Boundary Integral Operators on Curved Polygons (*).

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Summary. – *The operators of the single layer potential, double layer potential, and the Hilbert transform, acting on plane curves with corners, are members of the class of integral operators which we define in this paper. We use expansions in homogeneous kernels of increasing degree to define lower order symbols via local Mellin transforms. In this way we can study the mapping properties in (augmented) Sobolev spaces including Garding inequalities and higher regularity, thus generalizing the results of* [3] *and* [4] *from polygons to curved polygons.*

1. – Introduction.

Corner singularities of solutions of elliptic boundary value problems in two dimensions have been extensively studied in the literature, for references see [10], [11]. If the boundary near the corner consists of straight segments and the operators have constant coefficients then the available formulas are explicit enough to be used for precise numerical computations, and the available functional analysis is powerful enough to provide the corresponding error estimates. If the boundary is curved near the corner then the situation is less clear, although KONDRATIEV's fundamental paper [14] treated this case too (as well as higher dimensions).

The method of boundary integral equations is able to supply the necessary constructive ways of finding the singularities of higher order also for curved boundaries. To this end one has to expand the kernels of the operators near the corner into a sum of a principal part, lower order terms, and a remainder, where the principal part corresponds to a piecewise linear boundary, the lower order terms contain higher derivatives of the curve's parameter representation, and the principal part as well as the lower order terms are homogeneous kernels which can be studied using Mellin transformation.

Local Mellin transformation is a useful tool in the theory of pseudodifferential operators on piecewise smooth plane curves (and, more general, manifolds with conical points), see e.g. [19], [9], [13], [6], [7], [8], [2], [3], [4], [15], [16]. There are several attempts to define a general calculus of Mellin symbols comparable to that of pseudodifferential operators on smooth manifolds (see [13], [6], [18], [16]). So the operators considered here are contained in the class Op-$\Sigma^m_{c,d}$ of LEWIS & PARENTI [16]. Nevertheless, my results on higher regularity seem to be new. Only KOMEČ [13] used similar expansions (in higher dimensions).

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There are two kinds of parameters describing regularity: one at the corner point and another away from the corner. Only the second is analoguous to the Sobolev index in ordinary pseudodifferential operators and is usually studied when weighted Sobolev spaces are used for the description of regularity. Higher regularity at the corner requires an expansion into singular functions. This can be done using the SHAMIR - KONDRATIEV technique ([19], [14]) of analytic continuation and application of the residue theorem in the Mellin inversion integral. The functional analytic description then uses Sobolev spaces (even without weight) augmented by finite dimensional spaces of singular functions.

Higher regularity results of this kind are directly applicable in improved error estimates for numerical approximations of the respective integral equations, and for this reason our representation will be as explicit as possible. Also we do not try to achieve more generality than necessary for covering the main features of the relevant examples. Some obvious generalizations are indicated. In [5] we give an example of application to the Dirichlet, Neumann, and mixed problems of potential theory.

2. – Mellin transforms and some homogeneous kernels.

The Mellin transform $\tilde{f}$ of a function $f \in C_0^\infty(0, \infty)$ is defined by

$$\tilde{f}(\lambda) = \int_0^\infty x^{i\lambda - 1} f(x)\, dx \qquad (\lambda \in \boldsymbol{C})\,.$$

We start with the basic Mellin integral (see [19], [9], [16])

$$\int_0^\infty x^{i\lambda-1}(x \exp[i\omega] - 1)^{-1}\, dx = i\pi \frac{\exp[(\omega - \pi)\lambda]}{\sinh \pi\lambda} \,(\operatorname{Im} \lambda \in (-1, 0),\ \operatorname{Re} \omega \in (0, 2\pi))\,.$$

It allows to compute the Mellin symbol for the integral operator A_ω defined by

$$(2.1)\qquad \left\{\begin{aligned} &A_\omega f(x) := \int_0^\infty (y \exp[i\omega] - x)^{-1} f(y)\, dy \qquad (\omega \in (0, 2\pi)):\\ &\widetilde{A_\omega f}(\lambda) = -i\pi \exp[-i\omega] \frac{\exp[(\pi - \omega)\lambda]}{\sinh \pi\lambda} \tilde{f}(\lambda) \,(f \in C_0^\infty(0, \infty),\ \operatorname{Im} \lambda \in (-1, 0))\,. \end{aligned}\right.$$

The proof of (2.1) uses the convolution theorem for the Mellin transform and the fact that A_ω is a Mellin convolution, its kernel being homogeneous of degree -1.

From (2.1) we derive the Mellin transforms for a class of operators with rational kernels of the following type: Let

$$r_\omega(x, y) := y \exp[i\omega] - x\,,$$

$$R^{-n}_{\omega,k,l}(x, y) := r_\omega(x, y)^{-n} x^k y^l\,,$$

$$R^{-n}_{\omega,k,l} f(x) \quad := \int_0^\infty R^{-n}_{\omega,k,l}(x, y) f(y)\, dy\,.$$

(We will denote any integral operator and its kernel function by the same letter).

LEMMA 1. – *Let* $\omega \in (0, 2\pi)$, $k, l \in \boldsymbol{Z}$, $n \in \boldsymbol{N}$, $f \in C_0^\infty(0, \infty)$. *Then the Mellin integral for* $R^{-n}_{\omega,k,l} f$ *converges for* $\operatorname{Im} \lambda \in (k - n, k)$, *and*

$$\widehat{R^{-n}_{\omega,k,l} f}(\lambda) = \hat{R}^{-n}_\omega(\lambda - ki)\, \hat{f}(\lambda + (n - k - l - 1) i) \tag{2.2}$$

with

$$\hat{R}^{-n}_\omega(\lambda) = \frac{(-i)^n \pi}{(n-1)!} \exp[-in\omega](\lambda + i)(\lambda + 2i) \ldots (\lambda + (n-1) i)\, \frac{\exp[(\pi - \omega)]\lambda}{\sinh \pi\lambda}\,. \tag{2.3}$$

The Mellin inversion formula

$$R^{-n}_{\omega,k,l} f(x) = \frac{1}{2\pi} \int\limits_{\operatorname{Im}\lambda = h} x^{-i\lambda} \hat{R}^{-n}_\omega(\lambda - ki) \hat{f}(\lambda + (n - k - l - 1) i)\, d\lambda \tag{2.4}$$

holds for all $h \in (k - n, k)$.

PROOF. – If we denote by X and D the operators

$$Xg(x) = xg(x) \qquad \text{and} \qquad Dg(x) = \frac{dg(x)}{dx}$$

then

$$R^{-n}_{\omega,k,l} = \frac{1}{(n-1)!} X^k D^{n-1} A_\omega X^l\,.$$

Now for $g \in C^\infty(0, \infty)$ we have $\widehat{Xg}(\lambda) = \hat{g}(\lambda - i)$ and $\widehat{Dg}(\lambda) = -i(\lambda + i)\hat{g}(\lambda + i)$ on each line $\operatorname{Im} \lambda = h$ where the respective right hand side Mellin integrals converge absolutely. Thus (2.2) and (2.3) follow from (2.1) for $\operatorname{Im} \lambda \in (k - n, k - n + 1)$. As $\hat{f}$ and $\hat{R}^{-n}_\omega$ are exponentially decreasing on each line $\operatorname{Im} \lambda = h$, the Mellin inversion formula (2.4) holds for $h \in (k - n, k - n + 1)$, and the path of integration can be shifted to parallel lines as long as the integrand has no poles. $\hat{f}$ is an entire function, so poles come only from the factor $(\sinh \pi\lambda)^{-1}$ that has simple poles for $\lambda \in i\boldsymbol{Z}$. The poles at $\lambda = (k-1)i, (k-2)i, \ldots, (k - n + 1)i$ are killed by the polynomial factor in (2.3), so there are no poles in the strip $\operatorname{Im} \lambda \in (k - n, k)$. Thus (2.4) and hence (2.2) hold in this whole strip.

REMARKS. – 1) If one shifts the path of integration in (2.4) above $\operatorname{Im}\lambda = k$, one has to take into account the poles at $\lambda = i(k+m)$, $m \in \boldsymbol{N}_0$, yielding the contribution

$$(2.5) \qquad -i \operatorname*{Res}_{\lambda=i(k+m)} \{x^{-i\lambda}\hat{R}_\omega^{-n}(\lambda - ki)\hat{f}(\lambda + (n-k-l-1)i\} =$$

$$= -x^m \binom{m+n-1}{m} \exp[-i(m+n)\omega]\hat{f}(i(m+n-l-1)) =$$

$$= -\int_0^\infty x^k y^{l-n} \exp[-in\omega]\binom{m+n-1}{m}\left(\frac{x}{y}\exp[-i\omega]\right)^m f(y)\,dy$$

to the left hand side of (2.4). This corresponds just to the Taylor expansion with respect to x about $x=0$ for the kernel

$$R_{\omega,k,l}^{-n}(x,y) = x^k y^{l-n}\exp[-in\omega]\left(1 - \frac{x}{y}\exp[-i\omega]\right)^{-n}.$$

Similarly, shifting below $\operatorname{Im}\lambda = k - n$ involves poles corresponding to the Taylor expansion with respect to y about $y = 0$.

2) If regarded as « pseudodifferential operator of Mellin type » [16], $R_{\omega,k,l}^{-n}$ should be written as

$$R_{\omega,k,l}^{-n} f(x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=h} x^{-i\lambda}x^{k+l+1-n}\hat{R}_\omega^{-n}(\lambda + (l+1-n)i)\hat{f}(\lambda)\,d\lambda\,, \qquad h \in (-l-1, n-l-1)\,,$$

instead of (2.2)-(2.4). In this way it acquires an « x-dependent symbol » (our λ corresponds to $-iz$ in [16])

$$\sigma(x,\lambda) = x^{k+l+1-n}\hat{R}_\omega^{-n}(\lambda + (l+1-n)i)\,.$$

3. – A class of kernel functions on [0, 1].

The following is not the definition of the whole class of integral operators which we want to treat by Mellin transformation. It is only a description of the lower order terms. As principal parts (highest order terms) we will use the « constant coefficient » operators with kernels like $\log|x-y|$, $\log|y\exp[i\omega] - x|$, $pv(1/(x-y))$, which have been treated by Mellin transformation earlier (see [2], [3], [4], [15]).

Let $J := [0,1]$ and $Q := J\times J \setminus \{(0,0)\}$, so that $\operatorname{cl} Q = J \times J$.

DEFINITION 1. – *Let* $\omega \in (0, 2\pi)$ *and* $m, p \in \boldsymbol{Z}$ *with* $m \geqq -1$, $p \leqq m+1$. *Then*

$\mathcal{K}^{\omega}_{p,m}$ *is the class of complex valued functions* $K \in C^{m+1}(Q)$ *having a decomposition*

$$K = K_p + K_{p+1} + \ldots + K_m + R_m \tag{3.1}$$

with the following properties:

(i) *For each* $j = p, \ldots, m$, K_j *is homogeneous of degree* j *and has the special form*

$$K_j = \sum_{k=0}^{n_j+j} c_k^j R^{-n_j}_{\omega,k,n_j+j-k} \quad \text{for some } n_j \in \boldsymbol{N} \text{ and constants } c_k^j \in \boldsymbol{C}\,. \tag{3.2}$$

(ii) *For each multiindex* $\alpha \in \boldsymbol{N}_0^2$ *with* $|\alpha| \leq m+1$ *there is a* $C > 0$ *with*

$$|D^{\alpha} R_m(x, y)| \leq C(x+y)^{m+1-|\alpha|} \quad \text{for all } (x, y) \in Q\,. \tag{3.3}$$

For $p = m + 1$ *in* (3.1) *we mean the trivial sum, i.e.* $K = R_m$, *and condition* (i) *is void.*

REMARKS. – 1) The decomposition (3.1) is uniquely defined by (i) and (ii). If (ii) is replaced by the weaker requirement « $D^{\alpha}R_m$ bounded for $|\alpha| \leq m+1$ », then the K_j for $j \geq 0$ are unique only up to the Taylor polynomial of R_m. The class $\mathcal{K}^{\omega}_{p,m}$ remains the same in this case.

2) The numbers n_j and the representation (3.2) of K_j are not unique: One has e.g.

$$R^{-n}_{\omega,k,l} = \exp[i\omega] R^{-n-1}_{\omega,k,l+1} - R^{-n-1}_{\omega,k+1,l}\,. \tag{3.4}$$

3) An equivalent definition is:

$K \in \mathcal{K}^{\omega}_{p,m}$ iff there exist $n \in \boldsymbol{N}$ and a polynomial $P(x, y)$ of degree $\geq n + m$ with $D^{\alpha}P(0, 0) = 0$ for $|\alpha| < n + p$, such that $K - r_{\omega}^{-n}P$ has bounded derivatives of order up to $m + 1$.

4) $\mathcal{K}^{\omega}_{p_1,m_1} \subset \mathcal{K}^{\omega}_{p_2,m_2}$ if $p_1 \geq p_2$ and $m_1 \geq m_2$, and

$$\mathcal{K}^{\omega}_{p_1,m} = \{K \in \mathcal{K}^{\omega}_{p_2,m} | K_j = 0 \text{ for } j = p_2, \ldots, p_1 - 1\} \quad \text{for } p_2 \leq p_1\,.$$

5) Note that $R^{-n}_{\omega,k,l}$ uniquely determines ω, hence

$$\mathcal{K}^{\omega_1}_{p,m} \neq \mathcal{K}^{\omega_2}_{p,m} \quad \text{for } \omega_1 \neq \omega_2 \text{ and } p \leq m\,.$$

The following three lemmas are the main tools for showing that a given K is in $\mathcal{K}^{\omega}_{p,m}$.

LEMMA 2. – *$C^{m+1}(\mathrm{cl}\, Q) \subset \mathcal{K}^{\omega}_{0,m}$, and for $K \in C^{m+1}(\mathrm{cl}\, Q)$*

$$K_j(x, y) = \sum_{k+l=j} \frac{x^k y^l}{k!\, l!} \frac{\partial^j}{\partial x^k\, \partial y^l} K(0, 0) .$$

LEMMA 3. – *Let $K^{(l)} \in \mathcal{K}^{\omega}_{p_l, m_l}$ for $l = 1, 2$,*

$$m := \min \{m_1, m_2, m_1 + p_2, m_2 + p_1\} \geqq -1 , \quad \text{and} \quad p := \min \{p_1 + p_2, m + 1\} .$$

Then the (pointwise) product $K := K^{(1)} \cdot K^{(2)}$ is in $\mathcal{K}^{\omega}_{p,m}$, and

$$K_j = \sum_{q=p_1}^{j-p_2} (K^{(1)})_q (K^{(2)})_{j-q} \quad \text{for } p \leqq j \leqq m . \tag{3.5}$$

PROOF. – We may assume $p = p_1 + p_2 \leqq m + 1$. The K_j from (3.5) are of the form (3.2), so one has only to check the estimates (3.3) for the remainder

$$R_m = (K^{(1)}_{p_1} + \ldots + K^{(1)}_{m_2}) \cdot R^{(2)}_{m_2} + (K^{(2)}_{p_2} + \ldots + K^{(2)}_{m_2}) \cdot R^{(1)}_{m_1} + R^{(1)}_{m_1} \cdot R^{(2)}_{m_2} .$$

The worst members here are $K^{(1)}_{p_1} \cdot R^{(1)}_{m_1}$, consisting of summands $R^1(x, y) := x^k y^l r_{\omega}^{-n_{p_1}}(x, y) R^{(2)}_{m_2}(x, y)$, similarly $R^2(x, y)$ with sub- and superscripts 1 and 2 interchanged, and $R^3(x, y) := R^{(1)}_{m_1} \cdot R^{(2)}_{m_2}(x, y)$. For $|\alpha| \leqq m + 1$ we have by Leibniz' rule

$$|D^{\alpha} R^1(x, y)| \leqq \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} |D^{\beta} r_{\omega}^{-n_{p_1}}(x, y)| |D^{\alpha-\beta} x^k y^l R^{(2)}_{m_2}(x, y)| .$$

Now $D^{\beta} r_{\omega}^{-n_{p_1}}(x, y) = O\big((x + y)^{-n_{p_1} - |\beta|}\big)$ by homogeneity, and by (3.3) $D^{\alpha-\beta} x^k y^l R^{(2)}_{m_2}(x, y) = O\big((x + y)^{m_2 + 1 + k + l - |\alpha| + |\beta|}\big)$ for $(x, y) \in Q$, hence

$$D^{\alpha} R^1(x, y) = O\big((x + y)^{m_2 + 1 + k + l - n_{p_1} - |\alpha|}\big) = O\big((x + y)^{m_2 + p_1 + 1 - |\alpha|}\big) ,$$

where we used $k + l - n_{p_1} = p_1$ from (3.2). Thus for $m \leqq m_2 + p_1$ we have $D^{\alpha} R^1(x, y) = O\big((x + y)^{m + 1 - |\alpha|}\big)$ as desired. Similarly R^2 is estimated. For R^3 we obtain the estimate

$$\begin{aligned} |D^{\alpha} R^3(x, y)| &\leqq \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} |D^{\beta} R^{(1)}_{m_1}(x, y)| |D^{\alpha-\beta} R^{(2)}_{m_2}(x, y)| \\ &\leqq C \sum_{\beta \leqq \alpha} (x + y)^{m_1 + 1 - |\beta|} (x + y)^{m_2 + 1 - |\alpha| + |\beta|} = O\big((x + y)^{m_1 + m_2 + 2 - |\alpha|}\big) = O\big((x + y)^{m + 1 - |\alpha|}\big) , \end{aligned}$$

if in the whole sum we have $|\beta| \leqq m_1 + 1$ and $|\alpha - \beta| \leqq m_2 + 1$, i.e. if $|\alpha| \leqq \min \{m_1, m_2\} + 1$. By assumption this is true for all $|\alpha| \leqq m + 1$.

LEMMA 4. – *Let $K \in \mathcal{K}^{\omega}_{1,m}$ and F be holomorphic on a neighborhood of the compact set* $\operatorname{cl} K(Q) = K(Q) \cup \{0\}$. *Then* $F(K) \in \mathcal{K}^{\omega}_{0,m}$ *and*

$$F(K)_j = \sum_{k=0}^{j} \frac{1}{k!} F^{(k)}(0)(K^k)_j \quad \text{for } 0 \leqq j \leqq m \,. \tag{3.6}$$

PROOF. – F is defined on a neighborhood of the origin and thus has a Taylor expansion

$$F(z) = \sum_{k=0}^{m} \frac{1}{k!} F^{(k)}(0) z^k + z^{m+1} F_m(z)$$

with some holomorphic function F_m. Thus

$$F(K(x, y)) = \sum_{k=0}^{m} \frac{1}{k!} F^{(k)}(0) K^k(x, y) + K^{m+1}(x, y) F_m(K(x, y)) \,,$$

and by Lemma 3 the sum on the right hand side is in $\mathcal{K}^{\omega}_{0,m}$, so (3.6) holds if we can show that the last term $K^{m+1}F_m(K)$ belongs to the remainder, i.e. is contained in $\mathcal{K}^{\omega}_{m+1,m}$. This is the case for K^{m+1} by Lemma 3. Now from the definition of $\mathcal{K}^{\omega}_{1,m}$ follows that $D^\beta K(x, y) = O((x + y)^{1-|\beta|})$ for $|\beta| \leqq m + 1$. This together with the boundedness of the derivatives of F_m on the range of K yields with the chain rule the estimates

$$D^\beta F_m(K(x, y)) = O((x + y)^{1-|\beta|}) \quad \text{for } |\beta| \leqq m + 1 \,.$$

Because

$$D^{\alpha-\beta} K^{m+1}(x, y) = O((x + y)^{m+1-|\alpha|+|\beta|}) \,,$$

Leibniz' rule gives

$$D^\alpha[K^{m+1}(x, y) F_m(K(x, y))] = O((x + y)^{m+2-|\alpha|}) \quad \text{for } |\alpha| \leqq m + 1 \,,$$

which implies the desired $O((x + y)^{m+1-|\alpha|})$ estimate.

With these lemmas we are now able to show in several examples how $\mathcal{K}^{\omega}_{p,m}$ can be used to describe the lower order terms in natural expansions of some pseudo-differential operators on curves with corners.

Let Γ denote the neighborhood of one corner on such a curve, i.e. Γ is composed of two arcs Γ_- and Γ_+, intersecting only at the corner point and parametrized by functions

$$x_- \text{ resp. } x_+ \colon [0, 1] \to \boldsymbol{C} \,,$$

which we assume to belong to $C^{m+2}(J)$, $m \geqq 0$. We may further assume that the parameter is the arc length measured from the corner, and

$$x_-(0) = 0 = x_+(0); \quad \dot{x}_-(0) = \exp[i\omega]; \quad \dot{x}_+(0) = 1 .$$

Thus Γ_- and Γ_+ meet at the origin under the angle ω.

The following lemma gives the connection between the Taylor expansions of $x_\pm$ and expansions like (3.1) for the kernels in our examples.

Lemma 5. – *Let $A(s, t) = x_-(t) - x_+(s)$ and $B = r_\omega^{-1} \cdot A - 1$. Then $B \in \mathcal{K}^\omega_{1,m}$.*

Proof. – Clearly $A \in C^{m+2}(\mathrm{cl}\, Q)$, and $A(s, t) = t \exp[i\omega] - s + R_1(s, t)$ with $R_1 \in \mathcal{K}^\omega_{2,m+1}$. Now we have $r_\omega^{-1} \in \mathcal{K}^\omega_{-1,\infty}$, and we can apply Lemma 3 to find $B = r_\omega^{-1} \cdot R_1 \in \mathcal{K}^\omega_{1,m+1}$.

Any integral (or pseudodifferential) operator A on Γ gives rise to four operators A_{--}, A_{-+}, A_{+-}, A_{++} on $J = [0, 1]$, where the subscripts $\alpha, \beta \in \{-, +\}$ mean: integration along Γ_α, evaluation on Γ_β:

$$A_{\alpha\beta} f(s) = \int_0^1 A_{\alpha\beta}(s, t) f(t)\, dt = (Ag)\big(x_\beta(s)\big)$$

with

$$g\big(x_\alpha(t)\big) = f(t) \quad (t \in J),\ g = 0 \ \text{ on } \Gamma \setminus \Gamma_\alpha .$$

Example 1. – The operator V of the single layer potential on a curve Γ is defined by

$$Vg(z) = -\frac{1}{\pi} \int_\Gamma \log |z - \zeta| g(\zeta)|\, d\zeta| .$$

For the kernels $V_{\alpha\beta}(s, t) = -1/\pi \log |x_\beta(s) - x_\alpha(t)|$ we use the Taylor expansions of $x_\pm$ and obtain an expansion similar to (3.1). The « principal parts » are those obtained by replacing $\Gamma_\pm$ by their respective tangents at the corner point.

Proposition 1. – *For $\alpha, \beta \in \{-, +\}$ let*

$$V^1_{\alpha\beta}(s, t) := V_{\alpha\beta}(s, t) + \frac{1}{\pi} \log |s \dot{x}_\beta(0) - t \dot{x}_\alpha(0)| .$$

Then for $\alpha = \beta$, $V^1_{\alpha\beta} \in C^{m+1}(\mathrm{cl}\, Q)$; and for $\alpha \neq \beta$, $V^1_{\alpha\beta} \in \mathcal{K}^\omega_{1,m} + \mathcal{K}^{2\pi-\omega}_{1,m}$.

Remark. – The homogeneous kernels in the expansion (3.1) for $V^1_{\alpha\beta}$ can easily be computed from the following proof. For example, the next-to-highest term in V_{-+}, i.e. the term homogeneous of degree 1 in

$$V^1_{-+}(s, t) = V_{-+}(s, t) + \frac{1}{\pi} \log |t \exp[i\omega] - s| ,$$

is

$$-\frac{1}{2\pi}\operatorname{Im}\frac{\varkappa_+\exp[i\omega]t^2-\varkappa_-s^2}{t\exp[i\omega]-s}=-\frac{1}{4\pi i}\{\varkappa_+\exp[i\omega]R^{-1}_{\omega,0,2}(s,t)-\varkappa_-R^{-1}_{\omega,2,0}(s,t)-$$
$$-\varkappa_+\exp[-i\omega]R^{-1}_{2\pi-\omega,0,2}(s,t)+\varkappa_-R^{-1}_{2\pi-\omega,2,0}(s,t)\},$$

where $\varkappa_\pm$ are the curvatures on $\Gamma_\pm$ at the corner point [5].

PROOF OF THE PROPOSITION. – The argument of log in $V^1_{++}(s,t)=-1/\pi\log|(x_+(s)-x_+(t))/(s-t)|$ never vanishes and is in $C^{m+1}(\mathrm{cl}\,Q)$ yielding $V^1_{++}\in C^{m+1}(\mathrm{cl}\,Q)$. Similarly for V^1_{--}. From Definition 1 it is clear that for $K\in\mathcal{K}^\omega_{p,m}$ the complex conjugate kernel $\overline{K(s,t)}$ as well as the transpose $K^t(s,t)=K(t,s)$ are contained in $\mathcal{K}^{2\pi-\omega}_{p,m}$. But we have $V^1_{+-}=(V^1_{-+})^t$ and $V^1_{-+}(s,t)=-1/\pi\operatorname{Re}\Big(\log\big((x_+(s)-x_-(t))/(t\exp[i\omega]-s)\big)\Big)$, hence it suffices to show that $K\in\mathcal{K}^\omega_{1,m}$ with $K(s,t):=\log\big((x_+(s)-x_-(t))/r_\omega(s,t)\big)$. With the notation of Lemma 5 we have $K=\log(1+B)$ with $B\in\mathcal{K}^\omega_{1,m}$, and we can use Lemma 4 for the holomorphic function $F(z)=\log(1+z)$ and obtain $K\in\mathcal{K}^\omega_{0,m}$. That even $K\in\mathcal{K}^\omega_{1,m}$ holds, follows from (3.6) and $F(0)=0$.

EXAMPLE 2. – The operator K of the double layer potential on Γ is defined by

$$Kg(z)=-\frac{1}{\pi}\int_\Gamma\left(\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\right)g(\zeta)|\,d\zeta|,$$

where $\partial/\partial n_\zeta$ means derivative with respect to the outer normal at ζ. This gives for $K_{\alpha\beta}$ the representation

$$K_{\alpha\beta}(s,t)=-\frac{1}{\pi}\operatorname{Im}\frac{\alpha\dot{x}_\alpha(t)}{x_\alpha(t)-x_\beta(s)}\qquad(\alpha,\beta\in\{-,+\}).$$

The principal parts for K_{++} and K_{--} vanish, and for K_{-+} and K_{+-} they equal $K_\omega(s,t)=1/\pi\operatorname{Im}\big(\exp[i\omega]r^{-1}_\omega(s,t)\big)$. So they fit into an expansion like (3.1), and hence not only the lower order terms but the whole kernels possess such an expansion:

PROPOSITION 2. – $K_{++},K_{--}\in C^m(\mathrm{Cl}\,Q)$; $K_{-+},K_{+-}\in\mathcal{K}^\omega_{-1,m-1}+\mathcal{K}^{2\pi-\omega}_{-1,m-1}$.

PROOF. – For K_{-+} one has to show $\dot{x}_-(t)/(x_-(t)-x_+(s))\in\mathcal{K}^\omega_{-1,m-1}$. Writing this as $\dot{x}_-(t)\cdot A(s,t)^{-1}=\dot{x}_-(t)\cdot r^{-1}_\omega(s,t)\cdot(1+B(s,t))^{-1}$ in the notation of Lemma 5, one can use $\dot{x}_-\in C^{m+1}\subset\mathcal{K}^\omega_{0,m}$, $r^{-1}_\omega\in\mathcal{K}^\omega_{-1,\infty}$, and $(1+B)^{-1}\in\mathcal{K}^\omega_{0,m}$ according to Lemma 4 with $F(z)=(1+z)^{-1}$; and then apply Lemma 3 to see that the product is in $\mathcal{K}^\omega_{-1,m-1}$. Similarly for K_{-+}.

Since $\operatorname{Im}(t-s)^{-1}=0$, one has

$$\operatorname{Im}\frac{\dot{x}_+(t)}{x_+(t)-x_+(s)}=\operatorname{Im}\left\{\frac{t-s}{x_+(t)-x_+(s)}(t-s)^{-2}\big(\dot{x}_+(t)(t-s)-x_+(t)+x_+(s)\big)\right\}=$$
$$=\operatorname{Im}\left\{\frac{t-s}{x_+(t)-x_+(s)}\int_0^1\int_0^\tau\ddot{x}_+\big(t-(\tau-\sigma)(t-s)\big)\,d\sigma\,d\tau\right\}.$$

Here the first factor is in C^{m+1} and the second in C^m, hence $K_{++} \in C^m(\mathrm{cl}\, Q)$ and similarly $K_{--} \in C^m(\mathrm{cl}\, Q)$.

EXAMPLE 3. – The Hilbert transform (Cauchy singular integral) on Γ is given by

$$Sg(z) = \frac{1}{i\pi} \int_{\Gamma} \frac{1}{\zeta - z} g(\zeta)\, d\zeta\,,$$

where the integral is understood in the Cauchy principal value sense. One finds the kernels

$$S_{\alpha\beta}(s, t) = \frac{1}{i\pi} \frac{\alpha \dot{x}_\alpha(t)}{x_\alpha(t) - x_\beta(s)}$$

with the following principal parts:

For S_{++} and S_{--}: $1/(i\pi(t-s))$ and $-1/(i\pi(t-s))$, respectively; for S_{+-}: $(i\pi \exp[i\omega] r_{2\pi-\omega}(s, t))^{-1}$, and for S_{-+}: $(-i\pi \exp[-i\omega] r_\omega(s, t))^{-1}$.

If we write $S^1_{\alpha\beta} := S_{\alpha\beta} -$ (principal part), then a proof essentially identical to the preceding one shows

PROPOSITION 3. – $S^1_{++}, S^1_{--} \in C^m(\mathrm{cl}\, Q)$; $S_{+-} \in \mathcal{K}^{2\pi-\omega}_{-1,m-1}$; $S_{-+} \in \mathcal{K}^{\omega}_{-1,m-1}$.

EXAMPLE 4. – The fundamental solution of the Helmholtz equation is $-(i/4) H_0^{(1)}(k|z-\zeta|) =: \frac{1}{2} G(z, \zeta)$, where $H_0^{(1)}$ is a Hankel function. If one defines the single layer potential as in Example 1 with G instead of V, one finds the same principal parts as for V because the leading singularity of $H_0^{(1)}(z)$ is $2i/\pi \log z$.

Let us consider G_{+-}: $G^2 := G_{+-} - V_{+-}$ is clearly C^{m+2} outside the origin and of order $O((t+s)^2 \log(t+s))$ for $t + s \to 0$. Therefore we have $G^2 \in C^{1+\mu}(\mathrm{cl}\, Q)$ for all $\mu < 1$. Thus for $G^1 := G_{+-} -$ (principal part), we would find

$$G^1 \in \mathcal{K}^{\omega}_{1,\mu} + \mathcal{K}^{2\pi-\omega}_{1,\mu} \qquad \text{for all } 0 \leq \mu < 1$$

if we would generalize Definition 1 to non-integer values of m in the sense of Hölder continuity of R_m. Note that then for $\mu > 0$ in G^1 a term homogeneous of degree 1 coming from the expansion of V_{-+}, is seperated from the remainder, whereas for $\mu = 0$ there would be the remainder alone.

A second generalization of Definition 1 would be necessary for covering the further expansion of G^1 according to the expansion of $H_0^{(1)}(z)$ at $z = 0$. Because the latter involves terms $z^n \log z$, one would have to include into (3.1) and (3.2) pseudo-homogeneous kernels $r_\omega^{-j} \log r_\omega$. This same generalization would be appropriate for the treatment of general biharmonic or polyharmonic potentials, and it is indeed not difficult to treat such kernels by Mellin transformation, but we will not do this here.

4. – Mapping properties.

We want to study the mapping properties of operators with kernels from $\mathcal{K}^{\omega}_{p,m}$ in the Sobolev spaces $H^s(J)$ and $\tilde{H}^s(J)$ which we define as follows:

$H^s(J)$ consists of restrictions to $(0,1)$ of distributions in $H^s(\boldsymbol{R})$;

$\tilde{H}^s(J)$ consists of restrictions to $(0,1)$ of distributions in $H^s(\boldsymbol{R})$ which vanish on $(-\infty, 0)$;

the norms are the natural ones [17].

Thus both spaces are restrictions to $(0,1)$ of $H^s(\boldsymbol{R}_+)$ and $\tilde{H}^s(\boldsymbol{R}_+)$, respectively, as used in [3]. They both form interpolation scales, and for $s \geqq 0$, $s+\frac{1}{2} \notin \boldsymbol{N}$, $\tilde{H}^s(J)$ is a closed subspace of codimension $[s+\frac{1}{2}]$ of $H^s(J)$. Projecting a function in this subspace is done by subtracting its Taylor polynomial of degree $[s-\frac{1}{2}]$ at the origin.

The weighted Sobolev space $\mathring{W}^s_\alpha$ is defined as the completion of $C_0^\infty(0,\infty)$ under the norm

$$\|f\|^2_{\mathring{W}^s_\alpha} = \frac{1}{2\pi} \int\limits_{\operatorname{Im}\lambda = s-(\alpha+1)/2} (1+|\lambda|^2)^s |\hat{f}(\lambda)|^2 \, d\lambda \,. \tag{4.1}$$

We refer to KONDRATIEV [14] and AVANTAGGIATI [1] for properties of these spaces. It is also known [3] that $\tilde{H}^s(J)$ for $s \geqq 0$ and $H^s(J)$ for $s \leqq 0$ are restrictions of $\mathring{W}^s_0$ to $(0,1)$. We define therefore

$$\overset{\{\sim\}}{H}{}^s := \begin{cases} \tilde{H}^s & \text{for } s \geqq 0\,, \\ H^s & \text{for } s \leqq 0\,. \end{cases}$$

For the following we fix a cut-off function $\chi \in C_0^\infty[0,1)$ which is equal to one on a neighborhood of 0. We denote by χ also the operator of multiplication by χ. Then χ is continuous as operator from $\mathring{W}^s_0$ to $\overset{\{\sim\}}{H}{}^s(\boldsymbol{R}_+)$, from $\overset{\{\sim\}}{H}{}^s(\boldsymbol{R}_+)$ to $\overset{\{\sim\}}{H}{}^s(J)$, and from $\overset{\{\sim\}}{H}{}^s(J)$ to $\mathring{W}^s_0$. If the domain of definition, $\boldsymbol{R}_+$ or J, is of no importance, we simply omit it.

From the Parseval relation (4.1) one proves most easily the boundedness of operators such as $R^{-n}_{\omega,k,l}$ which are converted into multiplication operators by Mellin transformation. Thus Lemma 1 gives immediately

LEMMA 6. – $R^{-n}_{\omega,k,l}\colon \mathring{W}^s_\alpha \to \mathring{W}^t_\beta$ *is continuous for any* s, t, *and* α *satisfying* $s-(\alpha+1)/2 \in (-l-1, n-l-1)$. *Then* β *is given by* $\beta = \alpha + 2(t-s+n-k-l-1)$. *In particular,* $\alpha=\beta=0$ *is possible for* $s \in (-l-\frac{1}{2}, n-l-\frac{1}{2})$ *and* $t = s-n+k+l+1$. *Hence*

$$\chi R^{-n}_{\omega,k,l}\chi\colon \overset{\{\sim\}}{H}{}^s \to \overset{\{\sim\}}{H}{}^{s-n+k+l+1} \tag{4.2}$$

is continuous for $n \in \boldsymbol{N}$, $k, l \in \boldsymbol{Z}$, *and* $s \in (-l-\frac{1}{2}, n-l-\frac{1}{2})$.

Before using this result to study the mapping properties of the operators K_j from the expansion (3.1), we look at the remainder R_m.

LEMMA 7. – *Let R_m satisfy* (3.3). *Then*

$$R_m\colon\ H^s(J)' \to H^{m+1-s}(J) \quad \textit{and}$$

$$R_m\chi\colon \tilde{H}^{-s}(J) \to H^{m+1-s}(J) \quad \textit{are continuous for all } s \in [0, m+1]\,.$$

PROOF. – Let $f \in L^2(J)$. Then clearly $R_m f \in C^{m+1}(0, 1]$, and all derivatives up to order $m+1$ of $R_m f$ are bounded on $(0, 1]$. Hence $R_m f \in H^{m+1}(J)$. (Actually $R_m f \in C^{m+1}(J)$.) The continuity of the mapping $R_m\colon L^2(J) \to H^{m+1}(J)$ is obvious.

Now the formal adjoint kernel $R_m^*(x, y) = \overline{R_m(y, x)}$ also satisfies (3.3), hence defines a continuous operator $R_m^*\colon L^2(J) \to H^{m+1}(J)$. Taking adjoints, one finds that $R_m\colon H^{m+1}(J)' \to L^2(J)$ is continuous. (Here $R_m f$ of course is defined in the distributional sense: $R_m f(x) = \langle R_m(x,\cdot), f\rangle$, where $\langle\cdot,\cdot\rangle$ denotes duality between $H^{m+1}(J)$ and its dual space $H^{m+1}(J)'$.)

Interpolating these two cases of continuity of R_m, we see that $R_m\colon H^s(J)' \to H^{m+1-s}(J)$ for $s \in [0, m+1]$ is continuous.

For the second statement in the lemma we use the fact that $\chi\tilde{H}^{-s}(J)$ is continuously embedded in $H^s(J)'$.

REMARK. – In general, $R_m\chi$ is not even definable on the space $H^{-s}(J)$, nor does it map into $\tilde{H}^{m+1-s}(J)$ (except, of course, for $|s| < \frac{1}{2}$ or $|m+1-s| < \frac{1}{2}$, respectively, because for $|s| < \frac{1}{2}$ the spaces $H^s(J)$, $\tilde{H}^s(J)$, and $H^{-s}(J)'$ coincide). From (3.3) follows only that the coefficients $(1/k!)(d^k/dx^k)(R_m\chi f)(0)$ of the Taylor polynomial of degree n for $R_m\chi f$ are defined by linear functionals acting on f that are continuous on the spaces $H^{-s}(J)$ for $s < m + \frac{3}{2} - n$ $(1 \leqq n \leqq m)$ or $s \leqq m+1$ $(n = 0)$, respectively. If one needs in a particular application $R_m\chi u \in \tilde{H}^{m+1-s}$, then one has to check that these functionals vanish; compare the example in the following proof.

THEOREM 1. – *Let Γ be a curved polygon piecewise of class C^3, with corner angles different from* 0 *and* 2π. *Then the Gårding inequalities, shown in Theorems* 2.19 & 2.24 *of* [3] *and Theorem* 3.5 *of* [4] *for boundary integral operators in the case of a piecewise linear curve, remain valid.*

PROOF. – One uses the expansion (3.1) only for splitting off the principal parts of the operators. These principal parts are just those operators for which the Gårding inequalities in [3] and [4] have been proved.

What remains to show is that the remainders R_m from (3.1) yield compact operators in the respective Sobolev spaces. If $m = 1$, then according to Propositions 1 and 2 and Lemma 7 (for $s = 0$ and $m = 0$ or for $s = \frac{1}{2}$ and $m = 1$, respectively), these remainders map at least into H^1 which is always sufficient.

The only problem is to check the compatibility conditions at the corner. It is here that the above-mentioned Taylor expansions of $R_m f(x)$ at $x = 0$ come in. We will make this check here for the representative case of the double layer potential. One has to show (in order to find that K maps $H^1(\Gamma)$ into itself) that for $f \in C^\infty(\Gamma)$, $\chi K \chi f$ is continuous at the corner, where we may assume $\Gamma = \Gamma_- \cup \Gamma_+$ as in § 3. Because the principal parts satisfy this requirement, we have to look only at the operators $K^1_{\alpha\beta}$. Now from the definition we see that $K_{\alpha\beta}(0, t) = -\alpha/\pi \operatorname{Im} (\dot{x}_\alpha(t)/x_\alpha(t))$ is independent of β. This is then also true for the remainders $K^1_{\alpha\beta}$ which map $L^2(J)$ into $C(J)$. Independence of β means equal limits at 0 on Γ_- and on Γ_+, and hence $\chi K \chi -$ (principal parts) maps even arbitrary $L^2(\Gamma)$-functions to continuous ones.

Let us now look at the terms K_j in the expansion (3.1): In (3.2) there appear $R^{-n}_{\omega,k,l}$ with indices k and l ranging from 0 to $n + j$. Although the order $n - k - l - 1 = -j - 1$ of the operator in (4.2) depends only on the degree of homogeneity j of its kernel and thus all terms in (3.2) will have the same order, the conditions $s \in (-l - \frac{1}{2},\ n - l - \frac{1}{2})$ will not, in general, have an interval of joint validity.

This problem appears for all $j \geqq 0$. Namely, for $l_0 = 0$ and $l_1 = n + j$ one has $(-l_0 - \frac{1}{2},\ n - l_0 - \frac{1}{2}) \cap (-l_1 - \frac{1}{2},\ n - l_1 - \frac{1}{2}) = \emptyset$ For $j < 0$, the interval $(-\frac{1}{2}, -\frac{1}{2} - j)$ is contained in $(-l - \frac{1}{2},\ n - l - \frac{1}{2})$ for all $0 \leqq l \leqq n + j$, showing

$$\chi K_j \chi \colon \tilde{H}^s \to \overset{(\sim)}{H}{}^{s+j+1} \quad \text{is continuous for } j < 0,\ s \in (-\tfrac{1}{2}, -\tfrac{1}{2} - j)\,.$$

But even this will be insufficient for general purposes. In order to find reasonable domains of definition for K_j, we have to modify K_j by subtracting suitable polynomials in x, x^{-1}, y, and y^{-1}, which are defined in the following way:

First, choose an integer $l_0 \leqq 0$. Then use (3.4) in the form

$$R^{-n}_{\omega,k,l} = \exp[-i\omega](R^{-n}_{\omega,k+1,l-1} + R^{-n+1}_{\omega,k,l-1}) \tag{4.3}$$

in (3.2) repeatedly to achieve eventually $l = l_0$. This changes then (3.2) to the form

$$K_j = \sum_{n=0}^{n_j} \sum_{k=0}^{n+j-l_0} a^j_{nk} R^{-n}_{\omega,k,n+j-k} \quad \text{with } a^j_{nk} \neq 0 \text{ only for } n = 0 \text{ or } n + j - k = l_0\,.$$

Now define

$$Q^{l_0}_j(x, y) = \sum_{k=0}^{j-l_0} a^j_{0k} x^k y^{j-k} \quad \text{and} \quad K^{l_0}_j = \sum_{n=1}^{n_j} a^j_{nn+j-l_0} R^{-n}_{\omega,n+j-l_0,l_0}\,.$$

For $j < l_0$ one finds $Q^{l_0}_j = 0$.

Second, let $l_0 > j$ and $k_0 := j - l_0 + 1$. Then $k_0 \leqq 0$, and we can use (4.3) in the form

$$R^{-n}_{\omega,k,l} = \exp[i\omega] R^{-n}_{\omega,k-1,l+1} - R^{-n+1}_{\omega,k-1,l} \tag{4.4}$$

to reduce the index k to $k = k_0$. This changes (3.2) to the form

$$K_j = \sum_{n=0}^{n_j} \sum_{k=k_0}^{n+j} b^j_{nk} R^{-n}_{\omega,k,n+j-k} \quad \text{with } b^j_{nk} \neq 0 \text{ only for } n = 0 \text{ or } k = k_0 .$$

We define

$$Q^{l_0}_j(x, y) = \sum_{k=k_0}^{j} b^j_{0k} x^k y^{j-k} \quad \text{and} \quad K^{l_0}_j = \sum_{n=1}^{n_j} b^j_{nk_0} R^{-n}_{\omega,k_0,n+j-k_0} .$$

Third, let $0 < l_0 \leqq j$ and $k_0 = j - l_0 + 1$. Then $0 < k_0 \leqq j$. In (3.2) there appear two (not exclusive) cases for $R^{-n}_{\omega,k,l}$: $l \geqq l_0$ and $k \geqq k_0$. Because of $k + l = n + j > j = k_0 + l_0 - 1$ no other cases occur. In case $l \geqq l_0$ we use (4.3) to reduce to $l = l_0$, and in case $k \geqq k_0$ (4.4) to reduce to $k = k_0$. We find

$$K_j = \sum_{n=0}^{n_j} \sum_{k=0}^{n+j} d^j_{nk} R^{-n}_{\omega,k,n+j-k}$$

with $d^j_{nk} \neq 0$ only for $n = 0$ or $k = k_0$ or $n + j - k = l_0$

We define

$$Q^{l_0}_j(x, y) = \sum_{k=0}^{j} d^j_{0k} x^k y^{j-k}$$

and

$$K^{l_0}_j(x, y) = \sum_{n=1}^{n_j} (d^j_{nk_0} R^{-n}_{\omega,k_0,n+j-k_0} + d^j_{nn+j-l_0} R^{-n}_{\omega,n+j-l_0,l_0}) .$$

So in all three cases we have the decomposition $K_j = K^l_j + Q^l_j$, where Q^l_j is a polynomial in x, x^{-1}, y, y^{-1} with the following properties:

(i) For $j < l \leqq 0$: $Q^l_j = 0$.

(ii) For $l \leqq 0$ and $j \geqq l$: $y^{-l} Q^l_j(x, y)$ is a polynomial in x and y, homogeneous of degree $j - l$.

(iii) For $l > 0$ and $j < 1$: $x^{l-j-1} Q^l_j(x, y)$ is a polynomial in x and y, homogeneous of degree $l - 1$.

(iv) For $0 < l \leqq j$: Q^l_j is a polynomial in x and y, homogeneous of degree j.

There is an equivalent definition of Q^l_j in terms of Taylor expansions of K_j. Repeated use of (4.3), until terms with $n = 0$ appear, is equivalent to the expansion of $R^{-n}_{\omega,k,l}(x, y)$ with respect to x about $x = 0$. Similarly, the repeated use of (4.4) corresponds to Taylor expansion in y about $y = 0$. Comparing with (2.5), this gives e.g. for $l < 0$:

$$Q_j^l(x, y) = \sum_{k=0}^{\min\{j-1, j+n_j\}} c_k^j \sum_{p=k}^{j-1} \binom{p-k+n_j-1}{p-k} \exp\left[-i(n_j+p-k)\omega\right] x^p y^{j-p}$$

with n_e and c_k^j as in (3.2).

If we take into account Lemma 1 and the remark following it, we can see that the subtraction of Q_j^l corresponds to analytic continuation in (2.2), respectively shifting the path of integration in (2.4), in such a way that the finally reached strips for all summands in K_j contain the strip $\operatorname{Im}\lambda \in (j-l, j-l+1)$. Defining

$$\hat{K}_j(\lambda) := \sum_{k=0}^{n_j+j} c_k^j \hat{R}_\omega^{-n_j}(\lambda - ki)\,,$$

we conclude therefore

LEMMA 8. – *Let $f \in C_0^\infty(0, \infty)$ and $j, l \in \boldsymbol{Z}$. Then the Mellin transform of $K_j^l f$ exists for $\operatorname{Im}\lambda \in (j-l, j-l+1)$, and*

$$\widehat{K_j^l f}(\lambda) = \hat{K}_j(\lambda)\hat{f}(\lambda-(j+1)i) \quad \textit{for} \quad \operatorname{Im}\lambda \in (j-l, j-l+1)\,.$$

Now we can prove the following continuity result:

THEOREM 2. – *Let $j \in \boldsymbol{Z}$. Then the operators*

$$\chi K_j^l \chi\colon \overset{\{\sim\}}{H}{}^s \to \overset{\{\sim\}}{H}{}^{s+j+1} \quad \textit{for all } l \in \boldsymbol{Z} \textit{ and } s \in (-l-\tfrac{1}{2}, -l+\tfrac{1}{2}) \tag{4.5}$$

and

$$\chi K_j \chi\colon \tilde{H}^s \to H^{s+j+1} \quad \textit{for all } s \in \boldsymbol{R} \tag{4.6}$$

are continuous.

PROOF. – We use (4.2) and the definition of K_j^l. Let first $l \leq 0$. Then in the definition of K_j^l there appear $R_{\omega,k,p}^{-n}$ only for $n \geq 1$, $k = n+j-l$, and $p = l$. Hence (4.2) yields continuity of $\chi K_j^l \chi\colon \overset{\{\sim\}}{H}{}^s \to \overset{\{\sim\}}{H}{}^{s+j+1}$ for $s \in (-l-\frac{1}{2}, -l+\frac{1}{2})$. Now Q_j^l contains in this case only positive powers of x, and the lowest power of y is y^l, hence Q_j^l maps $\tilde{H}^s$ into $C^\infty[0, \infty)$ for all $s > -l-\frac{1}{2}$. So we also have $\chi K_j \chi = \chi K_j^l \chi + \chi Q_j^l \chi\colon \tilde{H}^s \to H^{s+j+1}$ for all $s \in (-l-\frac{1}{2}, -l+\frac{1}{2})$, $l \leq 0$. Thus (4.6) holds for $s \in \bigcup_{l=-\infty}^{0} (-l-\frac{1}{2}, -l+\frac{1}{2})$. From this follows (4.6) for all $s \in (-\frac{1}{2}, \infty)$ by interpolation. Now the formal adjoint kernel K_j^* is of the form (3.2) too, hence it satisfies also (4.6) for $s > -\frac{1}{2}$. By transposition we get $\chi K_j \chi\colon \tilde{H}^{-(s+j+1)} \to H^{-s}$ for $s \in (-\frac{1}{2}, \infty)$.

This is (4.6) for $s \in (-\infty, -j-\frac{1}{2})$. The gap $s \in [-j-\frac{1}{2}, \frac{1}{2}]$ remaining for $j \geqq 0$ is again filled by interpolation, and so (4.6) is shown for all $s \in \boldsymbol{R}$.

In the cases $j < l$ and $0 < l \leqq j$ remaining for (4.5), there appear in the definition of K_j^l only $R_{\omega,k,p}^{-n}$ with $n \geqq 1$, $p = l$, and $k = n + j - l$, or with $k = j - l + 1$ and $n = k + p - j \geqq 1$. In either case $s - n + k + l + 1 = s + j + 1$, $(-p-\frac{1}{2}, n-p-\frac{1}{2}) \supset (-l-\frac{1}{2}, -l+\frac{1}{2})$. Hence (4.2) shows that always $\chi R_{\omega,k,p}^{-n}\chi\colon \overset{\{\sim\}}{H}{}^s \to \overset{\{\sim\}}{H}{}^{s+j+1}$ is continuous for $s \in (-l-\frac{1}{2}, -l+\frac{1}{2})$ which proves (4.5).

5. – Regularity.

In order to describe the regularity of solutions of integral equations with kernels as above we introduce Sobolev spaces augmented by singular functions. A similar notation was introduced by KOMEČ [13].

Let us call a set $\Lambda \subset \boldsymbol{C}$ left-finite if for every $s \in \boldsymbol{R}$ the set

$$\Lambda^s := \Lambda \cap \{z \in \boldsymbol{C} | \operatorname{Re} z < s - \tfrac{1}{2}\}$$

is finite. For a left-finite set Λ and $s \in \boldsymbol{R}$ we make the

DEFINITION 2. – *$f \in H_\Lambda^s$ if and only if there exist numbers $p_\alpha \in \boldsymbol{N}_0$, $c_{\alpha p} \in \boldsymbol{C}$ for any $\alpha \in \Lambda^s$, $0 \leqq p \leqq p_\alpha$, such that*

$$f(x) = \sum_{\alpha \in \Lambda^s} \sum_{p=0}^{p_\alpha} c_{\alpha p} x^\alpha (\log x)^p + f_0(x) \qquad \textit{for } x \in (0,1) \tag{5.1}$$

with $f_0 \in \overset{\{\sim\}}{H}{}^s(J)$. The norm in H_Λ^s is defined by

$$\|f\|_{H_\Lambda^s}^2 := \sum_{\alpha \in \Lambda^s} \sum_{p=0}^{p_\alpha} |c_{\alpha p}|^2 + \|f_0\|_{\overset{\{\sim\}}{H}{}^s(J)}^2 .$$

In (5.1) all occurring exponents α have to satisfy $\operatorname{Re} \alpha < s - \frac{1}{2}$. For $\operatorname{Re} \alpha > s - \frac{1}{2}$ we would have $x^\alpha (\log x)^p \in \overset{\{\sim\}}{H}{}^s(J)$ which would make the coefficients in (5.1) non-unique, and the condition $s - \frac{1}{2} \notin \operatorname{Re} \Lambda^s$ is necessary to make H_Λ^s a Hilbert space. As an example, for $s \geqq 0$, $s - \frac{1}{2} \notin \boldsymbol{N}_0$ we find $H^s(J) \subset H_\Lambda^s$ if and only if $\{0, 1, \ldots, [s-\frac{1}{2}]\} \subset \Lambda$.

It is well known ([1], [12]) that H_Λ^s can be characterized via Mellin transforms as follows:

Let χ be a cut-off function as above, $f \in H_\Lambda^s$, and $\alpha_0 = \min\{\operatorname{Re} \alpha | \alpha \in \Lambda,\ c_{\alpha p} \neq 0$ for some $p\}$. Then the Mellin transform $\widehat{\chi f}(\lambda)$ exists for $\operatorname{Im} \lambda < \alpha_0$ and is holomorphic there. It has a meromorphic extension $g(\lambda)$ to $\operatorname{Im} \lambda < s - \frac{1}{2}$ with poles of order

$p_\alpha + 1$ at $\lambda = i\alpha$ ($\alpha \in \Lambda^s$), and for any $\delta > 0$ there is $C_\delta > 0$ such that for each $h < s - \frac{1}{2}$ with dist $(h, i\Lambda) > \delta$ there holds the estimate

$$\int_{\operatorname{Im}\lambda = h} (1 + |\lambda|^2)^s |g(\lambda)|^2 \, d\lambda \leqq C_\delta \, .$$

Conversely, if g is a meromorphic function with these properties, then there is $f \in \mathring{W}^s_\beta$, $\beta > 2(s - \alpha_0) - 1$, with $\hat{f} = g$ for $\operatorname{Im}\lambda < \alpha_0$ and $\chi f \in H^s_\Lambda$.

In these augmented Sobolev spaces, the continuity properties of our operators K^l_j are described as follows ($\Lambda + j + 1$ etc. are pointwise sums, i.e. $\Lambda + j + 1 = \{\alpha + j + 1 | \alpha \in \Lambda\}$):

LEMMA 9. – *Let $j, l \in \boldsymbol{Z}$, $s \in \boldsymbol{R}$, and Λ left-finite satisfy*

$$-l - 1 < s - \tfrac{1}{2} \notin \boldsymbol{Z} \quad \textit{and} \quad -l - 1 < \operatorname{Re}\alpha \quad \textit{for all } \alpha \in \Lambda \, .$$

Then

$$K^l_j \colon H^s_\Lambda \to H^{s+j+1}_{\tilde{\Lambda}} \quad \textit{is continuous with } \tilde{\Lambda} = (\Lambda + j + 1) \cup (\boldsymbol{N} + j - l) \, .$$

The multiplicities p_α are not increased except for $\alpha \in \boldsymbol{Z}$, in which case they may grow by one.

PROOF. – Let $l_s := [-s + \frac{1}{2}]$ so that $s \in (-l_s - \frac{1}{2}, -l_s + \frac{1}{2})$, and Theorem 2 shows $\chi K^{l_s}_j \chi \overset{\{\sim\}}{H}{}^s \subset \overset{\{\sim\}}{H}{}^{s+j+1}$. Now $K^l_j = K^{l_s}_j - (Q^l_j - Q^{l_s}_j)$. From $l \geqq l_s$ follows by Theorem 2 $\chi K^l_j \chi \overset{\{\sim\}}{H}{}^s \subset \overset{\{\sim\}}{H}{}^{-l+j+\frac{3}{2}-\varepsilon}$ for every $\varepsilon > 0$, hence $\chi (Q^l_j - Q^{l_s}_j) \chi \overset{\{\sim\}}{H}{}^s \subset \overset{\{\sim\}}{H}{}^{-l+j+\frac{3}{2}-\varepsilon}$. Therefore $Q^l_j(x, y) - Q^{l_s}_j(x, y)$ can contain powers y^m only for $m > -s - \frac{1}{2}$ and x^k only for $k > -l + j + \frac{2}{3} - \varepsilon - \frac{1}{2}$. Thus $\chi K^l_j \chi \overset{\{\sim\}}{H}{}^s \subset H^{s+j+1}_{\Lambda'}$ for $\Lambda' = \boldsymbol{N} + j - l$. The continuity of the mapping is obvious and the leftmost χ can be omitted because $K^l_j \chi f$ is a C^∞ function outside the support of χ.

It remains to look at $K^l_j \chi f$ with $f(x) = x^\alpha (\log x)^p$. Because $\operatorname{Re}\alpha + \frac{1}{2} > -l - \frac{1}{2}$, we have $f \in \overset{\{\sim\}}{H}{}^t(J)$ for some $t \in (-l - \frac{1}{2}, -l + \frac{1}{2})$. Hence we can use (4.5), and therefore the Mellin transform of $K^l_j \chi f$ from Lemma 8 is meromorphic for $\operatorname{Im}\lambda < \operatorname{Re}\alpha$. Now $\widehat{\chi f}(\lambda)$ has a pole of order $p + 1$ at $\lambda = i\alpha$ and is holomorphic otherwise, so we see that $\widehat{K^l_j \chi f}(\lambda)$ has a corresponding pole at $\lambda = i(\alpha + j + 1)$ of order $\leqq p + 1$ if $\hat{K}_j$ is holomorphic there. $\hat{K}_j$ has only first order poles at $\lambda = in$, $n \in \boldsymbol{Z}$. Therefore if $i\alpha \in \boldsymbol{Z}$, the order of pole may become $p + 2$. All other poles of $\widehat{K^l_j \chi f}$ are those of $\hat{K}_j$. We conclude

$$K^l_j \chi f(x) = \sum_{k=0}^{p+1} c_k x^{\alpha+j+1} (\log x)^k + \sum_{k \in \boldsymbol{Z} \cap (\operatorname{Re}\alpha + j + 1, 0)} d_k x^k + g_0(x)$$

with $g_0 \in C^\infty[0,1]$. This implies

$$K_j^l \chi f \in H^s_{\Lambda''} \quad \text{with} \quad \Lambda'' = \{\alpha + j + 1\} \cup \big(\boldsymbol{Z} \cap (\operatorname{Re}\alpha + j + 1,\, s + j + \tfrac{1}{2})\big) \subset \tilde{\Lambda}\,,$$

and the proof is complete.

Now we define the class of operators for which we want to prove a regularity theorem. To keep the notation clear, we will not try to achieve maximal generality, but instead describe a rather narrow class that is more like an example admitting, however, several obvious generalizations to be considered afterwards.

Our operators will be of the form $A + B$, where A is a «principal part» which we only assume to satisfy a certain ellipticity condition, and the «lower order part» B is an operator as investigated in the preceding paragraphs. More precisely we make the following

ASSUMPTION (A). – *A is elliptic in the following sense: There exist numbers $a \in \boldsymbol{Z}$ (the order of A), $m \in \boldsymbol{Z}$, and $s_0 \in \boldsymbol{R}$ satisfying $m \geqq \max\{-1, -a-1\}$ and $\min\{0, a\} - \frac{1}{2} < s_0 \leqq m + 1$, and a left-finite set $\Lambda(A) \subset \{z \in \boldsymbol{C} | \operatorname{Re} z > s_0 - \frac{1}{2}\}$ such that for every $s \in \boldsymbol{R}$ with $s \leqq m + 1 + \min\{-a, 0, s_0\}$ and $s + a - \frac{1}{2} \notin \Lambda(A)$, every left-finite $\Lambda \subset \{z \in \boldsymbol{C} | \operatorname{Re} z > s_0 - a - \frac{1}{2}\}$, and every $f \in H^s_\Lambda$ the equation*

$$Au = f \quad \text{for } u \in \tilde{H}^{s_0} \quad \text{with} \quad \operatorname{supp} u \subset [0, 1)$$

implies $u \in H^{s+a}_{\tilde{\Lambda}}$ with $\tilde{\Lambda} = (\Lambda + \boldsymbol{N}_0 + a) \cup \Lambda(A)$.

Furthermore, there holds the a-priori estimate

$$\|u\|_{H^{s+a}_{\tilde{\Lambda}}} \leqq C\big(\|f\|_{H^s_\Lambda} + \|u\|_{\tilde{H}^{s_0}}\big)\,.$$

This assumption is satisfied e.g. if A is of the following type:

(i) For $u \in C_0^\infty(0, \infty)$ the Mellin transform $\widehat{Au}(\lambda)$ exists in some strip $\operatorname{Im}\lambda \in (h_0, h_1)$ with $s_0 - a - \frac{1}{2} \in (h_0, h_1)$ and

$$\widehat{Au}(\lambda) = \hat{A}(\lambda)\hat{u}(\lambda + ai)$$

there.

(ii) The function $\hat{A}(\lambda)^{-1}$ is meromorphic in a strip $\operatorname{Im}\lambda \in (h_0, h_2)$ with $h_2 > m - a + \frac{1}{2}$ and has a finite set $i(\Lambda(A) - a)$ of poles there. In each substrip $\operatorname{Im}\lambda \in [c, d] \subset (h_0, h_2)$ which contains no poles, $\hat{A}^{-1}$ satisfies an estimate

$$|\hat{A}(\lambda)^{-1}| \leqq C(1 + |\lambda|^2)^{-a/2}\,.$$

That (A) follows from (i) and (ii) is concluded from the Parseval equation (4.1), Cauchy's residue theorem, and some simple cut-off considerations. This is the

SHAMIR-KONDRATIEV ([19], [14]) technique which was displayed in [3], [4] for several representative examples. To be on the safe side, one should include in $\Lambda(A)$ also the integer numbers in $[s_0 - a - \frac{1}{2}, m - a + \frac{1}{2}]$ which possibly do not correspond to poles of $\hat{A}^{-1}$.

Now we can state the announced regularity result.

THEOREM 3. – *Let A be elliptic in the sense of (A) and $B \in \mathcal{K}^{\omega}_{-a,m}$. Then $A + B$ is elliptic with the same a, m, and s_0, and with*

$$\Lambda(A + B) = \big(\Lambda(A) + \boldsymbol{N}_0\big) \cup (\boldsymbol{N}_0 + a) \cup \boldsymbol{N}\,.$$

REMARK. – This means that the perturbation by B causes each singular function of the form $\sum\limits_{p} c_p x^{\alpha}(\log x)^p$, appearing in the solutions of $Au = f$ for smooth f, to split into a sequence of singular functions with exponents $\alpha, \alpha + 1, \alpha + 2, \ldots$. In the generic case, i.e. $\alpha \notin \boldsymbol{Z}$ and $\alpha \notin \Lambda(A) + \boldsymbol{Z}$, and the singular function being just $c_0 x^{\alpha}$ near $x = 0$, one finds that the perturbed singular function is $c_0 x^{\alpha} P(x)$ with a polynomial P, and the coefficients of P can easily be computed from the symbols of A and B (see [5] for examples).

PROOF OF THE THEOREM. – We use induction on the parameter s in (A):

For $s \leqq s_0 - a$, the assertion of (A) is trivial because $\|u\|_{H^{s+a}_{\tilde{\Lambda}}} \leqq \|u\|_{\tilde{H}^{s_0}}$ for all $\tilde{\Lambda}$, hence (A) is satisfied for $A + B$ in this case.

Next assume that (A) for $A + B$ holds for any $s \leqq s_1$, where s_1 satisfies the side conditions

$$s_0 - a \leqq s_1 \leqq m + 1 + \min\{-a, 0, s_0\} \quad \text{and} \quad s_1 + a - \tfrac{1}{2} \notin \Lambda(A)\,.$$

We show that then (A) for $A + B$ even holds for all $s < s_1 + 1$ that satisfy these side conditions. This will complete the proof.

So let $f \in H^s_{\Lambda}$ with $s_1 \leqq s < s_1 + 1$ and $u \in \tilde{H}^{s_0}$ satisfy $(A + B)u = f$. Clearly $f \in H^{s_1}_{\Lambda}$, and by induction hypothesis this implies

$$u \in H^{s_1+a}_{\tilde{\Lambda}_1} \quad \text{with } \tilde{\Lambda}_1 = (\Lambda + \boldsymbol{N}_0 + a) \cup \Lambda(A + B)\,. \tag{5.2}$$

On the other hand, if we can show that

$$Bu \in H^s_{\Lambda_2} \quad \text{with } \Lambda_2 = (\Lambda + \boldsymbol{N}_0) \cup \big(\Lambda(A + B) - a\big) \tag{5.3}$$

holds, then clearly $f - Bu \in H^s_{\Lambda_2}$, and from the equation $Au = f - Bu$ and assumption (A) for A we will find $u \in H^{s+a}_{\tilde{\Lambda}_2}$ with $\tilde{\Lambda}_2 = (\Lambda + \boldsymbol{N}_0 + a) \cup \Lambda(A + B)$, which is just the assertion of (A) for $A + B$. (The a-priori estimate can be derived along the same lines or simply from the closed graph theorem.)

So we have to show (5.3). Write

$$B = \sum_{j=-a}^{m} K_j + R_m \,, \qquad \text{and split } K_j = K_j^l + Q_j^l \text{ with } l := \max\{0, a\}\,.$$

For $R_m u$ and $Q_j^l u$ we use the assumption $u \in \tilde{H}^{s_0}$, $\operatorname{supp} u \subset [0, 1)$: As $s_0 \geqq -m-1$, $s \leqq m+1$, and $s \leqq m+1+s_0$, we find from Lemma 7 that $R_m u \in H^s(J) \cap L^2(J) \subset H^s_{N_0} \subset H^s_{\Lambda_2}$.

For Q_j^l there are only three cases possible: $0 < l = -a \leqq j$, or $0 = l \leqq j$, in which cases Q_j^l contains no negative powers of x or y, or $0 = l > j$ implying $Q_j^l = 0$. Hence Q_j^l maps $\tilde{H}^{s_0}$ into any $H^t(J)$; hence $Q_j^l u \in H^s_{N_0} \subset H^s_{\Lambda_2}$.

For K_j^l we use the induction hypothesis (5.2) together with Lemma 9. To check its hypotheses, we note that for all $\alpha \in \tilde{\Lambda}_1$, $\operatorname{Re} \alpha > \min\{s_0 - \frac{1}{2}, a-1, 0\} > -l-1$ holds according to the conditions on $\Lambda(A)$, s_0, and Λ in (A) and the definitions of $\Lambda(A+B)$ and l. If $s_1 + a - \frac{1}{2} \in \boldsymbol{Z}$, replace s_1 by $s_1 - \varepsilon$ for $\varepsilon \in (0, s_1 + 1 - s)$. By the lemma,

$$K_j^l u \in H^{s_1+a+j+1}_{\Lambda_3} \qquad \text{with } \Lambda_3 = (\tilde{\Lambda}_1 + j + 1) \cup (\boldsymbol{N} + j - l)\,.$$

Now $j \geqq -a$ gives $s_1 + a + j + 1 \geqq s_1 + 1 > s$ and $\boldsymbol{N} + j - l \subset \boldsymbol{N} - a \subset \Lambda_2$ as well as $\tilde{\Lambda}_1 + j + 1 \subset (\Lambda + \boldsymbol{N}) \cup (\Lambda(A+B) - a + \boldsymbol{N}) \subset \Lambda_2$. Hence $K_j^l u \in H^s_{\Lambda_2}$, and the proof is complete.

REMARKS. – 1) In some cases one has only $u \in H^{s_0}$ instead of $\tilde{H}^{s_0}$ in (A). This requires a careful analysis of the steps of the above proof for this special case. For example, the energy norm for the single layer potential operator corresponds to $H^{-\frac{1}{2}}$ for the even part u_e of u (cf. [3]). Thus for the weak solution one has $u_e \in H^{-\frac{1}{2}}$. Then one has first to use regularity for the principal part together with some minimal smoothness of the remainder to conclude $u_e \in H^{-\frac{1}{2}+\varepsilon} = \tilde{H}^{-\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$. Hereafter, the theorem can be used for $s_0 = -\frac{1}{2} + \varepsilon$.

2) There are two obvious generalizations of the above definition (A) of ellipticity and, correspondingly, of Theorem 3:

First one can consider systems of equations, i.e. matrices of operators, and second one can admit for such matrices (A_{jk}) different orders a_{jk} for different matrix elements A_{jk}, as long as they satisfy the usual compatibility conditions for systems elliptic in the AGMON-DOUGLIS-NIRENBERG sense:

$$a_{ik} - a_{jk} - a_{il} + a_{jl} = 0 \qquad \text{for all } i, j, k, l\,.$$

In the applications to boundary integral equations on curved polygons, there appear always systems of size at least 2×2; compare the examples. The system corresponding to a mixed Dirichlet-Neumann problem for the Laplacian, which was studied in [3], has orders

$$(a_{jk})_{j,k=1,2} = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

3) The principal parts of the boundary integral operators in our examples satisfy (i) and (ii) above, and hence (A), for every $m \in \boldsymbol{N}_0$ (if any). Hence for any $s_0 > -\frac{1}{2} + \min\{0, a\}$ and any s, one can find an m such that (A) holds. This m has to satisfy

$$m + 1 \geqq \max\{s_0, s, s - s_0, s - a\}.$$

In view of Propositions 1, 2, and 3, one has to require that the curved polygon is piecewise C^{m+2} for the single layer potential and piecewise C^{m+3} for the double layer potential and the Hilbert transform in order to obtain the appropriate a-priori estimates for the corresponding boundary integral equations.

4) If the principal part A in the theorem satisfies (i) and (ii) above, then the Mellin transform of the equation $(A + B)u = f$ can be written as

$$\hat{A}(\lambda)\hat{u}(\lambda + ai) + \sum_{j=-a}^{m} \hat{K}_j(\lambda)\hat{u}(\lambda - (j+1)i) = (\widehat{f - R_m u})(\lambda).$$

Here all terms may be assumed to have Mellin transforms in some half-plane, $\operatorname{Im}\lambda <$ $< s_0 - \frac{1}{2}$ for the left hand side and $\operatorname{Im}\lambda < s - \frac{1}{2}$ for the right hand side. Then one can use this equation to find a meromorphic extension of $\hat{u}$ to $\operatorname{Im}\lambda < s + a - \frac{1}{2}$, and the inductive method of the above proof corresponds to the iterative solution of this difference equation, written as

$$\hat{u}(\lambda) = -\sum_{k=1}^{m+1+a} \hat{A}(\lambda - ai)^{-1}\hat{K}_j(\lambda - ai)\hat{u}(\lambda - ki) + (\widehat{f - R_m u})(\lambda - ai). \tag{5.4}$$

From this equation it is immediately clear where the (nontrivial) poles of $\hat{u}$, i.e. the singular parts of u at 0, come from:

First from the poles of $\hat{A}(\lambda - ai)^{-1}$ and of $\hat{f}(\lambda - ai)$, and second from the previously found poles of $\hat{u}$ at values $\lambda - i, \lambda - 2i, \ldots, \lambda - (m + 1 + a)i$.

Repeating these two steps, one obtains from a single pole of $\hat{A}^{-1}$ or $\hat{f}$ at λ_0 a sequence of poles at $\lambda_0 + i, \lambda_0 + 2i, \ldots$, and from (5.4) it is easy to determine the residues of $\hat{u}$ at these poles. (See [5] for an example of application.)

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REFERENCES

[1] A. AVANTAGGIATI - M. TROISI, *Spazi di Sobolev con peso e problemi ellittici in un angolo, I, II, III*, Ann. Mat. Pura Appl., **95** (1973), pp. 361-408; Ann. Mat. Pura Appl. **97** (1973), pp. 207-252; Ann. Mat. Pura Appl., **99** (1974), pp. 1-51.

[2] M. COSTABEL, *An inverse for the Gohberg-Krupnik symbol map*, Proc. Royal Soc. Edinburgh, **87**-A (1980), pp. 153-165.

[3] M. COSTABEL - E. STEPHAN, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, THD-preprint 593, Darmstadt, 1981, to appear at Banach Center Publications, Warsaw.

[4] M. COSTABEL - E. STEPHAN - W. L. WENDLAND, *On boundary integral equations of the first kind for the bi-Laplacian in a polygonal plane domain*, THD-preprint 670, Darmstadt, 1982, to appear in Ann. Sc. Norm. Sup., Pisa.

[5] M. COSTABEL - E. STEPHAN, *Curvature terms in the asymptotic expansions for solutions of boundary integral equations on curved polygons*, THD-preprint 673, Darmstadt, 1982, to appear in J. Integral Equations.

[6] R. V. DUDUČAVA, *On bisingular integral operators with discontinuous coefficients*, Math. USSR Sbornik, **30** (1976), pp. 515-537.

[7] R. V. DUDUČAVA, *On the index of bisingular integral operators I, II*, Math. Nachr., **91** (1979), pp. 431-460; Math. Nachr., **92** (1979), pp. 289-307. (In russian).

[8] R. DUDUCHAVA, *Integral equations with fixed singularities*, Teubner Texte zur Mathematik, B.G. Teubner, Leipzig, 1979.

[9] G. I. ÈSKIN, *Boundary value problems for elliptic pseudo-differential equations*, American Mathematical Society, Providence, Rh. I., 1981.

[10] P. GRISVARD, *Behaviour of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain*, In « Numerical Sol. of PDE III » (ed. B. HUBBARD, Synspade), Academic Press, New York, (1976), pp. 207-274.

[11] P. GRISVARD, *Boundary value problems in non-smooth domains*, Univ. of Maryland, MD20742, Lecture Notes, No. **19** (1980).

[12] J. JEANQUARTIER, *Transformation de Mellin et développements asymptotiques*, L'Enseignement Math., **25** (1979), pp. 285-308.

[13] A. I. KOMEČ, *Elliptic boundary value problems for pseudo-differential operators on manifolds with conical points*, Math. USSR Sbornik, **15** (1971), pp. 261-297.

[14] V. A. KONDRATIEV, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., **16** (1967), pp. 227-313.

[15] J. E. LEWIS - C. PARENTI, *Pseudodifferential operators and Hardy kernels on $L^p(\boldsymbol{R}^+)$*, Ann. Sc. Norm. Sup. Pisa, Ser. IV, **7** (1980), pp. 481-503.

[16] J. E. LEWIS - C. PARENTI, *Pseudodifferential operators of Mellin type*, to appear.

[17] J. L. LIONS - E. MAGENES, *Non-homogeneous boundary value problems and applications I*, Springer, Berlin, 1972.

[18] B. A. PLAMENEVSKII, *An algebra of pseudodifferential operators in spaces with weighted norms*, Math. USSR Sbornik, **34** (1978), pp. 841-865.

[19] E. SHAMIR, *Reduced Hilbert transforms and singular integral equations*, J. Analyse Math., **12** (1964), pp. 227-305.