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The Numerical Solution of Symm's Equation on Smooth Open Arcs by Spline Galerkin Methods

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Abstract—We consider the classical Fredholm linear integral equation of the first kind with logarithmic kernel on a smooth Jordan open arc. Applying the well-known cosine change of variable, the arc is reparametrized and the problem is transformed into a new integral equation.

We investigate the existence of an asymptotic expansion for the error of the Galerkin method with splines on a uniform mesh as test-trial functions. We also analyse a full discretization of the method based on the Galerkin collocation method using high order integration formulae to keep the optimal error estimates of the Galerkin method in weak norms. Asymptotic expansions of the error for this method are provided. Finally, we show how these expansions extend to the computation of the potential.

The expansions of the error in powers of the discretization parameter are useful to obtain *a posteriori* estimates of the error and to apply Richardson extrapolation for acceleration of convergence.

Keywords—Boundary elements, Galerkin methods, Logarithmic equations.

1. INTRODUCTION

In this paper we investigate the numerical solution of Symm's equation on a smooth open arc

$$-\frac{1}{\pi}\int_{\Gamma} \log|\mathbf{z}-\mathbf{y}|u(\mathbf{y})\,d\sigma_{\mathbf{y}} = f(\mathbf{z}), \qquad \mathbf{z}\in\Gamma.$$

This equation is related to potential problems and has been widely studied, both theoretically and numerically, in closed curves. However, the classical spline Galerkin approach on uniform meshes presents a deficient behaviour, since the solution of the equation has singularities at both endpoints of the curve, even for very smooth data.

In [1], the Galerkin method with piecewise constant functions is analyzed on a mesh which is graded near the extremes of the arc. With this method, it is possible to restore the convergence of the Galerkin method in smooth closed curves (see [2]). In [3], a cosine change of variable in the parametrization interval is analyzed from a theoretical point of view. It turns out that the resulting equation gives smooth solutions (scaled solutions) from smooth data, and it is then more adequate for numerical purposes. In fact, this corresponds to a new mesh grading in a regular parametrization. In [4], some convergence results for the Galerkin and collocation method are

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proven in the simplest case of piecewise constant functions. This analysis is easily extendable to splines of higher order. Finally, Galerkin methods with trigonometric polynomials are studied in [5].

We study the existence of asymptotic expansions of the error of the spline Galerkin method. Asymptotic expansions of the error are useful for the application of Richardson extrapolation as a way of accelerating the convergence of the method and also to obtain *a posteriori* estimates of the error. In the frame of pseudodifferential equations, extrapolation has been studied for the collocation method in smooth closed curves under the action of a smoothing functional in [6]. More recently, in [7], specializing to logarithmic kernel equations, some expansions in stronger norms have been derived.

In Section 2 we present the basic ideas of the cosine change of variable. In Section 3 we prove the existence of an asymptotic expansion of the error of the Galerkin method in powers of the discretization parameter. We take time to study in detail the spline spaces associated with the problem. From the expansions we are able to obtain some superconvergence results in midpoints (respect to the cosine parametrization, that is, slightly displaced from the geometric midpoints) of the grid. In Section 4 we show a full discretization of the Galerkin method, based on the Galerkin collocation method by Hsiao *et al.* [8]. We show that with adequate quadrature formulae, the optimal order of convergence of the Galerkin method can bé obtained, for smoother solutions though. In Section 5 we apply the previous results for the practical obtention of potentials away from the arc. Finally, we give a different approach to both methods in Section 6 as particular cases of methods for a modified logarithmic kernel equation.

An appendix is devoted to giving some matricial notations reflecting the inherent symmetries of the formulation.

2. STATEMENT OF THE PROBLEM

Let Γ be a simple open arc, having an infinitely differentiable representation $\mathbf{x} : [-1,1] \to \Gamma \subset \mathbb{R}^2$ such that

$$|\mathbf{x}'(t)| > 0, \qquad \forall\, t \in [-1,1].$$

The smoothness requirement can be relaxed up to a certain degree in all results. We will keep it for the sake of simplicity. Our concern is the following Fredholm integral equation of the first kind:

$$-\frac{1}{\pi}\int_\Gamma \log|\mathbf{z}-\mathbf{y}|u(\mathbf{y})\,d\sigma_{\mathbf{y}} = f(\mathbf{z}), \qquad \mathbf{z}\in\Gamma. \tag{1}$$

As in [4], we first apply a cosine change of variable to the equation (in fact, to the parametrization interval). Let $\mathbf{a} : [0,\pi] \to \Gamma$ be given by

$$\mathbf{a}(\theta) := \mathbf{x}(\cos\theta).$$

Note that since $\mathbf{a}'(\theta) = -\sin\theta\,\mathbf{x}'(\cos\theta)$, then $\mathbf{a}$ is not a regular parametrization of the arc. Let

$$\begin{aligned} w(\theta) &:= u(\mathbf{a}(\theta))|\mathbf{x}'(\cos\theta)|\,|\sin\theta|, \qquad \text{and}\\ g(\theta) &:= f(\mathbf{a}(\theta)). \end{aligned}$$

Note that even if w is smooth, u is likely to have singularities in both endpoints of Γ. Applying the change of variable $\theta = \arccos t$ in the integral equation, the problem is equivalent to

$$-\frac{1}{\pi}\int_0^\pi \log|\mathbf{a}(\cdot)-\mathbf{a}(\theta)|w(\theta)\,d\theta = g. \tag{2}$$

We denote also w and g to the (unique) even 2π-periodic extensions of w and g, respectively. Then, (2) is equivalent to the search for even solutions of

$$Lw = g, \tag{3}$$

where

$$Lw := -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\mathbf{a}(\cdot) - \mathbf{a}(\theta)| w(\theta)\, d\theta.$$

Let Γ_0 be the segment given by the parametrization

$$\mathbf{x}_0(t) := \mathbf{P}_0 + 2e^{-1}t\mathbf{v}_0,$$

where $\mathbf{P}_0$ is a fixed point in the plane, $\mathbf{v}_0$ is a vector with unit length and $t \in [-1, 1]$. Then, equation (3) on Γ_0 is written $Aw = g$, where

$$Aw := -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|2e^{-1}(\cos\cdot - \cos\theta)| w(\theta)\, d\theta.$$

The usual functional frame for these equations is that of the periodic even Sobolev spaces. Consider the space

$$\mathcal{D}(2\pi) := \{f : \mathbb{R} \to \mathbb{R} : f \in \mathcal{C}^\infty, f(t) = f(t + 2\pi),\ \forall t\},$$

endowed with the metrizable topology of uniform convergence of all derivatives. Its dual space, denoted $\mathcal{D}'(2\pi)$, can be shown to be isomorphic to the space of 2π-periodic distributions on the real line. Given $u \in \mathcal{D}'(2\pi)$, we define its Fourier coefficients

$$\hat{u}(k) := \frac{1}{\sqrt{2\pi}}\langle u, e_{-k}\rangle,$$

where $e_k(s) := e^{-iks}$ and $\langle\cdot,\cdot\rangle$ is the duality bracket between $\mathcal{D}'(2\pi)$ and $\mathcal{D}(2\pi)$. A periodic distribution $u \in \mathcal{D}'(2\pi)$ is said to be even if $\hat{u}(-k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$. For any $r \in \mathbb{R}$, consider the Sobolev spaces

$$H^r(2\pi) := \{u \in \mathcal{D}'(2\pi) : \|u\|_r < +\infty\},$$

where

$$\|u\|_r := \left(|\hat{u}(0)|^2 + \sum_{0\neq k\in\mathbb{Z}} |k|^{2r}|\hat{u}(k)|^2\right)^{1/2},$$

and their closed subspaces

$$H^r_e(2\pi) := \{u \in H^r(2\pi) : \hat{u}(-k) = \hat{u}(k),\ \forall\, k \in \mathbb{Z}\}.$$

The associated inner products (both in $H^r(2\pi)$ and $H^r_e(2\pi)$) will be denoted $(\cdot,\cdot)_r$. Note that for all $s \in \mathbb{R}$, the product $(\cdot,\cdot)_0$ can be extended to give a representation of the duality between $H^s_e(2\pi)$ and $H^{-s}_e(2\pi)$ (and also between $H^s(2\pi)$ and $H^{-s}(2\pi)$). Additional properties of these spaces can be found in [3].

The operator A is an isomorphism from $H^s_e(2\pi)$ onto $H^{s+1}_e(2\pi)$ for all s. Moreover, for all w even and integrable

$$Aw = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(4e^{-1}\sin^2\left(\frac{\cdot - t}{2}\right)\right) w(t)\, dt =: \Lambda w, \tag{4}$$

that is, A coincides with the single-layer operator on a circle with radius $e^{-1/2}$ (the Bessel operator). Therefore, A is $H^{-1/2}_e(2\pi)$-elliptic.

Then, it follows readily that L can be decomposed as

$$L = A + K,$$

where the integral operator K has a smooth periodic kernel. Equation (3) will be considered in $H^{1/2}_e(2\pi)$, that is, solutions will be looked for in $H^{-1/2}_e(2\pi)$. Note that L satisfies Gårding's inequality in $H^{-1/2}_e(2\pi)$. Moreover, if the logarithmic capacity of Γ (see [9] for its definition) is different from 1, then $L : H^s_e(2\pi) \to H^{s+1}_e(2\pi)$ is an isomorphism for all $s \in \mathbb{R}$.

3. SPLINE GALERKIN METHODS

3.1. The Discrete Spaces

Let us consider a uniform grid defined by a parameter $h := \pi/N$,

$$s_i := s_0 + ih, \qquad i \in \mathbb{Z} \setminus \{0\},$$

for some s_0 to be fixed. The points s_i will be taken as midpoints of the intervals of a partition given by $I_i := (s_i - h/2, s_i + h/2)$. We consider the set V_h of the 2π-periodic smoothest splines of degree m with respect to the partition $\{I_i : i \in \mathbb{Z}\}$, that is, for $m = 0$,

$$V_h := \{u_h \in H^0(2\pi); u_h|_{I_i} \in P_0, \forall\, i\},$$

and for $m \geq 1$,

$$V_h := \{u_h \in \mathcal{C}^{m-1}(2\pi) : u_h|_{I_i} \in P_m, \forall\, i\},$$

where P_k denotes the space of the polynomials of degree less than or equal to k and $\mathcal{C}^k(2\pi)$ is the class of 2π-periodic k times continuously differentiable functions of a real variable. We assume that $2N \geq m+1$. Clearly, the space V_h has finite dimension equal to $2N$. We are interested in the subspace

$$V_h^e := \{u_h \in V_h : u_h(-t) = u_h(t), \forall\, t\},$$

for some special choices of s_0.

Associated with the splines (or rather to the grid) there is a new grid of interpolation points (hereafter nodes): if m is even we define

$$z_i = s_i, \qquad \forall\, i \in \mathbb{Z},$$

whereas if m is odd,

$$z_i := \frac{1}{2}(s_{i-1} - s_i), \qquad \forall\, i \in \mathbb{Z}.$$

It will be the point z_0 which we will fix in order to determine both grids $\{s_i\}$ and $\{z_i\}$ and the spaces V_h and V_h^e. The cases we are interested in are as follows.

- Case A. The origin is a node, say $z_0 = 0$. Then, $\dim V_h^e = N + 1$.
- Case B. The origin falls in the middle of two nodes (for instance, $z_0 + z_1 = 0$). Then, $\dim V_h^e = N$.

See more comments about these choices in Section 3.4.

For $u \in \mathcal{C}^0(2\pi)$ we define $Q_h u$ as the unique element in V_h such that $Q_h u(z_i) = u(z_i)$ for all i. The interpolation operator Q_h is well defined by the Schoenberg-Withney theorem and has several interesting asymptotic properties. Some of them are listed in [10] and some comments are given in Section 3.4. The following result is proven in Section 3.4 and in a somewhat more involved way (nearer to the interpolation matrices) in the Appendix.

LEMMA 1. *Let $u \in \mathcal{C}_e^0(2\pi)$. Then, $Q_h u \in V_h^e$.*

3.2. The Galerkin Method

Given any of the two choices of the discrete space V_h^e (for fixed degree m of the splines), the spline Galerkin method for the approximate solution of (3) consists of the following family of finite-dimensional variational problems:

$$(P_h)\begin{cases} \text{find } w_h \in V_h^e \text{ such that} \\ (Lw_h, r_h)_0 = (g, r_h)_0, \qquad \forall\, r_h \in V_h^e. \end{cases}$$

For h sufficiently small, the problems (P_h) are uniquely solvable. Moreover, the method is stable, that is, if we denote $w_h = G_h w$ (G_h is the Galerkin projection), it follows that

$$\|G_h w\|_{-1/2} \leq C\|w\|_{-1/2},$$

and the sequence of solutions converges to the exact solution in the $H^{-1/2}(2\pi)$ norm. In [4], error estimates are proven for the special case $m = 0$ (they are easily extendable to $m \geq 1$). In the 'natural' $H^{-1/2}(2\pi)$ norm

$$\|w - w_h\|_{-1/2} \leq Ch^{m+3/2}\|w\|_{m+1},$$

whereas in a weaker norm we get the optimal convergence rate

$$\|w - w_h\|_{-m-2} \leq Ch^{2m+3}\|w\|_{m+1}.$$

For simplicity, we only deal with smooth solutions of (3). A more restricted analysis, for less smooth solutions and curves is valid. We also can obtain error estimates in stronger norms applying the inverse inequalities of the splines.

3.3. Asymptotic Expansion of the Error

Let

$$f(m) := \begin{cases} m+1, & \text{if } m \text{ is odd}, \\ m+2, & \text{if } m \text{ is even}. \end{cases}$$

PROPOSITION 2. *Let $w \in C_e^\infty(2\pi)$ and M be a positive integer. Then, there exist even periodic functions $\{e_k\}_{k=f(m)}^M$, as smooth as desired, such that*

$$(L(G_h - Q_h)w, r_h)_0 = \sum_{k=f(m)}^{M} h^k (e_k, r_h)_0 + \mathcal{O}\left(h^{M+1}\right)\|r_h\|_{-1/2},$$

for all $r_h \in V_h^e$.

PROOF. Let Λ be the operator in the right-hand side of (4). Then, applying the decomposition $L = A + K$, the definition of G_h, and (4),

$$(L(G_h - Q_h)w, r_h)_0 = (\Lambda(I - Q_h)w, r_h)_0 + (K(I - Q_h)w, r_h)_0, \tag{5}$$

for all $r_h \in V_h$, since $w, Q_h w$ (see Lemma 1) and $G_h w$ are even. The right-hand side of (5) has been studied in [7]. There, the existence of functions g_k, $k \geq f(m)$ as smooth as desired such that

$$(\Lambda(I - Q_h)w, r_h)_0 + (K(I - Q_h)w, r_h)_0 = \sum_{k=f(m)}^{M} h^k (g_k, r_h)_0 + \mathcal{O}\left(h^{M+1}\right)\|r_h\|_{-1/2},$$

for every $r_h \in V_h$ is proven. Thus, defining

$$e_k := \frac{1}{2}(g_k + g_k(-\cdot)),$$

which is even, and remarking that for all $r_h \in V_h^e$, $(e_k - g_k, r_h)_0 = 0$, the result follows. ∎

Let us consider the Sobolev spaces

$$W^{m,\infty} := \left\{u \in L^\infty : u^{(j)} \in L^\infty,\ 1 \leq j \leq m\right\},$$

with the usual essential supremum norm, denoted by $\|\cdot\|_{m,\infty}$. It follows from properties of the interpolation operator Q_h and some inverse inequalities that G_h is bounded from $W^{m+1,\infty}$ to $W^{m,\infty}$, uniformly in h.

THEOREM 3. *For all positive integers M, there exist even 2π-periodic functions, as smooth as desired, such that*

$$\left\| G_h w - Q_h w - \sum_{k=f(m)}^{M} h^k Q_h f_k \right\|_{m,\infty} \leq C h^{M+1}.$$

This theorem follows from Proposition 2 and stability properties of the Galerkin method, with the same proof as that of Theorem 2 in [7] for closed curves. Likewise, we obtain the same asymptotic expansions of the error in midpoints and knots, in addition to some superconvergence results in midpoints of the grid. ∎

REMARK. In the proof of Proposition 2 we do not use any of the special choices (Cases A and B) of V_h^e. Note that in fact we have two different Galerkin projections for fixed h (there are two possible discrete spaces). However, the functions e_k in Proposition 2 are independent of the choice of z_0, since they come from a weak expansion where only V_h is involved. In Theorem 3 and subsequent pointwise expansions, the particular choice of V_h^e appears in the fact that it is relevant whether a fixed point is a knot or a midpoint of the grid.

3.4. Some Remarks About the Space V_h^e

We define μ_m to be the $(m+1)$-fold convolution of

$$\mu_0 := \chi_{(-1/2,1/2)},$$

where χ_E denotes the characteristic function of the set E. Then, for $m \geq 1$, μ_m is an even continuous function, of class C^{m-1}, piecewise polynomial of degree m, with breakpoints in $\mathbb{Z}$ if m is odd and in $1/2 + \mathbb{Z}$ if m is even. Moreover,

$$\operatorname{supp} \mu_m = \left[-\frac{m+1}{2}, \frac{m+1}{2} \right],$$

and

$$\int_{\mathbb{R}} \mu_m(x)\, dx = 1.$$

Then, we define the functions ψ_i, $i \in \mathbb{Z}$ as the unique 2π-periodic functions such that

$$\psi_i(s) = \mu_m \left(\frac{s - z_i}{h} \right), \qquad s \in (z_i - \pi, z_i + \pi].$$

Obviously, $\psi_i = \psi_j$ if $i \equiv j$, modulo $2N$. Moreover, the set $\{\psi_1, \dots, \psi_{2N}\}$ is a basis of V_h. We are interested in the following cases.

- Case A. If $z_0 = 0$ (or, equivalently, 0 is a node), then $\dim V_h^e = N + 1$ since $\{\psi_0, \psi_1 + \psi_{-1}, \dots, \psi_{N-1} + \psi_{-N+1}, \psi_N\}$ is a basis of the space.
- Case B. If $z_0 + z_1 = 0$ (or equivalently, 0 falls in the middle of two nodes), then $\dim V_h^e = N$, since $\{\psi_1 + \psi_{2N}, \psi_2 + \psi_{2N-1}, \dots, \psi_N + \psi_{N+1}\}$ is a basis of the space.

In both cases, the set of the 2π-periodic odd splines with knots on the same grid is a supplementary space of V_h^e in V_h. In fact, with $2N$ intervals of equal length in $(-\pi, \pi)$, all other choices make $V_h^e = P_0$.

We now prove the following assertion (Lemma 1): *if $u \in C_e^0$, then $Q_h u \in V_h^e$.*

Let $\{a_i\}$ be the coefficients of $Q_h u$ in the basis $\{\psi_i\}$ of V_h. In Case A, the hypothesis means that for all i, $u(z_i) = u(z_{2N-i})$. Then, we define

$$u_h := \sum_{j=1}^{2N-1} a_{2N-j}\psi_j + a_{2N}\psi_{2N},$$

and remark that

$$u_h(z_i) = \sum_{j=1}^{2N-1} a_{2N-j}\psi_{2N-j}(z_{2N-i}) + a_{2N}\psi_{2N}(z_{2N-i}) = Q_h u(z_{2N-i}) = u(z_i),$$

since in this case $\psi_j(-z) = \psi_{-j}(z)$ for all j. Consequently, by the uniqueness of the interpolate, $Q_h u = u_h$ and it is easy to see that $Q_h u + u_h$ is even.

In Case B, we have $u(z_{2N+1-i}) = u(z_i)$ for all i. We now define

$$u_h := \sum_{j=1}^{2N} a_{2N+1-j}\psi_j,$$

and prove that $u_h(z_i) = u(z_i)$ for all i. This last property follows from the fact that now $\psi_j(-z) = \psi_{1-j}(z)$. The remainder of the proof for the previous case is also valid here. ∎

4. THE ASSOCIATED GALERKIN COLLOCATION METHOD

Let us consider any of the choices of discrete space V_h^e for fixed m and let us denote $d := \dim V_h^e$. Let $\{\varphi_1, \dots, \varphi_d\}$ be the basis of V_h^e given in Section 3.4. The Galerkin scheme (P_h) is the equivalent to a $d \times d$ linear system. The symmetric matrix of coefficients of the system is given by

$$(L\varphi_i, \varphi_j)_0 = (\Lambda\varphi_i, \varphi_j)_0 + (K\varphi_i, \varphi_j)_0, \qquad i, j = 1, \dots, d,$$

where Λ is the operator in the right-hand side of (4) and K is the integral operator with smooth kernel from the decomposition $L = A + K$. The right-hand side is given by

$$(g, \varphi_i)_0, \qquad i = 1, \dots, d.$$

Thus, we only need to approximate the double integrals

$$k_{i,j} := (K\psi_i, \psi_j)_0, \quad \lambda_{i,j} := (\Lambda\psi_i, \psi_j)_0, \qquad i, j = 1, \dots, 2N,$$

and the quadratures

$$g_i := (g, \psi_i)_0, \qquad i = 1, \dots, 2N,$$

$\{\psi_1, \dots, \psi_{2N}\}$ being the usual basis of V_h. Note that by the symmetries of the kernels and of the functions a great part of those elements appears more than once in each table. This fact has to be considered when implementing the method to avoid an excess of evaluations.

Let us consider a quadrature rule

$$L_m f := \sum_{j=-l}^{l} c_j f(x_j) \simeq \int_{\mathbb{R}} f(x)\mu_m(x)\, dx,$$

satisfying the symmetries

$$x_{-i} = -x_i, \quad c_{-i} = c_i, \qquad i = 0, \dots, l,$$

with possibly $c_0 = 0$ and $L_m f = f$ for all $f \in P_{2m+3}$. We also consider the corresponding bidimensional rule

$$L_m^2 F := \sum_{j,k=-l}^{l} c_j c_k F(x_j, x_k) \simeq \int_{\mathbb{R}} \int_{\mathbb{R}} F(x,y) \mu_m(x) \mu_m(y) \, dx \, dy.$$

4.1. Approximation of the Linear System

The terms related to the smooth kernel $K(\sigma, s)$ of the operator K are approximated as

$$k_{i,j} \simeq \tilde{k}_{i,j} := h^2 L_m^2 K(z_i + h\cdot, z_j + h\cdot),$$

and those of the right-hand side as

$$g_i \simeq \tilde{g}_i := h L_m g(z_i + h\cdot).$$

Obviously, $\tilde{k}_{i,j} = \tilde{k}_{j,i}$. We can then define a bilinear form $k_h : V_h \times V_h \to \mathbb{R}$ and a linear functional $g_h : V_h \to \mathbb{R}$ such that

$$k_h(\psi_i, \psi_j) = \tilde{k}_{i,j}, \qquad g_h(\psi_i) = \tilde{g}_i,$$

for all i, j.

The matrix $(\lambda_{i,j})_{i,j=1}^{2N}$ is circulant and symmetric, since

$$\lambda_{i,j} = -\frac{1}{2\pi} h^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \log\left(4e^{-1} \sin^2\left(\frac{(t-u+i-j)h}{2}\right)\right) \mu_m(t) \mu_m(u) \, dt \, du = \eta_{|i-j|}.$$

For the terms related to the logarithmic operator in the circle, we apply the usual Galerkin collocation scheme (see [7,8]). We thus decompose

$$-2\pi \lambda_{i,j} = h^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda(h(i-j+t-u)) \mu_m(t) \mu_m(u) \, dt \, du + h^2 \left(\log h^2 + B_{|i-j|,m}\right),$$

for $|i-j| \le N$, where $\lambda(\cdot)$ is the continuous extension of

$$\lambda(t) = \log\left(4e^{-1} t^{-2} \sin^2\left(\frac{t}{2}\right)\right).$$

The coefficients $B_{k,m}$ can be evaluated exactly. Note that by symmetry, $\lambda_{i,i-N} = \lambda_{i,i+N}$. When $j > i + N$, we give the corresponding decomposition for indices i and $j - 2N$, and we proceed similarly in other cases. The approximation of these elements is simply done by applying the rule L_m^2 to the double integral and denoted $\tilde{\lambda}_{i,j}$. Therefore, the classical Galerkin collocation approach for the circle has a symmetric circulant matrix. We finally define $\lambda_h : V_h \times V_h \to \mathbb{R}$ to be the bilinear form such that $\lambda_h(\psi_i, \psi_j) = \tilde{\lambda}_{i,j}$.

4.2. Asymptotic Expansion of the Error

The Galerkin collocation method turns out to be a modified Galerkin scheme, which can be written as

$$(P_h^*) \begin{cases} \text{find } w_h^* \in V_h^e \text{ such that} \\ \lambda_h(w_h^*, r_h) + k_h(w_h^*, r_h) = g_h(r_h), \qquad \forall r_h \in V_h^e. \end{cases}$$

Since for all $r_h, v_h \in V_h^e$,

$$|(Lr_h, v_h)_0 - \lambda_h(r_h, v_h) - k_h(r_h, v_h)| \le C h^{2m+3} \|r_h\|_{-1/2} \|v_h\|_{-1/2},$$

then the bilinear form $\lambda_h + k_h$ satisfies a uniform Babuška-Brezzi condition in V_h^e, and problem (P_h^*) is thus uniquely solvable for h small enough.

Convergence of the method follows readily from that of the Galerkin method. Note that if we are only interested in stronger error estimates than $H^{-1/2}(2\pi)$, we only need degree $m+2$ for the quadrature rule L_m. With stronger smoothness requirements on the exact solution than in the Galerkin method, we get

$$\|w - w_h\|_{-1/2} = \mathcal{O}(h^{m+3/2}), \qquad \|w - w_h\|_{-m-2} = \mathcal{O}(h^{2m+3}).$$

With similar proofs to those appearing in [7], we can prove the following asymptotic expansion of the error, which is the equivalent result to Theorem 3 for the Galerkin collocation method.

THEOREM 4. *For all positive integers M, there exist even 2π-periodic functions, as smooth as desired, such that*

$$\left\| w_h^* - Q_h w - \sum_{k=f(m)}^{M} h^k Q_h f_k \right\|_{m,\infty} \leq C h^{M+1}.$$

5. ADDITIONAL REMARKS

Once we have approximately solved equation (3) we can (theoretically) evaluate the associated potential in $\mathbb{R}^2 \setminus \Gamma$. Thus, we define

$$u(\mathbf{y}; \phi_h) := -\frac{1}{\pi} \int_0^{\pi} \log |\mathbf{y} - \mathbf{a}(s)| \phi_h(s)\, ds, \tag{6}$$

for $\mathbf{y} \in \mathbb{R}^2 \setminus \Gamma$ and ϕ_h equal to one of the discrete solutions. We can also define a fully discrete version of that potential

$$u_h(\mathbf{y}; \phi_h) := -\frac{1}{2\pi} h \sum_{i=1}^{2N} \phi_{h,i} L_m(\log |\mathbf{y} - \mathbf{a}(z_i + h\cdot)|), \tag{7}$$

ϕ_h being decomposed as

$$\phi_h = \sum_{i=1}^{2N} \phi_{h,i} \psi_i = \sum_{i=1}^{d} \phi_{h,i} \varphi_i,$$

in the notations of Section 4. Both expressions define approximations of

$$u(\mathbf{y}) := u(\mathbf{y}; w) = -\frac{1}{\pi} \int_0^{\pi} \log |\mathbf{y} - \mathbf{a}(s)| w(s)\, ds.$$

For both discrete methods (Galerkin and fully discrete Galerkin), we can prove the existence of an asymptotic expansion of the error for the potential and for the approximate potential.

THEOREM 5. *Let K be a compact set strictly contained in $\mathbb{R}^2 \setminus \Gamma$. Then let U_h be any of (6) or (7). Then, there exist functions, defined on K, $u_k, k \geq 2m+3$, independent of h, such that*

$$\left\| U_h - u - \sum_{k=2m+3}^{M} h^k u_k \right\|_{L^\infty(K)} \leq C h^{M+1}.$$

PROOF. The result follows from the theory of linear postprocessing of solutions given in [11] without substantial modifications. ∎

A very similar approach to that given in equation (1) applies also for the modified problem

$$-\frac{1}{\pi} \int_\Gamma \log |\mathbf{z} - \mathbf{y}| u(\mathbf{y})\, d\sigma_{\mathbf{y}} + c = f(\mathbf{z}), \qquad \mathbf{z} \in \Gamma,$$

$$\int_\Gamma u(\mathbf{y})\, d\sigma_{\mathbf{y}} = b,$$

where f and b are given. With the same notations as in Section 2, we can rewrite the problem as the search for an even function w and $c \in \mathbb{R}$ such that

$$Lw + c = g, \qquad (w,1)_0 = b.$$

This problem is uniquely solvable (see [9]) even when the logarithmic capacity of Γ equals 1. The adapted Galerkin and the Galerkin collocation schemes for this equation (now we look for the solution in $H_e^{-1/2}(2\pi) \times \mathbb{R}$) have the same properties. Since the system can be decoupled (see [9,10] for closed contours), it is easy to see that the foregoing analysis is valid here and that the numerical approximations of the real unknown c inherits the optimal convergence rates of the preceding Galerkin methods.

A particular case of this problem appears with $f = 0$, $b = 1$, that is,

$$Lw + d = 0, \qquad (w,1)_0 = 1.$$

Then $C_\Gamma := e^{\pi d}$ is the logarithmic capacity of Γ and w is the equilibrium distribution (see [9]), which, the data being smooth, is an infinitely differentiable periodic function. Moreover, if d_h is the Galerkin or Galerkin collocation approximation to d, it follows readily from the analysis of both methods and from the analyticity of the exponential function that

$$C_\Gamma - e^{\pi d_h} = \sum_{k=2m+3}^{M} h^k d_k + \mathcal{O}\left(h^{M+1}\right),$$

for some real constants d_k, independent of h.

6. AN ALTERNATIVE APPROACH

We have so far studied two discretization methods for Symm's equation with the cosine change of variable in the frame of periodic even Sobolev spaces. The distinction between Cases A and B, which give different dimensions for the discrete spaces, marked the only two possibilities of having nontrivial ($V_h \neq P_0$) spaces of periodic even splines with uniform meshes. However, it is clear in the proof of Proposition 2 that the consistency error expansions do not show the particular choice.

In this section, we rewrite the Galerkin and the Galerkin collocation methods for a modified equation and prove that Cases A and B turn out to be two particular cases of a more general method.

Notice first, that any even solution of $L(w) = g$ is an even solution of $\Lambda w + Kw = g$. The Bessel operator Λ maps $H_e^s(2\pi)$ onto $H_e^{s+1}(2\pi)$. Likewise, if we denote $H_o^r(2\pi)$ to the completion of the set of 2π-periodic indefinitely differentiable odd functions with the norm $\|\cdot\|_r$, Λ maps $H_o^s(2\pi)$ onto $H_o^{s+1}(2\pi)$. This can be easily deduced from the Fourier expansion of even and odd periodic distributions and the Fourier representation of Λ (see [3]).

PROPOSITION 6. *If $C_\Gamma \neq 1$, then $\Lambda + K : H^s(2\pi) \to H^{s+1}(2\pi)$ is an isomorphism for all $s \in \mathbb{R}$.*

PROOF. Since Λ is an isomorphism between $H^s(2\pi)$ and $H^{s+1}(2\pi)$ and K is compact, $\Lambda + K$ is a Fredholm operator of index 0. Thus, bijectivity follows from injectivity. If $\Lambda w + Kw = 0$, then $w = -\Lambda^{-1}Kw \in H_e^s(2\pi)$ and, therefore, $Lw = \Lambda w + Kw = 0$. Hence, $w = 0$. ∎

Hereafter, we assume that the logarithmic capacity of Γ is different from 1. Given $g \in H_e^{s+1}(2\pi)$, the unique solution of

$$(\Lambda + K)w = g \tag{8}$$

belongs to $H_e^s(2\pi)$. We also remark that $\Lambda + K$ satisfies a Gårding inequality in $H^{-1/2}(2\pi)$.

Given any s_0 (or equivalently, any z_0), we consider the space V_h. The Galerkin approximation to (8)

$$\begin{cases} \text{find } v_h \in V_h \text{ such that} \\ (\Lambda v_h + K v_h, r_h)_0 = (g, r_h)_0, \qquad \forall\, r_h \in V_h, \end{cases} \tag{9}$$

is well defined for h small enough. The method is stable and convergent.

PROPOSITION 7. *If $g \in H_e^{1/2}(2\pi)$, then in Cases A and B, the solution of (9) is even and coincides with $G_h w$.*

PROOF. Notice, that in both cases, $V_h = V_h^e \oplus V_h^o$, where V_h^o is the set of elements of V_h which are odd functions. Then $G_h w$ satisfies

$$((\Lambda + K)G_h w, r_h)_0 = (LG_h w, r_h)_0 = (g, r_h)_0,$$

for all $r_h \in V_h^e$. Since

$$((\Lambda + K)G_h w, r_h)_0 = 0 = (g, r_h)_0,$$

for all $r_h \in V_h^0$, then $G_h w$ satisfies the conditions of problem (9) and is thus its unique solution.

Then, the method explained in Section 3 is a particular case of (9). In all other cases, $V_h^e = P_0$ and the Galerkin approximation to (8) for even data is not even. Nevertheless, both branches of the solution $v_h|_{[0,\pi]}$ and $v_h|_{[\pi,2\pi]}$ approximate w, which is the even periodic extension of the actual solution, a function defined on $[0, \pi]$.

The fully discrete Galerkin method is then given by

$$\begin{cases} \text{find } v_h^* \in V_h \text{ such that} \\ \lambda_h(v_h^*, r_h) + k_h(v_h^*, r_h) = g_h(r_h), \qquad \forall\, r_h \in V_h. \end{cases} \tag{10}$$

Again, (10) is uniquely solvable for h small enough and has the same order of approximation as the Galerkin method for smoother data. Moreover, we regain the method explained in Section 5. ∎

PROPOSITION 8. *In Cases A and B, with even data g, the solution of (10) is even and coincides with w_h^*.*

PROOF. For all $r_h \in V_h^o$, we have that

$$g_h(r_h) = 0.$$

Also, for $r_h \in V_h^o$ and $w_h \in V_h^e$,

$$\lambda_h(w_h, r_h) = 0 = k_h(w_h, r_h).$$

Both results follow from the symmetries of the quadrature formula L_m. Then the proof of Proposition 7 remains essentially valid here. ∎

In a way, Propositions 7 and 8 reflect some symmetries of the matrices for the special case when the symmetry of nodes coincides with that of the functions. These properties can be proven in a language very near to that of the Appendix, where we study the symmetries of the interpolation operator.

The present approach gives equivalent systems in Cases A and B with more equations and unknowns ($2N$ against $d = N$ or $d = N + 1$), although additional information about the solution is known *a priori*. In fact, when 'assembling' the matrices of the original Galerkin collocation method, the matrices of this modified method are essentially assembled during the process. There is an additional advantage now: the matrix $(\lambda_{i,j})$ which determines the logarithmic part of the method is circulant, symmetric, and positive definite. Therefore, its inverse can be used as a preconditioner of the whole system.

APPENDIX
MATRICES AND SYMMETRIES OF INTERPOLATION

For $k \in \mathbb{Z}$ let us denote $[k]$ to the unique element in $\{-N+1, \dots, N\}$ congruent with k modulo $2N$. Then, we define the $2N \times 2N$ matrix $\mathbf{A}_h$ given by its elements

$$\mathbf{A}_{i,j} := \mu_m([i-j]).$$

Obviously, $\mathbf{A}_h$ is symmetric, Toeplitz, and circulant. Moreover, $\mathbf{A}_h$ is the matrix of the system whose solution determines the coefficients of the interpolate in basis $\{\psi_1, \dots, \psi_{2N}\}$.

Given $n \geq 1$, we consider the matrix $\mathbf{J}_n$ defined by

$$\mathbf{J}_{i,j} := \delta_{n+1}^{i+j},$$

δ_l^m being the Kronecker symbol. Note that $\mathbf{J}_n^{-1} = \mathbf{J}_n$.

Our aim is to prove Lemma 1 from a matrix point of view.

CASE B. In this case, 0 is the middle point of two nodes, say z_0 and z_1. Since u is 2π-periodic and even, $u(z_{2N-i+1}) = u(z_i)$, for all i. Then, denoting

$$\mathbf{u}_h := (u(z_1), \dots, u(z_{2N}))^\top,$$

it follows that $\mathbf{J}_{2N}\mathbf{u}_h = \mathbf{u}_h$. If we denote $\mathbf{A}_h^\# := \mathbf{J}_{2N}\mathbf{A}_h\mathbf{J}_{2N}$ (the converse of $\mathbf{A}_h$), then it is easy to see that

$$\mathbf{A}_{i,j}^\# = \mathbf{A}_{2N+1-i,2N+1-j} = \mu_m([2N+1-i-2N-1+j]) = \mu_m([i-j]) = \mathbf{A}_{i,j},$$

so $\mathbf{A}_h$ is self-converse (this fact holds for all Toeplitz symmetric matrices). Consequently, if $\mathbf{A}_h\mathbf{a}_h = \mathbf{u}_h$, we have

$$\mathbf{A}_h\mathbf{J}_{2N}\mathbf{a}_h = \mathbf{A}_h^\#\mathbf{J}_{2N}\mathbf{a}_h = \mathbf{J}_{2N}\mathbf{A}_h\mathbf{a}_h = \mathbf{J}_{2N}\mathbf{u}_h = \mathbf{u}_h,$$

and by the invertibility of $\mathbf{A}_h$, it follows that $\mathbf{J}_{2N}\mathbf{a}_h = \mathbf{a}_h$. Consequently,

$$Q_h u = \sum_{j=1}^{2N} a_j \psi_j = \sum_{j=1}^{N} a_j(\psi_j + \psi_{2N+1-j}),$$

and $Q_h u$ belongs to V_h^e.

CASE A. Now, 0 is a node. Suppose $z_0 = 0$. Consider the $2N \times 2N$ matrix given by

$$\mathbf{K}_{2N} := \begin{pmatrix} \mathbf{J}_{2N-1} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Then, clearly, $\mathbf{K}_{2N}^{-1} = \mathbf{K}_{2N}$. Since u is even and 2π-periodic, we have that $u(z_{N-i}) = u(z_{N+i})$, for all i. Thus, denoting again $\mathbf{u}_h := (u(z_1), \dots, u(z_{2N}))^\top$, we have $\mathbf{K}_{2N}\mathbf{u}_h = \mathbf{u}_h$.

Let us decompose the matrix $\mathbf{A}_h$ in the block form

$$\mathbf{A}_h = \begin{pmatrix} \mathbf{A}_{2N-1} & \mathbf{a} \\ \mathbf{a}^\top & a \end{pmatrix}.$$

Then,

$$\mathbf{K}_{2N}\mathbf{A}_h\mathbf{K}_{2N} = \begin{pmatrix} \mathbf{A}_{2N-1}^\# & \mathbf{J}_{2N-1}\mathbf{a} \\ (\mathbf{J}_{2N-1}\mathbf{a})^\top & a \end{pmatrix}.$$

Noticing that $\mathbf{A}_{2N-1}$ is self-converse (by the same reason as in the previous case) and that $\mathbf{J}_{2N-1}\mathbf{a} = \mathbf{a}$ since

$$(\mathbf{J}_{2N-1}\mathbf{a})_i = \sum_{k=1}^{2N-1} \delta_{2N}^{i+k}\mathbf{a}_k = \mathbf{a}_{2N-i} = \mathbf{A}_{2N-i,2N} = \mu_m([2N-i-2N]) = \mathbf{A}_{i,2N} = \mathbf{a}_i,$$

we obtain $\mathbf{K}_{2N}\mathbf{A}_h\mathbf{K}_{2N} = \mathbf{A}_h$. Finally, if $\mathbf{A}_h\mathbf{a}_h = \mathbf{u}_h$, then $\mathbf{A}_h\mathbf{K}_{2N}\mathbf{a}_h = \mathbf{u}_h$, and therefore $\mathbf{K}_{2N}\mathbf{a}_h = \mathbf{a}_h$. From there

$$Q_h u = \sum_{j=1}^{2N} a_j\psi_j = \sum_{j=1}^{N-1} a_j(\psi_{N-j} + \psi_{N+j}) + a_N\psi_N + a_{2N}\psi_{2N},$$

and consequently $Q_h u \in V_h^e$.

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