New preconditioners for Laplace and Helmholtz integral equations on open curves:

I. Theoretical framework and numerical results.

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Abstract

Studying wave scattering problems in 2D by open arcs leads to ill-conditioned systems. This is partly due to the singular behavior of the solutions near the edges of the arc. We introduce two new preconditioners to tackle this problem either with Dirichlet or Neumann boundary data, that take the form of square roots of local operators. We introduce an adapted analytical setting to analyze them and demonstrate the efficiency of this method on several numerical examples. A complete suitable pseudo-differential calculus for the complete study of such operators is postponed to the second part of this work.

1 Introduction

The problem of preconditionning the linear systems coming from the discretization of first kind integral equations has received considerable attention since two decades. Among the possible strategies are the so-called pseudo-differential preconditionners [3–5, 11, 15, 38]. Roughly speaking, if the original problem is written in the abstract form

$$\mathcal{L}u = f\,, \tag{1}$$

where $\mathcal{L}$ is a linear operator, the strategy consists in left multiplying equation (1) by an operator $\mathcal{K}$ and solve

$$\mathcal{K}\mathcal{L}u = \mathcal{K}f\,. \tag{2}$$

Now, if $\mathcal{K}\mathcal{L}$ is a compact perturbation of the identity, the condition number of the discretized underlying linear system becomes independent of the mesh size, enabling the efficient use of iterative methods such as GMRES [31]. This is in particular the case when $\mathcal{K}$ is a parametrix of $\mathcal{L}$, which is usually proven using tools from pseudo-differential calculus [36].

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Several strategies, depending on the problem to solve (e.g. Helmholtz or Maxwell equations) have been studied in the literature to propose such operators $\mathcal{K}$, that often turn out to be very effective in practice, when numerical applications are considered. Among those, we would like to emphasize the viewpoint first proposed in [3,4], where, for Helmholtz equation, the authors consider integral formulations of the problem that involve the Dirichlet-Neumann map DN which leads to well-conditioned systems after discretization. Combined with the approximation of DN proposed first in [5] under the form of the square root operator

$$DN \sim \sqrt{-\Delta_\Gamma - k^2}\,, \tag{3}$$

where Δ_Γ stands for the Laplacian operator on the surface Γ, the method (called GCSIE) yields a very impressive reduction of the number of iterations.

However, all the preceding results and theories are limited to smooth surfaces Γ and very little is known when the integral equation is posed on non-smooth domains, such as domains with corners (in 2D), wedges or conical points (in 3D). One of the reasons might be the fact that pseudo-differential calculus is difficult to generalize on such manifolds and the existing theories such as the ones presented in e.g. [24,33,34] do not seem to be adequate for the analysis of such problems.

Nevertheless, attempts to precondition the integral equations that come from the discretization of single-layer or double-layer potentials for Laplace equation on singular domains has started a few years ago [10,15–17,30] in dimension 2 or 3, but for very particular domains such as a straight and then curved segments in 2D and a unit disc in 3D. In most of these works, weighted versions of the usual integral operators are introduced, which enjoy better mapping properties than the standard operators, and the analysis is obtained "by hand" without any explicit use of pseudo-differential calculus.

The aim of this paper and the forthcoming [7] is twofold. Here we describe an analytical framework and a Galerkin method suited to the inversion of these weighted operators, with optimal rates of convergence. On the other hand, we build efficient preconditioners for the linear systems arising from the previous Galerkin method. The complete description and analysis of the method is postponed to [7] and we instead focus in the present paper on the description and the numerical implementation. Preliminary analytical results, that show the optimality of the approach for Laplace problems, are nevertheless included here.

The paper is organized as follows: in the first section, we introduce the notations and the usual boundary integral operators on an open curve. In the second section, we treat the case of the Laplace equation, where it is possible to derive explicit inverses of the weighted layer potentials in terms of square roots of local operators. We then generalize our results to non-zero frequency and non-flat arc in the third section. In the last section, we show the efficiency of this approach on several numerical examples.

2 The scattering problem outside an open curve

Let Γ be a smooth non-intersecting bounded open curve in $\mathbb{R}^2$, and let $k \geq 0$ the wave number. We seek a solution of the Helmholtz equation

$$-\Delta u - k^2 u = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma \tag{4}$$

when one considers furthermore Dirichlet or Neumann boundary conditions, namely

$$u = u_D, \text{ on } \Gamma \tag{5}$$

or

$$\frac{\partial u}{\partial n} = u_N \text{ on } \Gamma \tag{6}$$

respectively. In order for the problem to be well posed, we also need to impose suitable decay at infinity, given by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{7}$$

with $r = |x|$ for $x \in \mathbb{R}^2$. Notice that in equation (6), n stands for a smooth unit vector normal to Γ.

Existence and uniqueness of solutions to the previous problems are guaranteed by the following theorem.

Theorem 1. *[25, 37, 39]. Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (4,5,7) and (4,6,7) both possess a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ.*

For the definition of Sobolev spaces on smooth open curves, we follow [22] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma}) \right\}.$$

This definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover, we also define

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

The solution to (4,5,7) and (4,6,7) can be furthermore expressed in terms of the Green function associated to the problem

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi} \ln|z|, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{8}$$

where H_0 is the Hankel function of the first kind. The expressions involve the so-called single-layer, double-layer and hypersingular potentials that we recall now. The single-layer potential is defined by

$$\forall x \notin \Gamma, \quad \mathcal{S}_k \lambda(x) = \int_\Gamma G_k(x-y)\lambda(y) d\sigma_y \tag{9}$$

and the solution u of the Dirichlet problem (4,5,7) can be expressed as

$$u = \mathcal{S}_k \lambda \tag{10}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the unique solution to

$$S_k \lambda = u_D \,. \tag{11}$$

Here, the operator S_k is defined by $S_k := \gamma \mathcal{S}_k$, where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ [39, Theorem 1.8]. Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k \mu(x) = \int_\Gamma n(y) \cdot \nabla G_k(x-y) \mu(y) d\sigma_y$$

for any smooth function μ defined on γ. The normal derivative of $\mathcal{D}_k \mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \partial_n \mathcal{D}_k$. This latter operator admits the representation for $x \in \Gamma$

$$(N_k \mu)(x) = \lim_{\varepsilon \to 0^+} \int_\Gamma n(y) \cdot \nabla G_k(x + \varepsilon n(x) - y) \mu(y) d\sigma_y. \tag{12}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned} \langle N_k \mu, \nu \rangle \quad = \quad & \int_{\Gamma \times \Gamma} G_k(x-y) \mu'(x) \nu'(y) \\ & - k^2 G_k(x,y) \mu(x) \nu(y) n(x) \cdot n(y) d\sigma_x d\sigma_y \,. \end{aligned} \tag{13}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously [39, Theorem 1.4], and that the solution u to the Neumann problem (4,6,7) can be written as

$$u = \mathcal{D}_k \mu \tag{14}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ solves

$$N_k \mu = u_N \,. \tag{15}$$

It is also known that λ and μ are singular singular near the edges of the arc Γ their respective asymptotics are given by the following proposition.

Proposition 1. *[25, 37, 39]. Near the edges of the arc Γ, λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}} \right),$$

while μ is locally given by

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi.$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$ and C is a constant.

In both cases (Dirichlet or Neumann problems) numerical approximations of the solution can be sought by a suitable discretization of the equations (11) and (15) respectively. This amounts to solve linear systems which turn out to be ill-conditioned in practice, and the aim of the paper is to provide the reader with a formalism that enables to give preconditioned version of them.

3 Laplace equation on a flat segment

We first restrict our attention to the case where Γ is the segment

$$\Gamma = [-1, 1] \times \{0\},$$

and the wavenumber $k = 0$, by considering the equations

$$S_0\lambda = u_D \quad \text{and} \quad N_0\mu = u_N .$$

These problems have been already considered thoroughly in the literature, both in terms of analytical and numerical properties (see for instance [10, 18]). However, to the best of our knowledge, the natural inverses in terms of square roots of local operators have remained unnoticed. In order to derive suitable preconditonners for such equations and their corresponding expressions, we need to introduce the following analytical setting. As it is well known, Chebyshev polynomials of first and second kind play an important role.

3.1 Analytical setting

We introduce the Chebyshev polynomials of first and second kinds [21], respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}},$$

and we call ω the operator $\omega : u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$. We also denote by ∂_x the derivation operator. The Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)\partial_{xx}T_n - x\partial_x T_n + n^2 T_n = 0$$

and

$$(1-x^2)\partial_{xx}U_n - 3x\partial_x U_n + n(n+2)U_n = 0$$

which can be rewritten under the form

$$(\omega\partial_x)^2 T_n = -n^2 T_n , \tag{16}$$
$$(\partial_x\omega)^2 U_n = -(n+1)^2 U_n . \tag{17}$$

(Notice that by $(\partial_x\omega)f$ we mean $\partial_x(\omega f)$.) Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} f^2(x)\sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by $(\cdot,\cdot)_{\frac{1}{\omega}}$ and $(\cdot,\cdot)_{\omega}$ the inner products of $L^2_{\frac{1}{\omega}}$ and L^2_{ω},

$$(u,v)_{\frac{1}{\omega}} := \frac{1}{\pi}\int_{-1}^{1} \frac{u(x)\overline{v(x)}}{\omega(x)}dx\,, \quad (u,v)_{\omega} := \frac{1}{\pi}\int_{-1}^{1} u(x)\overline{v(x)}\omega(x)dx\,.$$

The Chebyshev polynomials satisfy

$$(T_n,T_m)_{\frac{1}{\omega}} = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } m = n = 0 \\ 1/2 \text{ otherwise} \end{cases} \tag{18}$$

and

$$(U_n,U_m)_{\omega} = \begin{cases} 0 \text{ if } n \neq m \\ 1/2 \text{ otherwise} \end{cases} \tag{19}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{20}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by $\hat{u}_n := \dfrac{(u_n,T_n)_{\frac{1}{\omega}}}{(T_n,T_n)_{\frac{1}{\omega}}}$. Similarly, any function $v \in L^2_{\omega}$ can be decomposed along the $(U_n)_{n\geq 0}$ as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by $\check{v}_n := \dfrac{(v,U_n)_{\omega}}{(U_n,U_n)_{\omega}}$. Those properties can be used to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^s \left|\hat{u}_n\right|^2 < +\infty \right\}.$$

Endowed with the scalar product

$$(u,v)_{T^s} = \hat{u}_0\overline{\hat{v}_0} + \frac{1}{2}\sum_{n=1}^{+\infty}(1+n^2)^s \hat{u}_n\overline{\hat{v}_n},$$

T^s is a Hilbert space for all $s \geq 0$. Similarly, we set

$$U^s = \left\{ u \in L^2_{\omega} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^s \left|\check{u}_n\right|^2 \right\}$$

which is a Hilbert space for the scalar product

$$(u,v)_{U^s} = \frac{1}{2}\sum_{n=0}^{+\infty}(1+(n+1)^2)^s \check{u}_n\overline{\check{v}_n}.$$

One can extend the definition of T^s ad U^s for $s \in \mathbb{R}$, in which case they form interpolating scales of Hilbert space, see [7] for details. For $s = \pm\frac{1}{2}$, those spaces have been analyzed (with different notation) e.g. in [17] and verify

$$T^{-1/2} = \omega \tilde{H}^{-1/2}(-1,1), \quad T^{1/2} = H^{1/2}(-1,1)\,, \tag{21}$$

$$U^{-1/2} = H^{-1/2}(-1,1), \quad U^{1/2} = \frac{1}{\omega}\tilde{H}^{1/2}(-1,1)\,. \tag{22}$$

Denoting by $T^\infty = \cap_{s\geq 0} T^s$ and similarly for U^∞, it is shown in [7] that

Lemma 1.

$$T^\infty = U^\infty = C^\infty([-1,1])\,.$$

3.2 Single layer equation

In the case of Dirichlet condition, we seek a solution to the equation $S_0\lambda = g$, with $\lambda \in \tilde{H}^{-1/2}(\Gamma)$, that is

$$-\frac{1}{2\pi}\int_{-1}^{1} \log|x-y|\lambda(y) = g(x), \quad \forall x \in (-1,1)\,. \tag{23}$$

This equation is sometimes called "Symm's integral equation" and its resolution has received a lot of attention in the 1990's. Numerical methods, using both the collocation and Galerkin paradigms have been presented and analyzed [6, 35, 41–43].

Our analysis lies on the following formula. (For a proof, see for example [21] Theorem 9.2. Note that this is also the main ingredient in several connected works, such as [10] and [18].)

Lemma 2. *For all $n \in \mathbb{N}$, we have*

$$-\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln|x-y|}{\sqrt{1-y^2}} T_n(y)dy = \sigma_n T_n(x)$$

where

$$\sigma_n = \begin{cases} \dfrac{\ln(2)}{2} & \text{if } n = 0 \\ \\ \dfrac{1}{2n} & \text{otherwise.} \end{cases}$$

Using the decomposition of g and of the logarithmic kernel on the basis $(T_n)_n$, we see that the solution λ to equation (23) admits the following expansion

$$\lambda(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{+\infty} \frac{\hat{g}_n}{\sigma_n} T_n(x)\,. \tag{24}$$

We deduce the following well-known fact:

Corollary 1. *If the data g is in $C^\infty([-1,1])$, the solution λ to the equation*

$$S_0\lambda = g$$

is of the form

$$\lambda(x) = \frac{\alpha(x)}{\sqrt{1-x^2}}$$

with $\alpha \in C^\infty([-1,1])$.

Proof. Let $\alpha(x) = \sqrt{1-x^2}\,\lambda(x)$ where λ is the solution of $S_0\lambda = g$. By Lemma 1, if $g \in C^\infty([-1,1])$, then $g \in T^\infty$, and by equation (24),

$$\hat{\alpha}_n = \frac{\hat{g}_n}{\sigma_n}\,,$$

from which we deduce that α also belongs to $T^\infty = C^\infty([-1,1])$. □

Following [10], we introduce the weighted single layer operator as the operator that appears in Lemma 2.

Definition 2. *Let $S_{0,\omega}$ be the weighted single layer operator defined by*

$$S_{0,\omega} : \alpha \in C^\infty([-1,1]) \longrightarrow -\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln|x-y|}{\omega(y)}\alpha(y)dy\,.$$

To obtain the solution of (23), we thus solve

$$S_{0,\omega}\alpha = u_D\,, \tag{25}$$

and let $\lambda = \frac{\alpha}{\omega}$, which indeed belongs to $\tilde{H}^{-1/2}$ by (21). From the previous considerations, the inverse for $S_{0,\omega}$ can be derived by building an operator R such that $RT_n = \frac{1}{\sigma_n}T_n$. It turns out that such an operator R has been characterized in [17,40], through explicit variational forms in closed forms (see also the recent paper [16] for the case of the unit disk in $\mathbb{R}^3$).

Here, we give an alternative form for the operator R, that we will then generalize for a non-zero frequency.

Theorem 2. *For any s, the operator $-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0$ is bicontinuous from T^{s+2} to T^s and*

$$S_{0,\omega}^2 = \frac{1}{4}\left(-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0\right)^{-1}\,.$$

Corollary 2. *The operator $S_{0,\omega}$ being non negative, its inverse can thus be equivalently expressed as*

$$S_{0,\omega}^{-1} = R := 2\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$$

where the square root of the operator is defined using standard spectral theory.

Proof. We simply notice that for all $n \in \mathbb{N}$, $(\omega\partial_x)T_n = -n^2 T_n$. Therefore

$$\left(-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0\right) T_n = \begin{cases} n^2 T_n \text{ if } n \neq 0 \\ \frac{1}{\ln(2)^2} \text{ otherwise} \end{cases} = \frac{1}{4\sigma_n^2} T_n \,.$$

□

Since the operator $\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$ is the inverse of $S_{0,\omega}$, in particular, it can be used as an efficient preconditioner for the weighted integral equation (25).

3.3 Hypersingular equation

We now turn our attention to the equation

$$N_0\mu = g\,. \tag{26}$$

Similarly to the previous section and following [10], we consider the weighted version of the hypersingular operator $N_{0,\omega} := N_0\omega$ defined by

$$N_{0,\omega}\mu = \lim_{\varepsilon\to 0} \int_{-1}^{1} n(y)\cdot\nabla G(x+\varepsilon n(x)-y)\sqrt{1-y^2}dy\,.$$

We can get the solution to equation (26) by solving

$$N_{0,\omega}\beta = u_N, \tag{27}$$

and letting $\mu = \omega\beta$. We now show that $N_{0,\omega}$ can be analyzed using this time the spaces U^s.

Lemma 3. *For any β, β', one has*

$$\langle N_{0,\omega}\beta, \beta'\rangle_\omega = \langle S_{0,\omega}\omega\partial_x\omega\beta, \omega\partial_x\omega\beta'\rangle_{\frac{1}{\omega}}\,.$$

Proof. We use the well-known integration by part formula

$$\langle N_0 u, v\rangle = \langle S_0\partial_x u, \partial_x v\rangle\,,$$

valid when u and v are regular enough and vanish at the extremities of the segment. For a smooth β, we thus have

$$\langle N_0(\omega\beta), (\omega\beta')\rangle = \langle S_0\partial_x(\omega\beta), \partial_x(\omega\beta')\rangle$$

which obviously implies the claimed identity. □

Lemma 4. *For all $n \in \mathbb{N}$, we have*

$$N_{0,\omega}U_n = \frac{n+1}{2}U_n.$$

Proof. From the identity $\partial_x T_{n+1} = (n+1)U_n$ and Equation (16) we obtain

$$\omega\partial_x\omega U_n = -(n+1)T_{n+1}.$$

Therefore, by Lemma 3

$$\begin{aligned}\langle N_{0,\omega}U_m, U_n\rangle_\omega &= (n+1)(m+1)\langle S_{0,\omega}T_{m+1}, T_{n+1}\rangle_{\frac{1}{\omega}} \\ &= \delta_{m=n}\frac{n+1}{2}. \end{aligned} \tag{28}$$

□

Moreover, recalling that $-(\partial_x\omega)^2 U_n = (n+1)^2 U_n$ leads to the following Theorem that expresses again the operator N_ω as a square root operator.

Theorem 3.

$$N_{0,\omega} = \frac{1}{2}\sqrt{-(\partial_x\omega)^2}\,. \tag{29}$$

4 Helmholtz equation

In this section, we aim at generalizing the preceding analysis to the case of Helmholtz equation on $\mathbb{R}^2 \setminus \Gamma$ with $\Gamma = [-1,1]\times\{0\}$, based on the explicit formulas presented in the previous section. Recall the definition of the single layer and hypersingular operators, S_k and N_k, given in (9) and (12), and the integral equations (11) and (15) for the Dirichlet and Neumann problems respectively. As before, let $S_{k,\omega} := S_k\frac{1}{\omega}$ and $N_{k,\omega} := N_k\omega$. The following commutation holds:

Theorem 4.

$$S_{k,\omega}\left[-(\omega\partial_x)^2 - k^2\omega^2\right] = \left[-(\omega\partial_x)^2 - k^2\omega^2\right]S_{k,\omega},$$

Proof. Since $(\omega\partial_x)^2$ is self adjoint and symmetric (with respect to the bilinear form $(\cdot,\cdot)_{\frac{1}{\omega}}$), we have

$$\left(S_{k,\omega}(\omega\partial_x)^2 u\right)(x) = \int_{-1}^{1}\frac{(\omega_y\partial_y)^2\left[G_k(x-y)\right]u(y)}{\omega(y)},$$

where we use the notation ω_y and ∂_y to emphasize the dependence in the variable y. Thus,

$$\left(\left(S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega}\right)u\right)(x) = \int_{-1}^{1}\frac{D_k(x,y)u(y)}{\omega(y)},$$

where $D_k(x,y) := \left[(\omega_y\partial_y)^2 - (\omega_x\partial_x)^2\right]\left[G_k(x-y)\right]$. A simple computation leads to

$$D_k(x,y) = \partial_{xx}G_k(x-y)(\omega_y^2 - \omega_x^2) + \partial_x G_k(x-y)(y+x).$$

Since G_k is a solution of the Helmholtz equation, we have for all $(x \neq y) \in \mathbb{R}^2$

$$\partial_x G_k(x-y) = (y-x)(\partial_{xx}G_k(x-y) + k^2 G(x-y)),$$

thus

$$D_k(x,y) = \partial_{xx} G_k(x-y)\left(\omega_y^2 - \omega_x^2 + y^2 - x^2\right) + k^2(y^2-x^2)G_k(x-y).$$

A careful analysis shows that no Dirac mass appears in the previous formula. Note that $y^2 - x^2 = \omega_x^2 - \omega_y^2$ so the first term vanishes and we find

$$S_{k,\omega}(\omega\partial_x)^2 - (\omega\partial_x)^2 S_{k,\omega} = k^2\left(\omega^2 S_{k,\omega} - S_{k,\omega}\omega^2\right)$$

as claimed. □

There also holds the following identity:

$$N_{k,\omega}\left[-(\partial_x\omega)^2 - k^2\omega^2\right] = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] N_{k,\omega}\,,$$

however, the proof of this commutation is quite heavy. We chose to not include it in the present work for the sake of conciseness.

Those commutations imply that the operators $S_{k,\omega}$ and $N_{k,\omega}$ share the same eigenvectors as, respectively, $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$ and $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]$. The eigenfunctions of the operator $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]$ thus provide us with a diagonal basis for $S_{k,\omega}$. They are the solutions to the differential equation

$$(1-x^2)\partial_{xx}y - x\partial_x y - k^2\omega^2 y = \lambda y\,.$$

Once we set $x = \cos\theta$, $\tilde{y}(\theta) = y(x)$, $q = \frac{k^2}{4}$, $a = \lambda + 2q$, $\tilde{y}$ is a solution of the standard Mathieu equation

$$\tilde{y}'' + (a - 2q\cos(2\theta))\tilde{y} = 0\,. \tag{30}$$

There exists a discrete set of values $a_{2n}(q)$ for which this equation possesses even and 2π periodic solutions, which are known as the Mathieu cosine functions, and usually denoted by ce_n. Here, we use the notation ce_n^k to emphasize the dependency in the parameter $k = \sqrt{2q}$ of those functions. The normalization is taken as

$$\int_{-\pi}^{\pi} \mathrm{ce}_n^k(\theta)^2 d\theta = \pi.$$

The Mathieu cosine functions also satisfy

$$\int_{-\pi}^{\pi} \mathrm{ce}_n^k(\theta)\mathrm{ce}_m^k(\theta) = \pi\delta_{m,n}$$

so that any even 2π periodic function in $L^2(-\pi,\pi)$ can be expanded along the functions ce_n, with the coefficients obtained by orthonormal projection. Setting

$$T_n^k := \mathrm{ce}_n^k(\arccos(x)),$$

in analogy to the zero-frequency case, we have

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] T_n^k = \lambda_{n,k}^2 T_n^k.$$

For large n, using the general results from the theory of Hill's equations (see e.g. [27, eq. (21), (28) and (29)]), we have the following asymptotic formula for $\lambda_{n,k}$:

$$\lambda_{n,k}^2 = n^2 - \frac{k^4}{16n^2} + o\left(n^{-2}\right).$$

The first commutation established in Theorem 4 implies that Mathieu cosine functions are also the eigenfunctions of the single-layer operator. (An equivalent statement is given in [9, Thm 4.2], if we allow the degenerate case $\mu = 0$.)

A similar analysis can be applied to the hypersingular operator. The eigenfunctions of $\left[-(\partial_x \omega)^2 - k^2\omega^2\right]$ are given by

$$U_n^k := \frac{\mathrm{se}_n^k(\arccos(x))}{\omega(x)}$$

where se_n^k are the so-called Mathieu sine functions, which also satisfy the Mathieu differential equation (30), but with the condition that they are 2π periodic and odd functions.

All the previous considerations highly suggest the following theorem:

Theorem 5. *There exists a compact operator K from T^s to T^{s+2} such that*

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega}^2 = \frac{I_d}{4} + K,$$

and a compact operator K' from U^s to U^{s-2} such that

$$N_{k,\omega}^2 = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] + K'.$$

The proof of this theorem, requires the introduction of additional analytic tools that are out of the scope of this introductory paper. It is thus omitted in the present work and given in full details in [7]. Nevertheless, Theorem 5 gives a generalization to Helmholtz equation of the result obtained precedingly for Laplace equation, namely, the operator $\left[-(\omega\partial_x)^2 - k^2\omega^2\right]^{1/2}$ is expected to be a good preconditioner for $S_{k,\omega}$, as well as $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]^{-1/2}$ for $N_{k,\omega}$.

Finally, we note that in the more general case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, the previous result extends, replacing ∂_x by ∂_τ the tangential derivative on Γ, $S_{k,\omega}$ and $N_{k,\omega}$ respectively by $S_{k,\omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}$, and $N_{k,\omega_\Gamma} := N_k\omega_\Gamma$, and ω by ω_Γ. Here ω_Γ is defined by $\omega_\Gamma(x) = \frac{|\Gamma|}{2}\omega(r(x))$, where $|\Gamma|$ is the length of the curve and $r : [-1, 1] \to \Gamma$ is such that for all x, $|\partial_x r(x)| = \frac{|\Gamma|}{2}$.

5 Numerical results

5.1 Galerkin setting

To solve numerically the integral equations presented earlier, various methods have been described and analyzed in the literature. The standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergence (see for example [32, Chap. 4]). Several strategies have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips (h-BEM) or increase the polynomial order in the trial space (p-BEM). The combination of the last two methods, known as h-p BEM, can achieve an exponential rate

of convergence with respect to the dimension of the trial space, see [29] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example in [10], and some results exist for piecewise linear functions in the colocation setting [12].

Here, we describe a simple Galerkin setting, suited to the spaces T^s and U^s. We use simple piecewise affine functions defined on a non-uniform mesh, which is refined towards the edges of the curve. More precisely, let $-1 = x_0 < x_1 < \cdots < x_N = 1$ and let $\theta_i := \arccos(x_i)$. We choose the points x_i such that $(\theta_i)_{0\leq i\leq N}$ are equispaced, that is to say $\theta_i = ih$ with $h = \pi/N$. The nodes of the mesh are then set to $X_i = r(x_i)$, where r is a smooth parametrization of the curve Γ. This turns out to be analogous to a graded mesh with a grading parameter equal to 2 which means that, near the edge, the width of the $i-th$ interval is approximately $(ih)^2$. Notice that it is known that this modification alone, i.e. using the h-BEM method with a polynomial order $p = 1$, is not sufficient to get an optimal rate of convergence. Indeed, it can be shown that it only leads to a convergence rate in $O(h)$ for the L^2 norm (cf. [29, Theorem 1.3]) instead of the expected $O(h^2)$ behavior.

The key ingredient, beside the graded mesh, to recover optimal convergence is to use a weighted L^2 scalar product (with weight $\frac{1}{\omega}$ or ω depending on the considered equation), in order to assemble the operators in their natural spaces. We state here the orders of convergence that one gets with this new method, and refer again the reader to [7] for rigorous proofs. To keep the exposition simple, we also restrict our presentation to the case where $\Gamma = [-1, 1] \times \{0\}$ and $k = 0$.

Dirichlet problem For the of resolution the single-layer equation (23) we use a variational formulation of (25) to compute an approximation α_h of α. Namely, let V_h the Galerkin space of (discontinuous) piecewise affine functions defined on the mesh $(x_i)_{0\leq i\leq N}$ defined above, and α_h the unique solution in V_h to

$$(S_{0,\omega}\alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (u_D, \alpha'_h)_{\frac{1}{\omega}}\,, \quad \forall \alpha'_h \in V_h\,.$$

We then compute $\lambda_h = \frac{\alpha_h}{\omega}$. Using the notation C to denote any constant that does not depend on the relevant parameters, we then have

Theorem 6. *(see [7]). If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2}\,\|\omega\lambda\|_{T^s} \leq Ch^{s+1/2}\,\|u_D\|_{T^{s+1}}\,.$$

In particular, when u_D is smooth, the solution $\alpha = \omega\lambda$ belongs to T^∞, and we get the optimal rate of convergence of the error in $O(h^{5/2})$.

Neumann problem. For the numerical resolution of (26), we use a variational form for equation (27) to compute an approximation β_h of β, and solve it using a Galerkin method with continuous piecewise affine functions. Introducing W_h the space of continuous piecewise affine functions on the mesh defined

by the points $(x_i)_{0\leq i\leq N}$, we denote by β_h the unique solution in W_h to the variational equation:

$$(N_\omega \beta_h, \beta_h')_\omega = (u_N, \beta_h')_\omega \,, \quad \forall \beta_h' \in W_h. \tag{31}$$

Then, the proposed approximation for μ, given by $\mu_h = \omega\beta_h$, satisfies the following error estimate.

Theorem 7. *(see [7]). If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \left\|\frac{\mu}{\omega}\right\|_{U^s} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}} \,.$$

In fact, in [7], we prove that, for the Dirichlet problem,

$$\forall s,t \in \left[-\frac{1}{2}, 2\right], \quad \|\alpha - \alpha_h\|_{T^t} \leq Ch^{s-t} \|\alpha\|_{T^s} \,.$$

The case $t = 0$ gives a convergence rate in $O(h^s)$ for the weighted L^2 error $e_1(h) = \|\alpha - \alpha_h\|_{L^2_{\frac{1}{\omega}}}$, where s is such that $\alpha \in T^s$. A numerical validation is shown in Figure 1. We consider here the case where $\alpha_1(x) = \omega(x) \in T^s$ for $s < \frac{3}{2}$ and $\alpha_1 \notin T^{3/2}$ and $\alpha_2 = \omega(x)^3 \in T^2$. We indeed observe the respective rates $O(h^{3/2})$ and $O(h^2)$ predicted by the theory.
Similarly for the Neumann case it is shown in [7] that

$$\forall s,t \in [0,2] \,, \quad \|\beta - \beta_h\|_{U^t} \leq Ch^{s-t} \|\beta\|_{U^s} \tag{32}$$

A numerical validation is shown in Figure 2 in the case $\beta = U_2 \in U^\infty$.The error in L^2_ω and U^1 norms are plotted, and the rates of convergence observed predicted by the theory are recovered in practice.

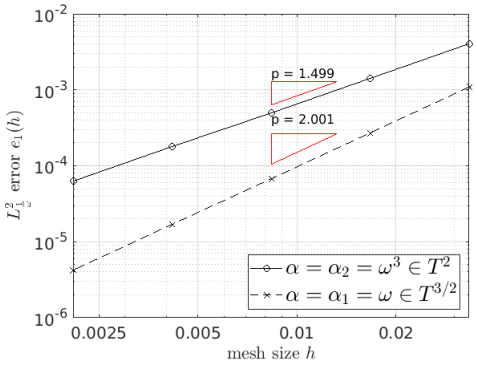

Figure 1: Effective order of convergence of the approximation of the solution α to (25) by the weighted Galerkin method. The blue data correspond to a case where $\alpha \in T^s$ for all $s < \frac{3}{2}$ but $\alpha \notin T^{3/2}$. In this case, the theory predicts a $O(h^{3/2})$ rate of convergence, which is what we observe in practice. The red data corresponds to a case where $\alpha \in T^s$. In this case, a $O(h^2)$ rate of convergence is predicted by the theory and indeed observed here.

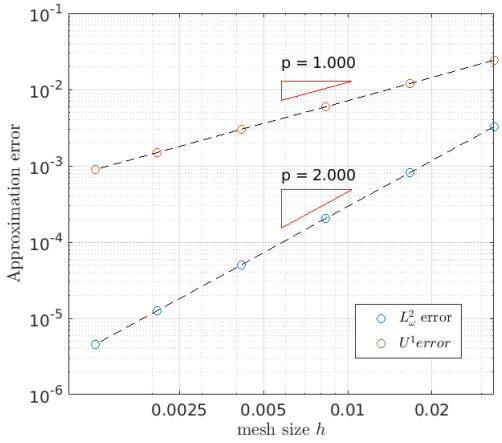

Figure 2: Effective order of convergence of the approximation of the solution β to (27) by the weighted Galerkin method. The second member is taken as $u_N = U_2 \in U^\infty$, and the solution is therefore $\beta = \frac{2}{3}U_2$. We plot in blue the L^2_ω error $\|\beta - \beta_h\|_\omega$ and in red the U^1 error $\|\beta - \beta_h\|_{U^1}$. Here the theory predicts the $O(h)$ order of convergence for the second curve. The $O(h^2)$ order of the first curve is not yet explained by our theory.

5.2 Preconditioning the linear systems

Let X_h the considered finite element space ($X_h = V_h$ or W_h), and $(\phi_i)_i$ the basis functions. For an operator A, we denote by $[A]_p$ the Galerkin matrix of the operator for the relevant weight $p(x) = \frac{1}{\omega(x)}$ or $\omega(x)$, defined by

$$[A]_{p,ij} = \int_\Gamma (A\phi_j)(x)\phi_i(x)p(x)\,dx\,.$$

When the operator BA is a compact perturbation of the identity (either in T^s or U^s) then, following [36], we precondition the linear system $[A]_p\, x = b$ by the matrix $[I_d]_p^{-1}\,[B]_p\,[I_d]_p^{-1}$, which amounts to solve

$$[I_d]_p^{-1}\,[B]_p\,[I_d]_p^{-1}\,[A]_p x = [I_d]_p^{-1}\,[B]_p\,[I_d]_p^{-1}\,b\,.$$

When B is the inverse of a local operator C, then it may be more convenient to compute $[C]_p$, and solve instead

$$[C]_p^{-1}\,[A]_p x = [C]_p^{-1}\,b\,.$$

The preconditioners introduced in this work are in the form of square roots of local operators. More precisely, we introduced two preconditioners P_1 and P_2 with

$$\begin{aligned} P_1(k) &= \left(-(\omega\partial_x)^2 - k^2\omega^2\right)^{1/2} \\ P_2(k) &= \left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{-1/2} \end{aligned}$$

For the second equation, we rewrite

$$P_2(k) = \left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{-1}\left(-(\partial_x\omega)^2 - k^2\omega^2\right)^{1/2},$$

which brings us back to computing the square root of a sparse matrix. When the frequency is 0, we use the method exposed in [14]. When the frequency is non-zero, the previous method fails since the spectrum of the matrix contains negative values. In [5], a method involving a Padé approximation of the square root, with a rotated branch cut, is used to compute the matrix of an operator of the form $\sqrt{X - k^2 I_d}$ where X is a positive definite operator. This method gives excellent results in our context when using $X = -(\partial\omega_x)^2 + k^2\left(I_d - \omega^2\right)$. Specifically, we build a rational approximation of the function $X \mapsto \sqrt{X - k^2}$ in the form

$$\sqrt{X - k^2} = a_0 + \sum_{i=0}^{N_p} \frac{a_i}{b_i + X}\,.$$

We then take

$$\left[\sqrt{X - k^2 I_d}\right]_p \approx a_0[I_d]_p + \sum_{i=0}^{N_p} a_i\left(b_i[I_d]_p + [X]_p\right)^{-1}\,.$$

5.3 Numerical results

All the numerical results exposed here are obtained on a personal laptop running on an eight cores intel i7 processor with a clock rate of 2.8GHz. The Galerkin method has been implemented in the language Matlab R2018.

Flat segment, Laplace-Dirichlet problem. In Table 1, we report the number of iterations for the numerical resolution of the Laplace problem (25) by the method detailed above, in section 5. Two cases are considered, first without any preconditioner, and then with a preconditioner given by $[I_d]_{\frac{1}{\omega}}^{-1}[B]_{\frac{1}{\omega}}[I_d]_{\frac{1}{\omega}}^{-1}$ where $[B]_{\frac{1}{\omega}}$ is the Galerkin matrix of the operator $\sqrt{-(\omega\partial_x)^2 + \frac{1}{\ln(2)^2}\pi_0}$. The right hand side in (25) is chosen as $u_D(x) = (x^2 + 0.001)^{-1/2}, x \in [-1, 1]$. A graph of the residual along the iterations is given in Figure 3 for a graded mesh with 1600 node points.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
50	7	0.11	35	0.21
200	7	0.14	53	0.55
800	7	0.29	76	2.0
3200	7	0.95	107	9.5

Table 1: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.

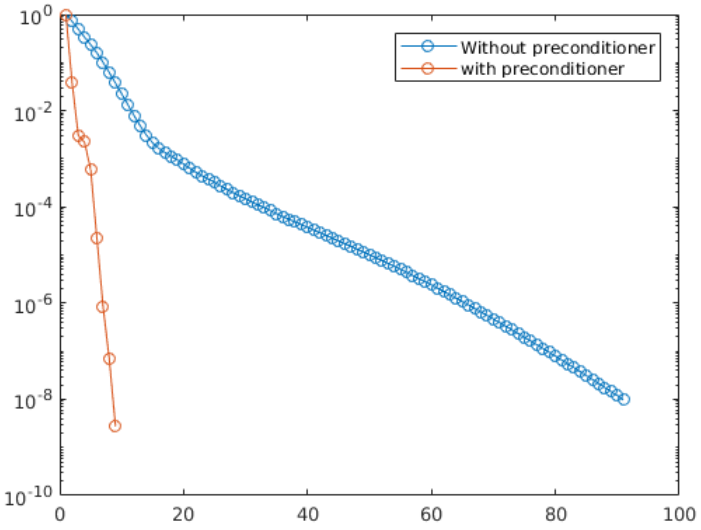

Figure 3: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N = 1600$.

Flat segment, Laplace-Neumann problem. For the Neumann problem (27), we also report in Table 2 the number of iterations for the numerical resolution with and without the preconditioner given by $[I_p]_\omega^{-1}[C]_\omega[I_p]_\omega^{-1}$ where $[C]_\omega$ is the Galerkin matrix of the operator $\sqrt{-(\partial_x\omega)^{-2}}$. The right hand side in (27) is chosen as $u_N(x) = (x^2+0.001)^{1/2}, x \in [-1,1]$. The decay of the residual along the iterations is shown in Fig. 4.

	with Prec.		without Prec.	
N	n_{it}	t(s)	n_{it}	t(s)
50	4	0.09	50	0.31
200	4	0.12	200	2.0
800	4	0.56	799	30
3200	4	17.7	3007	630

Table 2: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.

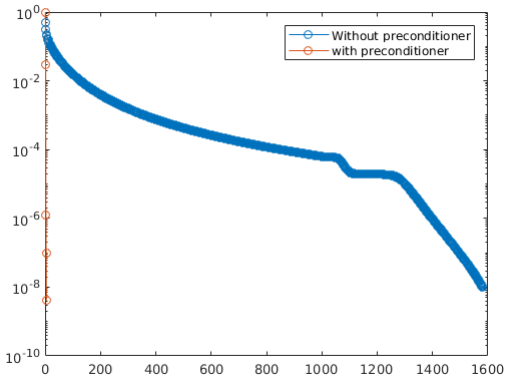

Figure 4: Number of iteration in the resolution of the hypersingular integral equation with a mesh of size $N = 1600$. The importance of preconditioning in this case is even more spectacular than in the case of the single-layer equation.

We observe in both Dirichlet and Neumann cases a drastic decay of the number of iteration which justifies the approach. We also see that, as expected, the number of iterations obtained with the preconditioned version does not depend on the mesh.

We now turn our attention to Helmholtz equation. In each case, in order to fully resolve the frequency, the number of segments in the discretization is set to $N \approx 10k$, where $k = \frac{\pi}{\lambda}$ is the wavenumber. In the GMRES iteration, we require a relative residual below 10^{-8}.

Flat segment, Helmholtz-Dirichlet problem. In Table 3 we report the number of GMRES iterations for the numerical resolution of Equation (11) on the segment $\Gamma = [-1, 1] \times \{0\}$, when the linear system is preconditioned by the operator $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$, as compared to the case where no preconditioner is used. We take, for the Dirichlet data, the plane wave $u_D(x) = e^{ikx}$. We also provide, in Figure 5, the value of the relative residual in the GMRES method along the iterations, with and without preconditioner, for a problem with $L = 800\lambda$. As before, we see that the number of iterations needed to reach a given precision decreases significantly but, this time, we observe a very slight increase with respect to the wavenumber.

	with Prec.		without Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	8	0.1	73	0.28
200	10	1.3	116	17
800	15	34	148	300

Table 3: Number of iterations and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.

en maillage à fréquence fixée, c'est bien indépentdant donc j'ai remplacé with respect to "mesh size" par "wavenumber"

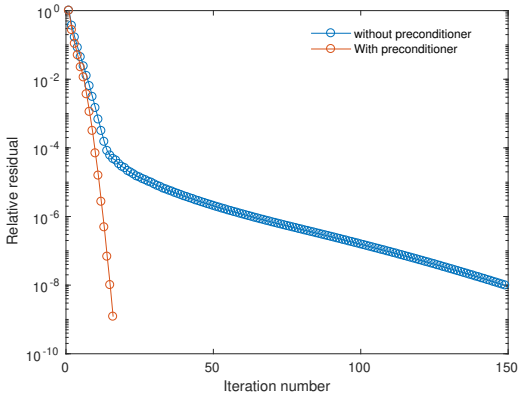

Figure 5: Number of iteration in the resolution of the single layer integral equation with a mesh of size $N \approx 3700$, $L = 800\lambda$.

Flat segment, Helmholtz-Neumann problem. Finally, to end up this validation stage, we run the same numerical comparisons, this time solving (15) and considering the preconditioning operator $\left(-(\partial_x \omega)^2 - k^2\omega^2\right)^{-1/2}$. Results are given in Table 4 for different meshes and in Figure 6 for the evolution of the residual along the iterations. Huge differences, btoh in time and number of iterations are shown in favor of the preconditioned system.

	with Prec.		without Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	8	0.08	785	9.4
200	10	3.6	> 2000	$>$ 2min
800	17	73	> 2000	$>$ 2min

Table 4: Number of iteration and time needed for the numerical resolution of (25) using Galerkin finite elements with and without preconditioner.

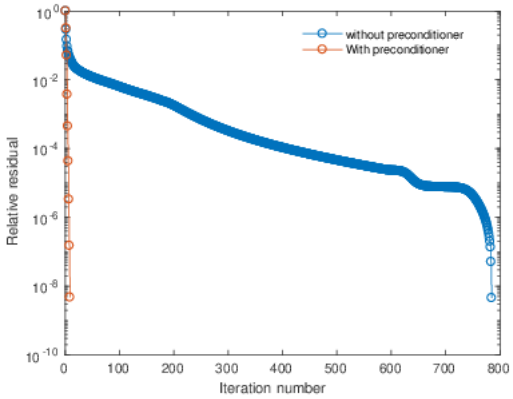

Figure 6: Number of iteration in the resolution of Hypersingular integral equation with a mesh of size $N \approx 800$, $L = 50\lambda$.

Non-flat arc. Here, we also report numerical results when the curve is a portion of spiral (see Figure 7), for both boundary conditions. This shows that the preconditioning strategy is also efficient in presence of non-zero curvature.

	With prec.		Without prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	23	0.6	785	9.4
200	27	9	> 2000	$>$ 2min
800	40	35	> 2000	$>$ 2min

Table 5: Computing times and number of iterations for the spiral-shaped arc.

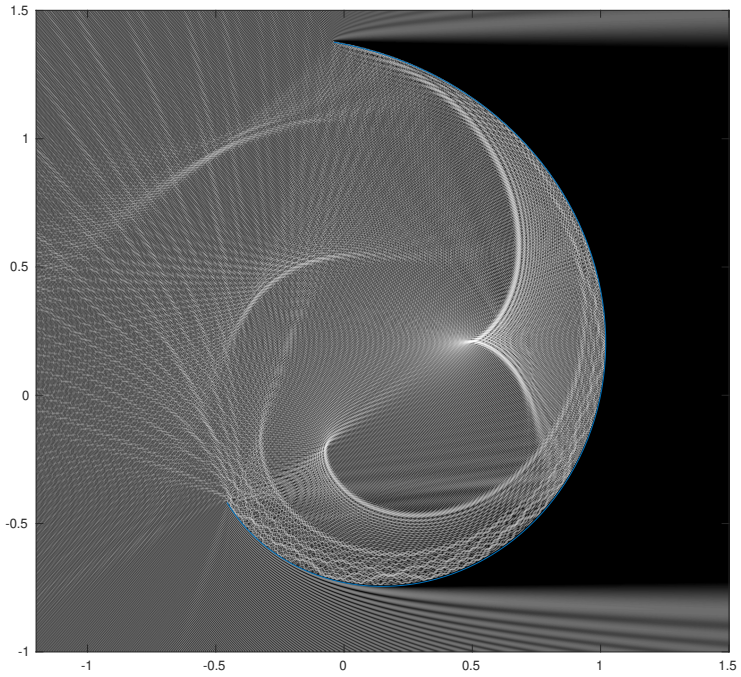

Figure 7: Sample diffraction pattern (Dirichlet boundary conditions) with left to right incidence for an arc of spiral of size $L = 800\lambda$. After the resolution of the integral equation, the computation of the image is accelerated by the EBD method [8].

Comparison with the generalized Calderon relations. Eventually, we test the idea that was presented in [10], namely to use $S_{k,\omega}$ and $N_{k,\omega}$ as mutual preconditioners. This alternative method is also very efficient in our numerical setting (here, we use simple piecewise affine functions, whereas in [10], spectral discretization with trigonometric polynomials is used). We report in Table 6 the number of iterations and computing times for the Neumann problem with an angle of incidence $\frac{\pi}{4}$ on the flat segment. The number of iterations is comparable for both methods. However, the matrix-vector product time in our method is

significantly slower as it only involves sparse operators. This leads to a faster resolution of the linear system.

	Calderon Prec.		Square root Prec.	
L/λ	n_{it}	t(s)	n_{it}	t(s)
50	14	0.1	8	0.1
200	15	7.5	11	3.6
800	16	130	15	70

Table 6: Number of iteration and time needed for the numerical resolution of (11) using Galerkin finite elements with and without preconditioner.

6 Conclusion

We have presented a new approach for the preconditioning of integral equations coming from the discretization of wave scattering problems in 2D by open arc domains. The methodology is very effective and proven to be optimal for Laplace problems on straight segments. It has three advantages:

- It generalizes the formulas mainly proposed in [5] for regular domains, which is only modified by a suitable weight.
- We can show that one can recover optimal error estimates, provided that the mesh is suitably graded near the edges.
- Eventually, a novel pseudo-differential approach, adapted to the corner singularities that appear in the problem is proposed. It is sketched in the present paper, but given in full details in [7].

We deeply believe that the methodology opens new perspectives for such problems. First, a generalization to specific 3D scattering problem, e.g. by a flat disc seems a simple generalization. We plan to extend the results presented here to such problems in the very near future. Second, the strategy that we used here seems very likely to be extended to the half line and hopefully to 2D sectors, giving, on the one hand a new pseudo-differential analysis more suitable than classical ones (see e.g. [24, 28, 33, 34]) for handling Helmholtz-like problems on singular domains, and, on the other hand, a completely new preconditioning technique adapted to the treatment of BEM operators on domains with corners or wedges in 3D. Eventually, the weighted square root operators that appeared in the present context might well be generalized to give suitable approximation of the exterior Dirichlet to Neumann map for the Helmholtz equation which is of particular importance in e.g. domain decomposition methods. Having such approximations might therefore lead to better methods in that context too.

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