**Definition 1.** *Let the M-transform be defined by*

$$\mathcal{M}\phi(\xi) := \int_0^{+\infty} \cos(2\pi\sqrt{\xi u})\frac{\phi(u)}{\sqrt{u}}du$$

**Definition 2.** *For any function $\phi$ defined on $\mathbb{R}^+$, let $C$ the operator defined by*

$$C\phi(t) = \phi(t^2), \quad t \in \mathbb{R}$$

*For any function even function $\phi$ defined on $\mathbb{R}$, $C^{-1}\phi$ is a function defined on $\mathbb{R}^+$ by*

$$C^{-1}\phi(u) = \phi(\sqrt{u})$$

**Definition 3.** *We note $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ the space of $\phi$ such that $C\phi \in \mathcal{S}(\mathbb{R})$. Let $S_p(\mathbb{R})$ the subspace of real even functions that belong to the Schwartz class.*

**Proposition 1.** $C\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right) = \mathcal{S}_p(\mathbb{R})$

**Proposition 2.** *If $\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, then*

$$\sqrt{x}\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$$

*For all polynomial $P$,*

$$C^{-1}P(dx^2)C\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$$

*Where $dx$ is the differentiation operator.*

**Proposition 3.** *The operator $\mathcal{M} : \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right) \longrightarrow \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ is an involution, and can be rewritten as*

$$\mathcal{M} = C^{-1}\mathcal{F}C$$

*where $\mathcal{F}$ is the Fourier transform defined on $\mathcal{S}_p(\mathbb{R})$*

$$\mathcal{F}u(\xi) = \int_{-\infty}^{+\infty} e^{-i2\pi x\xi}u(x)dx$$

*or equivalently*

$$\mathcal{F}u(\xi) = \int_0^{+\infty} 2\cos(2\pi\xi x)u(x)dx$$

*$\mathcal{F}$ is self-adjoint on $S_p(\mathbb{R})$.*

**Definition 4.** *For $\psi$ and $\phi$ in $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, we define the duality product*

$$\langle\phi, \psi\rangle_\omega = \int_0^{+\infty} \frac{\phi(x)\psi(x)}{\sqrt{x}}$$

**Proposition 4.**

$$\langle \phi, \psi \rangle_\omega = \langle C\phi, C\psi \rangle$$

**Proposition 5.** *For any $\phi$, $\psi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, one has*

$$\langle \mathcal{M}\phi, \psi \rangle_\omega = \langle \phi, \mathcal{M}\psi \rangle_\omega$$

**Definition 5.** *Let $\Delta_\omega$ the operator defined on $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ by*

$$\Delta_\omega \phi(x) = 2\sqrt{x}\left(2\sqrt{x}\phi'(x)\right)'$$

If we call $\Delta$ the usual Laplace operator defined on $S_p(\mathbb{R})$, we have

**Proposition 6.**

$$\Delta_\omega \phi = C^{-1}\Delta C\phi$$

**Corollary 1.** *$\Delta_\omega$ maps $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ on itself.*

**Corollary 2.** $\langle \Delta_\omega \phi, \psi \rangle_\omega = \langle \phi, \Delta_\omega \psi \rangle_\omega$

**Proposition 7.** *One has, for all $\xi \in \mathbb{R}^+$*

$$\mathcal{M}(\Delta_\omega \phi) = -\xi \mathcal{M}\phi$$

**Definition 6.** *For $s \in \mathbb{R}$, we define $\mathcal{M}^s(\mathbb{R})$ as*

$$f \in \mathcal{M}^s(\mathbb{R}) \iff \int_0^{+\infty} \frac{(1+\xi)^s}{\sqrt{\xi}} \left|\mathcal{M}f\right|^2(\xi) < +\infty$$

**Proposition 8.**

$$f \in \mathcal{M}^s(\mathbb{R}^+) \iff Cf \in H^s(\mathbb{R})$$

*and we have*

$$\|f\|_{\mathcal{M}^s} = \|Cf\|_{H^s}$$

**Definition 7.** *For $s = 0$, we note $L^2_\omega = \mathcal{M}^0(\mathbb{R}^+)$. It is a Hilbert space with the scalar product corresponding to $\langle \cdot, \cdot \rangle_\omega$ defined earlier.*

*Proof.* To prove that $L^2_\omega$ is complete, it suffices to show that any Cauchy sequence $f_n$ has a limit in $L^2_\omega$. Obviously, $g_n := \dfrac{f_n}{x^{1/4}}$ is a Cauchy sequence in $L^2(\mathbb{R}^+)$, so it admits a limit $g_\infty \in L^2(\mathbb{R}^+)$. Then $f_\infty := x^{1/4} g_\infty$ belongs to $L^2_\omega$ and we have

$$\|f_n - f_\infty\|_{L^2_\omega} = \|g_n - g_\infty\|_{L^2}$$

which ensures $f_n \to f_\infty$ in $L^2_\omega$. □

**Proposition 9.** *$\mathcal{M}^s(\mathbb{R}^+)$ is a closed subspace of $L^2_\omega$. It is also a Hilbert space for the scalar product defined by*

$$\langle u, v \rangle_{\omega,s} := \int_0^{+\infty} \frac{(1+\xi)^s}{\sqrt{\xi}} \mathcal{M}u(\xi)\mathcal{M}v(\xi)$$

*Proof.* Let $u_n$ a Cauchy sequence in $\mathcal{M}^s(\mathbb{R}^+)$. Then, by the same arguments as above,

$$v_n := \frac{(1+\xi)^{s/2}\mathcal{M}u_n(\xi)}{\xi^{1/4}}$$

has a limit $v_\infty$ in $L^2$, and

$$u_\infty = \mathcal{M}\left(\frac{\xi^{1/4}}{(1+\xi)^{s/2}}v_\infty\right)$$

is the limit of $u_n$ in $\mathcal{M}^s(\mathbb{R}^+)$. $\square$