Preuve non-stabilité par composition

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Lemma 1. *Let u_n and v_n two bounded sequences with either u_n or v_n absolutely converging. Then, for all $m \in \mathbb{N}$, there holds*

$$\sum_{n=0}^{+\infty} u_n \frac{v_{n+m} + v_{|n-m|}}{2} = \frac{u_0 v_m - u_m v_0}{2} + \sum_{n=0}^{+\infty} v_n \frac{u_{n+m} + u_{|n-m|}}{2}.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{+\infty} u_n \frac{v_{n+m} + v_{|n-m|}}{2} &= \sum_{n=0}^{+\infty} \frac{u_n}{2} v_{n+m} + \sum_{n=0}^{m-1} \frac{u_n}{2} v_{m-n} + \sum_{n=m}^{+\infty} \frac{u_n}{2} v_{n-m} \\ &= \sum_{n=m}^{+\infty} \frac{u_{n-m}}{2} v_n + \sum_{n=1}^{m} \frac{u_{m-n}}{2} v_n + \sum_{n=0}^{+\infty} \frac{u_{n+m}}{2} v_n\end{aligned}$$

and the conclusion follows. $\square$

Lemma 2. *Let α any C^∞ function, and let A the operator defined as $Au = \alpha(x)u(x)$. We assume that for any operator Λ defined by*

$$\Lambda u = \sum_{n=0}^{+\infty} \lambda_n \hat{u}_n T_n(x)$$

the composition ΛA can be expressed as an operator B satisfying

$$Bu = \sum_{n=0}^{+\infty} b(x,n) \hat{u}_n T_n(x)$$

where the functions $b(x,n)$ are C^∞. Then α is a constant function.

Proof. Let us write $\alpha = \sum_{i=0}^{+\infty} \alpha_i T_i(x)$. Fix $n \in \mathbb{N}$, we have

$$\begin{aligned}\alpha T_n &= \sum_{i=0}^{+\infty} \alpha_i T_n T_i \\ &= \sum_{i=0}^{+\infty} \alpha_i \frac{T_{n+i} + T_{|n-i|}}{2} \\ &= \frac{\alpha_0 T_n - \alpha_n}{2} + \sum_{i=0}^{+\infty} \frac{\alpha_{n+i} + \alpha_{|n-i|}}{2} T_i\end{aligned}$$

according to the previous lemma. Fix some $0 < i < n$ and choose $\lambda_k = 0$ except for $k = i$, with $\lambda_i = 1$. Then

$$\Lambda\alpha T_n = \lambda_i \frac{\alpha_{n+i} + \alpha_{n-i}}{2} T_i.$$

By assumption, there exists a function $b(x, n)$ that is C^∞ such that

$$b(x, n)T_n = \frac{\alpha_{n+i} + \alpha_{n-i}}{2} T_i(x).$$

Since T_n has strictly more roots than T_i, we can fix an x for which the left side is *zero* while $T_i \neq 0$. Thus, and this holds for all $1 < i < n$, $\alpha_{n+i} = -\alpha_{n-i}$. This implies that for all $n > 0$, $\alpha_n = 0$, and thus, $\phi(x) = \phi_0 T_0$ is a constant function. $\square$