

# Regularized Combined Field Integral Equations

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**Summary.** Many boundary integral equations for exterior boundary value problems for the Helmholtz equation suffer from a notorious instability for wave numbers related to interior resonances. The so-called combined field integral equations are not affected. However, if the boundary  $\Gamma$  is not smooth, the traditional combined field integral equations for the exterior Dirichlet problem do not give rise to an  $L^2(\Gamma)$ -coercive variational formulation. This foils attempts to establish asymptotic quasi-optimality of discrete solutions obtained through conforming Galerkin boundary element schemes.

This article presents new combined field integral equations on two-dimensional closed surfaces that possess coercivity in canonical trace spaces. The main idea is to use suitable regularizing operators in the framework of both direct and indirect methods. This permits us to apply the classical convergence theory of conforming Galerkin methods.

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## 1 Introduction

Given a bounded domain  $\Omega^- \subset \mathbb{R}^3$  we consider the exterior Dirichlet problem for the Helmholtz equation, cf. [30, Ch. 9]

$$(1.1) \quad \Delta U + \kappa^2 U = 0 \text{ on } \Omega^+ := \mathbb{R} \setminus \bar{\Omega}^-,$$

$$(1.2) \quad U = g \text{ on } \partial\Omega \text{ for some } g \in H^{\frac{1}{2}}(\Gamma),$$

$$(1.3) \quad \frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) = o(r^{-1}) \text{ uniformly as } r := |\mathbf{x}| \rightarrow \infty.$$

Here  $\kappa > 0$  stands for a fixed wave number. The above equations are a model for acoustic scattering at a sound-hard obstacle [14, Sect. 2.1].

We take for granted that the boundary  $\Gamma := \partial\Omega$  is Lipschitz continuous. Thus, it will possess an exterior unit normal vectorfield  $\mathbf{n} \in L^\infty(\Gamma)$  pointing from  $\Omega^-$  into  $\Omega^+$ . Numerical approximation in mind, we will even assume that  $\Gamma$  is a curvilinear Lipschitz polyhedron in the parlance of [16]. This will cover most geometric arrangements that occur in practical simulations. We emphasize that non-smooth geometries are the main focus of this paper.

It is well known that solutions of the above exterior boundary value are unique, see [30, Thm. 9.10]:

**Theorem 1.1** *The exterior Dirichlet problem (1.1), (1.2), and (1.3) for the Helmholtz equation has at most one weak solution  $U \in H_{\text{loc}}^1(\Omega^+)$ .*

Many different strategies are available to tackle (1.1)–(1.3) numerically: one could truncate  $\Omega^+$  and use standard finite elements in conjunction with absorbing boundary conditions [24]. An alternative is provided by infinite finite elements in  $\Omega^+$  [5, 4] or the method of fundamental solutions [20].

However, in this article we focus on integral equations methods, which reduce the problem (1.1)–(1.3) to equations on the bounded surface  $\Gamma$ . This class of methods comprises a variety of approaches, among them direct and indirect methods. Unfortunately, the formulations that can be derived from an integral representation formula for Helmholtz solutions in a straightforward fashion display a worrisome instability: if  $\kappa^2$  agrees with a Dirichlet or Neumann eigenvalue (resonant frequency) of the Laplacian in  $\Omega^-$ , then the integral equations may fail to possess a unique solution. In light of Thm. 1.1 this has been dubbed a spurious resonance phenomenon.

Spurious resonances are particularly distressing for numerical procedures based on the integral equations, because whenever  $\kappa^2$  is close to an interior resonant frequency the resulting linear systems of equations may not be useless, but will be extremely ill-conditioned: see the profound analysis of the impact of spurious resonances in the case of electromagnetic scattering given in [13].

One way to deal with spurious resonances is the use of integral operators with modified kernels [35, 26]. Here we will restrict our attention to another remedy, namely the widely used combined field integral equations (CFIE). They owe their name to the typical complex linear combination of different boundary integral operators on the left hand side of the final boundary integral equation. In the case of indirect schemes this trick has independently been discovered by Brakhage and Werner [6], Leis [29], and Panich [31] in 1965. In 1971 Burton and Miller used the same idea to obtain direct boundary integral equations without spurious resonances

[11]. Meanwhile, CFIEs have become the foundation for numerous numerical methods in direct and inverse acoustic and electromagnetic scattering [14, Ch. 3 & 6].

In terms of mathematical analysis many combined field integral equations are challenging. This is particularly true for non-smooth surfaces, for which the double layer integral operator is no longer a compact perturbation of the identity in  $L^2(\Gamma)$ . Thus, in the case of the exterior Dirichlet problem for non-smooth scatterers, Fredholm theory in  $L^2(\Gamma)$  can no longer be used to settle the issue of existence and uniqueness of weak solutions of the traditional CFIE. Of course, this also thwarts any attempt to establish convergence for Galerkin boundary element discretizations.

Hence, modified CFIE involving a regularizing operator have been suggested, mainly for theoretical purposes [14, 31], though. An exception known to us is [3] where regularization in the spirit of this paper was used to get rid of a hypersingular operator in the boundary integral equation for the Neumann problem. Yet, little analysis was provided and the authors did not aim at a mixed Galerkin formulation. A profound investigation of regularization strategies for indirect CFIEs with numerical applications in mind is given in the recent article [10]. There, in contrast to the current paper, regularization is based on smoothing integral operators, and no Galerkin error analysis is provided.

In this paper we take the cue from the idea to introduce regularizing operators. We derive new variational formulations that are coercive in natural trace spaces and cast them in mixed form, in order to avoid products of boundary integral operators. Galerkin boundary element discretizations are shown to yield asymptotically quasi-optimal approximate solutions. A similar analysis has successfully been applied to electromagnetic boundary integral equations [9].

The focus of the present article is the rigorous theoretical examination of the new combined field integral equations and of their Galerkin discretization. The optimal choices of parameters and actual numerical performance are not addressed yet and will be treated in future work. The same is true for the key issue of how the estimates for the Galerkin discretization error depend on the wave number  $\kappa$ . For certain (regularized) combined field integral equations on a sphere, this dependence is investigated in [10, 21].

## 2 Boundary integral operators

For the sake of completeness, in this section we review important properties of boundary integral operators related to Helmholtz' equation. The main

reference is the pioneering work by M. Costabel [15] and the textbooks [30, 33]. Throughout, for known results we will mainly refer to [30]. References to original work can be found in the bibliography of [30].

Without further explanation we will use Sobolev spaces  $H^s$ ,  $s \in \mathbb{R}$ , on domains and boundaries, in particular  $H^1(\Omega)$ ,  $H^{\frac{1}{2}}(\Gamma)$ , and  $H^{-\frac{1}{2}}(\Gamma)$ , cf. [1], [30, Ch. 2]. The corresponding Frechet spaces on unbounded domains will be tagged by a subscript loc, e.g.

$$H_{\text{loc}}^1(\Omega) := \{u : \Omega \mapsto \mathbb{C} : u|_{\Omega \cap K} \in H^1(\Omega \cap K) \text{ for any compact } K \subset \mathbb{R}^3\}.$$

Further, we adopt the notation  $H_{\text{comp}}^s(\Omega)$  for compactly supported distributions in  $H^s(\Omega)$ . For  $s < 0$  these space are to be read as dual spaces of  $H_{\text{loc}}^s(\Omega)$ .

Writing

$$H_{\text{loc}}(\Delta, \Omega) := \{U \in H_{\text{loc}}^1(\Omega), \Delta U \in L_{\text{loc}}^2(\Omega)\}.$$

for the domain of the Laplacian, we have continuous and surjective trace operators, cf. [15, Lemma 3.2],

$$\begin{aligned} \text{Dirichlet trace} \quad \gamma_D : H_{\text{loc}}^1(\Omega) &\mapsto H^{\frac{1}{2}}(\Gamma), \\ \text{Neumann trace} \quad \gamma_N : H_{\text{loc}}(\Delta, \Omega) &\mapsto H^{-\frac{1}{2}}(\Gamma) \end{aligned}$$

that generalize the following pointwise traces of smooth  $U \in C^\infty(\bar{\Omega})$ ,

$$(\gamma_D U)(\mathbf{x}) := U(\mathbf{x}) \quad \text{and} \quad (\gamma_N U)(\mathbf{x}) := \mathbf{grad} U(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}), \quad \mathbf{x} \in \Gamma,$$

respectively.

So far  $\Omega \subset \mathbb{R}^3$  has been a generic Lipschitz domain. Returning to our particular setting, superscripts  $+$  and  $-$  will tag traces from  $\Omega^+$  and  $\Omega^-$ , respectively. Jumps are defined as

$$[\gamma_D U]_\Gamma = \gamma_D^+ U - \gamma_D^- U, \quad [\gamma_N U]_\Gamma = \gamma_N^+ U - \gamma_N^- U.$$

Averages are denoted by

$$\{\gamma_D U\}_\Gamma = \frac{1}{2}(\gamma_D^+ U + \gamma_D^- U), \quad \{\gamma_N U\}_\Gamma = \frac{1}{2}(\gamma_N^+ U + \gamma_N^- U).$$

We recall that the bi-linear symmetric pairing

$$\langle \varphi, v \rangle_\Gamma := \int_\Gamma \varphi v \, dS, \quad \varphi, v \in L^2(\Gamma),$$

can be extended to a duality pairing on  $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ . We also recall the integration by parts formulas

$$(2.1) \quad \int_{\Omega^-} \mathbf{grad} U \cdot \mathbf{grad} V + \Delta U V \, d\mathbf{x} = \langle \gamma_N^- U, \gamma_D^- V \rangle_{\Gamma} ,$$

$$(2.2) \quad - \int_{\Omega^+} \mathbf{grad} U \cdot \mathbf{grad} V + \Delta U V \, d\mathbf{x} = \langle \gamma_N^+ U, \gamma_D^+ V \rangle_{\Gamma} ,$$

for  $U \in H_{\text{loc}}(\Delta, \Omega^\pm)$ ,  $V \in H_{\text{comp}}^1(\Omega^\pm)$ .

For fixed wavenumber  $\kappa > 0$  a distribution  $U$  in  $\mathbb{R}^3$  is called a *radiating Helmholtz solution*, if

$$(2.3) \quad \begin{aligned} \Delta U + \kappa^2 U &= 0 \quad \text{in } \Omega^- \cup \Omega^+ , \\ \frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) &= o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \rightarrow \infty . \end{aligned}$$

Based on the Helmholtz kernel

$$\Phi_\kappa(\mathbf{x}, \mathbf{y}) := \frac{\exp(i\kappa|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} .$$

we can state the transmission formula for radiating Helmholtz solution  $U$  [30, Thm. 6.10]

$$(2.4) \quad U = -\Psi_{\text{SL}}^\kappa([\gamma_N U]_{\Gamma}) + \Psi_{\text{DL}}^\kappa([\gamma_D U]_{\Gamma})$$

with potentials

$$\text{single layer potential:} \quad \Psi_{\text{SL}}^\kappa(\lambda)(\mathbf{x}) = \int_{\Gamma} \Phi_\kappa(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) \, dS(\mathbf{y}) ,$$

$$\text{double layer potential:} \quad \Psi_{\text{DL}}^\kappa(u)(\mathbf{x}) = \int_{\Gamma} \frac{\partial \Phi_\kappa(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u(\mathbf{y}) \, dS(\mathbf{y}) .$$

The potentials themselves provide radiating Helmholtz solutions, that is

$$(2.5) \quad (\Delta + \kappa^2) \Psi_{\text{SL}}^\kappa = 0 \quad , \quad (\Delta + \kappa^2) \Psi_{\text{DL}}^\kappa = 0 \quad \text{in } \Omega^- \cup \Omega^+ .$$

Moreover, they give rise to continuous mappings, see [30, Thm. 6.12],

$$(2.6) \quad \Psi_{\text{SL}}^\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H_{\text{loc}}^1(\mathbb{R}^3) \cap H_{\text{loc}}(\Delta, \Omega^- \cup \Omega^+) ,$$

$$(2.7) \quad \Psi_{\text{DL}}^\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H_{\text{loc}}(\Delta, \Omega^- \cup \Omega^+) .$$

This means that we can apply the trace operators to the potentials. This will yield the following four continuous boundary integral operators, cf. [30,

Thm. 7.1], but also [15] and [27].

$$\begin{aligned} \mathbf{V}_\kappa : H^s(\Gamma) &\mapsto H^{s+1}(\Gamma), \quad -1 \leq s \leq 0 \quad , \quad \mathbf{V}_\kappa := \{\gamma_D \Psi_{\text{SL}}^\kappa\}_\Gamma \quad , \\ \mathbf{K}_\kappa : H^s(\Gamma) &\mapsto H^s(\Gamma), \quad 0 \leq s \leq 1 \quad , \quad \mathbf{K}_\kappa := \{\gamma_D \Psi_{\text{DL}}^\kappa\}_\Gamma \quad , \\ \mathbf{K}_\kappa^* : H^s(\Gamma) &\mapsto H^s(\Gamma), \quad -1 \leq s \leq 0 \quad , \quad \mathbf{K}_\kappa^* := \{\gamma_N \Psi_{\text{SL}}^\kappa\}_\Gamma \quad , \\ \mathbf{D}_\kappa : H^s(\Gamma) &\mapsto H^{s-1}(\Gamma), \quad 0 \leq s \leq 1 \quad , \quad \mathbf{D}_\kappa := -\{\gamma_N \Psi_{\text{DL}}^\kappa\}_\Gamma \quad . \end{aligned}$$

By the *jump relations* [30, Thm. 6.11]

$$\begin{aligned} [\gamma_D \Psi_{\text{SL}}^\kappa(\lambda)]_\Gamma &= 0 \quad , \quad [\gamma_N \Psi_{\text{SL}}^\kappa(\lambda)]_\Gamma = -\lambda \quad , \quad \forall \lambda \in H^{-\frac{1}{2}}(\Gamma) \quad , \\ [\gamma_D \Psi_{\text{DL}}^\kappa(u)]_\Gamma &= u \quad , \quad [\gamma_N \Psi_{\text{DL}}^\kappa(u)]_\Gamma = 0 \quad , \quad \forall u \in H^{\frac{1}{2}}(\Gamma) \quad . \end{aligned}$$

we find

$$(2.8) \quad \begin{aligned} \gamma_D^- \Psi_{\text{DL}}^\kappa &= \mathbf{K}_\kappa - \frac{1}{2} Id \quad , \quad \gamma_D^+ \Psi_{\text{DL}}^\kappa = \mathbf{K}_\kappa + \frac{1}{2} Id \quad , \\ \gamma_N^- \Psi_{\text{SL}}^\kappa &= \mathbf{K}_\kappa^* + \frac{1}{2} Id \quad , \quad \gamma_N^+ \Psi_{\text{SL}}^\kappa = \mathbf{K}_\kappa^* - \frac{1}{2} Id \quad . \end{aligned}$$

Crucial will be the ellipticity of the single layer boundary integral operator in the natural trace norms [30, Cor. 8.13]

$$(2.9) \quad \exists c_V > 0 : \quad \langle \bar{\varphi}, \mathbf{V}_0 \varphi \rangle_\Gamma \geq c_V \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)}^2 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \quad .$$

**Lemma 2.1** *The operator  $\mathbf{V}_\kappa - \mathbf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ , is compact.*

*Proof* Note that

$$\frac{\exp(i\kappa|\mathbf{z}|) - 1}{4\pi|\mathbf{z}|} = |\mathbf{z}| G_1(|\mathbf{z}|^2) + i G_2(|\mathbf{z}|^2) \quad , \quad \mathbf{z} \in \mathbb{R}^3 \quad ,$$

with bounded analytic functions  $G_1, G_2 : \mathbb{R} \mapsto \mathbb{R}$ , This shows that all second derivatives

$$(2.10) \quad \frac{\partial^2}{\partial z_i \partial z_j} \frac{\exp(i\kappa|\mathbf{z}|) - 1}{4\pi|\mathbf{z}|} \quad , \quad i, j = 1, 2, 3 \quad ,$$

are weakly singular, that is, grow like  $O(|\mathbf{z}|^{-1})$  as  $|\mathbf{z}| \rightarrow 0$ . As a consequence, the kernels (2.10) will each give rise to an integral operator of order  $-2$ . Therefore, the Newton potential

$$(2.11) \quad (\tilde{\mathbf{N}}_\kappa f)(\mathbf{x}) := \int_{\mathbb{R}^3} \frac{\exp(i\kappa|\mathbf{x} - \mathbf{y}|) - 1}{4\pi|\mathbf{x} - \mathbf{y}|} f(\mathbf{y}) \, d\mathbf{y}$$

will provide a pseudo-differential operator of order  $-4$ , since, for any  $i, j = 1, 2, 3$ ,

$$\frac{\partial^2 \tilde{\mathbf{N}}_\kappa f}{\partial x_i \partial x_j}(\mathbf{x}) = \int_{\mathbb{R}^3} \left\{ \frac{\partial^2}{\partial z_i \partial z_j} \frac{\exp(i\kappa|\mathbf{z}|) - 1}{4\pi|\mathbf{z}|} \right\}_{\mathbf{z}=\mathbf{x}-\mathbf{y}} f(\mathbf{y}) d\mathbf{y}$$

belongs to  $H_{\text{loc}}^1(\mathbb{R}^3)$ , if  $f \in H_{\text{comp}}^{-1}(\mathbb{R}^3)$ . In other words, the Newton potential (2.11) provides a continuous mapping from  $H_{\text{comp}}^{-1}(\mathbb{R}^3)$  to  $H_{\text{loc}}^3(\mathbb{R}^3)$ .

From this we conclude the continuity of

$$\mathbf{V}_\kappa - \mathbf{V}_0 = \gamma_D \circ \tilde{\mathbf{N}}_\kappa \circ \gamma_D^* : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1(\Gamma) ,$$

where we have used a representation of the single layer boundary integral operators given, for instance, in [33, Sect. 3.1]. The compact embedding  $H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$  finishes the proof.  $\square$

### 3 Indirect boundary integral equations

We recall that indirect methods are based on potential representations for (exterior) radiating Helmholtz solutions in  $\Omega^+$ . By virtue of (2.5) we may set

$$(3.1) \quad U = \Psi_{\text{SL}}^\kappa(\phi), \quad \phi \in H^{-\frac{1}{2}}(\Gamma) \quad \text{or} \quad U = \Psi_{\text{DL}}^\kappa(u), \quad u \in H^{\frac{1}{2}}(\Gamma) .$$

Applying  $\gamma_D^+$  to (3.1) and using (2.8) we obtain the following integral equations for the exterior Dirichlet problem:

$$(3.2) \quad \mathbf{V}_\kappa(\phi) = g \quad \text{or} \quad (\mathbf{K}_\kappa + \tfrac{1}{2}Id)u = g .$$

However, these boundary integral equations are haunted by the problem of “resonant frequencies” [12, Sect. 7.7]: if  $\kappa^2$  is a Dirichlet eigenvalue of  $-\Delta$  in  $\Omega^-$ , then the Neumann traces of the corresponding eigenfunctions will belong to the kernel of  $\mathbf{V}_\kappa$ . Conversely, the kernel of  $\mathbf{K}_\kappa + \frac{1}{2}Id$  consists of the Dirichlet traces of interior Neumann eigenfunctions of  $-\Delta$ . This destroys injectivity of the operators in the boundary integral equations and bars us from applying the powerful Fredholm theory.

#### 3.1 Classical CFIE

As pointed out in the introduction, the awkward lack of uniqueness of solutions of (3.2) at resonant frequencies led to the development of the classical combined field integral equation [14, Sect. 3.2]. It can be obtained by an indirect approach starting from the trial expression

$$(3.3) \quad U = \Psi_{\text{DL}}^\kappa(u) + i\eta\Psi_{\text{SL}}^\kappa(u) ,$$

with real  $\eta \neq 0$ . Applying the exterior Dirichlet trace results in the boundary integral equation

$$(3.4) \quad g = (\tfrac{1}{2}Id + \mathbf{K}_\kappa)u + i\eta\mathbf{V}_\kappa u .$$

Actually, this equation is set in  $H^{\frac{1}{2}}(\Gamma)$  and the density  $u$  should be sought in  $H^{-\frac{1}{2}}(\Gamma)$ . Since we plug it into the double layer potential, this is not possible, unless we use a pairing in  $H^{-\frac{1}{2}}(\Gamma)$  to convert the equation into weak form. Yet, this will introduce products of non-local operators, which have to be discretized with great care. The fundamental difficulty is that we cannot use matching trial and test spaces, because in light of (2.6) and (2.7) the potentials involved in (3.1) should be applied to functions with different regularity. Hence, we have to abandon the framework of natural trace spaces, shift the equation (3.4) into  $L^2(\Gamma)$  and seek the unknown density  $u$  in  $L^2(\Gamma)$ , too.

A key argument in the theoretical treatment of (3.4) in  $L^2(\Gamma)$  is the compactness of the double layer potential operator  $\mathbf{K}_\kappa : L^2(\Gamma) \mapsto L^2(\Gamma)$  on smooth surfaces, which relegates the boundary integral operator associated with (3.4) to a compact perturbation of the identity. On non-smooth surfaces this argument is not available.

### 3.2 Regularized formulation

In order to remedy the mismatch of the regularity of the arguments in (3.3), regularization is a natural idea. It amounts to introducing another operator that lifts either the argument of the double layer potential or the argument of the single layer potential into the “right” trace space. Here, following [31], regularization will target the double layer potential. The objective is to force the boundary integral operator arising from the regularized double layer potential to become a compact perturbation of the single layer boundary integral operator.

First we adopt an abstract perspective and call an operator  $\mathbf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  a *regularizing operator*, if it satisfies

$$(3.5) \quad (i) \quad \mathbf{M} \text{ is compact and}$$

$$(3.6) \quad (ii) \quad \operatorname{Re}\{\langle \varphi, \mathbf{M}\bar{\varphi} \rangle_\Gamma\} > 0 \text{ for all } \varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\} .$$

The new regularized indirect method is based on the trial expression

$$(3.7) \quad U = \Psi_{\text{DL}}^\kappa(\mathbf{M}\varphi) + i\eta\Psi_{\text{SL}}^\kappa(\varphi) , \quad \varphi \in H^{-\frac{1}{2}}(\Gamma) ,$$

for a fixed  $\eta \in \mathbb{R} \setminus \{0\}$ . As above,  $U$  is a radiating Helmholtz solution in  $\Omega^- \cup \Omega^+$ . If we apply the Dirichlet trace, we arrive at the boundary integral equation

$$(3.8) \quad g = ((\tfrac{1}{2}Id + \mathbf{K}_\kappa) \circ \mathbf{M})(\varphi) + i\eta\mathbf{V}_\kappa\varphi \quad \text{in } H^{\frac{1}{2}}(\Gamma) .$$



This prompts us to introduce the boundary integral operator

$$\mathbf{S}_\kappa := (\tfrac{1}{2}Id + \mathbf{K}_\kappa) \circ \mathbf{M} + i\eta \mathbf{V}_\kappa .$$

Recalling (2.9) and the compactness of  $\mathbf{M}$ , it is clear that  $\mathbf{S}_\kappa$  will spawn a  $H^{-\frac{1}{2}}(\Gamma)$ -coercive bi-linear form in the sense that there is a compact bi-linear form  $k : H^{-\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \mapsto \mathbb{C}$  such that

$$\exists C > 0 : \quad |\langle \bar{\varphi}, \mathbf{S}_\kappa \varphi \rangle_\Gamma + k(\bar{\varphi}, \varphi)| \geq C \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)}^2 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) .$$

**Lemma 3.1** *The boundary integral operator  $\mathbf{S}_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  is injective.*

*Proof* The same idea as in the proof of injectivity of the boundary integral operators of the classical CFIE (3.4) can be applied: we assume that  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$  solves

$$\mathbf{S}_\kappa \varphi = ((\tfrac{1}{2}Id + \mathbf{K}_\kappa) \circ \mathbf{M})(\varphi) + i\eta \mathbf{V}_\kappa \varphi = 0 .$$

It is immediate from the jump relations that  $U$  given by (3.7) is a Helmholtz solution with  $\gamma_D^+ U = 0$ , which, by Thm. 1.1, implies  $U = 0$  in  $\Omega^+$ . Hence, the jump relations confirm that

$$\gamma_D^- U = -\mathbf{M}\varphi \quad , \quad \gamma_N^- U = i\eta \varphi .$$

Next, we use the integration by parts formula (2.1) and get

$$-i\eta \langle \varphi, \mathbf{M}\bar{\varphi} \rangle_\Gamma = \langle \gamma_N^- U, \gamma_D^- \bar{U} \rangle_\Gamma = \int_{\Omega^-} |\mathbf{grad} U|^2 - \kappa^2 |U|^2 d\mathbf{x} \in \mathbb{R}$$

Necessarily,  $\text{Re}\{\langle \mathbf{M}\varphi, \bar{\varphi} \rangle_\Gamma\} = 0$ , which, by requirement (3.6), can only be satisfied, if  $\varphi = 0$ .  $\square$

Eventually, Lemma 3.1 allows to deduce existence of solutions of (3.8) by means of a Fredholm argument, cf. [30, Thm. 2.34].

In the construction of our first concrete regularizing operator, we make explicit use of  $\Omega^-$  being a (curvilinear) Lipschitz polyhedron: denote by  $\Gamma_1, \dots, \Gamma_p$ ,  $p \in \mathbb{N}$ , its smooth (curved) polygonal faces and introduce the space

$$(3.9) \quad H_{\text{pw},0}^1(\Gamma) := H_0^1(\Gamma_1) \times \dots \times H_0^1(\Gamma_p) \subset H^1(\Gamma) .$$

Then define the regularizing operator  $\mathbf{M} : H^{-1}(\Gamma) \mapsto H_{\text{pw},0}^1(\Gamma)$  by

$$(3.10) \quad \langle \mathbf{grad}_\Gamma \mathbf{M}\varphi, \mathbf{grad}_\Gamma v \rangle_\Gamma = \langle \varphi, v \rangle_\Gamma \quad \forall v \in H_{\text{pw},0}^1(\Gamma), \varphi \in H^{-1}(\Gamma) .$$

In words,  $\mathbf{M}$  is a combination of inverse Laplace–Beltrami operators on the individual faces  $\Gamma_i$ ,  $i = 1, \dots, p$ . Continuity of  $\mathbf{M}$  is straightforward. The next lemma shows that  $\mathbf{M}$  is injective, when restricted to  $H^{-\frac{1}{2}}(\Gamma)$ .

**Lemma 3.2** *The space  $H_{\text{pw},0}^1(\Gamma)$  is dense in  $H^{\frac{1}{2}}(\Gamma)$ .*

*Proof* Denote by  $\Sigma$  the union of closed edges of  $\Omega^-$ . We can rely on Lemma 2.6 in [17] that claims that the embedding

$$C_\Sigma^\infty := \{u \in C^\infty(\bar{\Omega}^-), \supp u \cap \Sigma = \emptyset\} \subset H^1(\Omega^-)$$

is dense. Obviously,  $\gamma_D^-(C_\Sigma^\infty) \subset H_{\text{pw},0}^1(\Gamma)$  and the continuity of  $\gamma_D^- : H^1(\Omega^-) \mapsto H^{\frac{1}{2}}(\Gamma)$  finishes the proof.  $\square$

We conclude that for  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{aligned} \mathbf{M}\varphi = 0 &\Rightarrow \langle \varphi, v \rangle_\Gamma = 0 \quad \forall v \in H_{\text{pw},0}^1(\Gamma) \text{ by (3.10)} \\ &\Rightarrow \langle \varphi, v \rangle_\Gamma = 0 \quad \forall v \in H^{\frac{1}{2}}(\Gamma) \quad \text{by Lemma 3.2} \\ &\Rightarrow \varphi = 0 \quad \text{by duality of } H^{\frac{1}{2}}(\Gamma) \text{ and } H^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

In particular, this involves

$$(3.11) \quad \langle \varphi, \mathbf{M}\bar{\varphi} \rangle_\Gamma = |\mathbf{M}\varphi|_{H^1(\Gamma)}^2 > 0 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\},$$

which amounts to requirement (3.6).

Now, regard  $\mathbf{M}$  as an operator  $\mathbf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H_{\text{pw},0}^1(\Gamma)$ . As such it inherits its compactness from the embeddings  $H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$  and, consequently, meets requirement (3.5).

As the equation (3.8) is set in the space  $H^{\frac{1}{2}}(\Gamma)$ , a natural weak formulation can be obtained by testing with functions in  $H^{-\frac{1}{2}}(\Gamma)$ . However, in the context of a Galerkin discretization it is not entirely clear how to deal with the products of boundary integral operators occurring in the definition of  $\mathbf{S}_\kappa$ . The usual trick to avoid operator products is to switch to a mixed formulation. Here, this is done by introducing the new unknown  $u := \mathbf{M}\varphi \in H_{\text{pw},0}^1(\Gamma)$ . The definition of  $\mathbf{M}$  is used as second variational equation, which leads to the following saddle point problem: seek  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ ,  $u \in H_{\text{pw},0}^1(\Gamma)$ , such that

$$(3.12) \quad \begin{aligned} i\eta \langle \xi, \mathbf{V}_\kappa \varphi \rangle_\Gamma + \langle (\tfrac{1}{2}Id + \mathbf{K}_\kappa)u, \xi \rangle_\Gamma &= \langle g, \xi \rangle_\Gamma \quad \forall \xi \in H^{-\frac{1}{2}}(\Gamma), \\ -\langle \varphi, v \rangle_\Gamma + \langle \mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v \rangle_\Gamma &= 0 \quad \forall v \in H_{\text{pw},0}^1(\Gamma). \end{aligned}$$

It goes without saying that the first component of a solution  $(\varphi, u)$  of (3.12) will give us a solution of (3.8). Thus, Lemma 3.1 and the injectivity of  $\mathbf{M}$  confirm the uniqueness of solutions of (3.12).

Next, we aim to identify compact perturbations of the bi-linear form  $a : (H^{-\frac{1}{2}}(\Gamma) \times H_{\text{pw},0}^1(\Gamma)) \times (H^{-\frac{1}{2}}(\Gamma) \times H_{\text{pw},0}^1(\Gamma)) \mapsto \mathbb{C}$  associated with (3.12). First of all the term  $\langle \varphi, v \rangle_\Gamma : H_{\text{pw},0}^1(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \mapsto \mathbb{C}$  is compact

thanks to the compactness of the embedding  $H_{\text{pw},0}^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$ . Moreover, recall the continuity  $\mathbf{K}_\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  and take into account Lemma 2.1. Hence, up to compact perturbations we need only to examine the modified bi-linear form that comprises the principal parts of the diagonal terms of (3.12)

$$(3.13) \quad \tilde{a} \left( \begin{pmatrix} \lambda \\ u \end{pmatrix}, \begin{pmatrix} \mu \\ v \end{pmatrix} \right) := i\eta \langle \mu, \mathbf{V}_0 \lambda \rangle_\Gamma + \langle \mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v \rangle_\Gamma ,$$

$\lambda, \mu \in H^{-\frac{1}{2}}(\Gamma)$ ,  $u, v \in H_{\text{pw},0}^1(\Gamma)$ . By (2.9) and the Poincaré-Friedrichs inequalities on the faces  $\Gamma_i$ ,  $i = 1, \dots, p$ , it is obvious that  $\tilde{a}$  is elliptic on  $H^{-\frac{1}{2}}(\Gamma) \times H_{\text{pw},0}^1(\Gamma)$ . This permits us to conclude that the bilinear form belonging to (3.12) is coercive on  $H^{-\frac{1}{2}}(\Gamma) \times H_{\text{pw},0}^1(\Gamma)$ .

*Remark 3.1* It is also possible to use  $\mathbf{M} := (-\Delta_\Gamma + Id)^{-1}$ , where  $\Delta_\Gamma : H^1(\Gamma) \mapsto H^{-1}(\Gamma)$  is the Laplace-Beltrami operator on all of  $\Gamma$ . The rationale why we opted for a localized operator  $\mathbf{M}$  is explained in Sect. 5.

## 4 Direct boundary integral equations

The direct approach to the derivation of boundary integral equations uses the representation formula

$$(4.1) \quad U = \Psi_{\text{DL}}^\kappa(\gamma_D^+ U) - \Psi_{\text{SL}}^\kappa(\gamma_N^+ U) ,$$

valid for any exterior Helmholtz solution. Applying both the Dirichlet and Neumann trace operator to (4.1), we obtain the formulas of the *Calderón projector*

$$(4.2) \quad \gamma_D^+ U = (\mathbf{K}_\kappa + \tfrac{1}{2} Id)(\gamma_D^+ U) - \mathbf{V}_\kappa(\gamma_N^+ U) ,$$

$$(4.3) \quad \gamma_N^+ U = -\mathbf{D}_\kappa(\gamma_D^+ U) - (\mathbf{K}_\kappa^* - \tfrac{1}{2} Id)(\gamma_N^+ U) .$$

From these equations we can extract two boundary integral equations related to the exterior Dirichlet problem for the Helmholtz equation. Since, the boundary integral operators applied to the unknown Cauchy datum  $\gamma_N^+ U$  are the same as in (3.2), these boundary integral equations will also be affected by spurious resonances.

### 4.1 Classical CFIE

It was the idea of Burton and Miller in [11] to consider the following complex linear combination of the two equations (4.2) and (4.3)

$$(4.4) \quad (i\eta(\mathbf{K}_\kappa - \tfrac{1}{2} Id) - \mathbf{D}_\kappa)(\gamma_D^+ U) - (i\eta\mathbf{V}_\kappa + \tfrac{1}{2} Id + \mathbf{K}_\kappa^*)(\gamma_N^+ U) = 0 ,$$

where  $\eta \neq 0$  is a real parameter. Then, the boundary integral equation for the exterior Dirichlet problem integral equation reads

$$\varphi \in H^{-\frac{1}{2}}(\Gamma) : (i\eta \mathbf{V}_\kappa + \frac{1}{2}Id + \mathbf{K}_\kappa^*)(\varphi) = (i\eta(\mathbf{K}_\kappa - \frac{1}{2}Id) - \mathbf{D}_\kappa)(\gamma_D^+ U) .$$

As before this equation eludes a simple variational analysis in natural trace spaces, because it cannot be tested against functions in  $H^{-\frac{1}{2}}(\Gamma)$ . Lifting it to  $L^2(\Gamma)$  is a remedy only on smooth surfaces, cf. Sect. 3.1. Again, it takes regularization to get a coercive variational formulation.

#### 4.2 Regularized formulation

The strategy for regularization closely follows the “double layer regularization” of the indirect CFIE elaborated in Sect. 3.2. Again, we assume that a regularizing operator  $\mathbf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  that satisfies (3.5) and (3.6) is at our disposal.

Now, the trick is to apply  $\mathbf{M}$  to (4.3) before adding it to  $i\eta \cdot (4.2)$ . Doing so is strongly suggested by the fact that Dirichlet traces and Neumann traces belong to different spaces so that  $\gamma_N^+ U$  should be lifted into  $H^{\frac{1}{2}}(\Gamma)$  before adding it to  $i\eta \cdot \gamma_D^+ U$ . This yields the following boundary integral equation for the exterior Dirichlet problem

$$(4.5) \quad \mathbf{S}_\kappa(\varphi) = (i\eta(\mathbf{K}_\kappa - \frac{1}{2}Id) - \mathbf{M} \circ \mathbf{D}_\kappa)g ,$$

where  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$  is the unknown Neumann datum and

$$\mathbf{S}_\kappa := \mathbf{M} \circ (\mathbf{K}_\kappa^* + \frac{1}{2}Id) + i\eta \mathbf{V}_\kappa .$$

The first result corresponds to Lemma 3.1.

**Lemma 4.1** *The boundary integral operator  $\mathbf{S}_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  is injective.*

*Proof* We consider  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$  with  $\mathbf{S}_\kappa \varphi = 0$  and set  $U = \Psi_{\text{SL}}^\kappa(\varphi)$ . Thanks to (2.5) and the jump relations  $U|_{\Omega^-}$  is a solution of

$$(4.6) \quad \begin{aligned} \Delta U + \kappa^2 U &= 0 & \text{in } \Omega^- , \\ \mathbf{M}(\gamma_N^- U) + i\eta \gamma_D^- U &= 0 & \text{on } \Gamma . \end{aligned}$$

This is clear from the jump relations for the single layer potential. The Helmholtz solution  $U \in H^1(\Omega^-)$  satisfies

$$\int_{\Omega^-} \mathbf{grad} U \cdot \mathbf{grad} V - \kappa^2 U V \, d\mathbf{x} - \langle \gamma_N^- U, \gamma_D^- V \rangle_\Gamma = 0 \quad \forall V \in H^1(\Omega^-) .$$

Now, test with  $V = \bar{U}$  and use the boundary conditions from (4.6) to express  $\gamma_D^- U$

$$\int_{\Omega^-} |\mathbf{grad} U|^2 - \kappa^2 |U|^2 d\mathbf{x} + \frac{i}{\eta} \langle \gamma_N^- U, \mathbf{M}(\gamma_N^- \bar{U}) \rangle_\Gamma = 0 .$$

Equating imaginary parts and using the assumption (3.6) on  $\mathbf{M}$  we find  $\gamma_N^- U = 0$ , which implies  $\gamma_D^- U = 0$ . Hence,  $U|_{\Omega^-} \equiv 0$ . The jump relations for the single layer potential involve  $\gamma_D^+ U = 0$ , which, by Thm. 1.1, means  $U|_{\Omega^+} = 0$ . As another consequence of the jump relations,  $\varphi = -[\gamma_N U]_\Gamma = 0$ .  $\square$

This paves the way for applying the Fredholm alternative to (4.5): since

$$\mathbf{S}_\kappa = i\eta \mathbf{V}_0 + i\eta(\mathbf{V}_\kappa - \mathbf{V}_0) + \mathbf{M} \circ (\mathbf{K}_\kappa^* + \tfrac{1}{2} Id) ,$$

we can invoke Lemma 2.1, the continuity of  $\mathbf{K}_\kappa^*$ , and the compactness of  $\mathbf{M}$  to conclude that  $\mathbf{S}_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$  is bijective.

As a concrete incarnation of  $\mathbf{M}$  we could use the same operator as in Sect. 3.2, namely the one given by (3.10). Yet, in the case of the regularized direct CFIE there is no reason to eschew the choice  $\mathbf{M} = (-\Delta_\Gamma + Id)^{-1}$ , that is, this time we introduce  $\mathbf{M} : H^{-1}(\Gamma) \mapsto H^1(\Gamma)$  by

$$(4.7) \quad \langle \mathbf{grad}_\Gamma \mathbf{M}\varphi, \mathbf{grad}_\Gamma v \rangle_\Gamma + \langle \mathbf{M}\varphi, v \rangle_\Gamma = \langle \varphi, v \rangle_\Gamma \quad \forall v \in H^1(\Gamma) .$$

It goes without saying that this  $\mathbf{M}$  restricted to  $H^{-\frac{1}{2}}(\Gamma)$  satisfies both (3.5) and (3.6). This choice of  $\mathbf{M}$  makes it possible to switch to a simple mixed formulation by introducing the new unknown

$$(4.8) \quad u := \mathbf{M}((\tfrac{1}{2} Id + \mathbf{K}_\kappa^*)\varphi + \mathbf{D}_\kappa g) \in H^{\frac{1}{2}}(\Gamma) .$$

Actually,  $u$  is mislabelled, because it is by no means an unknown: recalling (4.3) we quickly realize that  $u = 0$ , if  $\varphi$  is the exact Neumann trace. What is the point of introducing  $u$ , nevertheless? The reason is that we aim to get a variational formulation suitable for Galerkin discretization and in the discrete setting the approximation of  $u$  does not necessarily vanish. The concrete variational problem reads: seek  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ ,  $u \in H^1(\Gamma)$  such that for all  $\xi \in H^{-\frac{1}{2}}(\Gamma)$ ,  $v \in H^1(\Gamma)$

$$(4.9) \quad \begin{aligned} i\eta \langle \xi, \mathbf{V}_\kappa \varphi \rangle_\Gamma + \langle \xi, u \rangle_\Gamma &= i\eta \langle \xi, (\mathbf{K}_\kappa - \tfrac{1}{2} Id)g \rangle_\Gamma , \\ - \langle (\tfrac{1}{2} Id + \mathbf{K}_\kappa^*)\varphi, v \rangle_\Gamma + \langle \mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v \rangle_\Gamma &= \langle \mathbf{D}_\kappa g, v \rangle_\Gamma . \end{aligned}$$

It is clear that  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$  solves (4.5) if and only if  $(\varphi, u)$ ,  $u$  given by (4.8), solves (4.9). Hence, Lemma 4.1 along with (3.6) also implies uniqueness of solutions of (4.9).

As in Sect. 3.2, it is immediate to see that the off-diagonal terms in (4.9) are compact. Eventually, up to compact perturbations, it turns out that the bi-linear form associated with (4.9) equals the  $H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$ -elliptic bi-linear form  $\tilde{a}$  defined in (3.13). Hence, it is a coercive bi-linear form on  $H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$  that underlies the variational problem (4.9).

## 5 Galerkin discretization

Both variational problems (3.12) and (4.9) have been shown to be based on bi-linear forms that are coercive in the respective function spaces. Moreover, we have established that they possess unique solutions. It is well known that these properties ensure asymptotic quasi-optimality of approximate Galerkin solutions in the sense that on sufficiently fine meshes the discretization error measured in the norm of  $H^{-\frac{1}{2}}(\Gamma) \times H^1_{\text{pw},0}(\Gamma)/H^1(\Gamma)$ , is bounded by a constant times the best approximation error of the trial spaces, see [36], but also [19, 34].

Conforming boundary element spaces for the approximation of functions in  $H^1(\Gamma)$ ,  $H^1_{\text{pw},0}(\Gamma)$ , and  $H^{-\frac{1}{2}}(\Gamma)$ , respectively, are standard. First, we equip  $\Gamma$  with a family  $\{\mathcal{T}_h\}_h$  of triangulations comprising (curved) triangles and/or quadrilaterals. The meshes  $\mathcal{T}_h$  have to resolve the shape of the curvilinear polyhedron  $\Omega^-$  in the sense that none of their elements may reach across an edge of  $\Omega^-$ . Then, we introduce boundary element spaces  $\mathcal{S}_h \subset H^1(\Gamma)$  and  $\mathcal{Q}_h \subset H^{-\frac{1}{2}}(\Gamma)$ , which contain piecewise polynomials of fixed total/maximal degree  $k+1$  and  $k$ ,  $k \in \mathbb{N}_0$ , respectively. We will restrict ourselves to these traditional boundary element spaces, but we remark that for smooth scatterers spectral discretization schemes are an attractive alternative [22].

For the Galerkin discretization of (3.12) we will need boundary element subspaces  $\mathcal{S}_h^{\text{pw}}$  of  $H^1_{\text{pw},0}(\Gamma)$ . They can be constructed as subspaces of  $\mathcal{S}_h$  by setting all degrees of freedom located on edges of  $\Gamma$  to zero.

Let  $h$  denote the meshwidth of  $\mathcal{T}_h$  and assume uniform shape-regularity, which, sloppily speaking, imposes a uniform bound on the distortion of the elements. Then we can find constants  $C_s, C_q > 0$  such that [7, Sect. 4.4]

$$(5.1) \quad \inf_{\phi_h \in \mathcal{Q}_h} \|\phi - \phi_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C_q h^{t+\frac{1}{2}} \|\phi\|_{H^t(\Gamma)} \\ \forall \phi \in H^t(\Gamma), \forall h, \quad 0 \leq t \leq k+1,$$

$$(5.2) \quad \inf_{v_h \in \mathcal{S}_h} \|v - v_h\|_{H^1(\Gamma)} \leq C_s h^{t-1} \|v\|_{H^t(\Gamma)} \\ \forall v \in H^t(\Gamma), \forall, \quad 1 \leq t \leq k+2.$$

Thus, the quantitative investigation of convergence entails establishing the Sobolev regularity of the continuous solutions. We start the investigation of it for the variational boundary integral equations (3.12) and (4.9).

It is useful to characterize the lifting properties of Neumann-to-Dirichlet maps for the interior/exterior Helmholtz problem by means of two real numbers  $\alpha^+/\alpha^-$ . In particular, let  $\alpha^-/\alpha^+$  be a real number such that for an interior/exterior Helmholtz solution  $\gamma_N^\pm U \in H^{s-\frac{1}{2}}(\Gamma)$  implies  $\gamma_D^\pm U \in H^{s+\frac{1}{2}}(\Gamma)$  for all  $s \leq \alpha^\pm$  and vice-versa. It is known that for mere Lipschitz domains  $\alpha^-, \alpha^+ \geq \frac{1}{2}$ , see [25] or [30, Thm. 4.24].

We first examine the indirect regularized formulation (3.12) introduced in Sect. 3.2. If  $\varphi \in H^{-\frac{1}{2}}(\Gamma)$  is the solution of (3.8) and the Helmholtz solution  $U$  is given by (3.7), the jump relations give us

$$(5.3) \quad [\gamma_D U]_\Gamma = \mathbf{M}\varphi \quad , \quad [\gamma_N U]_\Gamma = -i\eta\varphi \quad .$$

It is clear that the regularizing properties of  $\mathbf{M}$  will come into play. To measure them define for  $s \geq 1$

$$H_{\text{pw},0}^s(\Gamma) := \{v \in H_{\text{pw},0}^1(\Gamma), v|_{\Gamma_i} \in H^s(\Gamma_i), i = 1, \dots, p\} \quad .$$

We will write  $\beta$  for a real number such that  $\mathbf{M}v \in H^{s-1}(\Gamma)$  implies  $v \in H_{\text{pw},0}^{s+1}(\Gamma)$  for all  $s \leq \beta$ . From [23] we know that  $\beta > \frac{1}{2}$ , and that  $\beta \geq 1$  can be chosen, if all  $\Gamma_i$  are diffeomorphic images of convex polygons.

Assume that the Dirichlet boundary values  $g$  belong to  $H^{\sigma+\frac{1}{2}}(\Gamma)$ . This means that  $\gamma_N^+ U \in H^{-\frac{1}{2}+\min\{\sigma, \alpha^+\}}(\Gamma)$ . In addition,  $U|_{\Omega^-}$  satisfies the inhomogeneous boundary conditions

$$(5.4) \quad i\eta\mathbf{M}(\gamma_N^- U) - \gamma_D^- U = i\eta\mathbf{M}(\gamma_N^+ U) - g \quad .$$

Since  $\gamma_N^+ U \in H^{-\frac{1}{2}+\min\{\sigma, \alpha^+\}}(\Gamma)$ , using the mapping property of  $\mathbf{M}$ , we deduce that the right hand side of (5.4) belongs to  $H^r(\Gamma)$ , with  $r = \min\{\frac{3}{2} + \alpha^+, 1 + \beta, \frac{1}{2} + \sigma\}$ .

We first have that  $\gamma_D^- U \in H^{\min\{1, r\}}(\Gamma)$ , thus  $\gamma_N^- U \in H^{\min\{0, r-1, -\frac{1}{2}+\alpha^-\}}(\Gamma)$ . Now, a bootstrap argument can be used: by the shift theorem for  $\mathbf{M}$ , we obtain an improved regularity for  $\mathbf{M}(\gamma_N^- U)$ , namely,  $\mathbf{M}(\gamma_N^- U) \in H^{\min\{2, r+1, \frac{3}{2}+\alpha^-, 1+\beta\}}(\Gamma)$ .

Using again (5.4), we then have  $\gamma_D^- U \in H^{\min\{2, r, \frac{3}{2}+\alpha^-, 1+\beta\}}(\Gamma)$ . Thus, finally, recalling the definition of  $r$ , we have  $\gamma_N^- U \in H^{\min\{1, -\frac{1}{2}+\alpha^-, \beta, -\frac{1}{2}+\sigma\}}(\Gamma)$ .

By (5.3), this involves

$$(5.5) \quad \varphi \in H^{\min\{1, \beta, -\frac{1}{2}+\sigma, -\frac{1}{2}+\alpha^+, -\frac{1}{2}+\alpha^-\}}(\Gamma) \quad .$$

Note that, since  $\Omega$  is a polyhedron, either  $\alpha^+$  or  $\alpha^-$  is smaller than 1. Without loss of generality, we can then reduce (5.5) to:

$$(5.6) \quad \varphi \in H^{-\frac{1}{2}+\min\{\sigma, \alpha^+, \alpha^-\}}(\Gamma) \quad ,$$

which means that the regularity of  $\varphi$  depends only on the regularity of the Dirichlet datum and of the interior and exterior Dirichlet-to-Neumann maps.

For the mixed variational problem (3.12) convergence will also hinge on the regularity of the auxiliary variable  $u := \mathbf{M}\varphi$ . The regularity (5.6) of  $\varphi$  will directly translate into the regularity

$$(5.7) \quad u \in H_{\text{pw},0}^{\min\{1+\beta, \frac{3}{2}+\sigma, \frac{3}{2}+\alpha^+, \frac{3}{2}+\alpha^-\}}(\Gamma) .$$

We point out that the approximation estimate (5.2) remains true when we replace  $\mathcal{S}_h$  with  $\mathcal{S}_h^{\text{pw}}$  and  $H^t(\Gamma)$  with  $H_{\text{pw},0}^t(\Gamma)$ .

Summing up, in light of asymptotic quasi-optimality, by combining (5.6) and (5.7) with (5.1) and (5.1), we arrive at the following asymptotic a priori error estimate for the Galerkin boundary element solutions  $\varphi_h \in \mathcal{Q}_h$  and  $u_h \in \mathcal{S}_h$  of (4.9)

$$(5.8) \quad \|\varphi - \varphi_h\|_{H^{-\frac{1}{2}}(\Gamma)} + \|u - u_h\|_{H^1(\Gamma)} \leq Ch^{\min\{\beta, \sigma, \alpha^+, \alpha^-, k+1\}} ,$$

where  $C > 0$  does not depend on the meshwidth  $h$ .

*Remark 5.1* Estimate (5.8) highlights the need for good regularity of  $u$ , that is, the possibility to choose large  $\beta$ . This motivates the concrete choice of  $\mathbf{M}$  in Sect. 3.2. If we had opted for  $\mathbf{M} = (-\Delta_\Gamma + Id)^{-1}$ , cf. Rem. 3.1, Thm. 5.3 of [8] tells us that  $u$  may be only slightly more regular than merely belonging to  $H^1(\Gamma)$ . This could make  $u$  severely limit the overall rate of convergence.

A much simpler argument suffices for the direct formulation (4.9) for the exterior Dirichlet problem. The variational problem features the Cauchy datum  $\varphi := \gamma_N^+ U$  as the principal unknown. Assuming  $g \in H^{\frac{1}{2}+\sigma}(\Gamma)$ ,  $\sigma \geq 0$ , we conclude  $\varphi \in H^{\min\{-\frac{1}{2}+\sigma, -\frac{1}{2}+\alpha^+\}}(\Gamma)$ . Moreover, the exact solution for the auxiliary unknown  $u$  will be  $u = 0$ , which means that it does not affect the asymptotic convergence of the Galerkin scheme. Summing up, we find an a priori estimate  $O(h^{\min\{\sigma, \alpha^+, k+\frac{3}{2}\}})$  for the rate of convergence.

*Remark 5.2* What is the point of trying to “approximate”  $u = 0$  at all? If the variational problem (4.9) was truly elliptic, then a piecewise constant “approximation” of  $u$  on the connected components of  $\Gamma$  would really suffice. However, the considerations in [36] show that for merely coercive variational problems some minimal approximation power of the discrete trial spaces is required to allow a statement about existence and quality of Galerkin solutions. It is an interesting question what this barely sufficient approximation of  $u$  may be in the case of (4.9).

*Remark 5.3* The behavior of an exterior Helmholtz solution at edges and corners of  $\Gamma$  is well known [18]. This gives a lot of information about the local behavior of the Neumann trace  $\gamma_N^+ U$ . This knowledge can be exploited



to construct more efficient locally adapted approximations by means of local anisotropic refinement in conjunction with  $hp$ -adaptivity [32].

*Remark 5.4* An issue is still looming, namely the “optimal” choice of the parameter  $\eta$ . For the classical CFIE this has been addressed in [2] for a direct method and in [21, 28] for the indirect BIE (3.4). Investigations for a regularized formulations are carried out in [10]. In each case spectral analysis on a sphere was the main tool. It should be straightforward to apply it to the formulations proposed in this article.

## 6 Conclusion

We found that, in the case of non-smooth scatterers, the rigorous variational analysis of CFIEs for the exterior Dirichlet problem for the Helmholtz equation has to rely on special regularizing operators. For the indirect method regularization is aimed at the double layer potential. In the case of the direct method regularization amounts to a lifting of Neumann traces. In both cases, after introducing an auxiliary unknown, we end up with a coercive mixed variational formulation in natural trace spaces, for which conforming Galerkin boundary element schemes enjoy asymptotic quasi-optimality. On top of that the regularized direct CFIE instantly yields a priori error estimates that only depend on the regularity of the Neumann data: the auxiliary unknown has no influence. In this respect the direct method seems superior to the indirect approach. Still, comprehensive numerical experiments and comparisons will have to be carried out to bear out the practical viability of the proposed new CFIEs.

## References

1. Adams, R.: Sobolev Spaces. Academic Press, New York, 1975
2. Amini, S.: On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem. *Appl. Anal.*, **35**, 75–92 (1990)
3. Amini, S., Wilton, D.: An investigation of boundary element methods for the exterior acoustic problem. *Comp. Meth. Appl. Mech. Engr.* **54**, 49–65 (1986)
4. Astley, R.: Infinite elements for wave problems: A review of current formulations and an assessment of accuracy. *Int. J. Numer. Meth. Engr.* **49**, 951–976 (2000)
5. Bettess, P.: Infinite elements. *Int. J. Numer. Meth. Engr.* **11**, 53–64 (1977)
6. Brakhage, H., Werner, P.: Über das Dirichletsche Außenraumproblem für die Helmholtzsche Schwingungsgleichung. *Arch. der Math.* **16**, 325–329 (1965)
7. Brenner, S., Scott, R.: Mathematical theory of finite element methods. Texts in Applied Mathematics, Springer-Verlag, New York, 1994
8. Buffa, A., Costabel, M., Schwab, C.: Boundary element methods for Maxwell’s equations on non-smooth domains. *Numer. Math.* **92**, 679–710 (2002)
9. Buffa, A., Hiptmair, R.: A coercive combined field integral equation for electromagnetic scattering. *SIAM J. Numer. Anal.* **40**, 621–640 (2004)

10. Buffa, A., Sauter, S.: Stabilisation of the acoustic single layer potential on non-smooth domains. Preprint 19-2003, Institut für Mathematik, Universität Zürich, Zürich, Switzerland, 2003
11. Burton, A., Miller, G.: The application of integral methods for the numerical solution of boundary value problems. *Proc. R. Soc. London, Ser. A* **232**, 201–210 (1971)
12. Chen, G., Zhou, J.: *Boundary Element Methods*. Academic Press, New York, 1992
13. Christiansen, S.: Discrete Fredholm properties and convergence estimates for the electric field integral equation. *Math. Comp.* **73**, 143–167 (2004)
14. Colton, D., Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*. vol. 93 of *Applied Mathematical Sciences*, Springer, Heidelberg, 2nd ed., 1998
15. Costabel, M.: Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal.* **19**, 613–626 (1988)
16. Costabel, M., Dauge, M.: Maxwell and Lamé eigenvalues on polyhedra. *Math. Methods Appl. Sci.* **22**, 243–258 (1999)
17. Costabel, M., Dauge, M., Nicaise, S.: Singularities of Maxwell interface problems. *M<sup>2</sup>AN*, **33**, 627–649 (1999)
18. Dauge, M.: *Elliptic Boundary Value Problems on Corner Domains*. vol. 1341 of *Lecture Notes in Mathematics*, Springer, Berlin, 1988
19. Demkowicz, L.: Asymptotic convergence in finite and boundary element methods: Part I, Theoretical results. *Comput. Math. Appl.*, **27**, 69–84 (1994)
20. Fairweather, G., Karageorghis, A., Martin, P.: The method of fundamental solutions for scattering and radiation problems. *Eng. Anal. Bound. Elem.* **27**, 759–769 (2003)
21. Giebermann, K.: *Schnelle Summationsverfahren zur numerischen Lösung von Integralgleichungen für Streuprobleme im  $\mathbb{R}^3$*  (Fast summation methods for the numerical solution of integral equations for scattering problems in  $\mathbb{R}^3$ ), PhD thesis, Fakultät für Mathematik, Universität Karlsruhe, Karlsruhe, Germany, 1997
22. Graham, I., Sloan, I.: Fully discrete spectral boundary integral methods for helmholtz problems on smooth closed surfaces in  $\mathbb{R}^3$ . *Numer. Math.* **92**, 289–323 (2002)
23. Grisvard, P.: *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985
24. Hagstrom, T.: Radiation boundary conditions for the numerical simulation of waves. *Acta Numerica* **8**, 47–106 (1998)
25. Jerison, D., Kenig, C.: The inhomogeneous dirichlet problem in Lipschitz domains. *J. Funct. Anal.* **130**, 161–219 (1995)
26. Jones, D.: Integral equations for the exterior acoustic problem. *Quarterly J. Mech. Appl. Math.* **27**, 129–142 (1974)
27. Kenig, C.: Harmonic analysis techniques for second order elliptic boundary value problems. vol. 83 of *CBMS Regional Conference Series in Mathematics*, AMS, Providence, RI 1994
28. Kress, R.: Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. *Q. J. Mech. Appl. Math.* **38**, 323–341 (1985)
29. Leis, R.: Zur Dirichletschen Randwertaufgabe des Aussenraumes der Schwingungsgleichung. *Math. Z.* **90**, 205–211 (1965)
30. McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, Cambridge, UK, 2000
31. Panich, O.: On the question of the solvability of the exterior boundary-value problems for the wave equation and Maxwell's equations. *Usp. Mat. Nauk.* **20A**, 221–226 (1965) In Russian
32. Sauter, S., Schwab, C.: Quadrature for *hp*-Galerkin BEM in  $\mathbb{R}^3$ . *Numer. Math.* **78**, 211–258 (1997)
33. Sauter, S., Schwab, C.: *Randelementmethoden*, BG Teubner, Stuttgart, 2004

34. Schatz, A.: An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.* **28**, 959–962 (1974)
35. Ursell, F.: On the exterior problem of acoustics I. *Math. Proc. Cambridge Phil. Soc.* **74**, 117–125 (1973)
36. Wendland, W.: Boundary element methods for elliptic problems. In: *Mathematical Theory of Finite and Boundary Element Methods*, A. Schatz, V. Thomée, W. Wendland, (eds.), **15**, of DMV-Seminar, Birkhäuser, Basel, 1990, pp.219–276