Wavelet approximation methods for pseudodifferential equations II: Matrix compression and fast solution

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Abstract

This is the second part of two papers which are concerned with generalized Petrov–Galerkin schemes for elliptic periodic pseudodifferential equations in $\mathbb{R}^n$. This setting covers classical Galerkin methods, collocation, and quasi-interpolation. The numerical methods are based on a general framework of multiresolution analysis, i.e. of sequences of nested spaces which are generated by refinable functions. In this part, we analyse compression techniques for the resulting stiffness matrices relative to wavelet-type bases. We will show that, although these stiffness matrices are generally not sparse, the order of the overall computational work which is needed to realize a certain accuracy is of the form $O(N(\log N)^b)$, where N is the number of unknowns and $b \geq 0$ is some real number.

Keywords: Periodic pseudodifferential equations, pre-wavelets, biorthogonal wavelets, generalized Petrov–Galerkin schemes, wavelet representation, atomic decomposition, Calderón–Zygmund operators, matrix compression, error analysis.

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1. Introduction

In [17], we have proposed and analysed a rather general setting for the numerical solution of periodic pseudodifferential equations by means of generalized Petrov–Galerkin schemes. In particular, collocation and classical Galerkin methods are covered as special cases. These schemes are based on sequences of shift-invariant nested spaces generated by a single *refinable* function. We were able to characterize stability of these methods for the general case of variable symbols in terms of simple conditions on the Fourier transform of the generating refinable function and to estimate the convergence of the schemes. An essential ingredient of our analysis was a detailed information about the local approximation behavior of the various projection operators related to the numerical schemes. This knowledge will again play an important role in our present investigation. The objective of this paper is now to explore possibilities of efficiently solving the systems of linear equations induced by the above-mentioned Petrov–Galerkin schemes. The central problem is that the corresponding stiffness matrices are in general not sparse but full, a case which is, for instance, typically encountered for the special situation of boundary element methods.

Our approach is motivated by the recent interesting and intriguing paper by Beylkin et al. [4] (see also [2]). There, the key is to use matrix representations based on *wavelet* bases. Estimates for the decay of the corresponding matrix entries then lead to efficient approximate matrix vector multiplication. The central theme of the present paper is similar in that we investigate compression of stiffness matrices. It differs, however, in that we consider a much wider class of wavelet-type bases, and a larger class of numerical schemes covering Galerkin–Petrov and collocation schemes. While [4] treats only operators of order zero, we attempt to cover operators of positive and negative order as well, which will turn up a number of essential differences. More importantly, our goal here is to go beyond estimating only local truncation errors, but to establish for a possibly general framework rigorous convergence estimates for the final numerical approximations resulting from the compressed schemes in comparison with the exact solutions. To accomplish this, one needs the characterization of stability of the respective schemes established in [17] and stability of the compressed schemes to be established here. Since the present setting covers also the situation where boundary element methods apply, one should mention schemes like panel clustering [24], developed especially for this case. One hardly expects that the general schemes discussed here could do better in practice than such special schemes. But again stability results and hence convergence estimates are not, to our knowledge, available yet for panel clustering. The same refers to the multigrid approach in [6], which is closely related to the special case of collocation. At any rate, for the present setting we will establish that fixed prescribed accuracy of approximate solutions can be obtained at the expense of $O(N)$ operations, where N is the number of unknowns. Moreover, asymptotic error estimates of optimal order will be shown to hold when allowing an additional logarithmic factor. One should emphasize that we do not require any explicit knowledge about the structure of the kernels of the operators under consideration. Instead, our approach makes only use of asymptotic properties determining a rather wide class of operators covered by our analysis.

On the other hand, it is clear that one price we have to pay for such detailed information is to restrict the analysis to a class of model problems with regard to periodic boundary conditions. However, much of the analysis can be seen to remain valid under much more general assumptions, namely, everything based on local approximation results and therefore essentially the complete stability analysis as well as the estimates of the stiffness matrices. On the other hand, the periodic setting provides a convenient framework for establishing rigorous convergence estimates in the present generality, although for certain things it may occasionally even require a little more effort to work in the periodic case. Nevertheless, we find it worthwhile to stay on firm ground and attempt to give a possibly complete and rigorous analysis of the interplay of the essential ingredients involved in problems of this type. Hence, we feel that it is justified sticking with the periodic setting considered in [17].

The paper is organized as follows. In section 2, we briefly review the general setting from [17] which we will continue to work in here and collect a few facts on wavelets that will be frequently needed throughout the remainder of the paper. In particular, section 2 will be concluded with an essentially known characterization of Sobolev spaces but suitably extended to the somewhat more general wavelet-type expansions considered here. This will later provide the basis for preconditioning the stiffness matrices which, in turn, will be an essential ingredient for the intended compression.

In section 3, we describe the class of operator equations we deal with and define the corresponding generalized Petrov–Galerkin schemes.

Section 4 is devoted to deriving a number of auxiliary basic estimates for entries of stiffness matrices by combining some properties of Schwartz kernels of the operators under consideration with approximation properties of the linear projectors associated with the numerical schemes.

We will continue discussing two different kinds of compression strategies induced by different decompositions of the finite dimensional operators representing the underlying Petrov–Galerkin scheme. The first decomposition corresponds to blocks of the stiffness matrix relative to wavelet-type bases and will be referred to as *wavelet representation.* The second one is somewhat different and will be referred to as *atomic decomposition* because it is closely related to the atomic decomposition of Calderón–Zygmund operators studied in [34]. In [4] it is termed *non-standard representation.* In section 5, we deal with the wavelet representation, confining the discussion in this case to classical Petrov–Galerkin schemes, i.e. the test functionals are functions in L_2. After estimating first individual matrix entries, we will employ suitable versions of Schur's lemma to derive estimates on the norms of the compressed matrices as well as their inverses to ensure stability of the compressed schemes. This allows us to then prove that the solution of the compressed scheme can be made to deviate from the exact solution of the complete finite dimensional problem by no more than a prescribed tolerance. Here, the compressed matrices involve the same order of nonzero entries as unknowns. The corresponding constants, of course, depend on the given tolerance but not on the discretization level. We then proceed by modifying the compressions to finally prove quasioptimal overall asymptotic error estimates for the approximate solutions allowing for $O(N(\log N)^b)$ nonvanishing matrix entries, where N is the current number of unknowns and $b \geq 0$ is a fixed number.

In section 6, we analyse compression techniques based on the atomic decomposition. Here, we consider the full class of generalized Petrov–Galerkin schemes covering, for instance, also collocation. However, we confine ourselves to zero-order operators. In principle, one could extend these results also to operators of different orders, but this would require even further technical elaboration. In this case, it is relatively easy to realize fixed error tolerances at the expense of $O(N)$ operations when varying the compression rate depending on the discretization level. The treatment in [4,5], however, seems to suggest a slightly different type of compression, which we will study throughout the rest of the paper. We will show that in this way, any fixed accuracy can be achieved within linear complexity provided a BMO-type condition is satisfied. The analytical background can be traced back at least to David and Journé [21], establishing a boundedness criterion for generalized Calderón–Zygmund operators, the so-called T1 theorem. A wavelet formulation of this modern Calderón–Zygmund theory was later given by Meyer [32,34], who told us he has worked out similar results for the method proposed in [4]. We hasten to add, however, that the analysis shows that too naive compression strategies may fail in this context and substantiate this by an example.

We conclude in section 7 with a brief summary of estimates listed in a table, and some remarks on future work.

2. Refinable functions and wavelets

In this section, we collect some prerequisites concerning the general framework for the class of numerical schemes to be considered in the sequel. Since we will be interested in periodic problems, we will have to provide appropriate periodic trial spaces. A convenient way to construct such spaces is via periodization of functions defined on all of $\mathbb{R}^n$. Thus, we start by recalling from [17] a few facts about the central notion of refinable shift-invariant spaces and complement this material with further facts about wavelet bases which are relevant for our present purposes.

The main ingredient is a *refinable function* (sometimes called *scaling function*) $\varphi \in L_2(\mathbb{R}^n)$. By this, we mean that φ satisfies a *refinement equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} a_k \varphi(2x-k), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where the *mask* $\boldsymbol{a} = \{a_k\}_{k\in\mathbb{Z}^n}$ is some fixed sequence which typically belongs at least to $l_1(\mathbb{Z}^n)$. To stress the dependence on $\boldsymbol{a}$, we will sometimes say φ is $\boldsymbol{a}$-refinable. For our particular purposes here, we will always assume that φ has compact support and that $\boldsymbol{a}$ is finitely supported.

It then makes sense to work with the following notion of *(algebriac) linear independence*, which will be the second important property we will require. The integer translates of φ are called *algebraically linear independent* if the mapping

$$\lambda \mapsto \sum_{k\in\mathbb{Z}^n} \lambda_k \varphi(\cdot - k) \tag{2.2}$$

is injective on the space of *all* complex-valued sequences λ defined on $\mathbb{Z}^n$. Tensor products of cardinal B-splines or, more generally, certain cube splines are known to have this property (see e.g. [14]).

It is also known (see e.g. [27]) that algebraic linear independence implies stability in the sense that

$$\|\lambda\|_{l_2(\mathbb{Z}^n)} \sim \| \sum_{k\in\mathbb{Z}^n} \lambda_k \varphi(\cdot - k)\|_{L_2(\mathbb{R}^n)}, \tag{2.3}$$

where $A \sim B$ means that there exist two positive constants c_1, c_2 such that $c_1 A \le B \le c_2 A$ holds uniformly with respect to all parameters the quantities A, B may depend on. Here, $\|\lambda\|^2_{l_2(\mathbb{Z}^n)} = \sum_{k\in\mathbb{Z}^n} |\lambda_k|^2$ and $\|\cdot\|_{L_2(\mathbb{R}^n)}$ denotes the usual L_2-norm on $\mathbb{R}^n$.

Let $\langle x, y\rangle := \sum_{j=1}^n x_j \bar{y}_j$ denote the standard scalar product of $x, y \in \mathbb{C}^n$ so that $|x| := \langle x, x\rangle^{1/2}$ is the Euclidean distance. Defining the Fourier transform of $f \in L_1(\mathbb{R}^n)$ by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y\rangle} dx,$$

the stability of φ is well known to be equivalent to (cf. [27])

$$[\hat{\varphi}\overline{\hat{\varphi}}](\omega) := \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\omega + k)|^2 > 0 \quad \text{for all } \omega \in [0,1]^n. \tag{2.4}$$

Here for

$$(f, g) := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx,$$

we define in general

$$[\hat{f}\overline{\hat{g}}](\omega) := \sum_{k\in\mathbb{Z}^n} \hat{f}(\omega + k)\overline{\hat{g}(\omega + k)} = \sum_{k\in\mathbb{Z}^n} (f, g(\cdot\ k))\, e^{2\pi i\langle \omega, k\rangle}, \tag{2.5}$$

which, in particular, is well-defined when f, g have compact support but will be so as well under weaker assumptions (see theorem 3.1 in [27]).

Now let

$$\langle \varphi \rangle^j := \overline{\text{span}\,\{\varphi(2^j \cdot - \xi) : \xi \in \mathbb{Z}^n\}}, \tag{2.6}$$

where the closure is taken with respect to the L_2-norm. When φ is refinable, one clearly has $\langle\varphi\rangle^j \subset \langle\varphi\rangle^{j+1}$. We next wish to determine appropriate updates which complement $\langle\varphi\rangle^j$ in $\langle\varphi\rangle^{j+1}$. To this end, let $E := \{0, 1\}^n$ denote the standard set of representers of $\mathbb{Z}^n/2\mathbb{Z}^n$ and let

$$E_0 := E\backslash\{0\}.$$

The following facts are special cases of results established e.g. in [27].

THEOREM 2.1

Suppose $\varphi \in L_2(\mathbb{R}^n)$ is a stable refinable function of compact support with finitely supported mask $\boldsymbol{a} = \boldsymbol{a}^0$. Then there exist finitely supported masks $\boldsymbol{a}^e$, $e \in E_0$, such that the functions

$$\psi_e(x) := \sum_{\xi\in\mathbb{Z}^n} a_\xi^e \varphi(2x-\xi), \quad e \in E_0, \tag{2.7}$$

satisfy the following properties:

(i) $(\psi_e(\cdot-\xi), \psi_{e'}(\cdot-\xi')) = 0$ for $e, e' \in E$, $e \neq e'$, $\xi, \xi' \in \mathbb{Z}^n$, where we have set $\psi_0 := \varphi$.

(ii) The functions $\psi_e(\cdot-\xi)$, $e \in E_0$, $\xi \in \mathbb{Z}^n$ form an unconditional basis of the orthogonal complement of $\langle\varphi\rangle^0$ in $\langle\varphi\rangle^1$, i.e.

$$\langle\varphi\rangle^1 = \langle\varphi\rangle^0 \bigoplus_{e\in E_0} \langle\psi_e\rangle^0.$$

The functions ψ_e, $e \in E_0$, are called *pre-wavelets*. It is clear that they have the same regularity as the generator φ and that, under the above circumstances, they are also compactly supported. Explicit constructions of pre-wavelets can be found in [9,37]. If in addition the translates $\psi_e(\cdot-\xi)$ are also orthonormal, the ψ_e are called *wavelets*. Univariate compactly supported wavelets of arbitrary regularity are constructed in [18] (see also [8,19]).

It is often not necessary to deal with orthogonal decomposition. An alternative approach was developed in [10] which may be summarized for our purposes as follows. Suppose that in addition to a given $\boldsymbol{a}$-refinable function $\varphi \in L_2(\mathbb{R}^n)$ there exists another $\boldsymbol{d}$-refinable function $\zeta \in L_2(\mathbb{R}^n)$ such that

$$(\varphi, \zeta(\cdot-\xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n. \tag{2.8}$$

The issue is then to find additional masks $\boldsymbol{a}^e$, $\boldsymbol{d}^e$, $e \in E_0$, such that the functions

$$\psi_e := \sum_{\xi\in\mathbb{Z}^n} a_\xi^e \varphi(2\cdot-\xi), \quad \zeta_e := \sum_{\xi\in\mathbb{Z}^n} d_\xi^e \zeta(2\cdot-\xi), \quad e \in E_0, \tag{2.9}$$

satisfy

$$(\psi_e, \zeta_{e'}(\cdot-\xi)) = \delta_{e,e'}\delta_{0,\xi}, \quad e, e' \in E, \ \xi \in \mathbb{Z}^n, \tag{2.10}$$

where again we have set $\psi_0 = \varphi$, $\zeta_0 = \zeta$. We will refer to the ψ_e (and consequently to the ζ_e), $e \in E_0$, as *biorthogonal* wavelets. When dealing with biorthogonal wavelets, we will always make the following

ASSUMPTION

All the masks a^e, d^e, $e \in E$, are finitely supported, φ, ζ and hence ψ_e, ζ_e, $e \in E$, are compactly supported and ζ has as many continuous derivatives as φ.

For univariate examples satisfying these assumptions, see [10], and taking tensor products would, of course, preserve these properties. Let us record next some facts which will be important in the sequel. In particular, smoothness combined with refinability entails some important consequences. For instance, it is shown in [7] that when φ is refinable, has compact support, and belongs to $C^d(\mathbb{R}^s)$, then φ is exact *of degree* (at least) d, i.e. there exists for every $p \in \Pi_d(\mathbb{R}^n)$, the space of polynomials of degree at most d on $\mathbb{R}^n$, a unique polynomial $q \in \Pi_d(\mathbb{R}^n)$ such that

$$p(x) = \sum_{\xi \in \mathbb{Z}^n} q(\xi)\varphi(x-\xi), \quad x \in \mathbb{R}^n. \tag{2.11}$$

We will also make repeated use of the following additional properties.

PROPOSITION 2.1

Suppose $\varphi \in C^d(\mathbb{R}^n)$ is a compactly supported refinable function with linearly independent integer translates and let ψ_e, $e \in E_0$, be pre-wavelets or biorthogonal wavelets. Then, under the above assumptions, the following facts hold:

(i) The functions $2^{nj/2}\psi_e(2^j \cdot - \xi)$, $e \in E_0$, $\xi \in \mathbb{Z}^n$, $j \in \mathbb{Z}$, form a Riesz basis for $L_2(\mathbb{R}^n)$, i.e. every $f \in L_2(\mathbb{R}^n)$ has an expansion ($\psi_0 = \varphi$)

$$f = \sum_{j \in \mathbb{Z}} \sum_{e \in E_0} \sum_{\xi \in \mathbb{Z}^n} c_{j,e,\xi}(f) 2^{nj/2} \psi_e(2^j \cdot - \xi),$$

which converges strongly in $L_2(\mathbb{R}^n)$ and

$$\| f \|^2_{L_2(\mathbb{R}^n)} \sim \sum_{j,e,\xi} |c_{j,e,\xi}|^2, \quad f \in L_2(\mathbb{R}^n).$$

(ii) The following d^*th order moment conditions hold:

$$\int_{\mathbb{R}^n} x^\beta \psi_e(x)\,\mathrm{d}x = 0, \quad e \in E_0, \ \beta \in \mathbb{N}_0^n, \ |\beta| \le d^*,$$

where $d^* = d$ is the degree of exactness of φ when dealing with pre-wavelets and where d^* is the (possibly larger) degree of exactness of the dual refinable function ζ when the ψ_e are biorthogonal wavelets.

Proof

For pre-wavelets, (i) follows directly from theorem 2.1 and orthogonality. For biorthogonal wavelets, the claim is a consequence of the results in [15].

For pre-wavelets, (ii) follows immediately from (2.11) and orthogonality (ii) in theorem 2.1. In the case of biorthogonal wavelets, by our assumption (2.11) holds for φ, d replaced by ζ, d^* so that the assertion follows from (2.10). □

Note that the order d of moment conditions depends only on the degree of polynomials for which (2.11) holds which, in turn, is known to agree with the order d of the *Strang–Fix conditions*:

$$(\partial^\alpha \hat{\varphi})(k) = 0, \quad |\alpha| \le d, \ k \in \mathbb{Z}^n \setminus \{0\} \tag{2.12}$$

(see e.g. [7]), and dth order differentiability is only sufficient but not necessary for (2.12) to hold. This suggests introducing the following class of generators φ, which we will work with in the remainder of this paper.

The function φ is said to satisfy $\mathrm{C}_0^{d',d}$ for some d', $d \in \mathbb{N}_0$, $d' \le d$, if φ satisfies the following requirements:

- φ is refinable, has compact support and belongs to $C^{d'}(\mathbb{R}^n)$.
- The integer shifts of φ are algebraically linearly independent.
- φ satisfies Strang–Fix conditions of order d.

It is also known that for any φ satisfying $\mathrm{C}_0^{d',d}$, there exists a constant $c < \infty$ and some $\rho = \rho(\varphi) \in (0, 1)$ such that

$$|\partial^\beta \varphi(x) - \partial^\beta \varphi(y)| \le c\,|x - y|^\rho, \quad x, y \in \mathbb{R}^n, \ \beta \in \mathbb{N}_0^n, |\beta| = d' \tag{2.13}$$

Here, $\partial^\beta := \partial^{\beta_1}/\partial x_1^{\beta_1} \ldots \partial^{\beta_n}/\partial x_n^{\beta_n}$ and $|\beta| = \beta_1 + \ldots + \beta_n$ (cf. [13, 17]).

Finally, we mention that is it shown in [27] that many of the above properties still hold under much weaker assumptions on φ. In fact, for many purposes, it suffices to assume that φ belongs to the space

$$\mathcal{L}_2 := \{f \in L_2(\mathbb{R}^n) : \sum_{k \in \mathbb{Z}^n} |f(\cdot - k)| \in L_2([0,1]^n)\}.$$

It is clear that any function $\varphi \in L_2(\mathbb{R}^n)$ which has compact support or for which $\int_{k+[0,1]^n} |\varphi(x)|^2 dx$ decays exponentially, as $|k|$ tends to infinity, belongs to $\mathcal{L}_2$.

We will now turn to the analogous periodic setting introduced in [17]. Identifying one-periodic functions, i.e. functions f satisfying

$$f(x + k) = f(x), \quad \text{for all } k \in \mathbb{Z}^n,$$

with functions on the n-dimensional torus

$$\mathcal{T}^n := \mathbb{R}^n/\mathbb{Z}^n,$$

the periodization operator

$$[f](x) := \sum_{k \in \mathbb{Z}^n} f(x+k) \tag{2.14}$$

maps $L_2(\mathbb{R}^n)$ into $L_2(\mathcal{T}^n)$. Likewise, for notational convenience we will identify the cosets $[x] := x + \mathbb{Z}^n$, $x \in \mathbb{R}^n$, with its representer $x \in [0, 1]^n$. For any function $\phi \in \mathcal{L}_2$, we now define

$$\phi_k^j := 2^{jn/2}[\phi(2^j \cdot - k)], \quad k \in \mathbb{Z}^n. \tag{2.15}$$

Thus, setting for any two one-periodic functions $u, v \in L_2(\mathcal{T}^n)$

$$(u, v)_0 := \int_{[0,1]^n} u(x)\overline{v(x)}\, \mathrm{d}x,$$

we note that for any $g \in \mathcal{L}_2$, $u \in L_2(\mathcal{T}^n)$

$$([g], u)_0 = (g, u), \tag{2.16}$$

so that, for any $f, g \in \mathcal{L}_2$,

$$([f], [g])_0 = (f, [g]) = ([f], g). \tag{2.17}$$

Defining

$$\mathbb{Z}^{n,j} := \mathbb{Z}^n/(2^j \mathbb{Z}^n),$$

one easily derives from these facts the following observation (cf. [17]).

Remark 2.1

Let $f, g \in \mathcal{L}_2$ satisfy

$$(f, g(\,,\, - \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n.$$

Then

$$(f_k^j, g_l^j)_0 = \delta_{k,l}, \quad k, l \in \mathbb{Z}^{n,j}, \ j \in \mathbb{N}_0.$$

More generally, let η be any functional of compact support and define

$$\eta_k^j(v) := 2^{-nj/2}\eta(v(2^{-j}(\cdot + k))). \tag{2.18}$$

If for some $g \in \mathcal{L}_2$

$$\eta(g(\cdot + \xi)) = \delta_{0,\xi}, \quad \xi \in \mathbb{Z}^n,$$

then one also has

$$\eta_k^j(g_l^j) = \delta_{k,l}, \quad k, l \in \mathbb{Z}^{n,j}, \ j \in \mathbb{N}_0.$$

Hence, the previous orthogonality relations, refinability and remark 2.1 readily yield the following facts.

COROLLARY 2.1

For pre-wavelets ψ_e, $e \in E_0$, one has

$$(\psi_{e,k}^j, \psi_{e',m}^l)_0 = 0, \quad j,l \in \mathbb{N}_0, \ l \neq j, e, e' \in E_0, \ k \in \mathbb{Z}^{n,j}, \ m \in \mathbb{Z}^{n,l}, \tag{2.19}$$

while biorthogonal wavelets satisfy

$$(\psi_{e,k}^j, \zeta_{e',m}^l)_0 = \delta_{j,l}\delta_{e,e'}\delta_{k,m} \tag{2.20}$$

for $j, l \in \mathbb{N}_0$, $e, e' \in E, k \in \mathbb{Z}^{n,j}$ and $m \in \mathbb{Z}^{n,l}$.

For a given refinable function $\varphi \in \mathcal{L}_2$, we define now the spaces

$$V^j = \langle\varphi\rangle_0^j := \operatorname{span}\{\varphi_k^j : k \in \mathbb{Z}^{n,j}\} \tag{2.21}$$

Since by (2.1) and (2.15)

$$\varphi_k^j = 2^{-n/2} \sum_{m \in \mathbb{Z}^n} a_{m-2k} \varphi_m^{j+1},$$

we conclude

$$V^0 \subset V^1 \subset \ldots \subset V^j \subset V^{j+1} \subset \ldots \subset L_2(\mathcal{T}^n). \tag{2.22}$$

One can also show [12] that the stability (2.3) of $g \in \mathcal{L}_2$ is preserved under periodization in the following sense:

$$\|\lambda\|_{\ell_2(\mathbb{Z}^{n,j})} \sim \|\sum_{k \in \mathbb{Z}^{n,j}} \lambda_k g_k^j\|_{L_2(\mathcal{T}^n)}, \tag{2.23}$$

uniformly in $j \in \mathbb{N}$. Moreover, one now easily confirms from corresponding results on the non-periodic case [27] that, under the above assumptions,

$$\overline{\bigcup_{j \in \mathbb{N}_0} V^j} = \langle\varphi\rangle_0^0 \bigoplus_{j \in \mathbb{N}_0, e \in E_0} \langle\psi_e\rangle_0^j = L_2(\mathcal{T}^n) \tag{2.24}$$

holds for pre-wavelets and biorthogonal wavelets, which as before form Riesz bases for $L_2(\mathcal{T}^n)$.

The decomposition based on pre-wavelets and biorthogonal wavelets may be viewed as special instances of the following concept, which will be important for subsequent developments. Suppose γ is some refinable function and Q_j denotes a linear projector that maps for any $l \geq j$ the space $\langle\gamma\rangle_0^l$ onto $\langle\gamma\rangle_0^j$. It is clear that the spaces

$$W_j^Q := (Q_{j+1} - Q_j)\langle\gamma\rangle_0^{j+1}$$

give rise to a direct sum decomposition

$$\langle\gamma\rangle_0^{j+1} = \langle\gamma\rangle_0^j \oplus W_j^Q.$$

To be more specific about the form of the projectors Q_j, suppose that γ satisfies $C_0^{d^{*\prime},d^*}$, for some nonnegative integers $d^{*\prime}$, d^*, and η is a fixed compactly supported linear functional in the dual of $\langle\gamma\rangle^0$ such that

$$[\hat{\gamma}\bar{\hat{\eta}}](\omega) \neq 0, \quad \omega \in \mathscr{T}^n. \tag{2.25}$$

Here, $\hat{\eta}$ denotes the Fourier transform of η in the distributional sense. As pointed out in [17], the coefficients g_ξ in the trigonometric series

$$\frac{1}{[\hat{\gamma}\bar{\hat{\eta}}](\omega)} = \sum_{\xi \in \mathbb{Z}^n} g_\xi e^{2\pi i\langle\xi,\omega\rangle} \tag{2.26}$$

decay exponentially quickly. Thus, the Fourier transform of the function

$$\phi_0 := \sum_{\xi \in \mathbb{Z}^n} g_\xi \gamma(\cdot - \xi) \in \langle\gamma\rangle^0 \tag{2.27}$$

is given by

$$\hat{\phi}_0(\omega) = \frac{\hat{\gamma}(\omega)}{[\hat{\gamma}\bar{\hat{\eta}}](\omega)}. \tag{2.28}$$

Hence, ϕ_0 satisfies

$$[\hat{\phi}_0\bar{\hat{\eta}}](\omega) = 1, \quad \omega \in \mathscr{T}^n, \tag{2.29}$$

and therefore, as one easily confirms, in view of remark 2.1,

$$\eta_k^j(\phi_{0,m}^j) = \delta_{k,m}, \quad k, m \in \mathbb{Z}^{n,j}, \tag{2.30}$$

and

$$Q_j u := \sum_{k \in \mathbb{Z}^{n,j}} \eta_k^j(u)\phi_{0,k}^j \tag{2.31}$$

defines a projector onto $Y^j := \langle\gamma\rangle_0^j$. Clearly, when $\gamma = \varphi$ and $\eta(g) = (g, \zeta)$, where ζ is a biorthogonal refinable function (2.8), one has $\phi_0 = \varphi$ and Q_j takes the form

$$B_j u = \sum_{k \in \mathbb{Z}^{n,j}} (u, \zeta_k^j)_0 \varphi_k^j, \tag{2.32}$$

while for $\eta(g) := (g, \varphi)$ and $\gamma = \varphi$, the projector Q_j becomes the orthogonal projection P_{V^j} onto $V^j := \langle\varphi\rangle_0^j$ which is given by

$$P_{V^j} u = \sum_{k \in \mathbb{Z}^{n,j}} (u, \varphi_k^j)_0 \phi_k^j. \tag{2.33}$$

In this latter case, ϕ is given by

$$\hat{\phi}(\omega) = \frac{\hat{\varphi}(\omega)}{[|\hat{\varphi}|^2](\omega)}, \tag{2.34}$$

which, in view of the stability of φ, is well defined.

Likewise, we obtain for the differences

$$(B_{j+1} - B_j)u = \sum_{e\in E_0, k\in \mathbb{Z}^{n,j}} (u, \zeta^j_{e,k})_0 \psi^j_{e,k}, \tag{2.35}$$

in the case of biorthogonal wavelets, while for pre-wavelets,

$$(P_{V^{j+1}} - P_{V^j})u = \sum_{e\in E_0, k\in \mathbb{Z}^{n,j}} (u, \zeta^j_{e,k})_0 \psi^j_{e,k}, \tag{2.36}$$

where ζ_e is now given, in view of theorem 2.1 and remark 2.1, by

$$\hat{\zeta}_e(\omega) = \frac{\hat{\psi}_e(\omega)}{[|\hat{\psi}_e|^2](\omega)}. \tag{2.37}$$

Of course, in the case of orthogonal projections, the functions ϕ, ζ_e, $e \in E_0$, do not in general have compact support but decay exponentially quickly. The following fact established in [12] holds for the general case.

THEOREM 2.2

Let for η as above γ satisfy (2.25) and $C_0^{d^{*\prime}, d^*}$ for some $d^{*\prime}, d^* \in \mathbb{N}_0$ and let Q_j be defined by (2.31). Then there exist exponentially fast decaying coefficients g^e_ξ, q^e_ξ, $e \in E_0$, $\xi \in \mathbb{Z}^n$, such that for

$$\phi_e := \sum_{\xi\in\mathbb{Z}^n} g^e_\xi \gamma(2\cdot - \xi), \quad \eta_e(f) := \sum_{\xi\in\mathbb{Z}^n} q^e_\xi \eta(f(\tfrac{1}{2}(\cdot + \xi))), \tag{2.38}$$

one has

$$\eta^j_{e,k}(\phi^j_{e',k'}) = \delta_{e,e'}\delta_{k,k'}, \quad e, e' \in E_0, \ k, k' \in \mathbb{Z}^{n,j}, \tag{2.39}$$

and

$$(Q_{j+1} - Q_j)u = \sum_{e\in E_0} \sum_{k\in\mathbb{Z}^{n,j}} \eta^j_{e,k}(u)\phi^j_{e,k}. \tag{2.40}$$

Thus, the operators $Q_{j+1} - Q_j$ are projectors as well and therefore satisfy

$$Q_l Q_j = Q_l, \quad \text{for } l \le j. \tag{2.41}$$

For $s \in \mathbb{R}$ and any domain $\Omega \subseteq \mathbb{R}^n$, we denote by $H^s(\Omega)$ the usual L_2-Sobolev space of order s relative to Ω with norm $\|\cdot\|_s(\Omega)$. Whenever we work on the particular domain $\mathcal{T}^n$, we will drop any reference to the domain.

We will also make use of the following characterization of Sobolev spaces.

THEOREM 2.3

Let φ be a generator satisfying $C_0^{d',d}$ and let $\psi_{e,k}^l$ denote either pre-wavelets or biorthogonal wavelets satisfying the above assumptions. For any function $u \in H^s(\mathcal{T}^n) \cap H^{|s|}(\mathcal{T}^n)$ with $|s'| < d' + \rho$, one has a unique expansion of the form

$$u = \sum_{l=0}^{\infty} \sum_{k \in Z^{n,l}} \sum_{e \in E_0} d_k^{le}(u)\psi_{e,k}^l + \sum_{k \in Z^{n,0}} s_k^0(u)\varphi_k^0.$$

Moreover, one has the norm equivalence

$$\| u \|_s \sim \left(\sum_{l=0}^{\infty} \sum_{k \in Z^{n,l}} \sum_{e \in E_0} 2^{2ls} | d_k^{le}(u) |^2 + \sum_{k \in Z^{n,0}} | s_k^0(u) |^2 \right)^{1/2}. \tag{2.42}$$

For pre-wavelets and non-negative s, the above result is well-known (see e.g. [33]). For positive s and pre-wavelets as well as biorthogonal wavelets, this norm equivalence is a special case of Besov space characterizations given in [22, 13]. The case of biorthogonal wavelets and $s = 0$ is covered by proposition 2.1. The case $s < 0$ follows by duality.

3. Periodic pseudodifferential operators and generalized Petrov–Galerkin methods

We briefly recall the setting considered in [17] and introduce a class of periodic pseudodifferential equations which will be studied throughout the remainder of this paper.

Locally, these operators can be described in terms of pseudodifferential operators on $\mathbb{R}^n$ (cf. [3, 30]). We recall the following definition from [26, 28]. A pseudodifferential operator $A \in \Psi^r(\mathbb{R}^n)$ is a linear operator of the form

$$Au(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{2\pi i \langle \xi, x-y \rangle} \sigma(x, \xi) u(y) \, dy \right) d\xi, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{3.1}$$

where the symbol $\sigma(x, \xi)$ belongs to the symbol class $S^r(\mathbb{R}^n \times \mathbb{R}^n)$ containing all $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for each multi-indices α, β there exists some constant $c_{\alpha,\beta}$ with

$$| D_x^\beta D_\xi^\alpha \sigma(x, \xi) | \le c_{\alpha,\beta}(1 + |\xi|)^{r-|\alpha|}, \quad x, \xi \in \mathbb{R}^n. \tag{3.2}$$

Viewing $\mathcal{T}^n$ as a compact manifold, the corresponding classes $\Psi^r(\mathcal{T}^n)$ of periodic pseudodifferential operators are then defined via local partitions of unity. Alternatively, the elements of $\Psi^r(\mathcal{T}^n)$ may be represented in terms of Fourier series expansions, namely

$$\sigma(x,D)u(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i\langle\xi,x\rangle}\sigma(x,\xi)\tilde{u}(\xi), \quad u\in C^\infty(\mathcal{T}^n),$$

where

$$\tilde{u}(\xi) = \int_{\mathcal{T}^n} e^{-2\pi i\langle\xi,x\rangle}u(x)\,dx, \quad \xi\in\mathbb{Z}^n,$$

and σ belongs to a certain symbol class of periodic functions which is described in [17] (see also [3,30,31]).

Recall that a pseudodifferential operator $A\in\Psi^r(\mathcal{T}^n)$ maps

$$A: H^s(\mathcal{T}^n)\to H^{s-r}(\mathcal{T}^n), \quad s\in\mathbb{R}, \tag{3.3}$$

boundedly.

Our objective is to solve the pseudodifferential equation

$$Au=f \tag{3.4}$$

on $\mathcal{T}^n$ for $u\in H^s(\mathcal{T}^n)$, where $A\in\Psi^r(\mathcal{T}^n)$ and $f\in H^{s-r}(\mathcal{T}^n)$.

We will study a rather general class of numerical schemes for the solution of (3.4) based on a fixed compactly supported distribution

$$\eta\in H^{-s'}(\Gamma), \tag{3.5}$$

where $s'\geq 0$ satisfies $AV^j\subset H^{s'}(\mathcal{T}^n)$, and where $\Gamma\subset\mathbb{R}^n$ is some fixed ball with center zero. We will always assume in the sequel that the spaces V^j of "Ansatz-functions" are of the form

$$V^j = \langle\varphi\rangle_0^j,$$

where φ is a fixed function satisfying $C_0^{d',d}$ for some $d'\leq d\in\mathbb{N}_0$. As before, we define for $g\in H^{s'}(\mathbb{R}^n)$

$$\eta_k^j(g) := 2^{-nj/2}\eta(g(2^{-j}(\cdot+k))). \tag{3.6}$$

The corresponding Petrov–Galerkin scheme is then given by

$$\eta_k^j(Au^j) = \eta_k^j(f), \quad k\in\mathbb{Z}^{n,j}. \tag{3.7}$$

Specifically, the choice $\eta=\delta(\cdot-\omega_0)$, i.e.

$$\eta(g) := g(\omega_0), \tag{3.8}$$

gives rise to the type of collocation schemes studied in [39] for $n=1$ and in [35] for arbitrary spatial dimension and tensor product spline spaces, while

$$\eta(g) = (g,\overline{\varphi}) \tag{3.9}$$

corresponds to the standard Galerkin scheme. For further examples covered by this setting, the reader is referred to [17].

Following [17], we rephrase these schemes as projection methods. On the one hand, this helps to formulate a suitable stability concept. On the other hand, it will play a crucial role in our subsequent analysis of compression techniques. To describe this, let η be given as above and choose an appropriate compactly supported function γ satisfying (2.25). We can then define the projectors Q_j by (2.31), (2.26) and (2.27). It is clear that solving (3.7) is equivalent to finding $u^j \in V^j$ such that

$$Q_j A u^j = Q_j f. \tag{3.10}$$

We emphasize that for given η, we are free to choose γ appropriately to satisfy (2.25). In fact, the scheme (3.7) is, of course, completely defined independently of the choice of γ. With regard to the stability and convergence analysis for the scheme (3.7), γ serves only as an analytical device. Specifically, γ could be chosen to coincide with the generator φ of the Ansatz-functions but does not have to do so necessarily. For instance, when dealing with collocation, γ could be a tensor product B-spline for which cardinal interpolation (2.25) is well understood. However, we wish to emphasize already at this point that the choice of γ will be of essential importance later in connection with compressing the matrices arising from the scheme (3.7). Specifically, it will then be crucial to have the flexibility of choosing γ different from φ, namely so that γ satisfies also $\mathrm{C}_0^{d^{*\prime},d^*}$ but for some $d^{*\prime} \geq d'$ and $d^* \geq d$. In principle, the values of $d^{*\prime}$ and d^* could also differ from those for the biorthogonal wavelets. But in order to limit the number of parameters, we have deliberately chosen the same notation, that is, we will always assume that γ and the dual generator ζ from the biorthogonal wavelets setting have the same degree of exactness. This will later turn out to be justified, since both parameters will play the same role in those estimates that will govern our analysis of matrix compression techniques. So we will assume that γ satisfies $\mathrm{C}_0^{d^{*\prime},d^*}$ and we will set

$$Y^j := \langle \gamma \rangle_0^j = \langle \phi \rangle_0^j, \tag{3.11}$$

where ϕ is given by (2.26) and (2.27).

The scheme (3.7) is called (s, r)-stable (see remark 4.3 in [17]) if

$$\| Q_j A u^j \|_{s-r} \geq c \, \| u^j \|_s \quad \text{for all } u^j \in V^j, \tag{3.12}$$

uniformly in $j \in \mathbb{N}$. (s, r)-stability is characterized in [17] by the ellipticity of the so-called *numerical symbol* of the scheme (3.7) given by $[\sigma_y \hat{\varphi} \overline{\hat{\eta}}](\omega)$ (see theorem 6.2 in [17]).

4. Some basic estimates

In this section, we derive a number of basic estimates which will make systematic use of the approximation properties of the projections Q_j introduced in the previous section. For most of these properties, we will refer to [17], but recall at this point the following direct and inverse estimates because they will be used more frequently in the sequel. We will continue denoting by $\rho = \rho(\varphi) \in (0, 1)$ the Hölder exponent of the d'th order derivatives of φ (cf. (2.13)). The parameter ρ^* will have the analogous meaning with respect to γ (or ζ) and $d^{*'}$. We will always assume throughout the following that $d \le d^*$, $d' \le d^{*'}$ unless otherwise stated.

THEOREM 4.1

Let $-d' - 1 \le s < d' + \rho$, $-d' - \rho < t \le d + 1$ and $s \le t$. Then the *Jackson estimate*

$$\| u - P_{V^j} u \|_s \le c 2^{j(s-t)} \| u \|_t \tag{4.1}$$

holds for all $u \in H^t(\mathcal{T}^n)$, where c is independent of j and u.

Moreover, when $s \le t < d' + \rho$, there exists a constant c such that for all $u^j \in V^j$, $j \in \mathbb{N}_0$, the *Bernstein estimate*

$$\| u^j \|_t \le c 2^{j(t-s)} \| u^j \|_s \tag{4.2}$$

is valid.

Next, let

$$\theta(\omega) = (\theta_1(\omega), \ldots, \theta_n(\omega)), \;\; \theta_l(\omega) := \mathrm{e}^{2\pi \mathrm{i} \omega_l} - 1, \quad \omega \in \mathcal{T}^n, \tag{4.3}$$

and observe that for every fixed $a \in (0, 1)$ there exist finite positive constants c_1, c_2 such that

$$c_1 |\omega| \le |\theta(\omega)| \le c_2 |\omega|, \quad \omega \in [0, a]^n. \tag{4.4}$$

For given exponentially decaying coefficients c_ξ, $\xi \in \mathbb{Z}^n$, let

$$c_k^j := \sum_{\xi \in \mathbb{Z}^n} c_{2^j \xi + k}. \tag{4.5}$$

It is not difficult to show that there then exist some constant c and some $\delta \in (0, 1)$ such that for all $j \in \mathbb{N}$

$$|c_k^j| \le c\, \delta^{2^j |\theta(2^{-j} k)|}, \quad k \in \mathbb{Z}^{n,j}. \tag{4.6}$$

Moreover, suppose $h(t)$ is any positive strictly increasing function on $\mathbb{R}$ such that for all $0 < \delta < 1$ there exists a constant $c = c(\delta)$ with

$$h(2^j |\theta(2^{-j} k)|) \le c\, \delta^{-2^j |\theta(2^{-j} l)|} h(2^j |\theta(2^{-j}(k - l))|), \quad k, l \in \mathbb{Z}^{n,j}, \; j \in \mathbb{N}. \tag{4.7}$$

It is then clear from (4.6) and (4.7) that there exists some constant c such that for c_k^j as above

$$\Big| \sum_{l \in \mathbb{Z}^{n,j}} c_l^j h(2^j |\theta(2^{-j}(k-l))|)^{-1} \Big| \le c\, h(2^j |\theta(2^{-j}k)|)^{-1}, \quad k \in \mathbb{Z}^{n,j},\ j \in \mathbb{N}. \tag{4.8}$$

It is not difficult to verify that $h(t) := (1 + |t|)^{\tau}$ satisfies (4.7), which gives

LEMMA 4.1

Suppose the coefficients c_ξ, $\xi \in \mathbb{Z}^n$, decay exponentially quickly and let τ be any positive number. Then there exists a constant $c < \infty$ such that

$$\Big| \sum_{l \in \mathbb{Z}^{n,j}} c_l^j (1 + 2^j |\theta(2^{-j}(k-l))|)^{-\tau} \Big| \le c(1 + 2^j |\theta(2^{-j}k)|)^{-\tau}, \quad k \in \mathbb{Z}^{n,j},\ j \in \mathbb{N}.$$

It will be convenient to work with the Schwartz kernel representation of $A \in \Psi^r(\mathcal{T}^n)$ (see [30]). A corresponding further prerequisite is the following lemma which, in principle, is already known (see [42], p. 40). Since it plays an important role for our approach, we will sketch a proof here, following the treatment in [11].

LEMMA 4.2

The Schwartz kernel K_A of $A \in \Psi^r(\mathcal{T}^n)$ satisfies for $x \neq y$, $x, y \in \mathcal{T}^n$, the estimate

$$|\partial_x^\alpha K_A(x,y)| + |\partial_y^\alpha K_A(x,y)| \le c_\alpha |\theta(x-y)|^{-(n+r+|\alpha|)}, \quad n + r + |\alpha| > 0. \tag{4.9}$$

Proof

Since $\mathcal{T}^n$ is a compact manifold, every $A \in \Psi^r(\mathcal{T}^n)$ may be represented via partitions of unity in terms of elements from $\Psi^r(\mathbb{R}^n)$ so that it is sufficient to consider operators $A \in \Psi^r(\mathbb{R}^n)$ which are technically somewhat easier to work with.

Note first that for an operator $A \in \Psi^r(\mathbb{R}^n)$ with Schwartz kernel K_A, the quantities $\partial_x^\alpha K_A(x,y)$, $\partial_y^\alpha K_A(x,y)$ are Schwartz kernels of an operator A' (respectively, the transpose of such an operator) in $\Psi^{r+|\alpha|}(\mathbb{R}^n)$. Denoting by $\sigma_\alpha \in S^{r+|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n)$ the symbol of A', we recall next that for $x \neq y$, $K_{A'}(x,y) = K(x, x-y)$ is given by the *oscillatory integral* (see e.g. [28] for the precise definition)

$$K(x, x-y) = \int e^{2\pi i \langle \xi, x-y \rangle} \sigma_\alpha(x, \xi)\, d\xi. \tag{4.10}$$

Now let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function which is identically equal to one on $\{\xi \in \mathbb{R}^n : |\xi| \le 1\}$ and vanishes outside $\{\xi \in \mathbb{R}^n : |\xi| \le 2\}$. We set $\chi_R(\xi) = \chi(R\xi)$, choose $R = |x-y|$, and write the oscillatory integral (4.10) as

$$K(x, x-y) = K_1(x, x-y) + K_2(x, x-y),$$

where

$$K_1(x,y)=\int e^{2\pi i\langle\xi,x-y\rangle}\chi_R(\xi)\sigma_\alpha(x,\xi)\,d\xi,$$

$$K_2(x,y)=\int e^{2\pi i\langle\xi,x-y\rangle}(1-\xi_R(\xi))\sigma_\alpha(x,\xi)\,d\xi.$$

Of course, we wish to estimate K_1 and K_2 when R becomes small. Thus, we will assume that $R\le R_0$ for some fixed constant R_0. The first integral may be estimated by

$$\begin{aligned}|K_1(x,x-y)|&=\left|\int e^{2\pi i\langle\xi,x-y\rangle}\chi_R(\xi)\sigma_\alpha(x,\xi)\,d\xi\right|\le\int_{|\xi|<2R^{-1}}(1+|\xi|)^{r+|\alpha|}d\xi\\&\le cR^{-(n+r+|\alpha|)}=c|x-y|^{-(n+r+|\alpha|)},\end{aligned}\tag{4.11}$$

provided that $n+r+|\alpha|>0$.

In order to estimate the second kernel, we note that

$$\Delta_\xi^\beta e^{2\pi i\langle\xi,x\rangle}=(2\pi i)^{2\beta}|x|^{2\beta}e^{2\pi i\langle\xi,x\rangle},$$

where Δ_ξ denotes the Laplacian with respect to ξ, and apply partial integration to the oscillatory integral K_2. Thus, we obtain for sufficiently large β

$$\begin{aligned}|K_2(x,x-y)|&=\left|\int e^{2\pi i\langle\xi,x-y\rangle}(1-\chi_R(\xi))\sigma_\alpha(x,\xi)\,d\xi\right|\\&\le c(|x-y|)^{-2\beta}\left|\int e^{2\pi i\langle\xi,x-y\rangle}\Delta_\xi^\beta[(1-\chi_R)\sigma_\alpha](x,\xi)\,d\xi\right|\\&\le c|x-y|^{-2\beta}\int_{|\xi|>R^{-1}}|\xi|^{r+|\alpha|-2\beta}d\xi\\&\le c|x-y|^{-(n+r+|\alpha|)},\end{aligned}\tag{4.12}$$

where $c=c(R_0)$. The assertion now follows from (4.4). □

We remark that for $n+r+|\alpha|\le 0$, additional logarithmic terms appear in the estimate (4.9).

In addition, we need the following facts from [17] (cf. lemma 2.1 in [17]). Under the above assumptions on the function γ there exists some bounded domain $\Omega\subset\mathbb{R}^n$ and a linear functional F on $L_2(\mathbb{R}^n)$ supported on Ω such that

$$\begin{aligned}&F_k^j(\gamma_l^j)=\delta_{k,l},\quad k,l\in\mathbb{Z}^{n,j},\\&|F_k^j(v)|\le c\|v\|_0(\Omega_k^j),\end{aligned}\tag{4.13}$$

where for any domain $\Omega\subseteq\mathbb{R}^n$ we set $\Omega_k^j:=2^{-j}(k+\Omega)$. Hence,

$$G_j(v) := \sum_{k \in \mathbb{Z}^{n,j}} F_k^j(v)\gamma_k^j$$

is a projector onto Y^j.

The following notational convention will be convenient in the subsequent considerations. Given a domain Ω with center of gravity y, say, we will denote by Ω_* a set of the type $c(\Omega - y) + y$, where c is some constant which will always remain bounded independently of any other parameters involved. Thus, Ω_* represents an expanded version of Ω. Since the expanding factors c will not matter, we will denote any repeated expansion of Ω again by Ω_* as long as the number of expansions remains uniformly bounded. Thus, Ω_* may actually denote a different domain on each occurrence. However, under the above assumption, there will always exist a constant c such that $\operatorname{diam}\Omega_* \leq c \operatorname{diam}\Omega$. Also we will set

$$\tilde{\Omega}_k^j := \cup \{\square_m^j : \Omega_k^j \cap (\operatorname{supp}\gamma_m^j) \neq \varnothing\},$$

where $\square := [0, 1]^n$.

It is shown in [17] that when γ satisfies $C_0^{d^{*\prime}, d^*}$ and $0 \leq s \leq t \leq d^* + 1$, $s < d^{*\prime} + \rho(\gamma)$, then

$$\| G_j u - u \|_s(\Omega_k^j) \leq c\, 2^{-j(t-s)} \| u \|_t(\tilde{\Omega}_k^j), \quad u \in H^t(\mathcal{T}^n). \tag{4.14}$$

We will use these facts to derive the following estimates for the functionals η_k^j.

LEMMA 4.3

For any $u^l \in Y^l$ and $j \leq l$, one has

$$| \eta_k^j(u^l) | \leq c\, 2^{s'(l-j)} \| u^l \|_0(\tilde{\Gamma}_k^j), \tag{4.15}$$

where c is some constant independent of $j, l \in \mathbb{N}$ and $u^l \in Y^l$ and Γ is the domain in (3.5).

Proof

According to lemma 5.2 in [17], one has for any $s \geq s'$ and $u \in H^s(\mathcal{T}^n)$

$$| \eta_k^j(u) |^2 \leq c\Big(\| u \|_0^2(\Gamma_k^j) + 2^{-2sj} \| u \|_s^2(\Gamma_k^j) \Big). \tag{4.16}$$

Now let $u = u^l = \sum_{k \in \mathbb{Z}^{n,l}} c_k \gamma_k^l$ so that

$$\| u^l \|_s(\Gamma_k^j) = \| \sum_{m \in \Gamma_{k,\gamma}^{j,l}} c_m \gamma_m^l \|_s(\Gamma_k^j),$$

where

$$\Gamma_{k,\gamma}^{j,l} := \{ m \in \mathbb{Z}^{n,l} : \operatorname{supp}(\gamma_m^l) \cap \Gamma_k^j \neq \varnothing \}.$$

Thus, the inverse estimate (4.2) yields

$$\| u^l \|_s(\Gamma_k^j) \leq c\, 2^{ls} \sum_{m \in \Gamma_{k,\gamma}^{j,l}} | c_m | \| \gamma_m^l \|_0,$$

whence we conclude

$$\|u^l\|_s^2(\Gamma_k^j) \le c\, 2^{2ls} \sum_{m \in \Gamma_{k,\gamma}^{j,l}} |c_m|^2. \tag{4.17}$$

On the other hand, by (4.13)

$$|c_m|^2 = |F_m^l(u^l)| \le c\,\|u^l\|_0^2(\Omega_m^j),$$

so that the assertion follows upon summing over $m \in \Gamma_{k,\gamma}^{j,l}$ and using the stability of the γ_k^l. □

We are now in a position to prove the first estimate which will be used for our compression strategies.

THEOREM 4.2

Let $\eta_{e,k}^j$, $e \in E_0$, be the functionals defined in theorem 2.2, let φ, γ satisfy $C_0^{d',d}$, $C_0^{d^{*\prime},d^*}$, respectively, with $s' \le d' + \rho(\varphi) - r$ and let $A \in \Psi^r(\mathcal{T}^n)$. Then there exists a positive constant c independent of $j \in \mathbb{N}$, $k, k' \in \mathbb{Z}^{n,j}$ and $e \in E_0$ such that

$$|\eta_{e,k}^j(A\varphi_m^j)| \le c\, 2^{jr}(1 + 2^j|\theta(2^{-j}(k-m))|)^{-n-d^*-1-r}. \tag{4.18}$$

Proof

Let $u^j := (Q_{j+1} - Q_j)(A\varphi_m^j) \in Y^{j+1}$ so that, by (2.40), $\eta_{e,k}^j(A\varphi_m^j) = \eta_{e,k}^j(u^j)$. As in (4.5), let

$$q_k^{e,j} := \sum_{\xi \in \mathbb{Z}^n} q_{2^j\xi+k}^e, \quad e \in E_0,$$

where the q_ξ^e are given in theorem 2.2. One easily confirms that

$$\eta_{e,k}^j(v) = 2^{n/2} \sum_{e' \in E} \sum_{k' \in \mathbb{Z}^{n,j}} q_{e'+2k'}^{e,j+1}\, \eta_{e'+2(k'+k)}^{j+1}(v).$$

Thus, from theorem 2.2 and lemma 4.3 we therefore obtain

$$|\eta_{e,k}^j(A\varphi_m^j)| \le c \sum_{e' \in E} \sum_{k' \in \mathbb{Z}^{n,j}} |q_{e'+2k'}^{e,j+1}|\; \|(Q_{j+1} - Q_j)(A\varphi_m^j)\|_0(\tilde{\Gamma}_{e'+2(k'+k)}^{j+1})$$

$$\le c \sum_{k' \in \mathbb{Z}^{n,j}} |\tilde{q}_{k'}^{e,j}| \Big(\|(Q_j - I)(A\varphi_m^j)\|_0((\tilde{\Gamma}_{k'+k}^j)_*) + \|(I - Q_{j-1})(A\varphi_m^j)\|_0((\tilde{\Gamma}_{k'+k}^j)_*)\Big), \tag{4.19}$$

where $\tilde{q}_{k'}^{e,j} := \sum_{e' \in E} |q_{e'+2k'}^{e,j+1}|$. We will estimate in detail only the terms $\|(Q_j - I)(A\varphi_m^j)\|_0((\tilde{\Gamma}_{k'+k}^j)_*)$. The other ones can be treated in essentially the same way. To this end, let as before

$$(\tilde{\Gamma}_{k''}^j)_{*,\gamma} := \{v \in \mathbb{Z}^{n,j} : \operatorname{supp}(\gamma_v^j) \cap (\tilde{\Gamma}_{k''}^j)_* \ne \emptyset\},$$

and define

$$c_{k+k',m'}^j := \sum_{v \in (\tilde{\Gamma}_{k+k'}^j)_{*,\gamma}} |g_{m'+v}^j|, \tag{4.20}$$

where g_ξ are the coefficients from (2.26). The same arguments as in the proof of lemma 5.5 in [17] now yield

$$\begin{aligned}\|(Q_j - I)(A\varphi_m^j)\|_0((\tilde{\Gamma}_{k+k'}^j)_*) &\le \|(G_j - I)(A\varphi_m^j)\|_0((\tilde{\Gamma}_{k+k'}^j)_*) \\ &+ c \sum_{m' \in \mathbb{Z}^{n,j}} c_{k'+k,m'}^j\Big(\|(G_j - I)(A\varphi_m^j)\|_0((\tilde{\Gamma}_{m'}^j)_*) + 2^{-s'j}\|(G_j - I)(A\varphi_m^j)\|_{s'}((\tilde{\Gamma}_{m'}^j)_*)\Big).\end{aligned} \tag{4.21}$$

Since the cardinality of $(\tilde{\Gamma}_{k+k'}^j)_{*,\gamma}$ remains bounded independently of $k \in \mathbb{Z}^{n,j}$ and $j \in \mathbb{N}$, we infer from (4.20) and (4.6) that there exists some constant c and some $\delta \in (0, 1)$ such that

$$c_{k+k',m'}^j \le c\,\delta^{2^j|\theta(2^{-j}(k+k'-m'))|}. \tag{4.22}$$

Now let us abbreviate for any function f

$$S(f) := \operatorname{supp} f,$$

and suppose first that

$$S(\varphi_m^j) \cap (\tilde{\Gamma}_k^j)_* = \varnothing. \tag{4.23}$$

In this case one has, in view of (4.14), for every $t \in [0, s']$

$$\|(G_j - I)(A\varphi_m^j)\|_t(\tilde{\Gamma}_{k''}^j) \le c\,2^{-j(d^*+1-t)}|A\varphi_m^j|_{d^*+1}((\tilde{\Gamma}_{k''}^j)_*), \tag{4.24}$$

where $|v|_l^2(\Omega) = \sum_{|\alpha|=l}\|\partial^\alpha u\|_0^2(\Omega)$ denotes the usual Sobolev semi-norm of order l. Note now that lemma 4.2 yields for $|\alpha| = d^* + 1$

$$\begin{aligned}\|\partial_x^\alpha \int_{\mathbb{R}^n} K_A(\cdot, y)2^{nj/2}\varphi(2^j y - m)\,\mathrm{d}y\,\|_0((\tilde{\Gamma}_{k''}^j)_*) &\le c_\alpha\| \int_{2^{-j}(m+S(\varphi))} 2^{nj/2}|\theta(\cdot - y)|^{-(n+r+d^*+1)}\mathrm{d}y\,\|_0((\tilde{\Gamma}_{k''}^j)_*) \\ &\le c_\alpha 2^{-nj}|\theta(2^{-j}(m - k''))|^{-(n+1+r+d^*)},\end{aligned}$$

so that

$$2^{-jt}\|(G_j - I)(A\varphi_m^j)\|_t((\tilde{\Gamma}_{k''}^j)_*) \le c\,2^{jr}(1 + 2^j|\theta(2^{-j}(k'' - m))|)^{-(n+r+d^*+1)}, \tag{4.25}$$

whenever (4.23) holds. If (4.23) is not satisfied, we consider first the case $r < 0$ and use (4.14) to conclude that

$$\begin{aligned}\|(G_j - I)(A\varphi_m^j)\|_t((\tilde{\Gamma}_{k''}^j)_*) &\le c\,2^{j(t+r)}\|A\varphi_m^j\|_{-r}(\tilde{\Gamma}_{k''}^j)_*) \\ &\le c\,2^{j(t+r)}\|A\varphi_m^j\|_{-r} \qquad (4.26) \\ &\le c\,2^{j(t+r)}\|\varphi_m^j\|_0 \\ &\le c\,2^{jr}(1 + 2^j|\theta(2^{-j}(m - k''))|)^{-(n+r+d^*+1)}. \qquad (4.27)\end{aligned}$$

When $r \geq 0$, we define $\Omega^j_{k'',\gamma} = \{m'' \in \mathbb{Z}^{n,j} : (\tilde{\Gamma}^j_{k''})_* \cap \operatorname{supp}(\gamma^j_{m''}) \neq \emptyset\}$ and observe that

$$\begin{aligned}
\| G_j(A\varphi^j_m) \|_t((\tilde{\Gamma}^j_{k''})_*) &\leq \sum_{m'' \in \Omega^j_{k'',\gamma}} | F^j_{m''}(A\varphi^j_m) | \, \| \gamma^j_{m''} \|_t \\
&\leq c\, 2^{tj} \sum_{m'' \in \Omega^j_{k'',\gamma}} \| A\varphi^j_m \|_0(\Omega^j_{m''}) \\
&\leq c\, 2^{tj} \| A\varphi^j_m \|_0 \leq c\, 2^{tj} \| \varphi^j_m \|_r \\
&\leq c\, 2^{tj} 2^{jr} (1 + 2^j | \theta(2^{-j}(m - k'')) |)^{-(n+r+d^*+1)}, \qquad (4.28)
\end{aligned}$$

where we have also used in the last step the inverse estimate (4.2) and the fact that, since (4.23) is not satisfied, there exists some constant R such that $2^j | \theta(2^{-j}(m - k'')) | \leq R$. Similarly, one obtains

$$\begin{aligned}
\| A\varphi^j_m \|_t((\tilde{\Gamma}^j_{k''})_*) &\leq c \| \varphi^j_m \|_{t+r} \leq c\, 2^{j(t+r)} \| \varphi^j_m \|_0 \\
&\leq c\, 2^{tj} 2^{rj} (1 + 2^j | \theta(2^{-j}(m - k'')) |)^{-(n+r+1+d^*)}. \qquad (4.29)
\end{aligned}$$

Thus, summarizing (4.25), (4.26), (4.28), and (4.29) yields that for $t \in [0, s']$

$$2^{-jt} \| (G_j - I)(A\varphi^j_m) \|_t((\tilde{\Gamma}^j_{k''})_*) \leq c\, 2^{jr} (1 + 2^j | \theta(2^{-j}(k'' - m)) |)^{-(n+r+d^*+1)} \qquad (4.30)$$

holds in all the above cases for some constant c independent of $j \in \mathbb{N}$ and $m, k'' \in \mathbb{Z}^{n,j}$. Thus, substituting (4.30) in (4.21) and inserting this bound into (4.19) provides

$$\begin{aligned}
| \eta^j_{e,k}(A\varphi^j_m) | \leq c\, 2^{jr} \sum_{k' \in \mathbb{Z}^{n,j}} \tilde{q}^{e,j}_{k''} \Bigg\{ &(1 + 2^j | \theta(2^{-j}((k + k') - m)) |)^{-(n+r+1+d^*)} \\
&+ \sum_{m' \in \mathbb{Z}^{n,j}} c^j_{k+k',m'} (1 + 2^j | \theta(2^{-j}(m - m')) |)^{-(n+r+1+d^*)} \Bigg\}. \qquad (4.31)
\end{aligned}$$

A twofold application of lemma 4.1 to (4.31) now proves the assertion. □

A similar result holds for the dual situation. To this end, we recall that when dealing with biorthogonal wavelets, the dual generator ζ is supposed to satisfy $C_0^{d^{*\prime}, d^*}$, where d^* is possibly larger than d while, when the ψ_e are pre-wavelets, our default assumption is $d = d^*$.

THEOREM 4.3

Let $\psi^j_{e,k}$ denote either pre-wavelets or biorthogonal wavelets satisfying the assumptions listed in section 2. Then there exists a constant c such that

$$|\eta_k^j(A\psi_{e,l}^j)| \le c\, 2^{jr}(1+2^j|\theta(2^{-j}(k-l))|)^{-n-d^*-1-r}, \tag{4.32}$$

uniformly in $j \in \mathbb{N}_0$ and $l, k \in \mathbb{Z}^{n,j}$.

Proof

Suppose first that

$$\operatorname{dist}(S(\psi_{e,l}^j), \Gamma_k^j) > R2^{-j} \tag{4.33}$$

for some constant $R > 0$. Then, using Taylor's expansion, the moment conditions in proposition 2.1 and lemma 5.2 in [17], we obtain

$$\begin{aligned}|\eta_k^j(A\psi_{e,l}^j)| &= \left|\int_{\mathbb{R}^n} \left(\sum_{|\alpha|=d^*+1} \frac{(y-2^{-j}l)^\alpha}{\alpha!}\, \partial_y^\alpha(\eta_k^j K_A(\cdot,\tau(y)))\right) 2^{jn/2}\psi_e(2^jy-l)\,\mathrm{d}y\right|\\ &\le c\left(\|v_l\|_0(\Gamma_k^j) + 2^{-js}\|v_l\|_s(\Gamma_k^j)\right),\end{aligned}$$

where $s := \min\{w \in \mathbb{N}_0 : w \ge s'\}$, $\tau(y) \in 2^{-j}(l + S(\psi_e))$ and

$$v_l(x) := \int_{\mathbb{R}^n} \left(\sum_{|\alpha|=d^*+1} \frac{(y-2^{-j}l)^\alpha}{\alpha!}\, \partial_y^\alpha K_A(x,\tau(y))\right) 2^{jn/2}\psi_e(2^jy-l)\,\mathrm{d}y.$$

Thus, we infer from lemma 4.2 that for $|\beta| = s$

$$\begin{aligned}|\partial_x^\beta v_l(x)| &\le c\int_{\mathbb{R}^n} |y-2^{-j}l|^{d^*+1}|\theta(x-\tau(y))|^{-(n+r+s+d^*+1)}|2^{nj/2}\psi_e(2^jy-l)|\,\mathrm{d}y\\ &\le c\,2^{-nj/2}2^{-j(d^*+1)}|\theta(2^{-j}l)|^{-(n+r+s+d^*+1)}.\end{aligned} \tag{4.34}$$

Therefore,

$$\begin{aligned}&\|v_l\|_0(\Gamma_k^j) + 2^{-js}\|v_l\|_s(\Gamma_k^j)\\ &\le c\,2^{jr}\left((1+2^j|\theta(2^{-j}(k-l))|)^{-(n+r+d^*+1)} + (1+2^j|\theta(2^{-j}(k-l))|)^{-(n+r+d^*+1+s)}\right)\\ &\le c(1+2^j|\theta(2^{-j}(k-l))|)^{-(n+r+d^*+1)},\end{aligned}$$

which establishes our claim in the case (4.33).

Now suppose (4.33) does not hold and assume first $r < 0$. Lemma 5.2 in [17] gives for any $s \ge s'$

$$
\begin{aligned}
|\eta_k^j(A\psi_{e,l}^j)|^2 &\le c\left(\|A\psi_{e,l}^j\|_0^2 + 2^{-2js}\|A\psi_{e,l}^j\|_s^2\right) \\
&\le c\left(\|\psi_{e,l}^j\|_r^2 + 2^{-2js}\|\psi_{e,l}^j\|_{s+r}^2\right) \\
&\le c\|\psi_{e,l}^j\|_r^2, \qquad (4.35)
\end{aligned}
$$

where we have used (4.2) in the last step. When the $\psi_{e,l}^j$ are pre-wavelets, we have $\psi_{e,l}^j = (P_{V^{j+1}} - P_{V^j})\psi_{e,l}^j$ so that the direct estimate (4.1) yields

$$\|\psi_{e,l}^j\|_r \le c\, 2^{jr}\|\psi_{e,l}^j\|_0. \qquad (4.36)$$

When dealing with biorthogonal wavelets, one obtains $\psi_{e,l}^j = (B_{j+1} - B_j)\psi_{e,l}^j$, where the operators B_j are defined by (2.32). Since the B_j are uniformly bounded projectors on $L_2(\mathcal{T}^n)$, one could either argue directly or invoke theorem 5.2 in [17] to confirm that (4.36) remains valid in this case as well. Thus,

$$|\eta_k^j(A\psi_{e,l}^j)| \le c\, 2^{jr}\|\psi_{e,l}^j\|_0. \qquad (4.37)$$

If $r \ge 0$, (4.37) follows directly from (4.35) and (4.2), whence the assertion follows. □

We will now proceed by listing a few consequences of the above estimates.

COROLLARY 4.1

Suppose that $\phi \in L_2(\mathbb{R}^n)$ is any function of compact support. Under the previous assumptions on the (pre-wavelets or biorthogonal wavelets) $\psi_{e,l}^j$, one has for any $A \in \Psi^r(\mathcal{T}^n)$

$$|(A\psi_{e,k}^j, \phi_l^j)_0| \le c\, 2^{jr}(1 + 2^j|\theta(2^{-j}(k-l))|)^{-(n+r+d^*+1)}, \quad k, l \in \mathbb{Z}^{n,j}, \qquad (4.38)$$

where the constant c is independent of j, l, k.

Proof

Taking $\eta(g) := (g, \phi)$, the claim follows from theorem 4.3. □

COROLLARY 4.2

Suppose ϕ has the form (2.34), where φ has compact support and has stable integer translates. Then, under the remaining assumptions of corollary 4.1 the estimate (4.38) remains valid.

Proof

The assertion follows from corollary 4.1, theorem 2.2 and lemma 4.1. □

COROLLARY 4.3

Let $A \in \Psi^r(\mathcal{T}^n)$ and $r + n + d^* + 1 > 0$. Under the assumptions of corollary 4.1, one has

$$|(A\psi^j_{e',l}, \psi^j_{e,k})_0| + |(A\psi^j_{e',l}, \varphi^j_k)_0| + |(A\varphi^j_l, \psi^j_{e,k})_0|$$
$$\leq c\, 2^{jr}(1 + 2^j|\theta(2^{-j}(k - l))|)^{-(n+r+d^*+1)}, \tag{4.39}$$

where c is independent of e, e', k, l and j.

An estimate of the type (4.39) can also be found in [4] for the case $r = 0$ and for Daubechies wavelets.

Remark 4.1

The above reasoning reveals that the analysis is essentially based on two assumptions on the operators, namely local properties of the Schwartz kernels and boundedness properties of the pseudodifferential operators. Hence, for $r = 0$, the above results remain valid for the wider class of Calderón–Zygmund operators, which was already pointed out in [4].

We will make use of these prerequisites in subsequent sections to approximate the finite dimensional operators $Q_jAP_{V^j}$ corresponding to the numerical scheme (3.7) by an appropriate operator which has a sparse matrix representation relative to a suitable basis. These approximations will be based on the following decompositions of $Q_jAP_{V^j}$:

$$Q_jAP_{V^j} = \sum_{l,l'=0}^{j}(Q_l - Q_{l-1})A(P_{V^{l'}} - P_{V^{l'-1}}), \tag{4.40}$$

where $Q_{-1} = P_{V^{-1}} = 0$ and

$$Q_jAP_{V^j} = \sum_{l=0}^{j}(Q_lAP_{V^l} - Q_{l-1}AP_{V^{l-1}}). \tag{4.41}$$

Decomposition (4.40) corresponds to stiffness matrices relative to wavelet bases, while (4.41) will be referred to as *atomic decomposition.*

5. The wavelet representation

In this section, we will employ the decomposition (4.40) for approximating Q_jAP_j, where in the following $P_j := P_{V^j}$. However, in this case we confine the discussion to classical Petrov–Galerkin schemes, i.e. the functional η will be assumed to be a regular distribution represented by a function again denoted by η. At this point, η will be assumed to be continuous although this could be avoided at the expense of a little more technical elaboration. We will continue denoting by d, d^*

the respective degrees of exactness of φ on the one hand, and of γ (as well as of ζ when working with biothorgonal wavelets as trial functions) on the other hand. Of course, γ and η are assumed to satisfy (2.25) so that ϕ_e, η_e, $e \in E_0$ are given by theorem 2.2. As in the previous section, we will see that the decay rates will be governed by d^* and most of the underlying hypotheses will impose conditions on d^* not on the trial and test functions φ, η. Likewise, ρ, ρ^* will again be corresponding Hölder exponents. We recall from (2.41) that under all these circumstances the corresponding projectors Q_j satisfy

$$Q_{l'}Q_l = Q_{l'} \quad \text{for } l' < l. \tag{5.1}$$

This fact will facilitate exploiting the estimates from the previous section for now estimating quantities of the type $(A\psi^j_{e,k}, \eta^l_{e',k'})_0$ for $l \neq j$. Using appropriate versions of Schur's lemma, these results will then be used to estimate the norms of the compressed matrices and their inverse, as well as to relate these facts, by means of the norm equivalences from the end of section 2, to the stability concept developed in [17]. This, in turn, will allow us to establish error bounds for a fixed required accuracy, as well as asymptotic error bounds suggested by the convergence rates derived in [17].

Throughout the first two parts of this section, we will fix

$$s = r/2.$$

5.1. ESTIMATES FOR DIFFERENT LEVELS

For notational convenience, we introduce the following shorthand index notation:

$$\mathcal{J}_l := \{I : I = (l, e, k), k \in \mathbb{Z}^{n,l}, e \in E_0\}, \quad l \in \mathbb{N}_0,$$

$$\mathcal{J}_{-1} := \{(-1,\ 0,\ 0)\},$$

and

$$\mathcal{J}^j := \bigcup_{l=-1}^{j} \mathcal{J}_l .$$

Setting

$$|I| := 2^{-l} \quad \text{whenever } I \in \mathcal{J}_l,$$

we abbreviate at times

$$\psi_I = \begin{cases} \psi^l_{e,k}, & I = (l, e, k),\ l \in N_0, \\ \varphi^0_0, & I = (-1,\ 0,\ 0). \end{cases} \tag{5.2}$$

Thus, the stiffness matrix relative to the above wavelet type bases has the form

$$\mathbf{A}^j = (\eta_I(A\psi_J))_{I,J \in \mathcal{J}^j}, \tag{5.3}$$

and will be referred to as *wavelet representation*. For the special case that the ψ_I are orthonormal wavelets, it is called standard representation in [4].

Our first step is to *precondition* the matrix $\mathbf{A}^j$ which, on account of theorem 2.3, requires for $s = r/2$ only the block diagonal scaling

$$b_{I,J} = |I|^{r/2}|J|^{r/2}\eta_I(A\psi_J), \tag{5.4}$$

to guarantee, as will be detailed later in corollary 5.3, that the matrix

$$\mathbf{B}^j := (b_{I,J})_{I,J\in\mathcal{J}^j}$$

has uniformly bounded spectral condition numbers, provided the Petrov–Galerkin scheme is stable (cf. [17]).

In addition to the estimates in theorem 4.2 and theorem 4.3, we now have to consider also entries of the form $b_{I,J}$, where I, J stem from different scales.

LEMMA 5.1

Let $2(d' + \rho) > r$, $n + d^* + 1 + r > 0$ and that for some positive constant R

$$|\theta(2^{-l'}k' - 2^{-l}k)| \geq R\,2^{-\min\{l,l'\}}. \tag{5.5}$$

Then there exists a constant c depending only on R, r, n, d^* such that the coefficients of the matrices $\mathbf{B}^j, j \in \mathbb{N}$, defined by (5.4) satisfy

$$|b_{J,I}| = |(2^{-lr/2}A\psi_I, 2^{-l'r/2}\eta_J)_0| \leq c\,\frac{2^{-|l-l'|(\frac{n+r}{2}+d^*+1)}}{(1+2^{\min\{l,l'\}}|\theta(2^{-l}k-2^{-l'}k')|)^{n+d^*+1+r}}, \tag{5.6}$$

uniformly in $I \in \mathcal{J}_l$, $J \in \mathcal{J}_{l'}$, and $k \in \mathbb{Z}^{n,l}$, $k' \in \mathbb{Z}^{n,l'}$.

Proof

The case $l = l'$ has already been established in theorem 4.2. Therefore, suppose first that $l' < l$. We expand the function $Q_l(2^{-lr/2}A\psi^l_{e,k})$ as

$$g_{l,k} := Q_l(2^{-lr/2}A\psi^l_{e,k}) = \sum_{m\in\mathbb{Z}^{n,l}} c(m)\phi^l_m, \tag{5.7}$$

(see (2.31)), where for fixed $I = (l, e, k)$

$$c(m) = 2^{-lr/2}\eta^l_m(A\psi_I), \tag{5.8}$$

and ϕ is given by (2.28) or (2.34).

Now theorem 4.3 yields the estimate

$$|c(m)| \leq c\,2^{lr/2}(1 + 2^l|\theta(2^{-l}(k-m))|)^{-n-d^*-1-r} \quad \text{for } k, m \in \mathbb{Z}^{n,l}. \tag{5.9}$$

In order to estimate (2.38), we recall that ϕ_m^l decays exponentially and is continuous. Thus, combining lemma 4.1 with (5.9) yields

$$|g_{l,k}(x)| \le c\, 2^{lr/2} 2^{ln/2} (1 + 2^l |\theta(x - 2^{-l}k)|)^{-n-d^*-1-r} \quad \text{for } x - 2^{-l}k \in \mathcal{T}^n. \tag{5.10}$$

Next note that, in view of (5.1),

$$Q_{l'+1} - Q_{l'} = (Q_{l'+1} - Q_{l'})Q_l, \quad \text{for } l' < l.$$

Since

$$\eta_J(v) = \eta_J((Q_{l'+1} - Q_{l'})v),$$

we therefore obtain

$$\eta_J(A\psi_I) = \eta_J(Q_{l'+1} - Q_{l'})(Q_l A\psi_I) = \eta_J(Q_l A\psi_I) = \eta_J(g_{l,k}),$$

so that

$$|2^{-r(l+l')/2}\eta_J(A\psi_I)| = |(2^{-lr/2}A\psi_I, 2^{-l'r/2}\eta_J)_0| = |2^{-l'r/2}\eta_J(g_{l,k})|. \tag{5.11}$$

Now suppose $l' < l$ and recall that, by (2.38),

$$\eta_J = \sum_{\nu \in \mathbb{Z}^{n,l'+1}} q_{\nu-2k'}^{e',l'+1}\, \eta_\nu^{l'+1},$$

where $q_\nu^{e',l'+1} = \sum_{\mu \in \mathbb{Z}^n} q_{\nu+2^{l'+1}\mu}^{e'}$. Replacing for simplicity $l'+1$ by l', we estimate first $|2^{-l'r/2}\eta_\nu^{l'}(g_{l,k})|$. To this end, note that, on account of the continuity of θ, the fact that η has compact support, (5.5), and the estimate (5.10), we may conclude that

$$\begin{aligned}
|2^{-l'r/2}\eta_\nu^{l'}(g_{l,k})| &\le c\, 2^{r(l-l')/2} 2^{nl/2} (1 + 2^l |\theta(2^{-l'}\nu - 2^{-l}k)|)^{-n-r-d^*-1} \int_{\mathcal{T}^n} |\eta_\nu^{l'}(x)|\,dx \\
&\le c\, 2^{(l-l')(n+r)/2} (1 + 2^l |\theta(2^{-l'}\nu - 2^{-l}k)|)^{-n-d^*-1-r} \\
&\le c\, 2^{-(l-l')(\frac{n+r}{2}+d^*+1)} (1 + 2^{l'} |\theta(2^{-l'}\nu - 2^{-l}k)|)^{-n-d^*-1-r}.
\end{aligned} \tag{5.12}$$

The assertion then follows from lemma 4.1. When $l < l'$, we replace η_e by ψ_e and A by A^* and repeat the above reasoning. □

As a second step, we extract constants which, due to the nature of the wavelet-type basis, simply means to replace by zero all those entries which contain a scalar product with translates of a scaling function on the coarsest grid. The corresponding matrix is therefore

$$\mathrm{T}^j := (t_{I,J})_{I,J \in \mathcal{J}^j}$$

with

$$t_{I,J} = \begin{cases} b_{I,I'}, & \text{if } I, J \notin \mathcal{J}_{-1}, \\ 0, & \text{if otherwise.} \end{cases} \tag{5.13}$$

On account of the compact support of the scaling functions, the subtracted matrix has $O(N)$, $N = 2^{jn}$, nonzero coefficients. The resulting matrix is the stiffness matrix of the operator

$$A^{\#} = (I - Q_0)A(I - P_0).$$

Hence, the operator $A^{\#}$ and also its L_2-adjoint annihilate the space V^0. Furthermore, the matrices $\mathbf{T}^j$ and $\mathbf{B}^j$ differ only in $O(2^{jn})$ entries. In particular, periodization implies that V^0 consists of constant functions, so that

$$A^{\#}1 = (A^{\#})^{*}1 = 0. \tag{5.14}$$

This fact will allow us to establish estimates of the above type without requiring condition (5.5).

LEMMA 5.2

Suppose that $d' + \rho, d^* > r/2, d^* + 1 + r/2 > 0, n + d^* + r + 1 > 0$ and $\gamma \in C_0^{d^{*\prime}, d^*}$. Then there exist constants c and $\delta \in (0, 1]$ such that the coefficients of the matrices $\mathbf{T}^j$, $j \in \mathbb{N}$, defined by (5.13) and (5.4) satisfy the estimate

$$|t_{I,J}| = |2^{-l'r/2}\eta_J(2^{-lr/2}A^{\#}\psi_I)| \le c\, \frac{2^{-|l-l'|(n/2+\delta)}}{(1 + 2^{\min\{l,l'\}}|\theta(2^{-l}k - 2^{-l'}k')|)^{n+d^*+1+r}}, \tag{5.15}$$

uniformly in $I \in \mathcal{J}_l$, $J \in \mathcal{J}_{l'}$, and $k \in \mathbb{Z}^{n,l}$, $k' \in \mathbb{Z}^{n,l'}$.

Proof

In view of the definition of $A^{\#}$ and (5.13), the estimates in theorems 4.2 and 4.3 remain trivially valid when A is replaced by $A^{\#}$. Since the assertion has been already confirmed for any $\delta \in (0, d^* + 1 + r/2]$ in lemma 5.1 when (5.5) holds, we may assume here that

$$2^{\min\{l,l'\}}|2^{-l'}k' - 2^{-l}k| < \tfrac{1}{2}, \quad k, k' \in \mathbb{Z}^n. \tag{5.16}$$

We consider first the case $l' < l$ and begin by collecting a few preliminary facts. It will be convenient to consider the nonperiodic versions of the projectors Q_j, defined by

$$\tilde{Q}_j v := \sum_{\xi \in \mathbb{Z}^n} \eta(v(2^{-j}(\cdot + \xi)))\phi_0(2^j \cdot - \xi)$$

$$= \sum_{\xi \in \mathbb{Z}^n} (v, 2^{nj/2}\eta(2^j \cdot - \xi))2^{nj/2}\phi_0(2^j \cdot - \xi), \quad j \in \mathbb{Z}, \quad (5.17)$$

where as before, ϕ_0 and Q_j are given by (2.27) and (2.31), respectively. Note that, due to the exponential decay of ϕ_0, $\tilde{Q}_j$ is well defined on the space of all polynomials as well as on $L_2(\mathcal{T}^n)$. Now recall that by (2.16)

$$\eta_k^j(v) = 2^{-nj/2}\eta(v(2^{-j}(\cdot + k))) = (v, \eta_k^j)_0 = (v, 2^{nj/2}\eta(2^j \cdot - k)),$$

for $v \in H^{s'}(\mathcal{T}^n)$. One readily infers from these observations that

$$Q_j v = \tilde{Q}_j v, \quad v \in H^{s'}(\mathcal{T}^n). \quad (5.18)$$

We now have to estimate for $I = (e, l, k)$, $J = (e', l', k')$, $l' < l$ the quantities

$$(A^{\#}\psi_I, \eta_J)_0 = \eta_J(A^{\#}\psi_I) = \eta_J((Q_{l'+1} - Q_{l'})A^{\#}\psi_I)$$

$$= ((Q_{l'+1} - Q_{l'})A^{\#}\psi_I, \eta_J)_0 = ((Q_{l'+1} - Q_{l'})Q_l A^{\#}\psi_I, \eta_J)_0,$$

where we have used (5 .1) in the last step. Since $(I - P_0)\psi_I = \psi_I$ and, by (5.1), $(I - Q_0)^*(Q_{l'+1} - Q_{l'})^* = Q^*_{l'+1} - Q^*_{l'}$, we conclude from (5.18) that

$$(A^{\#}\psi_I, \eta_J)_0 = (Q_l A\psi_I, (Q_{l'+1} - Q_{l'})^*\eta_J)_0 = (g_{l,k}, (Q_{l'+1} - Q_{l'})^*\eta_J)_0, \quad (5.19)$$

where

$$g_{l,k} := Q_l A\psi_I = \sum_{m \in \mathbb{Z}^{n,l}} c_I(m)\phi_{0,m}^l, \quad (5.20)$$

i.e.,

$$c_I(m) = \eta_m^l(A\psi_I).$$

Recalling that

$$\phi_0 = \sum_{\xi \in \mathbb{Z}^n} g_\xi \varphi(\cdot - \xi),$$

straightforward calculations yield

$$\phi_{0,m}^l = 2^{nl/2} \sum_{\beta \in \mathbb{Z}^n} \phi_0(2^l(\cdot + \beta) - m) = \sum_{\mu \in \mathbb{Z}^{n,l}} g_\mu^l \varphi_{m+\mu}^l, \quad (5.21)$$

where

$$g_\mu^l = \sum_{\nu \in \mathbb{Z}^n} g_{\mu + 2^l \nu}.$$

Thus,

$$(g_{l,k}, (Q_{l'+1} - Q_{l'})^* \eta_J)_0 = \sum_{m \in \mathbb{Z}^{n,l}} c_l(m) (\phi^l_{0,m}, (Q_{l'+1} - Q_{l'})^* \eta_J)_0$$

$$= \sum_{m \in \mathbb{Z}^{n,l}} \sum_{\mu \in \mathbb{Z}^{n,l}} c_l(m) g^l_\mu (\varphi^l_{m+\mu}, (Q_{l'+1} - Q_{l'})^* \eta_J)_0. \tag{5.22}$$

We consider first an individual summand and use (5.18) and (2.11) to conclude that

$$\begin{aligned}(\varphi^l_{m+\mu}, (Q_{l'+1} - Q_{l'})^* \eta_J)_0 &= 2^{nl/2} (\varphi(2^l \cdot - (m+\mu)), (\tilde{Q}_{l'+1} - \tilde{Q}_{l'})^* \eta_J) \\ &= 2^{nl/2} (\varphi(2^l \cdot - (m+\mu)), (\tilde{Q}_{l'+1} - \tilde{Q}_{l'})^* (\eta_J + p)),\end{aligned} \tag{5.23}$$

where p is any polynomial of degree at most d^*. Specifically, we choose p to be the best polynomial approximation of η_J in some neighborhood of $2^{-l}(m+\mu)$ of diameter $2^{-l'}$, say, and set

$$H_{J,m+\mu} := \eta_J + p.$$

We will estimate the right-hand side of (5.23) by terms of type

$$(\varphi(2^l \cdot - (m+\mu)), \tilde{Q}^*_{l'} H_{J,m+\mu})$$

$$= \sum_{\nu \in \Omega^{l,l'}_{m+\mu}} (H_{J,m+\mu}, 2^{nl'/2} \phi_0(2^{l'} \cdot - \nu)) (\varphi(2^l \cdot - (m+\mu)), 2^{nl'/2} \eta(2^{l'} \cdot - \nu)), \tag{5.24}$$

where

$$\Omega^{l,l'}_{m+\mu} := \{\nu \in \mathbb{Z}^n : \operatorname{supp}(\eta(2^{l'} \cdot - \nu)) \cap \operatorname{supp}(\varphi(2^l \cdot - (m+\mu)) \neq \varnothing\}.$$

Clearly, when $l' < l$, one has

$$\# \Omega^{l,l'}_{m+\mu} \le c, \tag{5.25}$$

where c is independent of l, l', m and μ. Hence, the quantity in (5.23) can be bounded as follows:

$$|(\varphi^l_{m+\mu}, (\tilde{Q}_{l'+1} - \tilde{Q}_{l'})^* \eta_J)_0|$$

$$\le c\, 2^{nl/2} \max_{\nu \in \Omega^{l,l'}_{m+\mu}} |(H_{J,m+\mu}, 2^{nl'/2} \phi_0(2^{l'} \cdot - \nu))|\, |(\varphi(2^l \cdot - (m+\mu)), 2^{nl'/2} \eta(2^{l'} \cdot - \nu))|. \tag{5.26}$$

Next note that

$$\left| 2^{nl'/2} \int_{\mathbb{R}^n} \varphi(2^l x - (m+\mu)) \overline{\eta(2^{l'} x - \nu)}\, dx \right| \le c\, 2^{nl'/2} 2^{-nl}, \tag{5.27}$$

while

$$|(H_{J,m+\mu}, 2^{nl'/2}\phi_0(2^{l'}\cdot - \nu))| \le \sum_{\xi \in \mathbf{Z}^n} |g_{\xi-\nu}|\,|(H_{J,m+\mu}, 2^{nl'/2}\varphi(2^{l'}\cdot - \xi))|. \quad (5.28)$$

Noting that

$$|(H_{J,m+\mu}, 2^{nl'/2}\varphi(2^{l'}\cdot - \xi))| \le c\,2^{nl'/2}2^{-nl'}\|H_{J,m+\mu}\|_\infty(\operatorname{supp}(\varphi(2^{l'}\cdot - \xi))), \quad (5.29)$$

recalling from (2.38) that

$$\eta_{e'} = \sum_{\nu \in \mathbf{Z}^n} q_\nu^{e'}\eta(\cdot - \nu),$$

where η has compact support, and that $\eta_J = \eta_{e',l'}^{l'}$, classical Whitney-type estimates for local polynomial approximation yield

$$\|H_{J,m+\mu}\|_\infty(\operatorname{supp}(\varphi(2^{l'}\cdot - \xi))) \le c\,2^{nl'/2}|q_{\xi-k'}^{e'}|, \quad (5.30)$$

whenever $|2^{-l'}\xi - 2^{-l}(m+\mu)| \le c\,2^{-l'}$, while otherwise the polynomial growth of p can be bounded, in view of the normalization of η_J, by

$$\|H_{J,m+\mu}\|_\infty(\operatorname{supp}(\varphi(2^{l'}\cdot - \xi))) \le c\,2^{nl'/2}2^{l'd^*}|q_{\xi-k'}^{e'}|\,|2^{-l}(m+\mu) - 2^{-l'}\xi|^{d^*}. \quad (5.31)$$

Thus, we infer from (5.30) and (5.31) the bound

$$\|H_{J,m+\mu}\|_\infty(\operatorname{supp}(\varphi(2^{l'}\cdot - \xi))) \le c\,2^{nl'/2}|q_{\xi-k'}^{e'}|(1 + |2^{l'-l}(m+\mu) - \xi|)^{d^*}, \quad (5.32)$$

which holds for all ξ. Now we conclude from (5.28), (5.29) and (5.32) that

$$|(H_{J,m+\mu}, 2^{nl'/2}\phi_0(2^{l'}\cdot - \nu))| \le c\sum_{\xi \in \mathbf{Z}^n} |g_{\xi-\nu}|\,|q_{\xi-k'}^{e'}|(1 + |2^{l'-l}(m+\mu) - \xi|)^{d^*}, \quad (5.33)$$

so that (5.26), (5.27) and (5.33) provide

$$\begin{aligned} &|(\varphi_{m+\mu}^l, (\tilde{Q}_{l'+1} - \tilde{Q}_{l'})^*\eta_J)_0| \\ &\quad \le c\,2^{n(l'-l)/2}\max_{\nu \in \Omega_{m+\mu}^{l,l'}}\sum_{\xi \in \mathbf{Z}^n}|g_{\xi-\nu}q_{\xi-k'}^{e'}|(1 + |2^{l'-l}(m+\mu) - \xi|)^{d^*}, \end{aligned} \quad (5.34)$$

and therefore, on account of (5.22) and (5.34),

$$\begin{aligned} &|(g_{l,k}, (\tilde{Q}_{l'+1} - \tilde{Q}_{l'})^*\eta_J)_0| \\ &\quad \le c\,2^{n(l'-l)/2}\sum_{m \in \mathbf{Z}^{n,l}}\sum_{\mu \in \mathbf{Z}^{n,l}}|c_l(m)g_\mu^l|\max_{\nu \in \Omega_{m+\mu}^{l,l'}}\sum_{\xi \in \mathbf{Z}^n}|g_{\xi-\nu}q_{\xi-k'}^{e'}|(1 + |2^{l'-l}(m+\mu) - \xi|)^{d^*}. \end{aligned} \quad (5.35)$$

Recalling that

$$\nu \in \Omega^{l,l'}_{m+\mu} \quad \text{iff } |\nu - 2^{l'-l}(m+\mu)| \le c, \tag{5.36}$$

for some constant c, the right-hand side of (5.35) can be bounded by

$$c\, 2^{(l'-l)n/2} \sum_{m,\mu \in \mathbf{Z}^{n,l}} |c_l(m) g^l_\mu| \max_{\nu \in \Omega^{l,l'}_{m+\mu}} \sum_{\xi \in \mathbf{Z}^n} |g_{\xi-\nu} g^{e'}_{\xi-k'}| (1 + |\nu - \xi|)^{d^*}.$$

Since the coefficients $g_\xi, q^{e'}_\xi$ decay exponentially quickly, a nonperiodic counterpart of lemma 4.1 and (5.25) yields a bound of the type

$$c\, 2^{(l'-l)n/2} \sum_{m,\mu \in \mathbf{Z}^{n,l}} |c_l(m) g^l_\mu| \max_{\nu \in \Omega^{l,l'}_{m+\mu}} (\mathrm{e}^{-a|\nu-k'|}(1 + |\nu - k'|)^{d^*}),$$

where a is some positive real number. Applying lemma 4.1 and using (5.36) again, we can estimate the latter expression by

$$c\, 2^{(l'-l)n/2} \sum_{m \in \mathbf{Z}^{n,l}} |c_l(m)|\, \mathrm{e}^{-a'|2^{l'-l}m-k'|} (1 + |2^{l'-l}m - k'|)^{d^*}.$$

Since, by assumption, $|k' - 2^{l'-l}k| \le c$, and estimating the coefficients $c_l(m)$ in analogy to (5.9), we obtain the bound

$$\begin{aligned} &c\, 2^{(l'-l)n/2} \sum_{m \in \mathbf{Z}^n} 2^{rl}(1 + |k - m|)^{-n-1-d^*-r} 2^{d^*(l'-l)} (1 + |m - k|)^{d^*} \mathrm{e}^{-a'|2^{l'-l}m-k'|} \\ &\quad \le c\, 2^{(l'-l)(n/2+d^*)} 2^{rl}, \end{aligned}$$

where c is independent of l', l. Thus, in summary, we arrive at the estimate

$$|(2^{-rl/2} A^{\#} \psi_I, 2^{-rl'/2} \eta_J)_0| \le c\, 2^{(l'-l)(n/2+d^*-r/2)}, \tag{5.37}$$

which, in view of lemma 5.1, confirms the claim for $l' < l$ whenever $d^* > r/2$ with $\delta := d - r/2$.

When $l' \ge l$, we interchange the roles of η_I and ψ_J so that, when the ψ_J are biorthogonal or pre-wavelets, the Q_j are replaced by the operators B_j from (2.32) or by the orthogonal projectors P_{Vj} (2.33), respectively. Since we have assumed that γ and ζ have the same degree of exactness d^* and since we have also made the convention that $d = d^*$ when dealing with pre-wavelets, the proof is complete. □

Note that, when dealing with pre-wavelets, the decay rate of the matrix entries is directly tied to the accuracy of the Petrov–Galerkin scheme, which will be seen later to constrain the efficiency of the compression with regard to resulting overall accuracy of corresponding approximate solutions.

Note also that for $r < 0$, a simpler argument is available when assuming that η is Hölder continuous of order $\rho > 0$ and $\sum_{k \in \mathbf{Z}^n} \eta(\cdot - k) = 1$. Suppose again that (5.16) holds. Similarly as before, let

$$g_{l,k} = 2^{-lr/2} Q_l A^{\#} \psi^l_{e,k} = \sum_{m \in Z^{n,l}} c(m) \phi^l_m,$$

i.e. $c(m) = 2^{-lr/2}(A^{\#}\psi_l, \eta^l_m)_0$. Since $\sum_{k \in Z^{n,l}} \eta^l_k$ is constant, and since $(A^{\#})^* 1 = 0$, we conclude that

$$\sum_{m \in Z^{n,l}} c(m) = 0, \tag{5.38}$$

and

$$\int_{\mathcal{T}^n} g_{l,k}(x)\, \mathrm{d}x = 0. \tag{5.39}$$

Since η_e is assumed to be Hölder continuous with some Hölder coefficient $\rho > 0$ we may use (5.39), (5.10) to obtain

$$\begin{aligned}
&2^{-rl'/2} \Big| \int_{\mathcal{T}^n} \overline{g_{l,k}(x)}\, \eta^{l'}_{e',k'}(x)\, \mathrm{d}x \Big| \\
&\leq c\, 2^{-rl'/2} \Big| \int_{\mathcal{T}^n} \overline{g_{l,k}(x)}\, (\eta^{l'}_{e',k'}(x) - \eta^{l'}_{e',k'}(2^{-l}k))\, \mathrm{d}x \Big| \\
&\leq c\, 2^{(n-r)(l'+l)/2} \int_{\mathcal{T}^n} 2^{lr} (2^{l'} | \theta(2^{-l}k - x) |)^{\rho} (1 + 2^l | \theta(x - 2^{-l}k) |)^{-n-r-d^*-1} \mathrm{d}x \\
&\leq c\, 2^{(n-r)(l'+l)/2} \int_{\mathbb{R}^n} 2^{lr} | 2^{l'-l}k - 2^{l'}x |^{\rho} (1 + | 2^l x - k |)^{-n-r-d^*-1} \mathrm{d}x \\
&< c\, 2^{-(l-l')(\rho + (n-r)/2)},
\end{aligned} \tag{5.40}$$

where c does not depend on l, l', k, k'. By the same reasoning as used above at the beginning of the proof of lemma 5.1, we establish the assertion in the case $r \leq 0$ for $\delta := \rho - r/2$.

For the proof of the next result, we need a version of the well-known Schur lemma (cf. [34]).

LEMMA 5.3

Let $\mathrm{A} = (a_{ij})_{i,j \in \mathbb{N}}$ be an infinite matrix and $\gamma(i) > 0$, $i \in \mathbb{N}$. If for some positive constant c one has

$$\sum_{i \in \mathbb{N}} | a_{ij} | \gamma(i) \leq c\, \gamma(j) \quad \text{for all } j \in N \tag{5.41}$$

and

$$\sum_{i \in \mathbb{N}} | a_{ij} | \gamma(j) \leq c\, \gamma(i) \quad \text{for all } i \in N \tag{5.42}$$

then the operator $A : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is bounded and has operator norm less than or equal to c.

We are now ready for the first step towards compressing the matrix $\mathbf{B}^j$.

LEMMA 5.4

Let $d^*, d'+\rho > r/2$, $n+d^*+r+1>0$ and $d^*+1+r/2>0$. Furthermore, let $\mathbf{B}^j_\varepsilon$ be defined as follows:

$$b^\varepsilon_{I,J} = \begin{cases} 0 & \text{if } (I,J) \in \mathcal{R}_1(\varepsilon_1), \\ b_{I,J} & \text{otherwise,} \end{cases} \tag{5.43}$$

where

$$\mathcal{R}_1(\varepsilon_1) := \{(I,J) \in \mathcal{J}^j \times \mathcal{J}^j : I = (l,e,k),\ J = (l',e',k'),$$
$$l, l' > 0,\ 2^{\min\{l,l'\}} |\theta(2^{-l}k - 2^{-l'}k')| \geq \varepsilon_1^{-1}\}. \tag{5.44}$$

Then there exists a positive constant c such that

$$\| \mathbf{B}^j - \mathbf{B}^j_\varepsilon \|_{\mathcal{L}(\ell_2(\mathcal{J}^j))} \leq c \min\{1, \varepsilon_1^{d^*+1+r}\} \tag{5.45}$$

holds uniformly in $j \in \mathbb{N}_0$.

Proof

We wish to apply lemma 5.3 with $\gamma(I) = 2^{-ln/2}$ and $a_{I,J} := b_{I,J} - b^\varepsilon_{I,J}$. Our concern is to estimate

$$\sum_{l' \in \mathbb{N}_0} \sum_{\{k' \in \mathbb{Z}^{n,l'} : 2^M |\theta(2^{-l}k - 2^{-l'}k')| \geq \varepsilon_1^{-1}\}} 2^{-l'n/2} |b_{I,J}|, \tag{5.46}$$

where $M := \min\{l, l'\}$. Let us rewrite this expression as

$$\left\{ \sum_{l' \geq l} + \sum_{l' < l} \right\} \sum_{\{k' \in \mathbb{Z}^{n,l'} : 2^M |\theta(2^{-l}k - 2^{-l'}k')| \geq \varepsilon_1^{-1}\}} 2^{-l'n/2} |b_{I,J}| := \Sigma_+ + \Sigma_-.$$

Assume first that $l' \geq l$. According to lemma 5.1 and (4.4), we may estimate the summands for $k, k' \in \mathbb{Z}^n$, $2^{-l}k - 2^{-l'}k' \in [-1/2, +1/2]^n$ to conclude that Σ_+ is bounded by

$$\sum_{\{l' : l' \geq l\}} \sum_{\{k' : |2^{l-l'}k' - k| \geq \varepsilon_1^{-1}\}} \frac{2^{-l'n/2 - |l-l'|[n+r+2(d^*+1)]/2}}{(1 + |k - k'2^{l-l'}|)^{n+d^*+1+r}}. \tag{5.47}$$

Moreover, we obtain

$$\sum_{\{k' \in \mathbb{Z}^n : 2^l |2^{-l'}k' - 2^{-l}k| \geq \varepsilon_1^{-1}\}} 2^{-n(l'-l)} \left(\frac{1}{1 + |k - k'2^{l-l'}|} \right)^{n+d^*+1+r}$$
$$\leq c \int_{\{|x| > \varepsilon_1^{-1}\}} (1+|x|)^{-n-d^*-1-r} \mathrm{d}x$$
$$\leq c \min\{1, \varepsilon_1^{d^*+1+r}\}. \tag{5.48}$$

Inserting (5.48) into (5.47), we arrive at

$$\begin{aligned}\Sigma_+ &\le c\min\{1,\varepsilon_1^{d^*+1+r}\}\sum_{\{l':l'\ge l\}} 2^{-ln/2}2^{-|l-l'|(d^*+1+r/2)}\\ &\le c\,2^{-ln/2}\min\{1,\varepsilon_1^{d^*+1+r}\}. \end{aligned}\tag{5.49}$$

When $l'<l$, we set $D=l-l'$ and proceed as before, estimating first the interior sum in Σ_-. Noting that

$$\begin{aligned}&\sum_{\{k':|2^{-D}k-k'|\ge\varepsilon_1^{-1}\}}(1+|2^{-D}k-k'|)^{-n-d^*-1-r}\\ &\qquad\le c\int_{\{|2^{-D}k-x|\ge\varepsilon_1^{-1}\}}(1+|2^{-D}k-x|)^{-n-d^*-1-r}\mathrm{d}x\\ &\qquad\le c\min\{1,\varepsilon_1^{d^*+1+r}\},\end{aligned}\tag{5.50}$$

we estimate

$$\sum_{l'<l}\ \sum_{\{k'\in\mathbb{Z}^{n,l'}:2^{l'}|\theta(2^{-l}k-2^{-l'}k')|\ge\varepsilon_1^{-1}\}} 2^{-l'n/2}|b_{I,J}| \tag{5.51}$$

by

$$\begin{aligned}&\min\{1,\varepsilon_1^{d^*+1+r}\}\sum_{\{l':l'\le l\}}2^{-l'n/2}2^{-|l-l'|[n+2(d^*+1)+r]/2}\\ &\qquad\le c\min\{1,\varepsilon_1^{d^*+1+r}\}2^{-ln/2}\sum_{D\ge0}2^{-D(d^*+1+r)}\\ &\qquad\le c\min\{1,\varepsilon_1^{d^*+1+r}\}2^{-ln/2},\end{aligned}\tag{5.52}$$

which completes the proof. □

To examine the number of remaining nonzero entries in $\mathbf{B}_\varepsilon^j$, note that the matrices $\mathbf{A}^j$, $\mathbf{B}^j$, $\mathbf{T}^j$ have an obvious block structure induced by the different levels $-1\le l<j$. By the same arguments as detailed in section 5.4 below, one can show that inside each block, i.e. for fixed l, l', after the compression described by lemma 5.4 there remain at least $O(N)$, $N=2^{jn}$, nonzero entries. Adding over $j^2=(\log_2 N)^2$ different blocks, we end up with at most $O(N(\log_2 N)^2)$ nonzero coefficients.

Next we want to improve upon this compression in that we will get rid of the logarithmic terms.

THEOREM 5.1

Suppose d^*, $d'+\rho>r/2$, $d^*+1+r/2>0$, $n+d^*+r+1>0$ and let γ satisfy $\mathbf{C}_0^{d^{*\prime},d^*}$. Let $\mathbf{T}_\varepsilon^j=(t_{I,J}^\varepsilon)_{I,J\in\mathcal{J}^j}$ be defined by

$$t_{I,J}^{\varepsilon} = \begin{cases} 0 & \text{if } (I,J) \in \mathcal{R}_2(\varepsilon_2) \cup \mathcal{R}_1(\varepsilon_1), \\ t_{I,J} & \text{otherwise,} \end{cases} \tag{5.53}$$

where

$$\mathcal{R}_2(\varepsilon_2) := \{(I,J) \in \mathcal{J}^j \times \mathcal{J}^j : I = (l,e,k), J = (l',e',k'), |l - l'| \geq \varepsilon_2^{-1}\}.$$

Then there exist positive constants c, δ such that for all $j \in \mathbb{N}_0$ and $\varepsilon_1, \varepsilon_2 \geq 0$

$$\|\mathbf{T}^j - \mathbf{T}_\varepsilon^j\|_{\mathcal{L}(\ell_2(\mathcal{J}^j))} \leq c\left(\min\{1, \varepsilon_1^{d^*+1+r}\} + 2^{-\delta/\varepsilon_2}\right). \tag{5.54}$$

Letting $\varepsilon_1, \varepsilon_2$ tend to infinity, we readily conclude

COROLLARY 5.1

Under the assumptions of theorem 5.1, the operators $\mathbf{T}^j : \ell_2(\mathcal{J}^j) \to \ell_2(\mathcal{J}^j)$ are uniformly bounded.

Proof

Since, in contrast to lemma 5.4, our truncation now affects also levels which are far apart from each other, we have to estimate in addition the sum

$$\sum_{\{l' : |l-l'| \geq \varepsilon_2^{-1}\}} \sum_{k' \in \mathbb{Z}^{n,l'}} 2^{-l'n/2} |t_{I,J}|. \tag{5.55}$$

Restricting the summation in (5.49) and (5.52) to $D \geq \varepsilon_2^{-1}$, and letting ε_1 tend to infinity, yields the bound $c 2^{-ln/2} 2^{-\delta/\varepsilon_2}$, where δ is any positive number less than or equal to $d^* + 1 + r/2$. On account of lemma 5.4, this proves the assertion. □

It should be mentioned that similar arguments have been used by Meyer for his wavelet proof of the T1 theorem (see [34]).

Throughout the remainder of this section, we will assume that the hypotheses of theorem 5.1 are satisfied.

Obviously, the compressed matrices $\mathbf{T}_\varepsilon^j$ have $O(N)$, $N = 2^{jn}$, nonzero entries. Thus, in principle, theorem 5.1 provides a criterion for deciding beforehand which entries of the stiffness matrix $\mathbf{B}^j$ must be computed in order to guarantee a required accuracy without ever computing the full matrix.

Note that the matrix which is relevant for computations has the form

$$\mathbf{B}_\varepsilon^j = (b_{I,J}^\varepsilon)_{I,J \in \mathcal{J}^j}, \quad b_{I,J}^\varepsilon := \begin{cases} t_{I,J}^\varepsilon; & I, J \notin \mathcal{J}_{-1}, \\ b_{I,J}; & I, J \in \mathcal{J}_{-1}. \end{cases} \tag{5.56}$$

In order to relate the above discrete estimates to our stability concept, we will make frequent use of the following simple facts. For $u^j = (u_I^j)_{I \in \mathcal{J}^j}$, let

$$u^j := \sum_{I \in \mathcal{J}^j} u_I^j \psi_I \in V^j.$$

Then any matrix $\mathbf{C}^j$ on $\ell_2(\mathcal{J}^j)$ induces an operator C_j by

$$C_j u^j := \sum_{I \in \mathcal{J}^j} (\mathbf{C}^j u^j)_I \phi_I, \tag{5.57}$$

where for $I = (j, e, k)$ we set $\phi_I := \phi_{e,k}^j$ (cf. (2.40), (2.38)). On account of the stability of the wavelet basis, we have

$$\| C_j \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \sim \| \mathbf{C}^j \|_{\mathcal{L}(\ell_2(\mathcal{J}^j))} \tag{5.58}$$

uniformly in $j \in \mathbb{N}_0$. Moreover, it is convenient to introduce

$$\mathbf{D}^j := (|I|^{r/2} \delta_{I,J})_{I,J \in \mathcal{J}^j}, \tag{5.59}$$

and let D_j be the operator on V^j induced by $\mathbf{D}^j$ as in (5.58). Then we infer from theorem 2.3 that

$$\| (\mathbf{D}^j)^{-1} u^j \|_{\ell_2(\mathcal{J}^j)} \sim \| D_j^{-1} u^j \|_0 \sim \| u^j \|_{r/2}. \tag{5.60}$$

Utilizing theorem 2.3 again also yields

$$\begin{aligned} \| C_j u^j \|_{-r/2}^2 &\sim \sum_{I \in \mathcal{J}^j} |I|^r |(\mathbf{C}^j u^j)_I|^2 \\ &= \| (\mathbf{D}^j \mathbf{C}^j \mathbf{D}^j)((\mathbf{D}^j)^{-1} u^j) \|_{\ell_2(\mathcal{J}^j)}^2. \end{aligned} \tag{5.61}$$

Thus, combining (5.60) and (5.61), we conclude that

$$\| C_j \|_{\mathcal{L}(H^{r/2}(\mathcal{T}^n), H^{-r/2}(\mathcal{T}^n))} \sim \| \mathbf{D}^j \mathbf{C}^j \mathbf{D}^j \|_{\mathcal{L}(\ell_2(\mathcal{J}^j))} \sim \| D_j C_j D_j \|_{\mathcal{L}(L_2(\mathcal{T}^n))}. \tag{5.62}$$

Specifically, we may write in these terms

$$\mathbf{B}^j = \mathbf{D}^j \mathbf{A}^j \mathbf{D}^j, \tag{5.63}$$

where $\mathbf{A}^j$ is the stiffness matrix (5.3) relative to the wavelet-type basis and $\mathbf{B}^j$ was defined in (5.4). Of course, the induced operator A_j is also given by

$$A_j = Q_j A P_j. \tag{5.64}$$

Similarly, the compression of $\mathbf{A}^j$ is given, for $\mathbf{B}_\varepsilon^j$ defined in (5.56), by

$$\mathbf{A}_\varepsilon^j := (\mathbf{D}^j)^{-1} \mathbf{B}_\varepsilon^j (\mathbf{D}^j)^{-1}, \tag{5.65}$$

which, in turn, induces the operator A_j^ε via (5.57).

Now recall from (3.12) that the scheme (3.7) is called (s, r)-stable if

$$\|A_j u^j\|_{s-r} \geq c\|u^j\|_s \tag{5.66}$$

for some constant c independent of j [17].

As an immediate consequence of (5.62) and (5.63), we may now state

COROLLARY 5.2

The scheme (3.7) is $(r/2, r)$-stable if and only if

$$\|(\mathbf{B}^j)^{-1}\|_{\mathscr{L}(\ell_2(\mathscr{J}^j))} = O(1), \quad j \to \infty.$$

Thus, combining theorem 5.1 for sufficiently small ε with a simple perturbation argument based on Neumann series leads to the following conclusion.

COROLLARY 5.3

Suppose the scheme (3.7) is $(r/2, r)$-stable. Then there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$

$$\|(\mathbf{B}^j_\varepsilon)^{-1}\|_{\mathscr{L}(\ell_2(\mathscr{J}^j))} = O(1), \quad j \to \infty,$$

with a constant depending only on ε_0. Thus, also the compressed matrices $\mathbf{B}^j_\varepsilon$ have uniformly bounded condition numbers. In particular, the compressed schemes are $(r/2, r)$-stable as well.

The above observations also ensure that the compressed schemes can be efficiently treated by means of conjugate gradient like methods.

5.2. ERROR ESTIMATES FOR FIXED PRESCRIBED ACCURACY

We will now apply the above results to estimating the accuracy of the solutions to the compressed schemes relative to the solutions to (3.7).

Let $f^j_I = \eta_I(f)$ and $f^j = (f^j_I)_{I \in \mathscr{J}^j}$. By corollary 5.3, we know that the system of equations

$$\mathrm{A}^j_\varepsilon u^j_\varepsilon = f^j \tag{5.67}$$

has a unique solution $u^j_\varepsilon = (u^j_{\varepsilon,I})_{I \in \mathscr{J}^j}$ provided that $\varepsilon < \varepsilon_1$ for some sufficiently small $\varepsilon_1 > 0$. The corresponding approximate solution in V^j has therefore the form

$$u^j_\varepsilon = \sum_{I \in \mathscr{J}^j} u^j_{\varepsilon,I} \psi_I. \tag{5.68}$$

Of course, as said before, the solution u^j of (3.7) reads

$$u^j = \sum_{I \in \mathcal{J}^j} u_I^j \psi_I.$$

By construction, the compressed matrices $\mathbf{A}_\varepsilon^j$ have only $O(2^{nj})$ nonzero entries where the constant depends on ε but not on $j \in \mathbb{N}$. We are now in a position to compare the solution to the compressed schemes (5.67) to those of the exact scheme (3.7).

THEOREM 5.2

Suppose (3.7) is $(r/2, r)$-stable. Then there exists some $\varepsilon_0 > 0$, $\varepsilon_0 \leq \varepsilon_1$ and a function $q(\varepsilon)$ which tends to zero as ε tends to zero such that for every $\varepsilon > 0$, $\varepsilon < \varepsilon_0$ and for every $j \in \mathbb{N}_0$

$$\| u_\varepsilon^j - u^j \|_{r/2} \leq q(\varepsilon) \| u^* \|_{r/2}, \tag{5.69}$$

where u^* is the exact solution of (3.4).

Proof

We infer from (5.62), (5.63) and theorem 5.1 that

$$\begin{aligned} \| (A_j^\varepsilon - A_j) v^j \|_{-r/2} &\leq c \| \mathbf{B}^j - \mathbf{B}_\varepsilon^j \|_{\mathcal{L}(\ell_2(\mathcal{J}^j))} \| v^j \|_{r/2} \\ &\leq c\, q(\varepsilon) \| v^j \|_{r/2}, \qquad v^j \in V^j, \end{aligned} \tag{5.70}$$

where $q(\varepsilon)$ is given on the right-hand side of (5.54). Let us further introduce the operator

$$C_j^\varepsilon = A_j^{-1}(A_j - A_j^\varepsilon). \tag{5.71}$$

In view of (5.70) and $(r/2, r)$-stability, we obtain

$$\begin{aligned} \| C_j^\varepsilon v^j \|_{r/2} &\leq c \| (A_j - A_j^\varepsilon) v^j \|_{-r/2} \\ &\leq c\, q(\varepsilon) \| v^j \|_{r/2}. \end{aligned} \tag{5.72}$$

For sufficiently small $\varepsilon_0 > 0$, $\varepsilon_0 \leq \varepsilon_1$ and $\varepsilon < \varepsilon_0$, the operator C_j^ε therefore becomes a contraction. Since $u_\varepsilon^j \in V^j$ is easily seen to satisfy the relation

$$(I - C_j^\varepsilon) u_\varepsilon^j = u^j,$$

it can be expressed by a Neumann series

$$u_\varepsilon^j = \sum_{l \geq 0} (C_\varepsilon^j)^l u^j,$$

provided that $0 < \varepsilon < \varepsilon_0$. Therefore, we obtain

$$\begin{aligned}\| u_\varepsilon^j - u^j \|_{r/2} &= \| (C_j^\varepsilon \sum_{l=0}^{\infty} (C_j^\varepsilon)^l) u^j \|_{r/2} \\ &\le c\, q(\varepsilon) \| \sum_{l=0}^{\infty} C_j^\varepsilon u^j \|_{r/2} \\ &\le c\, q(\varepsilon) \| u^j \|_{r/2}.\end{aligned}$$

Finally, $(r/2, r)$-stability ensures the uniform bound $\| u^j \|_{r/2} \le c \| u^* \|_{r/2}$. □

In view of the estimates for $\| u^j - u^* \|_s$ established in [17], one can now also estimate the deviation of u_ε^j from the exact solution u^* of the original equation.

5.3. ASYMPTOTIC ERROR BOUNDS

The limited precision of the computer gives rise to fixed accuracy requirements as considered above. On the other hand, such error tolerances should be balanced with regard to the convergence behavior of the solutions of the exact discrete problem (3.7), i.e. the error between $\mathbf{B}^j$ and $\mathbf{B}_\varepsilon^j$ should be of the same order as the convergence rate of the exact solutions u^j of (3.7). We are aware that the following results are at that stage still of primarily theoretical nature. However, they should indicate the potential of multiscale techniques in the present context.

We will adhere to the above notation as well as to the various assumptions made before. In fact, for $\mathbf{u}^j = (u_J)_{J \in \mathcal{J}^j}$ and $u_J^j = u_J$, we put as before $u^j = \sum_J u_J \psi_J$. We may then rewrite the operator

$$B_j^\varepsilon u^j = \sum_{I \in \mathcal{J}^j} (\mathbf{B}_\varepsilon^j \mathbf{u}^j)_I \phi_I, \tag{5.73}$$

and recall that multiplying the coefficients of $\mathbf{B}_\varepsilon^j$ by $|I|^{-r/2} |J|^{-r/2}$ gives rise to its unpreconditioned counterpart

$$A_j^\varepsilon u^j = \sum_{I,J \in \mathcal{J}^j} (|I|\,|J|)^{-r/2} (\mathbf{B}_\varepsilon^j)_{I,J} (\mathbf{u}^j)_J \psi_I, \tag{5.74}$$

corresponding to (5.65).

The next lemma identifies the type of estimates needed to establish convergence rates.

LEMMA 5.5

Suppose the solution u^* of $Au = f$ belongs to $H^t(\mathcal{T}^n)$. Let $s \le t$, $-d' - \rho + r/2 < t \le d + 1$ and $-d - 1 + r/2 < s < d' + \rho$, where ρ denotes the Hölder exponent of φ and where $-d - 1 + r/2$ is the lower bound for the Sobolev scale in which one has quasi-optimal convergence for the Petrov–Galerkin scheme, see [17]. If A_j^ε is (s, r)-stable, i.e.

$$\| A_j^\varepsilon u^j \|_{s-r} \geq c \, \| u^j \|_s \quad \text{for all } u^j \in V^j$$

and if additionally

$$\| (A_j - A_j^\varepsilon) P_j u \|_{s-r} \leq w(\varepsilon) \, 2^{j(s-t)} \| u \|_t, \tag{5.75}$$

for some function $w(\varepsilon)$, then the solution u_ε^j of the equation

$$A_j^\varepsilon u^j = Q_j f$$

satisfies the error estimate

$$\| u^* - u_\varepsilon^j \|_s \leq c(1 + w(\varepsilon)) \, 2^{j(s-t)} \| u^* \|_t. \tag{5.76}$$

Moreover, when (5.75) holds for $t < d + \rho$ and $w(\varepsilon) \to 0$ as $\varepsilon \to 0$, one has for $s \leq \tau \leq t$

$$\| u^j - u_\varepsilon^j \|_s \leq c \, w(\varepsilon) 2^{j(s-\tau)} \| u^j \|_\tau \leq c \, w(\varepsilon) 2^{j(s-\tau)} \| u^* \|_\tau,$$

where u^j is the solution of (3.7).

Proof

Since A_j^ε is stable, we estimate in a standard fashion

$$\begin{aligned}
\| u^* - u_\varepsilon^j \|_s &\leq \| u^* - P_j u^* \|_s + c \, \| A_j^\varepsilon (P_j u^* - u_\varepsilon^j) \|_{s-r} \\
&\leq \| u^* - P_j u^* \|_s + c(\| Q_j f - f \|_{s-r} + \| A_j^\varepsilon P_j u^* - A u^* \|_{s-r}) \\
&\leq \| u^* - P_j u^* \|_s + c(\| Q_j f - f \|_{s-r} + \| A_j u^* - A u^* \|_{s-r} \\
&\quad + \| A_j^\varepsilon P_j u^* - A_j u^* \|_{s-r}).
\end{aligned} \tag{5.77}$$

The first three terms in (5.77), except the last one, have already been estimated in [17] (see (6.35)–(6.38) in the proof of theorem 6.3). It is shown there that they yield the desired optimal order of convergence. Condition (5.75) implies the same order for the remaining term, whence (5.76) follows.

As for the remaining part of the assertion, we again define $C_j^\varepsilon := A_j^{-1}(A_j - A_j^\varepsilon)$ and observe that (s, r)-stability combined with (5.75) yields for $v_j \in V^j$

$$\| C_j^\varepsilon v^j \|_s \leq c \, w(\varepsilon) 2^{j(s-t)} \| v^j \|_t.$$

Thus, applying the inverse estimate (4.2) to both sides of the above inequality, we obtain

$$\| C_j^\varepsilon P_j \|_{\mathcal{L}(H^\tau(\mathcal{T}^n))} := \sup_{\|u\|_\tau = 1} \| C_j^\varepsilon P_j u \|_\tau \le c\, w(\varepsilon)$$

for $s \le \tau \le t$. We may now follow the reasoning in the proof of theorem 5.2 to conclude that

$$\| u_\varepsilon^j - u^j \|_s = \| C_j^\varepsilon P_j \sum_{l=0}^{\infty} (C_j^\varepsilon P_j)^l u^j \|_s \le c\, w(\varepsilon) 2^{j(s-t)} \| \sum_{l=0}^{\infty} (C_j^\varepsilon P_j)^l u^j \|_t$$

$$\le c\, w(\varepsilon) 2^{j(s-t)} \sum_{l=0}^{\infty} (c\, w(\varepsilon))^l \| u^j \|_t \le c\, w(\varepsilon) 2^{j(s-t)} \| u^j \|_t,$$

whenever ε is sufficiently small so that $c\, w(\varepsilon) < 1$. Thus, the first estimate follows again from (4.2). Invoking theorem 6.3 from [17] completes the proof. □

Note that in the above proof, we have exploited the fact that the regularity of γ can be chosen independently of the actual scheme (3.7).

As mentioned before, the stiffness matrix in wavelet representation has an obvious block structure. It splits into blocks $\mathbf{B}^{l,l'} = (b_{I,J})_{I \in \mathcal{J}_l, J \in \mathcal{J}_{l'}}$. Each block gives rise to a block operator

$$B_{l,l'} u^j(x) = \sum_{I \in \mathcal{J}_l, J \in \mathcal{J}_{l'}} b_{I,J} u_J \phi_I. \tag{5.78}$$

In order to establish asymptotic estimates, we will again have to modify previous types of compression so as to ensure that accuracy improves with decreasing mesh size. Our next concern is then to estimate the effect of such a modified compression in each of the block matrices, respectively in the block operators (5.78).

LEMMA 5.6

Let $r \ge 0$, $t \le d + 1$. Set

$$M = \frac{t + r/2}{d^* + 1 + r/2} \tag{5.79}$$

and

$$\mathcal{R}_\varepsilon := \{ (I, J) \in \mathcal{J}^j \times \mathcal{J}^j : I = (l, e, k), J = (l', e', k'),$$

$$2^{\min\{l,l'\}} | \theta(2^{-l}k - 2^{-l'}k') | \ge 2^{M(j - \max\{l,l'\})} \varepsilon_1^{-1} \}, \tag{5.80}$$

where ε_1 will be a function of ε to be specified later. Let $\mathrm{T}_\varepsilon^j = (t_{I,J}^\varepsilon)_{I,J \in \mathcal{J}^j}$ be defined by

$$t_{I,J}^\varepsilon = \begin{cases} t_{I,J}, & \text{if } (I,J) \in \mathcal{R}_\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{5.81}$$

Furthermore, let the operators $T^{\varepsilon}_{l,l'}$ be defined in analogy to (5.78) by (5.57) relative to the matrix T^j. Then for $0 \le \delta \le \min\{d^* + 1 - t, d^* + 1 + r/2 - s\}$ and $0 \le s < d^* + 1 + r/2$, there exists a constant c independent of l, l' and j such that

$$2^{(l'-l)s} \| T^{\varepsilon}_{l,l'} \|_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, 2^{(l'-j)(t+r/2)} 2^{-|l-l'|\delta} \varepsilon_1^{d^*+1+r}. \tag{5.82}$$

Proof

Let $\delta := d^* + 1 - t$. In view of (5.58), the norms of the operators

$$2^{(l'-l)s} 2^{-l'(t+r/2)} T^{\varepsilon}_{l,l'} = 2^{(l'-l)s} 2^{-l'(d^*+1+r/2-\delta)} (Q_l - Q_{l-1}) T^{\varepsilon}_{l,l'} (P_{l'} - P_{l'-1}) \tag{5.83}$$

in $L_2(\mathscr{T}^n)$ can be bounded by estimating the norm of the corresponding matrices in l^2, which, in turn, will be accomplished by following the lines of lemma 5.4.

Let us first assume that $l' \le l$ and set $D = |l - l'| = l - l'$. Assume first $\delta > 0$, i.e. $M < 1$. Upon applying lemma 5.1, we obtain

$$\sum_{\{k' \in Z^{n,l'} : 2^{l'} |\theta(2^{-l}k - 2^{-l'}k')| \ge \varepsilon_1^{-1} 2^{M(j-l)}\}} 2^{-l'n/2} 2^{-l'(d^*+1+r/2-\delta)} 2^{(l'-l)s} |t_{I,J}|$$

$$\le c \sum_{\{k' : |2^{-D}k - k'| \ge (2^{M(l-j)}\varepsilon_1)^{-1}\}} 2^{-ln/2} 2^{-l(d^*+1+r/2-\delta)} 2^{(l'-l)\delta} (1 + |2^{-D}k - k'|)^{-n-d^*-1-r}$$

$$\le c\, 2^{-|l-l'|\delta} \int_{\{|2^{-D}k - x| \ge 2^{M(j-l)}\varepsilon_1^{-1}\}} 2^{-ln/2} 2^{-l(d^*+1+r/2-\delta)} |2^{-D}k - x|^{-n-d^*-1-r} dx$$

$$\le c\, 2^{-ln/2} \varepsilon_1^{d^*+1+r} 2^{(l-j)(t+r/2)\frac{d^*+1+r}{d^*+1+r/2}} 2^{(l'-l)\delta} 2^{-l(t+r/2)}$$

$$\le c\, 2^{-ln/2} \varepsilon_1^{d^*+1+r} 2^{-j(t+r/2)} 2^{(l-j)(t+r/2)\frac{r/2}{d^*+1+r/2}} 2^{(l'-l)\delta}. \tag{5.84}$$

We note that $l \le j$ and apply lemma 5.3 to conclude that

$$2^{(l'-l)s} 2^{-l'(t+r/2)} \| T^{\varepsilon}_{l,l'} \|_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, 2^{-|l-l'|\delta} \varepsilon_1^{d^*+1+r} 2^{-j(t+r/2)}. \tag{5.85}$$

When $M = 1$, we have $\delta = 0$ and obtain the bound

$$2^{(l'-l)s} 2^{-l'(d^*+1+r/2)} \| T^{\varepsilon}_{l,l'} \|_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, \varepsilon_1^{d^*+1+r} 2^{-j(d^*+1+r/2)}.$$

When $l' > l$, we again assume first that $M < 1$ and estimate (5.83) by

$$\sum_{\{k' \in \mathbb{Z}^{n,l'}: 2^l |\theta(2^{-l'}k' - 2^{-l}k)| \geq \varepsilon_1^{-1} 2^{M(j-l')}\}} \frac{2^{(s-n)(l'-l)} 2^{-l'(t+r/2)} 2^{(l-l')(d^*+1+r/2)}}{(1 + |k - k' 2^{l-l'}|)^{n+d^*+1+r}}$$

$$\leq c \sum_{\{k': |2^{l-l'}k' - k| \geq \varepsilon_1^{-1} 2^{M(j-l')}\}} 2^{-n(l'-l)} 2^{-l'(d^*+1+r/2-\delta)} 2^{(l-l')\delta} \left(\frac{1}{1 + |k - k' 2^{l-l'}|} \right)^{n+d^*+1+r}$$

$$\leq c\, 2^{-l'(d^*+1+r/2-\delta)} 2^{(l-l')\delta} \int_{\{|x| > \varepsilon_1^{-1} 2^{M(j-l')}\}} |x|^{-n-d^*-1-r} dx,$$

where we have now set, in view of our assumption $0 \leq s < d^* + 1 + r/2$, $\delta = \min\{d^* + 1 + r/2 - s,\ d^* + 1 - t\}$. Arguing in a similar fashion as before, we again obtain

$$2^{(l'-l)s} 2^{-l'(t+r/2)} \| T^{\varepsilon}_{l,l'} \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \leq c\, 2^{-|l-l'|\delta} \varepsilon_1^{d^*+1+r} 2^{-j(t+r/2)}. \tag{5.86}$$

When $\delta = 0$, i.e. $t = d^* + 1 = d + 1$, we estimate the difference (5.83) by

$$2^{(l'-l)s} 2^{-l'(d^*+1+r/2)} \| T^{\varepsilon}_{l,l'} \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \leq c\, \varepsilon_1^{d^*+1+r} 2^{-j(d^*+1+r/2)},$$

which completes the proof. □

Remark 5.1

The estimate (5.82) contains a geometrically decaying term $2^{-|l-l'|\delta}$ which will be important for estimating the convergence of the approximate solutions below. Note that the case $\delta = 0$ in the above proof can only occur when $d^* + 1 = t \leq d + 1$, which means $d = d^*$ and $M = 1$. Thus, employing biorthogonal wavelets with $d^* > d$ allows us to avoid this unfavorable effect which could happen when using pre-wavelets as basis functions for the trial spaces and when the right-hand side has high regularity.

In order to patch all the different blocks together, the following block variant of Schur's lemma will be helpful.

LEMMA 5.7

Suppose that $T_{l,l'}$ are given bounded linear operators on $L_2(\mathcal{T}^n)$ with operator norms $|T_{l,l'}| := \| T_{l,l'} \|_{\mathcal{L}(L_2(\mathcal{T}^n))} < \infty$. Defining

$$T_{(j)} := \sum_{l,l'=1}^{j} (Q_l - Q_{l-1}) T_{l,l'} (P_{l'} - P_{l'-1}),$$

one has for every function $u^j \in V^j$ and for any $s \in \mathbb{R}$

$$\|T_{(j)}u^j\|_0 \le c\left(\sup_{l\le j}\sum_{l'}2^{s(l-l')}|T_{l',l}|\right)^{1/2}\left(\sup_{l'\le j}\sum_{l}2^{s(l'-l)}|T_{l,l'}|\right)^{1/2}\|u^j\|_0. \quad (5.87)$$

Proof

We can write u^j as $u^j = P_0u^j + \sum_{l=1}^{j}(P_l - P_{l-1})u^j =: \sum_{l=0}^{j} w^l$. Likewise, noting that $T_{(j)}u^j$ has, by definition, no component in Y_0, we can rewrite $f^j := T_{(j)}u^j$ as $f^j = \sum_{l=1}^{j}(Q_l - Q_{l-1})f^j = \sum_{l=1}^{j} h^l$.

One can easily check that $h^l = \sum_{l'}(Q_l - Q_{l-1})T_{l,l'}w^{l'}$. Thus, Hölder's inequality and the boundedness of the projectors Q_l assumed in the present setting yields, for any $s \in \mathbb{R}$,

$$\|h^l\|_0 \le c\sum_{l'}(|T_{l,l'}|2^{l's/2}2^{-l's/2}\|w^{l'}\|_0$$

$$\le c\left(\sum_{l'}(|T_{l,l'}|2^{l's}\right)^{1/2}\left(\sum_{l'}2^{-l's}|T_{l,l'}|\,\|w^{l'}\|_0^2\right)^{1/2}$$

$$\le c\left(2^{-ls}\sum_{l'}|T_{l,l'}|2^{l's}\right)^{1/2}\left(2^{ls}\sum_{l'}2^{-l's}|T_{l,l'}|\,\|w^{l'}\|_0^2\right)^{1/2}.$$

Squaring and summing over $l = 1, \ldots, j$ gives

$$\sum_l\|h^l\|_0^2 \le c\left(\sup_{l\le j}\sum_{l'}2^{(l'-l)s}|T_{l,l'}|\right)\left(\sup_{l'\le j}\sum_{l}2^{(l-l')s}|T_{l,l'}|\right)\sum_{l'}\|w^{l'}\|_0^2.$$

Recalling that the ψ_I and ϕ_I both form Riesz bases for $L_2(\mathcal{T}^n)$, the assertion follows. □

To arrive at estimates for Sobolev norms, we will invoke theorem 2.3 and recall from [17] that for $s < d' + \rho$

$$\|u^j\|_s^2 \sim \|P_0u^j\|_0^2 + \sum_{0<l\le j}\|2^{ls}(P_l - P_{l-1})u^j\|_0^2, \quad (5.88)$$

while for $s < d + 1$

$$\|P_0u^j\|_0^2 + \sum_{0<l\le j}\|2^{ls}(P_l - P_{l-1})u^j\|_0^2 \le c\|u^j\|_s^2. \quad (5.89)$$

It will also be useful to recall certain Besov norms already used in [17] which are equivalent to the Sobolev norms for a slightly larger range of s. To this end, define for $h \in \mathbb{R}^n$ the ℓth order forward differences of u by

$$(\Delta_h^\ell u)(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^{\ell-j} u(x+jh).$$

The corresponding ℓth order L_2-modulus of continuity is then given as

$$\omega_\ell(u,t)_2 := \sup_{|h| \le t} \| \Delta_h^\ell u \|_0. \tag{5.90}$$

We are now ready to introduce the Besov norm

$$\| u \|_{B_{2,2}^t}^2 := \| u \|_0^2 + | u |_{(t)}^2, \tag{5.91}$$

where for any fixed $\ell \in \mathbb{N}$, $\ell > t$,

$$| u |_{(t)}^2 := \sum_{j=0}^{\infty} 2^{2jt} \omega_\ell(u, 2^{-j})_2^2. \tag{5.92}$$

It is known that for $0 < t < \ell$, the set of all functions in $L_2(\mathcal{T}^n)$ for which the above expression is finite, agrees with $H^t(\mathcal{T}^n)$ and that

$$\| \cdot \|_t \sim \| \cdot \|_{B_{2,2}^t}, \quad 0 < t < \ell \tag{5.93}$$

(see e.g. [17], section 5).

In a similar fashion as earlier in this section, we may now proceed to prove the following result.

THEOREM 5.3

Let $r \ge 0$, and $s < d' + \rho$. Furthermore, let $\mathbf{T}_\varepsilon^j = (t_{IJ}^\varepsilon)_{I,J \in \mathcal{J}^j}$ be defined by lemma 5.6 relative to $\varepsilon_1 = \varepsilon_1(\varepsilon)$ given by

$$\varepsilon_1 = \begin{cases} j^{-\frac{3}{2(d^*+1+r)}} \varepsilon & \text{if } M = 1, \text{ i.e. } d+1 = d^*+1 = t, \\ j^{-\frac{1}{2(d^*+1+r)}} \varepsilon & \text{if } d+1 = t < d^*+1, \\ \varepsilon & \text{if } 0 < M < 1, t < d+1. \end{cases} \tag{5.94}$$

Finally, suppose that the operators A_j^ε are defined by (5.74) relative to $\mathbf{T}_\varepsilon^j$. If $d^{*\prime} + \rho^* > 3/2r$, then there exists some $\varepsilon_0 > 0$ and a positive constant c such that for $-r/2 \le s \le t \le d+1$

$$\| (A_j - A_j^\varepsilon) P_j u \|_{s-r} \le c\, \varepsilon^{d^*+1+r} 2^{-j(t-s)} \| u \|_t \tag{5.95}$$

holds uniformly in $j \in \mathbb{N}_0$ and $\varepsilon_0 \ge \varepsilon \ge 0$.

Proof

Expanding P_j, Q_j in telescopic sums, we may write

$$Q_j(A_j - A_j^\varepsilon)P_j = \sum_{1 \le l,l' \le j} 2^{(l+l')r/2} T^\varepsilon_{l,l'}.$$

We will apply lemma 5.6 and lemma 5.7, for $s = r$, to $u^j = \sum_{l'=1}^{j} 2^{l't}(P_{l'} - P_{l'-1})u$. The norm equivalence asserted by theorem 2.3, the fact that the operators Q_j are uniformly bounded in $L_2(\mathcal{T}^n)$, the stability of the wavelet basis, and taking the definition of $T^\varepsilon_{l,l'}$ into account, yields

$$\begin{aligned} \| Q_j(A_j - A_j^\varepsilon)P_j u \|^2_{-3r/2} &\le c \sum_{l=1}^{j} 2^{-3rl} \| (Q_l - Q_{l-1}) Q_j (A_j - A_j^\varepsilon) P_j u \|_0^2 \\ &\le c \sum_{l=1}^{j} \| 2^{-lr} \sum_{l'=1}^{j} T^\varepsilon_{l,l'} 2^{l'r/2} (P_{l'} - P_{l'-1}) u \|_0^2 , \end{aligned} \tag{5.96}$$

where we have used that

$$T^\varepsilon_{p,q} = (Q_p - Q_{p-1}) T^\varepsilon_{p,q} = T^\varepsilon_{p,q}(P_q - P_{q-1}).$$

Using lemma 5.7 and the norm equivalence theorem 2.3 again, the right-hand side of (5.96) can be estimated by

$$\begin{aligned} & c \| \sum_{0<l,l' \le j} 2^{(l'-l)r} T^\varepsilon_{l,l'} 2^{-l'(t+r/2)} 2^{l't} (P_{l'} - P_{l'-1}) u \|_0^2 \\ & \le c \left(\left(\sup_{0<l\le j} \sum_{0<l' \le j} 2^{(l'-l)r} 2^{-l'(t+r/2)} | T^\varepsilon_{l,l'} | \right) \left(\sup_{0<l' \le j} \sum_{0<l\le j} 2^{(l'-l)r} 2^{-l'(t+r/2)} | T^\varepsilon_{l,l'} | \right) \right) \| u^j \|_0^2 . \end{aligned} \tag{5.97}$$

When $t = d + 1 = d^* + 1$, i.e. $M = 1$, lemma 5.6 yields

$$\begin{aligned} & \| Q_j(A_j - A_j^\varepsilon)P_j u \|^2_{-3r/2} \\ & \qquad \le (c\, \varepsilon_1^{d^*+1+r} j 2^{-j(d+1+r/2)})^2 \sum_{0<l\le j} (2^{l(d+1)} \| (P_l - P_{l-1}) u \|_0)^2 \\ & \qquad \le (c\, \varepsilon_1^{d^*+1+r} j 2^{-j(d+1+r/2)})^2 \sum_{0<l\le j} 2^{2l(d+1)} (\| (I - P_l) u \|_0^2 + \| (I - P_{l-1}) u \|_0^2) \\ & \qquad \le (c\, \varepsilon_1^{d^*+1+r} j 2^{-j(d+1+r/2)})^2 \sum_{0<l\le j} \| u \|_{d+1}^2 \\ & \qquad \le (c\, \varepsilon^{d^*+1+r} 2^{-j(d+1+r/2)} \| u \|_{d+1})^2 , \end{aligned} \tag{5.98}$$

where we have used the Jackson estimate (4.1) and the definition of ε_1 in the last two steps. To treat the case $M < 1$, we recall the Whitney type estimate

$$\| u - P_j u \|_0 \leq c\, \omega_{d+1}(u, 2^{-j})_2 \tag{5.99}$$

from (5.10) in [17]. We can then repeat the above reasoning in (5.98), where now the application of lemma 5.6 involves also the term $2^{-\delta|l-l'|}$, see remark 5.1. Thus we obtain, by (5.99),

$$\begin{aligned} \| Q_j(A_j - A_j^\varepsilon)P_j u \|^2_{-3r/2} &\leq (c\, \varepsilon_1^{d^*+1+r} 2^{-j(t+r/2)})^2 \sum_{0<l\leq j} (2^{lt} \| (P_l - P_{l-1})u \|_0)^2 \\ &\leq (c\, \varepsilon_1^{d^*+1+r} 2^{-j(t+r/2)})^2 \sum_{l=0}^{j} 2^{2lt} \omega_{d+1}(u, 2^{-l})_2^2 \\ &\leq (c\, \varepsilon^{d^*+1+r} 2^{-j(t+r/2)} \| u \|_t)^2, \end{aligned} \tag{5.100}$$

where we have used the norm equivalence (5.93) in the last step. The case $t = d+1 < d^* + 1$ is treated in an analogous fashion, where now $\sum_{l=0}^{j} 2^{2l(d+1)} \| (P_l - P_{l-1})u \|_0^2$ is estimated by $j \| u \|_{d+1}^2$. Employing the inverse estimate (4.2), we finally obtain

$$\| Q_j(A_j - A_j^\varepsilon)P_j u \|^2_{s-r} \leq (c\, 2^{2j(s+r/2)} \| Q_j(A_j - A_j^\varepsilon)P_j u \|^2_{-3r/2},$$

whence the assertion now follows from (5.100). □

We may now resort to lemma 5.5 to estimate $u^* - u_\varepsilon^j$.

THEOREM 5.4

Let $A \in \Psi^r(\mathcal{T}^n)$, $0 \leq 3r/2 < d^{*'} + \rho$, $-r/2 \leq s \leq t \leq d+1$, $s \leq d' + \rho$, and $f \in H^{t-r}(\mathcal{T}^n)$. Suppose that the Petrov–Galerkin scheme (3.7) is (s, r)-stable (see (3.12)). Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the compressed scheme $A_j^\varepsilon u_\varepsilon^j = Q_j f$ has a unique solution u_ε^j whose deviation from the exact solution u^* of the equation $Au^* = f$ may be estimated by

$$\| u^* - u_\varepsilon^j \|_s \leq c(1 + \varepsilon^{d^*+1+r}) 2^{-j(t-s)} \| u^* \|_t \tag{5.101}$$

uniformly in $j > N_0$, $0 < \varepsilon < \varepsilon_0$.

Proof

Setting $t = s$, theorem 5.3 provides

$$\| (A_j - A_j^\varepsilon)P_j u \|_{s-r} \leq c\, \varepsilon^{d^*+1+r} \| u \|_s. \tag{5.102}$$

If the Petrov–Galerkin scheme $A_j = Q_j A P_j$ is (s, r)-stable in the sense of (3.12), then (5.102) ensures that there exists $\varepsilon_0 > 0$ such that A_j^ε is also (s, r)-stable for $0 < \varepsilon < \varepsilon_0$.

Combining theorem 6.3 of [17] with theorem 5.3 verifies the assumptions of lemma 5.5 which, in turn, yields the desired result. □

Next, we consider the case $r < 0$. Unfortunately, it then seems difficult to establish optimal convergence rates for some range of Sobolev scales $H^s(\mathcal{T}^n)$. Nevertheless, one can still handle the case $s = 0$. Again, one has to slightly modify the compression strategy (5.80) and (5.81) by changing the constant M while leaving $\varepsilon_1 = \varepsilon_1(j)$ unchanged.

LEMMA 5.8

Let $r < 0$, $0 \le t \le d + 1$. Setting

$$M = \frac{t' + r}{d^* + 1 + r}, \tag{5.103}$$

where $t' = t + \tau$ for for some $0 \le \tau \le |r|$, let

$$\begin{aligned}\mathcal{R}_\varepsilon := \{(I, J) \in \mathcal{J}^j \times \mathcal{J}^j : I = (l, e, k), J = (l', e', k'),\\ 2^{\min\{l,l'\}}|\theta(2^{-l}k - 2^{-l'}k')| > 2^{M(j-\max\{l,l'\})}\varepsilon_1^{-1}\},\end{aligned} \tag{5.104}$$

(where ε_1 is to be viewed as a function of ε which will be specified later). Define $\mathrm{T}_\varepsilon^j = (t_{I,J}^\varepsilon)_{I,J\in\mathcal{J}^j}$ by

$$t_{I,J}^\varepsilon = \begin{cases} t_{I,J}, & \text{if } (I, J) \in \mathcal{R}_\varepsilon, \\ 0, & \text{otherwise.}\end{cases} \tag{5.105}$$

Then the corresponding operators $T_{l,l'}^\varepsilon$, defined in analogy to (5.78), satisfy

$$2^{-l't}2^{(l'-l)r/2}|T_{l,l'}^\varepsilon| \le c\, 2^{-j(t'+r)}2^{-|l-l'|\delta}\varepsilon_1^{d^*+1+r}, \tag{5.106}$$

where $\delta = d^* + 1 - t' \ge 0$ and the constant c is independent of j, l, l' and ε_1.

Proof

As before, let $\delta = d^* + 1 - t$. We will estimate the norm of the operators

$$2^{-l't}2^{(l'-l)r/2}T_{l,l'}^\varepsilon = 2^{-l't}2^{lr}2^{(l'+l)r/2}(Q_l - Q_{l-1})T_{l,l'}^\varepsilon(P_{l'} - P_{l'-1}) \tag{5.107}$$

in $L_2(\mathcal{T}^n)$.

In view of the fact that the underlying bases are Riesz bases, we may switch from the norm of the operators in $L_2(\mathcal{T}^n)$ to the norm of the corresponding matrices. In order to apply lemma 5.3, we wish to estimate

$$\sum_{e\in E_0}\ \sum_{\{k'\in \mathbb{Z}^{n,l'}:2^{l'}|\theta(2^{-l}k-2^{-l'}k')|\ge \varepsilon_1^{-1}2^{M(j-l)}\}} 2^{-l'n/2}2^{-l't}2^{(l'-l)r/2}|t_{I,J}| \tag{5.108}$$

and proceed as in the proof of lemma 5.6. Considering first the case $l'\le l$, (5.108) can be bounded by

$$c\sum_{\{k':|2^{-D}k-k'|\ge(2^{(l-j)M}\varepsilon_1)^{-1}\}} 2^{-ln/2}2^{-l't}2^{-lr}2^{l'r}2^{(l'-l)(d^*+1)}(1+|2^{-D}k-k'|)^{-n-d^*-1-r}$$

$$\le c\int_{\{|2^{-D}k-x|\ge 2^{M(j-l)}\varepsilon_1^{-1}\}} 2^{-ln/2}2^{-l(t'+r)}2^{l'\tau}2^{(l'-l)\delta}2^{l'r}|2^{-D}k-x|^{-n-d^*-1-r}\mathrm{d}x$$

$$\le c\,2^{-ln/2}\varepsilon_1^{d^*+1+r}2^{(l-j)M(d^*+1+r)}2^{-l(t'+r)}2^{(l'-l)\delta}$$

$$\le c\,2^{-ln/2}\varepsilon_1^{d^*+1+r}2^{-j(t'+r)}2^{-|l-l'|\delta},$$

where we have used that $\tau\le|r|$.

In the case $l'>l$, we estimate (5.108) by

$$c\sum_{\{k'\in \mathbb{Z}^{n,l'}:2^{l}|\theta(2^{-l'}k'-2^{-l}k)|\ge\varepsilon_1^{-1}2^{M(j-l')}\}} \frac{2^{-n(l'-l)}2^{-l't}2^{-lr}2^{lr}2^{(l-l')(d^*+1)}}{(1+|k-k'2^{l-l'}|)^{n+d^*+1+r}}$$

$$\le c\sum_{\{k':|2^{l-l'}k'-k|\ge\varepsilon_1^{-1}2^{M(j-l')}\}} 2^{-n(l'-l)}2^{-l't}2^{(l-l')\delta}\left(\frac{1}{1+|k-k'2^{l-l'}|}\right)^{n+d^*+1+r}$$

$$\le c\,2^{-l't}2^{(l-l')\delta}\int_{\{|x|>\varepsilon_1^{-1}2^{M(j-l')}\}}|x|^{-n-d^*-1-r}\mathrm{d}x$$

$$\le c\,\varepsilon_1^{d^*+1+r}2^{l'r}2^{-j(t'+r)}2^{-|l-l'|\delta}2^{l'\tau}$$

$$\le c\,\varepsilon_1^{d^*+1+r}2^{-j(t'+r)}2^{-|l-l'|\delta}. \tag{5.109}$$

We may now appeal again to lemma 5.3 with $\gamma(I)=2^{-ln/2}$ to prove the assertion (5.106). □

In order to assemble the block estimates, we will apply lemma 5.7.

LEMMA 5.9

Let $r<0$, $0\le t\le d+1$, $-r<d^{*'}+\rho^*$, and $\mathbf{T}_\varepsilon^j=(t_{I,J}^\varepsilon)_{I,J\in\mathcal{J}^j}$ be given by (5.105) in lemma 5.8 for

$$\varepsilon_1 = \begin{cases} j^{-\frac{3}{2(d^*+1+r)}}\varepsilon & \text{if } M = 1, \text{ i.e. } d+1 = d^*+1 = t, \\ j^{-\frac{1}{2(d^*+1+r)}}\varepsilon & \text{if } d+1 = t < d^*+1, \\ \varepsilon & \text{if } 0 < M < 1, t < d+1. \end{cases} \tag{5.110}$$

Finally, let A_j^ε be defined accordingly by (5.56) and (5.74). Then there exists a positive constant c such that

$$\|(A_j - A_j^\varepsilon)P_j u\|_{-r} \le c\,\varepsilon^{d^*+1+r}2^{-jt}\|u\|_t \tag{5.111}$$

holds uniformly in $j \in \mathbb{N}_0$, $u \in H^t(\mathcal{T}^n)$ and $\varepsilon_0 \ge \varepsilon \ge 0$.

Proof

We again first assume that $l' \le l$ and set $D = |l - l'| = l - l'$. Employing analogous arguments as in the proof of theorem 5.3 such as telescoping expansions as well as theorem 2.3 yields

$$\begin{aligned}\|Q_j(A_j - A_j^\varepsilon)P_j u\|_{-r}^2 &\le c \sum_{0<l,l'\le j} \|2^{-lr}2^{(l+l')r/2}T_{l,l'}^\varepsilon(P_{l'} - P_{l'-1})u\|_0^2 \\ &\le c\|\sum_{0<l,l'\le j} 2^{-l't}2^{-lr}2^{(l+l')r/2}T_{l,l'}^\varepsilon 2^{l't}(P_{l'} - P_{l'-1})u\|_0^2 .\end{aligned}$$

Now lemma 5.8 and lemma 5.7 enable us to argue exactly as in the proof of theorem 5.3. Thus, the exponentially decaying term appearing in the case $t < d+1$ yields, in view of (5.93) and (5.99), the estimate

$$\begin{aligned}\|Q_j(A_j - A_j^\varepsilon)P_j u\|_{-r}^2 &\le (c\,\varepsilon^{d^*+1+r}2^{-j(t'+r)})^2 \sum_{0<l\le j} (2^{lt}\|(P_l - P_{l-1})u\|_0)^2 \\ &\le (c\,\varepsilon^{d^*+1+r}2^{-j(t'+r)}\|u\|_t)^2,\end{aligned} \tag{5.112}$$

where here we choose $t' = t - r$ in (5.103). In the case $t = d+1 < d^*+1$, i.e. $M < 1$, we use (4.1) to derive the corresponding bound

$$\begin{aligned}\|Q_j(A_j - A_j^\varepsilon)P_j u\|_{-r}^2 &\le (c\,\varepsilon_1^{d^*+1+r}2^{-jt})^2 \sum_{0<l\le j} (2^{l(d+1)}\|(P_l - P_{l-1})u\|_0)^2 \\ &\le (c\,\varepsilon^{d^*+1+r}2^{-jt}\|u\|_{d+1})^2.\end{aligned}$$

The remaining case follows as before. This proves the desired result. □

We are now in a position to prove asymptotic convergence rates for $r < 0$.

THEOREM 5.5

Let $r < 0$, $A \in \Psi^r(\mathcal{T}^n)$, $-r < d^{*'} + \rho^*$, $0 \le t \le d+1$ and $f \in H^{t-r}(\mathcal{T}^n)$. Suppose that the Petrov–Galerkin scheme is (s, r)-stable for $s = 0$, cf. [17] (theorem 6.3). Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the compressed scheme $A_j^\varepsilon u^j = Q_j f$ has a unique solution u_ε^j which differs from the exact solution u^* of the equation $Au^* = f$ in the $L_2(\mathcal{T}^n)$ norm by

$$\| u^* - u_\varepsilon^j \|_0 \le c\,(1 + \varepsilon^{d^*+1+r}) 2^{-jt} \| u^* \|_t, \tag{5.113}$$

where the constant c does not depend on j.

Proof

Choosing $t = 0$, lemma 5.9 yields the bound

$$\| (A_j - A_j^\varepsilon) P_j u \|_{-r} \le c\, \varepsilon^{d^*+1+r} \| u \|_0. \tag{5.114}$$

Thus, if the Petrov–Galerkin scheme $A_j = Q_j A P_j$ is $(0, r)$-stable in the sense of (3.12) (cf. [17]), then there exists ε_0 such that A_j^ε is also $(0, r)$-stable for $0 < \varepsilon < \varepsilon_0$. Moreover, (5.114) implies condition (5.75) in lemma 5.5 for $s = 0$. The assertion now follows from [17] (theorem 6.3) and lemma 5.5. $\square$

Remark 5.2

The fact that the rates shown in (5.111) and (5.113) are asymptotically optimal relies on choosing $d^* > d$ which seems to rule out orthogonal (pre-)wavelets. For prewavelets, the corresponding bounds would involve an additional factor 2^{-rj}.

5.4. COMPUTATIONAL COSTS

We will briefly point out to what extent the stiffness matrices are being compressed by the above strategies.

PROPOSITION 5.1

The number of nonzero entries in the matrices $\mathbf{T}_\varepsilon^j$, defined in lemma 5.6 and lemma 5.8, is of the order

$$\begin{cases} O\left(j^{2+\frac{3n}{2(d^*+1+r)}}\, 2^{jn}\right) & \text{when } t = d+1 = d^*+1; \\ O\left(j^{2+\frac{n}{2(d^*+1+r)}}\, 2^{jn}\right) & \text{when } t = d+1 < d^*+1; \\ O(j 2^{jn}) & \text{when } t < d+1. \end{cases}$$

Proof

Due to the symmetry of the compression, it is sufficient to consider the case $l \geq l'$.

Let us first discuss the extreme case $t = d + 1 = d^* + 1$, i.e. $M = 1$. We claim that all block matrices $\mathrm{T}_\varepsilon^{l,l'}$ contain at most $O((j^{\frac{3}{2(d^*+1+r)}} 2^j)^n)$ nonzero entries. To see this, note first that the dimension of $W^j = (P_j - P_{j-1})V^j$ is $(2^n - 1)2^{(j-1)n}$. Thus, $\mathrm{T}_\varepsilon^{j,j}$ has $((2^n - 1)/2^n)2^{jn}$ different rows. The truncation criterion (5.80) or (5.104) ensures that each row contains at most $(c\,\varepsilon_1^{-1})^n = (c(\varepsilon^{-1} j)^{\frac{3}{2(d^*+1+r)}})^n$ nonzero entries so that $\mathrm{T}_\varepsilon^{j,j}$ has at most $O(j^{\frac{3n}{2(d^*+1+r)}} 2^{jn})$ nontrivial coefficients. This proves the above claim for $l = l' = j$. Decreasing the number l', each block matrix $\mathrm{T}_\varepsilon^{j,l'}$ has the same number of rows. We infer from the truncation (5.80), respectively (5.104), that each row contains asymptotically the same number of nonzero entries, namely $O((j^{\frac{3}{2(d^*+1+r)}})^n)$. This confirms our claim for $l = j$.

By the same reasoning, the diagonal block $\mathrm{T}_\varepsilon^{j-1,j-1}$ contains $((2^n - 1)/2^n)2^{(j-1)n}$ rows. But now we have, in view of the truncations (5.80), respectively (5.104), $O((2j^{\frac{3n}{2(d^*+1+r)}})^n)$ nonzero coefficients per row. This gives, for $\mathrm{T}_\varepsilon^{j-1,j-1}$, a total of at most $O((j^{\frac{3}{2(d^*+1+r)}} 2^j)^n)$ nonzero coefficients. By the same arguments as above, we verify the above claim for all block matrices $\mathrm{T}_\varepsilon^{j-1,l'}$, $l' < j - 1$.

By induction, we see that each block $\mathrm{T}_\varepsilon^{l,l'}$ contains at most $O(j^{\frac{3n}{2(d^*+1+r)}} 2^{jn})$ nonzero coefficients. Since we have j^2 different blocks, summation over all blocks gives $O(j^{2+\frac{3n}{2(d^*+1+r)}} 2^{jn})$ as an overall bound for the number of nonvanishing coefficients.

So far, we have examined the extreme case that the exact solution is sufficiently regular, i.e. $u \in H^{d+1}(\mathcal{T}^n)$. If this is not the case, say $u \in H^t(\mathcal{T}^n)$, where $t < d + 1$, or a lower convergence rate is provided, fewer coefficients are needed. Starting with $l = l' = j$, we see that $\mathrm{T}_\varepsilon^{j,j}$ contains $O(2^{jn})$ nonzero entries. Furthermore, we infer from the truncations (5.80) or (5.104) that $\mathrm{T}_\varepsilon^{l,l}$ contains $O(2^l 2^{M(j-l)})^n$ nonzero coefficients, where $M < 1$. The total number of entries in each block $\mathrm{T}_\varepsilon^{l,l'}$, $l' < l$, is constant with respect to l', if l is fixed. Therefore, the total number of nonvanishing entries is $O(2^j 2^{(j-l)(M-1)})^n)$. Consequently, summing over all blocks of $\mathrm{T}_\varepsilon^{l,l'}$ shows that in total at most $O(j2^{jn}) = O(N \log N)$ nonzero entries are required. $\square$

Note that, for $l + l' < j$, the full matrix $\mathrm{T}^{l,l'}$ has only the order of 2^{jn} entries. Thus, from an asymptotic point of view, it is not necessary to compress those blocks for which $l + l' < j$.

Recall that for operators of nonnegative order, the choice $d^* > d$ was used to obtain a more efficient compression for solutions of high regularity. However, this by itself does not seem to prevent the appearance of logarithmic terms in the count of nonvanishing entries since one cause for these terms seems to be the fact that the equivalence of Sobolev and certain Besov norms used above ceases to hold for certain ranges of t. For operators of negative order, however, the possibility of choosing $d^* > d$ seems to be essential for obtaining asymptotically optimal convergence rates for the solutions of the compressed schemes at all.

Remark 5.3

Finally, we mention that it is also possible to obtain accuracy bounds of order $O(2^{-\tau j})$, for $\tau < d' + \rho + r/2$ ($\tau < d' + \rho$), $r \geq 0$ ($r < 0$), at the expense of only $O(N)$ remaining nonvanishing entries in the compressed matrices. Thus, this convergence rate can also be achieved with linear complexity, as in the case of fixed error bounds. To accomplish this, one has to take in addition the truncation of terms into account which correspond to different levels which are far apart from each other. The complete analysis is rather technical, so here we decided to dispense with a proof which is essentially based on repeating previous arguments.

6. Atomic decomposition

We now turn to the second approach for compressing stiffness matrices based on the decomposition (4.41), although it should be kept in mind that the decomposition (4.41) itself does not provide a matrix representation of the operator A_j. However, sparse approximations to matrix representations of A_j can be obtained from a compression of the decomposition (4.41), whose primary purpose in [4] is to facilitate a fast evaluation of $A_j u^j$ for $u^j \in V^j$. As mentioned before, it is closely related to the *atomic decompositions* of Calderón–Zygmund operators studied by Meyer [34]. An analogous compression scheme for Galerkin schemes was proposed in [4,5], where the corresponding matrices were called *nonstandard representation*. However, here we will demonstrate that the atomic decomposition may be applied as well to the much wider class of generalized Petrov–Galerkin methods, as described in the first part [17]. In particular, it also applies to collocation methods. The latter example is closely related to a multigrid approach proposed in [6].

In this section, we treat only the case of zero-order operators, i.e. $r = 0$. In principle, our technique would still apply to the more general case of arbitrary order. But this would require still further technical elaboration, which should be left to a separate study. We will show that any fixed prescribed accuracy can be achieved by linear complexity provided a BMO-type condition (see theorem 6.2) for a certain paraproduct is satisfied. Moreover, we will point out that this type of condition can generally not be avoided, which indicates that too naive compression strategies may fail to produce acceptable results in this case. The corresponding analytical background was developed by David and Journé [21], establishing a boundedness criterion for generalized Calderón–Zygmund operators, the so-called celebrated T1 theorem. Later, Meyer gave a wavelet formulation of this modern Calderón–Zygmund theory [32,34]. Our present investigation also builds upon this theory. Recently, we were told by Meyer that he has also obtained similar results for the method proposed in [4].

The atomic decomposition of the finite dimensional operator $A_j = Q_j A P_j$, suggested in [4], is given by the following telescoping sum

$$A_j = Q_jAP_j = Q_0AP_0 + \sum_{l=1}^{j}(Q_lAP_l - Q_{l-1}AP_{l-1})$$

$$= Q_0AP_0 + \sum_{l=1}^{j}((Q_l - Q_{l-1})A(P_l - P_{l-1})$$

$$+(Q_l - Q_{l-1})AP_{l-1} + Q_{l-1}A(P_l^j - P_{l-1})). \tag{6.1}$$

Here, the operators Q_j are defined by (2.31), (2.27) and, when dealing with pre-wavelets, P_j denotes the orthogonal projector onto V^j while, in the case of biorthogonal wavelets, P_j has the form (2.32). However, for simplicity we will continue using the same notation P_j in both cases. In particular, the differences $Q_{j+1} - Q_j$, $P_{j+1} - P_j$ have the representations (2.40), (2.35) or (2.36), respectively.

For a given vector $u^l = (u_k)_{k \in \mathbb{Z}^{n,l}}$, we introduce the notation

$$u^l *^\circ \phi^l = \sum_{k \in \mathbb{Z}^{n,l}} u_k \phi_k^l.$$

In view of (6.1), we may write

$$A_j = \sum_{l=0}^{j-1}(H_l + G_l + D_l), \tag{6.2}$$

where the operators $H_{l-1} := (Q_l - Q_{l-1})A(P_l - P_{l-1})$, $G_{l-1} := (Q_l - Q_{l-1})AP_{l-1}$, and $D_{l-1} := Q_{l-1}A(P_l - P_{l-1})$, have the following Schwartz kernels:

$$H_l(x, y) = \sum_{e,e' \in E_0} \mathbf{H}_{e,e'}^l *^\circ (\phi_{e'}^l(x) \otimes \zeta_e^l(y))$$

$$= \sum_{e,e' \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} h_{k,k'}^{l,e,e'} \phi_{e',k'}^l(x)\zeta_{e,k}^l(y), \tag{6.3}$$

$$G_l(x, y) = \sum_{e' \in E_0} \mathbf{G}_{e'}^l *^\circ (\phi_{e'}^l(x) \otimes \phi^l(y))$$

$$= \sum_{e' \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} g_{k,k'}^{l,e'} \phi_{e',k'}^l(x)\phi_k^l(y), \tag{6.4}$$

$$D_l(x, y) = \sum_{e' \in E_0} \mathbf{D}^l *^\circ (\phi_0^l(x) \otimes \zeta_e^l(y))$$

$$= \sum_{e \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} d_{k,k'}^{l,e} \phi_{0,k'}^l(x)\zeta_{e,k}^l(y), \tag{6.5}$$

and where ϕ, ϕ_e, ζ_e, ψ_e are the functions from (2.34), (2.7), (2.38) and (2.37). Setting

$$\begin{aligned} h_{k,k'}^{l,e,e'} &:= \eta_{e',k'}^{l}(A\psi_{e,k}^{l}), \\ g_{k,k'}^{l,e'} &:= \eta_{e',k'}^{l}(A\varphi_{k}^{l}), \\ d_{k,k'}^{l,e} &:= \eta_{k'}^{l}(A\psi_{e,k}^{l}), \end{aligned} \tag{6.6}$$

the corresponding matrices are denoted by

$$\mathbf{H}_{e,e'}^{l} = (h_{k,k'}^{l,e,e'})_{k,k'\in\mathbf{Z}^{n,l}}, \quad \mathbf{G}_{e'}^{l} = (g_{k,k'}^{l,e'})_{k,k'\in\mathbf{Z}^{n,l}}, \quad \mathbf{D}_{e}^{l} = (d_{k,k'}^{l,e})_{k,k'\in\mathbf{Z}^{n,l}}. \tag{6.7}$$

6.1. ESTIMATES FOR PRESCRIBED ACCURACY ON VARIABLE BANDWIDTHS

The basic idea proposed in [4] for compressing A_j is to compress each individual component appearing in (6.1).

Here, we will consider a slightly more general scheme where we allow the compression to depend on the level of discretization. More precisely, the desired compression will be achieved by setting those entries to zero in all the above matrices for which

$$2^l|\theta(2^{-l}(k-k'))| \ge c(l)\varepsilon^{-1}, \quad 0 \le l < j, \tag{6.8}$$

where the cut off bandwidth $c(l)\varepsilon^{-1} > 0$ may increase in l to provide increasing accuracy on higher levels. This gives rise to perturbed operators A_j^{ε} defined by

$$A_j^{\varepsilon} := A_j - \sum_{l=0}^{j-1}(H_l^{\varepsilon} + G_l^{\varepsilon} + D_l^{\varepsilon}), \tag{6.9}$$

where the perturbations are given by

$$\begin{aligned} G_l^{\varepsilon}u(x) &= \sum_{e\in E_0} \sum_{\{k,k':2^l|\theta(2^{-l}(k-k'))|\ge c(l)\varepsilon^{-1}\}} g_{k,k'}^{l,e}(u,\phi_k^l)_0\phi_{e,k'}^l(x), \\ D_l^{\varepsilon}u(x) &= \sum_{e\in E_0} \sum_{\{k,k':2^l|\theta(2^{-l}(k-k'))|\ge c(l)\varepsilon^{-1}\}} d_{k,k'}^{l,e}(u,\zeta_{e,k}^l)_0\phi_{0,k'}^l(x), \\ H_l^{\varepsilon}u(x) &= \sum_{e,e'\in E_0} \sum_{\{k,k':2^l|\theta(2^{-l}(k-k'))|\ge c(l)\varepsilon^{-1}\}} h_{k,k'}^{l,e,e'}(u,\zeta_{e,k}^l)_0\phi_{e',k'}^l(x). \end{aligned} \tag{6.10}$$

On each single level l, $0 \le l \le j$, theorem 4.2 and theorem 4.3 combined with Schur's lemma will lead to the following bound.

PROPOSITION 6.1

Let $D_l^\varepsilon, G_l^\varepsilon, H_l^\varepsilon$ be defined by (6.10). Then there exists a constant c such that for all $j \in \mathbb{N}$

$$\| D_l^\varepsilon + G_l^\varepsilon + H_l^\varepsilon \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \le c \left(\frac{\varepsilon}{c(l)} \right)^{d^*+1}. \tag{6.11}$$

Proof

We confine our discussion to one term, say G_l^ε, since the other terms can be treated in the same way. Invoking the stability of the scaling functions on a fixed level, as well as the stability of the wavelets, we obtain for any $u \in L_2(\mathcal{T}^n)$

$$\begin{aligned}
\| G_l^\varepsilon u \|_0^2 &= \| G_l^\varepsilon (P_l u) \|_{L_2(\mathcal{T}^n)}^2 \\
&\le c \sum_{k' \in \mathbf{Z}^{n,l}} \sum_{e' \in E_0} \Big| \sum_{\{k \in \mathbf{Z}^{n,l} : 2^l | \theta 2^{-l}(k-k')| \ge c(l)\varepsilon^{-1}\}} g_{k,k'}^{l,e'} (u, \varphi_k^l)_0 \Big|^2 \\
&\le c \| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))}^2 \sum_{k \in \mathbf{Z}^{n,l}} |(u, \varphi_k^l)_0|^2 \le c \| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))}^2 \| P_l u \|_0^2 \\
&\le c \| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))}^2 \| u \|_0^2 .
\end{aligned}$$

The norm $\| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))}$ can be estimated with the aid of Schur's lemma.

$$\begin{aligned}
\| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))}^2 \le &\left(\sup_{k \in \mathbf{Z}^{n,l}} \sum_{\{k' \in \mathbf{Z}^{n,l} : 2^l |\theta(2^{-l}(k-k'))| \ge c(l)\varepsilon^{-1}\}} |g_{k,k'}^{l,e'}| \right) \\
&\left(\sup_{k' \in \mathbf{Z}^{n,l}} \sum_{\{k \in \mathbf{Z}^{n,l} : 2^l |\theta(2^{-l}(k-k'))| \ge c(l)\varepsilon^{-1}\}} |g_{k,k'}^{l,e'}| \right).
\end{aligned} \tag{6.12}$$

We may now invoke theorem 4.2 and theorem 4.3 to estimate the entries $g_{k,k'}^{l,e'}$ (and similarly $d_{k,k'}^{l,e'}, h_{k,k'}^{l,e,e'}$). Taking into account that the truncation yields

$$\| \mathbf{G}_\varepsilon^l \|_{\mathcal{L}(l_2(\mathbf{Z}^{n,l}))} \le c \left(\frac{\varepsilon}{c(l)} \right)^{(d^*+1)}, \tag{6.13}$$

which is the desired bound. □

After these prerequisites, we will show next that one can choose the bandwidth control $c(l)$ in such a way that a given error tolerance can be realized by means of compressed schemes involving $O(2^{jn}) = O(N)$ operations.

THEOREM 6.1

Suppose that the Petrov–Galerkin scheme $Q_jAP_j = A_j$ is (0, 0)-stable in the sense of (3.12) (cf. [17]). For some $t' \in (0, 1)$, let

$$M := \frac{t'}{d^* + 1}, \quad c(l) := 2^{M(j-l)}. \tag{6.14}$$

Then the corresponding perturbed operators A_j^ε, defined by (6.9) and (6.10), satisfy

$$\| A_j - A_j^\varepsilon \|_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, \varepsilon^{d^*+1}. \tag{6.15}$$

Moreover, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the compressed scheme $A_j^\varepsilon u_\varepsilon^j = f^j$ has a unique solution $u_\varepsilon^j \in V^j$ satisfying

$$\| u^j - u_\varepsilon^j \|_0 \le c\, \varepsilon^{d^*+1} \| u^j \|_0 \le c\, \varepsilon^{d^*+1} \| u^* \|_0, \tag{6.16}$$

where u^j, u^* are the solutions of (3.7), (3.4), respectively.

Proof

We again confine our discussion to one typical component of A_j^ε, namely G_l^ε. From proposition 6.1, we infer that

$$\begin{aligned} \| G_j^\varepsilon u \|_0 &\le c\, \varepsilon^{d^*+1} 2^{-(j-l)M(d^*+1)} \| u \|_0 \\ &\le c\, \varepsilon^{d^*+1} 2^{-t'(j-l)} \| u \|_0 , \end{aligned} \tag{6.17}$$

where $t' > 0$ is the constant from (6.14). Similar estimates hold for the operators $D_l^\varepsilon, H_l^\varepsilon$. Summing over $l = 1, \ldots, j$ therefore yields

$$\| A_j - A_j^\varepsilon \|_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, \varepsilon^{d^*+1} \sum_{l=0}^{j} 2^{(l-j)t'}, \tag{6.18}$$

proving (6.15).

Next, on account of (6.15), ε_0 can be chosen such that the compressed scheme A_j^ε also becomes (0, 0)-stable uniformly for $0 < \varepsilon < \varepsilon_0$. The rest of the assertion then follows from lemma 5.5. □

6.2. COMPUTATIONAL COSTS

One easily verifies that, when $t' < 1$, this procedure produces on a fixed level l only $O(2^{(j-l)t'n} 2^{ln})$ nonzero coefficients. Summing over all levels, one sees that there remains a total of $O(2^{jn}) = O(N)$ nonzero coefficients needed in order to achieve the desired accuracy.

6.3. ESTIMATES FOR CONSTANT BANDWIDTH

Next, we will investigate the extreme case where the bandwidth is almost constant over all levels, i.e. $c(l) = 1$. This kind of truncation has been suggested in [4,1] and has also been used by other authors, for example in [25]. In order to establish appropriate bounds for the perturbation operators in this case, we have to perform an explicit *extraction of constants*. For this reason, we introduce the following operators:

$$
\begin{aligned}
R_l^\varepsilon u(x) &:= \sum_{e\in E_0} \sum_{\{k,k'\in \mathbb{Z}^{n,l}:2^l|\theta(2^{-l}(k-k'))|\ge\varepsilon^{-1}\}} d_{k,k'}^{l,e}(u,\Phi_k^l)_0\phi_{e,k}^l(x),\\
S_l^\varepsilon u(x) &:= \sum_{e'\in E_0} \sum_{\{k,k'\in \mathbb{Z}^{n,l}:2^l|\theta(2^{-l}(k-k'))|\ge\varepsilon^{-1}\}} g_{k,k'}^{l,e'}(u,\zeta_{e',k'}^l)_0\Phi_{k'}^l(x),
\end{aligned}
\tag{6.19}
$$

where Φ denotes the cardinal multi-linear tensor product B-spline. Note that the above operators are defined by diagonal matrices with diagonal entries

$$
r_I^\varepsilon := r_{e,k}^{l,\varepsilon} := \sum_{\{k'\in \mathbb{Z}^{n,l}:2^l|\theta(2^{-l}(k-k'))|\ge\varepsilon^{-1}\}} d_{k,k'}^{l,e}, \quad I=(l,e,k) \tag{6.20}
$$

and

$$
s_J^\varepsilon := s_{e,k'}^{l,\varepsilon} := \sum_{\{k\in \mathbb{Z}^{n,l}:2^l|\theta(2^{-l}(k-k'))|\ge\varepsilon^{-1}\}} g_{k,k'}^{l,e}, \quad J=(l,e,k'). \tag{6.21}
$$

Using the fact that the translates of a scaling function build a partition of unity, i.e.

$$
\sum_{k\in \mathbb{Z}^{n,l}} \Phi_k^l = \sum_{k\in \mathbb{Z}^{n,l}} \varphi_k^l = \sum_{k\in \mathbb{Z}^{n,l}} \phi_k^l = \sum_{k\in \mathbb{Z}^{n,l}} \gamma_k^l = \sum_{k\in \mathbb{Z}^{n,l}} \phi_{0,k}^l = 2^{ln/2},
$$

so that

$$
\frac{(1,\varphi_k^l)_0}{(1,\Phi_k^l)_0} = \frac{(1,\gamma_{k'}^l)_0}{(1,\Phi_{k'}^l)_0} = 1,
$$

it is not difficult to see that the operators

$$
C_l^\varepsilon = H_l^\varepsilon + G_l^\varepsilon - S_l^\varepsilon + D_l^\varepsilon - R_l^\varepsilon, \tag{6.22}
$$

whose Schwartz kernels $C_l^\varepsilon(x,y)$ are defined by

$$
(C_l^\varepsilon f)(x) = \int_{\mathcal{T}^n} C_l^\varepsilon(x,y)f(y)\mathrm{d}y, \tag{6.23}
$$

satisfy

$$
C_l^\varepsilon 1 = 0, \quad (C_l^\varepsilon)^* 1 = 0. \tag{6.24}
$$

Thus, the definition in (6.19) may be viewed as an extraction of constants.

Theorem 4.2 and theorem 4.3 now yield for each level the following estimate.

PROPOSITION 6.2

For every $l \in \mathbb{N}$, $e, e' \in E_0$, the coefficients $r_{e,k}^{l,\varepsilon}$ and $s_{e,k}^{l,\varepsilon}$ defined by (6.20), (6.21) satisfy

$$|r_{e,k}^{l,\varepsilon}| + |s_{e',k}^{l,\varepsilon}| \leq c\, \varepsilon^{d^*+1}, \tag{6.25}$$

for some constant c independent of l.

Similar arguments as those used in the proof of proposition 6.1 now provide the following result.

PROPOSITION 6.3

Let C_l^ε be defined by (6.22). Then there exists a constant c such that for all $j \in \mathbb{N}$

$$\| C_l^\varepsilon \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \leq c\, \varepsilon^{d^*+1}.$$

We now proceed by analysing the above compression scheme. Since various aspects of the theory of *Calderón–Zygmund operators* will play an important role in this context, we recall the following definition (cf. [34]).

DEFINITION 6.1

Let T be a linear continuous operator $T : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$. Its Schwartz kernel $K(x, y)$ is called a *Calderón–Zygmund kernel*, if it satisfies the following conditions:

(i) The Schwartz kernel $K(x, y)$ of T is locally integrable on every open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n \backslash \{x \neq y\}$. Further, there exists $c > 0$ and $\delta \in (0, 1]$ such that $K(x, y)$ can be estimated by

$$|K(x, y)| \leq c\, |x - y|^n \tag{6.26}$$

(ii) and

$$|K(x', y) - K(x, y)| \leq c\, |x' - x|^\delta |x - y|^{-n-\delta} \quad \text{if } |x' - x| \leq \tfrac{1}{2}|x - y|, \tag{6.27}$$

$$|K(x, y') - K(x, y)| \leq c\, |y' - y|^\delta |x - y|^{-n-\delta} \quad \text{if } |y' - y| \leq \tfrac{1}{2}|x - y|, \tag{6.28}$$

for all $(x, y) \in \Omega$.

A bounded linear operator $T : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is called a *Calderón–Zygmund operator on* $\mathcal{T}^n$ if T is a bounded operator in $L^2(\mathcal{T}^n)$ and if its Schwartz kernel is locally a Calderón–Zygmund kernel. This means that, for a partition of unity Φ_j, $1 \leq j \leq N$, relative to a finite covering of $\mathcal{T}^n$, the canonically transported operators $\Phi_{j'} T \Phi_j$, $1 \leq j, j' \leq N$ (cf. e.g. [40]), are Calderón–Zygmund operators on $\mathbb{R}^n$.

To tie this concept into the present context, we will make use of the following further auxiliary facts. The subsequent first observation is formulated only for one typical configuration. The arguments which are being used cover all the other cases as well, since we will only exploit exponential decay.

LEMMA 6.1

Let

$$\phi_k^l := \sum_{k' \in \mathbb{Z}^{n,l}} g_{k-k'}^l \gamma_{k'}^l,$$

where the $g_{k'}^l$ are defined for exponentially decaying coefficients g_k by (4.5) and γ satisfies $C_0^{d^*,d^*}$. Then the estimate

$$|\phi_k^l(x) - \phi_k^l(x')| \le c\, 2^{ln/2} 2^{l\delta} |\theta(x-x')|^\delta (\tau^{2^l|\theta(x-2^{-l}k)|} + \tau^{2^l|\theta(x'-2^{-l}k)|}) \tag{6.29}$$

holds for some $\tau \in (0, 1)$, some constant c independent of l, n, x, x' and any $\delta \in (0, \rho^*]$ where $0 < \rho^* \le 1$ corresponds to the Hölder continuity of γ.

Proof

By (4.6), we obtain for some $\tau \in (0, 1)$

$$\begin{aligned} |\phi_k^l(x) - \phi_k^l(x')| &\le \sum_{k' \in \mathbb{Z}^{n,l}} |g_{k-k'}^l|\, |\gamma_{k'}^l(x) - \gamma_{k'}^l(x')| \\ &\le c \sum_{k' \in \mathbb{Z}^{n,l}} \tau^{2^l|\theta(2^{-l}(k-k'))|} |\gamma_{k'}^l(x) - \gamma_{k'}^l(x')|. \end{aligned} \tag{6.30}$$

In the case $|x - x'| \le c\, 2^{-l}$, we make use of the Hölder continuity of γ and the fact that γ has compact support to conclude that

$$\begin{aligned} |\phi_k^l(x) - \phi_k^l(x')| &\le c\, 2^{ln/2} (2^l|x-x'|)^\delta \sum_{\{k' : 2^l|\theta(2^{-l}k'-x)| \le c\}} \tau^{2^l|\theta(2^{-l}(k'-k))|} \\ &\le c\, 2^{ln/2} 2^{l\delta} |\theta(x-x')|^\delta (\tau^{2^l|\theta(x-2^{-l}k)|} + \tau^{2^l|\theta(x'-2^{-l}k)|}). \end{aligned}$$

If $|\theta(x-x')| > 2^{-l}$, we again take the compact support of γ into account to estimate

$$\begin{aligned} |\phi_k^l(x) - \phi_k^l(x')| &\le c \sum_{k' \in \mathbb{Z}^{n,l}} \tau^{2^l|\theta(2^{-l}(k-k'))|} (|\gamma_{k'}^l(x)| + |\gamma_{k'}^l(x')|) \\ &\le c\, 2^{ln/2} (\tau^{2^l|\theta(x-2^{-l}k)|} + \tau^{2^l|\theta(x'-2^{-l}k)|}) \\ &\le c\, 2^{ln/2} 2^{l\delta} |\theta(x-x')|^\delta (\tau^{2^l|\theta(x-2^{-l}k)|} + \tau^{2^l|\theta(x'-2^{-l}k)|}). \end{aligned} \tag{6.31}$$

In the last step, we used the assumption $1 < 2^{l\delta}|\theta(x-x')|^\delta$, where $\delta \in (0, 1]$. □

LEMMA 6.2

Let ϕ, ζ_e be defined by (2.34) and (2.37), respectively. Furthermore, let M_l have the form

$$M_l u(x) = \sum_{e \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} (u, \zeta^l_{e,k'})_0 m^{l,e}_{k,k'} \phi^l_k(x),$$

where for some $q > 0$

$$|m^{l,e}_{k,k'}| \leq c(1 + 2^l |\theta(2^{-l}(k - k'))|)^{-n-q}, \quad k, k' \in \mathbb{Z}^{n,l}, e \in E_0. \tag{6.32}$$

Then the Schwartz kernel $M_l(x, y)$ of M_l satisfies the following estimates:

$$|M_l(x, y)| \leq c\, 2^{ln}(1 + 2^l |\theta(x - y)|)^{-n-q}, \tag{6.33}$$

and, if in addition $\phi \in C^\delta(\mathbb{R}^n)$, one has

$$|M_l(x', y)| - M_l(x, y)| \leq c\, 2^{ln}(2^l |\theta(x - x')|)^\delta$$
$$((1 + 2^l |\theta(x - y)|)^{-n-q} + (1 + 2^l |\theta(x' - y)|)^{-n-q}). \tag{6.34}$$

Proof

By our assumptions on ϕ and ζ_e, (4.6) ensures that

$$|\phi^l_k(x)| \leq c\, 2^{ln/2} \tau^{2^l |\theta(x - 2^{-l}k)|}, \quad |\zeta^l_{e,k'}(y)| \leq c\, 2^{ln/2} \tau^{2^l |\theta(y - 2^{-l}k)|} \tag{6.35}$$

holds for some $\tau \in (0, 1)$. Since

$$M_l(x, y) = \sum_{k,k' \in \mathbb{Z}^{n,l}} \sum_{e \in E_0} m^{l,e}_{k,k'} \phi^l_k(x) \zeta^l_{e,k'}(y),$$

a twofold application of lemma 4.1 yields (6.33) for $x = 2^l k_0$, $y = 2^{-l}k'_0$. A continuity argument establishes (6.33) for any $x, y \in \mathcal{T}^n$.

Replacing $\phi^l_k(x)$ by $\phi^l_k(x) - \phi^l_k(x')$, we apply lemma 6.1:

$$|M_l(x, y) - M_l(x', y)| = |\sum_{e \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} \zeta^l_{e,k'}(y) m^{l,e}_{k,k'} (\phi^l_k(x) - \phi^l_k(x'))|$$

$$\leq c\, 2^{ln} \sum_{e \in E_0} \sum_{k,k' \in \mathbb{Z}^{n,l}} \tau^{2^l |\theta(y - 2^{-l}k')|} |m^{l,e}_{k,k'}| 2^{l\delta} |\theta(x - x')|^\delta$$
$$\times (\tau^{2^l |\theta(x - 2^{-l}k)|} + \tau^{2^l |\theta(x' - 2^{-l}k)|}).$$

Finally, (6.34) follows from lemma 4.1 and (6.32). □

One should note that the operators A_j locally have Calderón–Zygmund kernels.

PROPOSITION 6.4

The Schwartz kernels of the operators $A_j, H_j, G_j, D_j, j \in \mathbb{N}_0$, defined by (6.2), (6.3), (6.4), and (6.5) are locally Calderón–Zygmund kernels with constants c, δ in (6.26) and (6.27) not depending on $j \in \mathbb{N}_0$.

This fact can be established with the aid of the previous lemmas. The reasoning is implicitly contained in the proof of the following lemma, which gives more precise information needed later.

LEMMA 6.3

Let M_l^ε be any of the operators $(G_l^\varepsilon - S_l^\varepsilon)$, (H_l^ε), $(D_l^\varepsilon - R_l^\varepsilon)$ defined by (6.10) and (6.19). Then the Schwartz kernels and the transposed Schwartz kernels of the operators M_l^ε, $l \in \mathbb{N}_0$, are locally Calderón–Zygmund kernels satisfying

$$|M_l^\varepsilon(x,y)| \le c\,\varepsilon^{d^*+1}(\varepsilon 2^l)^n(1+\varepsilon 2^l|\theta(x-y)|)^{-n-d^*-1}, \tag{6.36}$$

and, for any $\varepsilon \in (0, 1)$,

$$|M_l^\varepsilon(x,y) - M_l^\varepsilon(x',y)| \le c\,\varepsilon^{d^*+1-\delta}(\varepsilon 2^l|\theta(x-x')|)^\delta$$
$$\times(\varepsilon 2^l)^n((1+\varepsilon 2^l|\theta(x-y)|)^{-n-d^*-1} + (1+\varepsilon 2^l|\theta(x'-y)|)^{-n-d^*-1}), \tag{6.37}$$

where $\delta \in (0, \rho^*]$, $\rho^* \le 1$, and the constants c in (6.39) and (6.40) do not depend on $l \in \mathbb{N}_0$.

Proof

By theorem 4.2, theorem 4.3, as well as by corollaries 4.1, 4.2 and 4.3, the corresponding entries $m_{k,k'}^{l,e'}$ can be bounded as follows:

$$|m_{k,k'}^{l,e'}| \le c(1+2^l|\theta(2^{-l}(k-k'))|)^{-n-d^*-1}. \tag{6.38}$$

One easily concludes from lemma 6.2 that

$$|M_l^\varepsilon(x,y)| \le c\,\varepsilon^{d^*+1}(\varepsilon 2^l)^n(\varepsilon 2^l|\theta(x-y)|)^{-n-d^*-1}, \tag{6.39}$$

which, in turn, yields (6.36) whenever $2^l|\theta(2^{-l}(k-k'))| \ge \varepsilon^{-1}$. Let us recall that the operators $R_l^\varepsilon, S_l^\varepsilon$ are represented by diagonal matrices. In view of the truncation (6.8) and by invoking proposition 6.2, we see that the bound

$$|m_{k,k'}^{l,e'}| \le c\,\varepsilon^{n+d^*+1} \tag{6.40}$$

is valid uniformly for all $k, k' \in \mathbb{Z}^{n,l}$. In the case $|\theta(x-y)| \le (2^l\varepsilon)^{-1}$, we observe that, in view of (6.40) and (4.6), summation provides for $x = 2^{-l}k$, $y = 2^{-l}k'$

$$|M_l^\varepsilon(x,y)| \le c\, 2^{ln} \varepsilon^{(n+d^*+1)}. \tag{6.41}$$

By the usual continuity argument, this confirms the first inequality (6.36) for all $x, y \in \mathcal{T}^n$.

For $\psi^l_{e',k'} \in C^\delta(\mathcal{T}^n)$, $\delta \in (0, \rho^*]$, lemma 6.3 applies and one concludes that

$$\begin{aligned} |M_l(x,y) - M_l(x',y)| &\le c\, 2^{l(n+\delta)} |\theta(x-x')|^\delta \\ &\quad ((2^l|\theta(x-y)|)^{-n-d^*-1} + (2^l|\theta(x'-y)|)^{-n-d^*-1}) \\ &\le c(\varepsilon 2^l)^n \varepsilon^{d^*+1-\delta} |\varepsilon\theta(x-x')|^\delta \\ &\quad ((\varepsilon 2^l|\theta(x-y)|)^{-n-d^*-1} + (\varepsilon 2^l|\theta(x'-y)|)^{-n-d^*-1}). \end{aligned} \tag{6.42}$$

Thus, if $|\theta(x-y)| \ge (2^l\varepsilon)^{-1}$ and $|\theta(x'-y)| \ge (2^l\varepsilon)^{-1}$ are satisfied, (6.37) follows. If only $|\theta(x'-y)| \ge (2^l\varepsilon)^{-1}$ is valid, one uses

$$\begin{aligned} |M_l(x,y) - M_l(x',y)| &\le c\, 2^{l(n+\delta)} |\theta(x-x')|^\delta (2^l|\theta(x'-y)|)^{-n-d^*-1} \\ &\le c(\varepsilon 2^l)^n \varepsilon^{d^*+1-\delta} |\varepsilon\theta(x-x')|^\delta (\varepsilon 2^l|\theta(x'-y)|)^{-n-d^*-1}. \end{aligned} \tag{6.43}$$

In the case that both $2^l|\theta(x-y)| \le \varepsilon^{-1}$ and $2^l|\theta(x'-y)| \le \varepsilon^{-1}$, we apply (6.40) as above to obtain

$$|M_l(x,y) - M_l(x',y)| \le c\, 2^{ln} |2^l\theta(x-x')|^\delta \varepsilon^{n+d^*+1}. \tag{6.44}$$

This confirms our claim. □

LEMMA 6.4

The Schwartz kernels $N_l^\varepsilon(x,y)$ of the operators $(R_l^\varepsilon + S_l^\varepsilon)$ are also Calderón–Zygmund kernels satisfying the estimates

$$|N_l^\varepsilon(x,y)| \le c\, \varepsilon^{d^*+1} 2^{ln} \tau^{2^l|\theta(x-y)|} \tag{6.45}$$

and

$$\begin{aligned} |N_l^\varepsilon(x,y) - N_l^\varepsilon(x',y)| &\le c\, \varepsilon^{d^*+1-\delta} (\varepsilon 2^l|\theta(x-x')|)^\delta \\ &\quad 2^{ln} (\tau^{2^l|\theta(x-y)|} + \tau^{2^l|\theta(x'-y)|}) \end{aligned} \tag{6.46}$$

for any $\varepsilon \in (0, 1)$, where $\delta \in (0, \rho]$, $\rho \le 1$ and constants c in (6.36) and (6.37) not depending on $l \in \mathbb{N}_0$.

Proof

By (6.19), one has the representation

$$N_l^\varepsilon(x, y) = \sum_{e,e'} \sum_{k \in Z^{n,l}} (r_{e,k}^{l,\varepsilon} \Phi_k^l(y) \phi_{e,k}^l(x) + s_{e',k}^{l,\varepsilon} \zeta_{e',k}^l(y) \Phi_k^l(x)).$$

We now infer from proposition 6.2 and lemma 4.1 that

$$|N_l^\varepsilon(x, y)| \le c\, \varepsilon^{d^*+1} \sum_{e,e'} \sum_{k \in Z^{n,l}} (|\Phi_k^l(y) \phi_{e,k}^l(x)| + |\zeta_{e',k}^l(y) \Phi_k^l(x)|)$$

$$\le c\, \varepsilon^{d^*+1} 2^{ln} \sum_{k \in Z^{n,l}} \tau^{2^l |\theta(2^{-l}k - x)|} \tau^{2^l |\theta(2^{-l}k - y)|}$$

$$\le c\, \varepsilon^{d^*+1} 2^{ln} \tau^{2^l |\theta(x-y)|}.$$

In order to prove (6.46), we employ analogous arguments as in the proof of lemma 6.2

$$|N_l^\varepsilon(x, y) - N_l^\varepsilon(x', y)| \le c\, \varepsilon^{d^*+1} (2^l |\theta(x - x')|)^\delta$$

$$\times 2^{ln} \sum_{k \in Z^{n,l}} \tau^{2^l |\theta(2^{-l}k - x)|} \tau^{2^l |\theta(2^{-l}k - y)|} + \tau^{2^l |\theta(2^{-l}k - x')|} \tau^{2^l |\theta(2^{-l}k - y)|}$$

$$\le c\, 2^{ln} \varepsilon^{d^*+1-\delta} (\varepsilon 2^l |\theta(x - x')|)^\delta (\tau^{2^l |\theta(x-y)|} + \tau^{2^l |\theta(x'-y)|}),$$

where we have used proposition 6.2 and lemma 4.1. □

Conversely, every Calderón–Zygmund operator A on $\mathcal{T}^n$ can be expanded in a series

$$A = P_0 A P_0 + \sum_{l=0}^{\infty} ((P_l - P_{l-1}) A (P_l - P_{l-1}) + (P_l - P_{l-1}) A P_{l-1} + P_{l-1} A (P_l - P_{l-1})). \tag{6.47}$$

The corresponding finite dimensional operators give rise to matrices satisfying the estimates of corollary 4.3 where $d^* + 1 + r$ is replaced by some $\delta \in (0, 1)$. This representation of Calderón–Zygmund operators in terms of wavelet expansions is the heart of Meyer's theory of Calderón–Zygmund operators (see [4,34]). It also provides the analytical background for the present investigations.

One should note that our analysis of the matrix compression carried out in the previous section used properties of pseudodifferential operators which, when considering the case $r = 0$, do not necessarily hold for Calderón–Zygmund operators, for instance, for Calderón–Zygmund operators, an estimate of type (4.32) is not automatically satisfied. Here, one would have to assume an additional condition, for instance, the so-called *weak cancellation property* (see [4]) or the *weak boundedness property* (see e.g. [34,20]), in order to obtain error estimates for compressed stiffness matrices relative to Galerkin schemes based on wavelet bases (cf. [4,34]).

In order to again prove in the present context that the computational complexity, needed for achieving a certain precision for the compressed operator, remains linear in the number of unknowns, we will make use of a celebrated lemma due to Cotlar and Stein [26,34].

LEMMA 6.5

Let A_j, $j \in \mathbb{Z}$, be a collection of bounded linear operators on a Hilbert space H such that for some constant M

$$\sum_{l\in\mathbb{Z}} \|(A_j)^* A_l\|^{1/2} \le M, \quad \sum_{l\in\mathbb{Z}} \| A_j (A_l)^* \|^{1/2} \le M \tag{6.48}$$

holds uniformly in $j \in \mathbb{N}$. Then the series

$$Tu = \sum_j A_j u, \quad u \in H, \tag{6.49}$$

converges in H and $\|T\|_{\mathcal{L}(H)} \le M$.

In order to apply lemma 6.5, we need the following lemma.

LEMMA 6.6

Let C_l^ε, $l \in \mathbb{N}_0$, be the operators on $L_2(\mathcal{T}^n)$ defined by (6.22). Then there exists a constant $c > 0$ such that

$$\| \sum_{l\ge 0} C_l^\varepsilon u \|_0 \le c\, \varepsilon^{d^*+1-\delta/2} \| u \|_0, \quad u \in L_2(\mathcal{T}^n), \tag{6.50}$$

where ε and δ are the constants appearing in lemma 6.3 and lemma 6.4.

Proof

According to (6.24), the Schwartz kernels $C_l^\varepsilon(x,y)$ satisfy the condition

$$(C_l^\varepsilon 1)(x) = \int_{\mathcal{T}^n} C_l(x,y)\mathrm{d}y = ((C_l^\varepsilon)^* 1)(x) = 0. \tag{6.51}$$

We decompose $C_l^\varepsilon(x,y) = M_l^\varepsilon(x,y) + N_l^\varepsilon(x,y)$, where we denote $M_l^\varepsilon(x,y) = H_l^\varepsilon(x,y) + G_l^\varepsilon(x,y) + D_l^\varepsilon(x,y)$ and $N_l^\varepsilon(x,y) = -(R_l^\varepsilon + S_l^\varepsilon)(x,y)$. By (6.51), the kernel of the operator $C_l^\varepsilon(C_{l'}^\varepsilon)^*$ can be expressed as

$$C_{l,l'}^\varepsilon(x,y) = C_l^\varepsilon(C_{l'}^\varepsilon)^*(x,y) = \int_{\mathcal{T}^n} C_l^\varepsilon(x,z)\overline{C_{l'}^\varepsilon(y,z)}\,\mathrm{d}z \tag{6.52}$$

$$= \int_{\mathcal{T}^n} (C_l^\varepsilon(x,z) - C_l^\varepsilon(x,y))\overline{C_{l'}^\varepsilon(y,z)}\,\mathrm{d}z$$

$$= \int_{\mathcal{T}^n} (M_l^\varepsilon(x,z) - M_l^\varepsilon(x,y))\overline{M_{l'}^\varepsilon(y,z)}\,\mathrm{d}z \tag{6.53}$$

$$+ \int_{\mathcal{T}^n} (N_l^\varepsilon(x,z) - N_l^\varepsilon(x,y))\overline{M_{l'}^\varepsilon(y,z)}\,\mathrm{d}z \tag{6.54}$$

$$+ \int_{\mathcal{T}^n} (M_l^\varepsilon(x,z) - M_l^\varepsilon(x,y))\overline{N_{l'}^\varepsilon(y,z)}\,\mathrm{d}z \tag{6.55}$$

$$+ \int_{\mathcal{T}^n} (N_l^\varepsilon(x,z) - N_l^\varepsilon(x,y))\overline{N_{l'}^\varepsilon(y,z)}\,\mathrm{d}z. \tag{6.56}$$

We wish to estimate

$$\int_{\mathcal{T}^n} |C_{l,l'}^\varepsilon(x,z)|\,\mathrm{d}y \quad \text{and} \quad \int_{\mathcal{T}^n} |C_{l,l'}^\varepsilon(x,y)|\,\mathrm{d}x \tag{6.57}$$

in order to apply the well-known Schur lemma.

We first investigate the expression (6.53), which will be denoted by $M\,M_{l,l'}^\varepsilon(x,y)$. The remaining terms will be treated in a similar fashion. Assuming now $l' \geq l$, we employ lemma 6.3 to obtain, for $\delta \in (0,1)$, and any $x, y \in \mathcal{T}^n$ such that $x - y \in [-\frac{1}{2}, \frac{1}{2}]^n$

$$\begin{aligned}|M\,M_{l,l'}^\varepsilon(x,y)| \leq c\,\varepsilon^{2(d^*+1)} \int_{\mathbb{R}^n} & \varepsilon^{-\delta}(\varepsilon|z-y|2^l)^\delta(\varepsilon^2 2^{(l+l')})^n \\ & \times((1+\varepsilon|2^l x - 2^l z|)^{-n-d^*-1} + (1+\varepsilon|2^l x - 2^l y|)^{-n-d^*-1}) \\ & \times(1+\varepsilon|2^{l'} z - 2^{l'} y|)^{-n-d^*-1}\mathrm{d}z.\end{aligned} \tag{6.58}$$

We are now in a position to estimate

$$\int_{\mathcal{T}^n} |M\, M^{\varepsilon}_{l,l'}(x,y)|\,\mathrm{d}y \quad \text{and} \quad \int_{\mathcal{T}^n} |M\, M^{\varepsilon}_{l,l'}(x,y)|\,\mathrm{d}x.$$

Bounding $|M\, M^{\varepsilon}_{l,l'}|$ by (6.58) and integrating (6.58) with respect to x over $\mathbb{R}^n$, while keeping in mind that the integral

$$\int_{\mathbb{R}^n} (\varepsilon 2^l)^n (1+\varepsilon|2^l(x-z)|)^{-n-d^*-1}\mathrm{d}x$$

is constant independent of z, it remains to estimate the integral

$$\begin{aligned}
\varepsilon^{2(d^*+1)} &\int_{\mathbb{R}^n} (|z-y|\varepsilon 2^l)^{\delta}\varepsilon^{-\delta}(\varepsilon 2^{l'})^n(1+\varepsilon|2^{l'}(y-z)|)^{-n-d^*-1}\mathrm{d}z \\
&\le c\,\varepsilon^{2(d^*+1)-\delta}2^{\delta(l-l')}\int_{\mathbb{R}^n}(1+|z|)^{-n-(d^*+1-\delta)}\mathrm{d}z \\
&\le c\,\varepsilon^{2(d^*+1)-\delta}2^{\delta(l-l')}.
\end{aligned}$$

By analogous arguments, the same result is obtained when integrating with respect to y.

Repeating the previous reasoning, we likewise obtain

$$\begin{aligned}
|M\, N^{\varepsilon}_{l,l'}(x,y)| \le c\,\varepsilon^{2(d^*+1)} &\int_{\mathbb{R}^n} \varepsilon^{-\delta}(\varepsilon|z-y|2^l)^{\delta}(\varepsilon 2^{(l+l')})^n \\
&\times((1+\varepsilon|2^l x-2^l z|)^{-n-d^*-1}+(1+\varepsilon|2^l x-2^l y|)^{-n-d^*-1}) \\
&\times \tau^{2^{l'}|\theta(z-y)|}\mathrm{d}z. \qquad (6.59)
\end{aligned}$$

Thus, one concludes that the integral $|\int M\, N^{\varepsilon}_{l,l'}(x,y)\mathrm{d}x|$ can be bounded by

$$\begin{aligned}
\varepsilon^{2(d^*+1)} &\int_{\mathbb{R}^n} (|z-y|\varepsilon 2^l)^{\delta}\varepsilon^{-\delta}2^{l'n}\tau^{|2^{l'}(y-z)|}\mathrm{d}z \\
&\le c\,\varepsilon^{2(d^*+1)-\delta}2^{\delta(l-l')}.
\end{aligned}$$

As for $|\int M\, N^{\varepsilon}_{l,l'}(x,y)\mathrm{d}y|$, we use

$$\int_{\mathbb{R}^n} 2^{l'n} \tau^{2^{l'}|z-y|} \mathrm{d}y = c,$$

to obtain the desired bound. The kernels $N\, M^{\varepsilon}_{l,l'}(x,y)$ and $N\, N^{\varepsilon}_{l,l'}(x,y)$ are treated in much the same way.

Summarizing the previous observations, one finally obtains

$$\| C^{\varepsilon}_{l}(C^{\varepsilon}_{l'})^{*} \|^{1/2}_{\mathscr{L}(L_2(\mathscr{T}^n))} \le c\, \varepsilon^{d^{*}+1-\delta/2} 2^{-\delta/2|l-l'|},$$

and the same type of estimate for the norm of the adjoints $\|(C^{\varepsilon}_{l})^{*} C^{\varepsilon}_{l'}\|^{1/2}_{\mathscr{L}(L_2(\mathscr{T}^n))}$. The assertion now follows from lemma 6.5. $\square$

In order to estimate the operator norm of $A_j - A^{\varepsilon}_{j}$, we have to find uniformly decreasing bounds for the operators $\sum_{0<l<j}(R^{\varepsilon}_{l} + S^{\varepsilon}_{l})$, $j > \mathbb{N}_0$, as $\varepsilon \to 0$. To this end, it will be convenient to use the following notation. Let Ω^{l}_{k} denote the interior of $\operatorname{supp} \Phi^{l}_{k}$ and define

$$r(\varepsilon)^2 := \sup_{l \in \mathbb{N}_0, k \in \mathbb{Z}^{n,l}} 2^{ln} \sum_{l'=l}^{\infty} \sum_{\{k' \in \mathbb{Z}^{n,l} : \Omega^{l'}_{k'} \subseteq \Omega^{l}_{k}\}} \sum_{e \in E_0} 2^{-l'n} |r^{l',\varepsilon}_{e,k'}|^2, \tag{6.60}$$

and

$$s(\varepsilon)^2 := \sup_{l \in \mathbb{N}_0, k \in \mathbb{Z}^{n,l}} 2^{ln} \sum_{l'=l}^{\infty} \sum_{\{k' \in \mathbb{Z}^{n,l} : \Omega^{l'}_{k'} \subseteq \Omega^{l}_{k}\}} \sum_{e \in E_0} 2^{-l'n} |s^{l',\varepsilon}_{e,k'}|^2. \tag{6.61}$$

The following theorem in its wavelet version is mainly due to Meyer. It can be traced back to David–Journée [21] and is a consequence of a celebrated lemma due to Carleson.

LEMMA 6.7

There exists a constant c such that

$$\begin{aligned} c^{-1} r(\varepsilon) &\le \Big\| \sum_{l \in \mathbb{N}_0} R^{\varepsilon}_{l} \Big\|_{\mathscr{L}(L^2(\mathscr{T}^n))} \le c\, r(\varepsilon), \\ c^{-1} s(\varepsilon) &\le \Big\| \sum_{l \in \mathbb{N}_0} S^{\varepsilon}_{l} \Big\|_{\mathscr{L}(L^2(\mathscr{T}^n))} \le c\, s(\varepsilon) \end{aligned} \tag{6.62}$$

holds for $0 < \varepsilon < \varepsilon_0$.

Proof

The proof is essentially given, for instance, in [34]. Because the result is important for our purposes and has to be slightly adapted to the present setting, we sketch the proof for the convenience of the reader. By the stability of the basis $\{\phi^l_{e,k}\}_{e,l,k}$ and (6.19), we have

$$c^{-1}\sum_{l}\sum_{e\in E_0}\sum_{k\in\mathbb{Z}^{n,l}}|r^{l,\varepsilon}_{e,k}(u,\Phi^l_k)_0|^2\le\|\sum_{l\in\mathbb{N}_0}R^\varepsilon_l u\|_0^2\le c\sum_{l}\sum_{e\in E_0}\sum_{k\in\mathbb{Z}^{n,l}}|r^{l,\varepsilon}_{e,k}(u,\Phi^l_k)_0|^2.$$

We verify the right-hand side inequality first. For $I\in\mathcal{J}^j$, we set $w(I)=2^{ln}|(\Phi^l_k,u)_0|^2$. Introducing the function

$$w(x)=\sup_{\{I\in\mathcal{J}^j:\,j\in\mathbb{N}_0,x\in\Omega^l_k\}}w(I),$$

one observes that

$$w(x)=\sup_{\{l\in\mathbb{N}_0,k\in\mathbb{Z}^{n,l}:x\in\Omega^l_k\}}2^{ln}|(u,\Phi^l_k)_0|^2\le c\left(\sup_{\{l,k:x\in\Omega^l_k\}}2^{ln}\int_{\Omega^l_k}|u(y)|\,\mathrm{d}y\right)^2$$

$$\le c\left(\sup_{\{\Omega:x\in\Omega\}}|\Omega|^{-1}\int_{\Omega}|u(x)|\,\mathrm{d}x\right)^2. \quad (6.63)$$

The right-hand side of (6.63) is the square of the Hardy–Littlewood maximal function denoted by $(M(u)(x))^2$. Thus, we conclude by the well-known maximal function theorem (see e.g. [41])

$$\int_{\mathcal{T}^n}w(x)\,\mathrm{d}x\le c\|M(u)\|_0^2\le c\|u\|_0^2. \quad (6.64)$$

The assertion is now a consequence of a lemma due to Carleson, cf. [34, p. 273], whose periodic variant says that

$$\sum_{I\in\mathcal{J}^l,l\ge0}|p(I)|w(I)\le c\,s(\varepsilon)\int_{\mathcal{T}^n}w(x)\,\mathrm{d}x, \quad (6.65)$$

provided that $p(I)$, $I=(l,e,k)$ and $J=(l',e',k')$ satisfy

$$\sup_{l\in\mathbb{N},I\in\mathcal{J}^l}2^{ln}\sum_{l'\ge l}\sum_{\{k'\in\mathbb{Z}^{n,l}:\Omega^{l'}_{k'}\subseteq\Omega^l_k\}}\sum_{e\in E_0}|p(J)|.$$

In fact, here we may choose $p(I):=2^{-ln}|r^{l,\varepsilon}_{e,k}|^2$ to then combine (6.64) with (6.65).

Incidentally, we have proved that the adjoint operators $\sum_l(S^\varepsilon_l)^*$ are bounded in $L^2(\mathcal{T}^n)$ with an operator norm less than or equal to $c\,r(\varepsilon)$. This immediately implies the corresponding result for $\sum_l S^\varepsilon_l$.

To verify the left-hand side inequality, we choose $u = \chi_{\Omega_k^l}$ to be the characteristic function of $\operatorname{supp}\Phi_k^l$. Thus, we have $\| u \|_0 = c\,2^{-ln}$ and $|(\Phi_{k'}^{l'}, u)|^2 \le \int_{\mathcal{T}^n} \Phi_{k'}^{l'}(x)\mathrm{d}x = c'2^{-l'n/2}$. Therefore, we conclude that for every $k \in \mathbb{Z}^{n,l}$ and $l \in \mathbb{N}$

$$\sup_{k\in \mathbb{Z}^{n,l}} \sum_{l'=l}^{\infty} \sum_{\{k' \in \mathbb{Z}^{n,l}:\Omega_{k'}^{l'} \subseteq \Omega_k^l\}} \sum_{e\in E_0} 2^{-l'n} |r_{e,k'}^{l',\varepsilon}|^2 \le \sum_{l'\ge l} \sum_{k'\in \mathbb{Z}^{n,l'}} \sum_{e\in E_0} |r_{e,k'}^{l',\varepsilon}|^2 |(u, \Phi_{k'}^{l'})_0|^2$$

$$\le c \| \sum_{l' \ge l} R_{l'}^{\varepsilon} u \|_0^2 \le c \| \sum_{l=0}^{\infty} R_l^{\varepsilon} \|_{\mathcal{L}(L_2(\mathcal{T}^n))}^2 \| u \|_0^2 \le c \| \sum_{l=0}^{\infty} R_l^{\varepsilon} \|_{\mathcal{L}(L_2(\mathcal{T}^n))}^2 2^{-ln}.$$

This completes the proof. □

We can now summarize the above results to prove the following counterpart to theorem 5.1.

THEOREM 6.2

Let A_j^{ε} be defined by (6.9). Then there exist $c > 0$ and $\delta \in (0, 1)$ such that for all $j \in \mathbb{N}$

$$\| A_j - A_j^{\varepsilon} \|_{\mathcal{L}(L_2(\mathcal{T}^n))} \le c(r(\varepsilon) + s(\varepsilon) + \varepsilon^{d^*+1-\delta/2}) \tag{6.66}$$

holds.

Proof

Recall that

$$A_j - A_j^{\varepsilon} = \sum_{l=1}^{j} (C_l^{\varepsilon} + R_l^{\varepsilon} + S_l^{\varepsilon}),$$

where $C_l^{\varepsilon}1 = (C_l^{\varepsilon})^*1 = 0$. Thus, lemma 6.6 gives the bound

$$\sum_l C_l^{\varepsilon} \le c\,\varepsilon^{d^*+1-\delta/2}.$$

The assertion now follows from lemma 6.7. □

If

$$\lim_{\varepsilon\to 0} r(\varepsilon) = \lim_{\varepsilon\to 0} s(\varepsilon) = 0, \tag{6.67}$$

one immediately concludes from (6.66) that the (0, 0)-stability of A_j implies (0, 0)-stability of A_j^{ε} for all $0 < \varepsilon \le \varepsilon_0$ provided ε_0 is sufficiently small. In the same way as before, we may now apply lemma 5.5 to prove the following result.

THEOREM 6.3

Suppose u^* is the exact solution of (3.4) and let u_ε^j denote the solution of the compressed scheme $A_j^{\varepsilon} u^j = Q_j f$, where A_j^{ε} is defined by (6.9). Furthermore, suppose that A_j is (0, 0)-stable. Then there exist some $\varepsilon_0 > 0$ and some constant c independent of $j \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_0)$, such that for all $0 < \varepsilon < \varepsilon_0$

$$\| u^j - u^j_\varepsilon \|_0 \le c(r(\varepsilon) + s(\varepsilon) + \varepsilon^{d^*+1-\delta/2}) \| u^* \|_0$$

holds uniformly in $j \in \mathbb{N}$, where u^j, u^* are the solutions of (3.7), (3.4), respectively.

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Note that the application of the operator A^ε_j requires only $O(2^{nj})$ operations.

Remark

Since

$$(A_j - A^\varepsilon_j)1 = \sum_{l=1}^{j} R^\varepsilon_l 1, \quad (A_j - A^\varepsilon_j)^* 1 = \sum_{l=1}^{j} (S^\varepsilon_l)^* 1,$$

relation (6.62) can be reformulated as

$$\lim_{j \to \infty} (\| (A_j - A^\varepsilon_j) 1 \|_{BMO} + \| (A_j - A^\varepsilon_j)^* 1 \|_{BMO}) \sim r(\varepsilon) + s(\varepsilon).$$

Here, $\| \cdot \|_{BMO}$ denotes the norm in the space of functions of *bounded mean oscillation* *BMO* (cf. [34,20]). Note that an impractical condition like (6.67) is not needed when working with the wavelet representation, since the operators A considered here are a priori known to be bounded in $L^2(\mathcal{T}^n)$.

On the other hand, for many interesting cases such a condition is, due to the structure of A, automatically satisfied or easy to verify. Trivial examples satisfying $s(\varepsilon) + r(\varepsilon) \to 0$ as $\varepsilon \to 0$ are given whenever $r^{l,\varepsilon}_{e,k} = s^{l,\varepsilon}_{e,k} = 0$, for $l > N_0$, $k \in \mathbb{Z}^{n,l}$, $e \in E_0$, or are geometrically decreasing with respect to l. Moreover, as we have seen in the previous subsection, sufficiently small $r(\varepsilon) + s(\varepsilon)$ can be easily achieved if one uses, for example, varying cut-off bandwidths on different levels.

EXAMPLE

We wish to conclude this section by pointing out that too simplistic compression strategies may indeed cause problems in connection with atomic representations. For the sake of simplicity, we will consider only the technical, somewhat simpler Euclidean space $\mathbb{R}^n$ which, however, already exhibits all the essential features. Let S be a singular integral operator given by

$$Su(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{2\pi i \langle \xi, x-y \rangle} \sigma(\xi) u(y) \mathrm{d}y \right) \mathrm{d}\xi, \quad u \in C^\infty_0(\mathbb{R}^n), \tag{6.68}$$

where $\xi \mapsto \sigma(\xi)$, $\xi \neq 0$, is assumed to be a bounded homogeneous function (of degree zero). Obviously, the operator S defines a bounded operator in $L^2(\mathbb{R}^n)$. Assume that φ generates a multiresolution analysis $\ldots \subset \langle\varphi\rangle \subset \langle\varphi\rangle^{j+1} \subset \ldots$ (cf. (2.6)). Since the operators are invariant under the action of the affine group, i.e. they commute with translation and dilation operators, straightforward computation yields

$$2^{ln}(S\psi_e(2^l \cdot -k), \varphi(2^l \cdot -k')) = d_{k,k'}^{l,e} = d_{e,k-k'}. \tag{6.69}$$

Here, the coefficients $d_{k,k'}^{l,e} = d_{e,k-k'}$ are independent of l, $k + k'$. The corresponding extraction of constants can be performed by subtracting operators $R_l^\varepsilon, S_l^\varepsilon$ analogously to the periodic case. Since $d_{k,k'}^{l,e}$ depends only on $e \in E_0$ and $k - k' \in \mathbb{Z}^n$, the operators $R_l^\varepsilon, S_l^\varepsilon$ are induced by diagonal matrices, e.g.,

$$R_l^\varepsilon u(x) = \sum_{e \in E_0} \sum_{k \in \mathbb{Z}^n} 2^{ln} r_\varepsilon(\zeta_e(2^l \cdot -k), u)\Phi_{k'}^l(2^l x - k), \tag{6.70}$$

where $r_\varepsilon = \sum_{e,k'} d_{e,k'}$ is independent of k and the level l. Let $r(\varepsilon)$ be defined in analogy to (6.60). Therefore, $r(\varepsilon) = 0$ if and only if $r_\varepsilon = 0$, whereas $r(\varepsilon)$ becomes unbounded otherwise. Indeed, for $j \in \mathbb{N}$, we have

$$\sum_{l'>l}^{j} \sum_{\Omega_{k'}^{l'} \subset \Omega_k^l} 2^{l'n} r_\varepsilon \geq c\, 2^{-ln}(j - l), \tag{6.71}$$

where, as above, Ω_k^l = interior $(\operatorname{supp}\Phi(2^l \cdot \; -k))$. Thus, (6.71) tends to infinity if $j \to \infty$ unless $r(\varepsilon) = 0$. As an aside, setting $\varepsilon = 0$ reproduces a known result, namely that for operators of type (6.68). $\sum_{e,k} d_{k,k'}^{l,e} = \sum_{e,k} d_{e,k} = 0$ as well as $\sum_{e',k} g_{k,k'}^{l,e'} = \sum_{e',k} g_{e',k} = 0$ necessarily hold for any k, k', e, l. Let us now consider constant compression on each level by discarding those entries for which $|k - k'| > \varepsilon^{-1}$ or alternatively discarding all entries with absolute value below a fixed threshold (cf. [4]). This seems to be natural in this case, since the coefficients do not depend on the levels l. As a consequence of theorem 6.2 and (6.71), our compression gives only small errors if the remaining coefficients satisfy the condition $\sum_{\{k:|k| \leq \varepsilon^{-1}\}} d_{e,k} = 0$ and $\sum_{\{k:|k| \leq \varepsilon^{-1}\}} g_{e',k} = 0$. Otherwise, the error increases with increasing j. Here the situation becomes even worse, since the norms of the compressed operators A_j^ε themselves tend to infinity as $j \to \infty$ according to the result of David–Journée [21].

7. Summary of estimates

The previous results are collected in table 1, showing the number of nonzero entries needed in the compressed matrices in order to achieve the desired optimal convergence rate or fixed error bound. Here, we set $N = 2^{jn}$, which is the total number of unknowns and recall that the mesh width is 2^{-j}. The first column just

Table 1

Error bound	ε	$N^{-\tau}$	$N^{-\tau}$	$N^{-\tau}$
		$r = 0$	$r > 0$	$r < 0$
Wavelet Representation	$O(N)$,	$O(N)$, $\tau < d' + \rho$ $O(N \log N)$, $\tau < d + 1$ $O(N \log^{2+\frac{n}{2(d^*+1)}} N)$, $\tau = d + 1, d < d^*$ $O(N \log^{2+\frac{3n}{2(d^*+1)}} N)$, $\tau = d + 1, d^* = d$	$O(N)$, $\tau < d' + \rho + r/2$ $O(N \log N)$, $\tau < d + 1 + r/2$ $O(N \log^{2+\frac{n}{2(d^*+1+r)}} N)$, $\tau = d + 1 + r/2, d < d^*$ $O(N \log^{2+\frac{3n}{2(d^*+1+r)}} N)$, $\tau = d + 1 + r/2, d^* = d$	$O(N)$, $\tau < d' + \rho$ $O(N \log N)$, $\tau < d' + 1$ $O(N \log^{2+\frac{n}{2(d'+1+r)}} N)$, $\tau = d + 1, d < d^*$ $O(N \log^{2+\frac{3n}{2(d^*+1+r)}} N)$, $\tau = d + 1, d^* = d$
Atomic representation	$O(N)$			

shows the order of nonvanishing entries needed to achieve a fixed error tolerance ε. The remaining three columns contain asymptotic error bounds. Here, the parameter τ in the convergence order $O(N^{-\tau})$ is the largest difference between the Sobolev scales covered by the estimates, and in this sense gives the highest possible convergence rate that can be achieved. For instance, in theorem 5.4 one can take $s = -r/2$ and $t = d + 1$ to obtain $\tau = d + 1 + r/2$. The second column refers to zero-order operators covering a larger class of operators, namely also Calderón–Zygmund operators. Finally, we recall that the rates $N^{-\tau}$ listed in table 1 as convergence rates for the solution corresponding to the compressed scheme differ from the respective consistency errors of the exact solution to (3.7) by the order of $\varepsilon^{1+d^*+r}N^{-\tau}$.

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