7

Pseudodifferential Operators

Introduction

In this chapter we discuss the basic theory of pseudodifferential operators as it has been developed to treat problems in linear PDE. We define pseudodifferential operators with symbols in classes denoted $S^m_{\rho,\delta}$, introduced by L. Hörmander. In §2 we derive some useful properties of their Schwartz kernels. In §3 we discuss adjoints and products of pseudodifferential operators. In §4 we show how the algebraic properties can be used to establish the regularity of solutions to elliptic PDE with smooth coefficients. In §5 we discuss mapping properties on L^2 and on the Sobolev spaces H^s. In §6 we establish Gårding's inequality.

In §7 we apply some of the previous material to establish the existence of solutions to hyperbolic equations. In §8 we show that certain important classes of pseudodifferential operators are preserved under the action of conjugation by solution operators to (scalar) hyperbolic equations, a result of Y. Egorov. We introduce the notion of wave front set in §9 and discuss the microlocal regularity of solutions to elliptic equations. We also discuss how solution operators to a class of hyperbolic equations propagate wave front sets. In §10 there is a brief discussion of pseudodifferential operators on manifolds.

We give some further applications of pseudodifferential operators in the next three sections. In §11 we discuss, from the perspective of the pseudodifferential operator calculus, the classical method of layer potentials, applied particularly to the Dirichlet and Neumann boundary problems for the Laplace operator. Historically, this sort of application was one of the earliest stimuli for the development of the theory of singular integral equations. One function of §11 is to provide a warm-up for the use of similar integral equations to tackle problems in scattering theory, in §7 of Chapter 9. Section 12 looks at general regular elliptic boundary problems and includes material complementary to that developed in §11 of Chapter 5. In §13 we construct a parametrix for the heat equation and apply this to obtain an asymptotic expansion of the trace of the solution operator. This expansion will be useful in studies of the spectrum in Chapter 8 and in index theory in Chapter 10.

Finally, in §14 we introduce the Weyl calculus. This can provide a powerful alternative to the operator calculus developed in §§1–6, as can be seen in [Ho4] and in Vol. 3 of [Ho5]. Here we concentrate on identities, tied to symmetries in the Weyl calculus. We show how this leads to a quicker construction of a parametrix for the heat equation than the method used in §13. We will make use of this in §10 of Chapter 10, on a direct attack on the index theorem for elliptic differential operators on two-dimensional manifolds.

Material in §§1–10 is taken from Chapter 0 of [T4], and the author thanks Birkhäuser Boston for permission to use this material. We also mention some books that take the theory of pseudodifferential operators farther than is done here: [Ho5], [Kg], [T1], and [Tre].

1. The Fourier integral representation and symbol classes

Using a slightly different convention from that established in Chapter 3, we write the Fourier inversion formula as

$$f(x) = \int \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi, \tag{1.1}$$

where $\hat{f}(\xi) = (2\pi)^{-n} \int f(x) e^{-ix\cdot\xi}\, dx$ is the Fourier transform of a function on $\mathbb{R}^n$. If one differentiates (1.1), one obtains

$$D^\alpha f(x) = \int \xi^\alpha \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{1.2}$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = (1/i)\,\partial/\partial x_j$. Hence, if

$$p(x, D) = \sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$$

is a differential operator, we have

$$p(x, D) f(x) = \int p(x, \xi) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \tag{1.3}$$

where

$$p(x, \xi) = \sum_{|\alpha|\le k} a_\alpha(x) \xi^\alpha.$$

One uses the Fourier integral representation (1.3) to define pseudodifferential operators, taking the function $p(x, \xi)$ to belong to one of a number of different classes of symbols. In this chapter we consider the following symbol classes, first defined by Hörmander [Ho2].

Assuming $\rho, \delta \in [0,1]$, $m \in \mathbb{R}$, we define $S^m_{\rho,\delta}$ to consist of C^∞-functions $p(x,\xi)$ satisfying

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}, \tag{1.4}$$

for all α, β, where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. In such a case we say the associated operator defined by (1.3) belongs to $OPS^m_{\rho,\delta}$. We say that $p(x,\xi)$ is the symbol of $p(x,D)$. The case of principal interest is $\rho = 1$, $\delta = 0$. This class is defined by [KN].

Recall that in Chapter 3, §8, we defined $P(\xi) \in S^m_1(\mathbb{R}^n)$ to satisfy (1.4), with $\rho = 1$, and with no x-derivatives involved. Thus $S^m_{1,0}$ contains $S^m_1(\mathbb{R}^n)$.

If there are smooth $p_{m-j}(x,\xi)$, homogeneous in ξ of degree $m-j$ for $|\xi| \ge 1$, that is, $p_{m-j}(x,r\xi) = r^{m-j} p_{m-j}(x,\xi)$ for $r, |\xi| \ge 1$, and if

$$p(x,\xi) \sim \sum_{j \ge 0} p_{m-j}(x,\xi) \tag{1.5}$$

in the sense that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}_{1,0}, \tag{1.6}$$

for all N, then we say $p(x,\xi) \in S^m_{cl}$, or just $p(x,\xi) \in S^m$. We call $p_m(x,\xi)$ the *principal symbol* of $p(x,D)$. We will give a more general definition of the principal symbol in §10.

It is easy to see that if $p(x,\xi) \in S^m_{\rho,\delta}$ and $\rho, \delta \in [0,1]$, then $p(x,D) : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. In fact, multiplying (1.3) by x^α, writing $x^\alpha e^{ix\cdot\xi} = (-D_\xi)^\alpha e^{ix\cdot\xi}$, and integrating by parts yield

$$p(x,D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.7}$$

Under one restriction, $p(x,D)$ also acts on tempered distributions:

Lemma 1.1. *If $\delta < 1$, then*

$$p(x,D) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.8}$$

Proof. Given $u \in \mathcal{S}'$, $v \in \mathcal{S}$, we have (formally)

$$\langle v, p(x,D)u \rangle = \langle p_v, \hat{u} \rangle, \tag{1.9}$$

where

$$p_v(\xi) = (2\pi)^{-n} \int v(x) p(x,\xi) e^{ix\cdot\xi} \, dx.$$

Now integration by parts gives

$$\xi^\alpha p_v(\xi) = (2\pi)^{-n} \int D_x^\alpha \big(v(x) p(x,\xi) \big) e^{ix\cdot\xi} \, dx,$$

so

$$|p_v(\xi)| \le C_\alpha \langle \xi \rangle^{m+\delta|\alpha|-|\alpha|}.$$

Thus if $\delta < 1$, we have rapid decrease of $p_v(\xi)$. Similarly, we get rapid decrease of derivatives of $p_v(\xi)$, so it belongs to $\mathcal{S}$. Thus the right side of (1.9) is well defined.

In §5 we will analyze the action of pseudodifferential operators on Sobolev spaces.

Classes of symbols more general than $S^m_{\rho,\delta}$ have been introduced by R. Beals and C. Fefferman [BF], [Be], and still more general classes were studied by Hörmander [Ho4]. These classes have some deep applications, but they will not be used in this book.

Exercises

1. Show that, for $a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n})$,

$$a(x,D)u = \int \hat{a}(q,p)\; e^{iq\cdot X} e^{ip\cdot D} u(x)\; dq\; dp, \tag{1.10}$$

where $\hat{a}(q,p)$ is the Fourier transform of $a(x,\xi)$, and the operators $e^{iq\cdot X}$ and $e^{ip\cdot D}$ are defined by

$$e^{iq\cdot X} u(x) = e^{iq\cdot x} u(x), \quad e^{ip\cdot D} u(x) = u(x+p).$$

2. Establish the identity

$$e^{ip\cdot D} e^{iq\cdot X} = e^{iq\cdot p}\; e^{iq\cdot X} e^{ip\cdot D}. \tag{1.11}$$

Deduce that, for $(t,q,p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n = \mathcal{H}^n$, the binary operation

$$(t,q,p) \circ (t',q',p') = (t+t'+p\cdot q', q+q', p+p') \tag{1.12}$$

gives a group and that

$$\tilde{\pi}(t,q,p) = e^{it} e^{iq\cdot X} e^{ip\cdot X} \tag{1.13}$$

defines a unitary representation of $\mathcal{H}^n$ on $L^2(\mathbb{R}^n)$; in particular, it is a group homomorphism: $\tilde{\pi}(z \circ z') = \tilde{\pi}(z)\tilde{\pi}(z')$. $\mathcal{H}^n$ is called the Heisenberg group.

3. Give a definition of $a(x-q, D-p)$, acting on $u(x)$. Show that

$$a(x-q, D-p) = \tilde{\pi}(0,q,p)\; a(x,D)\; \tilde{\pi}(0,q,p)^{-1}.$$

4. Assume $a(x,\xi) \in S^m_{\rho,\delta}$ and $b(x,\xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Show that $c(x,\xi) = (b * a)(x,\xi)$ belongs to $S^m_{\rho,\delta}$ ($*$ being convolution on $\mathbb{R}^{2n}$). Show that

$$c(x,D)u = \int b(y,\eta)\; a(x-y, D-\eta)\; dy\; d\eta.$$

5. Show that the map $\Psi(p,u) = p(x,D)u$ has a unique, continuous, bilinear extension from $S^m_{\rho,\delta} \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ to

$$\Psi : \mathcal{S}'(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n),$$

so that $p(x,D)$ is "well defined" for any $p \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$.

6. Let $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ be 1 for $|\xi| \le 1$, $\chi_\varepsilon(\xi) = \chi(\varepsilon\xi)$. Given $p(x,\xi) \in S^m_{\rho,\delta}$, let $p_\varepsilon(x,\xi) = \chi_\varepsilon(\xi)p(x,\xi)$. Show that if $\rho, \delta \in [0,1]$, then

$$u \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow p_\varepsilon(x,D)u \to p(x,D)u \ \text{ in } \mathcal{S}(\mathbb{R}^n). \tag{1.14}$$

If also $\delta < 1$, show that

$$u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow p_\varepsilon(x,D)u \to p(x,D)u \ \text{ in } \mathcal{S}'(\mathbb{R}^n), \tag{1.15}$$

where we give $\mathcal{S}'(\mathbb{R}^n)$ the weak* topology.

7. For $s \in \mathbb{R}$, define $\Lambda^s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\Lambda^s u(x) = \int \langle\xi\rangle^s \hat{u}(\xi)\, e^{ix\cdot\xi}\, d\xi, \tag{1.16}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Show that $\Lambda^s \in OPS^s$.

8. Given $p_j(x,\xi) \in S^{m_j}_{\rho,\delta}$, for $j \ge 0$, with $\rho, \delta \in [0,1]$ and $m_j \searrow -\infty$, show that there exists $p(x,\xi) \in S^{m_0}_{\rho,\delta}$ such that

$$p(x,\xi) \sim \sum_{j \ge 0} p_j(x,\xi),$$

in the sense that, for all k,

$$p(x,\xi) - \sum_{j=0}^{k-1} p_j(x,\xi) \in S^{m_k}_{\rho,\delta}.$$

2. Schwartz kernels of pseudodifferential operators

To an operator $p(x,D) \in OPS^m_{\rho,\delta}$ defined by (1.3) there corresponds a Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, satisfying

$$\begin{aligned} \langle u(x)v(y), K\rangle &= \iint u(x)p(x,\xi)\hat{v}(\xi)e^{ix\cdot\xi}\, d\xi\, dx \\ &= (2\pi)^{-n} \iiint u(x)p(x,\xi)e^{i(x-y)\cdot\xi} v(y)\, dy\, d\xi\, dx. \end{aligned} \tag{2.1}$$

Thus, K is given as an "oscillatory integral"

$$K = (2\pi)^{-n} \int p(x,\xi)e^{i(x-y)\cdot\xi}\, d\xi. \tag{2.2}$$

We have the following basic result.

Proposition 2.1. *If $\rho > 0$, then K is C^∞ off the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. For given $\alpha \ge 0$,

$$(x-y)^\alpha K = \int e^{i(x-y)\cdot\xi}\, D^\alpha_\xi p(x,\xi)\, d\xi. \tag{2.3}$$

This integral is clearly absolutely convergent for $|\alpha|$ so large that $m-\rho|\alpha| < -n$. Similarly, it is seen that applying j derivatives to (2.3) yields an absolutely convergent integral provided $m+j-\rho|\alpha| < -n$, so in that case $(x-y)^\alpha K \in C^j(\mathbb{R}^n \times \mathbb{R}^n)$. This gives the proof.

Generally, if T has the mapping properties

$$T : C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n), \quad T : \mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n),$$

and its Schwartz kernel K is C^∞ off the diagonal, it follows easily that

$$\text{sing supp } Tu \subset \text{ sing supp } u, \quad \text{for } u \in \mathcal{E}'(\mathbb{R}^n).$$

This is called the *pseudolocal property*. By (1.7)–(1.8) it holds for $T \in OPS^m_{\rho,\delta}$ if $\rho > 0$ and $\delta < 1$.

We remark that the proof of Proposition 2.1 leads to the estimate

$$|D^\beta_{x,y} K| \le C|x-y|^{-k}, \tag{2.4}$$

where $k \ge 0$ is any integer strictly greater than $(1/\rho)(m+n+|\beta|)$. In fact, this estimate is rather crude. It is of interest to record a more precise estimate that holds when $p(x,\xi) \in S^m_{1,\delta}$.

Proposition 2.2. *If $p(x,\xi) \in S^m_{1,\delta}$, then the Schwartz kernel K of $p(x,D)$ satisfies estimates*

$$|D^\beta_{x,y} K| \le C|x-y|^{-n-m-|\beta|} \tag{2.5}$$

provided $m+|\beta| > -n$.

The result is easily reduced to the case $p(x,\xi) = p(\xi)$, satisfying $|D^\alpha p(\xi)| \le C_\alpha \langle\xi\rangle^{m-|\alpha|}$, for which $p(D)$ has Schwartz kernel $K = \hat{p}(y-x)$. It suffices to prove (2.5) for such a case, for $\beta = 0$ and $m > -n$. We make use of the following simple but important characterization of such symbols.

Lemma 2.3. *Given $p(\xi) \in C^\infty(\mathbb{R}^n)$, it belongs to $S^m_{1,0}$ if and only if*

$$p_r(\xi) = r^{-m} p(r\xi) \text{ is bounded in } C^\infty(1 \le |\xi| \le 2), \quad \text{for } r \in [1,\infty). \tag{2.6}$$

Given this, we can write $p(\xi) = p_0(\xi) + \int_0^\infty q_\tau(e^{-\tau}\xi)\,d\tau$ with $p_0(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $e^{-m\tau} q_\tau(\xi)$ bounded in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, for $\tau \in [0,\infty)$. Hence $e^{-m\tau}\hat{q}_\tau(z)$ is bounded in $\mathcal{S}(\mathbb{R}^n)$. In particular, $e^{-m\tau}|\hat{q}_\tau(z)| \le C_N \langle z\rangle^{-N}$, so

$$\begin{aligned} |\hat{p}(z)| &\le |\hat{p}_0(z)| + C_N \int_0^\infty e^{(n+m)\tau}\big(1+|e^\tau z|\big)^{-N}\,d\tau \\ &\le C + C_N|z|^{-n-m} \int_{\log|z|}^\infty e^{(n+m)\tau}(1+e^\tau)^{-N}\,d\tau, \end{aligned} \tag{2.7}$$

which implies (2.5). We also see that in the case $m + |\beta| = -n$, we obtain a result upon replacing the right side of (2.5) by $C \log |x-y|^{-1}$, (provided $|x-y| < 1/2$).

We can get a complete characterization of $\hat{P}(x) \in \mathcal{S}'(\mathbb{R}^n)$, given $P(\xi) \in S^m_1(\mathbb{R}^n)$, provided $-n < m < 0$.

Proposition 2.4. *Assume $-n < m < 0$. Let $q \in \mathcal{S}'(\mathbb{R}^n)$ be smooth outside the origin and rapidly decreasing as $|x| \to \infty$. Then $q = \hat{P}$ for some $P(\xi) \in S^m_1(\mathbb{R}^n)$ if and only if $q \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and, for $x \neq 0$,*

$$|D^\beta_x q(x)| \le C_\beta \, |x|^{-n-m-|\beta|}. \tag{2.8}$$

Proof. That $P \in S^m_1(\mathbb{R}^n)$ implies (2.8) has been established above. For the converse, write $q = q_0(x) + \sum_{j\ge 0} \psi_j(x) q(x)$, where $\psi_0 \in C^\infty_0(\mathbb{R}^n)$ is supported in $1/2 < |x| < 2$, $\psi_j(x) = \psi_0(2^j x)$, $\sum_{j \ge 0} \psi_j(x) = 1$ on $|x| \le 1$. Since $|q(x)| \le C|x|^{-n-m}$, $m < 0$, it follows that $\sum \psi_j(x) q(x)$ converges in L^1-norm. Then $q_0 \in \mathcal{S}(\mathbb{R}^n)$. The hypothesis (2.8) implies that $2^{-nj-mj}\psi_j(2^{-j}x)q(2^{-j}x)$ is bounded in $\mathcal{S}(\mathbb{R}^n)$, and an argument similar to that used for Proposition 2.2 implies $\hat{q}_0(\xi) + \sum_{j=0}^\infty (\psi_j q)\hat{\ }(\xi) \in S^m_1(\mathbb{R}^n)$.

We will deal further with the space of elements of $\mathcal{S}'(\mathbb{R}^n)$ that are smooth outside the origin and rapidly decreasing (with all their derivatives) at infinity. We will denote this space by $\mathcal{S}'_0(\mathbb{R}^n)$.

If $m \le -n$, the argument above extends to show that (2.8) is a sufficient condition for $q = \hat{P}$ with $P \in S^m_1(\mathbb{R}^n)$, but, as noted above, there exist symbols $P \in S^m_1(\mathbb{R}^n)$ for which $q = \hat{P}$ does not satisfy (2.8). Now, given that $q \in \mathcal{S}'_0(\mathbb{R}^n)$, it is easy to see that

$$\nabla q \in \mathcal{F}\big(S^{m+1}_1(\mathbb{R}^n)\big) \Longleftrightarrow q \in \mathcal{F}\big(S^m_1(\mathbb{R}^n)\big). \tag{2.9}$$

Thus, if $-n-1 < m \le -n$, then Proposition 2.4 is almost applicable to ∇q, for $n \ge 2$.

Proposition 2.5. *Assume $n \ge 2$ and $-n-1 < m \le -n$. If $q \in \mathcal{S}'_0(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}$, then $q = \hat{P}$ for some $P \in S^m(\mathbb{R}^n)$ if and only if (2.8) holds for $|\beta| \ge 1$.*

Proof. First note that the hypotheses imply $q \in L^1(\mathbb{R}^n)$; thus $\tilde{q}(\xi)$ is continuous and vanishes as $|\xi| \to \infty$. In the proposition, we need to prove the "if" part. To use the reasoning behind Proposition 2.4, we need only deal with the fact that ∇q is not assumed to be in L^1_{loc}. The sum $\sum \psi_j(x)\nabla q(x)$ still converges in $L^1(\mathbb{R}^n)$, and so $\nabla q - \sum \psi_j(x) \nabla q$ is a sum of an element of $\mathcal{S}(\mathbb{R}^n)$ and possibly a distribution (call it ν) supported at 0. Thus $\hat{\nu}(\xi)$ is a polynomial. But as noted, $\hat{q}(\xi)$ is bounded, so $\hat{\nu}(\xi)$ can have at most linear growth. Hence

$$\xi_j \tilde{q}(\xi) = P_j(\xi) + \ell_j(\xi),$$

where $P_j \in S_1^{m+1}(\mathbb{R}^n)$ and $\ell_j(\xi)$ is a first-order polynomial in ξ. Since $\tilde{q}(\xi) \to 0$ as $|\xi| \to \infty$ and $m+1 \le -n+1 < 0$, we deduce that $\ell_j(\xi) = c_j$, a constant, that is,

$$\xi_j \tilde{q}(\xi) = P_j(\xi) + c_j, \quad P_j \in S_1^{m+1}(\mathbb{R}^n),\ m+1 < 0. \tag{2.10}$$

Now the left side vanishes on the hyperplane $\xi_j = 0$, which is unbounded if $n \ge 2$. This forces $c_j = 0$, and the proof of the proposition is then easily completed.

If we take $n = 1$ and assume $-2 < m < -1$, the rest of the hypotheses of Proposition 2.5 still yield (2.10), so

$$\frac{dq}{dx} = \hat{P}_1 + c_1\delta.$$

If we also assume q is continuous on $\mathbb{R}$, then $c_1 = 0$ and we again conclude that $q = \hat{P}$ with $P \in S_1^m(\mathbb{R})$. But if q has a simple jump at $x = 0$, then this conclusion fails.

Proposition 2.4 can be given other extensions, which we leave to the reader. We give a few examples that indicate ways in which the result does not extend, making use of results from §8 of Chapter 3. As shown in (8.31) of that chapter, on $\mathbb{R}^n$,

$$v = PF\ |x|^{-n} \Longrightarrow \hat{v}(\xi) = C_n\ \log|\xi|. \tag{2.11}$$

Now v is not rapidly decreasing at infinity, but if $\varphi(x)$ is a cut-off, belonging to $C_0^\infty(\mathbb{R}^n)$ and equal to 1 near $x = 0$, then $f = \varphi v$ belongs to $\mathcal{S}_0'(\mathbb{R}^n)$ and $\hat{f} = c\hat{\varphi} * \hat{v}$ behaves like $\log|\xi|$ as $|\xi| \to \infty$. One can then deduce that, for $n = 1$,

$$f(x) = \varphi(x)\ \log|x|\ \operatorname{sgn}|x| \Longrightarrow \hat{f}(\xi) \sim C\ \xi^{-1}\ \log|\xi|, \quad |\xi| \to \infty. \tag{2.12}$$

Thus Proposition 2.5 does not extend to the case $n = 1$, $m = -1$. However, we note that, in this case, $\hat{f}$ belongs to $S_1^{-1+\varepsilon}(\mathbb{R})$, for all $\varepsilon > 0$. In contrast to (2.12), note that, again for $n = 1$,

$$g(x) = \varphi(x)\ \log|x| \Longrightarrow \hat{g}(\xi) \sim C\,|\xi|^{-1}, \quad |\xi| \to \infty. \tag{2.13}$$

In this case, $(d/dx)\log|x| = PV(1/x)$.

Of considerable utility is the classification of $\mathcal{F}\big(S_{cl}^m(\mathbb{R}^n)\big)$. When $m = -j$ is a negative integer, this was effectively solved in §§8 and 9 of Chapter 3. The following result is what follows from the proof of Proposition 9.2 in Chapter 3.

Proposition 2.6. *Assume $q \in \mathcal{S}_0'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $j = 1, 2, 3, \dots$. Then $q = \hat{P}$ for some $P \in S_{cl}^{-j}(\mathbb{R}^n)$ if and only if*

$$q \sim \sum_{\ell \ge 0} \big(q_\ell + p_\ell(x) \log|x|\big), \tag{2.14}$$

where

$$q_\ell \in \mathcal{H}^{\#}_{j+\ell-n}(\mathbb{R}^n), \tag{2.15}$$

and $p_\ell(x)$ is a polynomial homogeneous of degree $j+\ell-n$; these log coefficients appear only for $\ell \geq n-j$.

We recall that $\mathcal{H}^{\#}_{\mu}(\mathbb{R}^n)$ is the space of distributions on $\mathbb{R}^n$, homogeneous of degree μ, which are smooth on $\mathbb{R}^n \setminus 0$. For $\mu > -n$, $\mathcal{H}^{\#}_{\mu}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. The meaning of the expansion (2.14) is that, for any $k \in \mathbb{Z}^+$, there is an $N < \infty$ such that the difference between q and the sum over $\ell < N$ belongs to $C^k(\mathbb{R}^n)$. Note that, for $n = 1$, the function $g(x)$ in (2.13) is of the form (2.14), but the function $f(x)$ in (2.12) is not.

To go from the proof of Proposition 9.2 of Chapter 3 to the result stated above, it suffices to note explicitly that

$$\varphi(x)x^\alpha \log|x| \in \mathcal{F}\big(S_1^{-n-|\alpha|}(\mathbb{R}^n)\big), \tag{2.16}$$

where φ is the cut-off used before. Since $\mathcal{F}$ intertwines D_ξ^α and multiplication by x^α, it suffices to verify the case $\alpha = 0$, and this follows from the formula (2.11), with x and ξ interchanged.

We can also classify Schwartz kernels of operators in $OPS^m_{1,0}$ and OPS^m_{cl}, if we write the kernel K of (2.2) in the form

$$K(x,y) = L(x, x-y), \tag{2.17}$$

with

$$L(x,z) = (2\pi)^{-n} \int p(x,\xi) e^{iz\cdot\xi}\, d\xi. \tag{2.18}$$

The following two results follow from the arguments given above.

Proposition 2.7. *Assume $-n < m < 0$. Let $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ be a smooth function of x with values in $\mathcal{S}_0'(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then (2.17) defines the Schwartz kernel of an operator in $OPS^m_{1,0}$ if and only if, for $z \neq 0$,*

$$|D_x^\beta D_z^\gamma L(x,z)| \leq C_{\beta\gamma}\, |z|^{-n-m-|\gamma|}. \tag{2.19}$$

Proposition 2.8. *Assume $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is a smooth function of x with values in $\mathcal{S}_0'(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Let $j = 1, 2, 3, \dots$. Then (2.17) defines the Schwartz kernel of an operator in OPS^{-j}_{cl} if and only if*

$$L(x,z) \sim \sum_{\ell \geq 0} \big(q_\ell(x,z) + p_\ell(x,z) \log|z|\big), \tag{2.20}$$

where each $D_x^\beta q_\ell(x,\cdot)$ is a bounded continuous function of x with values in $\mathcal{H}^{\#}_{j+\ell-n}$, and $p_\ell(x,z)$ is a polynomial homogeneous of degree $j+\ell-n$ in z, with coefficients that are bounded, together with all their x-derivatives.

Exercises

1. Using the proof of Proposition 2.2, show that, given $p(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, then
$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C' \langle \xi \rangle^{-|\alpha|+|\beta|}, \quad \text{for } |\beta| \le 1, \ |\alpha| \le n+1+|\beta|,$$
implies
$$|K(x,y)| \le C|x-y|^{-n} \quad \text{and} \quad |\nabla_{x,y} K(x,y)| \le C|x-y|^{-n-1}.$$
2. If the map κ is given by (2.2) (i.e., $\kappa(p) = K$) show that we get an isomorphism $\kappa : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$. Reconsider Exercise 3 of §1.
3. Show that κ, defined in Exercise 2, gives an isomorphism (isometric up to a scalar factor) $\kappa : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$. Deduce that $p(x,D)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, precisely when $p(x,\xi) \in L^2(\mathbb{R}^{2n})$.

3. Adjoints and products

Given $p(x,\xi) \in S^m_{\rho,\delta}$, we obtain readily from the definition that the adjoint is given by

$$p(x,D)^* v = (2\pi)^{-n} \int p(y,\xi)^* e^{i(x-y)\cdot\xi} v(y) \, dy \, d\xi. \tag{3.1}$$

This is not quite in the form (1.3), as the amplitude $p(y,\xi)^*$ is not a function of (x,ξ). We need to transform (3.1) into such a form.

Before continuing the analysis of (3.1), we are motivated to look at a general class of operators

$$Au(x) = (2\pi)^{-n} \int a(x,y,\xi) e^{i(x-y)\cdot\xi} u(y) \, dy \, d\xi. \tag{3.2}$$

We assume

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|} \tag{3.3}$$

and then say $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$. A brief calculation transforms (3.2) into

$$(2\pi)^{-n} \int q(x,\xi) e^{i(x-y)\cdot\xi} u(y) \, dy \, d\xi, \tag{3.4}$$

with

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-n} \int a(x,y,\eta) e^{i(x-y)\cdot(\eta-\xi)} \, dy \, d\eta \\ &= e^{iD_\xi \cdot D_y} a(x,y,\xi)\big|_{y=x}. \end{aligned} \tag{3.5}$$

Note that a formal expansion $e^{iD_\xi \cdot D_y} = I + iD_\xi \cdot D_y - (1/2)(D_\xi \cdot D_y)^2 + \cdots$ gives

$$q(x,\xi) \sim \sum_{\alpha \ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)\big|_{y=x}. \tag{3.6}$$

If $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$, with $0 \le \delta_2 < \rho \le 1$, then the general term in (3.6) belongs to $S^{m-(\rho-\delta_2)|\alpha|}_{\rho,\delta}$, where $\delta = \max(\delta_1, \delta_2)$, so the sum on the right is formally asymptotic. This suggests the following result:

Proposition 3.1. *If $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$, with $0 \le \delta_2 < \rho \le 1$, then (3.2) defines an operator*

$$A \in OPS^m_{\rho,\delta}, \quad \delta = \max(\delta_1, \delta_2).$$

Furthermore, $A = q(x,D)$, where $q(x,\xi)$ has the asymptotic expansion (3.6), in the sense that

$$q(x,\xi) - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_y a(x,y,\xi)\big|_{y=x} = r_N(x,\xi) \in S^{m-N(\rho-\delta_2)}_{\rho,\delta}.$$

To prove this proposition, one can first show that the Schwartz kernel

$$K(x,y) = (2\pi)^{-n} \int a(x,y,\xi) e^{i(x-y)\cdot\xi}\, d\xi$$

satisfies the same estimates as established in Proposition 2.1, and hence, altering A only by an operator in $OPS^{-\infty}$, we can assume $a(x,y,\xi)$ is supported on $|x-y| \le 1$. Let

$$\hat{b}(x,\eta,\xi) = (2\pi)^{-n} \int a(x,x+y,\xi) e^{-iy\cdot\eta}\, dy, \tag{3.7}$$

so

$$q(x,\xi) = \int \hat{b}(x,\eta,\xi+\eta)\, d\eta. \tag{3.8}$$

The hypotheses on $a(x,y,\xi)$ imply

$$|D^\beta_x D^\alpha_\xi \hat{b}(x,\eta,\xi)| \le C_{\nu\alpha\beta} \langle\xi\rangle^{m+\delta|\beta|+\delta_2\nu-\rho|\alpha|} \langle\eta\rangle^{-\nu}, \tag{3.9}$$

where $\delta = \max\,(\delta_1,\delta_2)$. Since $\delta_2 < 1$, it follows that $q(x,\xi)$ and any of its derivatives can be bounded by some power of $\langle\xi\rangle$.

Now a power-series expansion of $\hat{b}(x,\eta,\xi+\eta)$ in the last argument about ξ gives

$$\begin{aligned} &\Big|\hat{b}(x,\eta,\xi+\eta) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (iD_\xi)^\alpha \hat{b}(x,\eta,\xi)\eta^\alpha\Big| \\ &\qquad\qquad \le C_\nu |\eta|^N \langle\eta\rangle^{-\nu} \sup_{0\le t\le 1} \langle\xi+t\eta\rangle^{m+\delta_2\nu-\rho N}. \end{aligned} \tag{3.10}$$

Taking $\nu = N$, we get a bound on the left side of (3.10) by

$$C\langle\xi\rangle^{m-(\rho-\delta_2)N} \quad \text{if } \ |\eta| \le \frac{1}{2}|\xi|, \tag{3.11}$$

while taking ν large, we get a bound by any power of $\langle\eta\rangle^{-1}$ for $|\xi| \le 2|\eta|$. Hence

$$\Big|q(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}(iD_\xi)^\alpha D_y^\alpha a(x,x+y,\xi)\big|_{y=0}\Big| \le C\langle\xi\rangle^{m+n-(\rho-\delta_2)N}. \tag{3.12}$$

The proposition follows from this, plus similar estimates on the difference when derivatives are applied.

If we apply Proposition 3.1 to (3.1), we obtain:

Proposition 3.2. *If* $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, *then*

$$p(x,D)^* = p^*(x,D) \in OPS^m_{\rho,\delta}, \tag{3.13}$$

with

$$p^*(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha p(x,\xi)^*. \tag{3.14}$$

The result for products of pseudodifferential operators is the following.

Proposition 3.3. *Given* $p_j(x,D) \in OPS^{m_j}_{\rho_j,\delta_j}$, *suppose*

$$0 \le \delta_2 < \rho \le 1, \quad \textit{with } \rho = \min(\rho_1,\rho_2). \tag{3.15}$$

Then

$$p_1(x,D)p_2(x,D) = q(x,D) \in OPS^{m_1+m_2}_{\rho,\delta}, \tag{3.16}$$

with $\delta = \max(\delta_1,\delta_2)$, *and*

$$q(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi)\ D_x^\alpha p_2(x,\xi). \tag{3.17}$$

This can be proved by writing

$$p_1(x,D)p_2(x,D)u = p_1(x,D)p_2^*(x,D)^*u = Au, \tag{3.18}$$

for A as in (3.2), with

$$a(x,y,\xi) = p_1(x,\xi)p_2^*(y,\xi)^*, \tag{3.19}$$

and then applying Propositions 3.1 and 3.2, to obtain (3.16), with

$$q(x,\xi) \sim \sum_{\gamma,\sigma\ge 0} \frac{i^{|\sigma|-|\gamma|}}{\sigma!\gamma!} D_\xi^\sigma D_y^\sigma \Big(p_1(x,\xi) D_\xi^\gamma D_x^\gamma p_2(y,\xi)\Big)\Big|_{y=x}. \tag{3.20}$$

The general term in this sum is equal to

$$\frac{i^{|\sigma|-|\gamma|}}{\sigma!\gamma!} D_\xi^\sigma \Big(p_1(x,\xi) D_\xi^\gamma D_x^{\gamma+\sigma} p_2(x,\xi)\Big).$$

Evaluating this by the product rule

$$D_\xi^\sigma(uv) = \sum_{\alpha+\beta=\sigma} \binom{\sigma}{\alpha} D_\xi^\alpha u \cdot D_\xi^\beta v$$

gives

$$q(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) \sum_{\beta,\gamma} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!} D_\xi^{\beta+\gamma} D_x^{\beta+\gamma+\alpha} p_2(x,\xi). \tag{3.21}$$

That this yields (3.17) follows from the fact that, whenever $|\mu| > 0$,

$$\sum_{\beta+\gamma=\mu} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!} D_\xi^{\beta+\gamma} D_x^{\beta+\gamma+\alpha} p_2(x,\xi) = 0, \tag{3.22}$$

an identity we leave as an exercise.

An alternative approach to a proof of Proposition 3.3 is to compute directly that $p_1(x,D)p_2(x,D) = q(x,D)$, with

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-n} \int p_1(x,\eta)p_2(y,\xi)e^{i(x-y)\cdot(\eta-\xi)}\, d\eta\, dy \\ &= e^{iD_\eta \cdot D_y} p_1(x,\eta)p_2(y,\xi)\big|_{y=x,\eta=\xi}, \end{aligned} \tag{3.23}$$

and then apply an analysis such as used to prove Proposition 3.1. Carrying out this latter approach has the advantage that the hypothesis (3.15) can be weakened to

$$0 \le \delta_2 < \rho_1 \le 1,$$

which is quite natural since the right side of (3.17) is formally asymptotic under such a hypothesis. Also, the symbol expansion (3.17) is more easily seen from (3.23).

Note that if $P_j = p_j(x,D) \in OPS_{\rho,\delta}^{m_j}$ are scalar, and $0 \le \delta < \rho \le 1$, then the leading terms in the expansions of the symbols of P_1P_2 and P_2P_1 agree. It follows that the commutator

$$[P_1, P_2] = P_1P_2 - P_2P_1$$

has order lower than $m_1 + m_2$. In fact, the symbol expansion (3.17) implies

$$P_j \in OPS_{\rho,\delta}^{m_j} \text{ scalar } \Longrightarrow [P_1,P_2] \in OPS_{\rho,\delta}^{m_1+m_2-(\rho-\delta)}. \tag{3.24}$$

Also, looking at the sum over $|\alpha| = 1$ in (3.17), we see that the leading term in the expansion of the symbol of $[P_1, P_2]$ is given in terms of the Poisson bracket:

$$[P_1,P_2] = q(x,D), \quad q(x,\xi) = \frac{1}{i}\{p_1,p_2\}(x,\xi) \text{ mod } S_{\rho,\delta}^{m_1+m_2-2(\rho-\delta)}. \tag{3.25}$$

The Poisson bracket $\{p_1, p_2\}$ is defined by

$$\{p_1, p_2\}(x,\xi) = \sum_j \frac{\partial p_1}{\partial \xi_j}\frac{\partial p_2}{\partial x_j} - \frac{\partial p_1}{\partial x_j}\frac{\partial p_2}{\partial \xi_j}, \tag{3.26}$$

as in §10 of Chapter 1.

The result (3.25) plays an important role in the treatment of Egorov's theorem, in §8.

Exercises

1. Writing $a_j(x,D)$ in the form (1.10), that is,

$$a_j(x,D) = \int \hat{a}_j(q,p) e^{iq\cdot X} e^{ip\cdot D}\, dq\, dp, \tag{3.27}$$

use the formula (1.11) for $e^{ip\cdot D}e^{iq'\cdot X}$ to express $a_1(x,D)a_2(x,D)$ as a $4n$-fold integral. Show that it gives (3.20).

2. If $Q(x,x)$ is any nondegenerate, symmetric, bilinear form on $\mathbb{R}^n$, calculate the kernel $K_Q(x,y,t)$ for which

$$e^{itQ(D,D)}u(x) = \int\limits_{\mathbb{R}^n} K_Q(x,y,t)\, u(y)\, dy. \tag{3.28}$$

In case $x \in \mathbb{R}^n$ is replaced by $(x,\xi) \in \mathbb{R}^{2n}$, use this to verify (3.5). (*Hint*: Diagonalize Q and recall the treatment of $e^{it\Delta}$ in (6.42) of Chapter 3, giving

$$e^{-it\Delta}\delta(x) = (-4\pi it)^{-n/2}\, e^{|x|^2/4it}, \quad x \in \mathbb{R}^n.$$

Compare the treatment of the stationary phase method in Appendix B of Chapter 6.)

3. Establish the identity (3.22), used in the proof of Proposition 3.3. (*Hint*: The left side of (3.22) is equal to

$$\left(\sum_{\beta+\gamma=\mu} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!}\right) D_\xi^\mu D_x^{\mu+\alpha} p_2(x,\xi),$$

so one needs to show that the quantity in parentheses here vanishes if $|\mu| > 0$. To see this, make an expansion of $(z+w)^\mu$, and set $z = (i,\dots,i)$, $w = (-i,\dots,-i)$.)

4. Elliptic operators and parametrices

We say $p(x,D) \in OPS^m_{\rho,\delta}$ is elliptic if, for some $r < \infty$,

$$|p(x,\xi)^{-1}| \le C\langle\xi\rangle^{-m}, \quad \text{for } |\xi| \ge r. \tag{4.1}$$

Thus, if $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ is equal to 0 for $|\xi| \le r$, 1 for $|\xi| \ge 2r$, it follows easily from the chain rule that

$$\psi(\xi)p(x,\xi)^{-1} = q_0(x,\xi) \in S^{-m}_{\rho,\delta}. \tag{4.2}$$

As long as $0 \le \delta < \rho \le 1$, we can apply Proposition 3.3 to obtain

$$\begin{aligned} q_0(x,D)p(x,D) &= I + r_0(x,D), \\ p(x,D)q_0(x,D) &= I + \tilde{r}_0(x,D), \end{aligned} \tag{4.3}$$

with

$$r_0(x,\xi),\ \tilde{r}_0(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta}. \tag{4.4}$$

Using the formal expansion

$$I - r_0(x,D) + r_0(x,D)^2 - \cdots \sim I + s(x,D) \in OPS^0_{\rho,\delta} \tag{4.5}$$

and setting $q(x,D) = (I + s(x,D))q_0(x,D) \in OPS^{-m}_{\rho,\delta}$, we have

$$q(x,D)p(x,D) = I + r(x,D), \quad r(x,\xi) \in S^{-\infty}. \tag{4.6}$$

Similarly, we obtain $\tilde{q}(x,D) \in OPS^{-m}_{\rho,\delta}$ satisfying

$$p(x,D)\tilde{q}(x,D) = I + \tilde{r}(x,D), \quad \tilde{r}(x,\xi) \in S^{-\infty}. \tag{4.7}$$

But evaluating

$$\big(q(x,D)p(x,D)\big)\tilde{q}(x,D) = q(x,D)\big(p(x,D)\tilde{q}(x,D)\big) \tag{4.8}$$

yields $q(x,D) = \tilde{q}(x,D) \bmod OPS^{-\infty}$, so in fact

$$\begin{aligned} q(x,D)p(x,D) &= I \bmod OPS^{-\infty}, \\ p(x,D)q(x,D) &= I \bmod OPS^{-\infty}. \end{aligned} \tag{4.9}$$

We say that $q(x,D)$ is a *two-sided parametrix* for $p(x,D)$.

The parametrix can establish the local regularity of a solution to

$$p(x,D)u = f. \tag{4.10}$$

Suppose $u, f \in \mathcal{S}'(\mathbb{R}^n)$ and $p(x,D) \in OPS^m_{\rho,\delta}$ is elliptic, with $0 \le \delta < \rho \le 1$. Constructing $q(x,D) \in OPS^{-m}_{\rho,\delta}$ as in (4.6), we have

$$u = q(x,D)f - r(x,D)u. \tag{4.11}$$

Now a simple analysis parallel to (1.7) implies that

$$R \in OPS^{-\infty} \Longrightarrow R : \mathcal{E}' \longrightarrow \mathcal{S}. \tag{4.12}$$

By duality, since taking adjoints preserves $OPS^{-\infty}$,

$$R \in OPS^{-\infty} \Longrightarrow R : \mathcal{S}' \longrightarrow C^\infty. \tag{4.13}$$

Thus (4.11) implies

$$u = q(x,D)f \bmod C^\infty. \tag{4.14}$$

Applying the pseudolocal property to (4.10) and (4.14), we have the following elliptic regularity result.

Proposition 4.1. *If $p(x, D) \in OPS^m_{\rho,\delta}$ is elliptic and $0 \le \delta < \rho \le 1$, then, for any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\text{sing supp } p(x, D)u = \text{ sing supp } u. \tag{4.15}$$

More refined elliptic regularity involves keeping track of Sobolev space regularity. As we have the parametrix, this will follow simply from mapping properties of pseudodifferential operators, to be established in subsequent sections.

Exercises

1. Give the details of the implication (4.1) $\Rightarrow$ (4.2) when $p(x, \xi) \in S^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$. Include the case where $p(x, \xi)$ is a $k \times k$ matrix-valued function, using such identities as
$$\frac{\partial}{\partial x_j}\, p(x, \xi)^{-1} = -p(x, \xi)^{-1}\, \frac{\partial p}{\partial x_j}\, p(x, \xi)^{-1}.$$
2. On $\mathbb{R} \times \mathbb{R}^n$, consider the operator $P = \partial/\partial t - L(x, D_x)$, where
$$L(x, D_x) = \sum a_{jk}(x)\, \partial_j \partial_k u + \sum b_j(x)\, \partial_j u + c(x)u.$$
Assume that the coefficients are smooth and bounded, with all their derivatives, and that L satisfies the strong ellipticity condition
$$-L_2(x, \xi) = \sum a_{jk}(x)\xi_j\xi_k \ge C|\xi|^2, \quad C > 0.$$
Show that
$$\Bigl(i\tau - L_2(x, \xi) + 1\Bigr)^{-1} = E(t, x, \tau, \xi) \in S^{-1}_{1/2,0}.$$
Show that $E(t, x, D)P = A_1(t, x, D)$ and $PE(t, x, D) = A_2(t, x, D)$, where $A_j \in OPS^0_{1/2,0}$ are elliptic. Then, using Proposition 4.1, construct a parametrix for P, belonging to $OPS^{-1}_{1/2,0}$.
3. Assume $-n < m < 0$, and suppose $P = p(x, D) \in OPS^m_{cl}$ has Schwartz kernel $K(x, y) = L(x, x - y)$. Suppose that, at $x_0 \in \mathbb{R}^n$,
$$L(x_0, z) \sim a|z|^{-m-n} + \cdots, \quad z \to 0,$$
with $a \ne 0$, the remainder terms being progressively smoother. Show that
$$p_m(x_0, \xi) = b|\xi|^m, \quad b \ne 0,$$
and hence that P is elliptic near x_0.
4. Let $P = (P_{jk})$ be a $K \times K$ matrix of operators in OPS^*. It is said to be "elliptic in the sense of Douglis and Nirenberg" if there are numbers a_j, b_j, $1 \le j \le K$, such that $P_{jk} \in OPS^{a_j+b_k}$ and the matrix of principal symbols has nonvanishing determinant (homogeneous of order $\sum(a_j+b_j)$), for $\xi \ne 0$. If Λ^s is as in (1.17), let A be a $K\times K$ diagonal matrix with diagonal entries Λ^{-a_j},

and let B be diagonal, with entries Λ^{-b_j}. Show that this "DN-ellipticity" of P is equivalent to the ellipticity of APB in OPS^0.

5. L^2-estimates

Here we want to obtain L^2-estimates for pseudodifferential operators. The following simple basic estimate will get us started.

Proposition 5.1. *Let (X, μ) be a measure space. Suppose $k(x,y)$ is measurable on $X \times X$ and*

$$\int_X |k(x,y)|\ d\mu(x) \le C_1, \qquad \int_X |k(x,y)|\ d\mu(y) \le C_2, \tag{5.1}$$

for all y and x, respectively. Then

$$Tu(x) = \int k(x,y)u(y)\ d\mu(y) \tag{5.2}$$

satisfies

$$\|Tu\|_{L^p} \le C_1^{1/p}\ C_2^{1/q}\ \|u\|_{L^p}, \tag{5.3}$$

for $p \in [1, \infty]$, with

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{5.4}$$

This is proved in Appendix A on functional analysis; see Proposition 5.1 there. To apply this result when $X = \mathbb{R}^n$ and $k = K$ is the Schwartz kernel of $p(x, D) \in OPS^m_{\rho,\delta}$, note from the proof of Proposition 2.1 that

$$|K(x,y)| \le C_N |x-y|^{-N}, \quad \text{for } |x-y| \ge 1 \tag{5.5}$$

as long as $\rho > 0$, while

$$|K(x,y)| \le C|x-y|^{-(n-1)}, \quad \text{for } |x-y| \le 1 \tag{5.6}$$

as long as $m < -n + \rho(n-1)$. (Recall that this last estimate is actually rather crude.) Hence we have the following preliminary result.

Lemma 5.2. *If $p(x,D) \in OPS^m_{\rho,\delta}$, $\rho > 0$, and $m < -n + \rho(n-1)$, then*

$$p(x,D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 \le p \le \infty. \tag{5.7}$$

If $p(x,D) \in OPS^m_{1,\delta}$, then (5.7) holds for $m < 0$.

The last observation follows from the improvement of (5.6) given in (2.5).

Our main goal in this section is to prove the following.

Theorem 5.3. *If $p(x,D) \in OPS^0_{\rho,\delta}$ and $0 \le \delta < \rho \le 1$, then*

$$p(x,D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{5.8}$$

The proof we give, following [Ho5], begins with the following result.

Lemma 5.4. *If $p(x,D) \in OPS^{-a}_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, and $a > 0$, then (5.8) holds.*

Proof. Since $\|Pu\|^2_{L^2} = (P^*Pu, u)$, it suffices to prove that some power of $p(x,D)^*p(x,D) = Q$ is bounded on L^2. But $Q^k \in OPS^{-2ka}_{\rho,\delta}$, so for k large enough this follows from Lemma 5.2.

To proceed with the proof of Theorem 5.3, set $q(x,D) = p(x,D)^*p(x,D) \in OPS^0_{\rho,\delta}$, and suppose $|q(x,\xi)| \le M - b$, $b > 0$, so

$$M - \text{ Re } q(x,\xi) \ge b > 0. \tag{5.9}$$

In the matrix case, take Re $q(x,\xi) = (1/2)\big(q(x,\xi) + q(x,\xi)^*\big)$. It follows that

$$A(x,\xi) = \big(M - \text{ Re } q(x,\xi)\big)^{1/2} \in S^0_{\rho,\delta} \tag{5.10}$$

and

$$A(x,D)^*A(x,D) = M - q(x,D) + r(x,D), \quad r(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}. \tag{5.11}$$

Applying Lemma 5.4 to $r(x,D)$, we have

$$\begin{aligned} M\|u\|^2_{L^2} - \|p(x,D)u\|^2_{L^2} &= \|A(x,D)u\|^2_{L^2} - (r(x,D)u,u) \\ &\ge -C\|u\|^2_{L^2}, \end{aligned} \tag{5.12}$$

or

$$\|p(x,D)u\|^2 \le (M+C)\|u\|^2_{L^2}, \tag{5.13}$$

finishing the proof.

From these L^2-estimates easily follow L^2-Sobolev space estimates. Recall from Chapter 4 that the Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}. \tag{5.14}$$

Equivalently, with

$$\Lambda^s u = \int \langle\xi\rangle^s \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi; \quad \Lambda^s \in OPS^s, \tag{5.15}$$

we have

$$H^s(\mathbb{R}^n) = \Lambda^{-s} L^2(\mathbb{R}^n). \tag{5.16}$$

The operator calculus easily gives the next proposition:

Proposition 5.5. *If* $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, $m, s \in \mathbb{R}$, *then*

$$p(x,D) : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^m). \tag{5.17}$$

Given Proposition 5.5, one easily obtains the Sobolev regularity of solutions to the elliptic equations studied in §4.

Calderon and Vaillancourt sharpened Theorem 5.3, showing that

$$p(x,\xi) \in S^0_{\rho,\rho},\ 0 \le \rho < 1 \Longrightarrow p(x,D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{5.18}$$

This result, particularly for $\rho = 1/2$, has played an important role in linear PDE, especially in the study of subelliptic operators, but it will not be used in this book. The case $\rho = 0$ is treated in the exercises below.

Another important extension of Theorem 5.3 is that $p(x,D)$ is bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, when $p(x,\xi) \in S^0_{1,\delta}$. Similarly, Proposition 5.5 extends to a result on L^p-Sobolev spaces, in the case $\rho = 1$. This is important for applications to nonlinear PDE, and will be proved in Chapter 13.

Exercises

Exercises 1–7 present an approach to a proof of the Calderon-Vaillancourt theorem, (5.18), in the case $\rho = 0$. This approach is due to H. O. Cordes [Cor]; see also T. Kato [K] and R. Howe [How]. In these exercises, we assume that $U(y)$ is a (measurable) unitary, operator-valued function on a measure space Y, operating on a Hilbert space $\mathcal{H}$. Assume that, for $f, g \in \mathcal{V}$, a dense subset of $\mathcal{H}$,

$$\int_Y \Bigl|(U(y)f, g)\Bigr|^2 \, dm(y) = C_0 \|f\|^2 \ \|g\|^2. \tag{5.19}$$

1. Let $\varphi_0 \in \mathcal{H}$ be a unit vector, and set $\varphi_y = U(y)\varphi_0$. Show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$C_0^2 (Tf_1, f_2) = \int_Y \int_Y L_T(y,y')\ (f_1, \varphi_{y'})\ (\varphi_y, f_2)\ dm(y)\ dm(y'), \tag{5.20}$$

where

$$L_T(y,y') = (T\varphi_{y'}, \varphi_y). \tag{5.21}$$

(*Hint*: Start by showing that $\int (f_1,\varphi_y)(\varphi_y, f_2)\ dm(y) = C_0(f_1,f_2)$.)
A statement equivalent to (5.20) is

$$T = \iint L_T(y,y')\ U(y)\Phi_0 U(y')\ dm(y)\ dm(y'), \tag{5.22}$$

where Φ_0 is the orthogonal projection of $\mathcal{H}$ onto the span of φ_0.

2. For a partial converse, suppose L is measurable on $Y \times Y$ and

$$\int |L(y,y')|\, dm(y) \le C_1, \quad \int |L(y,y')|\, dm(y') \le C_1. \tag{5.23}$$

Define

$$T_L = \iint L(y,y')\, U(y)\Phi_0 U(y')^*\, dm(y)\, dm(y'). \tag{5.24}$$

Show that the operator norm of T_L on $\mathcal{H}$ has the estimate

$$\|T_L\| \le C_0^2 C_1.$$

3. If G is a trace class operator, and we set

$$T_{L,G} = \iint L(y,y')\, U(y)GU(y')^*\, dm(y)\, dm(y'), \tag{5.25}$$

show that

$$\|T_{L,G}\| \le C_0^2 C_1\, \|G\|_{\mathrm{TR}}. \tag{5.26}$$

(*Hint*: In case $G = G^*$, diagonalize G and use Exercise 2.)

4. Suppose $b \in L^\infty(Y)$ and we set

$$T^{\#}_{b,G} = \int b(y)\, U(y)GU(y)^*\, dm(y). \tag{5.27}$$

Show that

$$\|T^{\#}_{b,G}\| \le C_0 \|b\|_{L^\infty}\, \|G\|_{\mathrm{TR}}. \tag{5.28}$$

5. Let $Y = \mathbb{R}^{2n}$, with Lebesgue measure, $y = (q,p)$. Set $U(y) = e^{iq\cdot X} e^{ip\cdot D} = \tilde{\pi}(0,q,p)$, as in Exercises 1 and 2 of §1. Show that the identity (5.19) holds, for $f, g \in L^2(\mathbb{R}^n) = \mathcal{H}$, with $C_0 = (2\pi)^{-n}$. (*Hint*: Make use of the Plancherel theorem.)
6. Deduce that if $a(x,D)$ is a trace class operator,

$$\|(b * a)(x,D)\|_{\mathcal{L}(L^2)} \le C\|b\|_{L^\infty}\, \|a(x,D)\|_{\mathrm{TR}}. \tag{5.29}$$

(*Hint*: Look at Exercises 3–4 of §1.)

7. Suppose $p(x,\xi) \in S^0_{0,0}$. Set

$$a(x,\xi) = \psi(x)\psi(\xi), \quad b(x,\xi) = (1-\Delta_x)^k (1-\Delta_\xi)^k p(x,\xi), \tag{5.30}$$

where k is a positive integer, $\hat{\psi}(\xi) = \langle \xi \rangle^{-2k}$. Show that if k is chosen large enough, then $a(x,D)$ is trace class. Note that, for all $k \in \mathbb{Z}^+$, $b \in L^\infty(\mathbb{R}^{2n})$, provided $p \in S^0_{0,0}$. Show that

$$p(x,D) = (b * a)(x,D), \tag{5.31}$$

and deduce the $\rho = 0$ case of the Calderon-Vaillancourt estimate (5.19).

8. Sharpen the results of problems 3–4 above, showing that

$$\|T_{L,G}\|_{\mathcal{L}(\mathcal{H})} \le C_0^2\, \|L\|_{\mathcal{L}(L^2(Y))}\, \|G\|_{\mathrm{TR}}. \tag{5.32}$$

This is stronger than (5.26) in view of Proposition 5.1.

6. Gårding's inequality

In this section we establish a fundamental estimate, first obtained by L. Gårding in the case of differential operators.

Theorem 6.1. *Assume* $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, *and*

$$\text{Re } p(x,\xi) \ge C|\xi|^m, \quad \text{for } |\xi| \text{ large.} \tag{6.1}$$

Then, for any $s \in \mathbb{R}$, *there are* C_0, C_1 *such that, for* $u \in H^{m/2}(\mathbb{R}^n)$,

$$\text{Re } \big(p(x,D)u,u\big) \ge C_0\|u\|^2_{H^{m/2}} - C_1\|u\|^2_{H^s}. \tag{6.2}$$

Proof. Replacing $p(x,D)$ by $\Lambda^{-m/2}p(x,D)\Lambda^{-m/2}$, we can suppose without loss of generality that $m = 0$. Then, as in the proof of Theorem 5.3, take

$$A(x,\xi) = \Big(\text{Re } p(x,\xi) - \frac{1}{2}C\Big)^{1/2} \in S^0_{\rho,\delta}, \tag{6.3}$$

so

$$\begin{gathered} A(x,D)^*A(x,D) = \text{ Re } p(x,D) - \frac{1}{2}C + r(x,D), \\ r(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}. \end{gathered} \tag{6.4}$$

This gives

$$\begin{aligned} \text{Re } (p(x,D)u,u) &= \|A(x,D)u\|^2_{L^2} + \frac{1}{2}C\|u\|^2_{L^2} + \big(r(x,D)u,u\big) \\ &\ge \frac{1}{2}C\|u\|^2_{L^2} - C_1\|u\|^2_{H^s} \end{aligned} \tag{6.5}$$

with $s = -(\rho-\delta)/2$, so (6.2) holds in this case. If $s < -(\rho-\delta)/2 = s_0$, use the simple estimate

$$\|u\|^2_{H^{s_0}} \le \varepsilon\|u\|^2_{L^2} + C(\varepsilon)\|u\|^2_{H^s} \tag{6.6}$$

to obtain the desired result in this case.

This Gårding inequality has been improved to a sharp Gårding inequality, of the form

$$\text{Re } \big(p(x,D)u,u\big) \ge -C\|u\|^2_{L^2} \quad \text{when Re } p(x,\xi) \ge 0, \tag{6.7}$$

first for scalar $p(x,\xi) \in S^1_{1,0}$, by Hörmander, then for matrix-valued symbols, with Re $p(x,\xi)$ standing for $(1/2)\big(p(x,\xi) + p(x,\xi)^*\big)$, by P. Lax and L. Nirenberg. Proofs and some implications can be found in Vol. 3 of [Ho5], and in [T1] and [Tre]. A very strong improvement due to C. Fefferman and D. Phong [FP] is that (6.7) holds for scalar $p(x,\xi) \in S^2_{1,0}$. See also [Ho5] and [F] for further discussion.

Exercises

1. Suppose $m > 0$ and $p(x,D) \in OPS^m_{1,0}$ has a symbol satisfying (6.1). Examine the solvability of
$$\frac{\partial u}{\partial t} = p(x,D)u,$$
for $u = u(t,x)$, $u(0,x) = f \in H^s(\mathbb{R}^n)$.
(*Hint*: Look ahead at §7 for some useful techniques. Solve
$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon p(x,D) J_\varepsilon u_\varepsilon$$
and estimate $(d/dt)\|\Lambda^s u_\varepsilon(t)\|^2_{L^2}$, making use of Gårding's inequality.)

7. Hyperbolic evolution equations

In this section we examine first-order systems of the form

$$\frac{\partial u}{\partial t} = L(t,x,D_x)u + g(t,x), \quad u(0) = f. \tag{7.1}$$

We assume $L(t,x,\xi) \in S^1_{1,0}$, with smooth dependence on t, so

$$|D_t^j D_x^\beta D_\xi^\alpha L(t,x,\xi)| \le C_{j\alpha\beta}\langle\xi\rangle^{1-|\alpha|}. \tag{7.2}$$

Here $L(t,x,\xi)$ is a $K \times K$ matrix-valued function, and we make the hypothesis of symmetric hyperbolicity:

$$L(t,x,\xi)^* + L(t,x,\xi) \in S^0_{1,0}. \tag{7.3}$$

We suppose $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $g \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Our strategy will be to obtain a solution to (7.1) as a limit of solutions u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L J_\varepsilon u_\varepsilon + g, \quad u_\varepsilon(0) = f, \tag{7.4}$$

where

$$J_\varepsilon = \varphi(\varepsilon D_x), \tag{7.5}$$

for some $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 1$. The family of operators J_ε is called a *Friedrichs mollifier*. Note that, for any $\varepsilon > 0$, $J_\varepsilon \in OPS^{-\infty}$, while, for $\varepsilon \in (0,1]$, J_ε is bounded in $OPS^0_{1,0}$.

For any $\varepsilon > 0$, $J_\varepsilon L J_\varepsilon$ is a bounded linear operator on each H^s, and solvability of (7.4) is elementary. Our next task is to obtain estimates on u_ε, independent of $\varepsilon \in (0,1]$. Use the norm $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}$. We derive an estimate for

$$\frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} = 2\,\mathrm{Re}\,(\Lambda^s J_\varepsilon L J_\varepsilon u_\varepsilon, \Lambda^s u_\varepsilon) + 2\,\mathrm{Re}\,(\Lambda^s g, \Lambda^s u_\varepsilon). \tag{7.6}$$

Write the first two terms on the right as the real part of

$$2(L\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) + 2([\Lambda^s, L]J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon). \tag{7.7}$$

By (7.3), $L + L^* = B(t, x, D) \in OPS^0_{1,0}$, so the first term in (7.7) is equal to

$$\big(B(t, x, D)\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\big) \le C\|J_\varepsilon u_\varepsilon\|^2_{H^s}. \tag{7.8}$$

Meanwhile, $[\Lambda^s, L] \in OPS^s_{1,0}$, so the second term in (7.7) is also bounded by the right side of (7.8). Applying Cauchy's inequality to $2(\Lambda^s g, \Lambda^s u_\varepsilon)$, we obtain

$$\frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} \le C\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} + C\|g(t)\|^2_{H^s}. \tag{7.9}$$

Thus Gronwall's inequality yields an estimate

$$\|u_\varepsilon(t)\|^2_{H^s} \le C(t)\big[\|f\|^2_{H^s} + \|g\|^2_{C([0,t],H^s)}\big], \tag{7.10}$$

independent of $\varepsilon \in (0, 1]$. We are now prepared to establish the following existence result.

Proposition 7.1. *If (7.1) is symmetric hyperbolic and*

$$f \in H^s(\mathbb{R}^n), \quad g \in C(\mathbb{R}, H^s(\mathbb{R}^n)), \quad s \in \mathbb{R},$$

then there is a solution u to (7.1), satisfying

$$u \in L^\infty_{\text{loc}}(\mathbb{R}, H^s(\mathbb{R}^n)) \cap \text{ Lip } (\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{7.11}$$

Proof. Take $I = [-T, T]$. The bounded family

$$u_\varepsilon \in C(I, H^s) \cap C^1(I, H^{s-1})$$

will have a weak limit point u satisfying (7.11), and it is easy to verify that such u solves (7.1). As for the bound on $[-T, 0]$, this follows from the invariance of the class of hyperbolic equations under time reversal.

Analogous energy estimates can establish the uniqueness of such a solution u and rates of convergence of $u_\varepsilon \to u$ as $\varepsilon \to 0$. Also, (7.11) can be improved to

$$u \in C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{7.12}$$

To see this, let $f_j \in H^{s+1}$, $f_j \to f$ in H^s, and let u_j solve (7.1) with $u_j(0) = f_j$. Then each u_j belongs to $L^\infty_{\text{loc}}(\mathbb{R}, H^{s+1}) \cap \text{ Lip}(\mathbb{R}, H^s)$, so in particular each $u_j \in C(\mathbb{R}, H^s)$. Now $v_j = u - u_j$ solves (7.1) with $v_j(0) = f - f_j$, and $\|f - f_j\|_{H^s} \to 0$ as $j \to \infty$, so estimates arising in the proof of Proposition 7.1 imply that $\|v_j(t)\|_{H^s} \to 0$ locally uniformly in t, giving $u \in C(\mathbb{R}, H^s)$.

There are other notions of hyperbolicity. In particular, (7.1) is said to be *symmetrizable hyperbolic* if there is a $K \times K$ matrix-valued $S(t, x, \xi) \in S^0_{1,0}$

that is positive-definite and such that $S(t,x,\xi)L(t,x,\xi) = \tilde{L}(t,x,\xi)$ satisfies (7.3). Proposition 7.1 extends to the case of symmetrizable hyperbolic systems. Again, one obtains u as a limit of solutions u_ε to (7.4). There is one extra ingredient in the energy estimates. In this case, construct $S(t) \in OPS^0_{1,0}$, positive-definite, with symbol equal to $S(t,x,\xi)$ mod $S^{-1}_{1,0}$. For the energy estimates, replace the left side of (7.6) by

$$\frac{d}{dt}\big(\Lambda^s u_\varepsilon(t), S(t)\Lambda^s u_\varepsilon(t)\big)_{L^2}, \tag{7.13}$$

which can be estimated in a fashion similar to (7.7)–(7.9).

A $K \times K$ system of the form (7.1) with $L(t,x,\xi) \in S^1_{cl}$ is said to be *strictly hyperbolic* if its principal symbol $L_1(t,x,\xi)$, homogeneous of degree 1 in ξ, has K distinct, purely imaginary eigenvalues, for each x and each $\xi \neq 0$. The results above apply in this case, in view of:

Proposition 7.2. *Whenever (7.1) is strictly hyperbolic, it is symmetrizable.*

Proof. If we denote the eigenvalues of $L_1(t,x,\xi)$ by $i\lambda_\nu(t,x,\xi)$, ordered so that $\lambda_1(t,x,\xi) < \cdots < \lambda_K(t,x,\xi)$, then λ_ν are well-defined C^∞-functions of (t,x,ξ), homogeneous of degree 1 in ξ. If $P_\nu(t,x,\xi)$ are the projections onto the $-i\lambda_\nu$-eigenspaces of L_1^*,

$$P_\nu(t,x,\xi) = \frac{1}{2\pi i}\int_{\gamma_\nu} \big(\zeta - L_1(t,x,\xi)^*\big)^{-1}\, d\zeta, \tag{7.14}$$

where γ_ν is a small circle about $-i\lambda_\nu(t,x,\xi)$, then P_ν is smooth and homogeneous of degree 0 in ξ. Then

$$S(t,x,\xi) = \sum_j P_j(t,x,\xi)^* P_j(t,x,\xi) \tag{7.15}$$

gives the desired symmetrizer.

Higher-order, strictly hyperbolic PDE can be reduced to strictly hyperbolic, first-order systems of this nature. Thus one has an analysis of solutions to such higher-order hyperbolic equations.

Exercises

1. Carry out the reduction of a strictly hyperbolic PDE of order m to a first-order system of the form (7.1). Starting with

$$Lu = \frac{\partial^m u}{\partial y^m} + \sum_{j=0}^{m-1} A_j(y,x,D_x)\frac{\partial^j u}{\partial y^j},$$

where $A_j(y, x, D)$ has order $\leq m - j$, form $v = (v_1, \dots, v_m)^t$ with

$$v_1 = \Lambda^{m-1} u, \dots, v_j = \partial_y^{j-1} \Lambda^{m-j} u, \dots, v_m = \partial_y^{m-1} u,$$

to pass from $Lu = f$ to

$$\frac{\partial v}{\partial y} = K(y, x, D_x) v + F,$$

with $F = (0, \dots, 0, f)^t$. Give an appropriate definition of strict hyperbolicity in this context, and show that this first-order system is strictly hyperbolic provided L is.

2. Fix $r > 0$. Let $\gamma_r \in \mathcal{E}'(\mathbb{R}^2)$ denote the unit mass density on the circle of radius r:

$$\langle u, \gamma_r \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r \cos\theta, r \sin\theta) \, d\theta.$$

Let $\Gamma_r u = \gamma_r * u$. Show that there exist $A_r(\xi) \in S^{-1/2}(\mathbb{R}^2)$ and $B_r(\xi) \in S^{1/2}(\mathbb{R}^2)$, such that

$$\Gamma_r = A_r(D) \cos r\sqrt{-\Delta} + B_r(D) \, \frac{\sin r\sqrt{-\Delta}}{\sqrt{-\Delta}}. \tag{7.16}$$

(*Hint*: See Exercise 1 in §7 of Chapter 6.)

8. Egorov's theorem

We want to examine the behavior of operators obtained by conjugating a pseudodifferential operator $P_0 \in OPS^m_{1,0}$ by the solution operator to a scalar hyperbolic equation of the form

$$\frac{\partial u}{\partial t} = iA(t, x, D_x) u, \tag{8.1}$$

where we assume $A = A_1 + A_0$ with

$$A_1(t, x, \xi) \in S^1_{cl} \text{ real}, \quad A_0(t, x, \xi) \in S^0_{cl}. \tag{8.2}$$

We suppose $A_1(t, x, \xi)$ is homogeneous in ξ, for $|\xi| \geq 1$. Denote by $S(t, s)$ the solution operator to (8.1), taking $u(s)$ to $u(t)$. This is a bounded operator on each Sobolev space H^σ, with inverse $S(s, t)$. Set

$$P(t) = S(t, 0) P_0 S(0, t). \tag{8.3}$$

We aim to prove the following result of Y. Egorov.

Theorem 8.1. *If $P_0 = p_0(x, D) \in OPS^m_{1,0}$, then for each t, $P(t) \in OPS^m_{1,0}$, modulo a smoothing operator. The principal symbol of $P(t)$ (mod $S^{m-1}_{1,0}$) at a point (x_0, ξ_0) is equal to $p_0(y_0, \eta_0)$, where (y_0, η_0) is obtained from (x_0, ξ_0) by following the flow $\mathcal{C}(t)$ generated by the (time-dependent) Hamiltonian* *vector field*

$$H_{A_1(t,x,\xi)} = \sum_{j=1}^{n} \Bigl(\frac{\partial A_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial A_1}{\partial x_j} \frac{\partial}{\partial \xi_j} \Bigr). \tag{8.4}$$

To start the proof, differentiating (8.3) with respect to t yields

$$P'(t) = i[A(t,x,D), P(t)], \quad P(0) = P_0. \tag{8.5}$$

We will construct an approximate solution $Q(t)$ to (8.5) and then show that $Q(t) - P(t)$ is a smoothing operator.

So we are looking for $Q(t) = q(t,x,D) \in OPS^m_{1,0}$, solving

$$Q'(t) = i[A(t,x,D), Q(t)] + R(t), \quad Q(0) = P_0, \tag{8.6}$$

where $R(t)$ is a smooth family of operators in $OPS^{-\infty}$. We do this by constructing the symbol $q(t,x,\xi)$ in the form

$$q(t,x,\xi) \sim q_0(t,x,\xi) + q_1(t,x,\xi) + \cdots . \tag{8.7}$$

Now the symbol of $i[A, Q(t)]$ is of the form

$$H_{A_1} q + \{A_0, q\} + i \sum_{|\alpha| \ge 2} \frac{i^{|\alpha|}}{\alpha!} \Bigl(A^{(\alpha)} q_{(\alpha)} - q^{(\alpha)} A_{(\alpha)} \Bigr), \tag{8.8}$$

where $A^{(\alpha)} = D^\alpha_\xi A$, $A_{(\alpha)} = D^\alpha_x A$, and so on. Since we want the difference between this and $\partial q/\partial t$ to have order $-\infty$, this suggests defining $q_0(t,x,\xi)$ by

$$\Bigl(\frac{\partial}{\partial t} - H_{A_1} \Bigr) q_0(t,x,\xi) = 0, \quad q_0(0,x,\xi) = p_0(x,\xi). \tag{8.9}$$

Thus $q_0(t,x_0,\xi_0) = p_0(y_0,\eta_0)$, as in the statement of the theorem; we have $q_0(t,x,\xi) \in S^m_{1,0}$. The equation (8.9) is called a *transport equation*. Recursively, we obtain transport equations

$$\Bigl(\frac{\partial}{\partial t} - H_{A_1} \Bigr) q_j(t,x,\xi) = b_j(t,x,\xi), \quad q_j(0,x,\xi) = 0, \tag{8.10}$$

for $j \ge 1$, with solutions in $S^{m-j}_{1,0}$, leading to a solution to (8.6).

Finally, we show that $P(t) - Q(t)$ is a smoothing operator. Equivalently, we show that, for any $f \in H^\sigma(\mathbb{R}^n)$,

$$v(t) - w(t) = S(t,0)P_0 f - Q(t)S(t,0)f \in H^\infty(\mathbb{R}^n), \tag{8.11}$$

where $H^\infty(\mathbb{R}^n) = \cap_s H^s(\mathbb{R}^n)$. Note that

$$\frac{\partial v}{\partial t} = iA(t,x,D)v, \quad v(0) = P_0 f, \tag{8.12}$$

while use of (8.6) gives

$$\frac{\partial w}{\partial t} = iA(t,x,D)w + g, \quad w(0) = P_0 f, \tag{8.13}$$

where

$$g = R(t)S(t,0)w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n)). \tag{8.14}$$

Hence

$$\frac{\partial}{\partial t}(v-w) = iA(t,x,D)(v-w) - g, \quad v(0)-w(0) = 0. \tag{8.15}$$

Thus energy estimates for hyperbolic equations yield $v(t)-w(t) \in H^\infty$, for any $f \in H^\sigma(\mathbb{R}^n)$, completing the proof.

A check of the proof shows that

$$P_0 \in OPS^m_{cl} \Longrightarrow P(t) \in OPS^m_{cl}. \tag{8.16}$$

Also, the proof readily extends to yield the following:

Proposition 8.2. *With $A(t,x,D)$ as before,*

$$P_0 \in OPS^m_{\rho,\delta} \Longrightarrow P(t) \in OPS^m_{\rho,\delta} \tag{8.17}$$

provided

$$\rho > \frac{1}{2}, \quad \delta = 1-\rho. \tag{8.18}$$

One needs $\delta = 1-\rho$ to ensure that $p(\mathcal{C}(t)(x,\xi)) \in S^m_{\rho,\delta}$, and one needs $\rho > \delta$ to ensure that the transport equations generate $q_j(t,x,\xi)$ of progressively lower order.

Exercises

1. Let $\chi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that is a linear map outside some compact set. Define $\chi^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $\chi^* f(x) = f\Big(\chi(x)\Big)$. Show that

$$P \in OPS^m_{1,0} \Longrightarrow (\chi*)^{-1} P \chi^* \in OPS^m_{1,0}. \tag{8.19}$$

   (*Hint*: Reduce to the case where χ is homotopic to a linear map through diffeomorphisms, and show that the result in that case is a special case of Theorem 8.1, where $A(t,x,D)$ is a t-dependent family of real vector fields on $\mathbb{R}^n$.)

2. Let $a \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ be real-valued, and $\nabla\varphi \neq 0$ on supp a. If $P \in OPS^m$, show that

$$P\Big(a\, e^{i\lambda\varphi}\Big) = b(x,\lambda)\, e^{i\lambda\varphi(x)}, \tag{8.20}$$

   where

$$b(x,\lambda) \sim \lambda^m \Big[b_0^\pm(x) + b_1^\pm(x)\lambda^{-1} + \cdots\Big], \quad \lambda \to \pm\infty. \tag{8.21}$$

   (*Hint*: Using a partition of unity and Exercise 1, reduce to the case $\varphi(x) = x\cdot\xi$, for some $\xi \in \mathbb{R}^n \setminus 0$.)

3. If a and φ are as in Exercise 2 above and Γ_r is as in Exercise 2 of §7, show that, mod $O(\lambda^{-\infty})$,

$$\Gamma_r\Big(a\ e^{i\lambda\varphi}\Big) = \cos r\sqrt{-\Delta}\Big(A_r(x,\lambda)e^{i\lambda\varphi}\Big) + \frac{\sin r\sqrt{-\Delta}}{\sqrt{-\Delta}}\Big(B_r(x,\lambda)e^{i\lambda\varphi}\Big), \tag{8.22}$$

where

$$A_r(x,\lambda) \sim \lambda^{-1/2}\Big[a^{\pm}_{0r}(x) + a^{\pm}_{1r}(x)\lambda^{-1} + \cdots\Big],$$
$$B_r(x,\lambda) \sim \lambda^{1/2}\Big[b^{\pm}_{0r}(x) + b^{\pm}_{1r}(x)\lambda^{-1} + \cdots\Big],$$

as $\lambda \to \pm\infty$.

9. Microlocal regularity

We define the notion of wave front set of a distribution $u \in H^{-\infty}(\mathbb{R}^n) = \cup_s H^s(\mathbb{R}^n)$, which refines the notion of singular support. If $p(x,\xi) \in S^m$ has principal symbol $p_m(x,\xi)$, homogeneous in ξ, then the characteristic set of $P = p(x,D)$ is given by

$$\text{Char } P = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : p_m(x,\xi) = 0\}. \tag{9.1}$$

If $p_m(x,\xi)$ is a $K \times K$ matrix, take the determinant. Equivalently, (x_0,ξ_0) is noncharacteristic for P, or P is elliptic at (x_0,ξ_0), if $|p(x,\xi)^{-1}| \le C|\xi|^{-m}$, for (x,ξ) in a small conic neighborhood of (x_0,ξ_0) and $|\xi|$ large. By definition, a conic set is invariant under the dilations $(x,\xi) \mapsto (x,r\xi)$, $r \in (0,\infty)$. The wave front set is defined by

$$\text{WF}(u) = \bigcap\{\text{Char } P : P \in OPS^0,\ Pu \in C^\infty\}. \tag{9.2}$$

Clearly, $\text{WF}(u)$ is a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$.

Proposition 9.1. *If π is the projection $(x,\xi) \mapsto x$, then*

$$\pi(WF(u)) = \text{ sing supp } u.$$

Proof. If $x_0 \notin$ sing supp u, there is a $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ near x_0, such that $\varphi u \in C_0^\infty(\mathbb{R}^n)$. Clearly, $(x_0,\xi) \notin$ Char φ for any $\xi \neq 0$, so $\pi(WF(u)) \subset$ sing supp u.

Conversely, if $x_0 \notin \pi(WF(u))$, then for any $\xi \neq 0$ there is a $Q \in OPS^0$ such that $(x_0,\xi) \notin$ Char Q and $Qu \in C^\infty$. Thus we can construct finitely many $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$ and each (x_0,ξ) (with $|\xi| = 1$) is noncharacteristic for some Q_j. Let $Q = \sum Q_j^* Q_j \in OPS^0$. Then Q is elliptic near x_0 and $Qu \in C^\infty$, so u is C^∞ near x_0.

We define the associated notion of ES(P) for a pseudodifferential operator. Let U be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. We say that

$p(x,\xi) \in S^m_{\rho,\delta}$ has order $-\infty$ on U if for each closed conic set V of U we have estimates, for each N,

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta NV} \langle \xi \rangle^{-N}, \quad (x,\xi) \in V. \tag{9.3}$$

If $P = p(x,D) \in OPS^m_{\rho,\delta}$, we define the essential support of P (and of $p(x,\xi)$) to be the smallest closed conic set on the complement of which $p(x,\xi)$ has order $-\infty$. We denote this set by $\mathrm{ES}(P)$.

From the symbol calculus of §3, it follows easily that

$$\mathrm{ES}(P_1P_2) \subset \mathrm{ES}(P_1) \cap \mathrm{ES}(P_2) \tag{9.4}$$

provided $P_j \in OPS^{m_j}_{\rho_j,\delta_j}$ and $\rho_1 > \delta_2$. To relate $\mathrm{WF}(Pu)$ to $\mathrm{WF}(u)$ and $\mathrm{ES}(P)$, we begin with the following.

Lemma 9.2. *Let $u \in H^{-\infty}(\mathbb{R}^n)$, and suppose that U is a conic open set satisfying*

$$WF(u) \cap U = \emptyset.$$

If $P \in OPS^m_{\rho,\delta}$, $\rho > 0$, $\delta < 1$, and $ES(P) \subset U$, then $Pu \in C^\infty$.

Proof. Taking $P_0 \in OPS^0$ with symbol identically 1 on a conic neighborhood of $\mathrm{ES}(P)$, so $P = PP_0$ mod $OPS^{-\infty}$, it suffices to conclude that $P_0u \in C^\infty$, so we can specialize the hypothesis to $P \in OPS^0$.

By hypothesis, we can find $Q_j \in OPS^0$ such that $Q_ju \in C^\infty$ and each $(x,\xi) \in \mathrm{ES}(P)$ is noncharacteristic for some Q_j, and if $Q = \sum Q_j^*Q_j$, then $Qu \in C^\infty$ and $\mathrm{Char}\ Q \cap \mathrm{ES}(P) = \emptyset$. We claim there exists an operator $A \in OPS^0$ such that $AQ = P$ mod $OPS^{-\infty}$. Indeed, let $\tilde{Q}$ be an elliptic operator whose symbol equals that of Q on a conic neighborhood of $\mathrm{ES}(P)$, and let $\tilde{Q}^{-1}$ denote a parametrix for $\tilde{Q}$. Now simply set set $A = P\tilde{Q}^{-1}$. Consequently, (mod C^∞) $Pu = AQu \in C^\infty$, so the lemma is proved.

We are ready for the basic result on the preservation of wave front sets by a pseudodifferential operator.

Proposition 9.3. *If $u \in H^{-\infty}$ and $P \in OPS^m_{\rho,\delta}$, with $\rho > 0$, $\delta < 1$, then*

$$WF(Pu) \subset WF(u) \cap ES(P). \tag{9.5}$$

Proof. First we show $\mathrm{WF}(Pu) \subset \mathrm{ES}(P)$. Indeed, if $(x_0,\xi_0) \notin \mathrm{ES}(P)$, choose $Q = q(x,D) \in OPS^0$ such that $q(x,\xi) = 1$ on a conic neighborhood of (x_0,ξ_0) and $\mathrm{ES}(Q) \cap \mathrm{ES}(P) = \emptyset$. Thus $QP \in OPS^{-\infty}$, so $QPu \in C^\infty$. Hence $(x_0,\xi_0) \notin \mathrm{WF}(Pu)$.

In order to show that $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$, let Γ be any conic neighborhood of $\mathrm{WF}(u)$, and write $P = P_1 + P_2$, $P_j \in OPS^m_{\rho,\delta}$, with $\mathrm{ES}(P_1) \subset \Gamma$ and $\mathrm{ES}(P_2) \cap \mathrm{WF}(u) = \emptyset$. By Lemma 9.2, $P_2u \in C^\infty$. Thus $\mathrm{WF}(u) = \mathrm{WF}(P_1u) \subset \Gamma$, which shows $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$.

One says that a pseudodifferential operator of type (ρ, δ), with $\rho > 0$ and $\delta < 1$, is *microlocal*. As a corollary, we have the following sharper form of local regularity for elliptic operators, called *microlocal regularity*.

Corollary 9.4. *If $P \in OPS^m_{\rho,\delta}$ is elliptic, $0 \le \delta < \rho \le 1$, then*

$$WF(Pu) = WF(u). \tag{9.6}$$

Proof. We have seen that $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$. On the other hand, if $E \in OPS^{-m}_{\rho,\delta}$ is a parametrix for P, we see that $\mathrm{WF}(u) = \mathrm{WF}(EPu) \subset \mathrm{WF}(Pu)$. In fact, by an argument close to the proof of Lemma 9.2, we have for general P that

$$\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \text{ Char } P. \tag{9.7}$$

We next discuss how the solution operator e^{itA} to a scalar hyperbolic equation $\partial u/\partial t = iA(x, D)u$ propagates the wave front set. We assume $A(x, \xi) \in S^1_{cl}$, with real principal symbol. Suppose $\mathrm{WF}(u) = \Sigma$. Then there is a countable family of operators $p_j(x, D) \in OPS^0$, each of whose complete symbols vanishes in a neighborhood of Σ, but such that

$$\Sigma = \bigcap_j \{(x, \xi) : p_j(x, \xi) = 0\}. \tag{9.8}$$

We know that $p_j(x, D)u \in C^\infty$ for each j. Using Egorov's theorem, we want to construct a family of pseudodifferential operators $q_j(x, D) \in OPS^0$ such that $q_j(x, D)e^{itA}u \in C^\infty$, this family being rich enough to describe the wave front set of $e^{itA}u$.

Indeed, let $q_j(x, D) = e^{itA} p_j(x, D)e^{-itA}$. Egorov's theorem implies that $q_j(x, D) \in OPS^0$ (modulo a smoothing operator) and gives the principal symbol of $q_j(x, D)$. Since $p_j(x, D)u \in C^\infty$, we have $e^{itA} p_j(x, D)u \in C^\infty$, which in turn implies $q_j(x, D)e^{itA}u \in C^\infty$. From this it follows that $\mathrm{WF}(e^{itA}u)$ is contained in the intersection of the characteristics of the $q_j(x, D)$, which is precisely $C(t)\Sigma$, the image of Σ under the canonical transformation $C(t)$, generated by H_{A_1}. In other words,

$$\mathrm{WF}(e^{itA}u) \subset C(t)\mathrm{WF}(u).$$

However, our argument is reversible; $u = e^{-itA}(e^{itA}u)$. Consequently, we have the following result:

Proposition 9.5. *If $A = A(x, D) \in OPS^1$ is scalar with real principal symbol, then, for $u \in H^{-\infty}$,*

$$WF(e^{itA}u) = C(t)\,WF(u). \tag{9.9}$$

The same argument works for the solution operator $S(t, 0)$ to a time-dependent, scalar, hyperbolic equation.

Exercises

1. If $a \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ is real-valued, $\nabla\varphi \neq 0$ on supp a, as in Exercise 2 of §8, and $P = p(x, D) \in OPS^m$, so
$$P\Big(a\ e^{i\lambda\varphi}\Big) = b(x,\lambda)e^{i\lambda\varphi(x)},$$
as in (8.20), show that, mod $O(|\lambda|^{-\infty})$, $b(x,\lambda)$ depends only on the behavior of $p(x,\xi)$ on an arbitrarily small conic neighborhood of
$$C_\varphi = \Big\{\Big(x, \lambda d\varphi(x)\Big) : x \in \text{ supp } a, \lambda \neq 0\Big\}.$$
If C_φ^+ is the subset of C_φ on which $\lambda > 0$, show that the asymptotic behavior of $b(x,\lambda)$ as $\lambda \to +\infty$ depends only on the behavior of $p(x,\xi)$ on an arbitrarily small conic neighborhood of C_φ^+.
2. If Γ_r is as in (8.22), show that, given $r > 0$,
$$\Big(\cos r\sqrt{-\Delta}\Big)(a\ e^{i\lambda\varphi}) = \Gamma_r Q_r(a\ e^{i\lambda\varphi}), \quad \text{mod } O(\lambda^{-\infty}),\ \lambda > 0, \tag{9.10}$$
for some $Q_r \in OPS^{1/2}$. Consequently, analyze the behavior of the left side of (9.10), as $\lambda \to +\infty$, in terms of the behavior of Γ_r analyzed in §7 of Chapter 6.

10. Operators on manifolds

Let M be a smooth manifold. It would be natural to say that a continuous linear operator $P : C_0^\infty(M) \to \mathcal{D}'(M)$ is a pseudodifferential operator in $OPS^m_{\rho,\delta}(M)$ provided its Schwartz kernel is C^∞ off the diagonal in $M \times M$, and there exists an open cover Ω_j of M, a subordinate partition of unity φ_j, and diffeomorphisms $F_j : \Omega_j \to \mathcal{O}_j \subset \mathbb{R}^n$ that transform the operators $\varphi_k P \varphi_j : C^\infty(\Omega_j) \to \mathcal{E}'(\Omega_k)$ into pseudodifferential operators in $OPS^m_{\rho,\delta}$, as defined in §1.

This is a rather "liberal" definition of $OPS^m_{\rho,\delta}(M)$. For example, it poses no growth restrictions on the Schwartz kernel $K \in \mathcal{D}'(M \times M)$ at infinity. Consequently, if M happens to be $\mathbb{R}^n$, the class of operators in $OPS^m_{\rho,\delta}(M)$ as defined above is a bit larger than the class $OPS^m_{\rho,\delta}$ defined in §1. One negative consequence of this definition is that pseudodifferential operators cannot always be composed. One drastic step to fix this would be to insist that the kernel be properly supported, so $P : C_0^\infty(M) \to C_0^\infty(M)$. If M is compact, these problems do not arise. If M is noncompact, it is often of interest to place specific restrictions on K near infinity, but we won't go further into this point here.

Another way in which the definition of $OPS^m_{\rho,\delta}(M)$ given above is liberal is that it requires P to be locally transformed to pseudodifferential operators on $\mathbb{R}^n$ by *some* coordinate cover. One might ask if then P is necessarily so transformed by *every* coordinate cover. This comes down to asking if the class $OPS^m_{\rho,\delta}$ defined in §1 is invariant under a diffeomorphism

$F : \mathbb{R}^n \to \mathbb{R}^n$. It would suffice to establish this for the case where F is the identity outside a compact set.

In case $\rho \in (1/2, 1]$ and $\delta = 1 - \rho$, this invariance is a special case of the Egorov theorem established in §8. Indeed, one can find a time-dependent vector field $X(t)$ whose flow at $t = 1$ coincides with F and apply Theorem 8.1 to $iA(t, x, D) = X(t)$. Note that the formula for the principal symbol of the conjugated operator given there implies

$$p(1, F(x), \xi) = p_0(x, F'(x)^t \xi), \tag{10.1}$$

so that the principal symbol is well defined on the cotangent bundle of M.

We will therefore generally insist that $\rho \in (1/2, 1]$ and $\delta = 1 - \rho$ when talking about $OPS^m_{\rho,\delta}(M)$ for a manifold M, without a distinguished coordinate chart. In special situations, it might be natural to use coordinate charts with special structure. For instance, for a Cartesian product $M = \mathbb{R} \times \Omega$, one can stick to product coordinate systems. In such a case, we can construct a parametrix E for the hypoelliptic operator $\partial/\partial t - \Delta_x$, $t \in \mathbb{R}$, $x \in \Omega$, and unambiguously regard E as an operator in $OPS^{-1}_{1/2,0}(\mathbb{R} \times \Omega)$.

We make the following comments on the principal symbol of an operator $P \in OPS^m_{\rho,\delta}(M)$, when $\rho \in (1/2, 1]$, $\delta = 1 - \rho$. By the arguments in §8, the principal symbol is well defined, if it is regarded as an element of the quotient space:

$$p(x, \xi) \in S^m_{\rho,\delta}(T^*M)/S^{m-(2\rho-1)}_{\rho,\delta}(T^*M). \tag{10.2}$$

In particular, by Theorem 8.1, in case $P \in OPS^m_{1,0}(M)$, we have

$$p(x, \xi) \in S^m_{1,0}(T^*M)/S^{m-1}_{1,0}(T^*M). \tag{10.3}$$

If $P \in S^m_{cl}(M)$, then the principal symbol can be taken to be homogeneous in ξ of degree m, by (8.16). Note that the characterizations of the Schwartz kernels of operators in $OPS^m_{1,0}$ and in OPS^m_{cl} given in §2 also make clear the invariance of these classes under coordinate transformations.

We now discuss some properties of an elliptic operator $A \in OPS^m_{1,0}(M)$, when M is a compact Riemannian manifold. Denote by B a parametrix, so we have, for each $s \in \mathbb{R}$,

$$A : H^{s+m}(M) \longrightarrow H^s(M), \quad B : H^s(M) \longrightarrow H^{s+m}(M), \tag{10.4}$$

and $AB = I + K_1$, $BA = I + K_2$, where $K_j : \mathcal{D}'(M) \to C^\infty(M)$. Thus K_j is compact on each Sobolev space $H^s(M)$, so B is a two-sided Fredholm inverse of A in (10.4). In particular, A is a Fredholm operator; $\ker A = \mathcal{K}_{s+m} \subset H^{s+m}(M)$ is finite-dimensional, and $A\big(H^{s+m}(M)\big) \subset H^s(M)$ is closed, of finite codimension, so

$$\mathcal{C}_s = \{v \in H^{-s}(M) : \langle Au, v \rangle = 0 \text{ for all } u \in H^{s+m}(M)\}$$

is finite-dimensional. Note that $\mathcal{C}_s$ is the null space of

$$A^* : H^{-s}(M) \longrightarrow H^{-s-m}(M), \tag{10.5}$$

which is also an elliptic operator in $OPS^m_{1,0}(M)$. Elliptic regularity yields, for all s,

$$\mathcal{K}_{s+m} = \{u \in C^\infty(M) : Au = 0\}, \quad \mathcal{C}_s = \{v \in C^\infty(M) : A^*v = 0\}. \tag{10.6}$$

Thus these spaces are independent of s.

Suppose now that $m > 0$. We will consider A as an unbounded operator on the Hilbert space $L^2(M)$, with domain

$$\mathcal{D}(A) = \{u \in L^2(M) : Au \in L^2(M)\}. \tag{10.7}$$

It is easy to see that A is closed. Also, elliptic regularity implies

$$\mathcal{D}(A) = H^m(M). \tag{10.8}$$

Since A is closed and densely defined, its Hilbert space adjoint is defined, also as a closed, unbounded operator on $L^2(M)$, with a dense domain. The symbol A^* is also our preferred notation for the Hilbert space adjoint. To avoid confusion, we will *temporarily* use A^t to denote the adjoint on $\mathcal{D}'(M)$, so $A^t \in OPS^m(M)$, $A^t : H^{s+m}(M) \to H^s(M)$, for all s. Now the unbounded operator A^* has domain

$$\mathcal{D}(A^*) = \{u \in L^2(M) : |(u, Av)| \le c(u)\|v\|_{L^2}, \forall\ v \in \mathcal{D}(A)\}, \tag{10.9}$$

and then A^*u is the unique element of $L^2(M)$ such that

$$(A^*u, v) = (u, Av), \quad \text{for all } v \in \mathcal{D}(A). \tag{10.10}$$

Recall that $\mathcal{D}(A) = H^m(M)$. Since, for any $u \in H^m(M)$, $v \in H^m(M)$, we have $(A^tu, v) = (u, Av)$, we see that $\mathcal{D}(A^*) \supset H^m(M)$ and $A^* = A^t$ on $H^m(M)$. On the other hand, $(u, Av) = (A^tu, v)$ holds for all $v \in H^m(M)$, $u \in L^2(M)$, the latter inner product being given by the duality of $H^{-m}(M)$ and $H^m(M)$. Thus it follows that

$$u \in \mathcal{D}(A^*) \Longrightarrow A^*u = A^tu \in L^2(M).$$

But elliptic regularity for $A^t \in OPS^m_{1,0}(M)$ then implies $u \in H^m(M)$. Thus

$$\mathcal{D}(A^*) = H^m(M), \quad A^* = A^t\big|_{H^m(M)}. \tag{10.11}$$

In particular, if A is elliptic in $OPS^m_{1,0}(M)$, $m > 0$, and also symmetric (i.e., $A = A^t$), then the Hilbert space operator is *self-adjoint*; $A = A^*$. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\lambda I - A)^{-1} : L^2(M) \to \mathcal{D}(A) = H^m(M)$, so A has compact resolvent. Thus $L^2(M)$ has an orthonormal basis of eigenfunctions of A, $Au_j = \lambda_j u_j$, $|\lambda_j| \to \infty$, and, by elliptic regularity, each u_j belongs to $C^\infty(M)$.

Exercises

In the following exercises, assume that M is a smooth, compact, Riemannian manifold. Let $A \in OPS^m(M)$ be elliptic, positive, and self-adjoint, with $m > 0$. Let u_j be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of A, $Au_j = \lambda_j u_j$. Given $f \in \mathcal{D}'(M)$, form "Fourier coefficients" $\hat{f}(j) = (f, u_j)$. Thus $f \in L^2(M)$ implies

$$f = \sum_{j=0}^{\infty} \hat{f}(j) u_j, \tag{10.12}$$

with convergence in L^2-norm.

1. Given $s \in \mathbb{R}$, show that $f \in H^s(M)$ if and only if $\sum |\hat{f}(j)|^2 \langle \lambda_j \rangle^{2s/m} < \infty$.
2. Show that, for any $s \in \mathbb{R}$, $f \in H^s(M)$, (10.12) holds, with convergence in H^s-norm. Conclude that if $s > n/2$ and $f \in H^s(M)$, the series converges uniformly to f.
3. If $s > n/2$ and $f \in H^s(M)$, show that (10.12) converges *absolutely*. (*Hint*: Fix $x_0 \in M$ and pick $c_j \in \mathbb{C}$, $|c_j| = 1$, such that $c_j \hat{f}(j) u_j(x_0) \geq 0$. Now consider $\sum c_j \hat{f}(j) u_j$.)
4. Let $-L$ be a second-order, elliptic, positive, self-adjoint differential operator on a compact Riemannian manifold M. Suppose $A \in OPS^1(M)$ is positive, self-adjoint, and $A^2 = -L + R$, where $R : \mathcal{D}'(M) \to C^\infty(M)$. Show that $A - \sqrt{-L} : \mathcal{D}'(M) \to C^\infty(M)$.

One approach to Exercise 4 is the following.

5. Given $f \in H^s(M)$, form
$$u(y,x) = e^{-y\sqrt{-L}} f(x), \quad v(y,x) = e^{-yA} f(x),$$
for $(y,x) \in [0,\infty) \times M$. Note that
$$\Bigl(\frac{\partial^2}{\partial y^2} + L\Bigr) u = 0, \quad \Bigl(\frac{\partial^2}{\partial y^2} + L\Bigr) v = -Rv(y,x).$$
Use estimates and regularity for the Dirichlet problem for $\partial^2/\partial y^2 + L$ on $[0,\infty) \times M$ to show that $u - v \in C^\infty([0,\infty) \times M)$. Conclude that $\partial u/\partial y - \partial v/\partial y\Big|_{y=0} = (A - \sqrt{-L}) f \in C^\infty(M)$.
6. With L as above, use the symbol calculus of §3 to construct a self-adjoint $A \in OPS^1(M)$, with positive principal symbol, such that $A^2 + L \in OPS^{-\infty}(M)$. Conclude that Exercise 4 applies to A.
7. Show that $OPS^0_{1,0}(M)$ has a natural Fréchet space structure.

11. The method of layer potentials

We discuss, in the light of the theory of pseudodifferential operators, the use of "single- and double-layer potentials" to study the Dirichlet and Neumann boundary problems for the Laplace equation. Material developed here will be useful in §7 of Chapter 9, which treats the use of integral equations in scattering theory.

Let $\overline{\Omega}$ be a connected, compact Riemannian manifold with nonempty boundary; $n = \dim \Omega$. Suppose $\overline{\Omega} \subset M$, a Riemannian manifold of dimension n without boundary, on which there is a fundamental solution $E(x, y)$ to the Laplace equation:

$$\Delta_x E(x, y) = \delta_y(x), \tag{11.1}$$

where $E(x, y)$ is the Schwartz kernel of an operator $E(x, D) \in OPS^{-2}(M)$; we have

$$E(x, y) \sim c_n \ \mathrm{dist}(x, y)^{2-n} + \cdots \tag{11.2}$$

as $x \to y$, if $n \geq 3$, while

$$E(x, y) \sim c_2 \log \ \mathrm{dist}(x, y) + \cdots \tag{11.3}$$

if $n = 2$. Here, $c_n = -\big[(n-2)\mathrm{Area}(S^{n-1})\big]^{-1}$ for $n \geq 3$, and $c_2 = 1/2\pi$. The single- and double-layer potentials of a function f on $\partial\Omega$ are defined by

$$\mathcal{S}\ell \ f(x) = \int_{\partial\Omega} f(y) E(x, y) \ dS(y), \tag{11.4}$$

and

$$\mathcal{D}\ell \ f(x) = \int_{\partial\Omega} f(y) \ \frac{\partial E}{\partial \nu_y}(x, y) \ dS(y), \tag{11.5}$$

for $x \in M \setminus \partial\Omega$. Given a function v on $M \setminus \partial\Omega$, for $x \in \partial\Omega$, let $v_+(x)$ and $v_-(x)$ denote the limits of $v(z)$ as $z \to x$, from $z \in \Omega$ and $z \in M \setminus \overline{\Omega} = \mathcal{O}$, respectively, when these limits exist. The following are fundamental properties of these layer potentials.

Proposition 11.1. *For $x \in \partial\Omega$, we have*

$$\mathcal{S}\ell \ f_+(x) = \mathcal{S}\ell \ f_-(x) = Sf(x) \tag{11.6}$$

and

$$\mathcal{D}\ell \ f_\pm(x) = \pm\frac{1}{2} f(x) + \frac{1}{2} N f(x), \tag{11.7}$$

where, for $x \in \partial\Omega$,

$$Sf(x) = \int_{\partial\Omega} f(y) E(x, y) \ dS(y) \tag{11.8}$$

and

$$Nf(x) = 2 \int_{\partial\Omega} f(y) \frac{\partial E}{\partial \nu_y}(x, y) \ dS(y). \tag{11.9}$$

Note that $E(x,\cdot)\big|_{\partial\Omega}$ is integrable, uniformly in x, and that the conclusion in (11.6) is elementary, at least for f continuous; the conclusion in (11.7) is a bit more mysterious. To see what is behind such results, let us look at the more general situation of

$$v = p(x,D)(f\sigma), \tag{11.10}$$

where $\sigma \in \mathcal{E}'(M)$ is surface measure on a hypersurface (here $\partial\Omega$), $f \in \mathcal{D}'(\partial\Omega)$, so $f\sigma \in \mathcal{E}'(M)$. Assume that $p(x,D) \in OPS^m(M)$. Make a local coordinate change, straightening out the surface to $\{x_n = 0\}$. Then, in this coordinate system

$$\begin{aligned} v(x',x_n) &= \int \hat{f}(\xi')e^{ix'\cdot\xi'}p(x,\xi',\xi_n)e^{ix_n\xi_n}\,d\xi_n\,d\xi' \\ &= q(x_n,x',D_{x'})f, \end{aligned} \tag{11.11}$$

for $x_n \neq 0$, where

$$q(x_n,x',\xi') = \int p(x,\xi',\xi_n)e^{ix_n\xi_n}\,d\xi_n. \tag{11.12}$$

If $p(x,\xi)$ is homogeneous of degree m in ξ, for $|\xi| \geq 1$, then for $|\xi'| \geq 1$ we have

$$q(x_n,x',\xi') = |\xi'|^{m+1}\tilde{p}(x,\omega',x_n|\xi'|), \tag{11.13}$$

where $\omega' = \xi'/|\xi'|$ and

$$\tilde{p}(x,\omega',\tau) = \int p(x,\omega',\zeta)e^{i\zeta\tau}\,d\zeta.$$

Now, if $m < -1$, the integral in (11.12) is absolutely convergent and $q(x_n,x',\xi')$ is continuous in all arguments, even across $x_n = 0$. On the other hand, if $m = -1$, then, temporarily neglecting all the arguments of p but the last, we are looking at the Fourier transform of a smooth function of one variable whose asymptotic behavior as $\xi_n \to \pm\infty$ is of the form $C_1^\pm\xi_n^{-1} + C_2^\pm\xi_n^{-2} + \cdots$. From the results of Chapter 3 we know that the Fourier transform is smooth except at $x_n = 0$, and if $C_1^+ = C_1^-$, then the Fourier transform has a jump across $x_n = 0$; otherwise there may be a logarithmic singularity.

It follows that if $p(x,D) \in OPS^m(M)$ and $m < -1$, then (11.10) has a limit on $\partial\Omega$, given by

$$v\big|_{\partial\Omega} = Qf, \quad Q \in OPS^{m+1}(\partial\Omega). \tag{11.14}$$

On the other hand, if $m = -1$ and the symbol of $p(x,D)$ has the behavior that, for $x \in \partial\Omega$, ν_x normal to $\partial\Omega$ at x,

$$p(x,\xi \pm \tau\nu_x) = \pm C(x,\xi)\tau^{-1} + O(\tau^{-2}), \quad \tau \to +\infty, \tag{11.15}$$

then (11.10) has a limit from each side of $\partial\Omega$, and

$$v_\pm = Q_\pm f, \quad Q_\pm \in OPS^0(\partial\Omega). \tag{11.16}$$

To specialize these results to the setting of Proposition 11.1, note that

$$\mathcal{S}\ell\ f = E(x,D)(f\sigma) \tag{11.17}$$

and

$$\mathcal{D}\ell\ f = E(x,D)X^*(f\sigma), \tag{11.18}$$

where X is any vector field on M equal to $\partial/\partial\nu$ on $\partial\Omega$, with formal adjoint X^*, given by

$$X^*v = -Xv - (\text{div}\ X)v. \tag{11.19}$$

The analysis of (11.10) applies directly to (11.17), with $m = -2$. That the boundary value is given by (11.8) is elementary for $f \in C(\partial\Omega)$, as noted before. Given (11.14), it then follows for more general f.

Now (11.18) is also of the form (11.10), with $p(x,D) = E(x,D)X^* \in OPS^{-1}(M)$. Note that the principal symbol at $x \in \partial\Omega$ is given by

$$p_0(x,\xi) = -|\xi|^{-2}\langle\nu(x),\xi\rangle, \tag{11.20}$$

which satisfies the condition (11.15), so the conclusion (11.16) applies. Note that

$$p_0(x,\xi \pm \tau\nu_x) = -|\xi \pm \tau\nu_x|^{-2}\langle\nu_x, \xi \pm \tau\nu_x\rangle,$$

so in this case (11.15) holds with $C(x,\xi) = 1$. Thus the operators $Q_\pm$ in (11.16) have principal symbols $\pm$ const. That the constant is as given in (11.7) follows from keeping careful track of the constants in the calculations (11.11)–(11.13) (cf. Exercise 9 below).

Let us take a closer look at the behavior of $(\partial/\partial\nu_y)E(x,y)$. Note that, for x close to y, if $V_{x,y}$ denotes the unit vector at y in the direction of the geodesic from x to y, then (for $n \geq 3$)

$$\nabla_y E(x,y) \sim (2-n)c_n\ \text{dist}(x,y)^{1-n}V_{x,y} + \cdots. \tag{11.21}$$

If $y \in \partial\Omega$ and ν_y is the unit normal to $\partial\Omega$ at y, then

$$\frac{\partial}{\partial\nu_y}E(x,y) \sim (2-n)c_n\ \text{dist}(x,y)^{1-n}\langle V_{x,y},\nu_y\rangle + \cdots. \tag{11.22}$$

Note that $(2-n)c_n = -1/\text{Area}(S^{n-1})$. Clearly, the inner product $\langle V_{x,y},\nu_y\rangle = \alpha(x,y)$ restricted to $(x,y) \in \partial\Omega\times\partial\Omega$ is Lipschitz and vanishes on the diagonal $x = y$. This vanishing makes $(\partial E/\partial\nu_y)(x,y)$ integrable on $\partial\Omega\times\partial\Omega$. It is clear that in the case (11.7), $Q_\pm$ have Schwartz kernels equal to $(\partial/\partial\nu_y)E(x,y)$ on the complement of the diagonal in $\partial\Omega\times\partial\Omega$. In light of our analysis above of the principal symbol of $Q_\pm$, the proof of (11.7) is complete.

As a check on the evaluation of the constant c in $\mathcal{D}\ell\ f_\pm = \pm cf + (1/2)Nf$, $c = 1/2$, note that applying Green's formula to $\int_\Omega (\Delta 1)\cdot E(x,y)\,dy$ readily gives

$$\int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y)\, dS(y) = 1, \qquad \text{for } x \in \Omega,$$

$$0, \qquad \text{for } x \in \mathcal{O},$$

as the value of $\mathcal{D}\ell\ f_\pm$ for $f = 1$. Since $\mathcal{D}\ell\ f_+ - \mathcal{D}\ell\ f_- = 2cf$, this forces $c = 1/2$.

The way in which $\pm(1/2)f(x)$ arises in (11.7) is captured well by the model case of $\partial\Omega$ a hyperplane in $\mathbb{R}^n$, and

$$E\big((x', x_n), (y', 0)\big) = c_n\big[(x' - y')^2 + x_n^2\big]^{(2-n)/2},$$

when (11.22) becomes

$$\frac{\partial}{\partial y_n} E\big((x', x_n), (y', 0)\big) = (2-n)c_n x_n \big[(x' - y')^2 + x_n^2\big]^{-n/2},$$

though in this example $N = 0$.

The following properties of the operators S and N are fundamental.

Proposition 11.2. *We have*

$$S, N \in OPS^{-1}(\partial\Omega), \quad S \text{ elliptic.} \tag{11.23}$$

Proof. That S has this behavior follows immediately from (11.2) and (11.3). The ellipticity at x follows from taking normal coordinates at x and using Exercise 3 of §4, for $n \geq 3$; for $n = 2$, the reader can supply an analogous argument. That N also satisfies (11.23) follows from (11.22) and the vanishing of $\alpha(x,y) = \langle V_{x,y}, \nu_y\rangle$ on the diagonal.

An important result complementary to Proposition 11.1 is the following, on the behavior of the normal derivative at $\partial\Omega$ of single-layer potentials.

Proposition 11.3. *For $x \in \partial\Omega$, we have*

$$\frac{\partial}{\partial \nu} \mathcal{S}\ell\ f_\pm(x) = \frac{1}{2}\big(\mp f + N^\# f\big), \tag{11.24}$$

where $N^\# \in OPS^{-1}(\partial\Omega)$ is given by

$$N^\# f(x) = 2\int_{\partial\Omega} f(y)\, \frac{\partial E}{\partial \nu_x}(x,y)\, dS(y). \tag{11.25}$$

Proof. The proof of (11.24) is directly parallel to that of (11.7). To see on general principles why this should be so, use (11.17) to write $(\partial/\partial\nu)\mathcal{S}\ell\ f$ as the restriction to $\partial\Omega$ of

$$X\mathcal{S}\ell\ f = XE(x,D)(f\sigma). \tag{11.26}$$

Using (11.18) and (11.19), we see that

$$\begin{aligned}\mathcal{D}\ell\ f + X\mathcal{S}\ell\ f &= [X, E(x,D)](f\sigma) - E(x,D)(\mathrm{div}\ X)(f\sigma) \\ &= A(x,D)(f\sigma),\end{aligned} \tag{11.27}$$

with $A(x,D) \in OPS^{-2}(M)$, the same class as $E(x,D)$. Thus the extension of $A(x,D)(f\sigma)$ to $\partial\Omega$ is straightforward, and we have

$$\frac{\partial}{\partial\nu}\mathcal{S}\ell\ f_\pm = -\mathcal{D}\ell\ f_\pm + A(x,D)(f\sigma)\big|_{\partial\Omega}. \tag{11.28}$$

In particular, the jumps across $\partial\Omega$ are related by

$$\frac{\partial}{\partial\nu}\mathcal{S}\ell\ f_+ - \frac{\partial}{\partial\nu}\mathcal{S}\ell\ f_- = \mathcal{D}\ell\ f_- - \mathcal{D}\ell\ f_+, \tag{11.29}$$

consistent with the result implied by formulas (11.7) and (11.24).

It is also useful to understand the boundary behavior of $(\partial/\partial\nu)\mathcal{D}\ell\ f$. This is a bit harder since $\partial^2 E/\partial\nu_x\partial\nu_y$ is more highly singular. From here on, assume $E(x,y) = E(y,x)$, so also $\Delta_y E(x,y) = \delta_x(y)$. We define the *Neumann operator*

$$\mathcal{N} : C^\infty(\partial\Omega) \longrightarrow C^\infty(\partial\Omega) \tag{11.30}$$

as follows. Given $f \in C^\infty(\partial\Omega)$, let $u \in C^\infty(\overline{\Omega})$ be the unique solution to

$$\Delta u = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega, \tag{11.31}$$

and let

$$\mathcal{N}f = \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}, \tag{11.32}$$

the limit taken from within Ω. It is a simple consequence of Green's formula that if we form

$$\int_{\partial\Omega} \Big[f(y)\ \frac{\partial E}{\partial\nu_y}(x,y) - \mathcal{N}f(y)E(x,y)\Big]\,dS(y) = \mathcal{D}\ell\ f(x) - \mathcal{S}\ell\ \mathcal{N}f(x), \tag{11.33}$$

for $x \in M \setminus \partial\Omega$, then

$$\begin{aligned}\mathcal{D}\ell\ f(x) - \mathcal{S}\ell\ \mathcal{N}f(x) = u(x), &\quad x \in \Omega, \\ 0, &\quad x \in M \setminus \overline{\Omega},\end{aligned} \tag{11.34}$$

where u is given by (11.31). Note that taking the limit of (11.34) from within Ω, using (11.6) and (11.7), gives $f = (1/2)f + (1/2)Nf - S\mathcal{N}f$, which implies the identity

$$S\mathcal{N} = -\frac{1}{2}(I - N). \tag{11.35}$$

Taking the limit in (11.34) from $M \setminus \overline{\Omega}$ gives the same identity. In view of the behavior (11.23), in particular the ellipticity of S, we conclude that

$$\mathcal{N} \in OPS^1(\partial\Omega), \quad \text{elliptic.} \tag{11.36}$$

Now we apply $\partial/\partial\nu$ to the identity (11.34), evaluating on $\partial\Omega$ from both sides. Evaluating from Ω gives

$$\frac{\partial}{\partial\nu}\mathcal{D}\ell\, f_+ - \frac{\partial}{\partial\nu}\mathcal{S}\ell\,\mathcal{N}f_+ = \mathcal{N}f, \tag{11.37}$$

while evaluating from $M\setminus\overline{\Omega}$ gives

$$\frac{\partial}{\partial\nu}\mathcal{D}\ell\, f_- - \frac{\partial}{\partial\nu}\mathcal{S}\ell\,\mathcal{N}f_- = 0. \tag{11.38}$$

In particular, applying $\partial/\partial\nu$ to (11.34) shows that $(\partial/\partial\nu)\mathcal{D}\ell\, f_\pm$ exists, by Proposition 11.3. Furthermore, applying (11.24) to $(\partial/\partial\nu)\mathcal{S}\ell\,\mathcal{N}f_\pm$, we have a proof of the following.

Proposition 11.4. *For $x\in\partial\Omega$, we have*

$$\frac{\partial}{\partial\nu}\mathcal{D}\ell\, f_\pm(x) = \frac{1}{2}(I+N^{\#})\mathcal{N}f. \tag{11.39}$$

In particular, there is no jump across $\partial\Omega$ of $(\partial/\partial\nu)\mathcal{D}\ell\, f$.

We have now developed the layer potentials far enough to apply them to the study of the Dirichlet problem. We want an approximate formula for the Poisson integral $u = \mathrm{PI}\, f$, the unique solution to

$$\Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f. \tag{11.40}$$

Motivated by the Poisson integral formula on $\mathbb{R}^n_+$, we look for a solution of the form

$$u(x) = \mathcal{D}\ell\, g(x), \quad x\in\Omega, \tag{11.41}$$

and try to relate g to f. In view of Proposition 11.1, letting $x\to z\in\partial\Omega$ in (11.41) yields

$$u(z) = \frac{1}{2}(g+Ng), \quad \text{for } z\in\partial\Omega. \tag{11.42}$$

Thus if we define u by (11.41), then (11.40) is equivalent to

$$f = \frac{1}{2}(I+N)g. \tag{11.43}$$

Alternatively, we can try to solve (11.40) in terms of a single-layer potential:

$$u(x) = \mathcal{S}\ell\, h(x), \quad x\in\Omega. \tag{11.44}$$

If u is defined by (11.44), then (11.40) is equivalent to

$$f = Sh. \tag{11.45}$$

Note that, by (11.23), the operator $(1/2)(I+N)$ in (11.43) is Fredholm, of index zero, on each space $H^s(\partial\Omega)$. It is not hard to verify that S is elliptic of order -1, with real principal symbol, so for each s,

$$S : H^{s-1}(\partial\Omega) \longrightarrow H^s(\partial\Omega)$$

is Fredholm, of index zero.

One basic case when the equations (11.43) and (11.45) can both be solved is the case of bounded Ω in $M = \mathbb{R}^n$, with the standard flat Laplacian.

Proposition 11.5. *If $\overline{\Omega}$ is a smooth, bounded subdomain of $\mathbb{R}^n$, with connected complement, then, for all s,*

$$I + N : H^s(\partial\Omega) \longrightarrow H^s(\partial\Omega) \quad \text{and} \quad S : H^{s-1}(\partial\Omega) \longrightarrow H^s(\partial\Omega) \tag{11.46}$$

are isomorphisms.

Proof. It suffices to show that $I + N$ and S are injective on $C^\infty(\partial\Omega)$. First, if $g \in C^\infty(\partial\Omega)$ belongs to the null space of $I + N$, then, by (11.42) and the maximum principle, we have $\mathcal{D}\ell\ g = 0$ in Ω. By (11.7), the jump of $\mathcal{D}\ell\ g$ across $\partial\Omega$ is g, so we have for $v = \mathcal{D}\ell\ g|_{\mathcal{O}}$, where $\mathcal{O} = \mathbb{R}^n \setminus \overline{\Omega}$,

$$\Delta v = 0 \text{ on } \mathcal{O}, \quad v\big|_{\partial\Omega} = -g. \tag{11.47}$$

Also, v clearly vanishes at infinity. Now, by (11.39), $(\partial/\partial\nu)\mathcal{D}\ell\ g$ does not jump across $\partial\Omega$, so we have $\partial v/\partial\nu = 0$ on $\partial\Omega$. But at a point on $\partial\Omega$ where $-g$ is maximal, this contradicts Zaremba's principle, unless $g = 0$. This proves that $I + N$ is an isomorphism in this case.

Next, suppose $h \in C^\infty(\partial\Omega)$ belongs to the null space of S. Then, by (11.45) and the maximum principle, we have $\mathcal{S}\ell\ h = 0$ on Ω. By (11.24), the jump of $(\partial/\partial\nu)\mathcal{S}\ell\ h$ across $\partial\Omega$ is $-h$, so we have for $w = \mathcal{S}\ell\ h|_{\mathcal{O}}$ that

$$\Delta w = 0 \text{ on } \mathcal{O}, \quad \frac{\partial w}{\partial\nu}\Big|_{\partial\Omega} = h, \tag{11.48}$$

and w vanishes at infinity. This time, $\mathcal{S}\ell\ h$ does not jump across $\partial\Omega$, so we also have $w = 0$ on $\partial\Omega$. The maximum principle forces $w = 0$ on $\mathcal{O}$, so $h = 0$. This proves that S is an isomorphism in this case.

In view of (11.6), we see that (11.44) and (11.45) also give a solution to $\Delta u = 0$ on the *exterior* region $\mathbb{R}^n \setminus \overline{\Omega}$, satisfying $u = f$ on $\partial\Omega$ and $u(x) \to 0$ as $|x| \to \infty$, if $n \geq 3$. This solution is unique, by the maximum principle.

One can readily extend the proof of Proposition 11.5 and show that $I + N$ and S in (11.46) are isomorphisms in somewhat more general circumstances.

Let us now consider the Neumann problem

$$\Delta u = 0 \text{ on } \Omega, \quad \frac{\partial u}{\partial\nu} = \varphi \text{ on } \partial\Omega. \tag{11.49}$$

We can relate (11.49) to (11.40) via the Neumann operator:

$$\varphi = \mathcal{N} f. \tag{11.50}$$

Let us assume that Ω is connected; then

$$\text{Ker}\ \mathcal{N} = \{f = \text{ const. on } \Omega\}, \tag{11.51}$$

so dim Ker $\mathcal{N} = 1$. Note that, by Green's theorem,

$$(\mathcal{N}f, g)_{L^2(\partial\Omega)} = -(du, dv)_{L^2(\Omega)} = (f, \mathcal{N}g)_{L^2(\partial\Omega)}, \tag{11.52}$$

where $u = PI\ f$, $v = PI\ g$, so $\mathcal{N}$ is symmetric. In particular,

$$(\mathcal{N}f, f)_{L^2(\partial\Omega)} = -\|du\|^2_{L^2(\Omega)}, \tag{11.53}$$

so $\mathcal{N}$ is negative-semidefinite. The symmetry of $\mathcal{N}$ together with its ellipticity implies that, for each s,

$$\mathcal{N} : H^{s+1}(\partial\Omega) \longrightarrow H^s(\partial\Omega) \tag{11.54}$$

is Fredholm, of index zero, with both Ker $\mathcal{N}$ and $\mathcal{R}(\mathcal{N})^\perp$ of dimension 1, and so

$$\mathcal{R}(\mathcal{N}) = \Big\{\varphi \in H^s(\partial\Omega) : \int_{\partial\Omega} \varphi\ dS = 0\Big\}, \tag{11.55}$$

this integral interpreted in the obvious distributional sense when $s < 0$.

By (11.35), whenever S is an isomorphism in (11.46), we can say that (11.50) is equivalent to

$$(I - N)f = -2S\varphi. \tag{11.56}$$

We can also represent a solution to (11.49) as a single-layer potential, of the form (11.44). Using (11.24), we see that this works provided h satisfies

$$(I - N^\#)h = -2\varphi. \tag{11.57}$$

In view of the fact that (11.44) solves the Dirichlet problem (11.40) with $f = Sh$, we deduce the identity $\varphi = \mathcal{N}Sh$, or

$$\mathcal{N}S = -\frac{1}{2}(I - N^\#), \tag{11.58}$$

complementing (11.35). Comparing these identities, representing $S\mathcal{N}S$ in two ways, we obtain the intertwining relation

$$SN^\# = NS. \tag{11.59}$$

Also note that, under the symmetry hypothesis $E(x, y) = E(y, x)$, we have $N^\# = N^*$.

The method of layer potentials is applicable to other boundary problems. An application to the "Stokes system" will be given in Chapter 17, §A.

We remark that a number of results in this section do not make substantial use of the pseudodifferential operator calculus developed in the early sections; this makes it easy to extend such results to situations where the boundary has limited smoothness. For example, it is fairly straightforward to extend results on the double-layer potential $\mathcal{D}\ell$ to the case where $\partial\Omega$

is a C^{1+r}-hypersurface in $\mathbb{R}^n$, for any $r > 0$, and in particular to extend (partially) the first part of (11.46), obtaining

$$I + N : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \;\text{ invertible},$$

in such a case, thus obtaining the representation (11.41) for the solution to the Dirichlet problem with boundary data in $L^2(\partial\Omega)$, when $\partial\Omega$ is a C^{1+r}-surface. Results on S in (11.23) and some results on the Neumann operator, such as (11.36), do depend on the pseudodifferential operator calculus, so more work is required to adapt this material to C^{1+r}-surfaces, though that has been done.

In fact, via results of [Ca3] and [CMM], the layer potential approach has been extended to domains in $\mathbb{R}^n$ bounded by C^1-surfaces, in [FJR], and then to domains bounded by Lipschitz surfaces, in [Ver] and [DK]. See also [JK] for nonhomogeneous equations. Extensions to Lipschitz domains in Riemannian manifolds are given in [MT1] and [MT2], and extensions to "uniformly rectifiable" domains in [D], [DS], and [HMT]. We mention just one result here; many others can be found in the sources cited above and references they contain.

Proposition 11.6. *If Ω is a Lipschitz domain in a compact Riemannian manifold M, then*

$$\mathrm{PI} : L^2(\partial\Omega) \longrightarrow H^{1/2}(\Omega).$$

Exercises

1. Let M be a compact, connected Riemannian manifold, with Laplace operator L, and let $\overline{\Omega} = [0,1] \times M$, with Laplace operator $\Delta = \partial^2/\partial y^2 + L$, $y \in [0,1]$. Show that the Dirichlet problem
$$\Delta u = 0 \text{ on } \Omega, \quad u(0,x) = f_0(x), \quad u(1,x) = f_1(x)$$
has the solution
$$u(y,x) = e^{-y\sqrt{-L}}\varphi_0 + e^{-(1-y)\sqrt{-L}}\varphi_1 + \kappa y,$$
where κ is the constant $\kappa = (\text{vol } M)^{-1}\int_M (f_1 - f_0)\, dV$, and
$$\varphi_0 = (1 - e^{-2\sqrt{-L}})^{-1}(f_0 - e^{-\sqrt{-L}} f_1 - \kappa),$$
$$\varphi_1 = (1 - e^{-2\sqrt{-L}})^{-1}(f_1 - \kappa - e^{-\sqrt{-L}} f_0),$$
the operator $(1 - e^{-2\sqrt{-L}})^{-1}$ being well defined on $(\ker L)^{\perp}$.
2. If $\mathcal{N} f_0(x) = (\partial u/\partial y)(0,x)$, where u is as above, with $f_1 = 0$, show that
$$\mathcal{N} f_0 = -\sqrt{-L} f_0 + \mathcal{R} f_0,$$
where $\mathcal{R}$ is a smoothing operator, $\mathcal{R} : \mathcal{D}'(M) \to C^\infty(M)$. Using (11.36), deduce that these calculations imply
$$\sqrt{-L} \in OPS^1(M).$$

Compare Exercises 4–6 of §10.

3. If PI: $C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$ is the Poisson integral operator solving (11.40), show that, for $x \in \Omega$,

$$\mathrm{PI}\ f(x) = \int_{\partial\Omega} k(x,y) f(y)\ dS(y),$$

with

$$|k(x,y)| \le C\Big(d(x,y)^2 + \rho(x)^2\Big)^{-(n-1)/2},$$

where $n = \dim \Omega$, $d(x,y)$ is the distance from x to y, and $\rho(x)$ is the distance from x to $\partial\Omega$.

4. If $\overline{M}$ is an $(n-1)$-dimensional surface with boundary in $\overline{\Omega}$, intersecting $\partial\Omega$ transversally, with $\partial M \subset \partial\Omega$, and $\rho : C^\infty(\overline{\Omega}) \to C^\infty(\overline{M})$ is restriction to $\overline{M}$, show that

$$\rho \circ \mathrm{PI} : L^2(\partial\Omega) \longrightarrow L^2(M).$$

(*Hint*: Look at Exercise 2 in §5 of Appendix A on functional analysis.)

5. Given $y \in \Omega$, let G_y be the "Green function," satisfying

$$\Delta G_y = \delta_y, \quad G_y = 0 \text{ on } \partial\Omega.$$

Show that, for $f \in C^\infty(\partial\Omega)$,

$$\mathrm{PI}\ f(y) = \int_{\partial\Omega} f(x)\ \partial_\nu G_y(x)\ dS(x).$$

(*Hint*: Apply Green's formula to $(\mathrm{PI}\ f, \Delta G_y) = (\mathrm{PI}\ f, \Delta G_y) - (\Delta\ \mathrm{PI}\ f, G_y)$.)

6. Assume u is scalar, $\Delta u = f$, and w is a vector field on $\overline{\Omega}$. Show that

$$\begin{aligned}\int_{\partial\Omega} \langle \nu, w\rangle |\nabla u|^2\ dS &= 2\int_{\partial\Omega} (\nabla_w u)(\partial_\nu u)\ dS - 2\int_\Omega (\nabla_w u) f\ dV \\ &\quad + \int_\Omega (\mathrm{div}\ w)|\nabla u|^2\ dV - 2\int_\Omega (\mathcal{L}_w g)(\nabla u, \nabla u)\ dV,\end{aligned} \tag{11.60}$$

where g is the metric tensor on $\overline{\Omega}$. This identity is a "Rellich formula." (*Hint*: Compute $\mathrm{div}\Big(\langle \nabla u, \nabla u\rangle w\Big)$ and $2\ \mathrm{div}(\nabla_w u \cdot \nabla u)$, and apply the divergence theorem to the difference.)

7. In the setting of Exercise 6, assume w is a unit vector field and that $\langle \nu, w\rangle \ge a > 0$ on $\partial\Omega$. Deduce that

$$\begin{aligned}\frac{a}{2}\int_{\partial\Omega} |\nabla u|^2\ dS &\le \frac{2}{a}\int_{\partial\Omega} |\partial_\nu u|^2\ dS + \int_\Omega |f|^2\ dV \\ &\quad + \int_\Omega \Big\{|\mathrm{div}\ w| + 2|\mathrm{Def}\ w| + 1\Big\}|\nabla u|^2\ dV.\end{aligned} \tag{11.61}$$

When $\Delta u = f = 0$, compare implications of (11.61) with implications of (11.36).

See [Ver] for applications of Rellich's formula to analysis on domains with Lipschitz boundary.

8. What happens if, in Proposition 11.5, you allow $\mathcal{O} = \mathbb{R}^n \setminus \overline{\Omega}$ to have several connected components? Can you show that *one* of the operators in (11.46) is still an isomorphism?
9. Calculate $q(x_n, x', \xi')$ in (11.13) when $p(x, \xi) = \xi_j |\xi|^{-2}$. Relate this to the results (11.7) and (11.24) for $\mathcal{D}\ell\, f_\pm$ and $\partial_\nu\, \mathcal{S}\ell\, f_\pm$. (*Hint.* The calculation involves $\int (1+\zeta^2)^{-1} e^{i\zeta\tau}\, d\zeta = \pi e^{-|\tau|}$.)
10. Let N and $N^\#$ be the operators given by (11.9) and (11.25). Show that $N^\# = N^*$, the L^2-adjoint of N.

12. Parametrix for regular elliptic boundary problems

Here we shall complement material on regular boundary problems for elliptic operators developed in §11 of Chapter 5, including in particular results promised after the statement of Proposition 11.16 in that chapter.

Suppose P is an elliptic differential operator of order m on a compact manifold $\overline{M}$ with boundary, with boundary operators B_j of order m_j, $1 \le j \le \ell$, satisfying the regularity conditions given in §11 of Chapter 5. In order to construct a parametrix for the solution to $Pu = f$, $B_j u|_{\partial M} = g_j$, we will use pseudodifferential operator calculus to manipulate P in ways that constant-coefficient operators $P(D)$ were manipulated in that section. To start, we choose a collar neighborhood $\mathcal{C}$ of ∂M, $\mathcal{C} \approx [0,1] \times \partial M$; use coordinates (y, x), $y \in [0,1], x \in \partial M$; and without loss of generality, consider

$$Pu = \frac{\partial^m u}{\partial y^m} + \sum_{j=0}^{m-1} A_j(y, x, D_x) \frac{\partial^j u}{\partial y^j}, \tag{12.1}$$

the order of $A_j(y, x, D_x)$ being $\le m - j$. We convert $Pu = f$ to a first-order system using $v = (v_1, \dots, v_m)^t$, with

$$v_1 = \Lambda^{m-1} u, \dots, v_j = \partial_y^{j-1} \Lambda^{m-j} u, \dots, v_m = \partial_y^{m-1} u, \tag{12.2}$$

as in (11.41) of Chapter 5. Here, Λ can be taken to be any elliptic, invertible operator in $OPS^1(\partial M)$, with principal symbol $|\xi|$ (with respect to some Riemannian metric put on ∂M). Then $Pu = f$ becomes, on $\mathcal{C}$, the system

$$\frac{\partial v}{\partial y} = K(y, x, D_x) v + F, \tag{12.3}$$

where $F = (0, \dots, 0, f)^t$ and

$$K = \begin{pmatrix} 0 & \Lambda & & & \\ & 0 & \Lambda & & \\ & & \ddots & \ddots & \\ & & & & \Lambda \\ C_0 & C_1 & C_2 & \dots & C_{m-1} \end{pmatrix}, \tag{12.4}$$

where

$$C_j(y,x,D_x) = -A_j(y,x,D_x)\Lambda^{1-(m-j)} \tag{12.5}$$

is a smooth family of operators in $OPS^1(\partial M)$, with y as a parameter. As in Lemma 11.3 of Chapter 5, we have that P is elliptic if and only if, for all $(x,\xi) \in T^*\partial M \setminus 0$, the principal symbol $K_1(y,x,\xi)$ has no purely imaginary eigenvalues.

We also rewrite the boundary conditions $B_j u = g_j$ at $y = 0$. If

$$B_j = \sum_{k \le m_j} b_{jk}(x,D_x)\,\frac{\partial^k}{\partial y^k} \tag{12.6}$$

at $y = 0$, then we have for v_j the boundary conditions

$$\sum_{k \le m_j} \tilde{b}_{jk}(x,D_x)\Lambda^{k-m_j} v_{k+1}(0) = \Lambda^{m-m_j-1} g_j = h_j, \quad 1 \le j \le \ell, \tag{12.7}$$

where $\tilde{b}_{jk}(x,D)$ has the same principal symbol as $b_{jk}(x,D)$. We write this as

$$B(x,D_x)v(0) = h, \quad B(x,D_x) \in OPS^0(\partial M). \tag{12.8}$$

We will construct a parametrix for the solution of (12.3), (12.8), with $F = 0$.

Generalizing (11.52) of Chapter 5, we construct $E_0(y,x,\xi)$ for $(x,\xi) \in T^*\partial M \setminus 0$, the projection onto the sum of the generalized eigenspaces of $K_1(y,x,\xi)$ corresponding to eigenvalues of positive real part, annihilating the other generalized eigenspaces, in the form

$$E_0(y,x,\xi) = \frac{1}{2\pi i}\int_\gamma \bigl(\zeta - K_1(y,x,\xi)\bigr)^{-1} d\zeta, \tag{12.9}$$

where $\gamma = \gamma(y,x,\xi)$ is a curve in the right half-plane of $\mathbb{C}$, encircling all the eigenvalues of $K_1(y,x,\xi)$ of positive real part. Then $E_0(y,x,\xi)$ is homogeneous of degree 0 in ξ, so it is the principal symbol of a family of operators in $OPS^0(\partial M)$.

Recall the statement of Proposition 11.9 of Chapter 5 on the regularity condition for $(P, B_j, 1 \le j \le \ell)$. One characterization is that, for $(x,\xi) \in T^*\partial M \setminus 0$,

$$B_0(x,\xi) : V(x,\xi) \longrightarrow \mathbb{C}^\lambda \text{ isomorphically}, \tag{12.10}$$

where $V(x,\xi) = \ker\, E_0(0,x,\xi)$, and $B_0(x,\xi) : \mathbb{C}^\nu \to \mathbb{C}^\lambda$ is the principal symbol of $B(x,D_x)$. Another, equivalent characterization is that, for any $\eta \in \mathbb{C}^\lambda$, $(x,\xi) \in T^*\partial M \setminus 0$, there exists a unique bounded solution on $y \in [0,\infty)$ to the ODE

$$\frac{\partial \varphi}{\partial y} - K_1(0,x,\xi)\varphi = 0, \quad B_0(x,\xi)\varphi(0) = \eta. \tag{12.11}$$

In that case, of course, $\varphi(0) = \varphi(0, x, \xi)$ belongs to $V(x, \xi)$, so $\varphi(y, x, \xi)$ is actually exponentially decreasing as $y \to +\infty$, for fixed (x, ξ), and it is exponentially decreasing as $|\xi| \to \infty$, for fixed $y > 0, x \in \partial M$.

On a conic neighborhood Γ of any $(x_0, \xi_0) \in T^*\partial M \setminus 0$, one can construct $U_0(y, x, \xi)$ smooth and homogeneous of degree 0 in ξ, so that

$$U_0 K_1 U_0^{-1} = \begin{pmatrix} E_1 & 0 \\ 0 & F_1 \end{pmatrix}, \tag{12.12}$$

where $E_1(y, x, \xi)$ has eigenvalues all in $\operatorname{Re} \zeta < 0$ and F_1 has all its eigenvalues in $\operatorname{Re} \zeta > 0$. If we set $w^{(0)} = U_0(y, x, D)v$, then the equation $\partial v/\partial y = K(y, x, D_x)v$ is transformed to

$$\frac{\partial w^{(0)}}{\partial y} = \begin{pmatrix} E & \\ & F \end{pmatrix} w^{(0)} + Aw^{(0)} = Gw^{(0)} + Aw^{(0)}, \tag{12.13}$$

where $E(y, x, D_x)$ and $F(y, x, D_x)$ have E_1 and F_1 as their principal symbols, respectively, and $A(y, x, D_x)$ is a smooth family of operators in the space $OPS^0(\partial M)$.

We want to decouple this equation more completely into two pieces. The next step is to decouple terms of order zero. Let $w^{(1)} = (I + V_1)w^{(0)}$, with $V_1 \in OPS^{-1}$ to be determined. We have

(12.14)

$$\begin{aligned} \frac{\partial w^{(1)}}{\partial y} &= (I + V_1)G(I + V_1)^{-1}w^{(1)} + (I + V_1)A(I + V_1)^{-1}w^{(1)} + \cdots \\ &= Gw^{(1)} + (V_1 G - GV_1 + A)w^{(1)} + \cdots, \end{aligned}$$

where the remainder involves terms of order at most -1 operating on $w^{(1)}$. We would like to pick V_1 so that the off-diagonal terms of $V_1 G - GV_1 + A$ vanish. We require V_1 to be of the form

$$V_1 = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}.$$

If A is put into 2×2 block form with entries A_{jk}, we are led to require that (on the symbol level)

$$\begin{aligned} V_{12}E_1 - F_1 V_{12} &= -A_{12}, \\ V_{21}F_1 - E_1 V_{21} &= -A_{21}. \end{aligned} \tag{12.15}$$

That we have unique solutions $V_{jk}(y, x, \xi)$ (homogeneous of degree -1 in ξ) is a consequence of the following lemma.

Lemma 12.1. *Let $F \in M_{\nu\times\nu}$, the set of $\nu \times \nu$ matrices, and $E \in M_{\mu\times\mu}$. Define $\psi : M_{\nu\times\mu} \to M_{\nu\times\mu}$ by*

$$\psi(T) = TF - ET.$$

Then ψ is bijective, provided E and F have disjoint spectra.

Proof. In fact, if $\{f_j\}$ are the eigenvalues of F and $\{e_k\}$ those of E, it is easily seen that the eigenvalues of ψ are $\{f_j - e_k\}$.

Thus we obtain solutions V_{12} and V_{21} to (12.15). With such a choice of the symbol of K_1, we have

$$\frac{\partial w^{(1)}}{\partial y} = Gw^{(1)} + \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} w^{(1)} + Bw^{(1)}, \tag{12.16}$$

with $B \in OPS^{-1}$. To decouple the part of order -1, we try $w^{(2)} = (I + V_2)w^{(1)}$ with $V_2 \in OPS^{-2}$. We get

$$\frac{\partial w^{(2)}}{\partial y} = Gw^{(2)} + \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} w^{(2)} + (V_2 G - GV_2 + B)w^{(2)} + \cdots , \tag{12.17}$$

so we want to choose V_2 so that, on the symbol level, the off-diagonal terms of $V_2 G - GV_2 + B$ vanish. This is the problem solved above, so we are in good shape.

From here we continue, defining $w^{(j)} = (I+V_j)w^{(j-1)}$ with $V_j \in OPS^{-j}$, decoupling further out along the line. Letting $w = (I + V)v$, with

$$I + V \sim \cdots (I + V_3)(I + V_2)(I + V_1), \quad V \in OPS^{-1}, \tag{12.18}$$

we have

$$\frac{\partial w}{\partial y} = \begin{pmatrix} E' & \\ & F' \end{pmatrix} w, \quad \text{mod } C^\infty, \tag{12.19}$$

with $E' = E$, $F' = F$ mod OPS^0. The system (12.3) is now completely decoupled.

We now concentrate on constructing a parametrix for an "elliptic evolution equation"

$$\frac{\partial u}{\partial y} = E(y, x, D_x)u, \quad u(0) = f, \tag{12.20}$$

where E is a $k \times k$ system of first-order pseudodifferential operators, whose principal symbol satisfies

$$\text{spec } E_1(y, x, \xi) \subset \{\zeta \in \mathbb{C} : \text{Re } \zeta \le -C_0|\xi| < 0\}, \quad \xi \neq 0, \tag{12.21}$$

for some $C_0 > 0$. We look for the parametrix in the form (in local coordinates on ∂M)

$$u(y) = \int A(y, x, \xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{12.22}$$

with $A(y, x, \xi)$ in the form

$$A(y, x, \xi) \sim \sum_{j\ge 0} A_j(y, x, \xi), \tag{12.23}$$

and the $A_j(y,x,\xi)$ constructed inductively. We aim to obtain $A(y,x,\xi)$ bounded in $S^0_{1,0}$, for $y \in [0,1]$, among other things. In such a case,

$$\Bigl(\frac{\partial}{\partial y} - E\Bigr)u = (2\pi)^{-n} \int \Bigl(\frac{\partial A}{\partial y} - L(y,x,\xi)\Bigr) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{12.24}$$

where

$$L(y,x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} E^{(\alpha)}(y,x,\xi) A_{(\alpha)}(y,x,\xi). \tag{12.25}$$

We define $A_0(y,x,\xi)$ by the "transport equation"

$$\frac{\partial}{\partial y} A_0(y,x,\xi) = E(y,x,\xi) A_0(y,x,\xi), \quad A_0(0,x,\xi) = I. \tag{12.26}$$

If E is independent of y, the solution is

$$A_0(y,x,\xi) = e^{yE(x,\xi)}.$$

In general, $A_0(y,x,\xi)$ shares with this example the following important properties.

Lemma 12.2. *For $y \in [0,1]$, $k,\ell = 0,1,2,\dots,$ we have*

$$y^k D_y^\ell A_0(y,x,\xi) \text{ bounded in } S^{-k+\ell}_{1,0}. \tag{12.27}$$

Proof. We can take $C_2 \in (0, C_0)$ and M large, so that $E(y,x,\xi)$ has spectrum in the half-space $\operatorname{Re}\zeta < -C_2|\xi|$, for $|\xi| \geq M$. Fixing $K \in (0, C_2)$, if $S(y,\sigma,x,\xi)$ is the solution operator to $\partial B/\partial y = E(y,x,\xi)B$, taking $B(\sigma,x,\xi)$ to $B(y,x,\xi)$, then, for $y > \sigma$,

$$|S(y,\sigma,x,\xi)B| \leq C\, e^{-K(y-\sigma)|\xi|} |B|, \quad \text{for } |\xi| \geq M. \tag{12.28}$$

It follows that, for $y \in [0,1]$,

$$|A_0(y,x,\xi)| \leq C\, e^{-Ky|\xi|}, \tag{12.29}$$

which implies

$$|y^k A_0(y,x,\xi)| \leq C_k \langle\xi\rangle^{-k}\, e^{-Ky|\xi|/2}.$$

Now $A_{0j} = \partial A_0/\partial\xi_j$ satisfies

$$\frac{\partial}{\partial y} A_{0j} = E(y,x,\xi) A_{0j} + \frac{\partial E}{\partial \xi_j}(y,x,\xi) A_0, \quad A_{0j}(0,x,\xi) = 0,$$

so

$$A_{0j}(y,x,\xi) = \int_0^y S(y,\sigma,x,\xi) \frac{\partial E}{\partial \xi_j}(\sigma,x,\xi) A_0(\sigma,x,\xi)\, d\sigma, \tag{12.30}$$

which in concert with (12.28) and (12.29) yields

$$\Bigl|\frac{\partial}{\partial \xi_j} A_0(y,x,\xi)\Bigr| \leq Cy e^{-Ky|\xi|} \leq C\langle\xi\rangle^{-1} e^{-Ky|\xi|/2}. \tag{12.31}$$

Inductively, one obtains estimates on $D_\xi^\alpha D_x^\beta A_0(y,x,\xi)$ leading to the $\ell = 0$ case of (12.27), and then use of (12.26) and induction on ℓ give (12.27) in general.

For $j \geq 1$, we define $A_j(y,x,\xi)$ inductively by

$$\frac{\partial A_j}{\partial y} = E(y,x,\xi)A_j(y,x,\xi) + R_j(y,x,\xi), \quad A_j(0,x,\xi) = 0, \tag{12.32}$$

where

$$R_j(y,x,\xi) = \sum_{\ell<j,\ell+|\alpha|=j} \frac{1}{\alpha!} E^{(\alpha)}(y,x,\xi)A_{\ell(\alpha)}(y,x,\xi). \tag{12.33}$$

Then, if, as above, $S(y,\sigma,x,\xi)$ is the solution operator to the equation $\partial B/\partial y = E(y,x,\xi)B$, we have

$$A_j(y,x,\xi) = \int_0^y S(y,\sigma,x,\xi)R_j(\sigma,x,\xi)\, d\sigma, \quad j \geq 1. \tag{12.34}$$

The arguments used to prove Lemma 12.2 also establish the following result.

Lemma 12.3. *For $y \in [0,1]$, $k,\ell = 0,1,2,\dots, j \geq 1$, we have*

$$y^k D_y^\ell A_j(y,x,\xi) \text{ bounded in } S_{1,0}^{-j-k+\ell}. \tag{12.35}$$

A symbol satisfying the condition (12.35) will be said to belong to $\mathcal{P}^{-j}$. In fact, it is convenient to use the following stronger property possessed by the symbols $A_j(y,x,\xi)$, for $j \geq 0$. Given the hypothesis (12.21) on spec $E_1(y,x,\xi)$, let $0 < C_1 < C_0$. Then

$$A_j(y,x,\xi) = B_j(y,x,\xi)e^{-C_1 y\langle\xi\rangle}, \quad \text{with } B_j(y,x,\xi) \in \mathcal{P}^{-j}. \tag{12.36}$$

We will say $A_j(y,x,\xi) \in \mathcal{P}_e^{-j}$ if this holds or, more generally, if it holds modulo a smooth family of symbols $S(y) \in S^{-\infty}$, $y \in [0,1]$. The associated families of operators will be denoted $OP\mathcal{P}^{-j}$ and $OP\mathcal{P}_e^{-j}$, respectively.

Operators formed from such symbols have the following mapping property, recapturing the Sobolev space regularity established for solutions to regular elliptic boundary problems in Chapter 5.

Proposition 12.4. *If $A = A(y,x,D_x)$ has symbol*

$$A(y,x,\xi) = B(y,x,\xi)e^{-C_1 y\langle\xi\rangle}, \quad B(y,x,\xi) \in \mathcal{P}^{-j},$$

then, for $s \geq -j - 1/2$,

$$A : H^s(\partial M) \longrightarrow H^{s+j+1/2}(I \times \partial M). \tag{12.37}$$

Proof. First consider the case $s = -1/2, j = 0$. As $B(y, x, D_x)$ is bounded in $\mathcal{L}(L^2(\partial M))$ for $y \in [0, 1]$, we have, for $f \in H^{-1/2}(\partial M)$,

$$\begin{aligned}\int_0^1 \|A(y)f\|^2_{L^2(\partial M)}dy &\le C\int_0^1 \|e^{-C_1 y\Lambda}f\|^2_{L^2(\partial M)}dx \\ &= C_2\|\Lambda^{-1/2}f\|^2_{L^2(\partial M)} - C_2\|e^{-C_1\Lambda}\Lambda^{-1/2}f\|^2_{L^2(\partial M)},\end{aligned}$$

with $C_2 = C/(2C_1)$, since

$$(e^{-C_1 y\Lambda}f, e^{-C_1 y\Lambda}f) = -\frac{1}{2C_1}\frac{d}{dy}(e^{-2C_1 y\Lambda}f, \Lambda^{-1}f).$$

This proves (12.37) in this case. The extension to $s = k-1/2$ $(k = 1, 2, \dots)$, $j = 0$ is straightforward, and then the result for general $s \ge -1/2, j = 0$ follows by interpolation. The case of general j is reduced to that of $j = 0$ by forming $A(y, x, \xi)\langle\xi\rangle^{-j}$. One can take any $j \in \mathbb{R}$.

Having constructed operators with symbols in $\mathcal{P}^0_e$ as parametrices of (12.20), we now complete the construction of parametrices for the system (12.3), (12.8), when the regularity condition (12.10) holds. Using a partition of unity, write h as a sum $\sum h_j$, each term of which has wave front set in a conic set Γ_j on which the decoupling procedure (12.12) can be implemented. We drop the subscript j and just call the term h. Then, we construct a parametrix for $w = (I + V)U_0 v$, so that w solves (12.19), with $w(0) = (f, 0)^t$. Set $U = (I + V)U_0$, and let U^{-1} denote a parametrix of U. The solution $w(y)$ takes the form $w(y) = (w_1(y), 0)$, with

$$w_1(y) = A_1(y, x, D_x)f, \quad A_1(y, x, \xi) \in \mathcal{P}^0_e, \tag{12.38}$$

using the construction (12.22)–(12.34). Note that $v(0) = U^{-1}(f, 0)^t = U^{-1}J_1 f$, where here and below we set $J_1 f = (f, 0)^t$. Then

$$Bv(0) = BU^{-1}J_1 f, \tag{12.39}$$

so the boundary condition (12.8) is achieved (mod C^∞) provided f satisfies (mod C^∞)

$$BU^{-1}J_1 f = h. \tag{12.40}$$

The regularity condition (12.10) is precisely the condition that $BU^{-1}J_1$ is an elliptic $\lambda \times \lambda$ system, in $OPS^0(\partial M)$. Letting $Q \in OPS^0(\partial M)$ be a parametrix, we obtain

$$v(y) = U(y)^{-1}J_1 A_1(y)Qh = A^{\#}(y)h. \tag{12.41}$$

Recall that $Q \in OPS^0(\partial M)$, $U(y)^{-1}$ is a smooth family of operators in $OPS^0(\partial M)$, and $A_1(y) \in OP\mathcal{P}^0_e$. We can then say the following about the composition $A^{\#}(y) = A^{\#}(y, x, D_x)$.

Lemma 12.5. *Given $P_j(y)$, smooth families in $OPS^{m_j}(\partial M)$, and $A(y) \in OP\mathcal{P}_e^{\mu}$, we have*

$$P_1(y)A(y)P_2(y) = B(y) \in OP\mathcal{P}_e^{\mu+m_1+m_2}. \tag{12.42}$$

The proof is a straightforward application of the results on products from §3.

Consequently, we have a solution mod C^∞ to (12.3), (12.8), constructed in the form $v(y) = A^{\#}(y)h$, with $A^{\#}(y) \in OP\mathcal{P}_e^0$. Finally, returning to the boundary problem for P, we have:

Theorem 12.6. *If $(P, B_j, 1 \le j \le \ell)$ is a regular elliptic boundary problem, then a parametrix (i.e., a solution mod C^∞) for*

$$Pu = 0 \text{ on } M, \quad B_j u = g_j \text{ on } \partial M \tag{12.43}$$

is constructed in the form

$$u = \sum_{j=1}^{\ell} Q_j g_j, \tag{12.44}$$

where $Q_j g_j$ is C^∞ on the interior of $\overline{M}$, and, on a collar neighborhood $\mathcal{C} = [0,1] \times \partial M$,

$$Q_j g_j = Q_j(y) g_j, \quad Q_j(y) \in OP\mathcal{P}_e^{-m_j}. \tag{12.45}$$

Recall that m_j is the order of B_j. Here, the meaning of solution mod C^∞ to (12.43) is that if $u^{\#}$ is given by (12.44), then

$$Pu^{\#} \in C^\infty(\overline{M}), \quad B_j u^{\#} - g_j \in C^\infty(\partial M). \tag{12.46}$$

Of course, the regularity results of Chapter 5 imply that if u is a genuine solution to (12.43), then $u - u^{\#} \in C^\infty(\overline{M})$.

The following is an easy route to localizing boundary regularity results.

Proposition 12.7. *Take $A(y, x, \xi) \in \mathcal{P}^{-j}$. Let $\varphi, \psi \in C^\infty(\partial M)$, and assume their supports are disjoint. Then*

$$f \in \mathcal{D}'(\partial M) \Longrightarrow \varphi A(y, x, D)\psi f \in C^\infty([0,1] \times \partial M). \tag{12.47}$$

Proof. Symbol calculus gives

$$\varphi A(y, x, D)\psi \in \mathcal{P}^{-k}, \quad \forall\, k \ge 0.$$

Hence this is a smooth family of elements of $OPS^{-\infty}(\partial M)$. This readily gives (12.47).

Proposition 12.7 immediately gives the following.

Corollary 12.8. *In the setting of Theorem 12.6, if $\mathcal{O} \subset \partial M$ is open and $g_j \in C^\infty(\mathcal{O})$ for each j, then $u \in C^\infty$ on a neighborhood in $\overline{M}$ of $\mathcal{O}$.*

Exercises

1. Suppose $A(y) \in OP\mathcal{P}^m$. Show that

$$\frac{\partial^j}{\partial y^j} A(y)\Big|_{y=0} = Q_j f, \quad Q_j \in OPS^{m+j}_{1,0}(\partial M). \tag{12.48}$$

   If $A(y) \in OP\mathcal{P}^0_e$ is given by the construction (12.24)–(12.34), show that $Q_j \in OPS^j(\partial M)$.
2. Applying the construction of this section to the Dirichlet problem for Δ on $\overline{M}$, show that the Neumann operator $\mathcal{N}$, defined by (11.31)–(11.32), satisfies

$$\mathcal{N} \in OPS^1(\partial M), \tag{12.49}$$

   thus providing a proof different from that used in (11.36).
3. Show that $A(y, x, \xi)$ belongs to $\mathcal{P}^m_e$ if and only if, for some $\varepsilon > 0$ and all $N < \infty$,

$$|D^\ell_y D^\beta_x D^\alpha_\xi A_0(y, x, \xi)| \le C_{\alpha\beta\ell}\, e^{-\varepsilon y|\xi|}\, \langle\xi\rangle^{m+\ell-|\alpha|} + C_{N\alpha\beta\ell}\langle\xi\rangle^{-N}. \tag{12.50}$$

4. If $A(y, x, \xi) \in \mathcal{P}^{-j}_e$, show that, for some $\kappa > 0$, you can write

$$A(y, x, D) = e^{-\kappa y\Lambda} B(y, x, D), \quad B(y, x, \xi) \in \mathcal{P}^{-j}, \ y \in [0, 1], \tag{12.51}$$

   modulo a smooth family of smoothing operators.
5. If $u = \mathrm{PI}\, f$ is the solution to $\Delta u = 0$, $u\big|_{\partial\Omega} = f$, use Proposition 12.4 and Theorem 12.6 to show that

$$\mathrm{PI} : H^s(\partial\Omega) \longrightarrow H^{s+1/2}(\Omega), \quad \forall\, s \ge -\frac{1}{2}. \tag{12.52}$$

   Compare the regularity result of Propositions 11.14–11.15 in Chapter 5.

13. Parametrix for the heat equation

Let $L = L(x, D)$ be a second-order, elliptic differential operator, whose principal symbol $L_2(x, \xi)$ is a positive scalar function, though lower-order terms need not be scalar. We want to construct an approximate solution to the initial-value problem

$$\frac{\partial u}{\partial t} = -Lu, \quad u(0) = f, \tag{13.1}$$

in the form

$$u(t, x) = \int a(t, x, \xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{13.2}$$

for f supported in a coordinate patch. The amplitude $a(t,x,\xi)$ will have an asymptotic expansion of the form

$$a(t,x,\xi) \sim \sum_{j\ge 0} a_j(t,x,\xi), \tag{13.3}$$

and the $a_j(t,x,\xi)$ will be defined recursively, as follows. By the Leibniz formula, write

$$\begin{aligned} L(a\ e^{ix\cdot\xi}) &= e^{ix\cdot\xi} \sum_{|\alpha|\le 2} \frac{i^{|\alpha|}}{\alpha!} L^{(\alpha)}(x,\xi) D_x^\alpha a(t,x,\xi) \\ &= e^{ix\cdot\xi} \Big[L_2(x,\xi)a(t,x,\xi) + \sum_{\ell=1}^{2} B_{2-\ell}(x,\xi,D_x)a(t,x,\xi) \Big], \end{aligned} \tag{13.4}$$

where $B_{2-\ell}(x,\xi,D_x)$ is a differential operator (of order ℓ) whose coefficients are polynomials in ξ, homogeneous of degree $2-\ell$ in ξ.

Thus, we want the amplitude $a(t,x,\xi)$ in (13.2) to satisfy (formally)

$$\frac{\partial a}{\partial t} \sim -L_2 a - \sum_{\ell=1}^{2} B_{2-\ell}(x,\xi,D_x)a.$$

If a is taken to have the form (13.3), we obtain the following equations, called "transport equations," for a_j:

$$\frac{\partial a_0}{\partial t} = -L_2(x,\xi)a_0(t,x,\xi) \tag{13.5}$$

and, for $j \ge 1$,

$$\frac{\partial a_j}{\partial t} = -L_2(x,\xi)a_j(t,x,\xi) + \Omega_j(t,x,\xi), \tag{13.6}$$

where

$$\Omega_j(t,x,\xi) = -\sum_{\ell=1}^{2} B_{2-\ell}(x,\xi,D_x)a_{j-\ell}(t,x,\xi). \tag{13.7}$$

By convention we set $a_{-1} = 0$. So that (6.15) reduces to Fourier inversion at $t = 0$, we set

$$a_0(0,x,\xi) = 1, \quad a_j(0,x,\xi) = 0, \quad \text{for } j \ge 1. \tag{13.8}$$

Then we have

$$a_0(t,x,\xi) = e^{-tL_2(x,\xi)}, \tag{13.9}$$

and the solution to (13.6) is

$$a_j(t,x,\xi) = \int_0^t e^{(s-t)L_2(x,\xi)} \Omega_j(s,x,\xi)\, ds. \tag{13.10}$$

In view of (13.7), this defines $a_j(t,x,\xi)$ inductively in terms of $a_{j-1}(t,x,\xi)$ and $a_{j-2}(t,x,\xi)$.

We now make a closer analysis of these terms. Define $A_j(t,x,\xi)$ by

$$a_j(t,x,\xi) = A_j(t,x,\xi)e^{-tL_2(x,\xi)}. \tag{13.11}$$

The following result is useful; it applies to A_j for all $j \geq 1$.

Lemma 13.1. *If $\mu = 0,1,2,\dots$, $\nu \in \{1,2\}$, then $A_{2\mu+\nu}$ can be written in the form*

$$A_{2\mu+\nu}(t,x,\xi) = t^{\mu+1}A^{\#}_{2\mu+\nu}(x,\omega,\xi), \quad \text{with } \omega = t^{1/2}\xi. \tag{13.12}$$

The factor $A^{\#}_{2\mu+\nu}(x,\omega,\xi)$ is a polynomial in both ω and ξ. It is homogeneous of degree $2-\nu$ in ξ (i.e., either linear or constant). Furthermore, as a polynomial in ω, each monomial has even order; equivalently, $A^{\#}_{2\mu+\nu}(x,-\omega,\xi) = A^{\#}_{2\mu+\nu}(x,\omega,\xi)$.

To prove the lemma, we begin by recasting (13.10). Let $\Gamma_j(t,x,\xi)$ be defined by

$$\Omega_j(t,x,\xi) = \Gamma_j(t,x,\xi)e^{-tL_2(x,\xi)}. \tag{13.13}$$

Then the recursion (13.7) yields

$$\Gamma_j e^{-tL_2} = -\sum_{\ell=1}^{2} B_{2-\ell}(x,\xi,D_x)\big(A_{j-\ell}e^{-tL_2}\big). \tag{13.14}$$

Applying the Leibniz formula gives

$$\Gamma_j = -\sum_{\ell=1}^{2}\sum_{|\gamma|\leq\ell} \Lambda_\ell(x,\omega)B^{[\gamma]}_{2-\ell}(x,\xi,D_x)A_{j-\ell}(t,x,\xi), \tag{13.15}$$

evaluated at $\omega = t^{1/2}\xi$, where

$$e^{tL_2(x,\xi)}D_x^\gamma e^{-tL_2(x,\xi)} = \Lambda_\gamma(x,t^{1/2}\xi). \tag{13.16}$$

Clearly, $\Lambda_\gamma(x,t^{1/2}\xi)$ is a polynomial in ξ and also a polynomial in t; hence $\Lambda_\gamma(x,\omega)$ is an even polynomial in ω. Note also that the differential operator $B^{[\gamma]}_{2-\ell}(x,\xi,D_x)$ is of order $\ell-|\gamma|$, and its coefficients are polynomials in ξ, homogeneous of degree $2-\ell$, as were those of $B_{2-\ell}(x,\xi,D_x)$. The factor A_j is given by

$$A_j(t,x,\xi) = \int_0^t \Gamma_j(s,x,\xi)\, ds. \tag{13.17}$$

The recursion (13.15)–(13.17) will provide an inductive proof of Lemma 13.1.

To carry this out, assume the lemma true for A_j, for all $j < 2\mu + \nu$. We then have

$$\Gamma_{2\mu+\nu}(t,x,\xi) = \sum_{1\le\ell<\nu}\sum_{|\gamma|\le\ell} \Lambda_\ell(x,\omega) B^{[\gamma]}_{2-\ell}(x,\xi,D_x) A^{\#}_{2\mu+\nu-\ell}(x,\omega,\xi) t^{\mu+1}$$

$$+ \sum_{\nu\le\ell\le 2}\sum_{|\gamma|\le\ell} \Lambda_\ell(x,\omega) B^{[\gamma]}_{2-\ell}(x,\xi,D_x) A^{\#}_{2\mu+\nu-\ell}(x,\omega,\xi) t^{\mu}. \tag{13.18}$$

The first sum is empty if $\nu = 1$. In the first sum, $A^{\#}_{2\mu+\nu-\ell}(x,\omega,\xi)$ is homogeneous of degree $2+\ell-\nu$ in ξ, so in the first sum

$$t^{\mu+1}\Lambda_\gamma(x,\omega) B^{[\gamma]}_{2-\ell}(x,\xi,D_x) A^{\#}_{2\mu+\nu-\ell}(x,\omega,\xi) = t^{\mu+1} H^{\#}_{\mu\nu\ell\gamma}(x,\omega,\xi), \tag{13.19}$$

where $H^{\#}_{\mu\nu\ell\gamma}(x,\omega,\xi)$ is a polynomial in ξ, homogeneous of degree $4-\nu$, and an even polynomial in ω. We can hence write

$$t^{\mu+1} H^{\#}_{\mu\nu\ell\gamma}(x,\omega,\xi) = t^{\mu} H_{\mu\nu\ell\gamma}(x,\omega,\xi), \tag{13.20}$$

where $H_{\mu\nu\ell\gamma}(x,\omega,\xi)$ is a polynomial in ξ, homogeneous of degree $2-\nu$, and an even polynomial in ω.

In the last sum in (13.18), $A^{\#}_{2\mu+\nu-\ell}$ is homogeneous in ξ of degree $\ell-\nu$, so in this sum

$$t^{\mu}\Lambda_\gamma(x,\omega) B^{[\gamma]}_{2-\ell}(x,\xi,D_x) A^{\#}_{2\mu+\nu-\ell}(x,\omega,\xi) = t^{\mu} H_{\mu\nu\ell\gamma}(x,\omega,\xi), \tag{13.21}$$

where, as in (13.20), $H_{\mu\nu\ell\gamma}(x,\omega,\xi)$ is a polynomial in ξ, homogeneous of degree $2-\nu$, and an even polynomial in ω. Thus

$$\Gamma_{2\mu+\nu}(t,x,\xi) = t^{\mu}\sum_{\ell,\gamma} H_{\mu\nu\ell\gamma}(x,\omega,\xi) = t^{\mu} K_{\mu\nu}(x,\omega,\xi), \tag{13.22}$$

where $K_{\mu\nu}$ is a polynomial in ξ, homogeneous of degree $2-\nu$, and an even polynomial in ω. It follows that

$$A_{2\mu+\nu}(t,x,\xi) = \int_0^t s^{\mu} K_{\mu\nu}(x, s^{1/2}\xi, \xi)\, ds \tag{13.23}$$

has the properties stated in Lemma 13.1, whose proof is complete.

The analysis of (13.12) yields estimates on $a_j(t,x,\xi)$, easily obtained by writing (for $j = 2\mu+\nu$, $\nu = 1$ or 2)

$$a_j(t,x,\xi) = t^{\mu+1} A^{\#}_j(x,\omega,\xi) e^{-L_2(x,\omega)/2} e^{-tL_2(x,\xi)/2}, \tag{13.24}$$

and using the simple estimates

$$|\omega|^L e^{-L_2(x,\omega)/2} \le C_L, \quad \bigl(t|\xi|^2\bigr)^{\ell} e^{-tL_2(x,\xi)/2} \le C_{2\ell}. \tag{13.25}$$

Note that $t^{\mu+1} = t^{j/2}$ if j is even; if j is odd, then $t^{\mu+1} = t^{j/2}\cdot t^{1/2}$, and the factor $t^{1/2}$ can be paired with the linear factor of ξ in $A^{\#}_j$. Thus we have estimates

$$|a_j(t,x,\xi)| \le C_j t^{j/2} \tag{13.26}$$

and

$$|a_j(t,x,\xi)| \le C_j\langle\xi\rangle^{-j}. \tag{13.27}$$

Derivatives are readily estimated by the same method, and we obtain:

Lemma 13.2. *For $0 \le t \le T$, $k \ge -j$, we have*

$$t^{k/2} D_t^\ell a_j(t,x,\xi) \quad \text{bounded in } S^{2\ell-k-j}_{1,0}. \tag{13.28}$$

We can construct a function $a(t,x,\xi)$ such that each difference $a(t,x,\xi) - \sum_{\ell<j} a_\ell(t,x,\xi)$ has the properties (13.28), and then, for $u(t,x)$ given by (13.22), we have $u(0,x) = f(x)$ and

$$\Bigl(\frac{\partial}{\partial t} + L\Bigr)u(t,x) = r(t,x), \tag{13.29}$$

where $r(t,x)$ is smooth for $t \ge 0$ and rapidly decreasing as $t \searrow 0$. If the construction is made on a compact manifold M, energy estimates imply that the difference between $u(t,x)$ and $v(t,x) = e^{-tL}f(x)$ is smooth and rapidly decreasing as $t \searrow 0$, for all $f \in \mathcal{D}'(M)$. Consequently the "heat kernel" $H(t,x,y)$, given by

$$e^{-tL}f(x) = \int_M H(t,x,y)\, f(y)\, dV(y), \tag{13.30}$$

and the integral kernel $Q(t,x,y)$ of the operator constructed in the form (13.2) differ by a function $R(t,x,y)$, which is smooth on $[0,\infty) \times M \times M$ and rapidly decreasing as $t \searrow 0$.

Look at the integral kernel of the operator

$$Q_j(t,x,D)f = \int a_j(t,x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{13.31}$$

which is

$$Q_j(t,x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} a_j(t,x,\xi) e^{i(x-y)\cdot\xi}\, d\xi. \tag{13.32}$$

For $a_j(t,x,\xi)$ in the form (13.11)–(13.12), we obtain

$$Q_j(t,x,y) = t^{(j-n)/2}\, q_j\bigl(x, t^{-1/2}(x-y)\bigr), \tag{13.33}$$

where

$$q_0(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-L_2(x,\xi)} e^{iz\cdot\xi}\, d\xi \tag{13.34}$$

and, for $j \ge 1$,

$$q_j(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} A_j^{\#}(x,\xi,\xi) e^{-L_2(x,\xi)} e^{iz\cdot\xi}\, d\xi. \tag{13.35}$$

We can evaluate the Gaussian integral (13.34) via the method developed in Chapter 3. If, in the local coordinate system used in (13.2), $L_2(x,\xi) = \mathcal{L}(x)\xi \cdot \xi$, for a positive-definite matrix $\mathcal{L}(x)$, then

$$q_0(x,z) = \Big[\det\big(4\pi\mathcal{L}(x)\big)\Big]^{-1/2} \, e^{-\mathcal{G}(x)z\cdot z/4}, \tag{13.36}$$

where $\mathcal{G}(x) = \mathcal{L}(x)^{-1}$. Consequently,

$$Q_0(t,x,y) = (4\pi t)^{-n/2}\Big[\det \mathcal{L}(x)\Big]^{-1/2} e^{-\mathcal{G}(x)(x-y)\cdot(x-y)/4t}. \tag{13.37}$$

The integrals (13.35) can be computed in terms of

$$\begin{aligned}(2\pi)^{-n}\int \xi^\beta e^{-L_2(x,\xi)} e^{iz\cdot\xi} d\xi &= \Big[\det\big(4\pi\mathcal{L}(x)\big)\Big]^{-1/2} \, D_z^\beta e^{-\mathcal{G}(x)z\cdot z/4} \\ &= p_\beta(x,z) \; e^{-\mathcal{G}(x)z\cdot z/4},\end{aligned} \tag{13.38}$$

where $p_\beta(x,z)$ is a polynomial of degree $|\beta|$ in z. Clearly, $p_\beta(x,z)$ is even or odd in z according to the parity of $|\beta|$. Note also that, in (13.35), $A_j^\#(x,\xi,\xi)$ is even or odd in ξ according to the parity of j. We hence obtain the following result.

Proposition 13.3. *If L is a second-order, elliptic differential operator with positive scalar principal symbol, then the integral kernel $H(t,x,y)$ of the operator e^{-tL} has the form*

$$H(t,x,y) \sim \sum_{j\ge 0} t^{(j-n)/2} \; p_j\big(x, t^{-1/2}(x-y)\big) \; e^{-\mathcal{G}(x)(x-y)\cdot(x-y)/4t}, \tag{13.39}$$

where $p_j(x,z)$ is a polynomial in z, which is even or odd in z according to the parity of j.

To be precise about the strong sense in which (13.39) holds, we note that, for any $\nu < \infty$, there is an $N < \infty$ such that the difference $R_N(t,x,y)$ between the left side of (13.39) and the sum over $j \le N$ of the right side belongs to $C^\nu([0,\infty)\times M \times M)$ and vanishes to order ν as $t \searrow 0$.

In particular, we have

$$H(t,x,x) \sim \sum_{j\ge 0} t^{-n/2+j} \; p_{2j}(x,0), \tag{13.40}$$

since $p_j(x,0) = 0$ for j odd. Consequently, the trace of the operator e^{-tL} has the asymptotic expansion

$$\operatorname{Tr} e^{-tL} \sim t^{-n/2}\big(a_0 + a_1 t + a_2 t^2 + \cdots\big), \tag{13.41}$$

with

$$a_j = \int_M p_{2j}(x,0)\; dV(x). \tag{13.42}$$

Further use will be made of this in Chapters 8 and 10.

Note that the exponent in (13.39) agrees with $r(x,y)^2/4t$, up to $O(r^3/t)$, for x close to y, where $r(x,y)$ is the geodesic distance from x to y. In fact, when $L = -\Delta$, the integral operator with kernel

$$H_0(t,x,y) = (4\pi t)^{-n/2}\, e^{-r(x,y)^2/4t}, \quad t > 0, \tag{13.43}$$

is in some ways a better first approximation to e^{-tL} than is (13.2) with $a(t,x,\xi)$ replaced by $a_0(t,x,\xi) = e^{-tL_2(x,\xi)}$. (See Exercise 3 below.) It can be shown that

$$\Bigl(\frac{\partial}{\partial t} + L_x\Bigr) H_0(t,x,y) = Q(t,x,y), \quad t > 0, \tag{13.44}$$

is the integral kernel of an operator that is regularizing, and if one defines

$$H_0 \# Q(t,x,y) = \int_0^t \int\limits_M H_0(t-s,x,z) Q(s,z,y)\; dV(z)\; ds, \tag{13.45}$$

then a parametrix that is as good as (13.39) can be obtained in the form

$$\sim H_0 - H_0\#Q + H_0\#Q\#Q - \cdots . \tag{13.46}$$

This approach, one of several alternatives to that used above, is taken in [MS].

One can also look at (13.43)–(13.46) from a pseudodifferential operator perspective, as done in [Gr]. The symbol of $\partial/\partial t + L$ is $i\tau + L(x,\xi)$, and

$$H_0(x,\tau,\xi) = \Bigl(i\tau + L_2(x,\xi)\Bigr)^{-1} \in S^{-1}_{1/2,0}(\mathbb{R}\times M). \tag{13.47}$$

The operator with integral kernel $H_0(t-s,x,y)$ given by (13.43) belongs to $OPS^{-1}_{1/2,0}(\mathbb{R}\times M)$ and has (13.47) as its principal symbol. This operator has two additional properties; it is causal, that is, if v vanishes for $t < T$, so does $H_0 v$, for any T, and it commutes with translations. Denote by $\mathcal{C}^m$ the class of operators in $OPS^m_{1/2,0}(\mathbb{R}\times M)$ with these two properties. One easily has $P_j \in \mathcal{C}^{m_j} \Rightarrow P_1P_2 \in \mathcal{C}^{m_1+m_2}$. The symbol computation gives

$$\Bigl(\frac{\partial}{\partial t} + L\Bigr) H_0 = I + Q, \quad Q \in \mathcal{C}^{-1}, \tag{13.48}$$

and from here one obtains a parametrix

$$H \in \mathcal{C}^{-1}, \quad H \sim H_0 - H_0 Q + H_0 Q^2 - \cdots . \tag{13.49}$$

The formulas (13.46) and (13.49) agree, via the correspondence of operators and their integral kernels.

One can proceed to construct a parametrix for the heat equation on a manifold with boundary. We sketch an approach, using a variant of the double-layer-potential method described for elliptic boundary problems in §11. Let Ω be an open domain, with smooth boundary, in M, a compact

Riemannian manifold without boundary. We construct an approximate solution to

$$\frac{\partial u}{\partial t} = -Lu, \tag{13.50}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$, satisfying

$$u(0,x) = 0, \quad u(t,x) = h(t,x), \quad \text{for } x \in \partial\Omega, \tag{13.51}$$

in the form

$$u = \mathcal{D}\ell\ g(t,x) = \int_0^\infty \int\limits_{\partial\Omega} g(s,y)\ \frac{\partial H}{\partial \nu_y}(t-s,x,y)\ dS(y)\ ds, \tag{13.52}$$

where $H(t,x,y)$ is the heat kernel on $\mathbb{R}^+ \times M$ studied above. For $x \in \partial\Omega$, denote by $\mathcal{D}\ell\ g_+(t,x)$ the limit of $\mathcal{D}\ell\ g$ from within $\mathbb{R}^+ \times \Omega$. As in (11.7), one can establish the identity

$$\mathcal{D}\ell\ g_+ = \frac{1}{2}(I+N)g, \tag{13.53}$$

where $(1/2)Ng$ is given by the double integral on the right side of (13.52), with y and x both in $\partial\Omega$. In analogy with (11.23), we have

$$N \in OPS^{-1/2}_{1/2,0}(\mathbb{R}^+ \times \partial\Omega).$$

For u to solve (13.50)–(13.51), we need

$$h = \frac{1}{2}(I+N)g. \tag{13.54}$$

Thus we have a parametrix for (13.50)–(13.51) in the form (13.52) with

$$g \sim 2(I - N + N^2 - \cdots)h. \tag{13.55}$$

We can use the analysis of (13.50)–(13.55) to construct a parametrix for the solution operator to

$$\frac{\partial u}{\partial t} = \Delta u, \text{ for } x \in \Omega, \quad u(0,x) = f(x), \quad u(t,x) = 0, \text{ for } x \in \partial\Omega. \tag{13.56}$$

To begin, let v solve

$$\frac{\partial v}{\partial t} = \Delta v \text{ on } \mathbb{R}^+ \times M, \quad v(0) = \widetilde{f}, \tag{13.57}$$

where

$$\begin{aligned} \widetilde{f}(x) = f(x), &\quad \text{for } x \in \Omega, \\ 0, &\quad \text{for } x \in M \setminus \Omega. \end{aligned} \tag{13.58}$$

One way to obtain u would be to subtract a solution to (13.50)–(13.51), with $-L = \Delta$, $h = v\big|_{\mathbb{R}^+ \times \partial\Omega}$. This leads to a parametrix for the solution operator for (13.56) of the form

(13.59)

$$p(t,x,y) = H(t,x,y) - \int_0^\infty \int_{\partial\Omega} h(s,z,y) \frac{\partial H}{\partial \nu_z}(t-s,x,z)\, dS(z)\, ds,$$

$$h(s,z,y) \sim 2H(s,z,y) + \cdots,$$

where, as above, $H(t,x,y)$ is the heat kernel on $\mathbb{R}^+ \times M$.

We mention an alternative treatment of (13.56) that has some advantages. We will apply a reflection to v. To do this, assume that $\overline{\Omega}$ is contained in a compact Riemannian manifold M, diffeomorphic to the double of $\overline{\Omega}$, and let $R : M \to M$ be a smooth involution of M, fixing $\partial\Omega$, which near $\partial\Omega$ is a reflection of each geodesic normal to $\partial\Omega$, about the point where the geodesic intersects $\partial\Omega$. Pulling back the metric tensor on M by R yields a metric tensor that agrees with the original on $\partial\Omega$. Now set

$$u_1(t,x) = v(t,x) - v\big(t, R(x)\big), \quad x \in \Omega. \tag{13.60}$$

We see that u_1 satisfies

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + g, \quad u_1(0,x) = f, \quad u_1(t,x) = 0, \text{ for } x \in \partial\Omega, \tag{13.61}$$

where

$$g = L^b \widetilde{v}\big|_{\mathbb{R}^+ \times \Omega}, \quad \widetilde{v}(t,x) = v\big(t, R(x)\big), \tag{13.62}$$

and where L^b is a second-order differential operator, with smooth coefficients, whose principal symbol vanishes on $\partial\Omega$. Thus the difference $u - u_1 = w$ solves

$$\frac{\partial w}{\partial t} = \Delta w - g, \quad w(0) = 0, \quad w(t,x) = 0, \text{ for } x \in \partial\Omega. \tag{13.63}$$

Next let v_2 solve

$$\frac{\partial v_2}{\partial t} = \Delta v_2 - \widetilde{g} \text{ on } \mathbb{R}^+ \times M, \quad v_2(0) = 0, \tag{13.64}$$

where

$$\begin{aligned} \widetilde{g}(t,x) = g(t,x), &\quad \text{for } x \in \Omega, \\ 0, &\quad \text{for } x \in M \setminus \Omega, \end{aligned} \tag{13.65}$$

and set

$$u_2 = v_2\big|_{\mathbb{R}^+ \times \Omega}. \tag{13.66}$$

It follows that $w_2 = u - (u_1 + u_2)$ satisfies

$$\frac{\partial w_2}{\partial t} = \Delta w_2 \text{ on } \mathbb{R}^+ \times \Omega, \quad w_2(0) = 0, \quad w_2\big|_{\mathbb{R}^+ \times \partial\Omega} = -v_2\big|_{\mathbb{R}^+ \times \partial\Omega}. \tag{13.67}$$

Now we can obtain w_2 by the construction (13.52)–(13.55), with

$$h = -v_2\big|_{\mathbb{R}^+ \times \partial\Omega}.$$

To illustrate the effect of this construction using reflection, suppose that, in (13.56),

$$f \in H_0^1(\Omega). \tag{13.68}$$

Then, in (13.57)–(13.58), $\widetilde{f} \in H^1(M)$, so $v \in C\big(\mathbb{R}^+, H^1(M)\big)$, and hence

$$u_1 \in C\big(\mathbb{R}^+, H_0^1(\Omega)\big). \tag{13.69}$$

Furthermore, given the nature of L^b and that of the heat kernel on $\mathbb{R}^+ \times M \times M$, one can show that, in (13.62),

$$g \in C\big(\mathbb{R}^+, L^2(\Omega)\big), \tag{13.70}$$

that is, L^b effectively acts like a first-order operator on $\widetilde{v}$, when one restricts to Ω. It follows that $\widetilde{g} \in C\big(\mathbb{R}^+, L^2(M)\big)$ and hence, via Duhamel's formula for the solution to (13.64), that $v_2 \in C\big(\mathbb{R}^+, H^{2-\varepsilon}(M)\big)$, $\forall\, \varepsilon > 0$. Therefore,

$$u_2 \in C\big(\mathbb{R}^+, H^{2-\varepsilon}(\Omega)\big), \tag{13.71}$$

and, in (13.67), we have a PDE of the form (13.50)–(13.51), with $h \in C\big(\mathbb{R}^+, H^{3/2-\varepsilon}(\partial\Omega)\big)$, for all $\varepsilon > 0$. One can deduce from (13.52)–(13.55) that w_2 has as much regularity as that given for u_2 in (13.71).

It also follows directly from Duhamel's principle, applied to (13.63), that

$$w \in C\big(\mathbb{R}^+, H^{2-\varepsilon}(\Omega)\big), \tag{13.72}$$

so we can see without analyzing (13.52)–(13.55) that w_2 has as much regularity as mentioned above. Either way, we see that when f satisfies (13.68), the principal singularities of the solution u to (13.56) are captured by u_1, defined by (13.60). Constructions of u_2 and, via (13.52)–(13.55), of w_2 yield smoother corrections, at least when smoothness is measured in the spaces used above.

The construction (13.56)–(13.67) can be compared with constructions in §7 of Chapter 13.

Exercises

1. Let L be a positive, self-adjoint, elliptic differential operator of order $2k > 0$ on a compact manifold M, with scalar principal symbol $L_{2k}(x,\xi)$. Show that a parametrix for $\partial u/\partial t = -Lu$ can be constructed in the form (13.2)–(13.3), with $a_j(t,x,\xi)$ of the following form, generalizing (13.11)–(13.12):

$$a_j(t,x,\xi) = A_j(t,x,\xi)e^{-tL_{2k}(x,\xi)},$$

where $A_0(t,x,\xi) = 1$ and if $\mu = 0,1,2,\dots$ and $\nu \in \{1,\dots,2k\}$, then

$$A_{2k\mu+\nu}(t,x,\xi) = t^{\mu+1}A^{\#}_{2k\mu+\nu}(x,\omega,\xi), \quad \omega = t^{1/2k}\xi,$$

where $A^{\#}_{2k\mu+\nu}(x,\omega,\xi)$ is a polynomial in ξ, homogeneous of degree $2k-\nu$, whose coefficients are polynomials in ω, each monomial of which has degree (in ω) that is an integral multiple of $2k$, so $A^{\#}_{2k\mu+\nu}(x,e^{\pi i/k}\omega,\xi) = A^{\#}_{2k\mu+\nu}(x,\omega,\xi)$.

2. In the setting of Exercise 1, show that
$$\mathrm{Tr}\ e^{-tL} \sim t^{-n/2k}\Big(a_0 + a_1 t^{1/k} + a_2 t^{2/k} + \cdots\Big),$$
generalizing (13.41).
3. Let $g_{jk}(y,x)$ denote the components of the metric tensor at x in a normal coordinate system centered at y. Suppose $-Lu(x) = \Delta u(x) = g^{jk}(y,x)\,\partial_j\partial_k u(x) + b^j(y,x)\,\partial_j u(x)$ in this coordinate system. With $H_0(t,x,y)$ given by (13.43), show that
$$\begin{aligned}\Big(\frac{\partial}{\partial t} + L_x\Big) & H_0(t,x,y) \\ &= H_0(t,x,y)\Big\{(2t)^{-2}\Big[g^{jk}(x,x) - g^{jk}(y,x)\Big](x_j - y_j)(x_k - y_k) \\ &\qquad - (2t)^{-1}\Big[g^j{}_j(x,x) - g^j{}_j(y,x) - b^j(y,x)(x_j - y_j)\Big]\Big\} \\ &= H_0(t,x,y)\Big\{O\Big(\frac{|x-y|^4}{t^2}\Big) + O\Big(\frac{|x-y|^2}{t}\Big)\Big\}.\end{aligned}$$
Compare formula (2.10) in Chapter 5. Note that $g_{jk}(y,y) = \delta_{jk}$, $\partial_\ell g_{jk}(y,y) = 0$, and $b^j(y,y) = 0$. Relate this calculation to the discussion involving (13.43)–(13.49).
4. Using the parametrix, especially (13.39), show that if M is a smooth, compact Riemannian manifold, without boundary, then
$$e^{t\Delta} : C^k(M) \longrightarrow C^k(M)$$
is a strongly continuous semigroup, for each $k \in \mathbb{Z}^+$.

14. The Weyl calculus

To define the Weyl calculus, we begin with a modification of the formula (1.10) for $a(x,D)$. Namely, we replace $e^{iq\cdot X}e^{ip\cdot D}$ by $e^{i(q\cdot X + p\cdot D)}$, and set

$$a(X,D)u = \int \hat{a}(q,p)e^{i(q\cdot X+p\cdot D)}\ u\ dq\ dp, \tag{14.1}$$

initially for $a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n})$. Note that $v(t,x) = e^{it(q\cdot X+p\cdot D)}u(x)$ solves the PDE

$$\frac{\partial v}{\partial t} = \sum_j p_j \frac{\partial v}{\partial x_j} + i(q\cdot x)v, \quad v(0,x) = u(x), \tag{14.2}$$

and the solution is readily obtained by integrating along the integral curves of $\partial/\partial t - \sum p_j\,\partial/\partial x_j$, which are straight lines. We get

$$e^{i(q\cdot X+p\cdot D)}u(x) = e^{iq\cdot x + iq\cdot p/2}\ u(x+p). \tag{14.3}$$

Note that this is equivalent to the identity

$$e^{i(q\cdot X+p\cdot D)} = e^{iq\cdot p/2}\ e^{iq\cdot X}\ e^{ip\cdot D}. \tag{14.4}$$

If we plug (14.3) into (14.1), a few manipulations using the Fourier inversion formula yield

$$a(X,D)u(x) = (2\pi)^{-n} \int a\Big(\frac{x+y}{2},\xi\Big) e^{i(x-y)\cdot\xi}\ u(y)\ dy\ d\xi, \tag{14.5}$$

which can be compared with the formula (1.3) for $a(x,D)$. Note that $a(X,D)$ is of the form (3.2) with $a(x,y,\xi) = a\big((x+y)/2,\xi\big)$, while $a(x,D)$ is of the form (3.2) with $a(x,y,\xi) = a(x,\xi)$. In particular, Proposition 3.1 is applicable; we have

$$a(X,D) = b(x,D), \tag{14.6}$$

where

$$b(x,\xi) = e^{iD_\xi\cdot D_y}\ a\Big(\frac{x+y}{2},\xi\Big)\Big|_{y=x} = e^{(i/2)D_\xi\cdot D_x}\ a(x,\xi). \tag{14.7}$$

If $a(x,\xi) \in S^m_{\rho,\delta}$, with $0 \le \delta < \rho \le 1$, then $b(x,\xi)$ also belongs to $S^m_{\rho,\delta}$ and, by (3.6),

$$b(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} 2^{-|\alpha|}\ D_\xi^\alpha D_x^\alpha a(x,\xi). \tag{14.8}$$

Of course this relation is invertible; we have $a(x,\xi) = e^{-(i/2)D_\xi\cdot D_x} b(x,\xi)$ and a corresponding asymptotic expansion. Thus, at least on a basic level, the two methods of assigning an operator, either $a(x,D)$ or $a(X,D)$, to a symbol $a(x,\xi)$ lead to equivalent operator calculi. However, they are not identical, and the differences sometimes lead to subtle advantages for the Weyl calculus.

One difference is that since the adjoint of $e^{i(q\cdot X+p\cdot D)}$ is $e^{-i(q\cdot X+p\cdot D)}$, we have the formula

$$a(X,D)^* = b(X,D), \quad b(x,\xi) = a(x,\xi)^*, \tag{14.9}$$

which is somewhat simpler than the formula (3.13)–(3.14) for $a(x,D)^*$.

Other differences can be traced to the fact that the Weyl calculus exhibits certain symmetries rather clearly. To explain this, we recall, from the exercises after §1, that the set of operators

$$e^{it}\ e^{iq\cdot X}\ e^{ip\cdot D} = \tilde{\pi}(t,q,p) \tag{14.10}$$

form a unitary group of operators on $L^2(\mathbb{R}^n)$, a representation of the group $\mathcal{H}^n$, with group law

$$(t,q,p)\circ(t',q',p') = (t+t'+p\cdot q', q+q', p+p'). \tag{14.11}$$

Now, using (14.4), one easily computes that

$$e^{i(t+q\cdot X+p\cdot D)}\ e^{i(t'+q'\cdot X+p'\cdot D)} = e^{i(s+u\cdot X+v\cdot D)}, \tag{14.12}$$

with $u = q+q'$, $v = p+p'$, and

$$s = t+t'+\frac{1}{2}(p\cdot q' - q\cdot p') = t+t'+\frac{1}{2}\sigma\big((p,q),(p',q')\big), \tag{14.13}$$

where σ is the natural symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$. Thus

$$\pi(t, q, p) = e^{i(t+q\cdot X+p\cdot D)} \tag{14.14}$$

defines a unitary representation of a group we'll denote $\mathbf{H}^n$, which is $\mathbb{R}\times\mathbb{R}^{2n}$ with group law

$$(t, w) \cdot (t', w') = \Big(t + t' + \frac{1}{2}\sigma(w, w'), w + w'\Big), \tag{14.15}$$

where we have set $w = (q, p)$. Of course, the groups $\mathcal{H}^n$ and $\mathbf{H}^n$ are isomorphic; both are called the *Heisenberg group*. The advantage of using the group law (14.15) rather than (14.11) is that it makes transparent the existence of the action of the group $\mathrm{Sp}(n, \mathbb{R})$ of linear symplectic maps on $\mathbb{R}^{2n}$, as a group of automorphisms of $\mathbf{H}^n$. Namely, if $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear map preserving the symplectic form, so $\sigma(gw, gv) = \sigma(w, v)$ for $v, w \in \mathbb{R}^{2n}$, then

$$\alpha(g) : \mathbf{H}^n \to \mathbf{H}^n, \quad \alpha(g)(t, w) = (t, gw) \tag{14.16}$$

defines an automorphism of $\mathbf{H}^n$, so

$$(t, w) \cdot (t', w') = (s, v) \Rightarrow (t, gw) \cdot (t', gw') = (s, gv) \tag{14.17}$$

and $\alpha(gg') = \alpha(g)\alpha(g')$. The associated action of $\mathrm{Sp}(n, \mathbb{R})$ on $\mathcal{H}^n$ has a formula that is less clean.

This leads to an action of $\mathrm{Sp}(n, \mathbb{R})$ on operators in the Weyl calculus. Setting

$$a_g(x, \xi) = a\big(g^{-1}(x, \xi)\big), \tag{14.18}$$

we have

$$a(X, D)b(X, D) = c(X, D) \Rightarrow a_g(X, D)b_g(X, D) = c_g(X, D), \tag{14.19}$$

for $g \in \mathrm{Sp}(n, \mathbb{R})$.

In fact, let us rewrite (14.1) as

$$a(X, D) = \int \hat{a}(w)\pi(0, w)\, dw.$$

Then

$$\begin{aligned} &a(X, D)b(X, D) \\ &\quad= \iint \hat{a}(w)\hat{b}(w')\pi(0, w)\pi(0, w')\, dw\, dw' \\ &\quad= \iint \hat{a}(w)\hat{b}(w')e^{\sigma(w,w')/2}\pi(0, w + w')\, dw\, dw', \end{aligned} \tag{14.20}$$

so $c(X, D)$ in (14.19) has symbol satisfying

$$\hat{c}(w) = (2\pi)^{-n} \int \hat{a}(w - w')\hat{b}(w')e^{i\sigma(w,w')/2}\, dw'. \tag{14.21}$$

The implication in (14.19) follows immediately from this formula. Let us write $c(x,\xi) = (a \circ b)(x,\xi)$ when this relation holds.

From (14.21), one easily obtains the product formula

$$(a \circ b)(x,\xi) = e^{(i/2)(D_y \cdot D_\xi - D_x \cdot D_\eta)} a(x,\xi) b(y,\eta)\Big|_{y=x,\eta=\xi}. \tag{14.22}$$

If $a \in S^m_{\rho,\delta}$, $b \in S^\mu_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, we have the following asymptotic expansion:

$$(a \circ b)(x,\xi) \sim ab + \sum_{j \ge 1} \frac{1}{j!} \{a,b\}_j(x,\xi), \tag{14.23}$$

where

$$\{a,b\}_j(x,\xi) = \Big(-\frac{i}{2}\Big)^j \big(\partial_y \cdot \partial_\xi - \partial_x \cdot \partial_\eta\big)^j a(x,\xi) b(y,\eta)\Big|_{y=x,\eta=\xi}. \tag{14.24}$$

For comparison, recall the formula for

$$a(x,D) b(x,D) = (a\#b)(x,D) \tag{14.25}$$

given by (3.16)–(3.20):

$$\begin{aligned} (a\#b)(x,\xi) &= e^{iD_\eta \cdot D_\xi} a(x,\eta) b(y,\xi)\Big|_{y=x,\eta=\xi} \\ &\sim ab + \sum_{\alpha > 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi). \end{aligned} \tag{14.26}$$

In the respective cases, $(a \circ b)(x,\xi)$ differs from the sum over $j < N$ by an element of $S^{m+\mu-N(\rho-\delta)}_{\rho,\delta}$ and $(a\#b)(x,\xi)$ differs from the sum over $|\alpha| < N$ by an element of the same symbol class.

In particular, for $\rho = 1$, $\delta = 0$, we have

$$(a \circ b)(x,\xi) = a(x,\xi) b(x,\xi) + \frac{i}{2} \{a,b\}(x,\xi) \bmod S^{m+\mu-2}_{1,0}, \tag{14.27}$$

where $\{a,b\}$ is the Poisson bracket, while

$$(a\#b)(x,\xi) = a(x,\xi) b(x,\xi) - i \sum \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} \bmod S^{m+\mu-2}_{1,0}. \tag{14.28}$$

Consequently, in the scalar case,

$$\begin{aligned} [a(X,D), b(X,D)] &= [a(x,D), b(x,D)] \\ &= e(x,D) = e(X,D) \bmod OPS^{m+\mu-2}_{1,0}, \end{aligned} \tag{14.29}$$

with

$$e(x,\xi) = i\{a,b\}(x,\xi). \tag{14.30}$$

Now we point out one of the most useful aspects of the difference between (14.27) and (14.28). Namely, one starts with an operator $A = a(X,D) = a_1(x,D)$, maybe a differential operator, and perhaps one wants to construct a parametrix for A, or perhaps a "heat semigroup" e^{-tA}, under appropriate hypotheses. In such a case, the leading term in the symbol of the operator $b(X,D) = b_1(x,D)$ used in (14.20) or (14.25) is a *function* of $a(x,\xi)$, for example, $a(x,\xi)^{-1}$, or $e^{-ta(x,\xi)}$. But then, at least when $a(x,\xi)$ is scalar, the last term in (14.27) vanishes! On the other hand, the last term in (14.28) generally does not vanish. From this it follows that, with a given amount of work, one can often construct a more accurate approximation to a parametrix using the Weyl calculus, instead of using the constructions of the previous sections.

In the remainder of this section, we illustrate this point by reconsidering the parametrix construction for the heat equation, made in §13. Thus, we look again at

$$\frac{\partial u}{\partial t} = -Lu, \quad u(0) = f. \tag{14.31}$$

This time, set

$$Lu = a(X,D)u + b(x)u, \tag{14.32}$$

where

$$\begin{aligned} a(x,\xi) &= \sum g^{jk}(x)\xi_j\xi_k + \sum \ell_j(x)\xi_j \\ &= g(x,\xi) + \ell(x,\xi). \end{aligned} \tag{14.33}$$

We assume $g(x,\xi)$ is scalar, while $\ell(x,\xi)$ and $b(x)$ can be $K \times K$ matrix-valued. As the notation indicates, we assume (g^{jk}) is positive-definite, defining an inner product on cotangent vectors, corresponding to a Riemannian metric (g_{jk}). We note that a symbol that is a polynomial in ξ also defines a differential operator in the Weyl calculus. For example,

$$\begin{aligned} \ell(x,D)u &= \sum \ell_j(x)\,\partial_j u \Longrightarrow \\ \ell(X,D)u &= \sum \ell_j(x)\,\partial_j u + \frac{1}{2}\sum(\partial_j\ell_j)u \end{aligned} \tag{14.34}$$

and

$$\begin{aligned} a(x,D) &= \sum a_{jk}(x)\partial_j\partial_k u \Longrightarrow \\ a(X,D)u &= \sum \Big[a_{jk}(x)\partial_j\partial_k u + (\partial_j a_{jk})\partial_k u + \frac{1}{4}(\partial_j\partial_k a_{jk})u\Big] \\ &= \sum \Big[\partial_j(a_{jk}\partial_k u) + \frac{1}{4}(\partial_j\partial_k a_{jk})u\Big]. \end{aligned} \tag{14.35}$$

We use the Weyl calculus to construct a parametrix for (14.31). We will begin by treating the case when all the terms in (14.33) are scalar, and then we will discuss the case when only $g(x,\xi)$ is assumed to be scalar.

We want to write an approximate solution to (14.31) as

$$u = E(t,X,D)f. \tag{14.36}$$

We write

$$E(t,x,\xi) \sim E_0(t,x,\xi) + E_1(t,x,\xi) + \cdots \tag{14.37}$$

and obtain the various terms recursively. The PDE (14.31) requires

$$\frac{\partial}{\partial t} E(t,X,D) = -LE(t,X,D) = -(L \circ E)(t,X,D), \tag{14.38}$$

where, by the Weyl calculus,

$$(L \circ E)(t,x,\xi) \sim L(x,\xi)E(t,x,\xi) + \sum_{j \geq 1} \frac{1}{j!}\{L,E\}_j(t,x,\xi). \tag{14.39}$$

It is natural to set

$$E_0(t,x,\xi) = e^{-ta(x,\xi)}, \tag{14.40}$$

as in (13.9). Note that the Weyl calculus applied to this term provides a better approximation than the previous calculus, because

$$\{a, e^{-ta}\}_1 = 0!. \tag{14.41}$$

If we plug (14.37) into (14.39) and collect the highest order nonvanishing terms, we are led to define $E_1(t,x,\xi)$ as the solution to the "transport equation"

$$\frac{\partial E_1}{\partial t} = -aE_1 - \frac{1}{2}\{a, E_0\}_2 - b(x)E_0, \quad E_1(0,x,\xi) = 0. \tag{14.42}$$

Let us set

$$\Omega_1(t,x,\xi) = -\frac{1}{2}\{a, e^{-ta}\}_2 - b(x)e^{-ta(x,\xi)}. \tag{14.43}$$

Then the solution to (14.42) is

$$E_1(t,x,\xi) = \int_0^t e^{(s-t)a(x,\xi)}\Omega_1(s,x,\xi)\, ds. \tag{14.44}$$

Higher terms $E_j(t,x,\xi)$ are then obtained in a straightforward fashion. This construction is similar to (13.6)–(13.10), but there is the following important difference. Once you have $E_0(t,x,\xi)$ and $E_1(t,x,\xi)$ here, you have the first two terms in the expansion of the integral kernel of e^{-tL} on the diagonal:

$$K(t,x,x) \sim c_0(x)t^{-n/2} + c_1(x)t^{-n/2+1} + \cdots. \tag{14.45}$$

To get so far using the method of §13, it is necessary to go further and compute the solution $a_2(t,x,\xi)$ to the next transport equation. Since the formulas become rapidly more complicated, the advantage is with the method of this section. We proceed with an explicit determination of the first two terms in (14.45).

Thus we now evaluate the integral in (14.44). Clearly,

$$\int_0^t e^{(s-t)a(x,\xi)} b(x) e^{-sa(x,\xi)}\, ds = t b(x) e^{-ta(x,\xi)}. \tag{14.46}$$

Now, a straightforward calculation yields

$$\{a, e^{-sa}\}_2 = \frac{s}{2} Q(\nabla^2 a) e^{-sa} - \frac{s^2}{4} T(\nabla a, \nabla^2 a) e^{-sa}, \tag{14.47}$$

where

$$Q(\nabla^2 a) = \sum_{k,\ell} \Big\{ (\partial_{\xi_k}\partial_{\xi_\ell} a)(\partial_{x_k}\partial_{x_\ell} a) - (\partial_{\xi_k}\partial_{x_\ell} a)(\partial_{x_k}\partial_{\xi_\ell} a) \Big\} \tag{14.48}$$

and

$$\begin{aligned} &T(\nabla a, \nabla^2 a) \\ &\quad = \sum_{k,\ell} \Big\{ (\partial_{\xi_k}\partial_{\xi_\ell} a)(\partial_{x_k} a)(\partial_{x_\ell} a) \\ &\qquad + (\partial_{x_k}\partial_{x_\ell} a)(\partial_{\xi_k} a)(\partial_{\xi_\ell} a) - 2(\partial_{\xi_k}\partial_{x_\ell} a)(\partial_{x_k} a)(\partial_{\xi_\ell} a) \Big\}. \end{aligned} \tag{14.49}$$

Therefore,

$$\int_0^t e^{(s-t)a} \{a, e^{-sa}\}_2\, ds = \frac{t^2}{4} Q(\nabla^2 a) e^{-ta} - \frac{t^3}{12} T(\nabla a, \nabla^2 a) e^{-ta}. \tag{14.50}$$

We get $E_1(t, x, \xi)$ in (14.44) from (14.46) and (14.50).

Suppose for the moment that $\ell(x, \xi) = 0$ in (14.33), that is, $a(X, D) = g(X, D)$. Suppose also that, for some point x_0,

$$\nabla_x\, g^{jk}(x_0) = 0, \quad g^{jk}(x_0) = \delta_{jk}. \tag{14.51}$$

Then, at x_0,

$$\begin{aligned} Q(\nabla^2 a) &= \sum_{k,\ell} \big(\partial_{\xi_k}\partial_{\xi_\ell} a\big)\big(\partial_{x_k}\partial_{x_\ell} a\big) \\ &= 2 \sum_{j,k,\ell} \frac{\partial^2 g^{jk}}{\partial x_\ell^2}(x_0) \xi_j \xi_k \end{aligned} \tag{14.52}$$

and

$$\begin{aligned} T(\nabla a, \nabla^2 a) &= \sum_{k,\ell} \big(\partial_{x_k}\partial_{x_\ell} a\big)\big(\partial_{\xi_k} a\big)\big(\partial_{\xi_\ell} a\big) \\ &= 4 \sum_{j,k,\ell,m} \frac{\partial^2 g^{jk}}{\partial x_\ell \partial x_m}(x_0) \xi_j \xi_k \xi_\ell \xi_m. \end{aligned} \tag{14.53}$$

Such a situation as (14.51) arises if $g^{jk}(x)$ comes from a metric tensor $g_{jk}(x)$, and one uses geodesic normal coordinates centered at x_0. Now the

Laplace-Beltrami operator is given by

$$\Delta u = g^{-1/2} \sum \partial_j g^{jk} g^{1/2} \partial_k \, u, \tag{14.54}$$

where $g = \det(g_{jk})$. This is symmetric when one uses the Riemannian volume element $dV = \sqrt{g}\, dx_1 \cdots dx_n$. To use the Weyl calculus, we want an operator that is symmetric with respect to the Euclidean volume element $dx_1 \cdots dx_n$, so we conjugate Δ by multiplication by $g^{1/4}$:

$$-Lu = g^{1/4}\, \Delta\big(g^{-1/4} u\big) = g^{-1/4} \sum \partial_j g^{jk} g^{1/2}\, \partial_k \big(g^{-1/4} u\big). \tag{14.55}$$

Note that the integral kernel $k_L^t(x,y)$ of e^{tL} is $g^{1/4}(x) k_\Delta^t(x,y) g^{-1/4}(y)$; in particular, of course, the two kernels coincide on the diagonal $x = y$. To compare L with $g(X,D)$, note that

$$-Lu = \sum \partial_j g^{jk}\, \partial_k u + \Phi(x) u, \tag{14.56}$$

where

$$\Phi(x) = \sum \partial_j \big(g^{jk} g^{1/2}\, \partial_k g^{-1/4}\big) - \sum g^{jk} g^{1/2} \big(\partial_j g^{-1/4}\big)\big(\partial_k g^{-1/4}\big). \tag{14.57}$$

If $g^{jk}(x)$ satisfies (14.51), we see that

$$\Phi(x_0) = \sum_j \partial_j^2 g^{-1/4}(x_0) = -\frac{1}{4} \sum_\ell \partial_\ell^2 g(x_0). \tag{14.58}$$

Since $g(x_0 + h e_\ell) = \det\big(\delta_{jk} + (1/2) h^2\, \partial_\ell^2 g_{jk}\big) + O(h^3)$, we have

$$\Phi(x_0) = -\frac{1}{4} \sum_{j,\ell} \partial_\ell^2 g_{jj}(x_0). \tag{14.59}$$

By comparison, note that, by (14.35),

$$\begin{aligned} g(X,D)u &= -\sum \partial_j g^{jk}\, \partial_k u + \Psi(x) u, \\ \Psi(x) &= -\frac{1}{4} \sum \partial_j \partial_k g^{jk}(x). \end{aligned} \tag{14.60}$$

If x_0 is the center of a normal coordinate system, we can express these results in terms of curvature, using

$$\partial_\ell \partial_m g^{jk}(x_0) = \frac{1}{3} R_{j\ell km}(x_0) + \frac{1}{3} R_{jmk\ell}(x_0), \tag{14.61}$$

in terms of the components of the Riemann curvature tensor, which follows from formula (3.51) of Appendix C. Thus we get

$$\begin{aligned} \Phi(x_0) &= -\frac{1}{4} \cdot \frac{2}{3} \sum_{j,\ell} R_{j\ell j\ell}(x_0) = -\frac{1}{6} S(x_0), \\ \Psi(x_0) &= -\frac{1}{4} \cdot \frac{1}{3} \sum_{j,k} \big[R_{jjkk}(x_0) + R_{jkkj}(x_0)\big] = \frac{1}{12} S(x_0). \end{aligned} \tag{14.62}$$

Here S is the scalar curvature of the metric g_{jk}.

When $a(X,D) = g(X,D)$, we can express the quantities (14.52) and (14.53) in terms of curvature:

$$Q(\nabla^2 g) = 2 \cdot \frac{2}{3} \sum_{j,k,\ell} R_{j\ell k\ell}(x_0)\xi_j\xi_k = \frac{4}{3} \sum_{j,k} \mathrm{Ric}_{jk}(x_0)\xi_j\xi_k, \tag{14.63}$$

where Ric_{jk} denotes the components of the Ricci tensor, and

$$T(\nabla g, \nabla^2 g) = 4 \cdot \frac{2}{3} \sum_{j,k,\ell,m} R_{j\ell km}(x_0)\xi_j\xi_k\xi_\ell\xi_m = 0, \tag{14.64}$$

the cancellation here resulting from the antisymmetry of $R_{j\ell km}$ in (j,ℓ) and in (k,m).

Thus the heat kernel for (14.31) with

$$Lu = g(X,D)u + b(x)u \tag{14.65}$$

is of the form (14.36)–(14.37), with $E_0 = e^{-tg(x,\xi)}$ and

$$\begin{aligned} E_1(t,x,\xi) &= \Big(-tb(x) - \frac{t^2}{8}Q(\nabla^2 g) + \frac{t^3}{24}T(\nabla g, \nabla^2 g)\Big)e^{-tg} \\ &= -\Big(tb(x) + \frac{t^2}{6}\,\mathrm{Ric}(\xi,\xi)\Big)e^{-tg(x,\xi)}, \end{aligned} \tag{14.66}$$

at $x = x_0$. Note that $g(x_0,\xi) = |\xi|^2$.

Now the integral kernel of $E_j(t,X,D)$ is

$$K_j(t,x,y) = (2\pi)^{-n} \int E_j\Big(t, \frac{x+y}{2}, \xi\Big)e^{i(x-y)\cdot\xi}\, d\xi. \tag{14.67}$$

In particular, on the diagonal we have

$$K_j(t,x,x) = (2\pi)^{-n} \int E_j(t,x,\xi)\, d\xi. \tag{14.68}$$

We want to compute these quantities, for $j = 0, 1$, and at $x = x_0$. First,

$$K_0(t,x_0,x_0) = (2\pi)^{-n} \int e^{-t|\xi|^2}\, d\xi = (4\pi t)^{-n/2}, \tag{14.69}$$

since, as we know, the Gaussian integral in (14.69) is equal to $(\pi/t)^{n/2}$. Next,

$$\begin{aligned} &(2\pi)^n K_1(t,x_0,x_0) \\ &\quad = -tb(x_0)\int e^{-t|\xi|^2}d\xi - \frac{t^2}{6}\sum \mathrm{Ric}_{jk}(x_0)\int \xi_j\xi_k e^{-t|\xi|^2}d\xi. \end{aligned} \tag{14.70}$$

We need to compute more Gaussian integrals. If $j \neq k$, the integrand is an odd function of ξ_j, so the integral vanishes. On the other hand,

$$\begin{aligned}\int \xi_j^2 e^{-t|\xi|^2} d\xi &= \frac{1}{n} \int |\xi|^2 e^{-t|\xi|^2} d\xi \\ &= -\frac{1}{n} \frac{d}{dt} \int e^{-t|\xi|^2} d\xi = \frac{1}{2} \pi^{n/2} t^{-n/2-1}.\end{aligned} \tag{14.71}$$

Thus

$$K_1(t, x_0, x_0) = -(4\pi t)^{-n/2} \Big(t b(x_0) + \frac{t}{12} S(x_0) \Big), \tag{14.72}$$

since $\sum \mathrm{Ric}_{jj}(x) = S(x)$.

As noted above, the Laplace operator Δ on scalar functions, when conjugated by $g^{1/4}$, has the form (14.65), with $b(x_0) = \Phi(x_0) - \Psi(x_0) = -S(x_0)/4$. Thus, for the keat kernel $e^{t\Delta}$ on scalars, we have

$$K_1(t, x_0, x_0) = (4\pi t)^{-n/2} \frac{t}{6} S(x_0). \tag{14.73}$$

We now generalize this, setting

$$a(x, \xi) = g(x, \xi) + \ell(x, \xi), \quad \ell(x, \xi) = \sum \ell_j(x) \xi_j. \tag{14.74}$$

Continue to assume that $a(x, \xi)$ is scalar and consider $L = a(X, D) + b(x)$. We have

$$E_0(t, x, \xi) = e^{-ta(x,\xi)} = e^{-t\ell(x,\xi)} \; e^{-tg(x,\xi)}, \tag{14.75}$$

and $E_1(t, x, \xi)$ is still given by (14.42)–(14.50). A point to keep in mind is that we can drop $\ell(x, \xi)$ from the computation involving $\{a, e^{-ta}\}_2$, altering $K_1(t, x, x)$ only by $o(t^{-n/2+1})$ as $t \searrow 0$. Thus, mod $o(t^{-n/2+1})$, $K_1(t, x_0, x_0)$ is still given by (14.73). To get $K_0(t, x_0, x_0)$, expand $e^{-t\ell(x,\xi)}$ in (14.75) in powers of t:

$$E_0(t, x, \xi) \sim \Big[1 - t\ell(x, \xi) + \frac{t^2}{2} \ell(x, \xi)^2 + \cdots \Big] e^{-tg(x,\xi)}. \tag{14.76}$$

When doing the ξ-integral, the term $t\ell(x, \xi)$ is obliterated, of course, while, by (14.71),

$$\frac{t^2}{2} \int \ell(x_0, \xi)^2 e^{-t|\xi|^2} \, d\xi = \frac{1}{4} \pi^{n/2} t^{-n/2+1} \sum \ell_j(x_0)^2. \tag{14.77}$$

Hence, in this situation,

$$\begin{aligned}&K_0(t, x_0, x_0) + K_1(t, x_0, x_0) \\ &\quad = (4\pi t)^{-n/2} \Big[1 + t \Big(\sum \ell_j(x_0)^2 - b(x_0) - \frac{1}{12} S(x_0) \Big) + O(t^2) \Big].\end{aligned} \tag{14.78}$$

Finally, we drop the assumption that $\ell(x, \xi)$ in (14.74) be scalar. We still assume that $g(x, \xi)$ defines the metric tensor. There are several changes whose effects on (14.78) need to be investigated. In the first place, (14.41) is no longer quite true. We have

$$\{a, e^{-ta}\}_1 = \frac{i}{2}\sum\Big\{\frac{\partial a}{\partial x_j}\frac{\partial}{\partial \xi_j}e^{-ta} - \frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial x_j}e^{-ta}\Big\}. \tag{14.79}$$

In this case, with $a(x,\xi)$ matrix-valued, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}e^{-ta} &= -te^{-ta}\ \Xi\big(\mathrm{ad}(-ta)\big)\Big(\frac{\partial a}{\partial x_j}\Big)\\ &= -te^{-ta}\ \Xi\big(\mathrm{ad}(-t\ell)\big)\Big(\frac{\partial a}{\partial x_j}\Big),\end{aligned} \tag{14.80}$$

where $\Xi(z) = (1-e^{-z})/z$, so

$$\begin{aligned}\frac{\partial}{\partial x_j}e^{-ta} &= te^{-ta}\Big(\frac{\partial a}{\partial x_j} + \frac{t}{2}\big[\ell, \frac{\partial \ell}{\partial x_j}\big] + \cdots\Big)\\ &= -t\frac{\partial a}{\partial x_j} + O(t^2|\xi|)e^{-ta} + \cdots,\end{aligned} \tag{14.81}$$

and so forth. Hence

$$\{a, e^{-ta}\}_1 = -\frac{i}{2}t\sum\big[\frac{\partial \ell}{\partial x_j}, \frac{\partial \ell}{\partial \xi_j}\big]e^{-ta} + \cdots. \tag{14.82}$$

This is smaller than any of the terms in the transport equation (14.42) for E_1, so it could be put in a higher transport equation. It does not affect (14.78).

Another change comes from the following modification of (14.46):

$$\begin{aligned}&\int_0^t e^{(s-t)a(x,\xi)}b(x)e^{-sa(x,\xi)}\ ds\\ &\quad= \Big[\int_0^t e^{(s-t)\ell(x,\xi)}b(x)e^{-s\ell(x,\xi)}\ ds\Big]\cdot e^{-tg(x,\xi)}.\end{aligned} \tag{14.83}$$

This time, $b(x)$ and $\ell(x,\xi)$ may not commute. We can write the right side as

$$\begin{aligned}&\int_0^t e^{s\ \mathrm{ad}\ \ell(x,\xi)}\big[b(x)\big]\ ds\ e^{-t\ell(x,\xi)}e^{-tg(x,\xi)}\\ &\quad= t\Big\{b(x) - \frac{t}{2}\big(\ell(x,\xi)b(x) + b(x)\ell(x,\xi)\big) + \cdots\Big\}e^{-tg(x,\xi)}.\end{aligned} \tag{14.84}$$

Due to the extra power of t with the anticommutator, this does not lead to a change in (14.78).

The other change in letting $\ell(x,\xi)$ be nonscalar is that the quantity $\ell(x,\xi)^2 = \sum \ell_j(x)\ell_k(x)\xi_j\xi_k$ generally has noncommuting factors, but this also does not affect (14.78). Consequently, allowing $\ell(x,\xi)$ to be nonscalar does not change (14.78). We state our conclusion:

Theorem 14.1. *If $Lu = a(X,D)u + b(x)u$, with*

$$a(x,\xi) = \sum g^{jk}(x)\xi_j\xi_k + \sum \ell_j(x)\xi_j, \tag{14.85}$$

where (g^{jk}) is the inverse of a metric tensor (g_{jk}), and $\ell_j(x)$ and $b(x)$ are matrix-valued, and if $g_{jk}(x_0) = \delta_{jk}$, $\nabla g_{jk}(x_0) = 0$, then the integral kernel $K(t,x,y)$ of e^{-tL} has the property

$$K(t,x_0,x_0) = (4\pi t)^{-n/2}\Big[1 + t\Big(\sum \ell_j(x_0)^2 - b(x_0) - \frac{1}{12}S(x_0)\Big) + O(t^2)\Big]. \tag{14.86}$$

Exercises

1. If $a(x,\xi) = \sum a_\alpha(x)\xi^\alpha$ is a polynomial in ξ, so that $a(x,D)$ is a differential operator, show that $a(X,D)$ is also a differential operator, given by

$$\begin{aligned} a(X,D)u(x) &= \sum_\alpha D_y^\alpha\Big[a_\alpha\Big(\frac{x+y}{2}\Big)u(y)\Big]\Big|_{y=x} \\ &= \sum_\alpha \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} 2^{-|\gamma|}\, D^\gamma a_\alpha(x)\, D^\beta u(x). \end{aligned}$$

   Verify the formulas (14.34) and (14.35) as special cases.

2. If $p \in S^m_{1,0}$ and $q \in S^\mu_{1,0}$ are scalar symbols and $p \circ q$ is defined so that the product $p(X,D)q(X,D) = (p \circ q)(X,D)$, as in (14.22)–(14.23), show that

$$q \circ p \circ q = q^2 p \mod S^{m+2\mu-2}_{1,0}.$$

   More generally, if $p_{jk} \in S^m_{1,0}$, $p_{jk} = p_{kj}$, and $q_j \in S^\mu_{1,0}$, show that

$$\sum_{j,k} q_j \circ p_{jk} \circ q_k = \sum_{j,k} q_j p_{jk} q_k \mod S^{m+2\mu-2}_{1,0}.$$

   Relate this to the last identity in (14.35), comparing a second-order differential operator in the Weyl calculus and in divergence form.

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