

# Square root preconditioners for the Helmholtz integral equation

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## Abstract

We apply pseudo-differential operators theory to the first-kind integral equations on open curves, allowing us to analyze two new preconditioners and study the convergence orders of a Galerkin method on weighted  $L^2$  spaces.

## Introduction

### 1 Analytical setting

The Chebyshev polynomials of first and second kinds are respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sqrt{1-x^2}}$$

for  $x \in [-1, 1]$  []. They satisfy the ordinary differential equations

$$\begin{aligned} (1-x^2)T_n'' - xT_n' + n^2T_n &= 0 & (1) \\ (1-x^2)U_n'' - 3xU_n' + n(n+2)U_n &= 0 & (2) \end{aligned}$$

Let  $\omega$  the operator  $u(x) \mapsto \omega(x)u(x)$  with  $\omega(x) = \sqrt{1-x^2}$  and let  $\partial_x$  the derivation operator. Then (1) and (2) can be rewritten under the form

$$-(\omega\partial_x)^2T_n = n^2T_n, \quad (3)$$

$$-(\partial_x\omega)^2U_n = (n+1)^2U_n. \quad (4)$$

Notice that by  $(\partial_x\omega)f$  we mean  $(\omega f)'$ .

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## 1.1 Spaces $T^s$ and $U^s$

### 1.1.1 Definitions

Both  $T_n$  and  $U_n$  are polynomials of degree  $n$ , and form orthogonal families respectively of the Hilbert spaces

$$L_{\frac{1}{\omega}}^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 \frac{f^2(x)}{\sqrt{1-x^2}} dx < +\infty \right\}$$

and

$$L_{\omega}^2 := \left\{ u \in L_{\text{loc}}^1(-1, 1) \mid \int_{-1}^1 f^2(x) \sqrt{1-x^2} dx < +\infty \right\}.$$

We denote by  $\langle \cdot, \cdot \rangle_{\frac{1}{\omega}}$  and  $\langle \cdot, \cdot \rangle_{\omega}$  the inner products in  $L_{\frac{1}{\omega}}^2$  and  $L_{\omega}^2$  respectively. The Chebyshev polynomials satisfy

$$\langle T_n, T_m \rangle_{\frac{1}{\omega}} = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } m = n = 0 \\ \pi/2 & \text{otherwise} \end{cases} \quad (5)$$

and

$$\langle U_n, U_m \rangle_{\omega} = \begin{cases} 0 & \text{if } n \neq m \\ \pi/2 & \text{otherwise} \end{cases} \quad (6)$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any  $u \in L_{\frac{1}{\omega}}^2$  can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \quad (7)$$

where the Fourier-Chebyshev coefficients  $\hat{u}_n$  are given by

$$\hat{u}_n := \begin{cases} \frac{2}{\pi} \int_{-1}^1 \frac{u(x) T_n(x)}{\sqrt{1-x^2}} dx & \text{if } n \neq 0, \\ \frac{1}{\pi} \int_{-1}^1 \frac{u(x)}{\sqrt{1-x^2}} dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^1 \frac{u^2(x)}{\sqrt{1-x^2}} dx = \frac{\pi \hat{u}_0^2}{2} + \pi \sum_{n=1}^{+\infty} \hat{u}_n^2.$$

When  $u$  is furthermore a smooth function, the series (7) converges uniformly to  $u$ . Similarly, any function  $v \in L_{\omega}^2$  can be decomposed along the  $U_n$  as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients  $\check{v}_n$  are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^1 v(x) U_n(x) \sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^1 v^2(x) \sqrt{1-x^2} dx = \frac{\pi}{2} \sum_{n=0}^{+\infty} \check{v}_n^2.$$

The preceding analysis can be generalized to define Sobolev-like spaces.

**Definition 1.** For all  $s \geq 0$ , we may define

$$T^s = \left\{ u \in L_{\frac{1}{\omega}}^2 \left| \sum_{n=0}^{+\infty} (1+n^2)^s \hat{u}_n^2 < +\infty \right. \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u, v \rangle_{T^s} = \frac{\pi}{2} \hat{u}_0 \hat{v}_0 + \pi \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty} n^{2s} |\hat{u}_n|^2.$$

We denote by  $T^\infty$  the Fréchet space  $T^\infty := \bigcap_{s \in \mathbb{R}} T^s$ , and by  $T^{-\infty}$  the set of continuous linear forms on  $T^\infty$ . For  $l \in T^{-\infty}$ , we note  $\hat{l}_n = l(T_n)$ , so that for  $u \in T^\infty$ ,

$$l(u) = \frac{\pi}{2} \hat{l}_0 \hat{u}_0 + \pi \sum_{n=1}^{+\infty} \hat{l}_n \hat{u}_n.$$

We choose to identify the dual of  $L_{\frac{1}{\omega}}^2$  to itself using the previous bilinear form. With this identification, any element of  $T^s$  with  $s \geq 0$  can also be seen as an element of  $T^{-\infty}$ . Furthermore, the space  $T^{-s}$  can be defined for all  $s \geq 0$  as

$$T^{-s} = \left\{ u \in T^{-\infty} \left| \sum_{n=0}^{+\infty} (1+n^2)^{-s} \hat{u}_n^2 < \infty \right. \right\}.$$

Using the former identification  $T^{-s}$  becomes the dual of  $T^s$ . For  $s < t$ , the inclusion  $T^s \subset T^t$  is compact.

**Remark 1.** The spaces  $T^n$  correspond, up to a variable change, to the spaces  $H_e^n$  defined in [1, 3, 16, 17] among other works, that is, the restriction of the usual Sobolev space  $H^n$  to even periodic functions, as stated in Lemma 7.

In a similar fashion, we define the following spaces:

**Definition 2.** For all  $s \geq 0$ , we set

$$U^s = \left\{ u \in L_{\omega}^2 \left| \sum_{n=0}^{+\infty} (1+n^2)^s \check{u}_n^2 < \infty \right. \right\}.$$

We extend as before the definition to negative indices by setting  $U^{-s}$  to be the dual of  $U^s$  for  $s \geq 0$ , this time with respect to the duality  $\langle \cdot, \cdot \rangle_{\omega}$ .

### 1.1.2 Basic properties

Obviously, for any real  $s$ , if  $u \in T^s$  the sequence of polynomials

$$S_N(x) = \sum_{n=0}^N \hat{u}_n T_n(x)$$

converges to  $u$  in  $T^s$ . The same assertion holds for  $u \in U^s$  when  $T_n$  is replaced by  $U_n$ . Therefore

**Lemma 1.**  $C^\infty([-1, 1])$  is dense in  $T^s$  and  $U^s$  for all  $s \in \mathbb{R}$ .

The polynomials  $T_n$  and  $U_n$  are connected by the following formulas:

$$\forall n \geq 2, \quad T_n(x) = \frac{1}{2} (U_n - U_{n-2}), \quad (8)$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2 \sum_{j=0}^n T_{2j} - 1, \quad U_{2n+1} = 2 \sum_{j=0}^n T_{2j+1}. \quad (9)$$

We deduce the following inclusions:

**Lemma 2.** For all real  $s$ ,  $T^s \subset U^s$  and for all  $s > 1/2$ ,  $U^s \subset T^{s-1}$ .

Before starting the proof, we introduce the Cesàro operator  $C$  defined on  $l^2(\mathbb{N}^*)$  by

$$(Cu)_n = \frac{1}{n} \sum_{k=1}^n u_k.$$

As is well-known, this is a linear continuous operator on  $l^2(\mathbb{N}^*)$ . Its adjoint

$$(C^*u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k},$$

is therefore also continuous on  $l^2(\mathbb{N}^*)$ . In other words, for all  $u \in l^2(\mathbb{N})$ ,

$$\sum_{n=1}^{+\infty} \left( \sum_{k=n}^{+\infty} \frac{u_k}{k} \right)^2 \leq C \sum_{k=1}^{+\infty} u_k^2.$$

*Proof.* The first property is immediate from (8). When  $u \in U^s$  for  $s > 1/2$ , the series  $\sum |\check{u}_n|$  is converging, allowing to identify  $u$  to a function in  $T^{-\infty}$ , with, in view of (9),

$$\hat{u}_0 = 2 \sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2 \sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since  $u \in U^s$ , the sequence  $((1+n^2)^{s/2} |\check{u}|)_{n \geq 1}$  is in  $l^2(\mathbb{N}^*)$ . Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the se-

quence  $r_n := \left( \sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k| \right)_{n \geq 0}$  is in  $l^2(\mathbb{N})$ . But

$$\begin{aligned} \|u\|_{T^{s-1}}^2 &= \sum_{n=0}^{+\infty} (1+n^2)^{s-1} |\hat{u}_n|^2 \\ &\leq 4 \sum_{n=0}^{+\infty} (1+n^2)^{s-1} \left( \sum_{k=n}^{+\infty} |\check{u}_k| \right)^2 \\ &\leq 4 \sum_{n=0}^{+\infty} \left( \sum_{k=n}^{+\infty} (1+k^2)^{\frac{s-1}{2}} |\check{u}_k| \right)^2 \\ &= 4 \|r_n\|_{l^2}^2. \end{aligned}$$

□

One immediate consequence is that  $T^\infty = U^\infty$ . Moreover, we have the following result:

**Lemma 3.**

$$T^\infty = C^\infty([-1, 1]).$$

*Proof.* If  $u \in C^\infty([-1, 1])$ , then we can obtain by induction using integration by parts and (3), that for any  $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}} \int_{-1}^1 \frac{(\omega \partial_x)^{2k} u(x) T_n(x)}{\omega(x)} dx.$$

Noting that  $(\omega \partial_x)^2 = (1-x^2) \partial_x^2 - x \partial_x$ , the function  $(\omega \partial_x)^{2k} u$  is  $C^\infty$ , and since  $\|T_n\|_\infty = 1$ , the integral is bounded independently of  $n$ . Thus, the coefficients  $\hat{u}_n$  have a fast decay, proving that  $C^\infty([-1, 1]) \subset T^\infty$ .

For the converse inclusion, if  $u \in T^\infty$ , the series

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n T_n(x)$$

is normally converging since  $\|T_n\|_\infty = 1$ , so  $u$  is a continuous function. This proves  $T^\infty \subset C^0([-1, 1])$ . It suffices to show that  $u' \in T^\infty$  and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty} n u_n U_{n-1}(x).$$

Therefore,  $u'$  is in  $U^\infty = T^\infty$ . This proves the result. □

**Lemma 4.** For  $s \leq \frac{1}{2}$ , the functions of  $U^s$  cannot be identified to functions in  $T^{-\infty}$ .

*Proof.* Assume by contradiction that the functions of  $U^{\frac{1}{2}}$  can be identified to elements of  $T^{-\infty}$ . Then, there must exist a map  $I$  continuous from  $U^{\frac{1}{2}}$  to  $T^{-\infty}$  with the property

$$\forall u \in U^\infty, \quad Iu = u.$$

Now, let us consider for example the function  $u$  defined by  $\check{u}_n = \frac{1}{n \ln(n)}$ . Note that  $u$  is in  $U^{1/2}$ . Let  $u_N = \sum_{n=0}^N \check{u}_n U_n$ . Then  $(u_N)$  is a sequence of elements of  $U^\infty$  converging to  $u$  in  $U^{1/2}$ . By continuity of  $I$ , and since  $Iu_N = u_N$ , the sequence  $(\langle u_N, T_0 \rangle_{\frac{1}{\omega}})_{N \in \mathbb{N}}$  must converge with limit  $\langle Iu, T_0 \rangle$ . This is not the case since

$$\langle u_N, T_0 \rangle_{\frac{1}{\omega}} = \sum_{n=0}^N \check{u}_n \langle U_n, T_0 \rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2k \ln(2k)}$$

is diverging. The result for  $s \leq \frac{1}{2}$  comes from the fact that  $U^{\frac{1}{2}} \subset U^s$  for  $s \leq \frac{1}{2}$ .  $\square$

Two natural derivation operators arise in our context, that give another link between  $T^s$  and  $U^s$ . They are given by the identities

$$\partial_x T_n = n U_{n-1}, \quad (10)$$

$$-\omega \partial_x \omega U_n = (n+1) T_{n+1}. \quad (11)$$

The first one is obtained for example from the trigonometric definition of  $T_n$ . This combined with  $-(\omega \partial_x)^2 T_n = n^2 T_n$  gives the second identity.

**Definition 3.** For all real  $s$ , the operator  $\partial_x$  can be extended into a continuous map from  $T^{s+1}$  to  $U^s$  defined as

$$\forall v \in U^\infty, \quad \langle \partial_x u, v \rangle_\omega := -\langle u, \omega \partial_x \omega v \rangle_{\frac{1}{\omega}}.$$

In a similar fashion, the operator  $\omega \partial_x \omega$  can be extended into a continuous map from  $U^{s+1}$  to  $T^s$  defined as

$$\forall v \in T^\infty, \quad \langle \omega \partial_x \omega u, v \rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v \rangle_\omega.$$

*Proof.* Using the identities (10) and (11), one can check that the formulas indeed extend the usual definition of the two operators for smooth functions. We now show that the map  $\partial_x$  extended this way is continuous from  $T^{s+1}$  to  $U^s$ . The definition

$$\forall v \in U^\infty, \langle \partial_x u, v \rangle := -\langle u, \omega \partial_x \omega v \rangle$$

gives a sense to  $\partial_x u$  for all  $u$  in  $T^{-\infty}$ , as a duality  $T^{-\infty} \times T^\infty$  product, because if  $v \in U^\infty (= C^\infty)$ , then  $\omega \partial_x \omega v = (1 - x^2)v' - xv$  also lies in  $C^\infty (= T^\infty)$ . It remains to check the announced continuity. Letting  $w = \partial_x u$ , we have, by definition, for all  $n$

$$\check{w}_n = \langle w, U_n \rangle_\omega = -\langle u, \omega \partial_x \omega U_n \rangle_{\frac{1}{\omega}} = n \langle u, T_{n+1} \rangle_{\frac{1}{\omega}} = n \hat{u}_{n+1}$$

Obviously, this implies the announced continuity with

$$\|w\|_{U^s} \leq \|u\|_{T^{s+1}}.$$

The properties of  $\omega \partial_x \omega$  on  $T^s$  are established in a similar way.  $\square$

The operator  $\partial_x$  is not continuous from  $T^s$  to  $T^{s-1}$ . However, the following result holds:

**Corollary 1.** *The operator  $\partial_x$  is continuous from  $T^{s+2}$  to  $T^s$  for all  $s > -1/2$  and from  $U^{s+2}$  to  $U^s$  for all  $s > -3/2$ . On the other hand,  $\omega\partial_x\omega$  is continuous from  $T^{s+1}$  to  $T^s$  and from  $U^{s+1}$  to  $U^s$  for all  $s \in \mathbb{R}$ .*

*Proof.* For the continuity of  $\partial_x$  from  $T^{s+2}$  to  $T^s$ , we see the continuity of  $\partial_x$  from  $T^{s+2}$  to  $U^{s+1}$  and then of the identity from  $U^{s+1}$  to  $T^s$ . For the continuity of  $\partial_x$  from  $U^{s+2}$  to  $U^s$ , we use the same arguments in reverse order.

On the other hand, we have, for  $n \geq 2$ ,

$$\omega\partial_x\omega T_n = \omega\partial_x\omega \frac{U_n - U_{n-2}}{2} = \frac{(n+1)T_{n+1} - (n-1)T_{n-1}}{2}.$$

Therefore  $\omega\partial_x\omega$  is continuous from  $T^{s+1}$  to  $T^s$ . Finally,  $\omega\partial_x\omega$  is continuous from  $U^{s+1}$  to  $T^s$  and the inclusion  $T^s \subset U^s$  is continuous for all  $s$ .  $\square$

**Lemma 5.** *For all  $\varepsilon > 0$ , if  $u \in T^{\frac{1}{2}+\varepsilon}$ , then  $u$  is continuous and there exists a constant  $C$  such that for all  $x \in [-1, 1]$ ,*

$$|u(x)| \leq C \|u\|_{T^{\frac{1}{2}+\varepsilon}}.$$

*Similarly, if  $u \in U^{\frac{3}{2}+\varepsilon}$ , then  $u$  is continuous and*

$$|u(x)| \leq C \|u\|_{U^{\frac{3}{2}+\varepsilon}}.$$

*Proof.* Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all  $n$ ,  $\|T_n\|_{L^\infty} = 1$ . Cauchy-Schwarz's inequality then yields

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \|u\|_{T^{\frac{1}{2}+\varepsilon}}.$$

For the second statement, we use the inclusion  $U^s \subset T^{s-1}$  valid for  $s > 1/2$ , as established in Lemma 2.  $\square$

### 1.1.3 Characterization of $T^n$ and $U^n$ .

In this section, we provide a characterization of the spaces  $T^s$  and  $U^s$  in terms of  $L^2$  norms of the derivatives. In what follows,  $H_p^s(0, T)$  denotes the space of functions in  $H^s(\mathbb{T}_T)$ , where  $\mathbb{T}_T$  is the torus  $\mathbb{R}/(T\mathbb{Z})$ ,  $H_e^s(0, T)$  (resp.  $H_o^s(0, T)$ ) is the set of even (resp. odd) functions if  $H_p^s(0, T)$ .

**Lemma 6.** *The operator  $\omega\partial_x$  has a continuous extension from  $T^1$  to  $T^0$ . Similarly, the operator  $\partial_x\omega$  has a continuous extension from  $U^1$  to  $U^0$ .*

*Proof.* Obviously, the operator  $\omega$  maps  $L_\omega^2 = U^0$  to  $L_{\frac{1}{\omega}}^2 = T^0$ . This is in fact a bijective isometry with inverse  $\frac{1}{\omega}$ . Since  $\partial_x$  is continuous from  $T^1$  to  $U^0$ , we have the announced continuity of  $\omega\partial_x$ . For the second part, we write

$$\partial_x\omega = \frac{1}{\omega} (\omega\partial_x\omega).$$

Where  $\omega\partial_x\omega$  is continuous from  $U^1$  to  $T^0$ , and the multiplication by  $\frac{1}{\omega}$  is continuous from  $T^0$  to  $U^0$ .  $\square$

We can now state the main result of this paragraph. For a function  $u$  defined on  $[-1, 1]$ , we denote by  $Cu$  the function defined on  $\mathbb{T}_{2\pi}$  by

$$Cu(\theta) = u(\cos(\theta))$$

and by  $Su$  the function defined as

$$Su(\theta) := \sin(\theta)Cu(\theta).$$

**Lemma 7.** *A function  $u$  belongs to the space  $T^n$  if and only if  $u = \tilde{u} \circ \arccos$  for some function  $\tilde{u} \in H_e^n(0, 2\pi)$ . In this case,  $Cu = \tilde{u}$  and*

$$\|u\|_{T^n} \sim \|Cu\|_{H^n} \quad \text{and} \quad |u|_{T^n} \sim |Cu|_{H^n}.$$

*Similarly,  $u$  belongs to the space  $U^n$  if and only if  $u = \frac{1}{\sqrt{1-x^2}}\tilde{u} \circ \arccos$  for some function  $\tilde{u} \in H_o^n(0, 2\pi)$ . In this case,  $Su = \tilde{u}$  and*

$$\|u\|_{U^n} \sim |Su|_{H^n}.$$

*Moreover, if  $u \in T^n$ , then  $(\omega\partial_x)^n u$  is in  $L_{\omega}^2$  and*

$$|u|_{T^n}^2 \sim \int_{-1}^1 \frac{((\omega\partial_x)^n u)^2}{\omega}.$$

*Similarly, if  $u \in U^n$ , then  $(\partial_x\omega)^n u \in L_{\omega}^2$  and*

$$\|u\|_{U^n}^2 = \int_{-1}^1 \omega((\partial_x\omega)^n u)^2.$$

*Proof.* The first two equivalences stem from the fact that

$$\hat{u}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} Cu(\theta) \cos(n\theta), \quad \check{u}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} Su(\theta) \sin((n+1)\theta) d\theta,$$

which can be verified by using the change of variables  $x = \cos \theta$  in the definitions of  $\hat{u}_n$  and  $\check{u}_n$ . Now, let us show that if  $u \in T^n$ , then  $(\omega\partial_x)^n u$  is in  $L_{\omega}^2$ . The operator  $(\omega\partial_x)^2$  is continuous from  $T^s$  to  $T^{s-2}$  for all real  $s$  which implies the result if  $n$  is even. If  $n$  is odd, say  $n = 2k + 1$ , we write  $(\omega\partial_x)((\omega\partial_x)^2)^k$ , and conclude using Lemma 6. The same kind of proof also shows that if  $u \in U^n$ ,  $(\partial_x\omega)^n u \in L_{\omega}^2$ . The rest of the proof can be performed by computing the quantities for functions in  $C^\infty([-1, 1])$ , performing integrations by parts and concluding with the density of  $T^\infty$  in  $T^s$  and  $U^s$ .  $\square$

## 1.2 Spaces $T^s(\Gamma)$ and $U^s(\Gamma)$

Let  $\Gamma$  an smooth open curve in  $\mathbb{R}^2$  parametrized by a smooth  $C^\infty$  diffeomorphism  $r : [-1, 1] \rightarrow \Gamma$ . We choose  $r$  such that  $|r'(x)| = \frac{|\Gamma|}{2}$  for all  $x$ , where  $|\Gamma|$  is the length of  $\Gamma$ . For  $u : [-1, 1] \rightarrow \mathbb{R}$ , let  $R$  the operator defined by  $Ru(x) = u(r(x))$ . The tangential derivative on  $\Gamma$  is defined as  $\partial_\tau = \frac{2}{|\Gamma|} R\partial_x R^{-1}$ . Moreover, let  $\omega_\Gamma := \frac{|\Gamma|}{2} R^{-1}\omega(x)R$ . The previous results can be generalized to this context:



**Definition 4.** We say that a function  $u : \Gamma \rightarrow \mathbb{C}$  belongs to  $T^s(\Gamma)$  if the function  $R^{-1}u$  belongs to  $T^s$ . Similarly, we say that  $u \in U^s(\Gamma)$  if  $R^{-1}u \in U^s$ .

**Theorem 1.** For all  $s \in \mathbb{R}$ ,  $T^s(\Gamma)$  and  $U^s(\Gamma)$  are Hilbert spaces for the scalar products

$$\begin{aligned}\langle u, v \rangle_{T^s(\Gamma)} &= \langle R^{-1}u, R^{-1}v \rangle_{T^s}, \\ \langle u, v \rangle_{U^s(\Gamma)} &= \langle R^{-1}u, R^{-1}v \rangle_{U^s}.\end{aligned}$$

For  $s < t$ , the injections  $T^s(\Gamma) \subset T^t(\Gamma)$  and  $U^s(\Gamma) \subset U^t(\Gamma)$  are compact. For  $n \in \mathbb{N}$ , an equivalent scalar product on  $T^n$  and  $U^n$  is

$$\begin{aligned}\langle u, v \rangle_{T^n} &= \int_{\Gamma} \frac{u(x)v(x) + (\omega_{\Gamma}\partial_{\tau})^n u(x)(\omega_{\Gamma}\partial_{\tau})^n v(x)}{\omega_{\Gamma}(x)} d\sigma(x), \\ \langle u, v \rangle_{U^n} &= \int_{\Gamma} (\partial_{\tau}\omega_{\Gamma})^n u(x)(\partial_{\tau}\omega_{\Gamma})^n v(x)\omega_{\Gamma}(x) d\sigma(x),\end{aligned}$$

For all  $s \in \mathbb{R}$ ,  $T^s(\Gamma) \subset U^s(\Gamma)$  and for all  $s > \frac{1}{2}$ ,  $U^s(\Gamma) \subset T^{s+1}(\Gamma)$ . For  $s > \frac{1}{2}$ ,  $T^s(\Gamma) \subset C^0(\Gamma)$  and  $U^{s+1} \subset C^0(\Gamma)$ . Moreover,  $T^{\infty}(\Gamma) = U^{\infty}(\Gamma) = C^{\infty}(\Gamma)$ .

### 1.3 Periodic pseudo-differential operators

On the family of spaces  $H_p^s(0, T)$ , a class of periodic pseudo-differential operators (PPDO) has been introduced in [15], with symbolic calculus. A PPDO on  $H_p^s(0, T)$  of order  $p$  is an operator of the form

$$A : u \in H_p^s(0, T) \mapsto \sum_{n \in \mathbb{Z}} \sigma_A(t, n) \hat{u}_n e^{\frac{2in\pi t}{T}}.$$

for a "prolongated symbol"  $\sigma_A \in C^{\infty}(\mathbb{T}_T \times \mathbb{R})$  satisfying

$$\forall j, k \in \mathbb{N}, \exists C_{j,k} > 0 : \quad \left| \partial_t^j \partial_{\xi}^k \sigma_A(t, \xi) \right| \leq C_{j,k} (1 + |\xi|)^{p-k}. \quad (12)$$

Here,  $\hat{u}_n = \frac{1}{T} \int_0^T u(t) e^{-i\frac{2n\pi t}{T}} dt$  are the usual Fourier coefficients of  $u$  and

$$\partial_t := \frac{T}{2i\pi} \frac{\partial}{\partial t}, \quad \partial_{\xi} := \frac{T}{2i\pi} \frac{\partial}{\partial \xi},$$

with for  $j \geq 1$ ,  $\partial_t^{j+1} = \partial_t \partial_t^j$ , and  $\partial_{\xi}^{j+1} = \partial_{\xi} \partial_{\xi}^j$ . The class of symbols that satisfy (12) is denoted by  $\Sigma^{\alpha}$ , and  $\Sigma^{-\infty} := \cup_{\alpha \in \mathbb{Z}} \Sigma^{\alpha}$ . The operator corresponding to a symbol  $\sigma$  is denoted by  $Op(\sigma)$  and the set of PPDO of order  $\alpha$  is denoted by  $Op(\Sigma^{\alpha})$ .

The prolonged symbol is not unique but is determined uniquely at the integer values of  $\xi$  by

$$\sigma_A(t, n) = e^{-\frac{2in\pi t}{T}} A(e^{\frac{2in\pi t}{T}}). \quad (13)$$

as shown in [15], justifying the terminology of "prolongated symbol".  $A \in Op(\Sigma^{\alpha})$  if and only if those values satisfy

$$\forall j, k \in \mathbb{N}, \exists C_{j,k} > 0 : \quad \left| \partial_t^j \Delta_n^k \sigma_A(t, n) \right| \leq C_{j,k} (1 + |n|)^{\alpha-k},$$

where  $\Delta_n \phi(t, n) = \phi(t, n+1) - \phi(t, n)$  and for  $k \geq 1$ ,  $\Delta^{k+1} \phi = \Delta(\Delta^k \phi)$ . That is, if the symbol defined in (13) satisfies this condition, then there exists a prolonged symbol satisfying (12). Because of this, we will abusively write  $\sigma \in \Sigma^p$  for a function  $\sigma(t, n)$  that can be prolonged to a function  $\tilde{\sigma}(t, \xi) \in \Sigma^p$ . An operator in  $Op(\Sigma^\alpha)$  maps continuously  $H_p^s(0, T)$  to  $H_p^{s+\alpha}(0, T)$  for all  $s \in \mathbb{R}$ . The composition of two operators in  $Op(\Sigma^\alpha)$  and  $Op(\Sigma^\beta)$  gives rise to an operator in  $Op(\Sigma^{\alpha+\beta})$ . If two symbols  $a$  and  $b$  in  $\Sigma^{-\infty}$  satisfy  $a - b \in \Sigma^\alpha$ , we write  $a = b + \Sigma^\alpha$ .

**Definition 5.** Let  $a \in \Sigma^{-\infty}$ . If there exists a sequence of reals  $(p_j)_{j \in \mathbb{N}}$  such that  $p_j < p_{j+1}$  and a sequence of symbols  $a_j \in \Sigma^{p_j}$  such that for all  $N$ ,

$$a = \sum_{i=0}^N a_i + \Sigma^{p_{N+1}},$$

we then write

$$a = \sum_{i=0}^{+\infty} a_i.$$

This is called an asymptotic expansion of the symbol  $a$ .

The composition of two periodic pseudifferential operators  $A$  and  $B$  with symbols  $\sigma_A$  and  $\sigma_B$  has a symbol  $\sigma_C$  with the following asymptotic expansion

$$\sigma_C(t, \xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left( \frac{\partial}{\partial \xi} \right)^j \sigma_A(t, \xi) \partial_t^j \sigma_B(t, \xi). \quad (14)$$

We will use the following result, see [15]:

**Theorem 2.** Consider an integral operator  $K$  of the form

$$K : u \mapsto \frac{1}{T} \int_0^T a(t, s) \kappa(t-s) u(s) ds.$$

where  $a$  is  $T$ -periodic and  $C^\infty$  in both arguments and  $\kappa$  is a  $T$ -periodic distribution. Assume that the Fourier coefficients  $\hat{\kappa}(n)$  of  $\kappa$  can be prolonged to a function  $\hat{\kappa}(\xi)$  on  $\mathbb{R}$  such that

$$\forall k \in \mathbb{N}, \exists C_k > 0 : |\partial_\xi^k \hat{\kappa}(\xi)| \leq C_k (1 + |\xi|)^{\alpha-k}.$$

for some  $\alpha$ . Then  $K$  is in  $Op(\Sigma^\alpha)$  with a symbol satisfying the asymptotic expansion

$$\sigma_K(t, \xi) = \sum_{j=0}^{+\infty} \frac{1}{j!} \left( \frac{\partial}{\partial \xi} \right)^j \hat{\kappa}(\xi) \partial_s^j a(t, s)|_{s=t}.$$

In particular, taking  $\kappa = 1$ , we see that for all functions  $a \in C^\infty(\mathbb{T}_T^2)$

$$Ku = \frac{1}{T} \int_0^T a(t, s) u(s) ds$$

is in  $Op(\Sigma^{-\infty})$ .

## 1.4 Pseudo-differential operators on $T^s(\Gamma)$

**Lemma 8.** *Let  $A$  a PPDO that stabilizes the set of smooth even functions. Then  $A$  coincides on this set with the operator  $B$  with symbol given by*

$$\sigma_B(\theta, n) = \frac{\sigma_A(\theta, n) + \sigma_A(-\theta, -n)}{2}.$$

Moreover,  $\sigma_B$  admits the following decomposition:

$$\sigma_B(n, \theta) = a_1(n, \cos \theta) + i \sin(\theta) a_2(n, \cos \theta)$$

with

$$\begin{aligned} a_1(x, n) &= \frac{\sigma_B(\arccos(x), n) + \sigma_B(\arccos(x), -n)}{2} \\ a_2(x, n) &= \frac{\sigma_B(\arccos(x), n) - \sigma_B(\arccos(x), -n)}{2i\sqrt{1-x^2}} \end{aligned}$$

and  $a_1$  and  $a_2$  are  $C^\infty$ .

*Proof.* Since  $A$  stabilizes the set of even functions, for all even function  $u$  we have

$$Au(\theta) = \frac{Au(\theta) + Au(-\theta)}{2}.$$

Thus

$$Au(\theta) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sigma_A(n, \theta) \hat{u}_n e^{in\theta} + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sigma_A(n, -\theta) \hat{u}_n e^{-in\theta}.$$

Since  $u$  is even,  $\hat{u}_n = \hat{u}_{-n}$ , therefore,  $Au(\theta) = Bu(\theta)$  where  $B$  is the operator with symbol  $\sigma_B(n, \theta) = \frac{\sigma_A(n, \theta) + \sigma_A(-n, -\theta)}{2}$ . In particular, it satisfies the following symmetry:

$$\sigma_B(-n, -\theta) = \sigma_B(n, \theta).$$

We write  $\sigma_B(n, \theta) = f_B(n, \theta) + g_B(n, \theta)$  where  $f_B(n, \theta) = \frac{\sigma_B(n, \theta) + \sigma_B(-n, -\theta)}{2}$  and  $g_B(n, \theta) = \frac{\sigma_B(n, \theta) - \sigma_B(-n, -\theta)}{2}$ . Notice that  $f_B$  (resp.  $g_B$ ) is even (resp. odd) in both  $\theta$  and  $n$ . The functions  $a_1$  and  $a_2$  defined in the lemma satisfy

$$a_1(n, x) = f_B(n, \arccos(x)), \quad a_2(n, x) = \frac{g_B(n, \arccos(x))}{i\sqrt{1-x^2}}.$$

By Lemma 7, they lie in  $T^\infty$  since  $f_B$  (resp.  $g_B$ ) is a smooth even (resp. odd) function and  $f_B(n, \theta) = a_1(n, \cos \theta)$ , while  $g_B(n, \theta) = i \sin \theta a_2(n, \cos \theta)$ . Thus

$$\sigma_B(n, \theta) = a_1(n, \cos \theta) + i \sin \theta a_2(n, \cos \theta).$$

□

We use those results to transport the notion of periodic pseudo-differential operators by the change of variable  $x = \cos \theta$ . Let  $A$  an operator on  $T^{-\infty}$ , assume that there exists a couple of smooth functions  $a_1$  and  $a_2$  in  $C^\infty([-1, 1] \times \mathbb{N})$  such that for all  $n \in \mathbb{N}$ ,

$$AT_n = a_1(x, n)T_n - \omega^2 a_2(x, n)U_{n-1}.$$

Such a (non-unique) couple of functions is called a pair of symbols of  $A$ . For  $n \in \mathbb{Z}$  and  $\theta \in [0, 2\pi]$ , define the symbol  $\tilde{\sigma}(a_1, a_2)$  by

$$\tilde{\sigma}(a_1, a_2)(\theta, n) = a_1(\cos \theta, |n|) + i \sin \theta \operatorname{sign}(n) a_2(\cos \theta, |n|).$$

We say that  $(a_1, a_2) \in S_T^\alpha$  if  $\tilde{\sigma}(a_1, a_2) \in \Sigma^\alpha$ , and  $S_T^{-\infty} := \cup_{p \in \mathbb{Z}} S_T^p$ . The operator defined by a pair of symbols  $(a_1, a_2)$  is denoted by  $Op_T(a_1, a_2)$  and the set of pseudo-differential operators of order  $p$  in  $T^{-\infty}$  by  $Op(S_T^\alpha)$ . Then, we have the following properties of  $Op(S_T^\alpha)$ :

**Lemma 9.** *If  $A \in Op(S^p)$ , then letting  $\tilde{A} := Op(\tilde{\sigma}_A)$ , for  $u \in T^s$ , we have*

$$C(Au) = \tilde{A}(Cu),$$

where we recall that for any function  $v$  in  $T^s$ ,  $Cv(\theta) = v(\cos(\theta))$ . Reciprocally, if  $A$  is a linear operator that maps  $T^\infty$  to itself such that  $CA = \tilde{A}C$  where  $\tilde{A}$  is a PPDO of order  $p$ , then  $A$  is in  $Op(S^p)$  and if  $\sigma_{\tilde{A}}$  is the symbol of  $\tilde{A}$ , then  $A = Op(a_1, a_2)$  with the functions  $a_1$  and  $a_2$  of Lemma 8.

*Proof.* For the direct result, by linearity, it suffices to show the equality for  $u = T_n$  for some  $n \in \mathbb{N}$ . In this case, we have  $Cu(\theta) = T_n(\cos(\theta)) = \cos(n\theta)$ .

$$\tilde{A}(Cu)(\theta) = \frac{\tilde{A}e^{in\theta} + \tilde{A}e^{-in\theta}}{2}$$

which gives, by definition of  $\tilde{A}$  and using the determination of the symbol (13),

$$\tilde{A}(Cu)(\theta) = \tilde{\sigma}_A(\theta, n)e^{in\theta} + \tilde{\sigma}_A(\theta, -n)e^{-in\theta}.$$

By definition of  $\tilde{\sigma}_A$ , this gives

$$\tilde{A}(Cu)(\theta) = a_1(\cos \theta, n) \cos(n\theta) - \sin \theta a_2(\cos \theta, n) \sin(n\theta).$$

Using the identities  $\cos(n\theta) = T_n(\cos \theta)$  and  $\sin(n\theta) = \sin \theta U_{n-1}(\cos \theta)$  we obtain

$$\tilde{A}(Cu)(\theta) = a_1(\cos \theta, n)T_n(\cos \theta) - (1 - \cos^2 \theta)a_2(\cos \theta, n)U_{n-1}(\cos \theta),$$

as claimed. For the converse result, assume that  $CA = \tilde{A}C$  where  $\tilde{A}$  is a PPDO of order  $p$ . Then  $\tilde{A}$  stabilizes the set of smooth real even functions since  $CAu(\theta) = Au(\cos \theta)$  is real and even and  $A$  maps smooth functions to smooth functions. Let  $B$ ,  $a_1$  and  $a_2$  be defined as in ?? for the operator  $\tilde{A}$ . Using the same calculations as before, we find

$$A(T_n)(\cos \theta) = a_1(n, \cos \theta)T_n(\cos \theta) - \omega^2 a_2(n, \cos \theta)U_{n-1}(\cos \theta),$$

that is  $A = Op(a_1, a_2)$ . Since  $\sigma_B \in \Sigma^p$ , we get  $(a_1, a_2) \in S^p$  as claimed.  $\square$

**Definition 6.** *Let  $A : T^{-\infty}(\Gamma) \rightarrow T^{-\infty}(\Gamma)$ . We say that  $A$  is a pseudo-differential operator if  $RAR^{-1} \in Op(S_T^{-\infty})$ . Moreover, if  $RAR^{-1} \in Op(S_T^\alpha)$ , we say that  $A$  is a pseudo-differential operator of order  $\alpha$ . The set of pseudo-differential operators of order  $\alpha$  on  $\Gamma$  is denoted by  $S_T^\alpha(\Gamma)$ .*

**Definition 7.** *Let  $A$  and  $B$  in  $Op(S_T^{-\infty}(\Gamma))$ . If  $A - B \in Op(S_T^\alpha(\Gamma))$ , we write*

$$A = B + T_\alpha.$$

## 1.5 Pseudo-differential operators on $U^s(\Gamma)$

We may define similarly a class of pseudo-differential operators on the spaces  $U^s(\Gamma)$ . One can show the following result:

**Lemma 10.** *Let  $A$  a PPDO that stabilizes the set of smooth odd functions. Then  $A$  coincides on this set with the operator  $B$  with symbol given by*

$$\sigma_B(n, \theta) = \frac{\sigma_A(n, \theta) + \sigma_A(-n, -\theta)}{2}.$$

Moreover,  $\sigma_B$  admits the following decomposition

$$\sigma_B(n, \theta) = ia_1(\cos \theta, n) + \sin \theta a_2(\cos \theta, n)$$

with

$$\begin{aligned} a_1(x, n) &= \frac{\sigma_B(n, \arccos(x)) + \sigma_B(-n, \arccos(x))}{2i} \\ a_2(n, x) &= \frac{\sigma_B(n, \arccos(x)) - \sigma_B(-n, \arccos(x))}{2\sqrt{1-x^2}} \end{aligned}$$

and  $a_1$  and  $a_2$  are  $C^\infty$ .

Let  $A$  an operator on  $U^{-\infty}$ , assume that there exists a couple of smooth functions  $a_1$  and  $a_2$  in  $C^\infty([-1, 1] \times \mathbb{N})$  such that for all  $n \in \mathbb{N}$ ,

$$AU_n = a_1(x, n)U_n + a_2(x, n)T_{n+1}.$$

Such a (non-unique) couple of functions is called a pair of symbols of  $A$ . For  $n \in \mathbb{Z}$  and  $\theta \in [0, 2\pi]$ , define the symbol  $\tilde{\sigma}(a_1, a_2)$  by

$$\tilde{\sigma}(a_1, a_2)(\theta, n) = ia_1(\cos \theta, |n|) + \sin \theta \operatorname{sign}(n)a_2(\cos \theta, |n|).$$

We say that  $(a_1, a_2) \in S_U^\alpha$  if  $\tilde{\sigma}(a_1, a_2) \in \Sigma^\alpha$ , and  $S_U^{-\infty} := \cup_{p \in \mathbb{Z}} S_U^p$ . The operator defined by a pair of symbols  $(a_1, a_2)$  is denoted by  $Op_U(a_1, a_2)$  and the set of pseudo-differential operators of order  $p$  in  $T^{-\infty}$  by  $Op(S_U^\alpha)$ .

**Lemma 11.** *If  $A \in Op(S_U^\alpha)$ , then letting  $\tilde{A} := Op(\tilde{\sigma}_A)$ , for  $u \in U^s$ , we have*

$$S(Au) = \tilde{A}(Su),$$

where we recall that for any function  $v$  in  $U^s$ ,  $Sv(\theta) = \sin \theta v(\cos(\theta))$ . Reciprocally, if  $A$  is a linear operator that maps  $U^\infty$  to itself such that  $SA = \tilde{A}S$  where  $\tilde{A}$  is a PPDO of order  $\alpha$ , then  $A$  is in  $Op(S_U^\alpha)$  and if  $\sigma_{\tilde{A}}$  is the symbol of  $\tilde{A}$ , then  $A = Op_U(a_1, a_2)$  with the functions  $a_1$  and  $a_2$  of Lemma 10.

**Theorem 3.** *Let  $A \in Op(S_T^\alpha)$ . Then  $-\partial_x A \omega \partial_x \omega \in Op(S_U^{\alpha-2})$ , and there exists  $B \in S_U^{\alpha+2}$  such that  $A = -\omega \partial_x \omega B \partial_x$ . Similarly, let  $A \in Op(S_T^\alpha)$ . Then  $-\omega \partial_x \omega A \partial_x \in Op(S_T^{\alpha-2})$ , and there exists  $B \in S_T^{\alpha+2}$  such that  $A = -\partial_x B \omega \partial_x \omega + T^\infty$ .*

## 2 Application to preconditioning for the Helmholtz scattering problem

In this section, we apply the analytical tools introduced in the previous section to the study of the Helmholtz scattering problems. The two main results are ?? and ?. We start by introducing the notations.

## 2.1 The scattering problem for an open curve

Let  $\Gamma$  be a smooth non-intersecting open curve in  $\mathbb{R}^2$ , with a smooth unit normal vector  $n_\Gamma : \Gamma \rightarrow \mathbb{R}^2$  and let  $k \geq 0$  the wave number. We seek a solution to the two problems

$$-\Delta u_i - k^2 u_i = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma, \quad i = 1, 2 \quad (15)$$

with the following additional conditions

- Dirichlet or Neumann boundary conditions, respectively

$$u_1 = u_D, \text{ and } \frac{\partial u_2}{\partial n} = u_N \text{ on } \Gamma \quad (16)$$

where  $\frac{\partial u}{\partial n} = n_\Gamma \cdot \nabla u$ .

- Suitable decay at infinity, given for  $k > 0$  by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \quad (17)$$

with  $r = |x|$  for  $x \in \mathbb{R}^2$ .

When  $k = 0$ , the radiation condition must be replaced by an appropriate decay of  $u$  and  $\nabla u$  at infinity, see for example [13, 14], or [9, Chap. 7] **Vérifier le chapitre et la page**. Existence and uniqueness results are available for those problems, but the solutions fail to be regular even with smooth data  $u_D$  and  $u_N$ . More precisely, let  $\lambda = [\frac{\partial u_1}{\partial n}]_\Gamma$  and  $\mu = [u_2]_\Gamma$  where  $[\cdot]_\Gamma$  refers to the jump of a quantity across  $\Gamma$ , we have the following result.

**Theorem 4.** (see e.g. [10, 13, 14]) Assume  $u_D \in H^{1/2}(\Gamma)$ , and  $u_N \in H^{-1/2}(\Gamma)$ . Then problems (15, 16, 17) both possess a unique solution  $u_i \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \Gamma)$ , which is of class  $C^\infty$  outside  $\Gamma$ . Near the edges of the screen  $\Gamma$ ,  $\lambda$  is unbounded:

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

while  $\mu$  satisfies

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi$$

where  $\psi \in \tilde{H}^{3/2}(\Gamma)$ .

For the definition of Sobolev spaces on smooth open curves, we follow [9] by considering any smooth closed curve  $\tilde{\Gamma}$  containing  $\Gamma$ , and defining

$$H^s(\Gamma) = \{U|_\Gamma \mid U \in H^s(\tilde{\Gamma})\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve  $\tilde{\Gamma}$  containing  $\Gamma$ . Moreover,

$$\tilde{H}^s(\Gamma) = \{u \in H^s(\Gamma) \mid \tilde{u} \in H^s(\tilde{\Gamma})\}$$

where  $\tilde{u}$  denotes the extension by zero of  $u$  on  $\tilde{\Gamma}$ .

**Single-layer potential** We define the single-layer potential by

$$\mathcal{S}_k \lambda(x) = \int_{\Gamma} G_k(x-y) \lambda(y) d\sigma(y) \quad (18)$$

where  $G_k$  is the Green's function

$$\begin{cases} G_0(z) = -\frac{1}{2\pi} \ln |z|, & \text{if } k = 0, \\ G_k(z) = \frac{i}{4} H_0(k|z|), & \text{if } k > 0, \end{cases} \quad (19)$$

for  $x \in \mathbb{R}^2 \setminus \Gamma$ . Here  $H_0$  is the Hankel function of the first kind. For  $k > 0$ , the solution  $u_1$  to the Dirichlet problem admits the representation

$$u_1 = \mathcal{S}_k \lambda \quad (20)$$

where  $\lambda \in \tilde{H}^{-1/2}(\Gamma)$  is the jump of the normal derivative of  $u_1$  across  $\Gamma$  and is the unique solution to

$$S_k \lambda = u_D. \quad (21)$$

Here,  $S_k := \gamma \mathcal{S}_k$  where  $\gamma$  is the trace operator on  $\Gamma$ . The operator  $S_k$  maps continuously  $\tilde{H}^{-1/2}(\Gamma)$  to  $H^{1/2}(\Gamma)$ . When  $k = 0$ , the computation of  $u_1$  also involves the resolution of (21) but some subtleties arise in the representation of  $u_1$  by (20). On this topic, see [13, Theorem 1.4].

**Double-layer and hypersingular potentials** Similarly, we introduce the double layer potential  $\mathcal{D}_k$  by

$$\mathcal{D}_k \mu(x) = \int_{\Gamma} n_{\Gamma}(y) \cdot \nabla G_k(x-y) \mu(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \Gamma$$

for any smooth function  $\mu$  defined on  $\Gamma$ . The normal derivative of  $\mathcal{D}_k \mu$  is continuous across  $\Gamma$ , allowing us to define the hypersingular operator  $N_k = \frac{\partial}{\partial n} \mathcal{D}_k$ . This operator admits the following representation for  $x \in \Gamma$

$$N_k \mu(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{\partial}{\partial \varepsilon} \int_{\Gamma} n_{\Gamma}(y) \cdot \nabla G(x + \varepsilon n_{\Gamma}(x) - y) \mu(y) d\sigma(y). \quad (22)$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions  $\mu$  and  $\nu$  that vanish at the extremities of  $\Gamma$ :

$$\begin{aligned} \langle N_k \mu, \nu \rangle_{L^2(\Gamma)} &= \int_{\Gamma \times \Gamma} G_k(x-y) \partial_{\tau} \mu(x) \partial_{\tau} \nu(y) \\ &\quad - k^2 G_k(x-y) \mu(x) \nu(y) n_{\Gamma}(x) \cdot n_{\Gamma}(y) d\sigma(x) d\sigma(y). \end{aligned} \quad (23)$$

where  $\partial_{\tau}$  denotes the tangential derivative on  $\Gamma$  defined for  $u : \Gamma \rightarrow \mathbb{R}$  as follows. Let  $r : [a, b] \rightarrow \Gamma$  a smooth parametrization such that  $|r'(x)| \neq 0$  for  $x \in [a, b]$ . Then  $\partial_{\tau} u := \frac{1}{|r'(x)|} \frac{d}{dx} u(r(x))$ . It is also known that  $N_k$  maps  $\tilde{H}^{1/2}(\Gamma)$  to  $H^{-1/2}(\Gamma)$  continuously, and that the solution  $u_2$  to the Neumann problem can be written as

$$u_2 = \mathcal{D}_k \mu \quad (24)$$

where  $\mu \in \tilde{H}^{1/2}(\Gamma)$  is the jump of  $u_2$  across  $\Gamma$  and is the unique solution to

$$N_k \mu = u_N. \quad (25)$$

**Weighted layer potentials.** Theorem 4 implies that even if  $u_D$  and  $u_N$  are smooth, the solutions  $\lambda$  and  $\mu$  to the corresponding integral equations have singularities. As a remedy, we consider weighted versions of those integral operators. Let  $\omega_\Gamma$  the operator  $u \mapsto \omega_\Gamma(x)u(x)$  such that  $\omega_\Gamma(r(x)) := \frac{|\Gamma|}{2}\omega(x)$  where  $|\Gamma|$  is the length of  $\Gamma$ ,  $\omega(x) = \sqrt{1-x^2}$  as in the previous section, and  $r : [-1, 1] \rightarrow \Gamma$  is a smooth parametrization of  $\Gamma$  with constant speed  $|r'| = \frac{|\Gamma|}{2}$ . Let

$$R : L^2(\Gamma) \longrightarrow L^2(-1, 1)$$

defined by

$$Ru(x) = u(r(x))$$

Also let  $n = Rn_\Gamma$ . We have

$$\omega_\Gamma = \frac{|\Gamma|}{2}R^{-1}\omega R, \quad \frac{1}{\omega_\Gamma} = \frac{2}{|\Gamma|}R^{-1}\frac{1}{\omega}R, \quad \partial_\tau = \frac{2}{|\Gamma|}R^{-1}\partial_x R.$$

For  $u, v \in L^2(\Gamma)$ ,

$$\langle u, v \rangle_{L^2(\Gamma)} = \frac{|\Gamma|^2}{4} \langle Ru, Rv \rangle_{L^2(-1, 1)}.$$

**Definition 8.** The weighted layer potentials  $S_{k, \omega_\Gamma}$  and  $N_{k, \omega_\Gamma}$  are defined as follows:

$$S_{k, \omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}, \quad N_{k, \omega_\Gamma} := N_k \omega_\Gamma.$$

Solving the integral equations (21) and (25), is equivalent to solving

$$\begin{aligned} S_{k, \omega_\Gamma} \alpha &= u_D \\ N_{k, \omega_\Gamma} \beta &= u_N \end{aligned}$$

and letting  $\lambda = \frac{\alpha}{\omega_\Gamma}$ ,  $\mu = \omega_\Gamma u_N$ . Those weighted integral operators appear in many related works such as [3, 6, 7]. We also define the rescaled parametric representations  $S_{k, \omega}$  and  $N_{k, \omega}$  from  $[-1, 1]$  to  $\mathbb{C}$  by  $S_{k, \omega} := \frac{2}{|\Gamma|}RS_{k, \omega_\Gamma}R^{-1}$  and  $N_{k, \omega} := \frac{|\Gamma|}{2}RN_{k, \omega_\Gamma}R^{-1}$ .

**Lemma 12.** The operator  $N_{k, \omega}$  satisfies, for all  $\beta, \beta' \in C^\infty([-1, 1])$

$$\begin{aligned} \langle N_{k, \omega} \beta, \beta' \rangle_\omega &= \langle S_{k, \omega}(\omega \partial_x \omega) \beta, (\omega \partial_x \omega) \beta' \rangle_{\frac{1}{\omega}} \\ &\quad - k^2 \frac{|\Gamma|^2}{2} \int_{-1}^1 G_k(r(x) - r(y)) \omega(x) \beta(x) \omega(y) \beta'(y) n(x) \cdot n(y) dx dy \end{aligned}$$

*Proof.* By definition, we have

$$\begin{aligned} \langle N_{k, \omega} \beta, \beta' \rangle_\omega &= \langle N_{k, \omega} \beta, \omega \beta' \rangle_{L^2(-1, 1)} \\ &= \left( \frac{2}{|\Gamma|} \right)^2 \langle R^{-1} N_{k, \omega} \beta, R^{-1} \omega \beta' \rangle_{L^2(\Gamma)} \\ &= \left( \frac{2}{|\Gamma|} \right)^2 \langle N_k \omega_\Gamma R^{-1} \beta, \omega_\Gamma R^{-1} \beta' \rangle_{L^2(\Gamma)} \end{aligned}$$

which gives, using the identity (23),

$$\langle N_{k, \omega} \beta, \beta' \rangle_\omega = \left( \frac{2}{|\Gamma|} \right)^2 (I_1 - k^2 I_2),$$



where

$$I_1 = \langle S_k \partial_\tau \omega_\Gamma R^{-1} \beta, \partial_\tau \omega_\Gamma R^{-1} \beta' \rangle_{L^2(\Gamma)}$$

and

$$I_2 = \int_{\Gamma \times \Gamma} G_k(x-y) \omega_\Gamma(x) \beta(r^{-1}(x)) \omega_\Gamma(y) \beta'(r^{-1}(y)) n_\Gamma(x) \cdot n_\Gamma(y) dx dy.$$

Using the parametrization  $r$  of  $\Gamma$ , we can rewrite

$$I_2 = \left( \frac{|\Gamma|}{2} \right)^4 \int_{-1}^1 G_k(r(x) - r(y)) \omega(x) \beta(x) \omega(y) \beta(y) n(x) \cdot n(y) dx dy$$

For  $I_1$ , we write

$$I_1 = \frac{|\Gamma|^2}{4} \langle R S_k \partial_\tau \omega_\Gamma R^{-1} \beta, R \partial_\tau \omega_\Gamma R^{-1} \beta' \rangle_{L^2(-1,1)}$$

And we have

$$\begin{aligned} R S_k \partial_\tau \omega_\Gamma R^{-1} &= R S_k \frac{1}{\omega_\Gamma} R^{-1} R \omega_\Gamma R^{-1} R \partial_\tau R^{-1} R \omega_\Gamma R^{-1} \\ &= S_{k,\omega} \omega \partial_x \omega \end{aligned}$$

similarly,  $R \partial_\tau \omega_\Gamma R^{-1} = \partial_x \omega$ . Thus,

$$I_1 = \frac{|\Gamma|^2}{2} \langle S_{k,\omega}(\omega \partial_x \omega) \beta, \omega \partial_x \omega \beta' \rangle_{\frac{1}{\omega}},$$

and the identity is proved.  $\square$

## 2.2 Operators $S_\omega$ and $N_\omega$ on the flat segment

In this section, the wavenumber is equal to 0 and the curve  $\Gamma$  is the flat segment  $(-1, 1) \times 0$ . The parametrization  $r$  is the constant function equal to 1,  $\partial_\tau = \partial_x$  and  $\omega_\Gamma = \omega$ . In this simple context,  $S_\omega$  and  $N_\omega$  have elementary properties that allow us to characterize  $T^s$  and  $U^s$  for  $s = \pm \frac{1}{2}$ .

**Single layer potential** The operator  $S_\omega$  takes the form

$$S_\omega \alpha(x) = \int_{-1}^1 \frac{\ln|x-y| \alpha(y)}{\sqrt{1-y^2}} dy.$$

There holds

$$S_\omega T_n = \sigma_n T_n \tag{26}$$

where

$$\sigma_n = \begin{cases} \frac{\ln(2)}{2} & \text{if } n = 0 \\ \frac{1}{2n} & \text{otherwise.} \end{cases}.$$

In particular  $S_\omega$  is in the class  $Op(S_T^{-1})$ . As a consequence,  $S_\omega$  maps  $T^\infty$  to itself, so the image of a smooth function is a smooth function. This gives

the possibility to derive an explicit inverse of  $S_\omega$  as the square root of a local operator, as a prelude to the main theorem Theorem 5. Recall that

$$-(\omega\partial_x)^2 T_n = n^2 T_n$$

the operator  $-(\omega\partial_x)^2$  is thus in  $Op(S_T^2)$  and

$$-(\omega\partial_x)^2 S_\omega^2 = \frac{I_d}{4} + T_\infty. \quad (27)$$

This shows that  $\sqrt{-(\omega\partial_x)^2}$  and  $S_\omega$  are inverse operators modulo smoothing operators and that  $\sqrt{-(\omega\partial_x)^2}$  can thus be used as an efficient preconditioner for  $S_\omega$ . We now proceed to show the following characterization of  $T^{-1/2}$  and  $T^{1/2}$ , which is a reformulation of a result obtained in [6] **Ou bien [5], vérifier et citer le thm.**

**Lemma 13.** *We have  $T^{-1/2} = \omega\tilde{H}^{-1/2}(-1, 1)$  and for all  $u \in \tilde{H}^{-1/2}(-1, 1)$ ,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}.$$

*Moreover,  $T^{1/2} = H^{1/2}(-1, 1)$  and*

$$\|u\|_{H^{1/2}} = \|u\|_{T^{1/2}}$$

*Proof.* Since the logarithmic capacity of the segment is  $\frac{1}{4}$ , the (unweighted) single-layer operator  $S$  is positive and bounded from below on  $\tilde{H}^{-1/2}(-1, 1)$ , (see [9] chap. 8). Therefore the norm on  $\tilde{H}^{-1/2}(-1, 1)$  is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle Su, u \rangle}.$$

On the other hand, the explicit expression (26) imply that if  $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_\omega \alpha, \alpha \rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since  $\alpha = \omega u$ ,  $\langle S_\omega \alpha, \alpha \rangle_{\frac{1}{\omega}} = \langle Su, u \rangle$ . This proves the first result. For the second result, we know that,  $(\tilde{H}^{1/2}(-1, 1))' = \tilde{H}^{-1/2}(-1, 1)$  (taking the dual with respect to the usual  $L^2$  duality, [8] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v \neq 0} \frac{\langle u, v \rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}}.$$

According to the previous result, for all  $v \in \tilde{H}^{-\frac{1}{2}}$ , the function  $\alpha = \omega v$  is in  $T^{-1/2}$ , and  $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$ , while  $\langle u, v \rangle = \langle u, \alpha \rangle_\omega$ . Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha \neq 0} \frac{\langle u, \alpha \rangle_\omega}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the  $T^{1/2}$  norm of  $u$  since  $T^{1/2}$  is identified to the dual of  $T^{-1/2}$  for  $\langle \cdot, \cdot \rangle_\omega$ . □

**Hypersingular operator** For  $k = 0$  and when  $\Gamma = (-1, 1) \times \{0\}$ , the identity (2.1) becomes

$$\langle N_\omega \beta, \beta' \rangle_\omega = \langle S_\omega(\omega \partial_x \omega) \beta, (\omega \partial_x \omega) \beta' \rangle_{\frac{1}{\omega}}$$

Noticing that  $(\omega \partial_x \omega) U_n = -(n+1) T_{n+1}$ , we have for all  $n \neq m$

$$\langle N_\omega U_n, U_m \rangle_\omega = 0.$$

Therefore, we have

$$N_\omega U_n = \nu_n U_n$$

with  $\nu_n \|U_n\|_\omega^2 = (n+1)^2 \sigma_{n+1} \|T_{n+1}\|_{\frac{1}{\omega}}^2$ , that is,  $\nu_n = \frac{(n+1)}{2}$ . Thus  $N_\omega$  maps  $U^s$  to  $U^{s-1}$  for all  $s \in \mathbb{R}$ . In particular,  $N_\omega$  maps smooth functions to smooth functions. Here again, we can express the inverse of  $N_\omega$  in the form of the square root of a local operator. Recall that

$$-(\partial_x \omega)^2 U_n = (n+1)^2 U_n,$$

thus,

$$N_\omega = \frac{1}{2} \sqrt{-(\partial_x \omega)^2}. \quad (28)$$

As before, we obtain a characterization of  $U^s$  for  $s = \pm \frac{1}{2}$  from the previous formula:

**Lemma 14.** *We have  $U^{1/2} = \frac{1}{\omega} \tilde{H}^{1/2}(-1, 1)$  and for all  $u \in \tilde{H}^{1/2}(-1, 1)$ ,*

$$\|u\|_{\tilde{H}^{1/2}} \sim \left\| \frac{u}{\omega} \right\|_{U^{1/2}}.$$

Moreover,  $U^{-1/2} = H^{1/2}(-1, 1)$  and

$$\|u\|_{H^{1/2}} = \|u\|_{U^{1/2}}.$$

*Proof.* It suffices to remark that

$$\|u\|_{\tilde{H}^{1/2}} \sim \sqrt{\left\langle N_\omega \frac{u}{\omega}, \omega \frac{u}{\omega} \right\rangle} = \sqrt{\left\langle N_\omega \frac{u}{\omega}, \frac{u}{\omega} \right\rangle_\omega} \sim \left\| \frac{u}{\omega} \right\|_{U^{1/2}}.$$

The second equality follows from the same calculations that were done in Lemma 13, as well as the norm equivalence.  $\square$

In what follows, we show that (27) and (28) can be generalized to non-zero wavenumber  $k$  and arbitrary smooth and non-intersecting open curve  $\Gamma$ .

### 2.3 Non-flat arc and non-zero frequency

We first focus on the weighted single-layer operator problem with non-zero frequency, and establish the following result, announced in [2].

**Theorem 5.**  *$S_{k,\omega}$  is in  $S^1$  and*

$$[-(\omega_\Gamma \partial_\tau)^2 - k^2 \omega_\Gamma^2] S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + T_4.$$

**Remark 2.** The previous result also implies that

$$-(\omega_\Gamma \partial_\tau)^2 S_{k,\omega_\Gamma}^2 = \frac{I_d}{4} + R$$

where  $R$  is in  $Op(S_T^2(\Gamma))$ . This is also a compact perturbation of the identity. Nevertheless, we have  $R = k^2 \omega_\Gamma^2 S_{k,\omega_\Gamma}^2 + T_4$  and therefore, the term  $k^2 \omega^2 S_{k,\omega}^2$  is the leading first order correction accounting for the wavenumber. The inclusion of this term in the preconditioner leads to a drastic reduction of the number of GMRES iterations in numerical applications, as demonstrated in [2].

*Proof.* Our perturbation analysis hinges on the following property of the Hankel function:

$$H_0(z) = \frac{-1}{2\pi} \ln |z| J_0(z) + F_1(z^2) \quad (29)$$

where  $J_0$  is the Bessel function of first kind and order 0 and where  $F_1$  is analytic. One has

$$(S_{k,\omega} u)(x) = \int_{-1}^1 H_0(k|r(x) - r(y)|) \frac{u(y)}{\omega(y)} dy.$$

Using the change of variables  $x = \cos \theta$ ,  $y = \cos \theta'$ , we get

$$S_{k,\omega} u(\cos \theta) = \int_0^\pi H_0(k|r(\cos \theta) - r(\cos \theta')|) u(\cos(\theta)) d\theta.$$

We can rewrite this using (29)

$$\begin{aligned} S_{k,\omega} u(\cos \theta) &= \frac{-1}{2\pi} \int_0^\pi \ln |\cos \theta - \cos \theta'| J_0(k|r(\cos \theta) - r(\cos \theta')|) Cu(\theta) d\theta \\ &\quad + \int_0^\pi F_k(\cos \theta, \cos \theta') Cu(\theta) d\theta' \end{aligned}$$

where

$$F_2(x, y) = \ln \frac{|r(x) - r(y)|}{|x - y|} + F_1(k^2(x - y)^2)$$

is a  $C^\infty$  function. By parity, the second integral is equal to

$$\frac{1}{2} \int_{-\pi}^\pi F_2(\cos \theta, \cos \theta') Cu(\theta) d\theta.$$

By Theorem 2, this is of the form  $RCu$  where  $R$  is in  $Op(\Sigma^{-\infty})$ . For the first integral, we make the following classical manipulations. We first write  $\cos \theta - \cos \theta' = -2 \sin \frac{\theta+\theta'}{2} \sin \frac{\theta-\theta'}{2}$ . Thus  $\ln |\cos \theta - \cos \theta'| = \ln \left| \sqrt{2} \sin \frac{\theta+\theta'}{2} \right| + \ln \left| \sqrt{2} \sin \frac{\theta-\theta'}{2} \right|$ . Using the change of variables  $\theta \rightarrow -\theta$ , we get

$$C(S_{k,\omega} u)(\theta) = \frac{-1}{2\pi} \int_{-\pi}^\pi \ln \left| \sqrt{2} \sin \frac{\theta - \theta'}{2} \right| J_0(k|\cos \theta - \cos \theta'|) Cu(\theta') d\theta' + RCu.$$

Let  $\kappa := t \mapsto -\frac{1}{2\pi} \ln \left| \sqrt{2} \sin \frac{t}{2} \right|$ . It is well-known that  $\hat{\kappa}(n) = \frac{1}{2n}$  for  $n \neq 0$ . We may prolonge this by  $\kappa(\xi) = \frac{1}{2\xi}$  away from  $\xi = 0$ . Let  $a(t, s) =$

$J_0(k|r(\cos t) - r(\cos s)|)$ , which is a smooth function. Applying Theorem 2, the operator

$$\tilde{S}_k u \mapsto \int_{-\pi}^{\pi} \kappa(t-s)a(t,s)u(s)ds$$

is in  $Op(\Sigma^{-1})$  with its symbol satisfying, for  $\xi > 0$ ,

$$\sigma_{\tilde{S}_k}(t, \xi) = \frac{1}{2\xi} + \frac{\sin^2 t}{4\xi^3} + 3i \frac{k^2 \sin t \cos t}{4\xi^4} + \frac{4k^2(4\sin^2 t - 3\cos^2 t) + 3k^4 \sin^4 t}{16\xi^5} + \Sigma^6.$$

Since  $CS_{k,\omega} = \tilde{S}_k C$ , we have  $CS_{k,\omega}^2 = \tilde{S}_k CS_{k,\omega} = \tilde{S}_k^2 C$ . Applying symbolic calculus, the symbol of  $\sigma_2$  of  $\tilde{S}_k^2$  satisfies

$$\sigma_2 = \frac{1}{4\xi^2} + k^2 \frac{\sin^2 t}{4\xi^4} + ik^2 \frac{\sin t \cos t}{\xi^5} + k^2 \frac{13\sin^2 t - 11\cos^2 t}{8\xi^6} + k^4 \frac{\sin^4(t)}{4\xi^6} + \Sigma^7.$$

We can now notice that for  $u \in T^s$ , we have

$$C [-(\omega \partial_x)^2 - k^2 \omega^2] u = \left[ -\frac{\partial^2}{\partial \theta^2} - k^2 \sin^2 \theta \right] Cu.$$

Thus

$$C [-(\omega \partial_x)^2 - k^2 \omega^2] S_{k,\omega}^2 = \left[ -\frac{\partial^2}{\partial \theta^2} - k^2 \sin^2 \theta \right] \tilde{S}_k^2 C$$

Of course,  $-\frac{\partial^2}{\partial \theta^2} - k^2 \sin^2$  is a PPDO with symbol  $\sigma_{\Delta}(\theta, \xi) = \xi^2 - k^2 \sin^2(\theta)$ . We apply again symbolic calculus to find the symbol  $\sigma_3$  of  $\left[ -\frac{\partial^2}{\partial \theta^2} - k^2 \sin^2 \theta \right] \tilde{S}_k^2$ . We find

$$\sigma_3 = \frac{1}{4} + \frac{k^2}{8\xi^4} + \Sigma^6.$$

Using maple, we find that the order 5 and 7 terms are null and

$$\sigma_3 = \frac{1}{4} + \frac{k^2}{8\xi^4} + \frac{2k^4 \sin^2(\theta) + k^2}{8\xi^6} + \Sigma^8.$$

□

## 2.4 Neumann problem

Similarly, if we define  $N_{k,\omega} := N_k \omega$ , we have

**Theorem 6.**

$$N_{k,\omega}^2 = [-(\partial_x \omega)^2 - k^2 \omega^2] + U_2.$$

This result suggests  $[-(\partial_x \omega)^2 - k^2 \omega^2]^{-1/2}$  as a candidate preconditioner for  $N_{k,\omega}$ .

## 2.5 Non-flat arc

In the more general case of a  $C^\infty$  non-intersecting open curve  $\Gamma$  and non-zero frequency  $k$ , the results of the previous sections can be extended using again compact perturbations arguments. Essentially, in the decomposition Equation 29,  $x$  and  $y$  must be replaced by  $r(x)$  and  $r(y)$ , where the function  $r$  is a smooth, constant-speed parametrisation of  $\Gamma$  defined on  $[-1, 1]$  and satisfying  $|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4} |x - y|^2 + |x - y|^4 G(x, y)$  where  $|\Gamma|$  is the length of  $\Gamma$  and  $G$  is a  $C^\infty$  function on  $[-1, 1]$ . Let  $\omega_\Gamma(x) = |\Gamma| \omega(x)$ ,  $\partial_\tau$  the tangential derivative on  $\Gamma$  and  $S_{k, \omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}$ .

**Theorem 7.** *One has  $S_{k, \omega_\Gamma} \in S^{-1}$  and*

$$(-(\omega_\Gamma \partial_\tau)^2 - k^2 \omega_\Gamma^2) S_{k, \omega_\Gamma}^2 = \frac{I_d}{4} + S^4$$

*Similarly,*

$$N_{k, \omega_\Gamma}^2 = -(\partial_\tau \omega_\Gamma)^2 - k^2 \omega_\Gamma^2 + U_2.$$

*where  $U_2$  maps  $U^s$  to  $U^{s+2}$  for all  $s \in \mathbb{R}$ .*

## 3 Galerkin analysis

In this section, we describe and analyze the Galerkin scheme used to solve the integral equations in this work. To keep matters simple, we focus on equations (??) and (??) on the flat strip. The results extend to the general case using standard arguments in the theory of boundary element methods. Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [12, Chap. 4, ] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [11] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [3], and some results exist for piecewise linear functions in the collocation setting [4].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 8 and Theorem 9.

In what follows, we introduce a discretization of the segment  $[-1, 1]$  as  $-1 = x_0 < x_1 < \dots < x_N = 1$ , and let  $\theta_i := \arccos(x_i)$ . We define the parameter  $h$  of the discretization as

$$h := \min_{i=0 \dots N-1} |\theta_{i+1} - \theta_i|.$$

In practice, one should use a mesh for which  $|\theta_i - \theta_{i+1}|$  is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the  $i - th$  interval is approximately  $(ih)^2$ . In comparison, in the h-BEM method with  $p = 1$  polynomial order, this would only lead to a convergence rate in  $O(h)$  (cf. [11, Theorem 1.3]).

### 3.0.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution  $\lambda$  of (??). To achieve it, we use a variational formulation of (??) to compute an approximation  $\alpha_h$  of  $\alpha$ , and set  $\lambda_h = \frac{\alpha_h}{\omega}$ . Let  $V_h$  the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at  $x_i$ . Let  $\alpha_h$  the unique solution in  $V_h$  to

$$\langle S_\omega \alpha_h, \alpha'_h \rangle_{\frac{1}{\omega}} = -\langle u_D, \alpha'_h \rangle_{\frac{1}{\omega}}, \quad \forall \alpha'_h \in V_h.$$

We shall prove the following result:

**Theorem 8.** *If the data  $u_D$  is in  $T^{s+1}$  for some  $-1/2 \leq s \leq 2$ , then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2} \|u_D\|_{T^{s+1}}.$$

In particular, when  $u_D$  is smooth, it belongs to  $T^\infty$  so the rate of convergence is  $h^{5/2}$ . We start by proving an equivalent of Céa's lemma:

**Lemma 15.** *There exists a constant  $C$  such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

*Proof.* In view of the properties of  $S_\omega$  stated in ??, we have the equivalent norm

$$\|\alpha - \alpha_h\|_{T^{-1/2}}^2 \leq C \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \rangle.$$

Since  $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = -\langle u_D, \alpha'_h \rangle$  for all  $\alpha'_h \in V_h$ , we deduce

$$\|\alpha - \alpha_h\|_{T^{-1/2}}^2 \leq \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \rangle, \quad \forall \alpha'_h \in V_h.$$

By duality

$$\|\alpha - \alpha_h\|_{T^{-1/2}}^2 \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of  $S_\omega$  from  $T^{-1/2}$  to  $T^{1/2}$ .  $\square$

From this we can derive the rate of convergence for  $\alpha_h$  to the true solution  $\alpha$ . We use the  $L_{\frac{1}{\omega}}^2$  orthonormal projection  $\mathbb{P}_h$  on  $V_h$ , which satisfies the following properties:

**Lemma 16.** *For any function  $u$ ,*

$$\|(\mathbb{I} - \mathbb{P}_h)u\|_{L_{\frac{1}{\omega}}^2} \leq C \|u\|_{L_{\frac{1}{\omega}}^2},$$

$$\|(\mathbb{I} - \mathbb{P}_h)u\|_{L_{\frac{1}{\omega}}^2} \leq Ch^2 \|u\|_{T_2}.$$

The proof requires the following well-known result:

**Lemma 17.** *Let  $\tilde{u}$  in the Sobolev space  $H^2(\theta_1, \theta_2)$ , such that  $\tilde{u}(\theta_1) = \tilde{u}(\theta_2) = 0$ . Then there exists a constant  $C$  independent of  $\theta_1$  and  $\theta_2$  such that*

$$\int_{\theta_1}^{\theta_2} \tilde{u}(\theta)^2 \leq C(\theta_1 - \theta_2)^4 \int_{\theta_1}^{\theta_2} \tilde{u}''(\theta)^2 d\theta$$

*Proof.* The first inequality is obvious since  $\mathbb{P}_h$  is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the  $L^2_{\frac{1}{\omega}}$  norm,

$$\|I - \mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I - I_h u\|_{L^2_{\frac{1}{\omega}}}, \quad (30)$$

where  $I_h u$  is the piecewise affine (continuous) function that matches the values of  $u$  at the breakpoints  $x_i$ . By Lemma 7, on each interval  $[x_i, x_{i+1}]$ , the function  $\tilde{u}(\theta) := u(\cos(\theta))$  is in the Sobolev space  $H^2(\theta_i, \theta_{i+1})$  so we can apply Lemma 17:

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I}_h u)^2 \leq (\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I}_h u)''^2.$$

This gives

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} \leq 2h^4 \left( \int_{\theta_i}^{\theta_{i+1}} \tilde{u}''^2 + \int_{\theta_i}^{\theta_{i+1}} \tilde{I}_h u''^2 \right). \quad (31)$$

Before continuing, we need to establish the following result

**Lemma 18.** *There holds*

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I}_h u''^2 \leq C \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega}$$

*Proof.* The expression of  $I_h u$  is given by

$$\tilde{I}_h u(\theta) = u(x_i) + \frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} (\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I}_h u''^2 = \left( \frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left( \int_{x_i}^{x_{i+1}} u'(t) dt \right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change  $t = \cos(\theta)$  to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \leq \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in  $(\theta_i, \theta_{i+1})$ . Indeed, since  $\cos$  is injective on  $[0, \pi]$ , the only problematic case is the limit when  $\theta_i = \theta_{i+1}$ . It is easy to check that this limit is  $\cos(\theta_i)^2$ , which is indeed uniformly bounded in  $\theta_i$ .  $\square$



We can now conclude the proof of Lemma 16. Summing all inequalities (31) for  $i = 0, \dots, N+1$ , we get

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}}^2 \leq Ch^4 \left( \|u\|_{T^2}^2 + \|u'\|_{T_0}^2 \right).$$

By Corollary 1, the operator  $\partial_x$  is continuous from  $T^2$  to  $T^0$  which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T^2}.$$

Thanks to (30), this concludes the proof.  $\square$

We obtain the following corollary by interpolation:

**Corollary 2.** *The operator  $I - \mathbb{P}_N$  is continuous from  $L^2_{\frac{1}{\omega}}$  to  $T^s$  for  $0 \leq s \leq 2$  with*

$$\|(I - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s}.$$

We can now prove Theorem 8:

*Proof.* First, using Lemma 13, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}}.$$

Moreover, if  $u_D$  is in  $T^{s+1}$ , then  $\alpha = S_{\omega}^{-1} u_D$  is in  $T^s$  and  $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$ . By the analog of Céa's lemma, Lemma 15, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s}.$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since  $\mathbb{P}_h$  is an orthonormal projection on  $L^2_{\frac{1}{\omega}}$ ,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

Using Cauchy-Schwarz's inequality and Corollary 2 ( $s = \frac{1}{2}$ ),

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s}.$$

$\square$

### 3.0.2 Neumann problem

We now turn to the numerical resolution of (??). We use a variational form for equation (??), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce  $W_h$  the space of continuous piecewise affine functions with breakpoints at  $x_i$ , and we denote by  $\beta_h$  the unique solution in  $W_h$  to the variational equation:

$$\langle N_{\omega} \beta_h, \beta'_h \rangle_{\omega} = \langle u_N, \beta'_h \rangle_{\omega}, \quad \forall \beta'_h \in W_h. \quad (32)$$

Then,  $\mu_h = \omega \beta_h$  is the proposed approximation for  $\mu$ . We shall prove the following:

**Theorem 9.** *If  $u_N \in U^{s-1}$ , for some  $\frac{1}{2} \leq s \leq 2$ , there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Like before, we start with an analog of Céa's lemma:

**Lemma 19.** *There exists a constant  $C$  such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta'_h \in W_h} \|\beta - \beta'_h\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator  $I_h$ :

**Lemma 20.** *There holds*

$$\|u - I_h u\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_h u\|_{U^1} \leq Ch \|u\|_{U^2}$$

*Proof.* We only show the first estimation, the method of proof for the second being similar. Using again Lemma 17 on each segment  $[x_i, x_{i+1}]$ , one can write

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \omega(u - I_h u)^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_h u)''^2 \\ &\leq Ch^4 \left( 2 \int_{\theta_i}^{\theta_{i+1}} V u''^2 + 2 \int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \right) \end{aligned}$$

where we recall that for any function  $u$ ,  $Vu$  is defined as

$$Vu(\theta) = \sin(\theta)u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

**Lemma 21.**

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left( \|u\|_{U^2}^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

*Proof.* Using the expression of  $I_h$ , one can write

$$\begin{aligned} \int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 &\leq C \left( |u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \right. \\ &\quad \left. + \left( \frac{u(x_{i+1}) - u(x_i)}{\cos \theta_{i+1} - \cos \theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 (1 + \cos^2) \right) \quad (33) \end{aligned}$$

We can estimate the first term, thanks to Lemma 5:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned} (u(x_{i+1}) - u(x_i))^2 &= \left( \int_{x_i}^{x_{i+1}} \partial_x u \right)^2 \\ &\leq \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \int_{x_i}^{x_{i+1}} \frac{1}{\omega} \\ &= |\theta_{i+1} - \theta_i| \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2. \end{aligned}$$

to conclude, it remains to observe that the quantity

$$\frac{|\theta_{i+1} - \theta_i| \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2)}{(\cos(\theta_i) - \cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of  $\theta_i$  and  $\theta_{i+1}$ . Indeed, in the limit  $\theta_{i+1} \rightarrow \theta_i$ , the fraction has the value  $1 + \cos^2(\theta_i)$   $\square$

We now plug the estimate Lemma 21 in (33), and sum over  $i$ :

$$\|u - I_h u\|_{L_\omega^2}^2 \leq Ch^4 (\|u\|_{U^2}^2 + \|u'\|_{L_\omega^2}^2).$$

This implies the claim once we use the continuity of  $\partial_x$  from  $U^2$  to  $U^0$ , cf. Corollary 1.  $\square$

We can now prove Theorem 9

*Proof.* Let us denote by  $\Pi_h$  the Galerkin projection operator defined by  $\beta \mapsto \beta_h$ . Since it is an orthogonal projection on  $W_h$  with respect to the scalar product  $(\beta, \beta') := \langle N_\omega \beta, \beta' \rangle$ , it is continuous from  $U^{1/2}$  to itself, so we have for any  $u$  in  $U^{1/2}$ .

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq C \|u\|_{U^{1/2}}.$$

We are now going to show the estimate

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2}.$$

By the analog of Céa's lemma Lemma 19, one has

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq \|(I - I_h)u\|_{U^{1/2}}.$$

By interpolation, this norm satisfies

$$\|(I - I_h)u\|_{U^{1/2}} \leq C \sqrt{\|(I - I_h)u\|_{U^0}} \sqrt{\|(I - I_h)u\|_{U^1}},$$

which yields, applying Lemma 20,

$$\|(I - I_h)u\|_{U^{1/2}} \leq Ch^{3/2} \|u\|_{U^2}.$$

By interpolation, for all  $s \in [1/2, 2]$ , we get

$$\|(I - \Pi_h)u\|_{U^{1/2}} \leq Ch^{s-1/2} \|u\|_{U^s}.$$

In view of ??, we have  $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$ . In addition, since  $N_\omega$  is a continuous bijection from  $U^{s+1}$  to  $U^s$  for all  $s$ , there holds

$$\|\beta\|_{U^s} = \|N_\omega^{-1}u_N\|_{U^s} = \|u_N\|_{U^{s-1}}.$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-1/2} \|\beta\|_{U^s} \leq Ch^{s-1/2} \|u_N\|_{U^{s-1}}.$$

□

## 4 Conclusion

## 5 Proof of Theorem 6

From equation (23), we can deduce the following formula for the weighted operator:

$$N_{k,\omega} = -\partial_x S_{k,\omega} \omega \partial_x \omega - k^2 S_{k,\omega} \omega^2 \quad (34)$$

If we define  $L_n := -\partial_x O_{n+2} \omega \partial_x \omega$ , then using the mapping properties of  $\partial_x$  and  $\omega \partial_x \omega$  given by Definition 3, and since, by ??,  $O_{n+2}$  is of order  $n+2$  in the scale  $T^s$ , we deduce that  $L_n$  is of order  $n$  in the scale  $U^s$ . The expansion obtained for the weighted single-layer operator in ?? yields the following expansion for  $N_{k,\omega}$ .

**Lemma 22.**

$$N_{k,\omega} = N_\omega + k^2 \left( -\frac{L_1}{4} - S_\omega \omega^2 \right) + U_3$$

As a consequence,  $N_{k,\omega}$  is an operator of order  $-1$  in the scale  $U^s$ . Using equation (34), we have the following expression:

$$N_{k,\omega}^2 = N_\omega^2 - k^2 \left( \frac{L_1 N_\omega + N_\omega L_1}{4} + N_\omega S_\omega \omega^2 + S_\omega \omega^2 N_\omega \right) + U_2.$$

We have proved in By definition,  $L_1 = -\partial_x O_3 \omega \partial_x \omega$ , while  $N_\omega = -\partial_x S_\omega \omega \partial_x \omega$ , thus

$$L_1 N_\omega = \partial_x (O_3 (\omega \partial_x)^2 S_\omega) \omega \partial_x \omega.$$

Moreover,

$$N_\omega L_1 = \partial_x (S_\omega (\omega \partial_x)^2 O_3) \omega \partial_x \omega.$$

Adding these two inequalities and using ??, we get

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = \partial_x (S_\omega \omega^2 S_\omega) \omega \partial_x \omega + U_2.$$

Here again, we use the formula  $\partial_x S_\omega \omega^2 = S_\omega \omega \partial_x \omega$ , which yields

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = S_\omega \omega \partial_x \omega \partial_x S_\omega \omega^2 = \left( -\frac{I_d}{4} + T_\infty \right) \omega^2.$$

Since  $\omega^2$  is continuous from  $U^s$  to  $T^s$  by ?? and using the injections  $T^s \subset U^s$ , any operator of the form  $R\omega^2$  is smoothing in the scale  $U^s$  as soon as  $R$  is smoothing in the scale  $T^s$ . Therefore,

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = -\frac{\omega^2}{4} + U_\infty.$$

Moreover, we have

$$\begin{aligned} S_\omega \omega^2 N_\omega &= -S_\omega \omega^2 \partial_x S_\omega \omega \partial_x \omega \\ &= -S_\omega \omega^2 \partial_x^2 S_\omega \omega^2 \end{aligned}$$

using again ?. Since  $\omega^2 \partial_x^2 = (\omega \partial_x)^2 + x \partial_x$ , we get

$$S_\omega \omega^2 N_\omega = \frac{\omega^2}{4} - S_\omega x \partial_x S_\omega \omega^2 + U_\infty$$

Futhermore,

$$N_\omega S_\omega \omega^2 = -\partial_x S_\omega \omega \partial_x \omega S_\omega \omega^2.$$

We use  $\omega \partial_x \omega = \omega^2 \partial_x - x$ :

$$\begin{aligned} N_\omega S_\omega \omega^2 &= -\partial_x S_\omega \omega^2 \partial_x S_\omega \omega^2 + \partial_x S_\omega x S_\omega \omega^2 \\ &= \frac{\omega^2}{4} + \partial_x S_\omega x S_\omega \omega^2 \end{aligned}$$

Thus,

$$S_\omega \omega^2 N_\omega + N_\omega S_\omega \omega^2 = \frac{\omega^2}{2} + (\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2) + U_\infty.$$

We are done if we prove that the operator in parenthesis is of order 2 in the scale  $U^s$ . For this, we may compute the action of each one of them on  $U_n$ . Using the various identities at our disposal, we obtain on the one hand for  $n \geq 2$

$$\partial_x S_\omega x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8(n+2)} + \frac{U_n + U_{n-2}}{8n(n+2)}.$$

and on the other hand for  $n > 0$

$$S_\omega x \partial_x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8n}.$$

After substracting, this gives the rather surprising identity identity for  $n \geq 2$

$$(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2) U_n = \frac{U_n}{4n(n+2)}$$

which of course proves our claim.

## 6 Suggestion de découpage

J'y ai un tout petit peu réfléchi :

- Les analyses pseudo-diffs des espaces  $T^s$ , bien qu'intéressantes, sont trop longues et ne se justifient pas vraiment dans le simple but de faire une méthode numérique.
- La méthode de Galerkin est bien analysée et nouvelle (à ma connaissance) mais n'est pas vraiment essentielle pour le message.

Je pense qu'on pourrait envisager 3 articles. Un très concis sur la méthode numérique en elle-même. Utiliser le minimum d'info pour  $k=0$ , donner les inverses exacts, prouver la commutation des opérateurs pour  $k$  non nul, puis balancer les préconditionneurs, et mettre les figures.

Un article un peu à part sur la méthode de Galerkin, et tous les aspects numériques (bcp moins d'impact)

Un article (peut-être juste sur arxiv ?) sur les espaces  $T^s$  et  $U^s$ , qui donne toutes les justifications théoriques. (une sorte de version étendue de cet article.)

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