Gram Matrix in the SBD method

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In this document we give explicit formulas for the matrix A of size $P \times P$ defined as

$$A_{i,j} = \int_{\mathcal{A}(a,b)} \nabla e_i \cdot \nabla e_j, \quad i,j \in \{1, \cdots, P\}$$

where $e_i(r) = C_i J_0(\rho_i r)$ with C_i the normalization constant $C_i = \dfrac{\sqrt{2}}{\rho_i \left|J_1(\rho_i)\right|}$, J_α is the Bessel function of first kind and order α, ρ_i is the i-th root of J_0, $\mathcal{A}(a,b)$ is the ring $\{a < r < b\}$. We next study its conditioning. This is the Gram matrix that must be inverted to find the SBD coefficients. Recall that, if B is the unit ball in $\mathbb{R}^2$,

Theorem 1. *For all (i,j), one has*

$$\int_B \nabla e_i \cdot \nabla e_j = \delta_{i,j}$$

1 Explicit coefficients

The three identities are easy to obtain one from the previous.

Proposition 1. *For any $x < y$, we have the following identities*

(i) $\displaystyle\int_x^y u J_0(u) J_0'(u) du = -\frac{1}{2}\Big[u^2 J_0'(u)^2\Big]_x^y$

(ii) $\displaystyle\int_x^y u J_0(u)^2 du = \frac{1}{2}\Big[u^2 (J_0(u)^2 + J_0'(u)^2)\Big]_x^y$

(iii) $\displaystyle\int_x^y u J_0'(u)^2 du = \left[\frac{u^2}{2}\{J_0(u)^2 + J_0'(u)^2\} + u J_0(u) J_0'(u)\right]_x^y$

Theorem 2. *The extra-diagonal elements of A are given by*

$$A_{i,j} = 2\pi C_i C_j \frac{\rho_i \rho_j}{\rho_j^2 - \rho_i^2}\Big[r\{\rho_i J_0(\rho_i r) J_0'(\rho_j r) - \rho_j J_0(\rho_j r) J_0'(\rho_i r)\}\Big]_a^b$$

while the diagonal elements are

$$A_{i,i} = 2\pi C_i C_j \left[\frac{(\rho_i r)^2}{2} \{J_0(\rho_i r)^2 + J_0'(\rho_i r)^2\} + (\rho_i r) J_0(\rho_i r) J_0'(\rho_i r) \right]_a^b$$

2 Condition number

Here we provide a bound on the condition number of this matrix and perform some numerical tests.

Proposition 2. *Assume that* $0 < a < b < 1$*. The eigenvalues of* A *lie in* $(0, 1)$*.*

Proof. A is the matrix of a scalar product, thus positive definite, so its eigenvalues are above 0. Let $v = \{v_1, \cdots, v_P\}$ and

$$V = \sum_{p=1}^{P} v_p e_p.$$

Then, one has

$$\begin{aligned} v^T A v &= \int_{\mathcal{A}(a,b)} |\nabla V|^2 \\ &< \int_B |\nabla V|^2 \\ &= \sum_{p=1}^{P} v_p^2 \\ &= \|v\|_2^2 . \end{aligned}$$

proving that all eigenvalues of A are strictly less than 1. $\square$

We are now interested finding a lower bound for the smallest eigenvalue $\lambda_{\min}$ of A. For this, we introduce the matrix B that is equal to $I - A$. When $a = 0, b = 1$, $B = 0$. We call c the largest eigenvalue of B. Obviously,

$$\lambda_{\min} \geq 1 - c,$$

thus the condition number of A is bounded by $\dfrac{1}{1-c}$.

Proposition 3. *For all* $x \in \mathbb{R}^+$*,*

$$J_1(x) \leq \frac{1}{2} x$$

Proposition 4. *We have the following estimate:*

$$c \leq tatata$$