left\| \frac{dK\varphi}{ds} \right\|_{L^2(\Gamma)} \leq \tilde{C} \|\varphi\|_{H^{1/2}(\Gamma)}$$

with some constant \tilde{C} . Since K has continuous kernel (see Problem 6.1) it is bounded from $L^2(\Gamma)$ into $L^2(\Gamma)$. Therefore we conclude

$$\|K\varphi\|_{H^1(\Gamma)} \leq C \|\varphi\|_{H^{1/2}(\Gamma)}$$

for all $\varphi \in C^{1,\alpha}(\Gamma)$ with some constant C . This completes the proof of the boundedness of K from $H^{1/2}(\Gamma)$ into $H^1(\Gamma)$ since $C^{1,\alpha}(\Gamma)$ is dense in $H^{1/2}(\Gamma)$ by Theorem 8.2.

For $\varphi, \psi \in C^{1,\alpha}(\Gamma)$ clearly (8.24) is true and we have

$$\begin{aligned} |(\varphi, K'\psi)_{L^2(\Gamma)}| &= |(K\varphi, \psi)_{L^2(\Gamma)}| \\ &\leq \|K\varphi\|_{H^1(\Gamma)} \|\psi\|_{H^{-1}(\Gamma)} \leq C \|\varphi\|_{H^{1/2}(\Gamma)} \|\psi\|_{H^{-1}(\Gamma)}. \end{aligned}$$

It follows that

$$\|K'\psi\|_{H^{-1/2}(\Gamma)} \leq C \|\psi\|_{H^{-1}(\Gamma)}$$

for all $\psi \in C^{1,\alpha}(\Gamma)$ and, because $C^{1,\alpha}(\Gamma)$ is dense in $H^{-1}(\Gamma)$, also for all $\psi \in H^{-1}(\Gamma)$. Therefore K' is bounded from $H^{-1}(\Gamma)$ into $H^{-1/2}(\Gamma)$ and the statement on the compactness of K again follows from Theorems 2.15 and 8.3.

Finally, the validity of (8.24) for all $\varphi \in H^{1/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$ follows from the boundedness of K and K' by the denseness of $C^{1,\alpha}(\Gamma)$ in $H^{1/2}(\Gamma)$ and in $H^{-1/2}(\Gamma)$.

Theorem 8.21. *The operators $S : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ defined by the single-layer potential*

$$(S\varphi)(x) := \frac{1}{\pi} \int_{\Gamma} \varphi(y) \ln \frac{1}{|x - y|} ds(y), \quad x \in \Gamma,$$

and $T : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ defined by the normal derivative of the double-layer potential

$$(T\psi)(x) := \frac{1}{\pi} \frac{\partial}{\partial n(x)} \int_{\Gamma} \psi(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|x - y|} ds(y), \quad x \in \Gamma,$$

are bounded. The operator S is self adjoint with respect to the dual systems $(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))_{L^2(\Gamma)}$ and $(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))_{L^2(\Gamma)}$, that is,

$$(8.25) \quad (S\varphi, \psi)_{L^2(\Gamma)} = (\varphi, S\psi)_{L^2(\Gamma)}$$

for all $\varphi, \psi \in H^{-1/2}(\Gamma)$. The operator T is self adjoint with respect to the dual systems $(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))_{L^2(\Gamma)}$ and $(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))_{L^2(\Gamma)}$, that is,

$$(8.26) \quad (T\varphi, \psi)_{L^2(\Gamma)} = (\varphi, T\psi)_{L^2(\Gamma)}$$

for all $\varphi, \psi \in H^{1/2}(\Gamma)$.

Proof: We use a parametric representation $x(t)$, $t \in [0, 2\pi]$, of Γ . By Theorem 8.13, we may assume that the parameter t is given by the arclength on Γ multiplied by $2\pi/|\Gamma|$. Then, for $\varphi \in C^{0,\alpha}(\Gamma)$, we can write (see Problem 7.2)

$$(S\varphi)(x(t)) = \frac{|\Gamma|}{4\pi^2} \int_0^{2\pi} \left\{ -\ln 4 \sin^2 \frac{t - \tau}{2} + p(t, \tau) \right\} \varphi(x(\tau)) d\tau, \quad t \in [0, 2\pi],$$

where p is continuously differentiable. For the leading term on the right hand side we use the integrals

$$\frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t}{2} e^{imt} dt = \begin{cases} 0, & m = 0, \\ -\frac{1}{m}, & m = \pm 1, \pm 2, \dots, \end{cases}$$

to conclude that it represents a bounded operator from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. Since the kernel is continuously differentiable, using the Cauchy–Schwarz inequality and an integration by parts, the second term on the right is seen to be bounded from $H^0(\Gamma)$ into $H^1(\Gamma)$ and from $H^{-1}(\Gamma)$ into $H^0(\Gamma)$. Hence, by the interpolation Theorem 8.12, it is also bounded from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$.

For $\varphi, \psi \in C^{1,\alpha}(\Gamma)$ we use the relation (7.42) between S and T and a partial integration to write

$$(8.27) \quad (T\varphi, \psi)_{L^2(\Gamma)} = \left(\frac{d}{ds} S \frac{d\varphi}{ds}, \psi \right)_{L^2(\Gamma)} = - \left(S \frac{d\varphi}{ds}, \frac{d\psi}{ds} \right)_{L^2(\Gamma)}.$$

Now we can estimate

$$\begin{aligned} |(T\varphi, \psi)|_{L^2(\Gamma)} &\leq \left\| S \frac{d\varphi}{ds} \right\|_{H^{1/2}(\Gamma)} \left\| \frac{d\psi}{ds} \right\|_{H^{-1/2}(\Gamma)} \\ &\leq C \left\| \frac{d\varphi}{ds} \right\|_{H^{-1/2}(\Gamma)} \left\| \frac{d\psi}{ds} \right\|_{H^{-1/2}(\Gamma)} \leq C \|\varphi\|_{H^{1/2}(\Gamma)} \|\psi\|_{H^{1/2}(\Gamma)} \end{aligned}$$

with some constant C . Since $C^{1,\alpha}(\Gamma)$ is dense in $H^{1/2}(\Gamma)$ this implies that T is bounded from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.

The self adjoint property (8.25) follows as in the proof of Theorem 8.20 by extending it from the case of smooth functions. Finally, combining (8.25) and (8.27) yields (8.26).

Theorem 8.22. *The logarithmic single-layer potential defines a bounded linear operator from $H^{-1/2}(\Gamma)$ into $H^1(D)$. The logarithmic double-layer potential defines a bounded linear operator from $H^{1/2}(\Gamma)$ into $H^1(D)$.*

Proof: Let u be the single-layer potential with density $\varphi \in C^{0,\alpha}(\Gamma)$. Then, by Green's theorem and the jump relations of Theorem 7.26, we have

$$\int_D |\operatorname{grad} u|^2 dx = \int_\Gamma \bar{u} \frac{\partial u}{\partial n} ds = \frac{1}{4} \int_\Gamma S \bar{\varphi} (\varphi + K' \varphi) ds.$$

Therefore, by Theorems 8.20 and 8.21, we can estimate

$$\|\operatorname{grad} u\|_{L^2(D)}^2 \leq \frac{1}{4} \|S \bar{\varphi}\|_{H^{1/2}(\Gamma)} \|\varphi + K' \varphi\|_{H^{-1/2}(\Gamma)} \leq \tilde{C} \|\varphi\|_{H^{-1/2}(\Gamma)}^2.$$

Since S is bounded from $L^2(\Gamma)$ into $L^2(\Gamma)$, using Theorem 8.17, we see that

$$\|u\|_{H^1(D)} \leq C \|\varphi\|_{H^{-1/2}(\Gamma)}$$

with some constant C . Now the statement on the single-layer potential follows from the denseness of $C^{0,\alpha}(\Gamma)$ in $H^{-1/2}(\Gamma)$.

The case of the double-layer potential v with density ψ is treated analogously through

$$\int_D |\operatorname{grad} v|^2 dx = \frac{1}{4} \int_\Gamma T \bar{\varphi} (K \varphi - \varphi) ds.$$

Theorem 8.23. *The weak Dirichlet problem has a unique solution. The mapping taking the given boundary data $f \in H^{1/2}(\Gamma)$ into the solution $u \in H^1(D)$ is bounded.*

Proof: Analogous to the classical treatment we try to find the solution in the form of a double-layer potential

$$u(x) = \frac{1}{2\pi} \int_{\Gamma} \varphi(y) \frac{\partial}{\partial n(y)} \ln \frac{1}{|x - y|} ds(y), \quad x \in D.$$

If $\varphi \in C^{1,\alpha}(\Gamma)$ then, by Theorem 7.28, we have $u \in C^1(\bar{D})$ and, by Green's theorem and the jump relations, there holds

$$\int_D (\operatorname{grad} u, \operatorname{grad} v) dx = \frac{1}{2} \int_{\Gamma} T\varphi v ds$$

for all $v \in C^1(\bar{D})$ and

$$2u = K\varphi - \varphi \quad \text{on } \Gamma.$$

In the first equation, by Theorems 8.21 and 8.22 and the trace theorem, both sides are continuous bilinear mappings from $H^{1/2}(\Gamma) \times H^1(D) \rightarrow \mathbb{C}$. Hence, the double-layer potential with density $\varphi \in H^{1/2}(\Gamma)$ satisfies (8.21). In the second equation again both sides are bounded from $H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ by Theorems 8.20 and 8.22 and Corollary 8.16. Therefore u is a weak solution to the Dirichlet problem provided the density φ solves the integral equation

$$\varphi - K\varphi = -2f.$$

Let $\varphi \in H^{1/2}(\Gamma)$ be a solution to the homogeneous equation $\varphi - K\varphi = 0$. Then, since K has continuous kernel (see Problem 6.1), we have $\varphi \in C(\Gamma)$ and from the classical Theorem 6.16 we deduce that $N(I - K) = \{0\}$ in $H^{1/2}(\Gamma)$. Now existence of a solution in $H^{1/2}(\Gamma)$ to the inhomogeneous equation follows from the Riesz theory by Theorems 3.4 and 8.20.

The statement on the continuous dependence of the solution on the given boundary data is a consequence of the boundedness of the inverse of $I - K$ (Theorem 3.4) and Theorem 8.22.

Theorem 8.24. *The weak interior Neumann problem is solvable if and only if*

$$\int_{\Gamma} g ds = 0.$$

The operator mapping the given boundary data $g \in H^{-1/2}(\Gamma)$ into the unique solution $u \in H^1(D)$ satisfying the additional condition

$$\int_{\Gamma} u ds = 0$$

is bounded.

Proof: By Theorem 7.27, the single-layer potential

$$u(x) = \frac{1}{2\pi} \int_{\Gamma} \psi(y) \ln \frac{1}{|x - y|} ds(y), \quad x \in D,$$

with density $\psi \in C^{0,\alpha}(\Gamma)$ belongs to $C^1(\bar{D})$. It satisfies

$$\int_D (\operatorname{grad} u, \operatorname{grad} v) dx = \frac{1}{2} \int_{\Gamma} (\psi + K'\psi) v ds$$

for all $v \in C^1(\bar{D})$. As in the previous proof, both sides of this equation are continuous from $H^{-1/2}(\Gamma) \times H^1(D) \rightarrow \mathbb{C}$ by Theorems 8.20 and 8.22 and the trace theorem. Hence, the single-layer potential u with density $\psi \in H^{-1/2}(\Gamma)$ is a weak solution to the Neumann problem provided ψ solves the integral equation

$$\psi + K'\psi = 2g.$$

As in the proof of Theorem 8.23 from Theorem 6.16 we see that $N(I+K) = \operatorname{span}\{1\}$ in $H^{1/2}(\Gamma)$. Therefore, by Theorem 8.20 and the Fredholm alternative, the inhomogeneous equation $\psi + K'\psi = 2g$ is solvable in $H^{-1/2}(\Gamma)$ provided g satisfies the solvability condition $\int_{\Gamma} g ds = 0$. The necessity of the latter condition for the existence of a weak solution to the Neumann problem follows from (8.23) for $v = 1$.

The continuous dependence of the solution on the given boundary data is a consequence of Theorem 8.22 analogous to the classical case of Theorem 6.25. \square

It is also possible to obtain the weak solvability to the Dirichlet and Neumann problem through integral equations of the first kind (see Problem 8.5).

Our presentation clearly demonstrates that the results on weak solutions heavily rely on the classical analysis on single- and double-layer potentials with uniformly Hölder continuous densities. The mapping properties of the integral operators in the Sobolev spaces essentially follow from the corresponding properties in the Hölder spaces through applications of functional analytic tools, in particular denseness and continuation arguments.

The major advantage of the weak approach to the boundary integral equations, besides the fact that it allows less regular boundary data, stems from the fact that Sobolev spaces are Hilbert spaces rather than just normed spaces. This, as we have seen, adds some elegance to the analysis.

For a more detailed study of boundary integral equations in Sobolev space settings in the framework of *pseudodifferential operators* including the extensions of the analysis to exterior problems and to three-dimensional problems we refer to Wendland [5,6] and the literature therein. For extention of the approach presented here we refer to Kirsch [1].

Problems

- 8.1. Work out the proofs of Corollaries 8.6 and 8.7.
- 8.2. Prove Dini's Theorem: Let $D \subset \mathbb{R}^m$ be compact and let (φ_n) be a nondecreasing sequence of continuous real valued functions on D converging pointwise to a continuous function φ . Then (φ_n) converges uniformly on D to φ .
- 8.3. Let $K : [0, 2\pi] \times [0, 2\pi] \times \mathbb{R}$ be n times continuously differentiable. Show that the integral operator A with kernel K is compact from $H^p[0, 2\pi]$ into $H^q[0, 2\pi]$ for all $|p|, |q| \leq n$.

- 8.4. Formulate and prove the analog of the Noether theorems in the form of Corollary 7.24 in the Sobolev spaces $H^p[0, 2\pi]$.

Hint: Use

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{t}{2} e^{\pm imt} dt = \begin{cases} \pm i, & m = 1, 2, \dots, \\ 0, & m = 0. \end{cases}$$

- 8.5. Show that the operators K, K', S and T are related by

$$ST = -I + K^2$$

on $H^{1/2}(\Gamma)$ and

$$TS = -I + K'^2$$

on $H^{-1/2}(\Gamma)$. Use this result and Theorems 5.6, 8.20, 8.21 and 8.22 to solve the weak Dirichlet problem through a single-layer potential with $H^{-1/2}(\Gamma)$ -density and the weak Neumann problem through a double-layer potential with $H^{1/2}(\Gamma)$ -density.

Hint: Use Green's theorem and the jump relations to establish

$$\int_{\Gamma} T\varphi S\psi ds = \int_{\Gamma} (K - I)\varphi (K' + I)\psi ds$$

for all $\varphi \in C^{1,\alpha}(\Gamma)$ and $\psi \in C^{0,\alpha}(\Gamma)$. In the case of the Dirichlet problem the single-layer approach has to be amended analogously to Theorem 7.29.

9. The Heat Equation

The temperature distribution u in a homogeneous and isotropic heat conducting medium with conductivity k , heat capacity c and mass density ρ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u$$

where $\kappa = k/c\rho$. This equation is called the equation of heat conduction or, shortly, the *heat equation* and was first derived by Fourier. Simultaneously, the heat equation also occurs in the description of diffusion processes. The heat equation is the standard example for a *parabolic* differential equation. In this chapter we want to indicate the application of Volterra type integral equations for the solution of initial boundary value problems for the heat equation. Without loss of generality we assume the constant $\kappa = 1$.

9.1 Initial Boundary Value Problem: Uniqueness

The determination of the heat conduction in a medium with a given temperature distribution at some initial time and a given temperature distribution on the boundary of the medium for all times leads to the following initial boundary value problem for the heat equation.

Initial Boundary Value Problem. *Let $D \subset \mathbb{R}^m$, $m = 1, 2, 3$, be a bounded domain with boundary $\Gamma := \partial D$ and let T denote a positive number. Find a function $u \in C(\bar{D} \times [0, T])$ which is twice continuously differentiable with respect to the space variable x and continuously differentiable with respect to the time variable t in $D \times (0, T]$ and which satisfies the heat equation*

$$(9.1) \quad \frac{\partial u}{\partial t} = \Delta u \quad \text{in } D \times (0, T],$$

the initial condition

$$(9.2) \quad u(\cdot, 0) = w \quad \text{in } D$$

and the Dirichlet boundary condition

$$(9.3) \quad u = f \quad \text{on } \Gamma \times [0, T].$$

Here, $w \in C(\bar{D})$ and $f \in C(\Gamma \times [0, T])$ are given continuous functions subject to the compatibility condition

$$(9.4) \quad w = f(\cdot, 0) \quad \text{on } \Gamma.$$

Our aim is to establish that this initial boundary value problem has a unique solution depending continuously on the given initial and boundary data. As in the case of the Dirichlet problem for the Laplace equation, uniqueness and continuous dependence follow from a maximum-minimum principle.

Theorem 9.1 (Weak Maximum-Minimum Principle). *Let $u \in C(\bar{D} \times [0, T])$ be twice continuously differentiable with respect to the space variables, continuously differentiable with respect to the time variable and a solution of the heat equation in $D \times (0, T]$. Then u attains both its maximum and its minimum on the parabolic boundary*

$$B := \{(x, 0) : x \in D\} \cup \{(x, t) : x \in \Gamma, t \in [0, T]\}.$$

Proof: We confine ourselves to the proof for the maximum. Define a continuous function v on $\bar{D} \times [0, T]$ by

$$v(x, t) := u(x, t) + \varepsilon(T - t)$$

with $\varepsilon > 0$. Assume that v attains its maximum value in the *parabolic interior* $D \times (0, T]$, i.e., there exists a point $(x, t) \in D \times (0, T]$ such that

$$v(x, t) = \max_{(y, \tau) \in \bar{D} \times [0, T]} v(y, \tau).$$

Then the necessary conditions for a maximum

$$\frac{\partial^2 v}{\partial x_i^2}(x, t) \leq 0, \quad i = 1, \dots, m,$$

and

$$\frac{\partial v}{\partial t}(x, t) \geq 0$$

must be fulfilled. Hence, we have

$$\Delta v(x, t) - \frac{\partial v}{\partial t}(x, t) \leq 0$$

which is a contradiction to

$$\Delta v - \frac{\partial v}{\partial t} = \varepsilon > 0$$

throughout $D \times (0, T]$.

Therefore, v attains its maximum on the parabolic boundary B . Then we can conclude that for all $(x, t) \in \bar{D} \times [0, T]$ there holds

$$u(x, t) \leq v(x, t) \leq \max_{(y, \tau) \in B} v(y, \tau) \leq \max_{(y, \tau) \in B} u(y, \tau) + \varepsilon T.$$

As ε can be chosen arbitrarily it follows that

$$u(x, t) \leq \max_{(y, \tau) \in B} u(y, \tau),$$

and this ends the proof. \square

We wish to mention that analogous to the maximum-minimum principle Theorem 6.8 for harmonic functions there holds a strong maximum-minimum principle, stating that under the conditions of Theorem 9.1 a solution to the heat equation cannot assume its maximum or minimum in the parabolic interior unless it is constant. For a proof in the case $m = 1$ we refer to Cannon [1].

The maximum-minimum principle implies uniqueness and continuous dependence for the initial boundary value problem as formulated in the following two theorems.

Theorem 9.2. *The initial boundary value problem for the heat equation has at most one solution.*

Theorem 9.3. *The solution to the initial boundary value problem for the heat equation depends continuously in the maximum norm on the given initial and boundary data.*

In the statement of Theorem 9.3 we tacitly have assumed that we will be able to establish existence of the solution.

9.2 Heat Potentials

The function

$$(9.5) \quad G(x, t; y, \tau) := \frac{1}{\sqrt{4\pi(t-\tau)}^m} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\}, \quad t > \tau,$$

is called *fundamental solution* of the heat equation. Straightforward differentiation shows that G satisfies the heat equation with respect to the variables x and t . With the aid of this fundamental solution, as in the analysis of Laplace's equation in Chapter 6, we will define so called *heat potentials*. Then we shall use these potentials for solving the initial boundary value problem. For the remainder of this chapter, in the case $m = 2, 3$, we always will assume that the bounded domain $D \subset \mathbb{R}^m$ is of class C^2 . We will take care of the initial condition through the Poisson integral introduced in the following theorem.

Theorem 9.4. *Let w be a continuous function on \mathbb{R}^m with compact support, that is, w vanishes outside some compact set. Then the Poisson integral*

$$(9.6) \quad u(x, t) := \frac{1}{\sqrt{4\pi t}^m} \int_{\mathbb{R}^m} \exp\left\{-\frac{|x-y|^2}{4t}\right\} w(y) dy$$

defines an infinitely differentiable solution of the heat equation in $\mathbb{R}^m \times (0, \infty)$. It can be continuously extended into $\mathbb{R}^m \times [0, \infty)$ with initial values

$$(9.7) \quad u(\cdot, 0) = w.$$

Proof: Obviously, for all $x \in \mathbb{R}^m$ and $t \in (0, \infty)$ differentiation and integration can be interchanged to show that the Poisson integral is infinitely differentiable

and satisfies the heat equation. Substituting $y = x + 2\sqrt{t}z$, we obtain

$$\frac{1}{\sqrt{4\pi t}^m} \int_{\mathbb{R}^m} \exp\left\{-\frac{|x-y|^2}{4t}\right\} w(y) dy = \frac{1}{\sqrt{\pi}^m} \int_{\mathbb{R}^m} w(x + 2\sqrt{t}z) e^{-|z|^2} dz.$$

Since w has compact support it is bounded and uniformly continuous. Let M be a bound for w . Then for $\varepsilon > 0$ choose R such that

$$\frac{1}{\sqrt{\pi}^m} \int_{|z| \geq R} e^{-|z|^2} dz < \frac{\varepsilon}{4M}.$$

Because w is uniformly continuous there exists $\delta > 0$ such that

$$|w(x) - w(y)| < \frac{\varepsilon}{2}$$

for all x and y with $|x - y| < \delta$. Setting $\eta = \delta^2/4R^2$ and making use of the integral (see Problem 9.1)

$$(9.8) \quad \int_{\mathbb{R}^m} e^{-|z|^2} dz = \sqrt{\pi}^m,$$

we deduce that for all $t < \eta$ there holds

$$\begin{aligned} & \left| \frac{1}{\sqrt{4\pi t}^m} \int_{\mathbb{R}^m} \exp\left\{-\frac{|x-y|^2}{4t}\right\} w(y) dy - w(x) \right| \\ &= \left| \frac{1}{\sqrt{\pi}^m} \int_{\mathbb{R}^m} \{w(x + 2\sqrt{t}z) - w(x)\} e^{-|z|^2} dz \right| \\ &< \frac{\varepsilon}{2\sqrt{\pi}^m} \int_{|z| \leq R} e^{-|z|^2} dz + \frac{2M}{\sqrt{\pi}^m} \int_{|z| \geq R} e^{-|z|^2} dz < \varepsilon, \end{aligned}$$

since for $|z| \leq R$ we have $2\sqrt{t}z < 2\sqrt{\eta}R = \delta$. □

In order to deal with the boundary condition we will need *surface heat potentials*. Analogous to the terminology used for harmonic functions we define single- and double-layer potentials. For a function $\varphi \in C(\Gamma \times [0, T])$ the *single-layer heat potential* is given by

$$(9.9) \quad u(x, t) := \int_0^t \int_{\Gamma} G(x, t; y, \tau) \varphi(y, \tau) ds(y) d\tau$$

and the *double-layer heat potential* by

$$(9.10) \quad u(x, t) := \int_0^t \int_{\Gamma} \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \varphi(y, \tau) ds(y) d\tau$$

These potentials are well defined for $x \in D$ and $t \in (0, T]$ with the time integral to be understood as an improper integral with respect to the upper limit. For the double-layer potential we assume the unit normal vector n to the boundary surface Γ to be directed into the exterior of D .

In the case of one space dimension for an interval $D := (a, b)$ the two potentials have the form

$$(9.9') \quad \begin{aligned} u(x, t) &:= \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \exp \left\{ -\frac{(x-a)^2}{4(t-\tau)} \right\} \varphi(a, \tau) d\tau \\ &+ \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \exp \left\{ -\frac{(x-b)^2}{4(t-\tau)} \right\} \varphi(b, \tau) d\tau \end{aligned}$$

and

$$(9.10') \quad \begin{aligned} u(x, t) &:= \int_0^t \frac{1}{4\sqrt{\pi(t-\tau)}^3} (a-x) \exp \left\{ -\frac{(x-a)^2}{4(t-\tau)} \right\} \varphi(a, \tau) d\tau \\ &+ \int_0^t \frac{1}{4\sqrt{\pi(t-\tau)}^3} (x-b) \exp \left\{ -\frac{(x-b)^2}{4(t-\tau)} \right\} \varphi(b, \tau) d\tau. \end{aligned}$$

Standard analysis again shows that both the single- and the double-layer heat potential are infinitely differentiable solutions to the heat equation in $D \times (0, T]$. In addition they can be continuously extended into $D \times [0, T]$ with initial values $u(\cdot, 0) = 0$ in D . For the discussion of the boundary values we will confine our analysis to the double-layer potential.

Theorem 9.5. *The double-layer heat potential with continuous density φ can be continuously extended from $D \times (0, T]$ into $\bar{D} \times (0, T]$ with limiting values*

$$(9.11) \quad u(x, t) = \int_0^t \int_{\Gamma} \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \varphi(y, \tau) ds(y) d\tau - \frac{1}{2} \varphi(x, t)$$

for $x \in \Gamma$ and $t \in (0, T]$. Here the time integral exists as improper integral.

Proof: We carry out the proof for the cases $m = 2, 3$, and leave it to the reader to work out the details for the simpler case $m = 1$ (see Problem 9.3). For $x \in D$ we may interchange the integrations over Γ and $[0, t]$ since the integrand is continuous on $\Gamma \times [0, t]$ with value zero at the upper limit $\tau = t$. Then we substitute for the time integral

$$\sigma = \frac{|x-y|}{2\sqrt{t-\tau}}$$

to obtain

$$\begin{aligned} u(x, t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}^m} \int_{\Gamma} \frac{(n(y), x-y)}{2(t-\tau)} \exp \left\{ -\frac{(x-y)^2}{4(t-\tau)} \right\} \varphi(y, \tau) ds(y) d\tau \\ &= \frac{1}{\sqrt{\pi}^m} \int_{\Gamma} \frac{(n(y), x-y)}{|x-y|^m} \int_{|x-y|/2\sqrt{t}}^{\infty} \sigma^{m-1} e^{-\sigma^2} \varphi \left(y, t - \frac{|x-y|^2}{4\sigma^2} \right) d\sigma ds(y). \end{aligned}$$

Therefore, we can view the double-layer heat potential as a harmonic double-layer potential with the density

$$\begin{aligned}\psi(x, y, t) &:= \frac{1}{\sqrt{\pi^m}} \int_{|x-y|/2\sqrt{t}}^{\infty} \sigma^{m-1} e^{-\sigma^2} \varphi \left(y, t - \frac{|x-y|^2}{4\sigma^2} \right) d\sigma \\ &= |x-y|^m \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}^m} \frac{1}{2(t-\tau)} \exp \left\{ -\frac{|x-y|^2}{4(t-\tau)} \right\} \varphi(y, \tau) d\tau\end{aligned}$$

depending on the time t as a parameter. First, we will show that ψ is continuous on $\mathbb{R}^m \times \Gamma \times (0, T]$ with

$$(9.12) \quad \lim_{x \rightarrow y} \psi(x, y, t) = \gamma_m \varphi(y, t)$$

for all $y \in \Gamma$ and $t \in (0, T]$, where

$$\gamma_m := \frac{1}{\sqrt{\pi^m}} \int_0^{\infty} s^{m-1} e^{-s^2} ds$$

and where the limit holds uniformly on Γ and on compact subintervals of $(0, T]$. Clearly the function ψ is continuous for all $x \neq y$ and all $t \in (0, T]$. We establish the limit (9.12) by splitting

$$\begin{aligned}\psi(x, y, t) &= \frac{1}{\sqrt{\pi^m}} \int_{|x-y|/2\sqrt{t}}^{\sqrt{|x-y|}} \sigma^{m-1} e^{-\sigma^2} \varphi \left(y, t - \frac{|x-y|^2}{4\sigma^2} \right) d\sigma \\ &\quad + \frac{1}{\sqrt{\pi^m}} \int_{\sqrt{|x-y|}}^{\infty} \sigma^{m-1} e^{-\sigma^2} \left\{ \varphi \left(y, t - \frac{|x-y|^2}{4\sigma^2} \right) - \varphi(y, t) \right\} d\sigma \\ &\quad + \varphi(y, t) \frac{1}{\sqrt{\pi^m}} \int_{\sqrt{|x-y|}}^{\infty} \sigma^{m-1} e^{-\sigma^2} d\sigma =: I_1 + I_2 + I_3.\end{aligned}$$

Obviously we have

$$\lim_{x \rightarrow y} I_1(x, y, t) = 0$$

uniformly on Γ and on compact subintervals of $(0, T]$ and

$$\lim_{x \rightarrow y} I_3(x, y, t) = \gamma_m \varphi(y, t)$$

uniformly on $\Gamma \times [0, T]$. Since φ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\varphi(y, t_1) - \varphi(y, t_2)| < \varepsilon$$

for all $y \in \Gamma$ and all $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| < \delta$. Then for all $|x-y| < 4\delta$ and all $\sigma \geq \sqrt{|x-y|}$ we have

$$\frac{|x-y|^2}{4\sigma^2} \leq \frac{|x-y|}{4} < \delta$$

and therefore

$$\left| \varphi \left(y, t - \frac{|x-y|^2}{4\sigma^2} \right) - \varphi(y, t) \right| < \varepsilon.$$

Hence,

$$|I_2(x, y, t)| < \frac{\varepsilon}{\sqrt{\pi}^m} \int_{\sqrt{|x-y|}}^{\infty} \sigma^{m-1} e^{-\sigma^2} d\sigma \leq \gamma_m \varepsilon$$

and thus

$$\lim_{x \rightarrow y} I_2(x, y, t) = 0$$

uniformly on $\Gamma \times [0, T]$. Now, using $\gamma_2 = 1/2\pi$ and $\gamma_3 = 1/4\pi$, from Theorem 6.13 we can deduce that

$$(9.13) \quad \lim_{h \rightarrow 0} u(x - hn(x), t) = \int_0^t \int_{\Gamma} \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \varphi(y, \tau) ds(y) d\tau - \frac{1}{2} \varphi(x, t)$$

with uniform convergence on Γ and on compact subintervals of $(0, T]$. Note that we need a slightly generalized version of Theorem 6.13 where the density is allowed to depend also on x and a parameter t . It is left to the reader to go over a proof of Theorem 6.13 – for instance in Colton and Kress [1] – to see that it can be extended to this more general case.

Finally the statement on the continuity of the double-layer heat potential on $\bar{D} \times (0, T]$ follows from the fact that the right hand side of (9.13) is continuous on $\Gamma \times [0, T]$ by the following Theorem 9.6. \square

Consider the integral operator $H : C(\Gamma \times [0, T]) \rightarrow C(\Gamma \times [0, T])$ defined by

$$(9.14) \quad (H\varphi)(x, t) := 2 \int_0^t \int_{\Gamma} \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \varphi(y, \tau) ds(y) d\tau$$

for $x \in \Gamma$ and $t \in (0, T]$ with improper integral over $[0, t]$. For its kernel

$$2 \frac{\partial G(x, t; y, \tau)}{\partial n(y)} = \frac{1}{\sqrt{4\pi(t-\tau)}^m} \frac{(n(y), x-y)}{t-\tau} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\},$$

using Theorem 6.15, we can estimate

$$2 \left| \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \right| \leq \frac{L}{\sqrt{t-\tau}^m} \frac{|x-y|^2}{t-\tau} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\}, \quad t > \tau,$$

with some constant L . Applying the inequality (see Problem 9.2)

$$(9.15) \quad s^\beta e^{-s} \leq \beta^\beta e^{-\beta},$$

which is valid for all $0 < s, \beta < \infty$, with $s = |x-y|^2/4(t-\tau)$, $\beta = 1+m/2-\alpha$, we derive the further estimate

$$(9.16) \quad 2 \left| \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \right| \leq \frac{M}{(t-\tau)^\alpha |x-y|^{m-2\alpha}}, \quad t > \tau, \quad x \neq y,$$

for all $0 < \alpha < 1+m/2$ with some constant M depending on L and α . Hence, choosing $1/2 < \alpha < 1$, the kernel of H is seen to be weakly singular with respect to both the integrals over Γ and over time. Therefore, proceeding as in the proofs of Theorems 2.21 and 2.22, we can establish the following result.

Theorem 9.6. *The double-layer heat potential operator H is compact from $C(\Gamma \times [0, T])$ into $C(\Gamma \times [0, T])$.*

Summarizing, the double-layer heat potential with continuous density is continuous in $\bar{D} \times [0, T]$ with the exception of possible discontinuities on the boundary Γ for $t = 0$. From (9.16) we observe that $(H\varphi)(\cdot, 0) = 0$. Therefore we can expect continuity in $\bar{D} \times [0, T]$ only if $\varphi(\cdot, 0) = 0$ on Γ . And indeed, in this case we can extend the density φ continuously onto $\Gamma \times (-\infty, T]$ by setting $\varphi(\cdot, t) = 0$ for $t < 0$. Then, as a consequence of Theorem 9.5, we can state the following corollary.

Corollary 9.7. *The double-layer heat potential is continuous on $\bar{D} \times [0, T]$ provided the continuous density satisfies $\varphi(\cdot, 0) = 0$.*

9.3 Initial Boundary Value Problem: Existence

We now return to the initial boundary value problem. First we reduce it to the special case with initial condition zero. To this end consider

$$(9.17) \quad u(x, t) := v(x) + \frac{1}{\sqrt{4\pi t^m}} \int_{\mathbb{R}^m} \exp\left\{-\frac{|x-y|^2}{4t}\right\} \{w(y) - v(y)\} dy.$$

Here, v denotes the unique solution to the Dirichlet problem for Laplace's equation in D with boundary condition $v = f(\cdot, 0)$ on Γ (see Theorem 6.18). Because of the compatibility condition (9.4), the function $w - v$ can be continuously extended into \mathbb{R}^m by setting it zero outside D . Then, from Theorems 6.6 and 9.4, we deduce that (9.17) defines an infinitely differentiable solution to the heat equation in $D \times (0, T]$ which is continuous in $\bar{D} \times [0, T]$ and satisfies the initial condition $u(\cdot, 0) = w$ in D . Hence, by superposition, it suffices to treat the special case of the initial boundary value problem with initial condition

$$(9.18) \quad u(\cdot, 0) = 0 \quad \text{in } D$$

and boundary condition

$$(9.19) \quad u = f \quad \text{on } \Gamma \times [0, T]$$

where f satisfies the compatibility condition

$$(9.20) \quad f(\cdot, 0) = 0 \quad \text{on } \Gamma.$$

In order to deal with the boundary condition (9.19) we seek the solution in the form of a double-layer heat potential. Again we will leave the case $m = 1$ as an excercise for the reader (see Problem 9.4).

Theorem 9.8. *The double-layer heat potential (9.10) solves the initial boundary value problem (9.18) to (9.20) provided the continuous density φ for all $x \in \Gamma$ and $t \in (0, T]$ solves the integral equation*

$$(9.21). \quad \varphi(x, t) - 2 \int_0^t \int_{\Gamma} \frac{\partial G(x, t; y, \tau)}{\partial n(y)} \varphi(y, \tau) ds(y) d\tau = -2f(x, t)$$

Proof: From Theorem 9.5 and Corollary 9.7. The compatibility condition (9.20) for f ensures that $\varphi(\cdot, 0) = 0$ for solutions to (9.21). \square

By Theorem 9.6 and the Riesz theory Theorem 3.4, the inhomogeneous integral equation $\varphi - H\varphi = -2f$ is solvable provided the corresponding homogeneous integral equation $\varphi - H\varphi = 0$ has only the trivial solution. From the weak singularity (9.16) we see that for the integral operator H the integration over Γ corresponds to a compact operator. Therefore we can estimate for the maximum norm with respect to the space variable by

$$(9.22) \quad \|(H\varphi)(\cdot, t)\|_{\infty, \Gamma} \leq C \int_0^t \frac{1}{(t-\tau)^\alpha} \|\varphi(\cdot, \tau)\|_{\infty, \Gamma} d\tau$$

for all $0 < t \leq T$ with some constant C depending on Γ and α . Note that (see Problem 9.1)

$$(9.23) \quad \begin{aligned} \int_\tau^t \frac{d\sigma}{(t-\sigma)^\alpha(\sigma-\tau)^\beta} &= \frac{1}{(t-\tau)^{\alpha+\beta-1}} \int_0^1 \frac{ds}{s^\beta(1-s)^\alpha} \\ &\leq \frac{1}{(t-\tau)^{\alpha+\beta-1}} \int_0^1 \frac{ds}{[s(1-s)]^\alpha} \end{aligned}$$

for $\beta \leq \alpha < 1$. Then, from (9.22) by induction and by interchanging the integrals in

$$\int_0^t \int_0^\sigma \frac{d\tau}{(t-\tau)^\alpha(\sigma-\tau)^\beta} d\sigma = \int_0^t \int_\tau^t \frac{d\sigma}{(t-\sigma)^\alpha(\sigma-\tau)^\beta} d\tau,$$

we find that for all $k \in \mathbb{N}$ and all $0 < t \leq T$ there holds

$$\|(H^k \varphi)(\cdot, t)\|_{\infty, \Gamma} \leq C^k I^{k-1} \int_0^t \frac{1}{(t-\tau)^{k(\alpha-1)+1}} \|\varphi(\cdot, \tau)\|_{\infty, \Gamma} d\tau,$$

where

$$I := \int_0^1 \frac{ds}{[s(1-s)]^\alpha}.$$

Hence there exists an integer k such that for $\tilde{H} := H^k$ we have an estimate of the form

$$(9.24) \quad \|(\tilde{H}\varphi)(\cdot, t)\|_{\infty, \Gamma} \leq M \int_0^t \|\varphi(\cdot, \tau)\|_{\infty, \Gamma} d\tau$$

for all $0 \leq t \leq T$ with some constant M .

Now, let φ be a solution to the homogeneous equation $\varphi - H\varphi = 0$. Then, by iteration, φ also solves $\varphi - \tilde{H}\varphi = 0$. Proceeding as in the proof of Theorem 3.12, i.e., as for Volterra equations with continuous kernel, from (9.24) we can derive that

$$\|\varphi(\cdot, t)\|_{\infty, \Gamma} \leq \|\varphi\|_\infty \frac{M^n t^n}{n!}, \quad 0 \leq t \leq T,$$

for all $n \in \mathbb{N}$. Hence $\|\varphi(\cdot, t)\|_{\infty, \Gamma} = 0$ for all $t \in [0, T]$, i.e., $\varphi = 0$. Thus we have established the following existence theorem.

Theorem 9.9. *The initial boundary value problem for the heat equation has a unique solution.*

We wish to mention that the analysis of this chapter can be carried over to an initial boundary value problem where the Dirichlet boundary condition is replaced by a Neumann boundary condition. Here, the solution can be obtained by using a single-layer heat potential. For the one-dimensional case see Problem 9.5.

The idea to proceed along the same line in the case of the heat equation as for the Laplace equation essentially is due to Holmgren [1] and Gevrey [1,2]. The first rigorous existence proof was given by Müntz [1] using successive approximations for the integral equation in two dimensions. For a more comprehensive study of integral equations for the heat equation we refer to Cannon [1], Friedman [1], and Pogorzelski [1].

Problems

- 9.1. Prove the integrals (9.8) and (9.23).
- 9.2. Prove the inequality (9.15).
- 9.3. Carry out the proof of Theorem 9.5 in the one-dimensional case.
- 9.4. Show that the double-layer heat potential (9.10') solves the initial boundary value problem for the heat equation in $(a, b) \times (0, T]$ with homogeneous initial condition $u(\cdot, 0) = 0$ and Dirichlet boundary conditions $u(a, \cdot) = f(a, \cdot)$ and $u(b, \cdot) = f(b, \cdot)$ (with compatibility condition $f(a, 0) = f(b, 0) = 0$) provided the densities satisfy the system of Volterra integral equations

$$\varphi(a, t) - \int_0^t h(t, \tau) \varphi(b, \tau) d\tau = -2f(a, t)$$

$$\varphi(b, t) - \int_0^t h(t, \tau) \varphi(a, \tau) d\tau = -2f(b, t)$$

for $0 \leq t \leq T$ with kernel

$$h(t, \tau) := \frac{a-b}{2\sqrt{\pi}} \frac{1}{\sqrt{(t-\tau)^3}} \exp\left\{-\frac{(a-b)^2}{4(t-\tau)}\right\}, \quad 0 \leq \tau < t.$$

Establish existence and uniqueness for this system.

- 9.5. Show that the single-layer heat potential (9.9') solves the initial boundary value problem for the heat equation in $(a, b) \times (0, T]$ with homogeneous initial condition $u(\cdot, 0) = 0$ and Neumann boundary conditions $-\partial u(a, \cdot)/\partial x = g(a, \cdot)$ and $\partial u(b, \cdot)/\partial x = g(b, \cdot)$ (with compatibility condition $g(a, 0) = g(b, 0) = 0$) provided the densities satisfy the system of Volterra integral equations

$$\varphi(a, t) + \int_0^t h(t, \tau) \varphi(b, \tau) d\tau = 2g(a, t)$$

$$\varphi(b, t) + \int_0^t h(t, \tau) \varphi(a, \tau) d\tau = 2g(b, t)$$

for $0 \leq t \leq T$ with the kernel h given as in Problem 9.4. Establish existence and uniqueness for this system.

10. Operator Approximations

In the subsequent chapters we will study the numerical solution of integral equations. It is our intention to provide the basic tools for the investigation of approximate solution methods and their error analysis. We do not aim at a complete review of all the various numerical methods which have been developed in the literature. However, we will illustrate the principal ideas by a few instructive examples.

A fundamental concept for approximately solving an operator equation

$$A\varphi = f$$

with a bounded linear operator $A : X \rightarrow Y$ mapping a Banach space X into a Banach space Y is to replace it by an equation

$$A_n\varphi_n = f$$

with an approximating sequence of bounded linear operators $A_n : X \rightarrow Y$. For practical purposes, the approximating equations will be chosen such that they can be reduced to solving a finite dimensional linear system. In this chapter we will provide an error analysis for such general approximation schemes. In particular, we will derive convergence results and error estimates for the cases where we have either norm or pointwise convergence of the sequence $A_n \rightarrow A$, $n \rightarrow \infty$.

10.1 Approximations Based on Norm Convergence

Let X and Y be Banach spaces and let $A : X \rightarrow Y$ be a bounded linear operator. Assume the sequence $A_n : X \rightarrow Y$ of bounded linear operators to be norm convergent $\|A_n - A\| \rightarrow 0$, $n \rightarrow \infty$.

Theorem 10.1. *Assume that A has a bounded inverse $A^{-1} : Y \rightarrow X$. Then for sufficiently large n , more precisely, for all n with*

$$\|A^{-1}(A_n - A)\| < 1$$

the inverse operators $A_n^{-1} : Y \rightarrow X$ exist and are uniformly bounded by

$$(10.1) \quad \|A_n^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}.$$

For the solutions of the equations

$$A\varphi = f \quad \text{and} \quad A_n\varphi_n = f_n$$

there holds the error estimate

$$\|\varphi_n - \varphi\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|} \{ \|(A_n - A)\varphi\| + \|f_n - f\| \}.$$

Proof: If $\|A^{-1}(A_n - A)\| < 1$ then, by the Neumann series Theorem 2.8, the inverse $[I - A^{-1}(A - A_n)]^{-1}$ of $I - A^{-1}(A - A_n) = A^{-1}A_n$ exists and is bounded by

$$\|[I - A^{-1}(A - A_n)]^{-1}\| \leq \frac{1}{1 - \|A^{-1}(A_n - A)\|}.$$

But then obviously $[I - A^{-1}(A - A_n)]^{-1}A^{-1}$ is the inverse of A_n and bounded by (10.1). The error estimate follows from

$$A_n(\varphi_n - \varphi) = (A_n\varphi_n - A\varphi) + (A\varphi - A_n\varphi) = f_n - f + (A - A_n)\varphi$$

and the proof is complete. \square

In Theorem 10.1 from the unique solvability of the original equation we conclude unique solvability of the approximating equation provided the approximation is sufficiently close in the operator norm. The converse situation is described through the following

Theorem 10.2. *Assume there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ the inverse operators $A_n^{-1} : Y \rightarrow X$ exist and are uniformly bounded. Then the inverse operator $A^{-1} : Y \rightarrow X$ exists and is bounded by*

$$(10.2) \quad \|A^{-1}\| \leq \frac{\|A_n^{-1}\|}{1 - \|A_n^{-1}(A_n - A)\|}$$

for all n with $\|A_n^{-1}(A_n - A)\| < 1$. For the solutions of the equations

$$A\varphi = f \quad \text{and} \quad A_n\varphi_n = f_n$$

there holds the error estimate

$$\|\varphi_n - \varphi\| \leq \frac{\|A_n^{-1}\|}{1 - \|A_n^{-1}(A_n - A)\|} \{ \|(A_n - A)\varphi_n\| + \|f_n - f\| \}.$$

Proof: From Theorem 10.1 by interchanging the roles of A and A_n . \square

Note that Theorem 10.2 provides error bounds which, in principle, can be evaluated since they involve A_n^{-1} and φ_n but neither A^{-1} nor φ . Theorem 10.1 shows that the accuracy of the approximate solution depends on how well $A_n\varphi$ approximates $A\varphi$ for the exact solution. In Chapter 11 we will apply the error analysis of Theorems 10.1 and 10.2 to the approximation of integral equations of the second kind by replacing the kernels through so called degenerate kernels.

10.2 Uniform Boundedness Principle

In order to develop a similar error analysis for the case where the sequence (A_n) is merely pointwise convergent, that is, $A_n\varphi \rightarrow \varphi$, $n \rightarrow \infty$, for all φ , we will have to bridge the gap between norm and pointwise convergence. This goal will be achieved through compactness properties and one of the necessary tools will be provided by the following *principle of uniform boundedness*.

Theorem 10.3. *Let the sequence $A_n : X \rightarrow Y$ of bounded linear operators mapping a Banach space X into a normed space Y be pointwise bounded, that is, for each $\varphi \in X$ there exists a positive number C_φ depending on φ such that $\|A_n\varphi\| \leq C_\varphi$ for all $n \in \mathbb{N}$. Then the sequence (A_n) is uniformly bounded, that is, there exists some constant C such that $\|A_n\| \leq C$ for all $n \in \mathbb{N}$.*

Proof: In the first step, by an indirect proof, we establish that positive constants M and ρ and an element $\psi \in X$ can be chosen such that

$$(10.3) \quad \|A_n\varphi\| \leq M$$

for all $\varphi \in X$ with $\|\varphi - \psi\| \leq \rho$ and all $n \in \mathbb{N}$. Assume that this is not possible. Then, by induction, we construct sequences (n_k) in \mathbb{N} , (ρ_k) in \mathbb{R} and (φ_k) in X with the following properties:

- a) $0 < \rho_k \leq \frac{1}{2} \rho_{k-1}$,
- b) $\|\varphi_k - \varphi_{k-1}\| \leq \frac{1}{2} \rho_{k-1}$,
- c) $\|A_{n_k}(\varphi)\| \geq k$ for all φ with $\|\varphi - \varphi_k\| \leq \rho_k$,

for $k = 1, 2, 3, \dots$.

We initialize the induction by setting $n_0 = 1$, $\rho_0 = 1$ and $\varphi_0 = 0$. Assume $n_k \in \mathbb{N}$, $\rho_k > 0$ and $\varphi_k \in X$ given. Then there exist $n_{k+1} \in \mathbb{N}$ and $\varphi_{k+1} \in X$ satisfying $\|\varphi_{k+1} - \varphi_k\| \leq \rho_k/2$ and $\|A_{n_{k+1}}\varphi_{k+1}\| \geq k + 2$. Otherwise we would have $\|A_n\varphi\| \leq k + 2$ for all $\varphi \in X$ with $\|\varphi - \varphi_k\| \leq \rho_k/2$ and all $n \in \mathbb{N}$, and this contradicts our assumption. Set

$$\rho_{k+1} := \min \left(\frac{\rho_k}{2}, \frac{1}{\|A_{n_{k+1}}\|} \right) \leq \frac{\rho_k}{2}.$$

Then for all $\varphi \in X$ with $\|\varphi - \varphi_{k+1}\| \leq \rho_{k+1}$ there holds

$$\|A_{n_{k+1}}\varphi\| \geq \|A_{n_{k+1}}\varphi_{k+1}\| - \|A_{n_{k+1}}(\varphi - \varphi_{k+1})\| \geq k + 1,$$

since $\|A_{n_{k+1}}(\varphi - \varphi_{k+1})\| \leq \|A_{n_{k+1}}\| \rho_{k+1} \leq 1$.

For $j > k$, using the geometric series, we have

$$\|\varphi_k - \varphi_j\| \leq \|\varphi_k - \varphi_{k+1}\| + \dots + \|\varphi_{j-1} - \varphi_j\| \leq \frac{1}{2} \rho_k + \dots + \frac{1}{2} \rho_{j-1} \leq \rho_k.$$

Therefore (φ_k) is a Cauchy sequence and converges to some element φ in the Banach space X . From $\|\varphi_k - \varphi_j\| \leq \rho_k$ for all $j \geq k$ by passing to the limit

$j \rightarrow \infty$ we see that $\|\varphi_k - \varphi\| \leq \rho_k$ for all $k \in \mathbb{N}$. Therefore we have $\|A_{n_k}\varphi\| \geq k$ for all $k \in \mathbb{N}$ which is a contradiction to the boundedness of the sequence $(A_n\varphi)$.

Now, in the second step, from the validity of (10.3) we deduce for each $\varphi \in X$ with $\|\varphi\| \leq 1$ and for all $n \in \mathbb{N}$ the estimate

$$\|A_n\varphi\| = \frac{1}{\rho} \|A_n(\rho\varphi + \psi) - A_n\psi\| \leq \frac{2M}{\rho}.$$

This completes the proof. \square

As a simple consequence of the uniform boundedness principle we note that the limit operator of a pointwise convergent sequence of bounded linear operators again is bounded.

Corollary 10.4. *Let X be a Banach space and Y be a normed space and let the bounded linear operators $A_n : X \rightarrow Y$ be pointwise convergent with limit operator $A : X \rightarrow Y$. Then the convergence is uniform on compact subsets U of X , that is,*

$$\sup_{\varphi \in U} \|A_n\varphi - A\varphi\| \rightarrow 0, \quad n \rightarrow \infty.$$

Proof: For $\varepsilon > 0$ consider the open balls $B(\varphi; r) = \{\varphi \in X : \|\psi - \varphi\| < r\}$ with center $\varphi \in X$ and radius $r = \varepsilon/3C$. Clearly

$$U \subset \bigcup_{\varphi \in U} B(\varphi; r)$$

forms an open covering of U . Since U is compact, there exists a finite subcovering

$$U \subset \bigcup_{j=1}^m B(\varphi_j; r).$$

Pointwise convergence of (A_n) guarantees the existence of an integer $N(\varepsilon)$ such that

$$\|A_n\varphi_j - A\varphi_j\| < \frac{\varepsilon}{3}$$

for all $n \geq N(\varepsilon)$ and all $j = 1, \dots, m$. Now let $\varphi \in U$ be arbitrary. Then φ is contained in some ball $B(\varphi_j; r)$ with center φ_j . Hence for all $n \geq N(\varepsilon)$ there holds

$$\begin{aligned} \|A_n\varphi - A\varphi\| &\leq \|A_n\varphi - A_n\varphi_j\| + \|A_n\varphi_j - A\varphi_j\| + \|A\varphi_j - A\varphi\| \\ &\leq \|A_n\| \|\varphi - \varphi_j\| + \frac{\varepsilon}{3} + \|A\| \|\varphi_j - \varphi\| \leq 2Cr + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence the convergence is uniform on U .

10.3 Collectively Compact Operators

Motivated through Corollary 10.4 and following Anselone [1], we introduce the concept of collectively compact operators.

Definition 10.5. A set $\mathcal{A} = \{A : X \rightarrow Y\}$ of linear operators mapping a normed space X into a normed space Y is called *collectively compact* if for each bounded set $U \subset X$ the image set $\mathcal{A}(U) = \{A\varphi : \varphi \in U, A \in \mathcal{A}\}$ is relatively compact.

Clearly, every operator in a collectively compact set is compact. Each finite set of compact operators is collectively compact. A sequence (A_n) is called collectively compact when the corresponding set is. Pointwise convergence $A_n \rightarrow A$, $n \rightarrow \infty$, of a collectively compact sequence implies compactness of the limit operator since

$$A(U) \subset \overline{\{A_n\varphi : \varphi \in U, n \in \mathbb{N}\}}.$$

Theorem 10.6. Let X, Z be normed spaces and let Y be a Banach space. Let \mathcal{A} be a collectively compact set of operators mapping X into Y and let $L_n : Y \rightarrow Z$ be a pointwise convergent sequence of bounded linear operators with limit operator $L : Y \rightarrow Z$. Then

$$\|(L_n - L)A\| \rightarrow 0, \quad n \rightarrow \infty,$$

uniformly for all $A \in \mathcal{A}$, that is,

$$\sup_{A \in \mathcal{A}} \|(L_n - L)A\| \rightarrow 0, \quad n \rightarrow \infty.$$

Proof: The set $U := \{A\varphi : \|\varphi\| \leq 1, A \in \mathcal{A}\}$ is relatively compact. By Corollary 10.4 the convergence $L_n\psi \rightarrow L\psi$, $n \rightarrow \infty$, is uniform for all $\psi \in U$. Hence, for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$\|(L_n - L)\varphi\| < \varepsilon$$

for all $n \geq N(\varepsilon)$, all $\varphi \in X$ with $\|\varphi\| \leq 1$ and all $A \in \mathcal{A}$. Therefore

$$\|(L_n - L)A\| \leq \varepsilon$$

for all $n \geq N(\varepsilon)$ and all $A \in \mathcal{A}$.

Corollary 10.7. Let X be a Banach space and let $A_n : X \rightarrow X$ be a collectively compact and pointwise convergent sequence with limit operator $A : X \rightarrow X$. Then

$$\|(A_n - A)A\| \rightarrow 0, \quad n \rightarrow \infty,$$

and

$$\|(A_n - A)A_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

10.4 Approximations Based on Pointwise Convergence

The following error analysis for equations of the second kind takes advantage of the preceding Corollary 10.7 and is due to Brakhage [1] and Anselone and Moore [1]. Let X be a Banach space and let $A : X \rightarrow X$ be a compact linear operator. Assume the sequence $A_n : X \rightarrow X$ to be collectively compact and pointwise convergent $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in X$.

Theorem 10.8. *Assume that $I - A$ is invertible. Then for sufficiently large n , more precisely, for all n with*

$$\|(I - A)^{-1}(A_n - A)A_n\| < 1$$

the operators $I - A_n$ are invertible and uniformly bounded by

$$(10.4) \quad \|(I - A_n)^{-1}\| \leq \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}.$$

For the solutions of the equations

$$\varphi - A\varphi = f \quad \text{and} \quad \varphi_n - A_n\varphi_n = f$$

there holds the error estimate

$$\|\varphi_n - \varphi\| \leq \|(I - A)^{-1}\| \frac{\|(A_n - A)f\| + \|(A_n - A)A_n\varphi\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}.$$

Proof: By the Riesz theory Theorem 3.4 the inverse operator $(I - A)^{-1}$ is bounded. The identity

$$(I - A)^{-1} = I + (I - A)^{-1}A$$

suggests

$$B_n := I + (I - A)^{-1}A_n$$

as an approximate inverse for $I - A_n$. Elementary calculations yield

$$(10.5) \quad B_n(I - A_n) = I - S_n$$

where

$$S_n := (I - A)^{-1}(A_n - A)A_n.$$

From Corollary 10.7 we know that $\|S_n\| \rightarrow 0$, $n \rightarrow \infty$. For $\|S_n\| < 1$ the Neumann series Theorem 2.8 implies that $(I - S_n)^{-1}$ exists and is bounded by

$$\|(I - S_n)^{-1}\| \leq \frac{1}{1 - \|S_n\|}.$$

Now equation (10.5) first implies that $I - A_n$ is injective and therefore, since A_n is compact, by Theorem 3.4 the inverse $(I - A_n)^{-1}$ exists. Then (10.5) also yields $(I - A_n)^{-1} = (I - S_n)^{-1}B_n$, whence the estimate (10.4) follows.

Routine algebra yields

$$\begin{aligned} (I - A_n)^{-1} - (I - A)^{-1} &= (I - S_n)^{-1}\{B_n - (I - S_n)(I - A)^{-1}\} \\ &= (I - S_n)^{-1}\{(I - A)^{-1}(A_n - A) + S_n(I - A)^{-1}\}. \end{aligned}$$

From this we see that

$$\varphi_n - \varphi = (I - S_n)^{-1}\{(I - A)^{-1}(A_n - A)f + S_n\varphi\},$$

whence the error estimate follows. \square

Analogous to Theorem 10.1, in Theorem 10.8 from the unique solvability of the original equation we conclude unique solvability of the approximating equation provided the approximation is sufficiently close. The converse situation is described through the following

Theorem 10.9. *Assume there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ the inverse operators $(I - A_n)^{-1}$ exist and are uniformly bounded. Then the inverse $(I - A)^{-1}$ exists and is bounded by*

$$(10.6) \quad \|(I - A)^{-1}\| \leq \frac{1 + \|(I - A_n)^{-1}A\|}{1 - \|(I - A_n)^{-1}(A_n - A)A\|}$$

for all n with

$$\|(I - A_n)^{-1}(A_n - A)A\| < 1.$$

For the solutions of the equations

$$\varphi - A\varphi = f \quad \text{and} \quad \varphi_n - A_n\varphi_n = f$$

there holds the error estimate

$$\|\varphi_n - \varphi\| \leq \|(I - A_n)^{-1}\| \frac{\|(A_n - A)f\| + \|(A_n - A)A\varphi_n\|}{1 - \|(I - A_n)^{-1}(A_n - A)A\|}.$$

Proof: From Theorem 10.8 by interchanging the roles of A and A_n . \square

Note that Theorem 10.9 provides error bounds which, in principle, can be evaluated since they involve $(I - A_n)^{-1}$ and φ_n , but neither $(I - A)^{-1}$ nor φ . From Theorem 10.8 we have a uniform bound $\|(I - A_n)^{-1}\| \leq C$ for all $n \in \mathbb{N}$ with some constant C . Then writing

$$\varphi_n - \varphi = (I - A_n)^{-1}f - (I - A)^{-1}f = (I - A_n)^{-1}(A_n - A)\varphi$$

we obtain the estimate

$$(10.7) \quad \|\varphi_n - \varphi\| \leq C\|(A_n - A)\varphi\|.$$

This shows that the accuracy of the approximate solution essentially depends on how well $A_n\varphi$ approximates $A\varphi$ for the exact solution.

In Chapter 12 we will apply the error analysis of Theorems 10.8 and 10.9 to the approximation of integral equations of the second kind by using numerical quadratures.

10.5 Successive Approximations

We now briefly revisit the investigation of the convergence of successive approximations

$$\varphi_{n+1} := A\varphi_n + f$$

to solve the equation of the second kind

$$\varphi - A\varphi = f,$$

where $A : X \rightarrow X$ is a bounded linear operator in a Banach space X . So far we have used $\|A\| < 1$ as a sufficient condition for convergence (see Theorem 2.9). It is our aim now to show that similar to the case of finite dimensional linear equations convergence of successive approximations can be characterized through the spectral radius of the operator A . For the notion of the resolvent set $\rho(A)$, the resolvent $R(\lambda; A)$, the spectrum $\sigma(A)$, and the spectral radius $r(A)$ of a bounded linear operator A recall Definition 3.10.

We first note that the basic concepts of classical holomorphic function theory, i.e., derivative, integral, analyticity etc., and its fundamental theorems, i.e., Cauchy's integral theorem and integral formula, Taylor and Laurent series etc., can be canonically extended to functions f with values in a Banach space X depending on a complex variable varying on an open domain $D \subset \mathbb{C}$. In principle, all definitions and theorems, including their proofs, are obtained by replacing absolute values $|f(z)|$ in the classical theory through the norm $\|f(z)\|$ in X .

Theorem 10.10. *Let $A : X \rightarrow X$ be a bounded linear operator mapping the Banach space X into itself. Then the Neumann series*

$$(10.8) \quad (\lambda I - A)^{-1} = \sum_{k=0}^{\infty} \lambda^{-k-1} A^k$$

converges in the operator norm for all $|\lambda| > r(A)$ and diverges for all $|\lambda| < r(A)$.

Proof: Provided the series (10.8) converges it defines a bounded linear operator $S(\lambda)$. As in the proof of Theorem 2.8, this operator $S(\lambda)$ can be seen to be the inverse of $\lambda I - A$ since $|\lambda|^{-n-1} \|A^n\| \rightarrow 0$, $n \rightarrow \infty$, is necessary for the convergence of the series (10.8).

Let λ_0 belong to the resolvent set $\rho(A)$ of A . Then, by Theorem 2.8, for all λ with

$$|\lambda - \lambda_0| \|R(\lambda_0; A)\| < 1$$

the series

$$T(\lambda) := \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k R(\lambda_0; A)^{k+1}$$

converges in the operator norm and defines a bounded linear operator with

$$\begin{aligned} T(\lambda) &= R(\lambda_0; A) [I - (\lambda_0 - \lambda)R(\lambda_0; A)]^{-1} \\ &= [I - (\lambda_0 - \lambda)R(\lambda_0; A)]^{-1} R(\lambda_0; A). \end{aligned}$$

Hence,

$$\begin{aligned} (\lambda I - A)T(\lambda) &= [\lambda_0 I - A - (\lambda_0 - \lambda)I] T(\lambda) \\ &= [I - (\lambda_0 - \lambda)R(\lambda_0; A)] (\lambda_0 I - A)T(\lambda) = I \end{aligned}$$

and similarly $T(\lambda)(\lambda I - A) = I$. Therefore $\lambda \in \rho(A)$ and $T(\lambda) = R(\lambda; A)$. In particular, this means, that the resolvent set $\rho(A)$ is open and that the resolvent $R(\lambda; A)$ is an analytic function from the resolvent set into the Banach space $L(X, X)$ of bounded linear operators on X .

As in classical holomorphic function theory, we can expand $R(\lambda; A)$ into a uniquely determined Laurent expansion

$$R(\lambda; A) = \sum_{k=0}^{\infty} \lambda^{-k-1} A_k$$

with bounded linear operators A_k such that the series converges with respect to the operator norm for all $|\lambda| > r(A)$. For $|\lambda| > \|A\|$, again by Theorem 2.8, we know already that $R(\lambda; A)$ is given by the Neumann series (10.8). Therefore from the uniqueness of the Laurent expansion we conclude that the Neumann series is the Laurent expansion and hence converges for all $|\lambda| > r(A)$.

On the other hand, assume that the Neumann series converges for some $|\lambda_0| < r(A)$. Then there exists a constant M such that $|\lambda_0|^{-k-1} \|A^k\| \leq M$ for all $k \in \mathbb{N}$. Hence, for all λ with $|\lambda| > |\lambda_0|$ the Neumann series has a convergent geometric series as a majorant and thus it converges in the Banach space X (see Problem 1.5). Its limit represents the inverse of $\lambda I - A$, hence all $|\lambda|$ with $|\lambda| > |\lambda_0|$ belong to the resolvent set $\rho(A)$ which is a contradiction to $|\lambda_0| < r(A)$.

Theorem 10.11. *Let A have spectral radius $r(A) < 1$. Then for all $f \in X$ the successive approximations*

$$\varphi_{n+1} := A\varphi_n + f, \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in X$ converge to the unique solution of $\varphi - A\varphi = f$.

Proof: This follows as in the proof of Theorem 2.9 from the convergence of the Neumann series for $\lambda = 1 > r(A)$. \square

That the condition $r(A) < 1$ for convergence cannot be weakened is demonstrated by the following

Theorem 10.12. *Let A have spectral radius $r(A) > 1$. Then the successive approximations with $\varphi_0 = 0$ cannot converge for all $f \in X$.*

Proof: Assume the statement of the theorem is false. Convergence of successive approximations for all $f \in X$ is equivalent with pointwise convergence of

the Neumann series for $\lambda = 1$. Hence, by the uniform boundedness principle Theorem 10.3, there exists a positive constant C such that

$$\left\| \sum_{k=1}^n A^k \right\| \leq C$$

for all $n \in \mathbb{N}$. Then, by the triangle inequality, $\|A^n\| \leq 2C$ for all $n \in \mathbb{N}$. Since $r(A) > 1$ there exists $\lambda \in \sigma(A)$ with $|\lambda| > 1$. For the partial sums S_n of the Neumann series for this λ we have

$$\|S_m - S_n\| \leq \frac{2C|\lambda|^{-n-1}}{1 - |\lambda|}$$

for all $m > n$. Therefore (S_n) is a Cauchy sequence in the Banach space $L(X)$. This implies convergence of the Neumann series for λ which contradicts $\lambda \in \sigma(A)$. \square

If A is compact then the successive approximations, in general, cannot converge if $r(A) = 1$. By Theorem 3.11 all spectral values of a compact operator different from zero are eigenvalues. Hence, for $r(A) = 1$, there exists $f \in X$ with $f \neq 0$ and λ with $|\lambda| = 1$ such that $Af = \lambda f$. But then the successive approximations with $\varphi_0 = 0$ satisfy

$$\varphi_n = \sum_{k=0}^{n-1} \lambda^k f, \quad n = 1, 2, \dots,$$

and therefore they diverge.

Usually, in practical calculations, some approximation of the operator A will be used. Therefore we have to establish that our results on convergence remain stable under small perturbations. This is done through the following two theorems.

Theorem 10.13. *Let $A : X \rightarrow X$ be a bounded linear operator with $r(A) < 1$. Assume the sequence $A_m : X \rightarrow X$ of bounded linear operators to be norm convergent $\|A_m - A\| \rightarrow 0$, $m \rightarrow \infty$. Then for all sufficiently large m the equation $\varphi - A_m \varphi = f$ can be solved by successive approximations.*

Proof: Choose $\lambda_0 \in (r(A), 1)$. Then $R(\lambda; A) = (\lambda I - A)^{-1}$ exists and is bounded for all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$. Since $R(\lambda; A)$ is analytic and $\|R(\lambda; A)\| \rightarrow 0$, $\lambda \rightarrow \infty$ (this follows from the Neumann series expansion of the resolvent), we have

$$C := \sup_{|\lambda| \geq \lambda_0} \|(\lambda I - A)^{-1}\| < \infty.$$

Therefore, since $\|A_m - A\| \rightarrow 0$, $m \rightarrow \infty$, there exists an integer N such that $\|(\lambda I - A)^{-1}(A_m - A)\| < 1$ for all $m \geq N$ and all $|\lambda| \geq \lambda_0$. But then from Theorem 10.1, applied to $\lambda I - A$ and $\lambda I - A_m$, we deduce that $(\lambda I - A_m)^{-1}$ exists and is bounded for all $m \geq N$ and all $|\lambda| \geq \lambda_0$. Hence $r(A) < 1$ for all $m \geq N$ and the statement now follows from Theorem 10.11. \square

Similarly, based on Theorem 10.8, we can prove

Theorem 10.14. *Let $A : X \rightarrow X$ be a compact linear operator with $r(A) < 1$. Assume the sequence $A_m : X \rightarrow X$ of collectively compact linear operators to be pointwise convergent $A_m\varphi \rightarrow A\varphi$, $m \rightarrow \infty$, for all $\varphi \in X$. Then for all sufficiently large m the equation $\varphi - A_m\varphi = f$ can be solved by successive approximations.*

We conclude this section with two examples for the application of Theorem 10.11.

Theorem 10.15. *Volterra integral equations of the second kind with continuous or weakly singular kernels can be solved by successive approximations.*

Proof: Volterra integral operators have spectral radius zero. For continuous kernels this is a consequence of the Theorems 3.11 and 3.12. The case of a weakly singular kernel can be dealt with by reducing it to a continuous kernel through iteration as in the proof of Theorem 9.9. \square

Historically, the first existence proofs for Volterra equations of the second kind actually were obtained by directly establishing the convergence of successive approximations.

The following classical potential theoretic result goes back to Plemelj [2].

Theorem 10.16. *The boundary integral operators $K, K' : C(\Gamma) \rightarrow C(\Gamma)$, defined by (6.19) and (6.20), have spectrum*

$$-1 \in \sigma(K) = \sigma(K') \subset [-1, 1].$$

Proof: By Theorem 3.11, the spectrum $\sigma(K') \setminus \{0\}$ of the compact operator K' consists only of eigenvalues. Let $\lambda \neq -1$ be an eigenvalue of K' with eigenfunction φ and define the single-layer potential u with density φ . Then, from the jump relations of Theorem 6.13, we obtain $u_+ = u_-$ and

$$\frac{\partial u_\pm}{\partial n} = \frac{1}{2} K'\varphi \mp \frac{1}{2} \varphi = \frac{1}{2}(\lambda \mp 1)\varphi$$

on Γ . From this we see that

$$(\lambda + 1) \int_{\Gamma} \bar{u}_+ \frac{\partial u_+}{\partial n} ds = (\lambda - 1) \int_{\Gamma} \bar{u}_- \frac{\partial u_-}{\partial n} ds.$$

In addition, by the Gauss theorem Corollary 6.4, we deduce $\int_{\Gamma} \varphi ds = 0$. Therefore the single-layer potential u has the asymptotic behaviour

$$u(x) = O\left(\frac{1}{|x|^{m-1}}\right), \quad \text{grad } u(x) = O\left(\frac{1}{|x|^m}\right), \quad |x| \rightarrow \infty,$$

uniformly for all directions. Now we can apply the Green's Theorem 6.3 to get

$$(10.9) \quad (1 + \lambda) \int_{D_+} |\text{grad } u|^2 dx = (1 - \lambda) \int_{D_-} |\text{grad } u|^2 dx.$$

Assume that $\int_{D_+} |\operatorname{grad} u|^2 dx = \int_{D_-} |\operatorname{grad} u|^2 dx = 0$. Then $\operatorname{grad} u = 0$ in \mathbb{R}^m and from the jump relations we have $\varphi = 0$ which is a contradiction to the fact that φ is an eigenfunction. Hence, from (10.9) we conclude that λ is contained in $[-1, 1]$. From Theorem 6.16 we already know that -1 is an eigenvalue of K' whereas 1 is not an eigenvalue. Finally, $\sigma(K) = \sigma(K')$ is a consequence of the Fredholm alternative for the adjoint operators K and K' .

Corollary 10.17. *The successive approximations*

$$\varphi_{n+1} := \frac{1}{2} \varphi_n + \frac{1}{2} K\varphi_n - f, \quad n = 0, 1, 2, \dots,$$

converge uniformly for arbitrary φ_0 to the unique solution φ of the integral equation $\varphi - K\varphi = -f$ from Theorem 6.17 for the interior Dirichlet problem.

Proof: We apply Theorem 10.11 to the equation $\varphi - A\varphi = -f$ with the operator $A := (I + K)/2$. Then we have $\sigma(A) = 1/2 + \sigma(K)/2$ and therefore $r(A) < 1$ by Theorem 10.16. \square

We have already mentioned in Problem 6.5 that Neumann used the successive approximation scheme of Corollary 10.17 to give the first rigorous existence proof for the Dirichlet problem in two-dimensional convex domains.

Problems

- 10.1. Prove the Banach–Steinhaus theorem: Let $A : X \rightarrow Y$ be a bounded linear operator and let $A_n : X \rightarrow Y$ be a sequence of bounded linear operators from a Banach space X into a normed space Y . For pointwise convergence $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in X$ it is necessary and sufficient that $\|A_n\| \leq C$ for all $n \in \mathbb{N}$ with some constant C and that $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in U$ where U is some dense subset of X .
- 10.2. Show that a sequence $A_n : X \rightarrow Y$ of compact linear operators mapping a normed space X into a normed space Y is collectively compact if and only if for each bounded sequence (φ_n) in X the sequence $(A_n\varphi_n)$ is relatively compact in Y .
- 10.3. Let X and Y be Banach spaces, $S : X \rightarrow Y$ a bounded linear operator with bounded inverse $S^{-1} : Y \rightarrow X$ and let $A : X \rightarrow Y$ be compact. Formulate and prove extensions of Theorems 10.8 and 10.9 for the approximation of the equation

$$S\varphi - A\varphi = f$$

through equations

$$S_n\varphi_n - A_n\varphi_n = f_n$$

where the $A_n : X \rightarrow Y$ are collectively compact with pointwise convergence $A_n \rightarrow A$, $n \rightarrow \infty$, $S_n : X \rightarrow Y$ are bounded with pointwise convergence $S_n \rightarrow S$, $n \rightarrow \infty$, and with uniformly bounded inverses $S_n^{-1} : Y \rightarrow X$, and $f_n \rightarrow f$, $n \rightarrow \infty$.

- 10.4. Solve the Volterra integral equation

$$\varphi(x) - \int_0^x e^{x-y} \varphi(y) dy = f(x)$$

by successive approximations.

- 10.5. How can the integral equation from Theorem 6.21 for the interior Neumann problem be solved by successive approximations?

11. Degenerate Kernel Approximation

In this chapter we will consider the approximate solution of integral equations of the second kind by replacing the kernels by *degenerate kernels*, that is, by approximating a given kernel $K(x, y)$ through a sum of a finite number of products of functions of x alone by functions of y alone. In particular, we will describe the construction of appropriate degenerate kernels by interpolation of the given kernel and by orthonormal expansions. The corresponding error analysis will be settled by our results in Section 10.1.

11.1 Finite Dimensional Operators

Let $\langle X, Y \rangle$ be a dual system and let $A_n : X \rightarrow Y$ be a bounded linear operator with finite dimensional range. We assume that A_n can be expressed in the form

$$(11.1) \quad A_n\varphi = \sum_{j=1}^n \langle \varphi, b_j \rangle a_j$$

where the a_1, \dots, a_n are linearly independent elements of X and where the b_1, \dots, b_n are elements of Y . For the sake of notational clarity we restrict ourselves to the case where the index n for the operator A_n coincides with the number of elements a_1, \dots, a_n and b_1, \dots, b_n . The solution of the equation of the second kind

$$\varphi_n - A_n\varphi_n = f$$

reduces to solving a finite dimensional linear system.

Theorem 11.1. *Each solution of the equation*

$$(11.2) \quad \varphi_n - \sum_{j=1}^n \langle \varphi_n, b_j \rangle a_j = f$$

has the form

$$(11.3) \quad \varphi_n = f + \sum_{k=1}^n \gamma_k a_k$$

where the coefficients $\gamma_1, \dots, \gamma_n$ satisfy the linear system

$$(11.4) \quad \gamma_j - \sum_{k=1}^n \langle a_k, b_j \rangle \gamma_k = \langle f, b_j \rangle, \quad j = 1, \dots, n.$$

Conversely, for each solution $\gamma_1, \dots, \gamma_n$ of the linear system (11.4) there corresponds a solution φ_n of (11.2) defined by (11.3).

Proof: Let φ_n be a solution of (11.2). Writing $\gamma_k = \langle \varphi_n, b_k \rangle$ for $k = 1, \dots, n$, and solving for φ_n we obtain the form (11.3). Taking the bilinear form of (11.3) with b_j we find that the coefficients $\gamma_1, \dots, \gamma_n$ must satisfy the linear system (11.4). Conversely, let $\gamma_1, \dots, \gamma_n$ be a solution to the linear system (11.4) and define φ_n by (11.3). Then

$$\begin{aligned}\varphi_n - \sum_{j=1}^n \langle \varphi_n, b_j \rangle a_j &= f + \sum_{j=1}^n \gamma_j a_j - \sum_{j=1}^n \left\langle f + \sum_{k=1}^n \gamma_k a_k, b_j \right\rangle a_j \\ &= f + \sum_{j=1}^n \left\{ \gamma_j - \sum_{k=1}^n \langle a_k, b_j \rangle \gamma_k - \langle f, b_j \rangle \right\} a_j = f\end{aligned}$$

and the proof is complete. \square

For a given operator A we will use finite dimensional approximations A_n of the form (11.1) to obtain approximate solutions of equations of the second kind by employing Theorems 10.1 and 10.2 for the operators $I - A$ and $I - A_n$. We wish to mention that this approach via finite dimensional approximations also can be used to establish the Fredholm alternative for linear operators in Banach spaces which can be split into the sum of an operator of the form (11.1) and an operator with norm less than one (see Problems 11.1 and 11.2). For integral operators with continuous or weakly singular kernels this approximation can be done by Problem 2.3 and the approximation used in the proof of Theorem 2.21.

In the case of the dual system $(C(G), C(G))$ introduced in Theorem 4.3 the operator A_n has the form

$$(A_n \varphi)(x) = \int_G K_n(x, y) \varphi(y) dy, \quad x \in G,$$

of an integral operator with a *degenerate kernel*

$$K_n(x, y) = \sum_{j=1}^n a_j(x) b_j(y).$$

The solution of the integral equation of the second kind

$$(11.5) \quad \varphi_n(x) - \int_G \sum_{j=1}^n a_j(x) b_j(y) \varphi_n(y) dy = f(x), \quad x \in G,$$

with such a degenerate kernel is described through Theorem 11.1. We will use integral operators A_n with degenerate kernels K_n as approximations for integral operators A with continuous kernels K in the sense of Section 10.1. In view of Theorems 2.7, 10.1 and 10.2 we need approximations with the property that

$$\|A_n - A\|_\infty = \max_{x \in G} \int_G |K_n(x, y) - K(x, y)| dy$$

becomes small.

We do not attempt to give a complete account on the various possibilities for degenerate kernel approximations. Instead of this we will introduce into the basic ideas by considering a few simple examples.

11.2 Degenerate Kernels Via Interpolation

One important method to construct degenerate kernels approximating a given kernel is by interpolation. We recall the following basic existence and uniqueness result on interpolation.

Theorem 11.2. *Let $U_n \subset C(G)$ be an n -dimensional subspace and x_1, \dots, x_n be n points in G such that U_n is unisolvent with respect to x_1, \dots, x_n , that is, each function from U_n which vanishes on x_1, \dots, x_n vanishes identically. Then, given n values g_1, \dots, g_n , there exists a uniquely determined function $u \in U_n$ with the interpolation property*

$$u(x_j) = g_j, \quad j = 1, \dots, n.$$

With the data given by the values $g_j = g(x_j)$, $j = 1, \dots, n$, of a function $g \in C(G)$ the mapping $g \mapsto u$ defines a bounded linear operator $P_n : C(G) \rightarrow U_n$ called interpolation operator.

Proof: Let $U_n = \text{span}\{u_1, \dots, u_n\}$. Then, the solution to the interpolation problem is given by

$$u = \sum_{k=1}^n \gamma_k u_k$$

where the coefficients $\gamma_1, \dots, \gamma_n$ are determined through the uniquely solvable linear system

$$\sum_{k=1}^n \gamma_k u_k(x_j) = g_j, \quad j = 1, \dots, n.$$

Let L_1, \dots, L_n denote the Lagrange basis for U_n , that is, we have the interpolation property

$$L_j(x_k) = \delta_{jk}, \quad j, k = 1, \dots, n.$$

Then, from the representation

$$P_n g = \sum_{j=1}^n g(x_j) L_j$$

we conclude the linearity and boundedness of P_n . □

For integral equations, a given continuous kernel K is approximated by the kernel K_n interpolating K with respect to x , i.e., $K_n(\cdot, y) \in U_n$ and

$$K_n(x_j, y) = K(x_j, y), \quad j = 1, \dots, n,$$

for each y in G . Then we can write

$$K_n(x, y) = \sum_{j=1}^n L_j(x) K(x_j, y).$$

Hence, K_n is a degenerate kernel with $a_j = L_j$ and $b_j = K(x_j, \cdot)$. In this case the linear system (11.4) reads

$$(11.6) \quad \gamma_j - \sum_{k=1}^n \gamma_k \int_G K(x_j, y) L_k(y) dy = \int_G K(x_j, y) f(y) dy, \quad j = 1, \dots, n,$$

and the solution of the integral equation (11.5) is given by

$$\varphi_n = f + \sum_{k=1}^n \gamma_k L_k.$$

For the interpolation we may use polynomials, trigonometric polynomials or splines. We illustrate the method by two examples, where we confine ourselves to one-dimensional integral equations on an interval $[a, b]$.

First we consider linear splines, i.e., continuous piecewise linear functions. Let $x_j = a + jh$, $j = 0, \dots, n$, denote an equidistant subdivision with step-size $h = (b - a)/n$ and let U_n be the space of continuous functions on $[a, b]$ whose restrictions on each of the subintervals $[x_{j-1}, x_j]$, $j = 1, \dots, n$, are linear. Existence and uniqueness for the corresponding linear spline interpolation is evident. Here, the Lagrange basis is given by the hat functions

$$(11.7) \quad L_j(x) = \begin{cases} \frac{1}{h} (x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h} (x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0, & x \notin [x_{j-1}, x_{j+1}]. \end{cases}$$

For the interpolation of twice continuously differentiable functions g we have the following error estimate.

Theorem 11.3. *Let $g \in C^2[a, b]$. Then, for the error in piecewise linear interpolation there holds*

$$(11.8) \quad \|P_n g - g\|_\infty \leq \frac{1}{8} h^2 \|g''\|_\infty.$$

Proof: Obviously, the maximum of $|P_n g - g|$ in $[x_j, x_{j+1}]$ is attained at an interior point ξ where $g'(\xi) = \{g(x_{j+1}) - g(x_j)\}/h$. Without loss of generality we may assume that $\xi - x_j \leq h/2$. Then, using Taylor's formula, we derive

$$\begin{aligned} (P_n g)(\xi) - g(\xi) &= g(x_j) + \frac{1}{h} \{g(x_{j+1}) - g(x_j)\} (\xi - x_j) - g(\xi) \\ &= g(x_j) - g(\xi) - (x_j - \xi) g'(\xi) = \frac{1}{2} (x_j - \xi)^2 g''(\eta), \end{aligned}$$

with some $\eta \in (x_j, \xi)$. Hence,

$$\max_{x_j \leq x \leq x_{j+1}} |(P_n g)(x) - g(x)| \leq \frac{1}{8} h^2 \|g''\|_\infty$$

and the estimate (11.8) follows. \square

From Theorems 2.7 and 11.3, for the approximation of an integral operator A with a two times continuously differentiable kernel K via linear splines we conclude the estimate

$$(11.9) \quad \|A_n - A\|_\infty \leq \frac{1}{8} h^2(b-a) \left\| \frac{\partial^2 K}{\partial x^2} \right\|_\infty.$$

Therefore, by Theorem 10.1, the corresponding approximation of the solution to the integral equation will be of order $O(h^2)$. In principle, using Theorem 10.2, it is possible to derive computable error bounds based on the estimate (11.9). Usually these bounds will be difficult to evaluate in applications. In most practical problems it will be satisfactory to judge the accuracy of the computed solution by refining the subdivision and then comparing the results for the fine and the coarse grid with the aid of the convergence order, i.e., in the case of linear splines with the aid of the approximation order $O(h^2)$.

Example 11.4. Consider the integral equation

$$(11.10) \quad \varphi(x) - \frac{1}{2} \int_0^1 (x+1)e^{-xy} \varphi(y) dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \quad 0 \leq x \leq 1,$$

with exact solution $\varphi(x) = e^{-x}$. For the kernel $K(x, y) = (x+1)e^{-xy}/2$ we have

$$\max_{0 \leq x \leq 1} \int_0^1 |K(x, y)| dy = \sup_{0 < x \leq 1} \frac{x+1}{2x} (1 - e^{-x}) < 1.$$

Therefore, by Corollary 2.10, equation (11.10) is uniquely solvable.

In this special example the integrals for the coefficients and the right hand side of equation (11.6), in principle, can be carried out in closed form. In general, however, these integrals must be evaluated numerically. Therefore, we will use this example to indicate how the numerical quadrature can be performed without endangering the approximation order $O(h^2)$ for the solution to the integral equation.

For the evaluation of the matrix elements in (11.6) we need a quadrature formula for integrals of the form $\int_a^b K(x_j, y) L_k(y) dy$. To be consistent with our approximations, we replace $K(x_j, \cdot)$ by its linear spline interpolation, i.e., we approximate

$$(11.11) \quad \int_a^b K(x_j, y) L_k(y) dy \approx \sum_{m=0}^n K(x_j, x_m) \int_a^b L_m(y) L_k(y) dy,$$

for $j, k = 0, \dots, n$. Straightforward calculations yield the tridiagonal matrix

$$W = \frac{h}{6} \begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & 0 \\ & 1 & 4 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}$$

for the weights $w_{mk} = \int_a^b L_m(y) L_k(y) dy$. For the right hand side of (11.6) we simultaneously replace f and $K(x_j, \cdot)$ by its spline interpolations. This leads to the approximations

$$\int_a^b K(x_j, y) f(y) dy \approx \sum_{k,m=0}^n K(x_j, x_m) f(x_k) \int_a^b L_m(y) L_k(y) dy$$

for $j = 0, \dots, n$. We now investigate the influence of these approximations on the error analysis. For this we interpret the solution of the system (11.6) with the approximate values for the coefficients and the right hand sides as the solution $\tilde{\varphi}_n$ of an additional approximate equation $\tilde{\varphi}_n - \tilde{A}_n \tilde{\varphi}_n = \tilde{f}_n$, namely of the degenerate kernel equation

$$\tilde{\varphi}_n(x) - \int_a^b \tilde{K}_n(x, y) \tilde{\varphi}_n(y) dy = \tilde{f}_n(x), \quad a \leq x \leq b,$$

with

$$\tilde{K}_n(x, y) := \sum_{j,m=0}^n K(x_j, x_m) L_j(x) L_m(y)$$

and

$$\tilde{f}_n(x) := \sum_{m=0}^n f(x_m) L_m(x).$$

Provided the kernel K is twice continuously differentiable, using Theorem 11.3 and $\sum_{j=0}^n L_j(x) = 1$, we can estimate

$$\begin{aligned} |K(x, y) - \tilde{K}_n(x, y)| &\leq \left| K(x, y) - \sum_{j=0}^n K(x_j, y) L_j(x) \right| \\ &+ \left| \sum_{j=0}^n L_j(x) \left\{ K(x_j, y) - \sum_{m=0}^n K(x_j, x_m) L_m(y) \right\} \right| \\ &\leq \frac{1}{8} h^2 \left\{ \left\| \frac{\partial^2 K}{\partial x^2} \right\|_\infty + \left\| \frac{\partial^2 K}{\partial y^2} \right\|_\infty \right\} \end{aligned}$$

for all $a \leq x, y \leq b$. Hence, for the integral operator \tilde{A}_n with kernel \tilde{K}_n we have $\|\tilde{A}_n - A\|_\infty = O(h^2)$. When f is twice continuously differentiable, we also have $\|\tilde{f}_n - f\|_\infty = O(h^2)$. Now, Theorem 10.1 yields the error estimate $\|\tilde{\varphi}_n - \varphi\| = O(h^2)$. Therefore, the perturbed system (11.6) still approximates the original equation to order $O(h^2)$.

For Example 11.4 the Table 11.1 shows the error between the exact and the approximate solution at the points $x = 0, 0.25, 0.5, 0.75$ and 1 for various numbers n . It clearly exhibits the behaviour $O(h^2)$ which we expect from our error analysis.

Table 11.1. Numerical results for Example 11.4

n	$x = 0$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 1$
4	0.004808	0.005430	0.006178	0.007128	0.008331
8	0.001199	0.001354	0.001541	0.001778	0.002078
16	0.000300	0.000338	0.000385	0.000444	0.000519
32	0.000075	0.000085	0.000096	0.000111	0.000130

In the case of periodic functions it is preferable to use trigonometric interpolation. Let again $t_j = j\pi/n$, $j = 0, \dots, 2n - 1$, be an equidistant grid. Then, given values g_0, \dots, g_{2n-1} , there exists a unique trigonometric polynomial of the form

$$u(t) = \sum_{j=0}^n \alpha_j \cos jt + \sum_{j=1}^{n-1} \beta_j \sin jt$$

with the interpolation property $u(t_j) = g_j$, $j = 0, \dots, 2n - 1$. Its coefficients are given by

$$\alpha_0 = \frac{1}{2n} \sum_{k=0}^{2n-1} g_k,$$

$$(11.12) \quad \alpha_j = \frac{1}{n} \sum_{k=0}^{2n-1} g_k \cos jt_k, \quad \beta_j = \frac{1}{n} \sum_{k=0}^{2n-1} g_k \sin jt_k, \quad j = 1, \dots, n-1,$$

$$\alpha_n = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k g_k.$$

For the trigonometric interpolation of analytic functions we have the following error estimate (cf. Kress [4]).

Theorem 11.5. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be analytic and 2π -periodic. Then there exists a strip $D = \mathbb{R} \times (-s, s) \subset \mathbb{C}$ with $s > 0$ such that g can be extended to a holomorphic and 2π -periodic bounded function $g : D \rightarrow \mathbb{C}$. The remainder in trigonometric interpolation can be estimated uniformly on $[0, 2\pi]$ by*

$$\|P_n g - g\|_\infty \leq M \frac{\coth s/2}{\sinh ns},$$

where M denotes a bound for the holomorphic function g on D .

Proof: Since $g : \mathbb{R} \rightarrow \mathbb{R}$ is analytic, at each point $t \in \mathbb{R}$ the Taylor expansion provides a holomorphic extension of g into some open disk in the complex plane with radius $r(t) > 0$ and center t . The extended function again has period 2π ,

since the coefficients of the Taylor series at t and at $t + 2\pi$ coincide for the 2π -periodic function $g : \mathbb{R} \rightarrow \mathbb{R}$. The disks corresponding to all points of the interval $[0, 2\pi]$ provide an open covering of $[0, 2\pi]$. Since $[0, 2\pi]$ is compact, a finite number of these disks suffices to cover $[0, 2\pi]$. Then we have an extension into a strip D with finite width $2s$ contained in the union of the finite number of disks. Without loss of generality we may assume that g is bounded on D .

Using the real part of the geometric sum

$$1 + 2 \sum_{j=1}^{n-1} e^{ijt} + e^{int} = i(1 - e^{int}) \cot \frac{t}{2}, \quad 0 < t < 2\pi,$$

from the coefficients (11.12) we derive the form

$$(P_ng)(t) = \frac{1}{2n} \sin nt \sum_{k=0}^{2n-1} (-1)^k g(t_k) \cot \frac{t - t_k}{2}$$

for $0 \leq t < 2\pi$, $t \neq t_j$, $j = 0, \dots, 2n - 1$, of the trigonometric interpolation polynomial. From the last two equations, note that the Lagrange basis for the trigonometric interpolation is given by

$$\begin{aligned} L_j(t) &= \frac{1}{2n} \left\{ 1 + 2 \sum_{k=1}^{n-1} \cos k(t - t_j) + \cos n(t - t_j) \right\} \\ (11.13) \quad &= \frac{1}{2n} \sin n(t - t_j) \cot \frac{t - t_j}{2}, \quad t \neq t_j, \end{aligned}$$

for $j = 0, \dots, 2n - 1$.

Let $0 < \sigma < s$ be arbitrary. By Γ we denote the counterclockwise oriented boundary of the rectangle $[\pi/2n, 2\pi + \pi/2n] \times [-\sigma, \sigma]$. A straightforward application of the residue theorem yields

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\cot \frac{\tau - t}{2}}{\sin n\tau} g(\tau) d\tau = \frac{2g(t)}{\sin nt} - \frac{1}{n} \sum_{k=0}^{2n-1} (-1)^k g(t_k) \cot \frac{t - t_k}{2}$$

for $0 \leq t < 2\pi$, $t \neq t_j$, $j = 0, \dots, 2n - 1$. Hence, we obtain

$$g(t) - (P_ng)(t) = \frac{\sin nt}{4\pi i} \int_{\Gamma} \frac{\cot \frac{\tau - t}{2}}{\sin n\tau} g(\tau) d\tau,$$

where we obviously can drop the restriction that t does not coincide with an interpolation point. From the periodicity of the integrand and since, by the Schwarz reflection principle, g enjoys the symmetry property $g(\bar{\tau}) = \overline{g(\tau)}$, we find the representation

$$(11.14) \quad g(t) - (P_ng)(t) = \frac{1}{2\pi} \sin nt \operatorname{Re} \left\{ \int_{i\sigma}^{i\sigma+2\pi} \frac{i \cot \frac{\tau - t}{2}}{\sin n\tau} g(\tau) d\tau \right\}$$

with $0 < \sigma < s$ for the remainder in trigonometric interpolation of analytic functions. Because of $|\sin n\tau| \geq \sinh n\sigma$ and $|\cot \tau/2| \leq \coth \sigma/2$ for $\operatorname{Im} \tau = \sigma$ the bound (11.13) now follows immediately by estimating in the last equation and then passing to the limit $\sigma \rightarrow s$. \square

We can summarize Theorem 11.5 by the estimate

$$(11.15) \quad \|P_n g - g\|_\infty = O(e^{-ns})$$

for the trigonometric interpolation of periodic analytic functions with some positive constant s depending on g , i.e., the interpolation error decays at least exponentially.

From Theorems 2.7 and 11.5 for the approximation of an integral operator A with a periodic analytic kernel K via trigonometric interpolation we conclude the estimate

$$(11.16) \quad \|A_n - A\|_\infty = O(e^{-ns})$$

with some positive constant s depending on K . For the derivation of (11.16) we have to use the fact that for the kernel function, which is assumed to be analytic and periodic with respect to both variables, there exists a strip D with width $2s$ such that $K(\cdot, \tau)$ can be continued holomorphically and uniformly bounded into D for all $\tau \in [0, 2\pi]$. We leave it to the reader to go over the argument given in the proof of Theorem 11.5 to verify this continuation property. Now, by Theorem 10.1, the corresponding approximation of the solution to the integral equation also will be of order $O(e^{-ns})$.

Again we have to describe the numerical evaluation of the coefficients and the right hand side of the system (11.6). We proceed analogously to the preceding example. The matrix coefficients consist of integrals of the form $\int_0^{2\pi} K(t_j, \tau) L_k(\tau) d\tau$. We approximate these integrals by replacing $K(t_j, \cdot)$ by its trigonometric interpolation polynomial, i.e., we approximate

$$\int_0^{2\pi} K(t_j, \tau) L_k(\tau) d\tau \approx \sum_{m=0}^{2n-1} K(t_j, t_m) \int_0^{2\pi} L_m(\tau) L_k(\tau) d\tau$$

for $j, k = 0, \dots, 2n - 1$. Using (11.13), elementary integrations yield

$$(11.17) \quad \int_0^{2\pi} L_m(\tau) L_k(\tau) d\tau = \frac{\pi}{n} \delta_{mk} - (-1)^{m-k} \frac{\pi}{4n^2},$$

for $m, k = 0, \dots, 2n - 1$. For the right hand side of (11.6) we simultaneously replace f and $K(t_j, \cdot)$ by its trigonometric interpolation. This leads to

$$\int_0^{2\pi} K(t_j, \tau) f(\tau) d\tau \approx \sum_{k,m=0}^{2n-1} K(t_j, t_m) f(t_k) \int_0^{2\pi} L_m(\tau) L_k(\tau) d\tau$$

for $j = 0, \dots, 2n - 1$. Then we remain with a total error $O(e^{-ns})$ for the approximate solution of the integral equation with some $s > 0$ provided the kernel K is analytic and 2π -periodic with respect to both variables and the right hand

side f is also analytic and periodic. For deriving this result we have to use $\sum_{k=0}^{2n-1} |L_k(t)| \leq 2n$ and the fact that $ne^{-ns} = O(e^{-n\sigma})$ for all $0 < \sigma < s$. If K or f are not analytic, then the approximation order will be given through the corresponding approximation order in trigonometric interpolation of K and f depending on their smoothness. Note, that despite the global nature of the trigonometric interpolation and its Lagrange basis, due to the simple structure of the weights (11.17) in the quadrature rule, the computation of the matrix elements is not too costly.

Example 11.6. Consider the integral equation

$$(11.18) \quad \varphi(t) + \frac{ab}{\pi} \int_0^{2\pi} \frac{\varphi(\tau)d\tau}{a^2 + b^2 - (a^2 - b^2)\cos(t + \tau)} = f(t), \quad 0 \leq t \leq 2\pi,$$

corresponding to the Dirichlet problem for the Laplace equation in the interior of an ellipse with semiaxis $a \geq b > 0$ (see Problem 6.2). We numerically want to solve the case where the unique solution is given by

$$\varphi(t) = e^{\cos t} \cos(\sin t) = \operatorname{Re} \sum_{m=0}^{\infty} \frac{1}{m!} e^{imt}, \quad 0 \leq t \leq 2\pi.$$

Then, the right hand side becomes

$$f(t) = \varphi(t) + e^{c\cos t} \cos(c \sin t), \quad 0 \leq t \leq 2\pi,$$

where $c = (a - b)/(a + b)$ (compare also Problem 11.4).

Table 11.2 shows the error between the exact and the approximate solution for Example 11.6 at the three points $t = 0, \pi/2$ and π for various numbers n and depending on the ratio b/a . It clearly shows the exponential decay of the error: doubling the number of grid points doubles the number of correct digits in the approximate solution. The rate of the exponential decay depends on the parameters a and b which describe the location of the singularities of the integrands in the complex plane, i.e., they determine the value for the strip parameter s .

Table 11.2. Numerical results for Example 11.6

	n	$t=0$	$t = \pi/2$	$t = \pi$
$a=1$ $b=0.5$	4	-0.10752855	-0.03243176	0.03961310
	8	-0.00231537	0.00059809	0.00045961
	16	-0.000000044	0.00000002	-0.000000000
	32	0.000000000	0.000000000	0.000000000
$a=1$ $b=0.2$	4	-0.56984945	-0.18357135	0.06022598
	8	-0.14414257	-0.00368787	-0.00571394
	16	-0.00602543	-0.00035953	-0.00045408
	32	-0.00000919	-0.00000055	-0.00000069

11.3 Degenerate Kernels Via Expansions

A second possibility for constructing approximate degenerate kernels is by expansions, in particular, by orthonormal expansions. Let (\cdot, \cdot) denote a scalar product on $C(G)$ and let $\{u_1, u_2, \dots\}$ be a complete orthonormal system (recall Theorem 1.28). Then a given continuous kernel K is expanded with respect to x for each fixed $y \in G$, i.e., $K(x, y)$ is approximated by the partial sum

$$K_n(x, y) := \sum_{j=1}^n u_j(x)(K(\cdot, y), u_j)$$

of the Fourier series. For this degenerate kernel the linear system (11.4) reads

$$(11.19) \quad \gamma_j - \sum_{k=1}^n \gamma_k \int_G u_k(y)(K(\cdot, y), u_j) dy = \int_G f(y)(K(\cdot, y), u_j) dy$$

for $j = 1, \dots, n$. Usually, the scalar product will be given in terms of an integral. Therefore the system (11.19) requires a double integration for each coefficient and for each right hand side. Since it will turn out that the degenerate kernel method via orthonormal expansion is closely related to the Galerkin method (see Section 13.3) we omit a further discussion and a description of the numerical implementation.

For the orthonormal expansion degenerate kernel approximations, in principle, we can derive error estimates from Theorems 10.1 and 10.2 only with respect to the scalar product norm. In the subsequent analysis, for later theoretical use, we will describe how estimates in the maximum norm can be obtained under additional regularity assumptions on the kernel.

The *Tschebyscheff polynomials* of the first kind are defined by

$$(11.20) \quad T_n(z) := \cos(n \arccos z), \quad -1 \leq z \leq 1, \quad n = 0, 1, \dots$$

From $T_0(z) = 1$ and $T_1(z) = z$ and from the recursion formula

$$T_{n+1}(z) + T_{n-1}(z) = 2zT_n(z),$$

which follows from the cosine addition theorem, we observe that the T_n indeed are polynomials of degree n and therefore, in particular, well defined on the whole complex plane. A substitution $x = \cos t$ readily shows that

$$\int_{-1}^1 \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = m = 0, \\ \frac{\pi}{2}, & n = m \neq 0, \\ 0, & n \neq m, \end{cases}$$

i.e., the T_n form an orthogonal system with respect to the scalar product

$$(11.21) \quad (\varphi, \psi) := \int_{-1}^1 \frac{\varphi(x)\overline{\psi(x)}}{\sqrt{1-x^2}} dx$$

on $C[-1, 1]$. By the same substitution, the denseness of the trigonometric polynomials in $C[0, 2\pi]$ implies the denseness of $\text{span}\{T_0, T_1, \dots\}$ in $C[-1, 1]$ with respect to the scalar product (11.21), i.e., the T_n form a complete orthogonal system.

Theorem 11.7. *Let $g : [-1, 1] \rightarrow \mathbb{R}$ be analytic. Then there exists an ellipse E with foci at -1 and 1 such that g can be extended to a holomorphic and bounded function $g : D \rightarrow \mathbb{C}$, where D denotes the open interior of E . The orthonormal expansion with respect to the Tschebyscheff polynomials*

$$(11.22) \quad g = \frac{a_0}{2} T_0 + \sum_{n=1}^{\infty} a_n T_n$$

with coefficients

$$a_n = \frac{2}{\pi} \int_{-1}^1 \frac{g(x) T_n(x)}{\sqrt{1-x^2}} dx$$

is uniformly convergent with an error bound

$$\left\| g - \frac{a_0}{2} T_0 - \sum_{m=1}^n a_m T_m \right\|_{\infty} \leq \frac{2M}{R-1} R^{-n}.$$

Here, R is given through the semiaxis a and b of E by $R = a + b$ and M is a bound on g in D .

Proof: The existence of the holomorphic extension of the analytic function $g : [-1, 1] \rightarrow \mathbb{R}$ into the open interior D of some ellipse with foci at -1 and 1 is shown analogously to the proof of Theorem 11.5.

The function

$$z = \frac{w}{2} + \frac{1}{2w}$$

maps the annulus $1/R < |w| < R$ of the w -plane onto the interior D of the ellipse E of the z -plane. The circles $w = re^{it}$, $0 \leq t \leq 2\pi$, are transformed into the ellipses

$$z = \frac{1}{2} \left(r + \frac{1}{r} \right) \cos t + \frac{i}{2} \left(r - \frac{1}{r} \right) \sin t.$$

For each holomorphic function g on D the function \tilde{g} , defined by

$$\tilde{g}(w) := 2g \left(\frac{w}{2} + \frac{1}{2w} \right),$$

is holomorphic in the annulus. Therefore it can be expanded into a Laurent series

$$\tilde{g}(w) = \sum_{n=-\infty}^{\infty} a_n w^n$$

with coefficients

$$a_n = \frac{1}{\pi i} \int_{|w|=r} g \left(\frac{w}{2} + \frac{1}{2w} \right) \frac{dw}{w^{n+1}},$$

where $1/R < r < R$. By substituting $\tilde{w} = 1/w$ it follows that $a_n = a_{-n}$ for all $n \in \mathbb{N}$. Hence,

$$\tilde{g}(w) = a_0 + \sum_{n=1}^{\infty} a_n \left(w^n + \frac{1}{w^n} \right).$$

In particular, this Laurent expansion converges uniformly on the circle $|w| = 1$. Writing $w = e^{it}$, we derive

$$T_n \left(\frac{w}{2} + \frac{1}{2w} \right) = T_n(\cos t) = \cos nt = \frac{1}{2} (w^n + \frac{1}{w^n})$$

first on the unit circle, and then, since both sides of the equation represent rational functions, for all $w \neq 0$. Inserting this into the previous series, we obtain the expansion (11.22) and its uniform convergence on $[-1, 1]$.

Estimating the coefficients of the Laurent expansion and then passing to the limit $r \rightarrow R$ yields

$$|a_n| \leq \frac{2M}{R^n}, \quad n = 0, 1, 2, \dots,$$

with a bound M for g on D . Hence, using $|T_n(x)| \leq 1$ for all $-1 \leq x \leq 1$ and all $n \in \mathbb{N}$, the remainder can be estimated by

$$\left| \sum_{m=n+1}^{\infty} a_m T_m(x) \right| \leq \sum_{m=n+1}^{\infty} |a_m| \leq \frac{2M}{R-1} R^{-n}$$

which finishes the proof. \square

From Theorem 11.7, analogous to the derivation of the estimate (11.16), for the approximation of an integral operator A with an analytic kernel by orthonormal expansion with respect to Tschebyscheff polynomials we find the estimate

$$(11.23) \quad \|A_n - A\|_{\infty} = O(R^{-n})$$

with some constant $R > 1$ depending on the kernel of A .

In closing this chapter, we wish to mention the use of Taylor expansions as a further method to construct degenerate kernels which may be useful in special cases (see Problem 11.3).

The degenerate kernel approximation is the simplest method for numerically solving integral equations of the second kind as far as its understanding and its error analysis is concerned. Because the actual implementation requires the numerical evaluation of either a single or a double integral for each matrix coefficient and for each right hand side of the approximating linear system, in general, however it cannot compete in efficiency with other methods, for instance with the Nyström method of the next chapter.

Problems

- 11.1. Let $\langle X, Y \rangle$ be a dual system with two normed spaces X and Y and let $A_n : X \rightarrow X$ and $B_n : Y \rightarrow Y$ be adjoint finite dimensional operators of the form

$$A_n \varphi = \sum_{j=1}^n \langle \varphi, b_j \rangle a_j,$$

$$B_n \psi = \sum_{j=1}^n \langle a_j, \psi \rangle b_j$$

with linearly independent elements $a_1, \dots, a_n \in X$ and $b_1, \dots, b_n \in Y$. By reduction to linear systems as in Theorem 11.1, establish the validity of the Fredholm alternative for the operators $I - A_n$ and $I - B_n$.

- 11.2. Let $\langle X, Y \rangle$ be a dual system with two Banach spaces X and Y , let $A_n : X \rightarrow X$ and $B_n : Y \rightarrow Y$ be adjoint finite dimensional operators as in Problem 11.1, and let $S : X \rightarrow X$ and $T : Y \rightarrow Y$ be adjoint operators with norm less than one. With the aid of Theorem 2.8 and Problem 11.1 establish the validity of the Fredholm alternative for the operators $I - A$ and $I - B$ where $A = A_n + S$ and $B = B_n + T$.

Hint: Transform the equations with the operators $I - A$ and $I - B$ equivalently into equations with $I - (I - S)^{-1}A_n$ and $I - B_n(I - T)^{-1}$.

- 11.3. Solve the integral equation (11.10) approximately through degenerate kernels by using the Taylor series for e^{xy} .

- 11.4. Show that the eigenvalues of the integral equation (11.18) are described by

$$\frac{ab}{\pi} \int_0^{2\pi} \frac{e^{int} d\tau}{(a^2 + b^2) - (a^2 - b^2) \cos(t + \tau)} = \left(\frac{a - b}{a + b} \right)^n e^{-int}, \quad n = 0, 1, 2, \dots$$

- 11.5. Show that analogous to the estimate (11.23) for continuously differentiable kernels there holds $\|A_n - A\|_\infty = o(1/\sqrt{n})$ for the approximation through degenerate kernels via Tschebyscheff polynomials.

Hint: Use (8.3) and Dini's theorem from Problem 8.2.

12. Quadrature Methods

In this chapter we shall describe the *quadrature* or *Nyström method* for the approximate solution of integral equations of the second kind with continuous or weakly singular kernels. As we have pointed out in the previous chapter, the implementation of the degenerate kernel method, in general, requires some use of numerical quadrature. Therefore it is natural to try the application of numerical integration in a more direct approach to approximate integral operators by numerical integration operators. This will lead to a straightforward but widely applicable method for solving equations of the second kind. The reason we placed the description of the quadrature method after the degenerate kernel method is only because its error analysis is more involved.

12.1 Numerical Integration

We start with a brief account on the basics of numerical quadrature. In general, a quadrature formula is a numerical method for approximating an integral of the form

$$Q(g) := \int_G w(x)g(x)dx$$

where w is some *weight function*. Throughout this chapter $G \subset \mathbb{R}^m$ will be compact and Jordan measurable. We consider only quadrature rules of the form

$$Q_n(g) := \sum_{j=1}^n \alpha_j^{(n)} g(x_j^{(n)})$$

with *quadrature points* $x_1^{(n)}, \dots, x_n^{(n)}$ contained in G and real *quadrature weights* $\alpha_1^{(n)}, \dots, \alpha_n^{(n)}$. For the sake of notational clarity we restrict ourselves to the case where the index n of the quadrature rule coincides with the number of quadrature points. Occasionally, we also will write x_1, \dots, x_n instead of $x_1^{(n)}, \dots, x_n^{(n)}$ and $\alpha_1, \dots, \alpha_n$ instead of $\alpha_1^{(n)}, \dots, \alpha_n^{(n)}$. The basic numerical integrations are *interpolatory quadratures*. They are constructed by replacing the integrand g by an interpolation with respect to the quadrature points x_1, \dots, x_n , usually a polynomial, a trigonometric polynomial or a spline, and then integrating the interpolating function analytically. The classical Newton–Cotes rules are a special case of these interpolatory quadratures with polynomial interpolation on an equidistantly spaced subdivision of the interval $[a, b]$. Since the Newton–Cotes rules have unsatisfactory convergence behaviour as the degree of the interpolation increases, it is more practical to use so called *repeated rules*. These are

obtained by subdividing the interval of integration and then applying a fixed rule with low interpolation order to each of the subintervals. The most frequently used quadrature rules of this type are the *repeated trapezoidal rule* and the *repeated Simpson rule*. For convenience we shortly discuss their error analysis which we base on the representation of the remainder terms by means of *Peano kernels*. Let $x_j = a + jh$, $j = 0, \dots, n$, be an equidistant subdivision with stepsize $h = (b - a)/n$.

Theorem 12.1. *Let $g \in C^2[a, b]$. Then the remainder*

$$R_T(g) := \int_a^b g(x)dx - h \left[\frac{1}{2} g(x_0) + g(x_1) + \dots + g(x_{n-1}) + \frac{1}{2} g(x_n) \right]$$

for the trapezoidal rule can be estimated by

$$|R_T(g)| \leq \frac{1}{12} h^2(b - a) \|g''\|_\infty.$$

Proof: Define the Peano kernel for the trapezoidal rule by

$$K_T(x) := \frac{1}{2} (x - x_{j-1})(x_j - x), \quad x_{j-1} \leq x \leq x_j,$$

for $j = 1, \dots, n$. Then, straightforward partial integrations yield

$$\int_a^b K_T(x)g''(x)dx = -R_T(g).$$

Now the estimate follows from the observation that K_T is nonnegative on $[a, b]$ and

$$\int_a^b K_T(x)dx = \frac{h^2}{12} (b - a).$$

Theorem 12.2. *Let $g \in C^4[a, b]$ and let n be even. Then the remainder*

$$\begin{aligned} R_S(g) := \int_a^b g(x)dx - \frac{h}{3} [g(x_0) + 4g(x_1) + 2g(x_2) + \dots \\ + 2g(x_{n-2}) + 4g(x_{n-1}) + g(x_n)] \end{aligned}$$

for the Simpson rule can be estimated by

$$|R_S(g)| \leq \frac{1}{180} h^4(b - a) \|g^{(4)}\|_\infty.$$

Proof: This is proved analogously to Theorem 12.1 with the Peano kernel of the Simpson rule

$$K_S(x) := \begin{cases} \frac{h}{18} (x - x_{2j-2})^3 - \frac{1}{24} (x - x_{2j-2})^4, & x_{2j-2} \leq x \leq x_{2j-1}, \\ \frac{h}{18} (x_{2j} - x)^3 - \frac{1}{24} (x_{2j} - x)^4, & x_{2j-1} \leq x \leq x_{2j}, \end{cases}$$

for $j = 1, \dots, n/2$. Then

$$\int_a^b K_S(x)g^{(4)}(x)dx = -R_S(g).$$

Definition 12.3. A sequence (Q_n) of quadrature rules is called convergent if $Q_n(g) \rightarrow Q(g)$, $n \rightarrow \infty$, for all $g \in C(G)$, that is, if the sequence of linear functionals (Q_n) converges pointwise to the integral Q .

With the aid of the uniform boundedness principle we can state the following necessary and sufficient conditions for convergence.

Theorem 12.4. In order that the quadrature formulae (Q_n) converge it is necessary and sufficient that $Q_n(g) \rightarrow Q(g)$, $n \rightarrow \infty$, for all g in some dense subset $U \subset C(G)$, and that

$$\sup_{n \in \mathbb{N}} \sum_{j=1}^n |\alpha_j^{(n)}| < \infty.$$

Proof: It is left as an easy exercise to show that

$$\|Q_n\|_\infty = \sum_{j=1}^n |\alpha_j^{(n)}|.$$

Then the statement follows from the Banach–Steinhaus theorem (see Problem 10.1).

Corollary 12.5. Assume that $Q_n(1) \rightarrow Q(1)$, $n \rightarrow \infty$, and that the quadrature weights are all nonnegative. Then in order that the quadrature formulae converge it is necessary and sufficient that $Q_n(g) \rightarrow Q(g)$, $n \rightarrow \infty$, for all g in some dense subset $U \subset C(G)$.

Proof: From

$$\sum_{j=1}^n |\alpha_j^{(n)}| = \sum_{j=1}^n \alpha_j^{(n)} = Q_n(1) \rightarrow Q(1), \quad n \rightarrow \infty,$$

and the preceding theorem. \square

In particular, from Theorems 12.1 and 12.2 and Corollary 12.5 we observe that the trapezoidal and the Simpson rule are convergent.

We conclude our remarks on numerical quadrature by describing an error estimate for the integration of periodic analytic functions due to Davis [1] (see also Problem 12.2).

Theorem 12.6. Let g be as in Theorem 11.5. Then the error

$$R_T(g) := \frac{1}{2\pi} \int_0^{2\pi} g(t)dt - \frac{1}{2n} \sum_{j=0}^{2n-1} g\left(\frac{j\pi}{n}\right)$$

for the trapezoidal rule can be estimated by

$$|R_T(g)| \leq M(\coth ns - 1).$$

Proof: With the aid of

$$\frac{1}{2\pi} \int_0^{2\pi} \sin nt \cot \frac{\tau-t}{2} dt = -e^{int}, \quad \operatorname{Im} \tau > 0, \quad n = 1, 2, \dots,$$

we integrate the remainder term (11.14) for the trigonometric interpolation and obtain

$$R_T(g) = \frac{1}{2\pi} \operatorname{Re} \left\{ \int_{i\sigma}^{i\sigma+2\pi} (1 - i \cot n\tau) g(\tau) d\tau \right\}$$

for all $0 < \sigma < s$. This can also be shown directly through the residue theorem. Now the estimate follows from $|1 - i \cot n\tau| \leq \coth n\sigma - 1$ for $\operatorname{Im} \tau = \sigma$ and passing to the limit $\sigma \rightarrow s$. \square

We can summarize Theorem 12.6 by the estimate

$$(12.1) \quad R_T(g) = O(e^{-2ns})$$

for the trapezoidal rule for periodic analytic functions. The improvement by the factor 2 in the exponent as compared with (11.15) reflects the fact that the trapezoidal rule integrates trigonometric polynomials not only of degree less than or equal to n but also of degree less than or equal to $2n - 1$ exactly. Error estimates for the trapezoidal rule applied to differentiable functions can be obtained via the Euler–MacLaurin expansion (cf. Brosowski and Kress [1] or Isaacson and Keller [1]).

For a comprehensive study of numerical integration we refer the reader to Davis and Rabinowitz [1] and Engels [1].

12.2 Nyström's Method

We choose a convergent sequence (Q_n) of quadrature rules for the integral $Q(g) = \int_G g(x) dx$ and approximate the integral operator

$$(12.2) \quad (A\varphi)(x) := \int_G K(x, y) \varphi(y) dy, \quad x \in G,$$

with continuous kernel K , as introduced in Theorem 2.7, by a sequence of numerical integration operators

$$(12.3) \quad (A_n \varphi)(x) := \sum_{k=1}^n \alpha_k^{(n)} K(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in G.$$

Then the solution to the integral equation of the second kind

$$\varphi - A\varphi = f$$

is replaced by the solution of

$$\varphi_n - A_n \varphi_n = f,$$

which reduces to solving a finite dimensional linear system.

Theorem 12.7. *Let φ_n be a solution of*

$$(12.4) \quad \varphi_n(x) - \sum_{k=1}^n \alpha_k K(x, x_k) \varphi_n(x_k) = f(x), \quad x \in G.$$

Then the values $\varphi_j^{(n)} = \varphi_n(x_j)$, $j = 1, \dots, n$, at the quadrature points satisfy the linear system

$$(12.5) \quad \varphi_j^{(n)} - \sum_{k=1}^n \alpha_k K(x_j, x_k) \varphi_k^{(n)} = f(x_j), \quad j = 1, \dots, n.$$

Conversely, let $\varphi_j^{(n)}$, $j = 1, \dots, n$, be a solution of the system (12.5). Then the function φ_n defined by

$$(12.6) \quad \varphi_n(x) := f(x) + \sum_{k=1}^n \alpha_k K(x, x_k) \varphi_k^{(n)}, \quad x \in G,$$

solves equation (12.4).

Proof: The first statement is trivial. For a solution $\varphi_j^{(n)}$, $j = 1, \dots, n$, of the system (12.5) the function φ_n defined by (12.6) has values

$$\varphi_n(x_j) = f(x_j) + \sum_{k=1}^n \alpha_k K(x_j, x_k) \varphi_k^{(n)} = \varphi_j^{(n)}, \quad j = 1, \dots, n.$$

Inserting this into (12.6), we see that φ_n satisfies (12.4). \square

The formula (12.6) may be viewed as a natural interpolation of the values $\varphi_j^{(n)}$, $j = 1, \dots, n$, at the quadrature points to obtain the approximating function φ_n and goes back to Nyström [1].

The error analysis will be based on the following

Theorem 12.8. *Assume the quadrature formulae (Q_n) to be convergent. Then the sequence (A_n) is collectively compact and pointwise convergent $A_n \varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in C(G)$, but not norm convergent.*

Proof: Since the quadrature formulae (Q_n) are assumed to be convergent, by Theorem 12.4, there exists a constant C such that the weights satisfy

$$\sum_{k=1}^n |\alpha_k^{(n)}| \leq C$$

for all $n \in \mathbb{N}$. Then we can estimate

$$(12.7) \quad \|A_n \varphi\|_\infty \leq C \max_{x,y \in G} |K(x, y)| \|\varphi\|_\infty$$

and

$$(12.8) \quad |(A_n\varphi)(x_1) - (A_n\varphi)(x_2)| \leq C \max_{y \in G} |K(x_1, y) - K(x_2, y)| \|\varphi\|_\infty$$

for all $x_1, x_2 \in G$.

Now let $U \subset C(G)$ be bounded. Then from (12.7) and (12.8) we see that $\{A_n\varphi : \varphi \in U, n \in \mathbb{N}\}$ is bounded and equicontinuous because K is uniformly continuous on $G \times G$. Therefore, by the Arzelà–Ascoli Theorem 1.18, the sequence (A_n) is collectively compact.

Since the quadrature is convergent, for fixed $\varphi \in C(G)$ the sequence $(A_n\varphi)$ is pointwise convergent, i.e., $(A_n\varphi)(x) \rightarrow (A\varphi)(x)$, $n \rightarrow \infty$, for all $x \in G$. As a consequence of (12.8), the sequence $(A_n\varphi)$ is equicontinuous. Hence it is uniformly convergent $\|A_n\varphi - A\varphi\|_\infty \rightarrow 0$, $n \rightarrow \infty$, i.e., we have pointwise convergence $A_n\varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in C(G)$ (see Problem 12.1).

For $\varepsilon > 0$ choose a function $\psi_\varepsilon \in C(G)$ with $\|\psi_\varepsilon\|_\infty = 1$ such that $\psi_\varepsilon(x) = 1$ for all $x \in G$ with $\min_{j=1,\dots,n} |x - x_j| \geq \varepsilon$ and $\psi_\varepsilon(x_j) = 0$, $j = 1, \dots, n$. Then

$$\|A\varphi\psi_\varepsilon - A\varphi\|_\infty \leq \max_{x,y \in G} |K(x, y)| \int_G \{1 - \psi_\varepsilon(y)\} dy \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

for all $\varphi \in C(G)$ with $\|\varphi\|_\infty = 1$. Using this result, we derive

$$\begin{aligned} \|A - A_n\|_\infty &= \sup_{\|\varphi\|_\infty=1} \|(A - A_n)\varphi\|_\infty \geq \sup_{\|\varphi\|_\infty=1} \sup_{\varepsilon>0} \|(A - A_n)\varphi\psi_\varepsilon\|_\infty \\ &= \sup_{\|\varphi\|_\infty=1} \sup_{\varepsilon>0} \|A\varphi\psi_\varepsilon\|_\infty \geq \sup_{\|\varphi\|_\infty=1} \|A\varphi\|_\infty = \|A\|_\infty, \end{aligned}$$

whence we see that the sequence (A_n) cannot be norm convergent. \square

Theorem 12.8 enables us to apply the approximation theory of Theorems 10.8 and 10.9. In principle, using Theorem 10.9, it is possible to derive computable error bounds (see Problem 12.3). Since these, in general, will be too complicated to evaluate, as already mentioned in the previous chapter, in applications usually it will be sufficient to estimate the error by extrapolation from the convergence order. For the discussion of the error based on the estimate (10.7) we need the norm $\|(A - A_n)\varphi\|_\infty$. It can be expressed in terms of the error for the corresponding numerical quadrature by

$$\|(A - A_n)\varphi\|_\infty = \max_{x \in G} \left| \int_G K(x, y)\varphi(y) dy - \sum_{k=1}^n \alpha_k K(x, x_k)\varphi(x_k) \right|$$

and requires a uniform estimate for the error of the quadrature applied to the integration of $K(x, \cdot)\varphi$. Therefore, from Theorem 10.8 and the error estimate (10.7), it follows that under suitable regularity assumptions on the kernel K and the exact solution φ , the convergence order of the underlying quadrature formulae carries over to the convergence order of the approximate solutions to the integral equation. We illustrate this through the case of the trapezoidal rule. Under the assumption $\varphi \in C^2[a, b]$ and $K \in C^2([a, b] \times [a, b])$, by Theorem

12.1, we can write

$$\|(A - A_n)\varphi\|_\infty \leq \frac{1}{12} h^2(b-a) \max_{x,y \in G} \left| \frac{\partial^2}{\partial y^2} K(x,y) \varphi(y) \right|.$$

Example 12.9. We use the trapezoidal rule for approximately solving the integral equation (11.10) of Example 11.4. By the numerical results given in Table 12.1 the expected convergence rate $O(h^2)$ is clearly demonstrated.

Table 12.1. Numerical results for Example 12.9

n	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
4	0.007146	0.008878	0.010816	0.013007	0.015479
8	0.001788	0.002224	0.002711	0.003261	0.003882
16	0.000447	0.000556	0.000678	0.000816	0.000971
32	0.000112	0.000139	0.000170	0.000204	0.000243

Example 12.10. We use the Simpson rule for the integral equation of the preceding example. The numerical results in Table 12.2 show the convergence order $O(h^4)$, which we expect from Theorems 10.8 and 12.2.

Table 12.2. Numerical results for Example 12.10

n	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
4	0.00006652	0.00008311	0.00010905	0.00015046	0.00021416
8	0.00000422	0.00000527	0.00000692	0.00000956	0.00001366
16	0.00000026	0.00000033	0.00000043	0.00000060	0.00000086

Comparing the last two examples, we wish to emphasize on the major advantage of Nyström's method as compared with the degenerate kernel method of the previous chapter. The matrix and the right hand side of the linear system (12.5) are obtained by just evaluating the kernel K and the given function f at the quadrature points. Therefore, without any further computational effort we can improve considerably on the approximations by choosing a more accurate numerical quadrature. For example, in the degenerate kernel method, however, it is much more involved to design a scheme with convergence order $O(h^4)$ rather than $O(h^2)$ from the linear spline interpolation.

Example 12.11. For the integral equation (11.18) of Example 11.6 we use the trapezoidal rule. Since we are dealing with periodic analytic functions, from Theorem 12.6 we expect an exponentially decreasing error behaviour which is exhibited by the numerical results in Table 12.3. Note that for periodic analytic functions the trapezoidal rule, in general, yields better approximations than the Simpson rule.

Table 12.3. Numerical results for Example 12.11

	n	$t = 0$	$t = \pi/2$	$t = \pi$
a=1 b=0.5	4	-0.15350443	0.01354412	-0.00636277
	8	-0.00281745	0.00009601	-0.00004247
	16	-0.00000044	0.00000001	-0.00000001
	32	0.00000000	0.00000000	0.00000000
a=1 b=0.2	4	-0.69224130	-0.06117951	-0.06216587
	8	-0.15017166	-0.00971695	-0.01174302
	16	-0.00602633	-0.00036043	-0.00045498
	32	-0.00000919	-0.00000055	-0.00000069

We confine ourselves to these few examples for the application of the Nyström method. For a greater variety the reader is referred to Anderssen et al [1], Atkinson [2], Baker [1], Delves and Mohamed [1] and Golberg [1].

12.3 Nyström's Method for Weakly Singular Kernels

We will now describe the application of numerical quadrature to approximately solving integral equations with weakly singular kernels of the form

$$(12.9) \quad (A\varphi)(x) := \int_G w(|x - y|)K(x, y)\varphi(y)dy, \quad x \in G \subset \mathbb{R}^m.$$

Here, we assume the weight function $w : (0, \infty) \rightarrow \mathbb{R}$ to represent the weak singularity, i.e., w is continuous and satisfies $|w(t)| \leq Mt^{\alpha-m}$ for all $t > 0$ with some positive constants M and α . The remaining part K of the kernel is required to be continuous. We choose a sequence (Q_n) of quadrature rules

$$(Q_n g)(x) := \sum_{j=1}^n \alpha_j^{(n)}(x)g(x_j^{(n)}), \quad x \in G,$$

for the integral

$$(Qg)(x) := \int_G w(|x - y|)g(y)dy, \quad x \in G,$$

with quadrature weights depending on x . Then, we approximate the weakly singular integral operator by a sequence of numerical integration operators

$$(12.10) \quad (A_n \varphi)(x) := \sum_{k=1}^n \alpha_k^{(n)}(x)K(x, x_k^{(n)})\varphi(x_k^{(n)}), \quad x \in G.$$

Appropriately modified, Theorem 12.7 for the solution of the approximating equation of the second kind, of course, remains valid. The linear system (12.5) assumes the form

$$(12.11) \quad \varphi_j^{(n)} - \sum_{k=1}^n \alpha_k^{(n)}(x_j)K(x_j, x_k)\varphi_k^{(n)} = f(x_j), \quad j = 1, \dots, n.$$

For the error analysis we will assume that the sequence (Q_n) of quadrature formulae converges uniformly for all $x \in G$. Then applying the Banach–Steinhaus theorem to the sequence of linear operators $Q_n : C(G) \rightarrow C(G)$, we observe that for uniform convergence of the quadrature on G it is necessary and sufficient that $Q_n g \rightarrow Qg$, $n \rightarrow \infty$, uniformly on G for all g in some dense subset of $C(G)$, and that there exists a constant C such that the weights satisfy

$$\sum_{k=1}^n |\alpha_k^{(n)}(x)| \leq C$$

for all $n \in \mathbb{N}$ and all $x \in G$. Since the weights are not constant, this bound alone will not ensure collective compactness of the sequence (A_n) . Therefore, in addition we will assume that the weights satisfy

$$(12.12) \quad \lim_{y \rightarrow x} \sup_{n \in \mathbb{N}} \sum_{k=1}^n |\alpha_k^{(n)}(y) - \alpha_k^{(n)}(x)| = 0$$

uniformly for all $x \in G$. Then, writing

$$\begin{aligned} & |(A_n \varphi)(x_1) - (A_n \varphi)(x_2)| \\ & \leq \left| \sum_{k=1}^n \alpha_k^{(n)}(x_1) \{K(x_1, x_k^{(n)}) - K(x_2, x_k^{(n)})\} \varphi(x_k^{(n)}) \right| \\ & + \left| \sum_{k=1}^n \{\alpha_k^{(n)}(x_1) - \alpha_k^{(n)}(x_2)\} K(x_2, x_k^{(n)}) \varphi(x_k^{(n)}) \right| \\ & \leq C \max_{x \in G} |K(x_1, y) - K(x_2, y)| \|\varphi\|_\infty \\ & + \max_{x, y \in G} |K(x, y)| \sup_{n \in \mathbb{N}} \sum_{k=1}^n |\alpha_k^{(n)}(x_1) - \alpha_k^{(n)}(x_2)| \|\varphi\|_\infty, \end{aligned}$$

analogous to Theorem 12.8, we can prove the following result.

Theorem 12.12. *Assume the quadrature formulae (Q_n) to be convergent and to satisfy the condition (12.12). Then the sequence (A_n) is collectively compact and pointwise convergent $A_n \varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in C(G)$, but not norm convergent.*

For a systematic study of approximations satisfying condition (12.12) we refer to Sloan [2]. We confine ourselves to a special case and illustrate this method by considering a weakly singular operator

$$(12.13) \quad (A\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2} K(t, \tau) \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi,$$

with a logarithmic singularity in the space of periodic functions. The fundamental solutions to elliptic equations, for instance the Laplace or the Helmholtz equation, in two space dimensions usually contain a logarithmic singularity.

Therefore, the boundary integral equation approach to two-dimensional boundary value problems via a periodic parametrization of the boundary curve, in general, leads to such logarithmic singularities (see also Problem 7.2).

According to the general ideas as outlined in Section 12.1 and following Kussmaul [1] and Martensen [1], we construct numerical quadratures for the improper integral

$$(Qg)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2} g(\tau) d\tau$$

by replacing the continuous periodic function g by its trigonometric interpolation polynomial described in Section 11.2. Using the Lagrange basis we obtain

$$(12.14) \quad (Q_n g)(t) = \sum_{j=0}^{2n-1} R_j^{(n)}(t) g(t_j)$$

with the equidistant quadrature points $t_j = j\pi/n$ and the quadrature weights

$$(12.15) \quad R_j^{(n)}(t) = \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2} L_j(\tau) d\tau.$$

With the help of the integrals

$$(12.16) \quad \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{\tau}{2} e^{im\tau} d\tau = \begin{cases} 0, & m = 0, \\ -\frac{1}{m}, & m = 1, 2, \dots, \end{cases}$$

and the form (11.13) of the Lagrange basis we derive

$$R_j^{(n)}(t) = -\frac{1}{n} \left\{ \sum_{m=1}^{n-1} \frac{1}{m} \cos m(t-t_j) + \frac{1}{2n} \cos n(t-t_j) \right\}$$

for $j = 0, \dots, 2n-1$.

By construction this quadrature is uniformly convergent for all trigonometric polynomials, since Q_n integrates polynomials of degree less than or equal to n exactly. We will establish convergence for all 2π -periodic continuous functions and the validity of (12.12) with the aid of the uniform boundedness principle. Represent the trigonometric interpolation operator P_n in terms of the Lagrange basis

$$P_n g = \sum_{k=0}^{2n-1} g(t_k) L_k^{(n)}.$$

Then, with the help of (11.17) and the Cauchy–Schwarz inequality, we can estimate

$$\|P_n g\|_2^2 \leq \|g\|_\infty^2 \sum_{k,m=0}^{2n-1} \int_0^{2\pi} L_k^{(n)}(t) L_m^{(n)}(t) dt \leq 3\pi \|g\|_\infty^2.$$

Hence, the interpolation operators $P_n : C[0, 2\pi] \rightarrow L^2[0, 2\pi]$ are uniformly bounded. In passing, we note that this, by the Banach–Steinhaus theorem and

the denseness of the trigonometric polynomials in $C[0, 2\pi]$, implies mean square convergence of the trigonometric interpolation.

Now we consider the operators $F_n : L^2[0, 2\pi] \rightarrow C^*[0, 2\pi]$ from $L^2[0, 2\pi]$ into the dual space $C^*[0, 2\pi]$ of bounded linear functionals on $C[0, 2\pi]$, defined by

$$(F_n f)(g) := \int_0^{2\pi} f(t)(P_n g)(t) dt = \sum_{j=0}^{2n-1} g(t_j) \int_0^{2\pi} f(t) L_j^{(n)}(t) dt$$

for all $f \in L^2[0, 2\pi]$ and $g \in C[0, 2\pi]$. From

$$|(F_n)(g)| \leq \|f\|_2 \|P_n g\|_2 \leq \sqrt{3\pi} \|f\|_2 \|g\|_\infty$$

for all $g \in C[0, 2\pi]$ and the uniform boundedness principle Theorem 10.3, applied to the sequence $F_n f : C[0, 2\pi] \rightarrow \mathbb{C}$ of bounded linear functionals, it follows that

$$\|F_n f\|_\infty \leq C_f$$

for all $n \in \mathbb{N}$ with some constant C_f depending on f . The operators F_n are clearly linear and they are bounded since

$$\|F_n f\|_\infty = \sum_{j=0}^{2n-1} \left| \int_0^{2\pi} f(t) L_j^{(n)}(t) dt \right| \leq \sum_{j=0}^{2n-1} \|L_j^{(n)}\|_2 \|f\|_2.$$

Hence, on applying again the uniform boundedness principle, this time to the sequence $F_n : L^2[0, 2\pi] \rightarrow C^*[0, 2\pi]$, it follows that

$$\|F_n\| \leq C$$

for all $n \in \mathbb{N}$ with some constant C .

From the integrals (12.16) and Parseval's equality it is obvious, that for each $t \in [0, 2\pi]$ by

$$f_t(\tau) := \frac{1}{2\pi} \ln 4 \sin^2 \frac{t - \tau}{2}, \quad \tau \in \mathbb{R}, \quad \tau \neq t + 2n\pi,$$

we define a function $f_t \in L^2[0, 2\pi]$. Then, by construction of our quadrature rule,

$$\sum_{j=0}^{2n-1} |R_j^{(n)}(t)| = \sum_{j=0}^{2n-1} \left| \int_0^{2\pi} f_t(\tau) L_j^{(n)}(\tau) d\tau \right| = \|F_n f_t\| \leq C \|f_t\|_2,$$

which establishes the uniform convergence by the Banach-Steinhaus theorem. Similarly,

$$\sum_{j=0}^{2n-1} |R_j^{(n)}(t_1) - R_j^{(n)}(t_2)| \leq C \|f_{t_1} - f_{t_2}\|_2,$$

and again from (12.16) and Parseval's equality we see that

$$\|f_{t_1} - f_{t_2}\|_2^2 = \frac{1}{2\pi} \sum_{m=-\infty, m \neq 0}^{\infty} \frac{1}{m^2} |e^{imt_1} - e^{imt_2}|^2,$$

whence the validity of condition (12.12) follows. Therefore, we can state the following

Theorem 12.13. *The sequence (A_n) generated by the quadratures (12.14) is collectively compact and pointwise convergent to the integral operator A with logarithmic singularity given by (12.13).*

Therefore our general error analysis based on Theorems 10.8 and 10.9 is applicable. By Theorem 11.5, in the case of analytic functions the quadrature error is exponentially decreasing. This behaviour, by the estimate (10.7), is inherited by the approximate solution to the integral equation provided the kernel and the exact solution are analytic (see Problem 12.4). The matrix entering into (12.11) is a circulant matrix. We write

$$R_k^{(n)}(t_j) = R_{|j-k|}^{(n)}$$

with the weights

$$(12.17) \quad R_j^{(n)} := -\frac{1}{n} \left\{ \sum_{m=1}^{n-1} \frac{1}{m} \cos \frac{mj\pi}{n} + \frac{(-1)^j}{2n} \right\}$$

for $j = 0, \dots, 2n - 1$.

Example 12.14. We consider the Neumann boundary value problem for the reduced wave equation or *Helmholtz equation*

$$\Delta u + \kappa^2 u = 0$$

with a positive wave number κ in the interior D of the unit circle Γ with given normal derivative

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma.$$

We assume the unit normal n to be directed into the exterior of D . The fundamental solution to the two-dimensional Helmholtz equation is given by

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(\kappa|x - y|), \quad x \neq y,$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind and of order zero. We decompose

$$H_0^{(1)} = J_0 + i N_0,$$

where J_0 and N_0 are the Bessel and Neumann functions of order zero, and note the power series

$$(12.18) \quad J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k}$$

and

$$(12.19) \quad N_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + C \right) J_0(z) + \sum_{k=1}^{\infty} a_k z^{2k}$$

with real coefficients a_k and the Euler constant $C = 0.57721\dots$. From these expansions we deduce the asymptotics

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + \frac{i}{4} - \frac{1}{2\pi} \left(\ln \frac{\kappa}{2} + C \right) + O \left(|x - y|^2 \ln \frac{1}{|x - y|} \right)$$

for $|x - y| \rightarrow 0$. Therefore, the fundamental solution to the Helmholtz equation has the same singular behaviour as the fundamental solution of Laplace's equation. As a consequence the Green's representation Theorem 6.5 and the potential theoretic jump relations of Theorem 6.13 can be carried over to the reduced wave equation. For details we refer to Colton and Kress [1]. In particular, the boundary integral equation corresponding to (6.37) remains valid, that is, the unknown boundary values $\varphi = u$ on Γ of the solution u satisfy

$$(12.20) \quad \varphi(x) + 2 \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = 2 \int_{\Gamma} g(y) \Phi(x, y) ds(y), \quad x \in \Gamma.$$

With the exception of a countable set of wave numbers κ accumulating only at infinity, for which the homogeneous Neumann problem admits nontrivial solutions, this integral equation is uniquely solvable. For the normal derivative of the fundamental solution we use the relation

$$\frac{dH_0^{(1)}}{dz} = -H_1^{(1)} = -J_1 - iN_1,$$

where $H_1^{(1)}$ denotes the Hankel function of the first kind and of order one represented in terms of the Bessel and Neumann functions J_1 and N_1 of order one.

We use the representation $x(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$, for the unit circle and, by straightforward calculations, transform the integral equation (12.20) into the parametric form

$$(12.21) \quad \tilde{\varphi}(t) - \frac{1}{2\pi} \int_0^{2\pi} K(t, \tau) \tilde{\varphi}(\tau) d\tau = \frac{1}{2\pi} \int_0^{2\pi} L(t, \tau) \tilde{g}(\tau) d\tau$$

for $0 \leq t \leq 2\pi$. Here we have set $\tilde{\varphi}(t) := \varphi(x(t))$ and $\tilde{g}(t) := g(x(t))$ and the kernels are given by

$$K(t, \tau) := i\pi\kappa \left| \sin \frac{t - \tau}{2} \right| H_1^{(1)} \left(2\kappa \left| \sin \frac{t - \tau}{2} \right| \right)$$

and

$$L(t, \tau) := i\pi H_0^{(1)} \left(2\kappa \left| \sin \frac{t - \tau}{2} \right| \right)$$

for $t \neq \tau$. Observing the expansions (12.18) and (12.19) and their term by term derivatives, we can split the kernels into

$$(12.22) \quad K(t, \tau) = K_1(t, \tau) \ln 4 \sin^2 \frac{t - \tau}{2} + K_2(t, \tau)$$

and

$$(12.23) \quad L(t, \tau) = L_1(t, \tau) \ln 4 \sin^2 \frac{t - \tau}{2} + L_2(t, \tau)$$

where

$$K_1(t, \tau) := -\kappa \sin \frac{t-\tau}{2} J_1 \left(2\kappa \sin \frac{t-\tau}{2} \right),$$

$$K_2(t, \tau) := K(t, \tau) - K_1(t, \tau) \ln 4 \sin^2 \frac{t-\tau}{2},$$

$$L_1(t, \tau) := -J_0 \left(2\kappa \sin \frac{t-\tau}{2} \right),$$

$$L_2(t, \tau) := L(t, \tau) - L_1(t, \tau) \ln 4 \sin^2 \frac{t-\tau}{2}.$$

The kernels K_1 , K_2 , L_1 , and L_2 turn out to be analytic. In particular, from

$$\lim_{z \rightarrow 0} z H_1^{(1)}(z) = \frac{2}{\pi i}$$

and

$$\lim_{z \rightarrow 0} \left\{ H_0(z) - \frac{2i}{\pi} \ln z J_0(z) \right\} = \frac{2i}{\pi} \{C - \ln 2\} + 1$$

we deduce the diagonal terms

$$K_2(t, t) = K(t, t) = 1$$

and

$$L_2(t, t) = -2 \ln \frac{\kappa}{2} - 2C + i\pi$$

for $0 \leq t \leq 2\pi$. Note that despite the continuity of the kernel K , for numerical accuracy it is advantageous to incorporate the quadrature (12.14) because of the logarithmic singularities of the derivatives of the kernel K . Now, in the spirit of Theorem 12.7, we replace the integral equation (12.21) by the linear system

$$\begin{aligned} \tilde{\varphi}_j &= \sum_{k=0}^{2n-1} \left\{ R_{|j-k|}^{(n)} K_1(t_j, t_k) + \frac{1}{2n} K_2(t_j, t_k) \right\} \tilde{\varphi}_k \\ &= \sum_{k=0}^{2n-1} \left\{ R_{|j-k|}^{(n)} L_1(t_j, t_k) + \frac{1}{2n} L_2(t_j, t_k) \right\} \tilde{g}(t_k), \quad j = 0, \dots, 2n-1, \end{aligned}$$

for the approximating values $\tilde{\varphi}_j$ for $\tilde{\varphi}(t_j)$.

For a numerical example we consider the case in which the exact solution to the Neumann problem is given by $u(x) = N_0(\kappa|x - x_0|)$ where $x_0 = (q, 0)$ with $q > 1$. Then the right hand side is given by

$$\tilde{g}(t) = -\kappa \frac{1 - q \cos t}{v(t)} N_1(\kappa v(t))$$

and the exact solution of the integral equation by

$$\tilde{\varphi}(t) = N_0(\kappa v(t))$$

where $v(t) = 1 + q^2 - 2q \cos t$. The numerical results contained in Table 12.4 confirm the exponentially decreasing behaviour of the error which we expect from our general error analysis.

Table 12.4. Numerical results for Example 12.14

	n	$t = 0$	$t = \pi/2$	$t = \pi$
$\kappa = 1$	4	-0.07907728	0.12761991	0.24102137
	8	-0.01306333	0.00867450	0.01155067
	16	-0.00023494	0.00003924	0.00004517
	32	-0.00000019	0.00000001	0.00000000
$q = 2$	4	-0.13910590	0.39499045	0.68563472
	8	-0.07111636	0.07511294	0.10540051
	16	-0.00659502	0.00277753	0.00386074
	32	-0.00005924	0.00000558	0.00000616
$\kappa = 1.5$	8	0.35729406	0.21301358	-0.16596385
	16	0.01138634	0.00181974	0.00377600
	32	-0.00000558	-0.00000006	-0.00000010
	64	0.00000003	0.00000002	0.00000000
$\kappa = 5$	8	-0.54146680	-0.35298932	0.09706015
	16	-0.05669554	-0.02916764	-0.02480085
	32	-0.00021246	-0.00000877	-0.00001332
	64	-0.00000006	0.00000000	0.00000003

In closing this chapter, we wish to point out, that despite the fact that our numerical examples for the application of the Nyström method are all one-dimensional, this method of course can be applied to multi-dimensional integral equations provided reasonable numerical quadratures are available. The latter requirement puts some limits on the application to the weakly singular integral equations arising for boundary value problems in more than two space dimensions.

Problems

- 12.1. Let X and Y be normed spaces, let $U \subset X$ be compact and let (φ_n) be an equicontinuous sequence of functions $\varphi_n : U \rightarrow Y$ converging pointwise on U to some function $\varphi : U \rightarrow Y$. Then the sequence (φ_n) converges uniformly on U .
- 12.2. Let $D = \mathbb{R} \times (-s, s)$ denote a strip in the complex plane. Show that the space $H(D)$ of 2π -periodic holomorphic functions defined on D with the property

$$\lim_{\sigma \rightarrow s} \int_0^{2\pi} \{|f(t + i\sigma)|^2 + |f(t - i\sigma)|^2\} dt < \infty$$

is a Hilbert space with the scalar product

$$(f, g) := \lim_{\sigma \rightarrow s} \int_0^{2\pi} \{f(t + i\sigma)\overline{g(t + i\sigma)} + f(t - i\sigma)\overline{g(t - i\sigma)}\} dt.$$

Show that the functions

$$f_n(t) = \frac{1}{(4\pi \cosh 2ns)^{1/2}} e^{int}, \quad n \in \mathbb{Z},$$

form a complete orthonormal system in $H(D)$. Use Cauchy's integral formula to verify that the remainder for the trapezoidal rule is a bounded linear functional on $H(D)$. Apply the Riesz representation Theorem 4.8 and Parseval's equality from Theorem 1.28 to derive the error bound

$$|R_n(g)| \leq \frac{2M}{(e^{4ns} - 1)^{1/2}}$$

which is slightly better than the estimate given in Theorem 12.6 (cf. Engels [1] and Kress [5]).

- 12.3. Derive bounds on $\|(A_n - A)A\|_\infty$ and $\|(A_n - A)A_n\|_\infty$ for the numerical integration operators using the trapezoidal rule.
- 12.4. Let $D = \mathbb{R} \times (-s, s)$ denote a strip in the complex plane. Consider the Banach space $B(D)$ of 2π -periodic continuous functions defined on \bar{D} which are holomorphic in D , furnished with the maximum norm

$$\|f\|_{\infty, D} := \max_{t \in \bar{D}} |f(t)|.$$

Prove that – under suitable assumptions on K – the integral operator given by (12.13) is compact in $B(D)$. Use this result to deduce that solutions to integral equations of the second kind with the logarithmic kernel of (12.13) are analytic provided the kernel K and the right hand side are analytic.

Hint: By the Arzelà–Ascoli Theorem, show that integral operators with analytic kernels are compact in $B(D)$. Use

$$(A\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{s}{2} K(t, t + s) \varphi(t + s) ds$$

for the definition of A on $B(D)$. Approximate $\ln \sin^2 s/2$ by its Fourier series and use Theorem 2.16 and the Cauchy–Schwarz inequality.

- 12.5. Use a regular parametric representation $x(t) = (x_1(t), x_2(t))$, $0 \leq t \leq 2\pi$, for an arbitrary boundary curve Γ to derive a parametric form of the integral equation (12.20) for the Neumann problem for the reduced wave equation corresponding to (12.21). Describe an appropriate treatment of the logarithmic singularity as in Example 12.14 (cf. Kussmaul [1] and Kress [6]).

13. Projection Methods

The application of the quadrature method, in principle, is confined to equations of the second kind. In order to develop numerical methods which can also be used for equations of the first kind we now will describe projection methods as a general tool for approximately solving operator equations. After introducing into the principal ideas of projection methods and their convergence and error analysis we shall consider collocation and Galerkin methods as special cases. We do not intend to give a complete account of the numerous implementations of collocation and Galerkin methods for integral equations which have been developed in the literature. Our presentation is meant as an introduction into these methods by studying their basic concepts and by describing their numerical performance through a few typical examples.

13.1 The Projection Method

We describe the approximate solution of linear operator equations by projecting them onto subspaces, which for practical calculations we assume to be finite dimensional.

Definition 13.1. *Let X be a normed space and $U \subset X$ a nontrivial subspace. A bounded linear operator $P : X \rightarrow U$ with the property $P\varphi = \varphi$ for all $\varphi \in U$ is called a **projection operator** from X onto U .*

Theorem 13.2. *A nontrivial bounded linear operator is a projection operator if and only if it satisfies $P^2 = P$. There holds $\|P\| \geq 1$.*

Proof: Let $P : X \rightarrow U$ be a projection. Then, from $P\varphi \in U$ there follows that $P^2\varphi = P(P\varphi) = P\varphi$ for all $\varphi \in X$. Conversely, let $P^2 = P$ and set $U := P(X)$. Then, for all $\varphi \in U$ we may write $\varphi = P\psi$ with some $\psi \in X$ and obtain $P\varphi = \varphi$. Finally $P = P^2$, by Theorem 2.6, implies $\|P\| \leq \|P\|^2$, whence $\|P\| \geq 1$. \square

An important example for projection operators is given by the so called orthogonal projection, i.e., by the best approximation in pre-Hilbert spaces in the sense of Theorem 1.26.

Theorem 13.3. *Let U be a nontrivial complete subspace of a pre-Hilbert space X . Then the operator P mapping each element $\varphi \in X$ into its unique best*

approximation with respect to U is a projection operator. It is called orthogonal projection onto U and satisfies $\|P\| = 1$.

Proof: Trivially, we have $P\varphi = \varphi$ for all $\varphi \in U$. From the orthogonality condition of Theorem 1.25 for the best approximation in pre-Hilbert spaces we readily verify that P is linear and that

$$\|\varphi\|^2 = \|P\varphi + (\varphi - P\varphi)\|^2 = \|P\varphi\|^2 + \|\varphi - P\varphi\|^2 \geq \|P\varphi\|^2$$

for all $\varphi \in X$. Hence $\|P\| \leq 1$ and Theorem 13.2 yields $\|P\| = 1$. \square

A second important example for projections is given by interpolation operators.

Theorem 13.4. The interpolation operator introduced in Theorem 11.2 is a projection operator.

Note that for polynomial or trigonometric polynomial interpolation, because of the general nonconvergence results due to Faber combined with the uniform boundedness principle Theorem 10.3, the interpolation operators cannot be uniformly bounded with respect to the maximum norm (see Schönhage [1]).

Definition 13.5. Let X and Y be Banach spaces and let $A : X \rightarrow Y$ be an injective bounded linear operator. Let $X_n \subset X$ and $Y_n \subset Y$ be two sequences of subspaces with $\dim X_n = \dim Y_n = n$ and let $P_n : Y \rightarrow Y_n$ be projection operators. The projection method, generated by X_n and P_n , approximates the equation

$$A\varphi = f$$

by the projected equation

$$P_n A \varphi_n = P_n f.$$

This projection method is called convergent for the operator A if there exists an index N such that for each $f \in A(X)$ the approximating equation $P_n A \varphi_n = P_n f$ has a unique solution $\varphi_n \in X_n$ for all $n \geq N$ and if these solutions converge $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, to the unique solution φ of $A\varphi = f$.

In terms of operators, convergence of the projection method means that for all $n \geq N$ the finite dimensional operators $A_n := P_n A : X_n \rightarrow Y_n$ are invertible and that pointwise convergence $A_n^{-1} P_n A \varphi \rightarrow \varphi$, $n \rightarrow \infty$, holds for all $\varphi \in X$. In general, we can expect convergence only if the subspaces X_n possess the denseness property

$$(13.1) \quad \inf_{\psi \in X_n} \|\psi - \varphi\| \rightarrow 0, \quad n \rightarrow \infty,$$

for all $\varphi \in X$. Therefore, in the subsequent analysis we always will assume that this condition is fulfilled.

Since $A_n = P_n A$ is a linear operator between two finite dimensional spaces, carrying out the projection method reduces to solving a finite dimensional

linear system. In the following Sections 13.2 and 13.3 we shall describe the collocation and the Galerkin method as projection methods in the sense of this definition obtained via interpolation and orthogonal projection, respectively. Here, we first proceed with a general convergence and error analysis.

Theorem 13.6. *The projection method converges if and only if there exists an index N and a positive constant M such that for all $n \geq N$ the finite dimensional operators*

$$A_n := P_n A : X_n \rightarrow Y_n$$

are invertible and the operators $A_n^{-1} P_n A : X \rightarrow X$ are uniformly bounded

$$(13.2) \quad \|A_n^{-1} P_n A\| \leq M.$$

There holds an error estimate

$$(13.3) \quad \|\varphi_n - \varphi\| \leq (1 + M) \inf_{\psi \in X_n} \|\psi - \varphi\|.$$

Proof: Provided the projection method converges, the uniform boundedness (13.2) is a consequence of Theorem 10.3. Conversely, if the assumptions of the theorem are fulfilled we can write

$$\varphi_n - \varphi = (A_n^{-1} P_n A - I)\varphi.$$

Since for all $\psi \in X_n$, trivially, there holds $A_n^{-1} P_n A \psi = \psi$, we have

$$\varphi_n - \varphi = (A_n^{-1} P_n A - I)(\varphi - \psi).$$

Hence, we have the error estimate (13.3) and, with the aid of the denseness (13.1), the convergence follows. \square

The error estimate of Theorem 13.6 usually is referred to as Cea's lemma (see Cea [1]). It indicates that the error in the projection method is determined by how well the exact solution can be approximated by elements of the subspace X_n .

We now state the main stability property of the projection method.

Theorem 13.7. *Assume that $S : X \rightarrow Y$ is a bounded linear operator with a bounded inverse $S^{-1} : Y \rightarrow X$ and that the projection method is convergent for S . Let $A : X \rightarrow Y$ be a bounded linear operator satisfying either*

- (1) $\|A\|$ is sufficiently small or
- (2) A is compact and $S - A$ is injective.

Then the projection method also converges for $S - A$.

Proof: The operator S satisfies the conditions of Theorem 13.6, that is, the operators $S_n := P_n S$ are invertible for all sufficiently large n and satisfy $\|S_n^{-1} P_n S\| \leq M$ with some constant M . Since S has a bounded inverse, the pointwise convergence $S_n^{-1} P_n S \rightarrow I$, $n \rightarrow \infty$, on X implies pointwise convergence $S_n^{-1} P_n \rightarrow S^{-1}$, $n \rightarrow \infty$, on Y . We will show that for sufficiently large n the inverse operators of $I - S_n^{-1} P_n A : X \rightarrow X$ exist and are uniformly bounded if (1) or (2) is satisfied.

- (1) We apply the uniform boundedness principle Theorem 10.3 to the pointwise convergent sequence $(S_n^{-1}P_n)$. Then, by Theorem 10.1, the inverse operators $(I - S_n^{-1}P_nA)^{-1}$ exist and are uniformly bounded for all sufficiently large n provided A satisfies $\sup_{n \in \mathbb{N}} \|S_n^{-1}P_n\| \|A\| < 1$.
- (2) Since $S^{-1}A$ is compact, by the Riesz theory, $I - S^{-1}A : X \rightarrow X$ has a bounded inverse. From the pointwise convergence of the sequence $(S_n^{-1}P_n)$ and the compactness of A , by Theorem 10.6, we derive norm convergence $\|S^{-1}A - S_n^{-1}P_nA\| \rightarrow 0$, $n \rightarrow \infty$. Therefore, by Theorem 2.8, the inverse operators $(I - S_n^{-1}P_nA)^{-1}$ exist and are uniformly bounded for all sufficiently large n .

Note that $(I - S_n^{-1}P_nA)^{-1}$ maps X_n into itself. We abbreviate $\tilde{S} := S - A$ and $\tilde{S}_n := P_n\tilde{S}$. Then $\tilde{S}_n = S_n(I - S_n^{-1}P_nA) : X_n \rightarrow Y_n$ is invertible for sufficiently large n with the inverse given by

$$\tilde{S}_n^{-1} = (I - S_n^{-1}P_nA)^{-1}S_n^{-1}.$$

From $\tilde{S}_n^{-1}P_n\tilde{S} = (I - S_n^{-1}P_nA)^{-1}S_n^{-1}P_nS(I - S^{-1}A)$ we estimate

$$\|\tilde{S}_n^{-1}P_n\tilde{S}\| \leq \|(I - S_n^{-1}P_nA)^{-1}\| \|I - S^{-1}A\| M$$

and observe that the condition (13.2) is satisfied for \tilde{S} . This completes the proof. \square

For an equation of the second kind

$$(13.4) \quad \varphi - A\varphi = f$$

with a bounded linear operator $A : X \rightarrow X$ we need only a sequence of subspaces $X_n \subset X$ and projection operators $P_n : X \rightarrow X_n$. Then the projection method assumes the form

$$(13.5) \quad \varphi_n - P_nA\varphi_n = P_nf.$$

Note that each solution $\varphi_n \in X$ to (13.5) automatically belongs to X_n . When A is compact, from Theorem 13.7 we have the following convergence property.

Corollary 13.8. *Let $A : X \rightarrow X$ be compact, $I - A$ be injective and let the projections $P_n : X \rightarrow X_n$ converge pointwise $P_n\varphi \rightarrow \varphi$, $n \rightarrow \infty$, for all $\varphi \in X$. Then the projection method for $I - A$ converges.*

Proof: Apply the second part of Theorem 13.7 for $S = I$ and identify $X = Y$ and $X_n = Y_n$. \square

We wish to give an alternate proof of the latter convergence result based directly on Theorem 10.1. This will also give us the opportunity to point out that the projection method for equations of the second kind may converge without pointwise convergence of the projection operators on all of X .

Theorem 13.9. Let $A : X \rightarrow X$ be compact and $I - A$ be injective. Assume that the projections $P_n : X \rightarrow X_n$ satisfy $\|P_n A - A\| \rightarrow 0$, $n \rightarrow \infty$. Then, for sufficiently large n , the approximate equation (13.5) is uniquely solvable for all $f \in X$ and there holds an error estimate

$$(13.6) \quad \|\varphi_n - \varphi\| \leq M \|P_n \varphi - \varphi\|$$

with some positive constant M depending on A .

Proof: Theorems 3.4 and 10.1, applied to $I - A$ and $I - P_n A$, imply that for all sufficiently large n the inverse operators $(I - P_n A)^{-1}$ exist and are uniformly bounded. To verify the error bound, we apply the projection operator P_n to (13.4) and get

$$\varphi - P_n A \varphi = P_n f + \varphi - P_n \varphi.$$

Subtracting this from (13.5) we find

$$(I - P_n A)(\varphi_n - \varphi) = P_n \varphi - \varphi,$$

whence the estimate (13.6) follows. \square

Note that pointwise convergence $P_n \varphi \rightarrow \varphi$, $n \rightarrow \infty$, for all $\varphi \in X$, by Theorem 10.6 implies $\|P_n A - A\| \rightarrow 0$, $n \rightarrow \infty$. But $\|P_n A - A\| \rightarrow 0$, $n \rightarrow \infty$, may be satisfied without pointwise convergence of the projection sequence (P_n) as we will see in our discussion of the collocation method (see Theorems 13.14 and 13.15). Then, of course, we can assure convergence only for those cases where convergence $P_n \varphi \rightarrow \varphi$, $n \rightarrow \infty$, is satisfied for the exact solution φ .

In actual numerical calculations, usually instead of (13.5) an approximate version of the form

$$(13.7) \quad \varphi_n - P_n A_n \varphi_n = P_n f_n$$

will be solved where A_n is some approximation to A and f_n approximates f . Then, as readily seen, we can state the following

Corollary 13.10. Assume that $\|P_n A_n - A\| \rightarrow 0$, $n \rightarrow \infty$. Then, for sufficiently large n , the approximate equation (13.7) is uniquely solvable and there holds an error estimate

$$(13.8) \quad \|\varphi_n - \varphi\| \leq M \{ \|P_n \varphi - \varphi\| + \|P_n(A_n - A)\varphi\| + \|P_n(f_n - f)\| \}.$$

We conclude with a corollary which will enable us to derive convergence results for collocation and Galerkin methods for equations of the first kind. Consider the equation

$$(13.9) \quad S\varphi - A\varphi = f$$

where $S : X \rightarrow Y$ is bounded with bounded inverse $S^{-1} : Y \rightarrow X$ and where $A : X \rightarrow Y$ is compact and $S - A$ is injective. Then, for the projection method

$$(13.10) \quad P_n(S - A)\varphi_n = P_n f$$

with subspaces X_n and Y_n and projection operators $P_n : Y \rightarrow Y_n$ chosen as in Definition 13.5 we have the following

Corollary 13.11. *Assume that $Y_n = S(X_n)$ and $\|P_n A - A\| \rightarrow 0$, $n \rightarrow \infty$. Then, for sufficiently large n , the approximate equation (13.10) is uniquely solvable and there holds an error estimate*

$$\|\varphi_n - \varphi\| \leq M \|P_n S \varphi - S \varphi\|$$

with some positive constant M depending on S and A .

Proof: The equation (13.10) is equivalent to

$$S^{-1} P_n S (I - S^{-1} A) \varphi_n = S^{-1} P_n S S^{-1} f,$$

that is, to

$$Q_n (I - S^{-1} A) \varphi_n = Q_n S^{-1} f,$$

where $Q_n := S^{-1} P_n S : X \rightarrow X_n$ is obviously a projection operator. Using $\|Q_n S^{-1} A - S^{-1} A\| = \|S^{-1}(P_n A - A)\|$ and $Q_n \varphi - \varphi = S^{-1}(P_n S \varphi - S \varphi)$, the assertion now follows from Theorem 13.9. \square

The essence of Corollary 13.11 lies in the fact that the projection method is applied directly to the given equation whereas the convergence result is based on the regularized equation of the second kind. Note that the statement of Corollary 13.10 carries over to the situation treated in Corollary 13.11.

For a more exhaustive study of projection methods we refer to Krasnoselski [1], Prössdorf and Silbermann [1] and Michlin and Prössdorf [1].

13.2 The Collocation Method

The *collocation method* for approximately solving an equation

$$(13.11) \quad A\varphi = f,$$

roughly speaking, consists in seeking an approximate solution from a finite dimensional subspace by requiring that the equation (13.11) is satisfied only at a finite number of so called *collocation points*. To be more precise, let $Y = C(G)$ and $A : X \rightarrow Y$ be a bounded linear operator. Let $X_n \subset X$ and $Y_n \subset Y$ denote a sequence of subspaces with $\dim X_n = \dim Y_n = n$. Choose n points $x_1^{(n)}, \dots, x_n^{(n)}$ in G – we also will write x_1, \dots, x_n instead of $x_1^{(n)}, \dots, x_n^{(n)}$ – such that the subspace Y_n is unisolvant with respect to these points. Then the collocation method approximates the solution of (13.11) by an element $\varphi_n \in X_n$ satisfying

$$(13.12) \quad (A\varphi_n)(x_j) = f(x_j), \quad j = 1, \dots, n.$$

Let $X_n = \text{span}\{u_1, \dots, u_n\}$. Then we express φ_n as a linear combination

$$\varphi_n = \sum_{k=1}^n \gamma_k u_k$$

and immediately see that equation (13.12) is equivalent to the linear system

$$(13.13) \quad \sum_{k=1}^n \gamma_k (Au_k)(x_j) = f(x_j), \quad j = 1, \dots, n,$$

for the coefficients $\gamma_1, \dots, \gamma_n$. The collocation method can be interpreted as projection method with the interpolation operator $P_n : Y \rightarrow Y_n$ described in Theorem 13.4. Indeed, since the interpolating function is uniquely determined by its values at the interpolation points, equation (13.12) is equivalent to

$$P_n A \varphi_n = P_n f.$$

Hence, our general error and convergence results for projection methods apply to the collocation method.

For an equation of the second kind

$$\varphi - A\varphi = f$$

in $X = C(G)$ with a bounded linear operator $A : X \rightarrow X$ we need only one sequence of subspaces $X_n \subset X$ and assume X_n to be unisolvant with respect to the collocation points. Here, the equations (13.13) assume the form

$$(13.14) \quad \sum_{k=1}^n \gamma_k \{u_k(x_j) - (Au_k)(x_j)\} = f(x_j), \quad j = 1, \dots, n.$$

Note that the collocation method, of course, may also be applied in other function spaces than the space $C(G)$, for instance in Hölder spaces.

We shall first consider the collocation method for integral equations of the second kind

$$\varphi(x) - \int_G K(x, y) \varphi(y) dy = f(x), \quad x \in G,$$

with continuous or weakly singular kernel K . If we use the Lagrange basis for X_n and write

$$\varphi_n = \sum_{k=1}^n \gamma_k L_k$$

then, of course, $\gamma_j = \varphi_n(x_j)$, $j = 1, \dots, n$, and the system (13.14) becomes

$$(13.15) \quad \gamma_j - \sum_{k=1}^n \gamma_k \int_G K(x_j, y) L_k(y) dy = f(x_j), \quad j = 1, \dots, n.$$

From the last equation we observe that the collocation method for equations of the second kind is closely related to the degenerate kernel method via interpolation. Obviously, the matrix of the system (13.15) coincides with the matrix (11.6) from the degenerate kernel approach. But the right hand sides of both systems are different and the solutions $\gamma_1, \dots, \gamma_n$ have different meanings. However, if the right hand side f happens to belong to X_n , then $f = P_n f$ and we may express the collocation solution φ_n also in the form

$$\varphi_n = f + \sum_{k=1}^n \tilde{\gamma}_k L_k$$

Now, for the coefficients $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ we obtain the same system as (11.6), and in this case the collocation method and the degenerate kernel method via interpolation will yield the same numerical results.

There exists a broad variety of collocation methods corresponding to various choices for the subspaces X_n , for the basis functions u_1, \dots, u_n and for the collocation points x_1, \dots, x_n . We briefly discuss two possibilities based on linear splines and on trigonometric interpolation. When we use linear splines – or more general spline interpolations – then we have pointwise convergence of the corresponding interpolation operator as can be seen from Theorem 11.3 in connection with the Banach–Steinhaus theorem. Therefore, in this case, Corollary 13.8 applies and we can state

Theorem 13.12. *The collocation method with linear splines converges for integral equations of the second kind with continuous or weakly singular kernels.*

Provided the exact solution of the integral equation is twice continuously differentiable, then from Theorems 11.3 and 13.6 we derive an error estimate of the form

$$\|\varphi_n - \varphi\|_\infty \leq M \|\varphi''\|_\infty h^2$$

for the linear spline collocation approximate solution φ_n . Here, M denotes some constant depending on the kernel K .

In general, as in the degenerate kernel method, in most practical problems the evaluation of the matrix entries in (13.15) will require numerical integration. For twice continuously differentiable kernels, again, we suggest the quadrature rule (11.11). Then, according to our remarks concerning the connection between the equations (11.6) and (13.15), the collocation method will produce the same numerical results as the degenerate kernel method. So we do not have to repeat the numerical calculations for equation (11.10) of Example 11.4 using the quadrature (11.11). However, in order to illustrate that it does not pay off to evaluate the matrix elements more accurately, we give the numerical results for the spline collocation for equation (11.10) using the exact matrix entries. Comparing the Tables 11.1 and 13.1, we observe that the improvement in the accuracy for the solution of the integral equation is only marginal. In general, instead of investing computing efforts in a highly accurate evaluation of the matrix elements it is more efficient to increase the number of collocation points, i.e., to increase the size of the linear system.

Table 13.1. Collocation method for Example 11.4

n	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
4	0.003984	0.004428	0.004750	0.004978	0.005135
8	0.000100	0.001112	0.001193	0.001250	0.001291
16	0.000250	0.000278	0.000298	0.000313	0.000323
32	0.000063	0.000070	0.000075	0.000078	0.000081

We proceed discussing the collocation method based on trigonometric interpolation with equidistant knots $t_j = j\pi/n$, $j = 0, \dots, 2n - 1$. First, we establish a convergence result for the trigonometric interpolation of differentiable functions.

Theorem 13.13. *Let $g \in C^1[0, 2\pi]$. Then for the remainder in trigonometric interpolation there holds*

$$(13.16) \quad \|P_n g - g\|_\infty \leq c_n \|g'\|_2$$

with $c_n \rightarrow 0$, $n \rightarrow \infty$.

Proof: Consider the monomials $f_m(t) = e^{imt}$ and write $m = (2k + 1)n + q$ with $k \in \mathbb{Z}$ and $0 \leq q < 2n$. Since $f_m(t_j) = f_{q-n}(t_j)$ for $j = 0, \dots, 2n - 1$, the trigonometric interpolation polynomials for f_m and f_{q-n} coincide. Therefore we have

$$\|P_n f_m - f_m\|_\infty \leq 2$$

for all $|m| \geq n$. We expand g into a uniformly convergent Fourier series

$$g = \sum_{m=-\infty}^{\infty} a_m f_m$$

and, using the Cauchy–Schwarz inequality, with the help of the boundedness of P_n with respect to $\|\cdot\|_\infty$ and of (8.3), we derive

$$\|P_n g - g\|_\infty^2 \leq 4 \left\{ \sum_{|m|=n}^{\infty} |a_m| \right\}^2 \leq \frac{4}{\pi} \|g'\|_2^2 \sum_{m=n}^{\infty} \frac{1}{m^2}.$$

This implies (13.16). \square

Now, consider an integral equation of the second kind with 2π -periodic continuously differentiable kernel K and right hand side f . The corresponding integral operator A maps $C[0, 2\pi]$ into $C^1[0, 2\pi]$ with $\|(A\varphi)'\|_2 \leq M\|\varphi\|_\infty$ where $M = \sqrt{2\pi} \|\partial K/\partial t\|_\infty$. Therefore, making use of (13.16), we find

$$\|P_n A\varphi - A\varphi\|_\infty \leq c_n \|(A\varphi)'\|_2 \leq c_n M \|\varphi\|_\infty$$

for all $\varphi \in C[0, 2\pi]$. Hence, $\|P_n A - A\|_\infty \leq c_n M \rightarrow 0$, $n \rightarrow \infty$, and Theorem 13.9 can be applied to obtain

Theorem 13.14. *The collocation method with trigonometric polynomials converges for integral equations of the second kind with continuously differentiable kernels and right hand sides.*

One possibility for the implementation of the collocation method is to use the monomials f_m as basis functions. Then the integrals $\int_0^{2\pi} K(t_j, \tau) e^{ik\tau} d\tau$ have to be integrated numerically. Replacing the kernel by its trigonometric interpolation leads to the quadrature rule

$$\int_0^{2\pi} K(t_j, \tau) e^{ik\tau} d\tau \approx \frac{\pi}{n} \sum_{m=0}^{2n-1} K(t_j, t_m) e^{ikt_m}$$

for $j = 0, \dots, 2n - 1$. Using fast Fourier transform techniques (see Nussbaumer [1]) these quadratures can be carried out very rapidly. A second even more efficient possibility is to use the Lagrange basis as already described in connection with the degenerate kernel method. This leads to the quadrature rule (11.17) for the evalution of the matrix elements in (13.15). Because of the simple structure of (11.17), the only additional computational effort besides the kernel evalution is the computation of the row sums

$$\sum_{m=0}^{2n-1} (-1)^m K(t_j, t_m)$$

for $j = 0, \dots, 2n - 1$. Of course, with this quadrature the collocation method again will coincide with the degenerate kernel method via trigonometric interpolation. Therefore it is not necessary to repeat the numerical results for equation (11.18) of Example 11.6.

We now proceed to illustrate the application of the collocation method for equations of the second kind with weakly singular kernels. As in Section 12.3 we consider the operator

$$(13.17) \quad (A\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2} K(t, \tau) \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi,$$

where the kernel K is assumed 2π -periodic and twice continuously differentiable. Of course, again we use trigonometric interpolation. Consider the special cases

$$(A_0\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t-\tau}{2} \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi,$$

and

$$(A_1\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \sin(\tau - t) \ln 4 \sin^2 \frac{t-\tau}{2} \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi.$$

By Fourier expansion, with the aid of the integrals (12.16), we see that A_0 and A_1 map $C[0, 2\pi]$ boundedly into $H^1[0, 2\pi]$. The kernel

$$\tilde{K}(t, \tau) := \left\{ K(t, \tau) - K(t, t) - \frac{\partial K}{\partial \tau}(t, t) \sin(\tau - t) \right\} \ln 4 \sin^2 \frac{t-\tau}{2}, \quad t \neq \tau,$$

can be continued as a continuously differentiable function onto $[0, 2\pi] \times [0, 2\pi]$. Therefore, as shown above, the corresponding integral operator \tilde{A} maps $C[0, 2\pi]$ boundedly into $H^1[0, 2\pi]$. Combining all operators, it follows that A itself maps $C[0, 2\pi]$ boundedly into $H^1[0, 2\pi]$. Now, using (13.16), we can argue as in the case of the smooth kernel to show that Theorem 13.9 can be applied to equations of the second kind with the weakly singular operator (13.17).

Theorem 13.15. *The collocation method with trigonometric polynomials converges for integral equations of the second kind with logarithmic singularity of the form (13.17) provided the kernel K is twice continuously differentiable and the right hand side is continuously differentiable.*

For the evaluation of the matrix elements, as usual, we replace $K(t_j, \cdot)$ by its trigonometric interpolation. This yields the approximations

$$\begin{aligned}
 & \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t_j - \tau}{2} K(t_j, \tau) L_k(\tau) d\tau \\
 (13.18) \quad & \approx \sum_{m=0}^{2n-1} K(t_j, t_m) \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t_j - \tau}{2} L_k(\tau) L_m(\tau) d\tau \\
 & = \sum_{m=0}^{2n-1} S_{k-j, m-j} K(t_j, t_m).
 \end{aligned}$$

A practical numerical evaluation of the weights in (13.18) rests on the observation that the quadrature rule (12.14) with the number of grid points $2n$ doubled to $4n$ integrates trigonometric polynomials of degree less than or equal to $2n$ exactly provided the coefficient of the term $\sin 2nt$ vanishes. Therefore we have

$$S_{km} = \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{\tau}{2} L_k(\tau) L_m(\tau) d\tau = \sum_{j=0}^{4n-1} R_j^{(2n)} L_k(t_j^{(2n)}) L_m(t_j^{(2n)})$$

for $m, k = 0, \dots, 2n - 1$. In the case of an analytic kernel K , from Theorem 11.5, we deduce an exponentially decreasing quadrature error for (13.18).

With the quadrature rule (13.18) for the evaluation of the matrix elements, the system (13.15) actually corresponds to solving the equation (13.7) with

$$(A_n \varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t - \tau}{2} \sum_{m=0}^{2n-1} K(t, t_m) L_m(\tau) \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi.$$

Since the kernel K is assumed twice continuously differentiable, analogous to (13.16), we can derive an estimate $\|K_n - K\|_\infty = o(1/n)$ for the interpolating kernel K_n of A_n . Therefore $\|A_n - A\|_\infty = o(1/n)$, and $\|P_n\|_\infty \leq 2n$ implies $\|P_n(A_n - A)\|_\infty \rightarrow 0$, $n \rightarrow \infty$. Hence the error estimate (13.8) applies. In the case, when K is analytic, then from Theorem 11.5, we have $\|P_n(A_n - A)\|_\infty = O(e^{-ns})$ with some $s > 0$. Thus the convergence is exponentially if the solution φ is analytic.

Example 13.16. We apply the collocation method based on trigonometric polynomials for the integral equation (12.21) of Example 12.14. We use the splitting (12.22) of the kernel in a logarithmic and a smooth part, and then evaluate the corresponding matrix elements with the quadrature rules (13.18) and (11.17), respectively. The right hand side of the system (13.15) is evaluated by numerical integration as in Example 12.15. The numerical results show the exponential convergence, which we expect from our error analysis.

In general, the implementation of the collocation method as described by our two examples, can be used in all situations where the required numerical

Table 13.2. Numerical results for Example 13.16

	n	$t=0$	$t = \pi/2$	$t = \pi$
$\kappa=1$	4	-0.10828581	0.15071347	0.22404278
	8	-0.01368742	0.00827850	0.01129843
	16	-0.00023817	0.00003702	0.00004366
	32	-0.00000019	0.00000001	0.00000000
$\kappa=1.5$	4	-0.19642139	0.43455120	0.66382871
	8	-0.07271273	0.07400151	0.10462955
	16	-0.00662395	0.00275639	0.00384562
	32	-0.00005937	0.00000548	0.00000609
$\kappa=5$	8	0.18032618	0.02704260	-0.36058686
	16	0.00617287	-0.00852966	-0.00357971
	32	-0.00000686	-0.00000015	-0.00000009
	64	0.00000003	0.00000002	-0.00000000
$\kappa=5$	8	-0.17339263	0.01532368	0.46622097
	16	-0.04130869	-0.01351387	-0.01226315
	32	-0.00022670	-0.00000972	-0.00001335
	64	-0.00000006	0.00000000	0.00000003

quadratures for the matrix elements can be carried out in closed form for the chosen approximating subspace and collocation points. In all these cases, of course, the quadrature formulae which are required for the related Nyström method will also be available. Because the approximation order for both methods usually will be the same, Nyström's method is preferable since it requires the least computational effort for evaluating the matrix elements.

However, the situation drastically changes in cases where no straightforward quadrature rules for the application of Nyström's method are available. This, in particular, occurs for the boundary integral equations described in Chapter 6 in the case of three space dimensions. Here, the collocation method is the most important numerical approximation method. Usually, the boundary surface is subdivided into a finite number of segments, like curved triangles and squares. Then the approximation space is chosen to consist of some kind of low order polynomial splines with respect to these surface elements, which possess appropriate smoothness properties across the boundary curves of the segments. Within each segment, depending on the degree of freedom in the chosen splines, a number of collocation points is selected. Then, the integrals for the matrix elements in the collocation system (13.15) are evaluated using numerical integration. Since the integral equations usually will have a weakly singular kernel, the calculation of the improper integrals for the diagonal elements of the matrix,

where the collocation points and the surface elements coincide, needs special attention. With these few remarks we have given a somewhat vague outline of a very effective method known as *boundary element method*. For a detailed description and a review of the corresponding literature we refer the reader to Brebbia, Telles and Wrobel [1] and Wendland [5].

We complete our study of collocation methods by indicating its applications to integral equations of the first kind basing the convergence and error analysis on Corollary 13.11. Consider a singular integral equation of the first kind with Hilbert kernel

$$(13.19) \quad \frac{1}{2\pi} \int_0^{2\pi} \left\{ \cot \frac{\tau - t}{2} + K(t, \tau) \right\} \varphi(\tau) d\tau = f(t), \quad 0 \leq t \leq 2\pi,$$

as discussed in Corollary 7.25. Because of the Cauchy type singularity, this equation and correspondingly its error analysis has to be treated in the Hölder space $C^{0,\alpha}[0, 2\pi]$. For convenience, the 2π -periodic kernel K is assumed to be analytic with respect to both variables. We express the leading singular part of (13.19) in the form

$$(R\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \left\{ \cot \frac{\tau - t}{2} + 2 \right\} \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi.$$

For the constant kernel function $K = 2$, equation (13.19) corresponds to the Cauchy singular integral equation (7.48) for the unit circle. Therefore, from our analysis in the proof of Theorem 7.29 we conclude that the singular integral operator $R : C^{0,\alpha}[0, 2\pi] \rightarrow C^{0,\alpha}[0, 2\pi]$ has a bounded inverse.

Choose X_n to be the subspace of trigonometric polynomials of the form

$$(13.20) \quad u(t) = \sum_{j=0}^n \alpha_j \cos jt + \sum_{j=1}^{n-1} \beta_j \sin jt,$$

and note that X_n is unisolvant with respect to the equidistantly spaced grid $t_j = j\pi/n$, $j = 0, \dots, 2n - 1$. From the integrals

$$(13.21) \quad \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau}{2} e^{\pm im\tau} d\tau = \begin{cases} \pm i, & m = 1, 2, \dots, \\ 0, & m = 0, \end{cases}$$

it follows that $Y_n := R(X_n)$ is given by the trigonometric polynomials of the form

$$(13.22) \quad v(t) = \sum_{j=0}^{n-1} \tilde{\alpha}_j \cos jt + \sum_{j=1}^n \tilde{\beta}_j \sin jt.$$

Observe that Y_n is not unisolvant with respect to the t_j , $j = 0, \dots, 2n - 1$, that, however, it is unisolvant with respect to the interspersed set of points $\tilde{t}_j = t_j + \pi/2n$, $j = 0, \dots, 2n - 1$. Hence, we use the latter points as collocation points.

From the representation (11.14) for the error in trigonometric interpolation for analytic functions, it can be seen that the derivative of the remainder

term also decreases exponentially with the number n of interpolation points. Therefore, it follows that for the integral operator A with kernel K , analogous to (11.16), there holds an estimate of the form

$$(13.23) \quad \|\tilde{P}_n A - A\|_{0,\alpha} = O(e^{-ns}).$$

Here, $\tilde{P}_n : C^{0,\alpha}[0, 2\pi] \rightarrow Y_n$ stands for the trigonometric interpolation operator with respect to the points \tilde{t}_j , $j = 0, \dots, 2n - 1$, and s is some positive constant depending on the kernel K . It is left to the reader to establish a remainder representation for \tilde{P}_n corresponding to (11.14).

From Corollary 13.11 we now obtain that the collocation equations are solvable for sufficiently large n and that the error can be estimated by

$$\|\varphi_n - \varphi\|_{0,\alpha} \leq M \|\tilde{P}_n R\varphi - R\varphi\|_{0,\alpha}.$$

If the right hand side f of (13.19) is analytic, then $R\varphi$ is also analytic and therefore we can conclude exponential convergence of the collocation method in the Hölder norm. We summarize in

Theorem 13.17. *The collocation method with trigonometric polynomials converges exponentially for the integral equation of the first kind (13.19) with Hilbert kernel provided K and f are analytic.*

For a discussion of the collocation method for Hilbert kernels under weaker regularity assumptions we refer to Michlin and Prössdorf [1].

For the numerical implementation we need the integrals

$$(13.24) \quad \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - \tilde{t}_j}{2} L_k(\tau) d\tau = \frac{1}{2n} \{1 - \cos n(t_k - \tilde{t}_j)\} \cot \frac{t_k - \tilde{t}_j}{2},$$

for $j, k = 0, \dots, 2n - 1$, which can be easily derived from (11.13) and (13.21). The smooth part in (13.19) corresponding to K can be treated by using the quadrature (11.17).

Example 13.18. The integral equation (13.19) with kernel

$$(13.25) \quad K(t, \tau) = 2 - \frac{(a^2 - b^2) \sin(t + \tau)}{a^2 + b^2 - (a^2 - b^2) \cos(t + \tau)}$$

corresponds to the Cauchy singular integral equation (7.48) for an ellipse with semiaxis $a \geq b > 0$. Therefore, in particular, we have uniqueness. Table 13.3 gives some numerical results for the case where the exact solution is given by

$$\varphi(t) = e^{\cos t} \cos(\sin t) - 1, \quad 0 \leq t \leq 2\pi.$$

Then, setting $c = (a - b)/(a + b)$, the right hand side becomes

$$f(t) = e^{c \cos t} \sin(c \sin t) - e^{\cos t} \sin(\sin t), \quad 0 \leq t \leq 2\pi.$$

Table 13.3. Numerical results for Example 13.18

	n	$t=0$	$t = \pi/2$	$t = \pi$
$a=1$ $b=0.5$	4	-0.12246023	-0.03965981	0.03501397
	8	-0.00208608	0.00072332	0.00054052
	16	-0.00000040	0.00000004	0.00000001
	32	0.00000000	0.00000000	0.00000000
$a=1$ $b=0.2$	4	-0.15442954	-0.06593413	0.11953193
	8	-0.05638500	0.02304358	0.01123745
	16	-0.00267225	0.00071416	0.00022100
	32	-0.00000409	0.00000109	0.00000034

Finally, we consider an integral equation of the first kind with a logarithmic singularity of the type

$$(13.26) \quad \frac{1}{2\pi} \int_0^{2\pi} \left\{ \ln 4 \sin^2 \frac{t-\tau}{2} + K(t, \tau) \right\} \varphi(\tau) d\tau = f(t), \quad 0 \leq t \leq 2\pi,$$

where K again is assumed periodic and analytic. For applications of integral equations with such a logarithmic singularity we recall the remarks in Section 12.3 on two-dimensional boundary value problems and Problem 7.2. We express the leading singular part in the form

$$(S\varphi)(t) := \frac{1}{2\pi} \int_0^{2\pi} \left\{ \ln 4 \sin^2 \frac{t-\tau}{2} - 2 \right\} \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi,$$

which corresponds to the integral equation of the first kind (7.44) via a single-layer logarithmic potential for the Dirichlet problem in the case of the unit circle. Hence, by Theorem 7.28, the bounded operator $S : C^{0,\alpha}[0, 2\pi] \rightarrow C^{1,\alpha}[0, 2\pi]$ has a bounded inverse.

From the integrals (12.16) we see that S maps the space of trigonometric polynomials of the form (13.20) into itself. Hence, here we can use the set $t_j = j\pi/n$, $j = 0, \dots, 2n-1$, as collocation points. It is left to the reader to verify an estimate corresponding to (13.23) in the $C^{1,\alpha}$ -norm by considering the second derivative of the remainder term in trigonometric interpolation. Hence, using again Corollary 13.11, we can state

Theorem 13.19. *The collocation method with trigonometric polynomials converges exponentially for the integral equation of the first kind (13.26) with logarithmic kernel provided K and f are analytic.*

For the implementation of the collocation method we need the integrals

$$(13.27) \quad \frac{1}{2\pi} \int_0^{2\pi} \ln 4 \sin^2 \frac{t_j - \tau}{2} L_k(\tau) d\tau = R_{|j-k|}$$

with the weights R_j given by (12.17).

Example 13.20. The integral equation (13.26) with kernel

$$(13.28) \quad K(t, \tau) = \ln\{a^2 + b^2 - (a^2 - b^2)\cos(t + \tau)\} + 1$$

essentially corresponds to the integral equation (7.44) for an ellipse with semi-axis $0 < b \leq a \leq 1$. Therefore, in particular, it is uniquely solvable. Table 13.4 gives some numerical results for the case where the exact solution is given by

$$\varphi(t) = e^{\cos t}\{\cos t \cos(\sin t) - \sin t \sin(\sin t)\}, \quad 0 \leq t \leq 2\pi.$$

Then the right hand side becomes

$$f(t) = 2 - e^{\cos t} \cos(\sin t) - e^{c \cos t} \cos(c \sin t), \quad 0 \leq t \leq 2\pi,$$

with $c = (a - b)/(a + b)$.

Table 13.4. Numerical results for Example 13.20

	n	$t=0$	$t = \pi/2$	$t = \pi$
$a=1$ $b=0.5$	4	-0.62927694	-0.19858702	0.20711238
	8	-0.02649085	0.00602690	0.01355414
	16	-0.00000724	0.000000110	0.000000457
	32	0.000000000	0.000000000	0.000000000
$a=1$ $b=0.2$	4	-0.77913195	-0.22694775	0.20990539
	8	-0.08896065	0.02823001	0.03008962
	16	-0.00167596	0.00010607	0.00000226
	32	-0.00000114	0.00000009	-0.00000001

For a detailed numerical analysis for equations of the first kind with a logarithmic singularity using spline collocations we refer to Hsiao, Kopp and Wendland [1].

13.3 The Galerkin Method

For operator equations in Hilbert spaces the projection method via orthogonal projection into finite dimensional subspaces leads to the *Galerkin method*, named after the Russian engineer Galerkin. Let X and Y be Hilbert spaces and let $A : X \rightarrow Y$ be an injective bounded linear operator. Let $X_n \subset X$ and $Y_n \subset Y$ be subspaces with $\dim X_n = \dim Y_n = n$, and let $P_n : Y \rightarrow Y_n$ be the orthogonal projection described in Theorem 13.3. Then $\varphi_n \in X_n$ is a solution to the projection method, generated by X_n and P_n , for the equation $A\varphi = f$ if and only if

$$(13.29) \quad (A\varphi_n, g) = (f, g)$$

for all $g \in Y_n$. This follows immediately from the fact that (13.29), by Theorem 1.25, is equivalent to $P_n(A\varphi_n - f) = 0$. The equation (13.29) is called the *Galerkin equation*.

In the literature, the Galerkin method is also known as the *Petrov–Galerkin method* and the special case where $X = Y$ and $X_n = Y_n$ is also called *Bubnow–Galerkin method*. When the operator A is self adjoint and positive definite, as we will briefly explain, the Bubnow–Galerkin method coincides with the *Rayleigh–Ritz method*. A bounded linear operator $A : X \rightarrow X$ is called *self adjoint* if $A = A^*$; it is called *positive definite* if it is self adjoint and satisfies

$$(13.30) \quad (A\varphi, \varphi) > 0$$

for all $\varphi \in X$ with $\varphi \neq 0$. A positive definite operator clearly is injective. We can define an additional scalar product on X by

$$(13.31) \quad (\varphi, \psi)_E := (A\varphi, \psi),$$

with the corresponding norm $\|\cdot\|_E$ called *energy norm*. Consider the so called *energy functional*, defined on X by

$$(13.32) \quad E(\varphi) := (A\varphi, \varphi) - 2 \operatorname{Re}(f, \varphi)$$

for all $\varphi \in X$ with $f \in X$ a given element. For $f \in A(X)$ we can transform

$$E(\varphi) - E(A^{-1}f) = \|\varphi - A^{-1}f\|_E^2.$$

Hence, solving the equation $A\varphi = f$ is equivalent to minimizing the energy functional E over X . Based on the classical work of Rayleigh [1] from 1896 and Ritz [1] from 1909, the Rayleigh–Ritz method consists in an approximation by minimizing E only over a finite dimensional subspace X_n . This, obviously is equivalent to finding a best approximation to the exact solution with respect to X_n in the energy norm. By Theorems 1.25 and 1.26, the best approximation exists and is uniquely determined by the orthogonality condition

$$(\varphi_n, g)_E = (A^{-1}f, g)_E$$

for all $g \in X_n$. But, this coincides with the Galerkin equation (13.29), i.e., the Rayleigh–Ritz method is a special case of the Galerkin method. Note that we have convergence of the Rayleigh–Ritz method if the denseness condition (13.1) is satisfied with respect to the energy norm.

First Bubnow in 1913 and then, in more details, Galerkin [1] in 1915 approached and extended this approximation method without relying on a minimization formulation. Later, Petrov [1] first considered the general situation of the form (13.29) of the Galerkin method.

Assume that $X_n = \operatorname{span}\{u_1, \dots, u_n\}$ and $Y_n = \operatorname{span}\{v_1, \dots, v_n\}$. Then we express φ_n as a linear combination

$$\varphi_n = \sum_{k=1}^n \gamma_k u_k$$

and find that solving the equation (13.29) is equivalent to the linear system

$$(13.33) \quad \sum_{k=1}^n \gamma_k (Au_k, v_j) = (f, v_j), \quad j = 1, \dots, n,$$

for the coefficients $\gamma_1, \dots, \gamma_n$. Usually, the scalar product will have the form of an integral. Therefore, in the case of integral equations, the system (13.33) requires a double integration for each of the matrix elements. This is a disadvantage of the Galerkin method making it considerably inferior in numerical efficiency as compared with the collocation method. Its major advantage is based on the simplicity of the orthogonal projection and the exploitation of the Hilbert space structure. In particular, for integral equations of the second kind, Corollary 13.8 can always be applied since the denseness assumption (13.1) implies pointwise convergence of the orthogonal projections. Hence, we can state the following theorem.

Theorem 13.21. *Let $A : X \rightarrow X$ be a compact linear operator and assume that $I - A$ is injective. Then the Bubnow–Galerkin method converges.*

In contrast to our previous methods, the Galerkin method applies more easily to equations of the first kind. We proceed with a few general results.

Definition 13.22. *A bounded linear operator $A : X \rightarrow X$ in a Hilbert space X is called strictly coercive if there exists a constant $c > 0$ such that*

$$(13.34) \quad \operatorname{Re}(A\varphi, \varphi) \geq c\|\varphi\|^2$$

for all $\varphi \in X$.

Theorem 13.23 (Lax–Milgram). *In a Hilbert space X a strictly coercive operator $A : X \rightarrow X$ has a bounded inverse $A^{-1} : X \rightarrow X$.*

Proof: Using the Cauchy–Schwarz inequality, we can estimate

$$\|A\varphi\| \|\varphi\| \geq \operatorname{Re}(A\varphi, \varphi) \geq c\|\varphi\|^2,$$

whence

$$(13.35) \quad \|A\varphi\| \geq c\|\varphi\|$$

for all $\varphi \in X$ follows. Equation (13.35) implies that A is injective since $A\varphi = 0$ now yields $\varphi = 0$.

Next we show that the range $A(X)$ is closed. Let $\psi \in \overline{A(X)}$ and let (ψ_n) be a sequence from $T(X)$ with $\psi_n \rightarrow \psi$, $n \rightarrow \infty$. Then we can write $\psi_n = A\varphi_n$ with some $\varphi_n \in X$ and from (13.35) we find

$$c\|\varphi_n - \varphi_m\| \leq \|\psi_n - \psi_m\|$$

for all $n, m \in \mathbb{N}$. Therefore (φ_n) is a Cauchy sequence in X and converges $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, with some $\varphi \in X$. Then $\psi = A\varphi$ and $A(X) = \overline{A(X)}$ is proven.

Knowing now that $A(X)$ is complete, denote by $P : A \rightarrow A(X)$ the orthogonal projection. Let $f \in X$ be arbitrary. Then by Theorems 1.25, we have $Pf - f \perp A(X)$. In particular, $(Pf - f, A(Pf - f)) = 0$. Hence, from (13.34) we

see that $f = Pf \in A(X)$. Therefore, A is surjective. Finally, the boundedness of the inverse $\|A^{-1}\| \leq 1/c$ is a consequence of (13.35).

Theorem 13.24. *Let $A : X \rightarrow X$ be a strictly coercive operator. Then the Bubnow–Galerkin method converges.*

Proof: As in the previous proof we can estimate

$$\|P_n A\varphi\| \|\varphi\| \geq \operatorname{Re}(P_n A\varphi, \varphi) = \operatorname{Re}(A\varphi, \varphi) \geq c\|\varphi\|^2$$

for all $\varphi \in X$ since orthogonal projections are self adjoint (see Problem 13.1). Hence,

$$(13.36) \quad \|P_n A\varphi\| \geq c\|\varphi\|$$

for all $\varphi \in X_n$. This implies that $A_n = P_n A : X_n \rightarrow X_n$ is injective and consequently surjective. For all $\varphi \in X$, using (13.36) and Theorem 13.3, we can estimate

$$c\|A_n^{-1}P_n A\varphi\| \leq \|P_n A A_n^{-1}P_n A\varphi\| = \|P_n A\varphi\| \leq \|A\| \|\varphi\|.$$

Therefore, $\|A_n^{-1}P_n A\| \leq \|A\|/c$ for all $n \in \mathbb{N}$, and the statement follows from Theorem 13.6. \square

Theorems 13.23 and 13.24 can be extended to strictly coercive operators mapping a Hilbert space X into its dual space X^* (see Problem 13.4).

We conclude our presentation of the Galerkin method by describing one more special case.

Theorem 13.25. *Let X and Y be Hilbert spaces and let $A : X \rightarrow Y$ be an injective bounded linear operator. Let $X_n \subset X$ be a finite dimensional subspace. Then for each $f \in Y$ there exists a unique element $\varphi_n \in X_n$ such that*

$$\|A\varphi_n - f\| = \inf_{\psi \in X_n} \|A\psi - f\|.$$

It is called the least squares solution of $A\varphi = f$ with respect to X_n and coincides with the Petrov–Galerkin solution for the subspaces X_n and $Y_n := A(X_n)$.

Proof: Obviously, φ_n is a least squares solution of $A\varphi = f$ with respect to X_n if and only if $A\varphi_n$ is a best approximation to f with respect to Y_n . By Theorem 1.26, the best approximation exists and is unique. The injectivity of A then implies uniqueness for the least squares solution. By Theorem 1.25, the best approximation $A\varphi_n$ is characterized by the orthogonality condition $(A\varphi_n - f, g) = 0$ for all $g \in Y_n$. But this in turn is equivalent to the Galerkin equation (13.29) for the subspaces X_n and Y_n . \square

Note, that the numerical implementation of the least squares solution in the case of integral equations requires a triple integration for each matrix element and a double integration for each right hand side of the linear system

(13.33) with $g_j = Au_j$, $j = 1, \dots, n$. Therefore, in general, it cannot compete in efficiency with other methods.

We proceed to outline the use of the Galerkin method for integral equations. We start with the Bubnow–Galerkin method for equations of the second kind

$$\varphi(x) - \int_G K(x, y)\varphi(y)dy = f(x), \quad x \in G,$$

with continuous or weakly singular kernel K in the Hilbert space $L^2(G)$. Let $X_n = \text{span}\{u_1, \dots, u_n\}$ be an n -dimensional subspace of $L^2(G)$. Then we write

$$\varphi = \sum_{k=1}^n \gamma_k u_k$$

and the Bubnow–Galerkin equations assume the form

$$(13.37) \quad \begin{aligned} \sum_{k=1}^n \gamma_k \left\{ \int_G u_k(x) \overline{u_j(x)} dx - \int_G \int_G K(x, y) u_k(y) \overline{u_j(x)} dy dx \right\} \\ = \int_G f(x) \overline{u_j(x)} dx, \quad j = 1, \dots, n, \end{aligned}$$

for the coefficients $\gamma_1, \dots, \gamma_n$. For this system, usually the first term in the expression for the matrix elements can be evaluated in closed form, whereas the second term needs a double numerical integration. Before we proceed, we wish to point out that the matrix of the system (13.37) coincides with the system (11.19) obtained by the degenerate kernel method via orthogonal expansion.

Without entering too deeply into the numerical implementation, we just consider as an example the Galerkin method using linear splines. If we pursue the same idea as in the collocation method and use interpolatory quadratures with respect to both variables, it turns out that we solve the same approximate finite dimensional equation on the space of spline functions as in the degenerate kernel and in the collocation method via spline interpolation. Therefore the numerical results will be the same. Only the method of solution for this equation is different: in the collocation method we equate spline functions by requiring them to coincide at the interpolation grid, in the Galerkin method we require equality of scalar products with respect to a basis. In general, we may say that most of the numerical implementations of Galerkin methods somehow may be interpreted as implementations of a related collocation method. Therefore, in the literature this coincidence sometimes is indicated by denoting such procedures as *Galerkin collocation methods*.

For integral equations for periodic functions using trigonometric polynomials we note that instead of setting up the Bubnow–Galerkin method in $L^2[0, 2\pi]$ we also may use the Sobolev spaces $H^s[0, 2\pi]$ with $s \geq 0$. Since the trigonometric monomials are orthogonal in each of these Sobolev spaces, the orthogonal projection onto trigonometric polynomials always is given through truncation of the Fourier series. Therefore, the Bubnow–Galerkin equations are the same for all $s \geq 0$. In particular, this means that we automatically have convergence with respect to all Sobolev norms in the case of analytic kernels and right hand sides.

In principle, for the Galerkin method for equations of the second kind the same remarks as for the collocation method apply. As long as numerical quadratures are available, in general, the Galerkin method cannot compete in efficiency with the Nyström method. As compared with the collocation method, it is less efficient since its matrix elements require double integrations. Therefore, in practical problems, the collocation method is the most widely used numerical method for solving integral equations of the second kind, despite the fact that its error analysis often is less satisfactory than the error analysis for the Galerkin method.

We conclude with an indication of the Petrov–Galerkin method for equations of the form (13.19) and (13.26). In both cases, for simplicity, we assume K to be analytic. For the first equation we assume $f \in L^2[0, 2\pi]$ and as subspaces X_n and Y_n of $L^2[0, 2\pi]$ we choose trigonometric polynomials of the form (13.20) and (13.22), respectively. Since the leading part R maps X_n bijectively onto Y_n , the Petrov–Galerkin equation

$$P_n R \varphi_n = P_n f$$

has a unique solution $\varphi_n \in X_n$ given by $\varphi_n = R^{-1} P_n f$. Therefore, the Petrov–Galerkin method for R converges, and by the stability Theorem 13.7 it also converges for equation (13.19). Using the approximations (13.39) for the numerical quadratures will yield the same equations between the finite dimensional spaces X_n and Y_n as for the collocation method. Hence, we do not have to repeat numerical calculations for equation (13.19) from Example 13.18. The equation (13.26) with $f \in H^1[0, 2\pi]$ is treated analogously, using as subspaces X_n of $L^2[0, 2\pi]$ and Y_n of $H^1[0, 2\pi]$ trigonometric polynomials of the form (13.20). We summarize in

Theorem 13.26. *The Petrov–Galerkin method using trigonometric polynomials converges in the mean square norm for equations of the first kind with Hilbert or logarithmic kernels.*

Note, that the results on convergence in the Sobolev norms mentioned above for equations of the second kind also apply to the equations (13.19) and (13.26).

For equations of the first kind, again the collocation method is more efficient whereas the Galerkin method allows a more satisfactory error analysis.

Problems

13.1. Show that orthogonal projections are self adjoint.

13.2. Let $A : X \rightarrow X$ be a bounded positive selfadjoint operator in a Hilbert space X . Choose $u_0 \in X$ and define $u_j = Au_{j-1}$ for $j = 1, \dots, n-1$. Show that the Bubnow–Galerkin equations for $A\varphi = f$ with respect to the subspaces $X_n = \text{span}\{u_0, \dots, u_{n-1}\}$ are uniquely solvable for each $n \in \mathbb{N}$. Moreover, if f is in the closure of $\text{span}\{A^j u_0 : j = 0, 1, \dots\}$, then the Bubnow–Galerkin approximation φ_n converges to the solution of $A\varphi = f$.

Show that in the special case $u_0 = f$ the approximations φ_n can be computed iteratively by the formulae $\varphi_0 = 0$, $p_0 = f$ and

$$\begin{aligned}\varphi_{n+1} &= \varphi_n - \alpha_n p_n, \\ p_n &= r_n + \beta_{n-1} p_{n-1}, \\ r_n &= r_{n-1} - \alpha_{n-1} A p_{n-1}, \\ \alpha_{n-1} &= (r_{n-1}, p_{n-1}) / (A p_{n-1}, p_{n-1}), \\ \beta_{n-1} &= -(r_n, A p_{n-1}) / (A p_{n-1}, p_{n-1}).\end{aligned}$$

Here r_n is the residual $r_n = A\varphi_n - f$. This is the *conjugate gradient method* of Hestenes and Stiefel [1].

- 13.3. Under the assumptions of Theorem 13.21 let φ_n be the Bubnow–Galerkin solution. Consider the iterated Bubnow–Galerkin solution $\tilde{\varphi}_n$, defined by

$$\tilde{\varphi}_n := A\varphi_n + f$$

and show that

$$\|\tilde{\varphi}_n - \varphi\| \leq c_n \|\varphi_n - \varphi\|$$

where $c_n \rightarrow 0$, $n \rightarrow \infty$. This more rapid convergence of the iterated Bubnow–Galerkin solution is called *superconvergence* (see Sloan [1]).

Hint: Show first that $\tilde{\varphi}_n - AP_n \tilde{\varphi}_n = f$, then that $\|AP_n - A\| \rightarrow 0$, $n \rightarrow \infty$, and finally that $\tilde{\varphi}_n - \varphi = (I - AP_n)^{-1}(AP_n - A)(\varphi - P_n \varphi)$.

- 13.4. A bounded linear operator $A : X \rightarrow X^*$ mapping a Hilbert space X into its dual space X^* is called strictly coercive if there exists a constant $c > 0$ such that

$$\operatorname{Re}(A\varphi)(\varphi) \geq c \|\varphi\|^2$$

for all $\varphi \in X$. Formulate and prove extensions of Theorems 13.23 and 13.24.

- 13.5. Apply the results of Problem 13.4 to the integral equation (13.26) in the Sobolev space $H^{-1/2}[0, 2\pi]$.

14. Iterative Solution and Stability

The approximation methods for integral equations described in Chapters 11 to 13 lead to full linear systems. Only if the number of unknowns is reasonably small may these equations be solved by direct methods like Gaussian elimination. But, in general, a satisfying accuracy of the approximate solution to the integral equation will require a comparatively large number of unknowns, in particular for integral equations in more than one dimension. Therefore iterative methods for the resulting linear systems will be preferable. In this chapter, we will discuss some efficient iterative methods for solving linear systems obtained by discretizing operator equations based on the principal idea of the residual correction. In addition, at the end of this chapter we will briefly enter into the question of stability of the linear systems arising in the discretization of integral equations.

14.1 The Method of Residual Correction

Let X be a Banach space and let $A : X \rightarrow X$ be a bounded linear operator such that $I - A$ has a bounded inverse. As described in the previous chapters we replace the operator equation of the second kind

$$(14.1) \quad \varphi - A\varphi = f$$

by a sequence of approximating equations

$$(14.2) \quad \varphi_n - A_n\varphi_n = f_n$$

leading to finite dimensional linear systems. We assume the sequence (A_n) of bounded linear operators $A_n : X \rightarrow X$ to be either norm convergent or collectively compact and pointwise convergent such that either Theorem 10.1 or Theorem 10.8 may be applied to yield existence and uniqueness for the approximating equation (14.2) for all sufficiently large n . Our analysis includes projection methods in the setting of Theorem 13.9 by writing $A_n = P_n A$ and $f_n = P_n f$.

We note that the index n indicates different *levels* of discretization, i.e., different numbers of quadrature points in the Nyström method, or different dimensions of the approximating subspaces in the degenerate kernel or projection methods. Referring to the quadrature and collocation methods, we also will use the term *grid* instead of level. Deviating from the notation in the preceding chapters, here the number z_n of unknowns in the linear system on the level n will be different from n . Frequently, in particular for one-dimensional

integral equations, the number of unknowns z_n on the level n will duplicate the number of unknowns z_{n-1} on the preceding level $n - 1$ as in all our numerical tables in Chapters 11 to 13.

Assume we already have an approximate solution $\varphi_{n,0}$ of (14.2) with a *residual*

$$r_n := f_n - (I - A_n)\varphi_{n,0}.$$

Then we try to improve on the accuracy by writing

$$(14.3) \quad \varphi_{n,1} = \varphi_{n,0} + \delta_n.$$

In order that $\varphi_{n,1}$ solves equation (14.2) the correction term δ_n has to satisfy the *residual correction* equation

$$\delta_n - A_n\delta_n = r_n.$$

We observe that the correction term δ_n , in general, will be small as compared with $\varphi_{n,0}$, and therefore it is unnecessary to solve the residual correction equation exactly. Hence we write

$$\delta_n = B_n r_n,$$

where the bounded linear operator B_n is some approximation for the inverse $(I - A_n)^{-1}$ of $I - A_n$. Plugging this into (14.3) we obtain

$$\varphi_{n,1} = [I - B_n(I - A_n)]\varphi_{n,0} + B_n f_n$$

as our new approximate solution to (14.2). Repeating this procedure now yields the *defect correction iteration* defined by

$$(14.4) \quad \varphi_{n,i+1} := [I - B_n(I - A_n)]\varphi_{n,i} + B_n f_n, \quad i = 0, 1, 2, \dots,$$

for the solution of (14.2). By Theorem 2.9, the iteration (14.4) converges to the unique solution ψ_n of $B_n(I - A_n)\psi_n = B_n f_n$ provided

$$\|I - B_n(I - A_n)\| < 1,$$

or by Theorem 10.11, if the spectral radius of $I - B_n(I - A_n)$ is less than one. Since the unique solution φ_n of $\varphi_n - A_n\varphi_n = f_n$ trivially satisfies $B_n(I - A_n)\varphi_n = B_n f_n$, we then have convergence of the scheme (14.4) to the unique solution of (14.2). For a rapid convergence it is desirable that the norm or spectral radius is close to zero. For a more complete introduction into this residual correction principle we refer to Stetter [1].

Having in mind that the operators A_n approximate the operator A , an obvious choice for an approximate inverse B_n is given by the correct inverse $B_n = (I - A_m)^{-1}$ of $I - A_m$ for some level $m < n$ corresponding to a coarser discretization. In this case, we can simplify (14.4) by using

$$I - (I - A_m)^{-1}(I - A_n) = (I - A_m)^{-1}(A_n - A_m).$$

Following Brakhage [1] and Atkinson [1], we will consider the two special cases where we use either the preceding level $m = n - 1$ or the coarsest level $m = 0$.

Theorem 14.1. Assume the sequence of operators $A_n : X \rightarrow X$ to be either norm convergent $\|A_n - A\| \rightarrow 0$, $n \rightarrow \infty$, or collectively compact and pointwise convergent $A_n \varphi \rightarrow A\varphi$, $n \rightarrow \infty$, for all $\varphi \in X$. Then the defect correction iteration

$$(14.5) \quad \varphi_{n,i+1} := (I - A_{n-1})^{-1}\{(A_n - A_{n-1})\varphi_{n,i} + f_n\}, \quad i = 0, 1, 2, \dots,$$

using the preceding coarser level converges provided n is sufficiently large. The defect correction iteration

$$(14.6) \quad \varphi_{n,i+1} := (I - A_0)^{-1}\{(A_n - A_0)\varphi_{n,i} + f_n\}, \quad i = 0, 1, 2, \dots,$$

using the coarsest level converges provided the approximation A_0 is already sufficiently close to A .

Proof: Let (A_n) be norm convergent. Then from the uniform estimate (10.1) we observe that $\|(I - A_n)^{-1}\| \leq C$ for all sufficiently large n with some constant C . Then the statement on the scheme (14.5) follows from

$$(14.7) \quad \|(I - A_{n-1})^{-1}(A_n - A_{n-1})\| \leq C\|A_n - A_{n-1}\| \rightarrow 0, \quad n \rightarrow \infty.$$

For the scheme (14.6) we assume that the coarsest grid is chosen such that $\|A_n - A\| < 1/2C$ for all $n \geq 0$. Then $\|(I - A_0)^{-1}(A_n - A_0)\| < 1$ for all $n \in \mathbb{N}$. Note that

$$(14.8) \quad \|(I - A_0)^{-1}(A_n - A_0)\| \rightarrow \|(I - A_0)^{-1}(A - A_0)\| > 0, \quad n \rightarrow \infty.$$

Now assume that the sequence (A_n) is collectively compact and pointwise convergent. Then the limit operator A is also compact and, by Theorem 10.8, the inverse operators $(I - A_n)^{-1}$ exist and are uniformly bounded for sufficiently large n . Hence the sequence (\tilde{A}_n) with

$$\tilde{A}_n := (I - A_{n-1})^{-1}(A_n - A_{n-1})$$

is collectively compact. Since pointwise convergence $A_n \varphi - A_{n-1} \varphi \rightarrow 0$, $n \rightarrow \infty$, holds for all $\varphi \in X$, Theorem 10.6 yields

$$(14.9) \quad \|\{(I - A_{n-1})^{-1}(A_n - A_{n-1})\}^2\| \rightarrow 0, \quad n \rightarrow \infty.$$

Again by Theorem 10.6, we may choose the coarsest grid such that

$$\|(I - A_0)^{-1}(A_m - A)(I - A_0)^{-1}(A_n - A_0)\| < \frac{1}{2}$$

for all $m, n \geq 0$. This implies

$$(14.10) \quad \|\{(I - A_0)^{-1}(A_n - A_0)\}^2\| < 1$$

for all $n \in \mathbb{N}$. Now the statement follows from (14.9) and (14.10) and the variant of Theorem 2.9 discussed in Problem 2.2. \square

Note, that in view of Theorem 12.8, for the Nyström method we cannot expect that $\|(I - A_{n-1})^{-1}(A_n - A_{n-1})\| \rightarrow 0$, $n \rightarrow \infty$, instead of (14.9) or

that $\|(I - A_0)^{-1}(A_n - A_0)\| < 1$ instead of (14.10). Based on this observation, Brakhage [1] suggested a further variant of the defect correction iteration (see Problem 14.1).

Comparing the two variants we observe from (14.7) to (14.10) that, in general, we have a more rapid convergence in the first case. Actually the convergence rate for (14.5) tends to zero when $n \rightarrow \infty$. But for a value judgement we also have to take into account the computational effort for each iteration step. For the actual computation we write the defect correction equation in the form

$$(14.11) \quad \varphi_{n,i+1} - A_m \varphi_{n,i+1} = (A_n - A_m) \varphi_{n,i} + f_n,$$

indicating that, given $\varphi_{n,i}$, we have to solve a linear system for the unknown $\varphi_{n,i+1}$. From this we observe that for the evaluation of the right hand side in (14.11) we need to multiply a $z_n \times z_n$ -matrix with a z_n -vector requiring $O(z_n^2)$ operations. Then solving (14.11) for $\varphi_{n,i+1}$ means directly solving a linear system on the m -th level with z_m unknowns and requires $O(z_m^3)$ operations. Therefore, for the second variant, working with the coarsest grid we may neglect the computational effort for the solution of the linear system and remain with $O(z_n^2)$ operations for each iteration step. However, for the first variant, the effort for the solution of the linear system dominates: each iteration step needs $O(z_{n-1}^3)$ operations. In particular, when the discretization is set up such that the number of unknowns is doubled when we proceed from the level $n - 1$ to the next level n , then the computational effort for each iteration step in (14.5) is roughly 1/8-th of the effort for solving the linear system on level n directly. Hence, we have rapid convergence but each iteration step is still very costly.

We indicate the numerical implementation of the defect correction iteration for the Nyström method using the notations introduced in Chapter 12. Here the approximate operators are given by

$$(A_n \varphi)(x) = \sum_{k=1}^{z_n} \alpha_k^{(n)} K(x, x_k^{(n)}) \varphi(x_k^{(n)}), \quad x \in G.$$

Each iteration step first requires the evaluation of the right hand side

$$g_{n,i} := f_n + (A_n - A_m) \varphi_{n,i},$$

of (14.11) at the z_m quadrature points $x_j^{(m)}$, $j = 1, \dots, z_m$, on the level m and at the z_n quadrature points $x_j^{(n)}$, $j = 1, \dots, z_n$, on the level n . The corresponding computations

$$\begin{aligned} g_{n,i}(x_j^{(m)}) &= f_n(x_j^{(m)}) + \sum_{k=1}^{z_n} \alpha_k^{(n)} K(x_j^{(m)}, x_k^{(n)}) \varphi_{n,i}(x_k^{(n)}) \\ &\quad - \sum_{k=1}^{z_m} \alpha_k^{(m)} K(x_j^{(m)}, x_k^{(m)}) \varphi_{n,i}(x_k^{(m)}), \quad j = 1, \dots, z_m, \end{aligned}$$

and

$$\begin{aligned} g_{n,i}(x_j^{(n)}) &= f_n(x_j^{(n)}) + \sum_{k=1}^{z_n} \alpha_k^{(m)} K(x_j^{(n)}, x_k^{(n)}) \varphi_{n,i}(x_k^{(n)}) \\ &\quad - \sum_{k=1}^{z_m} \alpha_k^{(m)} K(x_j^{(n)}, x_k^{(m)}) \varphi_{n,i}(x_k^{(m)}), \quad j = 1, \dots, z_n, \end{aligned}$$

require $O(z_n^2)$ operations. Then, according to Theorem 12.7, we have to solve the linear system

$$\begin{aligned} \varphi_{n,i+1}(x_j^{(m)}) &- \sum_{k=1}^{z_m} \alpha_k^{(m)} K(x_j^{(m)}, x_k^{(m)}) \varphi_{n,i+1}(x_k^{(m)}) \\ &= g_{n,i}(x_j^{(m)}), \quad j = 1, \dots, z_m, \end{aligned}$$

for the values $\varphi_{n,i+1}(x_j^{(m)})$ at the z_m quadrature points $x_j^{(n)}$, $j = 1, \dots, z_m$. The direct solution of this system needs $O(z_m^3)$ operations. Finally, the values at the z_n quadrature points $x_j^{(n)}$, $j = 1, \dots, z_n$, are obtained from the Nyström interpolation

$$\varphi_{n,i+1}(x_j^{(n)}) = \sum_{k=1}^{z_m} \alpha_k^{(m)} K(x_j^{(n)}, x_k^{(m)}) \varphi_{n,i+1}(x_k^{(m)}) + g_{n,i}(x_j^{(n)}), \quad j = 1, \dots, z_n,$$

requiring $O(z_n z_m)$ operations. All together, in agreement with our previous operation count, we need $O(z_n^2) + O(z_m^3)$ operations. It is left as an exercise to set up the corresponding equations for the degenerate kernel and the collocation method (see Problem 14.2).

14.2 Multi-Grid Methods

Multi-grid methods have been developed as a very efficient iteration scheme for solving the sparse linear systems that arise from finite difference or finite element discretization of elliptic boundary value problems. Following Hackbusch [1] and Schippers [1] we now briefly sketch how multi-grid methods can also be applied for integral equations of the second kind. Whereas the two defect correction schemes described in Theorem 14.1 use only two levels the multi-grid methods use $n + 1$ levels.

Definition 14.2. *The multi-grid iteration is a defect correction iteration of the form (14.4) with the approximate inverses defined recursively by*

$$\begin{aligned} B_0 &:= (I - A_0)^{-1}, \\ (14.12) \quad B_n &:= \sum_{m=0}^{p-1} [I - B_{n-1}(I - A_{n-1})]^m B_{n-1}, \quad n = 1, 2, \dots, \end{aligned}$$

with some $p \in \mathbb{N}$.

Apparently, B_0 stands for the exact solution on the coarsest grid. The approximate inverse B_n for $n + 1$ levels represents the application of p steps of the multi-grid iteration B_{n-1} on n levels for approximately solving the residual correction equation starting with initial element zero, since it is given by the p -th partial sum of the Neumann series. This observation, simultaneously, is the motivation for the definition of the multi-grid method and the basis for its actual recursive numerical implementation. In practical calculations $p = 1, 2$ or 3 are appropriate values for the iteration number. Due to the forms of Figure 14.1, the iteration with $p = 1$ is called a V-cycle, while the iteration with $p = 2$ is called a W-cycle.

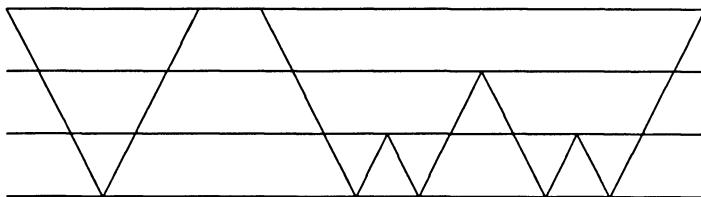


Figure 14.1. One multi-grid step for $p = 1, n = 3$ and $p = 2, n = 3$

It is our aim to illustrate that the multi-grid iteration combines the advantages of the two-grid methods of Theorem 14.1. It has the fast convergence rate of the scheme (14.5) and the computational effort for one step essentially is of the same order as for one step of the scheme (14.6). First we note that the approximate inverses satisfy

$$(14.13) \quad I - B_n(I - A_{n-1}) = [I - B_{n-1}(I - A_{n-1})]^p$$

for all $n \in \mathbb{N}$. For the iteration operator M_n corresponding to one step of the multi-grid iteration using $n + 1$ levels, from Definition 14.2 and from (14.13), we deduce

$$\begin{aligned} M_n &= I - B_n(I - A_n) = I - B_n(I - A_{n-1})(I - A_{n-1})^{-1}(I - A_n) \\ &= I - (I - A_{n-1})^{-1}(I - A_n) + \{I - B_n(I - A_{n-1})\}(I - A_n)^{-1}(I - A_n) \\ &= (I - A_{n-1})^{-1}(A_n - A_{n-1}) + M_{n-1}^p(I - A_{n-1})^{-1}(I - A_n). \end{aligned}$$

After introducing the two-grid iteration operator

$$(14.14) \quad T_n := (I - A_{n-1})^{-1}(A_n - A_{n-1}),$$

we can write the recursion for the iteration operator M_n in the form

$$\begin{aligned} (14.15) \quad M_1 &= T_1, \\ M_{n+1} &= T_{n+1} + M_n^p(I - T_{n+1}), \quad n = 1, 2, \dots \end{aligned}$$

Theorem 14.3. Assume that

$$(14.16) \quad \|T_n\| \leq q^{n-1}C$$

for all $n \in \mathbb{N}$ with some constants $q \in (0, 1]$ and

$$(14.17) \quad C \leq \frac{1}{2q}(\sqrt{1+q^2} - 1).$$

Then, if $p \geq 2$, for all $n \in \mathbb{N}$ there holds

$$\|M_n\| \leq 2q^{n-1}C < 1.$$

Proof: Note that $\sqrt{1+q^2} - 1 < q$ for all $q > 0$. Therefore the statement is correct for $n = 1$. Assume it is true for some $n \in \mathbb{N}$. Then, using the recurrence relation (14.15), it follows that

$$\begin{aligned} \|M_{n+1}\| &\leq \|T_{n+1}\| + \|M_n\|^p(1 + \|T_{n+1}\|) \\ &\leq q^nC + (2q^{n-1}C)^2(1 + q^nC) \\ &\leq q^nC\{1 + \frac{4}{q}C(1 + qC)\} \leq 2q^nC < 1 \end{aligned}$$

since $4t(1+qt) \leq q$ for all $0 \leq t \leq (\sqrt{1+q^2} - 1)/2q$. \square

Setting $q = 1$, we obtain that

$$t := \sup_{n \in \mathbb{N}} \|T_n\| \leq \frac{1}{2}(\sqrt{2} - 1) = 0.207\dots$$

implies

$$\|M_n\| \leq 2t < 1$$

for all $n \in \mathbb{N}$ and all $p \geq 2$. Roughly speaking, this ensures convergence of the multi-grid method provided the approximation on the coarsest level is sufficiently close. More generally, Theorem 14.3 states that the convergence rate of the multi-grid method has the same behaviour as the two-grid method based on the two finest grids. Consider as a typical case an approximation which is of order s in the number z_n of grid points on the n -th level

$$\|A_n - A\| = O\left(\frac{1}{z_n^s}\right)$$

and assume a relation

$$(14.18) \quad z_{n-1} = \zeta z_n$$

between the number of grid points on two consecutive levels with a constant $\zeta < 1$. Then

$$\|A_n - A_{n-1}\| = O(\zeta^{ns}).$$

Hence, in this case, (14.16) is satisfied with $q = \zeta^s$. In one dimension, the usual sequence will duplicate the number of grid points at each level, i.e., $\zeta = 1/2$. In this case for an approximation of second order $s = 2$ we have $q = 1/4$ and (14.17) requires

$$C \leq \frac{1}{2}(\sqrt{17} - 4) = 0.061\dots$$

For a discussion of the computational complexity, let a_n denote the number of operations necessary for the matrix multiplication involved in the application of A_n . In most cases we will have $a_n = az_n^2$ where a is a constant. Denote by b_n the number of operations for one step of the multi-grid iteration with $n+1$ levels. Then, from the recursive definition, we have

$$(14.19) \quad b_n = a_n + p b_{n-1}$$

since each step first requires the evaluation of the residual on the level n and then performs p steps of the multi-grid iteration with n levels. Neglecting the effort for the direct solutions on the coarsest grid, from (14.18) and (14.19) we obtain

$$(14.20) \quad b_n \leq \frac{a}{1-p\zeta^2} z_n^2$$

provided $p\zeta^2 < 1$. In particular, for the canonical case with $p = 2$ and $\zeta = 1/2$ we have $b_n = 2az_n^2$. In general, provided the number of grid points grows fast enough in relation to the iteration number p , then the computational complexity for the multi-grid method is of the same order as for the two-grid method using the finest and the coarsest grid.

Using iterative methods for the approximate solution of $\varphi_n - A_n\varphi_n = f_n$, in general it is useless to make the iteration error $\|\varphi_{n,i} - \varphi_n\|$ smaller than the discretization error $\|\varphi_n - \varphi\|$ to the exact solution of $\varphi - A\varphi = f$. Therefore, the number of iterations should be chosen such that both errors are of the same magnitude. Unfortunately, the quantitative size of the discretization error, in most practical situations, is not known beforehand, only its order of convergence. However, the *nested iteration* or full multi-grid scheme, which we will describe briefly, constructs approximations with iteration errors roughly of the same size as the discretization error. The basic idea is to provide a good initial element $\tilde{\varphi}_{n,0}$ for the multi-grid iteration on $n+1$ levels by multi-grid iterations on coarser levels as described in the following

Definition 14.4. *The full multi-grid scheme constructs a sequence $(\tilde{\varphi}_n)$ of approximations by*

1. initialize by setting $\tilde{\varphi}_0 := (I - A_0)^{-1}f_0$;
2. for $n = 1, 2, \dots$, obtain $\tilde{\varphi}_n$ by performing k steps of the multigrid iteration on $n+1$ levels using the preceding $\tilde{\varphi}_{n-1}$ as initial element.

First, we note that the computational complexity of the full multi-grid method is still of order $O(z_n^2)$. Using (14.19) and (14.20), the total number c_n of operations up to $n+1$ levels can be estimated by

$$c_n = k \sum_{m=1}^n b_m \leq k \frac{a}{1-p\zeta^2} \frac{1}{1-\zeta^2} z_n^2.$$

Theorem 14.5. Assume that the discretization error satisfies

$$(14.21) \quad \|\varphi_n - \varphi\| \leq Cq^n$$

with some $0 < q < 1$ and a positive constant C , and that $t := \sup_{n \in \mathbb{N}} \|M_n\|$ satisfies

$$(14.22) \quad t^k < q.$$

Then for the approximation $\tilde{\varphi}_n$ obtained by the full multi-grid method there holds

$$(14.23) \quad \|\tilde{\varphi}_n - \varphi_n\| \leq C \frac{(q+1)t^k}{q-t^k} q^n.$$

Proof: We set

$$\alpha := \frac{(q+1)t^k}{q-t^k}$$

and note that

$$(\alpha + 1 + q)t^k = q\alpha.$$

Trivially, (14.23) is true for $n = 0$. Assume it is true for some $n \geq 0$. Since $\tilde{\varphi}_{n+1}$ is obtained through k steps of the defect correction iteration (14.4) on the level $n+1$ with $\tilde{\varphi}_n$ as initial element we can write

$$\tilde{\varphi}_{n+1} = M_{n+1}^k \tilde{\varphi}_n + \sum_{m=0}^{k-1} M_{n+1}^m B_{n+1} f_{n+1}.$$

From $\varphi_{n+1} - A_{n+1}\varphi_{n+1} = f_{n+1}$ we deduce

$$B_{n+1} f_{n+1} = B_{n+1}(I - A_{n+1})\varphi_{n+1} = \varphi_{n+1} - M_{n+1}\varphi_{n+1}$$

and can insert this into the previous equation to arrive at

$$\tilde{\varphi}_{n+1} - \varphi_{n+1} = M_{n+1}^k (\tilde{\varphi}_n - \varphi_{n+1}).$$

Then we can estimate

$$\begin{aligned} \|\tilde{\varphi}_{n+1} - \varphi_{n+1}\| &= \|M_{n+1}^k (\tilde{\varphi}_n - \varphi_{n+1})\| \leq t^k \|\tilde{\varphi}_n - \varphi_{n+1}\| \\ &\leq t^k (\|\tilde{\varphi}_n - \varphi_n\| + \|\varphi_n - \varphi\| + \|\varphi - \varphi_{n+1}\|) \\ &\leq t^k C(\alpha + 1 + q) q^n = \alpha C q^{n+1}, \end{aligned}$$

and the proof is complete. \square

Now, indeed, Theorem 14.5 may be interpreted to mean that the iteration error, even for $k = 1$, is of the same size as the discretization error, provided the approximation on the coarsest level is sufficiently close to ensure (14.22) by Theorem 14.3. In the typical situation where $s = 2$ and $\zeta = 1/2$, an estimate of the form (14.21) is satisfied with $q = 1/4$.

In connection with the two-grid iterations of Theorem 14.1 we observed already that for the Nyström method we cannot expect $\|T_n\| \rightarrow 0$, $n \rightarrow \infty$. Therefore, in view of Theorem 14.3, modifications of the multi-grid scheme are necessary for the quadrature method. The multi-grid iteration proposed by Hemker and Schippers [1] replaces (14.12) by

$$\tilde{B}_0 := (I - A_0)^{-1}$$

$$\tilde{B}_n := \sum_{m=0}^{p-1} [I - \tilde{B}_{n-1}(I - A_{n-1})]^m \tilde{B}_{n-1}(I - A_{n-1} + A_n), \quad n = 1, 2, \dots$$

This means that at each level a kind of smoothing operation $I - A_{n-1} + A_n$ is included before the application of p steps of the multi-grid iteration at the preceding level. Here we write

$$\tilde{B}_n = Q_n(I - A_{n-1} + A_n)$$

and as above we have the property

$$I - Q_n(I - A_{n-1}) = [I - \tilde{B}_{n-1}(I - A_{n-1})]^p.$$

Proceeding as in the derivation of (14.15), it can be shown that the iteration operators satisfy the recurrence relation

$$\tilde{M}_1 = \tilde{T}_1,$$

$$\tilde{M}_{n+1} = \tilde{T}_{n+1} + \tilde{M}_n^p(I - \tilde{T}_{n+1}), \quad n = 1, 2, \dots,$$

where the two-grid iteration operator \tilde{T}_n is defined by

$$\tilde{T}_n := (I - A_{n-1})^{-1}(A_n - A_{n-1})A_n, \quad n = 1, 2, \dots$$

Now Theorem 14.3 can be applied with the help of Corollary 10.7.

For further variants and a detailed study of multi-grid methods for equations of the second kind including numerical examples we refer the reader to Hackbusch [1].

14.3 Stability of Linear Systems

We will now introduce a further important subject in connection with the actual numerical solution of linear systems arising from the discretization of linear operator equations. Usually, in the course of any numerical procedure for solving linear systems errors will be introduced. For example, in the collocation and Galerkin method errors occur through the numerical approximations of the matrix elements and the right hand side of the linear systems. We will call a linear system to be stable if small changes in the data of the system cause only small changes in the solution. We will make this notation more precise by introducing the concept of a condition number.

Definition 14.6. Let X and Y be Banach spaces and let $A : X \rightarrow Y$ be a bounded linear operator with bounded inverse $A^{-1} : Y \rightarrow X$. Then the condition number of A is defined as

$$(14.24) \quad \operatorname{cond} A := \|A^{-1}\| \|A\|.$$

Note that the condition number always satisfies

$$\operatorname{cond} A = \|A^{-1}\| \|A\| \geq \|A^{-1}A\| = 1.$$

The following theorem shows that it may serve as a measure for stability.

Theorem 14.7. Let A be as in Definition 14.6 and let the small perturbation $\delta A : X \rightarrow Y$ satisfy

$$\|\delta A\| < \frac{1}{\|A^{-1}\|}.$$

Let φ and $\varphi + \delta\varphi$ be the unique solutions of the equations

$$(14.25) \quad A\varphi = f$$

and

$$(14.26) \quad (A + \delta A)(\varphi + \delta\varphi) = f + \delta f,$$

respectively. Then

$$(14.27) \quad \frac{\|\delta\varphi\|}{\|\varphi\|} \leq \frac{\operatorname{cond} A}{1 - \operatorname{cond} A} \frac{\|\delta A\|}{\|A\|} \left(\frac{\|\delta f\|}{\|f\|} + \frac{\|\delta A\|}{\|A\|} \right).$$

Proof: By Theorems 2.9 and 10.1, the inequality $\|\delta A\| < 1/\|A^{-1}\|$ ensures the existence of the inverse operators $(I + A^{-1}\delta A)^{-1}$ and $(A + \delta A)^{-1}$ and the estimate

$$\|(I + A^{-1}\delta A)^{-1}\| \leq \frac{1}{1 - \|A^{-1}\|\|\delta A\|}.$$

We apply A^{-1} to both equations (14.25) and (14.26) and then combine them to obtain

$$(I + A^{-1}\delta A)(\varphi + \delta\varphi) = \varphi + A^{-1}\delta f.$$

Solving for $\delta\varphi$ yields

$$\delta\varphi = (I + A^{-1}\delta A)^{-1}A^{-1}(\delta f - \delta A\varphi).$$

Taking the norm of both sides and dividing by $\|\varphi\|$ we obtain

$$\frac{\|\delta\varphi\|}{\|\varphi\|} \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|\delta A\|} \left(\frac{\|\delta f\|}{\|\varphi\|} + \|\delta A\| \right).$$

Since $\|\varphi\| \geq \|f\|/\|A\|$ the estimate (14.27) now follows. \square

For finite dimensional approximations of a given operator equation we have to distinguish three condition numbers: namely, the condition numbers of the original operator and of the approximating operator as mappings from the Banach space X into the Banach space Y , and the condition number of the finite dimensional system which has to be set up for the actual numerical solution. This latter system we can influence, for example in the collocation and Galerkin methods by the choice of the basis for the approximating subspaces.

Consider an equation of the second kind $\varphi - A\varphi = f$ in a Banach space X and approximating equations $\varphi_n - A_n\varphi_n = f_n$ under the assumptions of Theorem 10.1, i.e., norm convergence, or Theorem 10.8, i.e., collective compactness and pointwise convergence. Then, by Theorems 10.1, 10.3 and 10.8, there exists a constant M such that

$$\text{cond}(I - A_n) \leq M \text{ cond}(I - A)$$

for all sufficiently large n . Hence, for the condition of the approximating scheme, we mainly have to be concerned with the condition of the linear system for the actual computation of the solution of $\varphi_n - A_n\varphi_n = f_n$.

For the Nyström method recall the notations of Section 12.2. We have to relate the condition number for the numerical quadrature operator

$$(A_n\varphi)(x) := \sum_{k=1}^n \alpha_k K(x, x_k) \varphi(x_k), \quad x \in G,$$

in the Banach space $C(G)$ to the condition number for the matrix operator $\tilde{A}_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (or $\tilde{A}_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$), given by

$$(\tilde{A}_n \Phi)_j := \sum_{k=1}^n \alpha_k K(x_j, x_k) \Phi_k, \quad j = 1, \dots, n.$$

For convenience, here we come back to assuming that the index n coincides with the number of quadrature points. For $F \in \mathbb{R}^n$ choose a function $f \in C(G)$ with $f(x_j) = F_j$, $j = 1, \dots, n$, and $\|f\|_\infty = \|F\|_\infty$. By Theorem 12.7, the solutions of the approximating equation $\varphi_n - A_n\varphi_n = f$ in $C(G)$ and of the linear system $\Phi_n - \tilde{A}_n \Phi_n = F$ in \mathbb{R}^n are related by $\varphi_n(x_j) = \Phi_{n,j}$, $j = 1, \dots, n$. Therefore,

$$\begin{aligned} \|(I - \tilde{A}_n)^{-1} F\|_\infty &= \|\Phi_n\|_\infty \leq \|\varphi_n\|_\infty \leq \|(I - A_n)^{-1}\|_\infty \|f\|_\infty \\ &= \|(I - A_n)^{-1}\|_\infty \|F\|_\infty, \end{aligned}$$

whence $\|(I - \tilde{A}_n)^{-1}\|_\infty \leq \|(I - A_n)^{-1}\|_\infty$. This, together with (see Problem 14.4)

$$\begin{aligned} \|I - \tilde{A}_n\|_\infty &\leq 1 + \max_{1 \leq j \leq n} \sum_{k=1}^n |\alpha_k K(x_j, x_k)| \\ (14.28) \quad &\leq 1 + \max_{x \in G} \sum_{k=1}^n |\alpha_k K(x, x_k)| = \|I - A_n\|_\infty, \end{aligned}$$

implies the following

Theorem 14.8. *For the Nyström method the condition number for the linear system satisfies*

$$\operatorname{cond}(I - \tilde{A}_n) \leq M \operatorname{cond}(I - A)$$

for all sufficiently large n with some constant M .

This theorem states that the quadrature method essentially preserves the condition number of the original integral equation.

For the collocation method recall the notations of Section 13.2. Here, we introduce matrix operators $E_n, \tilde{A}_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (or from \mathbb{C}^n into \mathbb{C}^n) by

$$(E_n \gamma)_j := \sum_{k=1}^n \gamma_k u_k(x_j)$$

and

$$(\tilde{A}_n \gamma)_j := \sum_{k=1}^n \gamma_k (Au_k)(x_j)$$

for $j = 1, \dots, n$. Since $Y_n = \operatorname{span}\{u_1, \dots, u_n\}$ is assumed to be unisolvant with respect to the collocation points x_1, \dots, x_n , the matrix operator E_n is invertible. For any $F \in \mathbb{R}^n$ we choose a function $f \in C(G)$ with $f(x_j) = F_j$, $j = 1, \dots, n$, and $\|f\|_\infty = \|F\|_\infty$. Then, by (13.12) and (13.14), for a vector $\gamma \in \mathbb{R}^n$ satisfying the linear system

$$(14.29) \quad (E_n - \tilde{A}_n)\gamma = F$$

the function $\varphi_n \in Y_n$ given by

$$\varphi_n = \sum_{k=1}^n \gamma_k u_k$$

solves the equation

$$(14.30) \quad \varphi_n - P_n A \varphi_n = P_n f.$$

Conversely, let $\varphi_n \in Y_n$ be a solution of (14.30). Then the vector

$$\gamma = E_n^{-1} \Phi_n$$

where $\Phi_{n,j} = \varphi_n(x_j)$, $j = 1, \dots, n$, solves (14.29). We may interprete φ_n as the interpolation of a continuous function assuming the values $\Phi_{n,j}$ at the collocation points x_j . Therefore

$$\|\varphi_n\|_\infty \leq \|P_n\|_\infty \|\Phi_n\|_\infty.$$

Hence we can estimate

$$\begin{aligned} & \| (E_n - \tilde{A}_n)\gamma \|_\infty = \|F\|_\infty = \|f\|_\infty \leq \|P_n f\|_\infty = \|P_n(I - A)\varphi_n\|_\infty \\ & \leq \|P_n\|_\infty^2 \|I - A\|_\infty \|\Phi_n\|_\infty \leq \|P_n\|^2 \|I - A\|_\infty \|E_n\|_\infty \|\gamma\|_\infty \end{aligned}$$

and, using Theorem 13.6 for the operator $I - A$,

$$\begin{aligned} & \| (E_n - \tilde{A}_n)^{-1} F \|_\infty = \|\gamma\|_\infty \leq \|E_n^{-1}\|_\infty \|\varPhi_n\|_\infty \\ & \leq \|E_n^{-1}\|_\infty \|\varphi_n\|_\infty = \|E_n^{-1}\|_\infty \|[P_n(I - A)]^{-1} P_n f\|_\infty \\ & \leq \|E_n^{-1}\|_\infty \|[P_n(I - A)]^{-1} P_n(I - A)(I - A)^{-1} f\|_\infty \\ & \leq M \|E_n^{-1}\|_\infty \|(I - A)^{-1}\|_\infty \|f\|_\infty = M \|E_n^{-1}\|_\infty \|(I - A)^{-1}\|_\infty \|F\|_\infty \end{aligned}$$

with some constant M . Thus we have proven

Theorem 14.9. *For the collocation method the condition number of the linear system satisfies*

$$\text{cond}(E_n - \tilde{A}_n) \leq M \|P_n\|_\infty^2 \text{cond } E_n \text{cond}(I - A)$$

for all sufficiently large n with some constant M .

This theorem suggests that the basis functions must be chosen with caution. For a poor choice, like monomials, the condition number of E_n can grow quite rapidly. However, for the Lagrange basis E_n is the identity matrix with condition number one. In addition, $\|P_n\|$ enters in the estimate on the condition number of the linear system, and, for example, for polynomial or trigonometric polynomial interpolation we have $\|P_n\| \rightarrow \infty$, $n \rightarrow \infty$.

A discussion of the condition number for the Galerkin method can be carried out quite similarly (see Problem 14.5).

Summarizing, we can state that the Nyström method is generically stable whereas the collocation and Galerkin methods may suffer from instabilities due to a poor choice of the basis for the approximating subspaces.

The stability of equations of the first kind will be a subject of the remaining part of the book.

Problems

14.1. For the Nyström method Brakhage [1] suggested to use

$$B_n = I + (I - A_m)^{-1} A_n$$

as an approximate inverse in the defect correction method (compare also the proof of Theorem 10.8). Prove a variant of Theorem 14.1 for this case.

14.2. Set up the equations for one step of the two-grid iteration for the collocation method.

14.3. Formulate and prove an analog of Theorem 14.3 for the case of exponential convergence $\|T_n\| = O(\exp(-sz_{n-1}))$ with some $s > 0$.

14.4. Prove inequality (14.28).

- 14.5. For the condition number of the Bubnow–Galerkin method for an equation of the second kind define matrix operators $E_n, \tilde{A}_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(E_n \gamma)_j := \sum_{k=1}^n (u_j, u_k) \gamma_k$$

and

$$(\tilde{A}_n \gamma)_j := \sum_{k=1}^n (u_j, A u_k) \gamma_k$$

for $j = 1, \dots, n$. Proceeding as in Theorem 14.9, show that in the Euclidean norm the condition number of the linear system satisfies

$$\text{cond}(E_n - \tilde{A}_n) \leq M \text{ cond } E_n \text{ cond } (I - A)$$

for all sufficiently large n with some constant M .

15. Equations of the First Kind

Compact operators cannot have a bounded inverse. Therefore, equations of the first kind with a compact operator provide a typical example for so called ill-posed problems. In this chapter we introduce methods for the stable solution of equations of the first kind in a Hilbert space setting.

15.1 Ill-Posed Problems

For problems in mathematical physics, in particular for initial and boundary value problems, Hadamard [1] postulated three properties:

1. Existence of a solution.
2. Uniqueness of the solution.
3. Continuous dependence of the solution on the data.

The third postulate is motivated by the fact that in all applications the data will be measured quantities. Therefore, one wants to make sure that small errors in the data will cause only small errors in the solution. A problem satisfying all three requirements is called well-posed. The potential theoretic boundary value problems of Chapter 6 and the initial boundary value problem for the heat equation of Chapter 9 are examples for well-posed problems. We will make Hadamard's concept of well-posedness more precise through the following

Definition 15.1. *Let $A : X \rightarrow Y$ be an operator from a normed space X into a normed space Y . The equation*

$$(15.1) \quad A\varphi = f$$

is called well-posed or properly posed if A is bijective and the inverse operator $A^{-1} : Y \rightarrow X$ is continuous. Otherwise the equation is called ill-posed or improperly posed.

According to this definition we may distinguish three types of ill-posedness. If A is not surjective then equation (15.1) is not solvable for all $f \in Y$ (*nonexistence*). If A is not injective then equation (15.1) may have more than one solution (*nonuniqueness*). Finally, if A^{-1} exists but is not continuous then the solution φ of equation (15.1) does not depend continuously on the data f (*instability*). The latter case of instability is the one of primary interest in the study of ill-posed problems. We note that the three properties, in general, are not independent. For example, if $A : X \rightarrow Y$ is a bounded linear operator mapping a Banach space X bijectively onto a Banach space Y , then by the

inverse mapping theorem (see Jörgens [1]) the inverse operator $A^{-1} : Y \rightarrow X$ is bounded and therefore continuous. For a long time the research on improperly posed problems was neglected since they were not considered relevant to the proper treatment of applied problems.

Note that the well-posedness of a problem is a property of the operator A together with the solution space X and the data space Y including the norms on X and Y . Therefore, if an equation is ill-posed one could try and restore stability by changing the spaces X and Y and their norms. But, in general, this approach is inadequate since the spaces X and Y including their norms are determined by practical needs. In particular, the space Y and its norm must be suitable to describe the measured data.

Example 15.2. A classical example for an ill-posed problem given by Hadamard is an initial value problem for the Laplace equation: Find a harmonic function u in $\mathbb{R} \times [0, \infty)$ satisfying the initial conditions

$$u(\cdot, 0) = 0, \quad \frac{\partial}{\partial y} u(\cdot, 0) = f,$$

where f is a given continuous function. If we choose as data

$$f_n(x) = \frac{1}{n} \sin nx, \quad x \in \mathbb{R},$$

for $n \in \mathbb{N}$ then we obtain the solution

$$u_n(x, y) = \frac{1}{n^2} \sin nx \sinh ny, \quad (x, y) \in \mathbb{R} \times [0, \infty).$$

Obviously, the data sequence (f_n) converges uniformly to zero, whereas the solution sequence (u_n) does not converge in any reasonable norm. Therefore, the solution to the initial value problem for harmonic functions is ill-posed. But a number of applications lead to such an initial value problem, for example the extrapolation of measured data for the gravity potential from parts of the surface of the earth to the space above or below this surface.

Example 15.3. A second typical example for an ill-posed problem is the backwards heat conduction. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

in a rectangle $[0, \pi] \times [0, T]$ subject to the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad 0 \leq t \leq T,$$

and the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq \pi,$$

where φ is a given function. For existence and uniqueness we refer back to Chapter 9. Here, the solution can be obtained by separation of variables in the form

$$(15.2) \quad u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx$$

with the Fourier coefficients

$$(15.3) \quad a_n = \frac{2}{\pi} \int_0^\pi \varphi(y) \sin ny \, dy$$

of the given initial values. This initial value problem is well-posed: The final temperature $f = u(\cdot, T)$ depends continuously on the initial temperature. For example in an L^2 -setting, using Parseval's equality, we have

$$\|f\|_2^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2 e^{-2n^2 T} \leq \frac{\pi}{2} e^{-2T} \sum_{n=1}^{\infty} a_n^2 = e^{-2T} \|\varphi\|_2^2.$$

However, the inverse problem, i.e., the determination of the initial temperature φ from the knowledge of the final temperature f is ill-posed. Here, we can write

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{n^2(T-t)} \sin nx$$

with the Fourier coefficients

$$b_n = \frac{2}{\pi} \int_0^\pi f(y) \sin ny \, dy$$

of the given final temperature. Then we have

$$\|\varphi\|_2^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2 e^{2n^2 T}$$

and there exists no positive constant C such that

$$\|\varphi\|_2^2 \leq C \|f\|_2^2 = C \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2.$$

We may interpret this inverse problem as an initial value problem for the backwards heat equation where t is replaced by $-t$. Then the ill-posedness reflects the fact that heat conduction is an irreversible physical process. \square

Plugging (15.3) into (15.2), we see that the last example can be put into the form of an integral equation of the first kind

$$\int_0^\pi K(x, y) \varphi(y) dy = f(x), \quad 0 \leq x \leq \pi,$$

where the kernel is given by

$$K(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 T} \sin nx \sin ny$$

for $0 \leq x, y \leq \pi$. In general, linear integral equations of the first kind with continuous or weakly singular kernels provide typical examples for ill-posed problems with respect to both the maximum and the mean square norm. They are special cases of the following

Theorem 15.4. *Let X and Y be normed spaces and let $A : X \rightarrow Y$ be a compact linear operator. Then the equation of the first kind $A\varphi = f$ is improperly posed if X is not of finite dimension.*

Proof: In view of Theorems 2.15 and 2.19 the compact operator A cannot have a continuous inverse. \square

The ill-posed nature of an equation, of course, has consequences for its numerical treatment. We may view a numerical approximation of a given equation as the solution to perturbed data. Therefore, straightforward application of the methods developed in the previous chapters to ill-posed equations of the first kind usually will generate numerical nonsense. In terms of the concepts introduced in Section 14.3, the fact that an operator A does not have a bounded inverse means that the condition numbers of its finite dimensional approximations grow with the quality of the approximation. Hence, a careless discretization of ill-posed problems leads to a numerical behaviour which only at a first glance seems to be paradoxical. Namely, increasing the degree of discretization, i.e., increasing the accuracy of the approximation for the operator A will cause the approximate solution to the equation $A\varphi = f$ to become less and less reliable.

15.2 Regularization of Ill-Posed Problems

Methods for a stable approximate solution of ill-posed problems are called *regularization methods*. Note that in the context of improperly posed problems the term regularization has a different meaning than in the theory of singular equations as described in Chapter 5. It is our aim to introduce a few of the more classical regularization concepts for linear equations of the first kind.

In the sequel, we mostly will assume that the linear operator A is injective. This is not a principal loss of generality since uniqueness for a linear equation always can be achieved by a suitable modification of the solution space X . We wish to approximate the solution φ to the equation $A\varphi = f$ from the knowledge of a perturbed right hand side f^δ with a known error level

$$(15.4) \quad \|f^\delta - f\| \leq \delta.$$

When f belongs to the range $A(X)$ then there exists a unique solution φ of $A\varphi = f$. For the perturbed right hand side, in general, we cannot expect $f^\delta \in A(X)$. Using the erroneous data f^δ we want to construct a reasonable approximation φ^δ to the exact solution φ of the unperturbed equation $A\varphi = f$. Of course, we want this approximation to be stable, i.e., we want φ^δ to depend continuously on the actual data f^δ . Therefore our task requires finding an approximation of the unbounded inverse operator $A^{-1} : A(X) \rightarrow X$ by a bounded linear operator $R : Y \rightarrow X$.

Definition 15.5. *Let X and Y be normed spaces and let $A : X \rightarrow Y$ be an injective bounded linear operator. A regularization scheme, in general, consists*

of a family of bounded linear operators $R_\alpha : Y \rightarrow X$, $\alpha > 0$, with the property of pointwise convergence

$$(15.5) \quad \lim_{\alpha \rightarrow 0} R_\alpha A \varphi = \varphi$$

for all $\varphi \in X$. The parameter α is called the regularization parameter.

Of course, (15.5) is equivalent to $R_\alpha f \rightarrow A^{-1}f$, $\alpha \rightarrow 0$, for all $f \in A(X)$. Occasionally, we will use regularization parameter sets different from the positive real numbers.

The following theorem illustrates properties which cannot be fulfilled by regularization schemes for compact operators.

Theorem 15.6. Let X and Y be normed spaces, let $A : X \rightarrow Y$ be a compact linear operator, and let $\dim X = \infty$. Then for a regularization scheme the operators R_α cannot be uniformly bounded with respect to α , and the operators $R_\alpha A$ cannot be norm convergent as $\alpha \rightarrow 0$.

Proof: For the first statement, assume $\|R_\alpha\| \leq C$ for all $\alpha > 0$ with some constant C . Then from $R_\alpha f \rightarrow A^{-1}f$, $\alpha \rightarrow 0$, for all $f \in A(X)$ we deduce $\|A^{-1}f\| \leq C\|f\|$, i.e., $A^{-1} : A(X) \rightarrow X$ is bounded. By Theorem 15.4 this is a contradiction to $\dim X = \infty$.

For the second statement, assume that we have norm convergence. Then there exists $\alpha > 0$ such that $\|R_\alpha A - I\| < 1/2$. Now for all $f \in A(X)$ we can estimate

$$\|A^{-1}f\| \leq \|A^{-1}f - R_\alpha A A^{-1}f\| + \|R_\alpha f\| \leq \frac{1}{2} \|A^{-1}f\| + \|R_\alpha\| \|f\|,$$

whence $\|A^{-1}f\| \leq 2\|R_\alpha\| \|f\|$ follows. Therefore $A^{-1} : A(X) \rightarrow X$ is bounded and we have the same contradiction as above. \square

The regularization scheme approximates the solution φ of $A\varphi = f$ by the regularized solution

$$(15.6) \quad \varphi_\alpha^\delta := R_\alpha f^\delta.$$

Then, for the approximation error, writing

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A \varphi - \varphi,$$

by the triangle inequality we have

$$(15.7) \quad \|\varphi_\alpha^\delta - \varphi\| \leq \delta \|R_\alpha\| + \|R_\alpha A \varphi - \varphi\|.$$

This estimate illustrates that the error consists of two parts: the first term reflects the influence of the incorrect data and the second term is due to the approximation error between R_α and A^{-1} . Under the assumptions of Theorem 15.6 the first term cannot be estimated uniformly with respect to α and the second term cannot be estimated uniformly with respect to φ . Typically, the

first term will be increasing as $\alpha \rightarrow 0$ due to the ill-posed nature of the problem whereas the second term will be decreasing as $\alpha \rightarrow 0$ according to (15.5). Every regularization scheme requires a strategy for choosing the parameter α in dependence on the error level δ in order to achieve an acceptable total error for the regularized solution. On one hand, the accuracy of the approximation asks for a small error $\|R_\alpha A\varphi - \varphi\|$, i.e., for a small parameter α . On the other hand, the stability requires a small $\|R_\alpha\|$, i.e., a large parameter α . An optimal choice would try and make the right hand side of (15.7) minimal. The corresponding parameter then kind of compromises between accuracy and stability. From a reasonable regularization strategy we expect the property that the regularized solution converges to the exact solution when the error level tends to zero. We express this requirement through the following

Definition 15.7. *A strategy for a regularization scheme R_α , $\alpha > 0$, that is, the choice of the regularization parameter $\alpha = \alpha(\delta)$ depending on the error level δ , is called regular if for all $f \in A(X)$ and all $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$ there holds*

$$R_{\alpha(\delta)} f^\delta \rightarrow A^{-1} f, \quad \delta \rightarrow 0.$$

We will describe some regularization schemes including regular strategies in a Hilbert space setting. Our approach will be based on the singular value decomposition.

15.3 Compact Self Adjoint Operators

Throughout the remainder of this chapter X and Y will always denote Hilbert spaces. From Theorem 4.9 recall that every bounded linear operator $A : X \rightarrow Y$ possesses an adjoint operator $A^* : Y \rightarrow X$, that is,

$$(A\varphi, \psi) = (\varphi, A^*\psi)$$

for all $\varphi \in X$ and $\psi \in Y$. For the norms there holds $\|A\| = \|A^*\|$.

Theorem 15.8. *For a bounded linear operator there holds*

$$A(X)^\perp = N(A^*) \quad \text{and} \quad N(A^*)^\perp = \overline{A(X)}.$$

Proof: $g \in A(X)^\perp$ means $(A\varphi, g) = 0$ for all $\varphi \in X$. This is equivalent to $(\varphi, A^*g) = 0$ for all $\varphi \in X$, which in turn is equivalent to $A^*g = 0$, that is, $g \in N(A^*)$. Hence, $A(X)^\perp = N(A^*)$. We abbreviate $U = A(X)$ and, trivially, have $\bar{U} \subset (U^\perp)^\perp$. Denote by $P : X \rightarrow \bar{U}$ the orthogonal projection. For arbitrary $\varphi \in (U^\perp)^\perp$, by Theorem 1.25, there holds orthogonality $P\varphi - \varphi \perp U$. But we also have $P\varphi - \varphi \perp U^\perp$ since we already know that $\bar{U} \subset (U^\perp)^\perp$. Therefore it follows that $\varphi = P\varphi \in \bar{U}$, whence $\bar{U} = (U^\perp)^\perp$, i.e., $\overline{A(X)} = N(A^*)^\perp$. \square

An operator $A : X \rightarrow X$ mapping a Hilbert space X into itself is called *self adjoint* if $A = A^*$, that is, if

$$(A\varphi, \psi) = (\varphi, A\psi)$$

for all $\varphi, \psi \in X$. Note, that for a self adjoint operator the scalar product $(A\varphi, \varphi)$ is real valued for all $\varphi \in X$ since $(\overline{A\varphi}, \varphi) = (\varphi, A\varphi) = (A\varphi, \varphi)$.

Theorem 15.9. *For a bounded self adjoint operator there holds*

$$(15.8) \quad \|A\| = \sup_{\|\varphi\|=1} |(A\varphi, \varphi)|.$$

Proof: We abbreviate the right hand side of (15.8) by q . By the Cauchy–Schwarz inequality, we find

$$|(A\varphi, \varphi)| \leq \|A\varphi\| \|\varphi\| \leq \|A\|$$

for all $\|\varphi\| = 1$, whence $q \leq \|A\|$ follows. On the other hand, for all $\varphi, \psi \in X$ there holds

$$(A(\varphi + \psi), \varphi + \psi) - (A(\varphi - \psi), \varphi - \psi) = 2\{(A\varphi, \psi) + (A\psi, \varphi)\} = 4 \operatorname{Re}(A\varphi, \psi).$$

Therefore we can estimate

$$4 \operatorname{Re}(A\varphi, \psi) \leq q\{\|\varphi + \psi\|^2 + \|\varphi - \psi\|^2\} = 2q\{\|\varphi\|^2 + \|\psi\|^2\}.$$

Now let $\|\varphi\| = 1$ and $A\varphi \neq 0$. Then choose $\psi = A\varphi/\|A\varphi\|$ to obtain

$$\|A\varphi\| = \operatorname{Re}(A\varphi, \psi) \leq q.$$

This implies $\|A\| \leq q$ and completes the proof. \square

Recall the spectral theoretic notions of Definition 3.10.

Theorem 15.10. *All eigenvalues of a self adjoint operator are real and eigenelements to different eigenvalues are orthogonal.*

Proof: $A\varphi = \lambda\varphi$ and $\varphi \neq 0$ imply $\lambda(\varphi, \varphi) = (A\varphi, \varphi) \in \mathbb{R}$, whence $\lambda \in \mathbb{R}$. Let $A\varphi = \lambda\varphi$ and $A\psi = \mu\psi$ with $\lambda \neq \mu$. Then $(\varphi, \psi) = 0$ follows from

$$(\lambda - \mu)(\varphi, \psi) = (A\varphi, \psi) - (\varphi, A\psi) = 0.$$

Theorem 15.11. *For the spectral radius of a bounded self adjoint operator A there holds*

$$(15.9) \quad r(A) = \|A\|.$$

If A is compact then there exists at least one eigenvalue with $|\lambda| = \|A\|$.

Proof: For each bounded linear operator there holds $r(A) \leq \|A\|$ since all $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ are regular by the Neumann series Theorem 2.8. By Theorem 15.9 there exists a sequence (φ_n) in X with $\|\varphi_n\| = 1$ such that

$$|(A\varphi_n, \varphi_n)| \rightarrow \|A\|, \quad n \rightarrow \infty.$$

We may assume that

$$(A\varphi_n, \varphi_n) \rightarrow \lambda, \quad n \rightarrow \infty,$$

where λ is real and $|\lambda| = \|A\|$. Then

$$\begin{aligned} 0 &\leq \|A\varphi_n - \lambda\varphi_n\|^2 = \|A\varphi_n\|^2 - 2\lambda(A\varphi_n, \varphi_n) + \lambda^2\|\varphi_n\|^2 \\ &\leq \|A\|^2 - 2\lambda(A\varphi_n, \varphi_n) + \lambda^2 = 2\lambda\{\lambda - (A\varphi_n, \varphi_n)\} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Therefore

$$(15.10) \quad A\varphi_n - \lambda\varphi_n \rightarrow 0, \quad n \rightarrow \infty.$$

This implies that λ is not a regular value because otherwise we would have the contradiction

$$1 = \|\varphi_n\| = \|(\lambda I - A)^{-1}(\lambda\varphi_n - A\varphi_n)\| \rightarrow 0, \quad n \rightarrow \infty.$$

Hence, $r(A) \geq |\lambda| = \|A\|$.

If A is compact, then the bounded sequence (φ_n) contains a subsequence $(\varphi_{n(k)})$ such that $A\varphi_{n(k)} \rightarrow \psi$, $k \rightarrow \infty$, with some $\psi \in X$. We may assume that $A \neq 0$ since for $A = 0$ the statement of the theorem is trivial. Then, from (15.10) it follows that $\varphi_{n(k)} \rightarrow \varphi$, $k \rightarrow \infty$, with some $\varphi \in X$, and $\|\varphi_{n(k)}\| = 1$ for all k implies $\|\varphi\| = 1$. Finally, again from (15.10), by the continuity of A , we obtain $A\varphi = \lambda\varphi$ and the proof is finished. \square

Now we are ready to summarize our results into the following *spectral theorem* for self adjoint compact operators.

Theorem 15.12. *Let X be a Hilbert space and let $A : X \rightarrow X$ be a self adjoint compact operator (with $A \neq 0$). Then all eigenvalues of A are real. A has at least one eigenvalue different from zero and at most a countable set of eigenvalues accumulating only at zero. All eigenspaces $N(\lambda I - A)$ for nonzero eigenvalues λ have finite dimension and eigenspaces to different eigenvalues are orthogonal. Assume the sequence (λ_n) of the nonzero eigenvalues to be ordered such that*

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$$

and denote by $P_n : X \rightarrow N(\lambda_n I - A)$ the orthogonal projection onto the eigenspace for the eigenvalue λ_n . Then there holds

$$(15.11) \quad A = \sum_{n=1}^{\infty} \lambda_n P_n$$

in the sense of norm convergence. Let $Q : X \rightarrow N(A)$ denote the orthogonal projection onto the nullspace $N(A)$. Then there holds

$$(15.12) \quad \varphi = \sum_{n=1}^{\infty} P_n \varphi + Q \varphi$$

for all $\varphi \in X$. When there are only finitely many eigenvalues then the series (15.11) and (15.12) degenerate into finite sums.

Proof: The first part of the statement is a consequence from the Theorems 3.1, 3.11, 15.10 and 15.11. The orthogonal projections P_n are self adjoint by

Problem 13.1 and bounded $\|P_n\| = 1$ by Theorem 13.3. Therefore, by Theorem 2.17, they are compact since the eigenspaces $N(\lambda_n I - A)$ have finite dimension. Hence the operators

$$A_m := A - \sum_{n=1}^m \lambda_n P_n$$

are self adjoint and compact.

Let $\lambda \neq 0$ be an eigenvalue of A_m with eigenelement φ , that is, $A_m \varphi = \lambda \varphi$. Then for $1 \leq n \leq m$ we have

$$\lambda P_n \varphi = P_n A_m \varphi = P_n (A \varphi - \lambda_n \varphi)$$

since $P_n P_k = 0$ for $n \neq k$. It follows that

$$\lambda^2 \|P_n \varphi\|^2 = (A \varphi - \lambda_n \varphi, P_n (A \varphi - \lambda_n \varphi)) = (\varphi, (A - \lambda_n I) P_n (A \varphi - \lambda_n \varphi)) = 0$$

since $P_n (A \varphi - \lambda_n \varphi) \in N(\lambda_n I - A)$. Therefore $P_n \varphi = 0$, whence $A \varphi = \lambda \varphi$. Conversely, let $\varphi \in N(\lambda_n I - A)$. Then $A_m \varphi = \lambda_n \varphi$ if $n > m$ and $A_m \varphi = 0$ if $n \leq m$. Therefore, the eigenvalues of A_m different from zero are given by $\lambda_{m+1}, \lambda_{m+2}, \dots$. Now Theorem 15.11 yields $\|A_m\| = |\lambda_{m+1}|$, whence (15.11) follows.

From

$$\left\| \varphi - \sum_{n=1}^m P_n \varphi \right\|^2 = \|\varphi\|^2 - \sum_{n=1}^m \|P_n \varphi\|^2$$

we observe that

$$\sum_{n=1}^{\infty} \|P_n \varphi\|^2 < \infty.$$

Hence, considering the partial sums we see that the series

$$\sum_{n=1}^{\infty} P_n \varphi$$

converges in the Hilbert space X . Then, by the continuity of A and (15.11), we get

$$A(\varphi - \sum_{n=1}^{\infty} P_n \varphi) = A\varphi - \sum_{n=1}^{\infty} \lambda_n P_n \varphi = 0.$$

Observing that $Q P_n = 0$ holds for all n now completes the proof of (15.12). \square

Let U be a finite dimensional subspace of a Hilbert space and let $\varphi_1, \dots, \varphi_m$ be an *orthonormal basis*, i.e., $(\varphi_n, \varphi_k) = \delta_{nk}$, $n, k = 1, \dots, m$. Then the orthogonal projection $P : X \rightarrow U$ has the representation

$$P\varphi = \sum_{n=1}^m (\varphi, \varphi_n) \varphi_n.$$

Hence, we can rewrite the series (15.11) and (15.12) into a more explicit form. For this, deviating from the numbering in Theorem 15.12, we repeat each eigenvalue in the sequence (λ_n) according to its *multiplicity*, i.e., according to the

dimension of the eigenspace $N(\lambda_n I - A)$. Assume (φ_n) to be a sequence of corresponding orthonormal eigenelements. Then for each $\varphi \in X$ we can expand

$$(15.11') \quad A\varphi = \sum_{n=1}^{\infty} \lambda_n (\varphi, \varphi_n) \varphi_n$$

and

$$(15.12') \quad \varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n) \varphi_n + Q\varphi.$$

By Theorem 15.12, the orthonormal eigenelements of a compact self adjoint operator, including those for the possible eigenvalue zero, are complete in the sense of Theorem 1.28.

Example 15.13. Consider the integral operator $A : L^2[0, \pi] \rightarrow L^2[0, \pi]$ with continuous kernel

$$K(x, y) := \begin{cases} \frac{1}{\pi} (\pi - x)y, & 0 \leq y \leq x \leq \pi, \\ \frac{1}{\pi} (\pi - y)x, & 0 \leq x \leq y \leq \pi. \end{cases}$$

This kernel is the so called Green's function to the boundary value problem for the simple ordinary differential equation

$$\varphi''(x) = -f(x), \quad 0 \leq x \leq \pi,$$

with homogeneous boundary condition $\varphi(0) = \varphi(\pi) = 0$. To each function $f \in C[0, \pi]$ there exists a unique solution $\varphi \in C^2[0, \pi]$ which is given by

$$(15.13) \quad \varphi(x) = \int_0^{\pi} K(x, y) f(y) dy, \quad 0 \leq x \leq \pi.$$

Uniqueness is obvious and straightforward differentiation shows that (15.13) indeed represents the solution. The compact integral operator A with this so called triangular kernel is self adjoint since its real valued kernel is symmetric $K(x, y) = K(y, x)$. The eigenvalue equation $A\varphi = \lambda\varphi$ is equivalent to the differential equation

$$\lambda\varphi''(x) + \varphi(x) = 0, \quad 0 \leq x \leq \pi,$$

with homogeneous boundary condition $\varphi(0) = \varphi(\pi) = 0$. In particular, taking $\lambda = 0$ yields a trivial nullspace $N(A) = \{0\}$. The only nontrivial solutions to the boundary value problem are given through

$$\lambda_n = \frac{1}{n^2}$$

and

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \quad 0 \leq x \leq \pi,$$

for $n \in \mathbb{N}$. Therefore, in this example (15.11) becomes

$$\int_0^\pi K(x, y)\varphi(y)dy = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx \int_0^\pi \sin ny \varphi(y)dy$$

corresponding to the Fourier expansion of the kernel

$$K(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx \sin ny.$$

We now wish to characterize the eigenvalues of compact self adjoint operators by a minimum-maximum principle. We confine our presentation to the case of a nonnegative operator. A self adjoint operator $A : X \rightarrow X$ is called *nonnegative* if

$$(A\varphi, \varphi) \geq 0$$

for all $\varphi \in X$. Note that nonnegative operators only have nonnegative eigenvalues.

Theorem 15.14. *Let X be a Hilbert space, let $A : X \rightarrow X$ be a nonnegative self adjoint compact operator and let (λ_n) denote the nonincreasing sequence of the nonzero eigenvalues repeated according to their multiplicity. Then*

$$(15.14) \quad \lambda_1 = \|A\| = \sup\{(A\varphi, \varphi) : \|\varphi\| = 1\}$$

and

$$(15.15) \quad \lambda_{n+1} = \inf_{\psi_1, \dots, \psi_n \in X} \sup\{(A\varphi, \varphi) : \varphi \perp \psi_1, \dots, \psi_n, \|\varphi\| = 1\},$$

for $n \in \mathbb{N}$.

Proof: We need only to prove (15.15). To this end, abbreviate its right hand side by μ_{n+1} . We choose the eigenelements $\psi_k = \varphi_k$, $k = 1, \dots, n$. Then, for all $\varphi \perp \psi_1, \dots, \psi_n$, from (15.11') and (15.12') we obtain that

$$(A\varphi, \varphi) = \sum_{k=n+1}^{\infty} \lambda_k |(\varphi, \varphi_k)|^2 \leq \lambda_{n+1} \sum_{k=n+1}^{\infty} |(\varphi, \varphi_k)|^2 \leq \lambda_{n+1} \|\varphi\|^2,$$

with equality holding for $\varphi = \varphi_{n+1}$. Therefore $\mu_{n+1} \leq \lambda_{n+1}$.

On the other hand, to each choice $\psi_1, \dots, \psi_n \in X$ there exists an element $\varphi \in \text{span}\{\varphi_1, \dots, \varphi_{n+1}\}$ with $\|\varphi\| = 1$ and $\varphi \perp \psi_1, \dots, \psi_n$. To see this we write

$$\varphi = \sum_{k=1}^{n+1} \gamma_k \varphi_k$$

and have to solve the linear system of n equations

$$\sum_{k=1}^{n+1} \gamma_k (\varphi_k, \psi_j) = 0, \quad j = 1, \dots, n,$$

for the $n+1$ unknowns $\gamma_1, \dots, \gamma_{n+1}$. This system always allows a solution which can be normalized such that

$$\|\varphi\|^2 = \sum_{k=1}^{n+1} |\gamma_k|^2 = 1.$$

Then, again from (15.11') and (15.12'), it follows that

$$(A\varphi, \varphi) = \sum_{k=1}^{n+1} \lambda_k |(\varphi, \varphi_k)|^2 \geq \lambda_{n+1} \sum_{k=1}^{n+1} |(\varphi, \varphi_k)|^2 = \lambda_{n+1} \|\varphi\|^2 = \lambda_{n+1}.$$

Hence $\mu_{n+1} \geq \lambda_{n+1}$ and the proof is complete.

15.4 Singular Value Decomposition

We now will describe modified forms of the expansions (15.11) and (15.12) for arbitrary compact operators in Hilbert spaces. From Theorem 4.10 we recall that the adjoint operator of a compact linear operator again is compact.

Definition 15.15. Let X and Y be Hilbert spaces, let $A : X \rightarrow Y$ be a compact linear operator and $A^* : Y \rightarrow X$ its adjoint. The nonnegative square roots of the eigenvalues of the nonnegative self adjoint compact operator $A^*A : X \rightarrow X$ are called singular values of A .

Theorem 15.16. Let (μ_n) denote the sequence of the nonzero singular values of the compact linear operator A (with $A \neq 0$) repeated according to their multiplicity, that is, according to the dimension of the nullspaces $N(\mu_n^2 I - A^*A)$. Then there exist orthonormal sequences (φ_n) in X and (g_n) in Y such that

$$(15.16) \quad A\varphi_n = \mu_n g_n, \quad A^*g_n = \mu_n \varphi_n$$

for all $n \in \mathbb{N}$. For each $\varphi \in X$ there holds the singular value decomposition

$$(15.17) \quad \varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n) \varphi_n + Q\varphi$$

with the orthogonal projection $Q : X \rightarrow N(A)$ and

$$(15.18) \quad A\varphi = \sum_{n=1}^{\infty} \mu_n (\varphi, \varphi_n) g_n.$$

Each system (μ_n, φ_n, g_n) , $n \in \mathbb{N}$, with these properties is called a singular system of A . When there are only finitely many singular values then the series (15.17) and (15.18) degenerate into finite sums. Note that for an injective operator the orthonormal system $\{\varphi_n : n \in \mathbb{N}\}$ provided by the singular system is complete in X .

Proof: Let (φ_n) denote an orthonormal sequence of the eigenelements of A^*A , that is,

$$A^*A\varphi_n = \mu_n^2 \varphi_n$$

and define a second orthonormal sequence by

$$g_n := \frac{1}{\mu_n} A\varphi_n.$$

Straightforward computations show that the system (μ_n, φ_n, g_n) , $n \in \mathbb{N}$, satisfies (15.16). Application of the expansion (15.12') to the self adjoint compact operator A^*A yields

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n) \varphi_n + Q\varphi$$

for all $\varphi \in X$ where Q denotes the orthogonal projection from X onto $N(A^*A)$. Let $\psi \in N(A^*A)$. Then $(A\psi, A\psi) = (\psi, A^*A\psi) = 0$ and this implies that $N(A^*A) = N(A)$. Therefore (15.17) is proven and (15.18) follows by applying A to (15.17). \square

Note that the singular value decomposition implies that

$$(15.19) \quad \|\varphi\|^2 = \sum_{n=1}^{\infty} |(\varphi, \varphi_n)|^2 + \|Q\varphi\|^2$$

and

$$(15.20) \quad \|A\varphi\|^2 = \sum_{n=1}^{\infty} \mu_n^2 |(\varphi, \varphi_n)|^2$$

for all $\varphi \in X$.

Theorem 15.17. *Let $A, B : X \rightarrow Y$ be compact linear operators. Then for the nonincreasing sequence of singular values there holds*

$$(15.21) \quad \mu_1(A) = \|A\| = \sup\{\|A\varphi\| : \|\varphi\| = 1\}$$

and

$$(15.22) \quad \mu_{n+1}(A) = \inf_{\psi_1, \dots, \psi_n \in X} \sup\{\|A\varphi\| : \varphi \perp \psi_1, \dots, \psi_n, \|\varphi\| = 1\}$$

for all $n \in \mathbb{N}$ and

$$(15.23) \quad \mu_{n+m+1}(A + B) \leq \mu_{n+1}(A) + \mu_{m+1}(B)$$

for all $n, m = 0, 1, 2, \dots$

Proof: (15.21) and (15.22) follow immediately from Theorem 15.14, since the squares μ_n^2 of the singular values of A are given by the eigenvalues of the nonnegative self adjoint operator A^*A , and since $(A^*A\varphi, \varphi) = \|A\varphi\|^2$ for all

$\varphi \in X$. Then (15.23) is obtained by

$$\begin{aligned}
& \mu_{n+m+1}(A + B) \\
&= \inf_{\psi_1, \dots, \psi_{n+m} \in X} \sup\{\|(A + B)\varphi\| : \varphi \perp \psi_1, \dots, \psi_{n+m}, \|\varphi\| = 1\} \\
&\leq \inf_{\psi_1, \dots, \psi_{n+m} \in X} \sup\{\|A\varphi\| : \varphi \perp \psi_1, \dots, \psi_n, \|\varphi\| = 1\} \\
&\quad + \inf_{\psi_1, \dots, \psi_{n+m} \in X} \sup\{\|B\varphi\| : \varphi \perp \psi_{n+1}, \dots, \psi_{n+m}, \|\varphi\| = 1\} \\
&= \inf_{\psi_1, \dots, \psi_n \in X} \sup\{\|A\varphi\| : \varphi \perp \psi_1, \dots, \psi_n, \|\varphi\| = 1\} \\
&\quad + \inf_{\psi_{n+1}, \dots, \psi_{n+m} \in X} \sup\{\|B\varphi\| : \varphi \perp \psi_{n+1}, \dots, \psi_{n+m}, \|\varphi\| = 1\} \\
&= \mu_{n+1}(A) + \mu_{m+1}(B).
\end{aligned}$$

In the following theorem we express the solution to an equation of the first kind with a compact operator in terms of a singular system.

Theorem 15.18 (Picard). *Let $A : X \rightarrow Y$ be a compact linear operator with singular system (μ_n, φ_n, g_n) . The equation of the first kind*

$$(15.24) \quad A\varphi = f$$

is solvable if and only if $f \in N(A^)^\perp$ and*

$$(15.25) \quad \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f, g_n)|^2 < \infty.$$

Then a solution is given by

$$(15.26) \quad \varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} (f, g_n) \varphi_n.$$

Proof: The necessity of $f \in N(A^*)^\perp$ can be seen from Theorem 15.8. If φ is a solution of (15.24) then

$$\mu_n(\varphi, \varphi_n) = (\varphi, A^*g_n) = (A\varphi, g_n) = (f, g_n)$$

and (15.19) implies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f, g_n)|^2 = \sum_{n=1}^{\infty} |(\varphi, \varphi_n)|^2 \leq \|\varphi\|^2,$$

whence the necessity of (15.25) follows.

Conversely, assume that $f \perp N(A^*)$ and (15.25) is fulfilled. Then, by considering the partial sums of (15.25) we see that the series (15.26) converges

in the Hilbert space X . We apply A to (15.26), use (15.17) with the singular system (μ_n, g_n, φ_n) of the operator A^* and observe $f \in N(A^*)^\perp$ to obtain

$$A\varphi = \sum_{n=1}^{\infty} (f, g_n) g_n = f.$$

This ends the proof. \square

Picard's theorem clearly demonstrates the ill-posed nature of the equation $A\varphi = f$. If we perturb the right hand side f by adding $f^\delta = \delta g_n$ we obtain a perturbation of the solution φ by $\varphi^\delta = \delta \varphi_n / \mu_n$. Hence, the ratio $\|\varphi^\delta\|/\|f^\delta\| = \delta/\mu_n$ can be made arbitrarily large due to the fact that the singular values tend to zero. Obviously, the influence of errors in the data f is controlled by the rate of this convergence. In this sense we may say that the equation is *mildly ill-posed* if the singular values decay slowly to zero and that it is *severely ill-posed* if they decay very rapidly.

Example 15.19. Consider the integral operator $A : L^2[0, 1] \rightarrow L^2[0, 1]$ defined by

$$(A\varphi)(x) := \int_0^x \varphi(y) dy, \quad 0 \leq x \leq 1.$$

Then, the inverse operator A^{-1} corresponds to differentiation. The adjoint operator is given by

$$(A^*\psi)(x) := \int_x^1 \psi(y) dy, \quad 0 \leq x \leq 1.$$

Hence

$$(A^*A\varphi)(x) := \int_x^1 \int_0^y \varphi(z) dz dy, \quad 0 \leq x \leq 1,$$

and the eigenvalue equation $A^*A\varphi = \mu^2\varphi$ is easily seen to be equivalent to the boundary value problem for the ordinary differential equation

$$\mu^2\varphi'' + \varphi = 0$$

with homogeneous boundary conditions $\varphi(1) = \varphi'(0) = 0$. A nontrivial solution exists only if

$$\mu_n = \frac{2}{(2n-1)\pi}$$

with $n \in \mathbb{N}$ and in this case

$$\varphi_n(x) = \sqrt{2} \cos \frac{2n-1}{2} \pi x.$$

The singular system is completed by $g_n = A\varphi_n / \mu_n$ through

$$g_n(x) = \sqrt{2} \sin \frac{2n-1}{2} \pi x.$$

In this example we have a decay $\mu_n = O(1/n)$ for an integral operator with a discontinuity of the kernel along the diagonal $x = y$. The kernel of the integral

operator in Example 15.13 has a discontinuity in the first derivatives along the diagonal and the singular values decay $\mu_n = O(1/n^2)$. \square

In general, for compact integral operators the smoothness of the kernel controls the degree of ill-posedness. Roughly speaking, the smoothness of the kernel of the operator A determines the smoothness of the range of A , and this effects the regularity condition on f for the solvability of $A\varphi = f$. We give an additional illustration for this statement by the following result due to Little and Reade [1].

Theorem 15.20. *Let $A : L^2[-1, 1] \rightarrow L^2[-1, 1]$ be an integral operator with analytic kernel on $[-1, 1] \times [-1, 1]$. Then the singular values of A decay at least exponentially*

$$\mu_n = O(R^{-n})$$

with some constant $R > 1$.

Proof: Let K denote the kernel and

$$K_n(x, y) = \frac{1}{2} T_0(x)a_0(y) + \sum_{m=1}^n T_m(x)a_m(y)$$

its approximation by the orthonormal Tschebyscheff expansion as considered in Section 11.3. For the integral operator A_n with the degenerate kernel K_n , by (11.23), we have $\|A_n - A\|_\infty = O(R^{-n})$ with some $R > 1$.

Since $\dim A_n(X) \leq n + 1$, from the singular value decomposition (15.18) we observe that A_n has at most $n + 1$ nonzero singular values. Therefore we can apply Theorem 15.17 to obtain

$$\mu_{n+2}(A) \leq \mu_1(A - A_n) + \mu_{n+2}(A_n) = \|A - A_n\|_2 = O(R^{-n}).$$

15.5 Regularization Schemes

As already mentioned, the Picard Theorem 15.18 illustrates the fact that the ill-posedness of an equation of the first kind with a compact operator stems from the behaviour of the singular values $\mu_n \rightarrow 0$, $n \rightarrow \infty$. This suggests to try to regularize the equation by damping out the influence of the factor $1/\mu_n$ in the solution formula (15.26).

Theorem 15.21. *Let $A : X \rightarrow Y$ be an injective compact linear operator with singular system (μ_n, φ_n, g_n) , $n \in \mathbb{N}$, and let $q : (0, \infty) \times (0, \|A\|] \rightarrow \mathbb{R}$ be a bounded function satisfying the following two conditions: For each $\alpha > 0$ there exists a positive constant $c(\alpha)$ such that*

$$(15.27) \quad |q(\alpha, \mu)| \leq c(\alpha)\mu$$

for all $0 < \mu \leq \|A\|$ and there holds

$$(15.28) \quad \lim_{\alpha \rightarrow 0} q(\alpha, \mu) = 1$$

for all $0 < \mu \leq \|A\|$. Then the bounded linear operators $R_\alpha : Y \rightarrow X$, $\alpha > 0$, defined by

$$(15.29) \quad R_\alpha f := \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha, \mu_n)(f, g_n) \varphi_n$$

for all $f \in Y$, describe a regularization scheme with

$$(15.30) \quad \|R_\alpha\| \leq c(\alpha).$$

Proof: From (15.19) and (15.27) we get

$$\|R_\alpha f\|^2 = \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} [q(\alpha, \mu_n)]^2 |(f, g_n)|^2 \leq [c(\alpha)]^2 \sum_{n=1}^{\infty} |(f, g_n)|^2 \leq [c(\alpha)]^2 \|f\|^2$$

for all $f \in Y$, whence the bound (15.30) follows. With the aid of

$$(R_\alpha A\varphi, \varphi_n) = \frac{1}{\mu_n} q(\alpha, \mu_n)(A\varphi, g_n) = q(\alpha, \mu_n)(\varphi, \varphi_n)$$

and the singular value decomposition for $R_\alpha A\varphi - \varphi$ we obtain

$$\|R_\alpha A\varphi - \varphi\|^2 = \sum_{n=1}^{\infty} |(R_\alpha A\varphi - \varphi, \varphi_n)|^2 = \sum_{n=1}^{\infty} [q(\alpha, \mu_n) - 1]^2 |(\varphi, \varphi_n)|^2.$$

Here we have used the fact that A is injective. Let $\varphi \in X$ with $\varphi \neq 0$ and $\varepsilon > 0$ be given and let M denote a bound for q . Then there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |(\varphi, \varphi_n)|^2 < \frac{\varepsilon}{2(M+1)^2}.$$

By the convergence condition (15.28) there exists $\alpha_0(\varepsilon) > 0$ such that

$$[q(\alpha, \mu_n) - 1]^2 < \frac{\varepsilon}{2\|\varphi\|^2}$$

for all $n = 1, \dots, N$ and all $0 < \alpha \leq \alpha_0$. Now, splitting the series in two parts and using (15.19), it follows that

$$\|R_\alpha A\varphi - \varphi\|^2 < \frac{\varepsilon}{2\|\varphi\|^2} \sum_{n=1}^N |(\varphi, \varphi_n)|^2 + \frac{\varepsilon}{2} \leq \varepsilon$$

for all $0 < \alpha \leq \alpha_0$. Thus we have established that $R_\alpha A\varphi \rightarrow \varphi$, $\alpha \rightarrow 0$, for all $\varphi \in X$ and the proof is complete.

Remark 15.22. If we replace condition (15.27) by the stronger condition

$$(15.27') \quad q(\alpha, \mu) \leq \tilde{c}(\alpha)\mu^2$$

for all $0 < \mu \leq \|A\|$ and denote by M a bound on q , then instead of (15.30) we have

$$(15.30') \quad \|R_\alpha\| \leq \sqrt{M\tilde{c}(\alpha)}.$$

Proof: From

$$AR_\alpha f = \sum_{n=1}^{\infty} q(\alpha, \mu_n) (f, g_n) g_n,$$

we find

$$\|AR_\alpha f\|^2 = \sum_{n=1}^{\infty} [q(\alpha, \mu_n)]^2 |(f, g_n)|^2 \leq M^2 \sum_{n=1}^{\infty} |(f, g_n)|^2 \leq M^2 \|f\|^2,$$

whence $\|AR_\alpha\| \leq M$ follows. For the operators $\tilde{R}_\alpha : Y \rightarrow Y$, $\alpha > 0$, defined by

$$\tilde{R}_\alpha f := \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} q(\alpha, \mu_n) (f, g_n) g_n$$

for all $f \in Y$, as in the proof of the preceding Theorem 15.21, the condition (15.27') implies $\|\tilde{R}_\alpha\| \leq \tilde{c}(\alpha)$. Now, using $A^* \tilde{R}_\alpha = R_\alpha$, by the Cauchy-Schwarz inequality we get

$$\|R_\alpha f\|^2 = (R_\alpha f, A^* \tilde{R}_\alpha f) = (AR_\alpha f, \tilde{R}_\alpha f) \leq \|AR_\alpha f\| \|\tilde{R}_\alpha f\| \leq M \tilde{c}(\alpha) \|f\|^2$$

for all $f \in Y$, whence (15.30') follows. \square

We now describe some classical regularization schemes by choosing the damping function q appropriately.

Theorem 15.23. *Let $A : X \rightarrow Y$ be an injective compact linear operator. Then for each $\alpha > 0$ the operator $\alpha I + A^* A : X \rightarrow X$ has a bounded inverse and*

$$R_\alpha := (\alpha I + A^* A)^{-1} A^*$$

describes a regularization scheme with $\|R_\alpha\| \leq 1/2\sqrt{\alpha}$.

Proof: For $\alpha > 0$ the operator $\alpha I + A^* A$ is injective. Let φ be a solution of the homogeneous equation $\alpha\varphi + A^* A\varphi = 0$. Then $\alpha(\varphi, \varphi) + (A\varphi, A\varphi) = (\varphi, \alpha\varphi + A^* A\varphi) = 0$ and this implies $\varphi = 0$. By the Riesz theory Theorem 3.4 we have a bounded inverse $(\alpha I + A^* A)^{-1}$ since $A^* A$ is compact.

Let (μ_n, φ_n, g_n) , $n \in \mathbb{N}$, be a singular system for A . Then the unique solution φ_α of

$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* f$$

can be written in the form

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{\mu_n}{\alpha + \mu_n^2} (f, g_n) \varphi_n.$$

Indeed, using $A^* A\varphi_n = \mu_n^2 \varphi_n$ and the singular value decomposition (15.18) applied to $A^* f$, we find

$$(\alpha I + A^* A)\varphi_\alpha = \sum_{n=1}^{\infty} \mu_n (f, g_n) \varphi_n = A^* f.$$

Hence, R_α can be brought into the form (15.29) with

$$q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2}.$$

This damping function is bounded by $0 < q(\alpha, \mu) < 1$ and satisfies the conditions (15.27) and (15.28) with

$$c(\alpha) = \frac{1}{2\sqrt{\alpha}}$$

because of the arithmetic-geometric mean inequality

$$\sqrt{\alpha}\mu \leq \frac{\alpha + \mu^2}{2}.$$

Now the proof is complete by Theorem 15.21. \square

The regularization described in Theorem 15.23 is called *Tikhonov regularization* since it was introduced by Tikhonov [1]. We will analyse it in more detail in the following Chapter 16.

Theorem 15.24. *Let $A : X \rightarrow Y$ be an injective compact linear operator and let $0 < a \leq 1/\|A\|^2$. Then the bounded linear operators*

$$R_m := a \sum_{k=0}^m (I - aA^*A)^k A^*$$

describe a regularization scheme with regularization parameter $m \rightarrow \infty$ and $\|R_m\| \leq \sqrt{a(m+1)}$.

Proof: Using a singular system for A and the singular value decomposition

$$A^*f = \sum_{n=1}^{\infty} \mu_n(f, g_n) \varphi_n,$$

we can write

$$R_m f = a \sum_{n=1}^{\infty} \mu_n \sum_{k=0}^m (1 - a\mu_n^2)^k (f, g_n) \varphi_n.$$

The corresponding damping function is

$$q(m, \mu) = a\mu^2 \sum_{k=0}^m (1 - a\mu^2)^k = 1 - (1 - a\mu^2)^{m+1}.$$

It is bounded by $0 \leq q(m, \mu) \leq 1$ for $0 < \mu \leq \|A\|$ and it satisfies the conditions (15.27') and (15.28) with

$$q(m, \mu) \leq a(m+1)\mu^2.$$

Hence, the proof is complete by Remark 15.22. \square

The evaluation of the approximation $\psi_m = R_m f$ corresponds to m steps of the iteration scheme

$$(15.31) \quad \begin{aligned} \psi_0 &:= a A^* f \\ \psi_k &:= (I - a A^* A) \psi_{k-1} + a A^* f, \quad k = 1, 2, \dots \end{aligned}$$

This scheme goes back to Landweber [1] and Friedman [1] and consequently is known as the *Landweber–Friedman iteration*. The regularization parameter is given by the number m of iteration steps. Accuracy of the approximation requires m to be large and stability requires m to be small.

Theorem 15.25. *Let $A : X \rightarrow Y$ be an injective compact linear operator with singular system (μ_n, φ_n, g_n) , $n \in \mathbb{N}$. Then the spectral cut-off*

$$R_\alpha f := \sum_{\mu_n^2 \geq \alpha} \frac{1}{\mu_n} (f, g_n) \varphi_n$$

describes a regularization scheme with $\|R_\alpha\| \leq 1/\sqrt{\alpha}$.

Proof: The damping function with $q(\alpha, \mu) = 1$ for $\mu^2 \geq \alpha$ and $q(\alpha, \mu) = 0$ for $\mu^2 < \alpha$ satisfies the conditions (15.27') and (15.28) with

$$q(\alpha, \mu) \leq \frac{1}{\alpha} \mu^2.$$

Here the regularization parameter α determines the number of terms in the truncation of the series (15.26). Accuracy of the approximation requires this number to be large and stability requires it to be small.

Theorem 15.26. *Under the assumptions of Theorem 15.21 the conditions*

$$(15.32) \quad \alpha(\delta) \rightarrow 0, \quad \delta \rightarrow 0,$$

and

$$(15.33) \quad c(\alpha(\delta)) \delta \rightarrow 0, \quad \delta \rightarrow 0,$$

are sufficient for regularity in the sense of Definition 15.7.

Proof: Using (15.30), the error estimate (15.7) reads

$$\|\varphi_\alpha^\delta - \varphi\| \leq \delta c(\alpha(\delta)) + \|R_{\alpha(\delta)} A \varphi - \varphi\|,$$

whence the statement follows. \square

Under the assumption (15.27') the condition (15.33) can be replaced by

$$(15.33') \quad \tilde{c}(\alpha(\delta)) \delta^2 \rightarrow 0, \quad \delta \rightarrow 0.$$

From the preceding theorems we can deduce the following three regularity conditions.

Corollary 15.27. *The Tikhonov regularization is regular if*

$$\alpha(\delta) \rightarrow 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta)} \rightarrow 0, \quad \delta \rightarrow 0.$$

The Landweber–Fridman iteration is regular if

$$m(\delta) \rightarrow \infty \quad \text{and} \quad m(\delta)\delta^2 \rightarrow 0, \quad \delta \rightarrow 0.$$

The spectral cut-off is regular if

$$\alpha(\delta) \rightarrow 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta)} \rightarrow 0, \quad \delta \rightarrow 0.$$

The regularity criterion for the spectral cut-off again illustrates the effect of the order with which the spectral values tend to zero on the degree of ill-posedness.

For a comprehensive study of regularization methods we refer the reader to Baumeister [1], Groetsch [1] and Tikhonov and Arsenin [1]. We shall proceed in the next chapter with a closer examination of the Tikhonov regularization method.

Problems

15.1. Prove the Riemann–Lebesgue lemma

$$\int_0^\pi K(\cdot, y) \sin ny dy \rightarrow 0, \quad n \rightarrow \infty,$$

in the mean square norm for a kernel $K \in L^2([0, \pi] \times [0, \pi])$. How can this result be used to illustrate the ill-posedness of integral equations of the first kind?

15.2. Let X be a Hilbert space and $A : X \rightarrow X$ a compact self adjoint operator. With the aid of the spectral Theorem 15.12, solve the equation of the second kind

$$\lambda\varphi - A\varphi = f, \quad \lambda \neq 0.$$

15.3. The Poisson integral

$$u(\rho, t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(t - \tau)} \varphi(\tau) d\tau$$

gives the solution to the Dirichlet problem for the Laplace equation $\Delta u = 0$ in the unit disk $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ with boundary values $u = \varphi$ on ∂D in polar coordinates $x = (\rho, t)$ for $0 \leq \rho < 1$ and $0 \leq t \leq 2\pi$. Therefore, the continuation of a harmonic function given on a concentric disk of radius $r < 1$ to a harmonic function on the unit disk is equivalent to the solution of the integral equation of the first kind

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \tau)} \varphi(\tau) d\tau = f(t), \quad 0 \leq t \leq 2\pi,$$

with $f = u(r, \cdot)$. Determine the singular values.

15.4. Determine the singular values of the integral operator $A : L^2[0, 1] \rightarrow L^2[0, 1]$ given by

$$(A\varphi)(x) := \int_0^x (x - y)\varphi(y)dy, \quad 0 \leq x \leq 1.$$

What is the inverse of A?

15.5. Consider the central difference quotient

$$(R_h f)(x) := \frac{1}{h} \left\{ f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \right\}, \quad 0 \leq x \leq 1,$$

for functions f which are odd with respect to $x = 0$ and even with respect to $x = 1$. Show that R_h , $h > 0$, can be interpreted as a regularization scheme in the sense of Theorem 15.21 for the integral operator of Example 15.19.

16. Tikhonov Regularization

This chapter will continue the study of Tikhonov regularization and will be based on its classical interpretation as a penalized residual minimization. We shall explain the concepts of quasi-solutions and minimum norm solutions as strategies for the selection of the regularization parameter.

16.1 The Tikhonov Functional

The following theorem presents another aspect of the Tikhonov regularization complementing its introduction in Theorem 15.23. Throughout this chapter X and Y will always denote Hilbert spaces.

Theorem 16.1. *Let $A : X \rightarrow Y$ be a bounded linear operator and let $\alpha > 0$. Then for each $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that*

$$(16.1) \quad \|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2 = \inf_{\varphi \in X} \{\|A\varphi - f\|^2 + \alpha\|\varphi\|^2\}.$$

The minimizer φ_α is given by the unique solution of the equation

$$(16.2) \quad \alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

and depends continuously on f .

Proof: From the equation

$$\begin{aligned} \|A\varphi - f\|^2 + \alpha\|\varphi\|^2 &= \|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2 \\ &\quad + 2\operatorname{Re}(\varphi - \varphi_\alpha, \alpha\varphi_\alpha + A^*(A\varphi_\alpha - f)) + \|A(\varphi - \varphi_\alpha)\|^2 + \alpha\|\varphi - \varphi_\alpha\|^2, \end{aligned}$$

which is valid for all $\varphi \in X$, we observe that the condition (16.2) is necessary and sufficient for φ_α to minimize the *Tikhonov functional* introduced by (16.1).

Consider the operator $T : X \rightarrow X$, given by $T := \alpha I + A^*A$. For all $\varphi \in X$ there holds

$$(16.3) \quad \alpha\|\varphi\|^2 \leq \alpha\|\varphi\|^2 + \|A\varphi\|^2 = \operatorname{Re}(T\varphi, \varphi).$$

Therefore, T is strictly coercive and, by the Lax–Milgram Theorem 13.23, has a bounded inverse $T^{-1} : X \rightarrow X$. \square

The equation (16.2), of course, coincides with the Tikhonov regularization introduced in Theorem 15.23. By the interpretation as minimizer of the

Tikhonov functional, its solution keeps the residual $\|A\varphi_\alpha - f\|^2$ small and is stabilized through the penalty term $\alpha\|\varphi_\alpha\|^2$. We can view this quadratic optimization problem as a *penalty method* for either one of the following *constraint* optimization problems:

- a) For given $\rho > 0$ minimize the defect $\|A\varphi - f\|$ subject to the constraint that the norm is bounded by $\|\varphi\| \leq \rho$.
- b) For given $\delta > 0$ minimize the norm $\|\varphi\|$ subject to the constraint that the defect is bounded by $\|A\varphi - f\| \leq \delta$.

The first interpretation leads to the concept of *quasi-solutions* and the second to the *method of residuals* or the *discrepancy principle*. For a study of these two stabilization procedures we need to introduce the notion of weak convergence.

16.2 Weak Convergence

Definition 16.2. A sequence (φ_n) of elements in a Hilbert space X is called weakly convergent to an element $\varphi \in X$ if

$$\lim_{n \rightarrow \infty} (\psi, \varphi_n) = (\psi, \varphi)$$

for all $\psi \in X$.

For a weakly convergent sequence we will write $\varphi_n \rightharpoonup \varphi$, $n \rightarrow \infty$. Note that norm convergence $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, always implies weak convergence $\varphi_n \rightharpoonup \varphi$, $n \rightarrow \infty$, whereas simple examples show that the converse of this statement, in general, is false. We leave it to the reader to verify that a sequence in a Hilbert space cannot weakly converge to two different elements.

Theorem 16.3. A weakly convergent sequence in a Hilbert space is bounded.

Proof: Let (φ_n) be a weakly convergent sequence. Then the sequence of bounded linear functionals $F_n : X \rightarrow \mathbb{C}$, defined by $F_n(\psi) := (\psi, \varphi_n)$ for $\psi \in X$ is pointwise convergent. Therefore, by the uniform boundedness principle Theorem 10.3 it is uniformly bounded, i.e., $\|F_n\| \leq C$ for all $n \in \mathbb{N}$ with some constant C . Hence from $\|\varphi_n\|^2 = F_n(\varphi_n) \leq \|F_n\| \|\varphi_n\| \leq C \|\varphi_n\|$ we have $\|\varphi_n\| \leq C$ for all $n \in \mathbb{N}$.

Theorem 16.4 Every bounded sequence in a Hilbert space contains a weakly convergent subsequence.

Proof: Let (φ_n) be a bounded sequence, i.e., $\|\varphi_n\| \leq C$ for all $n \in \mathbb{N}$ with some constant C . Then, for each $i \in \mathbb{N}$ the sequence (φ_i, φ_n) is bounded in \mathbb{C} . Therefore, by the standard diagonalization procedure (see the proof of Theorem 1.18) we can select a subsequence $(\varphi_{n(k)})$ such that $(\varphi_i, \varphi_{n(k)})$ converges in \mathbb{C} as $k \rightarrow \infty$ for each $i \in \mathbb{N}$. Hence, the linear functional F given by

$$F(\psi) := \lim_{k \rightarrow \infty} (\psi, \varphi_{n(k)})$$

is well defined on $U := \text{span}\{\varphi_i : i \in \mathbb{N}\}$ and by continuity it is also well defined on the closure \bar{U} . By decomposing an arbitrary $\psi \in X$ into the form $\psi = P\psi + \psi - P\psi$ where $P : X \rightarrow \bar{U}$ denotes the orthogonal projector, we finally find that F is well defined on all of X . Furthermore, F is bounded $\|F\| \leq C$. Therefore, by the Riesz representation Theorem 4.8, there exists a unique element $\varphi \in X$ such that

$$F(\psi) = (\psi, \varphi)$$

for all $\psi \in X$. Hence,

$$\lim_{k \rightarrow \infty} (\psi, \varphi_{n(k)}) = (\psi, \varphi)$$

for all $\psi \in X$, i.e., $(\varphi_{n(k)})$ converges weakly to φ as $k \rightarrow \infty$. \square

In an obvious meaning we may reformulate Theorem 16.4 by saying that in a Hilbert space each bounded set is *relatively weakly sequentially compact* (see Section 1.4).

16.3 Quasi-Solutions

The principal idea underlying the concept of quasi-solutions as introduced by Ivanov [1] is to stabilize an ill-posed problem by restricting the solution set to some subset $U \subset X$ exploiting suitable á-priori information on the solution of $A\varphi = f$. For perturbed right hand sides, in general, we cannot expect a solution in U . Therefore, instead of trying to solve the equation exactly, we minimize the residual. For simplicity we restrict our presentation to the case where $U = B[0; \rho]$ is a ball of radius ρ with some $\rho > 0$. This choice requires some á-priori knowledge on the norm of the solution.

Definition 16.5. Let $A : X \rightarrow Y$ be a bounded injective linear operator and let $\rho > 0$. For a given $f \in Y$ an element $\varphi_0 \in X$ is called a quasi-solution of $A\varphi = f$ with constraint ρ if $\|\varphi_0\| \leq \rho$ and

$$\|A\varphi_0 - f\| = \inf \{\|A\varphi - f\| : \|\varphi\| \leq \rho\}.$$

Note that φ_0 is a quasi-solution to $A\varphi = f$ with constraint ρ if and only if $A\varphi_0$ is a best approximation to f with respect to the set $V := A(B[0; \rho])$. It is obvious how the definition of a quasi-solution can be extended to more general constraint sets. The injectivity of the operator A is essential for uniqueness properties of the quasi-solution. For the sake of clarity, in the following analysis on quasi-solutions, we confine ourselves to the case where the operator A has dense range in Y .

Theorem 16.6. Let $A : X \rightarrow Y$ be a bounded injective linear operator with dense range in Y and let $\rho > 0$. Then for each $f \in Y$ there exists a unique quasi-solution of $A\varphi = f$ with constraint ρ .

Proof: Since A is linear the set V clearly is convex. By Theorem 1.27 there exists at most one best approximation to f with respect to V . Since A is injective this implies uniqueness for the quasi-solution.

If $f \in A(B[0; \rho])$ then there exists φ_0 with $\|\varphi_0\| \leq \rho$ and $A\varphi_0 = f$. Clearly φ_0 is quasi-solution. Therefore, we only need to be concerned with the case where $f \notin A(B[0; \rho])$. We will establish existence of the quasi-solution by constructing an element φ_0 which satisfies the sufficient condition for the best approximation given in Theorem 1.27. For approximation with respect to V this condition reads $\operatorname{Re}(f - A\varphi_0, A\varphi - A\varphi_0) \leq 0$, i.e.,

$$(16.4) \quad \operatorname{Re}(A^*(f - A\varphi_0), \varphi - \varphi_0) \leq 0$$

for all φ with $\|\varphi\| \leq \rho$. Obviously, any element φ_0 with

$$(16.5) \quad \|\varphi_0\| = \rho$$

and

$$(16.6) \quad \alpha\varphi_0 + A^*A\varphi_0 = A^*f,$$

such that $\alpha > 0$, satisfies the condition (16.4) and therefore provides a quasi-solution. For $f \notin A(B[0; \rho])$ we will show that α can be chosen such that the unique solution φ_0 of (16.6) (see Theorem 16.1) fulfills (16.5).

Define a function $F : (0, \infty) \rightarrow \mathbb{R}$ by

$$F(\alpha) := \|\varphi_\alpha\|^2 - \rho^2,$$

where φ_α denotes the unique solution of equation (16.2). We have to show that F has a zero. By a Neumann series argument, using Theorem 10.1, F can be seen to be continuous.

From (16.3) we obtain that $\alpha\|\varphi_\alpha\| \leq \|A^*f\|$, whence

$$(16.7) \quad \varphi_\alpha \rightarrow 0, \quad \alpha \rightarrow \infty,$$

follows. This implies $F(\alpha) \rightarrow -\rho^2 < 0$, $\alpha \rightarrow \infty$.

Since $A(X)$ is dense in Y , for every $\varepsilon > 0$ there exists $\varphi_\varepsilon \in X$ such that $\|A\varphi_\varepsilon - f\|^2 < \varepsilon/2$. Choose δ such that $\delta\|\varphi_\varepsilon\|^2 \leq \varepsilon/2$. Then, using Theorem 16.1, for all $\alpha < \delta$ we have

$$\|A\varphi_\alpha - f\|^2 \leq \|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2 \leq \|A\varphi_\varepsilon - f\|^2 + \alpha\|\varphi_\varepsilon\|^2 < \varepsilon.$$

This implies convergence

$$(16.8) \quad A\varphi_\alpha \rightarrow f, \quad \alpha \rightarrow 0.$$

Now assume that $\|\varphi_\alpha\| \leq \rho$ for all $\alpha > 0$. Then, by Theorem 16.4, we can choose a sequence (α_n) with $\alpha_n \rightarrow 0$, $n \rightarrow \infty$, such that we have weak convergence $\varphi_n := \varphi_{\alpha_n} \rightharpoonup \varphi$, $n \rightarrow \infty$, with some $\varphi \in X$. From

$$\|\varphi\|^2 = \lim_{n \rightarrow \infty} (\varphi_n, \varphi) \leq \rho\|\varphi\|$$

we obtain $\|\varphi\| \leq \rho$. Writing $(A\varphi_n, \psi) = (\varphi_n, A^*\psi)$ we conclude weak convergence $A\varphi_n \rightharpoonup A\varphi$, $n \rightarrow \infty$. Finally, we can use (16.8) to find

$$\|A\varphi - f\|^2 = \lim_{n \rightarrow \infty} (A\varphi_n - f, A\varphi - f) \leq \lim_{n \rightarrow \infty} \|A\varphi_n - f\| \|A\varphi - f\| = 0$$

which is a contradiction to $f \notin A(B[0; \rho])$. Therefore there exists α such that $F(\alpha) = \|\varphi_\alpha\|^2 - \rho^2 > 0$. Now the continuity of F implies the existence of a zero and the proof is completed. \square

The quasi-solution can be shown to restore stability in the sense that it depends weakly continuously on the right hand side f (see Problem 16.1).

In applications, generally, errors in the data f ensure that $f \notin A(B[0; \rho])$. Then, the quasi-solution with constraint ρ can be obtained numerically by Newton's method for solving $F(\alpha) = 0$. The derivative of F is given by

$$F'(\alpha) = 2 \operatorname{Re} \left(\frac{d\varphi_\alpha}{d\alpha}, \varphi_\alpha \right).$$

Differentiating (16.2) with respect to the parameter α yields

$$(16.9) \quad \alpha \frac{d\varphi_\alpha}{d\alpha} + A^* A \frac{d\varphi_\alpha}{d\alpha} = -\varphi_\alpha$$

as equation for $d\varphi_\alpha/d\alpha$ which has to be solved for the evaluation of $F'(\alpha)$.

We may view the quasi-solution as described in Theorem 16.6 as an á-posteriori strategy for the choice of the parameter α in the Tikhonov regularization: given a perturbed right hand side f^δ of an element $f \in A(X)$, we choose α such that

$$\varphi_\alpha^\delta = (\alpha I + A^* A)^{-1} A^* f^\delta$$

satisfies $\|\varphi_\alpha^\delta\| = \rho$ with some á-priori bound ρ on the norm of the exact solution. From equations (16.5) and (16.6), written down for φ_α^δ and f^δ , we can deduce that

$$\begin{aligned} \alpha \rho^2 &= (A\varphi_\alpha^\delta, f - A\varphi_\alpha^\delta) \leq \rho \|A\| \|f^\delta - A\varphi_\alpha^\delta\| \\ &\leq \rho \|A\| \|f^\delta - AA^{-1}f\| \leq \|A\| \rho \delta \end{aligned}$$

provided that $\|A^{-1}f\| \leq \rho$. Therefore, in this case, we have the estimate

$$(16.10) \quad \alpha \rho \leq \|A\| \delta,$$

which may serve as a starting value for the Newton iteration to find the zero of F . The following theorem answers the question for regularity in the sense of Definition 15.7 for this strategy.

Theorem 16.7. *Let $A : X \rightarrow Y$ be a bounded injective linear operator with dense range in Y , let $f \in A(X)$ and $\rho \geq \|A^{-1}f\|$. For $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$ let φ^δ denote the quasi-solution to $A\varphi = f^\delta$ with constraint ρ . Then there holds weak convergence*

$$\varphi^\delta \rightarrow A^{-1}f, \quad \delta \rightarrow 0.$$

If $\rho = \|A^{-1}f\|$ then there holds norm convergence

$$\varphi^\delta \rightarrow A^{-1}f, \quad \delta \rightarrow 0.$$

Proof: Let $g \in Y$ be arbitrary. Then, since $\|A^{-1}f\| \leq \rho$, we can estimate

$$(16.11) \quad \begin{aligned} |(A\varphi^\delta - f, g)| &\leq \{\|A\varphi^\delta - f^\delta\| + \|f^\delta - f\|\}\|g\| \\ &\leq \{\|AA^{-1}f - f^\delta\| + \|f^\delta - f\|\}\|g\| \leq 2\delta\|g\|. \end{aligned}$$

Hence, $(\varphi^\delta - A^{-1}f, A^*g) \rightarrow 0$, $\delta \rightarrow 0$, for all $g \in Y$. This implies weak convergence $\varphi^\delta \rightharpoonup A^{-1}f$, $\delta \rightarrow 0$, since for the injective operator A the range $A^*(Y)$ is dense in X by Theorem 15.8 and since φ^δ is bounded by $\|\varphi^\delta\| \leq \rho$.

When $\rho = \|A^{-1}f\|$, we then have

$$\begin{aligned} \|\varphi^\delta - A^{-1}f\|^2 &= \|\varphi^\delta\|^2 - 2\operatorname{Re}(\varphi^\delta, A^{-1}f) + \|A^{-1}f\|^2 \\ &\leq 2\operatorname{Re}(A^{-1}f - \varphi^\delta, A^{-1}f) \rightarrow 0, \quad \delta \rightarrow 0, \end{aligned}$$

and the proof is complete. \square

Note that we cannot expect weak convergence if $\rho < \|A^{-1}f\|$ since then we would have the contradiction

$$\|A^{-1}f\|^2 = \lim_{\delta \rightarrow 0} (\varphi^\delta, A^{-1}f) \leq \rho\|A^{-1}f\| < \|A^{-1}f\|^2.$$

In general, we also cannot expect norm convergence if $\rho > \|A^{-1}f\|$ because generically we will have $\|\varphi^\delta\| = \rho$ for all δ . Thus, for regularity we need an exact a-priori information on the norm of the exact solution.

Under additional conditions on f , which may be interpreted as regularity conditions, we can obtain results on the order of convergence.

Theorem 16.8. *Under the assumptions of Theorem 16.7 let $f \in AA^*(Y)$ and $\rho = \|A^{-1}f\|$. Then there holds*

$$\|\varphi^\delta - A^{-1}f\| = O(\delta^{1/2}), \quad \delta \rightarrow 0.$$

Proof: We can write $A^{-1}f = A^*g$ with some $g \in Y$. Therefore, the last inequality in the proof of Theorem 16.7 together with (16.11) yields

$$\|\varphi^\delta - A^{-1}f\|^2 \leq 2\operatorname{Re}(f - A\varphi^\delta, g) \leq 4\delta\|g\|,$$

and this is the desired result. \square

The following counterexample shows that the result of Theorem 16.8 is optimal. Let A be a compact operator with $\dim A(X) = \infty$ and singular system (μ_n, φ_n, g_n) . Consider $f = \mu_1 g_1$ and $f^{\delta_n} = \mu_1 g_1 + \delta_n g_n$ with $\delta_n = \mu_n^2$. Then $A^{-1}f = \varphi_1$ and

$$(16.12) \quad \begin{aligned} \varphi^{\delta_n} &= (\alpha_n I + A^*A)^{-1}A^*(\mu_1 g_1 + \delta_n g_n) \\ &= \frac{\mu_1^2}{\alpha_n + \mu_1^2} \varphi_1 + \frac{\delta_n \mu_n}{\alpha_n + \mu_n^2} \varphi_n, \end{aligned}$$

where α_n must satisfy

$$\frac{\mu_1^4}{(\alpha_n + \mu_1^2)^2} + \frac{\delta_n^2 \mu_n^2}{(\alpha_n + \mu_n^2)^2} = 1$$

in order that φ^{δ_n} is quasi-solution with constraint $\rho = 1$. Assume now that we have convergence order

$$\|\varphi^{\delta_n} - A^{-1}f\| = o(\delta_n^{1/2}), \quad n \rightarrow \infty.$$

Then, using $\delta_n = \mu_n^2$, from (16.12) we find

$$\frac{\delta_n}{\alpha_n + \delta_n} = \frac{\delta_n^{1/2} \mu_n}{\alpha_n + \delta_n} \rightarrow 0, \quad n \rightarrow \infty,$$

whence $\alpha_n/\delta_n \rightarrow \infty$, $n \rightarrow \infty$ follows. But this is a contradiction to the inequality (16.10).

For a compact integral operator $A : L^2[a, b] \rightarrow L^2[a, b]$, the Tikhonov residual functional corresponds to

$$(16.13) \quad \int_a^b |(A\varphi)(x) - f(x)|^2 dx + \alpha \int_a^b |\varphi(x)|^2 dx.$$

The regularized equation (16.2) is an integral equation of the second kind. For its numerical solution the methods of Chapters 12 and 13 are available.

Example 16.9. Consider the integral equation

$$\int_0^1 e^{xy} \varphi(y) dy = \frac{1}{x} (e^x - 1), \quad 0 < x \leq 1.$$

Its unique solution (see Problem 16.4) is given by $\varphi(x) = 1$. For the corresponding integral operator $A : L^2[0, 1] \rightarrow L^2[0, 1]$ elementary calculations yield

$$(A^*A\varphi)(x) = \int_0^1 H(x, y)\varphi(y) dy, \quad 0 \leq x \leq 1,$$

with

$$H(x, y) = \int_0^1 e^{(x+y)z} dz = \frac{1}{x+y} (e^{x+y} - 1)$$

for $x+y > 0$ and $H(0, 0) = 1$. Our numerical results are obtained by discretizing the integral equation of the second kind with $\alpha > 0$ through Nyström's method using the Simpson rule with 8 equidistant intervals. The integral for the right hand side is also evaluated numerically by Simpson's rule. We have assumed a regular error distribution $f_i^\delta = f_i + (-1)^i \delta$ at the grid points and used the norm $\rho = 1$ of the exact solution. The Table 16.1 gives the values of the regularization parameter α – obtained through Newton's method – and the mean square error $E := \|\varphi_\alpha^\delta - \varphi\|_2$ between the regularized solution φ_α^δ and the exact solution φ depending on the error level δ . In addition, the quotient $q := E/\delta^{1/2}$ is listed.

Table 16.1. Numerical results
for Example 16.9

δ	α	E	q
0.02	0.000055	0.112	0.795
0.04	0.000076	0.161	0.805
0.06	0.000093	0.198	0.809
0.08	0.000109	0.229	0.812
0.1	0.000123	0.257	0.814

16.4 Minimum Norm Solutions

The principal motivation for the minimum norm solution is based on the observation that, in general, for erroneous data it does not make too much sense to try and make the residual $\|A\varphi - f\|$ smaller than the error in f . Assume that we have some á-priori bound δ on the error in f . Then we look for elements φ satisfying $\|A\varphi - f\| \leq \delta$ and stabilize by making the norm $\|\varphi\|$ small.

Definition 16.10. Let $A : X \rightarrow Y$ be a bounded linear operator and let $\delta > 0$. For a given $f \in Y$ an element $\varphi_0 \in X$ is called a minimum norm solution of $A\varphi = f$ with discrepancy δ if $\|A\varphi_0 - f\| \leq \delta$ and

$$\|\varphi_0\| = \inf \{\|\varphi\| : \|A\varphi - f\| \leq \delta\}.$$

Note that φ_0 is a minimum norm solution to $A\varphi = f$ with discrepancy δ if and only if φ_0 is a best approximation to the zero element of X with respect to $U_f := \{\varphi \in X : \|A\varphi - f\| \leq \delta\}$. As in the discussion of quasi-solutions, for the sake of simplicity, in dealing with existence of minimum norm solutions we confine ourselves to operators with dense range.

Theorem 16.11. Let $A : X \rightarrow Y$ be a bounded linear operator with dense range in Y and let $\delta > 0$. Then for each $f \in Y$ there exists a unique minimum norm solution of $A\varphi = f$ with discrepancy δ .

Proof: From

$$\|A(\lambda\varphi_1 + (1 - \lambda)\varphi_2) - f\| \leq \lambda\|A\varphi_1 - f\| + (1 - \lambda)\|A\varphi_2 - f\|$$

for all $\varphi_1, \varphi_2 \in X$ and all $\lambda \in (0, 1)$ we observe that U_f is convex. Then, by Theorem 1.27, there exists at most one best approximation to the zero element with respect to U_f .

If $\|f\| \leq \delta$ then clearly $\varphi_0 = 0$ is the minimum norm solution with discrepancy δ . Therefore, we only need to consider the case where $\|f\| > \delta$. Since A has dense range in Y the set U_f is not empty and we again will establish existence of the minimum norm solution by constructing an element φ_0 which satisfies the sufficient condition for the best approximation given in Theorem 1.27. For approximation with respect to U_f this condition reads

$$(16.14) \quad \operatorname{Re}(\varphi_0, \varphi_0 - \varphi) \leq 0$$

for all $\varphi \in X$ with $\|A\varphi - f\| \leq \delta$. Let φ_0 be an element with

$$(16.15) \quad \|A\varphi_0 - f\| = \delta$$

and

$$(16.16) \quad \alpha\varphi_0 + A^*A\varphi_0 = A^*f$$

where $\alpha > 0$. Then,

$$\begin{aligned} \alpha \operatorname{Re}(\varphi_0, \varphi_0 - \varphi) &= \operatorname{Re}(A^*(f - A\varphi_0), \varphi_0 - \varphi) \\ &= \operatorname{Re}(A\varphi_0 - f, A\varphi - f) - \|A\varphi_0 - f\|^2 \leq \delta(\|A\varphi - f\| - \delta) \leq 0. \end{aligned}$$

Hence φ_0 satisfies the condition (16.14) and therefore is a minimum norm solution. We will show that α can be chosen such that the unique solution φ_0 of (16.16) (see Theorem 16.1) fulfills (16.15).

Define a function $G : (0, \infty) \rightarrow \mathbb{R}$ by

$$G(\alpha) := \|A\varphi_\alpha - f\|^2 - \delta^2,$$

where φ_α denotes the unique solution of equation (16.2). We have to show that the continuous function G has a zero. From the convergence (16.7) we observe that $G(\alpha) \rightarrow \|f\|^2 - \delta^2 > 0$, $\alpha \rightarrow \infty$. On the other hand, from (16.8) we obtain that $G(\alpha) \rightarrow -\delta^2 < 0$, $\alpha \rightarrow 0$. This completes the proof. \square

The minimum norm solution can be proven to depend weakly continuously on the right hand side f (see Problem 16.2).

In general, we will have data satisfying $\|f\| > \delta$, i.e., data exceeding the error level. Then, the minimum norm solution with discrepancy δ can be obtained numerically by Newton's method for solving $G(\alpha) = 0$. After rewriting

$$\|f - A\varphi_\alpha\|^2 = (f - A\varphi_\alpha, f) - (A^*(f - A\varphi_\alpha), \varphi_\alpha) = \|f\|^2 - (\varphi_\alpha, A^*f) - \alpha\|\varphi_\alpha\|^2,$$

we get

$$G(\alpha) = \|f\|^2 - (\varphi_\alpha, A^*f) - \alpha\|\varphi_\alpha\|^2 - \delta^2$$

and

$$G'(\alpha) = - \left(\frac{d\varphi_\alpha}{d\alpha}, A^*f \right) - \|\varphi_\alpha\|^2 - 2\alpha \operatorname{Re} \left(\frac{d\varphi_\alpha}{d\alpha}, \varphi_\alpha \right),$$

where the derivative $d\varphi_\alpha/d\alpha$ is again given by (16.9).

We may look at the minimum norm solution as described in Theorem 16.11 as an á-posteriori strategy for the choice of the parameter α in the Tikhonov regularization: given a perturbed right hand side f^δ of an element $f \in A(X)$ with a known error level $\|f^\delta - f\| \leq \delta < \|f^\delta\|$, we choose α such that

$$\varphi_\alpha^\delta = (\alpha I + A^*A)^{-1}A^*f^\delta$$

satisfies $\|A\varphi_\alpha^\delta - f^\delta\| = \delta$. From equations (16.15) and (16.16), applied to φ_α^δ and f^δ , we find

$$\|f^\delta\| - \delta = \|f^\delta\| - \|A\varphi_\alpha^\delta - f^\delta\| \leq \|A\varphi_\alpha^\delta\| = \frac{1}{\alpha} \|AA^*(f^\delta - A\varphi_\alpha^\delta)\| \leq \frac{\|A\|^2 \delta}{\alpha}$$

provided that $\|f^\delta\| > \delta$. Hence, we have established the estimate

$$(16.17) \quad \alpha(\|f^\delta\| - \delta) \leq \|A\|^2 \delta,$$

which we may use as a starting value for the Newton iteration to find the zero of G . The following theorem answers the question for regularity for this so called *discrepancy principle* as introduced by Morozow [1].

Theorem 16.12. *Let $A : X \rightarrow Y$ be a bounded injective linear operator with dense range in Y , let $\delta > 0$ and $f \in A(X)$. For $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta < \|f^\delta\|$ let φ^δ denote the minimum norm solution with discrepancy δ . Then there holds convergence*

$$\varphi^\delta \rightarrow A^{-1}f, \quad \delta \rightarrow 0.$$

Proof: Since $\|f^\delta\| > \delta$, from the proof of Theorem 16.11 we know that φ^δ minimizes the Tikhonov functional

$$\|A\varphi - f^\delta\|^2 + \alpha\|\varphi\|^2.$$

Therefore,

$$\begin{aligned} \delta^2 + \alpha\|\varphi^\delta\|^2 &= \|A\varphi^\delta - f^\delta\|^2 + \alpha\|\varphi^\delta\|^2 \\ &\leq \|AA^{-1}f - f^\delta\|^2 + \alpha\|A^{-1}f\|^2 \leq \delta^2 + \alpha\|A^{-1}f\|^2, \end{aligned}$$

whence

$$(16.18) \quad \|\varphi^\delta\| \leq \|A^{-1}f\|$$

follows. This inequality is also trivially satisfied when $\|f^\delta\| \leq \delta$ since in this case $\varphi^\delta = 0$.

Now let $g \in Y$ be arbitrary. Then, we can estimate

$$|(A\varphi^\delta - f, g)| \leq \{\|A\varphi^\delta - f^\delta\| + \|f^\delta - f\|\}\|g\| \leq 2\delta\|g\|.$$

As in the proof of Theorem 16.7, this now implies weak convergence $\varphi^\delta \rightharpoonup A^{-1}f$, $\delta \rightarrow 0$. Then using (16.18), we obtain

$$\begin{aligned} (16.19) \quad \|\varphi^\delta - A^{-1}f\|^2 &= \|\varphi^\delta\|^2 - 2\operatorname{Re}(\varphi^\delta, A^{-1}f) + \|A^{-1}f\|^2 \\ &\leq 2\{\|A^{-1}f\|^2 - \operatorname{Re}(\varphi^\delta, A^{-1}f)\} \rightarrow 0, \quad \delta \rightarrow 0, \end{aligned}$$

which finishes the proof.

Theorem 16.13. *Under the assumptions of Theorem 16.12 let $f \in AA^*(Y)$. Then there holds*

$$\|\varphi^\delta - A^{-1}f\| = O(\delta^{1/2}), \quad \delta \rightarrow 0.$$

Proof: Writing $A^{-1}f = A^*g$ with some $g \in Y$ from (16.19) we deduce

$$\begin{aligned} \|\varphi^\delta - A^{-1}f\|^2 &\leq 2\operatorname{Re}(A^{-1}f - \varphi^\delta, A^{-1}f) = 2\operatorname{Re}(f - A\varphi^\delta, g) \\ &\leq 2\{\|f - f^\delta\| + \|f^\delta - A\varphi^\delta\|\}\|g\| \leq 4\delta\|g\|. \end{aligned}$$

Using the same example as in connection with Theorem 16.8 it can be shown that the result of Theorem 16.13 is optimal (see Problem 16.3).

Example 16.14. We apply the minimum norm solution to the integral equation of Example 16.9. The Table 16.2 gives the values of the regularization parameter α and the mean square error $E := \|\varphi_\alpha^\delta - \varphi\|_2$ and the quotient $q := E/\delta^{1/2}$ depending on the error level δ .

Table 16.2. Numerical results
for Example 16.14

δ	α	E	q
0.02	0.0059	0.067	0.479
0.04	0.0148	0.108	0.542
0.06	0.0252	0.131	0.536
0.08	0.0359	0.145	0.514
0.1	0.0466	0.155	0.492

16.5 Classical Tikhonov Regularization

In our examples for integral equations of the first kind, so far, we have interpreted the integral operator as a mapping $A : L^2[a, b] \rightarrow L^2[a, b]$ corresponding to the Tikhonov residual functional in the form (16.13). Here, the choice of the data space $Y = L^2[a, b]$, in general, is determined by the need to adequately measure the error in the data. But we have more flexibility concerning the solution space X , in particular when additional regularity properties of the exact solution are a-priori known.

In his pioneering paper on integral equations of the first kind Tikhonov [1] in 1963 suggested to damp out highly oscillating parts in the approximate solution by incorporating the derivative into the penalty term, i.e., to replace (16.13) by

$$(16.20) \quad \int_a^b |(A\varphi)(x) - f(x)|^2 dx + \alpha \int_a^b \{|\varphi(x)|^2 + |\varphi'(x)|^2\} dx.$$

To include this approach into our general theory we need a Hilbert space with norm corresponding to the penalty term in (16.20). Since in Chapter 8 we have introduced the Sobolev spaces only via Fourier expansion, we briefly discuss the definition for the space $H^1[a, b]$ based on the concept of weak derivatives.

Definition 16.15. A function $\varphi \in L^2[a, b]$ is said to have a weak derivative $\varphi' \in L^2[a, b]$ if

$$(16.21) \quad \int_a^b \varphi \psi' dx = - \int_a^b \varphi' \psi dx$$

for all $\psi \in C^1[a, b]$ with $\psi(a) = \psi(b) = 0$.

Clearly, by partial integration, (16.21) is satisfied for functions $\varphi \in C^1[a, b]$. Hence, weak differentiability generalizes classical differentiability.

From the denseness of $\{\psi \in C^1[a, b] : \psi(a) = \psi(b) = 0\}$ in $L^2[a, b]$, or from the Fourier series for the odd extension for φ , it can be seen that the weak derivative, if it exists, is unique. From the denseness of $C[a, b]$ in $L^2[a, b]$, or from the Fourier series for the even extension of φ , it follows that each function with vanishing weak derivative must be constant (almost everywhere). The latter, in particular, implies

$$(16.22) \quad \varphi(x) = \int_a^x \varphi'(\xi) d\xi + \text{const}$$

for almost all $x \in [a, b]$, since by Fubini's theorem

$$\int_a^b \left(\int_a^x \varphi'(\xi) d\xi \right) \psi'(x) dx = \int_a^b \varphi'(\xi) \left(\int_\xi^b \psi'(x) dx \right) d\xi = - \int_a^b \varphi'(\xi) \psi(\xi) d\xi$$

and hence both sides of (16.22) have the same weak derivative.

Theorem 16.16. *The linear space*

$$H^1[a, b] := \{\varphi \in L^2[a, b] : \varphi' \in L^2[a, b]\}$$

endowed with the scalar product

$$(16.23) \quad (\varphi, \psi)_{H^1} := \int_a^b \{(\varphi, \bar{\psi}) + (\varphi', \bar{\psi}')\} dx$$

is a Hilbert space.

Proof: It is readily checked that $H^1[a, b]$ is a linear space and that (16.23) defines a scalar product. Let (φ_n) denote a H^1 -Cauchy sequence. Then (φ_n) and (φ'_n) are both L^2 -Cauchy sequences. From the completeness of $L^2[a, b]$ we obtain the existence of $\varphi \in L^2[a, b]$ and $\chi \in L^2[a, b]$ such that $\|\varphi_n - \varphi\|_2 \rightarrow 0$ and $\|\varphi'_n - \chi\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Then for all $\psi \in C^1[a, b]$ with $\psi(a) = \psi(b) = 0$ we can estimate

$$\begin{aligned} \int_a^b (\varphi \psi' + \chi \psi) dx &= \int_a^b \{(\varphi - \varphi_n)\psi' + (\chi - \varphi'_n)\psi\} dx \\ &\leq \|\varphi - \varphi_n\|_{L^2} \|\psi'\|_{L^2} + \|\chi - \varphi'_n\|_{L^2} \|\psi\|_{L^2} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Therefore, $\varphi \in H^1[a, b]$ with $\varphi' = \chi$ and $\|\varphi - \varphi_n\|_{H^1} \rightarrow 0$, $n \rightarrow \infty$, which completes the proof.

Theorem 16.17. *$C^1[a, b]$ is dense in $H^1[a, b]$.*

Proof: Since $C[a, b]$ is dense in $L^2[a, b]$ for each $\varphi \in H^1[a, b]$ and $\varepsilon > 0$ there exists $\chi \in C[a, b]$ such that $\|\varphi' - \chi\|_2 < \varepsilon$. Then we define $\psi \in C^1[a, b]$ by $\psi(x) := \varphi(a) + \int_a^x \chi(\xi) d\xi$ and, using (16.22), we have

$$\varphi(x) - \psi(x) = \int_a^x \{\varphi'(\xi) - \chi(\xi)\} d\xi.$$

By the Cauchy–Schwarz inequality this implies $\|\varphi - \psi\|_2 < (b-a)^{1/2}\varepsilon$ and the proof is complete.

Theorem 16.18. *$H^1[a, b]$ is contained in $C[a, b]$ with compact imbedding.*

Proof: From (16.22) we have

$$(16.24) \quad \varphi(x) - \varphi(y) = \int_y^x \varphi'(\xi) d\xi$$

whence, by the Cauchy–Schwarz inequality,

$$(16.25) \quad |\varphi(x) - \varphi(y)| \leq |x - y|^{1/2} \|\varphi'\|_2$$

follows for all $x, y \in [a, b]$. Therefore, every function $\varphi \in H^1[a, b]$ belongs to $C[a, b]$.

Choose $y \in [a, b]$ such that $|\varphi(y)| = \min_{a \leq x \leq b} |\varphi(x)|$. Then from

$$(b-a) \min_{a \leq x \leq b} |\varphi(x)|^2 \leq \int_a^b |\varphi(x)|^2 dx$$

and (16.24), again by the Cauchy–Schwarz inequality, we find that

$$(16.26) \quad \|\varphi\|_\infty \leq C \|\varphi\|_{H^1}$$

holds with some constant C . The latter inequality means that the H^1 –norm is stronger than the maximum norm (in one space dimension!).

The inequalities (16.25) and (16.26), in particular, imply that each bounded set in $H^1[a, b]$ is a bounded and equicontinuous subset of $C[a, b]$. Hence, by the Arzelà–Ascoli Theorem 1.18, the imbedding operator from $H^1[a, b]$ into $C[a, b]$ is compact. \square

Now we return to integral equations of the first kind and interprete the integral operator with smooth kernel as a mapping from $H^1[a, b]$ into $L^2[a, b]$. Then, our complete theory on regularization including convergence and regularity remains applicable in this setting. Since the imbedding from $H^1[a, b]$ into $L^2[a, b]$ clearly is bounded (by Theorem 16.18 it is even compact) the integral operator $A : H^1[a, b] \rightarrow L^2[a, b]$ is compact by Theorem 2.15. Only the adjoint operator $\tilde{A}^* : L^2[a, b] \rightarrow H^1[a, b]$ now looks differently. We denote it by \tilde{A}^* to distinguish it from the adjoint $A^* : L^2[a, b] \rightarrow L^2[a, b]$ of $A : L^2[a, b] \rightarrow L^2[a, b]$. We avoid its explicit calculation through the observation that the regularized equation

$$(16.27) \quad \alpha \varphi_\alpha + \tilde{A}^* A \varphi_\alpha = \tilde{A}^* f$$

is equivalent to

$$(16.28) \quad \alpha(\varphi_\alpha, \psi)_{H^1} + (A^* A \varphi_\alpha, \psi)_{L^2} = (A^* f, \psi)_{L^2}$$

for all $\psi \in H^1[a, b]$. This follows from the fact that

$$(\tilde{A}^* \chi, \psi)_{H^1} = (\chi, A \psi)_{L^2} = (A^* \chi, \psi)_{L^2}$$

for all $\chi \in L^2[a, b]$ and $\psi \in H^1[a, b]$. By Theorem 16.1, there exists a unique solution φ_α to (16.27). In the following theorem we will show that φ_α is the solution to a boundary value problem for an integro differential equation.

Theorem 16.19. *Assume that the integral operator A has continuous kernel. Then the unique solution φ_α of the regularized equation (16.27) belongs to $C^2[a, b]$ and satisfies the integro differential equation*

$$(16.29) \quad \alpha(\varphi_\alpha - \varphi''_\alpha) + A^*A\varphi_\alpha = A^*f$$

and the boundary condition

$$(16.30) \quad \varphi'_\alpha(a) = \varphi'_\alpha(b) = 0.$$

Proof: First we define $g \in C^1[a, b]$ by

$$g(x) := - \int_x^b \{(A^*A\varphi_\alpha)(\xi) + \alpha\varphi_\alpha(\xi) - (A^*f)(\xi)\} d\xi$$

and then $\psi \in H^1[a, b]$ by

$$\psi(x) := \int_a^x \{\alpha\varphi'_\alpha(\xi) - g(\xi)\} d\xi.$$

Then, by partial integration, since $g(b) = \psi(a) = 0$, we get

$$\begin{aligned} \|\alpha\varphi'_\alpha - g\|_{L^2}^2 &= \int_a^b (\alpha\varphi'_\alpha - g)\bar{\psi}' dx = \int_a^b (\alpha\varphi'_\alpha\bar{\psi}' + g'\bar{\psi}) dx \\ &= \alpha(\varphi_\alpha, \psi)_{H^1} + (A^*A\varphi_\alpha, \psi)_{L^2} - (A^*f, \psi)_{L^2} = 0. \end{aligned}$$

Note that by the denseness Theorem 16.17 partial integration can be carried over from $C^1[a, b]$ into $H^1[a, b]$. Our last equation implies $\alpha\varphi'_\alpha = g$, whence $\varphi_\alpha \in C^2[a, b]$ and the integro differential equation (16.29) follows. Plugging (16.29) into (16.28) and performing a partial integration yields

$$\varphi'_\alpha(b)\overline{\psi(b)} - \varphi'_\alpha(a)\overline{\psi(a)} = 0$$

for all $\psi \in H^1[a, b]$. This now yields that the boundary conditions (16.30) are fulfilled.

Note that by partial integration a solution of (16.29) and (16.30) also solves (16.28). Hence, the boundary value problem for the integro differential equation and the regularized equation (16.27) are equivalent.

Example 16.20. We compare the Tikhonov regularization based on the L^2 -norm and the H^1 -norm penalty term for the integral equation of Example 16.9. For the discretization of the regularized equation we use the Petrov–Galerkin method with linear splines.

Let $x_j = jh$, $j = 0, \dots, n$, be an equidistant grid with stepsize $h = 1/n$ and let L_j , $j = 0, \dots, n$, denote the corresponding Lagrange basis for linear spline

interpolation as introduced by (11.7). Then straightforward calculations yield the tridiagonal matrices

$$V = \frac{1}{h} \begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & 0 \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}$$

for the weights $v_{mk} = \int_0^1 L'_m(y) L'_k(y) dy$ and

$$W = \frac{h}{6} \begin{pmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & 0 \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & 1 & 4 & 1 \\ & & & & 1 & 2 & \end{pmatrix}$$

for the weights $w_{mk} = \int_0^1 L_m(y) L_k(y) dy$. Writing the approximate solution in the form

$$\varphi_n = \sum_{k=0}^n \gamma_k L_k,$$

we have to solve the linear system

$$\begin{aligned} \sum_{k=0}^n \gamma_k \{ \alpha(w_{jk} + v_{jk}) &+ \int_0^1 \int_0^1 H(x, y) L_j(x) L_k(y) dx dy \} \\ &= \int_0^1 \int_0^1 K(y, x) L_j(x) f(y) dx dy, \quad j = 0, \dots, n, \end{aligned}$$

in the case of the H_1 -norm penalty term. Here, K and H denote the kernels of the integral operators A and A^*A , respectively (see Example 16.9). In the case of the L^2 -norm the matrix V has to be omitted. Note that the weight matrix V indicates how oscillations in the approximate solution are penalized by the regularization employing the derivative. For the numerical evaluation of the matrix elements we apply interpolatory quadratures as described in detail in Chapters 11 and 13.

The Table 16.3 compares the mean square error between the exact and the approximate solution for both regularizations for an error distribution as in Example 16.9 with $\delta = 0.01$.

Table 16.3. Numerical results for Example 16.20

$\log \alpha$	-7	-6	-5	-4	-3	-2	-1
L^2 -penalty	41.83	6.541	3.268	1.567	0.250	0.040	0.107
H^1 -penalty	3.503	2.353	0.550	0.063	0.015	0.015	0.019

In closing this chapter, we wish to point out that in a paper predating that of Tikhonov, Phillips [1] proposed to penalize only by the L^2 -norm of the derivative, i.e., to replace (16.20) by

$$(16.31) \quad \int_a^b |(A\varphi)(x) - f(x)|^2 dx + \alpha \int_a^b |\varphi'(x)|^2 dx.$$

Note that this differs from the Tikhonov regularization because the penalty term is only a semi-norm rather than a norm. For an analysis of this approach we refer to Groetsch [1]. In view of our last Theorem 16.19 it is no surprise that minimizing (16.31) is equivalent to the boundary value problem for the integro differential equation

$$-\alpha\varphi''_\alpha + A^*A\varphi_\alpha = A^*f$$

with boundary condition

$$\varphi'_\alpha(a) = \varphi'_\alpha(b) = 0.$$

Problems

- 16.1. Show that the quasi-solution given by Theorem 16.6 depends weakly continuously on the right hand side, i.e., from norm convergence $f_n \rightarrow f$, $n \rightarrow \infty$, there follows weak convergence for the corresponding quasi-solutions $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$. Show that in the complement of $A(B[0; \rho])$ the quasi-solution depends continuously on the right hand side. How are these results related to the regularity Theorem 16.7?

Hint: By Theorem 16.4 the sequence (φ_n) contains a subsequence which converges weakly to some element ψ in X . Establish that $\|\psi\| \leq \rho$ and $\|A\psi - f\| \leq \|A\varphi - f\|$, i.e., ψ is quasi-solution and, by uniqueness, must coincide $\psi = \varphi$. Use this result to show that the (φ_n) itself converges weakly to φ . Finally, conclude norm convergence from $\|\varphi_n\| = \rho$.

- 16.2. Show that the minimum norm solution given by Theorem 16.11 depends weakly continuously on the right hand side. Show that in the complement of $B[0; \delta] \subset Y$ the minimum norm solution depends continuously on the right hand side. How are these results related to the regularity Theorem 16.12?

Hint: Show that there exists an element $\chi \in X$ with $\chi \in U_{f_n}$ for all sufficiently large n . Therefore the sequence (φ_n) is bounded and contains a weakly convergent subsequence with limit $\psi \in X$. Show that $\|A\psi - f\| \leq \delta$ and $\|\psi\| \leq \|\varphi\|$, i.e., ψ is minimum norm solution and, by uniqueness must coincide $\psi = \varphi$. Use this result to show that the sequence (φ_n) itself converges weakly to φ . Finally, conclude norm convergence from $\|A\varphi_n - f_n\| = \delta$.

- 16.3. Show that the convergence order given in Theorem 16.13 is optimal.

- 16.4. Show that the solution of the integral equation in Example 16.9 is unique.

Hint: Show that $\int_0^1 y^n \varphi(y) dy = 0$ for each solution φ of the homogeneous equation.

- 16.5. Show that a linear operator $A : X \rightarrow Y$ from a Hilbert space X into a Hilbert space Y is compact if and only if weak convergence $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$, implies norm convergence $A\varphi_n \rightarrow A\varphi$, $n \rightarrow \infty$.

17. Regularization by Discretization

We briefly return to the study of projection methods and will consider their application to ill-posed equations of the first kind. In particular we will present an exposition of the moment discretization method.

17.1 Projection Methods for Ill-Posed Equations

Numerically solving an equation of the first kind

$$(17.1) \quad A\varphi = f$$

with an injective compact operator $A : X \rightarrow Y$ from a Banach space X into a Banach space Y without regularization usually means approximately solving it by a projection method. The ill-posedness of the equation (17.1) then will cause the condition number of the discrete linear system to grow with the dimension n of the subspace used for the projection method. Increasing n will make the error due to the discretization smaller but due to the ill-conditioning the computation will be contaminated by errors in the given data f . Note that in actual numerical computations such errors will automatically occur because of round-off effects for the data. On the other hand, if n is small then the approximation is robust against errors in f but will be inaccurate due to a large discretization error. This dilemma calls for a compromise in the choice of the discretization parameter n .

We briefly recall the basic facts on projection methods from Chapter 13. Let $X_n \subset X$ and $Y_n \subset Y$ be two sequences of subspaces with $\dim X_n = \dim Y_n = n$ and let $P_n : Y \rightarrow Y_n$ be projection operators. For given $f \in A(X)$ the projection method approximates the solution $\varphi \in X$ of $A\varphi = f$ by the solution $\varphi_n \in X_n$ of the projected equation

$$(17.2) \quad P_n A \varphi_n = P_n f.$$

For the remainder of this chapter we assume the projection method to be chosen such that it is convergent for exact right hand sides $f \in A(X)$. This, by Theorem 13.6, implies that the finite dimensional operators

$$A_n := P_n A : X_n \rightarrow Y_n$$

are invertible and that the sequence of operators $A_n^{-1} P_n A : X \rightarrow X$ is uniformly bounded. Then the operators $R_n := A_n^{-1} P_n : Y \rightarrow X$ provide a regularization

scheme in the sense of Definition 15.5 with regularization parameter n . The approximation

$$(17.3) \quad \varphi_n^\delta := R_n f^\delta$$

corresponding to (15.6) just coincides with numerically performing the projection method with the inexact right hand side f^δ .

Theorem 17.1. *Let $A : X \rightarrow Y$ be an injective compact linear operator, let $f \in A(X)$ and let $f^\delta \in Y$ satisfy $\|f^\delta - f\| \leq \delta$. Then the approximation $\varphi_n^\delta := R_n f^\delta$ through the projection method satisfies an error estimate*

$$(17.4) \quad \|\varphi_n^\delta - A^{-1}f\| \leq \|R_n\| \delta + C \inf_{\psi \in X_n} \|\psi - A^{-1}f\|$$

with some constant C .

Proof: This follows from Theorem 13.6 by decomposing

$$\begin{aligned} \|\varphi_n^\delta - A^{-1}f\| &\leq \|R_n\| \|f^\delta - f\| + \|R_n f - A^{-1}f\| \\ &\leq \|R_n\| \delta + (1+M) \inf_{\psi \in X_n} \|\psi - A^{-1}f\|, \end{aligned}$$

where M is a bound on $\|R_n A\|$. □

The error estimate (17.4) corresponds to (15.7) and illustrates that the total error consists of two parts: the influence of the incorrect data and the discretization error of the projection method. Recall that for the compact operator A , by Theorem 15.6, the operators R_n cannot be uniformly bounded.

At this point we wish to emphasize the fact that for the analysis of the discretization error we may use any Banach space structure for the range space. In particular, we may choose the image space such that the equation becomes well-posed. Hence, we can apply the convergence results on projection methods given in Chapter 13. Only for the influence of the error in the right hand side we have to stick with an image space as dictated by the needs of the particular problem. We want to make these remarks more precise through the following

Theorem 17.2. *Let $A : X \rightarrow Y$ be an injective compact operator from a Banach space X into a Banach space Y and assume an additional Banach space $Z \subset Y$ with continuous imbedding such that $A : X \rightarrow Z$ has a bounded inverse. Let $X_n \subset X$ and $Y_n \subset Z \subset Y$ be two sequences of subspaces with $\dim X_n = \dim Y_n = n$ and let $P_n : Y \rightarrow Y_n$ be projection operators (bounded with respect to the norm on Y) and assume the projection method to be convergent for all $f \in A(X) = Z$. Then for the operators $R_n : Y \rightarrow X$ there holds*

$$\|R_n\| \leq C \|P_n\| \gamma_n,$$

where $\gamma_n := \sup\{\|f\|_Z : f \in Y_n, \|f\|_Y \leq 1\}$ and C is some constant.

Proof: Since, by Theorem 1.6, all norms on the finite dimensional subspace Y_n are equivalent, we have $\gamma_n < \infty$. Then, from $\|f\|_Z \leq \gamma_n \|f\|_Y$ for all $f \in Y_n$, it follows that

$$(17.5) \quad \|P_n f\|_Z \leq \gamma_n \|P_n f\|_Y \leq \gamma_n \|P_n\| \|f\|_Y \leq M \gamma_n \|P_n\| \|f\|_Z$$

for all $f \in Y$ with some constant M , since the imbedding from Z into Y is assumed to be bounded. Hence, the operators P_n are also bounded with respect to the norm on Z and we can interpret the projection scheme for $A : X \rightarrow Y$ also as a projection scheme in the structure $A : X \rightarrow Z$. Since we assume convergence for all $f \in A(X)$, by Theorem 13.6, we know that the operators $A_n^{-1} P_n A : X \rightarrow X$ are uniformly bounded. Therefore, for all $f \in Y$ with $\|f\|_Y = 1$, observing $A^{-1} : Z \rightarrow X$ and using (17.5), we can estimate

$$\begin{aligned} \|R_n f\| &= \|A_n^{-1} P_n f\| = \|A_n^{-1} P_n A A^{-1} P_n f\| \\ &\leq \|A_n^{-1} P_n A\| \|A^{-1}\| \|P_n f\|_Z \leq C \gamma_n \|P_n\| \end{aligned}$$

with some constant C . □

We illustrate the use of Theorem 17.2 by considering

Example 17.3. Let $A : L^2[0, 2\pi] \rightarrow L^2[0, 2\pi]$ be given by

$$(17.6) \quad (A\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \ln 4 \sin^2 \frac{t-\tau}{2} + K(t, \tau) \right\} \varphi(\tau) d\tau, \quad 0 \leq t \leq 2\pi,$$

where the kernel K is assumed to be analytic and 2π -periodic with respect to both variables (see equation (13.26)). By Theorem 13.26, we can interpret A as a bounded operator $A : L^2[0, 2\pi] \rightarrow H^1[0, 2\pi]$ with bounded inverse $A^{-1} : H^1[0, 2\pi] \rightarrow L^2[0, 2\pi]$, provided A is injective. From Theorem 13.26 we also know that the Petrov–Galerkin method using trigonometric polynomials converges for $A : L^2[0, 2\pi] \rightarrow H^1[0, 2\pi]$. In particular, if the right hand side also is analytic we have an exponentially decreasing discretization error. On the other hand, for the orthogonal projection, of course, there holds $\|P_n\| = 1$ and for the subspace Y_n of trigonometric polynomials of degree n we readily see that $\gamma_n = \sqrt{1+n^2}$. Hence, in this case we have $\|R_n\| = O(n)$ and the equation of the first kind with the operator (17.6) is only mildly ill-posed. Therefore, we conclude that it is numerically safe to apply the Petrov–Galerkin method in this case (see Example 13.20 and Problem 17.1). □

For projection methods the norm on the image space Y might not always be the most appropriate measure to describe the inaccuracy of the data. Consider the Petrov–Galerkin method for an operator $A : X \rightarrow Y$ between two Hilbert spaces X and Y . With subspaces $X_n = \text{span}\{u_1, \dots, u_n\}$ and $Y_n = \text{span}\{v_1, \dots, v_n\}$ we write

$$\varphi_n = \sum_{k=1}^n \gamma_k u_k$$

and then have to solve the Petrov–Galerkin equations

$$(17.7) \quad \sum_{k=1}^n \gamma_k (Au_k, v_j) = (f, v_j), \quad j = 1, \dots, n,$$

for the coefficients $\gamma_1, \dots, \gamma_n$. Here, it makes more sense to measure the error of the discrete data vector

$$F = ((f, v_1), \dots, (f, v_n))$$

in the Euclidean norm on \mathbb{C}^n (compare also the stability analysis of Section 14.3 for equations of the second kind). Let $F^d \in \mathbb{C}^n$ be a perturbed right hand side for (17.7) with

$$\|F^d - F\|_2 \leq d$$

and denote the Petrov–Galerkin solution to these inexact data by φ_n^d . After introducing the matrix operators $E_n, \tilde{A}_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$(E_n \gamma)_j := \sum_{k=1}^n (u_k, u_j) \gamma_k, \quad j = 1, \dots, n,$$

and

$$(\tilde{A}_n \gamma)_j := \sum_{k=1}^n (Au_k, v_j) \gamma_k, \quad j = 1, \dots, n,$$

we can formulate the following error estimate which replaces Theorem 17.1.

Theorem 17.4. *The error in the Petrov–Galerkin method can be estimated by*

$$(17.8) \quad \|\varphi_n^d - A^{-1}f\| \leq \frac{d}{\sqrt{\lambda_n \nu_n}} + C \inf_{\psi \in X_n} \|\psi - A^{-1}f\|,$$

where λ_n and ν_n denote the smallest singular values of \tilde{A}_n and $\tilde{A}_n E_n^{-1}$, respectively, and C is some constant.

Proof: Using the singular value decomposition Theorem 15.16, we can estimate

$$\begin{aligned} \|\varphi_n\|^2 &= \sum_{j, k=1}^n \gamma_j \bar{\gamma}_k (u_j, u_k) = (E_n \gamma, \gamma) = (E_n \tilde{A}_n^{-1} F, \tilde{A}_n^{-1} F) \\ &\leq \|E_n \tilde{A}_n^{-1}\|_2 \|\tilde{A}_n^{-1}\|_2 \|F\|_2^2 \leq \frac{1}{\lambda_n \nu_n} \|F\|_2^2. \end{aligned}$$

Now we apply this inequality to the difference $\varphi_n^d - \varphi_n$ and obtain (17.8) by decomposing the error as in the proof of Theorem 17.1. \square

For the least squares method as described in Theorem 13.25, that is, for $Y_n = A(X_n)$, the matrix operator \tilde{A}_n assumes the form

$$(\tilde{A}_n \gamma)_j := \sum_{k=1}^n (Au_k, Au_j) \gamma_k, \quad j = 1, \dots, n,$$

and is self adjoint and positive definite. In particular, for an orthonormal basis u_1, \dots, u_n of X_n , we have $E_n = I$. Hence, in this case we only need to be concerned with the eigenvalues of \tilde{A}_n . Let (μ_m, φ_m, g_m) , $m = 1, 2, \dots$, be a singular system of the compact operator A . Then we can choose an element

$$\varphi = \sum_{j=1}^n \gamma_j u_j$$

in X_n such that

$$(\varphi, \varphi_m) = 0, \quad m = 1, \dots, n-1,$$

and $\|\varphi\| = 1$. Now, by the singular value decomposition Theorem 15.16, we obtain

$$\mu_n^2 \geq (A^* A \varphi, \varphi) = (\tilde{A}_n \gamma, \gamma) \geq \lambda_n,$$

since $\|\gamma\|_2 = \|\varphi\| = 1$. Therefore,

$$\frac{1}{\lambda_n} \geq \frac{1}{\mu_n^2}.$$

This again exhibits the influence of the singular values on the degree of ill-posedness.

For the integral equation of Example 16.20, by Theorem 15.20, the singular values decay at least exponentially. Hence, projection methods cannot be recommended for this equation.

17.2 The Moment Method

We proceed with describing the moment method for approximately solving an equation of the first kind. Let $A : X \rightarrow Y$ be an injective bounded linear operator from a Hilbert space X into another Hilbert space Y . Choose an n -dimensional subspace $Y_n \subset Y$ and, given an element $f \in Y$, define the affine linear subspace $U_n \subset X$ by

$$U_n := \{\varphi \in X : (A\varphi, g) = (f, g), g \in Y_n\}.$$

We assume U_n to be non-empty. By the Riesz representation Theorem 4.8, bounded linear functionals in a Hilbert space can be equivalently expressed by scalar products. Therefore, we may interpret U_n as the set of all elements $\varphi \in X$ for which n linearly independent functionals vanish when applied to the residual $A\varphi - f$.

Definition 17.5. An element $\varphi_n \in U_n$ is called a moment solution to $A\varphi = f$ with respect to Y_n if

$$\|\varphi_n\| = \inf_{\varphi \in U_n} \|\varphi\|.$$

Note that φ_n is a moment solution with respect to Y_n if and only if it is a best approximation to the zero element in X with respect to the closed affine linear subspace U_n . Finding a best approximation with respect to an affine

linear subspace can be equivalently reduced to finding a best approximation with respect to a linear subspace. Therefore, by Theorem 1.26 there exists a unique moment solution, and by Theorem 1.25 it can be characterized as the unique element $\varphi_n \in U_n$ satisfying

$$(17.9) \quad (\varphi_n - \varphi, \varphi_n) = 0$$

for all $\varphi \in U_n$.

The following theorem gives an interpretation of the moment method as a projection method.

Theorem 17.6. *Let $A : X \rightarrow Y$ be an injective bounded linear operator with dense range in Y . Then the moment approximate solution of $A\varphi = f$ with respect to a subspace $Y_n \subset Y$ coincides with the Petrov–Galerkin approximation corresponding to the subspaces $X_n := A^*(Y_n) \subset X$ and $Y_n \subset Y$.*

Proof: By assumption there holds $\overline{A(X)} = Y$ and therefore, by Theorem 15.8, the adjoint operator $A^* : Y \rightarrow X$ is injective. Hence, $\dim X_n = \dim Y_n = n$. Let $P_n : Y \rightarrow Y_n$ be the orthogonal projection. We will show that the Petrov–Galerkin operator $A_n := P_n A : X_n \rightarrow Y_n$ is injective. Indeed, let $\varphi \in X_n$ satisfy $A_n \varphi = 0$. We can write $\varphi = A^*g$ with some $g \in Y_n$ and have

$$\|\varphi\|^2 = (A^*g, A^*g) = (AA^*g, g) = (P_n A \varphi, g) = 0.$$

The injectivity of A_n ensures that the Petrov–Galerkin approximation φ_n exists and is unique. It can be written in the form $\varphi_n = A^*g_n$ with $g_n \in Y_n$ and satisfies

$$(17.10) \quad (AA^*g_n, g) = (f, g)$$

for all $g \in Y_n$. This implies $\varphi_n \in U_n$ and

$$(\varphi_n - \varphi, \varphi_n) = (\varphi_n - \varphi, A^*g_n) = (A\varphi_n - A\varphi, g_n) = 0$$

for all $\varphi \in U_n$. Therefore, by (17.9), φ_n is the moment solution with respect to U_n . \square

When $f \in A(X)$ we write $A\varphi = f$ and have (17.10) to be equivalent to

$$(A^*g_n, A^*g) = (\varphi, A^*g)$$

for all $g \in Y_n$. This implies that the moment method corresponds to the least squares method for the adjoint (or dual) equation $A^*h = \varphi$ with respect to Y_n . Therefore it is also known as the *dual least squares method* (for the least squares method recall Theorem 13.25). We continue our analysis with a convergence result on the moment method. Note that the least squares method, in general, will not converge (see the counterexample in Problem 17.2).

Theorem 17.7. *Let A be as in Theorem 17.6 and let $f \in A(X)$. Let the sequence of subspaces $Y_n \subset Y$ possess the denseness property that*

$$(17.11) \quad \inf_{h \in Y_n} \|h - g\| \rightarrow 0, \quad n \rightarrow \infty,$$

for all $g \in Y$. Then the moment method converges

$$\varphi_n \rightarrow A^{-1}f, \quad n \rightarrow \infty.$$

Proof: Let $P_n : Y \rightarrow Y_n$ and $Q_n : X \rightarrow X_n = A^*(Y_n)$ be the orthogonal projections. We will show that (17.11) implies convergence

$$Q_n\varphi \rightarrow \varphi, \quad n \rightarrow \infty,$$

for all $\varphi \in X$. Since A is injective, by Theorem 15.8, the range $A^*(Y)$ is dense in X . Therefore, given $\varphi \in X$ and $\varepsilon > 0$, there exists $g \in Y$ such that $\|A^*g - \varphi\| < \varepsilon/2$. As a consequence of (17.11) we can choose $N \in \mathbb{N}$ such that $\|P_ng - g\| < \varepsilon/2\|A\|$ for all $n \geq N$. Since $A^*P_ng \in X_n$, for all $n \geq N$ we can estimate

$$\|Q_n\varphi - \varphi\| \leq \|A^*P_ng - \varphi\| \leq \|A^*(P_ng - g)\| + \|A^*g - \varphi\| < \varepsilon.$$

Now the assertion of the theorem follows from the fact that for $f \in A(X)$ the dual least squares solution φ_n is given by $\varphi_n = Q_nA^{-1}f$. Indeed, the orthogonal projection satisfies

$$(Q_nA^{-1}f, A^*g) = (A^{-1}f, A^*g)$$

for all $g \in Y_n$. This is equivalent to

$$(AQ_nA^{-1}f, g) = (f, g)$$

for all $g \in Y_n$. Hence $Q_nA^{-1}f$ solves the Petrov–Galerkin equation. \square

For the dual least squares method, from $X_n = A^*(Y_n)$ we observe that the matrix operators occurring in Theorem 17.4 are related by $E_n = \tilde{A}_n$. Therefore, in this case the error estimate (17.8) can be simplified to

$$(17.12) \quad \|\varphi_n^d - A^{-1}f\| \leq \frac{d}{\sqrt{\lambda_n}} + C \inf_{\psi \in A^*(Y_n)} \|\psi - A^{-1}f\|.$$

17.3 Hilbert Spaces with Reproducing Kernel

For an application of the above results to collocation methods for equations of the first kind, obviously, we need a Hilbert space in which the point evaluation functionals are bounded. Therefore, we briefly introduce the concept of Hilbert spaces with reproducing kernel.

Definition 17.8. Let H be a Hilbert space of real or complex valued functions f defined on an interval $[a, b]$. A function M on $[a, b] \times [a, b]$ is called a *reproducing kernel* if $h_x := M(x, \cdot)$ belongs to H for all $x \in [a, b]$ and

$$(17.13) \quad (f, h_x) = f(x)$$

for all $x \in [a, b]$ and all $f \in H$.

Theorem 17.9. *A Hilbert space H has a reproducing kernel if and only if the evaluation functionals*

$$f \mapsto f(x)$$

are bounded for all $x \in [a, b]$.

Proof: If H has a reproducing kernel, then (17.13) implies continuity for the evaluation functionals. Conversely, if the evaluation functionals are bounded, then by Theorem 4.8 for each $x \in [a, b]$ there exists an element $h_x \in H$ such that (17.13) holds for all $f \in H$. Hence $M(x, \cdot) := h_x$ is a reproducing kernel. \square

Obviously, $L^2[a, b]$ is not a Hilbert space with reproducing kernel, whereas $H^1[a, b]$ is, since by (16.26) the maximum norm is weaker than the H^1 -norm.

For each injective linear operator $A : X \rightarrow Y$ acting between two Hilbert spaces X and Y we can make the range $H := A(X)$ a Hilbert space with the scalar product

$$(17.14) \quad (f, g)_H = (A^{-1}f, A^{-1}g)_X$$

for all $f, g \in A(X)$. Indeed, since A is injective (17.14) defines a scalar product on $A(X)$. Assume (f_n) is a Cauchy sequence in H . Then we can write $f_n = A\varphi_n$ with $\varphi_n \in X$ and find that (φ_n) is a Cauchy sequence in X because

$$\|\varphi_n - \varphi_m\|_X = \|f_n - f_m\|_H.$$

Since X is a Hilbert space, there exists $\varphi \in X$ with $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$. Then $f = A\varphi \in H$ and

$$\|f_n - f\|_H = \|\varphi_n - \varphi\|_X \rightarrow 0, \quad n \rightarrow \infty.$$

Hence, the range $A(X)$ with scalar product given by (17.14) is a Hilbert space. By (17.14), the adjoint operator $A^* : H \rightarrow X$ of the bounded linear operator $A : X \rightarrow H$ clearly is given by $A^* = A^{-1}$.

In the case of an integral operator this Hilbert space turns out to have a reproducing kernel.

Theorem 17.10. *Let $A : L^2[a, b] \rightarrow L^2[a, b]$ denote an injective integral operator*

$$(A\varphi)(x) = \int_a^b K(x, y)\varphi(y)dy, \quad a \leq x \leq b,$$

with continuous kernel K . Then the range $H = A(L^2[a, b])$ furnished with the scalar product

$$(f, g)_H = (A^{-1}f, A^{-1}g)$$

is a Hilbert space with reproducing kernel

$$(17.15) \quad M(x, y) = \int_a^b \overline{K(x, z)}K(y, z)dz.$$

Proof: Define M by (17.15) and set $h_x := M(x, \cdot)$ and $k_x := \overline{K(x, \cdot)}$. Then

$$(Ak_x)(y) = \int_a^b K(y, z)\overline{K(x, z)}dz = M(x, y),$$

that is,

$$(17.16) \quad Ak_x = h_x$$

for all $x \in [a, b]$. Therefore, writing $f = A\varphi$, we find

$$\begin{aligned} (f, h_x)_H &= (A^{-1}f, A^{-1}h_x) = (\varphi, k_x) \\ &= \int_a^b K(x, y)\varphi(y)dy = (A\varphi)(x) = f(x). \end{aligned}$$

Hence H is a Hilbert space with reproducing kernel M . \square

As is easily seen, the statement of Theorem 17.10 remains valid for a weakly singular kernel K with the property that M , defined by (17.15), is continuous.

For a detailed study of Hilbert spaces with reproducing kernels we refer to Aronszajn [1] and Meschkowski [1].

17.4 Moment Collocation

For the numerical solution of the integral equation of the first kind

$$(17.17) \quad \int_a^b K(x, y)\varphi(y)dy = f(x), \quad a \leq x \leq b,$$

with continuous kernel K by moment collocation we choose n collocation points $x_j \in [a, b]$, $j = 1, \dots, n$. We approximate the solution φ of (17.17) by a function φ_n with minimal L^2 -norm satisfying the integral equation at the collocation points

$$(17.18) \quad \int_a^b K(x_j, y)\varphi_n(y)dy = f(x_j), \quad j = 1, \dots, n.$$

With the kernel function given in Theorem 17.10 we can rewrite (17.18) in the form

$$(17.19) \quad (A\varphi_n, h_{x_j})_H = (f, h_{x_j})_H, \quad j = 1, \dots, n.$$

Hence, the moment collocation coincides with the moment method applied to the integral operator $A : L^2[a, b] \rightarrow H$ with the subspaces $H_n \subset H$ given by $H_n = \text{span}\{h_{x_1}, \dots, h_{x_n}\}$. In particular, from the analysis in the two previous Sections 17.2 and 17.3 we conclude that a unique solution to (17.18) with minimal norm exists and is given by the orthogonal projection of the exact solution φ onto the subspace

$$X_n = A^*(H_n) = A^{-1}(H_n) = \text{span}\{\overline{K(x_1, \cdot)}, \dots, \overline{K(x_n, \cdot)}\}.$$

We may assume that the kernel functions $\overline{K(x_1, \cdot)}, \dots, \overline{K(x_n, \cdot)}$ are linearly independent, since otherwise the equations (17.18) would be linearly dependent. Then, writing

$$(17.20) \quad \varphi_n = \sum_{k=1}^n \gamma_k \overline{K(x_k, \cdot)}$$

we have to solve the linear system

$$(17.21) \quad \sum_{k=1}^n \gamma_k \int_a^b \overline{K(x_k, y)} K(x_j, y) dy = f(x_j), \quad j = 1, \dots, n,$$

with positive definite matrix for the coefficients $\gamma_1, \dots, \gamma_n$.

Theorem 17.11. *Assume that the sequence (x_n) of collocation points is dense in $[a, b]$. Then the corresponding moment collocation solutions converge*

$$\|\varphi_n - A^{-1}f\|_2 \rightarrow 0, \quad n \rightarrow \infty.$$

Proof: By Theorem 17.7, applied to $A : L^2[a, b] \rightarrow H$, it suffices to show that $U := \text{span}\{h_{x_n} : n \in \mathbb{N}\}$ is dense in H . Note that all $g \in H$ are continuous since A has continuous kernel K . Denote by $P : H \rightarrow \bar{U}$ the orthogonal projection. For $g \in H$, by Theorem 1.25, we have $Pg - g \perp U$. This implies $(Pg - g, h_{x_n}) = 0$ for all $n \in \mathbb{N}$, i.e., $(Pg)(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. From this, by the denseness of the collocation points and the continuity of Pg and g , it follows that $g = Pg \in \bar{U}$. \square

For the error we can apply the special case (17.12) of Theorem 17.4. Let $F^d \in \mathbb{C}^n$ be a perturbation of the data vector

$$F = (f(x_1), \dots, f(x_n))$$

with $\|F^d - F\|_2 \leq d$ and denote the moment collocation solution to these inexact data via (17.20) and (17.21) by φ_n^d . Then we have the following error estimate.

Theorem 17.12. *The error in the moment collocation can be estimated by*

$$(17.22) \quad \|\varphi_n^d - A^{-1}f\|_2 \leq \frac{d}{\sqrt{\lambda_n}} + C \inf_{\psi \in X_n} \|\psi - A^{-1}f\|_2,$$

where λ_n denotes the smallest eigenvalue of the positive definite matrix of the linear system (17.21) and C is some constant.

The quality of the discretization depends on how well the exact solution can be approximated by the subspace X_n , that is, – besides the smoothness of the exact solution – it depends on the smoothness of the kernel: the smoother the kernel, the better the approximation. On the other hand, the smoothness of the kernel also controls the asymptotics of the singular values of the integral operator and consequently also the behaviour of the eigenvalues λ_n (see Theorem 15.20). In general, the decay of the singular values increases with the smoothness of the kernel. Therefore, we can expect the moment collocation to deliver reliable approximations only for kernels with poor smoothness. For an example see Problem 17.5.

For more details on the moment collocation we refer to Groetsch [1], Nashed and Wahba [1] and Engl [1] and the literature therein. A somewhat different approach towards the discussion of regularization through discretization is taken by Baumeister [1] and Natterer [1].

Problems

- 17.1. Formulate and prove a variant of Theorem 17.4 for the collocation method of Section 13.2. Apply the result to equation (13.26).
- 17.2. Let $A : X \rightarrow X$ be a nonnegative self adjoint compact operator with one-dimensional nullspace $N(A) = \text{span}\{u_0\}$ and spectral decomposition

$$A\varphi = \sum_{j=1}^{\infty} \lambda_j(\varphi, u_j) u_j$$

with $\lambda_j \neq 0$ for all $j \in \mathbb{N}$. Consider the compact operator $S : X \rightarrow X$ with

$$S\varphi := A\varphi + (\varphi, u_0)g,$$

where $g \in X$ with $(g, u_0) = 1$ and $(g, u_j) \neq 0$ for all $j \in \mathbb{N}$. For the equation $S\varphi = f$ with right hand side $f = Ag + g$ and exact solution $\varphi = g$ show that the coefficients γ_j , $j = 0, \dots, n$, for the least squares approximation (see Theorem 13.25)

$$\varphi_n = \sum_{j=0}^n \gamma_j u_j$$

with respect to the subspace $X_n = \text{span}\{u_0, \dots, u_n\}$ satisfy

$$(\gamma_0 - 1)\{1 + \|g - P_n g\|^2\} = (Ag, g - P_n g)$$

and

$$\lambda_j\{\gamma_j - (g, u_j)\} = (1 - \gamma_0)(g, u_j), \quad j = 1, \dots, n.$$

Here $P_n : X \rightarrow X_n$ denotes the orthogonal projection. Use these results to establish the inequalities

$$\begin{aligned} \|\varphi_n - g\| &\geq |\gamma_n - (g, u_n)| \\ &\geq \frac{|(Ag, g - P_n g)| |(g, u_n)|}{\{1 + \|g\|^2\} |\lambda_n|} \geq \frac{|\lambda_{n+1}|}{|\lambda_n|} |(g, u_{n+1})|^2 |(g, u_n)|. \end{aligned}$$

Use this to show that A and g can be chosen such that the least squares solutions φ_n do not converge to the exact solution (see Seidman [1]).

- 17.3. Show that in a Hilbert space with reproducing kernel a weakly convergent sequence is pointwise convergent. Show that a norm convergent sequence converges uniformly provided the reproducing kernel is continuous.
- 17.4. Determine the reproducing kernel of the Sobolev space $H^1[a, b]$.
- 17.5. Apply the moment collocation to the Volterra integral operator of Example 15.19 with an equidistant mesh $x_j = jh$, $j = 1, \dots, n$, where $h = 1/n$. Show that the inverse matrix for the linear system (17.21) in this case is given by the tridiagonal matrix

$$n \left(\begin{array}{cccccc} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & 0 \\ & -1 & 2 & -1 & & & \\ . & . & . & . & . & . & . \\ 0 & & -1 & 2 & -1 & & \\ & & & & -1 & 1 & \end{array} \right)$$

and that the total error in the spirit of Theorem 17.12 can be estimated by

$$\|\varphi_n^d - A^{-1}f\|_2 \leq c_1 n^{1/2} d + \frac{c_2}{n}$$

with some constants c_1 and c_2 provided the exact solution is twice continuously differentiable.

18. Inverse Scattering Theory

To end this book we shall briefly indicate the application of ill-posed integral equations of the first kind to inverse acoustic scattering problems. Of course, in one single chapter it is impossible to give a complete picture of inverse scattering. Hence we shall content ourselves with developing some of the main ideas and will leave out much of the details. Before proceeding to the inverse scattering problem, as a motivation for our approach we shall indicate the use of ill-posed equations of the first kind in the solution of direct boundary value problems.

18.1 Ill-Posed Integral Equations in Potential Theory

In our treatment of the potential theoretic boundary value problems in Chapters 6 to 8 we have derived integral equations by seeking the solutions in the form of single- or double-layer potentials over the boundary of the domain in which the boundary value problem is posed. In the sequel we shall refer to the integral equations of the first and of the second kind obtained by this approach as boundary integral equations. The boundary integral equations of the second kind, of course, are well-posed, whereas the equations of the first kind are mildly ill-posed as indicated in Section 17.1. Due to the singularity of the fundamental solution, the boundary integral equations, in general, are weakly singular. Since this causes some difficulties with their numerical solution, it is certainly tempting to try and solve the boundary value problems by using potentials with densities on curves or surfaces different from the actual boundary of the underlying domain. We shall illustrate this idea by looking at the interior two-dimensional Dirichlet problem.

Let $D \subset \mathbb{R}^2$ be a bounded domain of class C^2 with boundary ∂D . Consider the Dirichlet problem for a solution $u \in C^2(D) \cap C(\bar{D})$ of Laplace's equation

$$(18.1) \quad \Delta u = 0 \quad \text{in } D$$

with boundary condition

$$(18.2) \quad u = f \quad \text{on } \partial D,$$

where f is a given continuous function. We choose a closed curve Γ of class C^2 such that the closure \bar{D} is contained in the interior of Γ and denote its outward unit normal by n . Then we seek the solution to the Dirichlet problem (18.1), (18.2) in the form of a double-layer potential

$$(18.3) \quad u(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \bar{D},$$

where

$$\Phi(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x - y|}, \quad x \neq y,$$

denotes the fundamental solution to Laplace's equation and $\varphi \in L^2(\Gamma)$ is an unknown density. The harmonic function u solves the Dirichlet problem provided the density φ is a solution to the integral equation of the first kind

$$(18.4) \quad \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = f(x), \quad x \in \partial D.$$

We introduce the double-layer operator $A : L^2(\Gamma) \rightarrow L^2(\partial D)$ by

$$(18.5) \quad (A\varphi)(x) := \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \partial D,$$

and rewrite the integral equation (18.4) in the short form $A\varphi = f$. Since the integral operator A has a continuous kernel it is compact and therefore our integral equation (18.4) is ill-posed. If both the boundary ∂D and the auxiliary curve Γ are analytic, then the kernel is analytic and (18.4) is severely ill-posed. This ill-posedness is also reflected by the observation that the double-layer potential (18.3) defines a harmonic function in all of the interior of Γ and not just only in D . Hence, for a given function f , the integral equation (18.4) can have a solution only if the solution to the Dirichlet problem in D with boundary values f can be continued across the boundary ∂D as a harmonic function into the interior of Γ . A closer examination shows that this condition together with the property that the boundary data of the continued function belong to $L^2(\Gamma)$ is also sufficient for the solvability. In general, however, there are no means available to decide for a given boundary condition whether such a continuation is possible or not.

In order to apply the regularization results of the preceding chapters we establish the following theorem.

Theorem 18.1. *The compact integral operator $A : L^2(\Gamma) \rightarrow L^2(\partial D)$, defined by (18.5), is injective and has dense range.*

Proof: Let φ satisfy $A\varphi = 0$ and define u in $\mathbb{R}^2 \setminus \Gamma$ by (18.3). Then $u = 0$ on ∂D and, by the uniqueness Theorem 6.10, it follows that $u = 0$ in D . The analyticity of harmonic functions from Theorem 6.6 then yields $u = 0$ in the interior of Γ . From the jump relation (6.40) for the double-layer potential with L^2 -densities we derive that $\varphi - K\varphi = 0$, where K denotes the double-layer boundary operator on the curve Γ . Now the continuity of the kernel of K (see Problem 6.1) and Theorem 6.16 imply that $\varphi = 0$. Hence A is injective.

The adjoint operator $A^* : L^2(\partial D) \rightarrow L^2(\Gamma)$ of A is given by

$$(A^*\psi)(x) := \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial n(x)} ds(y), \quad x \in \Gamma.$$

When $A^*\psi = 0$, then, with the aid of Example 6.14, we have

$$(1, \psi)_{L^2(\partial D)} = -(A1, \psi)_{L^2(\partial D)} = -(1, A^*\psi)_{L^2(\Gamma)} = 0,$$

i.e., $\int_{\partial D} \psi ds = 0$. Hence, the single-layer potential

$$v(x) = \int_{\partial D} \psi(y) \Phi(x, y) ds(y), \quad x \in \mathbb{R}^2,$$

vanishes at infinity. Because $A^* \psi = 0$ implies $\partial u / \partial n = 0$ on Γ , by the uniqueness Theorem 6.11, we have $v = 0$ in the exterior of Γ , and by analyticity $v = 0$ in the exterior of ∂D . Since logarithmic single-layer potentials with L^2 -densities are continuous in all of \mathbb{R}^2 (see Problem 18.1), the uniqueness for the interior Dirichlet problem tells us that $v = 0$ everywhere. Now the jump relations imply $\psi = 0$ (use Problems 6.1 and 8.5). Therefore A^* is injective, whence, by Theorem 15.8, the range of A is dense in $L^2(\partial D)$. \square

We now apply the Tikhonov regularization, i.e., we approximate (18.4) by the regularized equation

$$(18.6) \quad \alpha \varphi_\alpha + A^* A \varphi_\alpha = A^* f.$$

As is always the case with Tikhonov regularization, we can expect convergence of the solution φ_α to (18.6) as $\alpha \rightarrow 0$ only when the original equation (18.4) has a solution. Assume that $\varphi_\alpha \rightarrow \varphi \in L^2(\Gamma)$ for $\alpha \rightarrow 0$. Then, since A and A^* are continuous, it follows that $A^*(A\varphi - f) = 0$ and therefore $A\varphi = f$.

Fortunately, in our present situation, in order to obtain approximate solutions to the boundary value problem by

$$u_\alpha(x) = \int_{\Gamma} \varphi_\alpha(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \bar{D},$$

we do not need convergence of the densities φ_α . What we actually want is convergence $A\varphi_\alpha \rightarrow f$ as $\alpha \rightarrow 0$ on ∂D in order to satisfy the boundary condition approximately. Then, since the Dirichlet problem itself is well-posed, small deviations $u_\alpha - f$ in the boundary values ensure small deviations $u_\alpha - u$ in the solution in all of D . To make this comment more precise, let u_m be harmonic in D and continuous in \bar{D} and assume that

$$\|u_m\|_{L^2(\partial D)} \rightarrow 0, \quad m \rightarrow \infty.$$

Then, by Theorems 6.17 and 6.18, we may write u_m as double-layer potential over the boundary ∂D with continuous density φ_m . This density satisfies the boundary integral equation of the second kind $\varphi_m - K\varphi_m = -2u_m$ on ∂D with the double-layer integral operator K . Since K is compact from $L^2(\partial D)$ into $L^2(\partial D)$ and $I - K$ is injective, by the Riesz theory Theorem 3.4, the inverse operator $(I - K)^{-1} : L^2(\partial D) \rightarrow L^2(\partial D)$ is continuous. Therefore

$$\|\varphi_m\|_{L^2(\partial D)} \rightarrow 0, \quad m \rightarrow \infty.$$

Now from

$$u_m(x) = \int_{\partial D} \varphi_m(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in D,$$

using the Cauchy-Schwarz inequality, we see that the u_m (together with all derivatives) converge uniformly to zero on compact subsets of D .

For the convergence on the boundary we can state the following results for an injective compact operator with dense range.

Theorem 18.2. *For the unique solution φ_α to the regularized equation (18.6) there holds convergence*

$$\|A\varphi_\alpha - f\| \rightarrow 0, \quad \alpha \rightarrow 0.$$

Proof: By (16.8) this convergence is generally true for injective bounded linear operators with dense range. \square

In general, convergence of $A\varphi_\alpha \rightarrow f$, $\alpha \rightarrow 0$, will be slow. Therefore in order to obtain acceptable approximations for the solution to the boundary value problem we might have to choose the regularization parameter very small, leading to an unsatisfactory condition number for the regularized equation. However, convergence improves in cases where $A\varphi = f$ has a solution.

Theorem 18.3. *Assume $f \in A(L^2(\Gamma))$. Then*

$$\|A\varphi_\alpha - f\| = o(\sqrt{\alpha}), \quad \alpha \rightarrow 0.$$

Proof: We write $f = A\varphi$ and, by Theorem 15.23, we have $\varphi_\alpha \rightarrow \varphi$, $\alpha \rightarrow 0$. Then

$$\|A\varphi_\alpha - f\|^2 = (\varphi_\alpha - \varphi, A^*[A\varphi_\alpha - f]) = \alpha(\varphi - \varphi_\alpha, \varphi_\alpha),$$

which implies $\|A\varphi_\alpha - f\|^2 = o(\alpha)$ and the proof is complete. \square

The optimal rate of convergence is described in

Theorem 18.4. *The condition $f \in AA^*(L^2(\partial D))$ is necessary and sufficient for*

$$\|A\varphi_\alpha - f\| = O(\alpha), \quad \alpha \rightarrow 0.$$

Proof: Let (μ_n, φ_n, g_n) , $n = 1, 2, \dots$, be a singular system for the operator A . Since A^* is injective, by the singular value decomposition Theorem 15.6, the $\{g_n : n \in \mathbb{N}\}$ form a complete orthonormal system in $L^2(\partial D)$, that is, we can expand

$$f = \sum_{n=1}^{\infty} (f, g_n) g_n,$$

and from (15.31) we recall

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{\mu_n}{\alpha + \mu_n^2} (f, g_n) \varphi_n.$$

Both equations together imply

$$(18.7) \quad \|A\varphi_\alpha - f\|^2 = \sum_{n=1}^{\infty} \frac{\alpha^2}{(\alpha + \mu_n^2)^2} |(f, g_n)|^2.$$

Now let $f = AA^*g$ with some $g \in L^2(\partial D)$. Then

$$(f, g_n) = (AA^*g, g_n) = (g, AA^*g_n) = \mu_n^2(g, g_n),$$

and from (18.7) it follows that

$$\|A\varphi_\alpha - f\|^2 = \sum_{n=1}^{\infty} \frac{\alpha^2 \mu_n^4}{(\alpha + \mu_n^2)^2} |(g, g_n)|^2 \leq \alpha^2 \sum_{n=1}^{\infty} |(g, g_n)|^2 = \alpha^2 \|g\|^2.$$

Conversely, assume that $\|A\varphi_\alpha - f\| = O(\alpha)$, $\alpha \rightarrow 0$. Then from (18.7) we observe that there exists a constant $M > 0$ such that

$$\sum_{n=1}^{\infty} \frac{|(f, g_n)|^2}{(\alpha + \mu_n^2)^2} \leq M$$

for all $\alpha > 0$. Passing to the limit $\alpha \rightarrow 0$, this implies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^4} |(f, g_n)|^2 \leq M.$$

Hence,

$$g := \sum_{n=1}^{\infty} \frac{1}{\mu_n^2} (f, g_n) g_n$$

is well defined and $AA^*g = f$. \square

The conditions $f \in A(L^2(\Gamma))$ and $f \in AA^*(L^2(\partial D))$ of the two preceding theorems can be interpreted as regularity assumptions on f because they control how fast the Fourier coefficients (f, g_n) tend to zero as $n \rightarrow \infty$. They may also be considered as conditions on how far the solution to the Dirichlet problem can be continued across the boundary ∂D . The following example will shed some light on these remarks.

Example 18.5. Consider the simple situation where D is the unit disk and Γ a concentric circle with radius $R > 1$. We parametrize $x \in \partial D$ by

$$x(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi,$$

and $x \in \Gamma$ by

$$x(t) = (R \cos t, R \sin t), \quad 0 \leq t \leq 2\pi,$$

and write $\psi(t) := \varphi(x(t))$ and $g(t) := f(x(t))$. We transform the integral equation (18.4) into

$$(18.8) \quad \frac{R}{2\pi} \int_0^{2\pi} \frac{\cos(t-\tau) - R}{1 - 2R \cos(t-\tau) + R^2} \psi(\tau) d\tau = g(t), \quad 0 \leq t \leq 2\pi,$$

which we rewrite in operator notation

$$\tilde{A}\psi = g$$

with obvious meaning for the self adjoint operator $\tilde{A} : L^2[0, 2\pi] \rightarrow L^2[0, 2\pi]$. By decomposing

$$\frac{2(\cos t - R)e^{int}}{1 - 2R \cos t + R^2} = \frac{e^{int}}{e^{it} - R} + \frac{e^{int}}{e^{-it} - R},$$

we derive the integrals

$$\frac{R}{2\pi} \int_0^{2\pi} \frac{(\cos t - R)e^{int}}{1 - 2R \cos t + R^2} dt = \begin{cases} \frac{-1}{2R^{|n|}}, & n = \pm 1, \pm 2, \dots, \\ -1, & n = 0. \end{cases}$$

Hence, for $\psi_n(t) = e^{int}$ we have

$$\tilde{A}\psi_n = \frac{-1}{2R^{|n|}}(1 + \delta_{n0})\psi_n,$$

for all $n \in \mathbb{Z}$. Therefore, in polar coordinates $x = (r, \theta)$, a singular system (μ_n, φ_n, g_n) of A is given by

$$\mu_n = \frac{1}{2R^{|n|+1/2}}(1 + \delta_{n0}), \quad \varphi_n(x) = \frac{1}{\sqrt{2\pi}R}e^{in\theta}, \quad g_n(x) = \frac{-1}{\sqrt{2\pi}}e^{in\theta}$$

for $n = 0, \pm 1, \pm 2, \dots$. Then the approximate solution to the Dirichlet problem obtained by (18.6) can be written in the form

$$u_\alpha(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{\mu_n^2}{\alpha + \mu_n^2} (f, g_n) r^{|n|} e^{in\theta}$$

as compared with the exact separation of variables solution

$$(18.9) \quad u(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (f, g_n) r^{|n|} e^{in\theta}.$$

The solvability condition $f \in A(L^2(\Gamma))$ now corresponds to

$$\sum_{n=-\infty}^{\infty} R^{2|n|} |(f, g_n)|^2 < \infty,$$

and therefore, by the Cauchy-Schwarz inequality, the series (18.9) can be seen to converge uniformly in each disk with radius $\rho < R$. Therefore in this case the solution to the Dirichlet problem in the unit disk with boundary values f can be harmonically extended into the interior of the circle of radius R . The stronger condition $f \in AA^*(L^2(\partial D))$ corresponds to

$$\sum_{n=-\infty}^{\infty} R^{4|n|} |(f, g_n)|^2 < \infty,$$

and, similarly, implies a harmonic continuation into the interior of the circle of radius R^2 . \square

To illustrate our results numerically we consider the boundary condition given by

$$f(x) = \ln|x - x_0|,$$

where $x_0 = (q, 0)$ with $q > 1$. Here, the solution to the boundary value problem clearly allows a harmonic continuation into the open disk of radius q . Table 18.1

gives the mean square error between the exact solution and the Tikhonov regularization for (18.8), approximated by Nyström's method using the trapezoidal rule with 32 knots. The radius of Γ is chosen to be $R = 2$. The dependence of the error on the parameters α and q confirms the convergence behaviour predicted by our theoretical results.

Table 18.1. Numerical results for Example 18.5

$\log \alpha$	$q = 10$	$q = 4$	$q = 3$	$q = 2$	$q = 1.5$
-5	0.00006	0.00009	0.0001	0.001	0.01
-4	0.00060	0.00091	0.0014	0.006	0.02
-3	0.00642	0.00873	0.0125	0.030	0.08
-2	0.06255	0.07370	0.0955	0.162	0.27

This approach, of course, can be extended to cover the interior and exterior Dirichlet and Neumann problem in two and three space dimensions. Also, instead of the double-layer potential a single-layer potential can be used (see Problem 18.2). For an application of these ideas to the Helmholtz equation see Kress and Mohsen [1].

In Section 6.5 we saw that we can derive boundary integral equations from Green's representation formula. This is true also for the approach taken in this section here. By Green's Theorem 6.5, for each harmonic function u in $C^2(D) \cap C^1(\bar{D})$ there holds

$$\int_{\partial D} \left\{ \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right\} ds(y) = 0$$

for all $x \in \mathbb{R}^2 \setminus D$. From this we derive the ill-posed integral equation of the first kind

$$\int_{\partial D} \psi(y) \Phi(x, y) ds(y) = \int_{\partial D} f(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y), \quad x \in \Gamma,$$

for the unknown normal derivative $\psi = \partial u / \partial n$ on ∂D . This method to solve the potential theoretic boundary value problems is due to Kupradze [1]. A fairly complete description and extensive references are given by Christiansen [1].

The integral equations described in this section are simple since as opposed to the boundary integral equations they possess smooth kernels. Hence, they allow the application of simple quadrature rules without any need to treat singular integrals. On the other hand, in general, we can expect only poor convergence as the regularization parameter tends to zero. Therefore, this approach may serve as a very instructive example for the use of Tikhonov regularization but will not be a serious competitor for the numerically well established boundary integral equations. However, in Section 18.3 we will illustrate that integral equations of this nature have some merits in solving inverse problems where the actual boundary is not known.

18.2 An Inverse Acoustic Scattering Problem

The inverse scattering problem for acoustic waves we want to consider consists in the determination of the shape of a scatterer from the far-field pattern of the scattered wave. Inverse problems of this type occur in various applications such as remote sensing, nondestructive testing, ultrasound medicine and seismic imaging. They are difficult to solve since they are nonlinear and improperly posed. Of these two, it is the latter property that presents the more basic difficulty. Most of the methods available for approximately solving such inverse scattering problems are of an iterative nature and require the solution of the corresponding direct problem at each iteration step (see for example Kristensson and Vogel [1] and Roger [1]).

The two methods which we describe here and which were introduced by Kirsch and Kress and by Colton and Monk belong to a new group of schemes which do not need the solution of the direct problem. The principal idea of these methods is to stabilize the inverse scattering problem by reformulating it as a nonlinear optimization problem.

We consider acoustic wave propagation in a homogeneous isotropic medium in \mathbb{R}^2 with speed of sound c . The wave motion can be determined from a velocity potential $U = U(x, t)$ from which the velocity field v is obtained by

$$v = \text{grad } U$$

and the pressure p by

$$p - p_0 = -\frac{\partial U}{\partial t},$$

where p_0 denotes the pressure of the undisturbed medium, and where the gradient is taken with respect to the space variables. In the linearized theory, the velocity potential U satisfies the wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0.$$

Hence, for time-harmonic acoustic waves of the form $U(x, t) = u(x)e^{-i\omega t}$ with frequency $\omega > 0$, we deduce that the space dependent part u satisfies the *reduced wave equation* or *Helmholtz equation*

$$(18.10) \quad \Delta u + k^2 u = 0$$

where the wave number k is given by $k = \omega/c$ (see also Example 12.14). The mathematical description of the scattering of time-harmonic waves by an obstacle D leads to exterior boundary value problems for the Helmholtz equation. For example, prescribing the values of u on the boundary of the obstacle physically corresponds to prescribing the pressure of the acoustic wave on the boundary. Therefore scattering from sound-soft obstacles leads to a Dirichlet boundary value problem with the total pressure vanishing on the boundary. Consider the scattering of an incident wave u^i which we assume to be an entire solution to the Helmholtz equation, that is, u^i is defined and satisfies (18.10) everywhere in \mathbb{R}^2 . In particular, this includes the scattering of an incident plane wave u^i

given by $u^i(x) = e^{ik(d,x)}$ where d denotes a unit vector giving the direction of propagation of the incident wave. Then the *direct acoustic scattering problem* consists in determining the total field $u = u^i + u^s$ as a solution to the Helmholtz equation in the exterior domain $\mathbb{R}^2 \setminus \bar{D}$ with boundary condition

$$(18.11) \quad u = 0 \quad \text{on } \partial D.$$

The scattered wave u^s is required to satisfy the *Sommerfeld radiation condition*

$$(18.12) \quad \frac{\partial u^s}{\partial r} - iku^s = o\left(\frac{1}{\sqrt{|x|}}\right), \quad r = |x| \rightarrow \infty,$$

uniformly in all directions $\hat{x} := x/|x|$. This radiation condition ensures the uniqueness for the exterior boundary value problem by Rellich's theorem (see Colton and Kress [1]). In the sequel we shall refer to solutions of the Helmholtz equation satisfying the radiation condition as radiating solutions.

For smooth boundaries, for example ∂D of class C^2 , the well-posedness of (18.10) to (18.12) can be based on boundary integral equations. The fundamental solution to the Helmholtz equation in \mathbb{R}^2 is given by

$$(18.13) \quad \Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y,$$

where $H_0^{(1)}$ is the Hankel function of order zero and of the first kind. Then, extending the methods of Section 6.4 from Laplace's equation to the Helmholtz equation, a boundary integral equation approach using the fundamental solution (18.13) leads to integral equations of the second kind with continuous kernels. These can be used to establish the well-posedness of the direct scattering problem. In particular, it can be shown that for incident plane waves the scattered wave and its normal derivative on the boundary ∂D depend continuously in the maximum norm on the incident direction. For details we refer to Colton and Kress [1].

It is also possible to extend the Green's representation formula to the Helmholtz equation. As in Theorem 6.5, it can be shown that for each radiating solution $u^s \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ to the Helmholtz equation in the exterior of D there holds

$$(18.14) \quad u^s(x) = \int_{\partial D} \left\{ u^s(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \frac{\partial u^s}{\partial n}(y) \Phi(x, y) \right\} ds(y)$$

for all $x \in \mathbb{R}^2 \setminus \bar{D}$, where again n denotes the outward unit normal to the boundary ∂D . In the derivation of (18.14) the Sommerfeld radiation condition is incorporated (see Colton and Kress [1]).

Note the asymptotics (see Magnus et al. [1])

$$(18.15) \quad H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-n\pi/2-\pi/4)} \left\{ 1 + O\left(\frac{1}{z}\right) \right\}, \quad z \rightarrow \infty,$$

for the Hankel functions of order n and of the first kind, and the inequality

$$(18.16) \quad |x - y| = \sqrt{|x|^2 - 2(x, y) + |y|^2} = |x| - (\hat{x}, y) + O\left(\frac{1}{|x|}\right),$$

which is valid uniformly for all $y \in \partial D$. Then from (18.14) we derive that a radiating solution u^s to the Helmholtz equation has an asymptotic behaviour of the form

$$(18.17) \quad u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty,$$

uniformly in all directions $\hat{x} = x/|x|$. Here, the function u_∞ , defined on the unit circle Ω in \mathbb{R}^2 , is known as *far-field pattern* of the radiating wave u^s . The operator which maps u^s into the far-field pattern u_∞ is injective, that is, if $u_\infty = 0$ on the unit circle then $u^s = 0$ in the exterior of ∂D (see Problem 18.4).

The *inverse problem* now is, given the far-field pattern u_∞ of the scattered wave u^s on the unit circle for one or several incoming plane waves u^i with different directions, determine the shape of the scatterer D . For an overview on this inverse scattering problem we refer to Colton and Kress [1] and Ramm [1] and the survey papers by Colton [1] and Sleeman [1]. Based on a result due to Schiffer, the question of uniqueness has been addressed by Colton and Sleeman [1] and by Jones [1]. In principle, a finite number of incident plane waves with different directions uniquely determines the obstacle. As a challenging open problem we wish to point out that so far it is not known if one incoming plane wave for one single direction and one single wave number completely determines the scatterer. As opposed to the direct problem, the inverse problem is ill-posed. The solution – if it exists at all – does not depend continuously on the data in any reasonable norm. Therefore, the numerical treatment needs some kind of regularization.

18.3 Numerical Methods in Inverse Scattering

We now describe a method for approximately solving the inverse scattering problem which was suggested by Kirsch and Kress [1]. It is divided into two steps. In the first step (A1) the scattered field u^s is determined from the given far-field pattern. Then in the second step (A2) the boundary of the unknown scatterer is found as zero-level curve of the total field $u^i + u^s$.

Step (A1): We try to represent the scattered field u^s as an acoustic single-layer potential

$$(18.18) \quad u^s(x) = \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y)$$

with some unknown source density $\varphi \in L^2(\Gamma)$ on some auxiliary closed curve Γ contained in the unknown scatterer D . The knowledge of such an internal curve Γ requires a weak a-priori information about D . In the sequel we will always assume that the wave number k^2 is not a Dirichlet eigenvalue of the Laplacian in the interior of Γ , i.e., the homogeneous Dirichlet problem for the Helmholtz equation in the interior of Γ admits only the trivial solution. This is no loss of generality since the choice of Γ is at our disposal.

From (18.15) and (18.16) we see that

$$u^s(x) = \sigma \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ \int_{\Gamma} \varphi(y) e^{-ik(\hat{x}, y)} ds(y) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty,$$

where $\sigma = e^{i\frac{\pi}{4}}/\sqrt{8\pi k}$. Therefore, the far-field pattern of the single-layer potential is given by

$$(18.19) \quad u_{\infty}(\hat{x}) = \sigma \int_{\Gamma} \varphi(y) e^{-ik(\hat{x}, y)} ds(y), \quad \hat{x} \in \Omega.$$

Hence, if we define an integral operator $F : L^2(\Gamma) \rightarrow L^2(\Omega)$ by

$$(18.20) \quad (F\varphi)(\hat{x}) := \sigma \int_{\Gamma} \varphi(y) e^{-ik(\hat{x}, y)} ds(y), \quad \hat{x} \in \Omega,$$

then, given the far-field pattern u_{∞} , we are looking for a density φ satisfying the equation

$$(18.21) \quad F\varphi = u_{\infty}.$$

The integral operator F has an analytic kernel and therefore equation (18.21) is a severely ill-posed equation of the first kind. As for the integral equation (18.4), the solvability again can be related to the analytic continuation of the scattered field u^s across the boundary ∂D : the integral equation (18.21) is solvable if and only if u_{∞} is the far-field of a radiating solution to the Helmholtz equation in the exterior of Γ with boundary data in $H^1(\Gamma)$.

We illustrate the degree of ill-posedness by looking at the singular values of F . For the special case where Γ is the unit circle, from the integrals (see Problem 18.3)

$$(18.22) \quad \int_0^{2\pi} e^{iz \cos \theta - in\theta} d\theta = 2\pi i^n J_n(z), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z},$$

we deduce that the singular values of F are given by

$$\mu_n = 2\pi |\sigma J_n(k)|, \quad n = 0, 1, \dots$$

Therefore, from the power series expansion for the Bessel functions J_n we obtain the asymptotics

$$\mu_n = O\left(\frac{1}{n!} \left(\frac{k}{2}\right)^n\right), \quad n \rightarrow \infty.$$

Theorem 18.6. *The far-field integral operator F , defined by (18.20), is injective and has dense range.*

Proof: Let $F\varphi = 0$ and define the acoustic single-layer potential by (18.18). Then u^s has far-field $u_{\infty} = 0$, whence $u^s = 0$ in the exterior of Γ follows. By the continuity of the single-layer potential (see Problem 18.1), we can conclude that u^s solves the homogeneous Dirichlet problem in the interior of Γ . Hence, by our assumption on the wave number k , we have $u^s = 0$ everywhere in \mathbb{R}^2 .

Now, as in the proof of Theorem 18.1, the jump relations yield $\varphi = 0$, whence F is injective.

The adjoint operator $F^* : L^2(\Omega) \rightarrow L^2(\Gamma)$ of F is given by

$$(F^*g)(y) = \bar{\sigma} \int_{\Omega} g(\hat{x}) e^{ik(\hat{x}, y)} ds(\hat{x}), \quad y \in \Gamma.$$

Let $F^*g = 0$. Then

$$v(y) = \int_{\Omega} g(\hat{x}) e^{ik(\hat{x}, y)} ds(\hat{x}), \quad y \in \mathbb{R}^2,$$

solves the homogeneous Dirichlet problem in the interior of Γ and hence it vanishes there. Since v is analytic in \mathbb{R}^2 it follows that $v = 0$ everywhere. Now we pick $r = |y|$ such that $(kr)^2$ is not a Dirichlet eigenvalue for the Laplacian in the unit disk. Then, from the injectivity of F , with k set equal to kr and Γ chosen to be Ω , we see that $g = 0$. Therefore F^* is injective and by Theorem 15.8 the range of F is dense in $L^2(\Omega)$. \square

For later use we state a corresponding theorem for the acoustic single-layer operator $S : L^2(\Gamma) \rightarrow L^2(\Lambda)$ defined by

$$(18.23) \quad (S\varphi)(x) := \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y), \quad x \in \Lambda,$$

where Λ denotes a closed curve containing Γ in its interior. The proof is analogous to that of Theorem 18.6 and therefore left as an exercise to the reader.

Theorem 18.7. *The single-layer operator S , defined by (18.23), is injective and has dense range.*

The Tikhonov regularization technique may be applied for a numerical treatment of the ill-posed equation (18.21). Recall that solving the regularized equation

$$(18.24) \quad \alpha\varphi_\alpha + F^*F\varphi_\alpha = F^*u_\infty$$

with regularization parameter $\alpha > 0$ is equivalent to minimizing the penalized residual

$$A_1(\varphi; u_\infty, \alpha) := \|F\varphi - u_\infty\|_{L^2(\Omega)}^2 + \alpha \|\varphi\|_{L^2(\Gamma)}^2$$

over all $\varphi \in L^2(\Gamma)$. Note again that the unique solution φ_α to the regularized equation converges as $\alpha \rightarrow 0$ if and only if the original equation (18.21) is solvable.

Step (A2): After we have determined φ_α as the solution of (18.24) and the corresponding approximation u_α^s for the scattered field by the single-layer potential, we seek the boundary of the scatterer D as the location of the zeros of $u^i + u_\alpha^s$. We look for boundary curves which are starlike and can be parametrized in the form

$$(18.25) \quad x(\theta) = r(\theta)\hat{\theta}, \quad 0 \leq \theta \leq 2\pi,$$

where $\hat{\theta} = (\cos \theta, \sin \theta)$ and the function r belongs to some compact subset V of $C^{1,\beta}[0, 2\pi]$, $0 < \beta \leq 1$. To incorporate a-priori informations on the scatterer D we make a restriction $r_0 \leq r \leq r_1$ for all $r \in V$ with some given functions r_0 and r_1 . We assume that Γ is contained in the interior of the curve described by r_0 . Now introduce the functional

$$A_2(r, \varphi; u^i) = \int_0^{2\pi} |(u^i + S\varphi)(r(\theta)\hat{\theta})|^2 d\theta = \|(u^i + S\varphi) \circ r\|_{L^2(\Omega)}^2.$$

Then, given φ_α , we seek an approximation to ∂D by minimizing the defect $A_2(r, \varphi_\alpha; u^i)$ over all functions $r \in V$. Due to the compactness of V this minimization problem has a solution (see Problem 1.3).

Since, in general, we do not have convergence of the densities φ_α as $\alpha \rightarrow 0$ for a theoretically satisfactory reformulation of the inverse scattering problem we have to combine the two steps and minimize the sum

$$(18.26) \quad A(\varphi, r; u_\infty, u^i, \alpha) := A_1(\varphi; u_\infty, \alpha) + A_2(r, \varphi; u^i)$$

simultaneously over all $\varphi \in L^2(\Gamma)$ and $r \in V$, that is, we seek $\varphi \in L^2(\Gamma)$ and $r \in V$ such that

$$A(\varphi, r; u_\infty, u^i, \alpha) = M(u_\infty, u^i, \alpha)$$

where

$$M(u_\infty, u^i, \alpha) := \inf \{A(\psi, q; u_\infty, u^i, \alpha) : \psi \in L^2(\Gamma), q \in V\}.$$

For this final reformulation of the inverse scattering problem we can state the following results.

Theorem 18.8. *The optimization formulation of the inverse scattering problem given through (18.26) has a solution.*

Proof: Let (φ_n, r_n) be a minimizing sequence in $L^2(\Gamma) \times V$, i.e.,

$$(18.27) \quad \lim_{n \rightarrow \infty} A(\varphi_n, r_n; u_\infty, u^i, \alpha) = M(u_\infty, u^i, \alpha).$$

Since V is compact we can assume that $r_n \rightarrow r \in V$, $n \rightarrow \infty$. From

$$\alpha \|\varphi_n\|_{L^2(\Gamma)}^2 \leq A(\varphi_n, r_n; u_\infty, u^i, \alpha) \rightarrow M(u_\infty, u^i, \alpha), \quad n \rightarrow \infty,$$

and $\alpha > 0$ we conclude that the sequence (φ_n) is bounded. Hence, by Theorem 16.4, we can assume that it converges weakly $\varphi_n \rightharpoonup \varphi \in L^2(\Gamma)$ for $n \rightarrow \infty$. Since F and S represent compact operators, by Problem 16.5, it follows that $F\varphi_n \rightarrow F\varphi$ and $(S\varphi_n) \circ r \rightarrow (S\varphi) \circ r$ for $n \rightarrow \infty$. By the mean value theorem, we can estimate

$$|\Phi(r(\theta)\hat{\theta}, y) - \Phi(p(\theta)\hat{\theta}, y)| \leq C |r(\theta) - p(\theta)|$$

for all $x \in \Gamma$, all $\theta \in [0, 2\pi]$ and all $r, p \in V$. Here, C denotes a bound on $\text{grad}_x \Phi$ on $\{r(\theta)\hat{\theta} : r \in V, \theta \in [0, 2\pi]\} \times \Gamma$. Then, using the Cauchy-Schwarz inequality, we find

$$\|(S\varphi_n) \circ r_n - (S\varphi_n) \circ r\|_{L^2(\Omega)}^2 \leq 2\pi C |\Gamma| \|r_n - r\|_\infty \|\varphi_n\|_{L^2(\Gamma)} \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore, by the triangle inequality, we also have $(S\varphi_n) \circ r_n \rightarrow (S\varphi) \circ r$ for $n \rightarrow \infty$. This now implies

$$\alpha \|\varphi_n\|_{L^2(\Gamma)}^2 \rightarrow M(u_\infty, u^i, \alpha) - \|F\varphi - u_\infty\|_{L^2(\Omega)}^2 - \|(u^i + S\varphi) \circ r\|_{L^2(\Omega)}^2 \leq \alpha \|\varphi\|_{L^2(\Gamma)}^2$$

for $n \rightarrow \infty$. Since we already know weak convergence $\varphi_n \rightharpoonup \varphi$, $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi\|_{L^2(\Gamma)}^2 = \lim_{n \rightarrow \infty} \|\varphi_n\|_{L^2(\Gamma)}^2 - \|\varphi\|_{L^2(\Gamma)}^2 \leq 0,$$

i.e., we also have norm convergence $\varphi_n \rightarrow \varphi$, $n \rightarrow \infty$. Now finally, by continuity

$$A(\varphi, r; u_\infty, u^i, \alpha) = \lim_{n \rightarrow \infty} A(\varphi_n, r_n; u_\infty, u^i, \alpha) = M(u_\infty, u^i, \alpha),$$

and this completes the proof.

Theorem 18.9. *Let u_∞ be the far-field pattern of a domain D such that ∂D is described by some $r \in V$. Then*

$$(18.28) \quad \lim_{\alpha \rightarrow 0} M(u_\infty, u^i, \alpha) = 0.$$

Proof: By Theorem 18.7, given $\varepsilon > 0$ there exists $\varphi \in L^2(\Gamma)$ such that

$$\|(S\varphi + u^i) \circ r\|_{L^2(\Omega)} < \varepsilon.$$

Due to the well-posedness of the direct scattering problem, the far-field pattern of radiating solutions of the Helmholtz equation continuously depends on the boundary data. Therefore we can estimate

$$\|F\varphi - u_\infty\|_{L^2(\Omega)} \leq c \|(S\varphi - u^s) \circ r\|_{L^2(\Omega)}$$

with some constant c . From $u^i + u^s = 0$ on ∂D we then deduce

$$A(\varphi, r; u_\infty, u^i, \alpha) \leq (1 + c)\varepsilon + \alpha \|\varphi\|_{L^2(\Gamma)}^2 \rightarrow (1 + c)\varepsilon, \quad \alpha \rightarrow 0.$$

Since ε is arbitrary, (18.28) follows. \square

Based on Theorem 18.9 we can state the following convergence result.

Theorem 18.10. *Let (α_n) be a null sequence and let (φ_n, r_n) be a corresponding sequence of solutions to the optimization problem with regularization parameter α_n . Then there exists a convergent subsequence of the sequence (r_n) . Assume u_∞ is the exact far-field pattern of a domain D such that ∂D is described by some $r \in V$. Then every limit point of (r_n) represents a curve on which the total field vanishes.*

Proof: Existence of a convergent subsequence of (r_n) follows from the compactness of V . Let r^* be a limit point. Without loss of generality we can assume that $r_n \rightarrow r^*$, $n \rightarrow \infty$. Let u^* denote the unique solution to the direct scattering problem for the obstacle with boundary A^* described by r^* , i.e., the boundary condition reads

$$(18.29) \quad (u^* + u^i) \circ r^* = 0 \quad \text{on } \Omega.$$

We abbreviate $u_n := S\varphi_n$, $n = 1, 2, \dots$, and interpret u_n as the solution to an exterior Dirichlet problem with boundary values $S\varphi_n$ on the boundary Λ_n described by r_n . By Theorem 18.9, these boundary data satisfy

$$\| (u_n + u^i) \circ r_n \|_{L^2(\Omega)} \rightarrow 0, \quad n \rightarrow \infty.$$

It can be proven (see Angell, Colton and Kirsch [1]) that the far-field pattern depends continuously (in the uniform convergence norm) on the boundary data and on the boundary itself (in the $C^{1,\beta}$ -norm). Therefore, the last equation implies that the far-field patterns $F\varphi_n$ of u_n converge uniformly to the far-field pattern u_∞^* of u^* . But by the preceding Theorem 18.9 we also have $\|F\varphi_n - u_\infty\|_{L(\Omega)} \rightarrow 0$, $n \rightarrow \infty$. Therefore the far-field patterns coincide, i.e., $u_\infty = u_\infty^*$, whence $u^s = u^*$ follows. Because of (18.29) the total field $u^s + u^i$ now must vanish on Λ^* . \square

Since we do not have uniqueness either for the inverse scattering problem or for the optimization problem, in general, we cannot expect more than convergent subsequences. In addition, due to the lack of a uniqueness result for one wave number and one incoming plane wave, we cannot assure that we always have convergence towards the boundary of the unknown scatterer. The latter insufficiency can be remedied by using more incident waves u_1^i, \dots, u_n^i with different directions d_1, \dots, d_n with corresponding far-field patterns $u_{\infty,1}, \dots, u_{\infty,n}$. Then we have to minimize the sum

$$\sum_{j=1}^n A(\varphi_j, r; u_{\infty,j}, u_j^i, \alpha)$$

over all $\varphi_1, \dots, \varphi_n \in L^2(\Gamma)$ and all $r \in V$.

We also wish to mention that in the first step (A1), in principle, we may replace the approximation of the scattered field u^s through the single-layer potential by any other convenient approximation. For example Angell, Kleinman and Roach [1] have suggested to use an expansion with respect to spherical wave functions.

Finally, we want to work out the close connection of this approach to inverse acoustic scattering to another method which was introduced by Colton and Monk [1,2] and which is also divided into two steps. The first step (B1) approximately solves a boundary value problem in the interior of the unknown scatterer using Herglotz wave functions. Then in the second step (B2) the boundary of the scatterer is found as the curve where the boundary condition is satisfied.

Before we proceed we note that, using the boundary condition (18.11) and combining the Green's formula (18.14) and the representation corresponding to (6.4), we find

$$(18.30) \quad \int_{\partial D} \frac{\partial u}{\partial n}(y) \Phi(x, y) ds(y) = \begin{cases} u^i(x), & x \in D, \\ -u^s(x), & x \in \mathbb{R}^2 \setminus \bar{D}. \end{cases}$$

In particular, from this, as in (18.19) we derive

$$(18.31) \quad u_\infty(\hat{x}) = \gamma \int_{\partial D} e^{-ik(\hat{x}, y)} \frac{\partial u}{\partial n}(y) ds(y), \quad \hat{x} \in \Omega,$$

for the far-field pattern, where $\gamma = -\sigma$.

Step (B1): We associate the interior boundary value problem for the Helmholtz equation

$$(18.32) \quad \Delta v + k^2 v = 0 \quad \text{in } D$$

with boundary condition

$$(18.33) \quad v = \Phi(0, \cdot) \quad \text{on } \partial D$$

to the inverse scattering problem. Here we assume that the origin is contained in D . For simplicity, we restrict our presentation to the case where k^2 is not a Dirichlet eigenvalue for the Laplacian in D . Then the boundary value problem (18.32) to (18.33) admits a unique solution which we try to represent in the form of a *Herglotz wave function*

$$(18.34) \quad v(y) = \int_{\Omega} g(\hat{x}) e^{-ik(y, \hat{x})} ds(\hat{x}), \quad y \in \mathbb{R}^2,$$

with kernel $g \in L^2(\Omega)$. From (18.30) and (18.31), by interchanging the order of integration and using $u^i(0) = 1$, we readily derive the relation

$$\int_{\Omega} g(\hat{x}) u_\infty(\hat{x}; d) ds(\hat{x}) = \gamma$$

for all $d \in \Omega$. Here we have indicated the dependence of the far-field pattern on the direction d of the incident field. Therefore, given the far-field pattern u_∞ for all directions d , we define an integral operator $\mathcal{F} : L^2(\Omega) \rightarrow L^2(\Omega)$ by

$$(18.35) \quad (\mathcal{F}g)(d) = \int_{\Omega} g(\hat{x}) u_\infty(\hat{x}; d) ds(\hat{x}), \quad d \in \Omega.$$

Then, obviously we have to solve the integral equation

$$(18.36) \quad \mathcal{F}g = \gamma$$

for the Herglotz kernel g . Since the far-field pattern u_∞ is an analytic function (cf. Colton and Kress [1]), the integral equation (18.36) again is a severely ill-posed equation of the first kind (see also Problem 18.5).

Theorem 18.11. *The far-field integral operator \mathcal{F} , defined by (18.35), is injective and has dense range.*

Proof: With the aid of Green's Theorem 6.3 and the radiation condition we

derive

$$\begin{aligned}
& \int_{\partial D} e^{-ik(\hat{x}, y)} \frac{\partial u}{\partial n}(y; d) ds(y) \\
&= \int_{\partial D} \left\{ u^i(y; -\hat{x}) \frac{\partial u^i}{\partial n}(y; d) - u^s(y; -\hat{x}) \frac{\partial u^s}{\partial n}(y; d) \right\} ds(y) \\
&= \int_{\partial D} \left\{ u^i(y; d) \frac{\partial u^i}{\partial n}(y; -\hat{x}) - u^s(y; d) \frac{\partial u^s}{\partial n}(y; -\hat{x}) \right\} ds(y) \\
&= \int_{\partial D} e^{ik(d, y)} \frac{\partial u}{\partial n}(y; -\hat{x}) ds(y).
\end{aligned}$$

This, in view of (18.31) implies the *reciprocity relation*

$$u_\infty(\hat{x}; d) = u_\infty(-d; -\hat{x})$$

for all $\hat{x}, d \in \Omega$.

Let $\mathcal{F}g = 0$ and define the Herglotz wave function v by (18.34). Then, by superposition (use (18.30) and the continuous dependence of the normal derivatives of u^s on d),

$$w_\infty(d) := \int_{\Omega} u_\infty(d; -\hat{x}) g(\hat{x}) ds(\hat{x}), \quad d \in \Omega,$$

describes the far-field pattern of the radiating solution w to the Helmholtz equation with boundary condition $w = -v$ on ∂D . But, by the reciprocity relation, $\mathcal{F}g = 0$ implies $w_\infty = 0$. Therefore it follows that $w = 0$ in $\mathbb{R}^2 \setminus D$ and consequently $v = 0$ on ∂D . Now, Theorem 18.6 implies $g = 0$, that means \mathcal{F} is injective. Essentially the same argument also shows that the L^2 -adjoint \mathcal{F}^* of \mathcal{F} is injective. Hence, \mathcal{F} has dense range in $L^2(\Omega)$. \square

For an approximate solution of (18.36) we follow Colton and Monk [2] and use the method of quasi-solutions: by Theorem 16.6, for each choice of the regularization parameter $\rho > 0$ there exists a unique g_ρ which minimizes the residual

$$B_1(g; u_\infty, \rho) := \|\mathcal{F}g - \gamma\|_{L^2(\Omega)}^2$$

over all $g \in U_\rho := \{g \in L^2(\Omega) : \|g\|_{L^2(\Omega)} \leq \rho\}$.

Step (B2): The quasi-solution g_ρ determines an approximation v_ρ for the solution of the associated problem (18.32) to (18.33) by a Herglotz wave function corresponding to (18.34). Then the boundary of the scatterer D is sought as the location of the zeros of $v_\rho - \Phi(0, \cdot)$. Again we look for boundary curves of the form (18.25). We denote by Hg the Herglotz wave function with kernel g and introduce the functional

$$B_2(r, g) = \int_0^{2\pi} |(Hg)(r(\theta)\hat{\theta}) - \Phi(0, r(\theta))|^2 d\theta = \|(Hg - \Phi(0, \cdot)) \circ r\|_{L^2(\Omega)}^2.$$

Then, given g_ρ , we seek an approximation to ∂D by minimizing the defect $B_2(r, g_\rho)$ over all functions $r \in V$.

Since, in general, we do not have convergence of the densities g_ρ as $\rho \rightarrow \infty$, for a theoretically satisfactory reformulation of the inverse scattering problem we again have to combine the two steps and minimize the sum

$$B(g, r; u_\infty, \rho) := B_1(g; u_\infty, \rho) + B_2(r, g)$$

simultaneously over all $g \in U_\rho$ and $r \in V$. For this reformulation of the inverse scattering problem results similar to those given in Theorems 18.8 to 18.10 can be proven (see Colton and Monk [2]).

As a major difference between the two methods, we observe that in Method A the measured far-field patterns enter the integral equation on the right hand side, whereas in Method B they determine the kernel of the integral operator. From this remark it is obvious that Method B requires the knowledge of the far-field patterns for a large number of incident directions, while Method A performs well with the far-field patterns for only a few incident directions. On the other hand, Method B easily takes care of a large number of incoming waves, whereas in Method A the use of more incoming waves multiplies the computational effort. For numerical results and further comparisons of the two methods we refer to Kirsch, Kress, Monk and Zinn [1].

Problems

- 18.1. Show that the logarithmic single-layer potential with L^2 -density is continuous in all of \mathbb{R}^2 .
- 18.2. Analogous to Theorem 18.1, construct a single-layer operator which is injective and has dense range.

Hint: Take care of the logarithmic behaviour at infinity as in (7.43).

- 18.3. Use the power series expansion for the Bessel functions

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{z}{2}\right)^{2m+n}, \quad z \in \mathbb{R},$$

for $n = 0, 1, \dots$, and the relation $J_{-n} = (-1)^n J_n$ for $n \in \mathbb{N}$, to verify the integrals (18.22).

Hint: Substitute

$$\cos \theta = \frac{1}{2} \left(\zeta + \frac{1}{\zeta} \right)$$

and use the residue theorem.

- 18.4. The Bessel and Hankel functions satisfy the addition theorem (see Magnus et al. [1])

$$H_0^{(1)}(k|x-y|) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(kr) J_n(k\rho) e^{in(\theta-\psi)}$$

which, together with its term by term derivatives, is uniformly convergent in compact subsets of $|x| > |y|$ and where in polar coordinates $x = (r, \psi)$ and $y = (\rho, \theta)$. Use this

addition theorem and the representation formula (18.14) to show that each radiating solution to the Helmholtz equation allows an expansion

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(kr) e^{in\theta}$$

with respect to Hankel functions. Find a Fourier series for the far-field pattern u_∞ of u and show that $u_\infty = 0$ implies $u = 0$.

18.5. Use the Jacobi–Anger expansion

$$e^{iz \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}$$

and (18.15) to show that for scattering of a plane wave from the unit disk the far-field pattern is given by

$$u_\infty(\hat{x}; d) = 4i\sigma \sum_{n=-\infty}^{\infty} \frac{J_n(k)}{H_n^{(1)}(k)} e^{in(\theta-\phi)}$$

with $\hat{x} = (\cos \theta, \sin \theta)$ and $d = (\cos \phi, \sin \phi)$. From this derive the asymptotic behaviour of the singular values of \mathcal{F} by using the Laurent expansion

$$H_n^{(1)}(k) = \frac{(n-1)!}{i\pi} \left(\frac{2}{k}\right)^n + O(k^{-n+2}), \quad n \in \mathbb{N},$$

for the Hankel functions.

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Index

- adjoint operator 36
- Arzelà–Ascoli theorem 7
- Banach space 5
- Banach–Steinhaus theorem 153
- best approximation 10
- bijective 1
- bilinear form 36
 - nondegenerate 36
- boundary element method 196
- bounded linear functional 46
- bounded linear operator 13
- bounded set 4
 - totally 6
- Bubnow–Galerkin method 200
- Cauchy data 62
- Cauchy integral 85
- Cauchy integral operator 88
- Cauchy kernel 88, 94
- Cauchy principal value 85
- Cauchy sequence 5
- Cauchy–Schwarz inequality 9
- Cea's lemma 186
- closed ball 4
- closed set 4
- closure 4
- collocation method 189
- compact 5
 - relatively 7
 - sequentially 6
- compact operator 17
- complete orthonormal system 11
- complete set 5
- completion 5
- condition number 216
- conjugate gradient method 205
- constraint optimization 244
- continuous 2
 - uniformly 2
- continuous extension 23
- contraction 16
- convergence 2
 - mean square 3
 - norm 14
- pointwise 144
- uniform 3
- weak 244
- convex set 11
- defect correction iteration 207
- dense set 4
- diffeomorphism 118
- direct acoustic scattering problem 278
- Dirichlet integral 124
- Dirichlet problem 62
 - weak 124
- discrepancy 250
 - principle 252
- distance 2
- domain of class C^k 21
- double-layer heat potential 135
- double-layer potential 67
- dual least squares method 264
- dual space 46, 113
- dual system 36
 - positive 39
- duality pairing 114
- eigenelement 31
- eigenvalue 31
 - multiplicity 229
- far-field pattern 279
- Fourier series 11, 108
- Fredholm alternative 44
- Fredholm integral equation 1
- Fredholm theorems 41
- function 1
- function space
 - $C^k(D)$ 59
 - $C^k(\bar{D})$ 59
 - $C^k[0, 2\pi]$ 111
- fundamental solution
 - heat equation 134
 - Helmholtz equation 179, 278
 - Laplace equation 59
- Galerkin collocation 203
- Galerkin equation 199

- Galerkin method 199
- Green's formula 60
- Green's theorem 59
- Hölder continuous 82
- Hölder space $C^{0,\alpha}(G)$ 82
- harmonic function 58
- heat equation 132
- heat potential 134
- Helmholtz equation 179, 277
- Herglotz wave function 285
- Hilbert inversion formula 91
- Hilbert kernel 91
- Hilbert scale 115
- Hilbert space 9
- homeomorphic 112
- ill-posed problem 221
 - mildly 235
 - severely 235
- imbedding operator 84
- improperly posed problem 221
- index 56, 92
- injective 1
- integral equation
 - of the first kind 1
 - of the second kind 1
- integral operator 15
- integro differential equation 256
- interpolation 156
- interpolation operator 156
- interpolation space 115
- inverse operator 1
- inverse problem 279
- involution 38
- isometric 5
- isomorphic 5
- iterated operator 16
- Jordan measurable 14
- jump relation 68
- kernel 15
 - degenerate 155
 - weakly singular 20, 22
- Lagrange basis 156
- Landweber–Fridman iteration 240
- Laplace equation 58
- Lax theorem 40
- Lax–Milgram theorem 201
- least squares solution 202
- limit 2
- linear operator 13
- logarithmic double-layer potential 67
- logarithmic single-layer potential 67
- Lyapunov boundary 76
- mapping 1
- maximum-minimum principle
 - for harmonic functions 61
 - for heat equation 133
- mean value theorem 61
- minimum norm solution 250
- minimum-maximum principle 231
- moment collocation 267
- moment solution 263
- multi-grid iteration 210
 - full 213
- Neumann problem 62
 - weak 125
- Neumann series 16, 149
- Noether theorems 97
- nondegenerate 36
- norm 2
 - equivalent 3
 - maximum 2
 - mean square 3
 - stronger 12
 - weaker 12
- normal solvability 55
- normal vector 22
- normed space 2
- nullspace 25
- numerical integration operator 171
- Nyström method 171
- open ball 4
- open set 4
- operator 1
 - adjoint 36
 - bijective 1
 - bounded 13
 - compact 17
 - collectively 146
 - injective 1
 - linear 13
 - nonnegative 231
 - positive 200
 - self adjoint 200, 226
 - surjective 1
- operator equation
 - of the first kind 1, 20
 - of the second kind 1, 16, 20
- operator norm 14
- orthogonal 9
 - complement 44
 - projection 185
 - system 9

- orthonormal basis 229
- orthonormal system 9
 - complete 11
- parallel surfaces 63
- parametric representation 22
 - regular 22
- Parseval equality 11
- penalty method 244
- Petrov–Galerkin method 200
- Picard theorem 234
- piecewise linear interpolation 157
- Poisson integral 134, 241
- pre–Hilbert space 9
- projection method 185
- projection operator 30, 184
- properly posed problem 221
- quadrature 168
 - convergent 170
 - interpolatory 168
 - points 168
 - weights 168
- quasi-solution 245
- radiation condition 278
- range 1
- Rayleigh–Ritz method 200
- reduced wave equation 179, 277
- regular value 31
- regularization 224
 - parameter 225
 - scheme 224
 - strategy 226
 - regular 226
- regularizer 50
 - equivalent 51
 - left 50
 - right 50
- relatively compact 7
- reproducing kernel 265
- residual 207
- residual correction 207
- resolvent 31
- Riemann problem 91
- Riemann–Lebesgue lemma 241
- Riesz number 27
- Riesz theorems 25, 38
- scalar product 9
- Schauder theory 46
- sectionally holomorphic 85
- semi-norm 83
- sequentially compact 6
- series 12
- sesquilinear form 37
 - nondegenerate 38
 - symmetric 38
 - positive 38
- Simpson rule 169
- single-layer heat potential 135
- single-layer potential 67
- singular integral operator 94
- singular system 232
- singular value 232
 - decomposition 232
- Sobolev space
 - $H^1(D)$ 119
 - $H^p(\Gamma)$ 117
 - $H^p[0, 2\pi]$ 109
- Sokhotski–Plemelj theorem 88
- span 3
- spectral cut-off 240
- spectral radius 31
- spectral theorem 228
- spectrum 31
- spline interpolation 157
- successive approximations 17, 150
- superconvergence 205
- surface element 22
- surface patch 22
- surjective 1
- tangent vector 22
- Tikhonov functional 243
- Tikhonov regularization 239
- totally bounded 6
- trace operator 120
- trapezoidal rule 169
- trigonometric interpolation 160
- Tschebyscheff polynomial 164
- uniform boundedness principle 144
- unisolvent 156
- Volterra integral equations 33, 152
- weak derivative 253
- Weierstrass approximation theorem 4
- weight function 168
- well-posed problem 221

Applied Mathematical Sciences

cont. from page II

54. Wasow: **Linear Turning Point Theory.**
55. Yosida: **Operational Calculus: A Theory of Hyperfunctions.**
56. Chang/Howes: **Nonlinear Singular Perturbation Phenomena: Theory and Applications.**
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