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PSEUDODIFFERENTIAL OPERATORS OF MELLIN TYPE

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INTRODUCTION

This article studies several algebras of pseudo-differential operators (pdo's) on the half line R^+ . The operators of order 0 include the Hilbert transform on $L^p(R^+)$

$$Hf(t) = \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds$$

and the point of view is motivated by the representation of H as a Mellin multiplier,

$$Hf(t) = \frac{-1}{2\pi i} \int_{\text{Re } z = 1/p} t^{-z} \cot \pi z \tilde{f}(z) dz$$

where $f(z) = \int_0^\infty t^{z-1} f(t) dt$, $f \in C_0^\infty(R^+)$, is the Mellin transform of f . Since $(-t \frac{d}{dt} f)^\sim(z) = z \tilde{f}(z)$, we consider operators of the form

$$(1) \quad a\left(t, -t \frac{d}{dt}\right) f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(t, z) \tilde{f}(z) dz$$

where the symbol $a(t, z)$ satisfies suitable smoothness properties in t and holomorphy and growth properties in z . The spectral and continuity properties of the operators (1) on various L^p and Sobolev spaces depend on the domain of holomorphy of the functions $a(0, z)$ and $a(\infty, z)$.

In Chapter 1 we develop a C^∞ theory for the operators (1) which is motivated by the usual theory of pdo's on R^n . The basic operator is

$$\partial = -t \frac{d}{dt}$$

and we study the action of (1) on functions in $C^\infty(R^+)$ which satisfy Holder type continuity conditions at $t = 0$ and $t = \infty$. To include the Hilbert transform, H , we allow the symbols $a(\cdot, z)$ to have a sequence of poles $\{z_k\}$, $\operatorname{Re} z_k \rightarrow -\infty$; this makes the proof of the symbolic calculus theorems for compositions and adjoints more delicate than for the corresponding theorems on R^n . We develop algebras ample enough to include H as well as many Hardy kernel operators of the form

$$Kf(t) = \int_0^\infty k\left(\frac{t}{s}\right) f(s) \frac{ds}{s} = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} \tilde{k}(z) \tilde{f}(z) dz$$

and multiplication by cutoff functions $\chi(t) \in C_0^\infty(R)$, $\chi(t) = 1$ near $t = 0$.

In Chapter 2 we study the continuity properties of pdo's on various weighted and unweighted L^p and Sobolev spaces on the half line R^+ .

A principal symbol for a class of pdo's on $L^p(R^+)$ is introduced in Chapter 3. The notion of ellipticity on $L^p(R^+)$ of a pdo (i.e., the existence of a parametrix modulo compact operators) depends on p as in [LP1]. We obtain that a pdo is elliptic on $L^p(R^+)$ iff its principal symbol never vanishes. An index theorem for pdo's on $L^2(R^+)$ was proved by Cordes and Herman [CH]. We give an index theorem for an elliptic system on $L^p(R^+)$ which generalizes [CH] and [LP1].

In Chapter 4 we define an algebra of pdo's of order 0 on $L^p(I)$, where I is a finite interval. We work modulo compact operators on $L^p(I)$ to define a principal symbol and prove that a pdo is a Fredholm operator on $L^p(I)$ iff its principal symbol never vanishes. For an elliptic system of pdo's on $L^p(I)$, we prove an index theorem which relates the index of the system to the topological degree of the principal symbol. In this case the principal symbol may be identified with a map from the unit circle S^1 into $GL(N, C)$; of course the degree and hence the index depend on the L^p space under consideration.

In Chapter 5, we apply the theory of pdo's to the study of single and double layer potentials for

Laplace's equation in a plane polygon. We study the Fredholm and index properties of the integral equations which arise for Dirichlet, Neumann, and oblique derivative and mixed problems.

Various classes of pdo's on R^+ , halfspaces, and manifolds with boundary have been considered by many authors. Boutet de Monvel [B] studied pdo's with the transmission property. Melrose [M] defined a class of operators on a half-space via oscillatory integrals. Nourrigat [N1] considered symbols $a(\cdot, z)$ holomorphic in a left half plane and gave applications to Fuchs type equations in [N2]. Shamir [S1, S2] observed how the spectrum of a pdo on R^+ depends on the L^p space on which it acts. Cordes and Herman [CH] and Cordes [C] used Banach algebra techniques to introduce the notion of a principal symbol. Eskin [E] used the Mellin transform representation of the Hilbert transform and Hardy kernels to study mixed boundary problems. Plamenevskii [P1], using the Mellin transform, assigned a pdo on the sphere S^{n-1} (actually a meromorphic function of a complex parameter λ) to an operator on R^n . He then reduced the study of an algebra of operators on R^n to the investigation of an algebra of meromorphic operator-valued functions on S^{n-1} . The authors [LP1] developed an algebra which is properly included in the algebra $Op - \Sigma_{1/p}$ studied in Chapter 3.

For the index properties of boundary value problems in polynomial domains, see the extensive work and bibliographies of Grisvard [G], Kondratiev [K], and Avantaggiati [Av1]. The L^p theory of boundary value problems was studied by Merigot [Mer]. Double layer potentials in sectors and bounded domains were considered by Fabes, Jodeit, and Lewis [JFL]. Applications of the present theory to the biharmonic operator in a polygon have been given by Diomeda and Lisena [DL].

1. The C^∞ theory of pdo's on R^+ .

We shall denote the open half line $(0, \infty)$ as R^+ and $[0, \infty)$ as $\overline{R^+}$. The basic differential operator used will be

$$\partial = -t \frac{d}{dt}$$

and ∂_t will denote $-t \frac{\partial}{\partial t}$. If a is a real number, the integer $[a]$ will be defined as

$$[a] = \text{greatest integer} < a.$$

The following function spaces will be used for the C^∞ theory of pdo's on R^+ .

DEFINITION 1.1. Let $-\infty < a < b < \infty$. By $\mathcal{F}_{a,b}$ we denote the class of functions $f \in C^\infty(R^+)$ such that:

- (1) For $j = 0, \dots, (-a)$, there are scalars f_{j0}

such that for every k and every $\delta > 0$,

$$(1.1) \quad \partial^k \left(f(t) - \sum_{j=0}^{(-a]} \frac{1}{j!} f_{j0} t^j \right) = o(t^{-a-\delta}), \quad t \rightarrow 0^+,$$

(2) For $j = 0, \dots, (b]$, there are scalars $f_{j\infty}$ such that for every k and every $\delta > 0$,

$$(1.2) \quad \partial^k \left(f(t) - \sum_{j=0}^{(b]} \frac{1}{j!} f_{j\infty} t^{-j} \right) = o(t^{-b+\delta}), \quad t \rightarrow \infty.$$

$\mathfrak{F}_{a,b}$ is a Frechet space with the seminorms

$$(1.3) \quad \begin{aligned} & |f_{j0}|, \quad j = 0, \dots, (-a]; \quad |f_{j\infty}|, \quad j = 0, \dots, (b]; \\ & \sup_{0 < t < 2} \left| t^{a+\delta} \partial^k \left(f(t) - \sum_{j=0}^{(-a]} \frac{1}{j!} f_{j0} t^j \right) \right|; \\ & \sup_{1 < t < \infty} \left| t^{b-\delta} \partial^k \left(f(t) - \sum_{j=0}^{(b]} \frac{1}{j!} f_{j\infty} t^{-j} \right) \right|. \end{aligned}$$

We shall abbreviate (1.1) as $f(t) \sim \sum \frac{1}{j!} f_{j0} t^j$, $t \rightarrow 0^+$, and shall write (1.2) as $f(t) \sim \sum \frac{1}{j!} f_{j\infty} t^{-j}$, $t \rightarrow \infty$. If $b > 0$, we denote by $\mathfrak{F}'_{a,b}$ the space of functions $f \in \mathfrak{F}_{a,b}$ such that $f \sim 0$, $t \rightarrow \infty$; i.e., $\partial^k f(t) = o(t^{-b+\delta})$, $t \rightarrow \infty$, for every $\delta > 0$. Similarly $\mathring{\mathfrak{F}}_{a,b}$ consists of these functions $f \in \mathfrak{F}_{a,b}$ such that $f(t) \sim 0$, $t \rightarrow 0^+$. We define

$$\mathfrak{F}''_{a,b} = \mathring{\mathfrak{F}}_{a,b} \cap \mathfrak{F}'_{a,b}.$$

The space $\check{\mathcal{F}}_{a,b}$ is stable under multiplication by $\log t$, and $C_0^\infty(\mathbb{R}^+)$ is dense in $\check{\mathcal{F}}_{a,b}$. We use the convention that $\check{\mathcal{F}}_{-\infty,b} = \bigcap_{a < b} \check{\mathcal{F}}_{a,b}$ and note that

$$\check{\mathcal{F}}_{a,b} = \bigcap_{\delta > 0} \check{\mathcal{F}}_{a+\delta, b-\delta}. \quad \text{If we define}$$

$$(1.4) \quad \check{f}(t) = f\left(\frac{1}{t}\right),$$

then $f \in \check{\mathcal{F}}_{a,b}$ iff $f \in \check{\mathcal{F}}_{-b,-a}$.

The spaces $\check{\mathcal{M}}_{a,b}$ defined in Definition 1.2 will be the Mellin transforms of function in $\check{\mathcal{F}}_{a,b}$. By $S_{a,b}$ we denote the strip

$$(1.5) \quad S_{a,b} = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$$

and let Z_- be the set of nonpositive integers.

DEFINITION 1.2. Let $0 < b$. The space $\check{\mathcal{M}}_{a,b}$ consists of functions $g(z)$ defined and meromorphic on the strip $S_{a,b}$ such that

1. g has poles only at $z = -j \in Z_- \cap S_{a,b}$ which are at most simple; denote the residue at $z = -j$ by $(1/j!)g_j$.
2. $g(z)$ is rapidly decreasing as $|\operatorname{Im} z| \rightarrow \infty$ in $S_{a,b}$; i.e., on every strip with base $(a', b') \subset (a, b)$

$$(1.6) \quad |g(z)| = O(|\operatorname{Im} z|^{-N})$$

for every N as $|\operatorname{Im} z| \rightarrow \infty$, $z \in S_{a', b'}$.

The following Lemma follows from the arguments of Avantaggiati [Av2], Nourrigat [N2], or [LP2].

LEMMA 1.1. Let $0 < b$. For $f \in \mathcal{F}_{a, b}'$ and
 $\max(a, 0) < \operatorname{Re} z < b$, define

$$(1.7) \quad \tilde{f}(z) = \int_0^\infty t^{z-1} f(t) dt.$$

Then $f(z)$ may be extended as a meromorphic function
in $\mathcal{M}_{a, b}'$ with residues at $z = -j \in \mathbb{Z}_- \cap S_{a, b}$ given
by $(1/j!)f_{j0}$.

Conversely, given $g(z) \in \mathcal{M}_{a, b}'$, for any γ such
that $\max(a, 0) < \gamma < b$, define

$$(1.8) \quad f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} t^{-z} g(z) dz.$$

Then $f \in \mathcal{F}_{a, b}'$, $f \sim \sum_{j=0}^{(-a]} \frac{g_j}{j!} t^j$, $t \rightarrow 0^+$, and
 $\tilde{f}(z) = g(z)$, where \tilde{f} is defined by (1.7).

DEFINITION 1.3. For $-\infty \leq c < d \leq +\infty$ and m real
the class $\mathcal{S}_{c, d}^m$ consists of those functions $a(z)$
holomorphic in the strip $S_{c, d}$ such that for every
 $(c', d') \subset (c, d)$, for every k

$$(1.9) \quad \frac{d^k}{dz^k} a(z) = O(|\operatorname{Im} z|^{m-k})$$

as $|\operatorname{Im} z| \rightarrow \infty$, $z \in S_{c', d'}$.

$\mathcal{O}_{c, d}^m$ is a Frechet space with the seminorms

$$(1.10) \quad \sup_{z \in S_{c', d'}} |(1+|z|)^{k-m} \frac{d^k}{dz^k} a(z)|, (c', d') \subset (c, d).$$

For $-\infty \leq c < d \leq +\infty$ and $-\infty \leq a < b \leq +\infty$, $\mathcal{F}_{a, b}(\mathcal{O}_{c, d}^m)$ denotes the Frechet space $\mathcal{F}_{a, b}(\mathbb{R}^+; \mathcal{O}_{c, d}^m)$ of functions $a(t, z) \in \mathcal{F}_{a, b}$ as functions of t with values in $\mathcal{O}_{c, d}^m$.

We shall write $\mathcal{O}_{c, d}^{-\infty} = \bigcap_m \mathcal{O}_{c, d}^m$.

DEFINITION 1.4. For $-\infty \leq c < d \leq \infty$ and $m \in \mathbb{R}$, the symbol class $\Sigma_{c, d}^m$ is defined by

$$(1.11) \quad \Sigma_{c, d}^m = \bigcap_{(c', d') \subset (c, d)} \mathcal{F}_{c-c', d-d'}(\mathcal{O}_{c', d'}^m).$$

Also

$$(1.12) \quad \Sigma_{c, d}^m = \bigcap_{(c', d') \subset (c, d)} \mathcal{F}_{c-c', d-d'}(\mathcal{O}_{c', d'}^m).$$

REMARKS.

(1) If $\chi(t)$ is a cutoff function (i.e., $\chi(t) \in C_0^\infty(\mathbb{R})$, $\chi(t) = 1$ near $t = 0$), $\alpha > 0$, and $\operatorname{Re} z_0 = c$, then for any $a(z) \in \mathcal{O}_{c, d}^{-\infty}$, the function

(1.13) $a(t,z) = \chi(t)(\alpha t)^{z-z_0} a(z)$

is in $\Sigma_{c,d}^{0,-\infty}$.

2. If $a(t,z) \in \Sigma_{c,d}^m$ and $c < c' < \operatorname{Re} z < d$, we have the asymptotic expansion

(1.14) $a(t,z) \sim \sum_{j=0}^{(c'-c]} \frac{a_{j0}(z)}{j!} t^j, \quad t \rightarrow 0^+,$

where $a_{j0}(z) \in \mathcal{O}_{c+j,d}^m$.

We are now ready to define a class of pseudo-differential operators.

DEFINITION 1.5. Let $-\infty \leq c < d \leq +\infty$ and suppose that

(1.15) $J = (c,d) \cap (0,1) \neq \emptyset.$

If $a(t,z) \in \Sigma_{c,d}^m$ define the operator

$A = a(t,\partial) \in \operatorname{Op} - \Sigma_{c,d}^m$ as follows:

Choose $\gamma \in J$ and for $f \in \mathcal{S}'_{c,d}$, let

(1.16) $Af(t) = a(t,\partial)f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} t^{-z} a(t,z) \tilde{f}(z) dz.$

The definition (1.16) is independent of $\gamma \in J$.

THEOREM 1.1. If $a(t,z) \in \Sigma_{c,d}^m$, then

$$a(t, \partial) : \mathcal{S}'_{c,d} \rightarrow \mathcal{S}'_{c,d}$$

is continuous; moreover, if

$$(1.17) \quad f(t) \sim \sum_{j=0}^{(-c]} \frac{1}{j!} f_{j0} t^j, \quad t \rightarrow 0^+$$

then

$$(1.18) \quad Af(t) \sim \sum_{k=0}^{(-c]} \frac{1}{k!} \left\{ \sum_{j=0}^k \binom{k}{j} a_{k-j,0}(-j) f_{j0} \right\} t^k, \quad t \rightarrow 0^+,$$

where $a_{k-j,0}(z)$ is given by (1.14). Hence

$$(1.19) \quad a(t, \partial) : \mathcal{S}_{c,d} \rightarrow \mathcal{S}_{c,d}$$

is continuous.

PROOF: Let $f \in \mathcal{S}'_{c,d}$ satisfy (1.17). For all sufficiently small $\delta > 0$, we shift the contour of integration in (1.16) to $\operatorname{Re} z = d - \delta$ and rewrite (1.16) as

$$(1.20) \quad Af(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = d - \delta} t^{-z} a(t, z) \tilde{f}(z) dz.$$

This shows that $Af(t)$ is in $C^\infty(\mathbb{R}^+)$. Since for $\operatorname{Re} z = d - \delta$, $\partial_t^k (a(t, z) - a(\infty, z)) = O(t^{-\delta + \epsilon} (1 + |z|)^m)$, $t \rightarrow \infty$, we readily obtain from (1.20) that $\partial^k f(t) = O(t^{-d+\delta})$, $t \rightarrow \infty$.

To study the behavior of (1.16) near $t = 0$, we choose any small $\delta > 0$ such that $c + \delta$ is not a nonnegative integer. In (1.16) shift the contour of

integration to $\operatorname{Re} z = c + \delta$, and obtain that

$$Af(t) = \sum_{j=0}^{(-c]} \operatorname{Res}\{t^{-z}a(t,z)\tilde{f}(z); z = -j\} + I(t),$$

where $I(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = c+\delta} t^{-z}a(t,z)\tilde{f}(z)dz$. It is easy to show that $\partial^k I(t) = O(t^{-c+\delta})$, $t \rightarrow 0^+$. We now study the residue

$$(1.22) \quad R_{-j} = \operatorname{Res}\{t^{-z}a(t,z)\tilde{f}(z); z = -j\}.$$

Choose a small $\epsilon > 0$ such that $(-c-\epsilon] = (-c]$. Since $a(t,z) \in \mathfrak{F}_{c+j+\epsilon, d+j-\epsilon}(\mathfrak{S}_{-j-\epsilon, -j+\epsilon}^m)$, then for $|z+j| \leq \epsilon/2$,

$$\partial_t^k \{a(t,z) - \sum_{k=0}^{(-c-j]} \frac{1}{k!} a_{k0}(z)t^k\} = O(t^{-c-k-2\epsilon}),$$

with $a_{k0}(z) \in \mathfrak{S}_{-j-\epsilon, -j+\epsilon}^m$.

Using the representation

$$R_{-j} = \frac{1}{2\pi i} \int_{|z+j| = \epsilon/2} t^{-z}a(t,z)\tilde{f}(z)dz$$

for (1.22) we have that $\partial_t^k R_{-j}$ is

$$\partial_t^k \left\{ \sum_{k=0}^{(-c-j]} \frac{1}{k!} a_{k0}(-j) \frac{1}{j!} f_{j0} t^{k+j} \right\} + O(t^{-c-j-2\epsilon}).$$

This proves that $Af \in \mathfrak{F}_{c,d}'$ and formulas (1.18) and (1.19). q.e.d.

We now study the composition of two operators.

THEOREM 1.2. Let $a(t, z) \in \Sigma_{c, d}^m$ and $b(t, z) \in \Sigma_{c, d}^{m'}$.
Then there is a symbol $c(t, z) \in \Sigma_{c, d}^{m+m'}$ such that

1. For every N

$$(1.23) \quad c(t, z) - \sum_{k=0}^N \frac{1}{k!} \frac{\partial^k}{\partial z^k} b(t, z) (\partial_t^k a)(t, z) \in \Sigma_{c, d}^{m+m'-N-1}.$$

2. $b(t, \partial) \circ a(t, \partial) = c(t, \partial)$ (as operators on
 $\mathcal{S}_{c, d}'$).

PROOF: Fixing a cutoff function $\chi(t)$, we split the symbol of $a(t, z)$ as

$$(1.24) \quad a(t, z) = \chi(t)a(t, z) + (1-\chi(t))a(t, z) \\ = a_0(t, z) + a_\infty(t, z).$$

Consider the contribution of $b(t, \partial) \circ a_\infty(t, \partial)$. Since $a_\infty(t, z) \in \Sigma_{c, d}^m$, for $\operatorname{Re} z - d < \operatorname{Re} w < 0$ we define

$$(1.25) \quad m_{a_\infty}(w, z) = \int_0^\infty t^{w-1} a_\infty(t, z) dt.$$

Since $m_{a_\infty}(w, z)$ is the Mellin transform of the function $t \rightarrow a_\infty\left(\frac{1}{t}, z\right)$ evaluated at the point $(-w)$, and $t \rightarrow a_\infty\left(\frac{1}{t}, z\right)$ is a function in $\mathcal{S}_{\operatorname{Re} z - d, \operatorname{Re} z - c}'$, the function $m_{a_\infty}(w, z)$ is a meromorphic function of w in $S_z = S_{c - \operatorname{Re} z, \operatorname{Re} z}$ with simple poles at $\{0, 1, 2, \dots\} \cap S_z$ with residues given by $-a_{j_\infty}(z)/j!$; $j = 0, \dots, (d - \operatorname{Re} z]$; and we have the inversion formula

$$(1.26) \quad a_\infty(t, z) = \frac{1}{2\pi i} \int_{\operatorname{Re} w = \gamma'} t^{-w} m_{a_\infty}(w, z) dw,$$

$$\operatorname{Re} z - d < \gamma' < 0.$$

Fix $f \in \mathcal{F}_{c,d}$ and let $g(t) = a_{\infty}(t, \partial)f(t)$. If $\max(0, c) < \gamma < d$ and $c < \operatorname{Re} w = \gamma' < \gamma$, then

$$\begin{aligned} \tilde{g}(w) &= \int_0^{\infty} t^{w-1} \left\{ \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} t^{-z} a_{\infty}(t, z) \tilde{f}(z) dz \right\} dw \\ (1.27) \quad &= \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} m a_{\infty}(w-z, z) \tilde{f}(z) dz. \end{aligned}$$

The use of Fubini's Theorem in (1.27) is justified since $c - \operatorname{Re} z < \operatorname{Re} w - \operatorname{Re} z < 0$. Since $g(t) \in \mathcal{F}_{c,d}$ $\tilde{g}(w)$ has no pole in $S_{c,d}$; hence for any γ' , $c < \gamma' < d$, we have

$$(1.28) \quad b(t, \partial)g(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} w = \gamma'} t^{-w} b(t, w) \tilde{g}(w) dw.$$

Using (1.27) and Peetre's inequality, $(1+|w-z|)^{-N} \leq C(1+|w|)^{-N}(1+|z|)^N$, we apply Fubini's Theorem to obtain

$$(1.29) \quad b(t, \partial) \circ a_{\infty}(t, \partial)f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \gamma} t^{-z} c_{\infty}(t, z) \tilde{f}(z) dz,$$

where

$$(1.30) \quad c_{\infty}(t, z) = \frac{1}{2\pi i} \int_{\operatorname{Re} w = \gamma'} t^{z-w} b(t, w) m a_{\infty}(w-z, z) dw.$$

In (1.30) choose any $\gamma' \in (c, \operatorname{Re} z)$. We make the change of variables $w-z = v$, $\operatorname{Re} v = \gamma_1 \in (c-\operatorname{Re} z, 0)$,

and apply Taylor's formula with integral remainder and the inversion formula (1.26), so that

$$(1.31) \quad c_{\infty}(t, z) = \sum_{k=0}^N \frac{1}{k!} \frac{\partial^k}{\partial z^k} b(t, z) (\partial_t^k a_{\infty})(t, z) + R_{\infty}^N(t, z),$$

with

$$(1.32) \quad R_{\infty}^N(t, z)$$

$$= \frac{1}{N!} \int_0^1 (1-s)^N \frac{1}{2\pi i} \int_{\operatorname{Re} v = \gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t, z+sv) v^N m_{\infty}(v, z) dv ds$$

Fix s , $0 \leq s \leq 1$, and let

$$(1.33) \quad c_s(t, z) = \frac{1}{2\pi i} \int_{\operatorname{Re} v = \gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t, z+sv) v^N m_{\infty}(v, z) dv$$

where $\gamma_1 \in (c - \operatorname{Re} z, 0)$. The function $c_s(t, z)$, originally defined by (1.33) for $\operatorname{Re} z > 0$ may be prolonged as a holomorphic function of z in the entire strip $S_{c,d}$ by using formula (1.33) with small γ_1 ; shifting the contour of integration to $\operatorname{Re} v = c - \operatorname{Re} z + \delta$ or $d - \operatorname{Re} z - \delta$ and taking into account the poles $v^N m_{\infty}(v, a)$ at $v = 1, \dots, (d - \operatorname{Re} z]$, we may show that

$$c_s(t, z) \in \mathcal{E}_{c-c'}^0(\mathcal{O}_{c',d'}^{m+m'-N-1})$$

if $(c', d') \subset\subset (c, d)$. Hence $R_{\infty}^N(t, z) \in \Sigma_{c,d}^{m+m'-N-1}$.

A similar, but simpler argument shows that if $a_0(t, z)$ is defined as in (1.24) then

$$(1.34) \quad b(t, \partial) \circ a_0(t, \partial) f(t) = c_0(t, \partial) f(t),$$

where for $\max(0, c) < \operatorname{Re} z < d$

$$(1.35) \quad c_0(t, z) = \sum_{k=0}^N \frac{1}{k!} \frac{\partial^k}{\partial z^k} b(t, z) (\partial_t^k a_0)(t, z) + R_0^N(t, z).$$

and

$$(1.36) \quad R_0^N(t, z)$$

$$= \frac{1}{N!} \int_0^1 (1-s)^N \frac{1}{2\pi i} \int_{\operatorname{Re} v = \gamma_1} t^{-v} \frac{\partial^{N+1}}{\partial z^{N+1}} b(t, z+sv) v^N a_0^\sim(v, z) dv ds,$$

$$\text{where } \gamma_1 \in (0, d - \operatorname{Re} z) \text{ and } a_0^\sim(v, z) = \int_0^\infty t^{v-1} a_0(t, z) dt.$$

Again it follows that $R_0^N(t, z) \in \Sigma_{c,d}^{m+m'-N-1}$.

q.e.d.

We consider the transpose and the adjoint of an operator $A = a(t, \partial) \in \operatorname{Op} - \Sigma_{c,d}^m$. The transpose of the operator tA , and the adjoint of the operator, A^* , are defined by the relations:

$$(1.37) \quad \begin{aligned} &\text{For } f \in \mathcal{F}_{c,d}, \varphi \in C_0^\infty(\mathbb{R}^+), \\ &\int_0^\infty Af(t)\varphi(t)dt = \int_0^\infty f(t)({}^tA\varphi)(t)dt, \\ &\int_0^\infty Af(t)\overline{\varphi(t)}dt = \int_0^\infty f(t)\overline{A^*\varphi(t)}dt. \end{aligned}$$

THEOREM 1.3. Suppose that $(c, d) \cap (0, 1) \neq \emptyset$, and
that $a(t, z) \in \operatorname{Op} - \Sigma_{c,d}^m$. Then there are symbols
 ${}^ta(t, z)$ and $a^*(t, z) \in \Sigma_{1-d, 1-c}^m$, such that

$$(1.38) \quad {}^t a(t, z) - \sum_{k=0}^N \frac{(-1)^k}{k!} \left(\frac{\partial}{\partial w} \right)^k \left(\partial_t^k a \right)(t, w) \Big|_{w=1-z} \in \Sigma_{1-d, 1-c}^{m-N-1}$$

$$(1.39) \quad a^*(t, z) - \sum_{k=0}^N \frac{(-1)^k}{k!} \left\{ \left(\frac{\partial}{\partial w} \right)^k \left(\partial_t^k a \right)(t, w) \Big|_{w=1-\bar{z}} \right\}^* \in \Sigma_{1-d, 1-c}^{m-N-1}$$

and if $A = a(t, \partial) \in \text{Op} - \Sigma_{c, d}^m$, then

$$(1.40) \quad {}^t A = {}^t a(t, \partial) \in \text{Op} - \Sigma_{1-d, 1-c}^m,$$

$$(1.41) \quad A^* = a^*(t, \partial) \in \text{Op} - \Sigma_{1-d, 1-c}^m.$$

We sketch the proof of Theorem 5.2. Split $a(t, z)$ as in (1.24). If $m \leq -2$ and $\varphi \in C_0^\infty(\mathbb{R}^+)$ show that with $\max(0, c) < \gamma < 1$,

$$(1.42) \quad {}^t A_0 \varphi(s) = \frac{1}{2\pi i} \int_{\text{Re } w = 1-\gamma} s^{-w} \int_0^\infty t^{w-1} a_0(t, 1-w) \varphi(t) dt dw.$$

The proof then proceeds in the spirit of the proof of Theorem 5.1. See also Theorem 4 of [LP2]. In the case $m > -2$, choose $\lambda_0 > d$ and write $a(t, z) = a_1(t, z)(z - \lambda_0)^{m+2}$ with $a_1(t, z) \in \Sigma_{c, d}^{-2}$. Then $a(t, \partial) = a_1(t, \partial)(\partial - \lambda_0)^{m+2}$. The transpose of the differential operator $(\partial - \lambda_0)^{m+2}$ may be calculated directly and ${}^t a(t, \partial)$ is handled by the first part of the proof. We leave the details to the reader.

2. L^p estimates for pseudodifferential operators on \mathbb{R}^+ .

Let $1 < p < \infty$. We shall prove estimates for the operators considered in Chapter 1 on $L^p(\mathbb{R}^+)$ and on

various weighted and Sobolev spaces on R^+ . Let

$$\|f\|_p = \|f; L^p(R^+)\| = \left(\int_0^\infty |f(t)|^p dt \right)^{1/p}.$$

THEOREM 2.1. Suppose that $1 < p < \infty$ and that
 $c < 1/p < d$. If $a(t, z) \in \Sigma_{c,d}^0$ and

$$(2.1) \quad \begin{aligned} Af(t) &= a(t, \partial)f(t) \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(t, z) \tilde{f}(t) dt, \end{aligned}$$

for $f \in \mathcal{S}'_{c,d}$, then there is a constant $C = C(a, p)$
such that

$$(2.2) \quad \|Af\|_p \leq C \|f\|_p.$$

PROOF: For $f \in \mathcal{S}'_{c,d}$ define

$$(2.3) \quad T_p f(x) = F_p(x) = e^{-(1/p)x} f(e^{-x}) \in \mathcal{S}(R).$$

Then $\|F_p; L^p(R)\| = \|f; L^p(R^+)\|$ and

$$T_p Af(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} a(e^{-x}, 1/p + i\xi) \tilde{F}_p(\xi) d\xi.$$

The symbol $b_p(x, \xi) = a(e^{-x}, 1/p + i\xi)$ satisfies

$$\frac{\partial^{j+k}}{\partial x^j \partial \xi^k} b_p(x, \xi) = O((1 + |\xi|)^{-k})$$

and hence $T_p A T_p^{-1}$ is a classical Calderon-Zygmund

operator of class $S_{1,0}^0$, and hence

$$(2.4) \quad \|T_p A T_p^{-1} F_p; L^p(R)\| \leq C_p \|F_p; L^p(R)\|.$$

Cf. Meyer [M] or Coifman and Meyer [CM]. Inequality

(2.3) now follows from (2.4).

q.e.d.

Before studying the continuity of operator $a(t, \partial) \in Op - \Sigma_{c,d}^m$ on Sobolev spaces we remark that if $a(t, z) \in \Sigma_{c,d}^m$, then $a(t, z-1) \in \Sigma_{c+1,d+1}^m$ and that if $c+1 < d$, $(\partial/\partial t)a(t, z) \in \Sigma_{c+1,d}^m$.

For $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$, we let $W^{p,k}$ be the usual Sobolev space on R^+ :

$$W^{p,k} = \{f \in L^p(R^+); f^{(j)} \in L^p(R^+), j = 0, \dots, k\};$$

$$W^{p,k} = \{f \in W^{p,k}; f^{(j)}(0) = 0, j = 0, \dots, k-1\}.$$

If $k \in \mathbb{Z}_- = \{0, -1, -2, \dots\}$, then $W^{p,k}$ is the dual of $W^{q,-k}$, $1/p + 1/q = 1$, under the bilinear form

$$\langle f, \varphi \rangle = \int_0^\infty f(t) \varphi(t), \quad f \in C^\infty(\overline{R^+}), \quad \varphi \in C_0^\infty(R^+).$$

THEOREM 2.2. Suppose that $a(t, z) \in \Sigma_{c,d}^0$ and that $A = a(t, \partial)$ is defined by (2.1). Then

1. If $k \geq 0$ and $c+k < 1/p < d$,

$$(2.5) \quad a(t, \partial): W^{p,k} \rightarrow W^{p,k}$$

is continuous; moreover,

(2.6) $a(t,\partial): \dot{W}^{p,k} \rightarrow \dot{W}^{p,k}$

is continuous.

2. If $j \geq 0$ and $c < 1/p < d-j$, then

(2.7) $a(t,\partial): W^{p,-j} \rightarrow W^{p,-j}$

is continuous.

PROOF: If $c+1 < 1/p < d$ and $f \in \mathcal{F}'_{c,d}$, then

(2.8)
$$\begin{aligned} \frac{d}{dt} a(t,\partial)f(t) &= \frac{1}{2\pi i} \int_{\text{Re } z = 1/p} t^{-z-1} a(t,z) (-z\tilde{f}(z)) dz \\ &= \frac{1}{2\pi i} \int_{\text{Re } z = 1/p} t^{-z} (\partial/\partial t) a(t,z) \tilde{f}(z) dz. \end{aligned}$$

The first term on the r.h.s. of (2.8) is

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\text{Re } z = 1/p+1} t^{-z} a(t,z-1) [f']^{\sim}(t) dt \\ &= \frac{1}{2\pi i} \int_{\text{Re } z = 1/p} t^{-z} a(t,z-1) [f']^{\sim}(z) dz. \end{aligned}$$

The

second term on the r.h.s. of (2.8) is $(\partial/\partial t)a(t,\partial)f$.
Hence by Theorem 2.1,

(2.9) $\|\frac{d}{dt} a(t,\partial)f\|_p \leq C\|f'\|_p + C\|f\|_p.$

The continuity of (2.5) now follows by induction on k ; (2.6) follows since $a(t,\partial): \mathcal{F}'_{c,d} \rightarrow \mathcal{F}'_{c,d}$.

To prove 2., fix $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}^+)$. By Theorem 1.3,

$$\langle Af, \varphi \rangle = \langle f, {}^t A \varphi \rangle,$$

with ${}^t A = {}^t a(t, \partial)$, where ${}^t a(t, z) \in \Sigma_{1-d, 1-c}^0$. If $1/p + 1/q = 1$ and $1-d+j < 1/q < 1-c$, i.e., $c < 1/p < d-j$, then ${}^t A: W^{q,j} \rightarrow W^{q,j}$ is continuous by part 1 of the Theorem. Part 2 follows.

q.e.d.

To study operator $a(t, \partial) \in \text{Op} - \Sigma_{c,d}^m$ with $m \neq 0$, we introduce weighted Sobolev spaces following the ideas and notation of Nourrigat [N2].

DEFINITION 2.1. For $\lambda \in \mathbb{Z}_+$ define

$$(2.10) \quad W_\lambda^{p,k} = \{f \in W^{p,k}; \partial^j f \in W^{p,k}, j = 0, \dots, \lambda\},$$

and

$$(2.11) \quad \dot{W}_\lambda^{p,k} = \{f \in W^{p,k}; \partial^j f \in \dot{W}^{p,k}, j = 0, \dots, \lambda\}.$$

To define $W_\lambda^{p,k}$ with $\lambda \in \mathbb{Z}_-$, we introduce an invertible pdo $\Lambda = \lambda(\partial)$ of order 1. Let d_0 be a large positive number and define

$$(2.11) \quad \lambda(z) = z - d_0 \in \Sigma_{-\infty, \infty}^{-1},$$

$$\lambda^{-1}(z) = (z - d_0)^{-1} \in \Sigma_{-\infty, d_0}^{-1}$$

and let $\Lambda = \lambda(\partial)$, $\Lambda^{-1} = \lambda^{-1}(\partial)$. If $f \in \mathcal{S}'_{c,d}$, $d \leq d_0$

$$(2.12) \quad \Lambda^{-1}f(t) = \int_0^t \left(\frac{t}{s}\right)^{-d_0} f(s) \frac{ds}{s}.$$

and

$$(2.13) \quad \Lambda\Lambda^{-1}f = \Lambda^{-1}\Lambda f = f.$$

LEMMA 2.1. If $l \in \mathbb{Z}_+$, then $f \in W_l^{p,k}$ iff $\Lambda^l f \in W^{p,k}$; moreover, $f \in W_l^{p,k}$ iff $\Lambda^l f \in W^{p,k}$. Also

$$\|f; W_l^{p,k}\| \sim \|\Lambda^l f; W^{p,k}\|.$$

PROOF: If $f \in W_l^{p,k}$, then $\|\Lambda^l f; W^{p,k}\| \leq C \sum_{j=0}^l \|\partial^j f; W^{p,k}\|$
 $= C\|f; W_l^{p,k}\|.$

If $f \in C_0^\infty(\mathbb{R})$, then for $j = 0, \dots, l$, $\partial^j f = a_j(\partial)\Lambda^l f$
 where $a_j(z) = z^j(z - d_0)^{-l} \in \Sigma_{-\infty, d_0}^{j-l}$. By Theorem 2.2,
 $\|\partial^j f; W^{p,k}\| \leq C\|\Lambda^l f; W^{p,k}\|.$

q.e.d.

DEFINITION 2.2. For $l \in \mathbb{Z}_-$, a distribution $f \in \mathcal{S}'(\mathbb{R}^+)$ is in $W_l^{p,k}$ (respectively $\dot{W}_l^{p,k}$) iff $f = \Lambda^{-l}g$, $g \in W^{p,k}$ (respectively $\dot{W}^{p,k}$). The norm on $W_l^{p,k}$ is

$$(2.15) \quad \|f; W_{\ell}^{p,k}\| = \|\Lambda^{\ell} f; W^{p,k}\|.$$

THEOREM 2.3. Let $m \in \mathbb{Z}$ and $d \leq d_0$. Let $a(t, z) \in \Sigma_{c,d}^m$ and define $A = a(t, \partial)$. Then

1. If $k \in \mathbb{Z}_+$ and $c+k \leq 1/p < d$, then
for all $\ell \in \mathbb{Z}$,

$$(2.16) \quad A : W_{\ell}^{p,k} \rightarrow W_{\ell-m}^{p,k}$$

$$A : \dot{W}_{\ell}^{p,k} \rightarrow \dot{W}_{\ell-m}^{p,k}$$

is continuous.

2. If $j \geq 0$ and $c < 1/p < d-j$, then

$$(2.17) \quad A : W_{\ell}^{p,-j} \rightarrow W_{\ell}^{p,-j}$$

is continuous.

PROOF: If $f = \Lambda^{-\ell} g$, $g \in W^{p,k}$, then

$\Lambda^{\ell-m} a(t, \partial) f = \Lambda^{\ell-m} a(t, \partial) \Lambda^{-\ell} g$ and the operator

$$b(t, \partial) = \Lambda^{\ell-m} a(t, \partial) \Lambda^{-\ell}$$

is in $Op - \Sigma_{c,d}^0$ by the Symbolic Calculus Theorem

1.2. Hence by Theorem 2.2, $\|A f; W_{\ell-m}^{p,k}\| \sim \|\Lambda^{\ell-m} A f; W^{p,k}\|$

$$\leq C \|b(t, \partial) g; W^{p,k}\| \leq C \|g; W^{p,k}\| \sim \|f; W_{\ell}^{p,k}\|.$$

q.e.d.

3. An algebra of pseudodifferential operators on $L^p(\mathbb{R}^+)$.

Let $a(t,\partial) \in \text{Op} - \Sigma_{c,d}^0$ with $c < 1/p < d$. We wish to give conditions on the symbol $a(t,z)$ which determine whether the operator $a(t,\partial)$ is a Fredholm operator on $L^p(\mathbb{R}^+)$ and to link the index of a Fredholm operator to topological properties of the symbol. It may happen that the operator $a(t,\partial)$ is Fredholm on $L^p(\mathbb{R}^+)$ for some choices of $1/p \in (c,d)$ and not for others; the index of the operator (on $L^p(\mathbb{R}^+)$) will change as we cross a non-Fredholm value of $1/p \in (c,d)$.

Before defining the notion of principal symbol, we introduce the functions in $\mathcal{O}_{0,1}^0$;

(3.1)

$$\begin{aligned} h(z) &= -\cot \pi z, \\ \theta(z) &= (1 - e^{2\pi iz})^{-1}, \\ 1 - \theta(z) &= -e^{2\pi iz}(1 - e^{2\pi iz})^{-1}. \end{aligned}$$

Note that $\theta(1/p + i\infty) = 1$, $\theta(1/p - i\infty) = 0$, and that for $j \geq 1$, $(d/dz)^j \theta(z) \in \mathcal{O}_{0,1}^{-\infty}$. The corresponding operators in $\text{Op} - \Sigma_{0,1}^0$ have the representations:

(3.2)

$$\begin{aligned} Hf(t) &= h(\partial)f(t) = \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds, \\ \theta(\partial) &= \frac{1}{2} (I - iH), \\ I - \theta(\partial) &= \frac{1}{2} (I + iH); \end{aligned}$$

H is the Hilbert transform on $L^p(\mathbb{R}^+)$: Note that if \mathcal{H} is the Hilbert transform on $L^p(\mathbb{R}), (1/2)(I \pm i\mathcal{H})$ form a basis for singular integral operators on \mathbb{R} .

DEFINITION 3.1. A symbol $a(t, z)$ is in the class $\Sigma_{1/p}$ iff for some c, d with $0 \leq c < 1/p < d \leq 1$,

$$1. \quad a(t, z) \in \Sigma_{c, d}^0,$$

$$2. \quad \text{There are functions } a_+(t), a_-(t) \in \Sigma_{c-d, d-c} \text{ such that}$$

$$(3.3) \quad a(t, z) - a_+(t)\theta(z) - a_-(t)(1 - \theta(z)) \in \Sigma_{c, d}^{-1}.$$

If $a(t, z) \in \Sigma_{1/p}$, the functions $a_+(t), a_-(t)$ in (3.3) are uniquely determined by the relations

$$(3.4) \quad a_+(t) = a(t, 1/p + i\infty), \quad a_-(t) = a(t, 1/p - i\infty).$$

The function $a(t, z)$ is a continuous function defined on the compact rectangle:

$$(3.5) \quad R_{1/p} = \{(t, z) : 0 \leq t \leq \infty, \quad z = 1/p + i\xi, \quad -\infty \leq \xi \leq \infty\}.$$

DEFINITION 3.2. If $a(t, z) \in \Sigma_{1/p}$ and

$$(3.6) \quad \begin{aligned} Af(t) &= a(t, \partial)f(t) \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(t, z) \tilde{f}(z) dz, \end{aligned}$$

for $f \in C_0^\infty(\mathbb{R})$, then the principal symbol, $\sigma(A)$, of the operator A is the function $a(t, z)$ restricted to the boundary of the compact rectangle $R_{1/p}$.

REMARKS

1. $\sigma(A) = 0$ iff $a(t, z) \in \mathcal{F}_{c,d}^{-1}$ for some c, d .
2. If for some $\delta > 0$ we are given functions $a_+(t), a_-(t) \in \mathcal{F}_{-\delta, \delta}$, and $a_0(z), a_\infty(z)$ such that the functions

$$(3.7) \quad \begin{aligned} b_0(z) &= a_0(z) - a_+(0)\theta(z) - a_-(0)(1 - \theta(z)), \\ b_\infty(z) &= a_\infty(z) - a_+(\infty)\theta(z) - a_-(\infty)(1 - \theta(z)), \end{aligned}$$

are in $\mathcal{O}_{1/p - \delta, 1/p + \delta}^{-1}$, we select a cutoff function $\chi(t) \in C_0^\infty(\mathbb{R})$, $\chi(t) = 1$ near $t = 0$, and define

$$(3.8) \quad \begin{aligned} a(t, z) &= \chi(t)b_0(z) + (1 - \chi(t))b_\infty(z) + a_+(t)\theta(z) \\ &\quad + a_-(t)(1 - \theta(z)). \end{aligned}$$

Then $a(t, z) \in \Sigma_{1/p}$ and

$$(3.9) \quad \begin{cases} a(0, z) = a_0(z), \quad a(\infty, z) = a_\infty(z), \\ a(t, 1/p + i\infty) = a_+(t), \quad a(t, 1/p - i\infty) = a_-(t). \end{cases}$$

3. That the principal symbol, $\sigma(A)$, of an operator $A = a(t, \partial)$ is uniquely determined by the operator A is a consequence of the following observations. Fix

$f \in C_0^\infty(\mathbb{R}^+)$. Then for $\lambda \in \mathbb{R}$,

$$(3.10) \quad t^{i\lambda} A(t^{-i\lambda} f)(t) = \\ = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(t, z + i\lambda) \tilde{f}(z) dz.$$

As $\lambda \rightarrow \pm\infty$, $t^{i\lambda} A(t^{-i\lambda} f)$ converges in $L^p(\mathbb{R}^+)$ to $a_\pm(t)f(t)$. Next, for $\lambda > 0$, let $T_\lambda f(t) = \lambda^{-1/p} f(t/\lambda)$. Then $\|T_\lambda f\|_p = \|f\|_p$ and

$$(3.11) \quad (T_{1/\lambda} A T_\lambda) f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(\lambda t, z) \tilde{f}(z) dz.$$

Hence in $L^p(\mathbb{R}^+)$,

$$(3.12) \quad \lim_{\epsilon \rightarrow 0} (T_{1/\epsilon} A T_\epsilon f)(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(0, z) \tilde{f}(z) dz,$$

$$\lim_{\lambda \rightarrow \infty} (T_{1/\lambda} A T_\lambda f)(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = 1/p} t^{-z} a(\infty, z) \tilde{f}(z) dz.$$

The following result is fundamental in studying the Fredholm properties on $L^p(\mathbb{R}^+)$ operators defined by (3.6).

THEOREM 3.1. If $a(t, z) \in \Sigma_{1/p}$, then

$$A = a(t, \partial) : L^p \rightarrow L^p$$

is a compact operator iff the principal symbol, $\sigma(A)$, is identically 0; i.e., $a(t, z) \in \Sigma_{c,d}^{-1}$ for some c, d , $c < 1/p < d$.

PROOF: We first show that if $a(t,z) \in \Sigma_{c,d}^{-1}$ then $a(t,\partial)$ is compact on $L^p(\mathbb{R}^+)$. If $a(t,z) \in \Sigma_{c,d}^{-1}$, then $f \mapsto -t \frac{\partial}{\partial t} Af$ is bounded on $L^p(\mathbb{R}^+)$ by Theorems 1.2 and 2.1. Hence if $a(t,z) = 0$ for t outside $[\epsilon, T]$, $0 < \epsilon < T < \infty$, A maps L^p into $W^{p,1}(\epsilon, T)$, which is compactly embedded in L^p . If $a(t,z) \in \Sigma_{c,d}^{\alpha-1}$, and $\epsilon > 0$, let $\chi(t)$ be a cutoff function, and define

$$a_\epsilon(t,z) = (1 - \chi(t/\epsilon))\chi(\epsilon t)a(t,z);$$

$a_\epsilon(t,z)$ converges to $a(t,z)$ in $\Sigma_{c,d}^{\alpha-1}$ as $\epsilon \rightarrow 0^+$. The corresponding operators $a_\epsilon(t,\partial)$ are compact on L^p and converge in operator norm to $a(t,\partial)$.

Conversely, suppose $A = a(t,\partial) : L^p \rightarrow L^p$ is compact. We use the constructions of Remark 3. following Definition 3.2. Fix $f \in C_0^\infty(\mathbb{R}^+)$. Then as $\lambda \rightarrow \pm\infty$, $t^{-i\lambda}f(t)$ converges to 0 weakly in L^p , by the Riemann-Lebesgue Lemma, so that $t^{i\lambda}A(t^{-i\lambda}f)$ converges to 0 strongly in L^p . Hence $a_+(t) = a_-(t) = 0$. If $T_\lambda f(t) = \lambda^{-1/p}f(t/\lambda)$, as $\lambda \rightarrow 0^+$ or $\lambda \rightarrow \infty$, $T_\lambda f$ converges to 0 weakly in L^p , and $A(T_\lambda f)$ converges to 0 in L^p . Each of the operators in (3.11) is 0 so that $a(0,z) = a(\infty,z) = 0$.

q.e.d.

DEFINITION 3.4. Let $a(t,z) \in \Sigma_{1/p}$ and let $a(t,\partial) \in Op - \Sigma_{1/p}$ be defined by (3.6). Then A

is said to be elliptic on $L^p(\mathbb{R}^+)$ iff its principal symbol, $\sigma(A)$, satisfies

$$(3.13) \quad \sigma(A)(t, z) \neq 0, \quad (t, z) \in \partial R_{1/p}.$$

If $a(t, z)$ is the symbol of an elliptic operator on $L^p(\mathbb{R}^+)$, then for $0 \leq t \leq \infty$,

$$(3.14) \quad \begin{aligned} a_+(t) &= a(t, 1/p + i\infty) \neq 0, \\ a_-(t) &= a(t, 1/p - i\infty) \neq 0, \end{aligned}$$

which are conditions independent of p , and reflect the fact that $a(t, \partial)$, considered as a classical pdo of order 0 on the noncompact manifold \mathbb{R}^+ is elliptic. If $a(t, z) \in \Sigma_{1/p}$, the conditions

$$(3.15) \quad a(0, z) \neq 0, \quad a(\infty, z) \neq 0, \quad \operatorname{Re} z = 1/p$$

are then satisfied for all p such that $1/p$ lies outside a discrete set in (c, d) .

We now relate "ellipticity on $L^p(\mathbb{R}^+)$ " to the Fredholm properties of the operator $A = a(t, \partial)$.

THEOREM 3.2. Let $A = a(t, \partial) \in \mathcal{O}p - \Sigma_{1/p}$. The following conditions are equivalent:

1. $a(t, \partial)$ is elliptic on $L^p(\mathbb{R}^+)$.
2. For some c, d , $0 \leq c < 1/p < d \leq 1$, there is a symbol $b(t, z) \in \Sigma_{c,d}^0$ such that

$$a(t, \partial)b(t, \partial) - I, b(t, \partial)a(t, \partial) - I \in \text{Op} - \Sigma_{c,d}^{-1}.$$

3. There is a bounded operator B on $L^p(\mathbb{R}^+)$ such that $AB - I$ and $BA - I$ are compact operators on $L^p(\mathbb{R}^+)$.

4. For $f \in C_0^\infty(\mathbb{R}^+)$, there is an a priori estimate

$$(3.16) \quad \|f\|_p \leq C\|Af\|_p + \|Kf\|_p,$$

where K is a compact operator on $L^p(\mathbb{R}^+)$.

PROOF: 1 implies 2: Choose $c < 1/p < d$ such that $a(0, z)$ and $a(\infty, z)$ have no zeroes in the strip $S_{c,d}$. Let $b(t, z) \in \Sigma_{c,d}^0$ be such that for $c < 1/q < d$, $b(t, z)a(t, z) = a(t, z)b(t, z) = 1$ on the boundary of $R_{1/q}$; $b(t, \partial)$ could be constructed as in Remark 2 following Definition 3.2. Condition 2 now follows from Theorem 1.2.

2 implies 3: This follows from Theorem 3.1.

3 implies 4: This is obvious.

4 implies 1: We use the constructions of Remark 3 following Definition 3.2. Fix $f \in C_0^\infty(\mathbb{R}^+)$. Then if K is compact on L^p , $\|t^{i\lambda}K(t^{-i\lambda}f)\|_p \rightarrow 0$ as $\lambda \rightarrow \pm\infty$. Inequality (3.16) implies that

$$\|f\|_p \leq C\|t^{i\lambda}A(t^{-i\lambda}f)\|_p + \|t^{i\lambda}K(t^{-i\lambda}f)\|_p$$

and letting $\lambda \rightarrow \pm\infty$, we obtain the a priori inequality

$$(3.17) \quad \|f\|_p \leq C \|a_{\pm}(t)f\|_p$$

which implies that $|a_{\pm}(t)| \geq \frac{1}{C}$. If $T_{\lambda}f(t) = \lambda^{-1/p}f(t/\lambda)$, then

$$(3.18) \quad \|f\|_p \leq C \|T_{1/\lambda} A(T_{\lambda}f)\|_p + \|T_{1/\lambda} K T_{\lambda} f\|_p.$$

Letting $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ in (3.18) and applying (3.12) we obtain

$$(3.19) \quad \|f\|_p \leq C \|a(0, \partial)f\|_p, \quad \|f\|_p \leq C \|a(\infty, \partial)f\|_p,$$

for $f \in C_0^{\infty}(R^+)$. Inequalities (3.19) imply that the Mellin multiplier operators $a(0, \partial)$ and $a(\infty, \partial)$ have closed range and hence the symbols $a(0, z)$ and $a(\infty, z)$ have no zeroes on $\operatorname{Re} z = 1/p$.

q.e.d.

For an $N \times N$ system of operators in $Op - \Sigma_{1/p}$, the notion of ellipticity on $L^p([L^p(R^+)]^N)$ is that

$$(3.20) \quad \det[\sigma(A_{ij})] \neq 0, \quad (t, z) \in \partial R_{1/p},$$

where $[\sigma(A_{ij})]$ is the matrix of principal symbols. A theorem analogous to Theorem 3.2 may be proved for such a system.

Since $\partial R_{1/p}$ is homeomorphic to the unit circle S^1 , if A is an $N \times N$ system of operators which is

elliptic on L^p , then $[\sigma(A_{1j})]$, considered as a map from $\partial R_{1/p}$ to $GL(N, \mathbb{C})$, has a topological degree, $n_{1/p}$, defined as

$$(3.21) \quad \begin{cases} n_{1/p} = n_{1/p}(\sigma(A)) = \text{winding number } \det[\sigma(A_{1j})] \\ \quad = \frac{1}{2\pi} \Delta_{\partial R_{1/p}} \text{Arg } \det[\sigma(A_{1j})], \end{cases}$$

where the change in argument is calculated as $\partial R_{1/p}$ is traversed in the clockwise direction.

We emphasize that $n_{1/p}$ depends on $1/p$ and that if p is changed so that the line $\text{Re } z = 1/p$ crosses a zero of $\det[\sigma(A_{1j})](0, z)$ or $\det[\sigma(A_{1j})](\infty, z)$, the jump in $n_{1/p}$ may be calculated by the Argument Principle.

The analytical index of an $N \times N$ system $A = [a_{ij}(t, \partial)]$ of operators in $Op - \Sigma_{1/p}$ which is elliptic on L^p is defined as

$$(3.22) \quad \text{ind}_p(A) = \dim \ker A - \dim \ker A^*,$$

where in (3.22) we have considered A as an operator on $[L^p(\mathbb{R}^+)]^N$ and $A^* = [a_{ji}^*(t, \partial)]$ as an operator on $[L^q(\mathbb{R}^+)]^N$, $1/p + 1/q = 1$.

The index theorem we shall develop in Theorem 3.3 is that

$$(3.23) \quad \text{ind}_p(A) = n_{1/p}(\sigma(A)).$$

The first step in the proof of (3.23) is the construction of a canonical elliptic system A for which

$$(3.24) \quad \text{ind}_p(A) = 1 = n_{1/p}(\sigma(A)).$$

For this see the example of Cordes and Herman [CH,C], or consider the following example.

For $f \in L^p(\mathbb{R})$ define the (column) vector function $f = (f_1, f_2) \in [L^p(\mathbb{R}^+)]^2$ by

$$(3.25) \quad f_1(t) = f(t), \quad f_2(t) = f(-t).$$

The Hilbert transform on $L^p(\mathbb{R})$

$$(3.26) \quad \mathcal{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy,$$

may be represented as the 2×2 system of operators in $\mathcal{O}_p - \Sigma_{1/p}$:

$$(3.27) \quad \begin{aligned} (\mathcal{H}f)_1(t) &= Hf_1(t) + Sf_2(t) \\ (\mathcal{H}f)_2(t) &= -Sf_1(t) - Hf_2(t), \end{aligned}$$

when $H = h(\partial)$ is a Hilbert transform on $L^p(\mathbb{R}^+)$ defined in (3.1-2) and

$$(3.28) \quad Sf(t) = s(\partial)f(t) = \frac{1}{\pi} \int_0^{\infty} \frac{f(s)}{t+s} ds$$

is the Stieltjes transform on $L^p(\mathbb{R}^+)$ with symbol $s(z) = \csc \pi z \in \mathcal{O}_{0,1}^{-\infty}$. For $f \in L^p(\mathbb{R})$, define

$$(3.29) \quad f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-z} dx, \quad \text{Im } z \neq 0,$$

$$f^{\pm}(x) = \lim_{\epsilon \rightarrow 0^+} f(x \pm i\epsilon) = \frac{1}{2} (I \pm iN)f(x).$$

For $x \in \mathbb{R}$ let $w = w(x) = \frac{x-1}{x+1}$; then $f(x) \rightarrow M_w f(x) = w(x)f(x)$ has the system representation

$$(3.30) \quad (M_w f)_1(t) = w(t)f_1(t), \quad (M_w f)_2(t) = \bar{w}(t)f_2(t).$$

It is well known that if on $L^p(\mathbb{R})$ we define $Gf(x) = w(x)f^-(x) + f^+(x)$, G has kernel spanned by $f(x) = \frac{1}{w(x)} - 1$ and cokernel 0. When G is written as a 2×2 system of pseudodifferential operators we obtain the matrices of principal symbols

$$(3.31) \quad \sigma(G)\left(t, \frac{1}{p} + i\infty\right) = \begin{bmatrix} w(t) & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma(G)(0, z) = \begin{bmatrix} -ih & is \\ -is & ih \end{bmatrix}, \quad \sigma(G)(\infty, z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma(G)\left(t, \frac{1}{p} - i\infty\right) = \begin{bmatrix} 1 & 0 \\ 0 & \bar{w}(t) \end{bmatrix}.$$

Using (3.31) we have that $\text{ind}_p(G) = 1 = n_{1/p}(\sigma(G))$.

The second step in the proof of (3.23) is, given a system A of operators which is elliptic on L^p and such that $n_{1/p}(\sigma(A)) = 0$, to construct a

homotopy through elliptic operators to the identity operator.

Suppose that A_0 and A_1 are systems which are elliptic on L^p and there is a homotopy between the matrices of principal symbols

$$(3.32) \quad \sigma(A_s) : \partial R_{1/p} \rightarrow GL(N, \mathbb{C}), \quad 0 \leq s \leq 1$$

such that for each s , $\sigma(A_s)$ is a matrix of principal symbols, and

1. If $A_{s\pm}(t) = \sigma(A_s)(t, 1/p \pm i\infty)$, then

$$(3.33) \quad \lim_{s \rightarrow s'} \sup_{0 \leq t \leq \infty} \|A_{s\pm}(t) - A_{s'\pm}(t)\| = 0.$$

2. $\lim_{s \rightarrow s'} \sup_{\operatorname{Re} z = 1/p} \|\sigma(A_s)(0, z) - \sigma(A_{s'})(0, z)\| = 0,$

$$(3.34) \quad \lim_{s \rightarrow s'} \sup_{\operatorname{Re} z = 1/p} \left\| z \frac{d}{dz} [\sigma(A_s)(0, z) - \sigma(A_{s'})(0, z)] \right\| = 0.$$

3. $\lim_{s \rightarrow s'} \sup_{\operatorname{Re} z = 1/p} \|\sigma(A_s)(\infty, z) - \sigma(A_{s'})(\infty, z)\| = 0,$

$$(3.35) \quad \lim_{s \rightarrow s'} \sup_{\operatorname{Re} z = 1/p} \left\| z \frac{d}{dz} [\sigma(A_s)(\infty, z) - \sigma(A_{s'})(\infty, z)] \right\| = 0.$$

A homotopy (3.32) satisfying (3.33-35) will be called a homotopy through elliptic symbols; for each s we use the construction of (3.8) for a matrix of operators, A_s , with the given symbols $\sigma(A_s)$; by the 1-

dimensional Marcinkiewicz Theorem we have that

$s \rightarrow A_s$ is continuous from $[0,1]$ into the bounded operators on $[L^p(\mathbb{R}^+)]^N$. Hence $\text{ind}_p(A_0) = \text{ind}_p(A_1)$.

We now outline the proof of the Index Theorem. As a first step consider the scalar case.

LEMMA 3.1. Let $\sigma(A)$ be the symbol of an operator $A = a(t, \partial) \in \text{Op} - \Sigma_{1/p}$ which is elliptic on $L^p(\mathbb{R}^+)$ and suppose that

$$n_{1/p} = 0$$

where $n_{1/p} = n_{1/p}(\sigma(A))$ is defined by (3.21) with $N = 1$. Then

$$\text{ind}_p(A) = 0.$$

PROOF: The required homotopy through elliptic symbols is given by

$$\sigma(A_s)(t, z) = e^{s \log \sigma(A)}, \quad 0 \leq s \leq 1, \quad (t, z) \in \partial R_{1/p}.$$

q.e.d.

We now introduce a nonzero scalar function

$\varphi \in \mathcal{F}_{-\delta, \delta}$ such that $\varphi(0) = \varphi(\infty) = 1$ and $n(\varphi) = \frac{1}{2\pi} \Delta \int_0^\infty \text{Arg } \varphi = 1$. For $k \in \mathbb{Z}$ define the operator

$$(3.36) \quad A_k = a_k(t, \partial) = \varphi(t)^k \theta(\partial) + (I - \theta(\partial)).$$

Then A_k is elliptic on L^p for $1 < p < \infty$, and $n_{1/p}(\sigma(A_k)) = n(\sigma^k) = k$. Define

$$(3.37) \quad J = \text{ind}_p(A_1).$$

Then

$$(3.38) \quad \text{ind}_p(A_k) = k \text{ind}_p(A_1) = Jk.$$

(We will show in Theorem 3.3 that in fact, $J = 1$.)

The next Lemma is evident.

LEMMA 3.2. If $A = [\delta_{ij} a_{ij}(t, \partial)]$ is a diagonal sys-
tem of operators which is elliptic on L^p . Then

$$(3.39) \quad \text{ind}_p(A) = J n_{1/p}(\sigma(A)),$$

where J is defined as in (3.37).

We next calculate the index of a system whose symbols is the identity matrix on \mathbb{R}^n sides of $\partial R_{1/p}$.

LEMMA 3.3. Let $\sigma(A) = [\sigma(A_{ij})]$ be an $N \times N$ matrix
of principal symbols a system A which is elliptic
on L^p . Suppose moreover that:

$$(3.40) \quad \begin{aligned} \sigma(A)(t, 1/p + i\infty) &= \sigma(A)(t, 1/p - i\infty) = I, \\ \sigma(A)(\infty, z) &= I. \end{aligned}$$

Then

$$(3.41) \quad \text{ind}_p(A) = J n_{1/p}(\sigma(A)).$$

PROOF: Multiplying A by a diagonal system, B , of operators satisfying the hypotheses of Lemma 3.2, condition (3.40), and such that $n_{1/p}(\sigma(B)) = -n_{1/p}(\sigma(A))$, we may assume that $n_{1/p}(\sigma(A)) = 0$.

From condition (3.40) we may choose c, d , $0 \leq c < 1/p < d \leq 1$ such that $a_{ij}(0, z) = \delta_{ij} + g_{ij}(z)$, where $g_{ij} \in \mathcal{O}_{c,d}^{-1}$ and $[a_{ij}(0, z)]$ has nonvanishing determinant in $S_{c,d}$.

Given $\epsilon > 0$, shrinking (c, d) if necessary, by Runge's Theorem there are rational functions

$g_{ij\epsilon} \in \mathcal{O}_{c,d}^{-1}$ such that

$$\sup_{S_{c,d}} |g_{ij} - g_{ij\epsilon}| < \epsilon, \quad \sup_{S_{c,d}} \left| z \frac{d}{dz} \{g_{ij} - g_{ij\epsilon}\} \right| < \epsilon.$$

Let $a_{ij\epsilon}(0, z) = \delta_{ij} + g_{ij\epsilon}(z)$ and let A_ϵ be a system of operators with principal symbol, $\sigma(A_\epsilon)$

which satisfies (3.40) and such that

$\sigma(A_\epsilon)(0, z) = [a_{ij\epsilon}(0, z)]$. If ϵ is small, then $n_{1/p}(\sigma(A_\epsilon)) = 0$ and $\text{ind}_p A_\epsilon = \text{ind}_p A$. We have reduced the problem to the case that $\sigma(A_\epsilon)(0, z)$ consists of rational functions.

Make a fractional linear transformation, $z \rightarrow w$, which maps the half plane $\{\text{Re } z < 1/p\}$ onto the interior of the unit circle $\{|w| < 1\}$ and such that as z traverses $\text{Re } z = 1/p + i\xi$, $-\infty \leq \xi \leq \infty$, w tra-

verses $\{|w| = 1\}$ in the clockwise direction. Define $G(w) = [a_{ij}(0, z)]$. Then $G: S^1 \rightarrow GL(N, \mathbb{C})$ is a rational function. Now (Cf., e.g., [V, pp. 45-49]) there are matrices of rational functions, $G_0(w)$, $G_\infty(w)$ holomorphic in $\{|w| \leq 1\}$ and $\{|w| \geq 1\}$ respectively such that

$$(3.42) \quad G(w) = G_0(w) D G_\infty(w)$$

where

$$(3.43) \quad D = D(n_1, \dots, n_N) = [\delta_{ij} w^{n_i}]$$

is a diagonal matrix and

$$n_1 + \dots + n_N = 0 = \frac{1}{2\pi} \Delta_{|w|=1} \operatorname{Arg} \det G(w).$$

For $0 \leq s \leq 1$, define

$$(3.44) \quad G_s(w) = G_0^{-1}(s) G_0(sw) D G_\infty(w/s) G_\infty^{-1}(1/s).$$

Formula (3.44) gives a homotopy through elliptic symbols to a diagonal matrix of operators which satisfies the hypotheses of Lemma 3.2.

q.e.d.

It is now straightforward to prove the formula

$$(3.45) \quad \operatorname{ind}_p(A) = J n_{1/p}(\sigma(A))$$

in the case that $\sigma(A)$ is the principal symbol of an

elliptic operator whose principal symbol is the identity matrix I at the corners of $\partial R_{1/p}$; i.e.,

$$(3.46) \quad \sigma(A)(0, 1/p \pm i\infty) = \sigma(A)(\infty, 1/p \pm i\infty) = I.$$

To handle the general case we use the following Lemma whose proof we leave to the reader.

LEMMA 3.4. Given 4 matrices A_{0+} , A_{0-} , $A_{\infty+}$, $A_{\infty-}$ in $GL(N, \mathbb{C})$, there is a symbol, $\sigma(A)$, of an elliptic system on L^p such that

$$(3.47) \quad \begin{aligned} \sigma(A)\left(0, \frac{1}{p} \pm i\infty\right) &= A_{0, \pm} \\ \sigma(A)\left(\infty, \frac{1}{p} \pm i\infty\right) &= A_{\infty, \pm} \end{aligned}$$

and such that

$$(3.48) \quad \text{ind}_p(A) = J n_{1/p}(\sigma(A)).$$

We are now ready to prove (3.23).

THEOREM 3.3. (The Index Theorem). If A is a system of operators in $\text{Op } \Sigma_{1/p}$ which is elliptic on L^p , then

$$(3.23) \quad \text{ind}_p(A) = n_{1/p}(\sigma(A)).$$

PROOF: By Lemmas 3.1 to 3.4 we have that

$\text{ind}_p(A) = J n_{1/p}(\sigma(A))$. The example studied in (3.25) to (3.31) shows that $J = 1$. q.e.d.

4. Pseudodifferential operators on a finite interval.

We shall develop an algebra of pdo's on a finite interval $I = [0, \pi]$. For $f \in L^p(I)$, define

$$Tf(t) = f(\pi - t).$$

Then $T^* = T$ and $TT = I$. If A is a bounded operator on $L^p(I)$, the essential requirement for A to be in our class of pdo's is that, when acting on functions supported in $[0, \pi)$, both A and TAT , modulo compact operators, are in $Op - \Sigma_{1/p}$ on R^+ .

DEFINITION 4.1. A bounded operator A on $L^p(I)$ is a pdo of class $Op - \Sigma_{1/p}(I)$ iff

1. If $\varphi, \psi \in C_0^\infty(R)$ have disjoint supports then the map

$$(4.1) \quad f \rightarrow (\varphi A \psi)f = \varphi \chi_{[0, \pi]} A(\chi_{[0, \pi]} \psi f)$$

is a compact operator on $L^p(R^+)$.

2. If $\varphi, \psi \in C_0^\infty([0, \pi))$, there is an operator $A_{\varphi\psi} \in Op - \Sigma_{1/p}$ and a compact operator $K_{\varphi\psi}$ on $L^p(I)$ such that

$$(4.2) \quad \varphi A \psi = A_{\varphi\psi} + K_{\varphi\psi}.$$

3. The operator

$$(4.3) \quad f \rightarrow TATf = A_T f$$

with $Tf(t) = f(\pi - t)$ satisfies conditions 1 and 2.

An example of an operator of class $Op - \Sigma_{1/p}(I)$ is the finite Hilbert transform

$$(4.4) \quad Hf(t) = \text{p.v.} \frac{1}{\pi} \int_0^\pi \frac{f(s)}{t-s} ds, \quad (THT = -H),$$

and the finite Stieltjes transform

$$(4.5) \quad Sf(t) = \text{p.v.} \frac{1}{\pi} \int_0^\pi \frac{f(s)}{t+s} ds.$$

For $\varphi, \psi \in C_0^\infty([0, \pi])$, $\varphi TST\psi$ is a compact operator on $L^p(I)$.

To define the principal symbol of a pdo of class $Op - \Sigma_{1/p}(I)$, we first note that if $\varphi, \psi \in C_0^\infty([0, \pi])$ and $a_{\varphi\psi}(t, \partial) = A_{\varphi\psi} \in Op - \Sigma_{1/p}$ is defined so that

$$(4.6) \quad \varphi A\psi = A_{\varphi\psi} + K_{\varphi\psi}$$

as in (4.2), then there is a well defined function $a_0(t, z)$ defined on

$$(4.7) \quad \Lambda_{1/p} = \{(t, z) : (t, z) \in \partial R_{1/p}, 0 \leq t < \pi\}$$

such that

$$(4.8) \quad \sigma(a_{\varphi\psi}(t, \partial)) = \varphi(t)a_0(t, z)\psi(t)$$

for $(t, z) \in \Lambda_{1/p}$. The function $a_0(t, z)$ is defined by choosing $\varphi, \psi \in C_0^\infty([0, \pi])$ which are $= 1$ in a neighborhood of t and letting

(4.9) $a_0(t,z) = \sigma(a_{\varphi\psi})(t,z).$

The formula (4.9) uniquely determines $a_0(t,z)$ on $\Lambda_{1/p}$; if φ_1 and ψ_1 are also = 1 near t then $\varphi_1\varphi A\psi_1 = \varphi_1\psi_1 a_{\varphi\psi}(t,\partial) + K = \varphi\psi a_{\varphi_1\psi_1} + K_1$ where K and K_1 are compact operators on $L^p(\mathbb{R}^+)$, and by Theorem 3.1, $\varphi_1\psi_1 a_{\varphi\psi}$ and $\varphi\psi a_{\varphi_1\psi_1}$ have the same principal symbols.

The operator $A_T = TAT$ has a well defined principal symbol on $\Lambda_{1/p}$ which we denote by $a_\pi(t,z).$

To glue together the symbols $a_0(t,z)$ and $a_\pi(t,z)$ we will show that for $0 < t < \pi$, the following compatibility conditions are satisfied:

(4.10)
$$\begin{aligned} a_0(t, 1/p + i\infty) &= a_\pi(\pi - t, 1/p - i\infty), \\ a_0(t, 1/p - i\infty) &= a_\pi(\pi - t, 1/p + i\infty). \end{aligned}$$

To prove (4.10) we introduce the operators

(4.11)
$$\begin{aligned} \Theta &= \frac{1}{2} (I - iH) \\ I - \Theta &= \frac{1}{2} (I + iH) = T\Theta T \end{aligned}$$

with H the finite Hilbert transform given by (4.4).

Fix $\varphi \in C^\infty_0((0,\pi))$ such that $T\varphi(t) = \varphi(t)$ and $\varphi(t) = 1$ on $(\delta, \pi - \delta)$. As operators on $L^p(\mathbb{R}^+)$,

(4.12)
$$\begin{aligned} \varphi A\varphi &= \varphi\{a_0(t, 1/p + i\infty)\Theta(\partial) \\ &\quad + a_0(t, 1/p - i\infty)\} \varphi + K_0, \end{aligned}$$

and

(4.13)
$$\varphi T A T \varphi = \varphi \{ a_{\pi}(t, 1/p + i\infty) \theta(\partial) + a_{\pi}(t, 1/p - i\infty) \} \varphi + K_{\pi},$$

with K_0 and K_{π} compact on $L^p(\mathbb{R}^+)$ and $\theta(\partial)$ defined by (3.2). Modulo compact operators, on $L^p(I)$, we have the representation:

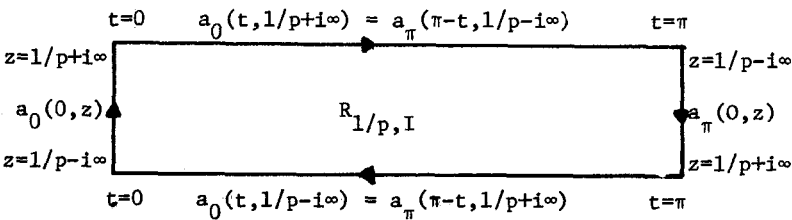
$$\begin{aligned} \varphi A \varphi &= \varphi (a_0(t, 1/p + i\infty) \Theta \\ &\quad + a_0(t, 1/p - i\infty) (I - \Theta)) \end{aligned}$$

and since $T \Theta T = I - \Theta$,

(4.14)
$$\begin{aligned} T \varphi A \varphi T &= \varphi \{ a_0(\pi - t, 1/p + i\infty) (I - \Theta) \\ &\quad + a_0(\pi - t, 1/p - i\infty) \Theta \} \varphi. \end{aligned}$$

Comparing (4.14) to (4.13) we obtain (4.10).

If A is a pdo of class $Op - \Sigma_{1/p}(I)$, the principal symbol of A , $\sigma(A)$, is defined as the pair of functions $a_0(t, z)$, $a_{\pi}(t, z)$ defined on $\wedge_{1/p}$ which satisfy the compatibility conditions (4.10). The natural setting for the principal symbol as a continuous function defined on the boundary of a compact rectangle $R_{1/p, I}$ in (4.15):



It can be shown that a pdo of class $Op - \Sigma_{1/p}(I)$ is compact iff its principal symbol is $\equiv 0$.

The principal symbol of an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ is defined as the matrix of principal symbols.

DEFINITION 4.2. An $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ is elliptic on $L^p = [L^p(I)]^N$ iff the determinant of the matrix of principal symbols does not vanish on $\partial R_{1/p, I}$.

THEOREM 4.1. Let $A = [A_{ij}]$ be an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$. The following are equivalent:

1. A is elliptic on L^p .
2. There is an $N \times N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$ such that

$$AB - I \text{ and } BA - I$$

are compact operators on $[L^p(I)]^N$.

3. There is a bounded operator B on $[L^p(I)]^N$ such that $BA - I$ is a compact operator on $[L^p(I)]^N$.

4. For $\vec{f} = (f_1, \dots, f_N)$ and N -tuple of func-
tions in $C_0^\infty((0,1))$, there is an a priori esti-
mate

(4.16)
$$\|f; L^p(I)\| \leq c \|Af : L^p(I)\| + \|Kf : L^p(I)\|$$

where K is a compact operator on $[L^p(I)]^N$.

PROOF: That 1. implies 2. implies 3. implies 4. is obvious. If (4.16) holds, let $f \in [C_0^\infty((\delta, \pi - \delta))]^N$, and let $\varphi \in C_0^\infty([0, \pi])$ be such that $\varphi = 1$ on $[0, \pi - \delta)$. Then $Af = \varphi A_0 \varphi f + K$, where K is a compact operator on $L^p(\mathbb{R}^+)$. Using the techniques in the proof of Theorem 3.2, we obtain the apriori estimates

(4.17)
$$\|f; L^p(\mathbb{R}^+)\| \leq c \|A_0(t; 1/p \pm i\epsilon)f; L^p(\mathbb{R}^+)\|,$$

(4.18)
$$\|f; L^p(\mathbb{R}^+)\| \leq c \|A_0(0, \delta)f; L^p(\mathbb{R}^+)\|.$$

Inequalities (4.17) and (4.18) are valid for all $\vec{f} \in [C_0^\infty((\delta, \pi - \delta))]^N$, with c independent of δ . Since $A_0(0, \delta)$ commutes with dilations, (4.18) is valid for all $\vec{f} \in [C_0^\infty(\mathbb{R}^+)]^N$ and therefore $\det[a_{1j0}(0, z)] \neq 0$, $z = 1/p + i\epsilon$, $-\infty \leq \epsilon \leq \infty$. The continuity of $\sigma(A)(t, z)$ and (4.17) show that $[a_{1j0}(t, z)]$ is nonsingular on $\Lambda_{1/p}$.

q.e.d.

If A is a system of pdo's of class $Op - \Sigma_{1/p}(I)$ which is elliptic on L^p we define

(4.19)
$$n_{1/p}(\sigma(A)) = \frac{1}{2\pi} \Delta_{\partial R_{1/p, I}} \text{Arg } \det(\sigma(A));$$

the change in argument being taken as $\partial R_{1/p, I}$ in (4.15) is traversed in the clockwise direction. The analytical index of the system A is defined as

$$(4.20) \quad \text{ind}_p(A) = \dim \ker A - \dim \ker A^*,$$

where A is considered as an operator on $[L^p(I)]^N$ and A^* is considered as an operator on $[L^q(I)]^N$, $1/p + 1/q = 1$.

Before proving the Index Theorem 4.2, i.e., that

$$(4.21) \quad \text{ind}_p(A) = n_{1/p}(\sigma(A)),$$

we consider a canonical system A for which we show directly that

$$(4.22) \quad \text{ind}_p(A) = 1 = n_{1/p}(\sigma(A)).$$

Let $S^1 = \{w \in \mathbb{C}; |w| = 1\}$ be the unit circle and for $f(w) = f(e^{i\varphi}) \in C^\infty(S^1)$ define for $z = e^{i\theta}$

$$(4.23) \quad \begin{aligned} Bf(z) &= \text{p.v.} \frac{1}{\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) d\varphi - \text{p.v.} \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(e^{i\varphi}) \cot\left(\frac{\theta - \varphi}{2}\right) d\varphi. \end{aligned}$$

Note that $f \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) d\varphi$ is a compact operator on $L^p(-\pi, \pi)$. Given $f \in L^p(S^1)$ we define the column vector function $\vec{f} = (f_1, f_2) \in [L^p(I)]^2$ by

$$\begin{aligned}
 (4.24) \quad f_1(t) &= f(e^{1t}) & 0 \leq t \leq \pi \\
 f_2(t) &= f(e^{-1t}) & 0 \leq t \leq \pi.
 \end{aligned}$$

We consider the operator

$$(4.25) \quad B_{11}f(t) = \text{p.v.} \frac{1}{\pi} \int_0^\pi f(s) \frac{1}{2} \cot\left(\frac{t-s}{2}\right) ds, \quad t \in [0, \pi].$$

If $\varphi \in C_0^\infty([0, \pi])$, $\varphi B_{11}\varphi f = \varphi H\varphi f + Kf$, where K is a compact operator on $L^p(0, \pi)$ and H is the finite Hilbert transform defined by (4.4). Now

$$TB_{11}T = -B_{11}$$

so that B_{11} differs from H by a compact operator.

Now consider

$$(4.26) \quad B_{12}f(t) = \frac{1}{\pi} \int_0^\pi f(s) \frac{1}{2} \cot\left(\frac{t+s}{2}\right) ds, \quad t \in [0, \pi].$$

The singularity of the kernel $\frac{1}{2} \cot\left(\frac{t+s}{2}\right)$ occurs when $t+s$ is near 0 or near 2π . If $\varphi \in C_0^\infty([0, \pi])$,

$$\begin{aligned}
 (4.27) \quad \varphi B_{12}\varphi f(t) &= \frac{1}{\pi} \int_0^\pi \frac{\varphi(t)\varphi(s)}{t+s} f(s) ds + Kf(t), \\
 &= (\varphi S\varphi)f + Kf,
 \end{aligned}$$

where K is compact on $L^p(I)$ and S is the finite Stieltjes transform defined by (4.5). Also

$$(4.28) \quad TB_{12}T = -B_{12}.$$

The operator B defined in (4.23) on $L^p(S^1)$, using the correspondence (4.24) may be represented,

modulo a compact operator, as a system on $L^p(I) \times L^p(I)$:

$$(4.29) \quad \begin{aligned} (Bf)_1(t) &= B_{11}f_1(t) + B_{12}f_2(t) \\ (Bf)_2(t) &= -B_{12}f_1(t) - B_{11}f_2(t), \end{aligned}$$

which is a 2×2 system of pdo's of class $Op - \Sigma_{1/p}(I)$

whose principal symbols are given in (4.30). (In

$$(4.30) \quad h(z) = -\cot \pi z, \quad s(z) = \csc \pi z.)$$

$$(4.30) \quad \begin{array}{ccc} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \\ & \downarrow & \\ \begin{bmatrix} h & s \\ -s & h \end{bmatrix} & \xrightarrow{\partial R_{1/p, I}} & \begin{bmatrix} -h & -s \\ s & h \end{bmatrix} \\ & \uparrow & \\ & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \end{array}$$

Given $f(e^{i\theta}) \in C^\infty(S^1)$, define for $z = e^{i\theta}$, let

$$f^+(z) = \lim_{r \uparrow 1} \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - rz} d\zeta, \quad \text{and} \quad f^-(z) = \lim_{r \uparrow 1} \oint \frac{f(\zeta)}{\zeta - rz} d\zeta.$$

Modulo a compact operator,

$$f^\pm(e^{i\theta}) = \frac{1}{2} \{f(e^{i\theta}) \pm iBf(e^{i\theta})\}.$$

Finally, note that the multiplication operator

$f(w) \rightarrow wf(w)$ has the representation as a system:

$$f_1 \rightarrow e^{it}f_1; \quad f_2 \rightarrow e^{-it}f_2.$$

The operator

$$(4.31) \quad Af(w) = wf^-(w) + f^+$$

has a one dimensional kernel spanned by $1/w - 1$ and cokernel $\{0\}$. As a system of pdo's of class $Op - \Sigma_{1/p}(I)$, a direct calculation shows that $n_{1/p}(\sigma(A)) = 1$ so that

$$(4.32) \quad \text{ind}_p(A) = 1 = n_{1/p}(\sigma(A)).$$

We are now ready to prove the Index Theorem.

THEOREM 4.2. If A is an $N \times N$ matrix of pdo's of class $\Sigma_{1/p}(I)$ which is elliptic on L^p , then

$$\text{ind}_p(A) = n_{1/p}(\sigma(A)).$$

PROOF: We may follow the proofs of Lemmas 3.1-3.4 with only a small change in notation to prove the same special cases. Hence there is an integer J_I such that

$$\text{ind}_p(A) = J_I n_{1/p}(\sigma(A)).$$

By the example considered above, $J_I = 1$.

q.e.d.

5. Single and double layer potentials in polygonal domains.

As an application of pdo's of class $\Sigma_{1/p}(I)$ we use single and double layer potentials to study the

Dirichlet, Neumann, and Oblique Derivative Problems
for Laplace's equation in a polygon.

Let Ω be a simply connected polygon in R^2 with N successive vertices labeled as $P_1, P_3, \dots, P_{2N-1} = P_1$ as $\partial\Omega$ is traversed in the counter clockwise direction. Label the interior angles at P_{2k-1} as θ_{2k-1} , $k = 1, \dots, N$. At the midpoint of $\overline{P_{2k-1}P_{2k+1}}$ we introduce a false vertex P_{2k} with interior angle $\theta_{2k} = \pi$ and parametrize the new half sides with $t \in [0, 1]$ so that $t = 0$ at P_{2k-1} or P_{2k+1} and $t = 1$ at P_{2k} ; i.e.:

For $t \in [0, 1]$, define

$$\begin{aligned} P_t &= P_{t, 2k-1} = tP_{2k} + (1-t)P_{2k-1} \in \overline{P_{2k-1}P_{2k}}, \\ (5.1) \quad P_t &= P_{t, 2k-2} = tP_{2k-2} + (1-t)P_{2k-1} \in \overline{P_{2k-1}P_{2k-2}} \end{aligned}$$

If l_1 is the length of $\overline{P_1P_{1+1}}$, the arclength $d\sigma$ is given by

$$(5.2) \quad d\sigma = (-1)^{1+l_1} l_1 dt.$$

For $\phi \in L^p(\partial\Omega)$ define the double layer potential

$$(5.3) \quad u(X) = \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle X-Q, n_Q \rangle}{|X-Q|^2} \phi(Q) d\sigma_Q,$$

where n_Q is the interior unit normal to a point $Q \in \partial\Omega$.

For $P \in \partial\Omega$, let

$$\begin{aligned}
 (5.4) \quad u^+(P) &= \lim_{X \rightarrow P, X \in \Omega} u(X) = \phi(P) + K\phi(P), \\
 u^-(P) &= \lim_{X \rightarrow P, X \in \mathbb{R}^2 - \Omega} u(X) = -\phi(P) + K\phi(P),
 \end{aligned}$$

where

$$(5.5) \quad K\phi(P) = \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle P-Q, n_Q \rangle}{|P-Q|^2} \phi(Q) d\sigma_Q,$$

and the limits in (5.4) are taken nontangentially in $L^p(\partial\Omega)$ as $X \rightarrow P \in \partial\Omega$. Using the correspondences given by (5.1), consider ϕ and $K\phi$ as functions in $[L^p(0,1)]^{2N}$, where $\phi_1(t) = \phi(P_t)$; $P_t \in \overline{P_1 P_{1+1}}$. Then K may be represented as a $2N \times 2N$ system of pdo's of class $Op - \Sigma_{1/p}(I)$, $I = [0,1]$. We obtain the following operators for $K = [K_{ij}]$:

$$\begin{aligned}
 (5.6) \quad K_{11} &= 0, \quad K_{2k-1,2k} = K_{2k,2k-1} = 0, \\
 K_{ij} &\text{ is compact if } \overline{P_i P_{i+1}} \text{ and } \overline{P_j P_{j+1}} \text{ do not touch,} \\
 K_{2k-2,2k-1}\varphi(t) &= \int_0^1 k_\theta(l_{2k-2}t/l_{2k-1}s)\varphi(s) \frac{ds}{s}, \\
 K_{2k-1,2k-2}\varphi(t) &= \int_0^1 k_\theta(l_{2k-1}t/l_{2k-2}s)\varphi(s) \frac{ds}{s}, \\
 &\text{with } \theta = \theta_{2k-1}
 \end{aligned}$$

where

$$(5.7) \quad k_\theta(t) = \frac{1}{\pi} \frac{t \sin \theta}{t^2 + 1 - 2t \cos \theta}.$$

The Mellin transform of the kernel $k_\theta(t)$ is given by

$$(5.8) \quad \tilde{k}_\theta(z) = \frac{\sin(\pi-\theta)z}{\sin \pi z} \in \mathcal{O}_{0,1}^{-\infty}.$$

The matrix of principal symbols of the operator $[b_{ij}I + K_{ij}]$ is the identity on the top, bottom, and r.h.s. of $\partial R_{1/p,I}$; at $t = 0$ its determinant can be calculated by using an even number of row and column transpositions to obtain a matrix with 2×2 blocks on the diagonal; the vertex P_{2k-1} contributes

$$(5.8) \quad \begin{bmatrix} 1 & b^z \tilde{k}_\theta(z) \\ b^{-z} \tilde{k}_\theta(z) & 1 \end{bmatrix}$$

where $b = \mu_{2k-2}/\mu_{2k-1}$ and $\theta = \theta_{2k-1}$. Hence

$$\det(\sigma(I+K))(0,z) = \prod_{j=1}^N \left(1 - [\tilde{k}_{\theta_{2j-1}}(z)]^2\right).$$

The zeroes of $1 - [\tilde{k}_\theta(z)]^2$ occur at $z = n\pi/\theta$ or $z = n\pi/(2\pi-\theta)$. If $\theta \neq \pi$ we define p_θ by

$$(5.9) \quad 1/p_\theta = \begin{cases} \pi/(2\pi-\theta), & 0 < \theta < \pi, \\ \pi/\theta, & \pi < \theta < 2\pi, \end{cases}$$

so that $z_\theta = 1/p_\theta$ is the simple root of $1 - k_\theta^2 = 0$ satisfying $0 < \operatorname{Re} z < 1$; in fact $1 < p_\theta < 2$. By the Argument Principle

$$(5.10) \quad n_{1/p}(\theta) = \frac{1}{2\pi} \Delta_{z=1/p+i\epsilon}^{\operatorname{Arg} (1 - [k_\theta]^2)} \quad -\infty \leq \epsilon \leq +\infty$$

$$= \begin{cases} 0, 0 < 1/p < 1/p_\theta \\ 1, 1/p_\theta < 1/p < 1 \end{cases}.$$

From the results of Chapter 4, the system

$$(5.11) \quad (I+K)\phi = \psi$$

is Fredholm on $L^p(\partial\Omega)$ iff $p \neq p_{\theta_1}, \dots, p_{\theta_{2N-1}}$; in case (5.11) is Fredholm on $L^p(\partial\Omega)$ we have

$$(5.12) \quad \text{ind}_p(I+K) = \sum_{p < p_{\theta_{2k-1}}} 1.$$

In particular, since $p_\theta < 2$, for $p \geq 2$, $\text{ind}_p(I+K) = 0$.

To study the kernel and cokernel of (5.11) we introduce the single layer potential

$$(5.13) \quad v(X) = \frac{1}{2\pi} \int_{\partial\Omega} |X-Q|^2 \psi(Q) d\sigma_Q.$$

Then

$$(5.14) \quad \frac{\partial v}{\partial n^+}(P) = \lim_{X \rightarrow P, X \in \Omega} \frac{\partial v}{\partial n_P}(X) = (I-K^*)\psi$$

$$\frac{\partial v}{\partial n^-}(P) = \lim_{X \rightarrow P, X \in \mathbb{R}^2 - \Omega} \frac{\partial v}{\partial n_P}(X) = -(I+K^*)\psi,$$

where

$$(5.15) \quad K^*\psi(P) = \frac{1}{\pi} \int_{\partial\Omega} \frac{\langle Q-P, n_P \rangle}{|P-Q|^2} \psi(Q) d\sigma_Q.$$

Let M be the (compact) mean value operator

$$M\phi = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \phi(Q) d\sigma_Q.$$

From, e.g., Petrovsky [Pe, §34], we recall the following facts for these operators on $L^p(\partial\Omega)$:

$\ker(I-M) = \{\text{constant functions}\}$; $(I+K)M = 2M$; and $\ker(I+K^*)(I-M) = \{\text{constant functions}\}$.

Define

$$(5.16) \quad p_0 = \max p_{\theta_{2k-1}}$$

with p_θ given by (5.9). Note that $p_0 < 2$.

LEMMA 5.1. For $p > p_0$ the kernel of $(I-M)(I+K)$ consists of constant functions.

PROOF: Constant functions are in the kernel of $(I-M)(I+K)$ and $\text{ind}_p(I-M)(I+K) = \text{ind}_p(I+K) = 0$.

Moreover, $(I+K^*)(I-M)$ has kernel of dimension 1.

q.e.d.

LEMMA 5.2. For $p > p_0$,

$$(5.17) \quad \ker(I+K) = \{0\}.$$

PROOF: If $(I+K)\phi = 0$ then $\phi = \text{constant}$ by Lemma 5.1. But then $(I+K)\phi = 2\phi = 0$.

q.e.d.

LEMMA 5.3. For $1 < q < \infty$

$$\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}.$$

PROOF: If $1 < q \leq 2$ and $1/p + 1/q = 1$, then $p > p_0$. Hence $\text{ind}_q(I+K^*) = 0$ and $\ker(I+K) \cap L^p(\partial\Omega) = \{0\}$. Hence $\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}$. If $2 \leq q$, $\ker(I+K^*) \cap L^q(\partial\Omega) \subset \ker(I+K^*) \cap L^2(\partial\Omega) = \{0\}$.
q.e.d.

THEOREM 5.1. If K is defined by (5.5) then the integral equation

$$(5.11) \quad (I+K)\phi = \psi$$

considered on $L^p(\partial\Omega)$ has the following properties:

1. (5.11) is Fredholm on $L^p(\partial\Omega)$ iff

$$(5.16) \quad p \neq p_{\theta_1}, \dots, p_{\theta_{2N-1}},$$

2. For $1 < q < \infty$, $\ker(I+K^*) \cap L^q(\partial\Omega) = \{0\}$.

3. For every p satisfying (5.16),

$$(5.17) \quad \dim \ker(I+K) \cap L^p(\partial\Omega) = \begin{cases} 0, & p \geq 2 \\ \text{no. of vertices with} \\ \text{angle } \theta \text{ such that} \\ p < p_\theta \end{cases}$$

(p_θ given by (5.9)).

The proof of Theorem 5.1 follows from the preceding discussion.

We next seek to resolve the Neumann problem $\Delta v = 0$ in Ω , $\frac{\partial v}{\partial n^+}(P) = g(P) \in L^q(\partial\Omega)$ in the form of a single layer potential given by (5.13); using (5.14) we study the integral equation

$$(5.18) \quad (I-K^*)\psi = g$$

on $L^q(\partial\Omega)$. The operator $I-K$ is a Fredholm operator on $L^p(\partial\Omega)$ iff $p \neq p_{\theta_1}, \dots, p_{\theta_{2N-1}}$ and for such values of p has the same index on $L^p(\partial\Omega)$ as the operator $I+K$. We show that the kernel of $(I-K^*)$ consists of constants.

We show that the kernel of $I-K^*$ has dimension 1. Let q be near 1 and $S_q = \ker(I-K^*) \cap L^q(\partial\Omega)$. Now $S_q \neq \{0\}$ since $\{\text{constants}\} \subset \ker(I-K)$ and $\text{ind}_q(I-K) = 0$. It is easy to show that if $\psi \in S_q$ and $M\psi = 0$, then $\psi = 0$. Hence there is a $\psi_1 \in S_q$ such that $M\psi_1 = 1$. Then if $\psi \in S_q$, $\psi = (M\psi)\psi_1$.

For (5.18) we have the following theorem.

THEOREM 5.2. The integral equation

$$(5.18) \quad (I-K^*)\psi = g$$

on $L^q(\partial\Omega)$ has the following properties.

1. For $1 < q < \infty$,

(5.19) $\dim \ker(I-K^*) \cap L^q(\partial\Omega) = 1.$

2. (5.18) is Fredholm on $L^q(\partial\Omega)$ iff

(5.20) $q \neq q_{\theta_1}, \dots, q_{\theta_{2N-1}}; (1/q_{\theta_1} + 1/p_{\theta_1} = 1).$

(p_{θ} given by (5.9)).

3. If q satisfies (5.20),

$$\operatorname{ind}_q(I-K^*) = \begin{cases} 0, & q \leq 2, \\ -(\text{no. of vertices with} \\ \text{angle } \theta \text{ for which} \\ q > q_{\theta}). \end{cases}$$

COROLLARY 5.1. The integral equation

(5.21) $(I-K)\phi = \psi$

on $L^p(\partial\Omega)$ has the following properties.

- 1. If $p > p_0$, $\ker(I-K) \cap L^p(\partial\Omega) = \{\text{constant functions}\}.$
- 2. For $p \neq p_{\theta_1}, \dots, p_{\theta_{2N-1}}$

$$\dim \ker(I-K) \cap L^p(\partial\Omega) = 1 + (\text{no. of vertices with} \\ \text{angle } \theta \text{ for which} \\ p < p_{\theta}).$$

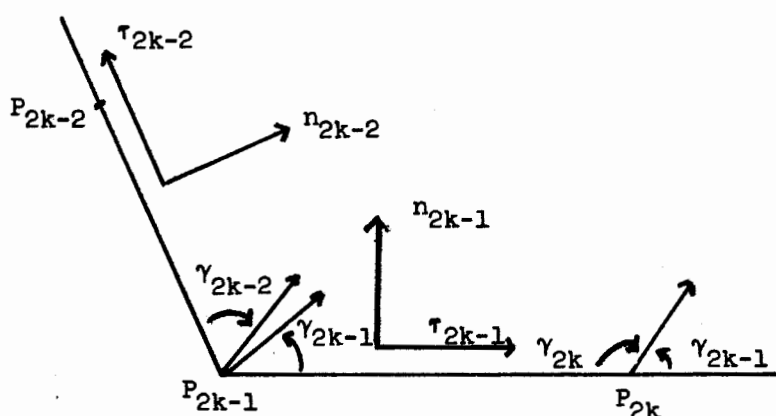
Finally we apply the pdo theory to study the
Oblique Derivative Problem or a Mixed Boundary Value

Problem in a polygon. We use the same notation as before, and use single layer potentials to study:

$$(5.22) \quad \begin{aligned} \Delta v &= 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} &= g(P) \in L^p(\partial\Omega), \end{aligned}$$

where on each segment $\overline{P_{2k-1}P_{2k+1}}$, $v = v(P)$ is a smooth nonvanishing vector field. Dirichlet data can be handled by taking v tangent to $\partial\Omega$; jump discontinuities in v may be considered by introducing a new vertex with angle $\theta = \pi$ at the jump.

For simplicity we assume that each segment $\overline{P_iP_{i+1}}$ has unit length. On each segment $\overline{P_iP_{i+1}}$ we introduce unit tangent vectors τ_i in the direction of increasing t as in (5.23):



Normalize so that $|v(P)| = 1$ and define smooth angles $\gamma_i = \gamma_i(t)$ so that on $\overline{P_iP_{i+1}}$;

$$v(P_t) = \cos \gamma_i(t) \tau_i + \sin \gamma_i(t) n_i.$$

The continuity of $v(P)$ at the even vertices may be expressed as:

$$(5.24) \quad \gamma_{2k-1}(1) + \gamma_{2k}(1) = \pi \pmod{2\pi}.$$

Consider the operators

$$(5.25) \quad A^\tau \psi(P) = \lim_{X \rightarrow P, X \in \Omega} \frac{\partial v}{\partial \tau_P}(X)$$

$$= \text{p.v.} \frac{1}{\pi} \int_{\partial \Omega} \frac{\langle P-Q, \tau_P \rangle}{|P-Q|^2} \psi(Q) d\sigma_Q,$$

$$(5.26) \quad A^n \psi(P) = (I-K^*)\psi(P) = \lim_{\epsilon \rightarrow 0^+} \frac{\partial v}{\partial n_P}(P + \epsilon n_P),$$

with v the single layer potential given by (5.13).

To represent the operators A^τ and A^n as systems of pdo's of class $\text{Op} - \Sigma_{1/p}(I)$, we introduce the Hardy kernels (and their Mellin transforms):

$$k_{\tau\theta}(t) = \frac{1}{\pi} \frac{t - \cos \theta}{t^2 + 1 - 2t \cos \theta}$$

$$\tilde{k}_{\tau\theta}(z) = - \frac{\cos((\pi-\theta)z + \theta)}{\sin \pi z}$$

$$(5.27) \quad k_{n\theta}(t) = \frac{1}{\pi} \frac{-\sin \theta}{t^2 + 1 - 2t \cos \theta}$$

$$\tilde{k}_{n\theta}(z) = - \frac{\sin((\pi-\theta)z + \theta)}{\sin \pi z}$$

$$s(t) = k_{\tau\pi}(t) = \frac{1}{\pi} \frac{1}{t+1}$$

$$\tilde{s}(z) = \frac{1}{\sin \pi z}$$

and the corresponding Hardy kernel operations:

(5.28)
$$K\psi(t) = \int_0^1 k\left(\frac{t}{s}\right) f(s) \frac{ds}{s}$$

which, as operators in $Op - \Sigma_{1/p}(I)$ have principal symbols, $\sigma(K)$, given by

(5.29)
$$\tilde{K}(z) \begin{array}{c} 0 \\ \boxed{R_{1/p,I}} \\ 0 \end{array} 0 \ .$$

For the operator $A^T = [A^T_{ij}]$, we obtain

1. $A^T_{11}\psi_1(t) = H\psi_1(t) = \text{p.v.} \frac{1}{\pi} \int_0^1 \frac{\psi_1(s)}{t-s} ds.$
2. $A^T_{2k-1,2k}\psi_{2k}(t) = \frac{1}{\pi} \int_1^2 \frac{1}{t-s} \psi_{2k}(2-s)ds$
 $= -TST\psi_{2k}(t).$
3. $A^T_{2k,2k-1}\psi_{2k-1}(t) = \frac{1}{\pi} \int_0^1 \frac{-1}{2-t-s} \psi_{2k-1}(s)ds$
 $= -TST\psi_{2k-1}(t).$
4. $A^T_{2k-2,2k-1} = A^T_{2k-1,2k-2} = K_{\tau\theta}, \theta = \theta_{2k-1}.$
5. A^T_{ij} is compact if $\overline{P_iP_{i+1}}$ and $\overline{P_jP_{j+1}}$ do not touch.

For the operator $A^n = [A_{ij}^n] = (I - K^*)$ we have

$$6. \quad A_{ii}^n = I.$$

$$7. \quad A_{2k-1, 2k}^n = A_{2k, 2k-1}^n = 0.$$

$$8. \quad A_{2k-2, 2k-1}^n = A_{2k-1, 2k-2}^n = K_{n\theta}.$$

$$9. \quad A_{ij}^n \text{ is compact if } \overline{P_1 P_{i+1}} \text{ and } \overline{P_j P_{j+1}} \text{ do not touch.}$$

The system which arises to study (5.22) is now

$$(5.30) \quad (\nu \cdot \tau_P) A^\tau \psi(P) + (\nu \cdot n_P) A^n \psi(P) = g(P).$$

on $L^p(\partial\Omega)$.

We now calculate the determinant of the matrices of principal symbols of (5.40) defined on $\partial R_{1/p, I}$.

On the r.h.s. of $\partial R_{1/p, I}$; i.e., when $t = 1$, we make an even number of row and column transpositions to obtain N 2×2 blocks along the diagonal; each even numbered vertex P_{2k} contributes with $\gamma = \gamma_{2k-1}(1) = \pi - \gamma_{2k}(1) \pmod{2\pi}$:

$$(5.31) \quad \begin{bmatrix} -\cos \gamma \cot \pi z + \sin \gamma & \cos \gamma \csc \pi z \\ -\cos \gamma \csc \pi z & \cos \gamma \cot \pi z + \sin \gamma \end{bmatrix}$$

which has determinant $= 1$ for $0 < \operatorname{Re} z < 1$. Hence on the r.h.s. of $\partial R_{1/p, I}$:

$$\det[\sigma(A)] = \prod_{j=1}^N 1 = 1.$$

To calculate the determinant of the matrix of principal symbols on $\partial R_{1/p, I}$, with $0 \leq t < 1$, we are again reduced to studying 2×2 blocks which arise from considering the operator near a vertex P_{2k-1} . Calling $i = 2k-1$, $\theta = \theta_i$, $\gamma_{i-1} = \gamma_{2k-2}(t)$, $\gamma_i = \gamma_{2k-1}(t)$ the matrix which comes from P_i is given by

$$(5.32) \quad \frac{-1}{\sin \pi z} \begin{bmatrix} \cos(\pi z + \gamma_{i-1}) & \cos((\pi - \theta)z + \theta - \gamma_{i-1}) \\ \cos((\pi - \theta)z + \theta - \gamma_i) & \cos(\pi z + \gamma_i) \end{bmatrix}$$

which has determinant

$$(5.33) \quad \frac{1}{2} \frac{\cos(2\pi z + 2\gamma) - \cos(2(\pi - \theta)z + 2\theta + 2\gamma)}{\sin^2 \pi z}$$

$$2\gamma = \gamma_{i-1} + \gamma_i.$$

The function in (5.33) may be expressed as

$$(5.34) \quad F_{\theta 2\gamma}(z) = \frac{\sin((2\pi - \theta)z + \theta + 2\gamma)\sin(\theta(1-z))}{\sin^2 \pi z}.$$

$F_{\theta 2\gamma}$ satisfies

$$(5.35) \quad F_{\theta 2\gamma}(1/p + i\infty) = -e^{-12\gamma}, \quad F_{\theta 2\gamma}(1/p - i\infty) = -e^{-12\gamma}$$

and has zeroes when

$$(5.36) \quad 1 - z = \frac{n\pi}{\theta}, \quad 1 - z = \frac{n\pi + 2\gamma}{2\pi - \theta}.$$

Hence

$$\det[\sigma(A)](t, 1/p + i\infty) = (-1)^N \prod_{j=1}^N e^{-1(\gamma_{2k-2}(t) + \gamma_{2k-1}(t))} \quad (5.37)$$

$$\det[\sigma(A)](t, 1/p - i\infty) = (-1)^N \prod_{j=1}^N e^{+1(\gamma_{2k-2}(t) + \gamma_{2k-1}(t))}.$$

Note that $\det[\sigma(A)](1, 1/p \pm i\infty) = 1$ by the continuity condition (5.24). We conclude that (5.40) is Fredholm on $L^p(\partial\Omega)$ iff $1/q = 1 - 1/p$ satisfies

$$\frac{1}{q} \neq \frac{n\pi}{\theta}, \quad \frac{1}{q} \neq \frac{n\pi + 2\gamma}{2\pi - \theta} \quad (5.38)$$

$$\theta = \theta_{2k-1}, \quad 2\gamma = \gamma_{2k-2}(0) + \gamma_{2k-1}(0), \quad k = 1, \dots, N.$$

If (5.38) is satisfied, the index of (5.30) on $L^p(\partial\Omega)$ can be readily calculated using the Index Theorem 4.2 and (5.37).

For an Index Theorem for the Oblique Derivative Problem in the case of a domain with smooth boundary, see Hormander [H]. We also refer the reader to the work of Costabel [Cos] who develops an algebra of singular integral operators on curves with corners which includes the double layer potential and proves an Index Theorem. His approach also allows weights such as t^β at the vertices of the polygon.

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