Integral Equations
and Operator Theory
Vol.5 (1982)

0378-620X/82/040562-11$01.50+0.20/0

INTEGRAL EQUATIONS WITH NON INTEGRABLE KERNELS

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We study here some integral equations linked to the Laplace or the Helmholtz equation, or to the system of elasticity equations. These equations lead to non integrable kernels only defined as finite parts, so that they are quite difficult to approximate. In each case, we introduce a variational formulation which avoids this difficulty and allow us to use stable finite element approximations for these problems.

1. Neumann's problem using a double layer representation

Let Ω be a bounded regular domain in $\mathbb{R}^3$ which boundary is the closed surface Γ. We denote by Ω' the domain exterior to Ω. We are interested in the following problem :

$$
(1.1) \quad \begin{cases} \text{Find the function } u \text{ such that} \\ \Delta u = 0 \quad \text{in } \Omega \text{ and in } \Omega' ; \\ \dfrac{\partial u_i}{\partial n}\Big|_\Gamma = \dfrac{\partial u_e}{\partial n}\Big|_\Gamma = g \quad \text{on } \Gamma ; \end{cases}
$$

where u_i is the interior limit of u in Ω, and u_e the exterior one, $\vec{n}$ being the unitary exterior normal to the surface Γ.

We suppose that

$$(1.2) \qquad \int_\Gamma g \, d\gamma = 0 \, .$$

For any function v , regular in Ω and in Ω' , we write

$$[v] = v_i - v_e \, .$$

We can then prove that the problem (1.1) has a unique solution in the following Hilbert space X :

$$(|x|^2 = \sum_{i=1}^{3} x_i^2)$$

$$W^1(\Omega') = \left\{ v \mid \frac{v}{(1+|x|^2)^{\frac{1}{2}}} \in L^2(\Omega') \; ; \; \nabla v \in (L^2(\Omega'))^3 \right\}$$

$$W = \left\{ v \in H^1(\Omega)/_{\mathbb{R}} \times W^1(\Omega') \; ; \; \Delta v = 0 \text{ in } \Omega \text{ and } \Omega' \; ; [\tfrac{\partial v}{\partial n}] = 0 \right\}$$

The solution of this problem admits a double-layer representation of density $\varphi(x) = [u(x)]$; $x \in \Gamma$. So that we have

$$(1.3) \qquad u(y) = -\frac{1}{4\pi} \int_\Gamma \varphi(x) \frac{\partial}{\partial n_x} \left(\frac{1}{|x-y|}\right) d\gamma(x) \; ; \; y \in \mathbb{R}^3 - \Gamma \, .$$

The expression of the normal derivative of such a potential is not an integral. It is a finite part expression :

$$(1.4) \qquad \frac{\partial u}{\partial n}(y) = -\frac{1}{4\pi} \oint_\Gamma \varphi(x) \frac{\partial^2}{\partial n_x \partial n_y} \left(\frac{1}{|x-y|}\right) d\gamma(y) \; ; \; y \in \Gamma \, .$$

Its kernel has a singularity in $\frac{1}{|x-y|^3}$ when x and y are close .

Variational formulation

Let φ be a function on the surface Γ , in the Hilbert space $H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}}$ (see LIONS-MAGENES [4] for the definition of this space). We can solve

$$(1.5) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \text{ and in } \Omega' \; ; \\ [u] = \varphi & \text{on } \Gamma \, . \end{cases}$$

This problem has the following variational formulation

$$\int_{\Omega} \nabla u . \nabla v \; dx + \int_{\Omega'} \nabla u . \nabla v \; dx = \int_{\Gamma} \varphi \frac{\partial v}{\partial n} \, d\gamma \; ; \; \forall \, v \in X \; .$$

There exists a unique solution in the Hilbert space X .

Now, let φ and ψ be in $H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}}$. We denote by $b(\varphi,\psi)$ the following bilinear form

$$(1.6) \qquad b(\varphi,\psi) = \int_{\Omega} \nabla u(\varphi) . \nabla u(\psi) \; dx + \int_{\Omega'} \nabla u(\varphi) . \nabla u(\psi) \; dx.$$

Then, a variational formulation of the integral equation (1.4) is

$$(1.7) \qquad b(\varphi,\psi) = \int_{\Gamma} g \; \psi \; d\gamma \; ; \; \forall \; \psi \in H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}} \; .$$

From the properties of the space X , it also results that

$$(1.8) \qquad \begin{cases} b(\varphi,\psi) = b(\psi,\varphi) \; ; \; \forall \; \varphi,\psi \in H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}} \; ; \\ b(\varphi,\varphi) \geqslant \beta \; \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}}} \; ; \; \forall \; \varphi \in H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}} \; ; \; \beta > 0 \; . \end{cases}$$

The formulation (1.7) will be useful if we can obtain an easy expression for the bilinear form $b(\varphi,\psi)$. This expression exists and, in order to get it, we need some differential operators on the surface Γ .

To any function φ on the surface Γ , we associate a function $\widetilde{\varphi}$ defined in a neighbourhood of Γ in $\mathbb{R}^3$ by

$$\widetilde{\varphi}(x) = \varphi(\mathscr{P}(x)),$$

where $\mathscr{P}(x)$ is the local projection of x onto Γ .

We then define

$$(1.9) \qquad \begin{cases} \vec{\text{grad}}_{\Gamma} \, \varphi(x) = \vec{\text{grad}} \, \widetilde{\varphi}(x) \\ \vec{\text{curl}}_{\Gamma} \, \varphi(x) = \vec{n}(x) \wedge \vec{\text{grad}}_{\Gamma} \, \varphi(x) . \end{cases}$$

To any tangential vector $\vec{J}$ of the surface Γ , we associate a vector $\widetilde{J}$ defined in a neighbourhood of Γ in $\mathbb{R}^3$ by

$$\tilde{J}(x) = \vec{J}(\mathscr{P}(x))$$

and

$$(1.10)\qquad \begin{aligned} \operatorname{curl}_\Gamma \vec{J}(x) &= \vec{n}(x).\overrightarrow{\operatorname{curl}}\, \tilde{J}(x) \\ \Delta_\Gamma \varphi &= \operatorname{curl}_\Gamma \overrightarrow{\operatorname{curl}}_\Gamma \varphi \end{aligned}$$

We have the following result :

<u>*THEOREM 1*</u> *: The bilinear form* $b(\varphi,\psi)$ *given by (1.6) has the expression*

$$(1.11)\qquad b(\varphi,\psi) = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\overrightarrow{\operatorname{curl}}_\Gamma\varphi(x).\overrightarrow{\operatorname{curl}}_\Gamma\psi(y)}{|x-y|}\, d\gamma(x)d\gamma(y)\,;\forall\, \varphi,\psi \in H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}}$$

We also have

$$(1.12)\qquad \frac{\partial u}{\partial n}(y) = -\frac{1}{4\pi}\int_\Gamma \frac{\Delta_\Gamma\varphi(x)}{|x-y|}\, d\gamma(x) - \frac{1}{4\pi}\int_\Gamma (\overrightarrow{\operatorname{curl}}_\Gamma\varphi(x)\wedge \overrightarrow{\operatorname{grad}}_x (\frac{1}{|x-y|})(\vec{n}_y-\vec{n}_x))d\gamma(x)\,.$$

<u>Proof</u> : Let us consider the vector $h = \nabla u$ in Ω and in Ω'. We have $\operatorname{div} h = 0$ in $\mathbb{R}^3$, so that $h = \operatorname{curl} A$. We can choose A unique such that $\operatorname{div} \vec{A} = 0$. We then have

$$\Delta\vec{A} = \operatorname{curl}\operatorname{curl} A - \operatorname{grad}\operatorname{div} A = \operatorname{curl} h\ .$$

But since $\operatorname{curl} h = 0$ in Ω and in Ω', $\operatorname{curl} h$ is a distribution on the surface Γ . Using Green's formula, it can be proved that

$$\operatorname{curl} h = \operatorname{curl}_\Gamma \varphi\, \delta_\Gamma$$

where δ_Γ represents the Dirac distribution of the surface Γ . We can then give the following representation of A :

$$A(y) = \frac{1}{4\pi}\int_\Gamma \frac{\operatorname{curl}_\Gamma \varphi(x)}{|x-y|}\, d\gamma(x)\ .$$

Using again Green's formula, we obtain

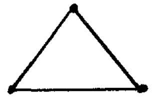
$$b(\varphi,\psi) = \int_\Gamma A(\varphi)\operatorname{curl}_\Gamma \psi\, d\gamma\ ,$$

and this proves the result. ∎

Numerical approximation

In order to approximate equation (1.7), we use finite element techniques to construct an approximate surface Γ_h and a finite dimensional subspace of $H^{\frac{1}{2}}(\Gamma_h)/_{\mathbb{R}}$.

In the simplest case, the surface Γ_h is built using the finite element $\mathbb{P}_1$ with three degrees of freedom



Γ_h is a union of plate triangles, adjacent by their edges and with a vertex on the surface Γ. In this case, the subspace V_h is exactly the space of continuous functions on the surface Γ_h which are polynomials of degree 1 on each triangle. The approximate problem is then

$$(1.13) \qquad \frac{1}{4\pi}\int_{\Gamma_h}\int_{\Gamma_h} \frac{\vec{\operatorname{curl}}_{\Gamma_h}\varphi_h(x)\cdot\vec{\operatorname{curl}}_{\Gamma_h}\psi_h(y)}{|x-y|}\,d\gamma(x)\,d\gamma(y) = \int_{\Gamma_h} g_h\psi_h\,d\gamma\ ;\ \forall\,\psi_h \in V_h/_{\mathbb{R}}.$$

The corresponding matrix has coefficients b_{ij} which are a simple combination of the following coefficients :

$$a_{k\ell} = \frac{1}{4\pi}\int_{T_k}\int_{T_\ell} \frac{d\gamma(x)\,d\gamma(y)}{|x-y|}\ .$$

In this case, it is possible to prove an error estimate of the following type

$$(1.14) \qquad |\varphi-\varphi_h|_{L^2(\Gamma)} \leqslant (c_1\,h + c_2\,h^2)\,\|\varphi\|_{H^2(\Gamma)}\ ,$$

where h is the maximum diameter of the triangles. The first term comes from the geometrical error and the second one, from the interpolation error.

2. Neumann's problem for the Helmholtz' equation

Let us use the same notations as in Section 1. Instead of (1.1), we consider the following problem

$$(2.1) \qquad \begin{cases} \Delta u + k^2 u = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \Omega' \ ; \\ \frac{\partial u}{\partial n}\Big|_\Gamma = g \ . \end{cases}$$

The function u is now complex and we add to (2.1) the so-called radiation condition

$$(2.2) \qquad \frac{\partial u}{\partial k} - iku = 0(\frac{1}{r}) \ , \text{ when } \ r \to \infty \ .$$

The double layer representation can be written under as

$$(2.3) \qquad u(y) = -\frac{1}{4\pi}\int_\Gamma \varphi(x) \frac{\partial}{\partial n_x} (\frac{e^{ik|x-y|}}{|x-y|}) \, d\gamma(x); \ \forall \ y \in \mathbb{R}^3 - \Gamma \ .$$

Again, the integral equation corresponding to this representation involves a finite part, analogous to the one of equation (1.4).

We have the following result (due to HAMDI [2]) :

THEOREM 2 : The equation (2.1) can be replaced by an integral equation using the representation (2.3). This equation has the following variational expression

$$(2.4) \qquad b(\varphi,\psi) = \int_\Gamma g \ \psi \ d\gamma \ ; \ \forall \ \psi \in H^{\frac{1}{2}}(\Gamma)/_{\mathbb{R}}$$

where the bilinear form b *is given by*

$$(2.5) \qquad b(\varphi,\psi) = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} \operatorname{curl}_\Gamma \varphi(x).\operatorname{curl}_\Gamma \psi(y) \, d\gamma(x)d\gamma(y)$$
$$- k^2 \int_\Gamma\int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} \varphi(x) \ \psi(y)(\vec{n}_x.\vec{n}_y) \ d\gamma(x)d\gamma(y).$$

The approximation of the equation (2.4) using finite element techniques is then similar to (1.13). We also obtain similar error estimates when k is not an eigenvalue of the Dirichlet interior problem.

3. Elasticity in $\mathbb{R}^3$

In this section, we consider the integral equations associated to the system of elasticity in $\mathbb{R}^3$, in the case corresponding to the double layer representation.

We are looking for a vector $\vec{u} = (u_1,u_2,u_3)$, defined in Ω and in Ω'. We define the tensors ε and σ by

$$\varepsilon_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) ; \quad 1 \leq i,j \leq 3 ; \tag{3.1}$$

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\left(\sum_{k=1}^{3} \varepsilon_{kk}\right) \delta_{ij} . \tag{3.2}$$

The equilibrium equations are then

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) = 0 \quad \text{in } \Omega \text{ and } \Omega' ; \; i = 1,2,3 . \tag{3.3}$$

Here, we are interested in the case where the boundary conditions are the forces, that is

$$\sum_{j=1}^{3} \sigma_{ij}\, n_j = g_i \; ; \; i = 1,2,3 \quad (\text{or } \sigma.\vec{n} = \vec{g}). \tag{3.4}$$

The double-layer representation is the solution of the following problem

$$\begin{cases} \displaystyle\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) = 0 \quad \text{in } \Omega \text{ and } \Omega' ; \; i = 1,2,3 ; \\ [\sigma.\vec{n}] = 0 ; \\ [\vec{u}] = \vec{\varphi} ; \end{cases} \tag{3.5}$$

Let $\mathscr{R}$ be the rigid body displacements in $\mathbb{R}^3$:

$$\mathscr{R} = \left\{ \vec{u} = \vec{\alpha} + \vec{\beta} \wedge \vec{x} \; ; \; \vec{\alpha}, \vec{\beta} = \text{constant vectors} \right\}. \tag{3.6}$$

Let X be the following Hilbert space

$$X = \left\{ \vec{u} \in (H^1(\Omega))^3/\mathscr{R} \times (W^1(\Omega'))^3 \; ; \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) = 0 \text{ in } \Omega \text{ and } \Omega' ; [\sigma.\vec{n}] = 0 \right\}$$

Then, using Korn's inequality, it can be proved that the problem (3.5) has a unique solution in the space X, when $\vec{\varphi}$ is given in the space $(H^{\frac{1}{2}}(\Gamma))^3/\mathscr{R}$.

A fundamental solution of the elasticity system is a symmetrical tensor U such that

$$(3.7) \qquad \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(U_{k.}) = \delta_{ki}\,\delta(0).$$

It is given by

$$(3.8) \qquad U_{k\ell}(x) = \frac{1}{4\pi\mu}\left(\frac{1}{|x|}\delta_{k\ell} - \frac{(\lambda+\mu)}{2(2\mu+\lambda)} \cdot \frac{\partial^2}{\partial x_k \partial x_\ell}(|x|)\right) .$$

Using Green's formula, we then obtain a representation of the solution of (3.5) written as

$$(3.9) \qquad u_k(y) = -\sum_{i,j=1}^{3}\int_\Gamma \sigma_{ij}(U_{k.}(y-x))\, n_j(x)\, \varphi_i(x)\, d\gamma(x) \; ; \; y \in \mathbb{R}^3 - \Gamma .$$

The integral equation linking $\vec{\varphi}$ and $\vec{g}$ is then

$$(3.10) \qquad -\sum_{i,k,\ell=1}^{3} \oint_\Gamma \Sigma_{ijk\ell}(y-x)\, n_k(x)\, n_\ell(y)\, \varphi_i(x)\, d\gamma(x) = g_j(y) \; ; \; y \in \Gamma ,$$

where Σ is the 3×3 symmetrical tensor of order 4

$$(3.11) \qquad \Sigma(x-y) = \sigma_x\, \sigma_y(U(x-y)).$$

This tensor has singular components which are in $\frac{1}{|x-y|^3}$ when x and y are close , so that the integral in (3.10) has in fact finite parts.

Variational formulation

We can obtain a variational formulation of the equation (3.10), using the problem (3.5). Let $\vec{u}$ be the solution of (3.5) associated to $\vec{\varphi}$ and let $\vec{v}$ be the solution associated to $\vec{\psi}$. Let us set

$$(3.12) \qquad b(\vec{\varphi},\vec{\psi}) = \sum_{i,j=1}^{3}\int_{\Omega\cup\Omega'} \sigma_{ij}(\vec{u})\, \varepsilon_{ij}(\vec{v})\, dx .$$

An equivalent formulation of equation (3.10) is then

(3.13) $$b(\vec{\varphi},\vec{\psi}) = \sum_{j=1}^{3} \int_\Gamma g_j\, \psi_j\, d\gamma \; ; \; \forall\, \vec{\psi} \in (H^{\frac{1}{2}}(\Gamma))^3 /_{\mathcal{R}} \, .$$

THEOREM 3 : The bilinear form b *given by (3.12) is symmetric, coercive on the space* $(H^{\frac{1}{2}}(\Gamma))^3/_{\mathcal{R}}$. *It is given by*

(3.14) $$b(\vec{\varphi},\vec{\psi}) = \sum_{i,j=1}^{3} \int_\Gamma \int_\Gamma G_{ij..}(x-y)\, \vec{\mathrm{curl}}_\Gamma\, \varphi_i(x) . \vec{\mathrm{curl}}_\Gamma\, \psi_j(y)\, d\gamma(x) d\gamma(y)$$

where $G_{ij..}$ *is a tensor of order 4 given by*

(3.15) $$G_{ij..} = - \vec{\mathrm{curl}}_x\, \vec{\mathrm{curl}}_y\, A_{i.j.}(x-y) ,$$

A *being a tensor of order 4 given by*

$$\left\{ \begin{aligned} A_{ijk\ell}(x) = {} & \frac{4\mu(\lambda+\mu)}{2\mu+\lambda} \cdot \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_\ell} \left(\frac{|x|^5}{2880\pi}\right) \\ & + \frac{2\lambda\mu}{2\mu+\lambda} \left(\frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{|x|^3}{96\pi}\right) \delta_{k\ell} + \frac{\partial^2}{\partial x_\ell \partial x_k} \left(\frac{|x|^3}{96\pi}\right) \delta_{ij} \right) \\ & + \mu \left(\frac{\partial^2}{\partial x_i \partial x_k} \left(\frac{|x|^3}{96\pi}\right) \delta_{\ell j} + \frac{\partial^2}{\partial x_j \partial x_k} \left(\frac{|x|^3}{96\pi}\right) \delta_{\ell i} \right. \\ & \quad \left. + \frac{\partial^2}{\partial x_j \partial x_\ell} \left(\frac{|x|^3}{96\pi}\right) \delta_{ki} + \frac{\partial^2}{\partial x_i \partial x_\ell} \left(\frac{|x|^3}{96\pi}\right) \delta_{kj} \right) \\ & + \mu \, \frac{|x|}{8\pi} \left(\delta_{ki}\, \delta_{\ell j} + \delta_{\ell i}\, \delta_{jk}\right) + \frac{2\mu\lambda}{2\mu+\lambda} \left(\frac{|x|}{8\pi}\right) \delta_{ij}\, \delta_{k\ell} \, . \end{aligned} \right.$$

We then have

(3.17) $$b(\vec{\varphi},\vec{\varphi}) \geqslant \beta \; \|\vec{\varphi}\|^2_{(H^{\frac{1}{2}}(\Gamma))^3/_{\mathcal{R}}} \; ; \; \beta > 0 \, .$$

4. Plane elasticity

We consider here the problem of elasticity in $\mathbb{R}^3$. We are looking for the displacement of the elastic body $\vec{u} = (u_1, u_2)$.

Exactly as in the previous section, the equations are

(4.1) $$\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \Omega' \; ; \; i = 1,2 \; ;$$

(4.2) $$\sum_{j=1} \sigma_{ij}\, n_j = g_i \quad \text{on} \quad \Gamma \; ; \; i = 1,2 \, .$$

A fundamental solution is the following tensor of order 2

$$(4.3) \qquad U_{k\ell}(x) = \frac{1}{\pi}\left(-\frac{1}{2\pi}\operatorname{Log}(|x|)\delta_{k\ell} + \frac{\lambda+\mu}{2\mu+\lambda}\frac{\partial^2}{\partial x_k \partial x_\ell}\left(\frac{|x|^2 \operatorname{Log}(|x|)}{8\pi}\right)\right).$$

Let Σ be the tensor of order 4 such that

$$(4.4) \qquad \Sigma(x-y) = \sigma_x\, \sigma_y\, (U(x-y)).$$

Then a representation of the solution as a "double-layer" potential is given by

$$(4.5) \qquad u_k(y) = -\sum_{i,j=1}^{2} \int \sigma_{ij}(U_{k.}(y-x))\, n_j(x)\, \varphi_i(x)\, d\gamma(x)\ ;\ y \in \mathbb{R}^3 - \Gamma.$$

The integral equation linking $\vec{\varphi}$ and $\vec{g}$ has the expression

$$(4.6) \qquad -\sum_{i,k,\ell=1}^{2} \int \Sigma_{ij\ \ell}(y-x)\, n_k(x)\, n_\ell(y)\, \varphi_i(x)\, d\gamma(x) = g_j(y)\ ;\ y\in\Gamma.$$

We have the following result [1] :

<u>THEOREM 4</u> : The equation (4.6) admits a variational formulation which is

$$(4.7) \qquad b(\vec{\varphi},\vec{\psi}) = \int \vec{g}\,\vec{\psi}\, d\gamma\ ;\ \forall\ \vec{\psi} \in (H^{\frac{1}{2}}(\Gamma))^2/\mathscr{R}$$

$$\mathscr{R} = \left\{\vec{\psi}\ ;\ \vec{\psi} = \vec{\alpha} + \beta \begin{vmatrix} -x_2 \\ +x_1 \end{vmatrix}\ ;\ \vec{\alpha}, \beta = \text{constants}\right\}\ ;$$

and the symmetric bilinear form b *is given by*

$$(4.8) \qquad b(\vec{\varphi},\vec{\psi}) = -\sum_{i,j=1}^{2} \int_\Gamma\int_\Gamma G_{ij}(x-y)\,\frac{d\varphi_i}{ds}(x)\,\frac{d\psi_j}{ds}(y)\, ds(x)\, ds(y)$$

$$(4.9) \qquad G_{ij}(x) = \frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)\mu}\left(\operatorname{Log}|x|\,\delta_{ij} - \frac{x_i\, x_j}{|x|^2}\right).$$

We also have

$$(4.10) \qquad b(\vec{\varphi},\vec{\psi}) \geqslant \beta \|\vec{\varphi}\|^2_{(H^{\frac{1}{2}}(\Gamma))^2/\mathscr{R}}\ ;\ \forall\ \vec{\varphi} \in (H^{\frac{1}{2}}(\Gamma))^2\ ;\ \beta > 0\ .$$

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