New preconditioners for Laplace and Helmholtz integral equation on open curves:

II. Theoretical analysis.

Martin Averseng*

February 1, 2019

Abstract

This theoretical paper is dedicated to establishing the results announced in the first part of this work, related to the weighted layer potentials appearing in the resolution of first kind integral equations on open curves in 2 space dimensions. Those include the optimal orders of convergence in a new Galerkin scheme for this problem, and the analysis of two new preconditioners, in the form of square root of differential operators.

Introduction

Je viens de faire un autre exemple d'intro, je laisse le précédent juste après, dis moi ce que tu en penses. J'avais envie de faire passer l'idée du point (i) et (ii), et d'expliquer clairement les rôles complémentaires des deux articles. Mais je ne sais pas si c'est pertinent donc je laisse l'intro précédente si tu préfères. Je commence par la nouvelle version

In the framework of finite element methods, two desirable features are: (i) fast convergence in terms of the mesh size and (ii) availability of an efficient preconditioner for the linear system if it is ill-conditioned. In the context of first kind integral equations on open curves in the plane, those two requirements are not easily fulfilled since on the one hand, the solutions to the integral equations have singularities near the edges of the curve, leading to poor rates of convergence in the Galerkin method, and on the other hand the usual preconditioning strategies break down in the presence of geometrical singularities on the curve. In the first part of this work [1], we recast those integral equations using weighted layer potentials, and described a Galerkin method with optimal orders of convergence for piecewise affine functions. Furthermore, two new efficient preconditioners for the weighted potentials were, thus providing a method with both features (i) and (ii).

The work [1] and the present have been thought with the following complementary roles:

*CMAP, Ecole polytechnique, Route de Saclay, 91128 Palaiseau Cedex.

- [1] aims at describing concisely the essential aspects of the method and showing its efficiency on numerical examples

- this work contains the complete theoretical analysis.

For this reason, we omit here the discussions on the physical nature of the problem and link with other numerical methods. We instead go straight to the proofs of the mathematical results.

The outline of the paper is as follows. We analyze in the first section the families of Hilbert spaces T^s and U^s, which are two interpolating Hilbert scales with interlacing properties. Those spaces are used in the second section to analyze the Galerkin method for the weighted layer potentials in weighted L^2 spaces, and prove optimal orders of convergence for piecewise affine functions. The linear systems appearing in the Galerkin method are ill-conditioned because, in some sense, the operators being discretized are not of order zero. We introduce in the third section a new concept of pseudo-differential operators on open curves that gives a precise meaning to this notion of order of an operator. This allows us to build parametrix for the layer potentials, which take the form of square roots of differential operators. Those parametrix are analyzed in the fourth and last section.

La version précédente, moins "risquée" :

In the first part of this work [1], some new preconditioners for the Laplace and Helmholtz integral equations are introduced and their numerical is demonstrated on several numerical examples. Here, we develop the theory to prove the main results that were announced there. To this aim, we analyze the spaces $(T^s)_{s\in\mathbb{R}}$ and $(U^s)_{s\in\mathbb{R}}$, which are two family of Hilbert spaces, formed of Chebyshev series defined on the unit segment. They provide two interpolating scales with special interlacing properties that are established in the first section of this paper. Those spaces are suited to the analysis of a new Galerkin setting for those integral equations, performed in the second section. There, the optimal orders of convergence announced in [1] for this scheme are established. In the third section, we build on the class of Periodic Pseudo-Differential Operators (studied for example in [8]), to define two classes of pseudo-differential operators respectively in the spaces T^s and U^s. Symbolic calculus is available in those classes, which combined to a formal calculator, allows us to analyze the efficiency of the preconditioners of [1]. In the first section, we establish some properties of the spaces T^s and U^s. After briefly collecting some facts on periodic pseudo-differential operators in the second section, we define the two new classes of pseudo-differential operators. In the third section, we apply this theory to the aforementioned preconditioners. Finally, the Galerkin analysis is exposed in the fourth section. We immediately start the presentation with some theoretical considerations. The numerical applications of this work are exposed in [1].

Remark 1. *Throughout all this article, we use repeatedly the letter C in estimates of the form $a \leq Cb$. From line to line, the value of the constant C may change but is independent of the relevant parameters defining a and b.*

1 Spaces T^s and U^s

1.1 Definitions

The Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$$

for $x \in [-1,1]$, see [4]. Letting ∂_x the derivation operator and ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$, T_n and U_n satisfy the following identities:

$$-(\omega\partial_x)^2 T_n = n^2 T_n\,, \tag{1}$$
$$-(\partial_x\omega)^2 U_n = (n+1)^2 U_n\,. \tag{2}$$

Notice that here and in the following, $\partial_x\omega$ denotes the composition of operators ∂_x and ω and not the function $x \mapsto \partial_x\omega(x)$. One can also check the identities

$$\partial_x T_n = nU_{n-1}\,, \tag{3}$$
$$-\omega\partial_x\omega U_n = (n+1)T_{n+1}\,. \tag{4}$$

The first one is obtained for example from the trigonometric definition of T_n. This combined with $-(\omega\partial_x)^2 T_n = n^2 T_n$ gives the second identity.

Both T_n and U_n are polynomials of degree n, and provide respectively a basis of the following Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{|u(x)|^2}{\sqrt{1-x^2}} dx < +\infty \right\},$$

$$L^2_{\omega} := \left\{ u \in L^1_{\text{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} |u(x)|^2 \sqrt{1-x^2} dx < +\infty \right\}.$$

Following the notations of [6], we denote the Banach duality products of $L^2_{\frac{1}{\omega}}$ and L^2_ω respectively by $\langle \cdot,\cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot,\cdot \rangle_\omega$ and the inner products respectively by $(\cdot,\cdot)_{\frac{1}{\omega}}$ and $(\cdot,\cdot)_\omega$, with the following normalization:

$$(u,v)_{\frac{1}{\omega}} = \langle u, \overline{v} \rangle_{\frac{1}{\omega}} := \frac{1}{\pi}\int_{-1}^{1} \frac{u(x)\overline{v(x)}}{\omega(x)} dx\,,$$

$$(u,v)_{\omega} = \langle u, \overline{v} \rangle_{\omega} := \frac{1}{\pi}\int_{-1}^{1} u(x)\overline{v(x)}\omega(x) dx\,.$$

The Chebyshev polynomials satisfy

$$(T_n, T_m)_{\frac{1}{\omega}} = \begin{cases} 0 \text{ if } n \neq m, \\ 1 \text{ if } m = n = 0, \\ 1/2 \text{ otherwise,} \end{cases} \tag{5}$$

and

$$(U_n, U_m)_\omega = \begin{cases} 0 \text{ if } n \neq m, \\ 1/2 \text{ otherwise,} \end{cases} \tag{6}$$

from which we obtain the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x)\,, \tag{7}$$

where the Fourier-Chebyshev coefficients of the first kind are given by $\hat{u}_n = \frac{(u,T_n)_{\frac{1}{\omega}}}{(T_n,T_n)_{\frac{1}{\omega}}}$ and satisfy the Parseval equality

$$\forall (u,v) \in L^2_{\frac{1}{\omega}} \qquad (u,v)_{\frac{1}{\omega}} = \hat{u}_0 \overline{\hat{v}}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \overline{\hat{v}}_n\,.$$

When u is furthermore a smooth function, one can check that the series (7) converges uniformly to u. Similarly, any function $v \in L^2_\omega$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the Fourier-Chebyshev coefficients of the second kind $\check{v}_n$ are given by $\check{v}_n := \frac{(v,U_n)_\omega}{(U_n,U_n)_\omega}$ with the Parseval identity

$$(u,v)_\omega = \frac{1}{2} \sum_{n=0}^{+\infty} \check{u}_n \overline{\check{v}_n}\,.$$

The preceding analysis can be used to define Sobolev-like spaces.

Definition 1. *We define T^s as the set of (formal) series*

$$u = \sum_{n \in \mathbb{N}} \hat{u}_n T_n$$

where the coefficients $\hat{u}_n$ satisfy

$$\sum_{n \in \mathbb{N}} (1+n^2)^s \left|\hat{u}_n\right|^2 < +\infty\,.$$

Let $T^\infty = \cap_{s \geq 0} T^s$ and $T^{-\infty} = \cup_{s \in \mathbb{R}} T^s$. For $u \in T^s$ when $s \geq 0$, the series defining u converges in $L^2_{\frac{1}{\omega}}$ and the Fourier-Chebysehv coefficients of the first kind of u coincide with $\hat{u}_n$, allowing to identify T^s to a subspace of $L^2_{\frac{1}{\omega}}$ with $T^0 = L^2_{\frac{1}{\omega}}$. For all $u \in T^s$, we define the linear form $\langle u, \cdot \rangle_{\frac{1}{\omega}}$ by

$$\forall \varphi \in T^\infty, \langle u, \varphi \rangle_{\frac{1}{\omega}} = \frac{1}{2} \hat{u}_0 \hat{\varphi}_0 + \frac{1}{2} \sum_{n=1}^{+\infty} \hat{u}_n \hat{\varphi}_n\,. \tag{8}$$

This linear form has a unique continuous continuation on T^{-s}, and the dual of T^s is the set of linear forms $\langle u, \cdot\rangle_{\frac{1}{\omega}}$ where $u \in T^{-s}$. Endowed with the scalar product

$$(u,v)_{T^s} := \hat{u}_0\overline{\hat{v}_0} + \frac{1}{2}\sum_{n=1}^{+\infty}(1+n^2)^s\hat{u}_n\overline{\hat{v}_n}\,,$$

T^s is a Hilbert space for all s. A homogeneous semi-norm on T^s can be defined as

$$|u|^2_{T^s} := \frac{1}{2}\sum_{n=1}^{+\infty} n^{2s}\left|\hat{u}_n\right|^2.$$

Definition 2. *In a similar fashion, we define U^s as the set of formal series*

$$u = \sum_{n\in\mathbb{N}}\check{u}_n U_n$$

where the coefficients $\check{u}_n$ satisfy

$$\sum_{n\in\mathbb{N}}(1+n^2)^s\left|\check{u}_n\right|^2 < +\infty\,.$$

Let $U^\infty = \cap_{s\in\mathbb{R}}U^s$ and $U^{-\infty} = \cup_{s\in\mathbb{R}}U^s$. For $u \in U^s$ when $s \geq 0$, the series defining u converges in L^2_ω and the Fourier-Chebyshev coefficients of the second kind of u coincide with $\check{u}_n$, allowing to identify U^s to a subspace of L^2_ω with $U^0 = L^2_\omega$. For all $u \in U^s$, we define the linear form $\langle u,\cdot\rangle_\omega$ by

$$\forall\varphi \in U^\infty, \langle u,\varphi\rangle_\omega := \frac{1}{2}\sum_{n=0}^{+\infty}\check{\varphi}_n\check{u}_n\,. \tag{9}$$

This linear form has a unique continuous extension on U^{-s}, and the dual of U^s may be identified to U^{-s} with respect to the bilinear form $\langle\cdot,\cdot\rangle_\omega$. Endowed with the scalar product

$$(u,v)_{U^s} := \frac{1}{2}\sum_{n\in\mathbb{N}}\left(1+(n+1)^2\right)^s\check{u}_n\overline{\check{v}}_n\,,$$

U^s is a Hilbert space for all $s \in \mathbb{R}$.

Let $s_1, s_2 \in_R$, $\theta \in (0,1)$ and let $s = \theta s_1 + (1-\theta)s_2$. It is easy to check that

$$\forall u \in T^\infty, \|u\|_{T^s} \leq \|u\|^\theta_{T^{s_1}}\|u\|^{1-\theta}_{T^{s_2}}$$

and

$$\forall u \in U^\infty, \|u\|_{U^s} \leq \|u\|^\theta_{U^{s_1}}\|u\|^{1-\theta}_{U^{s_2}}$$

Therefore, $(T^s)_{s\in\mathbb{R}}$ and $(U^s)_{s\in\mathbb{R}}$ are exact interpolation scales.

1.2 Basic properties

Links with smooth functions and continuous inclusions. For any real s, if $u \in T^s$, the sequence of polynomials

$$u_N(x) = \sum_{n=0}^{N}\hat{u}_nT_n(x)$$

converges to u in T^s. The same assertion holds for $u \in U^s$ when T_n is replaced by U_n. Therefore

Lemma 1. *$C^\infty([-1,1])$ is dense in T^s and U^s for all $s \in \mathbb{R}$.*

The polynomials T_n and U_n are connected by the following formulas:

$$T_0 = U_0, \quad T_1 = \frac{U_1}{2}, \quad \text{and} \quad \forall n \geq 2, \quad T_n = \frac{1}{2}\left(U_n - U_{n-2}\right), \tag{10}$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}. \tag{11}$$

This leads us to introduce the map

$$I : T^\infty \to U^\infty$$

defined by

$$\widecheck{I\varphi}_0 = \hat{\varphi}_0 - \frac{\hat{\varphi}_2}{2}, \quad \widecheck{I\varphi}_j = \frac{\hat{\varphi}_j - \hat{\varphi}_{j+2}}{2} \text{ for } j \geq 1.$$

I is bijective has the explicit inverse

$$\widehat{I^{-1}\varphi}_0 = \sum_{n=0}^{+\infty} \check{\varphi}_{2n}, \quad \widehat{I^{-1}\varphi}_j = 2\sum_{n=0}^{+\infty} \check{\varphi}_{j+2n} \text{ for } j \geq 1.$$

Lemma 2. *For all real s, I has a unique continuous extension from T^s to U^s and for $s > \frac{1}{2}$, I^{-1} has a continuous extension from U^s to T^{s-1}.*

Before starting the proof, we introduce the Cesàro operator C defined on $l^2(\mathbb{N}^*)$ by

$$(Cu)_n = \frac{1}{n}\sum_{k=1}^{n} u_k\,.$$

As is well-known, this is a linear continuous operator on the Hilbert space $l^2(\mathbb{N}^*)$. Its adjoint

$$(C^*u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k}\,,$$

is therefore also continuous on $l^2(\mathbb{N}^*)$. In other words, for all $(u_n)_n \in l^2(\mathbb{N})$,

$$\sum_{n=1}^{+\infty}\left(\sum_{k=n}^{+\infty}\frac{u_k}{k}\right)^2 \leq C\sum_{k=1}^{+\infty} u_k^2\,.$$

Proof. The first result is immediate from the definition of T^s, U^s and I. When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging thus $I^{-1}u$ is well defined. Since $u \in U^s$, the sequence $\left((1+n^2)^{s/2}\,|\check{u}_n|\right)_{n\geq 1}$ is in $l^2(\mathbb{N}^*)$. Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the sequence $(r_n)_n$ defined by

$$\forall n \geq 0, \quad r_n := \sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}\,|\check{u}_k|$$

is in $l^2(\mathbb{N})$ with a l^2 norm bounded by $\|u\|_{U^s}$. We now write

$$\begin{aligned}
\left\|I^{-1}u\right\|^2_{T^{s-1}} &= \sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left|\widehat{I^{-1}u}_n\right|^2\\
&\leq 4\sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left(\sum_{k=n}^{+\infty}|\check{u}_k|\right)^2\\
&\leq 4\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}|\check{u}_k|\right)\right)^2.\\
&= 4\left\|(r_n)_n\right\|^2_{l^2}
\end{aligned}$$

We saw that the last quantity is controlled by $\|u\|^2_{U^s}$ so the result is proved. $\square$

Digression à enlever (mais garder dans le manuscrit).

Lemma 3. *Let $s > 1/2$ and let $u \in U^s$. Then there exists $0 < \varepsilon < 1$ such that $\omega^{-\frac{1+\varepsilon}{2}}u \in L^2_\omega$ with*

$$\left\|\omega^{-\frac{1+\varepsilon}{2}}u\right\|_\omega \leq C\|u\|_{U^s}\,.$$

Proof. We start by showing the following estimate

$$\forall\varepsilon\in(0,1),\ \exists C_\varepsilon : \forall n\in\mathbb{N},\quad I_n := \int_{-1}^{1}U_n^2\omega^{-\varepsilon} \leq C_\varepsilon(n+1)^\varepsilon\,.$$

Fix $\varepsilon\in(0,1)$. Using the variable change $x = \cos\theta$ and the symmetry of the integrand with respect to the change $\theta\to\pi-\theta$, we transform the quantity to be estimated to

$$I_n = \int_0^{\frac{\pi}{2}}\frac{\sin((n+1)\theta)^2}{|\sin\theta|^{1+\varepsilon}}d\theta\,.$$

We split I_n into two parts. Let $I_{n,1} = \int_0^{\frac{\pi}{n+1}}\frac{\sin((n+1)\theta)^2}{|\sin\theta|^{1+\varepsilon}}d\theta$. On this interval, we use $\sin((n+1)\theta)\leq(n+1)\theta$ and $\sin\theta\geq\frac{2}{\pi}\theta$ to find

$$I_{n,1}\leq C(n+1)^2\int_0^{\frac{\pi}{n+1}}\theta^{1-\varepsilon}\leq C_{\varepsilon,1}(n+1)^\varepsilon\,.$$

Let $I_{n,2} = I - I_{n,1}$. On this interval, we estimate the numerator by

$$\sin((n+1)\theta)\leq 1$$

and use the same estimate as before for the denominator. One can check that this leads to $I_{n,2}\leq C_{\varepsilon,2}n^\varepsilon$. The proof of the main result is now as follows. Let $u\in U^s$ where $s>\frac{1}{2}$ and let $s=\frac{1}{2}+\varepsilon$. Then the series

$$\omega^{-\frac{1+\varepsilon}{2}}u = \sum_{n\in\mathbb{N}}\check{u}_n\frac{U_n}{\omega^{\frac{1+\varepsilon}{2}}}$$

converges in L^2_ω since

$$\begin{aligned}\sum_{n\in\mathbb{N}} \left\| \check{u}_n \frac{U_n}{\omega^{\frac{1+\varepsilon}{2}}} \right\|_{L^2_\omega} &\leq C \sum_{n\in\mathbb{N}} |\check{u}_n| \, (n+1)^{\frac{\varepsilon}{2}} \\ &\leq C \sqrt{\sum_{n\in\mathbb{N}} (n+1)^{2s} \, |\check{u}_n|^2} \sqrt{\sum_{n\in\mathbb{N}} (n+1)^{-1-\varepsilon}} \, . \\ &\leq C \, \|u\|_{U^s}\end{aligned}$$

Thus $\frac{u}{\omega^{\frac{1+\varepsilon}{2}}} \in L^2_\omega$ by normal convergence and the result is proved. □

Notice that for $\varphi \in C^\infty([-1,1])$, for all $\varepsilon > 0$, $\omega^{-\frac{3-\varepsilon}{2}}\varphi \in L^2_\omega$.

Corollary 1. *Let $u \in T^{-\infty}$. Then $Iu \in U^{-\infty}$ is characterized by*

$$\forall \varphi \in C^\infty([-1,1]), \quad \langle Iu, \varphi \rangle_\omega = \left\langle u, \omega^2 \varphi \right\rangle_{\frac{1}{\omega}} \, .$$

Let $u \in U^s$ with $s > \frac{1}{2}$. Let ε such that $\omega^{-\frac{1+\varepsilon}{2}} u \in L^2_\omega$. Then $I^{-1}u \in T^{-\infty}$ is characterized by

$$\forall \varphi \in C^\infty([-1,1]), \quad \left\langle I^{-1}u, \varphi \right\rangle_{\frac{1}{\omega}} = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega$$

Proof. We shall only treat the second statement, the first one being similar and simpler. By density of $C^\infty([-1,1])$ in U^s, we can fix a sequence of C^∞ functions u_N converging to u in U^s. Then, the sequence $\omega^{-\frac{1+\varepsilon}{2}} u_N$ converges to $\omega^{-\frac{1+\varepsilon}{2}} u$ in L^2_ω since, by the previous result,

$$\left\| \omega^{-\frac{1+\varepsilon}{2}} (u_N - u) \right\|_{L^2_\omega} \leq C \, \|u - u_N\|_{U^s} \, .$$

Thus, there holds

$$\lim_{N\to\infty} \left\langle \omega^{-\frac{1+\varepsilon}{2}} u_N, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega \, .$$

By continuity of I^{-1} from U^s to T^{s-1}, we also have

$$\lim_{N\to\infty} \left\langle I^{-1} u_N, \varphi \right\rangle_{\frac{1}{\omega}} = \left\langle I^{-1} u, \varphi \right\rangle_{\frac{1}{\omega}} \, .$$

For all N, $I^{-1}u_N = u_N \in C^\infty([-1,1])$. Therefore, we obviously have

$$\left\langle I^{-1} u_N, \varphi \right\rangle_{\frac{1}{\omega}} = \left\langle \omega^{-\frac{1+\varepsilon}{2}} u_N, \omega^{-\frac{3-\varepsilon}{2}} \varphi \right\rangle_\omega \, ,$$

from which the result follows. □

Fin d'une digression potentiellement à enlever.
Let

$$u = \sum_{n\in\mathbb{N}} \hat{u}_n T_n, \quad v = \sum_{n\in\mathbb{N}} \check{v}_n U_n \, .$$

When $Iu = v$, we identify u and v. The previous results have shown that this identification is compatible with the equality of functions in $L^2_{\frac{1}{\omega}}$ or L^2_ω. The mapping I is then the identity mapping and its properties just established can be rephrased in the following continuous inclusions:

Corollary 2. *For all $s \in \mathbb{R}$, $T^s \subset U^s$ and for all $s > \frac{1}{2}$, $U^s \subset T^{s-1}$ with continuous inclusions.*

One immediate consequence of the previous result is that $T^\infty = U^\infty$. Moreover, there holds

Lemma 4.

$$T^\infty = C^\infty([-1,1])\,.$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (1), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^{1} \frac{(\omega\partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, the function $(\omega\partial_x)^{2k}u$ is C^∞, and since $\|T_n\|_\infty = 1$, the integral is bounded independently of n. Thus, the coefficients $\hat{u}_n$ have a fast decay, proving that $C^\infty([-1,1]) \subset T^\infty$.

For the converse inclusion, if $u \in T^\infty$, the series

$$u(x) = \sum_{n=0} \hat{u}_n T_n(x)$$

is normally converging since $\|T_n\|_\infty = 1$, so u is a continuous function. This proves $T^\infty \subset C^0([-1,1])$. It suffices to show that $\partial_x u \in T^\infty$ and apply an induction argument. Applying term by term differentiation, since $\partial_x T_n = nU_{n-1}$ for all n (with the convention $U_{-1} = 0$),

$$\partial_x u(x) = \sum_{n=1}^{+\infty} n\hat{u}_n U_{n-1}(x).$$

Therefore, $\partial_x u$ is in $U^\infty = T^\infty$ which proves the result. $\square$

Lemma 5. *For $s \leq \frac{1}{2}$, the functions of U^s cannot be identified to functions in $T^{-\infty}$.*

Proof. Let $s \leq \frac{1}{2}$, and let us assume by contradiction that the functions of U^s can be identified to elements of $T^{-\infty}$. Then, there must exist a continuous map I from U^s to $T^{-\infty}$ with the property

$$\forall u \in C^\infty([-1,1]), \quad Iu = u.$$

We introduce the function u defined by $\check{u}_n = \frac{1}{n\ln(n)}$. One can check that $u \in U^{\frac{1}{2}} \subset U^s$, thus Iu must be element of $T^{-\infty}$. For all N, the function

$$u_N = \sum_{n=0}^{N} \check{u}_n U_n$$

is in U^∞ and $(u_N)_{N\in\mathbb{N}}$ converges to u in U^s. By continuity of I, the sequence $(\langle Iu_N, T_0\rangle_{\frac{1}{\omega}})_{N\in\mathbb{N}}$ must converge with limit $\langle Iu, T_0\rangle_{\frac{1}{\omega}}$. But since $Iu_N = u_N$,

$$\langle Iu_N, T_0\rangle_{\frac{1}{\omega}} = \langle u_N, T_0\rangle_{\frac{1}{\omega}} = \sum_{n=0}^{N} \check{u}_n \langle U_n, T_0\rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2}\rfloor} \frac{1}{2k\ln(2k)}\,.$$

This sum diverges to $+\infty$ when N goes to infinity, giving the contradiction. □

Derivation operators. We now extend the definition of the derivation operators ∂_x and $\omega\partial_x\omega$ appearing in eqs. (3) and (4).

Lemma 6. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s defined by*

$$\forall v \in C^\infty([-1,1]), \quad \langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}}\,.$$

In a similar fashion, the operator $\omega\partial_x\omega$ can be extended into a continuous map from U^{s+1} to T^s defined by

$$\forall v \in C^\infty([-1,1]), \quad \langle \omega\partial_x\omega u, v\rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v\rangle_\omega\,.$$

Proof. Using eqs. (3) and (4), one can check that the formulas indeed extend the usual definition of the two operators for smooth functions. We now show that the map ∂_x extended this way is continuous from T^{s+1} to U^s. The definition

$$\forall v \in U^\infty, \langle \partial_x u, v\rangle_\omega := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}}$$

gives a sense to $\partial_x u$ for all u in $T^{-\infty}$, as a duality $T^{-\infty}\times T^\infty$ product, because if $v \in U^\infty (= C^\infty([-1,1])$, then $\omega\partial_x\omega v = (1-x^2)v' - xv$ also lies in $C^\infty([-1,1]) (= T^\infty)$. Letting $w = \partial_x u$, we have by definition for all n

$$\check{w}_n = \langle w, U_n\rangle_\omega = -\langle u, \omega\partial_x\omega U_n\rangle_{\frac{1}{\omega}} = n\langle u, T_{n+1}\rangle_{\frac{1}{\omega}} = n\hat{u}_{n+1}\,.$$

Obviously, this implies the announced continuity with

$$\|w\|_{U^s} \le \|u\|_{T^{s+1}}\,.$$

The properties of $\omega\partial_x\omega$ on T^s are established similarly. □

Corollary 3. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$. On the other hand, $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s and from U^{s+1} to U^s for all $s \in \mathbb{R}$.*

Proof. For the continuity of ∂_x from T^{s+2} to T^s, we use the continuity of ∂_x from T^{s+2} to U^{s+1} and then of the identity from U^{s+1} to T^s. For the continuity of ∂_x from U^{s+2} to U^s, we use the same arguments in reverse order.
On the other hand, we have, for $n \ge 2$,

$$\omega\partial_x\omega T_n = \omega\partial_x\omega\frac{U_n - U_{n-2}}{2} = \frac{(n+1)T_{n+1} - (n-1)T_{n-1}}{2}\,.$$

Therefore $\omega\partial_x\omega$ is continuous from T^{s+1} to T^s. Finally, $\omega\partial_x\omega$ is continuous from U^{s+1} to T^s and the inclusion $T^s \subset U^s$ is continuous thus $\omega\partial_x\omega$ is continuous from U^{s+1} to U^s. □

Lemma 7. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and*

$$\exists C : \forall x \in [-1,1], \quad |u(x)| \leq C \|u\|_{T^{1/2+\varepsilon}} .$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$\exists C : \forall x \in [-1,1], \quad |u(x)| \leq C \|u\|_{U^{3/2+\varepsilon}} .$$

Proof. Let $x \in [-1,1]$. Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\|T_n\|_{L^\infty} = 1$. Applying Cauchy-Schwarz's inequality, one gets

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \|u\|_{T^{\frac{1}{2}+\varepsilon}} .$$

The second statement is deduced from the first and the continuous inclusion $U^s \subset T^{s-1}$ stated in Corollary 2. □

1.3 Equivalent norms on T^n and U^n

We now provide a characterization of the spaces T^n and U^n in terms of weighted L^2 norms of the derivatives and give equivalent norms on those spaces when n is an integer.

Lemma 8. *The operator ω is a bijective isometry from U^0 to T^0 with inverse $\frac{1}{\omega}$.*

Proof. This result follows from

$$\|\omega u\|_{\frac{1}{\omega}}^2 = \frac{1}{\pi} \int_{-1}^{1} \frac{|(\omega u)|^2}{\omega} = \frac{1}{\pi} \int_{-1}^{1} \omega |u|^2 = \|u\|_{\omega}^2 ,$$

valid for all $u \in L^2_\omega$. □

Definition 3. *For an even integer n, the operator $(\omega\partial_x)^n : T^{-\infty} \to T^{-\infty}$ is defined by*

$$(\omega\partial_x)^0 = I_d, \quad \forall k > 0, \ (\omega\partial_x)^{2k} := (\omega\partial_x\omega)\partial_x(\omega\partial_x)^{2k-2} .$$

The operator $(\partial_x\omega)^n : U^{-\infty} \to U^{-\infty}$ is defined in an analogous way.

Lemma 9. *Let n an even integer. For all $s \in \mathbb{R}$, $(\omega\partial_x)^n$ is continuous from T^s to T^{s-n} and $(\partial_x\omega)^n$ is continuous from U^s to U^{s-n}.*

Proof. Those results follow from the definition of the operators and by induction using the mapping properties of ∂_x and $\omega\partial_x\omega$ established in Lemma 6. □

Definition 4. *For an odd integer n, the operator $(\omega\partial_x)^n : T^n \to T^0$ is defined by*

$$(\omega\partial_x)^n := \omega\partial_x(\omega\partial_x)^{n-1}\,.$$

The operator $(\partial_x\omega)^n : T^n \to T^0$ is defined in an analogous way.

From Lemma 8, we deduce

Corollary 4. *The operators $(\omega\partial_x)^n$ and $(\partial_x\omega)^n$ are well defined and continuous respectively from T^n to T^0 and from U^n to U^0.*

Lemma 10. *Let $n \in \mathbb{N}$. If n is even,*

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; (\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}} \right\}\,.$$

If n is odd,

$$T^n = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \partial_x(\omega\partial_x)^{n-1} u \in L^2_{\omega} \right\}\,.$$

Moreover $u \mapsto \sqrt{\|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}}}$ defines an equivalent norm on T^n, and for all $u \in T^n$,

$$|u|_{T^n} = \|(\omega\partial_x)^n u\|_{L^2_{\frac{1}{\omega}}}\,.$$

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 9 and Corollary 4. For the converse inclusions, let u in $L^2_{\frac{1}{\omega}}$. If n is even, say $n = 2k$, the assumption is that $(\omega\partial_x)^n u \in L^2_{\frac{1}{\omega}}$. The Fourier-Chebyshev coefficients of $a = (\omega\partial_x)^n u$ are given for $j > 0$ by

$$\hat{a}_j = \frac{\left((\omega\partial_x)^{2k}u(x), T_j\right)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}} = \frac{\left(u(x), (\omega\partial_x)^{2k}T_j\right)_{\frac{1}{\omega}}}{(T_n, T_n)_{\frac{1}{\omega}}} = (-1)^k j^{2k}\hat{u}_j\,.$$

while for $j = 0$, $\hat{a}_j = 0$. Applying Parseval's equality to the function a, this gives

$$\frac{1}{2}\sum_{j>0} j^{2n}\left|\hat{u}_j\right|^2 = \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}}\,. \tag{12}$$

On the other hand, if n is odd, say $n = 2k+1$, let $b := \partial_x(\omega\partial_x)^{2k}u$. The assumption is now that $b \in L^2_\omega$, and by Lemma 8, $\omega b\,(= (\omega\partial_x)^n u) \in T^0$ with

$$\|\omega b\|_{\frac{1}{\omega}} = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}} = \|b\|_\omega\,.$$

One can write

$$\check{b}_j = 2\left(\partial_x(\omega\partial_x)^{2k}u, U_j\right) = -2\left(u, (\omega\partial_x)^{2k}(\omega\partial_x\omega)U_j\right)\,.$$

Using $-\omega\partial_x\omega U_j = (j+1)T_{j+1}$, we obtain

$$\check{b}_j = (-1)^k (j+1)^{2k+1}\hat{u}_{j+1}\,.$$

Parseval's equality then implies that (12) also hold for odd n. This establishes that $u \in T^n$ and $|u|_{T^n} = \|(\omega\partial_x)^n u\|_{\frac{1}{\omega}}$. For the norm equivalence, adding the Parseval equality for $u \in L^2_{\frac{1}{\omega}}$ to eq. (12), we get

$$|\hat{u}_0|^2 + \frac{1}{2}\sum_{j>0}(1+j^{2n})\,|\hat{u}_j|^2 = \|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}} \,. \tag{13}$$

There are two constants c and C such that $c(1+j^2)^n \leq (1+j^{2n}) \leq C(1+j^2)^n$. Injecting this in (13), we obtain

$$\frac{c}{2}\,\|u\|^2_{T^n} \leq \|u\|^2_{\frac{1}{\omega}} + \|(\omega\partial_x)^n u\|^2_{\frac{1}{\omega}} \leq C\,\|u\|^2_{T^n} \ ,$$

and the equivalence of the norms follows. □

Lemma 11. *Let $n \in \mathbb{N}$. If n is even, then*

$$U^n = \left\{u \in L^2_\omega \;\middle|\; (\partial_x\omega)^n u \in L^2_\omega\right\} \,.$$

If n is odd, then

$$U^n = \left\{u \in L^2_\omega \;\middle|\; \omega\partial_x\omega(\partial_x\omega)^{n-1}u \in L^2_{\frac{1}{\omega}}\right\} \,.$$

Moreover, $u \mapsto \sqrt{\int_{-1}^{1}\omega|(\partial_x\omega)^n u|^2}$ defines an equivalent norm on U^n.

Proof. The direct inclusions follow from the mapping properties established in Lemma 6, Lemma 9 and Corollary 4. For the converse inclusion, if n is even, let $a = (\partial_x\omega)^n u$, we assume that $a \in L^2_\omega$. One has

$$\check{a}_j = (-1)^k\,(1+j)^n\,\check{u}_j$$

so by Parseval's equality,

$$\frac{1}{2}\sum_{j=0}^{+\infty}(j+1)^{2n}\,|\check{u}_j|^2 = \|(\partial_x\omega)^n u\|^2_\omega \,. \tag{14}$$

If n is odd, the assumption is that $b = \omega\partial_x\omega(\partial_x\omega)^{n-1}u$ is in $L^2_{\frac{1}{\omega}}$. By calculations similar to those in the proof of the preceding lemma, we find that for $j > 0$,

$$\hat{b}_j = j^{2n}\check{u}_{j-1}\,.$$

while $\hat{b}_0 = 0$. By Lemma 8, $\frac{b}{\omega}\,(= (\partial_x\omega)^n u) \in U^0$. Applying Parseval's equality to b in $L^2_{\frac{1}{\omega}}$ and using $\left\|\frac{b}{\omega}\right\|_\omega = \|b\|_{\frac{1}{\omega}}$, we find that (14) also holds when n is odd, and thus the inclusion is proved. Finally, there exists two constants c and C such that for all $j \in \mathbb{N}$,

$$c(1+(j+1))^{2n} \leq (j+1)^{2n} \leq C(1+(j+1))^{2n}.$$

This implies the equivalence of the norms. □

1.4 Link with Periodic Sobolev spaces

We briefly recall here the definition of the periodic Sobolev spaces on the torus $\mathbb{T}_{2\pi} := \mathbb{R}/2\pi\mathbb{Z}$. A smooth function u on $\mathbb{T}_{2\pi}$ can be decomposed in Fourier series

$$u(\theta) = \sum_{n\in\mathbb{Z}} \mathcal{F}u(n)e^{in\theta}$$

with the Fourier coefficients defined by

$$\mathcal{F}u(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\theta)e^{-in\theta}d\theta.$$

For $n \in \mathbb{Z}$, let $e_n : \theta \mapsto e^{in\theta}$. We define the Fourier coefficients of any periodic distribution u on $\mathbb{T}_{2\pi}$, by $\mathcal{F}u(n) := u(e_{-n})$. For all s, the space H^s is the set of periodic distributions on $\mathbb{T}_{2\pi}$ for which

$$\|u\|_{H^s}^2 := \sum_{n\in\mathbb{Z}}(1+n^2)^s \left|\mathcal{F}u(n)\right|^2 < +\infty\,.$$

Introducing the duality product

$$\langle u, v\rangle_{\mathbb{T}_{2\pi}} = \sum_{n\in\mathbb{Z}} \mathcal{F}u(n)\mathcal{F}v(-n)\,, \tag{15}$$

H^s is identified to the dual of H^{-s} and $H^0 = L^2(\mathbb{T}_{2\pi})$. For $u, v \in H^0$, $\langle u, v\rangle_{\mathbb{T}_{2\pi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} uv$. The space H^s is the direct sum $H^s_e + H^s_o$ where

$$H^s_e := \left\{u \in H^s \mid \mathcal{F}u(n) = \mathcal{F}u(-n)\right\}\,,$$

$$H^s_o := \left\{u \in H^s \mid \mathcal{F}u(n) = -\mathcal{F}u(-n)\right\}\,.$$

Note that when u is continuous,

$$u \in H^s_e \iff \forall\theta \in \mathbb{T}_{2\pi},\quad u(-\theta) = u(\theta)\,,$$

$$u \in H^s_o \iff \forall\theta \in \mathbb{T}_{2\pi},\quad u(-\theta) = -u(\theta)\,.$$

Definition 5. *We define the operators $\mathcal{C} : T^{-\infty} \to H^{-\infty}_e$ by*

$$\forall n \in \mathbb{Z},\quad \mathcal{F}(\mathcal{C}u)(n) = \begin{cases} \hat{u}_0 & \text{if } n = 0,\\ \frac{\hat{u}_{|n|}}{2} & \text{otherwise}\end{cases}$$

and $\mathcal{S} : U^{-\infty} \to H^{-\infty}_o$ by

$$\forall n \in \mathbb{Z},\quad \mathcal{F}(\mathcal{S}u)(n) = \begin{cases} 0 & \text{if } n = 0,\\ \operatorname{sign}(n)\frac{\hat{u}_{|n|-1}}{2} & \text{otherwise.}\end{cases}$$

Lemma 12. *The operators $\mathcal{C}$ and $\mathcal{S}$ map smooth functions to smooth functions. For all $(u, v) \in T^{-\infty} \times T^{\infty}$,*

$$\langle u, v\rangle_{\frac{1}{\omega}} = \langle \mathcal{C}u, \mathcal{C}v\rangle_{\mathbb{T}_{2\pi}}\,.$$

For all $(u, v) \in U^{-\infty} \times U^{\infty}$,

$$\langle u, v\rangle_{\omega} = \langle \mathcal{S}u, \mathcal{S}v\rangle_{\mathbb{T}_{2\pi}}$$

Proof. The first assertion is obvious from the definition of $\mathcal{C}$ and $\mathcal{S}$. Let $(u,v) \in T^{-\infty} \times T^{\infty}$. By definition of $\langle \cdot,\cdot \rangle_{\frac{1}{\omega}}$ and $\langle \cdot,\cdot \rangle_{\mathbb{T}_{2\pi}}$ eqs. (8) and (15),

$$\begin{aligned}
\langle \mathcal{C}u, \mathcal{C}v \rangle_{\mathbb{T}_{2\pi}} &= \sum_{n\in\mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n)\mathcal{F}(\mathcal{C}v)(-n) \\
&= \hat{u}_0\hat{v}_0 + \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\hat{u}_{|n|}}{2}\frac{\hat{v}_{|n|}}{2} \\
&= \hat{u}_0\hat{v}_0 + \frac{1}{2}\sum_{n=1}^{+\infty} \hat{u}_n\hat{v}_n \\
&= \langle u, v \rangle_{\frac{1}{\omega}} \,.
\end{aligned}.$$

The second identity is proved similarly. □

Lemma 13. *For all $s \in \mathbb{R}$, the operators $\mathcal{C}$ and $\mathcal{S}$ induce bijective isometries respectively from T^s to H^s_e and from U^s to H^s_o. For $u \in C^\infty([-1,1])$,*

$$\mathcal{C}u(\theta) = u(\cos\theta) \quad \text{and} \quad \mathcal{S}u(\theta) = \sin\theta u(\cos\theta)\,. \tag{16}$$

Let $v, w \in C^\infty(\mathbb{T}_{2\pi})$, an even and an odd function respectively. Then

$$\mathcal{C}^{-1}v(x) = v(\arccos x) \quad \text{and} \quad \mathcal{S}^{-1}w(x) = \frac{w(\arccos x)}{\omega(x)}\,. \tag{17}$$

Proof. Let J^T_s, J^U_s and $\tilde{J}_s$ the linear continuous mappings defined respectively on $T^{-\infty}$, $U^{-\infty}$ and $H^{-\infty}$ by

$$J^T_s T_n = (1+n^2)^{\frac{s}{2}} T_n, \quad J^U_s U_{n-1} = (1+n^2)^{\frac{s}{2}} U_{n-1}, \quad \tilde{J}_s e_n = (1+n^2)^{\frac{s}{2}} e_n.$$

We recall that e_n is the function $\theta \mapsto e^{in\theta}$. One can check easily that for $u \in T^s$ and $v \in U^s$

$$\|u\|^2_{T^s} = \left\langle J^T_s u, \overline{J^T_s u} \right\rangle_{\frac{1}{\omega}} \quad \text{and} \quad \|v\|^2_{U^s} = \left\langle J^U_s v, \overline{J^U_s v} \right\rangle_{\omega},$$

while for $w \in H^s$,

$$\|w\|^2_{H^s} = \|u\|^2_{T^s} = \left\langle \tilde{J}_s u, \overline{\tilde{J}_s u} \right\rangle_{\mathbb{T}_{2\pi}}\,.$$

Moreover, the following identities hold:

$$\mathcal{C}J^T_s = \tilde{J}_s\mathcal{C}, \quad \mathcal{S}J^U_s = \tilde{J}_s\mathcal{S}\,.$$

The isometric property of $\mathcal{C}$ may now be deduced from Lemma 12 as follows. Let $u_N = \sum_{n=0}^N u_n T_n$. There holds

$$\begin{aligned}
\left\langle J^T_s u, \overline{J^T_s u_N} \right\rangle_{\frac{1}{\omega}} &= \left\langle \mathcal{C}J^T_s u, \overline{\mathcal{C}J^T_s u_N} \right\rangle_{\mathbb{T}_{2\pi}} \\
&= \left\langle \tilde{J}_s\mathcal{C}u, \overline{\tilde{J}_s\mathcal{C}u_N} \right\rangle_{\mathbb{T}_{2\pi}}\,.
\end{aligned}$$

Sending N to infinity, by continuity of J^T_s, $\tilde{J}^s$ and $\mathcal{C}$, this yields

$$\|u\|^2_{T^s} = \|\mathcal{C}u\|^2_{H^s}\,.$$

The isometric property of $\mathcal{S}$ is establish in a similar manner. Let us now prove (16). For the first identity, consider some smooth function u on $[-1,1]$. Since $\mathcal{C}u$ is smooth, the Fourier series of $\mathcal{C}u$ converges pointwise to $\mathcal{C}u$. Thus, for all $\theta \in \mathbb{T}_{2\pi}$,

$$\begin{aligned}
\mathcal{C}u(\theta) &= \sum_{n\in\mathbb{Z}} \mathcal{F}(\mathcal{C}u)(n)e^{in\theta} \\
&= \hat{u}_0 + \sum_{n\in\mathbb{Z},n\neq 0} \frac{\hat{u}_{|n|}}{2} e^{in\theta} \\
&= \hat{u}_0 + \frac{1}{2}\sum_{n=1}^{+\infty} \hat{u}_n \left(e^{in\theta} + e^{-in\theta}\right) \\
&= \sum_{n=0}^{+\infty} \hat{u}_n \cos(n\theta) \\
&= \sum_{n=0}^{+\infty} \hat{u}_n T_n(\cos\theta)
\end{aligned}$$

The last sum also converges pointwise to $u(\cos\theta)$ since $u \in T^\infty$. Similar calculations show that $\mathcal{S}u(\theta) = \sin\theta u(\cos\theta)$, using this time

$$\sin((n+1)\theta) = \sin\theta U_n(\cos\theta)\,.$$

To prove the bijectivity of $\mathcal{S}$ and $\mathcal{C}$, one can check that they have the explicit inverses $\mathcal{C}^{-1}$ and $\mathcal{S}^{-1}$ respectively defined on H^s_e and H^s_o as

$$\forall n \in \mathbb{N}, \quad \left(\widehat{\mathcal{C}^{-1}u}\right)_n = \begin{cases} \mathcal{F}u(0) & \text{if } n = 0, \\ 2\mathcal{F}u(n) & \text{otherwise} \end{cases}$$

and

$$\forall n \in \mathbb{N}, \quad \left(\widehat{\mathcal{S}^{-1}u}\right)_n = 2\mathcal{F}u(n+1)\,.$$

Finally, identities (17) are simply deduced by inverting (16). □

1.5 Generalization to a curve

Parametrization of the curve

We start by introducing some notation that will be extensively used throughout all the remainder of this work. Let Γ a smooth open curve in $\mathbb{R}^2$ parametrized by a smooth C^∞ diffeomorphism $r : [-1,1] \to \Gamma$. We assume that $|r'(x)| = \frac{|\Gamma|}{2}$ for all $x \in [-1,1]$, where $|\Gamma|$ is the length of Γ. This parametrization is related to the curvilinear abscissa $M(s)$ through

$$r(x) = M\left(\frac{|\Gamma|}{2}(1+x)\right)\,.$$

Let $R : C^\infty(\Gamma) \longrightarrow C^\infty(-1,1)$ defined by

$$Ru(x) = u(r(x))\,.$$

The tangent and normal vectors on the curve, τ and n, are respectively defined by

$$\tau(x) = \frac{\partial_x r(x)}{|\partial_x r(x)|}, \quad n(x) = \frac{\partial_x \tau(x)}{|\partial_x \tau'(x)|}\,.$$

Let $N : \Gamma \to \mathbb{R}^2$ such that $N(r(x)) = n(x)$, that is, $N = R^{-1}n$. Let $\kappa(x)$ the signed curvature of Γ at the point $r(x)$. Frenet-Serret's formulas give

$$\begin{aligned} r(y) &= r(x) + (y-x)\frac{|\Gamma|}{2}\tau(x) + \frac{(y-x)^2}{2}\frac{|\Gamma|^2}{4}\kappa(x)n(x) \\ &\quad + \frac{(x-y)^3}{6}\frac{|\Gamma|^3}{8}(\kappa'(x)n(x) - \kappa(x)^2\tau(x)) + O\left((x-y)^4\right)\,, \end{aligned}$$

so that

$$|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4}(y-x)^2 - \frac{(y-x)^4}{192}\,|\Gamma|^4\,\kappa(x)^2 + O(x-y)^5\,. \tag{18}$$

For $u, v \in L^2(\Gamma)$, we have by change of variables in the integral

$$\langle u, v\rangle_{L^2(\Gamma)} = \frac{|\Gamma|}{2}\,\langle Ru, Rv\rangle_{L^2(-1,1)}\,.$$

The tangential derivative ∂_τ on Γ satisfies

$$\partial_\tau = \frac{2}{|\Gamma|}R^{-1}\partial_x R\,. \tag{19}$$

Moreover, we define

$$\omega_\Gamma := \frac{|\Gamma|}{2}R^{-1}\omega R \tag{20}$$

the "weight" on the curve Γ. Finally, the uniform measure on Γ is denoted by $d\sigma$.

Spaces $T^s(\Gamma)$ and $U^s(\Gamma)$

The definition of the spaces T^s can be transported on the curve Γ, replacing the basis (T_n) and (U_n) by $(R^{-1}T_n)$ and $(R^{-1}U_n)$. The spaces $T^s(\Gamma)$ and $U^s(\Gamma)$ are thus defined as the sets of formal series respectively of the form

$$u = \sum_{n\in\mathbb{N}} \hat{u}_n R^{-1}T_n\,, \quad v = \sum_{n\in\mathbb{N}} \check{v}_n R^{-1}T_n\,,$$

where $Ru = \sum \hat{u}_n T_n \in T^s$ and $Rv = \sum \check{v}_n U_n \in U^s$. To u and v are associated the linear forms

$$\forall \varphi \in C^\infty(\overline{\Gamma}), \quad \langle u, \varphi\rangle_{\frac{1}{\omega_\Gamma}} := \langle Ru, R\varphi\rangle_{\frac{1}{\omega}}\,,$$

$$\forall \varphi \in C^\infty(\overline{\Gamma}), \quad \langle v, \varphi\rangle_{\omega_\Gamma} := \frac{|\Gamma|^2}{4}\,\langle Rv, R\varphi\rangle_{\omega}\,.$$

From the results of the previous section we deduce

Lemma 14. *For all $s \in \mathbb{R}$, $T^s(\Gamma)$ and $U^s(\Gamma)$ are Hilbert spaces for the scalar products*

$$(u,v)_{T^s(\Gamma)} = (Ru, Rv)_{T^s}\,,$$

$$(u,v)_{U^s(\Gamma)} = \frac{|\Gamma|^2}{2}(Ru, Rv)_{U^s}\,.$$

With these definitions,

$$(u,v)_{T^0(\Gamma)} = \langle u, \overline{v}\rangle_{\frac{1}{\omega_\Gamma}} = \int_\Gamma \frac{u(x)\overline{v(x)}}{\omega_\Gamma(x)}dx\,,$$

$$(u,v)_{U^0(\Gamma)} = \langle u, \overline{v}\rangle_{\omega_\Gamma} = \int_\Gamma \omega_\Gamma(x)u(x)\overline{v(x)}dx\,.$$

In particular $T^0(\Gamma) = L^2_{\frac{1}{\omega_\Gamma}}$ and $U^0(\Gamma) = L^2_{\omega_\Gamma}$. For $s \in \mathbb{R}$, the dual of $T^s(\Gamma)$ is the set of linear forms $\langle u, \cdot\rangle_{\frac{1}{\omega_\Gamma}}$ where $u \in T^{-s}(\Gamma)$, and the dual of $U^s(\Gamma)$ is the set of linear forms $\langle u, \cdot\rangle_{\omega_\Gamma}$ where $u \in U^{-s}(\Gamma)$. For $s < t$, the injections $T^t(\Gamma) \subset T^s(\Gamma)$ and $U^t(\Gamma) \subset U^s(\Gamma)$ are compact. $(T^s(\Gamma))_{s\in\mathbb{R}}$ and $(U^s(\Gamma))_{s\in\mathbb{R}}$ are two Hilbert interpolation scales. For an integer n, equivalent scalar products on T^n and U^n are given respectively by

$$(u,v) \mapsto \int_\Gamma \frac{u(x)\overline{v(x)} + (\omega_\Gamma\partial_\tau)^n u(x)(\omega_\Gamma\partial_\tau)^n\overline{v(x)}}{\omega_\Gamma(x)}d\sigma(x)\,,$$

$$(u,v) \mapsto \int_\Gamma (\partial_\tau\omega_\Gamma)^n u(x)(\partial_\tau\omega_\Gamma\partial)^n\overline{v(x)}\omega_\Gamma(x)d\sigma(x)\,,$$

For all $s \in \mathbb{R}$, $T^s(\Gamma) \subset U^s(\Gamma)$ and for all $s > \frac{1}{2}$, $U^s(\Gamma) \subset T^{s-1}(\Gamma)$ with continuous inclusions, as well as for $\varepsilon > 0$, $T^{1/2+\varepsilon}(\Gamma) \subset C^0(\Gamma)$ and $U^{3/2+\varepsilon} \subset C^0(\Gamma)$. Moreover, $T^\infty(\Gamma) = U^\infty(\Gamma) = C^\infty(\overline{\Gamma})$.

2 Application to Galerkin analysis

In this section, we introduce the notations for the first-kind integral equations under consideration. We then apply the theory of the first section to this problem in the case of a zero wavenumber and flat geometry.

2.1 First-kind integral equations

Recall the definition and parametrization of the curve Γ detailed in section 1.5. We consider the following boundary integral equations (BIEs)

$$S_k\lambda = u_D, \quad N_k\mu = u_N \tag{21}$$

where S_k and N_k are respectively the single-layer and hypersingular operators. We refer the reader to [1] and references therein for more details on the classical connection between eqs. (21) and the problem of wave scattering by the curve Γ. The operators S_k and N_k admit the integral representations

$$\begin{aligned}(S_k\lambda)(x) &= \int_\Gamma G_k(x-y)\lambda(y)d\sigma(y)\,,\\ (N_k\mu)(x) &= \lim_{\varepsilon\to 0^+}\int_\Gamma N(y)\cdot\nabla G_k(x+\varepsilon N(x)-y)\mu(y)d\sigma_y.\end{aligned} \tag{22}$$

for $x\in\Gamma$, with the Green function G_k defined by

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi}\ln|z|\,, & \text{if } k=0,\\ G_k(z) = \dfrac{i}{4}H_0(k|z|), & \text{if } k>0,\end{cases} \tag{23}$$

where H_0 is the Hankel function of the first kind. It is known that S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ [7, Theorem 1.8] and N_k maps continuously $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ [7, Theorem 1.4]. In the case $k=0$, the Helmholtz scattering reduces to the Laplace problem. The kernel of the hypersingular operator has a non-integrable singularity, but computations are facilitated by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned}\langle N_k\mu,\nu\rangle \quad = \quad & \int_{\Gamma\times\Gamma} G_k(x-y)\mu'(x)\nu'(y)\\ & \quad -k^2G_k(x,y)\mu(x)\nu(y)n(x)\cdot n(y)d\sigma_x d\sigma_y\,.\end{aligned} \tag{24}$$

For the geometry under consideration, the solutions λ and μ of the BIEs (21) have singularities (even for C^∞ data u_D and u_N) due to the presence of edges on the scatterer. It is now classical to introduce weighted versions of the usual layer potentials, known to enjoy better mapping properties than S_k and N_k. Namely, we define

$$S_{k,\omega_\Gamma} := S_k\frac{1}{\omega_\Gamma}\,,\quad N_{k,\omega_\Gamma} := N_k\omega_\Gamma\,, \tag{25}$$

and recast the BIEs (21) as

$$S_{k,\omega_\Gamma}\alpha = u_D\,,\quad N_{k,\omega_\Gamma}\beta = u_N\,. \tag{26}$$

where the unknowns α and β are related to λ and μ by

$$\lambda = \frac{\alpha}{\omega_\Gamma},\quad \mu = \omega_\Gamma\beta\,.$$

The relation between N_k and S_k can be rewritten in terms of N_{k,ω_Γ} and S_{k,ω_Γ}:

Lemma 15. *There holds*

$$N_{k,\omega_\Gamma} = -\partial_\tau S_{k,\omega_\Gamma}\omega_\Gamma\partial_\tau\omega_\Gamma - k^2V_k\omega_\Gamma^2$$

where V_k is the integral operator defined by

$$V_k u = \int_\Gamma \frac{G_k(x-y)N(x)\cdot N(y)u(y)}{\omega_\Gamma(y)}d\sigma(y)\,.$$

Proof. Eq. (24) can be rewritten equivalently as

$$N_k u = -\partial_\tau S_k \partial_\tau u - k^2 \int_\Gamma G_k(x-y) N(x) \cdot N(y) u(y) d\sigma(y)\,.$$

Using the definitions of N_{k,ω_Γ} and S_{k,ω_Γ}, the results follow from simple manipulations on this expression. $\square$

2.2 Weighted layer potentials on the flat segment

In this section, we consider the case where the wavenumber k is equal to 0 and $\Gamma = [-1,1] \times \{0\}$. The parametrization r is then the constant function equal to 1, $\partial_\tau = \partial_x$ and $\omega_\Gamma = \omega$. In this simple context, the weighted potentials are thus denoted by $S_{0,\omega}$ and $N_{0,\omega}$. They have elementary properties that allow us to characterize T^s and U^s for $s = \pm\frac{1}{2}$.

Single layer potential. The operator $S_{0,\omega}$ takes the form

$$S_{0,\omega}\alpha(x) = \int_{-1}^{1} \frac{\ln|x-y|\,\alpha(y)}{\sqrt{1-y^2}} dy\,.$$

There holds

$$S_{0,\omega} T_n = \sigma_n T_n \tag{27}$$

where

$$\sigma_n = \begin{cases} \dfrac{\ln(2)}{2} & \text{if } n = 0 \\ \dfrac{1}{2n} & \text{otherwise.} \end{cases}$$

Those identities are fundamental in our analysis. A proof can be found in [4, Theorem 9.2]. We deduce easily

Lemma 16. *The operator $S_{0,\omega}$ is a positive bicontinuous bijection from T^s to T^{s+1} for all $s \in \mathbb{R}$.*

In particular, $S_{0,\omega}$ maps T^∞ to itself, so the image of a smooth function by $S_{0,\omega}$ is a smooth function. We now proceed to show the following characterization of $T^{-1/2}$ and $T^{1/2}$. The next result, and Lemma 19 stated below are equivalent to results formulated in [2] (see equations (4.77-4.86), and Propositions 3.1 and 3.3 therein).

Lemma 17. *We have $T^{-1/2} = \omega \tilde{H}^{-1/2}(-1,1)$ and for all $u \in \tilde{H}^{-1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}\,.$$

Moreover, $T^{1/2} = H^{1/2}(-1,1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{T^{1/2}}$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the (unweighted) single-layer operator S_0 is positive and bounded from below on $\tilde{H}^{-1/2}(-1,1)$, (see [6] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1,1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle S_0 u, u\rangle}.$$

On the other hand, the explicit expression (27) imply that if $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_{0,\omega}\alpha, \alpha\rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since $\alpha = \omega u$, $\langle S_{0,\omega}\alpha, \alpha\rangle_{\frac{1}{\omega}} = \langle S_0 u, u\rangle$. This proves the first result. For the second result, we know that, $(H^{1/2}(-1,1))' = \tilde{H}^{-1/2}(-1,1)$ (taking the identification with respect to the usual L^2 duality denoted by $\langle\cdot,\cdot\rangle$, [5] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v\neq 0} \frac{\langle u, v\rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}}.$$

According to the previous result, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v\rangle = \langle u, \alpha\rangle_\omega$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha\neq 0} \frac{\langle u, \alpha\rangle_{\frac{1}{\omega}}}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle\cdot,\cdot\rangle_{\frac{1}{\omega}}$, showing the result. □

Hypersingular operator. For $k = 0$ and when $\Gamma = [-1,1]\times\{0\}$, the identity (15) takes the form

$$\langle N_{0,\omega}\beta, \beta'\rangle_\omega = \langle S_{0,\omega}(\omega\partial_x\omega)\beta, (\omega\partial_x\omega)\beta'\rangle_{\frac{1}{\omega}}.$$

Noticing that $(\omega\partial_x\omega)U_n = -(n+1)T_{n+1}$, we have, for all $n \neq m$,

$$\langle N_{0,\omega}U_n, U_m\rangle_\omega = 0.$$

Therefore, we have

$$N_{0,\omega}U_n = \nu_n U_n$$

with $\nu_n \|U_n\|_\omega^2 = (n+1)^2\sigma_{n+1}\|T_{n+1}\|_{\frac{1}{\omega}}^2$, that is, $\nu_n = \frac{(n+1)}{2}$. We deduce

Lemma 18. *The operator $N_{0,\omega}$ is a positive bicontinuous bijection from U^s to U^{s-1} for all $s \in \mathbb{R}$.*

In particular, $N_{0,\omega}$ maps smooth functions to smooth functions. As before, we obtain a characterization of U^s for $s = \pm\frac{1}{2}$ from the previous formula.

Lemma 19. *We have $U^{1/2} = \frac{1}{\omega}\tilde{H}^{1/2}(-1,1)$ and for all $u \in \tilde{H}^{1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{1/2}} \sim \left\|\frac{u}{\omega}\right\|_{U^{1/2}}.$$

Moreover, $U^{-1/2} = H^{1/2}(-1,1)$ and

$$\|u\|_{H^{1/2}} = \|u\|_{U^{1/2}}.$$

2.3 Galerkin method

A Galerkin method based on a refined mesh and weighted L^2 scalar products is described in [1] to solve eqs. (26) and orders of convergences are announced for the Laplace problem ($k = 0$) when $\Gamma = [-1, 1] \times \{0\}$. In this section we provide the proofs for those statements. Let us consider the following discretization of the segment $[-1, 1]$

$$-1 = x_0 < x_1 < \cdots < x_N = 1$$

where $x_i = \cos(i\frac{\pi}{N})$.

2.4 Dirichlet problem

Let V_h the Galerkin space of (discontinuous) piecewise affine functions on the mesh $(x_i)_{0\leq i\leq N}$ defined above, and α_h the unique solution in V_h to the variational problem

$$(S_{0,\omega}\alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (u_D, \alpha'_h)_{\frac{1}{\omega}}\,, \quad \forall \alpha'_h \in V_h\,. \tag{28}$$

Let $\lambda_h = \frac{\alpha_h}{\omega}$. We also write $\alpha_0 := \omega\lambda$.

Theorem 1. *If the data u_D is in T^{s+1} for some $-\frac{1}{2} \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2}\, \|\omega\lambda\|_{T^s} \leq Ch^{s+1/2}\, \|u_D\|_{T^{s+1}}\,.$$

Note that since $\mathcal{C}S_0 = S_{0,\omega}\mathcal{C}$, which is proved using the change of variables $\theta = \arccos(x)$, the variational problem (28) is equivalent, by Lemma 13, to

$$(S_0\varphi_h, \varphi'_h)_{\mathbb{T}_{2\pi}} = (\mathcal{C}u_D, \varphi'_h)_{\mathbb{T}_{2\pi}}\,, \quad \forall \alpha'_h \in \mathcal{C}V_h\,,$$

where $\varphi_h = \mathcal{C}\alpha_h$ and $(u, v)_{\mathbb{T}_{2\pi}} = \int_{-\pi}^{\pi} u(\theta)\overline{v(\theta)}dx$ is the usual L^2 scalar product on $\mathbb{T}_{2\pi}$. If instead of affine functions, V_h would contain affine functions of $\arccos(x)$, then φ_h and the test functions φ'_h would be piecewise affine (discontinuous) functions of θ. The standard Galerkin theory then gives

$$\|\varphi_h - \varphi\|_{H^{-1/2}} \leq Ch^{s+\frac{1}{2}}\, \|\mathcal{C}u_D\|_{H^s}\,,$$

where $\varphi = \mathcal{C}\alpha$. This implies Theorem 1 since $\|\mathcal{C}u_D\|_{H^s} = \|u_D\|_{T^s}$ and

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} = \|\varphi - \varphi_h\|_{H^{-1/2}}\,.$$

Here instead, φ'_h are piecewise affine functions of $\cos\theta$ and to the best knowledge of the author, no approximation theory is available for this kind of basis functions in H^s_e. This is the main difficulty in the proof of Theorem 1.

Proof. Let us denote by Π_h the operator that maps a function $\alpha \in T^{-1/2}$ to the element $\alpha_h \in V_h$ such that

$$(S_{0,\omega}\alpha_h, \alpha'_h)_{\frac{1}{\omega}} = (S_{0,\omega}\alpha, \alpha'_h)_{\frac{1}{\omega}} \quad \forall \beta'_h \in W_h\,.$$

Since $S_{0,\omega}$ is coercive in $T^{-1/2}$ by Lemma 16, we have an analog of Céa's lemma in our context:

$$\forall \alpha \in T^{-1/2}, \quad \|\alpha - \Pi_h \alpha\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}} . \tag{29}$$

From this we deduce, taking $\alpha'_h = 0$ in the infimum

$$\forall \alpha \in T^{-1/2} \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C \|\alpha\|_{T^{-1/2}} .$$

In addition, we are going to show

$$\forall \alpha \in T^2, \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{5/2} \|\alpha\|_{T^2} . \tag{30}$$

By interpolation, this implies for all $s \in \left[-\frac{1}{2}, 2\right]$

$$\forall \alpha \in T^s , \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s} .$$

The result then follows from this, since, on the one hand

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq C \|\alpha_0 - \alpha_h\|_{T^{-1/2}}$$

by Lemma 17, with $\alpha_h = \Pi_h \alpha_0$ and, on the other hand, $\omega u = \alpha_0 = S_{0,\omega}^{-1} u_D$ which by Lemma 16, gives

$$\|\alpha_0\|_{T^s} \leq C \|u_D\|_{T^{s+1}} .$$

As it is classical with piecewise affine discontinuous basis functions, we prove (30) by studying the properties of two particular operators: the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h and the interpolation operator I_h which maps a continuous function α to the (continuous) function of V_h that matches α at the breakpoints x_i. Because of Céa's lemma, we have

$$\forall \alpha \in T^{-1/2}, \quad \|(I_d - \Pi_h)\alpha\|_{T^{-1/2}} \leq C \|(I_d - \mathbb{P}_h)\alpha\|_{T^{-1/2}} .$$

Therefore, it suffices to show (30) where Π_h is replaced by $\mathbb{P}_h$ to establish the theorem. We shall first show that

$$\forall \alpha \in T^s, \quad \|(I_d - \mathbb{P}_h)\alpha\| \leq Ch^s \|\alpha\|_{T^s} \tag{31}$$

for $s \in [0, 2]$. The estimate in $T^{-1/2}$ norm is then deduced by the classical duality method:

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}} ,$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

This, by Cauchy-Schwarz equality and (31), gives

$$\forall \alpha \in T^2, \quad \|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^2 \|\alpha\|_{T^2} \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} ,$$

implying the desired etimate. The proof of (31) also follows the classical steps. First, it is obvious that

$$\forall \alpha \in L^2_{\frac{1}{\omega}}, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq C \|\alpha\|_{L^2_{\frac{1}{\omega}}} ,$$

as $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$. Using again an interpolation argument, it is sufficient to show

$$\forall \alpha \in T^2, \quad \|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} . \tag{32}$$

To this aim, we establish the following estimate:

$$\forall \alpha \in T^2, \quad \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|\alpha\|_{T^2} , \tag{33}$$

and conclude with $\|(I_d - \mathbb{P}_h)\alpha\|_{L^2_{\frac{1}{\omega}}} \leq \|(I_d - I_h)\alpha\|_{L^2_{\frac{1}{\omega}}}$ since $\mathbb{P}_h$ minimizes the $L^2_{\frac{1}{\omega}}$ error. Any function $\alpha \in T^2$ is continuous by Lemma 7, thus $I_h\alpha$ is well-defined. To prove (33), let us fix $\alpha \in T^2$ and $i \in [0, N-1]$. The function $\mathcal{C}\alpha$ belongs to H^2 by Lemma 13 and thus its restriction on $[\theta_i, \theta_{i+1}]$ belongs to $H^2(\theta_i, \theta_{i+1})$. The function $I_h\alpha$ being C^∞ on $[x_i, x_{i+1}]$, the restriction of $\mathcal{C}I_h\alpha$ to $[\theta_i, \theta_{i+1}]$ also belongs to $H^2(\theta_i, \theta_{i+1})$. Moreover, $\mathcal{C}(I_h - I_d)\alpha$ vanishes on θ_i and θ_{i+1}, and it is well-known that for a function v in $H^2(a, b)$ which vanishes on a and b, there holds

$$\int_a^b |v(\theta)|^2 \, d\theta \leq C(b-a)^4 \int_a^b |\partial_{\theta\theta} v(\theta)|^2 \, d\theta \,. \tag{34}$$

Thus, applying this result on $[\theta_i, \theta_{i+1}]$ to $v = (\mathcal{C}(I_d - I_h)u)_{|[\theta_i, \theta_{i+1}]}$ and summing those inequalities for $i = 0$ to $N-1$, we obtain

$$\int_0^\pi |\mathcal{C}(I_d - I_h)\alpha(\theta)|^2 \, d\theta \leq h^4 \int_0^\pi |\partial_{\theta\theta}\mathcal{C}(I_d - I_h)\alpha(\theta)|^2 ,$$

in other words $\|\mathcal{C}(I_d - I_h)\alpha(\theta)\|_{\mathbb{T}_{2\pi}} \leq Ch^2 |\mathcal{C}(I_d - I_h)\alpha(\theta)|_{H^2_e}$. This, by Lemma 13, implies

$$\|(I_d - I_h)\alpha\|_{\frac{1}{\omega}} \leq Ch^2 |(I_d - I_h)\alpha|_{T^2} . \tag{35}$$

If $\mathcal{C}I_h\alpha$ were affine, the proof would end here since in this case $|I_h\alpha|_{T^2} = |\mathcal{C}I_h\alpha|_{H^2} = 0$. This does not hold in our case and this is the main difference with the standard proof. Nevertheless, if we show that

$$|I_h\alpha|_{T^2} \leq C \|\alpha\|_{T^2} ,$$

then eq. (33) follows from eq. (35) by triangular inequality. The expression of $\mathcal{C}I_h\alpha$ on $[\theta_i, \theta_{i+1}]$ is given by

$$\mathcal{C}I_h\alpha(\theta) = \alpha(x_i) + \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)}(\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} |\partial_{\theta\theta}\mathcal{C}I_h\alpha|^2 = \left| \frac{\alpha(x_i) - \alpha(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right|^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 = \left| \int_{x_i}^{x_{i+1}} \partial_x \alpha(x) dx \right|^2 ,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$|\alpha(x_{i+1}) - \alpha(x_i)|^2 \leq \int_{x_i}^{x_{i+1}} \frac{|\partial_x \alpha(x)|^2}{\omega(x)} dx \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

Notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. We deduce

$$\int_0^\pi |\partial_{\theta\theta} \mathcal{C} I_h \alpha|^2 \leq C \left\| \partial_x \alpha \right\|_{\frac{1}{\omega}}^2 ,$$

that is $|I_h \alpha|_{T^2} \leq C \left\| \partial_x \alpha \right\|_{\frac{1}{\omega}}$. By Corollary 3, one has $\left\| \partial_x \alpha \right\|_{\frac{1}{\omega}} \leq C \left\| \alpha \right\|_{T^2}$, thus (33) is established, concluding the proof. □

Corollary 5. *For all $(s, t) \in [-\frac{1}{2}, 2]$, if $\alpha_0 \in T^t$, then*

$$\left\| \alpha_0 - \alpha_h \right\|_{T^s} \leq C h^{t-s} \left\| \alpha_0 \right\|_{T^t} .$$

Proof. Let us first establish the inverse estimate

$$\forall \alpha'_h \in V_h, \quad \forall s \in \mathbb{N}, \quad \left\| \alpha'_h \right\|_{T^s} \leq C h^{-s} \left\| \alpha'_h \right\|_{\frac{1}{\omega}} .$$

For this, we fix a function α'_h in V_h and restrict our attention on a fixed segment $[x_i, x_{i+1}]$. Let us first assume that s is an integer. Let $u_h(u) = \mathcal{C}\alpha'_h(\theta_i + uh)$ defined on $[0, 1]$. Notice that

$$\partial_u^k u_h(u) = h^k \partial_\theta^k \mathcal{C} \alpha'_h(\theta_i + hu) .$$

Since u_h belongs to a finite-dimensional space of dimension 2, all norms are equivalent, in particular

$$\left\| u_h \right\|_{H^s(0,1)} \leq C \left\| u_h \right\|_{L^2(0,1)}^2 .$$

Therefore

$$\begin{aligned}
|\mathcal{C}\alpha'_h|_{H^s(\theta_i, \theta_{i+1})}^2 &= h \int_0^1 h^{-2s} \left| \partial_u^s u_h(u) \right| du \\
&\leq C h^{-2s} \int_0^1 \left| \partial_u^t u_h(u) \right| h du \\
&\leq C h^{-2s} \left\| \mathcal{C}\alpha'_h \right\|_{L^2(\theta_i, \theta_{i+1})} .
\end{aligned}$$

as claimed. When s is not an integer, the result is deduced by interpolation. To establish the announced result it suffices to show that $\left\| \alpha_0 - \alpha_h \right\|_{T^2} \leq C \left\| \alpha_0 \right\|_{T^2}$, and conclude by interpolation with the estimate $\|\alpha_0 - \alpha_h\|_{T^{-1/2}} \leq C\|\alpha_0\|_{T^2}$ obtained in the proof of the previous theorem. One can write

$$\|\alpha_0 - \alpha_h\|_{T^2} \leq \|\alpha_0\|_{T^2} + \|I_h\alpha_0\|_{T^2} + \|I_h\alpha_0 - \alpha_h\|_{T^2} \ .$$

We have seen in the previous proof that $\|I_h\alpha_0\|_{T^2} \leq C\|\alpha_0\|_{T^2}$. For the third term, we can use the inverse estimate:

$$\|I_h\alpha_0 - \alpha_h\| \leq Ch^{-2}\|I_h\alpha_0 - \alpha_h\|_{\frac{1}{\omega}} \ .$$

Using triangular inequality

$$\|I_h\alpha_0 - \alpha_h\| \leq Ch^{-2}\left(\|\alpha_0 - I_h\alpha_0\|_{\frac{1}{\omega}} + \|\alpha_0 - \alpha_h\|_{\frac{1}{\omega}}\right) \ .$$

In the previous proof, we have shown that

$$\|\alpha_0 - \alpha_h\|_{\frac{1}{\omega}} \leq C\|\alpha_0 - I_h\alpha_0\|_{\frac{1}{\omega}} \leq Ch^2\|\alpha_0\|_{T^2}$$

and the proof is concluded. □

2.5 Neumann problem

Let W_h the Galerkin space of **continuous** piecewise affine functions on the mesh $(x_i)_{0\leq i\leq N}$ defined above, and β_h the unique solution in W_h to

$$(N_{0,\omega}\beta_h, \beta_h')_\omega = (u_N, \beta_h')_\omega \,, \quad \forall \beta_h' \in W_h \ . \tag{36}$$

Let $\mu_h = \beta_h\omega$. We also write $\beta_0 := \frac{\mu}{\omega}$.

Theorem 2. *If $u_N \in U^{s-1}$, for some $\frac{1}{2} \leq s \leq 2$, there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}}\left\|\frac{\mu}{\omega}\right\|_{U^s} \leq Ch^{s-\frac{1}{2}}\|u_N\|_{U^{s-1}} \ .$$

Proof. Like before, let us denote by Π_h the operator that maps a function $\beta \in U^{1/2}$ to the element $\beta_h \in W_h$ such that

$$(N_{0,\omega}\beta_h, \beta_h') = (N_{0,\omega}\beta, \beta_h') \quad \forall \beta_h' \in W_h \ .$$

The operator $N_{0,\omega}$ being coercive on $U^{1/2}$ by Lemma 19, we have a Céa's lemma:

$$\forall \beta \in U^{\frac{1}{2}}, \quad \|\beta - \Pi_h\beta\|_{U^{1/2}} \leq C \inf_{\beta_h' \in W_h} \|\beta - \beta_h'\|_{U^{1/2}} \ .$$

In particular taking $\beta_h' = 0$ in the infimum,

$$\forall \beta \in U^{1/2}, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq C\|\beta\|_{U^{1/2}} \ .$$

Once we prove

$$\forall \beta \in U^2, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{\frac{3}{2}}\|\beta\|_{U^2} \ , \tag{37}$$

we get by interpolation

$$\forall \beta \in U^s, \quad \|(I_d - \Pi_h)\beta\|_{U^{1/2}} \leq C h^{s-\frac{1}{2}} \|\beta\|_{U^s}$$

for all $s \in [\frac{1}{2}, 2]$. The result follows since on the one hand,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} = \|\beta_0 - \beta_h\|_{U^{1/2}} ,$$

by Lemma 19, with $\Pi_h \beta_0 = \beta_h$ and, on the other hand, $\frac{\mu}{\omega} = \beta = N_{0,\omega}^{-1} u_N$ which, by Lemma 18, implies

$$\|\beta\|_{U^s} \leq C \|u_N\|_{U^{s-1}} .$$

Like before, the proof of (37) involves the study of the interpolation operator I_h. Namely, if we have

$$\forall \beta \in U^2, \quad \|(I_d - I_h)\beta\|_{\omega} \leq C h^2 \|\beta\|_{U^2} , \quad \|(I_d - I_h)\beta\|_{U^1} \leq C h \|\beta\|_{U^2} , \tag{38}$$

then, by interpolation, we obtain

$$\|(I_d - I_h)\beta\|_{U^{1/2}} \leq C h^{3/2} \|u\|_{U^2} ,$$

which gives (37) after applying Céa's lemma. Let us show the first estimate in (38). Applying Lemma 13 and using again the property of H^2 functions vanishing at the boundary (34) one can write

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \omega \left|(I_d - I_h)\beta\right|^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} \left|\partial_{\theta\theta}(\mathcal{S}\beta - \mathcal{S}I_h\beta)\right|^2 \\ &\leq C h^4 \left(2 \int_{\theta_i}^{\theta_{i+1}} \left|\partial_{\theta\theta}\mathcal{S}\beta\right|^2 + 2 \int_{\theta_i}^{\theta_{i+1}} \left|\partial_{\theta\theta}\mathcal{S}I_h\beta\right|^2 \right) . \end{aligned}$$

Summing for $i = 0, \cdots, N-1$, by Lemma 13, we get

$$\|(I_d - I_h)\beta\|_{\omega} \leq C h^2 \left(\|\beta\|_{U^2} + |\mathcal{S}I_h\beta|_{H^2}\right) \tag{39}$$

As for the Dirichlet conditions, the proof would end here with another choice of basis functions, namely functions of the form

$$\phi_i(x) = \frac{a_i + b_i \arccos(x)}{\omega(x)} ,$$

because in this case, $\mathcal{S}I_h u$ would be affine and thus the second term in the right hand side would be 0. Here, we need to show that the second term is controlled by $\|\beta\|_{U^2}$. Using the expression of I_h, one can write

$$\begin{aligned} \int_{\theta_i}^{\theta_{i+1}} \left|\partial_{\theta\theta}\mathcal{S}I_h\beta\right|^2 \leq C \Bigg(&\left|\beta(x_i)\right|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta d\theta \\ &+ \left|\frac{\beta(x_{i+1}) - \beta(x_i)}{\cos\theta_{i+1} - \cos\theta_i}\right|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta(1 + \cos^2\theta) d\theta \Bigg) . \end{aligned} \tag{40}$$

We can estimate the first term, thanks to Lemma 7:

$$|\beta(x_i)| \leq C \|\beta\|_{U^2} ,$$

while for the second term, the numerator of the fraction is estimated as follows:

$$\begin{aligned}|\beta(x_{i+1}) - \beta(x_i)|^2 &= \left|\int_{x_i}^{x_{i+1}} \partial_x \beta\right|^2 \\ &\leq \int_{x_i}^{x_{i+1}} \omega \, |\partial_x \beta|^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\ &= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega \, |\partial_x \beta|^2 \, .\end{aligned}$$

Observe that the quantity

$$\frac{|\theta_{i+1} - \theta_i| \int_{\theta_i}^{\theta_{i+1}} \sin^2\theta (1 + \cos^2\theta) d\theta}{(\cos\theta_i - \cos\theta_{i+1})^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1} \to \theta_i$, the fraction has the value $1 + \cos^2\theta_i$. Therefore, (39) becomes

$$\|(I_d - I_h)\beta\|_\omega \leq Ch^2 \left(\|\beta\|_{U^2} + \|\partial_x \beta\|_\omega\right) .$$

Recalling Corollary 3, the second term in this estimate is controlled by $\|\beta\|_{U^2}$ and the first estimate of (38) is established. The second estimate of (38) can be shown in a similar manner, concluding the proof. $\square$

Using Aubin-Nitsche's duality technique and inverse estimates as in the previous paragraph, one can check that the following result holds.

Corollary 6. *For all* $(s, t) \in [0, 2]$,

$$\|\beta - \beta_h\|_{U^s} \leq Ch^{t-s} \|\beta\|_{U^s} \, .$$

3 Pseudo-differential operators

In this section, we introduce two classes of pseudo-differential operators on $T^s(\Gamma)$ and $U^s(\Gamma)$, which are based on a class of periodic pseudo-differential operators on the torus. We will see in the next section that the weighted layer potentials S_{k,ω_Γ} and N_{k,ω_Γ} are elements of those classes, of order -1 and 1 respectively. The symbolic calculus that we establish here will enable us to build parametrix for the aforementioned operators.

3.1 Periodic pseudo-differential operators

On the family of periodic Sobolev spaces H^s, a class of periodic pseudo differential operators (PPDO) is studied in [8]. We quickly review here the definitions and properties needed for our purposes. A PPDO of order α on H^s is an operator of the form

$$Au(\theta) = \sum_{n \in \mathbb{Z}} \sigma_A(\theta, n) \hat{u}_n e^{in\theta} \, .$$

for a "prolongated symbol" $\sigma_A \in C^\infty(\mathbb{T}_T \times \mathbb{R})$ satisfying

$$\forall j, k \in \mathbb{N}, \quad \exists C_{j,k} > 0 : \quad \left|D_\theta^j D_\xi^k \sigma_A(\theta, \xi)\right| \leq C_{j,k}(1 + |\xi|)^{\alpha - k} \, . \tag{41}$$

Here, $\hat{u}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)e^{-in\theta}d\theta$ are the usual Fourier coefficients of u and

$$D_\theta := \frac{1}{i}\frac{\partial}{\partial\theta}, \quad D_\xi := \frac{1}{i}\frac{\partial}{\partial\xi},$$

with for $j \geq 1$, $D_\theta^{j+1} = D_\theta D_\theta^j$, and $D_\xi^{j+1} = D_\xi D_\xi^j$. The class of symbols that satisfy (41) is denoted by Σ^α, and $\Sigma^{-\infty} := \cup_{\alpha\in\mathbb{Z}}\Sigma^\alpha$. The operator defined by a symbol σ is denoted by $Op(\sigma)$ and the set of PPDOs of order α is denoted by $Op(\Sigma^\alpha)$.

The prolongated symbol is not unique but determined uniquely at the integer values of ξ by

$$\sigma_A(\theta, n) = e_{-n}(\theta)Ae_n(\theta)\,, \tag{42}$$

where we recall the notation $e_n(\theta) = e^{in\theta}$, as shown in [8]. This justifies the terminology of "prolongated symbol". The operator A is in $Op(\Sigma^\alpha)$ if and only if

$$\forall j,k \in \mathbb{N}, \quad \exists C_{j,k} > 0 : \quad \left|D_\theta^j \Delta_n^k \sigma_A(\theta,n)\right| \leq C_{j,k}(1+|n|)^{\alpha-k}\,,$$

where $\Delta_n\phi(\theta,n) = \phi(\theta,n+1) - \phi(\theta,n)$ and for $k \geq 1$, $\Delta_n^{k+1}\phi = \Delta_n(\Delta_n^k\phi)$. That is, if the symbol defined in (42) satisfies this condition, then there exists a prolongated symbol satisfying (41). Because of this, we write $\sigma \in \Sigma^p$ for a symbol $\sigma(\theta,n)$ that can be prolongated to a symbol $\sigma(\theta,\xi) \in \Sigma^p$. An operator in $Op(\Sigma^\alpha)$ maps continuously H^s to $H^{s+\alpha}$ for all $s \in \mathbb{R}$. The composition of two operators in $Op(\Sigma^\alpha)$ and $Op(\Sigma^\beta)$ gives rise to an operator in $Op(\Sigma^{\alpha+\beta})$. If two symbols a and b in $\Sigma^{-\infty}$ satisfy $a - b \in \Sigma^\alpha$, we write $a = b + \Sigma^\alpha$.

Definition 6. *Let $a \in \Sigma^{-\infty}$. If there exists a sequence of reals $(p_j)_{j\in\mathbb{N}}$ such that $p_j < p_{j+1}$ and a sequence of symbols $a_j \in \Sigma^{p_j}$ such that for all N, $a = \sum_{i=0}^{N} a_i + \Sigma^{p_j+1}$, we write*

$$a = \sum_{i=0}^{+\infty} a_j\,.$$

This is called an asymptotic expansion of the symbol a.

The symbol of the composition of two PPDOs A and B is denoted by $\sigma_A\#\sigma_B$ and satisfies the asymptotic expansion

$$\sigma_A\#\sigma_B(t,\xi) = \sum_{j=0}^{+\infty}\frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \sigma_A(\theta,\xi)D_\theta^j\sigma_B(\theta,\xi)\,. \tag{43}$$

We will also use the following result, proved in [8]:

Proposition 1. *Consider an integral operator K of the form*

$$K : u \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta,\theta')h(\theta-\theta')u(\theta')d\theta'.$$

where a is 2π-periodic and C^∞ in both arguments and h is a 2π-periodic distribution. Assume that the Fourier coefficients $\hat{h}(n)$ of h can be prolonged to a function $\hat{h}(\xi)$ on $\mathbb{R}$ such that

$$\forall k \in \mathbb{N}, \quad \exists C_k > 0 : \quad \left|\partial_\xi^k \hat{h}(\xi)\right| \leq C_k(1+|\xi|)^{\alpha-k}\,.$$

for some α. Then K is in $Op(\Sigma^\alpha)$ with a symbol satisfying the asymptotic expansion

$$\sigma_K(\theta,\xi) = \sum_{j=0}^{+\infty} \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \hat{h}(\xi) D_t^j a(t,\theta)_{|t=\theta}\,. \tag{44}$$

In particular, taking $h \equiv 1$, we see that for all functions $a \in C^\infty(\mathbb{T}_T^2)$

$$K : u \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\theta,\theta')u(\theta')d\theta'$$

is in $Op\left(\Sigma^{-\infty}\right)$.

3.2 Pseudo-differential operators on $T^s(\Gamma)$

Lemma 20. *Let A a PPDO that stabilizes the set of smooth even functions. Then A coincides on this set with the operator B defined by the symbol*

$$\sigma_B(\theta,n) = \frac{\sigma_A(\theta,n)+\sigma_A(-\theta,-n)}{2}\,.$$

Moreover, σ_B admits the following decomposition:

$$\sigma_B(\theta,n) = a_1(\cos\theta,n) + i\sin(\theta)a_2(\cos\theta,n)$$

with

$$a_1(x,n) = \frac{\sigma_B(\arccos(x),n)+\sigma_B(\arccos(x),-n)}{2}$$

$$a_2(x,n) = \frac{\sigma_B(\arccos(x),n)-\sigma_B(\arccos(x),-n)}{2i\sqrt{1-x^2}}$$

and a_1 and a_2 are C^∞ in x. The functions a_1 and a_2 thus defined are denoted by $a_1^T(A)$ and $a_2^T(A)$.

Proof. For a smooth even function u, one has

$$Au(\theta) = \frac{Au(\theta)+Au(-\theta)}{2}\,,$$

thus

$$Au(\theta) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sigma_A(\theta,n)\hat{u}_n e^{in\theta} + \frac{1}{2}\sum_{n\in\mathbb{Z}}\sigma_A(-\theta,n)\hat{u}_n e^{-in\theta}\,.$$

Since u is even, $\hat{u}_n = \hat{u}_{-n}$, so that $Au(\theta) = Bu(\theta)$ where B is the operator with symbol $\sigma_B(\theta,n) = \frac{\sigma_A(\theta,n)+\sigma_A(-\theta,-n)}{2}$. In particular, it satisfies the following symmetry:

$$\sigma_B(-\theta,-n) = \sigma_B(\theta,n)\,.$$

We write $\sigma_B(\theta,n) = f_B(\theta,n) + g_B(\theta,n)$ where $f_B(\theta,n) = \frac{\sigma_B(\theta,n)+\sigma_B(\theta,-n)}{2}$ and $g_B(\theta,n) = \frac{\sigma_B(\theta,n)-\sigma_B(\theta,-n)}{2}$. Notice that f_B (resp. g_B) is even (resp. odd) in both θ and n. The functions a_1 and a_2 defined in the statement of the Lemma satisfy

$$a_1(x,n) = f_B(\arccos(x),n), \quad a_2(x,n) = \frac{g_B(\arccos(x),n)}{i\sqrt{1-x^2}}\,,$$

thus

$$\sigma_B(\theta, n) = a_1(\cos\theta, n) + i\sin\theta a_2(\cos\theta, n)\,.$$

For fixed n, there holds $a_1(\cdot, n) = \mathcal{C}^{-1} f_B(\cdot, n)$ and $a_2(\cdot, n) = -i\mathcal{S}^{-1} g_B$. By Lemma 13, a_1 and a_2 are thus C^∞ in x since f_B (resp. g_B) is a smooth even (resp. odd) function. □

We use this result to transport the notion of periodic pseudo-differential operators to the segment $[-1, 1]$ by the change of variable $x = \cos\theta$.

Definition 7. *Let A an operator on $T^{-\infty}$ and assume that there exists a couple of smooth functions a_1 and a_2 in $C^\infty([-1, 1] \times \mathbb{N})$ such that for all $n \in \mathbb{N}$,*

$$AT_n = a_1(x, n)T_n - \omega^2 a_2(x, n)U_{n-1}\,, \tag{45}$$

with, by convention, $U_{-1} = 0$. Such a (non-unique) couple of functions is called a pair of symbols of A. For $n \in \mathbb{Z}$ and $\theta \in [0, 2\pi]$, define the symbol $\tilde{\sigma}_T(a_1, a_2)$ by

$$\tilde{\sigma}_T(a_1, a_2)(\theta, n) := a_1(\cos\theta, |n|) + i\sin\theta\, sign(n) a_2(\cos\theta, |n|)\,.$$

We say that $(a_1, a_2) \in S^\alpha_T$ if $\tilde{\sigma}_T(a_1, a_2) \in \Sigma^\alpha$. We also take the notation $S^\infty_T := \cup_{\alpha\in\mathbb{R}} S^\alpha_T$. The operator defined by a pair of symbols (a_1, a_2) is denoted by $Op_T(a_1, a_2)$ and the set of pseudo-differential operators (of order α) in $T^{-\infty}$ by $Op(S^\infty_T)$ (by $Op(S^\alpha_T)$).

Recall the definition of the isometric mapping $\mathcal{C}$ from Lemma 13.

Theorem 1. *Let $(a_1, a_2) \in S^\alpha_T$ and $A = Op_T(a_1, a_2)$. There holds*

$$\mathcal{C}A = \tilde{A}\mathcal{C}$$

where $\tilde{A} = Op(\tilde{\sigma}_T(a_1, a_2))$. Reciprocally, let $A : T^\infty \to T^{-\infty}$ and assume that there exists a PPDO $\tilde{A}$ of order α such that

$$\forall u \in T^\infty, \quad \mathcal{C}Au = \tilde{A}\mathcal{C}u\,.$$

Then A has a unique linear continuous extension on $T^{-\infty}$ satisfying $\mathcal{C}A = \tilde{A}\mathcal{C}$ and this extension belongs to $Op(S^\alpha_T)$. Moreover, A admits the pair of symbols $(a^T_1(\tilde{A}), a^T_2(\tilde{A}))$.

Proof. For the direct result, we start by showing the equality for $u = T_n$ for all $n \in \mathbb{N}$. One has $\mathcal{C}T_n(\theta) = T_n(\cos(\theta)) = \cos(n\theta)$. Consequently,

$$\tilde{A}\left(\mathcal{C}T_n\right)(\theta) = \frac{\tilde{A}e^{in\theta} + \tilde{A}e^{-in\theta}}{2} \tag{46}$$

which, using the determination of the symbol (42), yields

$$\tilde{A}\left(\mathcal{C}T_n\right)(\theta) = \frac{\sigma(\theta, n)e^{in\theta} + \sigma(\theta, -n)e^{-in\theta}}{2}\,,$$

where $\sigma = \tilde{\sigma}_T(a_1, a_2)$. Replacing this definition in the former equation, one gets

$$\tilde{A}\left(\mathcal{C}u\right)(\theta) = a_1(\cos\theta, n)\cos(n\theta) - \sin\theta a_2(\cos\theta, n)\sin(n\theta).$$

Since $\cos(n\theta) = T_n(\cos\theta)$ and $\sin(n\theta) = \sin\theta U_{n-1}(\cos\theta)$,

$$\begin{aligned}\tilde{A}\left(\mathcal{C}T_n\right)(\theta) &= a_1(\cos\theta, n)T_n(\cos\theta) - (1-\cos^2\theta)a_2(\cos\theta, n)U_{n-1}(\cos\theta)\,,\\ &= \mathcal{C}(AT_n)\end{aligned} \tag{47}$$

as claimed. To show the general case, fix $u \in T^{-\infty}$ and $\tilde{v} \in H^{-\infty}$. One has, by linearity and continuity of A, $\tilde{A}$ and $\mathcal{C}$:

$$\begin{aligned}\langle \mathcal{C}Au, \tilde{v}\rangle_{\mathbb{T}_{2\pi}} &= \sum_{n=0}^{+\infty} \hat{u}_n \left\langle \mathcal{C}AT_n, \tilde{v}\right\rangle_{\mathbb{T}_{2\pi}}\\ &= \sum_{n=0}^{+\infty} \hat{u}_n \left\langle \tilde{A}\mathcal{C}T_n, \tilde{v}\right\rangle_{\mathbb{T}_{2\pi}}\\ &= \left\langle \tilde{A}\mathcal{C}u, \tilde{v}\right\rangle_{\mathbb{T}_{2\pi}}\,.\end{aligned}$$

The last identity shows that $\mathcal{C}Au = \tilde{A}\mathcal{C}u$ for all $u \in T^{-\infty}$, in other words, $\mathcal{C}A = \tilde{A}\mathcal{C}$. For the converse result, we now assume that for any $u \in T^{\infty}$, $\mathcal{C}Au = \tilde{A}\mathcal{C}u$ where $\tilde{A}$ is some PPDO of order α with a symbol $\sigma_{\tilde{A}}$. The previous computations show that any linear continuous extension of A satisfies

$$\mathcal{C}A = \tilde{A}\mathcal{C}\,. \tag{48}$$

This in turn defines uniquely the operator A on $T^{-\infty}$ since for $u \in T^{-\infty}$ and $v \in T^{\infty}$, one has, by Lemma 12,

$$\langle Au, v\rangle_{\frac{1}{\omega}} = \langle \mathcal{C}Au, \mathcal{C}v\rangle_{\mathbb{T}_{2\pi}}\,.$$

Let us show that A sends T^{∞} to T^{∞}. Let $u \in T^{\infty}$ and $s \in \mathbb{R}$. Using (48), the continuity of $\tilde{A}$ from $H^{s+\alpha}$ to H^s and the isometric property of $\mathcal{C}$,

$$\begin{aligned}\|Au\|_{T^s} &= \|\mathcal{C}Au\|_{H^s}\\ &= \left\|\tilde{A}\mathcal{C}u\right\|_{H^s}\\ &\leq C\,\|\mathcal{C}u\|_{H^{s+\alpha}}\\ &\leq C\,\|u\|_{T^{s+\alpha}}\,.\end{aligned}$$

The last quantity is finite since $u \in T^{\infty} \subset T^{s+\alpha}$. This proves that A sends T^{∞} to T^{∞}.
Eq (48) implies in particular that $\tilde{A}$ stabilizes the set of smooth even functions since $\mathcal{C}Au(\theta) = Au(\cos\theta)$ is even and A maps smooth functions to smooth functions. Lemma 20 can thus be applied. Let $a_1 = a_1^T(\tilde{A})$ and $a_2 = a_2^T(\tilde{A})$. Starting from eq. (48), the computations from eqs. (46) to (47) can be performed in reverse order to show

$$AT_n(\cos\theta) = a_1(n, \cos\theta)T_n(\cos\theta) - (1-\cos^2\theta)a_2(n,\cos\theta)U_{n-1}(\cos\theta)\,,$$

which, taking $x = \cos\theta$, leads to $A = Op_T(a_1, a_2)$. To establish that A is in $Op(S_T^{\alpha})$, we have to show $\tilde{\sigma}(a_1, a_2) \in \Sigma^{\alpha}$. By Lemma 20, $\tilde{\sigma}(a_1, a_2)$ is exactly the symbol σ_B defined by

$$\sigma_B(\theta, n) = \frac{\sigma_{\tilde{A}}(\theta, n) + \sigma_{\tilde{A}}(-\theta, -n)}{2}\,.$$

This is indeed in Σ^{α} since it is the case for $\sigma_{\tilde{A}}$ by assumption. □

Remark 2. *When $A \in Op(S_T^\alpha)$, there is an infinity of operators $\tilde{A}$ satisfying $\mathcal{C}A = \tilde{A}\mathcal{C}$. Indeed, if this holds for some $\tilde{A}$, it also holds for $\tilde{A} + B$ where B is any PPDO of order α with the property that $Bu = 0$ when u is even. This non-uniqueness is also reflected by the fact that the couple of symbols of an operator A in $Op(S_T^\alpha)$ is not unique, or in other words, the null operator has non-trivial pair of symbols in $S_T^{-\infty}$. For example take a_1 and a_2 as follows: fix $n_0 \in \mathbb{N}$ and let*

$$a_1(x, n_0) = -\omega^2 U_{n_0-1}(x)\,, \quad a_2(x, n_0) = T_{n_0}(x)$$

while $a_1(x, n) = a_2(x, n) = 0$ for $n \neq 0$. Obviously, $(a_1, a_2) \in S_T^{-\infty}$ and $Op_T(a_1, a_2) \equiv 0$. One idea to enforce uniqueness would be to take for $\tilde{A}$ the operator $\tilde{A}^*$ satisfying $\tilde{A}^* u = 0$ whenever u is odd. Such a condition would demand the following symmetry on the symbol $\sigma_{\tilde{A}^*}$:

$$\sigma_{\tilde{A}^*}(\theta, -n) = e^{2in\theta}\sigma_{\tilde{A}^*}(\theta, n)\,.$$

One can show that if $\mathcal{C}A = \tilde{A}\mathcal{C}$ for some operator $\tilde{A}$, then the symbol of $\tilde{A}^*$ must be given by

$$\sigma_{\tilde{A}^*}(\theta, n) = \sigma_{\tilde{A}}(\theta, n) + e^{-2in\theta}\sigma_{\tilde{A}}(\theta, -n)\,.$$

However, in general, this symbol is not in Σ^α because of the oscillatory term $e^{-2in\theta}$. In other words, one cannot always construct an operator $\tilde{A}^*$ satisfying the following three conditions

- *$\tilde{A}^*$ coincides on the set of even functions with some given PPDO $\tilde{A}$ of order α,*
- *$\tilde{A}^*$ vanishes on the set of odd functions,*
- *$\tilde{A}^*$ is a PPDO of order α.*

As a conclusion, it is not clear how to fix a natural representative in the class of pairs (a_1, a_2) that define the same operator A.

Definition 8. *Let $A : T^{-\infty}(\Gamma) \to T^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $T^{-\infty}(\Gamma)$ if $RAR^{-1} \in Op(S_T^\infty)$ ($\in Op(S_T^\alpha)$). The set of pseudo-differential operators of order α on $T^{-\infty}(\Gamma)$ is denoted by $Op(S_T^\alpha(\Gamma))$. We say that (a_1, a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1}.*

As a corollary of Theorem 1, we have he following properties

Corollary 7. *Let $A \in Op(S_T^\alpha(\Gamma))$. Then for all s, A is continuous from $T^s(\Gamma)$ to $T^{s-\alpha}(\Gamma)$. If B and C respectively belong to $Op(S_T^{\alpha_1}(\Gamma))$ and $Op(S_T^{\alpha_2}(\Gamma))$, with pairs of symbols (b_1, b_2) and (c_1, c_2), then BC is in $Op(S_T^{\alpha_1+\alpha_2}(\Gamma))$ and admits the pair of symbols $\left(a_1^T(\tilde{A}), a_2^T(\tilde{A})\right)$ where*

$$\tilde{A} = Op(\tilde{\sigma}_T(b_1, b_2))Op(\tilde{\sigma}_T(c_1, c_2)) = Op(\tilde{\sigma}_T(b_1, b_2)\#\tilde{\sigma}_T(c_1, c_2))\,.$$

Proof. Let $A \in Op(S_T^\alpha(\Gamma)$ and $s \in \mathbb{R}$. By Theorem 1, there exists $\tilde{A} \in Op(\Sigma^\alpha)$ such that

$$\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}\,.$$

Using the definition of the norm on $T^s(\Gamma)$, the isometric property of $\mathcal{C}$ and the continuity of $\tilde{A}$ from H^s to $H^{s-\alpha}$, we have for all $u \in T^s(\Gamma)$,

$$\begin{aligned}\|Au\|_{T^{s-\alpha}(\Gamma)} &= \|RAu\|_{T^{s-\alpha}} = \|\mathcal{C}RAu\|_{H^{s-\alpha}} = \left\|\tilde{A}\mathcal{C}Ru\right\|_{H^{s-\alpha}} \\ &\leq C\left\|\mathcal{C}Ru\right\|_{H^s} = C\left\|Ru\right\|_{T^s} = C\left\|u\right\|_{T^s(\Gamma)}\,.\end{aligned}$$

Let $B, C \in Op(S_T^{\alpha_1}(\Gamma)) \times Op(S_T^{\alpha_2}(\Gamma))$, with respective pairs of symbols (b_1, b_2) and (c_1, c_2). Let $\tilde{B} = Op(\tilde{\sigma}(b_1, b_2))$ and $\tilde{C} = Op(\tilde{\sigma}(c_1, c_2))$. We have

$$\mathcal{C}RBR^{-1} = \tilde{B}\mathcal{C}, \quad \text{and} \quad \mathcal{C}RCR^{-1} = \tilde{C}\mathcal{C}\,.$$

Therefore,

$$\mathcal{C}RBCR^{-1} = \tilde{A}\mathcal{C}$$

where $\tilde{A} = \tilde{B}\tilde{C}$. One has $\tilde{A} \in Op(\Sigma^{\alpha_1+\alpha_2})$. By Theorem 1, $RBCR^{-1}$ is in $Op(S^{\alpha_1+\alpha_2})$ and admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$. By definition, this means that $BC \in Op(S^{\alpha_1+\alpha_2}(\Gamma))$ and admits the pair of symbols $(a_1^T(\tilde{A}), a_2^T(\tilde{A}))$. □

Remark 3. *The previous result gives a method for a symbolic calculus on the class $S_T^\alpha(\Gamma)$ as follows. If B and C respectively admit the pair of symbols (b_1, b_2) and (c_1, c_2), then BC admits the pair of symbols*

$$(b_1, b_2)\#_T(c_1, c_2) := \left(a_1^T(\tilde{A}), a_2^T(\tilde{A})\right)$$

with $\tilde{A} = Op\left(\tilde{\sigma}(b_1, b_2)\#\tilde{\sigma}(c_1, c_2)\right)$. One can use (43) to compute an asymptotic expansion of $\tilde{\sigma}(b_1, b_2)\#\tilde{\sigma}(c_1, c_2)$ which, in turn, gives an asymptotic expansion of $(b_1, b_2)\#_T(c_1, c_2)$. The proofs of Theorem 3 and Theorem 4 rely on this method, but the details of the computation are omitted as they are quite heavy. In compensation, a Maple code giving procedures for the symbolic calculus is provided in [3].

3.3 Pseudo-differential operators on $U^s(\Gamma)$

We define similarly a class of pseudo-differential operators on the spaces $U^s(\Gamma)$. One can show the following result:

Lemma 21. *Let A a PPDO that stabilizes the set of smooth odd functions. Then A coincides on this set with the operator B with symbol given by*

$$\sigma_B(n, \theta) = \frac{\sigma_A(\theta, n) + \sigma_A(-\theta, -n)}{2}\,.$$

Moreover, σ_B admits the following decomposition

$$\sigma_B(n, \theta) = ia_1(\cos\theta, n) + \sin\theta a_2(\cos\theta, n)$$

with

$$\begin{aligned}a_1(x, n) &= \frac{\sigma_B(\arccos(x), n) + \sigma_B(\arccos(x), -n)}{2i} \\ a_2(n, x) &= \frac{\sigma_B(\arccos(x), n) - \sigma_B(\arccos(x), -n)}{2\sqrt{1-x^2}}\end{aligned}$$

and a_1 and a_2 are C^∞. The functions a_1 and a_2 thus defined are denoted by $a_1^U(A)$ and $a_2^U(A)$.

Let A an operator on $U^{-\infty}$ and assume that there exists a couple of smooth functions a_1 and a_2 in $C^\infty([-1,1]\times\mathbb{N})$ such that for all $n\in\mathbb{N}$,

$$AU_n = a_1(x,n)U_n + a_2(x,n)T_{n+1}\,.$$

Such a (non-unique) couple of functions is called a pair of symbols of A. For $n\in\mathbb{Z}$ and $\theta\in[0,2\pi]$, define the symbol $\tilde{\sigma}_U(a_1,a_2)$ by

$$\tilde{\sigma}_U(a_1,a_2)(\theta,n) = ia_1(\cos\theta,|n|) + \sin\theta\,\mathrm{sign}(n)a_2(\cos\theta,|n|)\,.$$

We say that $(a_1,a_2)\in S^\alpha_U$ if $\tilde{\sigma}_U(a_1,a_2)\in\Sigma^\alpha$, and $S^\infty_U := \cup_{\alpha\in\mathbb{Z}}S^\alpha_U$. The operator defined by a pair of symbols (a_1,a_2) is denoted by $Op_U(a_1,a_2)$ and the set of pseudo-differential operatos (of order α) in $U^{-\infty}$ by $Op(S^\infty_U)$ ($Op(S^\alpha_U)$). Recall the definition of the isometric mapping $\mathcal{S}$ from Lemma 13. Adapting the proof of Theorem 1, one can show

Theorem 2. *Let $(a_1,a_2)\in S^\alpha_U$ and $A = Op_U(a_1,a_2)$. There holds*

$$\mathcal{S}A = \tilde{A}\mathcal{S}$$

where $\tilde{A} = Op(\tilde{\sigma}_U(a_1,a_2))$. Reciprocally, let $A: U^\infty\to U^{-\infty}$ a linear operator such that there exist a PPDO $\tilde{A}$ of order α satisfying

$$\forall u\in U^\infty,\quad \mathcal{S}Au = \tilde{A}\mathcal{S}u\,.$$

Then A has a unique linear continuous extension on $U^{-\infty}$ satisfying $\mathcal{S}A = \tilde{A}\mathcal{S}$. This extension is in $Op(S^\alpha_U)$ and A admits the pair of symbols $\left(a_1^U(\tilde{A}), a_2^U(\tilde{A})\right)$.

Definition 9. *Let $A : U^{-\infty}(\Gamma)\to U^{-\infty}(\Gamma)$. We say that A is a pseudo-differential operator (of order α) on $U^{-\infty}(\Gamma)$ if $RAR^{-1}\in Op(S^\infty_U)$ ($\in Op(S^\alpha_U)$). The set of pseudo-differential operators of order α on $U^{-\infty}(\Gamma)$ is denoted by $Op(S^\alpha_U(\Gamma))$. We say that (a_1,a_2) is a pair of symbols of A if it is a pair of symbols of RAR^{-1}.*

Corollary 8. *Let $A\in Op(S^\alpha_U(\Gamma))$. Then for all s, A is continuous from U^s to $U^{s-\alpha}$. If B and C respectively belong to $Op(S^{\alpha_1}_U(\Gamma))$ and $Op(S^{\alpha_2}_U(\Gamma))$, with pairs of symbols (b_1,b_2) and (c_1,c_2), then BC is in $Op(S^{\alpha_1+\alpha_2}_U(\Gamma))$ and admits the pair of symbols $(a_1^U(\tilde{A}), a_2^U(\tilde{A}))$ where*

$$\tilde{A} = Op(\tilde{\sigma}_U(b_1,b_2))Op(\tilde{\sigma}_U(c_1,c_2)) = Op(\tilde{\sigma}_U(b_1,b_2)\#\tilde{\sigma}_U(c_1,c_2))\,.$$

Lemma 22. *Let $A\in Op(S^\alpha_T(\Gamma))$ and $B = -\partial_\tau A\omega_\Gamma\partial_\tau\omega_\Gamma$. Then $B\in Op(S^{\alpha+2}_U(\Gamma))$ and if $\tilde{A}$ is a PPDO such that $\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$, then $\mathcal{S}RBR^{-1} = -\partial_\theta\tilde{A}\partial_\theta\mathcal{S}$.*

Proof. On can check the following identities:

$$\begin{aligned}\partial_\theta\mathcal{S} &= -\mathcal{C}\omega\partial_x\omega\,,\\ \partial_\theta\mathcal{C} &= -\mathcal{S}\partial_x\,.\end{aligned}$$

Let $A' = RAR^{-1}$ and $B' = RBR^{-1}$. Assuming that $\mathcal{C}A' = \tilde{A}\mathcal{C}$, there holds

$$\begin{aligned}\mathcal{S}B' &= -\mathcal{S}R\partial_\Gamma A\omega_\Gamma\partial_\Gamma\omega_\Gamma R^{-1}\\ &= -\mathcal{S}\partial_x A'\omega\partial_x\omega\\ &= \partial_\theta\mathcal{C}A'\omega\partial_x\omega\\ &= \partial_\theta\tilde{A}\mathcal{C}\omega\partial_x\omega\\ &= -\partial_\theta\tilde{A}\partial_\theta\mathcal{S}\,.\end{aligned}$$

Since $\tilde{A}$ can be chosen as a PPDO of order α by Theorem 1, $\partial_\theta \tilde{A} \partial_\theta$ is then a PPDO of order $\alpha+2$ and by Theorem 2, we conclude that $B \in Op(S_U^{\alpha+2}(\Gamma))$. □

Lemma 23. *Let $A \in Op(S_T^\alpha(\Gamma))$ and $B = A\omega_\Gamma^2$. Then $B \in Op(S_U^\alpha(\Gamma))$ and if $\tilde{A}$ is a PPDO such that $\mathcal{C}RAR^{-1} = \tilde{A}\mathcal{C}$, then $\mathcal{S}RBR^{-1} = \sin \tilde{A} \sin \mathcal{S}$ where* sin *denotes the operator $f(\theta) \mapsto \sin(\theta) f(\theta)$.*

Proof. This follows from the identities

$$\mathcal{S} = \sin \mathcal{C}, \quad \mathcal{C}\omega^2 = \sin \mathcal{S}$$

and the same arguments as in the proof of Lemma 22. □

Definition 10. *Let A and B in $Op(S_T^\infty(\Gamma))$ (resp. $Op(S_U^\infty(\Gamma))$). If $A - B \in Op(S_T^\alpha(\Gamma))$ (resp. $Op(S_U^\alpha(\Gamma))$), we write $A = B + T_\alpha$ (resp. $A = B + U_\alpha$).*

4 Paramatrix for the layer potentials

We now apply the pseudo-differential theory on $T^s(\Gamma)$ and $U^s(\Gamma)$ to build parametrix for the weighted layer potentials introduced at the beginning of the second section.

4.1 Dirichlet problem

Lemma 24. *The operator S_{k,ω_Γ} belongs to $Op(S_T^{-1}(\Gamma))$, and satisfies*

$$\mathcal{C}RS_{k,\omega_\Gamma}R^{-1} = \tilde{S}_k \mathcal{C}$$

where the symbol of $\tilde{S}_k \in Op(\Sigma^{-1})$ has the asymptotic expansion

$$\begin{aligned}\sigma_{\tilde{S}_k}(\theta,\xi) = &\frac{1}{2\xi} + \frac{k^2\left|\Gamma\right|^2 \sin(\theta)^2}{16\xi^3} + \frac{3ik^2|\Gamma|^2 \sin\theta\cos\theta}{16\xi^4} \\ &+ \frac{-768k^2\kappa(\theta)^2\sin^4\theta + 64k^2\left|\Gamma\right|^2\sin^2\theta - 48k^2\left|\Gamma\right|^2\cos^2\theta + 3k^4\left|\Gamma\right|^4\sin^4\theta}{128\xi^5} \\ &+ \Sigma^{-6}\,.\end{aligned} \tag{49}$$

Proof. The Hankel function admits the following expansion

$$H_0(z) = \frac{-1}{2\pi}\ln|z|J_0(z) + F_1(z^2) \tag{50}$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. Let us define

$$S_{k,\omega} := RS_{k,\omega_\Gamma}R^{-1}\,.$$

We fix a smooth function $u \in T^\infty$. One has

$$\left(S_{k,\omega}u\right)(x) = \int_{-1}^{1} H_0\left(k\left|r(x) - r(y)\right|\right)\frac{u(y)}{\omega(y)}dy\,.$$

Using the variable changes $x = \cos\theta$, $y = \cos\theta'$, we get

$$S_{k,\omega}u(\cos\theta) = \int_0^\pi H_0(k\,|r(\cos\theta) - r(\cos\theta')|)u(\cos(\theta))d\theta\,,$$

which, in view of (50), can be rewritten as

$$\begin{aligned} S_{k,\omega}u(\cos\theta) =& \frac{-1}{2\pi}\int_0^\pi \ln|\cos\theta - \cos\theta'|\, J_0(k\,|r(\cos\theta) - r(\cos\theta')|)\mathcal{C}u(\theta)d\theta \\ &+ \int_0^\pi F_2(\cos\theta, \cos\theta')\mathcal{C}u(\theta)d\theta' \end{aligned}$$

where

$$F_2(x,y) = \ln\frac{|r(x) - r(y)|}{|x-y|} + F_1(k^2(x-y)^2)$$

is a C^∞ function. By parity, the second integral defines an operator

$$Ku(\theta) = \frac{1}{2}\int_{-\pi}^\pi F_2(\cos\theta, \cos\theta')\mathcal{C}u(\theta)d\theta.$$

There holds $K = \tilde{R}_1\mathcal{C}$ where, by Proposition 1, $R_1 \in Op(\Sigma^{-\infty})$. For the first integral, we make the following classical manipulations. We first write $\cos\theta - \cos\theta' = -2\sin\frac{\theta+\theta'}{2}\sin\frac{\theta-\theta'}{2}$. Thus $\ln|\cos\theta - \cos\theta'| = \ln\left|\sqrt{2}\sin\frac{\theta+\theta'}{2}\right| + \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right|$. We then integrate and apply the change of variables $\theta \to -\theta$ for the second term, yielding

$$S_{k,\omega}u(\cos\theta) = \left(\tilde{S}_{k,1} + \tilde{R}_1\right)\mathcal{C}u(\theta)$$

where

$$\tilde{S}_{k,1}u(\theta) = \frac{-1}{2\pi}\int_{-\pi}^\pi \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right| J_0(k\,|r(\cos\theta) - r(\cos\theta')|)u(\theta')d\theta'\,.$$

Let $g : \theta \mapsto -\frac{1}{2\pi}\ln\left|\sqrt{2}\sin\frac{\theta}{2}\right|$. It is well-known that $\hat{g}(n) = \frac{1}{2n}$ for $n \neq 0$. We may prolong this by $g(\xi) = \frac{1}{2\xi}$ away from $\xi = 0$. Let $a(\theta,\theta') = J_0(k\,|r(\cos\theta) - r(\cos\theta')|)$, which is a smooth function. By Proposition 1, the operator

$$\tilde{S}_{k,1}u(\theta) := \int_{-\pi}^\pi g(\theta-\theta')a(\theta,\theta')u(\theta')d\theta'$$

is in $Op(\Sigma^{-1})$. In particular, $\tilde{S}_{k,1}u$ is a smooth function, from which we deduce that $\theta \mapsto S_{k,\omega}u(\cos\theta)$ is a smooth (even) function. Lemma 13 then ensures

$$S_{k,\omega}u(\cos\theta) = \mathcal{C}S_{k,\omega}u(\theta)\,.$$

This establishes that $\mathcal{C}S_{k,\omega}u = \tilde{S}_k\mathcal{C}u$ for any smooth function u. By Theorem 1, this implies that $S_{k,\omega} \in Op\left(S_T^{-1}\right)$. We can compute the symbol of $\tilde{S}_{k,1}$ using the asymptotic expansion (44). The terms $\partial_s^j a(t,s)_{|t=s}$, can be related to the geometric characteristics of Γ through expansion (18). Using a computer calculator, we find that the the rhs of (49) is an asymptotic expansion of $\tilde{S}_{k,1}$. Obviously, this expansion also holds for $\tilde{S}_k := \tilde{S}_{k,1} + \tilde{R}_1$, which proves the results. $\square$

In particular, by Corollary 7,

Corollary 9. *S_{k,ω_Γ} is continuous from $T^s(\Gamma)$ to $T^{s+1}(\Gamma)$ for all $s \in \mathbb{R}$ and thus maps $C^\infty(\Gamma)$ to itself.*

Lemma 25. *The operator $-(\omega_\tau\partial_\tau)^2 - k^2\omega_\Gamma^2$ is in $Op(S_T^2(\Gamma))$ and satisfies*

$$\mathcal{C}R\left[-(\omega_\tau\partial_\tau)^2 - k^2\omega_\Gamma^2\right]R^{-1} = \tilde{D}_k\mathcal{C}$$

where $\tilde{D}_k \in Op(\Sigma^2)$ has the following symbol

$$\sigma_{\tilde{D}_k}(\theta,\xi) = |\xi|^2 - k^2\,|\Gamma|^2\sin^2(\theta)\,. \tag{51}$$

Proof. Recalling eqs. (19) and (20), one has

$$-(\omega_\Gamma)^2 - k^2\omega_\Gamma^2 = R^{-1}\left[-(\omega\partial_x)^2 - k^2\omega^2\right]R\,.$$

Letting $D_k = -(\omega\partial_x)^2 - k^2\omega^2$,

$$D_kT_n = (n^2 - k^2\,|\Gamma|^2\,\omega^2)T_n\,.$$

The result is then a consequence of Theorem 1. □

Theorem 3. *The operators $\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]$ and S_{k,ω_Γ} are respectively in $Op(S_T^2(\Gamma))$ and $Op(S_T^{-1}(\Gamma))$ and satisfy*

$$\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + T_{-4}.$$

Proof. We have shown that $\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]$ and S_{k,ω_Γ} are respectively in $Op(S_T^2(\Gamma))$ and $Op(S_T^{-1}(\Gamma))$ in the previous two lemmas. Using the method described in Remark 3, we can compute an asymptotic expansion of the symbol of the pseudo-differential operator

$$\left[-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right]S_{k,\omega_\Gamma}^2 - \frac{I_d}{4}\,.$$

The symbol of this operator is found to be in $S_T^{-4}(\Gamma)$, from which the result follows. □

Remark 4. *The previous theorem implies the following fact*

$$-(\omega_\Gamma\partial_\tau)^2S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + R$$

where R is in $Op(S_T^2(\Gamma))$. This is also a compact perturbation of the identity. Nevertheless, since $R = k^2\omega_\Gamma^2S_{k,\omega_\Gamma}^2 + T_{-4}$ the term $k^2\omega^2S_{k,\omega}^2$ can be viewed as the leading first order correction accounting for the wavenumber. The inclusion of this term in the preconditioner leads to a drastic reduction of the number of GMRES iterations in numerical applications, as demonstrated in [1].

4.2 Neumann problem

We saw in Lemma 15 that $N_{k,\omega_\Gamma} = N_1 - k^2 N_2$ where

$$N_1 = -\partial_\tau S_{k,\omega}\omega_\Gamma \partial_\tau \omega_\Gamma$$

and $N_2 = V_k \omega_\Gamma^2$ with

$$V_k u(x) = \int_\Gamma \frac{G_k(x-y) N(x) \cdot N(y) u(y)}{\omega_\Gamma(y)} d\sigma(y) \,.$$

Lemma 26. *The operator N_1 is in $Op(S_U^2(\Gamma))$ and*

$$\mathcal{S} R N_1 R^{-1} = \tilde{N}_1 \mathcal{S}$$

where $\tilde{N}_1$ is a PPDO with a symbol $\sigma_{\tilde{N}_1}$ satisfying

$$\sigma_{\tilde{N}_1}(\theta, \xi) = \frac{\xi}{2} + \frac{1}{16} \frac{k^2 \left|\Gamma\right|^2 \sin^2(\theta)}{\xi} + i \frac{k^2 L^2 \sin\theta \cos\theta}{16\xi^2} + \Sigma^{-3} \tag{52}$$

Proof. This result is obtained by symbolic calculus combining Lemma 24 and Lemma 22. ☐

A small adaptation of the proof of Lemma 24 yields the following result:

Lemma 27. *The operator V_k is in $Op(S_T^{-1}(\Gamma))$ and*

$$\mathcal{C} R V_k R^{-1} = \tilde{V}_k \mathcal{C}$$

where $\tilde{V}_k$ is a PPDO with a symbol $\sigma_{\tilde{N}_2}$ satisfying

$$\sigma_{\tilde{V}_k} = \frac{1}{2\xi} + \Sigma^{-3}$$

Applying Lemma 23, we deduce

Corollary 10. *The operator N_2 is in $Op(S_U^{-1}(\Gamma))$ and satisfies*

$$\mathcal{S} R N_2 R^{-1} = \tilde{N}_2 \mathcal{S}$$

where the symbol of $\tilde{N}_2$ has the asymptotic expansion

$$\sigma_{\tilde{N}_2} = \frac{\sin^2\theta}{2\xi} + i \frac{\sin\theta\cos\theta}{2\xi^2} + \Sigma^{-3} \,. \tag{53}$$

It is also easy to check that

Lemma 28. *The operator $\left[-(\partial_\tau \omega_\Gamma)^2 - k^2 \omega_\Gamma^2\right]$ is in $Op(S_U^2(\Gamma))$ and satisfies*

$$\mathcal{S} R \left[-(\partial_\tau \omega_\Gamma)^2 - k^2 \omega_\Gamma^2\right] R^{-1} = \tilde{D}_k \mathcal{S}$$

where $\tilde{D}_k$ is the operator defined in Lemma 25.

Theorem 4. *The operators* $\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$ *and* N_{k,ω_Γ} *are respectively in* $Op(S_U^2(\Gamma))$ *and* $Op(S_U^1(\Gamma))$*and satisfy*

$$N_{k,\omega_\Gamma}^2 = \left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right] + U_{-2}.$$

Proof. Gathering the previous lemmas, we have asymptotic expansions available for the symbols of the operators N_{k,ω_Γ} and $\left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$. We can thus, using the method of Remark 3, compute an asymptotic expansion of the symbol of the operator $N_{k,\omega_\Gamma}^2 - \left[-(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2\right]$ which turns out to be in $S_U^{-2}(\Gamma)$, giving the result. □

5 Conclusion

This work gives a complete mathematical analysis of the matters exposed in [1], where it is explained how the parametrix built for the layer potentials can be exploited to get efficient preconditioners in the resolution of the first kind integral equations by the Galerkin method of the second section. One could also derive expressions of parametrix to further orders, to include corrective terms accounting for the curvature of the curve. Moreover, it should be possible to extend the pseudo-differential theory exposed here to three space dimensions by using pseudo-differential operators on the sphere rather than the PPDOs considered here.

J'ai fait cette conclusion à la va vite, je me demande s'il faut en mettre une.

References

[1] François Alouges and Martin Averseng. New preconditioners for the first kind integral equations on open curves. *arXiv preprint :*.

[2] Carlos Jerez-Hanckes and Jean-Claude Nédélec. Explicit variational forms for the inverses of integral logarithmic operators over an interval. *SIAM Journal on Mathematical Analysis*, 44(4):2666–2694, 2012.

[3] Martin Averseng. Maple code for Symbolic calculus. `https://www.github.com/MartinAverseng/MapleSymbolicCalculus`.

[4] John C Mason and David C Handscomb. *Chebyshev polynomials*. CRC Press, 2002.

[5] W McLean. A spectral galerkin method for a boundary integral equation. *Mathematics of computation*, 47(176):597–607, 1986.

[6] William Charles Hector McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge university press, 2000.

[7] Ernst P Stephan and Wolfgang L Wendland. A hypersingular boundary integral method for two-dimensional screen and crack problems. *Archive for Rational Mechanics and Analysis*, 112(4):363–390, 1990.

[8] V Thrunen and Gennadi Vainikko. On symbol analysis of periodic pseudodifferential operators. *ZEITSCHRIFT FUR ANALYSIS UND IHRE ANWENDUNGEN*, 17:9–22, 1998.