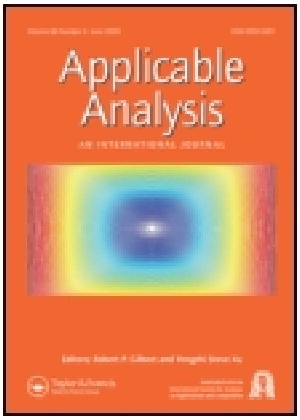

Applicable Analysis: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gapa20

Domain decomposition methods for boundary integral equations of the first kind: numerical results

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Version of record first published: 20 Jan 2011.

Applicable Analysis. Vol. 63 pp. 111–132
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Domain Decomposition Methods for Boundary Integral Equations of the First Kind: Numerical Results

Communicated by R.P. Gilbert

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Dedicated to Prof. G.C. Hsiao on the occasion of his sixtieth birthday

Abstract: We present numerical experiments for the additive Schwarz algorithms applied to the h- and p-version Galerkin boundary element methods to solve the Laplacian and Helmholtz boundary value problems in two dimensions. Both weakly singular and hypersingular integral equations covering Dirichlet and Neumann problems, respectively, are considered. In the case of Laplacian problems where the Galerkin scheme yields symmetric and positive definite stiffness matrices, we use the preconditioned CG method, whereas in the case of Helmholtz problems we have indefinite non Hermitian stiffness matrices, and therefore we use the preconditioned GMRES method. We find that the two level additive Schwarz methods yield only logarithmically growing condition numbers for both the h- and p-versions; thus only a fixed number of iterations is necessary to compute appropriate approximations of the Galerkin solutions. We also perform multilevel methods for integral equations belonging to the boundary value problems arising from the Laplacian and the Helmholtz equation. Here we observe (for both the weakly singular and hypersingular integral equations) that the condition numbers of the preconditioned systems grow like $p \log^3 p$ for the p-version. For the h-version we find in the case of the weakly singular equation (with the use of the Haar basis in the construction of the preconditioner) mildly increasing condition numbers, whereas for the hypersingular equation the multilevel additive Schwarz operator has bounded condition number and gives just the BPX preconditioner.

AMS: 65N55, 65N38

KEY WORDS: domain decomposition, additive Schwarz, boundary element method, conjugate gradient, GMRES

(Received for Publication December 1996)

1 Introduction.

In this paper we report on the numerical implementation of additive Schwarz methods for various boundary integral equations in two dimensions. In [7, 9, 10, 11, 13] we have analysed additive Schwarz methods for boundary integral equations; for corresponding results for the finite element methods we refer to [1, 2, 4, 15, 16, 17]. For simplicity we restrict our consideration to integral equations on intervals; the case of a polygon can be handled without additional difficulties. We consider the hypersingular integral equation

$$Dv(x) := -\frac{1}{\pi} f.p. \int_\Gamma \frac{v(y)}{|x-y|^2}\, ds_y = f(x), \quad x \in \Gamma = (-1,1), \tag{1}$$

and the weakly singular integral equation

$$Su(x) := -\frac{1}{\pi} \int_\Gamma \log|x-y| u(y)\, ds_y = f(x) \quad \text{for } x \in \Gamma = (-1,1). \tag{2}$$

Equations (1) and (2) result from the Neumann and the Dirichlet problems, respectively, for the Laplacian in the domain exterior to the slit Γ. As shown in [3], D and S are continuous and invertible from $\widetilde{H}^s(\Gamma)$ to $H^{-s}(\Gamma)$ for $s = 1/2$ and $s = -1/2$, respectively. For the definition of Sobolev spaces mentioned in this paper, we refer to [8, 7, 9, 10, 11, 13]. We consider a uniform mesh of size h on Γ as follows

$$x_j = -1 + jh, \quad h = \frac{2}{N_h}, \quad j = 0, \ldots, N_h. \tag{3}$$

We then define on this mesh the space V_h of continuous piecewise-linear functions on Γ which vanish at the endpoints of Γ. Then V_h is a subset of $\widetilde{H}^{1/2}(\Gamma)$. The h-version Galerkin boundary element method for Equation (1) reads as:
Find $u_h \in V_h$ such that

$$a(u_h, v_h) := \langle Du_h, v_h\rangle = \langle f, v_h\rangle \quad \textit{for any } v_h \in V_h. \tag{4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. Next we define on the mesh the space $\overline{V}_h$ of piecewise-constant functions. Then the h-version Galerkin boundary element method for Equation (2) reads as
Find $u_h \in \overline{V}_h$ such that

$$a(u_h, v_h) := \langle Su_h, v_h\rangle = \langle f, v_h\rangle \quad \textit{for any } v_h \in \overline{V}_h. \tag{5}$$

Both equations (4) and (5) yield linear systems of the form

$$A_N u = g$$

where the coefficient matrix A_N is dense, positive definite, and symmetric. Since the condition number of A_N grows like N, the conjugate gradient algorithm when used to

solve the above system yields the rate of convergence $\rho = 1 - O(1/N^{1/2})$ as $N \to \infty$. Therefore a preconditioner is necessary.
Here we present numerical experiments (cf. Tables 7,8) which show that the additive Schwarz operators yield preconditioned systems having almost bounded condition numbers, as proved in [13].
Furthermore, we consider the Galerkin scheme for the boundary integral equations arising from the boundary value problems for the Helmholtz equation. For these problems we have the hypersingular integral equation

$$D_k v(x) := -\frac{i}{2}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\left[H_0^1(k|x-y|)\right] v(y)\,ds_y = f(x), \quad x \in \Gamma, \tag{6}$$

and the weakly singular integral equation

$$S_k v(x) := -\frac{i}{2}\int_\Gamma \left[H_0^1(k|x-y|)\right] v(y)\,ds_y = f(x), \quad x \in \Gamma, \tag{7}$$

where H_0^1 is the Hankel function of the first kind and of order 0. Equations (6) and (7) result from the Neumann and Dirichlet problems for the Helmholtz equation in the domain exterior to the slit Γ.
Let $\alpha = 1/2$ in the case of the hypersingular equation, and $\alpha = -1/2$ in the case of the weakly singular equation. By writing both equations (6) and (7) in the abstract form

$$Bu = f,$$

we observe that B can be represented as (cf. [12, ?])

$$B = A + K,$$

where A and K satisfy

(i) A is positive definite, i.e., there exists $\mu > 0$ such that for any $u \in \widetilde{H}^\alpha(\Gamma)$,

$$A(u,u) \geq \mu \|u\|^2_{\widetilde{H}^\alpha(\Gamma)};$$

(ii) K is a compact operator from $\widetilde{H}^\alpha(\Gamma)$ into $H^{-\alpha}(\Gamma)$.

Here the bilinear form $A(\cdot,\cdot)$ is defined as $A(u,v) = \langle Au, v\rangle$ for any $u, v \in \widetilde{H}^\alpha(\Gamma)$. We also define the bilinear form $B(u,v) = \langle Bu, v\rangle$ for any $u, v \in \widetilde{H}^\alpha(\Gamma)$. It was proved [12, ?] that B satisfies the Gårding inequality, i.e., there exist $\gamma > 0$ and $\eta > 0$ such that the real part of $B(\cdot,\cdot)$ satisfies, for any $u \in \widetilde{H}^\alpha(\Gamma)$,

$$\Re(B(u,u)) \geq \gamma\|u\|^2_{\widetilde{H}^\alpha(\Gamma)} - \eta\|u\|^2_{\widetilde{H}^{\alpha-1/2}(\Gamma)}. \tag{8}$$

The corresponding h-version boundary element Galerkin schemes yield indefinite matrices, and therefore the conjugate gradient algorithm has to be substituted by the GMRES.

In [10] and [11] we proved that the rates of convergence of the GMRES method (for the h-version) when used to solve the systems preconditioned by additive Schwarz algorithms are bounded independently of h. In this paper we describe the implementation of the methods and present numerical experiments which underline the theoretical results.
As another refinement strategy, we can take the p-version of the Galerkin boundary element method. In that way we approximate the solutions of (1) ((2), respectively) by continuous piecewise polynomial functions of degree at most p, $p \geq 1$, (discontinuous piecewise polynomial functions of degree at most p, $p \geq 0$, respectively). In the first case we use in the implementation of the Galerkin scheme antiderivatives of Legendre polynomials, whereas in the second case we use Legendre polynomials. In this version, we increase the accuracy of the approximation not by reducing h which is fixed, but by increasing p. Since the condition number of the stiffness matrix grows at least like p^2 (see [9]), the CG algorithm when used to solve the linear system yields the rate of convergence $\rho = 1 - O(1/p)$ as $p \to \infty$. Our numerical results show that the additive Schwarz method yields only a logarithmically increasing condition number for the preconditioned system (cf. Tables 5, 6). Similar results hold for equations (6) and (7), where the GMRES algorithm is used instead of the CG method.
The paper is organised as follows. In §2 we first present the additive Schwarz method in an abstract setting. Then we introduce specific subspace decompositions for the h- and p-versions of both hypersingular and weakly singular boundary integral operators for the Laplace and the Helmholtz equations. We collect from [9, 13, 10, 11, 7] our theoretical results on the condition numbers of the additive Schwarz operators. In §3 we comment on implementation issues and present numerical experiments which clearly underline the theoretical results reported in §2. In the Appendix we list the preconditioned CG and GMRES algorithms for the convenience of the reader.

2 The additive Schwarz method.

Let us first describe abstractly the additive Schwarz method and consider the following abstract variational problem:
Given a finite dimensional space V find $u^* \in V$ such that

$$a(u^*, v) = f(v) \quad \forall v \in V.$$

The bilinear form $a(\cdot,\cdot)$ is required to satisfy a Gårding inequality but it is not necessarily Hermitian and positive definite. We assume there holds a decomposition of the space V into subspaces

$$V = V_0 + V_1 + \cdots + V_N,$$

where V_0 is usually the coarse grid space which connects all subspaces. We also assume that the following auxiliary problem can be solved efficiently. Let a bilinear form $b_i(\cdot,\cdot)$ be defined on $V_i \times V_i$. Then for an element $w \in V$ find $P_i w$ such that

$$b_i(P_i w, v) = a(w, v) \quad \forall v \in V_i.$$

If there holds $b_i(\cdot,\cdot) = a(\cdot,\cdot)$ then $P_i w$ is the orthogonal projection of w on V_i with respect to the scalar product $a(\cdot,\cdot)$. If u is an approximation to the solution u^* then the projection of the error $u - u^*$ onto the subspace V_i can be computed as follows

$$\begin{aligned} b_i(P_i(u^* - u), v) &= a(u^* - u, v) \\ &= f(v) - a(u,v) \quad \forall v \in V_i. \end{aligned}$$

Thus we can compute the projection of the error on the subspace without knowing the exact solution. The additive Schwarz method is defined as

$u^0 \rightarrow 0$

For $i = 0$ until the convergence criteria is satisfied, compute

$u^{i+1} \rightarrow u^i + \tau \sum_j P_j(u^* - u^i)$

End i

where τ is the scalar parameter for speeding up the convergence. This yields the additive Schwarz operator

$$P = \sum P_i$$

which gives preconditioners for the CG or GMRES method. The performance of these preconditioners is studied in this paper. For the use of the additive Schwarz method as a linear iterative solver see [6].

2.1 Hypersingular integral equation for the Laplacian.

2.1.1 The h-version.

¿From the abstract setting of the additive Schwarz method we see that the additive Schwarz operator is automatically defined as soon as we find an explicit subspace decomposition. It is essential that the decomposition is composed of a coarse grid subspace and finer grid spaces (cf. [15]). We first present the 2-level method.

For the h-version, $V = V_h$ is the space of continuous piecewise-linear functions vanishing at ± 1. Let V_H be the space of continuous piecewise-linear functions defined on an uniform mesh with mesh size $H = 2h$. We then decompose V_h as

$$V_h = V_H + V_{h,1} + \cdots + V_{h,N_h}, \tag{9}$$

where $V_{h,j} = \text{span}\{\phi_{h,j}\}$. Here $\phi_{h,j}$ is the hat function which takes the value 1 at the mesh point x_j and 0 at the other meshpoints.

With the decomposition (9) a lower bound and an upper bound for the minimum and maximum eigenvalues of the additive Schwarz operator P have been proved, c.f. [13, Lemmas 2.2 and 2.3].

Theorem 1 (2-level) *For $\epsilon > 0$, arbitrary, there exist positive constants C_1 and C_2 independent of h such that*

$$\lambda_{\min}(P) \geq C_1 h^\epsilon \quad \textit{and} \quad \lambda_{\max}(P) \leq C_2,$$

and therefore the additive Schwarz operator corresponding to the decomposition (9) has condition number bounded as

$$\kappa(P) \leq (C_2/C_1)h^{-\epsilon}.$$

Remark 1 *The term $h^{-\epsilon}$ is due to the singularity of the exact solution of the integral equation at the endpoints of the open curve Γ. In case Γ is a closed curve, such a term does not appear and therefore the condition number of P is bounded independent of h.*

If we continue the 2-level method for the global problem on V_H, we then end up with the multilevel method which was analysed in [13]. We have the following result

Theorem 2 (multilevel) *There exists a constant C independent of h_L and the number of levels L such that*

$$\kappa(P) \leq C h_L^{-\epsilon},$$

where h_L is the mesh size of the L-level and $\epsilon > 0$ arbitrary.

Remark 2 *For the hypersingular integral equation (1) and decomposition (9) the preconditioner of the multilevel additive Schwarz method differs from the BPX-preconditioner only by a constant factor.*

2.1.2 The p-version.

Let us first consider the 2-level method. The ansatz space V^p consists of continuous functions vanishing at ± 1 whose restrictions on $\Gamma_j = (x_{j-1}, x_j)$, $j = 1, \ldots, N_0$, are polynomials of degree at most p, $p \geq 1$. Here $N_0 = N_h$ is fixed. We denote by V^1 the space of continuous piecewise-linear functions which vanish at the endpoints ± 1. This space serves the same purpose as the coarse grid space in the h-version. To each subinterval Γ_j we associate the space

$$V_j^p = \operatorname{span}\{\mathcal{L}_{2,j}, \ldots, \mathcal{L}_{p,j}\},$$

where $\mathcal{L}_{q,j}$ is the affine image onto Γ_j of

$$\mathcal{L}_q(x) = \int_{-1}^{x} L_{q-1}(s)\, ds, \quad x \in [-1, 1].$$

Here L_{q-1} is the Legendre polynomial of degree $q-1$. We extend $\mathcal{L}_{q,j}$ by 0 outside Γ_j. It is easy to check that V^p can be decomposed as a direct sum

$$V^p = V^1 \oplus V_1^p \oplus \cdots \oplus V_{N_0}^p. \tag{10}$$

It was proved in [9, Theorem 2.1] that

Theorem 3 (2-level) *There exist constants C_1 and C_2 independent p such that*

$$\lambda_{\min}(P) \geq C_1/(1 + \log^3 p) \quad \textit{and} \quad \lambda_{\max}(P) \leq C_2,$$

and therefore the condition number of the additive Schwarz operator P associated with the direct sum decomposition (10) is bounded by $C_2(1 + \log^3 p)/C_1$.

Next we consider the multilevel method. We further decompose V_j^p as

$$V_j^p = \widetilde{V}_j^2 \oplus \widetilde{V}_j^3 \oplus \cdots \oplus \widetilde{V}_j^p,$$

where $\widetilde{V}_j^k = \text{span}\{\mathcal{L}_{k,j}\}$. Hence V^p can now be decomposed as

$$V^p = V^1 \oplus \left(\oplus_{k=2}^p \oplus_{j=1}^{N_0} \widetilde{V}_j^k\right).$$

With this direct sum decomposition we have (cf. [7, Corollary 3.3])
Theorem 4 (multilevel) *The additive Schwarz operator P associated with multilevel decomposition has condition number bounded as*

$$\kappa(P) \leq Cp(1 + \log^3 p).$$

The positive constant C is independent of p.
We note that even though the condition number of the multilevel operator grows faster than that of the 2-level operator, it is worth applying this method since it is the diagonal preconditioner. Therefore, it is actually a very simple and cheap preconditioner.

2.2 Weakly singular integral equation for the Laplacian.

2.2.1 The h-version.

Let us first look at the 2-level method. The ansatz space V_h now consists of piecewise-constant functions defined on the mesh (3). Let V_H be the space of piecewise-constant functions defined on a uniform mesh with mesh size $H = 2h$. We decompose V_h as

$$V_h = V_H + V_{h,0} + V_{h,1} + \cdots + V_{h,N_h-1}, \tag{11}$$

where $V_{h,j} = \text{span}\{\psi_{h,j}\}$, with $\psi_{h,j}$ being the derivative of the hat function $\phi_{h,j}$ defined in §§2.1.1 and $V_{h,0} = \text{span}\{1\}$ being the global constant. $\{1, \psi_{h,j}, j = 1, \ldots, N_h - 1\}$ is called the Haar basis.
With the decomposition (11) we proved a lower bound and an upper bound for the minimum and maximum eigenvalues of P, respectively, c.f. [13, Lemmas 3.3 and 3.4].
Theorem 5 (2-level) *There exist constants C_1 and C_2 independent of h such that*

$$\lambda_{\min}(P) \geq C_1 h^\epsilon \quad \textit{and} \quad \lambda_{\max}(P) \leq C_2,$$

and therefore the condition number of the additive Schwarz operator P associated with the direct sum decomposition (10) is bounded by $C_2 h^{-\epsilon}/C_1$ with $\epsilon > 0$ arbitrary.
Remark 3 *The same result holds for the multilevel method as in the case of the hyper-singular operator (cf. [13]).*
Remark 4 *For the weakly singular integral equation (2) BPX and multilevel additive Schwarz preconditioners yield different schemes (cf. Table 7). The multilevel additive Schwarz method needs only a bounded number of iterations whereas the iteration numbers for the BPX increase logarithmically with the number of unknowns. As stopping criterion we use that the relative change of the solution vector in the iterative scheme has to be less than 10^{-10}.*

2.2.2 The p-version.

Again we first look at the 2-level method. The ansatz space V^p now consists of functions (not necessarily continuous) whose restrictions on $\Gamma_j = (x_{j-1}, x_j)$, $j = 1, \ldots, N_0$, are polynomials of degree at most p, $p \geq 0$. Here $N_0 = N_h$ is fixed. We define on the same mesh the space V_0 of piecewise-constant functions. This space serves the same purpose as the coarse grid space in the h-version. To each subinterval Γ_j, we associate the space

$$V_j^p = \mathrm{span}\{L_{1,j}, \ldots, L_{p,j}\},$$

where $L_{q,j}$ is the affine image onto Γ_j of the Legendre polynomial of degree q. We extend $L_{q,j}$ by 0 outside Γ_j. It is easy to see that V^p can be decomposed as a direct sum

$$V^p := V_0 \oplus V_1^p \oplus \cdots \oplus V_{N_0}^p. \tag{12}$$

It was proved in [9, Theorem 3.1] that
Theorem 6 (2-level) *There exists a constant c independent of p such that the condition number of the additive Schwarz operator P associated with the direct sum decomposition (12) is bounded by*

$$\kappa(P) \leq c(1 + \log^3(p+1)).$$

For the multilevel method we further decompose V_j^p as

$$V_j^p = \widetilde{V}_j^1 \oplus \widetilde{V}_j^2 \oplus \cdots \oplus \widetilde{V}_j^p,$$

where $\widetilde{V}_j^k = \mathrm{span}\{L_{k,j}\}$. Hence V^p can now be decomposed as

$$V^p = V^0 \oplus \left(\oplus_{k=1}^p \oplus_{j=1}^{N_0} \widetilde{V}_j^k\right). \tag{13}$$

With this direct sum decomposition we have (cf. [7, Corollary 4.3]):
Theorem 7 (multilevel) *The additive Schwarz operator P associated with multilevel decomposition (13) has condition number bounded as*

$$\kappa(P) \leq Cp(1 + \log^3 p).$$

The positive constant C is independent of p.

2.3 The Helmholtz equation.

For both the weakly singular and hypersingular integral equations ((7) and (6)) belonging to the Helmholtz equation we have analysed in [10, 11] 2-level methods for the h- and p-versions. Here the corresponding Galerkin boundary element schemes yield indefinite and non Hermitian stiffness matrices. In [10], [11] we showed that the condition number of the additive Schwarz operator is bounded in the case of the h-version, and grows at most like $\log^3 p$ in the case of the p-version. This behaviour is clearly reflected by the numerical experiments presented in §3. For the Dirichlet problem see results in Table 2 and Figure 3 for the p-version, and Table 9 and Figure 8 for the h-version. For the Neumann problem see results in Table 4 and Figure 5 for the p-version, and Table 10 and Figure 9 for the h-version.

3 Implementation and numerical results.

To implement additive Schwarz methods we have to distinguish two basicly different cases, namely when the subspaces form a direct sum or when they are nested.

(a) When the subspace decomposition is given as a direct sum we proceed as follows. Let

$$V = \operatorname{span}\{\phi_j : j = 1, \ldots, \dim V\},$$

and

$$V_i = \operatorname{span}\{\phi_j : j = N_{i-1} + 1, \ldots, N_i\}.$$

Let A_i denote the Galerkin matrix corresponding to V_i, $i = 0, \ldots, N$. Then we have $P = \sum_i P_i = \sum_i C_i^T A_i^{-1} C_i A_V$, where the projection matrices $C_i = (c_{l,k}^{(i)})$ are diagonal matrices with $c_{l,k}^{(i)} = 1$ when $l = k = N_{i-1} + 1, \ldots, N_i$ and $c_{l,k}^{(i)} = 0$ otherwise. We also have $A_i = C_i^T A_V C_i$, where A_V is the Galerkin matrix corresponding to V. The simple form of the projection matrices C_i is due to the fact that the same basis functions are used in V and in the subspaces V_i. Note this is exactly the situation of the p-version. If the matrices A_i are simply the diagonal blocks of A_V, the preconditioner P can be written as

$$P = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{pmatrix}^{-1} A_V.$$

(b) When the subspaces form a nested sequence we proceed with the implementation as follows. Again let A_i denote the Galerkin matrix belonging to V_i, $i = 0, \ldots, N$. Note that now the subspaces V_i are spanned by different basis functions, $V_i = \operatorname{span}\{\phi_j^i : j = 1, \ldots, \dim V_i\}$. Here the projection matrices C_i are defined by $C_i = (c_{j,l}^i)$. Hence the projection to the lower level is given by

$$\phi_j^i = \sum_{l=1}^{\dim V_{i+1}} c_{j,l}^i \phi_l^{i+1},$$

and we have with a preconditioner B_{MAS} that $P = \sum_i P_i$ is given in algorithmic form by

$y = Px = B_{MAS} A_V x$
input: x
$x_N := A_V x$
for $i = N-1, \ldots, 0$
    $x_i = C_i x_{i+1}$

$y_0 = A_0^{-1} x_0$
for $i = 1, \ldots, N$
$\quad y_i = (\operatorname{diag} A_i)^{-1} x_i + C_i^T y_{i-1}$
$y := y_N$

where $A_i = C_i A_{i+1} C_i^T$ and $i = 0, \ldots, N-1$ and $A_N := A_V$. P can also be written in matrix form

$$\left\{ (\operatorname{diag} A_N)^{-1} + \ldots + C_1^T ((\operatorname{diag} A_1)^{-1} + C_0^T A_0^{-1} C_0) C_1 \cdots C_{N-1} \right\} A_V$$

Note that C_i are sparse matrices and therefore the action of C_i and C_i^T can be implemented in a very efficient way; therefore their multiplication with a vector costs only $O(N)$ operations in contrary to $O(N^2)$ in case of the full matrix.

In the case of hat functions on an uniform mesh the projection matrices C_i are given by (see Figure 1)

$$\phi_j^i = \frac{1}{2} \phi_{2j-1}^{i+1} + \phi_{2j}^{i+1} + \frac{1}{2} \phi_{2j+1}^{i+1}, \quad j = 1, \ldots, \dim V_i.$$

In the case of the haar basis there holds a similar result

$$\psi_0^i = \psi_0^{i+1} \equiv 1, \ \psi_j^i = \frac{1}{2} \psi_{2j-1}^{i+1} + \psi_{2j}^{i+1} + \frac{1}{2} \psi_{2j+1}^{i+1}, \quad j = 1, \ldots, \dim V_i - 1.$$

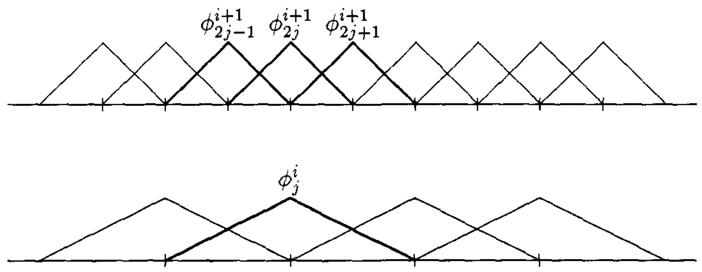

Figure 1: Construction of the coarser mesh

In the following we list graphs and tables for the additive Schwarz methods described above. The numerical experiments were performed on SUN-Sparcstation 4/470 at the Institute for Applied Mathematics at the University of Hannover.

	Number of iterations					
	$N_0 = 2$			$N_0 = 4$		
p	CG	multilevel	2-level	CG	multilevel	2-level
1	4	2	2	8	4	4
2	6	3	3	15	6	6
3	9	4	4	20	8	8
4	12	5	5	27	10	10
5	15	6	6	36	12	10
6	17	7	6	43	14	11
7	21	8	6	52	15	11
8	24	9	6	56	17	11
9	27	10	6	65	19	12
10	33	11	6	78	22	13

Table 1: WEAKLY SINGULAR INTEGRAL EQUATION (2) WITH $f(x) \equiv 1$: p-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS. N_0 = NUMBER OF INTERVALS.

	Condition number			Number of iterations				
	$N_0 = 2$			$N_0 = 2$			$N_0 = 4$	
p	A_N	2-level	multilevel	GMRES	2-level	multilevel	GMRES	2-level
1	9.70	2.84	2.35	2	2	2	4	4
2	29.03	4.17	3.71	3	3	3	6	6
3	61.02	5.05	4.84	4	4	4	9	8
4	108.33	5.84	6.14	5	5	5	12	10
5	174.00	6.49	7.25	6	6	6	14	11
6	261.44	7.09	8.44	8	6	7	17	11
7	373.92	7.62	9.65	9	7	8	20	12
8	514.89	8.12	11.01	10	7	9	22	12
9	687.62	8.58	12.26	12	7	10	25	13
10	895.54	9.01	13.60	13	7	11	29	14

Table 2: WEAKLY SINGULAR INTEGRAL EQUATION (7) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: p-VERSION, USING GMRES WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Number of iterations		
p	CG	multilevel	2-level
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	7	6	6
7	8	7	7
8	9	8	7
9	12	9	7
10	13	10	7

Table 3: HYPERSINGULAR INTEGRAL EQUATION (1) WITH $f(x) \equiv 1$: p-VERSION WITH $N_0 = 2$, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Condition number			Number of iterations		
p	A_N	2-level	multilevel	GMRES	2-level	multilevel
2	3.99	2.96	2.96	2	2	2
3	14.58	3.46	3.46	3	3	3
4	29.12	4.09	4.45	4	4	4
5	51.20	4.58	5.26	5	5	5
6	81.54	5.06	6.47	6	6	6
7	122.18	5.48	7.67	7	7	7
8	174.29	5.87	8.97	8	7	8
9	239.73	6.23	10.29	9	7	9
10	319.81	6.56	11.61	11	7	10

Table 4: HYPERSINGULAR INTEGRAL EQUATION (6) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: p-VERSION WITH $N_0 = 2$, USING GMRES WITH 2-LEVEL AND MULTILEVL ADDITIVE SCHWARZ PRECONDITIONERS.

	Minimum eigenvalue			Maximum eigenvalue			Condition number		
p	CG	2-level	multilev.	CG	2-level	multilev.	CG	2-lev	m-lev
1	5.41e-2	0.55	0.53	0.52	1.43	1.47	9.67	2.59	2.75
2	1.87e-2	0.39	0.38	0.52	1.43	1.47	27.90	3.65	3.88
3	9.10e-3	0.33	0.29	0.52	1.49	1.52	57.57	4.46	5.27
4	5.16e-3	0.29	0.23	0.52	1.49	1.52	101.53	5.15	6.50
5	3.22e-3	0.26	0.20	0.52	1.52	1.54	162.54	5.77	7.82
6	2.15e-3	0.24	0.17	0.52	1.52	1.54	243.82	6.32	9.06
7	1.50e-3	0.23	0.15	0.52	1.54	1.54	348.35	6.83	10.37
8	1.09e-3	0.21	0.13	0.52	1.55	1.55	479.38	7.29	11.62
9	8.19e-4	0.20	0.12	0.52	1.56	1.55	639.93	7.73	12.91
10	6.29e-4	0.19	0.11	0.52	1.56	1.55	833.20	8.14	14.16

Table 5: WEAKLY SINGULAR INTEGRAL EQUATION (2) WITH $f(x) \equiv 1$: p-VERSION WITH $N_0 = 2$, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Minimum eigenvalue			Maximum eigenvalue			Condition number		
p	CG	2-level	multilev.	CG	2-level	multilev.	CG	2-lev	m-lev
2	0.21	0.55	0.55	0.93	1.27	1.27	4.44	2.31	2.31
3	7.40e-2	0.39	0.39	0.94	1.39	1.39	12.75	3.55	3.55
4	3.62e-2	0.33	0.29	0.94	1.39	1.43	26.04	4.16	4.85
5	2.06e-2	0.29	0.24	0.94	1.40	1.47	45.88	4.84	6.14
6	1.29e-2	0.26	0.20	0.94	1.40	1.48	73.32	5.31	7.41
7	8.58e-2	0.24	0.17	0.94	1.41	1.50	109.94	5.84	8.69
8	6.01e-2	0.23	0.15	0.94	1.40	1.50	156.98	6.21	9.95
9	4.37e-3	0.21	0.13	0.94	1.40	1.51	215.97	6.61	11.24
10	3.27e-3	0.20	0.12	0.94	1.40	1.52	288.23	6.95	12.50

Table 6: HYPERSINGULAR INTEGRAL EQUATION (1) WITH $f(x) \equiv 1$: p-VERSION WITH $N_0 = 2$, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Condition number				Number of iterations			
N	A_N	2-level	multilevel	BPX	CG	2-lev	m-lev	BPX
16	15.7545	2.1310	5.1005	4.0543	8	8	8	8
32	32.6847	2.6175	6.0171	4.8520	20	12	11	12
64	65.1292	2.9767	6.9188	8.8454	33	16	14	17
128	129.6566	3.1926	7.8182	10.4593	45	19	16	19
256	259.8891	3.3153	8.7199	12.1835	66	21	17	22
512	517.1460	3.3777	9.6253	14.0330	85	21	19	23
1024	1036.1733	3.4108	10.5344	16.0186	125	22	19	24
2048	2066.6248	3.4255	11.4468	18.1483	168	22	19	26
4096	4119.1740	3.4338	12.3619	20.4271	226	23	19	29
8192		3.4361	13.2791	22.8595		23	19	30
16384		3.4370	14.1980	25.4481		23	19	31
32768		3.4379	15.1182	28.1890		23	19	32
65536		3.4406	16.0395	31.1033		24	19	33

Table 7: WEAKLY SINGULAR INTEGRAL EQUATION (2) WITH $f(x) \equiv 1$: h-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ AND BPX PRECONDITIONERS.

	Condition number			Number of iterations		
N	A_N	2-level	multilevel	CG	2-level	multilevel
15	7.7629	2.1475	3.0353	8	7	8
31	15.5445	2.2072	3.4613	11	11	11
63	31.1092	2.2162	3.7561	17	12	14
127	62.4163	2.2276	3.9714	26	13	16
255	125.0924	2.2299	4.1335	38	13	17
511	250.4733	2.2262	4.2578	55	12	17
1023	501.2394	2.2250	4.3545	78	12	17
2047	1002.7757	2.2236	4.4308	109	12	17
4095	2005.8634	2.2225	4.4917	154	12	18
8191		2.2160	4.5408		11	18
16383		2.2153	4.5808		11	18
32767		2.2150	4.6138		11	18
65535		2.2067	4.6413		10	18

Table 8: HYPERSINGULAR INTEGRAL EQUATION (1) WITH $f(x) \equiv 1$: h-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Condition number			Number of iterations		
N	A_N	2-level	multilevel	GMRES	2-level	multilevel
16	16.36	2.38	3.23	8	8	8
32	33.81	2.34	5.62	18	11	13
64	67.82	2.30	11.05	26	16	20
128	135.73	2.28	21.97	34	17	31
256	271.52	2.26	43.83	44	18	46
512	543.06	2.26	87.59	55	18	66

Table 9: WEAKLY SINGULAR INTEGRAL EQUATION (7) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: h-VERSION, USING GMRES WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

	Condition number			Number of iterations		
N	A_N	2-level	multilevel	GMRES	2-level	multilevel
16	9.04	2.17	3.20	8	8	8
32	12.46	2.22	3.62	12	11	12
64	24.90	2.24	3.89	19	14	16
128	49.77	2.26	4.06	29	15	18
256	99.50	2.26	4.19	43	15	20
512	198.96	2.26	4.27	62	15	21

Table 10: HYPERSINGULAR INTEGRAL EQUATION (6) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: h-VERSION, USING GMRES WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

Appendix.
For the convenience of the reader we list below the CG and GMRES algorithms. Note that in the iterative procedure, we need to implement the acting of preconditioner B_{MAS} on r_k, the residual. This procedure is given in Section 2. We also compute the condition numbers by using the Lanczos algorithm (see [5]).

CG ALGORITHM

input: $x = 0$, $r =$ RHS

output: $x =$ approximated solution, $r =$ residual, $k =$ number of iterations performed

$k = 0$, $x_0 = 0$, $r_0 =$ RHS

while $(r_k \neq 0)$ (test for convergence)

    $z_k = B_{MAS} r_k$ (call preconditioning routine, see Section 2)
    $k = k + 1$
    if $k = 1$
        $\beta_1 = 0$ and $p_1 = z_0$
    else
        $\beta_k = (r_{k-1}, z_{k-1})/(r_{k-2}, z_{k-2})$
        $p_k = z_{k-1} + \beta_k p_{k-1}$
    endif
    $\alpha_k = (r_{k-1}, z_{k-1})/(p_k, Ap_k)$
    $x_k = x_{k-1} + \alpha_k p_k$
    $r_k = r_{k-1} - \alpha_k Ap_k$

end

$x = x_k$

The pseudocode for the restarted GMRES algorithm with preconditioner B_{MAS} is as follows

GMRES

input: $x^{(0)} = 0$

for $j = 1, 2, \ldots$

    $z = B_{MAS}(b - Ax^{(j-1)})$
    $v^{(1)} = z/\|z\|_2$
    $s := \|z\|_2 e_1$

for $i = 1, 2, \ldots, m$

  $w = B_{MAS}Av^{(i)}$

  for $k = 1, \ldots, i$

    $h_{k,i} = (w, v^{(k)})$

    $w = w - h_{k,i}v^{(k)}$

  end

  $h_{i+1,i} = \|w\|_2$

  $v^{(i+1)} = w/h_{i+1,i}$

  apply $J_1, J_2, \ldots, J_{i-1}$ on $(h_{1,i}, \ldots, h_{i+1,i})$

  construct Givens rotation J_i acting on ith and $(i+1)$st component of $h_{.,i}$, such that $(i+1)$st component of $J_i h_{.,i}$ is 0

  $s := J_i s$

  if $s^{(i+1)}$ is small enough then (UPDATE($x^{(j)}, i$) and quit)

end

UPDATE($x^{(j)}, m$)

end

UPDATE($x^{(j)}, i$)

input: $x^{(j-1)}, s, H, v^{(k)}, k = 1, \ldots, i$

Compute y as the solution of $Hy = \tilde{s}$, in which the upper $i \times i$ triangular part of H has $h_{k,l}$ as its elements (in least squares sense if H is singular), $\tilde{s}$ represents the first i components of s

$x^{(j)} = x^{(j-1)} + y^{(1)}v^{(1)} + y^{(2)}v^{(2)} + \ldots + y^{(i)}v^{(i)}$

$s^{(i+1)} = \|b - Ax^{(j)}\|_2$

end

Acknowledgements

This work was carried out while the third author was visiting the Institut für Angewandte Mathematik at the University of Hannover. His visit was supported by Deutsche Forschungsgemeinschaft (DFG) and the Australian Research Council (ARC).

References

[1] J. Bramble, J. Pasciak: New estimates for multilevel algorithms including the V-cycle. *Math. Comp.* **60** (1993) 447–471.

[2] X. Cai, O. Widlund: Domain decomposition algorithms for indefinite elliptic problems. *SIAM J. Sci. Stat. Comput.* **13** (1992) 243–258.

[3] M. Costabel: Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal.* **19(3)** (1988) 613–626.

[4] M. Dryja, O. Widlund: Multilevel additive methods for elliptic finite element problems, in Parallel Algorithms for PDE's (Proc. of the 6. GAMM-Seminar, Kiel, Germany, January 19–21, 1990), W. Hackbusch, ed., Braunschweig, 1991, Vieweg, 58–69.

[5] G. Golub, C. V. Loan:*Matrix Computations.* The Johns Hopkins University Press, Baltimore-London, 1989.

[6] M. Hahne, E. P. Stephan: Schwarz iterations for the efficient solution of screen problems with boundary elements. *Computing* **56** (1996) 61–85.

[7] N. Heuer, E. P. Stephan, T. Tran: A multilevel additive Schwarz method for the *h-p* version Galerkin boundary element method. *Math. Comp.*(to appear).

[8] J. Lions, E. Magenes: *Non-Homogeneous Boundary Value Problems and Applications I.* Springer-Verlag, Berlin, 1972.

[9] T. Tran, E. P. Stephan: Additive schwarz algorithms for the *p*-version boundary element method. Applied Math. Report AMR95/13, University of New South Wales, Sydney, Australia, 1995 (submitted).

[10] T. Tran, E. P. Stephan: Domain decomposition algorithms for indefinite hypersingular integral equations–the *h* and *p*-versions. *SIAM J. Sci. Stat. Comput.* (to appear).

[11] T. Tran, E. P. Stephan: Domain decomposition algorithms for indefinite weakly singular integral equations–the *h* and *p*-versions. Preprint, University of Hannover, 1995 (submitted).

[12] E. P. Stephan, W. L. Wendland: An Augmented Galerkin Procedure for the Boundary Integral Method Applied to Two-Dimensional Screen and Crack Problems. *Appl. Anal.* **18** (1984) 183–219.

[13] T. Tran, E. P. Stephan: Additive schwarz methods for the *h*-version boundary element method. *Appl. Anal.* **60** (1996) 63–84.

[14] W. L. Wendland, E. P. Stephan: A Hypersingular Boundary Integral Method for Two-Dimensional Screen and Crack Problems. *Arch. Rational. Mech. Anal.* **112** (1990) 363–390.

[15] O. Widlund: Optimal iterative refinement methods, in Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, 1989, SIAM.

[16] J. Xu: Iterative methods by space decomposition and subspace correction. *SIAM Review* **34** (1992) 581–613.

[17] X. Zhang: Multilevel schwarz methods. *Numer. Math.* **63** (1992) 521–539.

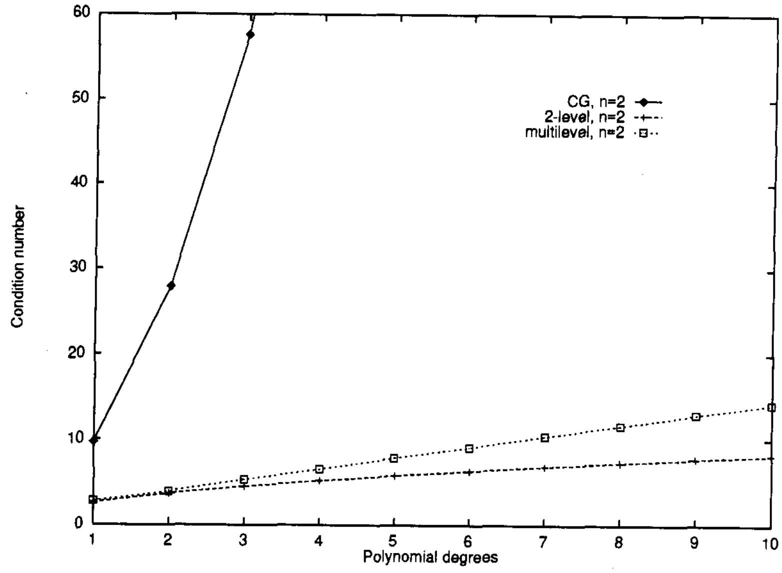

Figure 2: WEAKLY SINGULAR INTEGRAL EQUATION (2) WITH $f(x) \equiv 1$: p-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

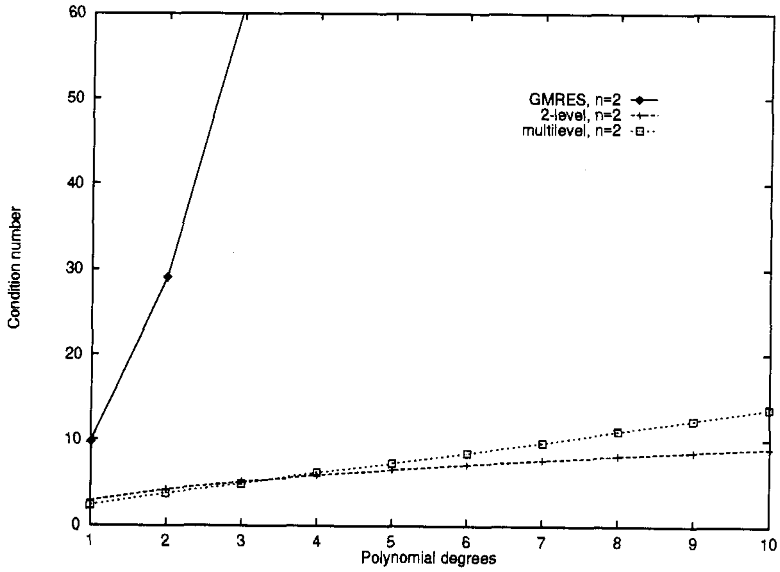

Figure 3: WEAKLY SINGULAR INTEGRAL EQUATION (7) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: p-VERSION, USING GMRES WITH ADDITIVE SCHWARZ PRECONDITIONERS.

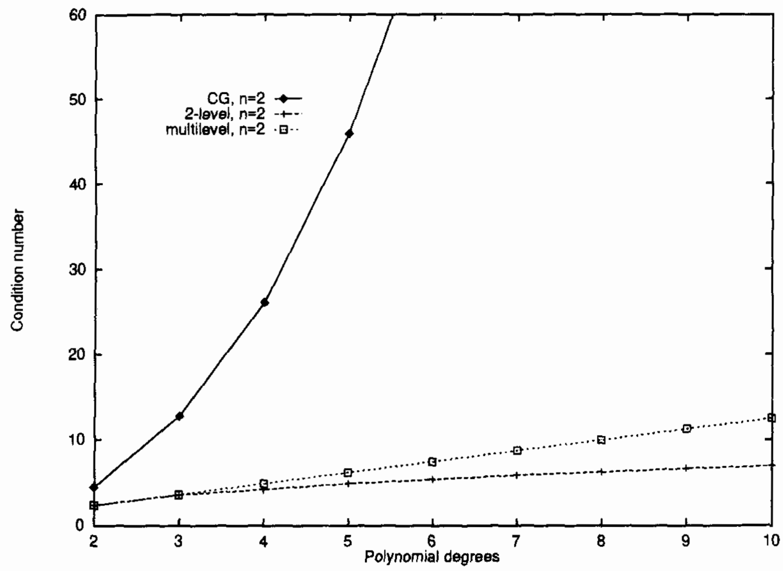

Figure 4: HYPERSINGULAR INTEGRAL EQUATION (1) WITH $f(x) \equiv 1$: p-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

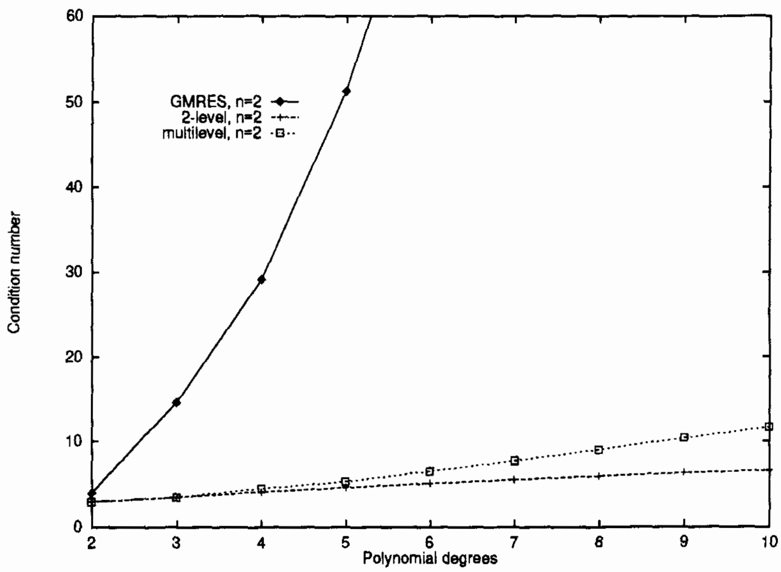

Figure 5: HYPERSINGULAR INTEGRAL EQUATION (6) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: p-VERSION, USING GMRES WITH ADDITIVE SCHWARZ PRECONDITIONERS.

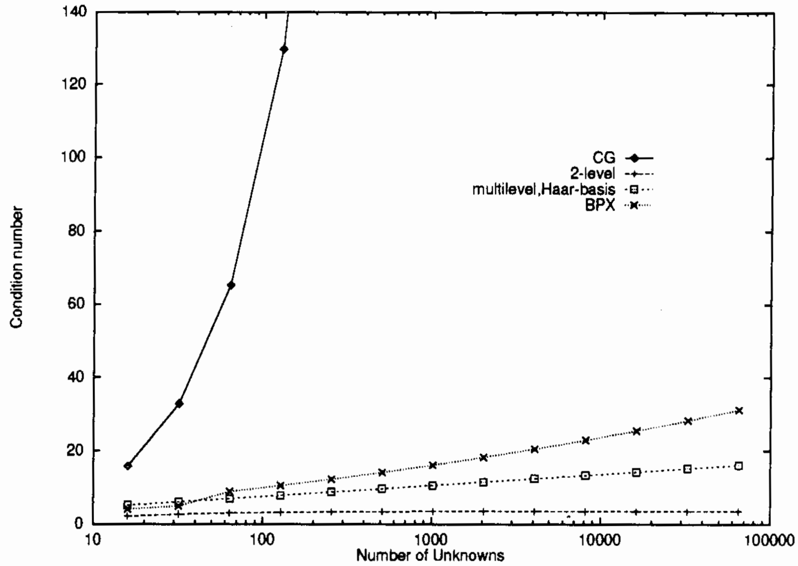

Figure 6: WEAKLY SINGULAR INTEGRAL EQUATION (2) WITH $f(x) \equiv 1$: h-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

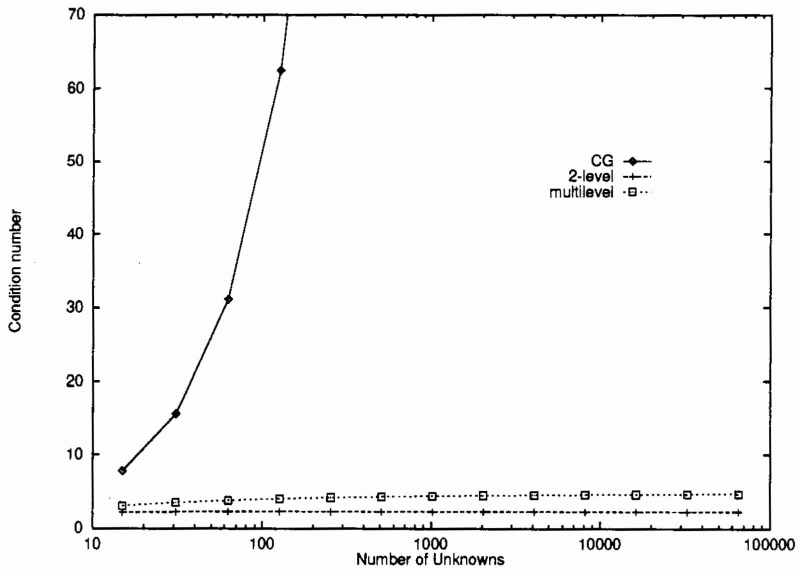

Figure 7: HYPERSINGULAR INTEGRAL EQUATION (1) WITH $f(x) \equiv 1$: h-VERSION, USING CG WITH 2-LEVEL AND MULTILEVEL ADDITIVE SCHWARZ PRECONDITIONERS.

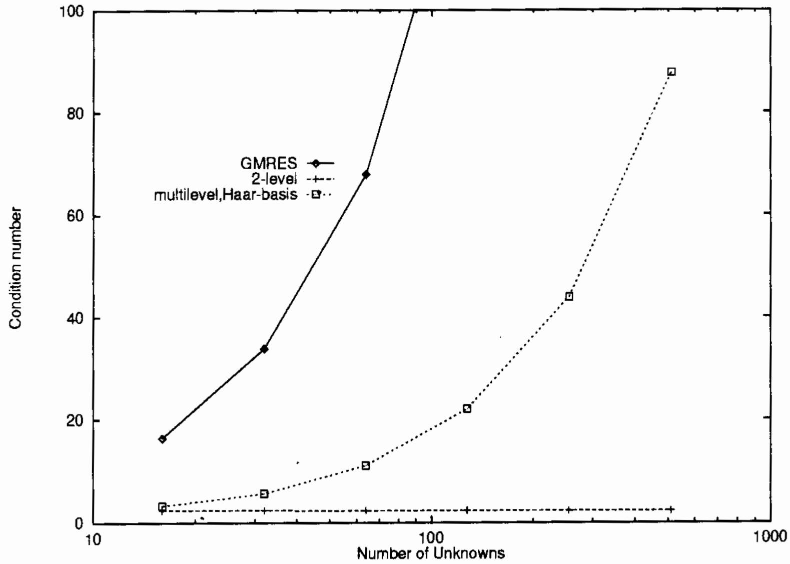

Figure 8: WEAKLY SINGULAR INTEGRAL EQUATION (7) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: h-VERSION, USING GMRES WITH ADDITIVE SCHWARZ PRECONDITIONERS.

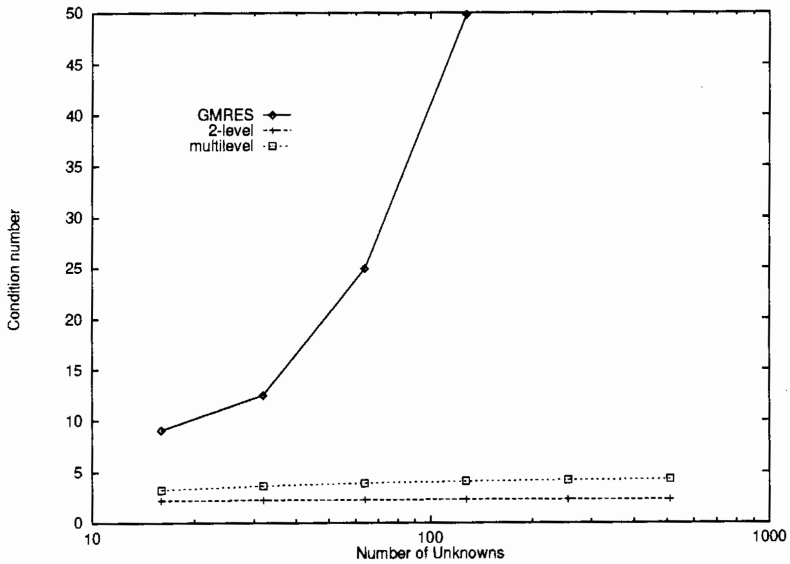

Figure 9: HYPERSINGULAR INTEGRAL EQUATION (6) WITH $f(x) \equiv 1$ AND WAVE NUMBER $k = 2$: h-VERSION, USING GMRES WITH ADDITIVE SCHWARZ PRECONDITIONERS.