Concis disque

Martin, François

February 19, 2019

Abstract

We do the same as for the segment but this time with the disk.

1 Laplace

We define
$$D = \left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1, \quad z = 0\right\}$$
the unit disk in the plane $z = 0$ and
$$\mathbb{S}^2 = \left\{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\right\}$$
the unit sphere in dimension 3.

Definition 1. *For $x \in D$, we put*
$$\omega(x) = \sqrt{1 - \rho^2}$$
where $\rho^2 = x_1^2 + x_2^2$.

We define the single layer operator $\mathcal{S}$ by
$$\mathcal{S}\varphi(x) = \int_D \frac{1}{4\pi \|x - y\|} \varphi(y) dD(y) \tag{1}$$
and $S := \gamma\mathcal{S}$, where $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ denotes the Euclidean norm of x and dD is the surface measure on D. The hypersingular operator is
$$N\varphi(x) := \text{f.p.} \int \frac{\partial^2}{\partial n_x n_y} \frac{1}{\|x - y\|} \varphi(y) dD(y) \tag{2}$$
We let $S_\omega = S\frac{1}{\omega}$ and $N_\omega = N\omega$.

Let us review some properties of the spherical harmonics Y_l^m.

Definition 2. *For $l \in \mathbb{N}$ and $-l \leq m \leq l$, the function $Y_l^m(\theta, \varphi)$ is defined by*

$$Y_l^m(\theta, \varphi) = \gamma_l^m P_{lm}(\cos\theta) e^{im\varphi} ,$$

where $P_{lm}(x)$ is the so-called "Associated Legendre" polynomial which is the solution of the differential equation

$$(1 - x^2)\frac{d^2}{dx^2} P_{lm}(x) - 2x\frac{d}{dx} P_{lm}(x) + \left[l(l+1) - \frac{m^2}{1 - x^2} \right] P_{lm}(x) = 0.$$

The spherical harmonics are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^2$ which takes the following form on the sphere:

$$\Delta_{\mathbb{S}^2} u(\theta, \varphi) = \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \frac{\partial}{\partial_\theta} u \right) - \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} .$$

With this definition, one has

$$-\Delta_{\mathbb{S}^2} Y_l^m = l(l+1) Y_l^m$$

Proposition 1. *The single-layer operator and the hypersingular operator $S_{\mathbb{S}^2}$ and $N_{\mathbb{S}^2}$ defined as in eqs.* (1) *and* (2) *replacing the domain of integration by $\mathbb{S}^2$, satisfy*

$$S_{\mathbb{S}^2} Y_l^m = \frac{1}{2l+1} Y_l^m$$

$$N_{\mathbb{S}^2} Y_l^m = -\frac{l(l+1)}{2l+1} Y_l^m$$

This proposition illustrates the fact that $S_{\mathbb{S}^2} N_{\mathbb{S}^2}$ is an order 0 operator on the unit sphere, with a spectrum concentrated towards $\frac{1}{4}$. Moreover, both operators have the property that they map smooth functions to smooth functions bijectively. This fails to be the case for the operators S and N on the disk. However, once they are appropriately weighted, we have explicit eigenfunctions and eigenvalues. Let us introduce the "projected spherical harmonics" PSH:

Definition 3. *We define the PSH y_l^m by*

$$y_{lm}(\rho, \varphi) = \eta_l^m e^{im\varphi} P_{lm}(\sqrt{1 - \rho^2}) \propto Y_{lm}(\pi/2 - \arccos(\rho), \varphi)$$

where ρ, φ is a system of cylindrical coordinates on D. The constant η_l^m is defined by

$$\eta_l^m = (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} .$$

If l_1 and m_1 and l_2 m_2 have the same parity, there holds

$$\int_D \frac{y_{l_1}^{m_1} y_{l_2}^{m_2}}{\omega} = \delta_{l_1 = l_2} \delta_{m_1 = m_2} .$$

Definition 4. *To reinforce the connection between this work and our previous work in 2d, we denote*

$$T_l^m(x) = y_l^m(x,\varphi)$$
$$U_l^m(x) = \frac{y_{l+1}^m(x,\varphi)}{\sqrt{1-x^2}}$$

where $l+m$ is even. Both T_l^m and U_l^m are polynomials of order l in ρ.

Proposition 2. *The functions T_l^m form an orthogonal basis of $L^2_{\frac{1}{\omega}}(D)$ and the functions U_l^m an orthogonal basis of $L^2_\omega(D)$.*

Proposition 3. *Let $l, m \in \mathbb{N}$ with $l+m$ even. Then*

$$S_\omega T_l^m = \frac{1}{2\lambda_l^m} T_l^m$$

and

$$N_\omega U_l^m = \frac{\lambda_{l+1}^m}{2} U_l^m$$

where

$$\lambda_l^m = 2\frac{\Gamma\left(\frac{l+m+2}{2}\right)\Gamma\left(\frac{l-m+2}{2}\right)}{\Gamma\left(\frac{l+m+1}{2}\right)\Gamma\left(\frac{l-m+1}{2}\right)}$$

Definition 5. *We define the "angular moments" $\mathcal{L}_+$ and $\mathcal{L}_-$ on the disk by the following formulas for smooth functions*

$$\mathcal{L}_\pm u(\rho,\varphi) := e^{\pm i\varphi}\left(\pm\frac{\partial u}{\partial \rho} + i\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\right)$$

One has

$$\nabla_D = \frac{1}{2}\begin{pmatrix} \mathcal{L}_+ - \mathcal{L}- \\ -i(\mathcal{L}_+ + \mathcal{L}_-) \end{pmatrix}$$

therefore, using $\Delta_D = \nabla_D \cdot \nabla_D$,

$$\Delta_D = -\frac{1}{2}(\mathcal{L}_+\mathcal{L}_- + \mathcal{L}_-\mathcal{L}_+)$$

Ici je pense qu'il y a une erreur dans la thèse de Pedro, les deux termes ne sont pas égaux (p. 62) Later on, we drop the subscript D.

Proposition 4. *Let l, m such that $l+m$ is even. Then*

$$\mathcal{L}_\pm T_{l,m} = \sqrt{(l\pm m)(l\pm m+1)}\,U_{l-1}^{m\pm 1}$$

while

$$\omega\mathcal{L}_\pm \omega U_l^m = \sqrt{(l\pm m)(l\pm m+1)}\,T_{l+1}^{m\pm 1}$$

Proof. Those identities are easily deduced from the formulas for $\mathcal{L}_+ y_l^m$ and $\mathcal{L}_- y_l^m$ established in [4] and summarized in [2]. $\square$

One can deduce the following identities

Corollary 1.

$$\omega\mathcal{L}_+\omega\mathcal{L}_- T_l^m = (l(l+1) - m^2 + m)T_{l,m}$$
$$\omega\mathcal{L}_-\omega\mathcal{L}_+ T_l^m = (l(l+1) - m^2 - m)T_{l,m}$$

while

$$\mathcal{L}_+\omega\mathcal{L}_-\omega U_l^m = ((l+1)(l+2) - m^2 + m)U_{l,m}$$
$$\omega\mathcal{L}_-\omega\mathcal{L}_+ U_l^m = ((l+1)(l+2) - m^2 - m)U_{l,m}$$

We further introduce two other operators, $(\omega\nabla\omega)\cdot\nabla$ and $\nabla\cdot(\omega\nabla\omega)$ which satisfy

$$(\omega\nabla\omega)\cdot\nabla = \frac{1}{2}\left(\omega\mathcal{L}_+\omega\mathcal{L}_- + \omega\mathcal{L}_-\omega\mathcal{L}_+\right)$$

and

$$\nabla\cdot(\omega\nabla\omega) = \frac{1}{2}\left(\mathcal{L}_+\omega\mathcal{L}_-\omega + \mathcal{L}_-\omega\mathcal{L}_+\omega\right)$$

Corollary 2. *One has, for $l+m$ even,*

$$(\omega\nabla\omega)\cdot\nabla T_l^m = (l(l+1) - m^2)T_{l,m}\,,$$

and

$$\nabla\cdot(\omega\nabla\omega)U_l^m = ((l+1)(l+2) - m^2)U_{l,m}\,.$$

Proposition 5. *Let $l\in\mathbb{N}$ and $-l \le m \le l$, such that l and m are not both zero. Then*

$$1 \le \frac{4\left(\lambda_l^m\right)^2}{l(l+1) - m^2} \le 2 \tag{3}$$

Proof. By Gauschi's inequality [1], there holds for all $x > 0$ and $s \in (0,1)$,

$$x^{1-s} \le \frac{\Gamma(x+1)}{\Gamma(x+s)} \le (x+1)^{1-s}$$

This inequality, applied with $s = \frac{1}{2}$ and $x = \frac{l\pm m}{2}$ yields

$$\frac{3}{4} \le \frac{4\left(\lambda_l^m\right)^2}{l(l+1) - m^2} \le 4\,.$$

Note that for $s = \frac{1}{2}$, both sides of Gautstchi's inequality remain valid for $x = 0$, so the cases $m = \pm l$ are permitted. To find the constants 1 and 2 instead of $\frac{3}{4}$ and 4, one can use the following improvement of Gautschi's inequality [3]:

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s}\,.$$

Again, this inequality remains valid in the case $x = 0$ for $s = \frac{1}{2}$. Taking again $x = \frac{l \pm m}{2}$, multiplying the squares of the resulting inequalities, and multiplying by $\frac{4}{l(l+1)-m^2}$, one finds

$$\frac{(l+\frac{1}{2})^2 - m^2}{l(l+1)-m^2} < \frac{4\,(\lambda_l^m)^2}{l(l+1)-m^2} < \frac{(l+\frac{\sqrt{3}-1}{2})^2 - m^2}{l(l+1)-m^2} \tag{4}$$

Some manipulations on those inequalities yields

$$1 + \frac{1}{4(l(l+1)-m^2)} < \frac{4\,(\lambda_l^m)^2}{l(l+1)-m^2} < 1 + \frac{(2\sqrt{3}-3)l}{l(l+1)-m^2} + \frac{4-2\sqrt{3}}{l(l+1)-m^2}$$

The left side of the latter inequality immediately implies the left inequality of (3). For the right side, we obtain the result after using $l(l+1) - m^2 \geq l$ for the denominator of the second term, and $l(l+1) - m^2 \geq 1$ for the third term. □

As it can be seen in the proof, the lower bound is sharp, since for $m = 0$ and in the limit $l \to \infty$, the two sides of (4) converge to 1. For the upper bound, it seems, based on numerical evidence, that the highest value of the quantity being estimated is reached for $l = 1$ and $m = \pm 1$. In this case, one has $\frac{4(\lambda_l^m)^2}{l(l+1)-m^2} = \frac{16}{\pi^2} \approx 1.6$.

2 Helmholtz

Much like in the 2-dimensional case, the following commutation holds.

Theorem 1. *There holds the commutations*

$$\left[(\omega\nabla\omega)\cdot\nabla - k^2\omega^2\right] S_{k,\omega} = S_{k,\omega}\left[(\omega\nabla\omega)\cdot\nabla - k^2\omega^2\right]$$

and

$$\left[\nabla\cdot(\omega\nabla\omega) - k^2\omega^2\right] N_{k,\omega} = N_{k,\omega}\left[\nabla\cdot(\omega\nabla\omega) - k^2\omega^2\right]$$

Proof. □

References

[1] Walter Gautschi. Some elementary inequalities relating to the gamma and incomplete gamma function. *Journal of Mathematics and Physics*, 38(1-4):77–81, 1959.

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[3] D Kershaw. Some extensions of w. gautschi's inequalities for the gamma function. *Mathematics of Computation*, pages 607–611, 1983.

[4] Pedro Ramaciotti Morales. *Theoretical and numerical aspects of wave propagation phenomena in complex domains and applications to remote sensing.* PhD thesis, Paris Saclay, 2016.