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Decompositions in Edge and Corner Singularities for the Solution of the Dirichlet Problem of the Laplacian in a Polyhedron

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Abstract. The solution of the three-dimensional Dirichlet problem for the Laplacian in a polyhedral domain has special singular forms at corners and edges. The main result of this paper is a "tensor-product" decomposition of those singular forms along the edges. Such a decomposition with both edge singularities, additional corner singularities and a smoother remainder refines known regularity results for the solution where either the edge singularities are of non-tensor product form or the remainder term belongs to an anisotropic Sobolev space for data given in an isotropic Sobolev space.

1. Introduction

In this paper we consider the Dirichlet problem for the Laplacian

(1.1) $$\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

where Ω is either a bounded domain in $\boldsymbol{R}^2$ with a polygonal boundary $\partial\Omega$ or an infinite wedge $\boldsymbol{R} \times K$ or a bounded domain in $\boldsymbol{R}^3$ with a polyhedral boundary $\partial\Omega$. We are interested in the regularity of u where f is given in the Sobolev space $H^{s-1}(\Omega)$, $s \geqq 0$. For a regular boundary we would have $u \in H^{s+1}(\Omega)$. For two-dimensional domains with a polygonal boundary, it is well known that in general the solution u of (1.1) exhibits so-called corner singularities which do not belong to $H^{s+1}(\Omega)$ (see [10], [3] for example).

The extension to three dimensions of this phenomena is not simple as can be seen easily in the case of an infinite wedge $\Omega = \boldsymbol{R} \times K$ whose cross-section K is a plane sector with opening ω. Consider $u \in \mathring{H}^1_{\text{loc}}(\boldsymbol{R} \times K)$ solution of (1.1) with $f \in L^2_{\text{loc}}(\boldsymbol{R} \times K)$. Obviously one likes to know whether u can be written as a sum of some smooth $u_0 \in H^2_{\text{loc}}(\boldsymbol{R} \times K)$ and a singular function u_s carrying the singular behavior of u along the edge $\boldsymbol{R}$. It seems natural to expect a decomposition

(1.2) $$u(x, y, z) = u_0(x, y, z) + c(z)\, S(x, y)$$

where $S(x, y) = \varrho^{\nu} \sin \nu\phi$, $\nu = \frac{\pi}{\omega}$, is the singularity term of the two-dimensional problem (1.1) where $\Omega = K$ and (ϱ, ϕ) denote the polar coordinates at the vertex of the sector K. Unfortunately the analysis in [8] shows that u_0 is not so regular in z as a consequence of the poor regularity of the stress intensity factor c. As shown by P. GRISVARD in [11] and M. DAUGE in [5], [6] there holds *instead of* (1.2) a non-tensor product decomposition for the solution (1.1) in $\Omega = \mathbf{R} \times K$: Namely, there exists $u_0 \in H^2_{\mathrm{loc}}(\mathbf{R} \times K)$ and $c \in H^{1-\nu}(\mathbf{R})$ such that

$$(1.3) \qquad u = u_0 + u_\Phi$$

with

$$(1.4) \qquad u_\Phi(x, y, z) = c \underset{z}{*} \Phi(z, \varrho)\, S(x, y)$$

where Φ is a regularizing sequence for $\varrho \to 0$ and $\underset{z}{*}$ denotes convolution with respect to z.

An improved numerical scheme to solve (1.1) approximately, for example, a GALERKIN scheme with finite elements, has obviously to make use of the special singular forms of the exact solution near corners and edges. For two-dimensional domains quasi-optimal error estimates for the GALERKIN solution are well-known, where high convergence rates are obtained either by augmenting the space of test and trial functions by special singular elements which imitate the corner behavior of the exact solution or by using appropriate mesh refinement in the vicinity of the corners.

One can regain (see [18], [22], [7]) those high convergence rates for the GALERKIN solution *to problem (1.1) in the case of a three-dimensional polyhedral domain* Ω if one knows instead of (1.3) a decomposition for the solution u of (1.1) where all appearing singular functions are in tensor product form.

It is the main purpose of this paper to prove that such a tensor product decomposition for u in (1.1) exists if higher regularity is assumed for the given data. In case of the infinite wedge $\Omega = \mathbf{R} \times K$ we show (see Theorem 3) that u in (1.1) is of the form (1.2) with $u_0 \in H^{1+2\nu-\varepsilon}_{\mathrm{loc}}(\Omega)$ for any $\varepsilon > 0$, $c \in H^{s-\nu}(\mathbf{R})$ for $f \in H^{s-1}(\Omega)$ with $s \geqq 3\nu + 1$ where $\nu = \frac{\pi}{\omega}$. Comparison with the results by M. DAUGE in [5], [6] shows (see Theorem 2) that for less regular data, $f \in L^2(\Omega)$ for example, one obtains only a non-tensor product decomposition (1.3). In most practical problems, however, f is smoother anyway, satisfying our regularity assumption.

Our main results in this paper are the tensor product decompositions for the solution of u of (1.1) in case of an infinite wedge (Theorem 3) and in case of a polyhedral cone (Theorem 6). Since away from the edges and corners, $u \in H^{s+1}$ for $f \in H^{s-1}(\Omega)$, the corresponding decomposition result for u in case of an arbitrary polyhedral domain follows by combination of Theorems 3 and 6. The assertions of those theorems are illustrated via representative examples of domains with special geometries.

The starting points of the proofs of Theorems 3 and 6 are the non-tensor product decompositions like (1.3) by M. DAUGE (see Theorems 2 and 4). As in [19] we replace in these proofs the regularized edge singularities of the form (1.4) by edge singularities in tensor product form as in (1.2) and analyze the difference. The regularity of these difference terms is determined by the smoothness of the stress intensity factors. Assuming additional regularity of the given data f in (1.1) the stress intensity factors become sufficiently smooth, and thus the difference terms can be included in the regular part v_0 of the decompositions (2.11) or (3.10) for u. Note that both in the case of a wedge and of a polyhedron this regular part v_0 belongs to an isotropic SOBOLEV space and not to an anisotropic SOBOLEV space as in [8], [9]. Our results for u in (1.1) differ also from the regularity results in weighted SOBOLEV spaces as given in [13], [14], [15], [17] for wedge shape domains or in [12] for conical domains.

The paper is organized as follows: In Section 2 we first review the well-known decomposition results for solution u of (1.1) where Ω is a plane domain with a polygonal boundary. Then we consider problem (1.1) in an infinite wedge in three dimensions. We state the result by M. DAUGE (Theorem 2) for this case and present as one of our main results the corresponding modification with tensor product singularities (Theorem 3). In Section 3 we consider a polyhedral cone with plane faces. Again we first quote the corresponding result by M. DAUGE (Theorem 4) where the singularities along the edges are not in tensor product form. Then we give as our major result the modified decomposition with tensor product singularities. We further present alternative decompositions which show the "total" edge singularities, i.e., when also the corner singularities are expanded in edge terms (Corollary 3) and which show the "physically relevant" stress intensity factors (Corollary 4).

In Section 4 we illustrate our results with examples for various geometries. They show the different behavior of edge and corner singularities. Example 2 shows that the physically relevant stress intensity factor may even blow up in the corners. Example 3 shows a case where the corner singularity is stronger than the edge singularity, whereas in most cases the opposite holds.

In Section 5 we present tensor product decompositions for the normal derivative $\frac{\partial u}{\partial n}$ of the solution u of the inhomogeneous DIRICHLET problem

$$\Delta u = 0 \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega \tag{1.5}$$

where g is given on the polyhedral boundary $\partial\Omega$ of Ω. The corresponding result given in Theorem 8 follows by a refined analysis from Theorem 6. The tensor product decomposition in Theorem 8 is important when problem (1.5) is solved via boundary integral equations, especially when boundary elements are used for the numerical solution of the integral equations. For screen problems and interior crack problems where Γ is an open surface piece with smooth edges a decomposition analogous to (5.3) in Theorem 7 can be also obtained by applying WIENER-HOPF techniques to the equivalent integral equations (see [4], [20]). We want to mention that the results of Theorems 7 and 8 can be used in a convergence analysis for improved boundary element methods based on mesh refinement or singular elements (see [18], [21]). Throughout the paper C denotes a generic constant.

2. Decomposition Theorems for an Infinite Wedge

First, we review the two-dimensional case. For simplicity we cite the corresponding decomposition theorem for a polygonal domain Ω whose boundary $\partial\Omega$ consists of straight sides. The interior angle at vertex t_j is denoted by ω_j where $1 \leqq j \leqq J$.

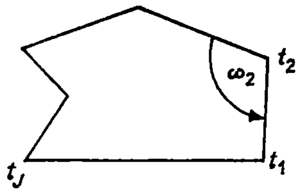


Fig. 1

Theorem 1 ([10], [5, Theorem 5.11]). *Let $f \in H^{s-1}(\Omega)$ and $s > 0$, $\nu_j = \frac{\pi}{\omega_j}$, $s \neq m\nu_j$, m integer, m odd for $\nu_j = \frac{1}{2}$. Then the weak solution $u \in H^1(\Omega)$ of problem* (1.1) *admits with C^∞ cut-off-functions χ_j concentrated near t_j the decomposition*

$$u = \sum_{j=1}^{J} \chi_j \sum_{0<m\nu_j<s} c_m^j S_m^j + u_0 \tag{2.1}$$

where $u_0 \in H^{s+1}(\Omega)$ and $c_m^j \in \boldsymbol{R}$ are constants depending on f. In (2.1) *we have $m \in \boldsymbol{N}$ for $\nu_j \neq \frac{1}{2}$, for $\nu_j = \frac{1}{2}$ the sum is taken over odd values of m. The singular functions S_m^j are given by*

$$S_m^j(\varrho_j, \phi_j) = \varrho_j^{m\nu_j} \sin m\nu_j\phi_j \quad \text{for} \quad m\nu_j \notin \boldsymbol{N} \tag{2.2}$$

$$S_m^j(\varrho_j, \phi_j) = \varrho_j^{m\nu_j}(\log \varrho_j \sin m\nu_j\phi_j + \phi_j \cos m\nu_j\phi_j) \quad \text{otherwise} \tag{2.3}$$

with polar coordinates (ϱ_j, ϕ_j) at t_j.

It is understood that the corresponding modifications for $\nu_j = \frac{1}{2}$ are incorporated in the assertions of Theorems 2 to 8.

Note that the term (2.3) does not satisfy the homogeneous Dirichlet condition on $\partial\Omega$. Nevertheless u in (2.1) satisfies the homogeneous boundary condition in (1.1) and furthermore $\Delta S_m^j \in H^{s-1}(\Omega)$.

Remark 1 (see eg. [5], Prop. 16.8). If we consider instead of Problem (1.1) the Dirichlet problem for the Helmholtz equation

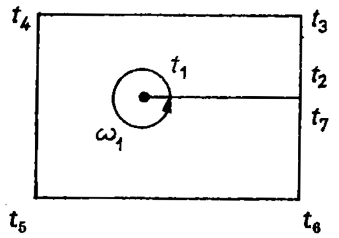
$$(\Delta + \mu)\, u = f \in H^{s-1}(\Omega), \qquad u = 0 \ \text{ on } \partial\Omega$$

with $\mu \in \boldsymbol{R}$, then the solution u is for $s > 0$, $s \neq m\nu_j$ of the form

$$u = \sum_{j=1}^{J} \chi_j \sum_{0<m\nu_j<s} c_m^j \sum_{0\leqq 2p<s-m\nu_j} \alpha_{m,p}\varrho^{2p} S_m^j(\varrho_j, \phi_j) + u_0$$

where $u_0 \in H^{s+1}(\Omega)$, $c_m^j \in \boldsymbol{R}$. The constants $\alpha_{m,p} \in \boldsymbol{R}$ do not depend on f.

Remark 2. The case of a slit can be understood as a degenerated polygon. Here the above theorem yields for $f \in H^{s-1}(\Omega)$, $s > 0$, $s \neq m\nu_j$ with $\nu_1 = \frac{1}{2}$, $\nu_2 = \cdots = \nu_7 = 2$:



$$u = \sum_{j=1}^{J} \chi_j \sum_{0 < m\nu_j < s} c_m^j S_m^j + u_0,$$
$$u_0 \in H^{s+1}(\Omega).$$

Fig. 2

Therefore for $f \in L^2(\Omega)$ and $s = \frac{3}{2} - \varepsilon$ Theorem 1 yields $u \in H^{3/2-\varepsilon}(\Omega)$ satisfying

$$u = c_1^1 \varrho^{1/2} \sin \frac{\phi}{2} \chi_1 + u_0, \qquad u_0 \in H^{5/2-\varepsilon}(\Omega).$$

Remark 3. For $f \in L^2(\Omega)$, the decomposition (2.1) shows $u \in H^{3/2-\varepsilon}(\Omega)$, at least. If Ω is a *convex* polygon then $u \in H^2(\Omega)$.

Next we consider problem (1.1) in the case when Ω is a dihedron, i.e., $\Omega = \boldsymbol{R} \times K$ where K is a plane sector with opening ω. We set $x = (y, z) \in \boldsymbol{R} \times K$ and let (ϱ, ϕ) denote the polar coordinates of z in K. We assume that

(2.4) $$u \in \mathring{H}^1(\Omega), \qquad \Delta u = f \in H^{s-1}(\Omega) \quad \text{with} \quad u = 0 \text{ for } |z| \geq 1.$$

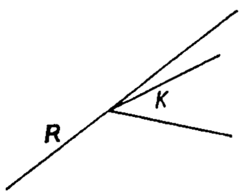


Fig. 3

Theorem 2 ([5], Thm. 16.9). *Let* $s \frac{\omega}{\pi} \notin \boldsymbol{Z}$ *and* $s \neq \frac{1}{2}$. *Then with* (2.4) *the solution* u *of problem* (1.1) *satisfies with* $\nu = \frac{\pi}{\omega}$ *and a cut-off function* $\chi(\varrho) \in C_0^\infty(\boldsymbol{R}^+)$, $\chi(\varrho) \equiv 1$ *for small* ϱ

(2.5) $$u = u_0 + \chi(\varrho) \sum_{0 < m\nu < s} Z_m(c_m)$$

where $u_0 \in H^{s+1}(\Omega)$, $c_m \in H^{s-m\nu}(\boldsymbol{R})$. *Here for* $m\nu \notin \boldsymbol{N}$ *the term* Z_m *is given via* S_m *in* (2.2) *by*

(2.6) $$Z_m(c_m)(y, \varrho, \phi) = \sum_{0 \leq 2p \leq s - m\nu} \left(D_y^{2p} c_m \underset{y}{*} \Phi\right)(y, \varrho)\, \alpha_{m,p} \varrho^{2p} S_m(\varrho, \phi),$$

whereas for $m\nu \in N$

$$(2.7)\qquad Z_m(c_m)(y, \varrho, \phi) = \sum_{0\le 2p\le s-m\nu} \left(D_y^{2p} c_m \underset{y}{*} \Phi\right)(y, \varrho)\, \alpha_{m,p} \varrho^{2p} S_m(\varrho, \phi) + \sum_{0\le 2p\le s-m\nu} \left([D_y^{2p} \operatorname{Op}(\Psi)(c_m)] \underset{y}{*} \Phi\right)(y, \varrho)\, \alpha_{m,p} \varrho^{2p} \tilde{S}_m(\varrho, \phi)$$

where S_m is given by (2.3) *and $\tilde{S}_m$ is given by* (2.2).

Remark 4. The coefficients $\alpha_{m,p} \in \boldsymbol{R}$ are independent of f. The symbol $\underset{y}{*}$ denotes convolution with respect to y. The function $\Phi(y, \varrho)$ is defined as

$$(2.8)\qquad \Phi(y, \varrho) = \frac{1}{\varrho}\, \psi\left(\frac{y}{\varrho}\right)$$

with $\psi \in C_0^\infty(\boldsymbol{R})$ satisfying $\hat{\psi}(0) = 1$. Here $\hat{\psi}(0) = \hat{\psi}(\xi)|_{\xi=0}$ where $\hat{\psi}$ is the FOURIER transform of ψ w.r.t. y. We furthermore require that the FOURIER transform $\hat{\psi}$ is tangential to 1 in 0 of order $[s] + 1$, i.e., $\hat{\psi}(\xi) - 1$ vanishes of order $[s] + 1$ at $\xi = 0$. The construction of such a function ψ is shown in Lemma 1 below. The FOURIER transform of Φ with respect to y is

$$(2.9)\qquad \mathcal{F}_{y\to\xi}\Phi(y, \varrho) = \hat{\psi}(\varrho\, |\xi|).$$

M. DAUGE uses in [5], for technical reasons, instead of $\Phi(y, \varrho)$ in (2.9) a modified function $\tilde{\Phi}(y, \varrho)$ defined by

$$(2.10)\qquad \mathcal{F}_{y\to\xi}\tilde{\Phi}(y, \varrho) = \hat{\psi}(\varrho \sup\{1, |\xi|\}).$$

Since (2.10) modifies Φ only for small $|\xi|$, it can be seen easily that the proofs in our paper (especially Lemma 3 and 4) also hold with $\tilde{\Phi}$ instead of Φ. It can be shown with a proof similar to the one of Lemma 4 below, that the difference between the term $Z_m(c_m)$ defined with Φ and the corresponding term defined with $\tilde{\Phi}$ has regularity H^{s+1}. Therefore, this difference changes only the regular part of the decompositions (2.5), (3.4), (3.9) in Theorems 2, 4, and 5 respectively. Thus we can use for simplicity the function $\Phi(y, \varrho)$ defined by (2.8) in the sequel.

Furthermore Op (Ψ) denotes the pseudodifferential operator with symbol $\Psi(\xi)$ where Ψ is a C^∞-function on $\boldsymbol{R}$ which is equal to $\log|\xi|$ outside a compact neighborhood of zero.

Remark 5. (i) If one replaces $D_y^{2p}c_m * \Phi$ by $D_y^{2p}c_m$ in (2.6), (2.7), then one obtains only $u_0 \in L^2(\boldsymbol{R}, H^{s+1}(K))$ in (2.5) (see [8], [9]).

(ii) For $\varrho \to 0$ the function Φ in (2.8) is a regularizing sequence, i.e., for $c_m \in H^\sigma(\boldsymbol{R})$, $\Phi \underset{y}{*} c_m \in C^\infty(\boldsymbol{R})$ for fixed $\varrho > 0$ and $\Phi \underset{y}{*} c_m \to c_m$ in $H^\sigma(\boldsymbol{R})$ as $\varrho \to 0$.

Now, contrary to (2.5) we derive a tensor product decomposition with a regular part in the isotropic space $H^{s+1}(\Omega)$ for the solution u of (1.1) in the infinite wedge (dihedron) $\Omega = \boldsymbol{R} \times K$.

Theorem 3. *Let $A = \{m\nu + 2p \mid m > 0, p \geqq 0 \text{ integers}\}$ with $\nu = \dfrac{\pi}{\omega}$ and let $s_0 \in A$ and $s_1 = \max\{t \in A \mid t < s_0\}$. Then for $f \in H^{s-1}(\Omega)$ with $s \geqq s_0 + s_1 + 1$ and $s \neq m\nu$ the solution u of problem* (1.1) *with* (2.4) *admits a decomposition*

$$(2.11)\qquad u = v_0 + \chi(\varrho) \sum_{m\nu+2p<s_0} b_{m,p} \varrho^{2p} S_m(\varrho, \phi)$$

with $v_0 \in H^{1+s_0-\varepsilon}(\Omega)$ *for any* $\varepsilon > 0$ *and* $b_{m,p} \in H^{s-m\nu-2p}(\boldsymbol{R})$ *where* S_m *as in* (2.2) *and* (2.3) *and* χ *as in Theorem* 2.

Here s_0 denotes the smallest singular exponent $m\nu + 2p$ *not* occurring in the decomposition (2.11).

The assumptions in the above theorem are natural: If we replace in (2.5) the regularization term $D^{2p}c_m * \Phi$ by $D^{2p}c_m$, we perturb $u_0 \in H^{1+s_0-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$ by an additional term $R_{m,p}(D^{2p}c_m)(y, z)$ (see (2.17)) which belongs to $H^{s-m\nu-2p}(\boldsymbol{R})$ for fixed z due to the given regularity of c_m (see Theorem 2). Also Lemma 4 below shows that for fixed y the term $R_{m,p}(D^{2p}c_m)(y, z)$ belongs to $H^{s-m\nu-2p}(K)$. Therefore, we obtain a regular part

$$v_0 = \sum_{m\nu+2p<s_0} R_{m,p}(D^{2p}c_m) + u_0$$

belonging to the isotropic space $H^{1+s_0-\varepsilon}(\Omega)$ if we choose s large enough such that $s - m\nu - 2p \geqq 1 + s_0 - \varepsilon$ for $m\nu + 2p$ occurring in (2.5)–(2.7). Since

$$s_1 = \max\{m\nu + 2p \mid m \in \boldsymbol{N}, p \in \boldsymbol{N}_0, m\nu + 2p < s_0\}$$

it is sufficient for (2.11) to choose $s > s_0 + s_1 + 1$ as in the hypothesis of the theorem.

For the proof of Theorem 3 we need the following lemmas. First we construct a convolution function ψ such that its Fourier transform $\hat{\psi}$ is tangential to 1 in 0 of sufficiently higher order:

Lemma 1. *For all positive integers N there exists a function $\psi_N \in C_0^\infty(\boldsymbol{R})$ such that*

$$\hat{\psi}_N(0) = 1, \quad \frac{d^k}{d\xi^k}\hat{\psi}_N(\xi)|_{\xi=0} = 0 \quad \textit{for} \quad k = 1, \ldots, N. \tag{2.12}$$

Proof. We use induction. For $N = 0$ define $a_0 := \int_{\boldsymbol{R}} F_0(y)\,dy \neq 0$ with some $F_0 \in C_0^\infty(\boldsymbol{R})$. Then $\psi_0 := a_0^{-1}F_0$ satisfies $\hat{\psi}_0(0) = \int_{\boldsymbol{R}} \psi_0(y)\,dy = 1$.

Now assume (2.12) is proven for some N. Define $\chi_N(x) = \psi_N\left(\frac{1}{2}x\right)$. Then

$$\hat{\chi}_N(\xi) = \int_{x\in\boldsymbol{R}} e^{-ix\xi}\psi_N\left(\frac{x}{2}\right)dx = \int_{y\in\boldsymbol{R}} 2e^{-2iy\xi}\psi_N(y)\,dy = 2\hat{\psi}_N(2\xi)$$

and

$$\frac{d^k}{d\xi^k}\hat{\chi}_N(\xi)|_\xi = 2^{k+1}\frac{d^k}{d\xi^k}\hat{\psi}_N(\xi).$$

Thus

$$\psi_{N+1}(x) := (1 - 2^{-N-1})^{-1}(\psi_N - 2^{-N-2}\chi_N)$$

satisfies the following relations for $k = 1, \ldots, N$ and also for $k = N + 1$.

$$\begin{aligned}\hat{\psi}_{N+1}(0) &= (1 - 2^{-N-1})^{-1}\left(\hat{\psi}_N(0) - 2^{-N-2}2\hat{\psi}_N(0)\right)\\ &= (1 - 2^{-N-1})^{-1}(1 - 2^{-N-1})\,\hat{\psi}_N(0) = 1\end{aligned}$$

$$\frac{d^k}{d\xi^k}\hat{\psi}_{N+1}(\xi)|_\xi = (1 - 2^{-N-1})^{-1}(1 - 2^{-N-2}2^{k+1})\frac{d^k}{d\xi^k}\hat{\psi}_N(\xi)|_{\xi=0} = 0. \quad \square$$

Lemma 2. *Let $\tilde{\Omega}$ be a polygonal domain, $f \in H^s(\tilde{\Omega})$ for $s > 0$. Let $s_1 = -[-s]$ denote the smallest integer greater than or equal to s and $g \in C^{s_1}(\tilde{\Omega})$. Then the norms satisfy*

$$\|fg\|_{H^s(\tilde{\Omega})} \leqq \|f\|_{H^s(\Omega)} \|g\|_{C^{s_1}(\tilde{\Omega})}.$$

Proof. For integer s the lemma is trivial. The other case follows by interpolation. □

The next lemma shows that the singularity terms $Z_m(c_m)$ in (2.6) and (2.7) have the same regularity $H^{1+m\nu-\epsilon}$ as the 2-dimensional singularities $S_m(\varrho, \phi)$ of Theorem 1. For the proof of Theorem 3 we need only the case $\eta = 0$, whereas the case $\eta \neq 0$ will be required later for the singularities on a polyhedral cone.

Lemma 3. *Let $\alpha > 0$ and $g \in C^\infty([0, \omega)]$, $e^{\eta t}c(t) \in L^2(\boldsymbol{R})$ with $\eta \in \boldsymbol{R}$ and ψ be defined as in Lemma 1 with the same integer N. Let $\chi(\varrho) \in C_0^\infty(\boldsymbol{R}^+)$ denote a cut-off function with $\chi(\varrho) \equiv 1$ for small ϱ. Then for $s < 1 + \alpha$ we have*

$$h(t, \varrho, \phi) := \chi(\varrho)\, e^{\eta t} \left[c(t) \underset{t}{*} \frac{1}{\varrho}\, \psi\left(\frac{t}{\varrho}\right)\right] \varrho^\alpha g(\phi) \in H^s(\Omega)$$

and

$$h_1(t, \varrho, \phi) := \chi(\varrho)\, e^{\eta t} \left[c(t) \underset{t}{*} \frac{1}{\varrho}\, \psi\left(\frac{t}{\varrho}\right)\right] \varrho^\alpha \log \varrho g(\phi) \in H^s(\Omega).$$

Here (t, ϱ, ϕ) denote cylindrical coordinates in $\Omega = K \times \boldsymbol{R}$.

Proof. Fourier transformation in t-direction gives

$$\hat{h}(\xi, \varrho, \phi) = \chi(\varrho)\, \hat{c}(\xi + i\eta)\, \Phi(\varrho(\xi + i\eta))\, \varrho^\alpha g(\phi).$$

Here $\hat{c}(\xi + i\eta)$ denotes the Fourier transform of $e^{\eta t}c(t)$. We first show that $h(t, \varrho, \phi) \in L^2(\boldsymbol{R}, H^s(K))$. There holds for the norm

$$\|h(t, \varrho, \phi)\|^2_{L_2(\boldsymbol{R}, H^s(K))} = \int_{\xi \in \boldsymbol{R}} \|\hat{h}(\xi, \varrho, \phi)\|^2_{H^s(K)}\, d\xi = \int_{\xi \in \boldsymbol{R}} |\hat{c}(\xi + i\eta)|^2\, \|b_1(\xi, \varrho, \phi)\|^2_{H^s(K)}\, d\xi$$

with $b_1(\xi, \varrho, \phi) = \chi(\varrho)\, \Phi(\varrho(\xi + i\eta))\, \varrho^\alpha g(\phi)$. If $\eta = 0$ and ξ tends to 0, then $b_1(\xi, \varrho, \phi)$ converges to $\chi(\varrho)\, \varrho^\alpha g(\phi)$ in $H^s(K)$. (This can be seen by Lemma 2.) Hence $\|b_1(\xi, \varrho, \phi)\|^2_{H^s(K)}$ is bounded for $\eta = 0$ and small ξ.

Now we suppose without loss of generality that $\operatorname{supp} \chi \subset [0, 1]$. Since $\Phi \in C^\infty$, we obtain by Lemma 2

$$\|b_1(\xi, \varrho, \phi)\|^2_{H^s(K)} \leqq C\, \|\Phi(\varrho(\xi + i\eta))\, \varrho^\alpha g(\phi)\|_{H^s(K \cap B_1(0))}$$

Here $B_1(0)$ denotes a ball of radius 1 with center 0.

Next, we introduce the change of variables $\tilde{\varrho} := |\xi + i\eta|\, \varrho$ and set $K_1 := K \cap B_{|\xi + i\eta|}(0)$. This yields

$$\begin{aligned}
&b_1\|(\xi, \varrho, \phi)\|^2_{H^s(K)} \\
&\leqq |\xi + i\eta|^{-2}\, |\xi + i\eta|^{2s} \left\|\Phi\left(\tilde{\varrho}\, \frac{\xi + i\eta}{|\xi + i\eta|}\right) \left(\frac{\tilde{\varrho}}{|\xi + i\eta|}\right)^\alpha g(\phi)\right\|^2_{H^s(K_1)} \\
&= |\xi + i\eta|^{2(s-1-\alpha)} \left\|\Phi\left(\tilde{\varrho}\, \frac{\xi + i\eta}{|\xi + i\eta|}\right) \tilde{\varrho}^\alpha g(\phi)\right\|^2_{H^s(K_1)} \\
&\leqq |\xi + i\eta|^{2(s-1-\alpha)} \left\|\Phi\left(\tilde{\varrho}\, \frac{\xi + i\eta}{|\xi + i\eta|}\right) (1 + \tilde{\varrho})^M\right\|^2_{C^{s_1}(K_1)} \|(1 + \tilde{\varrho})^{-M}\, \tilde{\varrho}^\alpha g(\phi)\|^2_{H^s(K)}.
\end{aligned}$$

Here we used Lemma 2 with $s_1 := -[-s]$, and M is a sufficiently large integer such that $\|\tilde{\varrho}^\alpha(1+\tilde{\varrho})^{-M}\|_{H^{s_1}(R^1)}$ is bounded.

Since $e^{\eta t}\psi(t) \in C_0^\infty(\boldsymbol{R})$ we have $\left(e^{\eta t}\psi(t)\right)^\wedge = \phi(\xi + i\eta) \in \mathscr{S}$, the space of rapidly decaying functions, hence there exist $C_{\beta,M}$ such that

$$|D^\beta\phi(\zeta)| \leqq C_{\beta,M}(1 + |\zeta|)^{-M} \quad \text{for } |\operatorname{Im}\zeta| \leqq \eta. \tag{2.13}$$

As $\tilde{\varrho}\,\dfrac{\xi + i\eta}{|\xi + i\eta|} = \varrho(\xi + i\eta)$ with $\varrho \in [0, 1]$ we have $|\operatorname{Im}\varrho(\xi + i\eta)| \leqq \eta$ and therefore

$$\left\|\phi\left(\tilde{\varrho}\,\frac{\xi + i\eta}{|\xi + i\eta|}\right)(1 + \tilde{\varrho})^M\right\|_{C^{s_1}(K_1)} \leqq C \quad \text{for all } \xi \in \boldsymbol{R}.$$

Since $s < 1 + \alpha$ and $\|b_1\|_{H^s(K)}$ is bounded for small ξ (as shown above) we have

$$\|h(t, \varrho, \phi)\|^2_{L^2(\boldsymbol{R},H^s(K))} \leqq C \int\limits_{\xi\in\boldsymbol{R}} |\hat{c}(\xi + i\eta)|^2\, d\xi = C\,\|e^{\eta t}c(t)\|^2_{L^2(\boldsymbol{R})}.$$

Next we show that $h(t, \varrho, \phi) \in H^s\big(\boldsymbol{R}, L^2(K)\big)$. First we have

$$\|h(t, \varrho, \phi)\|^2_{H^s(\boldsymbol{R},L^2(K))} = \int\limits_{\xi\in\boldsymbol{R}} (1 + |\xi|)^{2s}\, |\hat{c}(\xi + i\eta)|^2\, \|b_2(\xi, \varrho, \phi)\|^2_{L^2(K)}\, d\xi$$

with $b_2(\xi, \varrho, \phi) = \phi(\varrho\,|\xi + i\eta|)\,\varrho^\alpha g(\phi)\,\chi(\varrho)$. For $\eta = 0$ and small ξ the term $\|b_2\|_{L^2(K)}$ is bounded.

With the change of variables $\tilde{\varrho} = \varrho\,|\xi + i\eta|$ we get

$$\|b_2\|^2_{L^2(K)} \leqq |\xi + i\eta|^{-2}\left\|\phi\left(\tilde{\varrho}\,\frac{\xi + i\eta}{|\xi + i\eta|}\right)\left(\frac{\tilde{\varrho}}{|\xi + i\eta|}\right)^\alpha g(\phi)\,\chi\left(\frac{\tilde{\varrho}}{|\xi + i\eta|}\right)\right\|^2_{L^2(K)}.$$

By applying the estimates (2.13) as above we find

$$\|b_2\|^2_{L^2(K)} \leqq C\,|\xi + i\eta|^{-2(1+\alpha)}$$

and

$$\|h(t, \varrho, \phi)\|^2_{H^s(\boldsymbol{R},L^2(K))} \leqq C \int\limits_{\xi\in\boldsymbol{R}} (1 + |\xi|)^{2(s-1-\alpha)}\, |\hat{c}(\xi + i\eta)|^2\, d\xi \leqq C\,\|e^{\eta t}c(t)\|^2_{L^2(\boldsymbol{R})}.$$

Since $H^s(\boldsymbol{R} \times K) = L^2\big(\boldsymbol{R}, H^s(K)\big) \cap H^s\big(\boldsymbol{R}, L^2(K)\big)$, the first assertion of the lemma is shown. It only remains to treat the case with $\varrho^\alpha \log \varrho$. Here the change of variables yields

$$\left(\frac{\tilde{\varrho}}{|\xi + i\eta|}\right)^\alpha \log\left(\frac{\tilde{\varrho}}{|\xi + i\eta|}\right) = |\xi + i\eta|^{-\alpha}\tilde{\varrho}^\alpha \log\tilde{\varrho} - |\xi + i\eta|^{-\alpha}\log|\xi + i\eta|\,\tilde{\varrho}^\alpha.$$

Both terms on the right hand side can be treated as above. □

As a consequence of Lemma 3 we have the following regularity result for the singularity terms Z_m in (2.6) and (2.7).

Corollary 1. *For* $c_m \in L^2(\boldsymbol{R})$ *the singularity functions* Z_m *in* (2.6) *and* (2.7) *have the following regularity:*

$$\chi(\varrho)\, Z_m(c_m) \in H^{1+m\nu-\varepsilon}(\Omega).$$

For the proof of Theorem 3 we need furthermore the following Lemma 4 with $\eta = 0$.

Lemma 4. *Let $s > 0$, $\eta \in \boldsymbol{R}$, $e^{\eta t}c(t) \in H^s(\boldsymbol{R})$.*

$$R(c)\,(t, \varrho, \phi) := \left[c(t) \underset{t}{*} \Phi(t, \varrho) - c(t)\right] \varrho^\alpha \chi(\varrho)\, g(\phi) \tag{2.14}$$

with $g \in C^\infty[0, \omega]$ and $\alpha \geqq 0$. Here let $\Phi(t, \varrho) = \dfrac{1}{\varrho}\, \psi_N \left(\dfrac{t}{\varrho}\right)$ with $N > s_1 - \alpha - 2$ with $s_1 = -[-s]$ and ψ_N as defined in Lemma 1. Then there holds

$$\|e^{\eta t} R(c)\|_{H^s(\boldsymbol{R}\times K)} \leqq M\, \|e^{\eta t} c(t)\|_{H^s(\boldsymbol{R})}.$$

Furthermore the assertion also holds with $\varrho^\alpha \log \varrho$ instead of ϱ^α.

Proof. The proof is given in two steps. (i) We first estimate $\left|e^{\eta t} R(c)\right|_{H^s(\boldsymbol{R}, L^2(K))}$. Application of Fourier transformation with respect to t gives

$$\mathcal{F}_{t\to\xi}\left(e^{\eta t} R(c)\right) = \left(\hat\psi_N\left((\xi + i\eta)\, \varrho\right) - 1\right) \hat c(\xi + i\eta)\, \varrho^\alpha \chi(\varrho)\, g(\phi).$$

Since ψ_N has compact support, $\hat\psi_N$ has a holomorphic extension to $\boldsymbol{C}$. Let $\hat c(\xi + i\eta)$ be the Fourier transform of $e^{\eta t}c(t)$. Then

$$\begin{aligned} &\|e^{\eta t} R(c)\|^2_{H^s(\boldsymbol{R}, L^2(K))} \\ &= \int_{\xi\in\boldsymbol{R}} (1 + |\xi|)^{2s}\, |\hat c(\xi + i\eta)|^2 \left\|\left(\hat\psi_N\left((\xi + i\eta)\, \varrho\right) - 1\right) \varrho^\alpha \chi(\varrho)\, g(\phi)\right\|^2_{L^2(K)} d\xi. \end{aligned}$$

We assume without loss of generality that $\operatorname{supp} \chi \subset [0, 1]$. Since ψ_N has compact support, $\hat\psi_N\left((\xi + i\eta)\, \varrho\right)$ is bounded for $|\varrho| < 1$ and for fixed η, i.e.,

$$\begin{aligned} \left|\hat\psi_N\left((\xi + i\eta)\, \varrho\right)\right| &= \left|\int_{t\in\boldsymbol{R}} e^{-i\xi t}\, e^{\eta\varrho t} \psi(t)\, dt\right| \leqq \int_{t\in\boldsymbol{R}} e^{\eta\varrho t}\, |\psi(t)|\, dt \\ &\leqq e^{|\eta| t_*} \int_{t\in\boldsymbol{R}} |\psi(t)|\, dt \leqq C \end{aligned}$$

where $t_* = \sup \{|t| : t \in \operatorname{supp} \psi\}$. This gives

$$\begin{aligned} \|e^{\eta t} R(c)\|^2_{H^s(\boldsymbol{R}, L^2(K))} &\leqq \|\varrho^\alpha \chi(\varrho)\, g(\phi)\|^2_{L^2(K)} \int_{\xi\in\boldsymbol{R}} (1 + |\xi|)^{2s}\, |\hat c(\xi + i\eta)|^2\, d\xi \\ &\leqq M\, \|e^{\eta t} c(t)\|^2_{H^s(\boldsymbol{R})}. \end{aligned}$$

(ii) Next, we estimate $\|e^{\eta t} R(c)\|_{L^2(\boldsymbol{R}, H^s(K))}$. Let $s_1 = -[-s]$ be the smallest integer greater than or equal to s. We denote

$$a(\xi, \varrho, \phi) := \left[\hat\psi_N\left((\xi + i\eta)\, \varrho\right) - 1\right] \varrho^\alpha \chi(\varrho)\, g(\phi)$$

and obtain

$$\|e^{\eta t} R(c)\|^2_{L^2(\boldsymbol{R}, H^{s_1}(K))} = \int_{\xi\in\boldsymbol{R}} |\hat c(\xi + i\eta)|^2\, \|a(\xi, \varrho, \phi)\|^2_{H^{s_1}(K)}\, d\xi$$

$$\|a(\xi, \varrho, \phi)\|^2_{H^{s_1}(K)} \leqq C\, \|a(\xi, \varrho, \phi)\|^2_{L^2(K)} + C\, |a(\xi, \varrho, \phi)|^2_{H^{s_1}(K)}$$

with the seminorm

$$|a(\xi, \varrho, \phi)|^2_{H^{s_1}(K)} = \sum_{|\beta| = s_1} \|D^\beta a(\xi, \varrho, \phi)\|^2_{L^2(K)}.$$

Here D^β denotes a differential operator of order β with respect to the cartesian coordinates in K. Let B_{r_0} denote a ball of radius r_0. Now decompose the norm over K in two parts over $K_1 = K \cap B_{|\xi+i\eta|^{-1}}$ and $K_2 = K \setminus B_{|\xi+i\eta|^{-1}}$.

a) For K_1 we have $\varrho \leqq |\xi + i\eta|^{-1}$. By definition of ψ_N there exists a holomorphic function $\tilde{\psi}(\zeta)$ such that $\phi_N(\zeta) - 1 = \zeta^{N+1}\tilde{\psi}(\zeta)$. If $\zeta = (\xi + i\eta)\,\varrho$ and $|\varrho| \leqq |\xi + i\eta|^{-1}$ then $|\zeta| \leqq 1$ and there exists a constant C_0 such that $|D^\beta\tilde{\psi}(\zeta)| \leqq C_0$ for $|\beta| \leqq s_1$. Then the product rule of differentiation gives

$$D^\beta a(\xi, \varrho, \phi) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1\beta_2} D^{\beta_1}\big[\tilde{\psi}\big((\xi + i\eta)\,\varrho\big] D^{\beta_2}[\varrho^{N+1}\varrho^\alpha\chi(\varrho)\,g(\phi)]\,(\xi + i\eta)^{N+1}$$

$$|D^\beta a(\xi, \varrho, \phi)| \leqq \sum_{\beta_1+\beta_2=\beta} C_{\beta_1\beta_2}\,|\xi + i\eta|^{|\beta_1|}\,D^{\beta_1}\tilde{\psi}\big((\xi + i\eta)\,\varrho\big)\,C\varrho^{N+1+\alpha-|\beta_2|}\,|\xi + i\eta|^{N+1}.$$

Here we used that χ has bounded derivatives. Then

$$|a(\xi, \varrho, \phi)|^2_{H^{s_1}(K_1)} \leqq C \sum_{|\beta|=s_1}\ \sum_{\beta_1+\beta_2=\beta} |\xi + i\eta|^{2(|\beta_1|+N+1)} \int_{\varrho=0}^{\min(1,|\xi+i\eta|)^{-1})} \varrho^{2(N+1+\alpha-|\beta_2|)}\,\varrho\,d\varrho.$$

Here the integral exists if $2N + 2\alpha - 2\,|\beta_2| > -4$, which holds for $N > s_1 - \alpha - 2$. Therefore we have

$$|a(\xi, \varrho, \phi)|^2_{H^{s_1}(K_1)} \leqq C\,|\xi + i\eta|^{2(|\beta_1|+N+1)}\,(|\xi + i\eta|^{-1})^{2N+2\alpha-2|\beta_2|+4}$$
$$\leqq C\,|\xi + i\eta|^{2(|\beta_1|+|\beta_2|-\alpha-1)} = C\,|\xi + i\eta|^{2(s_1-\alpha-1)}.$$

b) For K_2 we have $\varrho > |\xi + i\eta|^{-1}$: Since $\psi_N(t) \in C_0^\infty$ the following estimates hold:

$$\left|\left(\frac{d}{d\zeta}\right)^{\beta_1} \phi_N(\zeta)\right| \leqq \tilde{C}_{\beta_1,M}\,|\zeta|^{-M} \quad \text{for}\ \ |\zeta| > 1,\ |\mathrm{Im}\,\zeta| \leqq \eta.$$

This gives with $\zeta = (\xi + i\eta)\,\varrho$ and $M = \beta_1$

$$\left|\left(\frac{d}{d\varrho}\right)^{\beta_1} \phi_N\big((\xi + i\eta)\varrho\big)\right| \leqq |\xi + i\eta|^{\beta_1}\,\tilde{C}_{\beta_1,M}\,|\xi + i\eta|^{-M}\varrho^{-M} \leqq \tilde{C}_{\beta_1,\beta_1}\varrho^{-\beta_1}.$$

Note that the same estimate holds for $\tilde{\psi}_N\big((\xi + i\eta)\,\varrho\big) - 1$ instead of $\phi_N\big((\xi + i\eta)\,\varrho\big)$. Then for $|\beta| = s_1$

$$|D^\beta a| \leqq \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\,\big|D^{\beta_1}\big[\phi_N\big((\xi + i\eta)\,\varrho\big) - 1\big]\,D^{\beta_2}[\varrho^\alpha\chi(\varrho)\,g(\phi)]\big|$$
$$\leqq \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\tilde{C}_{\beta_1,\beta_1}\varrho^{-|\beta_1|}\varrho^{\alpha-|\beta_2|} \leqq C\varrho^{\alpha-s_1}.$$

Note that $a(\xi, \varrho, \phi) \equiv 0$ for $|\xi + i\eta| \leqq 1$ and for $\varrho > |\xi + i\eta|^{-1}$ since $\chi(\varrho) \equiv 0$ for $\varrho > 1$. For $|\xi + i\eta| > 1$ we obtain

$$|a(\xi, \varrho, \phi)|^2_{H^{s_1}(K_2)} \leqq C \int_{|\xi+i\eta|^{-1}}^{1} \varrho^{2(\alpha-s_1)}\,\varrho\,d\varrho \leqq C(1 + |\xi + i\eta|^{2(s_1-\alpha-1)}).$$

In part (i) it is shown that $\|a(\xi, \varrho, \phi)\|_{L^2(K)} \leqq C$. Hence we get

$$\|a(\xi, \varrho, \phi)\|^2_{H^{s_1}(K)} \leqq C\,\|a(\xi, \varrho, \phi)\|^2_{L^2(K)} + C\,|a(\xi,\varrho,\phi)|^2_{H^{s_1}(K_1)} + C\,|a(\xi,\varrho,\phi)|^2_{H^{s_1}(K_2)}$$
$$\leqq C(1 + |\xi|)^{2\max(0,s_1-\alpha-1)}.$$

This yields with $\sigma = \max\{0, s_1 - \alpha - 1\}$

$$\|e^{\eta t} R(c)\|^2_{L^2(\boldsymbol{R}, H^{s_1}(K))} \leqq \int_{\boldsymbol{R}} |\hat{c}(\xi + i\eta)|^2 \, \|a(\xi, \varrho, \phi)\|^2_{H^{s_1}(K)} \, d\xi \leqq C \, \|e^{\eta t} c(t)\|^2_{H^\sigma(\boldsymbol{R})}. \tag{2.15}$$

Since $s_1 > s$ and $s_1 - \alpha - 1 < s$ the latter estimate implies

$$\|e^{\eta t} R(c)\|_{L^2(\boldsymbol{R}, H^s(K))} \leqq C \, \|e^{\eta t} c(t)\|_{H^s(\boldsymbol{R})}.$$

Finally a combination of parts (i) and (ii) completes the proof since

$$H^s(\boldsymbol{R} \times K) = H^s(\boldsymbol{R}, L^2(K)) \cap L^2(\boldsymbol{R}, H^s(K)).$$

We note that the remaining case with $\varrho^\alpha \log \varrho$ instead of ϱ^α is proved analogously. □

Proof of Theorem 3. We first use Theorem 2 for $f \in H^{s-1}(\Omega)$ yielding

$$u = u_0 + \chi(\varrho) \sum_{0 < m\nu < s} Z_m(c_m), \qquad u_0 \in H^{s+1}(\Omega), \qquad c_m \in H^{s-m\nu}(\boldsymbol{R}).$$

Furthermore, by Corollary 1, we have $Z_m(c_m) \in H^{1+m\nu-\varepsilon}(\Omega) \subset H^{1+s_0-\varepsilon}(\Omega)$ for $s_0 \leqq m\nu < s$. Hence

$$u = u_0^* + \chi(\varrho) \sum_{0 < m\nu < s_0} Z_m(c_m) \text{ with } u_0^* \in H^{1+s_0-\varepsilon}(\Omega), \quad c_m \in H^{s-m\nu}(\boldsymbol{R}). \tag{2.16}$$

Next, we assume that Z_m is of the form (2.6). We want to replace the terms $(D_y^{2p} c_m * \Phi)$ by $D_y^{2p} c_m$ and write therefore

$$\chi(\varrho) \, Z_m(c_m) = \sum_{0 \leqq 2p \leqq s - m\nu} \alpha_{m,p} [(D_y^{2p} c_m) \, \varrho^{2p} S_m(\varrho, \phi) \, \chi(\varrho) + R_{m,p}(D_y^{2p} c_m)] \tag{2.17}$$

with

$$R_{m,p}(c) \, (y, \varrho, \phi) := \left(c(y) \underset{y}{*} \Phi(y, \varrho) - c(y)\right) \varrho^{2p} S_m(\varrho, \phi) \, \chi(\varrho). \tag{2.18}$$

The terms in (2.17) with $s_0 - m\nu < 2p \leqq s - m\nu$ are elements of $H^{1+s_0-\varepsilon}(\Omega)$ by Lemma 3, denote the sum of those terms by u_1^*.

So we have only to consider the remaining terms in (2.17) with $0 \leqq 2p \leqq s_0 - m\nu$. We have $D_y^{2p} c_m \in H^{s-m\nu-2p}(\boldsymbol{R}) \subset H^{s-s_1}(\boldsymbol{R})$ since by definitions $s_1 = \max\{m\nu + 2p \mid m > 0,\ p \geqq 0 \text{ integer},\ m\nu + 2p < s_0\}$. If $s \geqq s_0 + s_1 + 1$, then $s - s_1 > s_0 + 1 - \varepsilon$ for all $\varepsilon > 0$ and hence $D_y^{2p} c_m \in H^{1+s_0-\varepsilon}(\boldsymbol{R})$. Therefore Lemma 4 implies that $R_{m,p}(D_y^{2p} c_m) \in H^{1+s_0-\varepsilon}(\Omega)$ and

$$v_0 := u_0^* + u_1^* + \sum_{0 \leqq 2p \leqq s_0 - m\nu} \alpha_{m,p} R_{m,p}(D_y^{2p} c_m) \in H^{1+s_0-\varepsilon}(\Omega) \tag{2.19}$$

$$b_{m,p} := \alpha_{m,p} D_y^{2p} c_m \in H^{s-m\nu-2p}(\boldsymbol{R}). \tag{2.20}$$

Thus (2.16) becomes with (2.17), (2.19), (2.20)

$$u = v_0 + \chi(\varrho) \sum_{m\nu + 2p < s_0} b_{m,p} \varrho^{2p} S_m(\varrho, \phi).$$

It remains to treat the case $m\nu \in \boldsymbol{N}$ where Z_m is defined by (2.7). Here additional terms with $\mathrm{Op}(\hat{\Psi})(c_m)$ instead of c_m occur. Due to the definition of $\mathrm{Op}(\hat{\Psi})$, $c_m \in H^{s-m\nu}(\boldsymbol{R})$ implies $\mathrm{Op}(\hat{\Psi})(c_m) \in H^{s-m\nu-\tilde{\varepsilon}}(\boldsymbol{R})$ for all $\tilde{\varepsilon} > 0$ and

$$b := D_y^{2p} \, \mathrm{Op}(\hat{\Psi})(c_m) \in H^{s-m\nu-2p-\tilde{\varepsilon}}(\boldsymbol{R}) \subset H^{1+s_0-\varepsilon}(\boldsymbol{R})$$

by choosing $\tilde{\varepsilon} < \varepsilon$. We have

$$(2.21)\qquad \left(b(y) \underset{y}{*} \Phi(y, \varrho)\right) \varrho^{2p} \tilde{S}_m(\varrho, \phi)\, \chi(\varrho) = b(y)\, \varrho^{2p} \tilde{S}_m(\varrho, \phi)\, \chi(\varrho) + R_{m,p}(b).$$

Now $\varrho^{2p}\tilde{S}_m(\varrho, \phi) \in C^\infty(K)$ gives $\varrho^{2p}\tilde{S}_m(\varrho, \phi)\, \chi(\varrho) \in H^{1+s_0-\varepsilon}(\boldsymbol{R} \times K)$. The second term $R_{m,p}(b)$ on the right hand side in (2.21) is also in $H^{1+s_0-\varepsilon}(\boldsymbol{R} \times K)$ by Lemma 3. Hence the additional terms in (2.7) can be included in the regular part v_0 and the proof of Theorem 3 is complete. □

3. Decomposition Theorems for a Polyhedron

Next we consider problem (1.1) in a polyhedron with plane faces. Here we encounter besides singularities near the edges also singularities in the corners. In order to describe those corner singularities we introduce local spherical polar coordinates (r, ϕ, θ) as in Fig. 4.

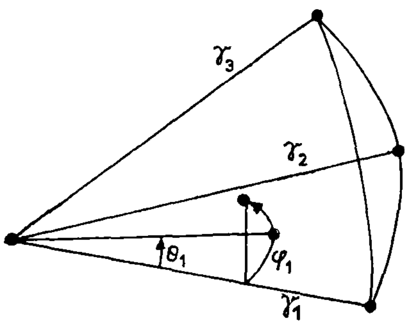


Fig. 4

As it is well known and can be easily seen by separation of variables those corner singularities are determined by the eigenvalues μ_l^j and eigenfunctions $v_l^j(\theta, \phi)$ of the Dirichlet problem for the Laplace-Beltrami operator $\Delta_{\theta,\phi}$ on the spherical cross-section S_j of Ω with a small sphere of radius r_0 centered at the respective vertex t_j. With

$$(3.1)\qquad \begin{cases} \Delta_{\theta,\phi} v_l^j := \dfrac{1}{\sin\theta} \dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial}{\partial\theta} v_l^j\right) + \dfrac{1}{\sin^2\theta} \dfrac{\partial^2}{\partial\phi^2} v_l^j = \mu_l^j v_l^j \text{ in } S_j \\[2ex] \lambda_l^j = -\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + \mu_l^j}, \qquad \mu_l^j > 0, \end{cases}$$

the corner singularities near t_j become

$$(3.2)\qquad c_l^j r^{\lambda_l^j} v_l^j(\theta, \phi) \quad \text{with} \quad c_l^j \in \boldsymbol{C}.$$

First we consider Ω being a polyhedral cone in $\boldsymbol{R}^3$ with spherical polar coordinates $r, \theta_j, \phi_j)$ $(j = 1, \ldots, J)$ (J = number of edges) with (θ_j, ϕ_j) oriented w.r.t. the edge γ_j with opening ω_j. Now we assume

$$(3.3)\qquad u \in \mathring{H}^1(\Omega), \qquad \Delta u = f \in H^{s-1}(\Omega) \quad \text{with supp } u \text{ compact}.$$

Then there holds the following theorem by M. Dauge ([5], Theorem 17.13).

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Theorem 4. *Let u satisfy (3.3) with $f \in H^{s-1}(\Omega)$, $s \in \boldsymbol{R}^+$, $s \neq \frac{1}{2}$, $s \neq m\nu_j$ $\left(\nu_j = \frac{\pi}{\omega_j}\right)$, $s \neq \lambda_k + \frac{1}{2}$ and λ_k as in (3.1) or λ_k an integer greater than 1. Then u satisfying (3.3) admits with suitable C^∞ cut-off functions χ, χ_j concentrated near the vertex and the edges, respectively, the decomposition*

$$u = u_0 + \chi(r) \sum_{0<\lambda_k<s-(1/2)} a_k r^{\lambda_k} v_k(\theta_1, \phi_1) + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j<s} \tilde{Z}^j_m(\tilde{c}^j_m)(r, \theta_j, \phi_j) \tag{3.4}$$

with

$$u_0 \in H^{s+1}(\Omega), \quad a_k \in \boldsymbol{R}, \quad \tilde{c}^j_m \in H^{s-m\nu_j}_{-m\nu_j}(\boldsymbol{R}^+), \quad \chi_j \in C_0^\infty[0, 2\pi]$$

supp $\tilde{c}^j_m$ *compact and $v_k \in H^1(S_0)$ as in (3.1). In case λ_k being an integer there are additional corner terms:*

$$\sum_{l=1}^{L(k)} \sum_{q=0}^{1} r^{\lambda_k} \log^q r \tilde{v}_{k,l,q}(\theta_1, \phi_1) \quad \textit{with} \quad \tilde{v}_{k,l,q} \in \mathring{H}^1(S_0). \tag{3.5}$$

Here S_0 is the intersection of Ω with a sphere centered at the vertex, and $L(k)$ is an integer depending on k.

Here with $t = \log r$ the term $\tilde{Z}^j_m(\tilde{c}^j_m)(r, \theta_j, \phi_j)$ is defined by

$$\tilde{Z}_m(\tilde{c}_m)(e^t, \theta_j, \phi_j) = \sum_{0\leqq 2p \leqq s-m\nu_j} c^{(2p)}_m(t) \underset{t}{*} \Phi(t, \theta_j)\, \alpha_{m,p} \theta_j^{2p} S^j_m(\theta_j, \phi_j) \tag{3.6}$$

where

$$c^{(2p)}_m(t) := D_t^{2p} \tilde{c}_m(e^t) \tag{3.7}$$

and $S^j_m(\theta_j, \phi_j)$ is given by (2.2), (2.3) with θ_j instead of ϱ. Here Φ is as in (2.8) with t instead of y and θ_j instead of ϱ. For $m\nu_j \in \boldsymbol{N}$ there are additional terms in (3.6), namely with S^j_m from (2.2) instead of (2.3)

$$\sum_{0\leqq 2p\leqq s-m\nu_j} \left[\operatorname{Op}\left(\hat{\Psi}(\xi)\, \chi(t-t')\right) c^{(2p)}_m(t')\right] \underset{t}{*} \Phi(t, \theta_j)\, \alpha_{m,p} \theta_j^{2p} S^j_m(t_j, \phi_j). \tag{3.8}$$

Here $\operatorname{Op}\left(\hat{\Psi}(\xi)\,\chi(t-t')\right)$ denotes the pseudodifferential operator with symbol $\hat{\Psi}(\xi)\,\chi(t-t')$. The function $\hat{\Psi}(\xi)$ is defined as in Theorem 2, and $\chi \in C_0^\infty(\boldsymbol{R})$ is a cut-off function with $0 \in \operatorname{supp} \chi$. Observe that with this definition supp $\tilde{c}_m$ compact implies supp $\tilde{Z}_m(\tilde{c}_m)$ compact.

The weighted Sobolev spaces $H^s_\gamma(C)$ on a cone C (e.g. $\boldsymbol{R}^+$ or Ω) appearing in Theorem 4 are defined as follows: *$f \in H^s_0(C)$ if and only if for all multi-indices β with* $|\beta| \leqq s$, $r^{|\beta|-s} D^\beta f \in L^2(C)$ and $D^\beta f \in H^\sigma(C)$ for $|\beta| = [s]$ if $\sigma := s - [s] > 0$. For $\gamma \in \boldsymbol{R}$ define $H^s_\gamma(C)$ as the space of functions f such that $r^\gamma f \in H^s_0(C)$. Here r denotes the distance to the vertex of C.

Note that for $\gamma \leqq 0$ there holds $H^s_\gamma(C) \subset H^s_{\mathrm{loc}}(C)$.

Remark 6. From $\tilde{c}^j_m \in H^{s-m\nu_j}_{-m\nu_j}(\boldsymbol{R}^+)$ follows $r^{-s}\tilde{c}^j_m \in L^2(\boldsymbol{R}^+)$.

Remark 7. Let Ω be a polyhedron and define

$$\sigma := \min\left\{\nu_j, \lambda_1^{(k)} + \frac{1}{2} : j \in \{\text{edges}\},\ k \in \{\text{vertices}\}\right\}.$$

Now we apply Theorem 4 for $s = \sigma - \varepsilon$: For $f \in H^{-1+\sigma}(\Omega)$ we get $u \in H^{1+\sigma-\varepsilon}(\Omega)$ and u is not smoother in general, even for $f \in C^\infty(\Omega)$. This implies that the rate of convergence of a finite element method with uniform mesh is $\sigma - \varepsilon$ when polynomials of sufficiently high degree are used.

In most cases the edge singularities are stronger than the vertex singularities, i.e., $\nu_{j_0} < \lambda_1 + \frac{1}{2}$. Example 3 below describes a case where $\lambda_1 + \frac{1}{2} < \nu_j$. For polyhedra without cracks one always has $\sigma > \frac{1}{2}$, and for *convex* polyhedra there holds $\sigma > 1$.

The eigenfunctions v_k of the Laplace-Beltrami operator also possess a decomposition into a regular part and singular terms like $S^j_m(\theta_j, \phi_j)$ located at the corners of the cross-section S_0. This is expressed in the following theorem.

Theorem 5 ([5], (17.14), (17.17)). *Under the assumptions of Theorem 4 there holds*

$$u = u_0^* + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j < s} \tilde{Z}^j_m(\tilde{d}^j_m)(r, \theta_j, \phi_j) \tag{3.9}$$

with $u_0^* \in H^{s+1}_s(\Omega)$ *and*

$$\tilde{d}^j_m = \tilde{c}^j_m + \chi(r) \sum_{\lambda_k < s-(1/2)} a^j_{m,k} r^{\lambda_k},$$

with $a^j_{k,m} \in \mathbf{R}$ *and* $\tilde{c}^j_m$, χ_j, χ *as in* (3.4). *For* λ_k *an integer there is an additional term* $\tilde{a}^j_{k,m} r^{\lambda_k} \log r$ *in* $\tilde{d}^j_m$.

Remark 8. For a general polyhedron Ω, we note that away from corners and edges, the solution u of the Dirichlet problem (1.1) is in $H^{s+1}(\tilde{\Omega})$ for $\Delta u \in H^{s-1}(\Omega)$ where $\tilde{\Omega}$ is compact subdomain of Ω. Standard arguments show that it is sufficient to investigate separately the behavior of the solution near an edge as in Theorem 2 and in a corner as in Theorem 4. Thus the decomposition of u for an arbitrary polyhedron is obtained by patching together the local expansions (see [5], Prop. 17.12).

As in the case of a dihedron (Theorem 3) we can also modify the decomposition (3.4) for the solution u of (1.1) in a polyhedral cone. We obtain a decomposition with singularity terms in tensor product form. (For notation compare with Theorem 4.)

Theorem 6. *Let* $\nu_j = \frac{\pi}{\omega_j}$, $A = \{m\nu_j + 2p \mid m > 0, p \geqq 0 \text{ integers}\}$, $B = \left\{\frac{1}{2} + \lambda_k : k \in \mathbf{N}\right\}$ *with* λ_k *as in* (3.1) *or* $\lambda_k \geqq 2$ *integer. Let* $s_0 \in A \cup B$, $s_1 = \max\{t \in A \mid t < s_0\}$ *and let* $f \in H^{-1+s}(\Omega)$ *with* $s \geqq s_0 + s_1 + 1$ *and* $s \neq m\nu_j$, $s \neq \lambda_k + \frac{1}{2}$, $s \neq \frac{1}{2}$. *Then with* (3.3) *the solution* $u \in \mathring{H}^1(\Omega)$ *of problem* (1.1) *satisfies*

$$u = v_0 + \chi(r) \sum_{\lambda_k + (1/2) < s_0} a_k r^{\lambda_k} v_k(\theta_1, \phi_1) \tag{3.10}$$

$$+ \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j + 2p < s_0} \tilde{b}^j_{m,p}(r)\, \theta_j^{2p} S_m(\theta_j, \phi_j)$$

with $v_0 \in H^{1+s_0-\varepsilon}(\Omega)$, $\varepsilon > 0$ *arbitrary*,

$$
\begin{aligned}
&\tilde{b}^j_{m,p} = \alpha_{m,p}\tilde{c}^{j,(2p)}_m + \sum_{s_0 \leqq \lambda_k + (1/2) < s} c^j_{m,p,k} r^{\lambda_k} \chi(r) \quad \text{with} \\
&\tilde{c}^{j,(2p)}_m \in H^{s-m\nu_j-2p}_{-m\nu_j-2p}(\mathbf{R}^+),\ c^j_{m,p,k} \in \mathbf{R},
\end{aligned}
\tag{3.11}
$$

$a_k \in \mathbf{R}$ *and* χ, χ_j, S_m *as in Theorem* 4. *For* λ_k *an integer, there are additional terms in* (3.10) *of the form* (3.5) *and there are terms of the form* $r^{\lambda_k} \log r$ *in* (3.11). *If* $Q := \{t \in A \mid t < s_0\} = \emptyset$ *and* $f \in H^{-1+s}(\Omega)$ *with* $s \geqq s_0$ *then* (3.10) *holds without edge singularities.*

Remark 9. (i) We note that in the case $Q = \emptyset$ the decompositions (3.4) and (3.10) coincide due to the absence of edge singularities (compare also Example 3).

(ii) From (3.11) follows $\tilde{b}^j_{m,p} \in H^{s_4}(\mathbf{R}^+)$ with

$$
s_4 = \min\left\{s - m\nu_j - 2p,\ \lambda_k + \frac{1}{2} - \varepsilon \mid \lambda_k \text{ with } s_0 \leqq \lambda_k + \frac{1}{2} < s\right\}
$$

and $\tilde{b}^j_{m,p} \in H^{s-m\nu_j-2p}_\gamma(\mathbf{R}^+)$ with $\gamma > -m\nu_j - 2p + s - \lambda_k - \dfrac{1}{2}$.

The proof of Theorem 6 is similar to the proof of Theorem 3. We have to consider the difference of the convolution singularities in (3.6) and the tensor product singularities

$$
\tilde{R}^j_{m,p}(\tilde{c}^{(2p)}_m)(e^t, \theta_j, \phi) := \left(\tilde{c}^{(2p)}_m(e^t) \underset{t}{*} \Phi(t, \theta_j) - \tilde{c}^{(2p)}_m(e^t)\right) \theta_j^{2p} S_m(\theta_j, \Phi)\, \chi_j(\theta_j).
\tag{3.12}
$$

The regularity of these terms determines the regularity of v_0 in (3.10). We use the following lemma (see [5], Thm. AA.3):

Lemma 5. *Let* $(r, \underset{\sim}{\theta})$ *be polar coordinates on the* n_1*-dimensional cone* Ω_1 *with spherical cross section* S_1 *and let* $\tilde{g}(r, \underset{\sim}{\theta})$ *be a function on* Ω_1. *Denote by* $g(t, \underset{\sim}{\theta})$ *the function on the cylinder* $\mathbf{R} \times S_1$ *which is defined by* $g(t, \underset{\sim}{\theta}) := \tilde{g}(e^t, \underset{\sim}{\theta})$. *Then with* $\eta = \gamma - s + \dfrac{n_1}{2}$ *there holds*

$$
\tilde{g} \in H^s_\gamma(\Omega_1) \text{ if and only if } e^{\eta t} g \in H^s(\mathbf{R} \times S_1).
$$

Now we prove the analogue of Corollary 1 for the polyhedron:

Corollary 2. *For* $\tilde{c}_m \in H^0_{-m\nu}(\mathbf{R}^+)$, supp $\tilde{c}_m$ compact, *the singularity functions* $\tilde{Z}_m$ *in* (3.6) *and* (3.8) *have the following regularity:*

$$
\tilde{Z}_m(\tilde{c}_m) \in H^{1+m\nu-\varepsilon}(\Omega).
$$

Proof. Lemma 5 with $n_1 = 1$, $\Omega_1 = \mathbf{R}^+$, $s = 0$ and $\tilde{c}_m \in H^0_{-m\nu}(\mathbf{R}^+)$ implies $e^{\eta t} c_m(t) \in L^2(\mathbf{R})$ with $\eta = -m\nu - 0 + \dfrac{1}{2}$ and $\tilde{c}_m(e^t) = c_m(t)$.

Lemma 3 gives

$$
e^{\eta t}\chi(\theta)\left(c(t) \underset{t}{*} \frac{1}{\theta}\psi\left(\frac{t}{\theta}\right)\right) S_m(\theta, \phi) \in H^{1+m\nu-\varepsilon}(\mathbf{R} \times S_0).
$$

Then we obtain with Lemma 5 for $n_1 = 3$, and $\Omega_1 = \Omega$ that

$$\chi(\theta)\left(c(t) \underset{t}{*} \frac{1}{\theta}\psi\left(\frac{t}{\theta}\right)\right)\Bigg|_{t=\log r} S_m(\theta, \phi) \in H_\gamma^{1+m\nu-\varepsilon}(\Omega)$$

with $\gamma = \eta + 1 + m\nu - \varepsilon - \dfrac{3}{2} = -\varepsilon$. Finally, this implies $\tilde{Z}_m(\tilde{c}_m) \in H^{1+m\nu-\varepsilon}$ since supp $\tilde{Z}_m(c_m)$ compact. □

Proof of Theorem 6. We apply Theorem 4 for $f \in H^{s-1}(\Omega)$ yielding

$$u = u_0 + \chi(r) \sum_{0<\lambda_k<s-(1/2)} a_k r^{\lambda_k} v_k(\theta_1, \phi_1) + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j<s} \tilde{Z}_m^j(\tilde{c}_m^j)(r, \theta_j, \phi_j) \tag{3.13}$$

with $u_0 \in H^{1+s}(\Omega)$, $\tilde{c}_m^j \in H_{-m\nu_j}^{s-m\nu_j}(\boldsymbol{R}^+)$, supp $\tilde{c}_m^j$ compact. First we assume that (3.13) contains no terms of the form (3.8). We will treat these terms at the end of the proof with the help of Lemma 6.

Since the functions $v_k(\theta, \phi)$ are eigenfunctions of the Laplace-Beltrami operator on the spherical polygon $S_0 := S \cap \Omega$ (S denotes a small sphere) there holds a decomposition as in Remark 1 with suitable s_2 to be chosen later.

$$v_k(\theta_1, \phi_1) = \chi_j(\theta_j) \sum_{j=1}^{J} \sum_{m\nu_j+2p<s_2} c_{m,p,k}^j \theta_j^{2p} S_m^j(\theta_j, \phi_j) + w_k(\theta_1, \phi_1), \tag{3.14}$$

$w_k \in H^{1+s_2}(S_0)$. We use decomposition (3.14) for all v_k in (3.13) with $s_0 \leqq \lambda_k + \dfrac{1}{2} < s$, and take $s_2 := s_0 - \varepsilon$. Then $w_k \in H^{1+s_0-\varepsilon}(S_0)$.

Next, we claim that $\chi(r)\, r^{\lambda_k} w_k \in H^{1+s_0-\varepsilon}(\Omega)$. Here we apply Lemma 5 with $\tilde{g}(r, \theta, \phi) := r^{\lambda_k} w_k(\theta, \phi)\, \chi(r)$. Then $g(t, \theta, \phi) = \tilde{g}(e^t, \theta, \phi) = e^{\lambda_k t} w_k(\theta, \phi)\, \chi(e^t)$ and $e^{\eta t} g(t, \theta, \phi) \in H^{1+s_0-\varepsilon}(\boldsymbol{R} \times S_0)$ for $\eta = -\lambda_k + \tilde{\varepsilon}$, $0 < \tilde{\varepsilon} < \varepsilon$. We get $\tilde{g} \in H_\gamma^{1+s_0-\varepsilon}(\Omega)$ with $\gamma = \eta + 1 + s_0 - \varepsilon - \dfrac{3}{2} = -\lambda_k + \tilde{\varepsilon} - \dfrac{1}{2} + s_0 - \varepsilon < 0$ since $\lambda_k + \dfrac{1}{2} \geqq s_0$. Thus (3.13) becomes

$$\begin{aligned} u = {} & \tilde{u}_0 + \chi(r) \sum_{0<\lambda_k<s_0-(1/2)} a_k r^{\lambda_k} v_k(\theta_1, \phi_1) \\ & + \sum_{\{k \mid s_0 \leqq \lambda_k+(1/2)<s\}} \chi(r) \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j+2p<s_0} c_{m,p,k}^j \theta_j^{2p} S_m^j(\theta_j, \phi_j)\, r^{\lambda_k} \\ & + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j<s} \tilde{Z}_m^j(\tilde{c}_m^j)(r, \theta_j, \phi_j) \end{aligned} \tag{3.15}$$

with $\tilde{u}_0 = u_0 + \chi(r) \sum_{s_0 \leqq \lambda_k+(1/2)<s} r^{\lambda_k} w_k \in H^{1+s_0-\varepsilon}(\Omega)$. Furthermore Corollary 2 yields

$$\chi_j(\theta_j)\, \tilde{Z}_m^j(\tilde{c}_m^j) \in H^{1+s_0-\varepsilon}(\Omega) \quad \text{for} \quad s_0 \leqq m\nu_j < s.$$

Then we can rewrite (3.15) as

$$\begin{aligned} u = {} & \tilde{\tilde{u}}_0 + \chi(r) \sum_{0<\lambda_k<s_0-(1/2)} a_k r^{\lambda_k} v_k(\theta, \phi) \\ & + \sum_{s_0 \leqq \lambda_k+(1/2)<s} \chi(r) \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j+2p<s_0} c_{m,p,k}^j r^{\lambda_k} \theta_j^{2p} S_m^j(\theta, \phi_j) \\ & + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j<s_0} \tilde{Z}_m^j(\tilde{c}_m^j)(r, \theta_j, \phi_j) \end{aligned} \tag{3.16}$$

with $\tilde{\tilde{u}}_0 = \tilde{u}_0 + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{s_0 \leqq m\nu_j < s} \tilde{Z}_m^j(\tilde{c}_m^j)(r, \theta_j, \phi_j) \in H^{1+s_0-\varepsilon}(\Omega)$. Now we write (see (3.6) and (3.12)) suppressing the upper index j

$$(3.17) \qquad \tilde{Z}_m(\tilde{c}_m)(e^t, \theta_j, \phi_j)\,\chi_j(\theta_j) = \sum_{0 \leqq 2p \leqq s - m\nu_j} \left(c_m^{(2p)}(t)\,\alpha_{m,p}\theta_j^{2p} S_m(\theta_j, \phi_j)\,\chi_j(\theta_j) + \alpha_{m,p}\tilde{R}_{m,p}(\tilde{c}_m)\right).$$

Next, we show with Lemma 5 that $\tilde{R}_{m,p}(\tilde{c}_m) \in H^{1+s_0-\varepsilon}(\Omega)$. First, we observe that the regularity of a function $h(\theta_j, \phi_j)\,\chi_j(\theta_j)$ near a corner of the spherical polygon S_0 with angle ω_j corresponds to the regularity of $h(\varrho, \phi)\,\chi(\varrho)$ on an infinite angular sector with angle ω_j and polar coordinates (ϱ, ϕ). Then Lemma 4 implies that $e^{\eta t}R_{m,p}(c_m^{(2p)}) \in H^{\eta_0}(\boldsymbol{R} \times S_0)$ if $e^{\eta t}c_m^{(2p)} \in H^{\eta_0}(\boldsymbol{R})$ with $R_{m,p}$ as in (2.18). (Note the definition of $c_m^{(2p)}$ in (3.7).) By (3.13) there holds $\tilde{c}_m \in H^{s-m\nu_j}_{-m\nu_j}(\boldsymbol{R}^+)$, thus $c_m(t) = \tilde{c}_m(e^t)$ satisfies $e^{\eta t}c_m(t) \in H^{s-m\nu_j}(\boldsymbol{R})$ with $\eta = s - \frac{1}{2}$ by Lemma 5. Hence $e^{\eta t}c_m^{(2p)} \in H^{s-m\nu_j-2p}(\boldsymbol{R})$ with $\eta = \frac{1}{2} - s$.

Thus Lemma 4 gives $e^{\eta t}R_{m,p}(c_m^{(2p)}) \in H^{s-m\nu_j-2p}(\boldsymbol{R} \times S_0)$ and therefore Lemma 5 shows that $\tilde{R}_{m,p}(\tilde{c}_m) \in H^{s-m\nu_j-2p}_{\gamma}(\Omega)$ where

$$\gamma = \eta + (s - m\nu_j - 2p) - \frac{3}{2} = -m\nu_j - 1 - 2p < 0.$$

By the hypotheses of Theorem 6 there holds $m\nu_j + 2p \leqq s_1$ and $s \geqq 1 + s_0 + s_1$, implying $s - m\nu_j - 2p \geqq 1 + s_0$ and $\tilde{R}_{m,p}(\tilde{c}_m) \in H^{1+s_0-\varepsilon}(\Omega)$.

Therefore all terms $\tilde{R}_{m,p}$ in (3.17) can be included into the regular part $v_0 \in H^{1+s_0-\varepsilon}(\Omega)$. Finally, we obtain (3.10) by setting

$$\tilde{c}_m^{j,(2p)}(r) := c_m^{j,(2p)}(\log r) \in H^{s-m\nu_j-2p}_{-m\nu_j-2p}(\boldsymbol{R}^+)$$

and

$$\tilde{b}_{m,p}^j := \alpha_{m,p}\tilde{c}_m^{j,(2p)} + \sum_{s_0 \leqq \lambda_k + (1/2) < s} c_{m,p,k}^j r^{\lambda_k}\chi(r)$$

with $c_{m,p,k}^j \in \boldsymbol{R}$.

It remains to treat the terms of the form (3.8) which arise for integer values of $m\nu_j$. We apply Lemma 6. Since $\psi(\xi) \in S^l(\boldsymbol{R})$ (see (3.19) below) for all $l > 0$ and $\tilde{c}_m^{(2p)} \in H^{s-m\nu-2p}_{-m\nu-2p}(\boldsymbol{R}^+)$, we obtain

$$\tilde{c}^*(e^t) := \mathrm{Op}\,[\psi(\xi)\,\chi(t - t')]\,c_m^{(2p)}(t)$$

satisfies $\tilde{c}^*(r) \in H^{s-m\nu-2p-\varepsilon'}_{-m\nu-2p-\varepsilon'}(\boldsymbol{R}^+)$ for all $\varepsilon' > 0$.

Then $c^*(t) := \tilde{c}^*(e^t)$ satisfies $e^{\eta t}c^*(t) \in H^{s-m\nu-2p-\varepsilon'}(\boldsymbol{R}^+)$ with $\eta = s - \frac{1}{2}$. We rewrite

$$(3.18) \qquad h(e^t, \theta_j, \phi_j) := c^*(t) \underset{t}{*} \Phi(t, \theta_j)\,\theta_j^{2p}S_m^j(\theta_j, \phi_j)\,\chi(\theta_j) = c^*(t)\,\theta_j^{2p}S_m^j(t_j, \phi_j)\,\chi(\theta_j) + R_{m,p}(c^*)(t_j, \phi_j, \theta_j).$$

By Lemma 4, $e^{\eta t}R_{m,p}(c^*) \in H^{s-m\nu-2p-\varepsilon'}(\boldsymbol{R} \times S_0)$ with $\eta = s - \frac{1}{2}$ and hence $\tilde{R}_{m,p}(\tilde{c}^*) \in H^{s-m\nu-2p-\varepsilon'}_{\gamma}(\Omega)$ with $\gamma = -m\nu - 1 - 2p - \varepsilon' < 0$ by Lemma 5. Since $m\nu \in \boldsymbol{N}$ and

S_m^j is defined by (2.2), the function $\theta_j^{2p} S_m^j \chi$ is in $C^\infty(S_0)$ and $e^{\eta t} c^*(t)\ \theta_j^{2p} S_m^j \chi \in H^{s-m\nu-2p-\varepsilon'}(\boldsymbol{R} \times S_0)$ with $\eta = s - \frac{1}{2}$. Thus $\tilde{c}^* \theta_j^{2p} S_m^j \chi \in H_\gamma^{s-m\nu-2p-\varepsilon'}(\Omega)$ with $\gamma = -m\nu - 1 - 2p - \varepsilon' < 0$ by Lemma 5. We use $\gamma < 0$ and $s - m\nu - 2p - \varepsilon' \geqq s - s_1 - \varepsilon' \geqq 1 + s_0 - \varepsilon$ for $\varepsilon' < \varepsilon$ to obtain h in (3.18) that $h(r, \theta_j, \phi_j) \in H^{1+s_0-\varepsilon}(\Omega)$. Therefore the terms of the form (3.8) can be included in the regular part v_0 in (3.10). This conpletes the proof of Theorem 6.

We now give Lemma 6 which was used in the proof of Theorem 6. First we need some definitions:

Definition. Define the space $S^l(\boldsymbol{R})$ by

$$S^l(\boldsymbol{R}) = \left\{\psi \in C^\infty(\boldsymbol{R}) : \forall\ \alpha\ \exists\ C_\alpha : |D^\alpha \psi(\xi)| \leqq C_\alpha (1 + |\xi^2|)^{\frac{l-|\alpha|}{2}}\right\}. \tag{3.19}$$

Let $\chi \in C_0^\infty(\boldsymbol{R}^+)$. Then the pseudodifferential operator $A = \mathrm{Op}^M\left(\chi\left(\frac{r}{r'}\right)\psi(\xi)\right)$ with Mellin symbol $\chi\left(\frac{r}{r'}\right)\psi(\xi)$ is defined as follows: For $\tilde{f} \in C_0^\infty(\boldsymbol{R}^+)$, let $f(t) = \tilde{f}(e^t)$. Then

$$A\tilde{f}(e^t) := (2\pi)^{-1/2} \int\int e^{i(t-t')\xi} \psi(\xi)\, \chi(e^{t-t'})\, f(t')\, dt'\, d\xi. \tag{3.20}$$

Lemma 6. *Let $\chi \in C_0^\infty(\boldsymbol{R}^+)$, $\psi \in S^l(\boldsymbol{R})$, $s, \gamma \in \boldsymbol{R}$. Then the operator*

$$A = \mathrm{Op}^M\left(\chi\left(\frac{r}{r'}\right)\psi(\xi)\right) : H^s_{\gamma+s}(\boldsymbol{R}^+) \to H^{s-l}_{\gamma+s-l}(\boldsymbol{R}^+)$$

is continuous.

Proof. Denote by G the Euler transformation: $(Gu)(t) := u(e^t)$. Then by Lemma 5 the continuity of $A : H^s_{\gamma+s}(\boldsymbol{R}^+) \to H^{s-l}_{\gamma+s-l}(\boldsymbol{R}^+)$ is equivalent to the continuity of $B = G \circ A \circ G^{-1} = \mathrm{Op}\left(\psi(\xi)\, \chi(e^{t-t'})\right) : H^s_{\gamma+(1/2)}(\boldsymbol{R}) \to H^{s-l}_{\gamma+(1/2)}(\boldsymbol{R})$. Here the space $H^{\tilde{s}}_{\tilde{\gamma}}(\boldsymbol{R})$ is defined by

$$u(t) \in H^{\tilde{s}}_{\tilde{\gamma}}(\boldsymbol{R}) \Leftrightarrow e^{\tilde{\gamma} t} u(t) \in H^{\tilde{s}}(\boldsymbol{R}). \tag{3.21}$$

The operator $B = \mathrm{Op}\left(\psi(\xi)\, \chi(e^{t-t'})\right)$ is defined by

$$(Bu)(t) = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} e^{i(t-t')\xi} \tilde{\chi}(t - t')\, \psi(\xi)\, u(t')\, dt'\, d\xi$$

with $\tilde{\chi}(t) := \chi(e^t) \in C_0^\infty(\boldsymbol{R})$. By Fourier transformation and the change of variables $z := t - t'$, we obtain

$$\begin{aligned}
(Bu)^\wedge(\eta) &= \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} e^{-i\eta t} e^{i(t-t')\xi} \tilde{\chi}(t - t')\, \psi(\xi)\, u(t')\, dt'\, d\xi\, dt \\
&= \int_{\boldsymbol{R}} e^{-i\eta t} \left[\int_{\boldsymbol{R}} \tilde{\chi}(z)\, u(t - z) \int_{\boldsymbol{R}} e^{iz\xi} \psi(\xi)\, d\xi (-dz)\right] dt \\
&= \int_{\boldsymbol{R}} e^{-i\eta t} \left[\int_{\boldsymbol{R}} \tilde{\chi}(z)\, u(t - z)\, \check{\psi}(z)\, (-dz)\right] dt \\
&= \int_{\boldsymbol{R}} e^{-i\eta t} [(\tilde{\chi}\check{\psi}) * u]\, (t)\, dt \\
&= (\tilde{\chi}\check{\psi})^\wedge\, (\eta)\, \hat{u}(\eta) = \int_{\boldsymbol{R}} \hat{\tilde{\chi}}(\zeta)\, \psi(\eta - \zeta)\, d\zeta \hat{u}(\eta).
\end{aligned}$$

Here $\check{u}(t)$ denotes the inverse FOURIER transform of $u(\xi)$. Now we use the definition (3.21) of $H^{s-l}_{\gamma+(1/2)}(\boldsymbol{R})$ and the relation

$$\int_{\boldsymbol{R}} e^{-it\xi}\left(e^{(\gamma+(1/2))t}u(t)\right) dt = \hat{u}\left(\xi + i\left(\gamma + \frac{1}{2}\right)\right).$$

Hence

$$\begin{aligned}
&\|Bu\|^2_{H^{s-l}_{\gamma+(1/2)}(\boldsymbol{R})} \\
&= \int_{\boldsymbol{R}} (1+|\xi|^2)^{s-l} \left|\hat{B}\hat{u}\left(\xi + i\left(\gamma+\frac{1}{2}\right)\right)\right|^2 d\xi \\
&= \int_{\boldsymbol{R}} (1+|\xi|^2)^{s-l} \left|\int_{\boldsymbol{R}} \hat{\tilde{\chi}}(\zeta)\,\psi\left(\xi + i\left(\gamma+\frac{1}{2}\right) - \zeta\right) d\zeta\right|^2 \\
&\quad \times \left|\hat{u}\left(\xi + i\left(\gamma+\frac{1}{2}\right)\right)\right|^2 d\xi \\
&\leqq \int_{\boldsymbol{R}} (1+|\xi|^2)^{s-l} \left(\int_{\boldsymbol{R}} |\hat{\tilde{\chi}}(\zeta)| \cdot \left|\psi\left(\xi + i\left(\gamma+\frac{1}{2}\right) - \zeta\right)\right| d\zeta\right)^2 \\
&\quad \times \left|\hat{u}\left(\xi + i\left(\gamma+\frac{1}{2}\right)\right)\right|^2 d\xi.
\end{aligned}$$

But $\psi \in S^l$ yields

$$\left|\psi\left(\xi + i\left(\gamma+\frac{1}{2}\right) - \zeta\right)\right| \leqq \left(1 + \left|\xi + i\left(\gamma+\frac{1}{2}\right) - \zeta\right|^2\right)^{l/2}.$$

Since $1 + |\xi-\eta|^2 \leqq (1+|\xi|^2)(1+|\eta|^2)$ we obtain with a constant C depending on γ

$$\left|\psi\left(\xi + i\left(\gamma+\frac{1}{2}\right) - \zeta\right)\right| \leqq C(1+|\xi|^2)^{l/2}(1+|\zeta|^2)^{l/2}.$$

By $\tilde{\chi} \in C_0^\infty(\boldsymbol{R})$ we have $|\hat{\tilde{\chi}}(\zeta)| \leqq C_N(1+|\zeta|^2)^{-N/2}$ for all $N \in \boldsymbol{N}$. Hence we obtain for N large enough

$$\begin{aligned}
\|Bu\|^2_{H^{s-l}_{\gamma+(1/2)}(\boldsymbol{R})} &\leqq C \int_{\boldsymbol{R}} (1+|\xi|^2)^{s-l}(1+|\xi|^2)^l \left|\hat{u}\left(\xi + i\left(\gamma+\frac{1}{2}\right)\right)\right|^2 d\xi \\
&\quad \times \int_{\boldsymbol{R}} (1+|\zeta|^2)^{(l/2)-(N/2)}\, d\zeta \\
&\leqq \tilde{C} \int_{\boldsymbol{R}} (1+|\xi|^2)^s \left|\hat{u}\left(\xi + i\left(\gamma+\frac{1}{2}\right)\right)\right|^2 d\xi = \tilde{C}\, \|u\|^2_{H^s_{\gamma+(1/2)}(\boldsymbol{R})}.
\end{aligned}$$

This completes the proof of Lemma 6. □

Next we derive the regularity results for $\tilde{b}^j_{m,p}$ claimed in Remark 9(ii) following Theorem 6. We note that $r^{\lambda_k}\chi(r) \in H^{\tilde{s}-mv_j}_{-mv_j}(\boldsymbol{R}^+)$ and $r^{\lambda_k}\chi(r) \in H^{\tilde{s}}(\boldsymbol{R}^+)$ if $\tilde{s} < \lambda_k + \frac{1}{2}$. Therefore with $s_3 = \min\left\{\lambda_k + \frac{1}{2} \,\middle|\, \lambda_k + \frac{1}{2} \geqq s_0\right\}$ we have in (3.11)

$$\tilde{b}^j_{p,m} \in H^{s_3-\varepsilon-mv_j-2p}_{-mv_j-2p}(\boldsymbol{R}^+) \subset H^{s_0-\varepsilon-mv_j-2p}_{-mv_j-2p}(\boldsymbol{R}^+)$$

and, furthermore,

$$\tilde{b}^j_{m,p} \in H^{s_4}(\boldsymbol{R}^+) \quad \text{with} \quad s_4 = \min\{s - mv_j - 2p, s_3 - \varepsilon\} \tag{3.22}$$

since $H^{s-mv_j-2p}_{-mv_j-2p}(\boldsymbol{R}^+) \subset H^{s-mv_j-2p}(\boldsymbol{R}^+)$. On the other hand, if we use that $r^{\lambda_k}\chi(r) \in H^{s-mv_j-2p}_{\gamma}(\boldsymbol{R}^+)$ for $\gamma > -mv_j - 2p + s - \lambda_k - \frac{1}{2}$ we get

$$\tilde{b}^j_{m,p} \in H^{s-mv_j-2p}_{\gamma}(\boldsymbol{R}^+). \tag{3.23}$$

Remark 10. As mentioned in the Remark 8 we now can formulate the decomposition for an arbitrary polyhedron Ω. Since the solution u of problem (1.1) has regularity H^{1+s} away from vertices and edges for given $f \in H^{s-1}(\Omega)$, we obtain a decomposition for u with a regular part $u_0 \in H^{1+s_0-\varepsilon}(\Omega)$ and singularities near edges as in (2.11) and near vertices as in (3.10). Thus the general case of an arbitrary polyhedron is covered completely by a combination of Theorems 3 and 6. This will be made clear by some explicit examples presented later on.

As in Theorem 5 we can interpret the vertex singularities as edge singularities which grow like r^{λ_k} towards the vertex. Since the functions $v_k(\theta, \phi)$ are eigenfunctions of the LAPLACE-BELTRAMI operator on the spherical polygon $S_0 := S \cap \Omega$ (S denotes a sphere), there holds on S_0 a decomposition as in Remark 1:

$$v_k(\theta, \phi) = \sum_{j=1}^{J} \sum_{mv_j+2p<s_0} c^j_{m,p,k}\theta_j^{2p} S^j_m(\theta_j, \phi_j) + w_k(\theta, \phi), \tag{3.24}$$

with $w_k \in H^{1+\sigma}(S_0)$ for $\sigma = \min\{mv_j + 2p \mid m, j \geqq 1, p \geqq 0 \text{ integers}, mv_j + 2p \geqq s_0\}$.

By inserting this form in Theorem 6 we thus obtain Corollary 3 and omit for brevity the remaining details of its proof.

Corollary 3. *Under the assumptions of Theorem* 6 *there holds for* λ_k *being not an integer with* w_k *as in* (3.24)

$$u = v_0 + \chi(r) \sum_{\lambda_k+(1/2)<s_0} a_k r^{\lambda_k} w_k(\theta_1, \phi_1) + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{mv_j+2p<s_0} e^j_{m,p}(r)\, \theta_j^{2p} S^j_m(\theta_j, \phi_j) \tag{3.25}$$

where $v_0 \in H^{s_0+1-\varepsilon}(\Omega)$, $a_k \in \boldsymbol{R}$,

$$w_k(\theta, \phi) \in H^{\tilde{s}_0+1-\varepsilon}(S_0),$$

$$\tilde{s}_0 = \min\{mv_j + 2p \mid m, j \geqq 1, p \geqq 0 \text{ integers}, mv_j \geqq s_0\} \geqq s_0$$

$$e^j_{m,p} = b^j_{m,p} + \chi(r) \sum_{\lambda_k+(1/2)<s} a^j_{m,p,k} r^{\lambda_k}, \qquad b^j_{m,p} \in H^{s-mv_j-2p}_{-mv_j-2p}(\boldsymbol{R}^+). \tag{3.26}$$

For λ_k an integer there are additional corner terms in (3.25) *of the form*

$$\chi(r) \sum_{l=1}^{L(k)} \sum_{q=0}^{1} r^{\lambda_k} \log^q r \tilde{w}_{k,l,q}(\theta_1, \phi_1) \tag{3.27}$$

with $L(k)$ as in (3.5) *and $\tilde{w}_{k,l,q} \in H^{\tilde{s}_0+1-\varepsilon}(S_0)$. Correspondingly, there are additional terms in the edge distributions* (3.26) *of the form*

$$\tilde{a}^j_{m,p,k} r^{\lambda_k} \log r$$

with $\tilde{a}^j_{m,p,k} \in \boldsymbol{R}$.

Note. In (3.27) we have used that the function $\tilde{v}_{k,l,q}$ in (3.5) allows a decomposition as in (3.24).

Alternatively, we can express the edge singularities in terms of the distance ϱ_j to the edge γ_j. Since $\varrho_j = r \sin \theta_j$ and $\sin \theta_j \sim \theta_j$ for small θ_j we obtain from (3.25) a new decomposition which contains explicitly the physically relevant stress intensity factors $h^j_{m,p}$. Again for brevity we omit the proof of Corollary 4.

Corollary 4. *We make the assumptions of Theorem* 6. *For simplicity we assume that $m\nu_j$ and λ_k are not integers, i.e., no logarithms occur as in decompositions* (2.11) *and* (3.25). *Then there holds*

$$u = v_0 + \chi(r) \sum_{\lambda_k + (1/2) < s_0} a_k r^{\lambda_k} w_k(\theta, \phi) + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j + 2p < s_0} h^j_{m,p}(r)\, \varrho_j^{2p} S^j_m(\varrho_j, \phi_j) \tag{3.28}$$

where $v_0 \in H^{s_0+1-\varepsilon}(\Omega)$, $a_k \in \boldsymbol{R}$,

$$\begin{aligned} & w_k(\theta, \phi) \in H^{\tilde{s}_0+1-\varepsilon}(S_0), \\ & \tilde{s}_0 = \min \{m\nu_j + 2p \mid m, j \geqq 1,\ p \geqq 0 \text{ integers},\ m\nu_j \geqq s_0\} \geqq s_0 \\ & h^j_{m,p} = \chi(r) \sum_{\lambda_k + (1/2) < s} a^j_{m,p,k} r^{\lambda_k - m\nu_j - 2p} + k^j_{m,p}, \quad k^j_{m,p} \in H_0^{s - m\nu_j - 2p}(\boldsymbol{R}^+) \end{aligned} \tag{3.29}$$

and $a^j_{m,p,k} \in \boldsymbol{R}$.

4. Examples

In the following we illustrate our theorems with some examples.

Example 1. In the cube $\Omega = \{x \in \boldsymbol{R}^3 \colon |x_j| \leqq 1\}$ we have locally as possibly dominant edge and vertex singularities: $\varrho^2(\log \varrho \sin 2\phi + \phi \cos 2\phi)$, $r^3 \log r v_1(\theta, \phi)$, $r^3 v_2(\theta, \phi)$ (see [5], Prop. 18.8). Taking $s_0 = 4$, $s_1 = 2$, $s = 7$ in Corollary 3, we obtain with $f \in H^6(\Omega)$ for the solution u of problem (1.1) near each corner (with $m = 1$, $p = 0$) the form:

$$u = v_0 + \sum_{j=1}^{3} \tilde{e}^j_{1,0} \theta_j^2 (\log \theta_j \sin 2\phi_j + \phi_j \cos 2\phi_j)\, \chi_j(\theta) + r^3 \log r \tilde{w}_1(\theta_1, \phi_1) + r^3 w_1 \tag{4.1}$$

with $v_0 \in H^{5-\varepsilon}(\Omega)$, $\tilde{e}^j_{1,0} = a^{(1)}_j r^3 \log r + a^{(2)}_j r^3 + \tilde{b}^j_{1,0}$, $\tilde{b}^j_{1,0} \in H^5_{-2}(\boldsymbol{R}^+)$, $a^{(k)}_j \in \boldsymbol{R}$.

The functions $w_1(\theta_1, \phi_1)$ and $\tilde{w}_1(\theta_1, \phi_1)$ belong to $H^{5-\varepsilon}(S_0)$ where S_0 denotes the intersection of Ω with a small sphere. If one omits the two terms $r^3 w_1$ and $r^3 \log r \tilde{w}_1$ then one gets a new decomposition for (4.1) with a regular part $v_0^* \in H^{9/2-\varepsilon}(\Omega)$.

Furthermore if f vanishes on all edges i.e., $f|_{\gamma_j} = 0$ ($j = 1, \ldots, 12$), then for $f \in H^{s-1}(\Omega)$, $2 < s < 4$, we have even $u \in H^{s+1}(\Omega)$, i.e., u has no terms like $\varrho^2 \log \varrho$, $r^3 \log r$ (cf. [5], Thm. 19.8).

Example 2. (Cube Ω with a lateral DIRICHLET crack as given in Fig. 5): We note that the solution u of the DIRICHLET problem (1.1) has edge singularities $\varrho^2 \log \varrho$, $\varrho^4 \log \varrho$, etc. for edges with $\omega_j = \frac{\pi}{2}$, i.e., $\nu_j = 2$, and singularities $\varrho^{1/2}$, $\varrho^{3/2}$, etc. for edges with $\omega_j = 2\pi$, i.e., $\nu_j = \frac{1}{2}$. There are three kinds of vertices (compare Fig. 5):

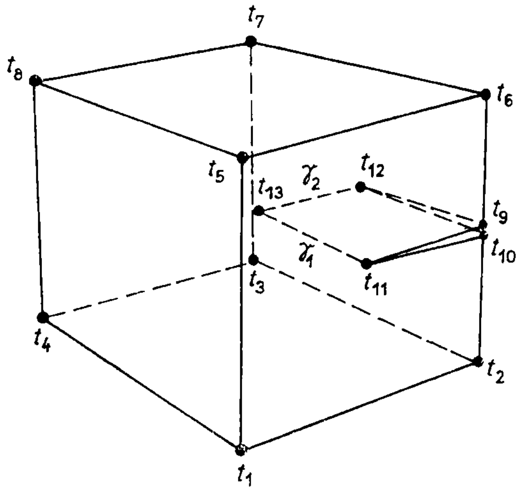

Fig. 5

(i) $t_1, \ldots, t_{10}$ coincide locally with the vertex of an octant where (see [5], Prop. 18.8)

$$\lambda_1 = 3, \quad \lambda_2 = 5, \quad \lambda_3 = 7, \ldots$$

(ii) t_{11}, t_{12} correspond locally to the corners of $\{x \in \boldsymbol{R}^3 \mid x_1 > 0\} \setminus \{x \in \boldsymbol{R}^3 \mid x_3 = 0, x_2 \geqq 0\}$, i.e., a halfspace with a halfplane removed from it. Here we have (see [5], Prop. 18.7)

$$\lambda_1 = \frac{3}{2}, \lambda_2 = \frac{7}{2}, \lambda_3 = \frac{11}{2}, \ldots$$

(iii) t_{13} corresponds to the origin of $\boldsymbol{R}^3 \setminus \{x \in \boldsymbol{R}^3 \mid x_3 = 0; x_1, x_2 \geqq 0\}$, i.e., the complement of a quarter plane. Here numerical results give [16], [1], [23], [2]

$$\lambda_1 = 0.2966, \quad \lambda_2 = 1.426, \quad \lambda_3 = 2.033, \ldots$$

The behavior of the solution u of problem (1.1) near a corner of type (i) is given in Example 1. For (ii) we choose $s_0 = \frac{3}{2}$, $s_1 = \frac{1}{2}$, and $s = 3$ in Corollary 3 and obtain with $f \in H^{s-1}(\Omega)$ a decomposition for u around t_{11} with one edge singularity only, located on γ_1 (compare Fig. 5), namely

$$u = v_0 + \chi_1(\theta_1)\, e_{1,0}^1(r)\, \theta_1^{1/2} \sin \frac{\phi_1}{2} \quad \text{with} \quad v_0 \in H^{5/2-\varepsilon}(\Omega).$$

Here the edge distribution satisfies

$$\tilde{e}^1_{1,0} = a_{1,1} r^{3/2} + b^1_{1,0}(r), \qquad b^1_{1,0} \in H^{5/2}_{-1/2}(\boldsymbol{R}^+).$$

In (iii) we choose again $s_0 = \frac{3}{2}$, $s_1 = \frac{1}{2}$ in Corollary 3 and obtain near t_{13} for $f \in H^2(\Omega)$

$$u = v_0 + \sum_{j=1}^{2} e^j_{1,0}(r)\, \theta_j^{1/2} \sin \frac{\phi_j}{2} + r^{0.2966} w_1(\theta_1, \phi_1), \tag{4.2}$$

with $v_0 \in H^{5/2-\varepsilon}(\Omega)$, $w_1(\theta_1, \phi_1) \in H^{5/2-\varepsilon}(S_0)$. Here the distributions on the edges γ_1, γ_2 meeting at t_{13} are

$$\begin{aligned} e^j_{1,0} &= a_{j,1} r^{0.2966} + a_{j,2} r^{1.426} + a_{j,3} r^{2.03} + b^j_{1,0}, \\ b^j_{1,0} &\in H^{5/2-1/2}_{-1/2}(\boldsymbol{R}^+), \qquad j = 1, 2. \end{aligned} \tag{4.3}$$

If we take $s_0 = \lambda_1 + \frac{1}{2} = 0.7966$, then $s_1 = \frac{1}{2}$ and we get for $s = 2.2966$ and $f \in H^{s-1}(\Omega)$

$$u = v_0^* + \sum_{j=1}^{2} e^j_{1,0}(r)\, \theta_j^{1/2} \sin \frac{\phi_j}{2} \quad \text{with} \quad v_0^* \in H^{1.7966-\varepsilon}(\Omega)$$

and $e^j_{1,0} = a_{j,1} r^{0.2966} + a_{j,2} r^{1.426} + b^j_{1,0}$, $b^j_{1,0} \in H^{1.7966}_{-1/2}(\boldsymbol{R}^+)$.

Example 2 (Continued). By applying Corollary 4 to the above situation, we obtain a decomposition with terms $h_j(y_j)\, \varrho_j^{1/2} \sin \frac{\phi_j}{2}$. (The coordinates y_j are defined as indicated in Fig. 6.) Here $h_j(y_j)$ are the physically relevant *stress intensity factors*. Furthermore, we can patch the local decomposition of (i)–(iii) together.

We consider $f \in H^{3/2}(\Omega)$, and take the origin of the coordinate system in t_{13}. Then if we define γ_1, γ_2 as in Example 2, (iii) and denote the variable along γ_j starting in t_{13} by y_j we have the *global* decomposition

$$u = v_0 + \sum_{j=1}^{2} \chi_j(\theta_j)\, h_j(y_j)\, \varrho_j^{1/2} \sin \frac{\phi_j}{2} + a_1 r^{0.2966} w(\theta_1, \phi_1)$$

where $v_0 \in H^{5/2-\varepsilon}(\Omega)$, $w(\theta_1, \phi_1) \in H^{5/2-\varepsilon}(S_0)$, θ_j, ϕ_j as in (4.2), $\chi_j(\theta_j)$ are cut-off functions and

$$h_j = a_1^j y_j^{-0.2034} + a_2^j y_j^{0.926} + a_3^j y_j^{1.53} + \tilde{a}_1^j (1 - y_j) + k_j(y)$$

with $k_j \in H^{5/2}(\gamma_j)$, $\chi(y)\, k_j(y) \in H_0^{5/2}(\boldsymbol{R}^+)$, $\chi(1 - y)\, k_j(y) \in H_0^{5/2}(\boldsymbol{R}^+)$.

Alternatively, we have for $f \in H^{1.2966}(\Omega)$ the decomposition

$$u = v_0^* + \sum_{j=1}^{2} h_j(y_j)\, \varrho_j^{1/2} \sin \frac{\phi_j}{2}$$

with $v_0^* \in H^{1.7966-\varepsilon}(\Omega)$.

Here the functions h_j are the physically relevant stress intensity factors which vanish at t_{12}, t_{11} and blow up at t_{13}. Note we identify t_{12} with $y_2 = 1$ and t_{11} with $y_1 = 1$.

Example 3. We consider a cube where a regular hexagonal pyramid with opening α less than 25° is removed. Let S_0 denote the intersection of Ω with a small sphere centered in P, and let $\tilde{S}_0$ denote the intersection with the exterior of a cone with opening α. Then the $\tilde{\lambda}_1$ corresponding to $\tilde{S}_0$ is calculated by Prop. 18.10 in [5] and satisfies

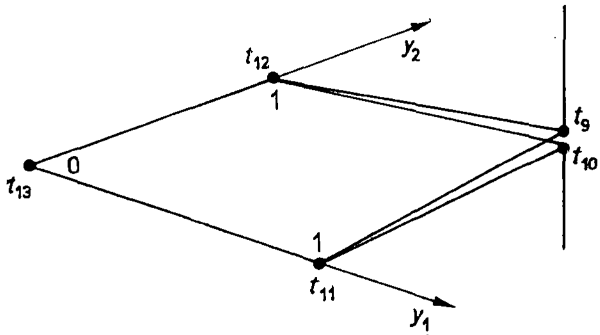

Fig. 6

$\tilde{\lambda}_1 < 0.25$ for $\alpha < 25°$. Since $\tilde{S}_0 \subset S_0$ we have $\lambda_1 \leqq \tilde{\lambda}_1$ (see e.g. Lemma 18.5 in [5]). The exterior angles at the edges of the pyramid are smaller than the exterior angles at the edges of a hexagonal cylinder which are $\frac{4}{3}\pi$. Hence $\nu_j = \frac{\pi}{\omega_j} > \frac{\pi}{\frac{4}{3}\pi} = 0.75$ and $\lambda_1 + \frac{1}{2} < \nu_j$. Thus the corner singularity is "stronger" than the edge singularities. By Theorem 4 and the following remark, we have $u \in H^{3/2+\lambda_1-\varepsilon}(\Omega)$ for $f \in H^{-1/2+\lambda_1}(\Omega)$. Now, if we use Theorem 4 with $s = \nu_1 - \varepsilon$ we obtain that for $f \in H^{-1+\nu_1}(\Omega)$ the solution u has the form

$$u = u_0 + r^{\lambda_1} v_1(\theta, \phi)\, \chi(r)$$

with $u_0 \in H^{1+\nu_1-\varepsilon}(\Omega)$, $v_1(\theta, \phi) \in H^{1+\nu_1-\varepsilon}(S_0)$.

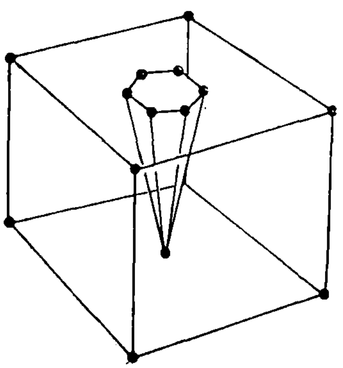

Fig. 7

It is also possible to take a triangular pyramid, if its opening is small enough. Since the numbers $\tilde{\lambda}_1^{(\alpha)}$ for the exterior of circular cones with opening α tand to 0 for $\alpha \to 0$, one can choose an α such that $\tilde{\lambda}_1 < 0.1$. Then the exponent λ_1 of the corresponding triangular pyramid satisfies $\lambda_1 + \frac{1}{2} < \nu_j$.

5. Decomposition Theorems for the Normal Derivative of the Solution

Often one considers instead of (1.1) the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

with given $g \in H^{1/2+s}(\partial\Omega)$. The space $H^{1/2+s}(\partial\Omega)$ is defined as the traces of functions in $H^{1+s}(\Omega)$.

$$H^{1/2+s}(\partial\Omega) = \{G|_{\partial\Omega} : G \in H^{1+s}(\Omega)\} \tag{5.2}$$

Problem (5.1) is equivalent to problem (1.1) in the following sense: Let $G \in H^{1+s}(\boldsymbol{R}^3)$ be a function with $G|_{\partial\Omega} = g$ and define $f := -\Delta G \in H^{-1+s}(\Omega)$. Then $\tilde{u} := u - G$ satisfies

$$\begin{aligned} \Delta\tilde{u} &= \Delta u - \Delta G = f \quad \text{in } \Omega \\ \tilde{u}|_{\partial\Omega} &= u|_{\partial\Omega} - G|_{\partial\Omega} = 0 . \end{aligned}$$

Hence we can apply the results of Section 2 and 3 to $\tilde{u}$ and thus get a decomposition for $u = \tilde{u} + G$, where $G \in H^{s+1}(\Omega)$ is included in the regular part.

If one solves problem (5.1) by a boundary integral equation, then one is interested in the unknown normal derivative $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$ rather than in u. One could obtain results for $\frac{\partial u}{\partial n}$ by taking the normal derivative of the decomposition for u in Sections 2 and 3. In this section we will show that a modification of the proofs in Sections 2 and 3 yields even stronger results.

We first state the result for an infinite wedge $\Omega = \boldsymbol{R} \times K$ with angle ω corresponding to Theorem 3. We denote the sides of the cross section K by Γ_1 and Γ_2, and define the faces of the wedge by $\tilde{\Gamma}_j := \boldsymbol{R} \times \Gamma_j$, $j = 1, 2$.

Theorem 7. *Let* $A = \{m\nu + 2p \mid m > 0, p \geqq 0 \text{ integers}\}$ *with* $\nu = \frac{\pi}{\omega}$ *and let* $s_0 \in A$ *and* $s_1 = \max\{t \in A \mid t < s_0\}$. *Then for* $g \in H^{1/2+s}(\partial\Omega)$ *with* $s \geqq s_0 + s_1 - \frac{1}{2}$ *and* $s \neq m\nu$ *the normal derivative* $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$ *of the solution* u *of problem* (5.1) *admits a decomposition*

$$\tilde{\chi}(\varrho)\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \psi^0 + \chi(\varrho) \sum_{m\nu+2p<s_0} b_{m,p}\varrho^{2p} \frac{\partial}{\partial n} S_m(\varrho, \phi)|_{\partial K} \tag{5.3}$$

with $\psi^0|_{\tilde{\Gamma}_j} \in H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_j)$ *for any* $\varepsilon > 0$ *and* $b_{m,p} \in H^{s-m\nu-2p}(\boldsymbol{R})$ *where* S_m *as in* (2.2) *and* (2.3), $\chi(\varrho)$ *and* $\tilde{\chi}(\varrho)$ *are* C^∞ *cut-off-functions concentrated at* $\varrho = 0$.

Remark 11. (i) The singularity functions $\frac{\partial}{\partial n} S_m(\varrho, \phi)|_{\Gamma_j}$ are explicitly given by

$$\frac{\partial}{\partial n} S_m|_{\Gamma_j} = \begin{cases} (-1)^j \, m\nu \varrho^{m\nu-1} & \text{for } m\nu \notin \boldsymbol{N} \\ (-1)^j \, \varrho^{m\nu-1} (m\nu \log \varrho + 1) & \text{for } m\nu \in \boldsymbol{N}. \end{cases}$$

(ii) The regularity of ψ^0 is given separately on each face $\tilde{\Gamma}_j$ of K, because in general the normal derivative is not continuous across the edge and therefore is not contained in $H^\sigma(\partial K)$ for $\sigma > \frac{1}{2}$.

Now we consider a polyhedral cone Ω with a spherical cross section S_0. We denote the sides of S_0 by Γ_j, $j = 1, \ldots, J$, and the faces of Ω by $\tilde{\Gamma}_j$, $j = 1, \ldots, J$. Then we obtain the following result corresponding to Theorem 6:

Theorem 8. *Let* $\nu_j = \frac{\pi}{\omega_j}$, $A = \{m\nu + 2p \mid m > 0, p \geqq 0, \textit{integers}\}$, $B = \left\{\frac{1}{2} + \lambda_k \mid k \in \boldsymbol{N}\right\}$ *with* λ_k *as in* (3.1) *or* $\lambda_k \geqq 2$ *integer. Let* $s_0 \in A \cup B$, $s_1 = \max\{t \in A \mid t < s_0\}$ *and let* $g \in H^{1/2+s}(\partial\Omega)$ *with* $s \geqq s_0 + s_1 - \frac{1}{2}$ *and* $s \neq m\nu_j$, $s \neq \lambda_k + \frac{1}{2}$ *and* $s \neq \frac{1}{2}$. *Then the normal derivative* $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$ *of the solution* u *of problem* (5.1) *satisfies*

$$\tilde{\chi}(r) \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \psi^0 + \chi(r) \sum_{\lambda_k + (1/2) < s_0} a_k r^{\lambda_k - 1} \frac{\partial}{\partial n} v_k|_{\partial S_0} + \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j + 2p < s_0} r^{-1} \tilde{b}^j_{m,p}(r) \, \theta_j^{2p} \frac{\partial}{\partial n} S_m(\theta_j, \phi_j)|_{\partial S_0} \tag{5.4}$$

with $\psi^0|_{\tilde{\Gamma}_i} \in H^{-(1/2)+s_0-\epsilon}(\tilde{\Gamma}_i)$, $\epsilon > 0$ *arbitrary*, $\tilde{b}^j_{m,p}$ *as in Theorem* 6, $a_k \in \boldsymbol{R}$, *where* χ, χ_j, S_m *as in Theorem* 4 *and* $\tilde{\chi}$ *is a* C^∞ *cut-off function concentrated near the vertex. For* λ_k *an integer, or* $\{t \in A \mid t < s_0\} = \emptyset$ *there are modifications corresponding to the ones of Theorem 6.*

The proofs of Theorems 7 and 8 are based on the following lemma for an infinite wedge $\boldsymbol{R} \times K$ which corresponds to Lemma 4. Let us recall that in this case $\tilde{\Gamma}_j = \boldsymbol{R} \times \Gamma_j$ with Γ_j being the sides of K.

Lemma 7. *Let* $s \geqq 0$, $\eta \in \boldsymbol{R}$, $e^{\eta t} c(t) \in H^s(\boldsymbol{R})$, $R(c)$ *as in* (2.14), $\alpha \geqq \frac{1}{2}$, $\alpha_2 = \frac{1}{2} - \left[\frac{1}{2} - \alpha\right]$, *and*

$$\tilde{s}_0 = \begin{cases} s + \alpha - \frac{1}{2} & \textit{for } s \geqq \alpha_2 - \alpha \\ \alpha_2 - \frac{3}{2} + \frac{s}{\alpha_2 - \alpha} & \textit{for } 0 \leqq s < \alpha_2 - \alpha \quad \textit{if } \alpha_2 \neq \alpha. \end{cases}$$

Then there holds for $j = 1, 2$

(i) $e^{\eta t} \frac{\partial}{\partial n} R(c)|_{\tilde{\Gamma}_j} \in H^s\big(\boldsymbol{R}, L^2(\Gamma_j)\big)$

(ii) $e^{\eta t} \dfrac{\partial}{\partial n} R(c)|_{\tilde{\Gamma}_j} \in L_2(\boldsymbol{R}, H^{\tilde{s}_0}(\Gamma_j))$

(iii) $e^{\eta t} \dfrac{\partial}{\partial n} R(c)|_{\tilde{\Gamma}_j} \in H^s(\tilde{\Gamma}_j)$.

Lemma 7 is a modification of Lemma 4. For the latter we give another formulation which is analogous to Lemma 7.

Lemma 4'. *Let $s \geqq 0$, $\eta \in \boldsymbol{R}$, $e^{\eta t}c(t) \in H^s(\boldsymbol{R})$, $R(c)$ as in* (2.14), $\alpha_1 := -[-\alpha]$ *and*

$$\tilde{s} = \begin{cases} s + \alpha + 1 & \text{for } s \geqq \alpha_1 - \alpha \\ \alpha_1 + \dfrac{s}{\alpha_1 - \alpha} & \text{for } 0 \leqq s \leqq \alpha_1 - \alpha \text{ if } \alpha_1 \neq \alpha . \end{cases}$$

Then there holds

(i) $e^{\eta t}R(c) \in H^s(\boldsymbol{R}, L^2(K))$

(ii) $e^{\eta t}R(c) \in L^2(\boldsymbol{R}, H^{\tilde{s}}(K))$

(iii) $e^{\eta t}R(c) \in H^s(\boldsymbol{R} \times K)$.

Proof. Assertion (i) was shown in part (i) of the proof of Lemma 4. Assertion (ii) follows by interpolation from (2.15) which was proved for integers s_1. Assertion (iii) was also shown in the proof of Lemma 4. ∎

Proof of Lemma 7. First we observe that

$$\frac{\partial}{\partial n} R(c)|_{\tilde{\Gamma}_j} = (-1)^j [c(t) * \Phi(t, \varrho) - c(t)]\, \varrho^{\alpha-1}\chi(\varrho)\, g'(\phi)_{\phi=\omega_j}$$

with $\omega_1 = 0$, $\omega_2 = \omega$. Now we repeat the proof of Lemma 4 with estimates on the boundary $\boldsymbol{R} \times \Gamma_j$ instead of $\boldsymbol{R} \times K$. Assertion (i) is shown by a straightforward modification of part (i) of the proof of Lemma 4. In order to show assertion (ii) we proceed as in part (ii) of the proof of Lemma 4. We observe that now with $\dfrac{\partial u}{\partial n}$ we have $\varrho^{\alpha-1}$ instead of ϱ^α for u. Correspondingly, the exponent α has to be replaced by $\alpha - 1$ and the measure $\varrho\, d\varrho$ has to be replaced by $d\varrho$ in the integrals appearing in the steps a) and b), since Γ_j is one-dimensional. This leads to the estimate

$$\left\| \frac{\partial}{\partial n} R(c)|_{\tilde{\Gamma}_j} \right\|_{L^2(\boldsymbol{R}, H^{s_1}(\Gamma_j))} \leqq C \, \|c\|_{H^{\tilde{\sigma}}(\boldsymbol{R})} \tag{5.5}$$

with $\tilde{\sigma} = \max \left\{0, s - \alpha + \dfrac{1}{2}\right\}$. After interpolation between $s_1 = -[-s]$ and $s_1 = -[-s] - 1$ and after renaming s we obtain assertion (ii). Now we comment in detail on the derivation of estimate (5.5). We denote

$$\tilde{a}(\xi, \varrho) := \left[\hat{\psi}\big((\xi + i\eta)\, \varrho\big) - 1\right] \varrho^{\alpha-1}\chi(\varrho)$$

and obtain as in part (ii) a) of the proof of Lemma 4 if $N > s_1 - \alpha + \frac{1}{2}$

$$\begin{aligned}(5.6)\qquad & |\tilde{a}(\xi,\varrho)|^2_{H^{s_1}(\tilde{\Gamma}_j \cap B_{|\xi+i\eta|^{-1}})} \\ & \leqq C \sum_{|\beta|=s_1} \sum_{\beta_1+\beta_2=\beta} |\xi + i\eta|^{2(|\beta_1|+N+1)} \int\limits_{\varrho=0}^{\min\{1,|\xi+i\eta|^{-1}\}} \varrho^{2(N+\alpha-|\beta_2|)}\, d\varrho \\ & \leqq C\, |\xi + i\eta|^{2(|\beta_1|+N+1)-(2N+2\alpha-2|\beta_2|+1)} = C\, |\xi + i\eta|^{2(s_1-\alpha+(1/2))}.\end{aligned}$$

Analogously we get like in part b) of the proof of Lemma 4

$$(5.7)\qquad |\tilde{a}(\xi,\varrho)|^2_{H^{s_1}(\tilde{\Gamma}_j \setminus B_{|\xi+i\eta|^{-1}})} \leqq C \int\limits_{|\xi+i\eta|^{-1}}^{1} \varrho^{2(\alpha-1-s_1)}\, d\varrho \leqq C(1 + |\xi + i\eta|^{2(s_1-\alpha+(1/2))}).$$

Now, the estimates (5.6), (5.7) together with $|\tilde{a}(\xi,\varrho)|_{L^2(\tilde{\Gamma}_j)} \leqq C$ give (5.5). To show assertion (iii) we note that for $\alpha = \frac{1}{2}$ we have $\alpha_2 = \frac{1}{2}$ and $\tilde{s}_0 = s$. For $\alpha > \frac{1}{2}$ we have $\alpha_2 \geqq \frac{3}{2}$ and hence $\tilde{s}_0 \geqq s$. Thus assertion (iii) is obtained by a combination of (i) and (ii). □

Now we can prove Theorem 7 for an infinite wedge.

Proof of Theorem 7. As in the proof of Theorem 3 we start with Theorem 2. Then we take the normal derivative of the decomposition (2.5) on $\tilde{\Gamma}_j$. The regular part u^0 has regularity $H^{1+s}(\Omega)$. Since $\frac{\pi}{\omega} > \frac{1}{2}$ the assumptions of Theorem 7 imply $s > \frac{1}{2}$. Hence we obtain $\frac{\partial}{\partial n}\, u^0|_{\tilde{\Gamma}_j} \in H^{-1/2+s}(\tilde{\Gamma}_j)$ by the trace lemma (see [10]). We proceed as in the proof of Theorem 3 and obtain the terms $\frac{\partial}{\partial n}\, R_{m,p}(D_y^{2p} c_m)|_{\tilde{\Gamma}_j}$ with $m\nu + 2p \leqq s_0$ and $R_{m,p}$ as defined in (2.18). Also, we have as in the proof of Theorem 3

$$D_y^{2p} c_m \in H^{s-m\nu-2p}(\boldsymbol{R}) \subset H^{s-s_1}(\boldsymbol{R}).$$

Then Lemma 7 (iii) implies with $\eta = 0$

$$(5.8)\qquad \frac{\partial}{\partial n}\, R_{m,p}(D_y^{2p} c_m)|_{\tilde{\Gamma}_j} \in H^{s-s_1}(\tilde{\Gamma}_j).$$

The condition $s \geqq s_0 + s_1 - \frac{1}{2}$ gives $H^{s-s_1}(\tilde{\Gamma}_j) \subset H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_j)$, hence the terms in (5.8) together with $\left.\frac{\partial u_0}{\partial n}\right|_{\tilde{\Gamma}_j}$ from (2.5) and the terms $\frac{\partial}{\partial n}\, Z_m(c_m)|_{\tilde{\Gamma}_j} \in H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_j)$ for $s_0 \leqq m\nu < s$ *also from (2.5) constitute the regular part ψ_j^0 in (5.3).* □

Proof of Theorem 8. We proceed as in the proof of Theorem 6. Using the normal derivative of $\tilde{Z}^j_m(\tilde{c}^j_m)$ in (3.17), we have to show that

$$\frac{\partial}{\partial n}\, \tilde{R}^j_{m,p}|_{\tilde{\Gamma}_i} \in H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_i)$$

with $\tilde{R}^j_{m,p}$ as in (3.12) and $m\nu_j + 2p \leqq s_1$. To simplify notation we suppress the index j in the following. We observe with $r = e^t$ and $\tilde{c}_m^{(2p)}(e^t) := c_m^{(2p)}(t) = D_t^{2p}\tilde{c}_m(e^t)$

$$\text{(5.9)} \qquad \frac{\partial}{\partial n}\tilde{R}_{m,p}(\tilde{c}_m^{(2p)})|_{\tilde{\Gamma}_i} = \frac{1}{r}\left[c_m^{(2p)}(t) \underset{t}{*} \Phi(t,\theta) - c_m^{(2p)}(t)\right]\theta^{2p}\left(\frac{\partial}{\partial n} S_m\right)\Big|_{\Gamma_i}\chi(\theta)$$

$$= e^{-t}\frac{\partial}{\partial n} R_{m,p}(c_m^{(2p)})|_{\boldsymbol{R}\times\Gamma_i}$$

where $R_{m,p}$ is given by (2.18).

As in the proof of Theorem 6 we have $e^{\eta t}c_m^{(2p)}(t) \in H^{s-m\nu-2p}(\boldsymbol{R})$ with $\eta = \frac{1}{2} - s$ due to Lemma 5. In (5.9) Γ_i is one side of the spherical polygon S_0. Since $\chi(\theta)$ has its support in a small neighborhood of a corner of S_0, regularity in $\boldsymbol{R}\times\Gamma_i$ corresponds to regularity in $\boldsymbol{R}\times\Gamma_i^*$, where Γ_i^* is a side of an angular sector K in $\boldsymbol{R}^2$. Therefore Lemma 7(iii) gives

$$h(t,\theta_j) := e^{\eta t}\frac{\partial}{\partial n} R_{m,p}(c_m^{(2p)})|_{\boldsymbol{R}\times\Gamma_i^*}(t,\theta_j) \in H^{s-m\nu-2p}(\boldsymbol{R}\times\Gamma_i^*)$$

with $\eta = \frac{1}{2} - s$. But, due to the foregoing remark we have the same assertion with $\boldsymbol{R}\times\Gamma_i$ instead of $\boldsymbol{R}\times\Gamma_i^*$. Next we define the function $\tilde{h}(r,\theta_j)$ on $\tilde{\Gamma}_i$ by $\tilde{h}(e^t,\theta_j) = h(t,\theta_j)$. Then application of Lemma 5 implies with $n_1 = 2$

$$\tilde{h}(r,\theta_j) \in H^{s-m\nu-2p}_{-m\nu-(1/2)-2p}(\tilde{\Gamma}_i).$$

Furthermore $\frac{\partial}{\partial n}\tilde{R}_{m,p}(\tilde{c}_m^{(2p)}) = \frac{1}{r}\tilde{h}(r,\theta_j)$ yields

$$\frac{\partial}{\partial n}\tilde{R}_{m,p}(\tilde{c}_m^{(2p)})|_{\tilde{\Gamma}_i} \in H^{s-m\nu-2p}_{-m\nu+(1/2)-2p}(\tilde{\Gamma}_i) \subset H^{s-m\nu-2p}(\tilde{\Gamma}_i)$$

since $-m\nu + \frac{1}{2} - 2p \leqq 0$. The conditions $m\nu + 2p \leqq s_1$ and $s \geqq s_0 + s_1 - \frac{1}{2}$ give $s - m\nu - 2p \geqq -\frac{1}{2} + s_0 - \varepsilon$. Hence the terms $\frac{\partial}{\partial n}\tilde{R}_{m,p}(\tilde{c}_m^{(2p)})$ can be included in the regular part ψ^0 of (5.4). ∎

Corresponding to Corollaries 3 and 4, we have the following results for the normal derivative $\frac{\partial u}{\partial n}$. In the same way as Corollaries 3 and 4 follow from Theorem 6 one can deduce Corollaries 5 and 6 from Theorem 8. For brevity we omit their proofs.

Corollary 5. *Under the assumptions of Theorem* 8 *there holds with* w_k *as in* (3.24)

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \psi^0 + \chi(r)\sum_{\lambda_k+(1/2)<s_0} a_k r^{\lambda_k-1}\frac{\partial}{\partial n} w_k|_{\partial S_0}$$

$$+ \sum_{j=1}^{J}\chi_j(\theta_j)\sum_{m\nu_j+2p<s_0} r^{-1}e^j_{m,p}(r)\,\theta_j^{2p}\frac{\partial}{\partial n} S^j_m|_{\partial S_0}$$

where $\psi^0|_{\tilde{\Gamma}_j} \in H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_i)$, $a_k \in \boldsymbol{R}$, $\frac{\partial}{\partial n} w_k|_{\Gamma_i} \in H^{-(1/2)+s_0-\varepsilon}(\Gamma_i)$ *and* $e^j_{m,p}$ *as in* (3.26). *For* λ_k *an integer there are corresponding modifications as in Corollary* 3.

Corollary 6. *We make the assumptions of Theorem* 8. *For simplicity we assume that* $m\nu_j$ *and* λ_k *are not integers. Then there holds with* w_k *as in* (3.24) *and with the notation of Corollary* 4

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \psi^0 + \chi(r) \sum_{\lambda_k+(1/2)<s_0} a_k r^{\lambda_k-1} \frac{\partial}{\partial n} w_k|_{\partial S_0}$$
$$+ \sum_{j=1}^{J} \chi_j(\theta_j) \sum_{m\nu_j+2p<s_0} h^j_{m,p}(r)\, \varrho_j^{2p} \left(\frac{\partial}{\partial n} S^j_m\right) \quad (\varrho_j, \phi_j)|_{\partial\Omega}$$

where $\psi^0|_{\tilde{\Gamma}_i} \in H^{-(1/2)+s_0-\varepsilon}(\tilde{\Gamma}_i)$, $a_k \in \boldsymbol{R}$, $\frac{\partial}{\partial n} w_k|_{\Gamma_i} \in H^{-(1/2)+s_0-\varepsilon}(\Gamma_i)$ *and* $h^j_{m,p}$ *as in* (3.29).

Next we present an example in order to illustrate the decompositions of Theorem 8 and Corollary 5.

Example 4. Let Ω be the exterior of the square

$$Q = \{(x_1, x_2, x_3)\colon -1 \leqq x_1 \leqq 1,\ -1 \leqq x_2 \leqq 1,\ x_3 = 0\} \quad \text{in } \boldsymbol{R}^3.$$

We consider problem (5.1) with identical Dirichlet data $g \in H^{1/2+s}(Q)$ on both sides of Q.

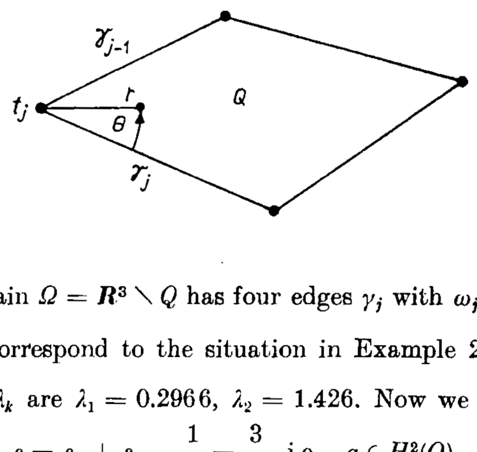

Fig. 8

The domain $\Omega = \boldsymbol{R}^3 \setminus Q$ has four edges γ_j with $\omega_j = 2\pi$ yielding $\nu_j = \frac{1}{2}$. The four corners t_j correspond to the situation in Example 2(iii). Hence the first two corner exponents λ_k are $\lambda_1 = 0.2966$, $\lambda_2 = 1.426$. Now we apply Theorem 8 with $s_0 = \frac{3}{2}$, $s_1 = \frac{1}{2}$ and $s = s_0 + s_1 - \frac{1}{2} = \frac{3}{2}$, i.e., $g \in H^2(Q)$. We obtain near each corner on the two sides Q_+ and Q_- of the square:

$$\left.\frac{\partial u}{\partial n}\right|_{Q_\pm} = \psi^0_\pm + \chi(r)\, a_1 r^{-.7034} \tilde{v}_\pm(\theta) + \chi(\theta)\, \tilde{b}^1_{1,0}(r)\, \theta^{-1/2}$$
$$+ \chi\left(\frac{\pi}{2} - \theta\right) \tilde{b}^2_{1,0}(r) \left(\frac{\pi}{2} - \theta\right)^{-1/2}$$

with $\psi^0_\pm \in H^{1-\varepsilon}(Q_\pm)$, $a_1 \in \boldsymbol{R}$, $\tilde{b}^j_{1,0} \in H^1_{-1/2}(\boldsymbol{R}^+)$.

Note that in this case we get the optimal regularity $\psi^0_\pm \in H^{-1/2+s-\varepsilon}(Q_\pm)$. Furthermore, Corollary 5 yields for $g \in H^2(Q)$ the decomposition

$$\left.\frac{\partial u}{\partial n}\right|_{Q_\pm} = \tilde{\psi}^0_\pm + \chi(r)\, a_1 r^{-.7034}\, \tilde{w}(\theta) + \chi(\theta)\, e^1_{1,0}(r)\, \theta^{-1/2}$$
$$+ \chi\left(\frac{\pi}{2} - \theta\right) e^2_{1,0}(r) \left(\frac{\pi}{2} - \theta\right)^{-1/2}$$

with $\tilde{\psi}^0_\pm \in H^{1-\varepsilon}(Q_\pm)$, $a_1 \in \boldsymbol{R}$, $\tilde{w} \in H^{1-\varepsilon}\left(\left[0, \frac{\pi}{2}\right]\right)$,

$$e^j_{1,0} = a^j_{11} r^{-.7034} + \tilde{b}^j_{1,0}, \qquad \tilde{b}^j_{1,0}(r) \in H^1_{-1/2}(\boldsymbol{R}^+).$$

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