Found. Comput. Math. (2002) 2:203–245

DOI: 10.1007/s102080010027

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Adaptive Wavelet Methods II—Beyond the Elliptic Case

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Abstract. This paper is concerned with the design and analysis of adaptive wavelet methods for systems of operator equations. Its main accomplishment is to extend the range of applicability of the adaptive wavelet-based method developed in [**17**] for symmetric positive definite problems to indefinite or unsymmetric systems of operator equations. This is accomplished by first introducing techniques (such as the least squares formulation developed in [**26**]) that transform the original (continuous) problem into an equivalent infinite system of equations which is now well-posed in the Euclidean metric. It is then shown how to utilize adaptive techniques to solve

Date received: November 21, 2000. Final version received: March 18, 2002. Communicated by Peter Olver. Online publication: May 14, 2002.
AMS classification: 41A25, 41A46, 65F99, 65N12, 65N55.
Key words and phrases: Operator equations, Indefinite problems, Adaptive methods, Convergence rates, Quasi-sparse matrices and vectors, Best N-term approximation, Fast matrix vector multiplication.

the resulting infinite system of equations. This second step requires a significant modification of the ideas from [**17**]. The main departure from [**17**] is to develop an iterative scheme that directly applies to the *infinite-dimensional* problem rather than finite subproblems derived from the infinite problem. This rests on an *adaptive application of the infinite-dimensional operator* to finite vectors representing elements from finite-dimensional trial spaces. It is shown that for a wide range of problems, this new adaptive method performs with asymptotically optimal complexity, i.e., it recovers an approximate solution with desired accuracy at a computational expense that stays proportional to the number of terms in a corresponding wavelet-best N-term approximation. An important advantage of this adaptive approach is that it *automatically* stabilizes the numerical procedure so that, for instance, compatibility constraints on the choice of trial spaces, like the LBB condition, no longer arise.

1. Introduction

Adaptivity is a key ingredient in modern numerical computation. In the context of operator equations, which is the subject of this paper, adaptivity is used in state of the art numerical solvers. For example, finite element methods (FEMs) based on adaptive grid refinements have been shown to experimentally outperform comparable fixed grid methods for many boundary value problems. On the other hand, there is, in the context of information-based complexity (IBC), a theorem of Traub and Wozniakowski [**36**] which brings the advantage of adaptivity into question. Sparing the details for a moment, the Traub–Wozniakowski theorem states that adaptivity gives no advantage over nonadaptive methods in approximating certain classes of target functions. So, does adaptivity really help and, if so, how can one explain the Traub–Wozniakowski theorem in the framework of numerical computation?

Some of these issues were settled in paper [**17**] in which an adaptive wavelet-based algorithm for solving certain elliptic operator equations was introduced and proven to perform optimally in the context of nonlinear approximation (to be described in more detail below). That paper proves (in the context of wavelet-based approximation) that adaptivity gives performance not attainable by nonadaptive wavelet methods. In spite of this achievement, several questions remain. First, there is the question of the scope of the applicability of wavelet-based methods. The algorithm developed in [**17**] applies only to symmetric positive definite problems. Moreover, it still remains to explain how the Traub–Wozniakowski theorem should be viewed in the context of numerical approximation.

The purpose of this present paper is mainly to address the first of these issues. We shall advance a significant modification of the adaptive methods of [**17**] in order to achieve applicability to a wide range of operator equations, both unsymmetric and indefinite, with the same optimal performance as the original algorithm. Actually, in trying to extend the applicability of adaptive wavelet-based methods, we had to rethink to some extent the adaptive algorithm in [**17**]. The end result is an algorithm

that has many advantages over that in [**17**] even when applied to symmetric positive definite problems. Moreover, it is also much easier to conceptually understand the adaptive algorithm of the present paper and the reason for its optimal performance.

While it is not the main purpose of this present paper, at the end of the following section, we shall say a few words about the possible interpretation of the Traub–Wozniakowski theorem in the context of numerical computation. But before turning to that issue, we shall give a more detailed description of adaptive wavelet-based methods and the results of the present paper.

2. Motivation and Background

In FEMs adaptivity occurs in the context of grid refinement. Standard (nonadaptive) FEMs fix the computational grids in advance and this choice is not influenced by the performance of the numerical computation. In adaptive methods, decisions on a further refinement of the mesh are based on preceding numerical computations, usually in the form of utilizing local bounds on residuals to determine where grids should be refined. Because adaptive schemes may give rise to highly varying mesh sizes, the classical error analysis in terms of the largest diameter of the mesh cells is no longer meaningful. Instead, a more reasonable measure for performance is obtained by relating the error ε produced by the adaptive scheme to the *number N of degrees of freedom* in the adaptively generated approximation. Thus, one is interested in the decay of ε as a function of N. The advantage of adaptivity could be ascertained by comparing the error bounds for adaptive schemes with the more classical bounds for fixed grid methods using the same number of degrees of freedom.

In wavelet methods for solving operator equations, the desired solution (henceforth called the *target function*) is approximated numerically by a linear combination of elements from the wavelet basis. In nonadaptive methods, the choice of basis elements is fixed in advance and the whole procedure can be viewed as approximating the solution by elements from linear spaces which are prescribed in advance. In adaptive wavelet-based methods, the basis functions to be used are not chosen in advance but, rather, are determined adaptively at each step of the computation by examining the current residual. In the wavelet context, the number of degrees of freedom is just the number of terms in the finite approximate wavelet expansion of the numerical solution.

In order to be able to appraise the merit of an adaptive wavelet-based method, one could compare its performance with what could be ideally accomplished. Thus a natural bench mark is the *best N-term approximation*. In this type of approximation one picks (assuming full knowledge of the target function) those N wavelets from the complete basis such that a suitable linear combination minimizes the error among all linear combinations of N candidates. Therefore, an asymptotically optimal convergence rate for an adaptive wavelet-based algorithm means that the algorithm realizes the error of best N-term approximation within a certain range

of convergence orders depending on the wavelet basis. To our knowledge an analogous result for conventional discretization settings, relating the achieved accuracy to the number of used degrees of freedom, is to date not available. Of course, the above description is not completely satisfactory from a numerical viewpoint since the computational work is a more important measure than the number of basis elements.

The adaptive wavelet-based method for symmetric, strongly elliptic problems introduced in [**17**] was shown to be optimal in the above sense. It is also remarkable that the computational work could be arranged to remain proportional (up to an additional log factor only for sorting operations) to the number N of terms. Moreover, since it is well-known precisely what regularity conditions (in terms of membership in certain Besov spaces) determine approximation order for both wavelet approximation from fixed linear spaces as well as N-term approximation, it is easy to give examples of target functions (and likewise solutions to elliptic operator equations) for which adaptive methods provide definitively better convergence rates than their nonadaptive method counterparts.

2.1. *Objectives and Conceptual Outline*

Although the scope of problems covered by [**17**] includes differential as well as singular integral operators (possibly of negative order), the role of symmetry and positive definiteness was essential for the analysis in [**17**]. The main purpose of the present paper is to extend the applicability of such adaptive schemes to possibly unsymmetric and indefinite problems while preserving their optimality in the above sense.

Well-Posedness and Scope of Problems. The type of problems we have in mind covers *systems* of operator equations of the following type: mixed formulations of elliptic boundary value problems, transmission problems or multiple field formulations where, in particular, essential (inhomogeneous) boundary conditions are incorporated in the operator formulation. A typical example is the system of Stokes equations augmented by inhomogeneous boundary conditions. Our approach will be shown to work on any such example by showing that the problem induces an operator which is an isomorphism from a certain Hilbert space into its dual, henceforth referred to as *well-posedness*.

An Equivalent Well-Posed ℓ_2-Problem. Our improvement on the adaptive concepts from [**17**] rests on two cornerstones. The first of these is to derive for indefinite problems an equivalent infinite matrix problem on ℓ_2. To explain this transformation of the problem, we recall that in the case of an elliptic operator equation the original problem is transformed into an *equivalent* infinite system of linear equations by simply expanding the solution and the right-hand side in their primal

(respectively, dual) wavelet decomposition. The resulting infinite system matrix **A** is the representation of the operator with respect to this (properly scaled) wavelet basis. Due to the nature of the underlying elliptic problem, the matrix **A** is symmetric positive definite and for suitable wavelet bases gives an automorphism of ℓ_2. Symmetry and positive definiteness were essential for guaranteeing a fixed error reduction by each adaptive refinement step based on inspecting the current residual. The matrix **A** also inherits certain decay properties from the ellipticity and the decay properties of wavelets. These special properties of **A** are essential in establishing the optimal performance of the adaptive scheme introduced in [**17**].

One should note that the fact that the original elliptic operator equation has an equivalent formulation as an infinite system of equations on ℓ_2 seems to be the biggest advantage of wavelet-based methods vis-à-vis FEMs. Wavelet-based methods reduce the original operator equation to the linear algebra problem of solving certain infinite systems of equations. In contrast, in the finite element approach, there is no such infinite system of equations but rather only a hierarchy of finite systems of equations which resolve the problem with better and better accuracy. Thus, in wavelet-based methods, adaptivity corresponds to taking larger and larger sections of a *fixed infinite-dimensional matrix*, whereas in adaptive FEMs, the systems of equations that arise are not naturally embedded in one fixed infinite system.

We wish to extend the procedure of [**17**] to systems of operator equations that are not necessarily definite nor symmetric. However, we will still insist on transforming the original continuous problem into an infinite system which is well-posed in the Euclidean metric. Indeed, combining the well-posedness of the operator equation with the fact that wavelet expansions induce norm equivalences for the involved function spaces, one sees that the wavelet representation of the operator establishes such a well-posed system. However, the resulting infinite matrix will generally be neither symmetric nor definite. Therefore, in this case, we have to apply a further transformation so that iterative schemes for the infinite-dimensional problem like gradient or conjugate gradient iterations apply, in principle, with fixed error reduction per step. One way to do this is to employ a *wavelet least squares* formulation of the original problem which was recently introduced and studied in the context of linear solution schemes in [**26**]. This will be shown to lead to asymptotically optimal schemes for the above-mentioned full scope of problems. When dealing with the more restricted class of *saddle point problems* we will also explore alternatives that avoid a least squares formulation, (hopefully) in favor of a quantitative reduction of condition numbers.

It is important to note that in both of these cases, the infinite matrix arising in the discrete ℓ_2-problem is, in general, no longer the wavelet representation of the original operator, as it was in [**17**].

Application of Infinite-Dimensional Operators. This brings us to the second major conceptual cornerstone which centers on how to resolve numerically the infinite matrix problem. The approach in [**17**] was to the finite problems obtained from

the infinite matrix by selecting a certain finite section of the matrix by choosing a set of rows and columns. The adaptivity of the method enters into how updates this selection at each iteration.

Our new approach will proceed differently. Rather than reduce $\mathbf{A}$ to a finite submatrix, we instead retain $\mathbf{A}$ in its entirety and concentrate rather on numerical schemes for computing the action of the full matrix $\mathbf{A}$ on a finite vector. At each stage the matrix $\mathbf{A}$ will be applied with dynamically updated accuracy to a current finite-dimensional approximation. The numerical efficiency of our new algorithm then relies on fast matrix/vector multiplication modified from that given in [**17**]. At this point several conceptual distinctions from [**17**] arise which distinguish the new adaptive method from the old one. In the new method, we can fully dispense with solving intermediate systems of equations. Moreover, the a-posteriori refinement strategy disappears. The incorporation of additional degrees of freedom is solely dictated by the adaptive application of the infinite-dimensional operator to the current approximation and the update through the iteration, which results in a conceptually much simpler structure. In fact, applications to different types of problems will be seen to require only two modifications in the adaptive algorithm of [**17**]. The first is to change the subroutine whose role is to resolve the right-hand side of the infinite system. The second change occurs in the routine for matrix/vector multiplication.

Stabilization. We note one final important feature of our approach. In treating indefinite problems by conventional schemes the choice of trial spaces is typically constrained by compatibility conditions like the LBB condition or else additional stabilization techniques are necessary. Similarly when approximately evaluating "difficult" norms like the H^{-1}-norm in the context of finite element least squares formulations the "ideal" least squares functionals are perturbed so that the issue of stability requires some precautions. Therefore, it might, at first glance, be surprising that the necessity for such precautions does *not* arise in the theory we put forward. Roughly speaking, the adaptive procedure stabilizes automatically by inheriting the stability properties of the underlying infinite-dimensional problem.

2.2. *The Traub–Wozniakowski Theorem*

Let us return to the Traub–Wozniakowski theorem and its role in numerical computation. This theorem concerns approximating a given class $\mathcal{F}$ of functions f. It puts into competition two possible methods. The nonadaptive method is to fix linear spaces $X_1 X_2, \ldots$ with X_n of dimension n and to take for each n as an approximation to a given $f \in \mathcal{F}$ an element from X_n. The second method, which is adaptive, allows us to query the target function f in the form of a sequence of n questions to be interpreted as asking for the values of linear functionals (which can be chosen at the user's wish) applied to f. Adaptivity occurs in that the nth question can depend on the answers to the previous $n-1$ questions. The Traub–

Wozniakowski theorem states that, for certain classes of functions $\mathcal{F}$, such an adaptive scheme cannot do better than a well-chosen nonadaptive scheme. That is, given $\mathcal{F}$ and given the rules for the adaptive scheme, it is possible to find linear spaces X_n, $n = 1, 2, \ldots$, such that the nonadaptive scheme with this choice of spaces performs at least as well as the adaptive scheme.

The most important observation to make about this theorem is that it is an existence theorem. While it proves the existence of the linear spaces X_n, $n = 1, 2, \ldots$, it does not tell us how to find them. An example might be illustrative. In elliptic problems with domains with rough boundaries, for example, corners, the solution is known to exhibit loss of regularity near the boundary. Adaptive finite element schemes generate finer grids near the boundary, while wavelet schemes introduce more wavelet basis functions there. In some cases, our analytical knowledge of the solution near the boundary is so strong that it is possible to prescribe the grids in advance which will improve the computation. In this sense, a linear scheme with these fixed grids would perform as well as an adaptive scheme. But, of course, for general domains there is no such analytic theory rich enough to prescribe in advance the nature of the optimal gridding. Adaptive numerical schemes find these optimal grids, or in the wavelet case, the optimal selection of basis functions utilizing only knowledge about the given data and information acquired in the course of the solution process.

2.3. *Contents of This Paper*

The present paper is organized as follows. In Section 3 we briefly indicate the scope of problems that can be treated with our new adaptive algorithm. In Section 4 we introduce our new adaptive algorithm and analyze its accuracy. We assume at this point that the original system of operator equations has been transformed into a well-posed ℓ_2-problem. Section 5 is devoted to a complexity analysis of our algorithm by using the notion of *best N-term approximation* as a bench mark. We identify conditions under which the iteration exhibits asymptotically optimal complexity in the following sense. For any desired target accuracy, our algorithm provides an approximate solution in terms of a finitely supported sequence involving asymptotically the same order of nonzero terms as the best N-term approximation of the true solution. Moreover, the computational work remains (up to additional logarithms for sort operations) proportional to the number of degrees of freedom. We conclude the general analysis in Section 5.4 by exhibiting an important class of matrices for which the above key requirements can be shown to be satisfied. In particular, the fast matrix/vector multiplication arising in this context will serve as a core ingredient in subsequent developments. Section 6 is concerned with wavelet discretization which provides the foundation for realizing the general concept from Section 4. In Section 6.1 we collect those wavelet prerequisites that are needed to facilitate the above-mentioned transformation of the original problem presented in Section 6.2. We briefly revisit in Section 6.3 the scalar elliptic case and present a

first application of the previous findings. In Section 7 a wavelet least squares formulation is shown to provide a transformation of the original variational problem into a well-posed ℓ_2-problem of the desired type for the whole scope of problems described in Section 3.1. In fact, this always works whenever a basic mapping property of the original continuous problem has been established. The general results are then applied to this case yielding asymptotically optimal schemes for this framework. Nevertheless, we discuss in Section 8 the special class of saddle point problems and propose alternatives that avoid the squaring of condition numbers.

We stress that our interest concerns the principal conceptual aspects and not so much the quantitatively optimal realizations which in each special case would certainly require much further specialized considerations. Therefore we will not care so much about optimal step size parameters and will often assume that estimates for the norms of the involved operators are given. The quality of these estimates will not effect asymptotics but may, of course, be essential for the quantitative numerical behavior of the schemes.

Throughout the rest of this paper we will employ the following notational convention. Unless specific constants have to be identified the relation $a \sim b$ will always stand for $a \lesssim b$ and $b \lesssim a$, the latter inequality meaning that b can be bounded by some constant times a uniformly in all parameters upon which a and b may depend.

3. The Scope of Problems

We begin by describing a class of problems to which the subsequent analysis will apply. In particular, we are interested in concepts that will also work for nonsymmetric or indefinite problems.

3.1. *A General Variational Formulation*

In this section we follow [**26**] to describe first the format of problems for which we will then construct adaptive solvers. In general we will be dealing with *systems* of operator equations given in variational form. Thus the solution will be vector valued with components living in certain spaces $H_{i,0}$, $i = 1, \dots, m$. These spaces will always be closed subspaces of Hilbert spaces H_i endowed with the inner products $\langle \cdot, \cdot \rangle_{H_i}$. The space $H'_{i,0}$ is the normed dual of $H_{i,0}$ endowed with the norm

$$\|v\|_{H'_{i,0}} := \sup_{0 \neq w \in H_{i,0}} \frac{\langle v, w \rangle}{\|w\|_{H_i}}. \tag{3.1.1}$$

The H_i will be Sobolev spaces on possibly different types of domains such as open domains $\Omega \subseteq \mathbb{R}^d$ or boundary manifolds $\Gamma = \partial\Omega$. Thus we have either

$$H_i \subseteq L_2 \subseteq H'_{i,0} \qquad \text{or} \qquad H'_{i,0} \subseteq L_2 \subseteq H_i, \tag{3.1.2}$$

where $L_2 = L_2(\Omega)$ or $L_2 = L_2(\Gamma)$. Of course, some of the H_i may actually be equal to L_2. The subspaces $H_{i,0}$ will typically be determined by homogeneous boundary conditions or zero mean values.

As before, the inner product defining a Hilbert space H will be denoted by $\langle\cdot,\cdot\rangle_H$. Only when $H = L_2$ will we drop subscripts and write $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L_2}$, $\|\cdot\| = \|\cdot\|_{L_2}$ provided there is no risk of confusion. Given $H_{i,0}$, $i = 1,\dots,m$, let $\mathcal{H} := \prod_{i=1}^m H_{i,0}$ where $\langle\cdot,\cdot\rangle_{\mathcal{H}} := \sum_{i=1}^m \langle\cdot,\cdot\rangle_{H_i}$ is the canonical inner product on $\mathcal{H}$.

We shall denote the elements $V \in \mathcal{H}$ by $V = (v_1,\dots,v_m)^T$. Then,

$$\|V\|_{\mathcal{H}} := \left(\sum_{i=1}^m \|v_i\|_{H_i}^2\right)^{1/2}, \qquad \|W\|_{\mathcal{H}'} = \sup_{\|V\|_{\mathcal{H}}=1} \langle V, W\rangle, \tag{3.1.3}$$

where $\langle V, W\rangle := \sum_{i=1}^m \langle v_i, w_i\rangle$.

Now suppose that $A_{i,l}(\cdot,\cdot)$ are bounded bilinear forms on $H_i \times H_l$,

$$A_{i,l}(v,w) \lesssim \|v\|_{H_i}\|w\|_{H_l}, \qquad i,l = 1,\dots,m, \tag{3.1.4}$$

and define, for $V \in \mathcal{H}$,

$$A_i(w,V) := \sum_{l=1}^m A_{i,l}(w,v_l), \qquad w \in H_{i,0}, \qquad i = 1,\dots,m. \tag{3.1.5}$$

We will be concerned with the following variational problem: Given $f_i \in H_{i,0}'$, $i = 1,\dots,m$, find $U \in \mathcal{H}$ such that

$$A_i(w,U) = \langle w, f_i\rangle, \qquad w \in H_{i,0}, \qquad i = 1,\dots,m. \tag{3.1.6}$$

The crucial property that will have to be verified in each concrete example is that (3.1.6) is *well-posed*. To explain what this means, let the operators

$$\mathcal{L}_{i,l}\colon H_{l,0} \to H_{i,0}', \qquad \mathcal{L}_i\colon \mathcal{H} \to H_{i,0}',$$

be defined by

$$\langle w, \mathcal{L}_{i,l}v\rangle = A_{i,l}(w,v), \qquad \langle w, \mathcal{L}_i U\rangle := A_i(w,U), \qquad w \in H_{i,0}. \tag{3.1.7}$$

Thus (3.1.6) is equivalent to the operator equation: Given $F = (f_1,\dots,f_m)^T \in \mathcal{H}'$ find $U = (u_1,\dots,u_m)^T \in \mathcal{H}$ such that

$$\sum_{j=1}^m \mathcal{L}_{i,j}u_j =: \mathcal{L}_i U = f_i, \qquad i = 1,\dots,m, \tag{3.1.8}$$

or, briefly,

$$\mathcal{L}U = F, \tag{3.1.9}$$

where $\mathcal{L} = (\mathcal{L}_{i,j})_{i,j=1}^m$.

We say that (3.1.9) is *well-posed* if $\mathcal{L}$ is an *isomorphism* from $\mathcal{H}$ to $\mathcal{H}'$, i.e., there exist finite positive constants $0 < c_{\mathcal{L}} \le C_{\mathcal{L}} < \infty$ such that

$$c_{\mathcal{L}}\|V\|_{\mathcal{H}} \le \|\mathcal{L}V\|_{\mathcal{H}'} \le C_{\mathcal{L}}\|V\|_{\mathcal{H}}, \qquad V \in \mathcal{H}. \tag{3.1.10}$$

3.2. *Examples*

We will outline next several examples covered by the above setting. For a more detailed discussion and the verification of (3.1.10) in each case we refer to [**26**].

Scalar Elliptic Problems. The simplest case arises when $m = 1$ and $A_{1,1}(\cdot,\cdot) = a(\cdot,\cdot)$ is a symmetric bilinear form on the Hilbert space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}} := \langle\cdot,\cdot\rangle_{\mathcal{H}}^{1/2}$ which is $\mathcal{H}$-elliptic, i.e., $a(v,v) \sim \|v\|_{\mathcal{H}}^2$, $v \in \mathcal{H}$. Thus the operator $\mathcal{L}$ defined by $\langle v, \mathcal{L}u\rangle = a(v,u)$ ($\langle\cdot,\cdot\rangle$ denoting the dual pairing for $\mathcal{H}\times\mathcal{H}'$) satisfies (3.1.10).

Typical examples are the single layer potential operator on a boundary manifold Γ with $\mathcal{H} = H^{-1/2}(\Gamma)$, the double layer or hypersingular operator with $\mathcal{H} = L_2(\Gamma)$, $\mathcal{H} = H^{1/2}(\Gamma)$, respectively, or, of course, Poisson's equation on some domain Ω,

$$-\Delta u + \beta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \Gamma_D, \tag{3.2.1}$$

where the function β is nonnegative on Ω and $\Gamma_D \subseteq \Gamma := \partial\Omega$. In this case, $\mathcal{H} = H^1_{0,\Gamma_D}(\Omega)$ is the closure of all smooth functions on Ω that vanish on Γ_D.

Mixed Formulations. It will be convenient to use boldface notation for vector valued quantities and corresponding spaces. For instance, denoting always by d the spatial dimension we set $\mathbf{L}_2(\Omega) = (L_2(\Omega))^d$ or $\mathbf{H}^1(\Omega) := (H^1(\Omega))^d$.

Consider again the simple elliptic problem (3.2.1). One is often primarily interested in the fluxes $\boldsymbol{\theta} := -\mathbf{a}\nabla u$ which can be directly accessed through the *mixed formulation* of the above second-order elliptic scalar boundary value problem as a first-order system

$$\begin{aligned} \langle\boldsymbol{\theta},\boldsymbol{\eta}\rangle + \langle\boldsymbol{\eta},\mathbf{a}\nabla u\rangle &= 0, \qquad && \forall\,\boldsymbol{\eta}\in\mathbf{L}_2(\Omega),\\ -\langle\boldsymbol{\theta},\nabla v\rangle + \langle ku, v\rangle &= \langle f, v\rangle, && \forall v\in H^1_{0,\Gamma_D}(\Omega). \end{aligned} \tag{3.2.2}$$

Clearly Galerkin discretizations of (3.2.2) will no longer give rise to definite linear systems. Nevertheless, (3.2.2) is of the form (3.1.6) and the induced operator $\mathcal{L}$ can be shown to satisfy (3.1.10) with

$$\mathcal{H} := \mathbf{L}_2(\Omega)\times H^1_{0,\Gamma_D}(\Omega). \tag{3.2.3}$$

Transmission Problems. The next example involves differential *and* integral operators. Let $\Omega_0 \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary Γ. Furthermore, suppose that Ω_1 is another domain with piecewise smooth boundary Γ_D and that the closure of Ω_1 is contained in the interior of Ω_0 so that $\Omega_0\backslash\Omega_1$ is an angular domain with inner and outer boundaries Γ_D (respectively, Γ). On the unbounded domain $\mathbb{R}^d\backslash\Omega_1$ we consider the following second-order

boundary value problem

$$\begin{aligned} -\nabla\cdot(\mathbf{a}\nabla u)+k\,u &= f \quad \text{in } \Omega_0\backslash\Omega_1,\\ -\Delta u &= 0 \quad \text{in } \mathbb{R}^d\backslash\Omega_0,\\ u|_{\Gamma_D} &= 0. \end{aligned} \tag{3.2.4}$$

When $d=2$ one also needs a suitable decay condition at infinity. For simplicity of exposition, we therefore confine the discussion in the following to the case $d\geq 3$. Denoting by $\partial_{\mathbf{n}}u$ the normal derivative with respect to the outer normal $\mathbf{n}$ of Ω_0 on Γ, we impose on the interface Γ between $\mathbb{R}^d\backslash\Omega_0$ and Ω_0 the *interface* or *transmission conditions* $u^-=u^+$, $(\partial_{\mathbf{n}})u^-=(\partial_{\mathbf{n}})u^+$, where the superscripts $+$ (respectively, $-$), denote the limits $\mathbf{y}\to\mathbf{x}\in\Gamma$, $\mathbf{y}\in\Omega_1$ (respectively, $\mathbf{y}\in\Omega_0$). The weak formulation of (3.2.4) reads: find $U=(u,\sigma)\in H^1_{0,\Gamma_D}(\Omega_0)\times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \langle\mathbf{a}\nabla u,\nabla v\rangle_{L_2(\Omega_0)}+\langle\mathcal{W}u-(\tfrac{1}{2}\mathcal{I}-\mathcal{K}')\sigma,v\rangle_{L_2(\Gamma)} &= \langle f,v\rangle_{L_2(\Omega_0)},\\ v\in H^1_{0,\Gamma_D}(L_2(\Omega_0)),&\\ \langle(\tfrac{1}{2}\mathcal{I}-\mathcal{K})u,\delta\rangle_{L_2(\Gamma)}+\langle\mathcal{V}\sigma,\delta\rangle_{L_2(\Gamma)} &= 0,\\ \delta\in H^{-1/2}(\Gamma),& \end{aligned} \tag{3.2.5}$$

where for $\mathcal{E}(\mathbf{x},\mathbf{y})=(4\pi|\mathbf{x}-\mathbf{y}|)^{-1}$ the involved boundary integral operators on the interface boundary Γ are given by

$$\begin{aligned} \mathcal{V}\sigma(\mathbf{x}) &:= \int_\Gamma \mathcal{E}(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})\,ds_{\mathbf{y}}, \qquad \mathcal{K}\sigma(\mathbf{x}) := \int_\Gamma(\partial_{\mathbf{n}_{\mathbf{y}}}\mathcal{E}(\mathbf{x},\mathbf{y}))\sigma(\mathbf{y})\,ds_{\mathbf{y}},\\ \mathcal{K}'\sigma(\mathbf{x}) &:= \int_\Gamma(\partial_{\mathbf{n}_{\mathbf{x}}}\mathcal{E}(\mathbf{x},\mathbf{y}))\sigma(\mathbf{y})\,ds_{\mathbf{y}},\\ \mathcal{W}\sigma(\mathbf{x}) &:= -\partial_{\mathbf{n}_{\mathbf{x}}}\int_\Gamma(\partial_{\mathbf{n}_{\mathbf{y}}}\mathcal{E}(\mathbf{x},\mathbf{y}))\sigma(\mathbf{y})\,ds_{\mathbf{y}}. \end{aligned} \tag{3.2.6}$$

Here $\mathcal{V}$, $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{W}$ are the *single layer, double layer, adjoint of the double layer potential*, and *hypersingular operator*, respectively [**31**]. Again (3.2.5) is of the form (3.1.6) and (3.1.10) can be shown to hold for the corresponding $\mathcal{L}$ with

$$\mathcal{H}:=H^1_{0,\Gamma_D}(\Omega_0)\times H^{-1/2}(\Gamma),$$

see [**26**] and the references cited therein.

The Stokes Problem with Inhomogeneous Boundary Conditions. Moreover, well-posed *saddle point problems* fall into the present setting. They will be addressed later in more detail. More generally, multiple field problems are obtained when also incorporating, in addition to the divergence constraint, boundary conditions.

As an example, consider the Stokes system

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}|_\Gamma &= \mathbf{g}, \qquad \int_\Omega p\,dx = 0, \end{aligned} \tag{3.2.7}$$

where the boundary function $\mathbf{g}$ satisfies $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,ds = 0$. To this end let, as usual, $L_{2,0}(\Omega) := \{q \in L_2(\Omega) : \int_\Omega q\,dx = 0\}$ and note that the gradient of $\mathbf{v} \in \mathbf{H}^1(\Omega)$ is now a $(d\times d)$-matrix $\nabla\mathbf{v} := (\nabla v_1, \ldots, \nabla v_d)$. Moreover, for any such $(d\times d)$-matrix valued functions $\underline{\boldsymbol{\theta}}, \underline{\boldsymbol{\eta}}$, whose columns are denoted by $\boldsymbol{\theta}_i, \boldsymbol{\eta}_i$, we define

$$\langle\underline{\boldsymbol{\theta}}, \underline{\boldsymbol{\eta}}\rangle := \sum_{i=1}^d \langle\boldsymbol{\theta}_i, \boldsymbol{\eta}_i\rangle = \sum_{i=1}^d \int_\Omega \boldsymbol{\theta}_i\cdot\boldsymbol{\eta}_i\,dx.$$

The so-called weak *three field formulation* of the Stokes problem (3.2.7) is then the following: given $\mathbf{f} \in (\mathbf{H}^1(\Omega))'$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ as above, find

$$U = (\mathbf{u}, \boldsymbol{\lambda}, p) \in \mathcal{H} := \mathbf{H}^1(\Omega)\times\mathbf{H}^{-1/2}(\Gamma)\times L_{2,0}(\Omega)$$

such that

$$\begin{aligned} \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_{\mathbf{L}_2(\Omega)} + \langle\gamma\mathbf{v}, \boldsymbol{\lambda}\rangle + (\operatorname{div}\mathbf{v}, p)_{L_2(\Omega)} &= \langle\mathbf{f}, \mathbf{v}\rangle, \qquad \mathbf{v} \in \mathbf{H}^1(\Omega), \\ \langle\gamma\mathbf{u}, \boldsymbol{\mu}\rangle &= \langle\mathbf{g}, \boldsymbol{\mu}\rangle, \quad \boldsymbol{\mu} \in \mathbf{H}^{-1/2}(\Gamma), \\ (\operatorname{div}\mathbf{u}, q)_{L_2(\Omega)} &= 0, \qquad q \in L_{2,0}(\Omega). \end{aligned} \tag{3.2.8}$$

The system (3.2.8) can be shown to induce an operator $\mathcal{L}$ which satisfies (3.1.10) for

$$\mathcal{H} := \mathbf{H}^1(\Omega)\times\mathbf{H}^{-1/2}(\Gamma)\times L_{2,0}(\Omega),$$

see, e.g., [**26**], [**34**]. The above formulation is of particular interest when boundary values act as *control variables* in an optimal control problem [**34**]. The Lagrange multiplier $\boldsymbol{\lambda}$ can be interpreted as boundary stresses.

First-Order Stokes System. One way of treating indefinite problems of the above type is to employ least squares formulations. In order to avoid squaring the order of the operator one can also turn (3.2.7) into a first-order system

$$\begin{aligned} \underline{\boldsymbol{\theta}} + \nabla\mathbf{u} &= \underline{\mathbf{0}} \quad \text{in } \Omega, \\ -\nu(\operatorname{div}\underline{\boldsymbol{\theta}})^T + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{3.2.9}$$

where, as before, $\underline{\boldsymbol{\theta}} := -\nabla \mathbf{u} := -(\nabla u_1, \ldots, \nabla u_d)$, one obtains the weak formulation

$$\begin{aligned} \langle \underline{\boldsymbol{\theta}}, \underline{\boldsymbol{\eta}} \rangle + \langle \underline{\boldsymbol{\eta}}, \nabla \mathbf{u} \rangle &= 0, & \underline{\boldsymbol{\eta}} &\in \underline{\mathbf{L}}_2(\Omega), \\ \nu \langle \underline{\boldsymbol{\theta}}, \nabla \mathbf{v} \rangle - \langle p, \operatorname{div} \mathbf{v} \rangle - \langle \boldsymbol{\lambda}, \mathbf{v} \rangle_{L_2(\Gamma)} &= \langle \mathbf{f}, \mathbf{v} \rangle, & \mathbf{v} &\in \mathbf{H}^1(\Omega), \\ \langle \operatorname{div} \mathbf{u}, q \rangle &= 0, & q &\in L_{2,0}(\Omega), \\ \langle \boldsymbol{\mu}, \mathbf{u} \rangle_{L_2(\Gamma)} &= \langle \boldsymbol{\mu}, \mathbf{g} \rangle_{L_2(\Gamma)}, & \boldsymbol{\mu} &\in \mathbf{H}^{-1/2}(\Gamma). \end{aligned} \tag{3.2.10}$$

In this case (3.1.10) holds with

$$\mathcal{H} := \underline{\mathbf{L}}_2(\Omega) \times \mathbf{H}^1(\Omega) \times L_{2,0}(\Omega) \times \mathbf{H}^{-1/2}(\Omega),$$

see [**13**], [**26**].

Moreover, the same results hold when Γ is not the boundary of Ω but of some smaller domain Ω_0 which is strictly contained in Ω. The restriction of the solution to Ω_0 can be shown to solve the original problem on Ω_0, see, e.g., [**33**]. In this case Ω can be chosen as a "fictitious" very simple domain like a cube which can be kept fixed even when the boundary Γ moves. On Ω one could therefore employ fixed simple and efficient discretizations. This will be seen later to support the use of wavelet concepts in such a context.

4. The Basic Concept

In this section we shall describe an abstract framework that will later be applied to the above examples.

4.1. *A General ℓ_2-Problem*

Recall that all examples discussed above can be recast as a weakly defined operator equation

$$\mathcal{L}U = F \tag{4.1.1}$$

which is well-posed in the sense that for some Hilbert space $\mathcal{H}$ one has

$$\|V\|_{\mathcal{H}} \sim \|\mathcal{L}V\|_{\mathcal{H}'}. \tag{4.1.2}$$

It will be seen later that, once (4.1.2) has been established, wavelet discretizations will allow us to transform (4.1.1) into an *equivalent* problem on ℓ_2,

$$\mathbf{MU} = \mathbf{F}, \tag{4.1.3}$$

which has an analogous mapping property, namely one has, for some positive finite constants c_M, C_M,

$$c_M \|\mathbf{V}\|_{\ell_2} \le \|\mathbf{MV}\|_{\ell_2} \le C_M \|\mathbf{V}\|_{\ell_2}, \qquad \mathbf{V} \in \ell_2. \tag{4.1.4}$$

Moreover, each $\mathbf{V} \in \ell_2$ will be associated with a unique $V \in \mathcal{H}$ such that

$$\|V\|_{\mathcal{H}} \sim \|\mathbf{V}\|_{\ell_2}. \tag{4.1.5}$$

Thus approximation in ℓ_2 is directly related to approximation in $\mathcal{H}$. The sequences in ℓ_2 will be indexed over sets $\mathcal{J}$ whose precise nature will be explained later.

At this point we will not yet specify the concrete way how to get from (4.1.1) to (4.1.3). We merely insist on the existence of some norm $\|\cdot\|$ on ℓ_2 satisfying

$$\hat{c}\|\mathbf{V}\|_{\ell_2} \le \|\mathbf{V}\| \le \hat{C}\|\mathbf{V}\|_{\ell_2}, \qquad \mathbf{V} \in \ell_2, \tag{4.1.6}$$

with positive constants $\hat{c}, \hat{C}$ independent of $\mathbf{V} \in \ell_2$, and that one has, for the associated operator norm $\|\mathbf{M}\| := \sup_{\|\mathbf{V}\|=1} \|\mathbf{M}\mathbf{V}\|$ and some positive number α,

$$\rho := \|\mathbf{id} - \alpha\mathbf{M}\| < 1. \tag{4.1.7}$$

We will assume throughout this section that (an estimate for) $\rho < 1$ is given. Supposing for the moment that (4.1.7) is valid, we will consider the following iterative scheme for the full *infinite-dimensional* problem (4.1.3),

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \alpha(\mathbf{F} - \mathbf{M}\mathbf{U}^n), \tag{4.1.8}$$

so that, by (4.1.7), an exact execution of each step reduces the current error by ρ,

$$\|\mathbf{U}^{n+1} - \mathbf{U}\| \le \rho\|\mathbf{U}^n - \mathbf{U}\|, \tag{4.1.9}$$

thereby ensuring convergence.

In some instances the matrix $\mathbf{M}$: $\ell_2 \to \ell_2$ will actually be a *symmetric positive definite* (infinite) matrix although $\mathcal{L}$ will generally not have this property, i.e., the bilinear form $\langle V, \mathcal{L}W\rangle_{\mathcal{H}}$ will in general neither be symmetric nor definite. In this case (4.1.4) can be reexpressed in the form

$$c_M\|\mathbf{V}\|_{\ell_2}^2 \le \mathbf{V}^*\mathbf{M}\mathbf{V} \le C_M\|\mathbf{V}\|_{\ell_2}^2, \tag{4.1.10}$$

with the constants c_M, C_M from (4.1.4). Moreover, $\|\cdot\| := \|\cdot\|_{\ell_2}$ is then the canonical choice so that in this case $\hat{c} = \hat{C} = 1$.

Under the latter assumption on $\mathbf{M}$ a natural choice would be the conjugate gradient method instead of the simple gradient iteration. Nevertheless, to keep things as simple as possible we confine the discussion to the iteration (4.1.8).

At any rate, pursuing this strategy, one faces two principal tasks. First one has to find for the scope of problems outlined in Section 3 transformations from (4.1.1) into (4.1.3) with the above properties. Second the ideal iteration (4.1.8) has to be turned into a practicable numerical scheme whose complexity has to be estimated. We will systematically address both issues in the subsequent sections starting with the latter one.

4.2. *Approximate Iterations*

Of course, (4.1.8) is not realistic since we can neither evaluate the infinite sequence $\mathbf{F}$ nor the application of $\mathbf{M}$ to $\mathbf{U}^n$. Therefore both ingredients have to be *approximated*. At this point we will assume we have the following routines at our disposal:

RHS $[\eta, \mathbf{F}] \to (\mathbf{F}_\eta, \Lambda)$
determines for any positive tolerance η a finitely supported sequence $\mathbf{F}_\eta$ with support $\Lambda \subset \mathcal{J}$ satisfying

$$\|\mathbf{F} - \mathbf{F}_\eta\|_{\ell_2} \le \eta. \tag{4.2.1}$$

APPLY $[\eta, \mathbf{M}, \mathbf{V}] \to (\mathbf{W}, \Lambda)$
determines for any given finitely supported vector $\mathbf{V}$ a vector $\mathbf{W}$ supported on some finite set $\Lambda \subset \mathcal{J}$ satisfying

$$\|\mathbf{MV} - \mathbf{W}\|_{\ell_2} \le \eta. \tag{4.2.2}$$

Our objective is to realize the required accuracy (4.2.1) and (4.2.2) at the expense of as few entries in the vectors $\mathbf{F}_\eta$ and $\mathbf{W}_\eta$ as possible. However, performing simply a perturbed analog to (4.1.8) with the exact evaluations replaced by the above approximations might eventually accumulate too many coefficients. To avoid this we will borrow the following thresholding strategy from [**17**]. To describe this we need to be a bit more specific about the sequences $\mathbf{V}$. In view of the structure of the variational problem (3.1.6), and the corresponding examples, the sequences $\mathbf{V}$ will be of the form $\mathbf{V} = (\mathbf{v}^1, \dots, \mathbf{v}^m)$ where each $\mathbf{v}^i$ will be an ℓ_2-sequence indexed over sets $\mathcal{J}_i$. (Accordingly, $\mathbf{M}$ is an $(m \times m)$-block matrix, each block also being infinite.) Setting $\mathcal{J} := \mathcal{J}_1 \times \cdots \times \mathcal{J}_m$, it is clear that $\|\mathbf{V}\|^2_{\ell_2(\mathcal{J})} = \sum_{i=1}^m \|\mathbf{v}^i\|^2_{\ell_2(\mathcal{J}_i)}$. Moreover, finite subsets of $\mathcal{J}$ will always be denoted by $\mathbf{\Lambda} = (\Lambda^1 \times \cdots \times \Lambda^m)$ where $\Lambda^i \subset \mathcal{J}_i$. To describe the next routine, which is to *coarsen* a given finitely supported sequence, let $\mathbf{v}^* = \{v_n^*\}_{n\in\mathbb{N}}$ denote a *nonincreasing rearrangement* of the entries in *all* individual sequences $\mathbf{v}^i$. Thus $v_n^* \ge v_{n+1}^*$ and $v_n^* = v_{\lambda_n}^{i_n}$ for some $i_n \in \{1, \dots, m\}$, $\lambda_n \in \mathcal{J}_{i_n}$. The following routine is a straightforward extension of a scalar version from [**17**] to the present setting:

COARSE $[\eta, \mathbf{W}] \to (\bar{\mathbf{W}}, \mathbf{\Lambda})$

(i) *Define $N := \#(\operatorname{supp}\mathbf{W})$ and sort the nonzero entries of $\mathbf{W}$ into decreasing order which gives $\mathbf{w}^*$ as described above. Then compute $\|\mathbf{W}\|^2_{\ell_2(\mathcal{J})} = \sum_{l=1}^N |w_l^*|^2$.*

(ii) *For $k = 1, 2, \dots$, form the sum $\sum_{j=1}^k |w_{i_j,\lambda_j}|^2$ in order to find the smallest value k such that this sum exceeds $\|\mathbf{W}\|^2_{\ell_2(\mathcal{J})} - \eta^2$. For this k, define $K := k$ and assemble $\mathbf{\Lambda}$ from the index pairs $(i_l, \lambda_l), l = 1, \dots, K$, for which $w_l^* = w_{\lambda_l}^{i_l}$. Define $\bar{\mathbf{W}}$ by $\bar{\mathbf{W}} := \mathbf{W}|_\Lambda$.*

The following properties of **COARSE** are easily derived, see Lemma 5.1 and Corollary 5.2 in [**17**].

Remark 4.1. For any finitely supported $\mathbf{W} \in \ell_2(\mathcal{J})$ with $N := \#(\operatorname{supp} \mathbf{W})$, and for any $\eta > 0$, we need at most $2N$ **arithmetic operations** and $N \log N$ operations spent on **sorting**, to compute the output $\bar{\mathbf{W}}$ of **COARSE** which, by construction, satisfies

$$\|\mathbf{W} - \bar{\mathbf{W}}\|_{\ell_2(\mathcal{J})} \le \eta. \tag{4.2.3}$$

Having these routines at hand we will now construct *finitely supported* sequences $\bar{\mathbf{U}}^n$, $n = 0, 1, 2, \ldots$, which approximate the solution $\mathbf{U}$ of (4.1.3). To this end, let

$$K := \min \left\{ l \in \mathbb{N} : \rho^{l-1}(2\alpha \hat{C} l + \rho) \le \left(\frac{8\hat{C}}{\hat{c}} + 2 \right)^{-1} \right\}. \tag{4.2.4}$$

Recall the constants $\hat{c}$, $\hat{C}$ from (4.1.6).

SOLVE $[\varepsilon, \mathbf{M}, \mathbf{F}] \to (\bar{\mathbf{U}}(\varepsilon), \mathbf{\Lambda}(\varepsilon))$
determines, for a given right-hand side $\mathbf{F} \in \ell_2(\mathcal{J})$, $\mathbf{M}$ *satisfying* (4.1.10) *and any target accuracy* ε *an approximate solution* $\bar{\mathbf{U}}(\varepsilon)$ *of* (4.1.3) *supported on some finite set* $\mathbf{\Lambda}(\varepsilon)$ *as follows*:

(i) *Initialization*: *Fix the target accuracy* ε *and set*

$$\bar{\mathbf{U}}^0 = \mathbf{0}, \qquad \mathbf{\Lambda}_0 := \emptyset, \qquad \varepsilon_0 := c_M^{-1} \hat{C} \|\mathbf{F}\|_{\ell_2}, \qquad j = 0. \tag{4.2.5}$$

(ii) *If* $\varepsilon_j \le \hat{c}\varepsilon$, *stop and accept* $\bar{\mathbf{U}}^j$ *as solution. Otherwise set* $\mathbf{V}^0 := \bar{\mathbf{U}}^j$.

(ii.1) *For* $l = 0, \ldots, K-1$, *compute*

RHS $[\rho^l \varepsilon_j, \mathbf{F}] \to (\mathbf{F}_l, \mathbf{\Lambda}_{l,F})$,
APPLY $[\rho^l \varepsilon_j, \mathbf{M}, \mathbf{V}^l] \to (\mathbf{W}^l, \mathbf{\Lambda}_{l,M})$,

as well as

$$\mathbf{V}^{l+1} := \mathbf{V}^l + \alpha(\mathbf{F}_l - \mathbf{W}^l). \tag{4.2.6}$$

(ii.2) *Apply* **COARSE** $[(4/\hat{c})(8\hat{C}/\hat{c} + 2)^{-1}\varepsilon_j, \mathbf{V}^K] \to (\bar{\mathbf{U}}^{j+1}, \mathbf{\Lambda}_{j+1})$, *set* $\varepsilon_{j+1} := \varepsilon_j/2$, $j+1 \to j$, *and go to* (ii).

Under the assumptions (4.2.1) and (4.2.2) the accuracy of the above iteration can be estimated as follows:

Proposition 4.2. *The iterates* $\bar{\mathbf{U}}^j$ *produced by* **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ *satisfy*

$$\|\mathbf{U} - \bar{\mathbf{U}}^j\| \le \varepsilon_j, \qquad j \in \mathbb{N}_0, \tag{4.2.7}$$

and hence, by (4.1.6),

$$\|\mathbf{U} - \bar{\mathbf{U}}^j\|_{\ell_2} \le \hat{c}^{-1}\varepsilon_j, \qquad j \in \mathbb{N}_0, \tag{4.2.8}$$

where as above $\varepsilon_j := 2^{-j}\varepsilon_0$.

Proof. Clearly, by (4.1.4),

$$\hat{C}^{-1}\|\mathbf{U}-\bar{\mathbf{U}}^0\| \le \|\mathbf{U}-\bar{\mathbf{U}}^0\|_{\ell_2} = \|\mathbf{U}\|_{\ell_2} \le \|\mathbf{M}^{-1}\|\|\mathbf{F}\|_{\ell_2} \le c_M^{-1}\|\mathbf{F}\|_{\ell_2} = \varepsilon_0/\hat{C}.$$

Now suppose that for some $j \ge 0$ one has $\|\mathbf{U}-\bar{\mathbf{U}}^j\| \le \varepsilon_j$. Denoting as before by $\mathbf{U}^j(\mathbf{V})$ the exact iterates from (4.1.8) with initial guess $\mathbf{V}$. Then one readily verifies that

$$\mathbf{V}^{l+1}-\mathbf{U}^{l+1}(\bar{\mathbf{U}}^j) = (\mathbf{id}-\alpha\mathbf{M})(\mathbf{V}^l-\mathbf{U}^l(\bar{\mathbf{U}}^j))+\alpha((\mathbf{F}_l-\mathbf{F})+(\mathbf{M}\mathbf{V}^l-\mathbf{W}^l)). \quad (4.2.9)$$

Thus, we infer from (4.1.9), (4.2.1), (4.2.2), (4.1.6), and step (ii) in **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ that

$$\begin{aligned}\|\mathbf{V}^{l+1}-\mathbf{U}^{l+1}(\bar{\mathbf{U}}^j)\| &\le \rho\|\mathbf{V}^l-\mathbf{U}^l(\bar{\mathbf{U}}^j)\|+\alpha(\|\mathbf{F}_l-\mathbf{F}\|+\|\mathbf{M}\mathbf{V}^l-\mathbf{W}^l\|)\\ &\le \rho\|\mathbf{V}^l-\mathbf{U}^l(\bar{\mathbf{U}}^j)\|+2\alpha\hat{C}\eta_l\\ &\le \rho^{l+1}\|\mathbf{V}^0-\mathbf{U}^0(\bar{\mathbf{U}}^j)\|+2\alpha\hat{C}\sum_{k=0}^{l}\eta_k\rho^{l-k},\end{aligned}$$

where $\eta_l := \rho^l\varepsilon_j$.Thus, recalling the definition of K from (4.2.4) and that $\mathbf{U}^0(\bar{\mathbf{U}}^j) = \bar{\mathbf{U}}^j = \mathbf{V}^0$, we conclude

$$\|\mathbf{V}^K-\mathbf{U}^K(\bar{\mathbf{U}}^j)\| \le 2\alpha\hat{C}K\rho^{K-1}\varepsilon_j. \quad (4.2.10)$$

Therefore, bearing in mind our assumption on $\bar{\mathbf{U}}^j$, we obtain

$$\begin{aligned}\|\mathbf{V}^K-\mathbf{U}\| &\le \|\mathbf{V}^K-\mathbf{U}^K(\bar{\mathbf{U}}^j)\|+\|\mathbf{U}^K(\bar{\mathbf{U}}^j)-\mathbf{U}\|\\ &\le 2\alpha\hat{C}K\rho^{K-1}\varepsilon_j+\rho^K\|\bar{\mathbf{U}}^j-\mathbf{U}\|\\ &\le (2\alpha\hat{C}K+\rho)\rho^{K-1}\varepsilon_j \le \left(\frac{8\hat{C}}{\hat{c}}+2\right)^{-1}\varepsilon_j, \quad (4.2.11)\end{aligned}$$

where we have used (4.2.4) in the last line. Using (4.1.6) and (4.2.11), the approximation $\bar{\mathbf{U}}^{j+1}$ satisfies now by the definition of **COARSE**

$$\begin{aligned}\|\bar{\mathbf{U}}^{j+1}-\mathbf{U}\| &\le \|\bar{\mathbf{U}}^{j+1}-\mathbf{V}^K\|+\|\mathbf{V}^K-\mathbf{U}\|\\ &\le \hat{C}\|\bar{\mathbf{U}}^{j+1}-\mathbf{V}^K\|_{\ell_2}+\left(\frac{8\hat{C}}{\hat{c}}+2\right)^{-1}\varepsilon_j\\ &\le \frac{4\hat{C}}{\hat{c}}\left(\frac{8\hat{C}}{\hat{c}}+2\right)^{-1}\varepsilon_j+\left(\frac{8\hat{C}}{\hat{c}}+2\right)^{-1}\varepsilon_j = \frac{\varepsilon_j}{2},\end{aligned}$$

which completes the proof. ∎

The above scheme should be viewed as a possibly simple representative of a *class* of schemes. We mention briefly two fairly obvious ways of improving the performance in practical applications. First, the constant K could be large and the required accuracy could be achieved at an earlier stage in the loop (ii.1). This can be monitored via the approximate residuals $\mathbf{F}_l - \mathbf{W}^l$ which are computed anyway. In fact, let $\kappa := (8\hat{C}/\hat{c} + 2)$. Since by (4.2.1), (4.2.2), $\|\mathbf{M}(\mathbf{V}^l - U) - (\mathbf{W}^l - \mathbf{F}_l)\|_{\ell_2} \leq 2\rho^l \varepsilon_j$, we infer from (4.1.4) and (4.1.6) that

$$\|\mathbf{V}^l - \mathbf{U}\| \leq \hat{C} c_M^{-1} (\|\mathbf{W}^l - \mathbf{F}_l\|_{\ell_2} + 2\rho^l \varepsilon_j).$$

Thus as soon as l is large enough to ensure that $2\rho^l < c_M/\kappa\hat{C}$ we have

$$\|\mathbf{W}^l - \mathbf{F}_l\|_{\ell_2(\mathcal{J})} \leq \varepsilon_j \left(\frac{c_M}{\hat{C}\kappa} - 2\rho^l \right) \Rightarrow \|\mathbf{V}^l - \mathbf{U}\| \leq \frac{\varepsilon_j}{\kappa},$$

which also ensures the estimate (4.2.11). Thus when $2\kappa\rho^l < c_M/\hat{C}$ holds for some $l < K$ and the approximate residual $\|\mathbf{W}^l - \mathbf{F}_l\|_{\ell_2}$ is sufficiently small one can branch to the coarsening step (ii.2) before K is reached.

Second, when M is symmetric positive definite one would of course not use a global damping parameter α in (4.1.8) based on estimates for the constants in (4.1.6). Setting $\mathbf{R}^l := \mathbf{F} - \mathbf{M}\mathbf{V}^l$ the optimal damping parameter in a gradient iteration would be $\alpha_l := ((\mathbf{R}^l)^*\mathbf{R}^l)/((\mathbf{R}^l)^*\mathbf{M}\mathbf{R}^l)$. Again the approximate residuals produced by the scheme provide approximations to α_l. In fact, denoting by $\bar{\mathbf{R}}^l := \mathbf{F}_l - \mathbf{W}^l$ the computed approximate residuals, one has, again by (4.1.4) and (4.1.6),

$$\begin{aligned}(\bar{\mathbf{R}}^l)^*\bar{\mathbf{R}}^l &\leq (\mathbf{R}^l)^*\mathbf{R}^l + 4\rho^{2l}\varepsilon_j^2 + 2\rho^l\varepsilon_j\|\mathbf{R}^l\|_{\ell_2(\mathcal{J})} \\ &\leq (\mathbf{R}^l)^*\mathbf{R}^l + 4\rho^{2l}\varepsilon_j^2 + 2\rho^l\frac{C_M}{\hat{c}}\varepsilon_j^2.\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}(\bar{\mathbf{R}}^l)^*\mathbf{M}\bar{\mathbf{R}}^l &\geq (\mathbf{R}^l)^*\mathbf{M}\mathbf{R}^l - \|\mathbf{M}\|_{\ell_2\to\ell_2}4\rho^{2l}\varepsilon_j^2 - 2\|\mathbf{M}\|_{\ell_2\to\ell_2}\rho^l\varepsilon_j\|\mathbf{R}^l\|_{\ell_2(\mathcal{J})} \\ &\geq (\mathbf{R}^l)^*\mathbf{M}\mathbf{R}^l - 4C_M\rho^{2l}\varepsilon_j^2 - 2\rho^l\frac{C_M^2}{\hat{c}}\varepsilon_j^2.\end{aligned}$$

Hence

$$\frac{(\bar{\mathbf{R}}^l)^*\bar{\mathbf{R}}^l - 4\rho^{2l}\varepsilon_j^2 - 2\rho^l C_M/\hat{c}\varepsilon_j^2}{(\bar{\mathbf{R}}^l)^*\mathbf{M}\bar{\mathbf{R}}^l + 4C_M\rho^{2l}\varepsilon_j^2 + 2\rho^l C_M^2/\hat{c}\varepsilon_j^2} \leq \alpha_l$$

yields conservative estimates for computable damping parameters at the expense of an additional application of $\mathbf{M}$. An even better error reduction would be obtained when employing conjugate directions.

Nevertheless, in order to keep matters as transparent as possible we dispense with these options here and confine the discussion to the simplified situation described above.

5. Convergence Rates and Computational Cost

We will investigate next the computational cost of determining the iterates $\bar{\mathbf{U}}^j$ produced by **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$. In particular, this will be related to the size of the supports of the iterates generated by the above scheme.

5.1. *Best N-Term Approximation*

As mentioned before our objective will be to generate approximate iterates $\bar{\mathbf{U}}^j$ satisfying (4.2.8) at the expense of *as few* nonzero entries as possible. This brings in the concept of *best N-term approximation*. For the Euclidean metric this notion is well-understood. Clearly for a given scalar sequence $\mathbf{v} \in \ell_2$ keeping its N largest (in modulus) coefficients minimizes the error among all finitely supported sequences with at most N nonzero entries.

Now we have to say what we mean by a best N-term approximation $\mathbf{V}_N$ to $\mathbf{V}$. Recall that $\mathbf{v}^* = \{v_n^*\}_{n\in\mathbb{N}}$ denotes a *nonincreasing rearrangement* of the entries in *all* individual sequences $\mathbf{v}^i$. Thus $v_n^* \geq v_{n+1}^*$ and $v_n^* = v_{\lambda_n}^{i_n}$ for some $i_n \in \{1, \ldots, m\}$, $\lambda_n \in \mathcal{J}_{i_n}$. $\mathbf{V}_N$ is then obtained by redistributing the v_n^* to their original components. Hence $\mathbf{V}_N$ has the form

$$\mathbf{V}_N = (\mathbf{v}_{N_1}^1, \ldots, \mathbf{v}_{N_m}^m), \qquad N_1 + \cdots + N_m = N, \tag{5.1}$$

where the $\mathbf{v}_{N_i}^i$ are best N_i-term approximations from the component sequences $\mathbf{v}_i$ of $\mathbf{V}$ and hence consist of the first N_i entries of the nonincreasing rearrangement $\mathbf{v}_i^*$ of $\mathbf{v}_i$.

The known characterization of best N-term approximation for *scalar* ℓ_2-sequences (see, e.g., [**30**], [**17**]) easily carries over into the present setting. To this end, let, for $0 < \tau < 2$,

$$|\mathbf{V}|_{\ell_\tau^w(\mathcal{J})} := \sup_{n\in\mathbb{N}} n^{1/\tau} |v_n^*|, \qquad \|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})} := \|\mathbf{V}\|_{\ell_2(\mathcal{J})} + |\mathbf{V}|_{\ell_\tau^w(\mathcal{J})}. \tag{5.2}$$

It is easy to see that

$$\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})} \leq 2\|\mathbf{V}\|_{\ell_\tau}, \tag{5.3}$$

so that by Jensen's inequality, in particular, $\ell_\tau^w(\mathcal{J}) \subset \ell_2(\mathcal{J})$.

We are interested in those cases where the error of best N-term approximation decays like N^{-s} for some positive s. In fact, these are the rates that can be achieved by approximation systems based on finite elements, splines, and wavelet bases. The following characterization can be inferred from the scalar case [**17**], [**30**]:

Proposition 5.1. *Let*

$$\frac{1}{\tau} = s + \tfrac{1}{2}, \tag{5.4}$$

then $\mathbf{V}$ *belongs to* $\ell_\tau^w(\mathcal{J})$ *if and only if* $\|\mathbf{V}-\mathbf{V}_N\|_{\ell_2(\mathcal{J})} = \mathcal{O}(N^{-s})$ *and one has the estimate*

$$\|\mathbf{V}-\mathbf{V}_N\|_{\ell_2(\mathcal{J})} \lesssim N^{-s}\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}. \tag{5.5}$$

Obviously there is a close interrelation between thresholding or coarsening and best N-term approximation in ℓ_2. In particular, it will be crucial to exploit that one can coarsen a given approximation in such a way that, on one hand, the resulting ℓ_2-error is not increased too much while, on the other hand, the reduced number of degrees of freedom is in accordance with the rate of best N-term approximation as described next, see [**17**].

Proposition 5.2. *For any threshold* $\eta > 0$ *one has by definition that the output* $\bar{\mathbf{W}}$ *of* **COARSE** $[\eta, \mathbf{V}]$ *satisfies* $\|\bar{\mathbf{W}}\|_{\ell_\tau^w(\mathcal{J})} \le \|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}$. *Moreover, assume that a sequence* $\mathbf{V}$, *a tolerance* $0 < \eta \le \|\mathbf{V}\|_{\ell_2(\nabla)}$, *and a finitely supported approximation* $\mathbf{W}$ *to* $\mathbf{V}$ *are given that satisfies*

$$\|\mathbf{V}-\mathbf{W}\|_{\ell_2(\mathcal{J})} \le \eta/5.$$

Assume in addition that $\mathbf{V} \in \ell_\tau^w(\mathcal{J})$ *with* $\tau = (s+\frac{1}{2})^{-1}$, *for some* $s > 0$. *Then the algorithm* **COARSE** $[\mathbf{W}, 4\eta/5]$ *produces a new approximation* $\bar{\mathbf{W}}$ *to* $\mathbf{V}$, *supported on* $\mathbf{\Lambda}$, *which satisfies* $\|\mathbf{V}-\bar{\mathbf{W}}\|_{\ell_2(\mathcal{J})} \le \eta$ *and has the following properties. There exists an absolute constant* C *which depends only on* s *when* s *tends to infinity such that*:

(i) *The cardinality of the support* $\mathbf{\Lambda}$ *of* $\bar{\mathbf{W}}$ *is bounded by*

$$\#(\mathbf{\Lambda}) \le C\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}^{1/s}\eta^{-1/s}. \tag{5.6}$$

(ii) *One has*

$$\|\mathbf{V}-\bar{\mathbf{W}}\|_{\ell_2(\mathcal{J})} \le C\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}\#(\mathbf{\Lambda})^{-s}. \tag{5.7}$$

(iii) *And*

$$\|\bar{\mathbf{W}}\|_{\ell_\tau^w(\mathcal{J})} \le C\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}. \tag{5.8}$$

5.2. *The Key Requirements*

Our next goal is to extract conditions under which **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ exhibits asymptotically optimal complexity. More precisely, we will ask the following question. Suppose that the solution $\mathbf{U}$ of (4.1.3) admits a best N-term approximation whose error decays like N^{-s} for some positive s. When does the output $\bar{\mathbf{U}}(\varepsilon)$ of **SOLVE** involve only the order of $\varepsilon^{-1/s}$ nonzero coefficients and the corresponding number of floating point operations stays of the same order? Due to the inherent role of sorting in the above concepts one has to expect in addition corresponding log terms for sort operations.

We will formulate in this section some *requirements* on the above routines **RHS** and **APPLY**. These requirements are, on one hand, motivated by the above properties of the coarsening routine. In fact, we will require that both routines **RHS** and **APPLY** exhibit the same *work/accuracy balance* as **COARSE** given in Proposition 5.2. Roughly speaking, for elements in $\ell_\tau^w(\mathcal{J})$ with $\tau^{-1} = s + \frac{1}{2}$ for $s < s^*$, accuracy η is achieved at the expense of an amount of work $\lesssim \eta^{-1/s}$. The validity of these requirements has indeed been established for a special case in [**17**]. We will then show first that, when these requirements on **RHS** and **APPLY** are satisfied, the asymptotically optimal complexity of **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ follows for a certain range of solutions $\mathbf{U}$. In later applications we will have to identify suitable routines **RHS** and **APPLY** and show that these properties do indeed hold in each respective case. We begin with the routine **APPLY**.

Requirements 5.3. *There exists some $s^* > 0$ such that, given any tolerance $\eta > 0$ and any vector $\mathbf{V}$ with finite support, the output $(\mathbf{W}_\eta, \mathbf{\Lambda}_\eta)$ of* **APPLY** $[\eta, \mathbf{M}, \mathbf{V}]$ *has for each $0 < s < s^*$ and $\tau = (s + \frac{1}{2})^{-1/2}$ $s^* > 0$, the following properties. There exists a positive constant C depending only on s when s tends to infinity such that*:

(i) *The size of the output $\mathbf{\Lambda}_\eta$ is bounded by*

$$\#(\mathbf{\Lambda}_\eta) \le C\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}^{1/s}\eta^{-1/s}. \tag{5.2.1}$$

(ii) *The number of* **arithmetic operations** *needed to compute $\mathbf{W}_\eta$ does not exceed $C\{\eta^{-1/s}\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}^{1/s} + N\}$ with $N := \#\operatorname{supp}\mathbf{V}$.*

(iii) *The number of* **sorts** *needed to compute $\mathbf{W}_\eta$ does not exceed $CN\log N$.*

(iv) *The output vector $\mathbf{W}_\eta$ satisfies*

$$\|\mathbf{W}\|_{\ell_\tau^w(\mathcal{J})} \le C\|\mathbf{V}\|_{\ell_\tau^w(\mathcal{J})}. \tag{5.2.2}$$

Definition 5.4. Any matrix $\mathbf{M}$ for which a routine **APPLY** exists that satisfies (4.2.2) and has the properties listed in Requirements 5.3 for some positive s^* is called *s^*-admissible*. The class of s^*-admissible matrices is denoted by $\mathcal{A}_{s^*}$.

Remark 5.5. In view of (4.2.2) and Proposition 5.1, every s^*-admissible matrix defines for every $s < s^*$ a bounded mapping on $\ell_\tau^w(\mathcal{J})$ when s and τ are related by (5.4).

An analogous requirement will be imposed on the routine **RHS**.

Requirements 5.6. *There exists some $s^* > 0$ such that, whenever for some positive $s < s^*$ the solution $\mathbf{U}$ of (4.1.3) belongs to $\ell_\tau^w(\mathcal{J})$ with $\tau = (s+1/2)^{-1}$ for any $\eta > 0$, the output $(\mathbf{F}_\eta, \mathbf{\Lambda}_\eta)$ of* **RHS** $[\eta, \mathbf{F}]$ *satisfies*

$$\|\mathbf{F}_\eta\|_{\ell_\tau^w(\mathcal{J})} \le C\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})}, \tag{5.2.3}$$

and

$$\#(\Lambda_\eta) \le C\|\mathbf{F}\|^{1/s}_{\ell^w_\tau(\mathcal{J})}\eta^{-1/s}, \tag{5.2.4}$$

where C depends only on s when $s \to \infty$, Moreover, the number of arithmetic operations needed to compute $\mathbf{F}_\eta$ remains proportional to $\#(\Lambda_\eta)$. The number of sort operations stays bounded by some fixed multiple of $\#(\Lambda_\eta)\log(\#(\Lambda_\eta))$.

In view of Proposition 5.1, the above requirements mean that the best N-term approximation to the data $\mathbf{F}$ can be realized at asymptotically optimal cost. We will comment more on this fact later in connection with more specific realizations.

5.3. *The Complexity of* **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$

We are now in a position to show that the validity of the Requirements 5.6 and 5.3 for some s^* implies asymptotically optimal complexity of the scheme **SOLVE** for that range of convergence rates in the following sense:

Theorem 5.7. *Assume that* (4.1.6) *and* (4.1.7) *hold. Under the assumptions* (4.2.1), (4.2.2) *for any target accuracy ε* **SOLVE** *determines an approximate solution $\bar{\mathbf{U}}(\varepsilon)$ to* (4.1.3) *after a finite number of steps such that*

$$\|\mathbf{U} - \bar{\mathbf{U}}(\varepsilon)\|_{\ell_2(\mathcal{J})} \le \varepsilon. \tag{5.3.1}$$

Moreover, if the routines **APPLY** *and* **RHS** *used in* **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ *have the properties stated in Requirements* 5.3 *and* 5.6 *for some $s^* > 0$ and if in addition the solution $\mathbf{U}$ of* (4.1.3) *has an error of best N-term approximation $\lesssim N^{-s}$, for some $s < s^*$, then the following statements are true*:

(i) *The support size of $\bar{\mathbf{U}}$ is bounded by*

$$\#(\operatorname{supp}\bar{\mathbf{U}}(\varepsilon)) = \#\Lambda(\varepsilon) \le C\|\mathbf{U}\|^{1/s}_{\ell^w_\tau(\mathcal{J})}\varepsilon^{-1/s}, \tag{5.3.2}$$

where C depends only on s when s approaches s^* and on the constants in (4.1.10) *and Properties* 5.6 *and* 5.3.

(ii) *One has*

$$\|\bar{\mathbf{U}}(\varepsilon)\|_{\ell^w_\tau(\mathcal{J})} \le C\|\mathbf{U}\|_{\ell^w_\tau(\mathcal{J})}, \tag{5.3.3}$$

where C as above is independent of ε.

(iii) *The number of arithmetic operations needed to compute $\bar{\mathbf{U}}$ is bounded by $C\varepsilon^{-1/s}$ arithmetic operations and $C\varepsilon^{-1/s}|\log\varepsilon|$ sort operations with C as above.*

Proof. The first part (5.3.1) of the assertion follows immediately from Proposition 4.2. Now assume that the error of best N-term approximation of $\mathbf{U}$ is $\le CN^{-s}$

for some $0 < s < s^*$, which by Proposition 5.1 means that $\mathbf{U} \in \ell_\tau^w(\mathcal{J})$. It also follows from Remark 5.5 that $\mathbf{F} \in \ell_\tau^w(\mathcal{J})$ and

$$\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} \lesssim \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})}. \tag{5.3.4}$$

It suffices then to show that for every j the iterates $\bar{\mathbf{U}}^j$ satisfy (i)–(iii) above with ε, $\bar{\mathbf{U}}(\varepsilon)$, $\mathbf{\Lambda}(\varepsilon)$ replaced by ε_j, $\bar{\mathbf{U}}^j$, $\mathbf{\Lambda}_j$, respectively. Obviously this is true for $j = 0$. Now suppose the validity of (i)–(iii) for some $j \geq 0$. By Requirements 5.6, the applications of **RHS** in step (ii.1) in **SOLVE** require at most

$$N_j^{\mathrm{RHS}} := C \left(\sum_{l=0}^{K} \rho^{-l/s} \right) \|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})}^{1/s} \, \varepsilon_j^{-1/s}$$

floating point operations and the order of $N_j^{\mathrm{RHS}} \log N_j^{\mathrm{RHS}}$ sort operations. Likewise, by Requirements 5.3(ii), (iii), the **APPLY** routine takes at most

$$N_j^{\mathrm{APPLY}} := C \left\{ \sum_{l=0}^{K} \rho^{-l/s} (\|\mathbf{V}^l\|_{\ell_\tau^w(\mathcal{J})}^{1/s} + \#(\operatorname{supp} \mathbf{V}^l)) \right\} \varepsilon_j^{-1/s}$$

flops again with additional $N_j^{\mathrm{APPLY}} \log N_j^{\mathrm{APPLY}}$ sort operations. Furthermore, by Requirements 5.3(iv) and 4.2.6, we know that

$$\begin{aligned} \|\mathbf{V}^{l+1}\|_{\ell_\tau^w(\mathcal{J})} &\leq \|\mathbf{V}^l\|_{\ell_\tau^w(\mathcal{J})} + \alpha(\|\mathbf{F}_l\|_{\ell_\tau^w(\mathcal{J})} + \|\mathbf{W}^l\|_{\ell_\tau^w(\mathcal{J})}) \\ &\leq C_1(\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} + \|\mathbf{V}^l\|_{\ell_\tau^w(\mathcal{J})}), \end{aligned} \tag{5.3.5}$$

where we have used that $\mathbf{V}^0 = \bar{\mathbf{U}}^j$ and also (5.2.3) in the last step. Hence, since K is finite we conclude that

$$\max_{l \leq K} \|\mathbf{V}^l\|_{\ell_\tau^w(\mathcal{J})} \leq C_2(\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} + \|\bar{\mathbf{U}}^j\|_{\ell_\tau^w(\mathcal{J})}) \leq C_3(\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} + \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})}), \tag{5.3.6}$$

where we have used our induction assumption (ii) on $\bar{\mathbf{U}}^j$. Therefore, in summary we obtain

$$N_j^{\mathrm{RHS}}, \; N_j^{\mathrm{APPLY}} \leq C_4(\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} + \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})}) \varepsilon_j^{-1/s}. \tag{5.3.7}$$

By Requirements 5.3(i), 5.6, (5.2.4), and (5.3.6), we are sure that the sizes of the intermediate supports stay controlled according to

$$\#(\operatorname{supp} \mathbf{V}^l) \leq C_5(\|\mathbf{F}\|_{\ell_\tau^w(\mathcal{J})} + \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})}) \varepsilon_j^{-1/s}, \qquad l \leq K, \tag{5.3.8}$$

where the constant C_5 depends only on K.

Recall now that after at most K approximate iterations in step (ii.1) of the scheme **SOLVE** the routine **COARSE** is applied in step (ii.2) to $\mathbf{V}^K$ to produce $\bar{\mathbf{U}}^{j+1}$. Bearing in mind the estimate (4.2.11), Proposition 5.2(ii) and (iii) yield

$$\|\bar{\mathbf{U}}^{j+1}\|_{\ell_\tau^w(\mathcal{J})} \leq C_6 \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})} \tag{5.3.9}$$

as well as

$$\#(\mathbf{\Lambda}_{j+1}) \le C_7 \|\mathbf{U}\|^{1/s} \varepsilon_{j+1}^{-1/s} \tag{5.3.10}$$

with C_6, C_7 independent of j. Moreover, the constants C_6, C_7 depend only on s when s becomes large. In particular, they are *independent* of the constants arising in the above previous estimates (5.3.5)–(5.3.8). In that sense the coarsening step *sets back* the constants. On the other hand, comparing (5.3.8) and (5.3.10) shows that the reduction caused by coarsening is always of constant order. This finishes the proof. ∎

One expects that the application of **SOLVE** may benefit from a previously computed approximation to $\mathbf{U} = \mathbf{M}^{-1}\mathbf{F}$. Thus instead of taking $\bar{\mathbf{U}}^0 = \mathbf{0}$ as an initial guess one might as well use any given, possibly better, initial guess $\bar{\mathbf{V}}$. The version

SOLVE $[\varepsilon, \mathbf{M}, \mathbf{F}, \delta, \bar{\mathbf{V}}] \to (\bar{\mathbf{U}}(\varepsilon), \mathbf{\Lambda}(\varepsilon))$
computes for any given initial guess $\bar{\mathbf{V}}$ *satisfying*

$$\|\mathbf{U} - \bar{\mathbf{V}}\| \le \delta, \tag{5.3.11}$$

with $\varepsilon < \delta$ *an approximate solution* $\bar{\mathbf{U}}(\varepsilon)$ *to* (4.1.3) *satisfying* (5.3.1). *It differs from* **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}]$ *only in the initialization step* (i) *where* $\bar{\mathbf{U}}^0 = \mathbf{0}$, $\varepsilon_0 = c_M^{-1}\hat{C}\|\mathbf{F}\|_{\ell_2(\mathcal{J})}$ *are replaced by* $\bar{\mathbf{V}}$, δ.

Theorem 5.7 remains valid for **SOLVE** $[\varepsilon, \mathbf{M}, \mathbf{F}, \delta, \bar{\mathbf{V}}]$ where the involved constants are expected to be, of course, more favorable.

5.4. *Compressible Matrices*

In the applications to be discussed below it now remains to verify the validity of Requirements 5.6 and 5.3 for each respective version of $\mathbf{F}$ and $\mathbf{M}$. In particular, a crucial role will be played by the following class of matrices.

Definition 5.8. Let $s^* > 0$. A matrix $\mathbf{C}$ is called s^*-*compressible* if for each $0 < s < s^*$ and for some summable positive sequence $\{\alpha_j\}_{j\in\mathbb{N}_0}$ there exists for each $j \in \mathbb{N}_0$ a matrix $\mathbf{C}_j$ having at most $\alpha_j 2^j$ nonzero entries per row and column such that

$$\|\mathbf{C} - \mathbf{C}_j\|_{\ell_2} \le \alpha_j 2^{-sj}, \qquad j \in \mathbb{N}_0. \tag{5.4.1}$$

The class of s^*-compressible matrices is denoted by $\mathcal{C}_{s^*}$.

Note that $\mathbf{C}$ may map $\ell_2(\mathcal{J})$ into $\ell_2(\mathcal{J}')$ where $\mathcal{J}' \neq \mathcal{J}$.

The following result has been proved in [**17**]:

Proposition 5.9. *Any* $\mathbf{C} \in \mathcal{C}_{s^*}$ *is bounded on* $\ell_\tau^w(\mathcal{J})$ *with* $\tau = (s + \frac{1}{2})^{-1}$ *when* $s < s^*$.

The above assertion actually follows from Proposition 5.1 combined with the following way of approximating $\mathbf{CV}$ for finitely supported $\mathbf{V}$. To this end, let

$$\mathbf{V}_{[j]} := \mathbf{V}_{2^j} - \mathbf{V}_{2^{j-1}}, \qquad \mathbf{V}_{[0]} := \mathbf{V}_{2^0},$$

where $\mathbf{V}_{2^j}$ denotes the best 2^j-term approximation to $\mathbf{V} \in \ell_2(\mathcal{J})$. A key concept from [**17**] is to build for any finitely supported sequence $\mathbf{V}$ approximations to $\mathbf{AV}$ of the form

$$\mathbf{W}_j := \mathbf{C}_j \mathbf{V}_{[0]} + \mathbf{C}_{j-1}\mathbf{V}_{[1]} + \cdots + \mathbf{C}_0 \mathbf{V}_{[j]}. \tag{5.4.2}$$

In fact, one has the error estimate

$$\|\mathbf{CV} - \mathbf{W}_k\|_{\ell_2(\mathcal{J})} \le c_2 \|\mathbf{V} - \mathbf{V}_{2^k}\|_{\ell_2(\mathcal{J})} + \sum_{j=0}^{k} a_j \|\mathbf{V}_{[k-j]}\|_{\ell_2(\mathcal{J})}, \tag{5.4.3}$$

where a_j is a bound for $\|\mathbf{C} - \mathbf{C}_j\|_{\ell_2(\mathcal{J})} \le \alpha_j 2^{-sj}$. This is the basis for the following routine, see [**17**]:

MULT $[\eta, \mathbf{C}, \mathbf{V}] \to (\mathbf{W}, \mathbf{\Lambda})$

(i) *Sort the nonzero entries of the vector* $\mathbf{V}$ *and form the vectors* $\mathbf{V}_{[0]}$, $\mathbf{V}_{[j]}$, $j = 1, \ldots, \lfloor \log N \rfloor$, *with* $N := \#\mathrm{supp}\, \mathbf{V}$. *Define* $\mathbf{V}_{[j]} := \mathbf{0}$ *for* $j > \log N$.

(ii) *Compute* $\|\mathbf{V}_{[j]}\|^2_{\ell_2(\mathcal{J})}$, $j = 0, \ldots, \lfloor \log N \rfloor + 1$, *and* $\|\mathbf{V}\|^2_{\ell_2(\mathcal{J})} = \sum_{j=0}^{N} \|\mathbf{V}_{[j]}\|^2_{\ell_2(\mathcal{J})}$.

(iii) *Set* $k = 0$.
  (a) *Compute the right-hand side* R_k *of* (5.4.3) *for the given value of* k.
  (b) *If* $R_k \le \eta$ *stop and output* k; *otherwise replace* k *by* $k + 1$ *and return to* (a).

(iv) *For the output* k *of* (iii) *and for* $j = 0, 1, \ldots, k$, *compute the nonzero entries in the matrices* $\mathbf{C}_{k-j}$ *which have a column index in common with one of the nonzero entries of* $\mathbf{V}_{[j]}$.

(v) *For the output* k *of* (iii), *compute* $\mathbf{W}_k$ *as in* (5.4.2) *and take* $\mathbf{W} := \mathbf{W}_k$ *and* $\mathbf{\Lambda} = \mathrm{supp}\, \mathbf{W}$.

The following facts have been established in [**17**]:

Proposition 5.10. *Given a tolerance* $\eta > 0$ *and a vector* $\mathbf{V}$ *with finite support, the algorithm* **MULT** *produces a vector* $\mathbf{W}$ *which satisfies* (4.2.2), *i.e.*,

$$\|\mathbf{CV} - \mathbf{W}\|_{\ell_2(\mathcal{J})} \le \eta. \tag{5.4.4}$$

Moreover, whenever $\mathbf{C} \in \mathcal{C}_{s^*}$, *then the scheme*

$$\mathbf{APPLY}\,[\eta, \mathbf{C}, \mathbf{V}] := \mathbf{MULT}\,[\eta, \mathbf{C}, \mathbf{V}] \tag{5.4.5}$$

satisfies Requirements 5.3 *for all* $s < s^*$.

Corollary 5.11. *One has* $\mathcal{C}_{s^*} \subseteq \mathcal{A}_{s^*}$, *i.e., any* s^**-compressible matrix* $\mathbf{A}$ *that belongs to* $\mathcal{C}_{s^*}$ *also belongs to* $\mathcal{A}_{s^*}$, *see Definition* 5.4.

6. Wavelet Discretization

The transformation of the original operator equation (3.1.9) into a well-posed ℓ_2-problem will rely on suitable wavelet bases.

6.1. *Some Wavelet Prerequisites*

We begin by briefly collecting some relevant properties of wavelet bases. To keep the notation simple we describe these prerequisites here first only for the scalar case, i.e., the following quantities like the index set $\mathcal{J}$ represent a generic component that may appear in the setting described in Section 3.1. Accordingly, lowercase letters are used here. Suppose that on a given domain or manifold we have a wavelet basis

$$\Psi = \{\psi_\lambda\colon\ \lambda \in \mathcal{J}\} \subset \mathcal{H}$$

at our disposal. We are not going to address in detail here any technical issues of constructing wavelet bases with the properties mentioned below but will be content with the following brief remarks that should provide some feeling for the nature of such tools and refer to the literature for details, see, e.g., [**15**], [**18**], [**27**], [**28**], [**29**]. Of course, the simplest case concerns wavelets on the whole real line $\mathbb{R}$. In this case, the ψ_λ have the form $\psi_\lambda = 2^{j/2}\psi(2^j \cdot -k)$, i.e., the index λ, representing (j, k), absorbs two types of information, namely the *discretization scale* j and the *spatial location* $2^{-j}k$. In this case, the index set $\mathcal{J}$ consists of such tuples (j, k). In essence this is the case in all practically relevant situations and it will be helpful to picture the index set $\mathcal{J}$ always in such a *scale-space* diagram. The scale level j associated with the index $\lambda \in \mathcal{J}$ will be denoted by $|\lambda|$. More precisely, when $\lambda = (j, k)$ we set $|\lambda| = j$.

It will also be convenient to employ the following notational conventions. Whenever Φ, Θ are (at most countable) collections of functions in L_2, the (possibly infinite) matrix with entries $\langle\varphi, \theta\rangle$, $\varphi \in \Phi, \theta \in \Theta$ will be denoted by $\langle\Phi, \Theta\rangle$. In particular, $\langle\varphi, \Theta\rangle$ and $\langle\Phi, \theta\rangle$ are then a row (respectively, a column) vector. Thus, denoting by $\tilde{\Psi}$ the *dual basis* to Ψ the corresponding biorthogonality relations $\langle\psi_\lambda, \tilde{\psi}_{\lambda'}\rangle$, $\lambda, \lambda' \in \mathcal{J}$, are then conveniently expressed as $\langle\Psi, \tilde{\Psi}\rangle = \mathbf{I}$, and $\langle v, \tilde{\Psi}\rangle\Psi =: \sum_{\lambda\in\mathcal{J}}\langle v, \tilde{\psi}_\lambda\rangle\psi_\lambda$ denotes the expansion of v with respect to Ψ.

Beyond that we require here the validity of the following three properties, see, e.g., [**16**], [**23**], [**24**]:

Locality. The wavelets in Ψ will always be assumed to be *local* in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda) \sim 2^{-|\lambda|}. \tag{6.1.1}$$

Norm Equivalences. Wavelet expansions induce isomorphisms between relevant function and sequence spaces. By this we mean that there exist finite positive constants c_1, C_2, and a diagonal matrix $\mathbf{D} = \operatorname{diag}(d_\lambda\colon \lambda \in \mathcal{J})$ such that, for any $\mathbf{u} = \{u_\lambda\colon \lambda \in \mathcal{J}\}$,

$$c_1\|\mathbf{u}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{u}^T\mathbf{D}^{-1}\Psi\|_{\mathcal{H}} \le C_1\|\mathbf{u}\|_{\ell_2(\mathcal{J})}. \tag{6.1.2}$$

Remark 6.1.1. By duality (6.1.2) implies that

$$C_1^{-1}\|\mathbf{D}^{-1}\mathbf{u}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{u}^T\tilde{\Psi}\|_{\mathcal{H}'} \le c_1^{-1}\|\mathbf{D}^{-1}\mathbf{u}\|_{\ell_2(\mathcal{J})}. \tag{6.1.3}$$

Cancellation Property. If the spaces $S(\tilde{\Psi}^j)$ spanned by all dual wavelets $\tilde{\psi}_\lambda$ with $|\lambda| < j$ realize approximation order $2^{-j\tilde{m}}$, one has

$$\langle v, \psi_\lambda\rangle \lesssim 2^{-|\lambda|(d/2+\tilde{m})}|v|_{W^{\tilde{m}}_\infty(\operatorname{supp}\psi_\lambda)}. \tag{6.1.4}$$

Thus integrating against a wavelet has the effect of taking an $\tilde{m}$th-order difference where $\tilde{m}$ is the order of the dual wavelets. In particular, when polynomials are defined on the domain of the wavelets, (6.1.4) implies $\tilde{m}$th-order vanishing moments. This will imply quasi-sparse representations of a wide class of operators.

6.2. *The ℓ_2-Formulation of Well-Posed Systems*

We shall describe now how to transform (4.1.1) into a discrete equivalent system (4.1.3) with the aid of wavelet transforms. To this end, we associate with each of the component spaces $H_{i,0}$ appearing in $\mathcal{H}$ a domain (or boundary manifold) Ω_i of spatial dimension d_i. In the above examples one encounters bounded domains in $\mathbb{R}^d$ as well as corresponding boundary manifolds Γ of dimension $d-1$. We will assume that for each $i = 1, \dots, m$, we have a wavelet basis Ψ^i satisfying (6.1.2) (for simplicity with the same constants c_1, C_1) with suitable diagonal matrices $\mathbf{D}_i$. Due to the possibly different nature of the domains Ω_i the bases Ψ^i may refer to different index sets $\mathcal{J}_i$, $i = 1, \dots, m$. As before, we will always abbreviate

$$\mathcal{J} := \mathcal{J}_1 \times \cdots \times \mathcal{J}_m.$$

Moreover, it will be convenient to catenate the individual bases Ψ^i, the corresponding coefficient sequences $\mathbf{v}_i$, and the scaling matrices $\mathbf{D}_i$ to

$$\Psi := \begin{pmatrix} \Psi^1 \\ \vdots \\ \Psi^m \end{pmatrix}, \qquad \mathbf{V} := \begin{pmatrix} \mathbf{v}^1 \\ \vdots \\ \mathbf{v}^m \end{pmatrix}, \qquad \mathbf{D} := \operatorname{diag}(\mathbf{D}_1, \dots, \mathbf{D}_m). \tag{6.2.1}$$

Thus, in particular, one has $\|\mathbf{V}\|^2_{\ell_2(\mathcal{J})} = \sum_{i=1}^m \|\mathbf{v}^i\|^2_{\ell_2(\mathcal{J}_i)}$. The following fact follows from (6.1.2) and (3.1.3):

Remark 6.2.1. Let $U = \mathbf{U}^*\mathbf{D}^{-1}\Psi \in \mathcal{H}$ and let $\mathbf{U}_N$ denote the best N-term approximation of $\mathbf{U}$ defined by (5.1) in Section 5.1. Then one has, by (6.1.2),

$$\|\mathbf{U} - \mathbf{U}_N\|_{\ell_2(\mathcal{J})} \sim \inf_{\substack{\mathbf{V} \in \ell_2(\mathcal{J}) \\ \sum_{i=1}^m \#\operatorname{supp}\mathbf{V}^i \le N}} \|U - \mathbf{V}^*\mathbf{D}^{-1}\Psi\|_{\mathcal{H}}.$$

Furthermore, setting

$$\mathbf{A}^{i,l} := A_{i,l}(\mathbf{D}_i^{-1}\Psi^i, \mathbf{D}_l^{-1}\Psi^l) = \mathbf{D}_i^{-1}\langle \Psi^i, \mathcal{L}_{i,l}\Psi^l\rangle \mathbf{D}_l^{-1}, \tag{6.2.2}$$

the wavelet representation of $\mathcal{L}$ is now given by the $(m \times m)$-block matrix $\mathbf{L} := (\mathbf{A}_{i,l})_{i,l=1}^m$ where each block $\mathbf{A}^{i,l}$ is an infinite matrix representing the operator $\mathcal{L}_{i,l}$ in wavelet coordinates. Moreover, defining

$$\mathbf{G} := \mathbf{D}^{-1} \begin{pmatrix} \langle \Psi^1, f_1\rangle \\ \vdots \\ \langle \Psi^m, f_m\rangle \end{pmatrix}, \tag{6.2.3}$$

the original problem (3.1.6) or (3.1.9) is equivalent to the infinite system

$$\mathbf{L}\mathbf{U} = \mathbf{G}. \tag{6.2.4}$$

Recall that well-posedness (3.1.10) together with the norm equivalences (6.1.2) imply that

$$c_L\|\mathbf{V}\|_{\ell_2(\mathcal{J})} \le \|\mathbf{L}V\|_{\ell_2(\mathcal{J})} \le C_L\|\mathbf{V}\|_{\ell_2(\mathcal{J})}, \tag{6.2.5}$$

where

$$c_L := c_1^2 c_{\mathcal{L}}, \qquad C_L := C_1^2 C_{\mathcal{L}}, \tag{6.2.6}$$

and $c_1, C_1, c_{\mathcal{L}}, C_{\mathcal{L}}$ are the constants from (6.1.2) and (3.1.10), respectively, see, e.g., [**26**], [**24**].

In principle, (6.2.5) already realizes (3.1.10) of our general program from Section 4.1. However, in general, the contraction property (4.1.7) will not hold for $\mathbf{M} = \mathbf{L}$. Before addressing this issue we note that, aside from the above stable ℓ_2-reformulation of the original variational problem, the wavelet representation

will also be seen to facilitate the efficient application of the involved operators. To this end, note that in all the examples from Section 3.2 (as well as in many other important cases) the operators $\mathcal{L}_{i,l}$ represented by the blocks $\mathbf{A}^{i,l}$ in $\mathbf{L}$ are of one of the following two types: Either they are *local* in the sense that

$$\langle \psi_\nu^i, \mathcal{L}_{i,l}\psi_\lambda^l\rangle = A_{i,l}(\psi_\nu^i, \psi_\lambda^l) = 0 \qquad \text{if} \qquad \operatorname{supp}\psi_\lambda^i \cap \operatorname{supp}\psi_\nu^l = \emptyset, \quad (6.2.7)$$

like trace or differential operators, in which case they will be referred to as *type* (*I*). Or they belong to the second class of interest, called *type* (*II*), which consist of operators with a global *Schwartz kernel*,

$$(\mathcal{L}_{i,l}v)(x) = \int_\Gamma K(x, y)v(y)\,dy,$$

with the following Calderón–Zygmund property

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim \operatorname{dist}(x, y)^{-(d+2t+|\alpha|+|\beta|)}. \quad (6.2.8)$$

Here $2t$ is the order of the operator $\mathcal{L}_{i,l}$ and $d = d_l$ is the spatial dimension of the domain Ω_l. One can show (see, e.g., [**23**] and the literature cited therein when $\Psi^i = \Psi^l$ and [**26**] for the general situation) that in either case one has the following estimates, for any such block matrix $\mathbf{A} = \mathbf{A}^{i,l}$,

$$|\mathbf{A}_{\lambda,\lambda'}| \lesssim \frac{2^{-||\lambda|-|\lambda'||\sigma}}{(1 + 2^{\min(|\lambda|,|\lambda'|)}\operatorname{dist}(\Omega_\lambda, \Omega_{\lambda'}))^\beta}, \quad (6.2.9)$$

where $\Omega_\lambda, \Omega_{\lambda'}$ denote the support of $\psi_\lambda^i, \psi_{\lambda'}^l$, respectively, $\sigma > d$ depends on the regularity of the wavelets in Ψ^i, Ψ^l, and β depends on the order of $\mathcal{L}_{i,l}$, the spatial dimensions of the respective domains Ω_i, Ω_l and the orders $\tilde{m}_i, \tilde{m}_l$ of *cancellation properties* of the wavelets in Ψ^i (respectively Ψ^l), see [**26**] for details. The following consequence of estimates of the type (6.2.9) has been established in [**17**].

Proposition 6.2.2. *For σ and β in* (6.2.9) *let*

$$s^* := \min\left\{\frac{\sigma}{d} - \tfrac{1}{2}, \tfrac{\beta}{d} - 1\right\}. \quad (6.2.10)$$

Then any matrix $\mathbf{A}$ *satisfying* (6.2.9) *belongs to $\mathcal{C}_{s^*}$ and, in view of* (6.5) *and Corollary* 5.11, *also to $\mathcal{A}_{s^*}$.*

From the currently known constructions of wavelet bases, it is known that (for a fixed multiresolution corresponding to the primal wavelets) the order $\tilde{m}$ of cancellation properties and hence β in (6.2.10) can always, in principle, be chosen as large as one wishes. Thus the regularity of the primal wavelets expressed by σ may be the limiting factor for s-admissibility. A construction where σ can, in principle, also be arranged to be as large as one wishes is given in [**24**]. Other

constructions like [**15**], [**18**], [**23**] provide wavelets with arbitrarily high regularity only inside macro patches. It should be mentioned though that the bound s^* in (6.2.10) is not best possible. For constant coefficient partial differential operators and spline wavelets of order m, $s^* = m - \frac{3}{2}$ can be verified [**3**].

The above findings may now be summarized as follows:

Remark 6.2.3.

(i) If there exists an $s^* > 0$ such that each block $\mathbf{A}_{i,l}$ of $\mathbf{L}$ from (6.2.2) belongs to $\mathcal{C}_{s^*}$ then also $\mathbf{L} \in \mathcal{C}_{s^*}$ as well as $\mathbf{L}^* \in \mathcal{C}_{s^*}$.

(ii) In each of the examples from Section 3.2 the operators $\mathcal{L}_{i,l}$ are either of type (I) or (II), see Section 6.3. The compressibility of the corresponding blocks $\mathbf{A}_{i,l}$ has already been established in [**26**]. In particular, it follows from Proposition 6.2.2 that a proper choice of wavelet bases allows one to realize any degree s^* of compressibility in these cases.

In quite the same spirit as in [**17**] we will now formulate two assumptions all our subsequent conclusions will be based upon. Their validity has to be verified in each concrete application.

Assumption 1. *We have full information about the entries in the arrays* $\mathbf{D}_i^{-1}\langle\Psi^i, f_i\rangle$ *which we view as* given *data provided by the user. We assume that for any given tolerance* $\eta > 0$, *we are provided with the set* $\Lambda := \Lambda(f, \eta)$ *of minimal size such that for each* $i = 1, \ldots, m$ *the array* $\bar{\mathbf{f}}_i := \mathbf{f}_i|_\Lambda$ *satisfies*

$$\|\mathbf{f}_i - \bar{\mathbf{f}}_i\|_{\ell_2(\mathcal{J}_i)} \le \eta. \tag{6.2.11}$$

For the purpose of our asymptotic analysis, we could actually replace "minimal" by "nearly minimal," in the sense that the following property holds: if $\mathbf{f}_i$ is in $\ell_\tau^w(\mathcal{J}_i)$ for some $\tau < 2$, then we have the estimate

$$\#(\Lambda) \le C\eta^{-1/s}\|\mathbf{f}_i\|^{1/s}_{\ell_\tau^w(\mathcal{J}_i)}, \tag{6.2.12}$$

with $s = \frac{1}{2} - 1/\tau$ and C a constant that depends only on s as s tends to $+\infty$. This modified assumption is much more realistic, since in practice one can only have approximate knowledge of the index set corresponding to the largest coefficients in $\mathbf{f}_i$, using some a-priori information on the smooth and singular parts of the function f_i. However, in order to simplify the notation and analysis, in what follows we shall assume that the set Λ is minimal.

Assumption 2. *We assume that the entries of the wavelet representation* $\mathbf{L}$ *can be computed* (*up to roundoff*) *at unit cost*.

As pointed out already in [**17**] this is realistic, e.g., for constant coefficient differential operators and piecewise polynomial wavelets. In general this is a much

more delicate problem and we refer to [**6**], [**4**] for possible approaches that justify this assumption for more general situations.

6.3. *The Elliptic Case Revisited*

It will be instructive to specialize the above development first to the first example from Section 3.2, namely a scalar elliptic (self-adjoint) problem which will also give the simplest instance of the transition from the original problem (3.1.9) to the ℓ_2-formulation (4.1.3). Recall that in this case $a(\cdot,\cdot)$ is a symmetric bilinear form on the Hilbert space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}} := \langle\cdot,\cdot\rangle_{\mathcal{H}}^{1/2}$. More precisely, $a(\cdot,\cdot)$ is $\mathcal{H}$-elliptic, i.e., $a(v,v) \sim \|v\|_{\mathcal{H}}^2$, $v \in \mathcal{H}$. Thus the operator $\mathcal{L}$ defined by $\langle v, \mathcal{L}u\rangle = a(v,u)$ ($\langle\cdot,\cdot\rangle$ denoting the dual pairing for $\mathcal{H}\times\mathcal{H}'$) satisfies (3.1.10). The wavelet representation $\mathbf{L}$ of $\mathcal{L}$ consists now of a single (infinite) block

$$\mathbf{A} := a(\mathbf{D}^{-1}\Psi, \mathbf{D}^{-1}\Psi). \tag{6.1}$$

Setting

$$\mathbf{f} := \mathbf{D}^{-1}\langle\Psi, f\rangle, \tag{6.2}$$

the problem of finding $u \in \mathcal{H}$ such that

$$a(v,u) = \langle v, f\rangle, \qquad v \in \mathcal{H}, \tag{6.3}$$

is then equivalent to the following specialization of (6.2.4):

$$\mathbf{Au} = \mathbf{f}, \tag{6.4}$$

where the sequence $\mathbf{u} \in \ell_2(\mathcal{J})$ is related to the solution u of $\mathcal{L}u = f$ by $u = \mathbf{u}^*\mathbf{D}^{-1}\Psi$.

While (6.2.5) can be rephrased as

$$\|\mathbf{A}\|_{\ell_2\to\ell_2}\|\mathbf{A}^{-1}\|_{\ell_2\to\ell_2} < \infty, \tag{6.5}$$

the $\mathcal{H}$-ellipticity and symmetry of the form $a(\cdot,\cdot)$ implies in this case in addition that $\mathbf{A}$ is *symmetric and positive definite*. In view of (6.5), this realizes (4.1.3), (4.1.10) with $\mathbf{M} = \mathbf{A}$. Thus (4.1.7) and hence (4.1.9) hold with $\|\cdot\| = \|\cdot\|_{\ell_2(\mathcal{J})}$. Moreover, Remark 6.2.1 ensures that an asymptotically optimal scheme for the solution of (6.4) provides asymptotically optimal approximations to the solution u of (6.3).

So far Proposition 6.2.2 says that for the above classes (I) and (II) of scalar elliptic problems the scheme $\mathbf{APPLY}_{ell}[\eta, \mathbf{M}, \mathbf{v}] := \mathbf{MULT}[\eta, \mathbf{A}, \mathbf{v}]$ defined by (5.4.5) in Proposition 5.10 for $\mathbf{M} = \mathbf{A}$ given by (6.1) satisfies Requirements 5.3 for some s^* which, in principle, can be made as large as one wishes by raising the regularity of the wavelets.

So it remains to discuss the evaluation of the right-hand side data. Under **Assumption 1** this reduces to the application of the routine **COARSE**.

Remark 6.1. Defining for any $\eta \geq \bar{\eta}$ the routine $\mathbf{RHS}_{ell}[\eta, \mathbf{f}] \to (\mathbf{f}_\eta, \Lambda_{\eta,f})$ by

$$\mathbf{RHS}_{ell}[\eta, \mathbf{f}] := \mathbf{COARSE}\,[\eta - \bar{\eta}, \bar{\mathbf{f}}], \tag{6.6}$$

it has been shown in [**17**] that under **Assumption 1** the Requirements 5.6 are satisfied.

Combining the above observations with Theorem 5.7 yields the following result:

Theorem 6.2. *Let* $\mathbf{A}, \mathbf{f}$ *be given by* (6.1) (*respectively* (6.2)) *and assume that* $\mathbf{A} \in \mathcal{C}_{s^*}$ *for some* $s^* > 0$. *Let the scheme* $\mathbf{SOLVE}_{ell}\,[\varepsilon, \mathbf{A}, \mathbf{f}]$ *be defined as in Section* 4.2 *with* $\mathbf{RHS}_{ell}$ *and* $\mathbf{APPLY}_{ell}$ *defined by* (6.6) (*respectively* (5.4.5)). *Then the scheme* $\mathbf{SOLVE}_{ell}[\varepsilon, \mathbf{A}, \mathbf{f}]$ *applied to the problem* (6.4) *exhibits asymptotically optimal complexity in the following sense. If the solution* $u = \mathbf{u}^*\mathbf{D}^{-1}\Psi$ *of* (6.3) *satisfies*

$$\inf_{\substack{\mathbf{v} \in \ell_2(\mathcal{J}) \\ \#\mathrm{supp}\,\mathbf{v} \leq N}} \|u - \mathbf{v}^*\mathbf{D}^{-1}\Psi\|_{\mathcal{H}} \lesssim N^{-s},$$

for some $s < s^*$, *then for any* $\varepsilon > 0$ *the approximation* $\mathbf{u}(\varepsilon)$ *produced by* **SOLVE** *satisfies*

$$\|u - \mathbf{u}(\varepsilon)^*\mathbf{D}^{-1}\Psi\|_{\mathcal{H}} \leq \varepsilon,$$

where $\#\mathrm{supp}\,\mathbf{u}(\varepsilon) \leq C\varepsilon^{-1/s}$ *and* C *depends only on* $s^* - s$ *and the constants in* (6.1.2) *and the ellipticity constants. Moreover, the number of flops needed to compute* $\mathbf{u}(\varepsilon)$ *is bounded by* $C\varepsilon^{-1/s}$ *with additional* $C\varepsilon^{-1/s}\log\varepsilon$ *operations spent on sorting with the same dependence of* C *as above.*

By Proposition 6.2.2, Theorem 6.2 applies, in particular, to elliptic boundary value problems and singular integral equations, see the beginning of Section 3.2. The result is analogous to the one from [**17**]. However, the algorithm $\mathbf{SOLVE}_{ell}$ is different from the one given in [**17**]. In the present case the a-posteriori refinement strategy is completely missing. The adaptivity relies completely on the *application* of the involved (infinite-dimensional) operators and the solution of intermediate Galerkin problems is avoided.

Our next objective is to establish an analogous result for the full scope of problems described in Section 3.1.

7. Least Squares Formulations

Since in general the wavelet representation $\mathbf{L}$ of $\mathcal{L}$ will neither be symmetric nor positive definite we cannot always simply take $\mathbf{M} := \mathbf{L}$ in the ℓ_2-formulation (4.1.3). In this section we present a conceptually simple way of turning (3.1.9) into an equivalent symmetric positive definite system which is well-posed on $\ell_2(\mathcal{J})$ that works for the full scope of well-posed problems covered by Section 3.1. The idea is to solve (6.2.4) in a *least squares sense*, see [**26**].

Theorem 7.1. *Let*

$$\mathbf{Q} := \mathbf{L}^*\mathbf{L}, \qquad \mathbf{F} := \mathbf{L}^*\mathbf{G}, \tag{7.1}$$

where $\mathbf{L}, \mathbf{G}$ *are given by* (6.2.2) *and* (6.2.3), *respectively. Then* $U \in \mathcal{H}$ *solves* (3.1.6) (*or equivalently* (3.1.9)) *if and only if* $\mathbf{U} := \mathbf{U}^*\mathbf{D}^{-1}\Psi$ *solves*

$$\mathbf{QU} = \mathbf{F}. \tag{7.2}$$

Moreover, $\mathbf{Q}$ *is symmetric positive definite and one has*

$$\mathrm{cond}_2(\mathbf{Q}) := \|\mathbf{Q}\|_{\ell_2(\mathcal{J})}\|\mathbf{Q}^{-1}\|_{\ell_2(\mathcal{J})} \le C_L^2/c_L^2. \tag{7.3}$$

Proof. The assertion is an immediate consequence of (6.2.5) and the equivalence of (6.2.4) and (3.1.9) derived above. ∎

Theorem 7.1 says that the concept from Section 3.1 is realized with $\mathbf{M} := \mathbf{Q}$ constructed above and $\|\cdot\| = \|\cdot\|_{\ell_2(\mathcal{J})}$, while Remark 6.2.1 establishes the desired interrelation between approximations to $\mathbf{U} \in \ell_2(\mathcal{J})$ to corresponding approximations to $U \in \mathcal{H}$.

The application of **SOLVE** to (7.2) now requires identifying suitable routines **RHS** and **APPLY**, see Section 4.2. They will simply be given as compositions of building blocks introduced before.

To begin with **RHS** we recall from **Assumption 1** that all coefficients of $\mathbf{G}$ are accessible and have been stored in decreasing order of magnitude so that, in particular, for some sufficiently small $\bar\eta$, one has $\|\mathbf{G} - \bar{\mathbf{G}}\|_{\ell_2(\mathcal{J})} \le \bar\eta$. As before in Remark 6.1 we will always briefly write

$$\mathbf{COARSE}\,[\eta, \mathbf{G}] := \mathbf{COARSE}\,[\eta - \bar\eta, \bar{\mathbf{G}}], \qquad \text{whenever} \quad \bar\eta < \eta,$$

so that (4.2.3) remains valid. Now for $\eta/2C_L > \bar\eta$ the scheme

$\mathbf{RHS}_{ls}\,[\eta, \mathbf{F}] \to (\mathbf{F}_\eta, \mathbf{\Lambda}_{\eta,F})$
is given by

$$\mathbf{MULT}\left[\frac{\eta}{2}, \mathbf{L}^*, \mathbf{COARSE}\left[\frac{\eta}{2C_L}, \mathbf{G}\right]\right] \to (\mathbf{F}_\eta, \mathbf{\Lambda}_{\eta,F}). \tag{7.4}$$

Proposition 7.2. *The output* $\mathbf{F}_\eta$ *of* $\mathbf{RHS}_{ls}$ *defined above satisfies* (4.2.1). *Moreover, whenever* $\mathbf{L}$ *belongs to* $\mathcal{C}_{s^*}$ *for some* $s^* > 0$, *then Requirements* 5.6 *are satisfied for* $s < s^*$.

Proof. Let $\mathbf{G}_0$ denote the finitely supported sequence produced by **COARSE** $[\eta/2C_L, \mathbf{G}]$. Clearly one has

$$\|\mathbf{F} - \mathbf{F}_\eta\|_{\ell_2(\mathcal{J})} \le \|\mathbf{L}^*\|_{\ell_2(\mathcal{J})}\|\mathbf{G} - \mathbf{G}_0\| + \|\mathbf{L}^*\mathbf{G}_0 - \mathbf{F}_\eta\|_{\ell_2(\mathcal{J})}.$$

By (4.2.3) and the preceding remarks the first summand can be estimated, in view of (6.2.5), by $\eta/2$. Moreover, (5.4.4) says that the second summand is bounded by $\eta/2$ as well which confirms the validity of (4.2.1). As for the rest of the assertion, by assumption, $\mathbf{L} \in \mathcal{C}_{s^*}$ and is, therefore, on account of Proposition 5.9, bounded on $\ell_\tau^w(\mathcal{J})$ for $\tau^{-1} = s + \frac{1}{2}$ and $s < s^*$. Thus, $\mathbf{G} \in \ell_\tau^w(\mathcal{J})$. Therefore, by Remark 4.1, one has $\#(\operatorname{supp} \mathbf{G}_0) \lesssim \eta^{-1/s}$, $\|\mathbf{G}_0\|_{\ell_\tau^w(\mathcal{J})} \lesssim \|\mathbf{G}\|_{\ell_\tau^w(\mathcal{J})} \lesssim \|\mathbf{U}\|_{\ell_\tau^w(\mathcal{J})}$. Morover, by definition, $\mathbf{L}^*$ also belongs to $\mathcal{C}_{s^*}$ and so is bounded on $\ell_\tau^w(\mathcal{J})$ as above. Thus $\mathbf{F}$ belongs to $\ell_\tau^w(\mathcal{J})$ for τ related to $s < s^*$ as above. The assertion follows now from Proposition 5.10. ∎

In view of (7.1), we define **APPLY** as follows:

APPLY$_{ls}$ $[\eta, \mathbf{Q}, \mathbf{V}] \to (\mathbf{W}_\eta, \mathbf{\Lambda}_{\eta,Q})$
is given by

$$\mathbf{MULT}\left[\frac{\eta}{2}, \mathbf{L}^*, \mathbf{MULT}\left[\frac{\eta}{2C_L}, \mathbf{L}, \mathbf{V}\right]\right] \to (\mathbf{W}_\eta, \mathbf{\Lambda}_{\eta,Q}). \tag{7.5}$$

Proposition 7.3. *The output* $\mathbf{W}_\eta$ *of* **APPLY**$_{ls}$ *satisfies* (4.2.2) *for* $\mathbf{M} = \mathbf{Q}$, *i.e.*,

$$\|\mathbf{QV} - \mathbf{W}_\eta\|_{\ell_2(\mathcal{J})} \le \eta. \tag{7.6}$$

Moreover, whenever $\mathbf{L} \in \mathcal{C}_{s^*}$ *for some* $s^* > 0$ *then* **APPLY**$_{ls}$ *satsifies Requirements* 5.3. *In particular,* $\mathbf{Q} \in \mathcal{A}_{s^*}$.

Proof. Let $\mathbf{W}_1$ denote the finitely supported approximation to $\mathbf{LV}$ produced by **MULT** $[\eta/2C_L, \mathbf{L}, \mathbf{V}]$. By (5.4.4) we have, on account of (6.2.5),

$$\begin{aligned}\|\mathbf{QV} - \mathbf{W}_\eta\|_{\ell_2(\mathcal{J})} &\le \|\mathbf{L}^*\|_{\ell_2(\mathcal{J})}\|\mathbf{LV} - \mathbf{W}_1\|_{\ell_2(\mathcal{J})} + \|\mathbf{L}^*\mathbf{W}_1 - \mathbf{W}_\eta\|_{\ell_2(\mathcal{J})} \\ &\le \frac{C_L\eta}{2C_L} + \frac{\eta}{2} = \eta,\end{aligned}$$

which confirms (7.6). The remaining part of the claim follows from Theorem 7.1, again by the properties of **MULT** stated in Proposition 5.10, and analogous arguments as in the proof of Proposition 7.2 above. ∎

Combining Theorem 5.7 with the above observations, we can now state the following result:

Theorem 7.4. *Assume that* (3.1.10) *holds and that the blocks* $\mathbf{A}_{i,l}$ *from* (6.2.2) *belong to* $\mathcal{C}_{s^*}$ *for some* $s^* > 0$. *Let the scheme* **SOLVE**$_{ls}$ *be defined as in Section* 4.2 *with the routines* **RHS**$_{ls}$ *and* **APPLY**$_{ls}$ *defined by* (7.4) *and* (7.5). *Then* **SOLVE**$_{ls}$ *when applied to* (7.2) *is work/accuracy optimal of any order* $s < s^*$ *with constants*

depending on $s^* - s$ in the following sense: *If the solution $U = \mathbf{U}^*\mathbf{D}^{-1}\Psi$ of* (3.1.6) *satisfies*

$$\inf_{\substack{\mathbf{V} \in \ell_2(\mathcal{J}) \\ \sum_{i=1}^m \#\mathrm{supp}\, \mathbf{v}^i \le N}} \|U - \mathbf{V}^*\mathbf{D}^{-1}\Psi\|_{\mathcal{H}} \lesssim N^{-s},$$

then for any $\varepsilon > 0$ the approximation $\mathbf{U}(\varepsilon)$ produced by **SOLVE** *satisfies*

$$\|U - \mathbf{U}(\varepsilon)^*\mathbf{D}^{-1}\Psi\|_{\mathcal{H}} \le \varepsilon,$$

where $\sum_{i=1}^m \#\mathrm{supp}\, \mathbf{u}^i(\varepsilon) \le C\varepsilon^{-1/s}$ and where C depends only on $s^* - s$, the constants in (6.1.2) *and the constants in* (3.1.10). *Moreover, the number of flops needed to compute $\mathbf{U}(\varepsilon)$ is bounded by $C\varepsilon^{-1/s}$ with additional $C\varepsilon^{-1/s}\log\varepsilon$ operations spent on sorting with the same dependence of C as above.*

In view of Remark 6.2.3, one has

Corollary 7.5. *The assertion of Theorem* 7.4 *holds for each of the examples in Section* 3.2 *for any $s^* > 0$ provided that sufficiently regular wavelets are used.*

It should be noted that the present least squares formulation does not require higher regularity of the involved wavelet bases than standard Galerkin discretizations since the conformity of the trial functions depends only on $\mathcal{L}$ not on the product $\mathcal{L}^*\mathcal{L}$. Therefore the current interest in more regular wavelets is merely to increase the range of compressibility of $\mathbf{L}$ (not of $\mathbf{L}^*\mathbf{L}$). In addition, using low-order formulations as the mixed formulation of the second-order boundary value problem or the first-order Stokes formulation also avoids squaring the order on the discrete level.

We conclude this section with a few remarks on the formulation (7.2) of the original problem (3.1.6). This is to relate the present approach to recent activities around least squares schemes, see, e.g., [**10**], [**8**], [**9**], [**11**], [**13**], [**14**]. Obviously, (7.2) is just the (infinite) system of *normal equations* for the *least squares problem*

$$\min_{\mathbf{V} \in \ell_2(\mathcal{J})} \|\mathbf{L}\mathbf{V} - \mathbf{G}\|^2_{\ell_2(\mathcal{J})}, \tag{7.7}$$

which by (6.2.5) possesses a unique solution. Moreover, as pointed out in [**26**], a natural least squares formulation of the operator equation (3.1.9) is given by

$$\min_{V \in \mathcal{H}} \|\mathcal{L}V - F\|^2_{\mathcal{H}'}, \tag{7.8}$$

that is, residuals are measured in the dual norm. This imposes only the natural minimal regularity requirements on the solution U of (7.8) (and equivalently (3.1.6)) and gives rise to optimal error estimates in $\mathcal{H}$.

The search for suitable least squares functionals is reflected by many investigations mainly in the finite element context, see, e.g., [**1**], [**8**], [**9**], [**13**], [**14**] and the literature cited in [**26**]. While, in principle, the choice (7.8) is at least implicitly

treated in many papers but is usually directly intertwined with discretizations and the (often severe) problem of numerically evaluating the dual norms $\|\cdot\|_{H'_{i,0}}$. In fact, the examples in Section 3.2 reveal that one usually encounters broken trace norms or the H^{-1}-norm. Once wavelet bases on the repective domains are available these problems disappear when replacing the dual norms by equivalent ones offered by Remark 6.1.1, [**26**].

Remark 7.6. Defining

$$\langle v, w\rangle_i := \sum_{\lambda\in\mathcal{J}_i} d_{i,\lambda}^{-2}\langle v, \psi_\lambda^i\rangle\langle\psi_\lambda^i, w\rangle = \langle v, \Psi^i\rangle\mathbf{D}_i^{-2}\langle\Psi^i, w\rangle, \tag{7.9}$$

(6.1.3) says that

$$C_1^{-2}\langle\cdot,\cdot\rangle_i \le \|\cdot\|^2_{H'_{i,0}} \le c_1^{-2}\langle\cdot,\cdot\rangle_i. \tag{7.10}$$

With this substitution of dual norms the least squares functional (7.8) in wavelet coordinates takes the equivalent form (7.7). In this sense, the formulation (7.2) represents a natural least squares formulation of the original problem (3.1.9) [**26**].

The system (7.2) still represents the *infinite-dimensional* problem. As soon as one turns to finite-dimensional discretizations the problem of stability may, in principle, arise again, since the least squares functional from the infinite-dimensional case can usually not be evaluated exactly. In the finite element context this is, for instance, reflected by approximately solving auxiliary Neumann problems [**9**]. In the present context, whenever a trial space for the problem (3.1.6) is a-priorily *fixed*, e.g., as $S(\Psi_\Lambda) := \mathrm{span}\{(\psi^1_{\lambda_1},\dots,\psi^m_{\lambda_m})\colon\ (\lambda_1,\dots,\lambda_m)\in\mathbf{\Lambda}\subset\mathcal{J}\}$ the corresponding Galerkin matrix $\mathbf{Q}_\Lambda$ is still the product of two *infinite* matrices $\mathbf{L}_\Lambda^*\mathbf{L}_\Lambda$ where $\mathbf{L}_\Lambda$ consists of those (infinite) columns of $\mathbf{L}$ which correspond to the finite index set $\mathbf{\Lambda}$. Thus $\mathbf{Q}_\Lambda$ cannot be computed exactly but has to be approximated, e.g., by discarding all but finitely many rows of $\mathbf{L}_\Lambda$. This corresponds to truncating the infinite sums over $\mathcal{J}_i$ in (7.9). It is shown in [**26**] how to truncate for any given $\mathbf{\Lambda}$ so as to preserve stability. A similar idea is used in [**5**] where suitably truncated versions of weighted ℓ_2-norms are used to stabilize semidefinite problems.

It is important to note that in the present context this stability problem does *not* arise. The main point is that the *adaptive* application of the full infinite-dimensional operator in **SOLVE** automatically inherits the stability of the infinite-dimensional problem. In this sense adaptivity stabilizes. We will encounter another example of this type in the next section.

8. Saddle Point Problems

On one hand, the least squares formulation from Section 7 offers a unified treatment in essentially all cases of well-posed problems in the sense of (3.1.10). On the other hand, in spite of the asserted asymptotic optimality of **SOLVE**, the least

squares formulation still entails a squaring of the condition of the underlying system. This might well affect the performance of a numerical scheme when the ratio C_L/c_L in (6.2.5) is large, see (7.3). We will therefore point out next how to arrive at admissible discrete formulations of the form (4.1.3) for the restricted class of *saddle point problems* without a least squares formulation. To be specific, in the terms of Section 3.1, this case corresponds to $m = 2$, $A_{1,1}(\cdot,\cdot) = a(\cdot,\cdot)$, $A_{1,2}(\cdot,\cdot) = A_{2,1}(\cdot,\cdot) =: b(\cdot,\cdot)$, and $A_{2,2}(\cdot,\cdot) = 0$. Following the usual convention we will also set $X := H_{1,0}$ and $M := H_{2,0}$. According to (3.1.4) the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $X \times X$ (respectively, $X \times M$). In addition, we assume that now $a(\cdot,\cdot)$ is symmetric positive semidefinite on $X \times X$.

Under these circumstances it is well-known that the present specialization

$$\begin{aligned} a(u,v) + b(q,v) &= \langle f, v\rangle_X, \qquad \forall\, v \in X, \\ b(p,u) &= \langle p, g\rangle_M, \qquad \forall\, p \in M, \end{aligned} \tag{8.1}$$

of (3.1.6) has a unique solution $U = (u,q) \in X \times M$ if and only if

$$\inf_{w\in\ker B}\sup_{v\in\ker B}\frac{a(v,w)}{\|v\|_X\|w\|_X} > \alpha, \qquad \inf_{p\in M}\sup_{v\in X}\frac{b(v,p)}{\|v\|_X\|p\|_M} > \beta, \tag{8.2}$$

for some positive numbers α, β where here $B := \mathcal{L}_{1,2}\colon X \to M$, see, e.g., [**12**]. Clearly the solution component u of (8.1) minimizes the quadratic form $a(\cdot,\cdot)$ under linear constraints given by $b(\cdot,\cdot)$.

Abbreviating $A^0 := \mathcal{L}_{1,1}\colon X \to X'$ (induced by $a(v,w) = \langle v, A^0 w\rangle_X, v \in X$), (8.1) is equivalent to the 2×2 block operator equation

$$\mathcal{L}U := \begin{pmatrix} A^0 & B' \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} f^0 \\ g \end{pmatrix} =: F, \tag{8.3}$$

where $\mathcal{L}$ is an isomorphism from $X \times M$ into its dual $X' \times M'$, i.e., (3.1.10) holds if and only if (8.2) holds.

The mixed formulation (3.2.2) can be recast into this form when $k = 0$. The Stokes problem (3.2.8) with homogeneous boundary conditions incorporated in the trial spaces for X is another example. The following third example is particularly interesting for wavelet concepts. Suppose for simplicity that the domain Ω is contained in the unit cube $\square = (0,1)^d$ and consider again the classical Dirichlet problem (3.2.1) now with possibly inhomogeneous boundary conditions $u = g$ on $\Gamma = \partial\Omega$. It is well-known that its solution agrees on Ω with the solution of the saddle point problem

$$\begin{aligned} \langle \nabla v, \nabla u\rangle + \beta\langle v, u\rangle + \langle v, q\rangle &= \langle v, \hat{f}\rangle, \qquad v \in H^1(\square), \\ \langle u, p\rangle &= \langle g, p\rangle, \qquad p \in H^{-1/2}(\Gamma), \end{aligned} \tag{8.4}$$

when $\hat{f}$ is a suitable extension of f^0 to all of $\square$, see, e.g, [**33**]. As in the case of the Stokes problem the first condition in (8.2) follows from the stronger condition

$$a(v,v) \sim \|v\|_X^2, \qquad v \in X, \tag{8.5}$$

when $\beta \geq c_0 > 0$ on Ω, which means that the operator A^0 in (8.3) is invertible on all of X.

8.1. *The Discrete Formulation*

Assuming, as before, that wavelet bases $\Psi_X := \Psi^1 \subset X$, $\Psi_M := \Psi^2 \subset M$ indexed over $\mathcal{J}_X$ (respectively $\mathcal{J}_M$), are available satisfying the corresponding norm equivalences (6.1.2), (6.2.4) now takes the form

$$\mathbf{L}^0\mathbf{U} := \begin{pmatrix} \mathbf{A}^0 & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{f}^0 \\ \mathbf{g} \end{pmatrix} =: \mathbf{G}. \tag{8.1.1}$$

Here (8.2) implies (6.2.5). By assumption $\mathbf{A}^0$ is symmetric but, e.g., when $\beta = 0$ in (8.4), *not* necessarily definite on all of $\ell_2(\mathcal{J}_X)$.

In what follows it will be important to ensure invertibility on all of $\ell_2(\mathcal{J}_X)$. This can always be achieved with the aid of an analog to what is generally referred to as *augmented Lagrangian formulation*. In fact, setting

$$\mathbf{A} := \mathbf{A}^0 + c\mathbf{B}^*\mathbf{B}, \tag{8.1.2}$$

where c is any fixed positive constant. Then (8.1.1) is equivalent to the system

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}, \tag{8.1.3}$$

where, for $\mathbf{A}$ given by (8.1.2),

$$\mathbf{f} := \mathbf{f}^0 + c\mathbf{B}^*\mathbf{g}. \tag{8.1.4}$$

Now $\mathbf{A}$ is invertible over $\ell_2(\mathcal{J}_X)$ (see [**21**]), i.e., (by the inverse mapping theorem) there exist positive constants c_A, C_A such that

$$c_A\|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)} \leq \|\mathbf{A}\mathbf{v}\|_{\ell_2(\mathcal{J}_X)} \leq C_A\|\mathbf{v}\|_{\ell_2(\mathcal{J}_X)}, \qquad \mathbf{v} \in \ell_2(\mathcal{J}_X). \tag{8.1.5}$$

Moreover, since the operator

$$\mathbf{L} := \begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$$

arises from $\mathbf{L}^0$ in (8.1.1) through a boundedly invertible transformation on $\ell_2(\mathcal{J}_X) \times \ell_2(\mathcal{J}_M)$ (see [**21**]), (6.2.5) remains valid for $\mathbf{L}$.

By block elimination the system (8.1.3) is equivalent to

$$\mathbf{S}\mathbf{q} = \mathbf{B}\mathbf{A}^{-1}\mathbf{f} - \mathbf{g} =: \mathbf{F}^1, \tag{8.1.6}$$

$$\mathbf{A}\mathbf{u} = \mathbf{f} - \mathbf{B}^*\mathbf{q} =: \mathbf{F}^2, \tag{8.1.7}$$

where $\mathbf{S} := \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^*$ is the Schur complement. Clearly one has

$$c_S\|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \le \|\mathbf{S}\mathbf{q}\|_{\ell_2(\mathcal{J}_M)} \le C_S\|\mathbf{q}\|_{\ell_2(\mathcal{J}_M)}, \qquad \mathbf{q} \in \ell_2(\mathcal{J}_M). \tag{8.1.8}$$

It is natural to apply the above adaptive concepts to the reduced system (8.1.6), i.e., to take $\mathbf{M} := \mathbf{S}$. In fact, a version of the (inexact) Uzawa scheme can be viewed as a gradient iteration for (8.1.6) intertwined with the approximate solution of (8.1.7). This latter part can be treated as in Section 6.3 combined with a suitable right-hand side scheme for $\mathbf{F}^2$ which, in turn, can be based on the routines **COARSE** and **MULT**. A variant of such an approach was first proposed in [**22**] where an adaptive outer Uzawa-type iteration is shown to converge. While [**22**] offers no complexity estimates the asymptotic optimality of the work/accuracy balance in the sense of Theorem 7.4 for the adaptive Uzawa scheme is shown in [**21**].

Here we wish to pursue yet another line which is neither based on the Schur complement nor on the least squares formulation. It is based on an idea from [**10**] which in combination with wavelet preconditioning is also used in [**35**].

8.2. *A Positive Definite System*

Taking up an idea of Bramble and Pasciak [**10**] one can turn the saddle point problem into a positive definite one. It requires a good preconditioner for the block $\mathbf{A}$ in (8.1.3). In view of (8.1.5) it suffices here to take just a constant multiple of the identity matrix which again exploits the fact that properly scaled wavelet representations are automatically well-conditioned. Moreover, all other blocks of $\mathbf{L}$ are by (6.2.5) bounded in ℓ_2. Now, fixing any positive $\gamma < c_A$, we readily infer from (8.1.5) that

$$(c_A - \gamma)\|\mathbf{v}\|^2_{\ell_2(\mathcal{J}_X)} \le \mathbf{v}^*(\mathbf{A} - \gamma\mathbf{id})\mathbf{v} \le \mathbf{v}^*\mathbf{A}\mathbf{v} \le C_A\|\mathbf{v}\|^2_{\ell_2(\mathcal{J}_X)}. \tag{8.2.1}$$

Multiplying the system (8.1.3) by the operator

$$\begin{pmatrix} \gamma^{-1}\mathbf{id} & \mathbf{0} \\ \gamma^{-1}\mathbf{B} & -\mathbf{id} \end{pmatrix},$$

which is boundedly invertible on $\ell_2(\mathcal{J}_X) \times \ell_2(\mathcal{J}_M)$, yields the equivalent system

$$\begin{pmatrix} \gamma^{-1}\mathbf{A} & \gamma^{-1}\mathbf{B}^* \\ \mathbf{B}(\gamma^{-1}\mathbf{A} - \mathbf{id}) & \gamma^{-1}\mathbf{B}\mathbf{B}^* \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \gamma^{-1}\mathbf{f} \\ \gamma^{-1}\mathbf{B}\mathbf{f} - \mathbf{g} \end{pmatrix},$$

which, in turn, is equivalent to

$$\hat{\mathbf{L}}\begin{pmatrix} \mathbf{u} \\ \mathbf{q} \end{pmatrix} := \begin{pmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B}(\mathbf{A} - \gamma\mathbf{id}) & \mathbf{B}\mathbf{B}^* \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{B}\mathbf{f} - \gamma\mathbf{g} \end{pmatrix}. \tag{8.2.2}$$

The main observation from [**10**] is that $\hat{\mathbf{L}}$ is *symmetric positive definite* with respect to the inner product $[\cdot,\cdot]$ on $(\ell_2(\mathcal{J}_X)\times\ell_2(\mathcal{J}_M))\times(\ell_2(\mathcal{J}_X)\times\ell_2(\mathcal{J}_M))$ defined by

$$\left[\begin{pmatrix}\mathbf{v}\\ \mathbf{p}\end{pmatrix},\begin{pmatrix}\mathbf{w}\\ \mathbf{r}\end{pmatrix}\right]:=\mathbf{w}^*(\mathbf{A}-\gamma\mathbf{id})\mathbf{v}+\mathbf{r}^*\mathbf{p}.$$

Hence $\hat{\mathbf{L}}$ has a bounded strictly positive spectrum and its maximum is given by $\|\hat{\mathbf{L}}\|$ where $\|\cdot\|^2:=[\cdot,\cdot]$ is the norm induced by this inner product. Clearly, by (8.2.1), this norm satisfies (4.1.6) with $\hat{c}:=\sqrt{c_A-\gamma}$, $\hat{C}:=\sqrt{C_A}$. Therefore we can find a fixed positive number ω such that

$$\|\mathbf{id}-\omega\hat{\mathbf{L}}\|\le\rho<1, \tag{8.2.3}$$

i.e., (4.1.7) holds for $\mathbf{M}=\hat{\mathbf{L}}$.

It remains to identify suitable routines **RHS** and **APPLY**. Assuming that $\|\mathbf{B}\|_{\ell_2\to\ell_2}\le B$ we set $\mathbf{f}_1:=\mathbf{COARSE}\,[\eta/C_B3\sqrt{2},\mathbf{f}]$ and $\mathbf{g}_1:=\mathbf{COARSE}\,[\eta/3\gamma\sqrt{2},\mathbf{g}]$,

$$\mathbf{RHS}\left[\eta,\begin{pmatrix}\mathbf{f}\\ \mathbf{Bf}-\gamma\mathbf{g}\end{pmatrix}\right]:=\begin{pmatrix}\mathbf{f}_1\\ \mathbf{MULT}\,[\eta/3\sqrt{2},\mathbf{B},\mathbf{f}_1]-\gamma\mathbf{g}_1\end{pmatrix}. \tag{8.2.4}$$

Remark 8.2.1. The routine **RHS** defined by (8.2.4) satisfies (4.2.1). Moreover, whenever $\mathbf{B}\in\mathcal{C}_{s^*}$ for some $s^*>0$, then **RHS** satisfies Requirements 5.6 for $s<s^*$.

Proof. The assertions are straightforward consequences of Propositions 5.2 and 5.3.1. ∎

Similarly the **APPLY** routine uses **MULT** as the main building block.

$\mathbf{APPLY}\,[\eta,\hat{\mathbf{L}},\binom{\mathbf{v}}{\mathbf{p}}]\to\binom{\mathbf{w}_1}{\mathbf{w}_2}$
is defined as follows:

(i) Compute

$$\mathbf{MULT}\left[\frac{\eta}{2\sqrt{2}},\mathbf{A},\mathbf{v}\right]\to\mathbf{w},\qquad\mathbf{MULT}\left[\frac{\eta}{2\sqrt{2}},\mathbf{B}^*,\mathbf{p}\right]\to\mathbf{z}.$$

(ii) Set

$$\mathbf{APPLY}\left[\eta,\hat{\mathbf{L}},\begin{pmatrix}\mathbf{v}\\ \mathbf{p}\end{pmatrix}\right]:=\begin{pmatrix}\mathbf{w}+\mathbf{z}\\ \mathbf{MULT}\,[\eta/2C_B\sqrt{2},\mathbf{B},\mathbf{w}-\gamma\mathbf{v}]+\mathbf{MULT}\,[\eta/2C_B\sqrt{2},\mathbf{B},\mathbf{z}]\end{pmatrix}.$$

Again we can use Proposition 5.10 to verify the following claim:

Remark 8.2.2. The routine **APPLY** defined by (8.2.5) satisfies (4.2.2). Moreover, when **A** and **B** belong to $\mathcal{C}_{s^*}$ for some $s^* > 0$, then **APPLY** satisfies Requirements 5.3 for $s < s^*$.

The above findings can be summarized as follows:

Theorem 8.2.3. *The scheme* **SOLVE** *applied to the system* (8.2.2) (*for* $\mathbf{M} := \hat{\mathbf{L}}$) *with* **RHS** *and* **APPLY** *defined above is asymptotically work/accuracy optimal in the sense of Theorem* 7.4.

Note again that no compatibility conditions between the bases for the components X and M like the LBB condition are needed in the above context. The adaptive application of the (infinite-dimensional) operators inherits the stability properties of the infinite-dimensional problem.

Of course, in a practical realization the parameter ω in (8.2.3) should depend on the iteration step. Or better yet, a conjugate gradient iteration with respect to the inner product $[\cdot, \cdot]$ should be employed.

Acknowledgments

This work has been supported in part by the Deutsche Forschungsgemeinschaft grants Da 117/8-2, the Office of Naval Research Contract N0014-91-J1343, and the TMR network "Wavelets in Numerical Simulation."

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