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BOUNDARY INTEGRAL EQUATIONS FOR SCREEN PROBLEMS IN $\mathbb{R}^3$

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Here we present a new solution procedure for Helmholtz and Laplacian Neumann screen or Dirichlet screen problems in $\mathbb{R}^3$ via boundary integral equations of the first kind having as unknown the jump of the field or of its normal derivative, respectively, across the screen S. Under the assumption of local finite energy we show the equivalence of the integral equations and the original boundary value problems. Via the Wiener-Hopf method in the halfspace, localization and the calculus of pseudodifferential operators we derive existence, uniqueness and regularity results for the solution of our boundary integral equations together with its explicit behavior near the edge of the screen. We give Galerkin schemes based on our integral equations on S and obtain high convergence rates by using special singular elements besides regular splines as test and trial functions.

1. INTRODUCTION

This paper presents a solution procedure for both the Dirichlet and the Neumann screen problems for the scalar Helmholtz equation (with small wave number k) via boundary integral equations on the screen surface S. The problems under consideration are the following ones. For given g or h on S find u in $\Omega_S := \mathbb{R}^3 \setminus \bar{S}$ satisfying

$$(\Delta + k^2)u = 0 \text{ in } \Omega_S \tag{1.1}$$

$$\left.\begin{array}{l} u = g \text{ on } S \text{ (Dirichlet) or} \\ \dfrac{\partial u}{\partial n} = h \text{ on } S \text{ (Neumann) and} \end{array}\right\} \tag{1.2}$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}) \text{ for } k \neq 0 \text{ or } u = O(r^{-1}) \text{ for } k = 0 \quad \text{as } r := |x| \to \infty. \tag{1.3}$$

Such problems appear in scattering of acoustic fields u by obstacles of different hardness (see [8]): The Dirichlet condition represents a soft screen whereas the Neumann condition represents a hard screen. For $k = 0$ the above Dirichlet problem describes the electrostatic field u of an electrified screen and one looks for the charge density (see [1]).

We make the general assumption:

$$\text{S is a bounded, simply connected, orientable smooth, open surface in } \mathbb{R}^3 \text{ with a smooth boundary curve } \gamma \text{ which does not intersect itself (see Fig. 1).} \tag{1.4}$$

Our solution procedure is to derive boundary integral equations of the first kind on S for the jump of the normal derivative $[\frac{\partial u}{\partial n}]$ across the screen S in the Dirichlet case and for the jump of the field $[u]$ across S in the Neumann case, respectively. In Section 2 we reduce the Dirichlet problem by use of Green's formula to the weakly singular boundary integral equation

$$V_S[\frac{\partial u}{\partial n}](x) := \frac{1}{2\pi} \int_S \frac{e^{ik|x-y|}}{|x-y|} [\frac{\partial u}{\partial n}](y)dS_y = 2g(x), \qquad x \in S \tag{1.5}$$

Similarly, the Neumann screen problem is reduced to the hypersingular boundary integral equation

$$D_S[u](x) := \frac{1}{2\pi} \int_S \frac{\partial^2}{\partial n_x \partial n_y} \frac{e^{ik|x-y|}}{|x-y|} [u](y)dS_y = -2h(x), \qquad x \in S \tag{1.6}$$

Both integral equations (1.5), (1.6) have been already suggested by physical arguments in [8]. Although the integral operators in (1.5), (1.6) possess quite different properties in the classical theory of integral equations, they both belong to the larger class of pseudodifferential operators. The solvability of the equations (1.5) and (1.6) hinges on the fact that V_S and D_S are strongly elliptic, that is they satisfy a Gårding inequality in suitable Sobolev spaces (Lemma 2.8). Therefore, V_S, D_S are Fredholm operators of index zero and, thus, the injectivity of the operators implies their bijectivity. Since the integral equation (1.5) is equivalent to the Dirichlet problem (1.1)-(1.3) and the integral equation (1.6) to the corresponding Neumann problem, respectively, our integral equations have no eigensolutions for $\operatorname{Im} k \geq 0$. Following [16] the analysis of the integral equations using the calculus of pseudodifferential operators and Wiener-Hopf technique provides the explicit edge behavior of the unknown densities $[\frac{\partial u}{\partial n}]$ and $[u]$ (Theorem 2.9). In case of the Dirichlet problem we obtain that the jump in the normal derivative of the acoustic field behaves near the edge like $\rho^{-1/2}$ even for C^∞ data where ρ denotes the Euclidean distance to the edge γ of the screen S. This is in agreement with the "edge condition" in physics (see [8]). Therefore, in general, the solution of (1.5) is not continuous for continuous data g contrary to Hayashi's claim [5], [6]. Both integral equations (1.5) and (1.6) are derived under the only assumption that the acoustic field u in (1.1)-(1.3) has local finite energy, i.e., $u \in H^1_{loc}$ near S -- which is the physically relevant property. For the Neumann problem we obtain the explicit edge behavior for the solution of (1.6) which improves the results by Durand [3].

The knowledge of the explicit edge behavior of the solutions of (1.5) and (1.6) is important for an effective numerical scheme (based on our integral equations) in order to diminish the pollution effect caused by the edge singularity. Solving (1.5), (1.6) approximately with regular splines we obtain only low convergence rates in the energy norm for the boundary element Galerkin solutions (Theorem 3.1). This is in accordance to the standard variational finite element method for (1.1)-(1.3) in the domain. In order to improve the convergence rate we incorporate in the Galerkin scheme as test and trial functions special singular elements which simulate the edge behavior of the exact solutions of the integral equations. With this augmentation method we obtain higher convergence rates (Theorem 3.2) as in the two-dimensional case [19].

2. BOUNDARY INTEGRAL EQUATIONS

Before we derive the integral equations (1.5) and (1.6) we reformulate the original screen problems: Let $g \in H^{1/2}(S)$ $(h \in H^{-1/2}(S))$ be given, we look for $u \in L_S$ satisfying the Dirichlet (Neumann) boundary condition in (1.2) where

$$L_S = \{u \in H^1_{loc}(\Omega_S): (\Delta+k^2)u = 0 \text{ in } \Omega_S,\ u \text{ satisfies } (1.3)\} \quad (2.1)$$

For completeness, let us introduce the Sobolev spaces $H^s(\Omega)$, $H^s(\Gamma)$, and $H^s(S)$, $s \in \mathbb{R}$, where Ω is a bounded domain with a smooth boundary Γ and a smooth boundary piece $S \subset \Gamma$. We recall from [9], [11] the function spaces used which are incorporated with their natural norms:

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^3)\} \qquad (s \in \mathbb{R})$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+\frac{1}{2}}(\mathbb{R}^3) & (s > 0) \\ L^2(\Gamma) & (s = 0) \\ H^{-s}(\Gamma))' \text{ (dual space)} & (s < 0) \end{cases}$$

$$H^s(S) = \{u|_S : u \in H^s(\Gamma)\} \qquad (s \geq 0)$$

$$\tilde{H}^s(S) = \{u \in H^s(\Gamma) : \operatorname{supp} u \subset \bar{S}\}, \qquad H^s(S) = H^s(\Gamma)/\tilde{H}^s(\Gamma\backslash\bar{S})$$

As in the two-dimensional case (see [19]) we prove uniqueness of the screen Dirichlet and the screen Neumann problem (1.1)-(1.3) for Im $k \geq 0$ by transforming them into appropriate transmission problems.

LEMMA 2.1. *For* Im $k \geq 0$ *the homogeneous screen Dirichlet (Neumann) problem has at most the trivial solution in* L_S.

For the proof and further investigation we extend S to an arbitrary smooth, simply connected, closed, orientable manifold (surface) ∂G_1 enclosing a bounded domain G_1 with boundary ∂G_1 (see Fig. 1). Let $\frac{\partial}{\partial n}$ denote the exterior normal derivative to ∂G_1. Let $[v]$ denote the jump $v_- - v_+$ where the subscript + (-) means the limit from $\mathbb{R}^3\backslash\bar{G}_1$ (from G_1) to ∂G_1. Furthermore, let B denote a sufficiently large ball with radius R including $\bar{G}_1$ and let $G_2 := B \cap (\mathbb{R}^3\backslash\bar{G}_1)$ and ∂B denote the boundary of B.

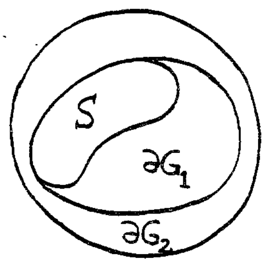

Fig. 1.

PROOF. Let us consider first the homogeneous Dirichlet problem; thus $u \in L_S$ with $u|_S = g = 0$. Then $u \in H^1(G_1) \cup H^1_{loc}(\mathbb{R}^3\setminus\bar{G}_1)$ solves the following transmission problem:

$$u = u_1 \text{ in } G_1 \text{ with } \Delta u_1 + k^2 u_1 = 0 \text{ in } G_1$$
$$u = u_2 \text{ in } \mathbb{R}^3\setminus\bar{G}_1 \text{ with } \Delta u_2 + k^2 u_2 = 0 \text{ in } \mathbb{R}^3\setminus\bar{G}_1$$
satisfying (1.3) and
$$u_1 = u_2 \text{ on } \partial G_1 \text{ and } \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial G_1\setminus\bar{S}. \tag{2.2}$$

Application of the Green's formula in G_1 and G_2 then yields with the transmission conditions in (2.2) by eliminating the integral over S

$$\int_{\partial B} \frac{\partial u_2}{\partial n}\bar{u}_2 ds = -k^2 \int_{G_1} |u_1|^2 dx - k^2 \int_{G_2} |u_2|^2 dx + \int_{G_1} |\nabla u_1|^2 dx + \int_{G_2} |\nabla u_2|^2 dx \tag{2.3}$$

Note that (2.3) holds with the traces of u_j since for $u \in L_S$ we also have $\Delta u_j \in L^2(G_j)$, $j = 1,2$. For Im $k > 0$ or $k = 0$, the left hand side in (2.3) tends to zero for $R \to \infty$. Hence, both imaginary and real parts of the right hand side in (2.3) vanish which implies $u_1 \equiv 0$ and $u_2 \equiv 0$. For Im $k = 0$, $k > 0$, we use the radiation condition (1.3) and then take the imaginary part of (2.3) obtaining

$$k \int_{\partial B} |u_2|^2 ds + o(1) = 0.$$

This gives with Rellich's theorem [23, Theorem 4.2] $u_2 \equiv 0$ and with (2.2) also $u_1 \equiv 0$.

The uniqueness for the Neumann screen problem follows by the same arguments if one considers (2.2) with the new transmission conditions

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ on } \partial G_1 \quad \text{and} \quad u_1 = u_2 \text{ on } \partial G_1\setminus\bar{S}. \qquad \square$$

For the boundary integral equations on S we shall need some properties of the traces of the solution $u \in L_S$ of (1.1)-(1.3). As in the two-dimensional case (see [19]) we obtain for the screen Dirichlet (Neumann) problem the following result on the jump of the normal derivative (of the trace) of the weak solution.

LEMMA 2.2. *Let* $u \in H^1_{loc}(\Omega_S)$ *be a weak solution of the screen problem* (1.1)-(1.3). *Then for the Dirichlet problem we have*

$$[\frac{\partial u}{\partial n}]|_S \in \tilde{H}^{-1/2}(S) = (H^{1/2}(S))' \tag{2.4}$$

and for the Neumann problem we have

$$[u]|_S \in \tilde{H}^{1/2}(S). \tag{2.5}$$

PROOF. Since $(\Delta+k^2)u = 0$ in Ω_S and $B \subset \Omega_S := \mathbb{R}^3\setminus\bar{S}$ from Green's formula

$$\int_{G_j} v \cdot \Delta u \, dx + \int_{G_j} \nabla v \cdot \nabla u \, dx = (-1)^j \langle v, \frac{\partial u}{\partial n}\rangle_{L^2(\partial G_j)} \qquad (j = 1,2)$$

we have $v \in H^{1/2}(\partial G_j)$ and $\frac{\partial u}{\partial n} \in H^{-1/2}(\partial G_j)$ by duality. Here we can take v with supp $v \subset\subset B$ such that the integral along ∂B vanishes.

Hence the jump $[\frac{\partial u}{\partial n}]$ belongs to $H^{-1/2}(\partial G_1)$. By assumption $u \in H^1_{loc}(\Omega_S)$, therefore away from S the relation $(\Delta+k^2)u = 0$ implies $u \in H^3_{loc}(\Omega_S)$, and repeating this argument implies even $u \in C^\infty$ away from S. Thus $[\frac{\partial u}{\partial n}] = 0$ on $\partial G_1\setminus\bar{S}$. This together with $[\frac{\partial u}{\partial n}]|_S \in H^{-1/2}(S)$ yields (2.4) because

$$\tilde{H}^{-1/2}(S) = \{w \in H^{-1/2}(\partial G_1): \text{supp } w \subset \bar{S}\} = \overline{C_0^\infty(S)}^{H^{-1/2}(\partial G_1)}$$

The assertion (2.5) follows by similar arguments with

$$\tilde{H}^{1/2}(S) = \overline{C_0^\infty(S)}^{H^{1/2}(\partial G_1)} .$$

□

The above properties of the Cauchy data enable us to derive boundary integral equations to solve (1.1)-(1.3) even for weak solutions $u \in L_S$.

Before we give our solution procedure let us briefly recall the idea of layer potentials by introducing the fundamental solution ϕ of $(\Delta+k^2)u = 0$ in G_j (see Fig. 1).

$$\phi(z,\zeta) := \begin{cases} -\dfrac{1}{4\pi|z-\zeta|} & \text{for } k = 0 \\[2ex] \dfrac{e^{ik|z-\zeta|}}{-4\pi|z-\zeta|} & \text{for } k \neq 0 \end{cases} \tag{2.6}$$

DEFINITION 2.3. Let $u \in C^\infty(\Gamma)$, Γ a bounded closed C^∞-surface. Then

$$\left.\begin{aligned} V_{G_j}u(z) &:= -2 \int_\Gamma \phi(z,\zeta)u(\zeta)dS_\zeta \\ K_{G_j}u(z) &:= -2 \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \phi(z,\zeta)dS_\zeta \end{aligned}\right\} \quad (z \in G_j) \tag{2.7}$$

The same definition of the potential of the simple and the double layer is valid for arbitrary distributions u on Γ since for $z \notin \Gamma$ the above kernels are C^∞ functions on Γ.

These potentials give the following representation formula for the solution of $(\Delta+k^2)u = 0$ in G_j.

LEMMA 2.4. *For* $u \in H^1_{loc}(G_j)$ $(j = 1,2)$ *with Cauchy data* $v := u|_{\partial G_j}$, $\psi := \left.\frac{\partial u}{\partial n}\right|_{\partial G_j}$ *and for* $z \in G_j$ *we have*

$$u(z) = (-1)^j \frac{1}{2} (K_{G_j}v(z) - V_{G_j}\psi(z)) \tag{2.8}$$

PROOF. This representation formula is well-known for smooth boundaries ∂G_j and smooth layers v and ψ. Since the operators in (2.7) have C^∞ kernels, formula (2.8) remains valid for the Cauchy data of $u \in H^1_{loc}(G_j)$, $j = 1,2$ (compare Fig. 1).□

Next, we give a representation of the weak solution of (1.1)-(1.3) via the operator of the single layer potential for the Dirichlet problem and via the operator of the normal derivative of the double layer potential in case of the Neumann problem.

Let us first consider the <u>Dirichlet problem</u> and proceed as in [18]. Using Green's formula we shall end up with an integral equation of the first kind for the jump of the normal derivative of the solution (1.1)-(1.3) on the open surface piece S. When we use the geometrical situation of Fig. 1 application of the representation formula (2.8) yields for $x \in G_1$

$$\begin{aligned} u(x) &= -\frac{1}{2} (K_{G_1}u(x) - V_{G_1}\frac{\partial u}{\partial n}(x)), \\ 0 &= -\frac{1}{2} (K_{G_2}u(x) - V_{G_2}\frac{\partial u}{\partial n}(x)), \end{aligned} \tag{2.9}$$

where K_{G_j}, V_{G_j}, $j = 1,2$, are defined according to (2.7).
Note that $u|_{\partial G_j} \in H^{1/2}(\partial G_j)$ and $\frac{\partial u}{\partial n}\Big|_{\partial G_j} \in H^{-1/2}(\partial G_j)$, $j = 1,2$.
Hence addition of (2.9) yields with the outer boundary $\partial B = \{y \in \mathbb{R}^3 : |y| = R\}$ and the fundamental solution ϕ in (2.6)

$$u(x) = \int_{|y|=R} u(y) \frac{\partial}{\partial n_y} \phi(x,y)dS_y - \int_{|y|=R} \frac{\partial u}{\partial n}(y)\phi(x,y)dS_y - \int_S [\frac{\partial u}{\partial n}](y)\, \phi(x,y)dS_y,$$

since $[\frac{\partial u}{\partial n}]\Big|_{\partial G_j \setminus \bar{S}} = 0$ according to Lemma 2.2.

For $x \to S$ the trace theorem yields together with the boundary condition $u|_S = g$ the relation

$$g(x) = \int_{|y|=R} \{u(y) \frac{\partial}{\partial n_y} \phi(x,y) - \frac{\partial u}{\partial n}(y)\phi(x,y)\}dS_y - \int_S [\frac{\partial u}{\partial n}](y)\, \phi(x,y)dS_y. \tag{2.10}$$

Since the radiation condition (1.3) holds for u and ϕ, the integral in (2.10) over $|y| = R$ vanishes as $R \to \infty$ and therefore (2.10) becomes for $x \in S$

$$g(x) = -\int_S [\frac{\partial u}{\partial n}](y)\, \phi(x,y)dS_y \tag{2.11}$$

Now, let $[\frac{\partial u}{\partial n}] \in \tilde{H}^{-1/2}(S)$ (compare Lemma 2.2) be a solution of (2.11). Then the potential of the single layer

$$u(x) = -\int_S [\frac{\partial u}{\partial n}](y)\, \phi(x,y)dS_y \tag{2.12}$$

is well defined for every $x \notin S$ since $\phi(x,y) \in H^{1/2}(S)$. The same holds for any derivative $(\frac{\partial}{\partial x})^{\ell}(\frac{\partial}{\partial y})^m \phi(x,y)|_S$. Now the single layer potential V in (2.12) is a ψdo of order $-3/2$ as a mapping from functions on the boundary ∂G_1 into functions on the bounded domain G_1 and on its complement $G_2 = \mathbb{R}^3 \setminus \bar{G}_1$ [4, §8]. Thus for $\psi^* \in H^{-1/2}(\partial G_1)$ we have $V\psi^* \in H^1(G_1) \cup H^1_{loc}(G_2)$ where $\psi^* = 0$ on $\partial G_1 \setminus \bar{S}$ and $\psi^* = [\frac{\partial u}{\partial n}]$ on S.

For $x \notin S$ we can interchange differentiation and integration in (2.12) obtaining

$$(\Delta + k^2)u = 0 \text{ in } \Omega_S = \mathbb{R}^3 \setminus \bar{S}.$$

For $|x| \geq R$ the potential (2.12) is C^∞ and satisfies the radiation condition (1.3). Hence the potential u in (2.12) belongs to $H^1_{loc}(\Omega_S)$.

The above results we sum up as follows:

THEOREM 2.5. *$u \in H^1_{loc}(\Omega_S)$ is the solution of the screen Dirichlet problem (1.1)-(1.3) if and only if the jump $[\frac{\partial u}{\partial n}]|_S \in \tilde{H}^{-1/2}(S)$ is the solution of the weakly singular integral equation*

$$V_S[\frac{\partial u}{\partial n}](x) := -2 \int_S [\frac{\partial u}{\partial n}](y)\phi(x,y)dS_y = 2g(x) \qquad (2.13)$$

for $x \in S$ with ϕ as in (2.6) and given $g \in H^{1/2}(S)$.

Next we consider the screen <u>Neumann problem</u>. Taking in (2.9) the normal derivative we obtain for $x \in G_1$

$$\frac{\partial u}{\partial n}(x) = \int_{|y|=R} \{u(y) \frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y}\phi(x,y) - \frac{\partial u}{\partial n}(y)\frac{\partial}{\partial n_x}\phi(x,y)\}dS_y$$
$$- \int_S [u](y) \frac{\partial^2}{\partial n_x \partial n_y}\phi(x,y)dS_y$$

since $[u]\Big|_{\partial G_j \setminus \bar{S}} = 0$. Since ϕ and its derivatives satisfy (1.3) we obtain for $x \to S$ and $R \to \infty$ together with $\frac{\partial u}{\partial n}\Big|_S = h$ the relation

$$h(x) = -\int_S [u](y) \frac{\partial^2}{\partial n_x \partial n_y}\phi(x,y)dS_y \qquad (2.14)$$

Conversely, let $[u] \in \tilde{H}^{1/2}(S)$ be a solution of (2.14). Then for $x \in \Omega_S =: \mathbb{R}^3 \setminus \bar{S}$ the potential of the double layer

$$u(x) = -\int_S [u](y) \frac{\partial}{\partial n_y}\phi(x,y)dS_y \qquad (2.15)$$

is well defined since $\frac{\partial}{\partial n_y}\phi(x,y) \in H^{-1/2}(S)$ together with all derivatives with respect to x and y. The double layer potential K in (2.15) is a pseudodifferential operator of order -1/2 as a mapping from functions on the boundary ∂G_1 into functions on the domains G_1 and $G_2 = \mathbb{R}^3 \setminus \overline{G_1}$ [4, §8]. Thus for $v^* \in H^{1/2}(\partial G_1)$ we have $Kv^* \in H^1(G_1) \cup H^1_{loc}(G_2)$ where $v^* = 0$ on $\partial G_1 \setminus \bar{S}$ and $v^* = [u]$ on S.

For $x \notin S$ we can interchange differentiation and integration obtaining $(\Delta + k^2)u = 0$ in Ω_S and for $|x| = R \to \infty$ the potential (2.15) satisfies the radiation condition (1.3).

Summing up, we have the following equivalence:

THEOREM 2.6. $u \in H^1_{loc}(\Omega_S)$ *is the solution of the screen Neumann problem* (1.1)-(1.3) *if and only if the jump* $[u]|_S \in \tilde{H}^{1/2}(S)$ *is the solution of the hypersingular integral equation*

$$D_S[u](x) := 2 \int_S [u](y) \frac{\partial^2}{\partial n_x \partial n_y} \phi(x,y)\, dS_y = -2h(x) \qquad (2.16)$$

for $x \in S$ *with* ϕ *as in* (2.6) *and given* $h \in H^{-1/2}(S)$.

We note that for the derivation of the integral equations (2.13) and (2.16) we only assumed local finite energy of the solution u of (1.1)-(1.3), i.e., $u \in H^1_{loc}(\Omega_S)$. We need no additional regularity assumptions. Using the calculus of pseudodifferential operators we derive the following existence results for the solutions of (2.13) and (2.16), respectively.

THEOREM 2.7. *Let* $\mathrm{Im}\, k \geq 0$. *Then there holds:*
(i) For given $g \in H^{1/2}(S)$ *there exists exactly one solution* $\psi \in \tilde{H}^{-1/2}(S)$, $\psi := [\frac{\partial u}{\partial n}]|_S$, *of the integral equation* (2.13).
(ii) For given $h \in H^{-1/2}(S)$ *there exists exactly one solution* $v \in \tilde{H}^{1/2}(S)$, $v = [u]|_S$, *of the integral equation* (2.16).

The proof of Theorem 2.7 is based on the following lemma showing the coerciveness of the operators V_S and D_S in the form of a Gårding inequality in the appropriate trace spaces.

LEMMA 2.8. *(i) The mappings* $V_S: \tilde{H}^s(S) \to H^{s+1}(S)$ *and* $D_S: \tilde{H}^s(S) \to H^{s-1}(S)$ *are continuous for any real number* s.
(ii) There exist constants $\gamma_i > 0$ $(i = 1,2)$ *and an operator* C_1 *from* $\tilde{H}^{-1/2}(S)$ *into* $H^{\frac{1}{2}+\varepsilon}(S)$ *and an operator* C_2 *from* $\tilde{H}^{1/2}(S)$ *into* $H^{-\frac{1}{2}+\varepsilon}$ *for some* $\varepsilon > 0$ *such that for all* $\psi \in \tilde{H}^{-1/2}(S)$ *and* $v \in \tilde{H}^{1/2}(S)$

$$\langle (V_S+C_1)\psi,\psi\rangle_{L^2(S)} \geq \gamma_1 \|\psi\|^2_{\tilde{H}^{-1/2}(S)} \qquad (2.17)$$

$$\langle (D_S+C_2)v,v\rangle_{L^2(S)} \geq \gamma_2 \|v\|^2_{\tilde{H}^{1/2}(S)} \qquad (2.18)$$

PROOF. For any $\psi \in \tilde{H}^s(S)$ the extension ψ^* by zero on $\Gamma\backslash\bar{S}$ belongs to $H^s(\Gamma)$ with Γ being the closed smooth bounded manifold ∂G_1 in Fig. 1. Thus, the assertion (i) follows since

V and D are continuous mappings from $H^s(\Gamma)$ into $H^{s+1}(\Gamma)$ or into $H^{s-1}(\Gamma)$, respectively, because V and D are pseudodifferential operators of order minus one and plus one respectively (see [2], [10], [15]). The Gårding inequalities (2.17), (2.18) for V_S and D_S are consequences of the strong ellipticity of V and D, since their respective principal symbols are

$$\sigma(V)(\xi) = |\xi|^{-1}, \quad \sigma(D)(\xi) = |\xi| \tag{2.19}$$

where $|\xi| = \sqrt{\xi_1^2 + \xi_2^2} \neq 0$. By use of a partition of unity the screen S is locally mapped to the halfspace $\mathbb{R}^2_+$ and therefore (2.17), (2.18) follow from (2.19) by standard arguments (see [16]).

The first expression in (2.19) is easily obtained by applying the Fourier transformation to the kernel ϕ in (2.6) yielding the Fourier transformed kernel (see [12])

$$\hat{\phi}(\xi) = (|\xi|^2 - k^2)^{-1/2}$$

having the asymptotic expansion

$$(|\xi|^2-k^2)^{-1/2} \sim |\xi|^{-1}\{1 + 2k^2|\xi|^{-2} + \tfrac{3}{8}k^4|\xi|^{-4} + \ldots\} \quad \text{for } |\xi| > |k| \tag{2.20}$$

This expansion into homogeneous functions of decreasing degree in the Fourier transformed variable ξ shows that the operator of the single layer potential is a pseudodifferential operator of order -1 (see also [16]). $\square$

PROOF OF THEOREM 2.7. From (2.17) and (2.18) follows that V and D are Fredholm operators of index zero from $\tilde{H}^{-1/2}(S)$ into $H^{1/2}(S)$ and from $\tilde{H}^{1/2}(S)$ into $H^{-1/2}(S)$, respectively. On the other hand, the integral equations (2.13) and (2.16) are equivalent to the screen Dirichlet problem and the screen Neumann problem, respectively. Due to Lemma 2.1 both problems do have <u>no</u> eigensolutions. Hence the above mappings V_S and D_S are injective and therefore bijective. $\square$

We now come to the point of our main concern -- the singularity of the densities of the integral equations (2.13), (2.16) near the edge γ of the screen S.

The following results give the asymptotic behavior of the exact solution $[\frac{\partial u}{\partial n}]$ of the integral equation (2.13) and [u] of (2.16) near the edge γ. The analysis here follows the procedure in [16] by (i) mapping locally S onto $\mathbb{R}^2_+$, (ii) applying the Wiener-Hopf technique in the halfspace $\mathbb{R}^2_+$ and (iii) patching together the local results.

THEOREM 2.9. *(i) Let* $g \in H^{3/2+\sigma}(S)$ *be given. Then the solution of the integral equation* (2.13) *has the form*

$$[\frac{\partial u}{\partial n}] = \beta(s)\rho^{-1/2}\chi(\rho) + \psi_r \text{ on } S \tag{2.21}$$

with $\beta \in H^{\frac{1}{2}+\sigma}(\gamma)$, $\psi_r \in \tilde{H}^{\frac{1}{2}+\sigma'}(S)$, $0 < \sigma' < \sigma < 1/2$.
(ii) Let $h \in H^{\frac{1}{2}+\sigma}(S)$ *be given. Then the solution of the integral equation* (2.16) *has the form*

$$[u] = \alpha(s)\rho^{1/2}\chi(\rho) + v_r \text{ on } S \tag{2.22}$$

with $\alpha \in H^{\frac{1}{2}+\sigma}(\gamma)$, $v_r \in L^2(I;H^{\frac{1}{2}+\sigma}(\gamma)) \cap \tilde{H}^{3/2+\sigma'}(I;L^2(\gamma))$, $0 < \sigma' < \sigma < 1/2$, *where* S *is identified with* $I \times \gamma$, $I = [0,1]$.

(Here s denotes the parameter of arclength of γ, ρ corresponds to the Euclidean distance to γ, χ is a C^∞ cut-off function with $\chi \equiv 1$ for $|\rho| < 1/2$ and $\chi \equiv 0$ for $|\rho| > 1$.)

PROOF. As a first step to prove the decompositions (2.21), (2.22) we discuss the halfspace case where $\Gamma = \partial G_1$ (compare Fig. 1) coincides with the plane $x_3 = 0$ and S coincides with $\mathbb{R}^2_+$ given by $x_3 = 0$ and $x_2 > 0$ such that its boundary γ becomes $\mathbb{R}^1$.

Following the ideas in [16], [17] the key of our analysis is to consider instead of (2.13), (2.16) the following equations which can be solved by the Wiener-Hopf technique:

$$p_+\hat{V}\psi_+ = 2g, \quad p_+\hat{D}_+v_+ = -2h \text{ on } \mathbb{R}^2_+ \tag{2.23}$$

for given data

$$g \in H^{3/2+\sigma}(\mathbb{R}^2_+), \quad h \in H^{\frac{1}{2}+\sigma}(\mathbb{R}^2_+), \quad |\sigma| < 1/2 \tag{2.24}$$

Here $\hat{V}(\hat{D})$ is the pseudodifferential operator with symbol $(\xi_2 + (|\xi_1| + 1)^2)^{(\mp)\frac{1}{2}}$, respectively, where ξ_1,ξ_2 are the dual variables to x_1,x_2 of the Fourier transformation. p_+ denotes the restriction to the halfspace $\mathbb{R}^2_+$. The operators $\hat{V}$ and $\hat{D}$ have the suitable form to perform the Wiener-Hopf technique according to [4, §7]. In [16], [17] we derive via the factorizations $\hat{V} = \hat{V}_+\hat{V}_- = \hat{V}_-\hat{V}_+$ and $\hat{D} = \hat{D}_+\hat{D}_- = \hat{D}_-\hat{D}_+$ the solutions of (2.23) as

$$\psi_+ = 2\hat{V}_+^{-1}p_+\hat{V}_-^{-1}\ell g$$

and

$$v_+ = -2\hat{D}_+^{-1}p_+\hat{D}_-^{-1}\ell h$$

where $\hat{V}_{(\pm)}$ has the symbol $(\xi_{2}\,{}_{(\pm)}\,i(|\xi_1| + 1))^{-1/2}$ and $\hat{D}_{(\pm)}$ has the symbol $(\xi_{2}\,{}_{(\pm)}\,i(|\xi_1| + 1)^{\frac{1}{2}}$.

Under the assumption (2.24) we can improve the above decompositions to

$$\psi_+ = \psi_s + \psi_r, \begin{cases} \psi_s = 2\hat{V}_+^{-1}\Lambda_+^{-1}ip'\hat{V}_-^{-1}\ell g \\ \psi_r = \hat{V}_+^{-1}\hat{\Lambda}_+^{-1}p_+\hat{\Lambda}_+\hat{V}_-^{-1}\ell g \end{cases} \tag{2.25}$$

$$v_+ = v_s + v_r\ , \begin{cases} v_s = -2\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}ip'\hat{D}_-^{-1}\ell h \\ v_r = -2\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}p_+\hat{\Lambda}_+\hat{D}_-^{-1}\ell h \end{cases} \tag{2.26}$$

where p' denotes the restriction to $\mathbb{R}^1$ and $\hat{\Lambda}_{(\pm)}$ denotes the pseudodifferential operator with symbol $(\xi_{2}\,{}_{(\pm)}\,i(|\xi_1| + 1))$.

From (2.25), (2.26) follows $\psi_r \in \tilde{H}^{\frac{1}{2}+\sigma}(\mathbb{R}_+^2)$, $v_r \in \tilde{H}^{3/2+\sigma}(\mathbb{R}_+^2)$. Since both $\hat{V}_+^{-1}\hat{\Lambda}_+^{-1}$ and $\hat{D}_+^{-1}\hat{\Lambda}_+^{-1}$ do <u>not</u> have the transmission property, ψ_s and v_s do not belong to those spaces: ψ_s and v_s are less regular, namely $\psi_s \in \tilde{H}^{-1/2}(S)$, $v_s \in \tilde{H}^{1/2}(S)$.

As shown in [16], [17] with application of Fourier transformation we deduce from (2.25), (2.26)

$$\psi_s(x) = b(x_1,x_2)\theta^+x_2^{-1/2}, \qquad v_s(x) = a(x_1,x_2)\theta^+x_2^{1/2} \tag{2.27}$$

with the Heaviside function $\theta^+(x_2) = 1$ for $x_2 > 0$ and $\theta^+(x_2) = 0$ for $x_2 < 0$. Here

$$\begin{aligned} b(\xi_1,x_2) &= c\ \exp[-x_2(1 + |\xi_1|)]\tilde{\beta}(\xi_1) \\ a(\xi_1,x_2) &= c\ \exp[-x_2(1 + |\xi_1|)]\tilde{\alpha}(\xi_1) \end{aligned} \tag{2.28}$$

where

$$\beta := 2p'\hat{V}_-^{-1}\ell g, \qquad \alpha := -2p'\hat{D}_-^{-1}\ell h, \tag{2.29}$$

and c denotes a generic constant.

We observe that the singular terms in (2.27) are <u>not</u> tensor products in x_1 and x_2. We derive in [16], [17] the claimed decompositions (2.21) and (2.22) by rewriting the singular terms, for example,

$$\tilde{\psi}_s(\xi_1,x_2) = x_{2+}^{-1/2}\tilde{\beta}(\xi_1) + r(\xi_1,x_2)$$
$$r(\xi_1,x_2) = x_{2+}^{-1/2}[b(\xi_1,x_2) - \tilde{\beta}(\xi_1)]$$

and studying the regularity of the remainders in anisotropic Sobolev spaces. Applying Theorem 1.4.4 in [16] we obtain the claimed decompositions (2.21)-(2.22) for the halfspace case $S = \mathbb{R}_+^2$.

In order to prove those decompositions for a smooth, bounded, open surface S we proceed as follows: First, we observe V and $\hat{V}$ have the same principal symbols. Therefore, they differ by a pseudodifferential operator C_1 of order $-1-\varepsilon$ for some $\varepsilon > 0$. Similarly $C_2 = D-\hat{D}$ is a pseudodifferential operator of order $1-\varepsilon$ for some $\varepsilon > 0$. Hence we can rewrite the integral equations (2.13), (2.16) as Riesz-Schauder equations

$$\begin{aligned}[I + \hat{V}^{-1}(V-\hat{V})]\psi &= 2\hat{V}^{-1}g \\ [I + \hat{D}^{-1}(D-\hat{D})]v &= -2\hat{D}^{-1}h\end{aligned} \tag{2.30}$$

Note that due to Rellich's embedding theorem the operators $V^{-1}(V-\hat{V})$ and $D^{-1}(D-\hat{D})$ are compact perturbations of the identity on S. The detailed analysis in [16] shows that the decompositions (2.21), (2.22) are not altered by such compact perturbations. Therefore, using standard localization techniques and patching the local results together completes the proof of Theorem 2.9. □

3. A BOUNDARY ELEMENT GALERKIN METHOD

In this section we solve the boundary integral equations (2.13), (2.16) in finite dimensional subspaces S_h^1, S_h^2 of the Sobolev spaces $\tilde{H}^{-1/2}(S)$ and $\tilde{H}^{1/2}(S)$, respectively. For conformity of our boundary element method, we assume that the families of finite element subspaces S_h^1, S_h^2 satisfy for integers t,k

$$S_h^1 = S_h^{t-1,k-1}(S) \subset H^{k-1}(S) \subset \tilde{H}^{-1/2}(S),$$
$$S_h^2 = S_h^{t,k}(S) \subset H^k(S) \cap \tilde{H}^1(S) \subset \tilde{H}^{1/2}(S), \quad t > k \geq 1 \tag{3.1}$$

In agreement with our general assumption (1.4) on S we can assume that S is given by local representations such that regular partitions in the parameter domains are mapped onto a corresponding partition of S. On the partitions in the parameter domains we use a regular (t,k)-system, called $S_h^{t,k}$, of finite elements. Then the local representation of S transplants these finite element functions onto S. In their coordinates the finite elements appear as simple functions over the parameter domains. The parameters in $S_h^{t,k}$ have the following meanings: h, $0 < h \leq h_0$, is the mesh size of the partition of S, for example h stands for the longest side of a triangle in a uniform triangulation; t-1 is the degree of piecewise polynomials constituting the corresponding finite element, k describes the conformity $S_h^{t,k} \subset H^k(S)$.

Now the Galerkin procedures to (2.13) and (2.16) read as follows:

For the Dirichlet screen problem find $\psi_h \in S_h^{t-1,k-1}(S)$ such that

$$\langle V_S\psi_h, \underset{\sim}{\phi}\rangle_{L^2(S)} = \langle 2g, \underset{\sim}{\phi}\rangle_{L^2(S)} \tag{3.2}$$

for all $\underset{\sim}{\phi} \in S^{t-1,k-1}$ with t,k as in (3.1).

For the Neumann screen problem find $v_h \in S_h^{t,k}(S)$ such that

$$\langle D_S v_h, w_h\rangle_{L^2(S)} = -\langle 2h, w_h\rangle_{L^2(S)} \tag{3.3}$$

for all $w_h \in S_h^{t,k}(S)$ with t,k as in (3.1).

The solvability of the above Galerkin equations and the convergence of the procedures are based on the Gårding inequalities (2.17), (2.18) and the uniqueness of the integral equations (2.13), (2.16). Application of the general results on the Galerkin procedure for strongly elliptic pseudodifferential equations ([7], [18], [21], [22]) yields the following theorem.

THEOREM 3.1. *(i) There exists a meshwidth* $h_0 > 0$ *such that the Galerkin equations* (3.2), (3.3) *are uniquely* *solvable for any* h, $0 < h \le h_0$.

(ii) *For decreasing meshsize* $h \to 0$ *the Galerkin solutions* $\psi_h \in S^{t-1,k-1}(S)$ *and* $v_h \in S_h^{t,k}(S)$ *converge to the exact solutions of* (2.13), (2.16), $\psi \in \tilde{H}^{-1/2}(S)$ *and* $v \in \tilde{H}^{1/2}(S)$. *Furthermore, we have the quasi-optimal asymptotic error estimates*

$$\|\psi-\psi_h\|_{-1/2,S} \le c \inf_{\underset{\sim}{\chi}\in S_h^{t-1,k-1}} \|\psi-\chi\|_{-1/2,S} \tag{3.4}$$

$$\|v-v_h\|_{1/2,S} \le c \inf_{\underset{\sim}{v}\in S_h^{t,k}} \|v-\underset{\sim}{v}\|_{1/2,S} \tag{3.5}$$

where $\|\ \|_{r,S}$ *denotes the norm in* $\tilde{H}^r(S)$ *and the constant* c *is independent of* ψ, ψ_h, v, v_h *and* h.

Due to Theorem 2.9, the exact solutions ψ,v of $V_S\psi = 2g$ and of $D_S v = -2h$ behave like $\rho^{-1/2}$ and $\rho^{1/2}$, respectively, where ρ is the distance to the boundary γ of the screen S, i.e., ψ is not square integrable on S. Since $\rho^{1/2} \in H^{1-\varepsilon}$ for some $\varepsilon > 0$ the estimates (3.4), (3.5) give at most convergence of order $\frac{1}{2} - \varepsilon$. As in the two-dimensional case [19], we can improve the asymptotic rate of convergence by using so-called singular elements as test and trial functions in the Galerkin procedures (3.2) and (3.3). Thus we augment the standard finite element spaces $S_h^{t,k}$ by special global singular elements according to the form of the exact solution given in Theorem 2.9. This gives the augmented finite element spaces $Z_h^{1/2}(S)$, $Z_h^{3/2}(S)$ on S:

$$Z_h^{1/2}(S) := \{\underset{\sim}{\psi} = \underset{\sim}{\psi}_r + \underset{\sim}{\beta}\rho^{-1/2}\chi:\ \underset{\sim}{\beta} \in S_h^{t',\ell}(\gamma),\ \underset{\sim}{\psi}_r \in \mathring{S}_h^{t-1,k-1}(S)\}$$
$$Z_h^{3/2}(S) := \{\underset{\sim}{v} = \underset{\sim}{v}_r + \underset{\sim}{\alpha}\rho^{1/2}\chi:\ \underset{\sim}{\alpha} \in S_h^{t',\ell}(\gamma),\ \underset{\sim}{v}_r \in \mathring{S}_h^{t,k}(S)\} \tag{3.6}$$

where $\underset{\sim}{\alpha},\underset{\sim}{\beta} \in S_h^{t',\ell}(\gamma) \subset H^1(\gamma)$; $\underset{\sim}{\psi}_r \in \mathring{S}_h^{t-1,k-1}(S) \subset H^{k-1}(S) \cap \tilde{H}^1(S)$;

$$\underset{\sim}{v}_r \in \mathring{S}_h^{t,k}(S) \subset H^k(S) \cap \tilde{H}^2(S). \tag{3.7}$$

with $t' > \ell \ge 1$; $t',\ell \in \mathbb{N}$ and t,k as in (3.1).

The inclusions (3.7) yield $t \ge 3$, $k \ge 2$, i.e., the simplest conform elements are continuous piecewise linear, one-dimensional elements $\underset{\sim}{\alpha}, \underset{\sim}{\beta}$ on γ, piecewise linear elements[+] $\underset{\sim}{\psi}_r$ on S with $\underset{\sim}{\psi}_r \in C^0(S)$ and piecewise quadratics $\underset{\sim}{v}_r$ on S with $\underset{\sim}{v}_r \in C^1(S)$ satisfying on γ

$$\underset{\sim}{v}_r = \frac{\partial}{\partial x_1} \underset{\sim}{v}_r = \frac{\partial}{\partial x_2} \underset{\sim}{v}_r = 0 \quad \text{and} \quad \underset{\sim}{\psi}_r = 0 \text{ on } \gamma.$$

If we introduce the notation $Z^{\frac{1}{2}+\sigma}$, $Z^{3/2+\sigma}$ for the set of all distributions having the form (2.21), (2.22) with the assigned regularity in Theorem 2.9, and with $\alpha \in H^{3/2+\sigma}(\gamma)$ then there holds the following conformity for the augmented finite element spaces $Z_h^{1/2}(S)$, $Z_h^{3/2}(S)$ in (3.6):

$$Z_h^{1/2}(S) \subset Z^{\frac{1}{2}+\sigma}(S) \subset \tilde{H}^{-1/2}(S) \tag{3.8}$$

and

$$Z_h^{3/2}(S) \subset Z^{3/2+\sigma}(S) \subset \tilde{H}^{1/2}(S)$$

Our <u>improved Galerkin schemes</u> now read as:

<u>For the Dirichlet screen problem find</u> $\psi_h = \beta_h \rho^{-1/2}\chi + \psi_h^r \in Z_h^{1/2}(S)$ <u>such that</u>

$$\langle V_s \psi_h, \underset{\sim}{\phi} \rangle_{L^2(S)} = \langle 2g, \underset{\sim}{\phi} \rangle_{L^2(S)} \tag{3.9}$$

<u>for all</u> $\underset{\sim}{\phi} = \underset{\sim}{\beta} \rho^{-1/2}\chi + \underset{\sim}{\phi}^r \in Z_h^{1/2}(S)$.

<u>For the Neumann screen problem find</u> $v_h = \alpha_h \rho^{1/2}\chi + v_h^r \in Z_h^{3/2}(S)$ <u>such that</u>

$$\langle D_s v_h, \underset{\sim}{v} \rangle_{L^2(S)} = - \langle 2h, \underset{\sim}{w} \rangle_{L^2(S)} \tag{3.10}$$

<u>for all</u> $\underset{\sim}{w} = \underset{\sim}{\alpha} \rho^{1/2}\chi + \underset{\sim}{v}^r \in Z_h^{3/2}(S)$.

Now the above Galerkin equations with test functions $\underset{\sim}{w} \in Z_h^{3/2}(S)$, $\underset{\sim}{\phi} \in Z_h^{1/2}(S)$ define quadratic systems of linear equations for the unknown coefficients of $\alpha_h, \beta_h \in S_h^{t',\ell}(\gamma)$, $v_h^r \in \mathring{S}_h^{t,k}(S)$ and $\psi_h^r \in \mathring{S}_h^{t-1,k-1}(S)$. There holds the following result:

THEOREM 3.2. *(i) There exists a meshwidth* $h_0 > 0$ *such that the Galerkin equations* (3.9), (3.10) *in the augmented finite element spaces* $Z_h^{1/2}(S)$, $Z_h^{3/2}(S)$, *are uniquely solvable for any* h, $0 < h \le h_0$.

[+] $\underset{\sim}{\psi}_r$ is a linear polynomial in two variables on each triangle of a quasi-uniform triangulation of S.

(ii) For decreasing meshsize $h \to 0$ *we have for the exact solutions* ψ *of* (2.13) *and* v *of* (2.16) *and for the Galerkin solutions*

$$\psi_h = \beta_h\, \rho^{-1/2}\chi + \psi_h^r \in Z_h^{1/2}(S), \qquad v_h = \alpha_h \rho^{1/2}\chi + v_h^r \in Z_h^{3/2}(S)$$

of (3.9), (3.10)

$$\|\psi-\psi_h\|_{-1/2,S} \le c \inf_{\eta\in Z_h^{1/2}(S)} \|\psi-\eta\|_{-1/2,S} \le ch^{1+\sigma}\, \|\psi\|_{Z^{\frac12+\sigma}(S)},$$

$$\|v-v_h\|_{1/2,S} \le c \inf_{\underset{\sim}{u}\in Z_h^{3/2}(S)} \|v-\underset{\sim}{u}\|_{1/2,S} \le ch^{1+\sigma}\, \|v\|_{Z^{3/2+\sigma}(S)}. \tag{3.11}$$

The arising norms are defined via the notation in Theorem 3.1 *as follows* ($\|\cdot\|_{q,\gamma}$ denotes the Sobolev norm in $H^q(\gamma)$):

$$\|v\|_{Z^q(S)} := \begin{cases} \|\alpha\sqrt{\rho}\chi(\rho) + v_r\|_{q,S}, & \frac{1}{2} \le q < 1-\varepsilon, \varepsilon>0 \text{ arbitrary} \\ \|\alpha\|_{q,\gamma} + \|v_r\|_{q,S}, & 1 \le q \le \frac{3}{2} + \sigma. \end{cases} \tag{3.12}$$

$$\|\psi\|_{Z^p(S)} := \begin{cases} \|-\frac{\alpha}{2}\rho^{-1/2} + \psi_r\|_{p,S}, & -\frac{1}{2} \le p < -\varepsilon, \\ \|\alpha\|_{p,\gamma} + \|\psi_r\|_{p,S}, & 0 \le p \le \frac{1}{2} + \sigma. \end{cases}$$

PROOF. (i) By the inclusions (3.8) the augmented Galerkin schemes (3.9), (3.10) are uniquely solvable for small enough h since the Gårding inequalities for V_s and D_s hold in the respective energy space $\tilde{H}^{-1/2}(S)$ and $\tilde{H}^{1/2}(S)$. Furthermore, the integral equations (2.13), (2.16) are uniquely solvable in thos spaces.

The estimate in (ii) is a consequence of Theorem 3.1 due to the conformity (3.8) and of the following convergence property (3.13), (3.14) of the augmented finite element spaces $Z_h^{3/2}(S)$, $Z_h^{1/2}(S)$. □

LEMMA 3.3. *The finite element spaces* $Z_h^{3/2}(S)$, $Z_h^{1/2}(S)$ *have the approximation properties: For any* $U = \alpha\sqrt{\rho}\chi(\rho) + v_r \in Z^s(S)$ *there exists a* $\tilde{U} = \underset{\sim}{\alpha}\sqrt{\rho}\chi(\rho) + \underset{\sim}{v}^r \in Z_h^{3/2}(S)$ *with* $t \ge s$ *and a* *constant* c > 0 *independent of* h *and* U *such that for* $q \le \min\{k,s\}$

$$\|U-\tilde{U}\|_{Z^q(S)} \le ch^{s-q}\,\|U\|_{Z^s(S)}\;, \quad \frac{1}{2} \le q \le s \le 2-\varepsilon \qquad (3.13)$$

For any $\Psi = \beta\rho^{-1/2}\chi(\rho) + \psi_r \in Z^{s'}(S)$ *there exists a* $\tilde{\Psi} = \underset{\sim}{\beta}\rho^{-1/2}\chi(\rho) + \underset{\sim}{\psi}^r \in Z_h^{1/2}(S)$ *with* $t-1 \ge s'$ *and a constant* $c > 0$ *independent of* h *and* Ψ *such that for* $q' \le \min\{k-1,s'\}$

$$\|\Psi-\tilde{\Psi}\|_{Z^{q'}(S)} \le ch^{s-q'}\,\|\Psi\|_{Z^{s'}(S)}\,. \qquad (3.14)$$

PROOF. For $|\sigma| < 1/2$ and arbitrary $\varepsilon > 0$ there holds with S substituted by $I \times \gamma$, $I = [0,1]$,

$$\begin{aligned}\|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{L^2(I,H^{\frac{1}{2}+\sigma}(\gamma))} &\le c\,\|\alpha-\underset{\sim}{\alpha}\|_{H^{\frac{1}{2}+\sigma}(\gamma)}\\ \|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{H^{\frac{1}{2}+\sigma}(I,L^2(\gamma))} &\le c\,\|\alpha-\underset{\sim}{\alpha}\|_{H^{\varepsilon}(\gamma)}\,.\end{aligned} \qquad (3.15)$$

The estimates (3.15) imply for $\frac{1}{2} \le q < 1-\varepsilon$ and $1-\varepsilon \le s < 2-\varepsilon$ with (3.12) and the triangle inequality

$$\begin{aligned}\|U-\tilde{U}\|_{Z^q(S)} &\le \|\sqrt{\rho}(\alpha-\underset{\sim}{\alpha})\|_{q,S} + \|v_r-\underset{\sim}{v}^r\|_{q,S}\\ &\le ch^{s-q}\{(1+h^{q-\varepsilon})\,\|\alpha\|_{s,\gamma} + \|v_r\|_{s,S}\}\\ &\le ch^{s-q}\,\|U\|_{Z^s(S)}\end{aligned} \qquad (3.16)$$

Here we have used $\|\alpha-\underset{\sim}{\alpha}\|_{H^q(\gamma)} \le ch^{s-q}\,\|\alpha\|_{H^s(\gamma)}$ and $\|v_r-\underset{\sim}{v}^r\|_{q,S} \le ch^{s-q}\,\|v_r\|_{s,S}$ which hold by the standard approximation property of $S_h^{t',\ell}(\gamma)$ and $\mathring{S}_h^{t,k}(S)$, respectively [21].

For $1 \le q \le s \le 3/2 + \sigma$ we have with (3.12)

$$\begin{aligned}\|U-\tilde{U}\|_{Z^q(S)} &= \|\alpha-\underset{\sim}{\alpha}\|_{q,\gamma} + \|v_r-\underset{\sim}{v}^r\|_{q,S}\\ &\le ch^{s-q}(\|\alpha\|_{s,\gamma} + \|v_r\|_{s,S})\\ &= ch^{s-q}\,\|U\|_{Z^s(S)}\,.\end{aligned}$$

For $s < 1-\varepsilon$ we take $\tilde{U} = \underset{\sim}{v}^r \in \mathring{S}_h^{t,k}(S)$ in (3.16) completing (3.13). The estimate (3.14) holds analogously. Finally we note that the estimates (3.15) are verified as in the proof of Theorem 1.4.4 in [16]. And for completeness we recall from [11] for any real number $s \geq 0$

$$H^s(I \times \gamma) = \{u: u \in L^2(I;H^s(\gamma)),\ u \in H^s(I;L^2(\gamma))\},$$

with the equivalence of norms

$$\|u\|^2_{H^s(I\times\gamma)} \approx \|u\|^2_{L^2(I;H^s(\gamma))} + \|u\|^2_{H^s(I;L^2(\gamma))}. \qquad \square$$

Finally, we remark that for problems in acoustics we have in (1.2)

$$g = -u_i \ , \quad h = -\frac{\partial u_i}{\partial n}$$

where u_i denotes the incident field with a source away from the scattering screen S. Therefore the given data for the scattered field u in (1.2) are infinitely smooth. In this case the proof of Theorem 2.9 (compare (2.25)-(2.29)) shows that the unknowns in our integral equations (2.13), (2.16) have still the form (2.21), (2.22) but with smooth line layers $\alpha,\beta \in C^\infty(\gamma)$ for a smooth edge curve γ. Following our above analysis we expect further improvement for the Galerkin method if further singular elements like $\rho^{3/2},\rho^{5/2},\ldots$ are used. This is indicated for the two-dimensional case in [19]. For an analysis of errors arising from a finite element approximation of the screen surface S we refer to [13].

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