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The Galerkin Method for Integral Equations of the First Kind with Logarithmic Kernel: Theory

I. H. SLOAN

Department of Applied Mathematics, University of New South Wales, Sydney, NSW 2033, Australia

AND

A. SPENCE

School of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK

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The aim of this paper is to develop a straightforward analysis of the Galerkin method for two-dimensional boundary integral equations of the first kind with logarithmic kernels. A distinctive feature of the analysis is that no appeal is made to 'coercivity', as a result of which some existence questions cannot be answered directly. In return, however, the analysis has no special difficulty in handling corners, cusps, or open arcs. Instead of coercivity, the central feature of the analysis is the positive-definite property of the integral operator for small enough contours. Rates of convergence are predicted theoretically and, in particular, certain linear functionals are shown to exhibit 'superconvergence'. Numerical results supporting the theory are given in the companion paper Sloan & Spence (1987) for problems on both open and closed polygonal arcs.

1. Introduction

IN THIS PAPER we consider the theory of the Galerkin method applied to integral equations of the form

$$-\int_{\Gamma} \log|t-s|\, y(s)\, \mathrm{d}l_s = f(t) \quad (t \in \Gamma), \tag{1.1}$$

where Γ is a rectifiable arc (either open or closed) in the plane, $|t-s|$ is the Euclidean distance, $\mathrm{d}l_s$ is the element of arc length, and f is a continuous function on Γ. We discuss numerical aspects in the companion paper devoted to applications (Sloan & Spence 1988).

We shall also consider equations of the related form (see Hsiao & MacCamy, 1973, and Jaswon & Symm, 1977),

$$-\int_{\Gamma} \log|t-s|\, z(s)\, dl_s + \omega = f(t) \quad (t \in \Gamma), \qquad \int_{\Gamma} z(s)\, dl_s = b, \tag{1.2}$$

in which the unknowns are the function z and the scalar ω, and b is a given real number.

Our concern in this paper is to develop a robust yet conceptually simple analysis of the Galerkin method for such problems: robust in that it copes easily if, for example, Γ is an open arc, or the boundary of a region with corners and cusps, or even the union of pieces of both kinds. Technically, unlike other studies of the Galerkin method for such problems (Leroux, 1974; Hsiao & Wendland, 1977; Richter, 1978, Wendland, 1983; Chandler 1984) we are prepared to abandon the 'coercivity' property. The effect is that we are freed from concern about many intricacies relating to function spaces in the presence of corners (see, for example, Costabel & Stephan, 1981, and McLean, 1985). The price is that the existence of solutions can no longer be inferred from the variational formulation of the problem.

An important special case is that in which Γ is the piecewise smooth boundary of a bounded open simply connected region Ω. In this situation, equations (1.1) and (1.2) arise in connection with the (interior) Dirichlet problem for the region Ω: if u is a harmonic function on Ω with boundary values f on Γ, then (1.1) arises if one seeks a 'single-layer' representation for u:

$$u(\boldsymbol{t}) = -\int_\Gamma \log|\boldsymbol{t}-\boldsymbol{s}|\, y(\boldsymbol{s})\, \mathrm{d}l_s \quad (\boldsymbol{t}\in\Omega), \tag{1.3}$$

where y is a 'charge density', yet to be determined. Similarly, (1.2) arises if one seeks a representation of the form

$$u(\boldsymbol{t}) = -\int_\Gamma \log|\boldsymbol{t}-\boldsymbol{s}|\, z(\boldsymbol{s})\, \mathrm{d}l_s + \omega \quad (\boldsymbol{t}\in\Omega) \tag{1.4}$$

with the constant ω fixed by specifying that the total 'charge' $\int_\Gamma z(\boldsymbol{s})\,\mathrm{d}l_s$ is some number b (e.g. $b=0$).

Integral equations of the above forms also arise for the exterior Dirichlet problem, that is, for the Dirichlet problem for the complement of $\bar{\Omega}$. If the potential u is required to be bounded at infinity, then the appropriate representation (see Jaswon & Symm, 1977, p. 55) is (1.4), with $b=0$ so that the final equation to be solved is (1.2) with $b=0$. It should also be observed that the exterior Dirichlet problem is well defined for the degenerate situation in which Γ is an open arc. Numerical examples associated with interior and exterior Dirichlet problems are considered in Sloan & Spence (1988), where special attention is paid to the approximation of potentials given by (1.3) and (1.4).

It may also be remarked that integral equations of the form (1.1) arise in the problem of the conformal mapping of an open simply connected region Ω with boundary Γ onto the interior of the unit circle. (For a discussion see, for example, Gaier, 1976, Jaswon & Symm, 1977, §4.5, and Hough & Papamichael, 1981.) In this case

$$f(\boldsymbol{t}) = -\log|\boldsymbol{t}-\boldsymbol{\tau}|,$$

where $\boldsymbol{\tau}$ is the point in Ω which is to be mapped to the centre of the circle. Also, it is worth noting that a good understanding of this equation is important in the discussion of the convergence rates of approximations to (1.3) and (1.4).

2. The equilibrium distribution and the transfinite diameter

An important special case of (1.1), and one that plays a key role in the present work, is that in which $f(t) \equiv 1$. Assuming for the moment that a solution of this problem exists and is unique, we have, using the special notation $y(s) = \lambda_\Gamma(s)$ for the solution in this case,

$$-\int_\Gamma \log|t-s|\,\lambda_\Gamma(s)\,dl_s = 1 \quad (t \in \Gamma). \tag{2.1}$$

A closely related quantity is the so-called equilibrium distribution $e_\Gamma(s)$: it is the solution of (1.1) with a constant right-hand side, but with the constant adjusted so that the integral of e_Γ is 1:

$$-\int_\Gamma \log|t-s|\,e_\Gamma(s)\,dl_s = u_\Gamma \quad (t \in \Gamma), \tag{2.2}$$

where u_Γ is such that

$$\int_\Gamma e_\Gamma = 1. \tag{2.3}$$

Note that the latter equations are a special case of (1.2), with $\omega = -u_\Gamma$, $f = 0$, $b = 1$. It is clear, if $u_\Gamma \neq 0$, that λ_Γ and e_Γ are related by $\lambda_\Gamma(s) = u_\Gamma^{-1} e_\Gamma(s)$, and that if λ_Γ is known then u_Γ can be computed from

$$u_\Gamma = \left(\int_\Gamma \lambda_\Gamma\right)^{-1}. \tag{2.4}$$

We shall see subsequently that, for certain contours Γ, a solution e_Γ of (2.2) and (2.3) exists for which $u_\Gamma = 0$. In that situation, (2.1) has no solution, since otherwise the double integral

$$-\int_\Gamma \int_\Gamma \log|t-s|\,e_\Gamma(s)\,dl_s\,\lambda_\Gamma(t)\,dl_t$$

would have the value 0 if integrated first with respect to s, but the value 1 if integrated first with respect to t. In the usage of Jaswon & Symm (1977), Γ is then a 'Γ-contour'.

Physically, the equilibrium distribution e_Γ is the charge density on a two-dimensional conductor, if the conductor has a total charge of 1. The constant u_Γ is the resulting potential on the conductor, with the required potential at an arbitrary point t being defined by

$$u(t) = -\int_\Gamma \log|t-s|\,e_\Gamma(s)\,dl_s \quad (t \in \Omega).$$

An important quantity associated with Γ is the 'transfinite diameter' or 'logarithmic capacity' C_Γ (Hille, 1962, Ch. 16). Assuming that the equilibrium distribution exists, and that u_Γ is defined by (2.2) and (2.3), we may define C_Γ by (see Hille, 1962, pp. 280–287)

$$C_\Gamma = \exp(-u_\Gamma). \tag{2.5}$$

The quantity C_Γ may be interpreted as a length associated with Γ, since it scales in the same way as Γ: if $\Gamma = a\Gamma'$, where a is a constant, it follows easily from (2.2) and (2.3) that $u_{\Gamma'} = u_\Gamma + \log a$, and hence $C_\Gamma = aC_{\Gamma'}$. (Scaling considerations are considered more fully in Section 8.)

The importance of C_Γ for our story is easily seen: If $C_\Gamma = 1$ then $u_\Gamma = 0$, so that the solution of (1.1), if it exists at all, is not unique, since to any solution can be added an arbitrary multiple of $e_\Gamma(s)$. Subsequently we shall see that, when $C_\Gamma < 1$, a particularly simple analysis, based on positivity considerations, is possible.

We now consider more carefully the question of the existence of the equilibrium distribution. The quantity u_Γ has an alternative variational definition (see (2.6) below), and hence C_Γ exists for any closed bounded set Γ. A thorough discussion of the transfinite diameter of closed bounded subsets of the plane is given by Hille (1962), Ch. 16. (The concepts of transfinite diameter and logarithmic capacity are shown there to be equivalent—see Theorem 16.4.4). Among the useful properties are:

the transfinite diameter of Γ does not exceed its Euclidean diameter;
if Γ lies inside Γ', then $C_\Gamma \leq C_{\Gamma'}$;
the transfinite diameter of a circle of radius a is a;
the transfinite diameter of an interval of length l is $\frac{1}{4}l$.

In the variational definition (see Hille, 1962, p. 280), u_Γ (known as Robin's constant) is expressed as an infimum over all normalized positive measures μ defined on Γ:

$$u_\Gamma = \inf\left(-\int_\Gamma\int_\Gamma \log|t-s|\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s)\right), \tag{2.6}$$

where $\mu \geq 0$, and $\int_\Gamma \mathrm{d}\mu = 1$. Loosely, we may think of $\mathrm{d}\mu(t)$ as the positive charge on the interval $\mathrm{d}l_t$ with (in general) point charges being allowed. (For precise definitions see Hille, 1962, §16.3.) In the case of a well behaved (absolutely continuous) measure μ we have $\mathrm{d}\mu(s) = y(s)\,\mathrm{d}l_s$, where y is a non-negative and integrable charge density.

If $0 < C_\Gamma < \infty$, corresponding to $\infty > u_\Gamma > -\infty$, then Hille (1962), Theorem 16.4.3, shows that there exists a unique normalized positive measure μ_e that achieves the infimum in (2.6). Moreover, except possibly for t in a set of transfinite diameter zero, one has (see Hille, Theorem 16.4.8)

$$-\int_\Gamma \log|t-s|\,\mathrm{d}\mu_e(s) = u_\Gamma \quad (t \in \Gamma). \tag{2.7}$$

(The result (2.7) holds, in fact, for all $t \in \Gamma$, without having to exclude a set of transfinite diameter zero, if Γ is the boundary of a simply connected region and satisfies mild regularity conditions, which even permit cusps pointing into Ω. See Hille (1962), p. 291, for details.) Thus μ_e can be identified as the measure corresponding to the classical equilibrium distribution e_Γ, if the latter exists. It follows that *the equilibrium distribution always exists in the sense of a measure*, even if not as a classical distribution function. Existence results in an appropriate

L_p sense are indicated in the following section. We shall show subsequently (see Sections 5 and 8) that e_Γ is uniquely determined by the defining equations (2.2) and (2.3), and correspondingly that, for $C_\Gamma \neq 1$, the function λ_Γ is uniquely determined by (2.1).

3. Equivalence and existence results for (1.1) and (1.2)

It is convenient to write (1.1) as

$$Ky = f, \tag{3.1}$$

where K is the integral operator on the left-hand side of (1.1). In considering this equation we shall always assume that $C_\Gamma \neq 1$, or equivalently that $u_\Gamma \neq 0$, where C_Γ is the transfinite diameter defined in the preceding section, and $u_\Gamma = \log C_\Gamma^{-1}$. Then (2.1) states that $K\lambda_\Gamma = 1$, where 1 is the function on Γ with the constant value 1, and $\lambda_\Gamma = u_\Gamma^{-1} e_\Gamma$.

In a similar way, (1.2) may be written as the pair of equations

$$Kz + \omega 1 = f, \qquad \int z = b. \tag{3.2}$$

The systems (3.1) and (3.2) are in a certain sense equivalent, provided that $C_\Gamma \neq 1$ (cf. Jaswon & Symm 1977, §4.3). To see this, observe that, if the pair (z, ω) satisfies (3.2), then clearly y given by

$$y := z + \omega\lambda_\Gamma \tag{3.3}$$

satisfies (3.1); conversely, if y satisfies (3.1), then (z, ω) given by

$$z := y - \omega\lambda_\Gamma, \qquad \omega := \left(\int \lambda_\Gamma\right)^{-1}\left(\int y - b\right) = u_\Gamma\left(\int y - b\right), \tag{3.4}$$

satisfies (3.2).

Note that the above equivalence hinges on the existence of the equilibrium distribution. This equivalence (and the analogous result for the Galerkin methods) will allow us in Sections 6 and 7 to analyse the Galerkin method for (1.2) in terms of the Galerkin method for (1.1).

The existence of solutions to (1.1) and (1.2) in L_p spaces can be studied if suitable conditions are imposed on Γ and f. If Ω is a bounded open subset of $\mathbb{R}^2$ with boundary Γ formed by a finite number of piecewise smooth curves which meet at nonzero angles, Theorem 1.1 of McLean (1986) shows the existence of a solution of (3.2) in $L_2(\Gamma) \times \mathbb{R}$, if and only if f has a weak tangential derivative $\mathrm{D}f$ in $L_2(\Gamma)$. A corresponding result follows for (3.1) by our equivalence argument, provided that $C_\Gamma \neq 1$. For the case of the arc $\Gamma = [-1, 1]$, Jörgens (1982, p. 370) shows that, for $1 < p < 2$, (3.1) has a solution $y \in L_p(\Gamma)$ if and only if $\mathrm{D}f \in L_p(\Gamma)$, and existence results for more general smooth arcs with $C_\Gamma \neq 1$ can readily be deduced from this result.

In the sequel, we shall assume that Γ and f are such that y satisfies

$$-\int_\Gamma\int_\Gamma y(t)\log|t-s|\,y(s)\,\mathrm{d}l_t\,\mathrm{d}l_s < \infty. \tag{3.5}$$

The two situations described in the preceding paragraph provide important examples of situations in which this holds.

4. The Galerkin method

The Galerkin method for (1.1), or equivalently for (3.1), may be defined in a standard way, as follows: Select S^h, a finite-dimensional space of continuous or piecewise continuous functions on Γ. Then the Galerkin approximation $y^h \in S^h$ satisfies

$$(Ky^h, \chi^h) = (f, \chi^h) \quad \forall \chi^h \in S^h, \tag{4.1}$$

where

$$(w, v) = \int_\Gamma w(s)v(s)\, dl_s.$$

Under the assumptions that $0 < C_\Gamma < 1$, that (1.1) has a solution y satisfying $(Ky, y) < \infty$ as in (3.5), and that

$$(K\chi^h, \chi^h) < \infty \quad \forall \chi^h \in S^h,$$

we shall show in Section 6 that the Galerkin approximation y^h exists and is unique. Moreover, in Section 8, the existence and uniqueness are shown to hold even if $C_\Gamma \geq 1$, provided only that $C_\Gamma^h \neq 1$, where C_Γ^h is the Galerkin approximation to C_Γ.

The quantity C_Γ^h is defined in terms of e_Γ^h, the Galerkin approximation to the equilibrium distribution e_Γ, which satisfies (cf. (2.2)) $e_\Gamma^h \in S^h$ and

$$(Ke_\Gamma^h, \chi^h) = u_\Gamma^h(1, \chi^h) \quad \forall \chi^h \in S^h, \tag{4.2}$$

with u_Γ^h determined by requiring

$$(e_\Gamma^h, 1) = 1. \tag{4.3}$$

We shall see subsequently (in Section 8) that e_Γ^h exists and is unique for every contour Γ, so that u_Γ^h is well defined. Then C_Γ^h is defined by the analogue of (2.5):

$$C_\Gamma^h = \exp(-u_\Gamma^h). \tag{4.4}$$

Closely related to e_Γ^h is λ_Γ^h, the Galerkin approximation to λ_Γ (see (2.1)), which satisfies $\lambda_\Gamma^h \in S^h$ and

$$(K\lambda_\Gamma^h, \chi^h) = (1, \chi^h) \quad \forall \chi^h \in S^h. \tag{4.5}$$

If $C_\Gamma^h \neq 1$, or equivalently if $u_\Gamma^h \neq 0$, then it follows from comparing (4.5) with (4.2) that λ_Γ^h and e_Γ^h are related by $\lambda_\Gamma^h = (u_\Gamma^h)^{-1} e_\Gamma^h$. Thus, if λ_Γ^h is known, then u_Γ^h may be computed from the analogue of (2.4),

$$u_\Gamma^h = (\lambda_\Gamma^h, 1)^{-1}, \tag{4.6}$$

and C_Γ^h then computed from (4.4). We shall see subsequently (in Section 7 and in Sloan & Spence, 1988) that this method of computing an approximate transfinite diameter has attractive convergence properties.

Now consider the Galerkin approximation to (1.2), or equivalently (3.2). The Galerkin approximation to the pair (z, ω), denoted by $(z^h, \omega^h) \in S^h \times \mathbb{R}$, satisfies

$$(Kz^h + \omega^h 1, \chi^h) = (f, \chi^h) \quad \forall \chi^h \in S^h, \qquad (z^h, 1) = b. \tag{4.7}$$

The Galerkin equations (4.1) and (4.7) are equivalent, provided that $C_\Gamma^h \neq 1$, in a way that exactly parallels the equivalence of the exact equations noted in the preceding section. In fact, if (z^h, ω^h) satisfies (4.7), then y^h given by

$$y^h := z^h + \omega^h \lambda_\Gamma^h \tag{4.8}$$

clearly satisfies (4.1). Conversely, if y^h satisfies (4.1), then the pair (z^h, ω^h) given by

$$z^h := y^h - \omega^h \lambda_\Gamma^h, \qquad \omega^h := (\lambda_\Gamma^h, 1)^{-1}((y^h, 1) - b) = u_\Gamma^h((y^h, 1) - b) \tag{4.9}$$

satisfies (4.7).

We shall show in the next section that, if $0 < C_\Gamma < 1$, then K is a positive-definite operator in the sense that

$$(Kw, w) \geqslant 0, \tag{4.10}$$

with equality if and only if $w = 0$. Here w may be any absolutely integrable function on Γ, so long as it is understood that the left-hand side may have the value $+\infty$.

Under this circumstance, the analysis of the Galerkin approximation (4.1) becomes straightforward (see Section 6). Also, because of the correspondence between y^h and (z^h, ω^h) established above, the properties of the Galerkin approximation (4.7) will then follow immediately.

5. Positive-definiteness of the logarithmic potential

The positive-definite property of K, under the assumption $0 < C_\Gamma < 1$, is touched on in many places in the potential-theory literature, but there does not seem to be an easily accessible proof. In this section we prove the result and indicate some immediate consequences.

The result is stated here in a form more general than (4.10):

THEOREM 1. *Let μ be any signed measure (or 'charge') defined on Γ (for definition, see Hille,* 1962, *pp.* 288–9). *Then, for* $0 < C_\Gamma < 1$,

$$-\int_\Gamma \int_\Gamma \log|t - s| \, d\mu(s) \, d\mu(t) \geqslant 0, \tag{5.1}$$

with equality only if $\mu = 0$.

This property is proved later in this section. However, because it may give useful insight, we first sketch a classical proof of more limited validity. A similar argument (limited to the case in which Γ is the smooth boundary of a simply connected region Ω) has been given by Hsiao & Wendland (1977).

In this preliminary discussion we restrict the charge to the classical form $d\mu(s) = y(s)\, dl_s$, where y is an absolutely integrable function on Γ. Then the property to be proved is that $(Ky, y) \geqslant 0$ with equality only if $y = 0$. We write

$$y = \alpha e_\Gamma + y_0,$$

where e_Γ is the equilibrium distribution, and $\alpha = (y, 1)$, so that $y_0 = y - \alpha e_\Gamma$ satisfies

$$(y_0, 1) = 0. \tag{5.2}$$

Then, with the aid of (2.2), which can be written as $Ke_\Gamma = u_\Gamma 1$, we obtain

$$Ky = \alpha K e_\Gamma + Ky_0 = \alpha u_\Gamma 1 + Ky_0,$$

and hence

$$\begin{aligned}(Ky, y) &= \alpha u_\Gamma (1, y) + (Ky_0, y) = \alpha^2 u_\Gamma + (y_0, Ky)\\ &= \alpha^2 u_\Gamma + (y_0, \alpha u_\Gamma 1 + Ky_0) = \alpha^2 u_\Gamma + (y_0, Ky_0),\end{aligned}$$

so that finally

$$(Ky, y) = \alpha^2 u_\Gamma + (Ky_0, y_0). \tag{5.3}$$

Since $u_\Gamma = \log C_\Gamma^{-1} > 0$, it follows from (5.3) that the desired result holds if we can show the same result for y_0, i.e. if we can show that $(Ky_0, y_0) \geqslant 0$, with equality only if $y_0 = 0$.

Let u denote the potential corresponding to the charge density y:

$$u(t) = -\int_\Gamma \log |t - s|\, y(s)\, \mathrm{d}l_s \quad (t \in \mathbb{R}^2). \tag{5.4}$$

Then it is well known (see, for example, Kellogg, 1929), under sufficiently strong regularity assumptions on Γ, y, and u, that the charge density is related to the discontinuity in the normal derivatives across Γ:

$$y(t) = -\frac{1}{2\pi}\left(\frac{\partial u}{\partial n_+}(t) - \frac{\partial u}{\partial n_-}(t)\right) \quad (t \in \Gamma), \tag{5.5}$$

where $\partial u/\partial n_\pm$ denote the limiting values of the normal derivatives if $t \in \Gamma$ is approached from the positive and negative sides (with respect to the normal) of Γ.

In a similar way, if u_0, given by

$$u_0(t) = -\int_\Gamma \log |t - s|\, y_0(s)\, \mathrm{d}l_s \quad (t \in \mathbb{R}^2), \tag{5.6}$$

is the potential corresponding to the charge density y_0, then

$$y_0(t) = -\frac{1}{2\pi}\left(\frac{\partial u_0}{\partial n_+}(t) - \frac{\partial u_0}{\partial n_-}(t)\right) \quad (t \in \Gamma). \tag{5.7}$$

It follows, with the aid of the divergence theorem, that

$$\begin{aligned}(Ky_0, y_0) = (u_0, y_0) &= -\frac{1}{2\pi}\left(u_0, \frac{\partial u_0}{\partial n_+} - \frac{\partial u_0}{\partial n_-}\right)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus\Gamma} \operatorname{div}(u_0 \nabla u_0)\, \mathrm{d}\tau = \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus\Gamma} (\nabla u_0)^2\, \mathrm{d}\tau \geqslant 0,\end{aligned}$$

where we have used the fact that u_0 is harmonic in $\mathbb{R}^2\backslash\Gamma$ and regular at infinity (Kellogg, 1929, p. 248). We also see, remembering that u_0 is continuous across Γ, that (Ky_0, y_0) vanishes only if u_0 is a constant. Since u_0 vanishes at infinity, we conclude that for (Ky_0, y_0) to vanish it is necessary that $u_0 = 0$, implying $y_0 = 0$. Thus the positive-definite property of K, under the given assumptions, is established.

The most delicate points of this classical argument are the divergence theorem and the formula (5.5). The latter is proved by Kellogg (1929) only under very restrictive assumptions: that the first and second derivatives of u have limits as Γ is approached from either side, that the charge density y is continuous, and additionally that Γ is a 'regular curve' (see Kellogg, 1929, p. 99). These classical conditions can be weakened in various ways; but, in any event, the proof given below does not rely on (5.5).

Before leaving these classical considerations, we note that similar arguments can be used to give a constructive proof of the existence of a solution to (1.1), under appropriate regularity conditions on Γ and f, and provided that $C_\Gamma \neq 1$. Let w denote the solution of the Dirichlet problem for the region $\mathbb{R}^2\backslash\Gamma$ with boundary data f, assuming that (as will almost always be the case in practice) such a solution exists. That is, w is the unique harmonic function in $\mathbb{R}^2\backslash\Gamma$ that is regular at infinity and converges to f as Γ is approached. With $w(\infty)$ denoting the limit of w at infinity, we define

$$u_0(t) = w(t) - w(\infty) \quad (t \in \mathbb{R}^2\backslash\Gamma), \tag{5.8}$$

so that u_0 is harmonic in $\mathbb{R}^2\backslash\Gamma$, vanishes at infinity, and converges to f_0, where

$$f_0(t) = f(t) - w(\infty) \quad (t \in \Gamma),$$

as Γ is approached. Green's third identity (or Green's formula, Jaswon & Symm, 1977, p. 57) then yields

$$u_0(t) = \frac{1}{2\pi}\int_\Gamma \log|t-s|\left(\frac{\partial u_0}{\partial n_+}(s) - \frac{\partial u_0}{\partial n_-}(s)\right) \mathrm{d}l_s \quad (t \in \mathbb{R}^2\backslash\Gamma).$$

Thus, if we *define* y_0 by (5.7), then (5.6) is satisfied. On letting t in (5.6) approach Γ, we obtain $f_0 = Ky_0$. Finally, since $C_\Gamma \neq 1$, we may define

$$y(s) = y_0(s) + w(\infty)\lambda_\Gamma(s) \quad (s \in \Gamma). \tag{5.9}$$

This is the desired solution of (1.1), since

$$Ky = Ky_0 + w(\infty)K\lambda_\Gamma = f_0 + w(\infty)1 = f.$$

Proof of Theorem 1. Let μ_e be the equilibrium charge defined in Section 2, i.e. the normalized positive measure that satisfies (2.7). Then we may write an arbitrary signed measure μ in the form

$$\mu = \alpha\mu_e + \mu_0,$$

where

$$\alpha = \int_\Gamma \mathrm{d}\mu,$$

and hence $\mu_0 = \mu - \alpha\mu_e$ satisfies

$$\int_\Gamma d\mu_0 = 0.$$

Letting $[\mu, \nu]$ (the 'mutual energy' of the charges μ and ν) be defined by

$$[\mu, \nu] = -\int_\Gamma \int_\Gamma \log|t-s| \, d\mu(s) \, d\nu(t) = [\nu, \mu],$$

it follows from (2.7) that

$$[\mu_e, \nu] = u_\Gamma \int_\Gamma d\nu.$$

thus

$$\begin{aligned}[\mu, \mu] &= \alpha^2[\mu_e, \mu_e] + 2\alpha[\mu_e, \mu_0] + [\mu_0, \mu_0] \\ &= \alpha^2 u_\Gamma + [\mu_0, \mu_0].\end{aligned}$$

Doob (1984), pp. 248–9, proves that $[\mu_0, \mu_0] \geq 0$, with equality if and only if μ_0 is the zero charge. Since $u_\Gamma = \log C_\Gamma^{-1} > 0$, the desired result now follows immediately. □

Finally, we indicate some immediate consequences of the positive-definite property for $0 < C_\Gamma < 1$. First, the equation (1.1) has at most one solution: for, if $Ky = f$ and $KY = f$, then $K(y - Y) = 0$, implying $(K(y - Y), y - Y) = 0$, and hence $y - Y = 0$. In particular, (2.1) has just one solution, so that λ_Γ and, in turn, e_Γ are uniquely determined by their respective integral equations. It also follows from the arguments in Section 3 that (1.2) has at most one solution (z, ω), namely that given by (3.4). In Section 8, scaling arguments will be used to extend these uniqueness results to values of $C_\Gamma > 1$.

6. Analysis of the Galerkin method in the case $0 < C_\Gamma < 1$

The arguments in this section follow closely on the well known arguments for the finite-element method for elliptic partial differential equations, except that all properties that follow from 'coercivity' are absent.

We shall assume the existence of an absolutely integrable function y satisfying (1.1) and

$$(Ky, y) < \infty. \tag{6.1}$$

To ensure that the Galerkin approximation similarly has finite energy, the condition

$$(K\chi^h, \chi^h) < \infty \quad \forall \chi^h \in S^h, \tag{6.2}$$

is imposed on the subspace S^h.

As is well known, the exact solution y of (1.1) is the unique minimizer of the quadratic functional

$$I(w) = (Kw, w) - 2(f, w),$$

the minimum being taken over all integrable functions of finite energy. In a similar way, the Galerkin approximation y^h, i.e. the element of S^h that satisfies (4.1), is the unique minimizer of $I(w^h)$ if w^h is restricted to the subspace S^h. (Because S^h is a finite-dimensional space, an element that minimizes $I(w^h)$ over S^h certainly exists.) Thus, the stability of the Galerkin method is proved in a rather straightforward manner.

In particular, on setting $f = 1$, it follows that λ_Γ^h, the solution to (4.5), exists and is unique. Moreover, on setting $\chi^h = \lambda_\Gamma^h$ in (4.5) we obtain

$$(\lambda_\Gamma^h, 1) = (K\lambda_\Gamma^h, \lambda_\Gamma^h),$$

so that the positive-definite property together with (4.6) and (4.4) yields

$$(\lambda_\Gamma^h, 1) > 0 \quad \text{and} \quad u_\Gamma^h > 0 \quad \text{for} \quad 0 < C_\Gamma^h < 1.$$

It now follows from (4.5) that (4.2) and (4.3) have the unique solution $e_\Gamma^h = u_\Gamma^h \lambda_\Gamma^h$, so that the Galerkin approximation e_Γ^h to the equilibrium distribution exists and is unique for $C_\Gamma < 1$.

To obtain an error analysis, it is convenient, in the usual way, to define an 'energy inner product'

$$\langle w, v \rangle = (Kw, v) = -\int_\Gamma \int_\Gamma \log |t - s| \, w(s) v(t) \, dl_s \, dl_t. \tag{6.3}$$

This clearly satisfies the inner-product axioms, given the positive-definite property of K.

Then, from the Galerkin approximation (4.1) and the exact equation (3.1), we have

$$(K(y^h - y), \chi^h) = 0 \quad \forall \chi^h \in S^h,$$

or equivalently

$$\langle y^h - y, \chi^h \rangle = 0 \quad \forall \chi^h \in S^h,$$

so that y^h is the orthogonal projection (with respect to the energy inner product) of y onto the space S^h.

With $\|\bullet\|$ denoting the 'energy norm', i.e.

$$\|v\| = \langle v, v \rangle^{1/2} = \left(-\int_\Gamma \int_\Gamma \log |t - s| \, v(s) v(t) \, dl_s \, dl_t \right)^{1/2},$$

it follows from the properties of orthogonal projections that y^h is the best approximation to y in the sense of the energy norm, i.e.

$$\|y^h - y\| = \min_{\chi^h \in S^h} \|\chi^h - y\|. \tag{6.4}$$

We summarize the results from the above analysis in the following theorem.

THEOREM 2. *Assume* (6.2). *For* $0<C_\Gamma<1$, *the Galerkin approximation* y^h *obtained from* (4.1) *exists and is unique. Moreover,* y^h *is the optimal approximation to the exact solution* y *in the sense of the energy norm.*

Now consider the Galerkin approximation z^h to the solution of (1.2). It follows from the equivalences established in Sections 3 and 4 that z and z^h exist and are unique. Then from (3.4) and (4.9) we have

$$\begin{aligned}\|z^h - z\| &= \|y^h - \omega^h\lambda_\Gamma^h - y + \omega\lambda_\Gamma\| \\ &\leqslant \|y^h - y\| + |\omega|\,\|\lambda_\Gamma^h - \lambda_\Gamma\| + |\omega^h - \omega|\,\|\lambda_\Gamma^h\|. \end{aligned} \tag{6.5}$$

An estimate of the last term, to be obtained in the next section (see 7.7), shows it to have a higher order of convergence than the second term.

7. Galerkin calculation of linear functionals in the case $0<C_\Gamma<1$

In this section, we take up the fact that often one is not interested directly in the solution y of (1.1), but rather in some linear functional (y, g), where g is a smooth function on Γ. This is the case, for example, if the real objective is to solve a Dirichlet problem in the manner indicated in Section 1: according to (1.3) or (1.4), the final computation of the potential at a point $t \notin \Gamma$ involves such an inner product, with $g(s) = -\log|t-s|$ $(s \in \Gamma)$. We shall see that the Galerkin approximation to (y, g) will often converge faster than $\|y^h - y\|$, and so may be said to exhibit 'superconvergence'.

The following argument is essentially that used for the Aubin–Nitsche lemma. We assume that the equation

$$Kw = g \tag{7.1}$$

has a solution w of finite energy. (In practive the existence of w can usually be inferred by the argument leading to (5.9), with f replaced by g and y by w.) The solution w is, of course, unique. Denote by $w^h \in S^h$ the corresponding Galerkin solution, i.e.

$$(Kw^h, \chi^h) = (g, \chi^h) \quad \forall\, \chi^h \in S^h.$$

Then

$$\begin{aligned}(y^h, g) - (y, g) &= (y^h - y, g) = (y^h - y, Kw) \\ &= (K(y^h - y), w) = (K(y^h - y), w - w^h) \\ &= -\langle y^h - y, w^h - w\rangle. \end{aligned} \tag{7.2}$$

The Cauchy–Schwarz inequality then proves the following theorem.

THEOREM 3. *Assume that* g *and* w *are related by* (7.1), *and that* $0<C_\Gamma<1$. Then

$$|(y^h, g) - (y, g)| \leqslant \|y^h - y\|\,\|w^h - w\|. \tag{7.3}$$

This result indicates the possibility of superconvergence, since $\|y^h - y\|\,\|w^h - w\|$ will generally tend to zero faster than $\|y - y^h\|$, with the precise improvement depending on the smoothness of w. Numerical results for several examples are given in Sloan & Spence (1987), where comparisons are made with predictions based on (7.2) and (7.3).

An important special case arises in the approximate calculation of the transfinite diameter via (4.4)–(4.6). In this case, the first step is to compute the Galerkin approximation to $(\lambda_\Gamma, 1)$, where λ_Γ is the solution of (1.1) with f replaced by 1. So, in this case, $g = 1$, and therefore the solution of (7.1) is $w = y = \lambda_\Gamma$. Thus (7.2) yields

$$(\lambda_\Gamma^h, 1) - (\lambda_\Gamma, 1) = -\|\lambda_\Gamma^h - \lambda_\Gamma\|^2, \tag{7.4}$$

so that the error in the inner product is the *square* of $\|\lambda_\Gamma^h - \lambda_\Gamma\|$. It follows then from (4.6) and (2.4) that

$$u_\Gamma^h - u_\Gamma = u_\Gamma^2 \|\lambda_\Gamma^h - \lambda_\Gamma\|^2 + O(\|\lambda_\Gamma^h - \lambda_\Gamma\|^4), \tag{7.5}$$

and similarly from (4.4) and (2.5) that

$$C_\Gamma^h - C_\Gamma = -C_\Gamma u_\Gamma^2 \|\lambda_\Gamma^h - \lambda_\Gamma\|^2 + O(\|\lambda_\Gamma^h - \lambda_\Gamma\|^4). \tag{7.6}$$

Finally, we consider the Galerkin approximation (z^h, ω^h) to the solution of (1.2). It follows from (4.9) and (3.4) that

$$\omega^h - \omega = u_\Gamma((y^h, 1) - (y, 1)) + (u_\Gamma^h - u_\Gamma)((y^h, 1) - b),$$

so that, from (7.5) and (7.3) (with $g = 1$), we obtain

$$|\omega^h - \omega| = O(\|y^h - y\| \, \|\lambda_\Gamma^h - \lambda_\Gamma\| + \|\lambda_\Gamma^h - \lambda_\Gamma\|^2). \tag{7.7}$$

In a similar way, we have

$$(z^h, g) - (z, g) = (y^h, g) - (y, g) - \omega^h((\lambda_\Gamma^h, g) - (\lambda_\Gamma, g)) - (\omega^h - \omega)(\lambda_\Gamma, g); \tag{7.8}$$

thus from (7.3) and (7.7) we obtain

$$|(z^h, g) - (z, g)| = O\big((\|y^h - y\| + \|\lambda_\Gamma^h - \lambda_\Gamma\|)(\|w^h - w\| + \|\lambda_\Gamma^h - \lambda_\Gamma\|)\big). \tag{7.9}$$

In Sloan & Spence (1988), the theoretical results in this section are used to derive rates of convergence in terms of powers of h, where h is a steplength parameter in the Galerkin method.

8. Scaling

So far we have assumed, in discussing the Galerkin method, that $0 < C_\Gamma < 1$. We now consider equations (1.1) and (1.2) for contours Γ that may not satisfy this condition. The first point to note is that an upper bound on C_Γ is always available, since, if Γ lies strictly inside a circle of radius a, then $C_\Gamma < a$ (see Section 2). We suppose, therefore, that $C_\Gamma < a$, and define a transformed contour Γ' by

$$\Gamma' = \Gamma / a.$$

Then, because C_Γ scales in the same way as Γ (see Section 2), we have $C_{\Gamma'} < 1$, so that the results of the preceding sections can be applied to equations over the contour Γ'.

The next step is to transform the boundary integral equations over Γ to equations over Γ'. It proves to be convenient to begin with the equations defining the equilibrium distribution e_Γ, since in this case the result turns out to be particularly simple. On writing $\boldsymbol{t} = a\boldsymbol{t}'$ and $\boldsymbol{s} = a\boldsymbol{s}'$, (2.2) becomes

$$-a\int_{\Gamma'} (\log a + \log|\boldsymbol{t}' - \boldsymbol{s}'|)e_\Gamma(a\boldsymbol{s}')\, dl_{s'} = u_\Gamma \quad (\boldsymbol{t}' \in \Gamma'),$$

so that (2.2) and (2.3) together are equivalent to

$$-a\int_{\Gamma'} \log|\boldsymbol{t}' - \boldsymbol{s}'|\, e_\Gamma(a\boldsymbol{s}')\, \mathrm{d}l_{s'} = u_\Gamma + \log a \quad (\boldsymbol{t}' \in \Gamma'), \qquad a\int_{\Gamma'} e_\Gamma(a\boldsymbol{s}')\, \mathrm{d}l_{s'} = 1.$$

It follows immediately, from the already established existence and uniqueness property (see Section 5) for the equations for the equilibrium distribution on a contour of transfinite diameter <1, that the unique solution of (2.2) and (2.3) is $e_{\Gamma'}$, where

$$e_\Gamma(a\boldsymbol{s}') = \frac{1}{a} e_{\Gamma'}(\boldsymbol{s}') \quad (\boldsymbol{s}' \in \Gamma') \tag{8.1}$$

and

$$u_\Gamma = u_{\Gamma'} - \log a. \tag{8.2}$$

The first result confirms that the equilibrium distribution exists (uniquely), no matter what the linear scale of Γ, and shows that it transforms under a change of scale in the simplest conceivable way. The second expresses, with the aid of (2.5), the already noted scaling property $C_\Gamma = aC_{\Gamma'}$.

If $C_\Gamma \neq 1$, or equivalently $u_\Gamma \neq 0$, it follows by a similar argument that the solution of (2.1) exists and is unique. Hence $\lambda_\Gamma = u_\Gamma^{-1} e_\Gamma$ has the scaling transformation

$$\lambda_\Gamma(a\boldsymbol{s}') = \frac{1}{a}\frac{u_{\Gamma'}}{u_\Gamma} \lambda_{\Gamma'}(\boldsymbol{s}') \quad (\boldsymbol{s}' \in \Gamma'). \tag{8.3}$$

We now turn to the scaling transformation for the general equation (1.1), under the assumption that a solution of the equation exists. On writing $\boldsymbol{t} = a\boldsymbol{t}'$ and $\boldsymbol{s} = a\boldsymbol{s}'$, (1.1) becomes

$$-a\int_{\Gamma'} \log|\boldsymbol{t}' - \boldsymbol{s}'|\, y(a\boldsymbol{s}')\, \mathrm{d}l_{s'} = f(a\boldsymbol{t}') + (\log a)\int_\Gamma y \quad (\boldsymbol{t}' \in \Gamma'). \tag{8.4}$$

Since the second term on the right-hand side is constant, we may write the solution as

$$y(a\boldsymbol{s}') = a^{-1}Y(\boldsymbol{s}') + (a^{-1}\log a)\Big(\int_\Gamma y\Big)\lambda_{\Gamma'}(\boldsymbol{s}') \quad (\boldsymbol{s}' \in \Gamma'), \tag{8.5}$$

where Y is the solution (unique because $C_{\Gamma'} < 1$) of

$$-\int_{\Gamma'} \log|\boldsymbol{t}' - \boldsymbol{s}'|\, Y(\boldsymbol{s}')\, \mathrm{d}l_{s'} = f(a\boldsymbol{t}') \quad (\boldsymbol{t}' \in \Gamma'). \tag{8.6}$$

It is clear that y given by (8.5) is the unique solution of (1.1) if and only if (8.5) determines a unique consistent value for $\int_\Gamma y$. On integrating (8.5) we obtain

$$\left(1-(\log a)\int_{\Gamma'}\lambda_{\Gamma'}\right)\int_{\Gamma} y=\int_{\Gamma'} Y;$$

and since

$$1-(\log a)\int_{\Gamma'}\lambda_{\Gamma'}=1-(\log a)u_{\Gamma'}^{-1}=u_\Gamma u_{\Gamma'}^{-1},$$

it follows that (1.1) has a unique solution if and only if $u_\Gamma \neq 0$, or equivalently $C_\Gamma \neq 1$. Under that assumption, the scaling transformation for the solution is

$$y(as')=\frac{1}{a}Y(s')+\frac{\log a}{a}\frac{u_{\Gamma'}}{u_\Gamma}\left(\int_{\Gamma'}Y\right)\lambda_{\Gamma'}(s')\quad (s'\in\Gamma'), \tag{8.7}$$

where Y is the unique solution of (8.6).

We now show that analogous scaling transformations hold for the Galerkin method. Let S_Γ^h denote the finite-dimensional space of functions defined on Γ. We shall require that the functions of this space transform in the obvious way as Γ is scaled: that is, if $\Gamma=a\Gamma'$, then $\chi'\in S_{\Gamma'}^{h'}$, where $h'=h/a$, if and only if χ defined by $\chi(s)=\chi'(s/a)$ belongs to S_Γ^h.

We begin with the equation defining e_Γ^h, the Galerkin approximation to the equilibrium distribution. On writing $s=as'$ and $t=at'$, (4.3) and (4.4) together are equivalent to

$$-a\int_{\Gamma'}\int_{\Gamma'}\log|t'-s'|\,e_\Gamma^h(as')\,\mathrm{d}l_{s'}\,\chi^h(at')\,\mathrm{d}l_{t'}$$

$$=(u_\Gamma^h+\log a)\int_{\Gamma'}\chi^h(at')\,\mathrm{d}l_{t'}\quad \forall\,\chi^h\in S^h,$$

$$a\int_{\Gamma'}e_\Gamma^h(as')\,\mathrm{d}l_{s'}=1.$$

Since these are essentially the Galerkin equations for the equilibrium distribution on Γ', where $C_{\Gamma'}<1$, it follows from the existence and uniqueness result established in Section 6 that $e_\Gamma^h\in S_\Gamma^h$ exists, is unique, and satisfies

$$e_\Gamma^h(as')=a^{-1}e_{\Gamma'}^{h'}(s')\quad (s'\in\Gamma'),$$

with

$$u_\Gamma^h=u_{\Gamma'}^{h'}-\log a.$$

The first result ensures that the Galerkin approximation to the equilibrium distribution exists (uniquely), no matter what the scale of Γ, and transforms under a change of scale in the same way as the exact e_Γ. The second is equivalent, given (4.4), to

$$C_\Gamma^h=aC_{\Gamma'}^{h'}. \tag{8.8}$$

Thus the approximate transfinite diameter scales in exactly the same way as the exact C_Γ.

If $C_\Gamma^h \neq 1$, or equivalently $u_\Gamma^h \neq 0$, it follows by a similar argument that the solution λ_Γ^h of (4.5) exists and is unique. Thus $\lambda_\Gamma^h = (u_\Gamma^h)^{-1} e_\Gamma^h$ has the scaling transformation

$$\lambda_\Gamma^h(as') = \frac{1}{a} \frac{u_{\Gamma'}^{h'}}{u_\Gamma^h} \lambda_{\Gamma'}^{h'}(s') \quad (s' \in \Gamma'),$$

which is analogous to (8.3).

It now follows, by arguments parallel to those for the exact solution, that the Galerkin equation (4.1) has a unique solution y^h if and only if $C_\Gamma^h \neq 1$; and moreover that the scaling transformation for y^h is

$$y^h(as') = \frac{1}{a} Y^{h'}(s') + \frac{\log a}{a} \frac{u_{\Gamma'}^{h'}}{u_\Gamma^h} \left(\int_{\Gamma'} Y^{h'} \right) \lambda_{\Gamma'}^{h'}(s') \quad (s' \in \Gamma'), \tag{8.9}$$

where $Y^{h'} \in S^{h'}$ is the Galerkin approximation to (8.6).

Thus far, we have used scaling considerations merely to establish existence and uniqueness properties of the exact and Galerkin equations for all contours except those with $C_\Gamma = 1$ or $C_\Gamma^h = 1$, as appropriate. It is also possible, if C_Γ is greater than 1 or close to 1, to make practical use of the scaling transformation (8.7), before applying the Galerkin approximation. Thus one could find the Galerkin approximation $Y^{h'}$ and $\lambda_{\Gamma'}^{h'}$ to Y and $\lambda_{\Gamma'}$ and then use (8.9). In strictly mathematical terms, there is no point in doing this, since the final result is equivalent (as we have seen) to the direct Galerkin approximation of (1.1). However, the preliminary analytic transformation may be desirable if C_Γ is close to 1, since otherwise the Galerkin matrix may be nearly singular, and so lead to an ill conditioned set of linear equations. Also, for $C_\Gamma < 1$, the Galerkin matrix is symmetric positive-definite and the Choleskii algorithm may be used to factorize the matrix.

Finally, we consider (1.2), which, as we shall see, can be scaled to an equation over the contour Γ', no matter what the size of C_Γ. From this follows a result of Hsiao & MacCamy (1973), that (1.2) has a solution (for suitable f) for all values of C_Γ—that is, there are no anomolous contours for the z equation.

On writing $\boldsymbol{t} = a\boldsymbol{t}'$ and $\boldsymbol{s} = a\boldsymbol{s}'$, (1.2) is equivalent to

$$-a \int_{\Gamma'} \log |\boldsymbol{t}' - \boldsymbol{s}'| \, z(a\boldsymbol{s}') \, dl_{s'} + \omega - b \log a = f(a\boldsymbol{t}') \quad (\boldsymbol{t}' \in \Gamma'),$$

$$a \int_{\Gamma'} z(a\boldsymbol{s}') \, dl_{s'} = b.$$

Since $C_{\Gamma'} < 1$, it follows from the corresponding uniqueness result in Section 5 that z and ω are uniquely determined by (1.2), and transform under the change of scale according to

$$z(a\boldsymbol{s}') = -a^{-1} Z(\boldsymbol{s}') \quad (\boldsymbol{s}' \in \Gamma'), \qquad \omega = \Omega + b \log a, \tag{8.10}$$

where the pair (Z, Ω) is the unique solution of

$$-\int_{\Gamma'} \log|t'-s'|\, Z(s')\, dl_{s'} + \Omega = f(at') \quad (t' \in \Gamma'), \qquad \int_{\Gamma'} Z(s')\, dl_{s'} = b,$$

which together constitute a version of (1.2) for the contour Γ'.

The Galerkin approximation to (1.2) can be analysed in a similar manner, to show that a solution of (4.7) exists and is unique no matter what the linear scale of Γ. The scaling transformation is the obvious analogue of (8.10).

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