

1 Introduction

Let $\Gamma \subset \mathbb{R}^2$ a C^∞ open curve in \mathbb{R}^2 not intersecting itself. Denote by Ω_Γ the set $\mathbb{R}^2 \setminus \Gamma$. The aim of this work is to provide an efficient method for the numerical solution of exterior Dirichlet and Neumann problems for the Laplace operator, with the boundary condition given on Γ , namely:

- The Dirichlet problem:

$$\begin{cases} \Delta u_D = 0 & \text{in } \Omega_\Gamma, \\ u_D = u^0 & \text{on } \Gamma, \\ u_D(x) = O(1) & \text{as } |x| \rightarrow \infty \end{cases} \quad (1)$$

- The Neumann problem:

$$\begin{cases} \Delta u_N = 0 & \text{in } \Omega_\Gamma, \\ \partial_n u_N = g & \text{on } \Gamma, \\ u_N(x) = O(|x|^{-1}) & \text{as } |x| \rightarrow \infty \end{cases} \quad (2)$$

Here, $\Delta = \partial_1^2 + \partial_2^2$ is the usual Laplace operator. The asymptotic conditions for $|x| \rightarrow \infty$ in the two problems are necessary for the solutions to be unique when they exist and allow to interpret u_D and u_N as "outgoing" solutions of the equations, in the sense that they are the superposition of divergent spherical waves.

Existence and uniqueness for the Dirichlet problem:

We use the usual definition of the Sobolev spaces, and we introduce, as in [?], the space

$$H_c^1(\Omega_\Gamma) = \overline{\mathbb{R} \oplus C_0^\infty(\mathbb{R}^2)}^{||\cdot||_{1,c}}, \quad (3)$$

with the norm

$$||f||_{1,c} = \left(\int_{\Omega_\Gamma} |\nabla f|^2 + \int_\Gamma |f|^2 \right)^{1/2} \quad (4)$$

The following result is shown in [?]

Theorem 1. *For each $u_0 \in H^{1/2}(\Gamma)$, there exists a unique solution $u_D \in H_c^1(\Omega_\Gamma)$ of problem (1).*

Uniqueness for the Neumann problem

We shall prove here that the solutions of (2) are unique. We will later show existence of the solution using potential methods. Let us introduce the following spaces:

Definition 1. *Let $g \in H^{-1/2}(\Gamma)$. A function $u \in H_{loc}^1(\Delta, \Omega_\Gamma)$ is called a generalized solution of the Laplace problem with Neumann boundary condition g if for any $v \in H_{loc}^1(\Omega_\Gamma)$ compactly supported,*

$$\int_{\Omega_\Gamma} \nabla u \cdot \nabla v = \langle g, v_1 - v_2 \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)}, \quad (5)$$

and if it satisfies the same asymptotic conditions as in (2) (the limit must be taken uniformly in $|x|$). In this definition, v_2 is trace of v when the limit is taken from the side "exterior" to Γ as indicated by the normal vector, and v_1 is the trace on the opposite side.

Obviously, the first condition implies that $\Delta u = 0$ on Ω_Γ , so it must be C^∞ on Ω_Γ . This justifies that the asymptotic conditions in (2) make sense also for a generalized solution. The following lemma shows how this definition captures the boundary condition.

Lemma 1. *For each $g \in H^{-1/2}(\Gamma)$, any solution u of the generalized problem with Neumann boundary condition g , satisfies $\frac{\partial u}{\partial n}\Big|_\Gamma = g$ in $H^{-1/2}(\Gamma)$.*

Proof. Fix a smooth domain G enclosing Γ , and extend Γ into a smooth curve that splits G into two parts G_1 and G_2 . Let $\phi \in \tilde{H}^{1/2}(\Gamma)$, and $\tilde{\phi}$ denote its extension by 0 on ∂G_1 . By definition of $\tilde{H}^{1/2}(\Gamma)$, we have $\tilde{\phi} \in H^{1/2}(\partial G_1)$. Thus, there exists $v_1 \in H^1(G_1)$ such that the trace of v_1 on ∂G_1 is equal to $\tilde{\phi}$. Let v the extension of v_1 by 0 on \mathbb{R}^2 . Since v_1 vanishes on $\partial G_1 \setminus \Gamma$, standard results for traces in sobolev spaces ensure that $v \in H^1(\Omega_\Gamma)$. Applying Green's identity and using 5, we get

$$\left\langle \frac{\partial u}{\partial n}, \phi \right\rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} = \langle g, \phi \rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} \quad (6)$$

□

Adapting the proof of [?, The 4.1 p. 56], one obtains:

Lemma 2. *For any $g \in H^{-1/2}(\Gamma)$, there exists at most one generalized solution of the Laplace problem with Neumann boundary conditions g .*

Equivalent formulation as integral equations

Integral equation for the Dirichlet problem

Problems (1) and (2) can be reformulated in the form of integral equations. To this end, we introduce the Green kernel

$$G(x, x') = -\frac{1}{2\pi} \ln(|x - x'|).$$

We introduce the classical single-layer potential acting on any smooth function u defined on Γ as:

$$\text{SL } \lambda(x) = \int_\Gamma G(x, x') \lambda(x') d\Gamma(x'), \quad x \notin \Gamma. \quad (7)$$

Let $S = \gamma \mathcal{S}$ where γ is the trace operator. We have the following result:

Theorem 2. *Fix a smooth cutoff function $\chi \in C_0^\infty(\mathbb{R}^2)$. For any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, The following mappings are continuous*

$$- \chi \text{SL} : \tilde{H}^{-\frac{1}{2}+\sigma}(\Gamma) \rightarrow H^{1+\sigma}(\Omega_\Gamma),$$

$$- S : \tilde{H}^{-\frac{1}{2}+\sigma}(\Gamma) \rightarrow H^{\frac{1}{2}+\sigma}(\Gamma).$$

Furthermore, for any function $\lambda \in L^2(\Gamma)$, one has the following integral representation with a weakly singular kernel

$$S\lambda(x) = \int_{\Gamma} G(x, x') \lambda(x') d\Gamma(x'), \quad x \in \Gamma \quad (8)$$

Proof. It is well-known (see for example [?, chap. 6]) that when Γ is a closed (Lipschitz) curve, the corresponding single-layer operator and its trace are continuous from $H^{-\frac{1}{2}+\sigma}(\Gamma)$ to $H^{1+\sigma}(\Gamma)$ and $H^{-\frac{1}{2}+\sigma}(\Gamma)$ to $H^{\frac{1}{2}+\sigma}(\Gamma)$. Therefore, extend Γ to some smooth closed curve ∂G , enclosing a bounded domain G_1 and let $G_2 = \mathbb{R}^2 \setminus G_1$. Let $SL_{\partial G}$ and $S_{\partial G}$ denote the corresponding single-layer potential and its trace on ∂G . Fix any $\lambda \in \tilde{H}^{-\frac{1}{2}+\sigma}$, denote by $\tilde{\lambda}$ its zero extension on ∂G . Since $\tilde{\lambda} \in H^{1/2+\sigma}(\partial G_1)$, we get $SL_G \lambda \in H^{1+\sigma}(G_1)$ with $\|SL_G \tilde{\lambda}\|_{H^{1+\sigma}(G_1)} \leq C \|\tilde{\lambda}\|_{H^{-\frac{1}{2}+\sigma}(\Gamma)}$. Obviously, $SL \lambda = SL_{\partial G} \tilde{\lambda}$ and $\|\lambda\|_{\tilde{H}^{-\frac{1}{2}+\sigma}(\Gamma)} = \|\tilde{\lambda}\|_{H^{-\frac{1}{2}+\sigma}(\partial G)}$. This result being true for any choice of the smooth curve ∂G , we obtain $SL \lambda \in H_{\text{loc}}^1(\Omega_{\Gamma})$. Similarly, we know that $S_{\partial G} \tilde{\lambda} \in H^{\frac{1}{2}+\sigma}(\partial G)$ thus, by restriction, $S\lambda \in H^{\frac{1}{2}+\sigma}(\Gamma)$ with continuity.

The last property is easily obtained from the corresponding result for single-layer potentials on closed curves. \square

Theorem 3. For any $u_0 \in H^{\frac{1}{2}}(\Gamma)$, if the integral equation

$$S\lambda = u_0 \quad (9)$$

has a solution λ in $\tilde{H}^{-\frac{1}{2}}(\Gamma)$, then the solution u_D of (1) can be obtained from λ by

$$u_D(x) = SL \left[\lambda - \frac{1}{|\Gamma|} \langle 1, \lambda \rangle_{\Gamma} \right] (x) \quad \text{in } \Omega_{\Gamma}, \quad (10)$$

where $\langle 1, \lambda \rangle_{\Gamma} = \int_{\Gamma} \lambda$.

Proof. Assume the equation $S\lambda = u_0$ has a solution in $\tilde{H}^{-\frac{1}{2}}$ (we will show later that it is always the case). Then $v = SL \left[\lambda - \frac{1}{|\Gamma|} \langle 1, \lambda \rangle_{\Gamma} \right]$ satisfies:

1. $\Delta v = 0$ in Ω_{Γ} .
2. $\gamma v = u_0$ on Γ .

Moreover, a limited expansion of (7) shows that for any $\psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$, one has

$$SL \psi = \frac{1}{2\pi} \langle 1, \psi \rangle \log(x) + O(|x|^{-1})$$

This implies that v is indeed a solution of (1). \square

Integral equation for the Neumann problem

We introduce the classical double-layer operator defined by

$$DL \mu(x) = \int_{\Gamma} \frac{\partial G}{\partial n'_x}(x, x') \mu(x') d\Gamma(x'), \quad x \notin \Gamma \quad (11)$$

We define the classical hypersingular operator H as

$$N\mu = \frac{\partial}{\partial n} \text{DL} \mu.$$

Using the same arguments as in Theorem 2

Theorem 4. Fix a smooth cutoff function $\chi \in C_0^\infty(\mathbb{R}^2)$. For any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, The following mappings are continuous

- $\chi \text{DL} : \tilde{H}^{\frac{1}{2}+\sigma}(\Gamma) \rightarrow H^{1+\sigma}(\Omega_\Gamma)$,
- $N : \tilde{H}^{\frac{1}{2}+\sigma}(\Gamma) \rightarrow H^{-\frac{1}{2}+\sigma}(\Gamma)$.

Furthermore, for any function $\mu \in \tilde{H}^{\frac{1}{2}}(\Gamma)$, one has the following integral representation with a singular kernel

$$N\mu(x) = \text{f.p.} \int_{\Gamma \setminus B_\varepsilon(x)} \frac{\partial^2}{\partial n_x \partial n'_x} G(x, x') \mu(x') d\Gamma(x'), \quad x \in \Gamma \quad (12)$$

Theorem 5. For any $g \in H^{-\frac{1}{2}}(\Gamma)$, assume that there exists a solution $\mu \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ to the equation

$$N\mu = g \quad (13)$$

then the solution of (2) can be obtained from μ by

$$u_N(x) = \text{DL} \mu(x) \quad \text{in } \Omega_\Gamma$$

Proof. Assume the equation $N\mu = g$ has a solution in $\tilde{H}^{\frac{1}{2}}(\Gamma)$ (we will show later that it is always the case). Then $v = \text{DL} \mu$ satisfies:

1. $\Delta v = 0$ in Ω_Γ .
2. $\partial_n v = g$ on Γ .

Moreover, a limited expansion of (11) shows that for any $\psi \in \tilde{H}^{\frac{1}{2}}(\Gamma)$, one has

$$\text{DL} \psi = O(|x|^{-1}).$$

□

Numerical resolution of the integral equations

The advantage of the equivalent formulations of (1) and (2) in terms of integral equations is that the latter only involve the unknown functions on the bounded domain Γ . However, because of the singularity of the domain (the tips of the open curve Γ), the solutions u_D and u_N possess well-known singularities. More precisely,

Theorem 6. Assume the data u^0 in eq. (9) is in the space $H^s(-1, 1)$. Then the solution f is of the form

$$f = \frac{\alpha}{\omega}$$

where α belongs to H^s .

When the Galerkin method is employed, the rate of convergence of the numerical solution to the exact solution is limited by this default of regularity. Several methods have been proposed in order to recover the optimal rate of convergence. Here, we follow closely the method of [?], but using piecewise constant discretization instead of spectral discretization and prove that optimal rates of convergence occur. The analysis is based on a "cosine change of variables" as in [?, ?, ?], which recasts the single-layer integral equation into the Symm's equation. It turns out that the operator in the former equation is diagonal in the Fourier basis, which makes its study straightforward.

1.1 Eigenvectors of S and N when $\Gamma = (-1, 1)$

We first consider S_0 and N_0 the single-layer potential on the segment $(-1, 1)$, as the operators in the general case are obtained as compact perturbations of S_0 and N_0 . We denote by $T_n(x)$ and $U_n(x)$ the n -th Chebyshev's polynomial of first and second kind respectively. Let $\omega(x) = \sqrt{1 - x^2}$, and recall the following orthogonality properties:

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\omega(x)} dx = t_m \delta_{m,n} \quad (14)$$

where $\delta_{n,m}$ stands for the Kroenecker symbol and $t_0 = \pi$ and $t_m = \frac{\pi}{2}, m \geq 1$.

$$\int_{-1}^1 U_m(x)U_n(x)\omega(x)dx = u_m \delta_{m,n} \quad (15)$$

where $u_m = \frac{\pi}{2}$ for all $m \geq 0$.

Let us also define some spaces of functions on the interval $(-1, 1)$ that will help describe the operators S and N . For any real-valued and locally integrable function α on the interval $(-1, 1)$, and for any $s \geq 0$, we define the following norm :

$$\|\alpha\|_s := |\alpha_0|^2 + \sum_{n=1}^{+\infty} |\alpha_n|^2 n^{2s},$$

where for all $n \geq 0$,

$$\alpha_n := \frac{1}{\sqrt{t_n}} \int_{-1}^1 \frac{\alpha(x)T_n(x)}{\omega(x)} dx.$$

Using the change of variables $x = \cos \theta$ and the properties of Chebyshev's polynomials, it can be easily seen that the coefficients α_n are closely related to the Fourier coefficients of the even function $\tilde{\alpha}(\theta) := \alpha(\cos \theta)$

$$\alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \tilde{\alpha}(\theta) e^{-in\theta} d\theta, \quad n \geq 1,$$

$$\alpha_0 = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{\pi} \tilde{\alpha}(\theta) d\theta$$

This encourages us to define a scale of weighted Sobolev spaces X^s based on the norm $\|\cdot\|_s$:

$$X_s := \{\alpha \mid \|\alpha\|_s < +\infty\}$$

For any $s \geq 0$, X^s is a Hilbert space for the scalar product

$$(\alpha|\beta)_{X^s} = \alpha_0\beta_0 + \sum_{n=1}^{+\infty} \alpha_n\beta_n n^{2s}$$

Moreover, the spaces X^{-s} can be defined as the dual of X^s , using the bilinear form

$$b(\alpha, \beta) = \int_{-1}^1 \frac{\alpha(x)\beta(x)}{\omega(x)} dx = \alpha_0\beta_0 + \sum_{n=1}^{+\infty} \alpha_n\beta_n.$$

Standard Fourier arguments show that $\cap_{s \in \mathbb{R}} X^s = C^\infty(-1, 1)$. We will later investigate further the link between X^s and more usual spaces.

The following theorem is the basis of the subsequent analysis.

Theorem 7. *For all $n \geq 0$,*

$$S_0 \frac{T_n}{\omega} = \lambda_n T_n \tag{16}$$

$$N_0 \omega U_n = \mu_n U_n \tag{17}$$

with $\lambda_0 = \frac{\ln(2)}{2}$, $\lambda_n = \frac{1}{2n}$ for $m \geq 1$, and $\mu_n = \frac{n+1}{2}$ for all n .

Remark 1. *It was proved in [?] that $S \frac{1}{\omega}$ and N_ω are Fredholm operators and that their product has zero index, meaning they can be efficiently used as mutual preconditioners.*

The following formula was obtained in [?]:

Corollary 1. *For almost all $(x, y) \in (-1, 1)^2$, one has:*

$$\ln(|x - y|) = -\ln(2) - \sum_{n=1}^{+\infty} \frac{2}{n} T_n(x) T_n(y). \tag{18}$$

Corollary 2. *S_0 and N_0 are symmetric operators, invertible and bounded below.*

In particular, this implies that problems (1) and (2) are always solvable.

Let Y the operator defined by

$$Y\varphi(x) = \text{f.p.} \int_{\varepsilon \rightarrow 0} \int_{(-1,1) \setminus (-\varepsilon, \varepsilon)} \frac{1 - xy}{\omega(x)\omega(y)|x - y|^2} \varphi(y) dy$$

By applying the operator $(\omega(x)D_x)^2$ on both sides of (18), we obtain

Corollary 3. *Y is well defined for $\varphi \in H^{\frac{1}{2}}(\Gamma)$ and is the inverse of S .*

Corollary 4. *Assume that $u^0 \in X^s$ for some $s \in \mathbb{R}$. Then the solution f to equation (9) is of the form*

$$f = \frac{\alpha}{\omega},$$

with $\alpha \in X^{s+1}$.

In other words, S^{-1} maps X^s into $\frac{1}{\omega} X^{s+1}$. In particular, it maps C^∞ into $\frac{1}{\omega} C^\infty$.

Theorem 8. *We have the following characterization of some Sobolev spaces on $\tilde{H}^s(-1, 1)$ in terms of X^s :*

$$f \in \tilde{H}^{-1/2}(-1, 1) \iff \alpha := \omega f \in X^{-1/2} \quad (19)$$

Moreover,

$$\|f\|_{\tilde{H}^{-1/2}(-1, 1)} \sim \|\alpha\|_{-1/2}$$

2 Cosine change of variable and numerical resolution of the Dirichlet problem

In eq. (9), we use the change of variable $x = \cos \theta$ to transform the equation into the well-known Symm's integral equation:

$$-\frac{1}{2\pi} \int_0^\pi \ln |\cos \theta - \cos \theta'| \tilde{\alpha}(\theta') d\theta' = \tilde{u}^0(\theta) \quad (20)$$

Once $\tilde{\alpha}$ is obtained, the original solution f is given by $u_D(x) = \frac{\tilde{\alpha}(\arccos(x))}{\omega(x)}$. In order to solve the Dirichlet problem, we use a Galerkin approximation of the unique solution to equation (9). Using Galerkin approximation on Symm's equation (20) instead of eq. (9) is more advantageous because this time the unknown α belongs to $H^s(-\pi, \pi)$ whenever the data u^0 is in X^s . We will show that this allows to regain optimal rates of convergence.

For this, we introduce a regular mesh of the interval x_0, \dots, x_N of $(0, \pi)$ and define the finite element space as the spaces of piecewise constant functions on each (x_i, x_{i+1}) . We introduce $\tilde{\alpha}_h(\theta) = \alpha_h(\cos \theta)$ the solution of the variational formulation

$$\forall \tilde{\beta}_h \in V_h, \quad \frac{-1}{2\pi} \int_0^\pi \int_0^\pi \ln |\cos \theta - \cos \theta'| \tilde{\alpha}_h(\theta) \tilde{\beta}_h(\theta') d\theta d\theta' = \int_0^\pi \tilde{u}^0(\theta) \tilde{\beta}_h(\theta) d\theta \quad (21)$$

Finally, we set $u_h(x) = \frac{\alpha_h(x)}{\omega(x)}$. Let us investigate the rate of convergence of this approximation with respect to h :

Proposition 1. *There exists a constant C such that for all h small enough,*

$$\|u_D - u_h\|_{\tilde{H}^{-1/2}(-1, 1)} \leq C \|\tilde{\alpha}\|_1 h^{3/2}$$

Proof. We first prove a Céa type lemma: by definition of α_h , we have, for all $\tilde{\beta}_h \in V_h$, letting $\beta_h(x) = \tilde{\beta}_h(\arccos(x))$,

$$\left\langle S \frac{\alpha_h}{\omega}, \frac{\beta_h}{\omega} \right\rangle = \left\langle u^0, \frac{\beta_h}{\omega} \right\rangle$$

We now repeat the classical argument leading to Céa's lemma. Since the former inequality is also true when we replace α_h by α , we obtain, for any $\tilde{\beta}_h \in V_h$,

$$\left\langle S \frac{\alpha - \alpha_h}{\omega}, \frac{\beta_h}{\omega} \right\rangle = 0$$

Using the fact that S is positive and bounded below on $\tilde{H}^{-1/2}(-1, 1)$, this leads to

$$\|u_D - u_h\|_{\tilde{H}^{-1/2}} \leq C \inf_{\tilde{\beta}_h \in V_h} \left\| \frac{\alpha - \beta_h}{\omega} \right\|_{\tilde{H}^{-1/2}}$$

According to theorem 8, we conclude that for some constant C ,

$$\|u_D - u_h\|_{\tilde{H}^{-1/2}} \leq C \inf_{\tilde{\beta}_h \in V_h} \|\tilde{\alpha} - \tilde{\beta}_h\|_{-1/2}$$

the result announced follows from classical interpolation results on the uniform mesh x_1, x_2, \dots, x_N for $\tilde{\alpha}$. \square

Remark 2. *From the proof of the above theorem, it can be observed that for the result to hold, it is sufficient to have a quasi-uniform mesh of the interval $(0, \pi)$. After change of variable, we can see that this amounts to a local mesh grading near the edges of the segment $(-1, 1)$ with a local mesh width $h(x) \sim d(x, \partial\Omega)^2$. If a uniform mesh of the segment $(-1, 1)$ is chosen instead, the theoretical rate of convergence is divided by 2.*

References