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WAVELET METHODS FOR FAST RESOLUTION OF ELLIPTIC PROBLEMS*

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Abstract. This paper shows that the use of wavelets to discretize an elliptic problem with Dirichlet or Neumann boundary conditions has two advantages: an explicit diagonal preconditioning makes the condition number of the corresponding matrix become bounded by a constant and the order of approximation is *locally* of spectral type (in contrast with classical methods); using a conjugate gradient method, one thus obtains fast numerical algorithms of resolution. A comparison is also drawn between wavelet and classical methods.

Key words. wavelet bases, preconditioning, elliptic equations, Galerkin methods

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1. Introduction. Let us recall at first some properties of Galerkin methods based on finite elements or of finite difference methods for the resolution of elliptic problems in a bounded domain.

In all these methods, once the problem has been properly discretized, we are led to solve a system which is ill conditioned. Typically, for a second-order elliptic problem in two dimensions, we obtain a matrix M such that

$$\kappa = \|M\| \, \|M^{-1}\| = O(1/h^2),$$

where h is the size of the discretization (see [8], [14] and [17]). Such ill conditioning has two drawbacks: it leads to numerical instabilities and to slow convergence for iterative resolution algorithms. In order to avoid this problem, we usually use a preconditioning, which amounts to finding an easily invertible matrix D such that $D^{-1}MD^{-1}$ (or $D^{-1}M$, depending on the method used) will have a better condition number κ. For the example we considered, the usual preconditioning methods on general domains (SSOR or DKR on a conjugate gradient method, for instance) make κ becomes $O(1/h)$ (see [8, Chap. 8.4]). We will show that, if we use a wavelet method, then the simplest conceivable preconditioning, namely, when D is a diagonal matrix, yields a $\kappa = O(1)$ in any dimension (provided that the domain has a Lipschitz boundary). A question that then arises naturally is how well a diagonal preconditioning can improve the condition number for a finite elements or a finite differences method. We will show that for such methods, a diagonal preconditioning always yields a condition number that tends to infinity as h tends to zero as, at best, $C/(h|\log h|)$.

These properties show that wavelet methods should provide an efficient numerical alternative to classical methods, as the first numerical results already show (see [3]).

In § 2 we consider the following Dirichlet type boundary value problem:

$$\text{(A)} \qquad \begin{aligned} &-\nabla\cdot(a\,\nabla\, u)+bu=g \quad \text{with } g\in H^{-1}(\Omega),\\ &u(x)=h \quad \text{for } x\in\partial\Omega \quad \text{with } h\in H^{1/2}(\partial\Omega). \end{aligned}$$

We suppose that Ω is a bounded domain of $\mathbf{R}^n$ with a Lipschitz boundary and that

$$(1) \qquad \begin{aligned} &0< C_1 \leqq a(x) \leqq C_2 < +\infty,\\ &0\leqq b(x)\leqq C_3<+\infty. \end{aligned}$$

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We will show that, using the Galerkin method in a wavelet basis, we have an explicit diagonal preconditioning, independent of the function a, such that the preconditioned matrix has a condition number $\kappa = O(1)$. This property will be a direct consequence of the characterization of Sobolev spaces on the wavelet coefficients.

In § 3, we consider the approximation problem when the smoothness of g is not everywhere the same. We will introduce some kind of adapted mesh refinement method in order to have everywhere the same precision in the solution, and we will show that, for this method, the order of approximation is locally of spectral type.

We will then consider the following Neumann type boundary value problem:

$$\text{(B)} \qquad \begin{aligned} -\Delta u &= g \quad \text{with } g \in H^{-1}(\Omega), \\ \frac{\partial u}{\partial n} &= h \quad \text{on } \partial\Omega \text{ with } h \in H^{1/2}(\partial\Omega). \end{aligned}$$

We suppose that Ω is a bounded domain of $\mathbf{R}^n$ with a Lipschitz, piecewise C^1 boundary and

$$\int_\Omega g = \int_{\partial\Omega} h.$$

We look for a solution u such that $\int_\Omega u = 0$.

We consider here a less general operator than in the Dirichlet case in order to show in this example how to deal with the condition of the vanishing integral. Of course, the method we give can be used for more general operators.

We will need to construct a new wavelet basis "adapted" to the Neumann problem, which will be done in § 5 (section 4 will be devoted to some prerequisites on orthogonalization methods), and we will show in § 6 that, for a Galerkin method using this basis, the same diagonal preconditioning as for the Dirichlet problem leads also to a bounded condition number in the resolution.

In § 7 we will show how sparse the matrices of the problems discretized in wavelet bases are. We will give a few numerical illustrations of these results in § 8. Section 9 will be devoted to some remarks and generalizations. In § 10 we will give a lower bound on the best possible condition number obtained using a diagonal preconditioning on a finite elements or a finite difference method. This will surprisingly be a consequence of the "too good" decay of the entries of the discretization matrices for such local methods.

Further numerical experiments are being performed and will be the subject of a forthcoming paper.

2. The wavelet method for a Dirichlet boundary problem. In order to solve (A), we use the wavelet basis constructed in [7]. We shall not repeat this construction, but rather recall some of its properties.

We consider the space V_p of functions that are C^{2m-2}, vanish outside Ω, and are polynomials of degree $2m-1$ in each variable in the cubes

$$k2^{-p} + 2^{-p}[0, 1]^n \quad \text{with } k \in \mathbf{Z}^n$$

(thus, the discretization step is $h = 2^{-p}$). Then there exists an $\mathbf{L}^2$-orthonormal basis of V_p composed of functions $\psi_{j,k}$ $(j \leqq p)$ such that

$$|\partial^\alpha \psi_{j,k}(x)| \leqq C 2^{j\alpha} 2^{nj/2} \exp(-\gamma 2^j |x - k2^{-j}|)$$

for $|\alpha| \leqq 2m-2$, and a positive γ.

The wavelets are indexed by j $(0 \leqq j \leqq p)$ and by the $k \in \mathbf{Z}_n$ such that $k2^{-j} + (m+1)2^{-j}[0, 1]^n \subset \bar{\Omega}$. The decay estimates show that $\psi_{j,k}$ and its partial derivatives are essentially centered around $k2^{-j}$ with a width 2^{-j}. In the following, the point $k2^{-j}$ will sometimes be denoted λ.

Remark that we make a slight change of notation from [7]. The basis we use here is composed of the $\phi_{0,k}$, and the $\psi_{j,k}$ for $j>0$ defined in [7]. The $\phi_{0,k}$ are thus renamed here $\psi_{0,k}$ in order to simplify the notations.

We shall need the following characterisation, proved in [7].

If a function f belongs to $H_0^1(\Omega)$, the following condition on its wavelet coefficients $c_{j,k}$ $(=\langle f, \psi_{j,k}\rangle)$ holds:

$$C\sum|2^j c_{j,k}|^2 \leqq \|f\|^2_{H_0^1} \leqq C'\sum|2^j c_{j,k}|^2. \tag{2}$$

We now use the Galerkin method to solve (A) (a related method has also been proposed in [18] when $\Omega=\mathbf{R}^n$). As usual, we first reduce (A) to a homogeneous problem by supposing that we can find a smooth function h' that extends h inside Ω. Then $u'=u-h'$ will be the solution of the following problem (A′), if we replace in (A′) g by $g'=g-\nabla\cdot(a\nabla h')+bh'$.

$$\text{(A}'\text{)}\qquad \begin{array}{ll} -\nabla\cdot(a\nabla u)+bu=g & \text{with } g\in L^2(\Omega), \\ u(x)=0 & \text{for } x\in\partial\Omega. \end{array}$$

The variational form of this problem is

$$\forall v\in H_0^1,\quad \int_\Omega a\nabla u\cdot\nabla v+\int_\Omega buv=\int_\Omega gv.$$

The Galerkin approximation will consist in finding u in V_j such that

$$\forall v\in V_j,\quad \int_\Omega a\nabla u\cdot\nabla v+\int_\Omega buv=\int_\Omega gv;$$

which, once the functions are expressed by their coordinates in the wavelet basis, amounts to solving the problem $MX=Y$, where the matrix M is given by

$$M_{(j,k),(j',k')}=\langle a\nabla\psi_{j,k}\,|\,\nabla\psi_{j',k'}\rangle+\langle b\psi_{j,k}\,|\,\psi_{j',k'}\rangle,$$

$\langle\ |\ \rangle$ denotes the $\mathbf{L}^2$ scalar product, and the vectors X and Y are $X=(\langle u\,|\,\psi_{j,k}\rangle)$ and $Y=(\langle g\,|\,\psi_{j,k}\rangle)$.

Let A be the vector $(c_{j,k})$, and $f=\sum c_{j,k}\psi_{j,k}$. Then

$$A^tMA=\langle a\nabla f\,|\,\nabla f\rangle+\langle bf\,|\,f\rangle.$$

Thus, because of (1),

$$C_1\|f\|^2_{H_0^1}\leqq A^tMA\leqq(C_2+C_3C'')\|f\|^2_{H_0^1},$$

where C'' is the smallest number such that

$$\|f\|^2_{L^2}\leqq C''\|\nabla f\|^2_{L^2}\quad (=\|f\|^2_{H_0^1}).$$

Using (2), we get

$$CC_1\sum|2^jc_{j,k}|^2\leqq A^tMA\leqq C'(C_2+C''C_3)\sum|2^jc_{j,k}|^2.$$

Hence, if D is the diagonal matrix defined by

$$D_{(j,k),(j',k')}=2^j\delta_{(j,j')}\delta_{(k,k')},$$

for any vector A,

$$CC_1\|A\|^2\leqq A^tD^{-1}MD^{-1}A\leqq C'(C_2+C_3)\|A\|^2.$$

We have thus proved the following theorem.

THEOREM 1. *For a Galerkin method using wavelets to approximate the solution of* (A), *the condition number of* $D^{-1}MD^{-1}$ *is bounded by* $C'(C_2+C''C_3)/CC_1$ (*and is thus independent of the mesh size*).

Remarks. The coefficients in D are known a priori and do not depend on the specific second-order elliptic operator considered, and κ depends only on the constants appearing in the Sobolev equivalences and on the functions a and b.

The regularity of the wavelet basis used has not been used up to now, but will be important in §§ 3 and 7.

From a mathematical point of view, the method described amounts to writing the inverse of an elliptic operator of order α on a domain as the product of a pseudodifferential operator of order zero, which is bounded and invertible on $\mathbf{L}^2$, and some kind of fractional integration diagonal in the wavelet basis, which consists in multiplying the wavelet $\psi_{j,k}$ by $2^{-\alpha j}$.

3. "Mesh refinement" and order of approximation. When computing a numerical solution of (A), it is desirable to obtain a better approximation in certain regions (near the boundary, or where the functions a, b, and g are less smooth, for instance). The usual method in finite elements is to use mesh refinements. A drawback is that the condition number (and hence the number of iterations needed in the conjugate gradient method) is the same as if the mesh had been refined everywhere (see [1]). A method which has the same purpose as mesh refinements but using wavelets is the following. Recall that, in the Galerkin method described in § 2, we already have a wavelet basis of V_j. We can add in the region of refinement functions $\psi_{l,k}$ such that $j \leqq l < j'$, and $k2^{-l}$ is inside or near the region considered. This is actually the same as if we had locally a basis of $V_{j'}$ and a mesh of size $2^{-j'}$.

We shall investigate this "mesh refinement" in the following framework. Suppose that the function g of problem (A) is smooth in certain regions. Because of the localization of the wavelets, we can expect the order of approximation of the method to be higher in the region of smoothness than in other regions. We can add high-order wavelets (with large j's) where g is not smooth in order to keep a global good approximation. The purpose of this section is to show how this program can be performed and to prove the optimality of the corresponding order of approximation. This will show that wavelet methods give a "local spectral" order of approximation.

Thus, we apply the Galerkin method in a space V_J spanned by wavelets $\psi_{j,k}$, where we take all the wavelets for $j \leqq J$ and, for higher values of j, only those centered where f is not smooth. The estimate on the condition number of the method is obviously not changed and, since the H^1 order of approximation satisfies

$$\|u-u_j\| \leqq C \inf_{v_j \in V_J} \|u-v_j\|$$

(following a classical lemma; see [14], for instance), we must only estimate $\inf_{v_j \in V_J} \|u-v_j\|$. Before constructing V_J and computing the order of approximation, we would like at first to recall how a priori information on the solution u can be obtained. Such results are usually based on two-microlocalization arguments; we refer to [6] for these properties and their proofs, and we will only give an example here.

A function f is $C^{\alpha}_{x_0}$ if there exists a polynomial P such that

$$|f(x)-P(x)| \leqq C|x-x_0|^{\alpha}.$$

It is C^{α} if it is $C^{\alpha}_{x_0}$ uniformly in x_0. Suppose that u is a solution of

$$\Delta u = f \quad \text{in } \Omega,$$

and that u is C^ε, for an $\varepsilon > 0$, in a subset $\Omega' \subset \Omega$. Let $\alpha > 0$, and suppose that the wavelets are at least $C^{\alpha+2}$. If f is $C^\alpha_{x_0}$ for an $x_0 \in \Omega'$, the wavelet coefficients of u verify the following (optimal) estimate:

$$|\langle u \mid \psi_{j,k}\rangle| \leqq C 2^{-[(n/2)+\alpha+2]j}(1+|k-2^j x_0|)^{\alpha+2},$$

which gives the decay of the coefficients near the regularity points of the second member of the equation. Actually, similar estimates exist when using Sobolev spaces instead of Hölder spaces, and the Laplacian can be replaced by a smooth elliptic second-order operator.

We now make some hypotheses. We consider u solution of (A) in dimension 2 or 3, with a function $a(x)$ smooth, and we take for boundary value $g(x)=0$ on $\partial\Omega$. We also make the following minimal global regularity assumption. We suppose in dimension 2 that u is in $H^2(\Omega) \cap H^1_0(\Omega)$ (which holds for instance when the operator is a Laplacian, if Ω is convex, or if $\partial\Omega$ is C^2, since f is in L^2; see [4]), and in dimension 3 that u is in $W^{2,p}(\Omega) \cap H^1_0(\Omega)$ for a $p > 2$. We shall also suppose that u is actually smoother in some region, namely, that u is in $H^s(\Omega')$ for a certain Ω', the boundary of which is Lipschitz and lies in Ω.

We now define some sets of indices Θ_j and Λ_j:

- If $j \leqq J$, Θ_j is empty.
- If $J < j \leqq 2sJ/\varepsilon$, Θ_j is the set of all $\lambda = k2^{-j}$ such that $d(\lambda, \partial\Omega) \geqq ((2s-\varepsilon)\log 2/\gamma) j 2^{-j}$ (where ε will be defined later).
- If $J > 2sJ/\varepsilon$, Θ_j is the set of all the $\lambda = k2^{-j}$.
- If $j \leqq J$, Λ_j is empty.
- If $j > J$, Λ_j is the set of all λ such that $d(\lambda, \partial\Omega') \leqq ((2s+n+1)\log 2/\gamma)2^{-j}j$.

The space V_J will be spanned by all the wavelets that do not belong to any Θ_j or Λ_j. Essentially, in the region where u is H^s we take all the wavelets up to the index J; where u is H^1 we take all the wavelets up to the index sJ, and we have to add a layer of higher-frequency wavelets near the boundary, since the H^2 order of approximation is lost near the boundary (this is because we approximate u by functions which have vanishing derivatives at the boundary, while $\partial u/\partial n$ does not vanish on $\partial\Omega$). Remark, however, that this "layer of wavelets" contains a few elements: for a discretization step $h = 2^{-j}$, the total number of wavelets used is $C2^{nj}$, and the number of layer wavelets is bounded by $C' j^2 2^{(n-1)j}$.

PROPOSITION 1. *Under the preceding hypotheses, if u is the exact solution of* (A), *and u_J the approximate solution obtained by the Galerkin method in V_J,*

$$\|u - u_J\|_{H^1} \leqq C2^{-(s-1)J}.$$

In order to prove this proposition, we must estimate

$$\left\| \sum_j \sum_{\lambda \in \Theta_j \cup \Lambda_j} C_{j,k}\psi_{j,k} \right\|_{H^1} \quad (\text{if } u = \sum C_{j,k}\psi_{j,k}),$$

which is equivalent to

$$\sum_j \sum_{\lambda \in \Theta_j \cup \Lambda_j} \left| 2^j C_{j,k} \right|^2.$$

We first prove some (technical) lemmas.

LEMMA 1. *The wavelet coefficients $C_{j,k}$ of u can be written as a sum of two terms $C^1_{j,k}$ and $C^2_{j,k}$ such that*

$$\sum |2^{2j} C^1_{j,k}|^2 < \infty$$

and

$$|C^2_{j,k}| \leqq C2^{-\alpha j} \exp(-\gamma 2^j d(\lambda, \partial\Omega))$$

with $\alpha > (n+1)/2$.

Proof. We can write (see [7]) $\psi_{j,k}$ as a sum of two wavelets $\psi^1_{j,k}$ and $\psi^2_{j,k}$ such that $\psi^1_{j,k}$ and its derivatives have the same size estimates as $\psi_{j,k}$ and have vanishing moments, and $\psi^2_{j,k}$ is supported outside Ω and such that

$$|\psi^2_{j,k}| \leqq C2^{nj/2} \exp(-\gamma 2^j (d(\lambda, \partial\Omega) + |x-\lambda|)).$$

Let u_1 be an extension of u that belongs to $H^2(R^n)$. Let $C^1_{j,k} = \int u_1 \psi^1_{j,k}$ and $C^2_{j,k} = \int u_1 \psi^2_{j,k}$. Then $C_{j,k} = C^1_{j,k} + C^2_{j,k}$. Following the terminology of [12], the $\psi^1_{j,k}$ are "vaguelettes," and thus

$$\sum |2^{2j} C^1_{j,k}|^2 \leqq C \|u_1\|^2_{H^2} \leqq C \|u\|^2_{H^2}.$$

We now estimate $C^2_{j,k}$. In dimension 2, u_1 belongs to H^2; thus, it belongs to $W^{1,p}$ for any $p < \infty$, and in dimension 3 it belongs to $W^{1,p}$ for a $p > 6$. Since u_1 vanishes on $\partial\Omega$, one easily checks that it can be written

$$u_1(x) = d(x, \partial\Omega) g(x)$$

with g in L^p. Thus

$$\begin{aligned}\left|\int u_1 \psi^2_{j,k}\right| &\leqq C2^{nj/2} \exp(-\gamma 2^j d(\lambda)) \\ &\quad \cdot \int d(x, \partial\Omega) g(x) \exp(-\gamma 2^j |x-\lambda|)\, dx \\ &\leqq C2^{nj/2} \exp(-\gamma 2^j d(\lambda, \partial\Omega)) \|g\|_{L^p} \\ &\quad \cdot \left\{\int (d(x, \partial\Omega) \exp(-\gamma 2^j |x-\lambda|))^q\, dx\right\}^{1/q}\end{aligned}$$

(with $1/p + 1/q = 1$)

$$\leqq C2^{[(n/2)-(n/q)-1]j} \exp(-\gamma 2^j d(\lambda, \partial\Omega)) \|u\|_{H^2} \quad \text{for any } q > 1.$$

The lemma is thus proved when $n = 2$. The proof in dimension 3 is exactly the same. The only change appears when it is necessary to use the Sobolev injection. Here, if we made the hypothesis $u \in H^2$, we would get $p = 6$, which would not be sufficient to conclude the problem. The slightly stronger hypothesis we make allows us to choose a $p > 6$ and thus a $q < 6/5$, which yields the right coefficient.

We note $\varepsilon = 2\alpha - n - 1$.

LEMMA 2. *The following estimate holds*

$$\sum_{sJ}^{\infty} \sum_{\lambda \in \Theta_j} |2^j C_{j,k}|^2 \leqq 2^{-2sJ}.$$

Proof. We use the decomposition of $C_{j,k}$ in $C^1_{j,k} + C^2_{j,k}$. We have

$$\sum_{sJ}^{\infty} |2^j C^1_{j,k}|^2 \leqq \sum_{sJ}^{\infty} |2^{2j-sJ} C^1_{j,k}|^2 \leqq C2^{-2sJ}.$$

We estimate now $\sum_{2sJ/\varepsilon}^{\infty} |2^j C^2_{j,k}|^2$. For a fixed j, we divide the set of λ's into the sets

$$n \leqq 2^j d(\lambda, \partial\Omega) < n+1.$$

Each of these sets contains at most $C2^{n-1}$ terms. Thus

$$\sum_{2sJ/\varepsilon}^{\infty} |2^j C^2_{j,k}|^2 \leqq \sum_{2sJ/\varepsilon}^{\infty} \sum_n C2^{-\varepsilon j} \exp(-2\gamma n) \leqq C2^{-2sJ}.$$

The last term to estimate is $\sum_{sJ}^{2sJ/\varepsilon}\sum_{\lambda\in\Theta_j}|2^jC_{j,k}|^2$. Taking the same subdivision of the λ's it is bounded by

$$\sum_{sJ}^{\infty}\sum_{n\geqq aj} C2^{-\varepsilon j}\exp(-\gamma n)\quad\left(\text{with } a=\frac{(2s-\varepsilon)\log 2}{\gamma}\right)\leqq C2^{-2sJ}.$$

LEMMA 3. *The following estimate holds*

$$\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}|2^jC_{j,k}|^2\leqq C2^{-(s-1)j}.$$

Proof. We first prove that

$$\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}|2^{sj}C_{j,k}|^2<\infty.$$

We extend u in a function u' which belongs to $H_0^s(\Omega)$. The wavelet coefficients of u' satisfy

$$\sum|2^{sj}\langle u'|\psi_\lambda\rangle|^2<\infty,$$

so that we must only prove the lemma for the coefficients of $u-u'$. Since $u-u'$ vanishes outside Ω',

$$\begin{aligned}\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}&|2^{sj}\langle u-u'|\psi_\lambda\rangle|^2\\ &\leqq C\|u-u'\|_2\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}2^{2sj}\exp(-\gamma 2^jd(\lambda,\partial\Omega'))\\ &\leqq\sum_{J}^{\infty}2^{2sj}2^{nj}\exp\left(-\gamma\left(\frac{(2s+n+1)\log 2}{\gamma}\right)j\right)<\infty.\end{aligned}$$

Thus

$$\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}|2^{sj}C_{j,k}|^2<\infty,$$

but

$$\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}|2^{sj}C_{j,k}|^2\geqq\sum_{J}^{\infty}\sum_{\lambda\in\Lambda_j}|2^j2^{J(s-1)}C_{j,k}|^2.$$

Hence Lemma 3. Proposition 1 is the immediate consequence of the two last lemmas.

We proved Proposition 1 in the case of two different degrees of smoothness in two regions in order to show, in a model case, what can be expected and how the proofs work. This can of course be generalized to more complicated settings (with more regions or C^α-type smoothness).

We shall now construct a wavelet basis adapted to problem (B). The basis we considered already could not be used since it is not a basis of the Sobolev space $\mathbf{H}^1(\Omega)$, which is the space associated to the variational formulation of (B). Some prerequisites are needed for this construction, and are given in the next section.

4. Some results on orthonormalization procedures. The technique to construct wavelet bases on domains is basically the following. We start with spaces of spline functions, or of finite elements defined on the domain, and then we apply an orthonormalization procedure. We now describe the one we will use since, though classical (see [15]), it is not as widely known as the Gram–Schmidt algorithm.

Let H be an Hilbert space, and (e_n) a Riesz basis of H, that is, a basis such that, for any sequence (a_n) in $\mathbf{l}^2$,

$$C_1 \sum |a_n|^2 \leqq \|\sum a_n e_n\|^2 \leqq C_2 \sum |a_n|^2 \tag{3}$$

for two positive constants C_1 and C_2.

Let G be the operator defined on H by

$$G(f) = \sum \langle f \mid e_n \rangle e_n.$$

Then we have Lemma 4.

LEMMA 4. *The operator G is a self-adjoint positive definite operator and, if we define h_n by*

$$h_n = G^{-1/2}(e_n),$$

the h_n form an orthonormal basis of H.

Proof. The operator G is self-adjoint positive because

$$\langle G(f) \mid g \rangle = \sum \langle f \mid e_n \rangle \langle g \mid e_n \rangle.$$

From (3), we obtain

$$C_1 \sum \langle f \mid e_n \rangle^2 \leqq \|\sum \langle f \mid e_n \rangle e_n\|^2 \leqq C_2 \sum \langle f \mid e_n \rangle^2,$$

so that

$$C_1 \langle G(f) \mid f \rangle \leqq \|G(f)\|^2 \leqq C_2 \langle G(f) \mid f \rangle;$$

hence G is positive definite and

$$C_1 \mathbf{I}d \leqq G \leqq C_2 \mathbf{I}d.$$

Let us prove that the h_n form an orthonormal basis of H. Since G is positive definite, $G^{-1/2}$ can be defined and is an isomorphism on H. Hence, the $G^{-1/2}(e_n)$ are a basis of H. We first check that

$$\forall x \in H, \quad \sum \langle G^{-1/2}(e_n) \mid x \rangle G^{-1/2}(e_n) = x. \tag{4}$$

We have

$$G(y) = \sum \langle e_n \mid y \rangle e_n,$$

so that

$$G^{1/2}(G^{1/2}(y)) = \sum \langle G^{-1/2}(e_n) \mid G^{1/2}(y) \rangle e_n$$

and

$$G^{1/2}(y) = \sum \langle G^{-1/2}(e_n) \mid G^{1/2}(y) \rangle G^{-1/2}(e_n).$$

Hence we have (3) if we take $x = G^{1/2}(y)$ (which is possible since $G^{1/2}$ is an isomorphism). The $G^{-1/2}(e_n)$ are a basis of H. If we take $x = G^{-1/2}(e_m)$ in (4), we obtain

$$\sum_{n \neq m} \langle G^{-1/2}(e_n) \mid G^{-1/2}(e_m) \rangle G^{-1/2}(e_n) + (\|G^{-1/2}(e_m)\|^2 - 1) G^{-1/2}(e_m) = 0.$$

Since the e_n are a basis of H, this implies that

$$\langle G^{-1/2}(e_n) \mid G^{-1/2}(e_m) \rangle = \delta_{n,m}$$

and that the h_n are an orthonormal basis of H.

We shall apply this construction in cases in which the e_n will be localized functions, and we shall show that the orthonormal basis thus constructed has a similar localization. It will be a consequence of the following lemma (see [5]).

LEMMA 5. *Let T be a discrete subset of $\mathbf{R}^n$ such that*

$$\forall k, l \in T, \quad \inf_{k \neq l} |k-l| \geqq C > 0.$$

Let G be a positive matrix indexed by $T \times T$ such that

$$C_1 \mathbf{I}d \leqq G \leqq C_2 \mathbf{I}d.$$

and

$$|G(k, l)| \leqq C' \exp(-\gamma |k-l|)$$

for a positive γ. Then

$$|G^{-1/2}(k, l)| \leqq C'' \exp(-\gamma' |k-l|),$$

where C'' and γ' depends only on γ, C, C', C_1, and C_2.

5. A new construction of wavelets. Call H the subspace of $H^1(\Omega)$ composed of functions v such that $\int_\Omega v = 0$. The variational formulation of (B) is

$$\forall v \in H, \quad \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} gv \, d\sigma = \int_\Omega fv.$$

If we want to follow the technique used in § 1, we must have a wavelet basis of the space H. For that, we shall construct an increasing sequence of imbedded spaces, each of them being generated by finite elements. This construction follows the idea of multiresolution analysis developed by Mallat in [10]. In the case of a polygonal domain in $\mathbf{R}^2$, a similar construction has been independently introduced by Lemarié in [9].

The values of the constants C, C', C'', and γ used in the following can change from a line to the next, but will always be independent of j.

We define a sequence of embedded spaces V_j composed of continuous piecewise linear functions on simplices $S_{j,k}$. These simplices (indexed by a $j \geqq 0$ and $k \in \Omega$) will be constructed by induction and will have the following usual compatibility conditions:

- For all $j \bigcup_k \overline{S_{j,k}} = \bar{\Omega}$, and $S_{j,k} \cap S_{j,k'}$ is empty if $k \neq k'$.
- Every $S_{j,k}$ is the union of a number of $S_{j+1,k}$ bounded by a constant independent of j.

Let $\beta_{j,k}$ be the radius of the biggest ball included in $S_{j,k}$, and let $\alpha_{j,k}$ be the radius of the smallest ball in which $S_{j,k}$ is included. Then

- There exists $\sigma > 0$ such that $\sup_k \alpha_{j,k} \leqq \sigma \inf_k \beta_{j,k}$ (which implies that a simplex cannot have arbitrarily small angles, and thus each vertex is the vertex of at most K simplices $S_{j,k}$ (for a fixed j), with K depending only on σ and thus not on j).
- There exists $\alpha, \beta > 0$ such that $\alpha 2^{-j} \leqq \sup_k \alpha_{j,k} \leqq \beta 2^{-j}$.
- Every vertex of a simplex $S_{j,k}$ is also a vertex of all the simplices $S_{j,k'}$ in the boundary on which it occurs.

Note that the boundary of each $S_{j,k}$ will be composed either of pieces of hyperplanes, or, if $S_{j,k}$ is situated at the boundary of Ω, of a piece of that boundary. Such meshes are now currently obtained by iterative mesh generating algorithms.

We now give a possible construction of the $S_{j,k}$, and of the corresponding spaces V_j.

Our space V_0 is spanned by the constant function

$$\phi_0(x) = \mathrm{Vol}\,(\Omega)^{-1/2}.$$

In order to construct V_1, we divide Ω in a set of simplices $S_{1,k}$ that follow the compatibility conditions for $j=1$. The possibility of such a division of Ω is proved in [19]. The space V_1 will then be composed of the functions that are continuous and piecewise linear on each simplex. A basis of V_1 is supplied by the functions $g_{1,l}$ (indexed by the set of all vertices of the simplices $S_{1,k}$) such that, for each vertex m of a simplex,

$$g_{1,l}(m) = \delta_{l,m}.$$

The spaces V_j are then constructed inductively. Suppose that a subdivision of Ω in simplices $S_{j,k}$ verifying the above conditions is constructed. There are many ways to obtain the next generation of simplices. A standard one is the following.

If $S_{j,k}$ is not situated on the boundary of Ω (i.e., if $\overline{S_{j,k}}$ does not intersect $\partial\Omega$), we then divide it in 2^n smaller simplices, by introducing new vertices at the middle of the edges. Else, we introduce a new vertex at the middle of each straight edge and other vertices on the edges on the boundary of Ω, such that the above compatibility conditions are satisfied. We then obtain a new subdivision $S_{j+1,k}$ of Ω in simplices verifying the compatibility conditions.

We define V_j as the space of continuous and piecewise linear functions on the $S_{j,k}$. It is immediate that the union of the V_j is dense in $L^2(\Omega)$. We again have a basis of V_j composed of the functions $g_{j,l}$ (indexed by points $l \in \Omega$) such that, for each vertex m of a simplex,

$$g_{j,l}(m) = \delta_{l,m}.$$

This function is supported by all the simplices that share l as a vertex.

We now construct an orthonormal basis of V_j following the procedure described in the previous section. In order to apply this orthonormalization algorithm, we must prove the following lemma.

LEMMA 6. *The functions $2^{nj/2} g_{j,l}$ form a Riesz basis of V_j such that*

$$C_1 \sum_l |c_l|^2 \leqq \left\| \sum_l c_l 2^{nj/2} g_{j,l} \right\|^2 \leqq C_2 \sum_l |c_l|^2$$

with C_1 and C_2 independent of j.

Proof. We first prove the second inequality. We saw that each vertex M is the vertex of at most K simplices (K independent of j), so that $\langle g_{j,l} | g_{j,l'} \rangle \neq 0$ for at most K vertices l'.

On the other hand,

$$0 \leqq g_{j,l}(x) \leqq 1_{B(l,\alpha_{j,l})}(x)$$

(where $B(x, r)$ is the ball centered on x and of radius r), so that

$$\|g_{j,l}(x)\|^2 \leqq C 2^{-nj}.$$

Thus

$$\begin{aligned} \left\| \sum_l c_l 2^{nj/2} g_{j,l} \right\|^2 &\leqq \sum_l \sum_{l'} c_l c_{l'} 2^{nj} \langle g_{j,l} | g_{j,l'} \rangle \\ &\leqq \sum_l \sum_{l'} (|c_l|^2 + |c_{l'}|^2) 2^{nj} \langle g_{j,l} | g_{j,l'} \rangle \\ &\leqq 2 \sum_l \sum_{l'} |c_l|^2 2^{nj} \langle g_{j,l} | g_{j,l'} \rangle \\ &\leqq 2KC \sum |c_l|^2. \end{aligned}$$

We now prove the first inequality of Lemma 6.

Each edge of a simplex has a length at least $\beta_{j,k}$, so that

$$\forall x, \quad |\nabla g_{j,l}(x)| \leqq C2^j.$$

On the other hand,

$$\left\| \sum_l c_l 2^{nj/2} g_{j,l} \right\|^2 \geqq \sum_{l'} \int_{B(l',\varepsilon)} \left| \sum_l c_l 2^{nj/2} g_{j,l} \right|^2$$

(provided that $\varepsilon \leqq \alpha_{j,k}/2$).

On $B(l', \varepsilon)$, all the $g_{j,l}$ vanish except $g_{j,l'}$ and the $g_{j,l}$ such that l and l' are the vertices of a same simplex. There are at most K of them. For such a vertex l ($\neq l'$),

$$\int_{B(l',\varepsilon)} |g_{j,l}|^2 \leqq \int_{B(l',\varepsilon)} |x-l'|^2 (\sup |\nabla g_{j,l}|)^2$$

(since $g_{j,l}$ vanishes at l').

Thus

$$\int_{B(l',\varepsilon)} |g_{j,l}|^2 \leqq C' \varepsilon^{n+2} 2^{2j}. \tag{5}$$

And

$$\int_{B(l',\varepsilon)} |g_{j,l'}|^2 \geqq \int_{B(l',\varepsilon)} (1 - |x-l'| (\sup |\nabla g_{j,l'}|))^2$$

(since $g_{j,l'}(l') = 1$)

$$\geqq \int_{B(l',\varepsilon)} (1 - C|x-l'| 2^{-j})^2.$$

Thus, if ε is such that $C\varepsilon 2^j \leqq \frac{1}{2}$,

$$\int_{B(l',\varepsilon)} |g_{j,l'}|^2 \geqq C'' \varepsilon^n, \tag{6}$$

so that

$$\left\| \sum_l c_l 2^{nj/2} g_{j,l} \right\|^2 \geqq \sum_{l'} \int_{B(l',\varepsilon)} \left| \sum_l c_l 2^{nj/2} g_{j,l} \right|^2$$

$$\geqq \sum_{l'} \left\{ \int_{B(l',\varepsilon)} \frac{c_{l'}^2 2^{nj}}{2} |g_{j,l}|^2 - \int_{B(l',\varepsilon)} \left| \sum_{l \neq l'} c_l 2^{nj/2} g_{j,l} \right|^2 \right\}$$

(because $\|a-b\|^2 \geqq \|a\|^2/2 - \|b\|^2$)

$$\geqq \sum_{l'} \left\{ \frac{c_{l'}^2 2^{nj}}{2} C'' \varepsilon^n - K \sum_{l \neq l'} \int_{B(l',\varepsilon)} |c_l 2^{nj/2} g_{j,l}|^2 \right\}$$

(because $|\sum_{i=1}^K a_i|^2 \leqq K \sum_{i=1}^K |a_i|^2$)

$$\geqq \frac{C'' 2^{nj} \varepsilon^n}{2} \sum_{l'} c_{l'}^2 - K^2 C' \varepsilon^{n+2} 2^{(n+2)j} \sum_{l'} |c_l|^2$$

if we choose $\varepsilon = 2^{-j}\eta$ with η small enough so that

$$C''/2 - K^2 C'\eta^2 > 0.$$

Hence we have proved Lemma 6.

We can now apply to the functions $2^{nj/2} g_{j,l}$ the orthonormalization procedure described in § 4. We introduce the operator G_j defined on V_j by

$$G_j(f) = \sum_l \langle f \,|\, 2^{nj/2} g_{j,l} \rangle 2^{nj/2} g_{j,l}.$$

We then define

$$\phi_{j,l} = G^{-1/2}(2^{nj/2} g_{j,l}).$$

The following proposition will now be easy to prove.

PROPOSITION 2. *The $\phi_{j,l}$ form an orthonormal basis of V_j and satisfy the following estimates, for a $\gamma > 0$ and $|\alpha| = 0$ or 1,*

$$|\partial^\alpha \phi_{j,l}(x)| \leqq C 2^{(\alpha + n/2)j} \exp(-\gamma 2^j |x - l|).$$

Proof. Lemma 6 shows that the functions $2^{nj/2} g_{j,l}$ verify the assumptions of Lemma 4. Hence the orthonormality of the $\phi_{j,l}$. We now want to obtain the decay estimates for the $\phi_{j,l}$. Let M be the matrix of G in the basis $g_{j,l}$ (M is the Gram matrix of the $g_{j,l}$, and we forget in the notation the dependency of M on j for the sake of simplicity). In order to apply Lemma 5, we now must consider that the indexation set appearing in Lemma 5 is composed of the $l' = 2^j l$ because then

$$\inf_{k \neq l} |2^j l - 2^j k| \geqq 2^j \inf \beta_{j,k} \geqq \frac{\alpha}{\sigma}.$$

We shall use in the following the notations $l' = 2^j l$, $k' = 2^j k$ when suitable. The other assumption of Lemma 5 holds because

$$\langle 2^{nj/2} g_{j,l} \,|\, 2^{nj/2} g_{j,k} \rangle = 0 \quad \text{if } |2^j l - 2^j k| \geqq 2^j 2 \sup \alpha_{j,k},$$

and thus $M_{k,l} = 0$ if $|2^j l - 2^j k| \geqq 2\beta$.

Let $A = (c_{j,l})$. Then

$$\langle M(A) \,|\, A \rangle = \|\sum c_{j,l} g_{j,l}\|^2.$$

Thus

$$C_1 \mathbf{I}d \leqq M \leqq C_2 \mathbf{I}d.$$

We can now apply Lemma 5 to M; we obtain

$$|M_{k,l}^{-1/2}| \leqq C \exp(-\gamma |k' - l'|).$$

For $|\alpha| = 0$ or 1,

$$|\partial^\alpha g_{j,k}(x)| \leqq 2^{nj/2} 2^{|\alpha| j} 1_{B(k, \alpha_{j,k})}(x) \leqq C 2^{nj/2} 2^{|\alpha| j} \exp(-\gamma 2^j |x - k|).$$

Thus

$$\begin{aligned}
|\partial \alpha \phi_{j,l}(x)| &= |G^{-1/2}(2^{nj/2} g_{j,l})| \\
&\leqq \sum_k C \exp(-\gamma |k' - l'|) 2^{nj/2} 2^{|\alpha| j} \exp(-\gamma |k' - 2^j x|) \\
&\leqq \sum_k C 2^{nj/2} 2^{|\alpha| j} \exp\left(-\frac{\gamma}{2} |k' - l'|\right) \exp\left(-\frac{\gamma}{2} |l' - 2^j x|\right) \\
&\leqq C' 2^{nj/2} 2^{|\alpha| j} \exp\left(-\frac{\gamma}{2} 2^j |l - x|\right).
\end{aligned}$$

We then define the space W_j as the orthogonal complement of V_{j-1} in V_j. We shall now construct an orthonormal basis of W_j.

We start with the functions $2^{nj/2}g_{j,l}$ such that l belongs to the set L of the vertices of the $S_{j,k}$ which are not a vertex of any of the simplices $S_{j-1,k'}$. These functions are a Riesz basis of the space O_j that they generate, because, by Lemma 6, the set of all the $2^{nj/2}g_{j,l}$ is a Riesz basis of V_j, and (3) holds with the same constants when restricted to a smaller set of vectors. Now let $h_{j,l}$ be the orthogonal projection of $2^{nj/2}g_{j,l}$ on W_j. We shall prove the following lemma.

LEMMA 7. *The functions $h_{j,l}$ form a Riesz basis of W_j such that*

$$C_1 \sum_l |c_l|^2 \leqq \left\| \sum_l c_l h_{j,l} \right\|^2 \leqq C_2 \sum_l |c_l|^2$$

with C_1 and C_2 independent of j, and satisfy the following estimates, for a $\gamma > 0$ and $|\alpha| = 0$ or 1,

$$|\partial^\alpha h_{j,l}(x)| \leqq C 2^{(\alpha+n/2)j} \exp(-\gamma 2^j |x-l|).$$

Proof. We start by proving that, if $f \in V_{j-1}$ and $g \in O_j$, then, for a positive δ independent of j,

$$\|f-g\| \geqq \delta(\|f\|+\|g\|). \tag{7}$$

We shall first prove that $\|f-g\| \geqq \eta \|f\|$. Let

$$f = \sum_m c_m g_{j-1,m},$$

where m belongs to the set of vertices of the $S_{j-1,k}$, and

$$g = \sum_{l \in L} c_l g_{j,l}.$$

Since $g_{j,l}$ vanishes at the points m, using (6), we have

$$\int_{B(m,\varepsilon)} |g_{j-1,m}|^2 \geqq C\varepsilon^n.$$

Using (5),

$$\int_{B(m,\varepsilon)} |g_{j,l}|^2 \leqq C\varepsilon^{n+2} 2^{2j};$$

and

$$\int_{B(m,\varepsilon)} |g_{j-1,m'}|^2 \leqq C\varepsilon^{n+2} 2^{2j}$$

for $m' \neq m$.

Thus

$$\begin{aligned} \|f-g\|^2 &\geqq \sum_m \int_{B(m,\varepsilon)} \|f-g\|^2 \\ &\geqq \sum_m \int_{B(m,\varepsilon)} \left| \sum_{m'} c_{m'} g_{j-1,m'} - \sum_{l\in L} c_l g_{j,l} \right|^2. \end{aligned}$$

The same calculation as in Lemma 6 yields that, for ε small enough,

$$\|f-g\|^2 \geqq C \sum_m |c_m|^2.$$

Since the $g_{j-1,m}$ are a Riesz basis of V_{j-1},

$$\|f\|^2 \leqq C' \sum_m |c_m|^2$$

and we obtain

$$\|f-g\| \geqq \eta \|f\|$$

with $\eta = \sqrt{CC'}$.

We also have, of course, $\|f-g\|+\|f\| \geqq \|g\|$; thus, by multiplying this last inequality by $\eta/2$ and adding up, we obtain

$$\|f-g\| \geqq \frac{\eta}{\eta+2}(\|f\|+\|g\|);$$

hence (7). Since $\sum c_l h_{j,l}$ is an orthogonal projection of $\sum c_l 2^{nj/2} g_{j,l}$, we have

$$\|\sum c_l 2^{nj/2} g_{j,l}\| \geqq \|\sum c_l h_{j,l}\|;$$

thus

$$\|\sum c_l 2^{nj/2} g_{j,l}\| \geqq \|\sum c_l h_{j,l}\| \geqq \delta \|\sum c_l 2^{nj/2} g_{j,l}\|,$$

and, from Lemma 6, we get

$$C_1 \sum_l |c_l|^2 \leqq \left\| \sum_l c_l h_{j,l} \right\|^2 \leqq C_2 \delta \sum_l |c_l|^2$$

and the first part of Lemma 7.

We now want to prove that we do not "lose localization" when projecting on W_j. Since, for f in V_j, the sum of the projections on V_{j-1} and W_j are the identity, it is sufficient to prove this result for the projection P_j on V_{j-1}. The $\phi_{j-1,k}$ being an orthonormal basis of V_{j-1}, we have

$$\begin{aligned}
|P_j(h_{j,k})(x)| &= \left| \sum_l \langle h_{j,k} | \phi_{j-1,l} \rangle \phi_{j-1,l}(x) \right| \\
&\leqq C \sum_l \int 2^{nj} \exp(-\gamma 2^j |k-y|) \exp(-\gamma 2^j |l-y|)\, dt\, 2^{nj/2} \exp(-\gamma 2^j |x-l|) \\
&\leqq C \sum_l \int 2^{nj} \exp\left(-\frac{\gamma}{2} 2^j |k-y|\right) \exp\left(-\frac{\gamma}{2} 2^j |l-k|\right) dt\, 2^{nj/2} \exp(-\gamma 2^j |x-l|) \\
&\leqq C' \sum_l 2^{nj/2} \exp\left(-\frac{\gamma}{2} 2^j |k-l|\right) 2^{nj/2} \exp(-\gamma 2^j |x-l|) \\
&\leqq C'' 2^{nj/2} \exp\left(-\frac{\gamma}{2} 2^j |k-x|\right).
\end{aligned}$$

As in Proposition 2, the estimates on the derivatives are obtained by the same proof.

By applying to the functions $h_{j,l}$ the orthonormalization procedure described in § 4, and using Lemmas 4 and 5, we obtain an orthonormal basis $\psi_{j,l}$ of W_j with the following properties.

PROPOSITION 3. *For a* $\gamma > 0$ *and* $|\alpha| = 0$ *or* 1,

$$|\partial^\alpha \psi_{j,l}(x)| \leqq C 2^{(\alpha+n/2)j} \exp(-\gamma 2^j |x-l|).$$

Because of the definition of W_j, *if we take the basis of* V_0, *i.e., the function* $\psi_{0,0}(=\phi_0) = \operatorname{Vol}(\Omega)^{-1/2}$, *and all the functions* $\psi_{j,l}$ *for* $j \geqq 1$, *we obtain an orthonormal basis of* $L^2(\Omega)$.

6. Properties of the wavelets and the Neumann boundary condition problem. We shall first prove that the wavelets we have constructed are well adapted to the variational problem for the Neumann boundary condition, i.e., that they are a basis of $H^1(\Omega)$.

PROPOSITION 4. *The functions $\psi_{j,l}$ for $j \geqq 0$ form an unconditional basis of $H^1(\Omega)$, and, if f belongs to $H^1(\Omega)$, the following condition on its wavelet coefficients $c_{j,l} (= \langle f \mid \psi_{j,l} \rangle)$ holds*:

$$C \sum |2^j c_{j,l}|^2 \leqq \|f\|^2_{H^1(\Omega)} \leqq C' \sum |2^j c_{j,l}|^2$$

with $0 < C \leqq C' < \infty$.

Proof. Because the affine functions belong to V_1, the moments of order zero or 1 of the wavelets $\psi_{j,l}$ vanish (provided that $j \geqq 1$).

Suppose that $\sum |2^j c_{j,l}|^2 < \infty$. We can extend the wavelets $\psi_{j,l}$ into functions $\theta_{j,l}$, defined on a larger bounded domain Ω', which are in $H^1_0(\Omega')$ and have the same vanishing moments and the same estimates as the $\psi_{j,l}$. The convergence in $H^1_0(\Omega')$ of the series $\sum c_{j,l}\theta_{j,l}$ is then a standard computation (see [11], for example). Hence, after restricting to Ω, the convergence of $\sum c_{j,l}\psi_{j,l}$ in $H^1(\Omega)$.

Conversely, suppose that $f \in H^1(\Omega)$, and let $c_{j,l}$ be its wavelet coefficients. We can extend f in a function of $H^1_0(\Omega')$ on a larger bounded domain Ω'. Let Δ be the Laplacian on $\mathbf{R}^n$, and $\omega_{j,l} = (-\Delta)^{-1/2}(2^j \psi_{j,l})$. Then

$$2^j c_{j,l} = \langle f \mid 2^j \psi_{j,l} \rangle = \langle (-\Delta)^{1/2} f \mid \omega_{j,l} \rangle.$$

Because of the vanishing moments and the decay estimates for the $\psi_{j,l}$, we easily obtain that the moments of order at most 1 of the $\omega_{j,l}$ vanish, and that, for $|\alpha| = 0$ or 1,

$$|\partial^\alpha \omega_{j,l}(x)| \leqq C 2^{(\alpha + n/2)j}(1 + 2^j|x - l|)^{-(n+1)}.$$

Since $(-\Delta)^{1/2} f \in L^2$, we obtain (by a standard proof, see [11] or [12], for example) that $\sum |2^j c_{j,l}|^2 < \infty$, and the proposition is proved (see also [9]).

Now that we have obtained a wavelet basis adapted to the space $H^1(\Omega)$, we immediately obtain a basis for H, since H is the subspace of $H^1(\Omega)$ orthogonal to V_0 (for the L^2 *and* the H^1 norm). Thus the $\psi_{j,l}$, for $j \geqq 1$ form an L^2 orthonormal basis of H, and the characterization given by Proposition 3 holds for H. Namely, we have the following corollary.

COROLLARY 1. *The functions $\psi_{j,l}$ for $j \geqq 1$ form an unconditional basis of H, and, if f belongs to H, the following condition on its wavelet coefficients $c_{j,l}$ holds*:

$$C \sum |2^j c_{j,l}|^2 \leqq \|f\|^2_{H^1(\Omega)} \leqq C' \sum |2^j c_{j,l}|^2$$

with $0 < C \leqq C' < \infty$.

As in § 3, we can now use this wavelet basis in a Galerkin method to solve (B) numerically. We are looking for a function u in $V'_p = V_p \cap H$ such that

$$\forall v \in V'_p \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} hv \, d\sigma = \int_\Omega gv.$$

This, once the functions are expressed by their coordinates in the wavelet basis, amounts to inverting the matrix

$$M_{(j,l),(j',l')} = \langle a \nabla \psi_{j,l} | \nabla \psi_{j'.l'} \rangle.$$

Hence, if D is the diagonal matrix defined by

$$D_{(j,l),(j',l')} = 2^j \delta_{(j,j')} \delta_{(l,l')},$$

because of Corollary 1, we again have the following property.

THEOREM 2. *For a Galerkin method using wavelets to approximate the solution of* (B), *after a preconditioning by the matrix D, the condition number of* $D^{-1}MD^{-1}$ *is bounded by* C'/C (*the constants appearing in Proposition* 3), *and is thus independent of the size of the discretization.*

7. Sparsity of the matrices in the wavelet method. An advantage of the finite elements or finite difference methods is the sparsity of the matrices associated to the discretized problems.

The wavelets do not have a localization as sharp as, say, finite elements (in fact, we saw that it is a key point in the improvement of the conditioning). Hence there are more entries in the matrix G to store and more computations (for a given step in an inversion procedure). Because this seems to be an important drawback, we shall study it in detail and give precise decay estimates for the entries of G. We follow the notations of § 2. The results are the same for the Neumann problem.

The decay estimates will depend on the smoothness of the function a. We recall that, in § 2, the wavelets are indexed by $j \geqq 1$ and $k \in \mathbf{Z}^n$, and are such that $k2^{-j} \in \Omega$; in the following, we note $\lambda = k2^{-j}$ and $\lambda' = k2^{-j'}$. Then we have Proposition 5.

PROPOSITION 5. *Suppose that the functions a and b belong to* $\mathbf{W}^m_\infty$; *then there exist* C *and* γ *positive such that*

$$|G_{(j,k),(j',k')}| \leqq C2^{-|j-j'|(m+n/2)} \exp(-\gamma 2^{\inf(j,j')}|\lambda - \lambda'|).$$

This estimate shows, for a given accuracy, which entries of the matrix G do not need to be calculated. In this proposition, the wavelets are supposed to be at least $\mathbf{C}^m$. Recall that $\mathbf{W}^m_\infty$ is the subspace of $\mathbf{L}^\infty$ composed of functions that have their derivatives of order at most m in $\mathbf{L}^\infty$.

Proof. We must estimate

$$M_{(j,k),(j',k')} = \int a \nabla \psi_{j,k} \nabla \psi_{j',k'}$$

(the estimate for $\int b\psi_{j,k}\psi_{j',k'}$ is similar). We can suppose that $j \geqq j'$. By Taylor's formula, there exist α_k such that

$$\left| a \nabla \psi_{j',k'} - \sum_{k<m} \alpha_k (x-\lambda')^k \right| \leqq C|x-\lambda|^m \sup_{B(\lambda,|x-\lambda|)} |\partial^m (a \nabla \psi_{j',k'})|.$$

For the wavelets used in the Dirichlet problem, the moments of $\psi_{j,k}$ of order at most m do not exactly vanish. However, following a trick already used in [7], we can suppose that they do disappear when computing $M_{(j,k),(j',k')}$, because we can extend $\psi_{j,k}$ outside Ω so that its moments vanish but without changing the uniform estimates. This does not change $M_{(j,k),(j',k')}$, since $\psi_{j',k'}$ vanishes outside Ω. Hence

$$\begin{aligned} |M_{(j,k),(j',k')}| \leqq\ & C2^{(n/2+1)j}2^{(n/2+m+1)j'} \\ & \cdot \int_\Omega |x-\lambda|^m \exp(-\gamma 2^j|x-\lambda|) \\ & \cdot \inf(1, \exp(-\gamma 2^{j'}(|\lambda-\lambda'|-|\lambda-x|))). \end{aligned}$$

If $x \in B(\lambda, |\lambda-\lambda'|/2)$,

$$\begin{aligned} |M_{(j,k),(j',k')}| &\leqq C2^{(m+n/2)(j'-j)}2^{j+j'} \\ &\quad \cdot \exp(-(\gamma/2)2^{j'}|\lambda-\lambda'|) \int_\Omega 2^{nj}(2^j|x-\lambda|)^m \exp(-\gamma 2^j|x-\lambda|) \\ &\leqq C2^{j+j'}2^{-(m+n/2)|j-j'|} \exp(-\gamma' 2^{j'}|\lambda-\lambda'|). \end{aligned}$$

If $x \notin B(\lambda, |\lambda - \lambda'|/2)$,

$$|M_{(j,k),(j',k')}| \leqq C \int_\Omega |x-\lambda|^m 2^{(n/2+1)j} \exp(-(\gamma/2)2^j|x-\lambda|) \cdot \exp(-(\gamma/2)2^j|\lambda-\lambda'|)2^{(n/2+m+1)j'}$$
$$\leqq C 2^{j+j'} 2^{-(m+n/2)|j-j'|} \exp(-\gamma' 2^j |\lambda - \lambda'|).$$

Hence Proposition 1, since

$$G_{(j,k),(j',k')} = 2^{-j} 2^{-j'} M_{(j,k),(j',k')}.$$

Remark that, if V_p is a space of dimension N (hence, if G is a $N \times N$ matrix), for a given accuracy ε, there are no more than $CN|\log \varepsilon|$ entries larger than ε; which is the same order of magnitude than if the entries of G were decreasing exponentionally fast away from the diagonal. In fact, these entries can be calculated using fast wavelet algorithms, starting with the entries $\langle a(x)\phi_{j,k}(x) | \phi_{j,l}(x)\rangle$, and adapting the cascade algorithms introduced by Mallat in [10] (see also [2]).

Actually, using smooth wavelets improves the sparsity of the matrix G (as was just proved), but it does not improve the order of convergence of the method. This is so because the derivatives of smooth wavelets vanish on $\partial\Omega$, which is, in general, not the case of the solutions of (A). Hence, the approximation cannot hold for a topology stronger than H_0^1 (at least near the boundary).

8. Some numerical results. The main purpose of the first numerical experiments is to check if, numerically, the condition number of the matrix to be inverted is effectively bounded and if this bound is reasonable.

We have so far emphasized the very good condition number that can be obtained using wavelets. This is important for two reasons. First because ill conditioning leads to numerical instabilities (a small error in the entries of the matrix or in the second term can lead to large errors in the solution).

The second reason is that, when using an iterative method to solve a linear system, ill conditioning can make the convergence become very slow. For example, in the conjugate gradient method, the number of iterations needed for a given accuracy is proportional to $\sqrt{\kappa}$ (see [1]). This actually implies that, if a conjugate gradient method is used after discretization in a wavelet basis, the number of iterations needed is bounded by a constant that depends neither on the dimension of space nor on the fineness of the discretization, in sharp contrast with other methods. As was already mentioned, the usual preconditionings (which are usually not diagonal) yield a condition number which is $O(1/h)$.

We present results for the second derivative in the one-dimensional case. The first example (Table 1) is piecewise linear wavelets and Dirichlet boundary conditions. In

TABLE 1
Percentage of coefficients

Points of discretization	Condition number	$<10^{-2}$	$<10^{-3}$
8	4.12	0	0
16	5.73	0.27	0.1
32	6.56	0.53	0.31
64	7.25	0.73	0.54
128	7.70	0.84	0.72
256	7.97	0.92	0.84

the first column we give the number of points of discretization, in the second the condition number of the matrix after preconditioning. In the two last columns we check the sparseness of the matrix, giving the percentage of coefficients smaller than 10^{-2} and 10^{-2}.

These results show that the condition number increases very slowly when N is large. The percentage of coefficients that are not numerically vanishing is quite high, and a direction of research is to find out for which wavelets the differential operators are "as diagonal as possible."

In the second example (Table 2) we follow the adaptative technique described in § 3, keeping high-frequency wavelets only near a point (here the middle of the segment). In the first column, we give the number of points of discretization in the finest region, in the second the condition number of the matrix after preconditioning, and in the last column, the percentage of wavelets that are kept for this computation.

We see that the condition number is not altered by this local refinement technique.

The third example (Table 3) is similar to the first, but the wavelets are adapted to the Neumann problem. The slight improvement in the condition number is probably due to the fact that the wavelets have more cancellation here.

9. Remarks. In this part, we will go a little further in the comparison between classical and wavelet methods.

Complexity of calculations. A drawback of the wavelet method is the following. The wavelets are not as easy to calculate as finite elements, and thus the entries of the matrix G are also more difficult to calculate. However, for the Dirichlet problem, it should be noted that the asymptotic behavior of the wavelets (see [5] for theoretical estimates and [16] for a numerical validation) makes this computation not as bad as it might appear at first sight. Also note that computing the wavelets for a given domain is done once and for all. Hence, the method may be costly for the resolution of one problem, but, if we keep in mind that the numerical resolution of many problems use an iterative procedure involving the resolution of a large number of linear elliptic problems *on the same domain*, the method we propose will in such cases become numerically efficient.

TABLE 2

Points of discretization	Condition number	Percentage of wavelets
64	7.13	0.5
128	7.65	0.32
256	7.93	0.18

TABLE 3

Points of discretization	Condition number
8	3.80
16	5.32
32	5.89
64	6.82
128	7.41
256	7.53

In contrast with the other methods, a priori knowledge on the smoothness of the solution indicates which wavelet coefficients we actually have to calculate, and which can be disregarded for a given accuracy (as was noticed in § 3).

Generalizations. Another drawback of the method is that the functions u and v that appear in the variational formulation need to be in the same space. This prevents the direct use of this method for problems that do not have such a variational formulation. In some cases this drawback can be reversed. It was, for instance, the case in the nonhomogeneous Dirichlet problem (A), the direct variational formulation of which yields a u and v that are not in the same space. The transformation of this problem into a homogeneous one corrected this difficulty.

Another way to turn the boundary problem would be to use wavelets in conjunction with penalization methods, because in these methods the boundary conditions do not appear in the definition of the space where the variational problem takes place.

It should be noticed that the only property of the differential operator that we use is the equivalence between the norm associated with this operator and the H^1 or the H_0^1 norm. Thus the wavelet method that we described can be used without any change for more general second-order elliptic operators. This is especially the case for the Neumann boundary problem; we considered only a model problem where the operator is a Laplacian mainly in order to show how to handle the condition of vanishing integral in the variational formulation. In other problems where this condition does not appear we must use the whole collection of wavelets, including the first constant one, and the technique is the same as in our model problem.

Solving higher-order problems requires smooth wavelet bases. Those constructed in [7] can only be used if the variational problem is set in H_0^m (the closure of $D(\Omega)$ for the H^m topology). An example of this setting is given by the following plane clamped plate problem:

$$-\Delta^2 u = f,$$
$$u(x) = \frac{\partial u}{\partial n}(x) = 0 \quad \text{for } x \in \partial\Omega.$$

A variational formulation is

$$\forall v \in H_0^2, \qquad \int \Delta u \, \Delta v = \int fv.$$

The method described in § 2 can be applied without any change, since, provided that the wavelets are smooth enough, we have

$$C \sum |2^{2j} c_{j,k}|^2 \leqq \|f\|_{H_0^2}^2 \leqq C' \sum |2^{2j} c_{j,k}|^2.$$

The Galerkin method described in § 2, if using as preconditioning the diagonal matrix

$$D_{(j,k),(j',k')} = 2^{2j} \delta_{(j,j')} \delta_{(k,k')},$$

will provide a discretization matrix, the condition number of which is bounded by a constant.

Constructions of arbitrarily smooth wavelets on general domains which would be bases of the H^m spaces do not seem to exist now (to our knowledge). However, for some specific geometries (such as cubes), such wavelets have recently been constructed by Meyer [13].

10. Annex: Diagonal conditioning for classical methods. The purpose of this section is to show that the condition number associated to a finite elements or finite differences method preconditioned by a diagonal preconditioning is at best in $O(1/h \log h)$. This should not be interpreted as a comparison argument with wavelet methods since diagonal preconditionings are not used in conjunction with these methods. But it shows in a way that wavelets are a "better approximation" of eigenfunctions of differential operators, and it explains the importance of dealing with functions at different scales in order to obtain the best condition numbers, which is characteristic of wavelet methods.

We shall now prove the following theorem.

THEOREM 3. *Suppose that we use a diagonal preconditioning on a Galerkin method in the resolution of problem* (A) *or* (B) *by finite elements or finite differences. Then under general hypotheses always verified in practice* (*and which will be explicited in the course of the proof*) *the resulting conditioning of the preconditioned matrix is always larger than* $C/(h|\log h|)$ (*where h denotes the size of the discretization*).

Call M $(=M_{k,l})$ the matrix (indexed by points of Ω) obtained by the discretization method. We can suppose under very general assumptions on the method that M is symmetric definite positive, and that, after a division by $\|M\|$,

$$Ch^a Id \leqq M \leqq Id$$

for a positive a (for the Laplacian, $a=2$, see [8]).

We are looking for a matrix G $(=G_{k,l})$, with a better condition number than M, such that

$$G = D^{-1} M D^{-1},$$

with D diagonal. After perhaps multiplying D by a constant, we can suppose that

$$Ch^b Id \leqq G \leqq Id,$$

where b is smaller that a (if not, we have not gained anything!). We note d_k the (diagonal) elements of D. We first prove the following lemmas.

LEMMA 8. *The following estimate holds*:

$$Ch^{b/2} \leqq d_k \leqq C' h^{-a/2}.$$

This is immediate since

$$h^b \leqq M_{k,k} \leqq 1,$$

$$h^a \leqq G_{k,k} \leqq 1,$$

and

$$G_{k,k} = d_k^{-2} M_{k,k}.$$

LEMMA 9. *Let κ be the condition number of G. Then*

$$|M_{k,l}^{-1}| \leqq Ch^{-a} \kappa \exp\left(-C'|k-l|/(h\kappa)\right).$$

Proof. G is a band matrix such that

$$G_{k,l} = 0 \quad \text{if } |k-l| > mh,$$

where m depends on the discretization chosen (for example, for a five-point discretization of the Laplacian on a regular grid, $m=1$).

We saw that

$$G = Id - R \quad \text{with} \quad \|R\|_{\mathbf{L}^2} = r < 1$$

($r = 1 - 1/\kappa$, where κ is the condition number of G). Then

$$G^{-1}_{k,l} = \sum_{n \geqq 0} R^n_{k,l}.$$

But $|R^n_{k,l}| \leqq r^n$ and $R^n_{k,l} = 0$ if $|k - l| > nmh$. Thus

$$\begin{aligned} |G^{-1}_{k,l}| &\leqq \sum_{n \geqq |k-l|/(mh)} r^n \\ &\leqq \frac{r^{|k-l|/mh}}{1-r} \\ &= \kappa \exp(|k-l| \log r/mh), \end{aligned}$$

which is equivalent to $\kappa \exp(|k-l|/mh\kappa)$.

Thus, by Lemma 8,

$$\begin{aligned} |M^{-1}_{k,l}| &= |d_k^{-1} G^{-1}_{k,l} d_l^{-1}| \\ &\leqq Ch^{-a}\kappa \exp(-C\kappa^{-1}|k-l|/mh). \end{aligned}$$

Hence Lemma 9.

The model problems we started with, once discretized, amount to solving $MX = F$, where $F = (f_k)$ is a certain discretization of the function f (and, in the Neumann case, also of the boundary condition; but we shall stick to the Dirichlet problem, for the sake of simplicity). We suppose that the discretization operator $f \to F$ is local, i.e., that f_k depends only on the values of f in a ball B_h centered on k and of radius Ch. More precisely, we suppose that

$$f_k \leqq Ch^d \int_{B_h} |f| \quad \text{or} \quad f_k \leq Ch^d \sup_{B_h} |f|$$

for a certain exponent d.

For finite elements, f_k is equal to $\langle f | e_k \rangle$ where e_k is the corresponding finite element. And for finite differences, f_k is usually the value of f at k or an average over a domain included in some B_h, multiplied by a correct normalization factor. Thus our assumption is always fulfilled.

We shall prove the following lemma.

LEMMA 10. *Let* $f \in \mathbf{L}^1_{\text{loc}}$ *and* u_h *be the solution of the discretized problem* (*with a size of discretization* h), *and* $k \in \Omega$. *Then* $u_h \to 0$ *uniformly in the complementary of* supp $(f) + B(k, \varepsilon)$ *for any* ε *if*

$$\kappa \leqq \frac{C(\varepsilon)}{h|\log h|}$$

(*for a certain* $C(\varepsilon)$ *that could be made explicit*).

Clearly, this lemma shows that, if the method used converges (in any sense) toward the solution of the continuous problem, and if simultaneously

$$\kappa \leqq \frac{C}{h|\log h|} \quad \text{for any } C > 0, \tag{8}$$

then the Green function $g(x, y)$ of our model problem would be a distribution concentrated on the diagonal $x = y$, which is false. Hence condition (8) cannot hold and

$$\kappa \geqq \frac{C}{h|\log h|}$$

for a sufficiently small C.

It only remains to prove Lemma 10.

Let k be such that $d(k, \operatorname{supp} f) \geqq \varepsilon$. Because of the assumptions made on f, $|f_l| \leqq Ch^d$, and, for h small enough, $f_l = 0$ in $B(k, \varepsilon/2)$. Then

$$\left|\sum M_{k,l}^{-1} f_l\right| \leqq \sum_{d(k,l) \geqq \varepsilon/2} Ch^{-a} \kappa \exp\left(-C|k-l|/\kappa h\right) f_l.$$

There are $O(h^{-n})$ points l (in dimension n); thus, for a certain (negative) e,

$$\left|\sum M_{k,l}^{-1} f_l\right| \leqq Ch^e \kappa \exp\left(-C\varepsilon/2\kappa h\right),$$

which tends to zero if $\kappa \leqq (C'/h|\log h|)$ for a sufficiently small C'. Hence we have proved Lemma 10.

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