Mathematical Methods in the Applied Sciences, Vol. 13, 43–53 (1990)
MOS subject classification: 65 F 15, 65 N 30, 65 N 45, 47 G 05

An Effective Direct Solution Method for Certain Boundary Element Equations in 3D

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Communicated by W. Wendland

The boundary element method for the Dirichlet problem in a three-dimensional rotational domain leads to a system of linear equations with a full dense matrix having a special block structure. A direct solution method for such systems is presented, which requires $O(N^{3/2}\ln N)$ arithmetical operations only, using a Fast Fourier Transformation (FFT), where N denotes the number of unknowns on the boundary surface.

0. Introduction

In this paper we consider the integral equation of the first kind for an unknown function $v(x)$:

$$\frac{1}{4\pi}\int_\Gamma \frac{v(x)}{|x-y|}\,\mathrm{d}F_x = f(y), \qquad x, y \in \Gamma \subset \mathbb{R}^3. \tag{0.1}$$

Here, Γ is a surface in $\mathbb{R}^3$, and $|x-y|$ denotes the Euclidean distance in $\mathbb{R}^3$. Let Ω be a bounded simply connected domain with sufficiently smooth boundary Γ. The integral equation (0.1) has applications in connection with the numerical solution of the Dirichlet problem,

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega \subset \mathbb{R}^3, \\ u(x) = u^*(x), & x \in \Gamma = \partial\Omega, \end{cases} \tag{0.2}$$

with the help of the boundary element method (BEM).

The unknown function $v(x)$ denotes the normal derivative of the function $u(x)$ from (0.2):

$$v(x) = \frac{\partial u(x)}{\partial n_x}$$

where n_x $(x \in \Gamma)$ is the unit outward normal of the surface Γ at x.

1. Basic properties

Throughout this paper, the following properties will be assumed for the equation (0.1) (see [11, 7, 17]).

0170–4214/90/020043–11$05.50 *Received 29 May 1989*

Lemma 1.1. *The pseudodifferential operator*

$$(\mathcal{A}v)(y) = \frac{1}{4\pi}\int_\Gamma \frac{v(x)}{|x-y|}\,dF_x \tag{1.1}$$

defines a continuous bijective mapping

$$\mathcal{A}: \mathbb{H}^{-1/2} \to \mathbb{H}^{1/2}.$$

For each $f \in \mathbb{H}^{1/2}$ *the equation* (0.1) *has a unique solution* $v \in \mathbb{H}^{-1/2}$ *with*

$$\langle \mathcal{A}v, w\rangle_{L_2(\Gamma)} = \langle f, w\rangle_{L_2(\Gamma)}$$

for all $w \in \mathbb{H}^{-1/2}$. *The solution* v *satisfies the a priori estimate* $\|v\|_{-1/2} \leqslant C\|f\|_{1/2}$.

$\mathbb{H}^s = \mathbb{H}^s(\Gamma)$ denotes the Sobolev space [1] on the boundary Γ, that is the closure of $C^\infty(\Gamma)$ (all smooth real functions on Γ) with respect to the norm

$$\|f\|_s^2 = \begin{cases} \displaystyle\sum_{|\beta|\leqslant s}\int_\Gamma |D^\beta f(x)|^2\,dF_x, & s\in\mathbb{N}\cup\{0\}, \\ \displaystyle\|f\|_{[s]}^2 + \sum_{|\beta|=[s]}\int_\Gamma\int_\Gamma \frac{|D^\beta f(x) - D^\beta f(y)|^2}{|x-y|^{2+2(s-[s])}}\,dF_x\,dF_y, & s>0,\ s\notin\mathbb{N}\cup\{0\}, \\ \displaystyle\left(\sup_{w\in\mathbb{H}^{-s}} \frac{\langle w, f\rangle_{L_2(\Gamma)}}{\|w\|_{-s}}\right)^2, & s<0. \end{cases}$$

Lemma 1.2. *The operator* $\mathcal{A}$ *of the equation* (0.1) *satisfies the strong coercivity estimate (Gårding inequality):*

$$\langle \mathcal{A}v, v\rangle_{L_2(\Gamma)} \geqslant \gamma\|v\|_{-1/2}^2, \qquad \forall v\in\mathbb{H}^{-1/2},$$

where $\gamma > 0$ *is a constant.*

Let Γ be given by a parametric representation,

$$\Gamma = \left\{x\in\mathbb{R}^3 : x = x(t,z) = \begin{pmatrix} R(z)\cos 2\pi t \\ R(z)\sin 2\pi t \\ z \end{pmatrix},\quad 0\leqslant z\leqslant 1,\ 0\leqslant t<1\right\}, \tag{1.2}$$

where the function $R(z)$ satisfies

$$R(0) = R(1) = 0;$$

$$R(z) > 0, \qquad z\in(0,1);$$

$$R'(z)\to\infty,\ z\to 0;$$

$$R'(z)\to -\infty,\ z\to 1;$$

$$\lim_{z\to 0}|R'(z)R(z)| < \infty;$$

$$\lim_{z\to 1}|R'(z)R(z)| < \infty.$$

That is, Γ originates from the rotation of the curve

$$\{R(z), 0\leqslant z\leqslant 1\} \tag{1.3}$$

around the z-axis.

Obviously by (1.2), any function on Γ is identified with a function on $[0, 1) \times [0, 1]$. With the help of (1.2) and (1.3) we write (0.1) as an integral equation over $[0, 1) \times [0, 1]$:

$$\frac{1}{4\pi} \int_0^1 \int_0^1 \frac{v(t, z)}{|x(t, z) - x(\tau, w)|} J(t, z)\, dt\, dz = f(\tau, w), \tag{1.4}$$

$$\forall (\tau, w) \in [0, 1) \times [0, 1],$$

where $y = x(\tau, w)$, $v(t, z) = v(x(t, z))$, $f(\tau, w) = f(x(\tau, w))$ and

$$J(t, z) = \sqrt{\left[\left| \frac{\partial x}{\partial t} \right|^2 \left| \frac{\partial x}{\partial z} \right|^2 - \left(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z} \right)^2 \right]}$$

is for the rotation surface (1.2) independent of t:

$$J(t, z) = J(z) = 2\pi R(z) \sqrt{\{1 + [R'(z)]^2\}}. \tag{1.5}$$

For the $L_2(\Gamma)$ inner product, this yields:

$$\langle u, v \rangle_{L_2(\Gamma)} = \int_\Gamma u(y) v(y)\, dF_y = \int_0^1 \int_0^1 J(w) u(\tau, w) v(\tau, w)\, d\tau\, dw. \tag{1.6}$$

The distance $|x - y|$ between the points $x, y \in \Gamma$ is

$$\begin{aligned} |x - y|^2 &= |x(t, z) - x(\tau, w)|^2 \\ &= (z - w)^2 + R^2(z) + R^2(w) - 2R(z)R(w) \cos 2\pi(t - \tau). \end{aligned} \tag{1.7}$$

Hence, the kernel $K(t, z, \tau, w)$ of (1.4) is symmetric with respect to (t, τ) and to (z, w):

$$K(t, z, \tau, w) = K(\tau, z, t, w) = K(t, w, \tau, z) = K(\tau, w, t, z), \tag{1.8}$$

furthermore,

$$K(t, z, \tau, w) = K(z, w, |t - \tau|). \tag{1.9}$$

The kernel $K(t, z, \tau, w)$ is 1-periodic with respect to t and τ:

$$K(t, z, \tau, w) = K(t + m, z, \tau + l, w), \qquad \forall m, l \in \mathbb{Z}. \tag{1.10}$$

2. The Galerkin method

The solution of (0.1) will be approximated with the help of the boundary element method on the surface Γ. The functions on Γ are identified with functions on $[0, 1) \times [0, 1]$, 1-periodic in the first argument t. We divide the domain $[0, 1) \times [0, 1]$ into rectangles using the nodes

$$\{(t_k, z_l) = (h_t(k - 1), h_z(l - 1)), \qquad k = 1, \ldots, n, l = 1, \ldots, m\},$$

where $h_t = 1/n$ and $h_z = 1/(m - 1)$. The images of the rectangles

$$\{(t, z) : t_k \leqslant t \leqslant t_{k+1}, z_l \leqslant z \leqslant z_{l+1}, \qquad k = 1, \ldots, n, l = 1, \ldots, m\}$$

on Γ are the boundary elements.

For the approximation of $v(t, z)$ we introduce an $(N = n*m)$-dimensional subspace of functions $\mathbb{H}_N$:

$$\mathbb{H}_N = \left\{ v_N(t, z) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{m} y_{(i_1,i_2)} \varphi_{(i_1,i_2)}(t, z) = \Phi y, \qquad y \in \mathbb{R}^N \right\}.$$

Here the Φ-system of basis functions is given by

$$\Phi = (\varphi_{(1,1)}(t, z), \varphi_{(2,1)}(t, z), \ldots, \varphi_{(n,1)}(t, z), \ldots, \ldots, \varphi_{(n,m)}(t, z)). \tag{2.1}$$

As

$$\varphi_{(1,l)}(t, z) \in \mathbb{H}^{-1/2}, \tag{2.2}$$

one may use the piecewise constant, linear or quadratic functions. The functions $\varphi_{(1,l)}(t, z)$ are even in the first argument:

$$\varphi_{(1,l)}(t, z) = \varphi_{(1,l)}(-t, z). \tag{2.3}$$

The basis functions $\varphi_{(k,l)}(t, z)$ are defined by shifting

$$\varphi_{(k,l)}(t, z) = \varphi_{(1,l)}(t - t_k, z), \qquad k = 1, \ldots, n \tag{2.4}$$

having the 1-periodic extensions

$$\varphi_{(k,l)}(t, z) = \varphi_{(k,l)}(t + m, z), \qquad \forall m \in \mathbb{Z},\ k = 1, \ldots, n. \tag{2.5}$$

It should be noted here that the assumptions on Φ made in (2.2) to (2.5) are satisfied for most choices of local piecewise polynomials on the given surface. On the other hand, the properties defined above guarantee a special structure of the resulting matrix, enabling the construction of a very effective solution strategy.

Lemma 2.1. *The functions $\varphi_{(k,l)}(t, z)$ have the following properties:*

$$\varphi_{(k,l)}(t_k - t, z) = \varphi_{(k,l)}(t_k + t, z); \tag{2.6}$$

$$\varphi_{(k_1,l)}(t, z) = \varphi_{(k_2,l)}(-t + t_{k_1} + t_{k_2}, z). \tag{2.7}$$

Proof. The property (2.6) follows from (2.3) and (2.4):

$$\varphi_{(k,l)}(t_k - t, z) = \varphi_{(1,l)}(-t, z) = \varphi_{(1,l)}(t, z) = \varphi_{(k,l)}(t_k + t, z).$$

Using (2.4) and (2.6) we obtain (2.7):

$$\varphi_{(k_1,l)}(t, z) = \varphi_{(k_2,l)}(t - t_{k_1} + t_{k_2}, z) = \varphi_{(k_2,l)}(-t + t_{k_1} + t_{k_2}, z).$$

■

The Galerkin method for (0.1) leads to:

Find $v_N \in \mathbb{H}_N$ such that the Galerkin equations

$$\langle \mathcal{A} v_N, w \rangle_{L_2(\Gamma)} = \langle f, w \rangle_{L_2(\Gamma)}, \tag{2.8}$$

are satisfied for all $w \in \mathbb{H}_N$.

Using (1.4), (1.5) and (1.6) the equations (2.8) are replaced by the algebraic system

$$Ay = b, \ A \in \mathbb{R}^{N*N}, \qquad y, b \in \mathbb{R}^N \tag{2.9}$$

with

$$a_{(i_1,i_2),(j_1,j_2)} = \langle \mathcal{A}\varphi_{(j_1,j_2)}, \varphi_{(i_1,i_2)} \rangle_{L_2(\Gamma)}$$

$$= \frac{1}{4\pi} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{J(w)J(z)}{|x(t,z) - x(\tau,w)|} \varphi_{(i_1,i_2)}(\tau, w)\varphi_{(j_1,j_2)}(t, z)\,dt\,dz\,d\tau\,dw, \quad (2.10)$$

$$b_{(i_1,i_2)} = \langle f, \varphi_{(i_1,i_2)} \rangle_{L_2(\Gamma)} = \int_0^1 \int_0^1 J(w)\varphi_{(i_1,i_2)}(\tau, w) f(\tau, w)\,d\tau\,dw. \quad (2.11)$$

3. The properties of the matrix

It is well known that the boundary element method leads to an algebraic system of equations with a full dense matrix. The Gaussian elimination is usually used for solving the linear system (2.9). Then the number of arithmetical operations would be

$$O(N^3) = O(n^3 m^3).$$

Given that the matrix A will generally be large, more efficient methods for the solution of (2.9) would be necessary. In this paper we construct a direct method for the solution of the system (2.9), requiring $O(m^2 n \ln n)$ arithmetical operations only. This leads to the idea of an iterative solution of (2.9) for more general domains, requiring $O(m^2 n^2)$ operations.

First, the properties of the matrix A with the elements (2.10) are discussed.

Lemma 3.1. *The matrix A is symmetric and positive definite.*

Proof. The symmetry of the matrix A follows from the symmetry of the kernel (1.8) and from the definition (2.10).

For each vector $y \in \mathbb{R}^N$, we have with (2.1), (2.2) and Lemma 1.2

$$(Ay, y) = \langle \mathcal{A}\Phi y, \Phi y \rangle_{L_2(\Gamma)} = \langle \mathcal{A}\varphi, \varphi \rangle_{L_2(\Gamma)} > 0,$$

where $\varphi = \Phi y \in \mathbb{H}_N \subset \mathbb{H}^{-1/2}$.

■

Lemma 3.2. *The matrix A has a block structure*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm} \end{bmatrix} \quad (3.1)$$

where $A_{i_2,j_2} \in \mathbb{R}^{n \times n}$ are symmetric matrices.

Proof. Lemma 2.1 yields

$$\varphi_{(i_1,i_2)}(\tau, w) = \varphi_{(j_1,i_2)}(-\tau + t_{i_1} + t_{j_1}, w);$$

$$\varphi_{(j_1,j_2)}(t, z) = \varphi_{(i_1,j_2)}(-t + t_{i_1} + t_{j_1}, z).$$

By the substitutions

$$t' = -t + t_{i_1} + t_{j_1}, \qquad dt' = -dt;$$

$$\tau' = -\tau + t_{i_1} + t_{j_1}, \qquad d\tau' = -d\tau;$$

in the integral (2.10) we obtain with (1.9) and (1.10)

$$a_{(i_1,i_2),(j_1,j_2)} = a_{(j_1,i_2),(i_1,j_2)},$$

because the kernel K and the basis functions are 1-periodic. ■

Lemma 3.3. *The matrices* A_{i_2,j_2} *from* (3.1) *are circulant*:

$$a_{(i_1,i_2),(j_1,j_2)} = a_{(i_1+1,i_2),(j_1+1,j_2)}, \qquad 1 \leqslant i_1, j_1 \leqslant n-1, \tag{3.2}$$

$$a_{(i_1,i_2),(n,j_2)} = a_{(i_1+1,i_2),(1,j_2)}, \qquad 1 \leqslant i_1 \leqslant n-1. \tag{3.3}$$

Proof. We use the definitions (2.4) and (2.10), and the property (1.8) for the proof of (3.2)

$$\varphi_{(i_1,i_2)}(\tau, w) = \varphi_{(i_1+1,i_2)}(\tau + h_t, w), \qquad 1 \leqslant i_1 \leqslant n-1,$$

$$\varphi_{(j_1,j_2)}(t, w) = \varphi_{(j_1+1,j_2)}(t + h_t, w), \qquad 1 \leqslant j_1 \leqslant n-1,$$

$$\varphi_{(n,j_2)}(t, w) = \varphi_{(1,j_2)}(t + h_t, w).$$

Using the substitutions

$$t' = t + h_t, \qquad \mathrm{d}t' = \mathrm{d}t,$$

$$\tau' = \tau + h_t, \qquad \mathrm{d}\tau' = \mathrm{d}\tau,$$

and the 1-periodic property of the kernel and the basis functions we obtain (3.2) and (3.3). ■

We introduce the most simple circulant matrix as

$$J = \begin{bmatrix} 010\ldots00 \\ 0010\ldots0 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ 0\ldots0001 \\ 10\ldots000 \end{bmatrix}, \qquad J \in \mathbb{R}^{n*n}. \tag{3.4}$$

It is not difficult to see that each circulant matrix $B \in \mathbb{R}^{n*n}$ satisfies

$$B = \sum_{l=1}^{n} b_{1l} J^{l-1} = b(J); \tag{3.5}$$

thus B is a polynomial function of the matrix J.

From Lemma 3.3 and formula (3.5) we obtain that the matrix A of the system (2.9) has a block structure (3.1), where each block is a polynomial function of the matrix J from (3.4):

$$A = \begin{bmatrix} a_{11}(J) \ldots a_{1m}(J) \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}(J) \ldots a_{mm}(J) \end{bmatrix}, \tag{3.6}$$

where

$$a_{kl}(J) = a_{lk}(J), \tag{3.7}$$

because $A = A^T$ and $A_{kl} = a_{kl}(J) = (a_{kl}(J))^T = A_{lk}$.

4. The algorithm

We use the form (3.6) of the matrix A to construct an efficient direct method for the solution of the system (2.9). Let us prove the following lemma:

Lemma 4.1. *Let $A \in \mathbb{R}^{m*m}$ have the structure*

$$A = A(\omega) = \begin{bmatrix} a_{11}(\omega) \dots a_{1m}(\omega) \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{m1}(\omega) \dots a_{mm}(\omega) \end{bmatrix}, \tag{4.1}$$

where $a_{ij}(\omega)$ are any functions. Let $B \in \mathbb{R}^{n*n}$ have the eigenvalue ω_k with the corresponding eigenvector x_k:

$$Bx_k = \omega_k x_k.$$

Furthermore, $A(\omega)$ has the eigenvalue $\lambda_l(\omega)$ with the eigenvector $y_l(\omega)$:

$$A(\omega)y_l(\omega) = \lambda_l(\omega)y_l(\omega),$$

then the matrix $A(B) \in \mathbb{R}^{N*N}$, $N = m*n$ has the eigenvalue $\lambda_l(\omega_k)$ with the eigenvector $y_l(\omega_k) \otimes x_k$ where $a \otimes b$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ denotes the Kronecker product

$$a \otimes b = (a_1 b^T, a_2 b^T, \dots, a_m b^T)^T \in \mathbb{R}^N, \qquad N = n*m.$$

Proof. Using the property of the Kronecker product

$$(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$$

we obtain with $E_{ij} = e_i e_j^T \in \mathbb{R}^{m*m}$

$$\begin{aligned} A(B)(y_l(\omega_k) \otimes x_k) &= \sum_{i,j} (E_{ij} \otimes a_{ij}(B))(y_l(\omega_k) \otimes x_k) \\ &= \sum_{i,j} (E_{ij} y_l(\omega_k)) \otimes (a_{ij}(B)x_k) \\ &= \sum_{i,j} a_{ij}(\omega_k)(E_{ij} y_l(\omega_k)) \otimes x_k \\ &= \left(\sum_{i,j} (a_{ij}(\omega_k) E_{ij}) y_l(\omega_k) \right) \otimes x_k \\ &= (A(\omega_k) y_l(\omega_k)) \otimes x_k \\ &= \lambda_l(\omega_k)(y_l(\omega_k) \otimes x_k). \end{aligned}$$

■

The matrix A from (2.9) is of the structure (3.6), which is identical with (4.1); J is substituted by ω. The eigenvalues and eigenvectors of the matrix J from (3.4) are known:

$$\begin{aligned} Jf_k &= \omega_k f_k, \\ \omega_k &= \exp[\mathrm{i}(2\pi/n)(k-1)], \qquad \mathrm{i}^2 = -1, \\ f_k &= (\omega_k^0, \omega_k^1, \ldots, \omega_k^{n-1})^{\mathrm{T}}, \qquad k = 1, \ldots, n, \\ f_i^* f_j &= n\delta_{ij}, \end{aligned} \tag{4.2}$$

where δ_{ij} is the Kronecker symbol.

We denote the real eigenvalues and eigenvectors of the matrices $A(\omega_k)$ by

$$\begin{aligned} &\lambda_l(\omega_k) \in \mathbb{R}, \qquad x_l(\omega_k) \in \mathbb{R}^m, \qquad (x_i(\omega_k))^{\mathrm{T}} x_j(\omega_k) = \delta_{ij}, \\ &A(\omega_k) x_l(\omega_k) = \lambda_l(\omega_k) x_l(\omega_k). \end{aligned} \tag{4.3}$$

The existence is guaranteed from

Lemma 4.2. *The matrices $A(\omega_k)$, $k = 1, \ldots, n$ are real and symmetric.*

Proof. The elements $a_{ij}(\omega_k)$ of the matrices $A(\omega_k)$ are the eigenvalues of the matrices $a_{ij}(J)$. They are all symmetric (Lemma 3.2) and have only real eigenvalues.

The symmetry of the matrices $A(\omega_k)$ follows from (3.7). ■

Without loss of generality, we assume that n is even.

Lemma 4.3. *The matrices $A(\omega_k)$ have the property*

$$A(\omega_k) = A(\omega_{n-k+2}), \qquad k = 2, \ldots, n/2. \tag{4.4}$$

Proof. At first we will prove that each eigenvalue of a real symmetric circulant matrix B of even dimension n (except λ_1 and $\lambda_{n/2+1}$) is double. Using (4.2) and (3.5) we obtain

$$Bf_k = \lambda_k f_k, \qquad B = B^{\mathrm{T}}, \qquad \lambda_k \in \mathbb{R},$$

or

$$B\bar{f}_k = \lambda_k \bar{f}_k, \qquad B = \bar{B}, \qquad \lambda_k = \bar{\lambda}_k.$$

With

$$\bar{f}_k = f_{n-k+2}, \qquad k = 2, \ldots, n/2$$

we obtain

$$\lambda_k = \lambda_{n-k+2}, \qquad k = 2, \ldots, n/2. \tag{4.5}$$

The elements $a_{ij}(\omega_k)$ of the matrix $A(\omega_k)$ are the eigenvalues of the matrices $a_{ij}(J)$. They are all real, symmetric and circulant (see (3.6)). With (4.5) this yields

$$a_{ij}(\omega_k) = a_{ij}(\omega_{n-k+2}), \qquad k = 2, \ldots, n/2. \qquad \blacksquare$$

Summarizing this, we have the following spectral properties for $A(J)$:

(i) $\lambda_l(\omega_1)$, simple eigenvalues with the eigenvectors $x_l(\omega_1) \otimes f_1$

(ii) $\lambda_l(\omega_k)$, double eigenvalues with the eigenvectors $x_l(\omega_k) \otimes f_k$ and $x_l(\omega_k) \otimes f_{n-k+2}$

(iii) $\lambda_l(\omega_{n/2+1})$, simple eigenvalues with the eigenvectors

$$x_l(\omega_{n/2+1}) \otimes f_{n/2+1},$$

$$l = 1, \ldots, m, \qquad k = 2, \ldots, n/2.$$

The matrix $A(J)$ can be given by the formula:

$$A = A(J) = \frac{1}{n} \sum_{k,l} \lambda_l(\omega_k)(x_l(\omega_k) \otimes f_k)(x_l(\omega_k) \otimes f_k)^*. \tag{4.6}$$

The solution of (2.9) can be obtained with the help of (4.6) as:

$$y = A^{-1}b = \frac{1}{n} \sum_{k,l} \frac{1}{\lambda_l(\omega_k)}(x_l(\omega_k) \otimes f_k)(x_l(\omega_k) \otimes f_k)^* b. \tag{4.7}$$

Now we are able to formulate the algorithm for computing the solution y. The vectors $y, b \in \mathbb{R}^N$ are written as

$$y = (y_1^T, y_2^T, \ldots, y_m^T)^T, \qquad b = (b_1^T, b_2^T, \ldots, b_m^T)^T, \qquad y_i, b_i \in \mathbb{R}^n. \tag{4.8}$$

Furthermore, with the help of (4.8) we define

$$X_l = (x_l(\omega_1) \vdots \ldots \vdots x_l(\omega_n)) \in \mathbb{R}^{m*n}, \qquad l = 1, \ldots, m;$$

$$F = (f_1 \vdots \ldots \vdots f_n) \in \mathbb{C}^{n*n};$$

$$B = (b_1 \vdots \ldots \vdots b_m) \in \mathbb{R}^{n*m};$$

$$Y = (y_1 \vdots \ldots \vdots y_m) \in \mathbb{R}^{n*m};$$

$$e_k = (0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0)^T \in \mathbb{R}^m.$$

Algorithm 4.1

1. $C := (F^*B)^T \in \mathbb{C}^{m*n}$;
2. $Y := 0$;
3. *for* $l = 1, \ldots, m$
   3.1. $D_l = \operatorname{diag}(d_1, \ldots, d_n)$ *with*

$$d_k = \frac{x_l^T(\omega_k) C e_k}{n\lambda_l(\omega_k)};$$

   3.2. $Y := Y + \operatorname{Re}(FD_l X_l^T)$.

We use the Fast Fourier Transformation (FFT) for the steps 1 and 3.2. It is well known that the matrix–vector multiplication with the matrix F of the discrete Fourier Transformation needs $O(n \ln n)$ arithmetical operations only (see [2]), leading to $O(m^2 n \ln n)$ arithmetical operations for the entire Algorithm 4.1.

Remark 4.1. The Algorithm 4.1 is written for a general rotational surface (1.2) by using the Galerkin method (2.8) with a very general system of basis functions (2.1)–(2.5). More analytic information about the spectrum of the matrices $A(\omega_k)$ in (4.3) can be obtained for special surfaces and basis functions only.

Remark 4.2. In general, all eigenvalues and eigenvectors of the matrices $A(\omega_k) \in \mathbb{R}^{m*m}$, $k = 1, \ldots, n/2 + 1$ can be calculated with the help of Housholder tridiagonalizing $A(\omega_k)$ and using the implicit QL-algorithm (see [3, 16]), requiring $O(nm^3)$ operations.

Remark 4.3. For general arbitrary domains Ω (not being rotationally), the Galerkin approach still yields a symmetric, positive-definite matrix A of the block structure (3.1), without having the additional special properties of the Lemmas 3.2 and 3.3. In order to solve the resulting linear system of equations, we are able to use a direct method (such as Gaussian elimination, better Cholesky decomposition here, $O(n^3m^3)$ operations) or the preconditioned conjugate gradient (CG) method [6, 15]. Here, one step of the iteration requires the matrix–vector multiplication $u := A*s$ ($O(m^2n^2)$ operations) and one solution of an appropriate preconditioning system

$$Cw = r. \tag{4.9}$$

One idea for the construction of C could be the Galerkin matrix of a rotational domain Ω^*, near to Ω in some sense. In this case the solution of (4.9) is calculated with the help of Algorithm 4.1 ($O(nm^2 \ln n)$ operations). Hence, the work per iteration is mainly dominated by the work for the matrix–vector multiplication. The decrease of the latter work is investigated in [4] for the so-called Panel method.

The total number of CG iterations is dependent on the spectral condition number of $C^{-1}A$, which can be expected to be a fixed number (independent of nm), so the entire process would require $O(n^2m^2)$ arithmetical operations.

Remark 4.4. The more simple two-dimensional case has been considered in [8, 10, 12, 13, 14]. Here a circulant preconditioning matrix C occurred, which leads to a drastic decrease of the numerical work for the solution of the resulting linear system of equations.

Remark 4.5. The FFT in connection with boundary integral equations has been considered in [5, 9], where the Galerkin method with finite sections of the Fourier series as trial and test functions has been used.

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