Chapter 4

Pseudodifferential Operators and Their Fredholm Properties

Pseudodifferential operators are a natural extension of linear integral and partial differential operators. The theory of such operators grew out of the study of singular integral operators by Giraud, Mikhlin, Calderòn and Zygmund, among others. It developed rapidly after 1965 with the systematic studies of Kohn and Nirenberg [114], Hörmander, and other researchers. This theory has found many fields of application. In particular, all the boundary integral operators corresponding to the elliptic boundary value problems studied in this book are such operators. By using such a theory, the analysis of boundary integral equations and boundary element methods can be either greatly simplified or presented in a more general and elegant form.

In this chapter, we present some basic facts and properties of pseudo-differential operators for our subsequent applications. The interested reader can find a general account of the theory in Grubb [85], Kumano-go [116], Taylor [177], and Treves [179, 180], for example. A concise account given in Gilkey [80] is also helpful.

4.1 Symbol class S^m

A linear partial differential operator is a polynomial expression

$$P(x,D) = \sum_{|\alpha|\leqslant m} a_\alpha(x) D_x^\alpha, \quad \alpha\text{s are multi-indices.} \tag{4.1}$$

The *symbol* of P is denoted by

$$\sigma P = p(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha(x)(i\xi)^\alpha. \tag{4.2}$$

Since

$$u(x) = \left[\overline{\mathscr{F}}(\mathscr{F}u)\right](x) = \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle}\widehat{u}(\xi)\,d\xi,$$

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we have

$$D_x^\alpha u(x) = \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle} \left[(2\pi i\xi)^\alpha \widehat{u}(\xi)\right] d\xi,$$

thus

$$\begin{aligned} P(x,D)u(x) &= \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle} \left[\sum_{|\alpha|\leqslant m} a_\alpha(x)(2\pi i\xi)^\alpha\right] \widehat{u}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle} p(x,2\pi\xi)\widehat{u}(\xi)\, d\xi. \end{aligned} \tag{4.3}$$

We use this expression to define the action of a pseudodifferential operator (ψDO) for a larger class of functions $p(x,\xi)$ than polynomials.

Remark 4.1. Had we defined the Fourier transform by (2.8), then formula (4.3) would now be

$$P(x,D)u(x) = \int_{\mathbb{R}^N} e^{i\langle x,\xi\rangle} p(x,\xi)\widehat{u}(\xi) \frac{d\xi}{(2\pi)^{N/2}}, \tag{4.4}$$

appearing somewhat more elegant than (4.3). Using the definition of the Fourier transform as in (2.8) throughout all our discussions, the only adjustment we need to make is to use $p(x,2\pi\xi)$ instead of $p(x,\xi)$, noting the correspondence between (4.3) and (4.4). □

Definition 4.1. $p(x,\xi)$ is said to be a symbol of order m, $m \in \mathbb{R}$, denoted by $p \in S^m$, of a pseudodifferential operator $P(x,D)$ defined through (4.6), if it satisfies

(i) $p(x,\xi)$ is C^∞ in x and ξ;
(ii) $p(x,\xi)$ has compact x-support, i.e., $p(x,\xi)$ is zero for all sufficiently large x; and
(iii) for all multi-indices α, β, there is a constant $C_{\alpha\beta}$ such that

$$\left|D_x^\alpha D_\xi^\beta p(x,\xi)\right| \leqslant C_{\alpha\beta}\,(1+|\xi|)^{m-|\beta|}. \tag{4.5}$$

□

Let $a_\alpha \in C_0^\infty(\mathbb{R}^N)$ in (4.1). Then it is easy to check that the symbol of $P(x,D)$ is in S^m if it is given by (4.1).

Theorem 4.1. *Let $p \in S^m$ for some $m \in \mathbb{R}$. Then*

$$P\colon\ H^s(\mathbb{R}^N) \longrightarrow H^{s-m}(\mathbb{R}^N)$$

defined by

$$(Pu)(x) = \int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle} p(x,2\pi\xi)\widehat{u}(\xi)\, d\xi \tag{4.6}$$

is a continuous linear operator for all $s \in \mathbb{R}$.

Proof. From (4.3),

$$\widehat{Pu}(\eta) = \iint e^{2\pi i\langle x,\xi-\eta\rangle} p(x, 2\pi\xi)\widehat{u}(\xi)\, d\xi\, dx. \tag{4.7}$$

Define

$$\widetilde{p}(\lambda, \xi) = \int e^{-2\pi i\langle x,\lambda\rangle} p(x, 2\pi\xi)\, dx. \tag{4.8}$$

Then

$$\widehat{Pu}(\eta) = \int \widetilde{p}(\eta - \xi, \xi)\widehat{u}(\xi)\, d\xi. \tag{4.9}$$

Now let $\phi \in H^{m-s}(\mathbb{R}^N)$. We have

$$\begin{aligned}\int \widehat{Pu}(\eta)\widehat{\phi}(\eta)\, d\eta &= \iint \widetilde{p}(\eta - \xi, \xi)\widehat{\phi}(\eta)\widehat{u}(\xi)\, d\xi\, d\eta \\ &= \iint \widetilde{p}(\eta - \xi, \xi)(1+|\xi|)^{-s}(1+|\eta|)^{s-m}\widehat{\phi}(\eta)\cdot \\ &\qquad (1+|\eta|)^{m-s}\widehat{u}(\xi)(1+|\xi|)^{s}\, d\xi\, d\eta. \end{aligned} \tag{4.10}$$

Define

$$K(\xi, \eta) = \widetilde{p}(\eta - \xi, \xi)(1+|\xi|)^{-s}(1+|\eta|)^{s-m}. \tag{4.11}$$

We claim that K satisfies

$$\int |K(\xi, \eta)|\, d\xi \leqslant C, \qquad \int |K(\xi, \eta)|\, d\eta \leqslant C, \tag{4.12}$$

for some constant C independent of $\xi, \eta \in \mathbb{R}^N$. From (4.5), $p \in S^m$ implies

$$|D_x^\alpha p| \leqslant C_1(1+|\xi|)^m, \quad \text{for any multi-index } \alpha.$$

Since p has compact x-support,

$$\begin{aligned}\left|\widetilde{D_x^\alpha p}(\lambda, \xi)\right| &= \left|\int e^{-2\pi i\langle x,\lambda\rangle} D_x^\alpha p(x, 2\pi\xi)\, dx\right| \\ &\leqslant C_2(1+|\xi|)^m. \end{aligned} \tag{4.13}$$

Integrating by parts $|k|$ times for the above integral, we obtain

$$\left|(-2\pi i\lambda)^k \int e^{-2\pi i\langle x,\lambda\rangle} D_x^{\alpha-k} p(x, 2\pi\xi)\, dx\right| \leqslant C_2(1+|\xi|)^m.$$

Therefore

$$\left|\int e^{-2\pi i\langle x,\lambda\rangle} D_x^{\alpha-k} p(x, 2\pi\xi)\, dx\right| \leqslant C_3(1+|\xi|)^m(1+|\lambda|)^{-|k|}. \tag{4.14}$$

Since the RHS above is independent of α, we obtain

$$\left|\widetilde{D_x^\alpha p}(\lambda, \xi)\right| \leqslant C_4(k)(1+|\xi|)^m(1+|\lambda|)^{-|k|}. \tag{4.15}$$

From (4.11) and (4.15), with $\alpha = (0,\dots,0)$ in (4.15), we get

$$\begin{aligned} |K(\xi,\eta)| &= \left|\widetilde{p}(\eta-\xi,\xi)(1+|\xi|)^{-s}(1+|\eta|)^{s-m}\right| \\ &\leqslant C_5(k)\frac{(1+|\xi|)^{m-s}}{(1+|\eta|)^{m-s}}(1+|\xi-\eta|)^{-|k|}. \end{aligned} \tag{4.16}$$

We apply the inequality

$$\frac{1+|a|}{1+|b|} \leqslant 1+|a-b| \tag{4.17}$$

to (4.16) and get

$$|K(\xi,\eta)| \leqslant C_5(k)(1+|\xi-\eta|)^{-|k|+m-s}.$$

By choosing $|k|$ sufficiently large, we obtain

$$\begin{aligned} \int |K(\xi,\eta)|\,d\xi &\leqslant C_5(k)\int (1+|\xi-\eta|)^{-|k|+m-s}\,d\xi < \infty, \\ \int |K(\xi,\eta)|\,d\eta &\leqslant C_5(k)\int (1+|\xi-\eta|)^{-|k|+m-s}\,d\xi < \infty, \end{aligned}$$

so (4.12) has been verified.

We continue from (4.10):

$$\begin{aligned} \left|\int \widehat{Pu}(\eta)\widehat{\phi}(\eta)\,d\eta\right| &= \left|\iint K(\xi,\eta)(1+|\eta|)^{m-s}\widehat{\phi}(\eta)(1+|\xi|)^{s}\widehat{u}(\xi)\,d\xi\,d\eta\right| \\ &\leqslant \left\{\iint |K(\xi,\eta)|\left[(1+|\eta|)^{m-s}|\widehat{\phi}(\eta)|\right]^2 d\xi\,d\eta\right\}^{1/2} \\ &\quad\cdot\left\{\iint |K(\xi,\eta)|\,[(1+|\xi|)^{s}|\widehat{u}(\xi)|]^2\,d\xi\,d\eta\right\}^{1/2} \\ &\leqslant C\|\phi\|_{H^{m-s}}\|u\|_{H^s}, \quad \text{by (4.12)}. \end{aligned}$$

Since H^{m-s} is dual to H^{s-m}, this implies that

$$\|Pu\|_{H^{s-m}} \leqslant C\|u\|_{H^s}.$$

Therefore P is a bounded operator from H^s into H^{s-m}. □

Exercise 4.1. Prove the inequality (4.16). □

Example 4.1. By Definition 4.1,

$$p(x,\xi) = c(x)(1+|\xi|^2)^{m/2} \in S^m, \tag{4.18}$$

where

$$c \in C_0^\infty(\mathbb{R}^N);$$

m here is not necessarily an integer. Then the ψDO P associated with p satisfies

$$P\colon\ H^s(\mathbb{R}^N) \longrightarrow H^{s-m}(\mathbb{R}^N), \quad \forall s\in\mathbb{R}.$$

For example, when $m=-2, P\colon\ H^s \to H^{s+2}$; thus P makes functions smoother, which is not the case with differential operators. □

Remark 4.2. In (ii) of Definition 4.1, the requirement that $p(x,\xi)$ has compact x–support is very restrictive. For example, $p(x,\xi) = (1+|\xi|^2)^{m/2}$ does not have compact x–support. This is somewhat inconvenient. Actually, we can remove the requirement (ii) in Definition 4.1, and prove that the operator P defined by (4.6) satisfies

$$P\colon\ H^s_{\text{loc}}(\mathbb{R}^N) \to H^{s-m}_{\text{loc}}(\mathbb{R}^N)$$

is continuous. See [179]. From now on, we will deemphasize the restriction of compact x-support of symbols in S^m. □

4.2 Products and adjoints of pseudo-differential operators. Asymptotic expansions of a symbol

As an example, let us consider the following ψDOs:

$$\left.\begin{array}{ll} P = \dfrac{1}{i}a(x)\dfrac{\partial}{\partial x}, & Q = b(x),\ x\in\mathbb{R}, \quad a,b,\in C_0^\infty(\mathbb{R}), \\ \sigma P = a(x)\xi, & \sigma Q = b(x). \end{array}\right\} \tag{4.19}$$

For $u \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned} (PQ)u = P(Qu) &= \frac{1}{i}a(x)\frac{\partial}{\partial x}[b(x)u(x)] \\ &= \frac{1}{i}a(x)\left[b'(x)u(x)+b(x)u'(x)\right] \\ &= \left(\frac{1}{i}ab\frac{\partial}{\partial x}+\frac{1}{i}ab'\right)u. \end{aligned}$$

Thus

$$PQ = \frac{1}{i}\,ab\,\frac{\partial}{\partial x}+\frac{1}{i}\,ab'$$

and

$$\begin{aligned} \sigma(PQ) = ab\xi + \frac{1}{i}ab' &= \sigma P\cdot\sigma Q + i^{-1}D_\xi(\sigma P)D_x(\sigma Q) \\ &= \sum_{\alpha=0,1} i^{-|\alpha|}\left[D_\xi^\alpha(\sigma P)\right][D_x^\alpha(\sigma Q)]. \end{aligned} \tag{4.20}$$

Let P^* denote the adjoint of P with respect to the L^2 inner product, i.e.,

$$\langle P^*u, v\rangle = \langle u, Pv\rangle. \tag{4.21}$$

Let us compute P^* for the P in (4.19):

$$\begin{aligned} \int_{\mathbb{R}}\left\{\left[\frac{1}{i}a(x)\frac{\partial}{\partial x}\right]^* u\right\}\overline{v}\,dx &= \int u\overline{\left[\frac{1}{i}a(x)\frac{\partial v}{\partial x}\right]}\,dx \\ &= i\int u\overline{av'}\,dx \\ &= -i\int(\overline{a}u)'\overline{v}\,dx \quad \text{(integration by parts).} \end{aligned}$$

Therefore

$$\left(\frac{1}{i}a\frac{\partial}{\partial x}\right)^* u = \frac{1}{i}(\overline{a}u)'$$

$$\left(\frac{1}{i}a\frac{\partial}{\partial x}\right)^* = \frac{1}{i}a^*\frac{\partial}{\partial x} + \frac{1}{i}(a^*)', \quad a^* \equiv \overline{a}.$$

So

$$\sigma(P^*) = a^*\xi + \frac{1}{i}(a^*)' \tag{4.22}$$

$$= (\sigma P)^* + i^{-1}D_\xi D_x(\sigma P)^* \tag{4.23}$$

$$= \sum_{\alpha=0,1} i^{-|\alpha|}D_\xi^\alpha D_x^\alpha(\sigma P)^*. \tag{4.24}$$

Next, we consider how to compute $\sigma(PQ)$ and $\sigma(P^*)$ for general ψDO's P and Q.
In the following discussion, the corresponding ψDO of a symbol denoted by a particular lower-case letter will be denoted by the corresponding capital letter. We introduce an equivalence relation on the class of symbols:

$$a \sim b \quad \text{iff} \quad a - b \in S^m \quad \forall\, m \in \mathbb{R}. \tag{4.25}$$

Consequently,

$$A - B\colon\ H^s \longrightarrow H^{s'} \quad \forall s, s' \in \mathbb{R}$$

$$A - B\colon\ H^s \longrightarrow \cap_{s'>0} H^{s'}.$$

Then a consequence of the Sobolev imbedding theorem (Corollary 2.1) gives

$$A - B\colon\ H^s \longrightarrow C^\infty. \tag{4.26}$$

(Note that C^∞ is equipped with the topology of uniform convergence of any finite-order derivatives on compact sets.) Hence $A-B$ is an infinitely smoothing operator, whose effect can be ignored for our purposes later on and whose symbol is thus quotiented out in the equivalence relation.

Definition 4.2. Given symbols a, $\{a_j\}_{j=0}^\infty$, we write

$$a \sim \sum_{j=0}^{\infty} a_j \tag{4.27}$$

provided that for any $m \in \mathbb{R}^+$ there exists $N_0 \in \mathbb{Z}^+$ such that

$$a - \sum_{j=0}^{n} a_j \in S^{-m} \quad \text{for } n \geqslant N_0. \tag{4.28}$$

That is, a can be approximated up to an arbitrarily smoothing symbol by taking finitely many terms in $\{a_j\}_{j=0}^\infty$. □

Let K be a compact set in $\mathbb{R}^N$. We denote by Ψ_K the class of ψDOs restricted to (act on) functions with support in K. We first prove a technical lemma.

Lemma 4.1. *Let $r(x,y,\xi)$ be of compact x– and y–support in a compact set K and satisfy the condition that for all multi-indices α, β, γ, there exists a constant $C_{\alpha\beta\gamma} > 0$ such that*

$$\left|D_x^\alpha D_y^\beta D_\xi^\gamma r(x,y,\xi)\right| \leqslant C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\gamma|} \tag{4.29}$$

for some fixed $m \in \mathbb{R}$. If

$$(Ru)(x) \equiv \iint e^{2\pi i\langle x-y,\xi\rangle} r(x,y,2\pi\xi)u(y)\,dy\,d\xi, \tag{4.30}$$

then $R \in \Psi_K$ and

$$\sigma R \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D_y^\alpha D_\xi^\alpha r(x,y,\xi)\,\big|_{y=x}. \tag{4.31}$$

Proof.

$$\int e^{-2\pi i\langle y,\xi\rangle} r(x,y,2\pi\xi)u(y)\,dy = \mathscr{F}_y\left(r(x,\cdot,2\pi\xi)u(\cdot)\right)(\xi), \tag{4.32}$$

where $\mathscr{F}_y$ is the Fourier transform with respect to the y variable. By (2.15)

$$\int e^{-2\pi i\langle y,\xi\rangle} r(x,y,2\pi\xi)u(y)\,dy = ([\mathscr{F}_y r(x,\cdot,2\pi\xi)] * \widehat{u})(\xi).$$

Therefore

$$(Ru)(x) = \iint e^{2\pi i\langle x,\xi\rangle}\,[\mathscr{F}_y r(x,\cdot,2\pi\xi)]\,(\xi-\eta)\widehat{u}(\eta)\,d\eta\,d\xi. \tag{4.33}$$

Because $r(x,y,\xi)$ and $u(y)$ have compact y-support, performing an estimation such as in (4.8)–(4.17), we obtain

$$\left.\begin{aligned} &\left|[\mathscr{F}_y r(x,\cdot,2\pi\xi)]\,(\xi-\eta)\right| \leqslant C_1(k)(1+|\xi|)^m(1+|\xi-\eta|)^{-k},\\ &|\widehat{u}(\eta)| \leqslant C_2(k)(1+|\eta|)^{-k}, \end{aligned}\right\} \tag{4.34}$$

for any $k \in \mathbb{N}$, where $C_1(k)$ and $C_2(k)$ depend only on k. So we can apply Fubini's theorem to the RHS of (4.33) and obtain

$$(Ru)(x) = \iint e^{2\pi i\langle x,\xi-\eta\rangle}\,[\mathscr{F}_y r(x,\cdot,2\pi\xi)]\,(\xi-\eta)\widehat{u}(\eta)\,d\xi\,d\eta. \tag{4.35}$$

Now define

$$p(x,2\pi\eta) = \int e^{2\pi i\langle x,\xi-\eta\rangle}\,[\mathscr{F}_y r(x,\cdot,2\pi\xi)]\,(\xi-\eta)\,d\xi.$$

Then

$$(Ru)(x) = \int e^{2\pi i\langle x,\eta\rangle} p(x,2\pi\eta)\widehat{u}(\eta)\,d\eta. \tag{4.36}$$

Note that $p(x,\eta) \in S^m$ because of (4.29) and (4.34), from which, (4.36) and the property of r it follows that $R \in \Psi_K$. In the RHS of (4.33), if we change the variable ξ to $\xi+\eta$, then

$$p(x,2\pi\eta) = \int e^{2\pi i\langle x,\xi\rangle}\,[\mathscr{F}_y r(x,\,\cdot\,,2\pi(\xi+\eta))]\,(\xi)\,d\xi.$$

But $\mathscr{F}_y r$ allows a Taylor series expansion

$$[\mathscr{F}_y r(x,\cdot,2\pi(\xi+\eta))]\,(\xi) = \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!} D_\eta^\alpha\,[\mathscr{F}_y r(x,\cdot,2\pi\eta)]\,(\xi)(2\pi\xi)^\alpha$$
$$+ \text{remainder.}$$

Thus

$$\begin{aligned}
p(x,2\pi\eta) &= \sum_{|\alpha|\leqslant k} \int e^{2\pi i\langle x,\xi\rangle} \cdot \frac{1}{\alpha!} D_\eta^\alpha\,[\mathscr{F}_y r(x,\cdot,2\pi\eta)]\,(\xi)(2\pi\xi)^\alpha\,d\xi \\
&\quad + \text{remainder} \\
&= \sum_{|\alpha|\leqslant k} D_\eta^\alpha \left\{ \int e^{2\pi i\langle x,\xi\rangle} \frac{1}{\alpha!}\,[\mathscr{F}_y r(x,\cdot,2\pi\eta)]\,(\xi)(2\pi\xi)^\alpha\,d\xi \right\} \\
&\quad + \text{remainder} \\
&= \sum_{|\alpha|\leqslant k} D_\eta^\alpha \left\{ \left(\frac{1}{i}\right)^\alpha \frac{1}{\alpha!} D_y^\alpha \int e^{2\pi i\langle y,\xi\rangle}\,[\mathscr{F}_y r(x,\cdot,2\pi\eta)]\,(\xi)\,d\xi \Big|_{y=x} \right\} \\
&\quad + \text{remainder} \\
&= \sum_{|\alpha|\leqslant k} D_\eta^\alpha \left(\frac{1}{i} D_y\right)^\alpha \frac{1}{\alpha!} r(x,y,2\pi\eta)\Big|_{y=x} + \text{remainder.}
\end{aligned}$$

It is easy to show that the remainder belongs to S^{m-k}. Hence (4.31) has been proved. $\square$

Using the technical lemma, we can now prove the following crucial theorem.

Theorem 4.2. *For $P \in \Psi_K$, let P^* denote its adjoint defined by*

$$\langle P^* u, v\rangle = \langle u, Pv\rangle \quad \forall u,v \in C_0^\infty(K).$$

Then

(i)

$$P^* \in \Psi_K \text{ and } \sigma P^* \sim \sum_\alpha \frac{i^{-|\alpha|} D_\xi^\alpha D_x^\alpha p^*}{\alpha!}, \tag{4.37}$$

where $p^*(x,\xi) = \overline{p(x,\xi)}$;

(ii)

$$P,Q \in \Psi_K \text{ implies } PQ \in \Psi_K \text{ and } \sigma(PQ) \sim \sum_\alpha i^{-|\alpha|} D_\xi^\alpha p D_x^\alpha \frac{q}{\alpha!}. \tag{4.38}$$

Proof. **(i)**

$$\begin{aligned}\langle u, Pv\rangle &= \iint u(x)e^{-2\pi i\langle x,\xi\rangle}\overline{p}(x,2\pi\xi)\overline{\widehat{v}}(\xi)\,d\xi\,dx\\ &= \iiint e^{-2\pi i\langle x-y,\xi\rangle}u(x)\overline{v}(y)\overline{p}(x,2\pi\xi)\,dy\,dx\,d\xi.\end{aligned}$$

We approximate $p(x,2\pi\xi)$ by functions of compact ξ-support as in (3.66), and apply Fubini's theorem in this limit process to obtain

$$\langle u, Pv\rangle = \iiint e^{-2\pi i\langle x-y,\xi\rangle}u(x)\overline{v}(y)\overline{p}(x,2\pi\xi)\,dx\,d\xi\,dy.$$

Now define

$$(P^*u)(y) = \iint e^{-2\pi i\langle x-y,\xi\rangle}\overline{p}(x,2\pi\xi)u(x)\,dx\,d\xi.$$

Then

$$\langle P^*u, v\rangle = \langle u, Pv\rangle.$$

Choose $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi = 1$ on K. Then

$$(P^*u)(x) = \iint e^{-2\pi i\langle y-x,\xi\rangle}\phi(x)\overline{p}(y,2\pi\xi)u(y)\,dy\,d\xi, \tag{4.39}$$

and

$$r(x,y,\xi) \equiv \phi(x)\overline{p}(y,\xi)$$

satisfies the conditions of Lemma 4.1 so that $P^* \in \Psi_K$ and

$$\begin{aligned}\sigma P^* &\sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}D_\xi^\alpha D_y^\alpha\,[\phi(x)\overline{p}(y,\xi)]\,|_{y=x}\\ &= \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}D_\xi^\alpha D_y^\alpha\overline{p}(y,\xi)\,|_{y=x},\end{aligned}$$

so (4.37) is proved.

(ii) From (4.39),

$$(Q^*u)(x) = \iint e^{-2\pi i\langle y-x,\xi\rangle}\overline{q}(y,2\pi\xi)u(y)\,dy\,d\xi.$$

Thus

$$\begin{aligned}(Qu)(x) &= [(Q^*)^*u]\,(x)\\ &= \iint e^{-2\pi i\langle y-x,\xi\rangle}\widetilde{q}(y,2\pi\xi)u(y)\,dy\,d\xi,\end{aligned} \tag{4.40}$$

where

$$\widetilde{q}(y,\xi) = \overline{\sigma Q^*}. \tag{4.41}$$

Observe that the RHS of (4.40) above is simply $\overline{\mathscr{F}}_\xi[\int e^{-2\pi i\langle y,\xi\rangle}\widetilde{q}(y,2\pi\xi)u(y)dy]$; therefore

$$\widehat{Qu}(\xi) = \int e^{-2\pi i\langle y,\xi\rangle}\widetilde{q}(y,2\pi\xi)u(y)dy.$$

We thus get

$$\begin{aligned}
[(PQ)u](x) &= P[Qu(x)] \\
&= \int e^{2\pi i\langle x,\xi\rangle} p(x,2\pi\xi)\widehat{Qu}(\xi)\,d\xi \\
&= \iint e^{2\pi i\langle x-y,\xi\rangle} p(x,2\pi\xi)\widetilde{q}(y,2\pi\xi)u(y)\,dy\,d\xi.
\end{aligned}$$

But now

$$r(x,y,\xi) \equiv p(x,\xi)\widetilde{q}(y,\xi)$$

satisfies condition (4.29) of Lemma 4.1; therefore $PQ \in \Psi_K$ and

$$\begin{aligned}
\sigma(PQ) &\sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha r(x,y,\xi)|_{y=x} \\
&= \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha\,[p(x,\xi)\widetilde{q}(y,\xi)]\,|_{y=x} \\
&= \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha\,[p(x,\xi)D_x^\alpha\widetilde{q}(x,\xi)]\,.
\end{aligned} \tag{4.42}$$

The Leibnitz rule says that

$$D_\xi^\alpha(fg) = \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{(\alpha_1!)(\alpha_2!)}(D_\xi^{\alpha_1} f)(D_\xi^{\alpha_2} g). \tag{4.43}$$

Applying this to (4.42), with $f = p(x,\xi)$ and $g = D_x^\alpha\widetilde{q}(x,\xi)$, we obtain

$$\begin{aligned}
\sigma(PQ) &\sim \sum_{\alpha_1+\alpha_2=\alpha} \frac{i^{-|\alpha|}\alpha!}{\alpha_1!\alpha_2!\alpha!}[D_\xi^{\alpha_1} p(x,\xi)][D_\xi^{\alpha_2} D_x^{\alpha_1} D_x^{\alpha_2}\widetilde{q}(x,\xi)] \\
&= \sum_{\alpha_1} \frac{i^{-|\alpha_1|}}{\alpha_1!}[D_\xi^{\alpha_1} p(x,\xi)] D_x^{\alpha_1}\left[\sum_{\alpha_2}\frac{i^{-|\alpha_2|}}{\alpha_2!} D_\xi^{\alpha_2} D_x^{\alpha_2}\widetilde{q}(x,\xi)\right] \\
&\sim \sum_{\alpha_1} \frac{i^{-|\alpha_1|}}{\alpha_1!}[D_\xi^{\alpha_1} p(x,\xi)][D_x^{\alpha_1} q(x,\xi)] \quad \text{(by (4.43) and (4.37))} \\
&= \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}[D_\xi^\alpha p(x,\xi)][D_x^\alpha q(x,\xi)].
\end{aligned} \tag{4.44}$$

$\square$

Thus, we see that (4.20) and (4.22) are special consequences of Theorem 4.2.

We are now in a position to prove the asymptotic expansion of a symbol.

Theorem 4.3. *Let $p_j \in S^{m_j}$, $j = 0, 1, 2, \ldots$, where $m_j \in \mathbb{R}$ decreases monotonically to $-\infty$, and the x-support of p_j is contained in some fixed compact set for all j. Then there is a unique $p \in S^{m_0}$ (up to an infinitely smoothing symbol) such that*

$$p \sim \sum_{j=0}^{\infty} p_j. \tag{4.45}$$

Proof. Let $\psi\colon \mathbb{R}^N \to [0,1]$ be C^∞ such that $\psi \equiv 0$ on $|\xi| < 1$ and $\psi \equiv 1$ on $|\xi| > 2$. That is, $1 - \psi(x)$ is a cutoff function as used earlier in (3.66).

Choose a sequence $t_j \in \mathbb{R}_+$ such that $t_j \uparrow \infty$.

We first observe that the sum $\sum_{j=0}^{\infty} \psi(\xi/t_j)p_j(x,\xi)$ is well-defined, since for any fixed (x,ξ) there are only finitely many nonzero terms.

Let $j \in \mathbb{Z}^+$. Then

$$\begin{aligned}\left|p_j(x,\xi)\right| &\leqslant C(1+|\xi|)^{m_j}\\ &= C(1+|\xi|)^{m_j-m_0}(1+|\xi|)^{m_0}\\ &\leqslant \frac{1}{2^j}(1+|\xi|)^{m_0} \quad \text{for sufficiently large } |\xi|,\\ &\qquad\qquad \text{because } m_j - m_0 < 0.\end{aligned}$$

Also,

$$\begin{aligned}\left|D_\xi^\alpha p_j(x,\xi)\right| &\leqslant C(1+|\xi|)^{m_j-|\alpha|}\\ &= C(1+|\xi|)^{m_j-m_0}(1+|\xi|)^{m_0-|\alpha|}\\ &\leqslant \frac{1}{2^j}(1+|\xi|)^{m_0-|\alpha|} \quad \text{for sufficiently large } \xi.\end{aligned}$$

Using a subsequence $\{t_{01}, t_{02}, t_{03}, \ldots\}$ of the t_js we can assume that

$$\left|D_x^\alpha D_\xi^\beta\left[\psi(\xi/t_{0j})p_j(x,\xi)\right]\right| \leqslant \frac{1}{2^j}(1+|\xi|)^{m_0-|\beta|} \quad \forall \xi \in \mathbb{R}^N.$$

Therefore

$$\begin{aligned}\left|D_x^\alpha D_\xi^\beta\left[\sum_{j=0}^{\infty} \psi(\xi/t_{0j})p_j(x,\xi)\right]\right| &\leqslant (1+|\xi|)^{m_0-|\beta|}\sum_{j=0}^{\infty}\frac{1}{2^j}\\ &\leqslant C(1+|\xi|)^{m_0-|\beta|} \text{ for some } C > 0.\end{aligned}$$

This shows that

$$\sum_{j=0}^{\infty} \psi(\xi/t_{0j})p_j(x,\xi) \in S^{m_0}.$$

Inductively, one can show that for any $k \in \mathbb{Z}^+$, we have

$$\sum_{j=k}^{\infty} \psi(\xi/t_{kj})p_j(x,\xi) \in S^{m_k}$$

for a suitably chosen subsequence $\{t_{kj} \mid j = 1, 2, \ldots\}$ of $\{t_{k-1,j} \mid j = 1, 2, \ldots\}$.

Now define

$$p(x,\xi) = \sum_{j=0}^{\infty} \psi(\xi/t_{jj}) p_j(x,\xi).$$

We see that $p \in S^{m_0}$ by the way it was constructed. Let $\phi = \psi - 1$. For each j, $\psi(\xi/t_{jj})p_j(x,\xi) - p_j(x,\xi)$ has compact ξ-support and therefore lies in S^k for all k, so it is infinitely smoothing. Now

$$\begin{aligned} p(x,\xi) - p_0(x,\xi) &= \phi(\xi/t_{00})p_0(x,\xi) + \sum_{j=1}^{\infty} \psi(\xi/t_{jj})p_j(x,\xi) - p_0(x,\xi) \\ &\sim \sum_{j=1}^{\infty} \psi(\xi/t_{jj})p_j(x,\xi) \in S^{m_1}, \end{aligned}$$

and

$$\begin{aligned} p(x,\xi) - \sum_{j=0}^{k} p_k(x,\xi) &= \sum_{j=0}^{k} \phi(\xi/t_{jj})p_j(x,\xi) + \sum_{j=k+1}^{\infty} \psi(\xi/t_{jj})p_j(x,\xi) \\ &\sim \sum_{j=k+1}^{\infty} \psi(\xi/t_{jj})p_j(x,\xi) \in S^{m_{k+1}}. \end{aligned}$$

This proves (4.45). □

4.3 Elliptic operators

Most of the boundary integral operators in the boundary element methods in this book are elliptic pseudodifferential operators. We give the following important definition.

Definition 4.3. Let $p(x,\xi) \in S^m$.

(i) p is said to be *elliptic* of order m if there exist $R > 0$ and $C > 0$ such that

$$|p(x,\xi)| \geqslant C(1+|\xi|)^m, \quad \forall\, |\xi| \geqslant R. \tag{4.46}$$

(ii) p is said to be *strongly elliptic* of order m if there exist $R > 0$ and $C > 0$ such that

$$\operatorname{Re} p(x,\xi) \geqslant C(1+|\xi|)^m, \quad \forall\, |\xi| \geqslant R. \tag{4.47}$$

A ψDO is said to be (strongly) elliptic if its symbol p is (strongly) elliptic. □

Example 4.2.

(i) The negative Laplacian $-\Delta$ on functions in $\mathbb{R}^N$,

$$-\Delta f = -\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2}, \quad \sigma(-\Delta) = |\xi|^2,$$

is strongly elliptic of order 2.

(ii) The operator $\partial/\partial\bar{z}$ on functions in $\mathbb{C}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \quad \sigma\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}(i\xi - \eta)$$

is elliptic of order 1.

(iii) The biharmonic operator Δ^2 on functions in $\mathbb{R}^N$,

$$\Delta^2 f = \left(\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}\right)^2 f, \quad \sigma(\Delta^2) = |\xi|^4,$$

is strongly elliptic of order 4. □

Given an elliptic ψDO, we want to construct an *approximate inverse*, called a *parametrix*, of this ψDO.

Lemma 4.2. *Let $p(x,\xi) \in S^m$ be elliptic of order m. Then there exists $q(x,\xi) \in S^{-m}$ such that $pq \sim qp \sim 1$.*

Proof. Let $R > 1$ be such that

$$|p(x,\xi)| \geqslant C(1+|\xi|)^m \qquad \text{for } |\xi| \geqslant R.$$

Choose a C^∞ function $\psi\colon\ \mathbb{R}^N \to [0,1]$ such that $\psi \equiv 0$ for $|\xi| < R-1$ and $\psi \equiv 1$ for $|\xi| \geqslant R$. Define a C^∞ function $q(x,\xi)$ such that

$$q(x,\xi) = \begin{cases} 0 & \text{for } |\xi| < R-1, \\ \psi(\xi)p(x,\xi)^{-1} & \text{for } |\xi| \geqslant R. \end{cases} \tag{4.48}$$

Then $q(x,\xi)$ has compact x-support because p does, and both $pq-1$ and $qp-1$ are nonzero at most on a compact subset of $|\xi| \leqslant R$. Therefore, $pq-1$ and $qp-1$ are C^∞ smoothing symbols. It remains to show that $q \in S^{-m}$.

Clearly

$$|q(x,\xi)| \leqslant C(1+|\xi|)^{-m} \ \forall\, \xi \in \mathbb{R}^N, \text{ for some } C > 0,$$

and

$$D_\xi^\alpha q(x,\xi) = \begin{cases} 0 & \text{for } |\xi| < R-1, \\ \dfrac{D_\xi \psi}{p} - \dfrac{\psi(D_\xi p)}{p^2} = -\dfrac{D_\xi p}{p^2} & \text{for } |\xi| \geqslant R,\ |\alpha| = 1. \end{cases}$$

But $D_\xi p \in S^{m-1}$ and $p^2 \in S^{2m}$, so

$$\frac{D_\xi p}{p^2} \in S^{m-1-2m} = S^{-(m+1)}.$$

Therefore $D_\xi q \in S^{-(m+1)}$. Similarly, one can show that $D_x D_\xi q(x,\xi) \in S^{-(m+1)}$, i.e., differentiating with respect to x leaves the order unchanged. Computing higher-order derivatives inductively yields

$$\left| D_x^\alpha D_\xi^\beta q(x,\xi) \right| \leqslant C_{\alpha\beta}(1+|\xi|)^{-m-|\beta|}. \tag{4.49}$$

□

Theorem 4.4. *If P is an elliptic ψDO then there is a unique (up to an infinitely smoothing operator) ψDO Q such that*

$$PQ - I \sim QP - I \sim 0.$$

Proof. By modifying $p(x,\xi)$ on $|\xi| \leqslant R$, we can assume that $p(x,\xi)^{-1}$ exists and is C^∞ for all ξ. We construct Q by a series as follows:

$$\sigma Q = q = q_0 + q_1 + \cdots, \quad q_i \in S^{-m-i}, \quad i \in \mathbb{N},$$

where

$$q_0 = \frac{1}{p} \in S^{-m} \quad \text{by Lemma 4.2.}$$

Assume that q_j has been defined for $j < k$. We define

$$q_k = -\frac{1}{p} \sum_{\substack{|\alpha|+j=k \\ j<k}} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha p D_x^\alpha q_j \in S^{-m-k}.$$

By (4.38), we have

$$\begin{aligned}
\sigma(PQ) &\sim \sum_{\alpha,j} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha p D_x^\alpha q_j \\
&= \sum_{k=0}^{\infty} \sum_{|\alpha|+j=k} \frac{i^{-|\alpha|}}{\alpha!} D_\xi^\alpha p D_x^\alpha q_j \\
&= pq + \text{lower-order terms} \\
&\approx 1.
\end{aligned}$$

Therefore, $PQ - I \sim 0$. Similarly, one can construct $\widetilde{Q}$ such that $\widetilde{Q}P - I \sim 0$. But

$$\widetilde{Q} \sim \widetilde{Q}(PQ) \sim (\widetilde{Q}P)Q \sim Q,$$

so Q is unique up to an infinitely smoothing operator. □

The following lemma builds up the machinery for proving the Gårding inequality.

Lemma 4.3. *Let $p \in S^0$ be strongly elliptic of order 0, i.e.,*

$$\operatorname{Re} p(x,\xi) \geqslant C > 0 \quad \forall\, x,\xi, \text{ for some } C.$$

Then there exists B such that $b \in S^0$ and

$$\sigma\,(\operatorname{Re} P - B^*B) \in S^{-\infty}, \tag{4.50}$$

where $\operatorname{Re} P = \frac{1}{2}(P + P^*)$.

Proof. We construct B's symbol $b \sim \sum_{j=0}^{\infty} b_j$, $b_j \in S^{-j}$, inductively:

$$b_0(x,\xi) \equiv [\operatorname{Re} p(x,\xi)]^{1/2}.$$

It is straightforward to verify that $b_0 \in S^0$. The formulas for adjoints and products (Theorem 4.2) show that

$$\sigma(\operatorname{Re} P - B_0^* B_0) \equiv r_1 \in S^{-1}.$$

Assume that we have $b_0, b_1, \ldots, b_j$. We want $b_{j+1} \in S^{-(j+1)}$ such that

$$\begin{aligned}&\sigma\{\operatorname{Re} P - [(B_0^* + B_1^* + \cdots + B_j^*) + B_{j+1}^*]\cdot\\ &[(B_0 + B_1 + \cdots + B_j t) + B_{j+1}]\} \equiv r_{j+1} \in S^{-(j+1)}.\end{aligned} \tag{4.51}$$

But the LHS is just

$$r_j - \sigma\left[(B_0^* + \cdots + B_j^*)B_{j+1} + B_{j+1}^*(B_0 + B_1 + \cdots + B_j) + B_{j+1}^* B_{j+1}\right]. \tag{4.52}$$

Let us choose

$$B_{j+1} = -\frac{1}{2}(B_0^*)^{-1} R_j.$$

Then

$$b_{j+1} \in S^{-j}, \tag{4.53}$$

$$B_{j+1}^* = -\frac{1}{2} R_j (B_0)^{-1} \qquad (\text{because } R_j = R_j^*), \tag{4.54}$$

so

$$R_j - (B_0^* B_{j+1} + B_{j+1}^* B_0) = 0, \tag{4.55}$$

and from (4.51)–(4.55) we obtain

$$\begin{aligned}r_{j+1} &= \sigma[-(B_1^* + \cdots + B_j^*)B_{j+1} - B_{j+1}^*(B_1 + \cdots + B_j) - B_{j+1}^* B_{j+1}]\\ &\in S^{-(j+1)}.\end{aligned}$$

The proof is complete. □

We define operators

$$\Lambda^\sigma u = \int (1+|\xi|^2)^{\sigma/2} e^{2\pi i\langle x,\xi\rangle} \widehat{u}(\xi)\, d\xi, \qquad \sigma \in \mathbb{R}. \tag{4.56}$$

By Theorem 4.1 and the Fourier inversion formula,

$$\Lambda^\sigma :\ H^s(\mathbb{R}^N) \longrightarrow H^{s-\sigma}(\mathbb{R}^n) \quad \text{isomorphically.} \tag{4.57}$$

Theorem 4.5 (Gårding's inequality). *Let $p \in S^m$ be strongly elliptic of order m. Then for any $s \in \mathbb{R}$ and for any bounded $\Omega \subset \mathbb{R}^N$, there are $C_1 > 0$, $C_2 > 0$ such that*

$$\operatorname{Re}\langle Pu,u\rangle \geqslant C_1 \|u\|^2_{H^{m/2}} - C_2\|u\|^2_{H^s}, \ \ \forall u \in C_0^\infty(\Omega). \tag{4.58}$$

Sketch of Proof. By Theorem 4.2, it is easy to verify that

$$\sigma(\Lambda^{-m/2} P \Lambda^{-m/2}) \in S^0 \text{ is strongly elliptic of order } 0.$$

Without loss of generality, we can replace P by $\Lambda^{-m/2}P\Lambda^{-m/2}$ and assume that $m = 0$. By making a C^∞ modification of p, we can assume that

$$\operatorname{Re} p(x,\xi) \geqslant C_0 > 0 \quad \forall x,\xi, \text{ for some } C_0.$$

We apply Lemma 4.3 to $q(x,\xi) \equiv \operatorname{Re} p(x,\xi) - \frac{1}{2}C_0$ to yield $b \in S^0$ satisfying

$$\sigma(Q - B^*B) = r \in S^{-\infty};$$

hence

$$\begin{aligned}
\operatorname{Re}\langle Pu,u\rangle - \frac{1}{2}C_0\langle u,u\rangle &= \langle B^*Bu,u\rangle + \operatorname{Re}\langle Ru,u\rangle \\
&= \langle Bu,Bu\rangle + \operatorname{Re}\langle Ru,u\rangle, \\
\operatorname{Re}\langle Pu,u\rangle &= \frac{1}{2}C_0\langle u,u\rangle + \langle Bu,Bu\rangle + \operatorname{Re}\langle Ru,u\rangle \\
&\geqslant \frac{1}{2}C_0\|u\|^2 + \operatorname{Re}\langle Ru,u\rangle.
\end{aligned}$$

Since $r \in S^{-\infty}$ and R is infinitely smoothing, we can apply an interpolation inequality in the form (cf. (2.33))

$$|\operatorname{Re}\langle Ru,u\rangle| \leqslant \varepsilon\|u\|^2 + C(\varepsilon,s)\|u\|^2_{H^s}$$

to all u such that $\operatorname{supp} u$ is compactly contained in Ω. Hence

$$\begin{aligned}
\operatorname{Re}\langle Pu,u\rangle &\geqslant \frac{1}{2}(C_0 - 2\varepsilon)\|u\|^2 - C(\varepsilon,s)\|u\|^2_{H^s} \\
&\equiv C_1\|u\|^2 - C_2\|u\|^2_{H^s}.
\end{aligned}$$

□

Let M be a compact manifold without boundary in $\mathbb{R}^N$. By changing the $L^2(\Omega)$–inner product $\langle\cdot,\cdot\rangle$ on the LHS of (4.58) to the $H^j(M)$–inner product $\langle\cdot,\cdot\rangle_{H^j}$ and by refining the proof of Theorem 4.5, one can further show that

$$\operatorname{Re}\langle Pu,u\rangle_{H^j} \geqslant C_0\|u\|^2_{H^{j+m/2}} - \langle Ku,u\rangle_{H^j} \;\; \forall u \in H^{j+m/2}(M), \tag{4.59}$$

where $K\colon\ H^{j+m/2}(M) \to H^{j-m/2}(M)$ is a compact operator (thus $\langle Ku,u\rangle_{H^j}$ is considered as a duality pairing in a sense similar to Theorem 2.4). The above is the form of Gårding's inequality we will refer to in the future.

4.4 Calculation of the principal symbols of boundary integral operators representing multiple-layer potentials

Boundary element methods are based upon the use of boundary integral operators representing multiple-layer potentials. The regularity properties of these operators are important in the later development of the theory of boundary integral equations and error estimates. It is known [170] that these boundary integral operators are all pseudodifferential operators. In this section, we will calculate the principal symbols of several multilayer potentials of the Laplacian, thereby determining the regularity properties of these operators.

Let $E(x,\xi)$ be the fundamental solution of the Laplacian as given by (1.3). Let Ω be a bounded open domain in $\mathbb{R}^N$ with C^∞ boundary $\partial\Omega$. For $\phi \in C^\infty(\partial\Omega)$, define boundary layer potentials

$$V_1\phi(x) = \int_{\partial\Omega} E(x,y)\phi(y)\,d\sigma_y, \tag{4.60}$$

$$V_2\phi(x) = \int_{\partial\Omega} \frac{\partial E(x,y)}{\partial n_y}\phi(y)\,d\sigma_y, \tag{4.61}$$

$$V_3\phi(x) = \frac{\partial}{\partial n_x}\int_{\partial\Omega} \frac{\partial E(x,y)}{\partial n_y}\phi(y)\,d\sigma_y \tag{4.62}$$

for $x \in \Omega$, where the normal vector field n_y has been extended in a C^∞ way to a neighborhood of $\partial\Omega$ through a Gaussian normal coordinate system (see [64, 23.48.4], and [107, §5.2], as in § 2.2). Then $V_1\phi$, $V_2\phi$ and $V_3\phi$ are in $C^\infty(\overline{\Omega})$. Define three boundary integral operators by the traces

$$A_1\phi = V_1\phi|_{\partial\Omega}, \tag{4.63}$$

$$A_2\phi = V_2\phi|_{\partial\Omega}, \tag{4.64}$$

$$A_3\phi = V_3\phi|_{\partial\Omega}. \tag{4.65}$$

The pseudodifferential operators considered so far are defined on the free space $\mathbb{R}^N$. In this section, we will study instead pseudodifferential operators on Ω. Thus we first define the class of symbols $S^m_{1,0}(\Omega,\mathbb{R}^N)$ to consist of C^∞ functions $p(x,\xi)$ satisfying the estimates

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \leqslant C_{\alpha\beta}(x)(1+|\xi|)^{m-|\beta|}, \qquad \forall (x,\xi)\in\Omega\times\mathbb{R}^N$$

where $C_{\alpha\beta}(x)$ is a continuous function of $x\in\Omega$. $S^m(\Omega,\mathbb{R}^N)$, a subclass of $S^m_{1,0}(\Omega,\mathbb{R}^N)$, is defined to consist of *polyhomogeneous symbols*, which are symbols possessing an asymptotic expansion as a sum of symbols $p_{m-l}(x,\xi)$, $l\in\mathbb{N}$, such that p_{m-l} is *homogeneous* in ξ, for $|\xi|\geqslant 1$, and

$$p-\sum_{l<k} p_{m-l}\in S^{m-k}_{1,0}(\Omega,\mathbb{R}^N)\ \ \forall\ k\in\mathbb{N}.$$

For a pseudodifferential operator P on $\mathbb{R}^N$, we restrict it to Ω by defining

$$P_\Omega u = r^+ P e^+ u$$

for $u\in\mathscr{D}'(\Omega)$, where e^+ is the "extension by zero" operator,

$$e^+:\ \mathscr{D}'(\Omega)\to\mathscr{D}'(\mathbb{R}^N),$$
$$e^+u(x)=\begin{cases} u(x) & \text{for } x\in\Omega,\\ 0 & \text{for } x\in\mathbb{R}^N\setminus\Omega,\end{cases}$$

and r^+ is the "restriction" operator,

$$r^+:\ \mathscr{D}'(\mathbb{R}^N)\to\mathscr{D}'(\Omega),$$
$$r^+u = u|_\Omega\in\mathscr{D}'(\Omega).$$

We are now in a position to study boundary integral operators. Denote

$$\begin{aligned}\mathbb{R}^N_+ &= \{x\in\mathbb{R}^N\mid x=(x_{\widehat{N}},x_N)\text{ such that}\\ &\qquad x_{\widehat{N}}=(x_1,\ldots,x_{N-1})\in\mathbb{R}^{N-1}, x_N>0\},\\ \mathbb{R}^N_- &= \mathbb{R}^N\setminus\overline{\mathbb{R}}^N_+.\end{aligned}$$

We can locally identify the smooth boundary $\partial\Omega$ with $(x_{\widehat{N}},0)$ in the sense illustrated in Fig. 4.1.

The neighborhood $\mathscr{N}(y_0)$ of a point $y_0\in\partial\Omega$ is mapped into $\mathbb{R}^N$ by a local diffeomorphism ν such that y_0 is mapped into the origin of $\mathbb{R}^N$ and $\mathscr{N}(y_0)\cap\overline{\Omega}$ is mapped into $\overline{\mathbb{R}}^N_+$ such that the directions $n(x)$ for $x\in\mathscr{N}(y_0)\cap\partial\Omega$ are mapped into the direction $\vec{e}_N=(0,\ldots,0,1)^{\mathrm{Tr}}$. In this fashion, $\partial\Omega$ is locally identified with (part of) $\mathbb{R}^{N-1}$, and ν induces a normal coordinate variable x_N.

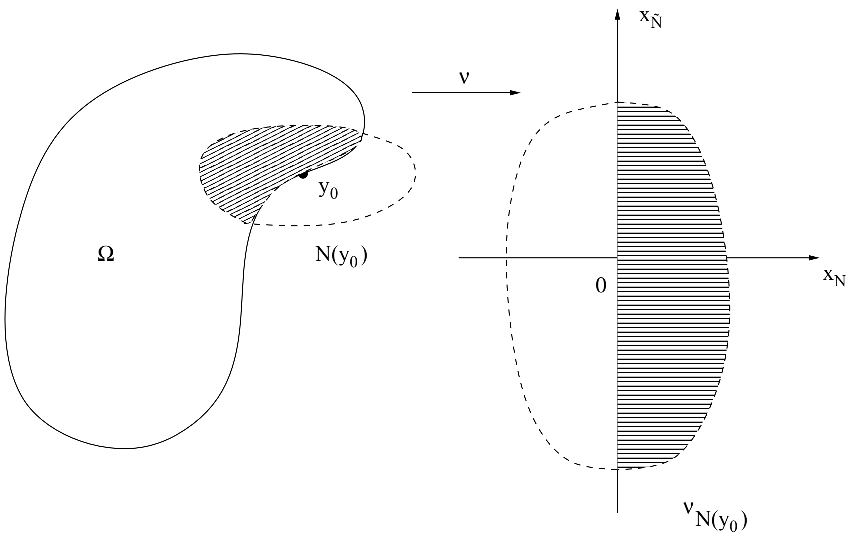

Figure 4.1: Flattening out $\partial\Omega$ locally.

Let γ_0 be the boundary trace operator:

$$\gamma_0\psi = \psi|_{\partial\Omega} \quad \forall \psi \in C^\infty(\overline{\Omega}). \tag{4.66}$$

Then γ_0 can be extended such that $\gamma_0\colon\ H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ is continuous for $s > \frac{1}{2}$, by Theorem 2.9.

Now, we study the operators A_1, A_2 and A_3 in (4.63)–(4.65) sequentially. First note that the fundamental solution E of the Laplacian satisfies

$$\Delta_x E(x,0) = -\delta(x), \quad x \in \mathbb{R}^N.$$

Using the Fourier transform, we obtain

$$\begin{aligned} |2\pi\xi|^2 \widehat{E}(\xi,0) &= 1, \\ E(x,0) &= \mathscr{F}^{-1}\left(\frac{1}{|2\pi\xi|^2}\right) \\ &= \mathscr{F}^{-1}\left(\frac{1}{4\pi^2}\frac{1}{|\xi_{\widehat{N}}|^2 + \xi_N^2}\right), \end{aligned}$$

The operator A_1 defined in (4.63) and (4.60) is a convolution of E with ϕ on $\partial\Omega$. Therefore A_1, as a pseudodifferential operator on $\mathbb{R}^{N-1} \leftrightarrow \partial\Omega$, through the correspondence shown in Fig. 4.1, can be written as

$$A_1\phi(x_{\widehat{N}}) = \gamma_0 r^+ \int_{\mathbb{R}^{2N}} e^{2\pi i\langle x-y,\xi\rangle} \frac{1}{4\pi^2} \frac{\phi(y_{\widehat{N}})\delta(y_N)}{|\xi_{\widehat{N}}|^2 + \xi_N^2}\, dy\, d\xi \tag{4.67}$$

for $x \in \mathbb{R}^N_+$, by the Fourier convolution formula (2.15). Now writing

$$\frac{1}{|\xi_{\widehat{N}}|^2+\xi_N^2} = \frac{1}{2|\xi_{\widehat{N}}|}\left(\frac{1}{|\xi_{\widehat{N}}|+i\xi_N}+\frac{1}{|\xi_N|-i\xi_N}\right), \tag{4.68}$$

we see that the above symbol is split into two parts: the first part has singularities at $\xi_N = i|\xi_{\widehat{N}}|$ and is thus analytic in the lower half plane $\operatorname{Im}\xi_N < 0$, and the second part has singularities at $\xi_N = -i|\xi_{\widehat{N}}|$ and is analytic in the upper half plane $\operatorname{Im}\xi_N > 0$, where these two parts are regarded as functions of the complex variable ξ_N. From the distribution of singularities $-i|\xi_{\widehat{N}}|$ and $i|\xi_{\widehat{N}}|$ on the RHS of (4.68), it is known (see e.g., [191, Chapt. VI, §4]) by an application of the inverse Fourier transform with respect to the y variable and the Paley–Wiener theorem that we have

$$\operatorname{supp}\left[\frac{1}{4\pi^2}\int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle}\left(\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N}\right)(\mathscr{F}\phi)(\xi_{\widehat{N}})\,d\xi\right] \subseteq \overline{\mathbb{R}^N_+}$$

and

$$\operatorname{supp}\left[\frac{1}{4\pi^2}\int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle}\left(\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|-i\xi_N}\right)(\mathscr{F}\phi)(\xi_{\widehat{N}})\,d\xi\right] \subseteq \overline{\mathbb{R}^N_-}.$$

Therefore (4.67) gives

$$\begin{aligned} A_1\phi(x_{\widehat{N}}) &= \gamma_0 r^+\left[\frac{1}{4\pi^2}\int_{\mathbb{R}^N} e^{2\pi i\langle x,\xi\rangle}\left(\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N}\right)(\mathscr{F}\phi)(\xi_{\widehat{N}})\,d\xi\right] \\ &= \int_{\mathbb{R}^{N-1}} e^{2\pi i\langle x_{\widehat{N}},\xi_{\widehat{N}}\rangle} p_1(2\pi\xi_{\widehat{N}})(\mathscr{F}\phi)(\xi_{\widehat{N}})\,d\xi_{\widehat{N}},\ \ x \in \mathbb{R}^N_+, \end{aligned}$$

where $p_1(\xi_{\widehat{N}})$ is the symbol of A_1, given by

$$p_1(2\pi\xi_{\widehat{N}}) = (2\pi)^{-2}\gamma_0 r^+\int_{\mathbb{R}} e^{2\pi i x_N\xi_N}\left(\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N}\right)d\xi_N.$$

Since γ_0 imposes the restriction to $x_N = 0$, the above is just

$$p_1(2\pi\xi_{\widehat{N}}) = (2\pi)^{-2}\oint_{C_+}\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N}\,d\xi_N, \tag{4.69}$$

where $\oint_{C_+}$ is a contour integral along a counterclockwise closed path C_+ in the upper half plane enclosing the singularity $\xi_N = i|\xi_{\widehat{N}}|$. We need only calculate the residue at the singularity $\xi_N = i|\xi_{\widehat{N}}|$. Since

$$\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N} = \frac{1}{2|\xi_{\widehat{N}}|i}\left(\frac{1}{\xi_N-i|\xi_{\widehat{N}}|}\right), \tag{4.70}$$

from (4.69) we get

$$p_1(2\pi\xi_{\widehat{N}}) = \frac{1}{2\pi}\frac{1}{2\pi i}\oint_{C^+}\frac{1}{2|\xi_{\widehat{N}}|}\frac{1}{\xi_N-i|\xi_{\widehat{N}}|}\,d\xi_N = \frac{1}{2}\frac{1}{|2\pi\xi_{\widehat{N}}|}. \tag{4.71}$$

Hence the principal symbol p_1 is found to be

$$p_1(\xi_{\widehat{N}}) = \frac{1}{2}|\xi_{\widehat{N}}|^{-1}, \tag{4.72}$$

which, after mapping from $\mathbb{R}^N$ back to Ω, implies that the principal symbol of A_1 is

$$\sigma(A_1) = \frac{1}{2}|\xi|^{-1}, \qquad \xi \in \partial\Omega, \tag{4.73}$$

where ξ is the dual variable of x for x restricted to $\partial\Omega$, i.e., the "cotangent vector" on $\partial\Omega$, and, as a consequence,

$$A_1 : \ H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \quad \forall s \in \mathbb{R}. \tag{4.74}$$

We proceed to study the principal symbols of A_2 and A_3. We note that normal differentiation $\partial/\partial n$ on $\partial\Omega$ corresponds to $\partial/\partial x_N$ on $\mathbb{R}^{N-1} = \partial\mathbb{R}^N_+$, and, after the Fourier transformation, $\partial/\partial n_y$ and $\partial/\partial n_x$ become respectively the multiplication by $2\pi i \xi_N$ and $-2\pi i\xi_N$ of the integrand of (4.67), i.e., for $x \in \mathbb{R}^N_+$,

$$A_2\phi(x_{\widehat{N}}) = \gamma_0 r^+ \left[\int_{\mathbb{R}^{2N}} e^{2\pi i\langle x-y,\xi\rangle} \frac{1}{4\pi^2} \frac{2\pi i\xi_N}{|\xi_{\widehat{N}}|^2 + \xi_N^2} \phi(y_{\widehat{N}})\delta(y_N)\, dy\, d\xi\right], \tag{4.75}$$

$$A_3\phi(x_{\widehat{N}}) = \gamma_0 r^+ \left[-\int_{\mathbb{R}^{2N}} e^{2\pi i\langle x-y,\xi\rangle} \frac{1}{4\pi^2} \frac{(2\pi i\xi_N)^2}{|\xi_{\widehat{N}}|^2 + \xi_N^2} \phi(y_{\widehat{N}})\delta(y_N)\, dy\, d\xi\right]. \tag{4.76}$$

Therefore, our primary task is to find a splitting for

$$\frac{i\xi_N}{|\xi_{\widehat{N}}|^2 + \xi_N^2} \tag{4.77}$$

and

$$\frac{-\xi_N^2}{|\xi_{\widehat{N}}|^2 + \xi_N^2} \tag{4.78}$$

such as (4.68). But this is straightforward, since

$$\frac{(i\xi_N)^k}{|\xi_{\widehat{N}}|^2 + \xi_N^2} = \frac{(i\xi_N)^k}{2|\xi_{\widehat{N}}|}\left(\frac{1}{|\xi_{\widehat{N}}| + i\xi_N} + \frac{1}{|\xi_{\widehat{N}}| - i\xi_N}\right), \ k = 1,2,$$

for (4.77) and (4.78). The parts

$$\frac{(i\xi_N)^k}{|\xi_{\widehat{N}}| + i\xi_N} \ \text{ and } \ \frac{(i\xi_N)^k}{|\xi_{\widehat{N}}| - i\xi_N}$$

have singularities at $\xi_N = i|\xi_{\widehat{N}}|$ and $\xi_N = -i|\xi_{\widehat{N}}|$, respectively, and thus are analytic on the lower and upper half planes respectively. By the same arguments as in the preceding paragraphs, the calculations of the symbols $p_2(\xi_{\widehat{N}})$ and $p_3(\xi_{\widehat{N}})$ for A_2 and A_3 respectively boil down to the following contour integrals:

$$p_k(2\pi\xi_{\widehat{N}}) = (2\pi)^{-2}(-1)^k \oint_{C_+} \frac{(2\pi i\xi_N)^{k-1}}{2|\xi_{\widehat{N}}|} \frac{1}{|\xi_{\widehat{N}}| + i\xi_N}\, d\xi_N, \ k = 2, 3. \tag{4.79}$$

We need only calculate the residues. Since

$$\begin{aligned}\frac{2\pi i\xi_N}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N} &= \frac{\pi\xi_N}{|\xi_{\widehat{N}}|}\frac{1}{\xi_N - i|\xi_{\widehat{N}}|}\\ &= \frac{\pi}{|\xi_{\widehat{N}}|}\left(1+\frac{i|\xi_{\widehat{N}}|}{\xi_N - i|\xi_{\widehat{N}}|}\right),\end{aligned} \tag{4.80}$$

the coefficient of the singular term $(\xi_N - i|\xi_{\widehat{N}}|)^{-1}$ is πi. Thus, from (4.79) and (4.80),

$$\begin{aligned}p_2(2\pi\xi_{\widehat{N}}) &= \frac{1}{(2\pi)^2}\oint_{C_+}(\pi i)\frac{1}{\xi_N - i|\xi_{\widehat{N}}|}\,d\xi_N\\ &= -\frac{1}{2\pi i}\oint_{C_+}\frac{1}{2}\frac{1}{\xi_N - i|\xi_{\widehat{N}}|}\,d\xi_N\\ &= -\frac{1}{2}.\end{aligned} \tag{4.81}$$

Therefore the leading symbol of A_2 is

$$\sigma(A_2) = -\frac{1}{2}. \tag{4.82}$$

A_2 is a pseudodifferential operator on $\partial\Omega$ of order 0, satisfying

$$A_2 \colon H^s(\partial\Omega) \to H^s(\partial\Omega) \quad \text{continuously,} \quad \forall s \in \mathbb{R}.$$

Finally,

$$\begin{aligned}\frac{(2\pi i\xi_N)^2}{2|\xi_{\widehat{N}}|}\frac{1}{|\xi_{\widehat{N}}|+i\xi_N} &= -\frac{2\pi^2}{i|\xi_{\widehat{N}}|}\left(\xi_N + \frac{i\xi_N|\xi_{\widehat{N}}|}{\xi_N - i|\xi_{\widehat{N}}|}\right)\\ &= -\frac{2\pi^2}{i|\xi_{\widehat{N}}|}\left[\xi_N + i|\xi_{\widehat{N}}|\left(1+\frac{i|\xi_{\widehat{N}}|}{\xi_N - i|\xi_{\widehat{N}}|}\right)\right];\end{aligned}$$

so the coefficient of the singular term $(\xi_N - i|\xi_N|)^{-1}$ is $-i2\pi^2|\xi_{\widehat{N}}|$, yielding

$$\begin{aligned}p_3(2\pi\xi_{\widehat{N}}) &= -\frac{1}{(2\pi)^2}\oint_{C_+}(-i2\pi^2)|\xi_{\widehat{N}}|\frac{1}{\xi_N - i|\xi_{\widehat{N}}|}\,d\xi_N\\ &= -\frac{1}{2\pi i}\oint_{C_+}\frac{|2\pi\xi_{\widehat{N}}|}{2}\frac{1}{\xi_N - i|\xi_{\widehat{N}}|}\,d\xi_N\\ &= -\frac{1}{2}|2\pi\xi_{\widehat{N}}|.\end{aligned}$$

Therefore the leading symbol of A_3 is

$$\sigma(A_3) = p_3(\xi) = -\frac{1}{2}|\xi|, \tag{4.83}$$

where ξ is the dual variable of x for x restricted to $\partial\Omega$. Consequently,

$$A_3 \colon H^s(\partial\Omega) \to H^{s-1}(\partial\Omega) \quad \text{is continuous } \forall s \in \mathbb{R}. \tag{4.84}$$

The operators A_1, A_2 and A_3 will be shown to be Fredholm, see § 4.6 and Chapter 6.

4.5 The Calderòn projector

There are some simple relationships between the boundary integral operators representing various layer potentials. They are obtainable through an elegant algebraic operator called *the Calderòn projector* [29, 64].

As before, let Ω be a bounded open domain in $\mathbb{R}^N$ with C^∞ boundary $\partial\Omega$, with outward unit normal vector field n on $\partial\Omega$. Let u_1 and u_2 be respectively C^∞ functions on $\overline{\Omega}$ and $\overline{\Omega^c}$. We first extend $n(x)$ to a C^∞ vector field on an open neighborhood of $\partial\Omega$ through a Gaussian normal coordinate system off $\partial\Omega$ and then define the kth normal derivative trace operators

$$\gamma_k u_i = \left[\left(\frac{\partial}{\partial n}\right)^k u_i\right]\Bigg|_{\partial\Omega}, \qquad i = 1, 2.$$

Let u be a C^∞ function on $\mathbb{R}^N \setminus \partial\Omega$ such that

$$u(x) = \begin{cases} u_1(x) \ \text{on}\ \Omega, \\ u_2(x) \ \text{on}\ \Omega^c, \end{cases} \tag{4.85}$$

where $u_1 \in C^\infty(\overline{\Omega})$ and $u_2 \in C^\infty(\overline{\Omega^c})$. Define the jump of $\gamma_k u$ across $\partial\Omega$ by

$$[\gamma_k u] \equiv \gamma_k u_2 - \gamma_k u_1 \quad \text{on}\ \partial\Omega, \tag{4.86}$$

as well as the jump vector (of order $2m, m \in \mathbb{N}$) across $\partial\Omega$ by

$$[\gamma u] \equiv ([\gamma_0 u], [\gamma_1 u], \ldots, [\gamma_{2m-1} u])^{\mathrm{Tr}}.$$

Note that the above definitions also are meaningful if u is a vector-valued function.

Let P be a given linear partial differential operator with C^∞ coefficients. Then we can write P as

$$P = \sum_{j=0}^{2m} P_j \partial_n^j, \tag{4.87}$$

where

$$P_j = \sum_{|l| \leqslant 2m-j} a_{jl}(x) D^l \qquad (x \in \partial\Omega) \tag{4.88}$$

is a partial differential operator of order $2m - j$ involving only tangential derivatives on $\partial\Omega$ [64, 23.48.13.3] (or more generally, on hypersurfaces parallel to $\partial\Omega$ through a local coordinate system on a neighborhood of $\partial\Omega$). Naturally, we may also consider P that are $q \times q$ matrices of operators acting on vector-valued functions u.

Let χ denote the characteristic function of Ω^c, i.e.,

$$\chi(x) = \begin{cases} 0, \ x \in \overline{\Omega}, \\ 1, \ x \in \Omega^c. \end{cases}$$

From § 3.1, it is easy to verify that

$$\partial_n \chi = \frac{\partial}{\partial n}\chi = \delta_{\partial\Omega} \tag{4.89}$$

in the sense of distributions, where $\delta_{\partial\Omega}$ is the Dirac delta distribution concentrated on $\partial\Omega$ defined by

$$\int_{\mathbb{R}^N} \delta_{\partial\Omega}(x)\phi(x)\,dx = \int_{\partial\Omega} \phi|_{\partial\Omega}\,d\sigma \quad \forall\phi \in \mathscr{D}(\mathbb{R}^N).$$

Let $v \in L^1(\partial\Omega)$. We define the tensorial distribution $v \otimes \delta_{\partial\Omega} \in \mathscr{D}'(\mathbb{R}^N)$ by

$$\int_{\mathbb{R}^N} (v \otimes \delta_{\partial\Omega})(x)\phi(x)\,dx = \int_{\partial\Omega} v\phi|_{\partial\Omega}\,d\sigma \quad \forall\phi \in \mathscr{D}(\mathbb{R}^N). \tag{4.90}$$

Lemma 4.4 ([64, 23.48.13.2]). *, Let u be C^∞ on $\mathbb{R}^N \setminus \partial\Omega$ and on $\overline{\Omega^c}$, and let P_0 be an lth-order operator of the form*

$$P_0 = D^j \partial_n^k, \qquad j+k=l,$$

where D^j is a tangential operator of order j. Then

$$P_0(u\chi) = (P_0 u)\chi + \sum_{i=0}^{k-1} D^j \partial_n^i \left[(\partial_n^{k-i-1} u)\big|_{\partial\Omega^+} \otimes \delta_{\partial\Omega} \right], \tag{4.91}$$

where in the above χ is the characteristic function of Ω^c and $(\partial_n^{k-i-1}u)\big|_{\partial\Omega^+}$ denotes the $(k-i-1)$th-order normal derivative trace taken from Ω^c.

Proof. First, we easily verify that

$$\partial_n(u\chi) = (\partial_n u)\chi + u(\partial_n\chi) = (\partial_n u)\chi + u|_{\partial\Omega} \otimes \delta_{\partial\Omega},$$

by (4.89) and (4.90). Repeating the above, we have

$$\begin{aligned}
\partial_n^2(u\chi) &= \partial_n[(\partial_n u)\chi + u|_{\partial\Omega} \otimes \delta_{\partial\Omega}] \\
&= (\partial_n^2 u)\chi + (\partial_n u)|_{\partial\Omega} \otimes \delta_{\partial\Omega} + \partial_n(u|_{\partial\Omega} \otimes \delta_{\partial\Omega}) \\
&\ \vdots \\
\partial_n^k(u\chi) &= (\partial_n^k u)\chi + \sum_{i=0}^{k-1} \partial_n^i[(\partial_n^{k-i-1}u)|_{\partial\Omega} \otimes \delta_{\partial\Omega}].
\end{aligned}$$

Since $u|_{\partial\Omega} \in C^\infty(\partial\Omega)$, for a tangential operator D^j we have

$$D^j\partial_n^k(u\chi) = (D^j\partial_n^k u)\chi + \sum_{i=0}^{k-1} D^j\partial_n^i[(\partial_n^{k-i-1}u)|_{\partial\Omega} \otimes \delta_{\partial\Omega}],$$

and (4.91) has been verified. □

Let $u \in C^\infty(\mathbb{R}^N \setminus \partial\Omega)$ satisfy (4.85) with $u_1 \in C^\infty(\overline{\Omega})$ and $u_2 \in C^\infty(\overline{\Omega^c})$. On $\mathbb{R}^N \setminus \partial\Omega$, let

$$Pu(x) = f(x) = \begin{cases} f_1(x), \, x \in \Omega, \\ f_2(x), \, x \in \Omega^c, \end{cases} \tag{4.92}$$

define f. Then, for P given by (4.87) (subject to (4.88)), from Lemma 4.4 we have

$$\begin{aligned} Pu &= P[u_2\chi + u_1(1-\chi)] \\ &= (Pu_2)\chi + (Pu_1)(1-\chi) \\ &\quad + \sum_{j=0}^{2m}\sum_{i=0}^{j-1}\{P_j\partial_n^i[(\partial_n^{j-i-1}u)|_{\partial\Omega^+} \otimes \delta_{\partial\Omega}] \\ &\quad - P_j\partial_n^i[(\partial_n^{j-i-1}u)|_{\partial\Omega} \otimes \delta_{\partial\Omega}]\} \\ &= f + \sum_{j=0}^{2m}\sum_{i=0}^{j-1}[P_j\partial_n^i([\partial_n^{j-i-1}u] \otimes \delta_{\partial\Omega})]. \end{aligned} \tag{4.93}$$

By making the change of indices $j - i - 1 = l$, the above becomes

$$Pu = f + \sum_{j=0}^{2m-1}\sum_{l=0}^{2m-j-1} P_{l+i+1}\partial_n^i([\partial_n^l u] \otimes \partial_{\partial\Omega}),$$

which is equation (1.3) of [55, p. 38]. Each term $\partial_n^i([\partial_n^l u] \otimes \delta_{\partial\Omega})$ is a distribution (also called an *i-fold layer*), satisfying

$$\int_{\mathbb{R}^N}([\partial_n^l u] \otimes \delta_{\partial\Omega})\phi(x)\,dx = \int_{\partial\Omega}[\partial_n^l u](\phi|_{\partial\Omega})\,d\sigma \quad \forall\phi \in \mathscr{D}(\mathbb{R}^N),$$

for $i = 0$. For $i = 1$, letting $\psi_l \equiv [\partial_n^l u]$ on $\partial\Omega$, noting that $n(x)$ has been extended outside $\partial\Omega$, we have

$$\begin{aligned} \int_{\mathbb{R}^N}[\partial_n([\partial_n^l u] \otimes \delta_{\partial\Omega})]\phi(x)\,dx &= \int_{\mathbb{R}^N}\left[\sum_{i=1}^N n_i(x)\frac{\partial}{\partial x_i}(\psi_l \otimes \delta)\right]\phi(x)\,dx \\ &= -\int_{\mathbb{R}^N}(\psi_l \otimes \delta)(x)\sum_{i=1}^N \frac{\partial}{\partial x_i}[n_i(x)\phi(x)]\,dx \\ &= -\int_{\mathbb{R}^N}(\psi_l \otimes \delta)\left[\frac{\partial}{\partial n}\phi + (\operatorname{div} n)\phi\right]dx \\ &= -\int_{\partial\Omega}\psi_l[(\partial_n\phi) - 2H(\phi|_{\partial\Omega})]\,d\sigma, \\ &= -\int_{\partial\Omega}[\partial_n^l u](\partial_n\phi - 2H\phi|_{\partial\Omega})\,d\sigma, \end{aligned}$$

where it is known [107, p. 112] that for $x \in \partial\Omega$,

$$H(x) = -\frac{1}{2}\operatorname{div} n(x) \tag{4.94}$$

is the mean curvature of the hypersurface $\partial\Omega$ at x. Similarly, for $i=2$,

$$\int_{\mathbb{R}^N} [\partial_n^2([\partial_n^l u]\otimes\delta_{\partial\Omega})]\phi(x)\,dx = \int_{\partial\Omega} [\partial_n^l u]\{\partial_n^2\phi - 4H\cdot\partial_n\phi + [-2\partial_n(H)+4H^2](\phi|_{\partial\Omega})\}\,d\sigma,$$

etc. Following the notation of [55], we define the distribution

$$v\otimes(\partial_n^i\delta_{\partial\Omega}) \equiv \partial_n^i(v\otimes\delta_{\partial\Omega}).$$

Then, for any test function $\phi\in\mathscr{D}(\mathbb{R}^N)$, we have

$$\begin{aligned}\langle v\otimes\partial_n^i\delta_{\partial\Omega},\phi\rangle &= \int_{\mathbb{R}^N}(v\otimes\partial_n^i\delta_{\partial\Omega})\phi\,dx\\ &= \int_{\mathbb{R}^N}[\partial_n^i(v\otimes\delta_{\partial\Omega})]\phi\,dx\\ &= \int_{\partial\Omega} v[(\partial_n^i)^*\phi]\,d\sigma,\end{aligned}$$

where $(\partial_n^i)^* = (\partial_n^*)^i$ is the adjoint operator of ∂_n^i satisfying

$$\partial_n^* = -\partial_n + 2H, \tag{4.95}$$

$$\partial_n^{*^2} = (-\partial_n+2H)^2 = \partial_n^2 - 2H\partial_n + [4H^2 - 2(\partial_n H)],$$

$$\vdots$$

$$(\partial_n^*)^k = (-1)^k\partial_n^k + \text{lower-order terms}. \tag{4.96}$$

Equation (4.93) can now be rewritten as

$$Pu = f + \sum_{i=0}^{2m-1}\sum_{l=0}^{2m-i-1} P_{l+i+1}[\partial_n^l u]\otimes(\partial_n^i\delta_{\partial\Omega}). \tag{4.97}$$

To explain Calderòn's ideas [29], we now assume that the operator P in (4.87) is elliptic with a fundamental solution G, i.e., G satisfies

$$GP\psi = PG\psi = \psi \tag{4.98}$$

for all distributions $\psi\in\mathscr{D}'(\mathbb{R}^N)$ with compact support. It is known [170] that G is a pseudodifferential operator of order $-2m$ and as a consequence has a locally integrable kernel $G(x,\xi)$, $x,\xi\in\mathbb{R}^N$, which is C^∞ for $x\neq\xi$ and

$$(G\phi)(x) = \int_{\mathbb{R}^N} G(x,\xi)\phi(\xi)\,d\xi, \quad \phi\in\mathscr{D}(\mathbb{R}^N).$$

The application of G to an i-fold layer $\phi\otimes\partial_n^i\delta_{\partial\Omega}$ gives rise to an i-fold layer potential

$$V_{i+1}\phi \equiv G(\phi\otimes\partial_n^i\delta_{\partial\Omega}). \tag{4.99}$$

For $x\notin\partial\Omega$, (4.99) gives

$$(V_{i+1}\phi)(x) = \int_{\partial\Omega}\partial_{n(\xi)}^{*^i} G(x,\xi)\phi(\xi)\,d\sigma_\xi, \quad \phi\in\mathscr{D}(\mathbb{R}^N). \tag{4.100}$$

It is known [64, §23.53]

$$V_{i+1}\phi \in H_{\mathrm{loc}}^{-1/2+2m-i-\varepsilon}(\mathbb{R}^N) \quad \forall \varepsilon > 0, \phi \in H_{\mathrm{loc}}^{0}(\mathbb{R}^N). \tag{4.101}$$

Now, applying G to (4.97) and utilizing (4.98), we obtain a general representation formula for the solution of the linear PDE (4.92) on $\mathbb{R}^N \setminus \partial\Omega$,

$$u = Gf + \sum_{\substack{i+l+1\leqslant 2m \\ i,l\geqslant 0}} V_{i+1}(P_{l+i+1}[\partial_n^l u]), \tag{4.102}$$

for u, which is equal to respectively u_1 on Ω and u_2 on Ω^c, with $u_1 \in C_0^\infty(\overline{\Omega})$ and $u_2 \in C_0^\infty(\overline{\Omega^c})$, and where $f = Pu$ on $\mathbb{R}^N \setminus \partial\Omega$.

Let $u_2 \equiv 0$ on Ω^c. Then (cf. (4.86))

$$[\gamma u] = -(\gamma_0 u_1, \gamma_1 u_1, \ldots, \gamma_{2m-1} u_1)^{\mathrm{Tr}}. \tag{4.103}$$

The RHS above is called the *Cauchy data* of u_1. Substituting (4.103) into (4.102) and then taking the Cauchy data of u in (4.102), we obtain an identity for the Cauchy data of u_1:

$$\gamma u_1 = \gamma G f_1 - \sum_{\substack{i+l+1\leqslant 2m \\ i,l\geqslant 0}} \gamma[(V_{i+1}(P_{l+i+1}\gamma_l u_1))|_\Omega] \tag{4.104}$$

$$\equiv \gamma G f_1 + \mathscr{C}_1(\gamma u_1) \quad \forall u_1 \in C_0^\infty(\overline{\Omega}).$$

The operator $\mathscr{C}_1$ is *the Calderòn projector*. More precisely, define the matrices of operators

$$\mathscr{P} \equiv [\mathscr{P}_{il}] = [P_{i+l+1}]_{i,l=0,\ldots,2m-1}, \tag{4.105}$$

$$\text{with } P_t = 0 \text{ for } t > 2m,$$

$$\mathscr{K}_j \equiv [\mathscr{K}_{ik}^{(j)}] = [\gamma_i V_{k+1}^{(j)}]_{i,k=0,\ldots,2m-1}, \tag{4.106}$$

with

$$V_{k+1}^{(j)}(\phi) = \begin{cases} V_{k+1}(\phi)|_{\overline{\Omega}}, & j = 1, \\ V_{k+1}(\phi)|_{\overline{\Omega^c}}, & j = 2. \end{cases}$$

Then

$$\mathscr{C}_1 = -\mathscr{K}_1 \mathscr{P}. \tag{4.107}$$

Similarly, setting $u_1 \equiv 0$ in (4.102), we obtain

$$\gamma u_2 = G f_2 + \mathscr{C}_2(\gamma u_2) \tag{4.108}$$

for any $u_2 \in C_0^\infty(\overline{\Omega^c})$, where $f_2 = Pu|_{\Omega^c}$ and

$$\mathscr{C}_2 = \mathscr{K}_2 \mathscr{P}. \tag{4.109}$$

Note that all the operators $\mathscr{P}, \mathscr{K}_j$ and $\mathscr{C}_j$, $j=1,2$, in (4.107) and (4.109) are well defined if $[\gamma u]$, γu_1 and γu_2 respectively are replaced by an arbitrary (vector-valued) function $\phi \in C^\infty(\partial\Omega;\mathbb{C}^{2m})$. According to [64, 23.4.7.3.1 and 23.48.14.3], it is known that $\mathscr{K}_j$ and $\mathscr{C}_j$ are pseudodifferential operators with

$$\begin{aligned}
\text{order } \mathscr{K}_j &= [\text{order } K_{ik}^{(j)}] \\
&= [i+k-2m+1]_{i,k=0,\ldots,2m-1}, \qquad (4.110)\\
\text{order } \mathscr{C}_j &= [\text{order } (\mathscr{C}_j)_{ik}] \\
&= [i-k]_{i,k=0,\ldots,2m-1}. \qquad (4.111)
\end{aligned}$$

with also the explicit formulas for the principal symbols given therein. A purely algebraic method for computing the principal symbols of $\mathscr{C}_j$ in terms of the symbol of P can be found in [86, Appendix].

Now we prove the following proposition, which dates back to Seeley [169].

Proposition 4.1 ([55, Lemma 1.1], [169, Lemma 5]). *Denote*

$$[\mathscr{K}] \equiv \mathscr{K}_2 - \mathscr{K}_1.$$

Then

$$[\mathscr{K}] = \mathscr{P}^{-1}, \tag{4.112}$$

where $\mathscr{P}^{-1}$ is the inverse matrix of tangential operators to $\mathscr{P}$. Equivalently, for the Calderòn operators,

$$\mathscr{C}_1 + \mathscr{C}_2 = I. \tag{4.113}$$

Proof. By (4.105), the matrix operator $\mathscr{P}$ is given by

$$\mathscr{P} = \begin{bmatrix} P_1 & P_2 & P_3 & \ldots & P_{2m} \\ P_2 & P_3 & \ldots & P_{2m} & 0 \\ P_3 & \vdots & & \vdots & \vdots \\ \vdots & P_{2m} & \ldots & \ldots & 0 \\ P_{2m} & 0 & \ldots & \ldots & 0 \end{bmatrix}.$$

It is upper-triangular with respect to the transverse diagonal, every entry of which is equal to P_{2m}. Since P_{2m} is of order zero and hence is just a function, P_{2m}^{-1} exists due to the ellipticity of P. Consequently, it is straightforward to verify that $\mathscr{P}$ is invertible, with $\mathscr{P}^{-1}$ being a lower-triangular (with respect to the transverse diagonal) matrix of tangential differential operators.

Thus, it suffices to show that $[\mathscr{K}]$ is a right inverse to $\mathscr{P}$. Let $v \in C^\infty(\partial\Omega)$ and $i \in \{0,1,\ldots,2m-1\}$. Define

$$u = V_{i+1}(v) = G(v \otimes \partial_n^i \delta_{\partial\Omega}).$$

Then u is C^∞ on $\overline{\Omega}$ and $\overline{\Omega^c}$, and by (4.97)

$$Pu = v \otimes \partial_n^i \delta_{\partial\Omega}.$$

Since $v \otimes \partial_n^i \delta_{\partial\Omega}$ is concentrated on $\partial\Omega$, we have

$$f = Pu|_{\mathbb{R}^N \setminus \partial\Omega} \equiv 0.$$

Apply (4.97) to the above u, yielding

$$v \otimes \partial_n^i \delta_{\partial\Omega} = \sum_{k=0}^{2m-1} \sum_{l=0}^{2m-k-1} P_{k+l+1}[\gamma_l u] \otimes \partial_n^k \delta_{\partial\Omega}. \tag{4.114}$$

Noting that for $\phi_k \in C^\infty(\partial\Omega)$ that do not vanish identically, the distributions

$$\{\phi_k \otimes \partial_n^k \delta_{\partial\Omega} \mid k = 0,\ldots,2m-1\}$$

are linearly independent, we obtain from (4.114) that

$$\sum_{l=0}^{2m-k-1} P_{k+l+1}[\gamma_l V_{i+1}(v)] = \begin{cases} v, & k = i, \\ 0, & k \neq i. \end{cases} \tag{4.115}$$

Since v is arbitrary, (4.115) gives

$$\mathscr{P}[\mathscr{K}] = I.$$

The equivalence between (4.112) and (4.113) is easily obtained after observing (4.107) and (4.109), and noting that $[\mathscr{K}]$ is a two-sided inverse of $\mathscr{P}$. □

The Calderòn projectors $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy the following useful properties as shown in the next theorem. We see that they are indeed the *projection operators into the space of Cauchy data* of solutions on Ω and Ω^c respectively.

Theorem 4.6. *For any $v = (v_0, v_1, \ldots, v_{2m-1})^{Tr} \in [C^\infty(\partial\Omega)]^{2m}$, we have*

$$\mathscr{C}_1\mathscr{C}_2 v = 0, \;\; \mathscr{C}_2\mathscr{C}_1 v = 0.$$

Consequently, $\mathscr{C}_1$ and $\mathscr{C}_2$ satisfy

$$\left.\begin{aligned} \mathscr{C}_1 + \mathscr{C}_2 = I, \;\; \mathscr{C}_1^2 = \mathscr{C}_1, \\ \mathscr{C}_2^2 = \mathscr{C}_2, \mathscr{C}_1\mathscr{C}_2 = \mathscr{C}_2\mathscr{C}_1 = 0. \end{aligned}\right\} \tag{4.116}$$

Proof. For any given $v = (v_0, v_1, \ldots, v_{2m-1}) \in [C^\infty(\partial\Omega)]^{2m}$, define u on $\mathbb{R}^N \setminus \partial\Omega$ by

$$\begin{aligned} u &= \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} V_{k+1}(P_{k+l+1}v_l) \\ &= \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} G((P_{k+l+1}v_l)\otimes \partial_n^k \delta_{\partial\Omega}). \end{aligned} \tag{4.117}$$

Define

$$u_1 = u|_\Omega, \quad u_2 = u|_{\Omega^c}.$$

Then u_1 and u_2 can be extended respectively to be C^∞ on $\overline{\Omega}$ and $\overline{\Omega^c}$. Using (4.98), we obtain, on $\mathbb{R}^N \setminus \partial\Omega$,

$$Pu = \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} P_{k+l+1}v_l \otimes \partial_n^k \delta_{\partial\Omega}.$$

Hence $Pu_1 = 0$ and $Pu_2 = 0$ on Ω and Ω^c respectively. Now extend u_1 and u_2 by 0 on Ω^c and Ω respectively, and denote the extended functions as $\widetilde{u}_1$ and $\widetilde{u}_2$. We can apply (4.102), (4.104) and (4.108) to get

$$\widetilde{u}_1 = -\sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} V_k(P_{l+k+1}\gamma_l u_1), \tag{4.118}$$

$$\widetilde{u}_2 = \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} V_k(P_{l+k+1}\gamma_l u_2), \tag{4.119}$$

$$\gamma(\widetilde{u}_1|_\Omega) = \gamma u_1 = \mathscr{C}_1(\gamma u_1), \tag{4.120}$$

$$\gamma(\widetilde{u}_2|_{\Omega^c}) = \gamma u_2 = \mathscr{C}_2(\gamma u_2). \tag{4.121}$$

It is obvious that

$$\mathscr{C}_2(\gamma u_1) = \mathscr{C}_2\mathscr{C}_1(\gamma u_1) = \mathscr{C}_2(\gamma(\widetilde{u}_1|_{\Omega^c})) = 0, \tag{4.122}$$

$$\mathscr{C}_1(\gamma u_2) = \mathscr{C}_1\mathscr{C}_2(\gamma u_2) = \mathscr{C}_1(\gamma(\widetilde{u}_2|_\Omega)) = 0. \tag{4.123}$$

Since u_1 and u_2 are defined from u in (4.117), by (4.102) we have

$$\begin{aligned} u &= \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} G((P_{k+l+1}v_l)\otimes \partial_n^k \delta_{\partial\Omega}) \\ &= \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} G((P_{k+l+1}[\gamma_l u]\otimes \partial_n^k \delta_{\partial\Omega}) \\ &= \sum_{\substack{k+l+1\leqslant 2m\\ k,l\geqslant 0}} G((P_{k+l+1}(\gamma_l u_2 - \gamma_l u_1))\otimes \partial_n^k \delta_{\partial\Omega}). \end{aligned}$$

Using the same arguments as in the proof of Proposition 4.1, we obtain

$$P_{k+l+1}v_l = P_{k+l+1}(\gamma_l u_2 - \gamma_l u_1) \quad \forall k,l \geqslant 0 \text{ with } k+l+1 \leqslant 2m.$$

Choosing $k+l+1 = 2m$, we have

$$P_{2m}v_l = P_{2m}(\gamma_l u_2 - \gamma_l u_1).$$

For the same reason as before, we get

$$v_l = \gamma_l u_2 - \gamma_l u_1, \text{ for } l = 0,1,\ldots,2m-1.$$

Hence

$$v = \gamma u_2 - \gamma u_1 = \mathscr{C}_2(\gamma u_2) - \mathscr{C}_1(\gamma u_1),$$

by (4.121) and (4.120). Therefore, by (4.120)–(4.123),

$$\begin{aligned}
\mathscr{C}_1 v &= \mathscr{C}_1\mathscr{C}_2(\gamma u_2) - \mathscr{C}_1^2(\gamma u_1) \\
&= 0 - \mathscr{C}_1^2(\gamma u_1) \\
&= -\gamma u_1, \\
\mathscr{C}_2 v &= \mathscr{C}_2\mathscr{C}_2(\gamma u_2) - \mathscr{C}_2\mathscr{C}_1(\gamma u_1) \\
&= \gamma u_2, \\
\mathscr{C}_1\mathscr{C}_2 v = \mathscr{C}_1(\gamma u_2) = 0, \mathscr{C}_2\mathscr{C}_1 v &= \mathscr{C}_2(\gamma u_1) \;=\; 0.
\end{aligned}$$

The proof is complete. $\square$

It is easy to see that for $w \in [C^\infty(\partial\Omega)]^{2m}$ the following are equivalent

(i) $w = \gamma u$ for some u satisfying $Pu = 0$ on $\mathbb{R}^N \setminus \partial\Omega, u$ being sufficiently smooth on $\overline{\Omega}$ and $\overline{\Omega^c}$;

(ii) $\mathscr{C}_j w = w, j = 1$ or 2;

(iii) $w = \mathscr{C}_j g$ for some $g \in [C^\infty(\partial\Omega)]^{2m}, j = 1$ or 2.

We now illustrate an application of the above theory to the Laplacian [55].

Example 4.3. The Calderòn projectors for Δ.

Let Ω be a bounded open domain in $\mathbb{R}^N$ with C^∞ boundary $\partial\Omega$. Consider $P = -\Delta = -\sum_{i=1}^N \partial^2/\partial x_i^2$. It is known (see e.g., [107, p. 124, last line]) that P can be written as

$$P = -\Delta = -\Delta_0 + 2H\partial_n - \partial_n^2, \text{ with } \mathscr{P} = \begin{bmatrix} 2H & -1 \\ -1 & 0 \end{bmatrix},$$

where H is mean curvature as in (4.94), and Δ_0 is the Laplace–Beltrami operator on $\partial\Omega$. (For example, when Ω is a sphere with center at the origin, we have

$$P = -\Delta = -\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right] - \frac{2}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2},$$

where $\partial/\partial r = \partial/\partial n$ is the normal derivative).

From (4.95)

$$\partial_n^* = -\partial_n + 2H,$$

and the layer potentials (4.100) take the forms

$$\text{(simple layer)} \qquad V_1\phi(x) = \int_{\partial\Omega} E(x,\xi)\phi(\xi)\,d\sigma_\xi,$$

$$\text{(double layer)} \qquad V_2\phi(x) = -\int_{\partial\Omega}\left[\frac{\partial}{\partial n_\xi}E(x,\xi)\right]\phi(\xi)\,d\sigma_\xi + 2\int_{\partial\Omega} H(\xi)E(x,\xi)\phi(\xi)\,d\sigma_\xi, \quad x \in \mathbb{R}^N \setminus \partial\Omega,$$

where $E(x,\xi)$ is given by (1.3).

In comparison with the classical double-layer potential (4.61) to be used and studied later on, V_2 above is actually the sum of the classical double-layer potential with a simple-layer potential due to the form of ∂_n^*. To obtain the classical double-layer potential and, more generally, the classical multiple-layer potentials

$$\widetilde{V}_{k+1}\phi \equiv G(\phi \otimes \partial_n^{*k}\delta_{\partial\Omega}) = \int_{\partial\Omega}\left(\frac{\partial}{\partial n_\xi}\right)^k E(x,\xi)\phi(\xi)\,d\sigma_\xi,$$

we can interchange the roles of ∂_n and ∂_n^* in (4.100) and (4.87):

$$P = \Sigma\widetilde{P}_j\partial_n^{*j}.$$

For the special case $P = -\Delta$, there is an easier and more straightforward way. We have, from (4.92),

$$\begin{aligned} -\Delta u &= f + (2H[\gamma_0 u] - [\gamma_1 u]) \otimes \delta_{\partial\Omega} + (-[\gamma_0 u]) \otimes \partial_n\delta_{\partial\Omega} \\ &= f - [\gamma_1 u] \otimes \delta_{\partial\Omega} + [\gamma_0 u] \otimes \partial_n^*\delta_{\partial\Omega}. \end{aligned}$$

Thus (4.102) and the above give the direct Green's representation formula

$$u(x) = \int_{\mathbb{R}^N} E(x,\xi)f(\xi)\,d\xi + \int_{\partial\Omega}\frac{\partial E(x,\xi)}{\partial n_\xi}[\gamma_0 u](\xi)\,d\sigma_\xi - \int_{\partial\Omega} E(x,\xi)[\gamma_1 u](\xi)\,d\sigma_\xi.$$

As will be shown later (cf. Theorem 6.3, with a slight change of notation), the jump conditions of potentials yield

$$\gamma_0 u_1 = \frac{1}{2}(1 - 2\widetilde{V}_2)(\gamma_0 u_1) + V_1(\gamma_1 u_1), \tag{4.124}$$

$$\gamma_1 u_1 = -\widetilde{V}_2^d(\gamma_0 u_1) + \frac{1}{2}(1 + 2\widetilde{V}_2^*)(\gamma_1 u_1), \tag{4.125}$$

where $u_1 = u|_{\Omega}$, and for $x \in \partial\Omega$,

$$V_1\phi(x) = \int_{\partial\Omega} E(x,\xi)\phi(\xi)\,d\sigma_\xi,$$

$$\widetilde{V}_2\phi(x) = \int_{\partial\Omega} \left[\frac{\partial}{\partial n_\xi} E(x,\xi)\right] \phi(\xi)\,d\sigma_\xi,$$

$$\widetilde{V}_2^*\phi(x) = \int_{\partial\Omega} \left[\frac{\partial}{\partial n_x} E(x,\xi)\right] \phi(\xi)\,d\sigma_\xi,$$

$$\widetilde{V}_2^d\phi(x) = \int_{\partial\Omega} \left[\frac{\partial^2}{\partial n_x \partial n_\xi} E(x,\xi)\right] \phi(\xi)\,d\sigma_\xi,$$

i.e., V_1, $\widetilde{V}_2$, $\widetilde{V}_2^*$ and $\widetilde{V}_2^d$ are respectively, the simple-layer, double-layer, the adjoint of the double-layer, and the normal derivative of the double-layer potential operators. Hence the first Calderòn operator, which projects into the space of Cauchy data of $Pu = 0$ on Ω, by (4.124) and (4.125), is

$$\mathscr{C}_1 = \begin{bmatrix} \frac{1}{2} - \widetilde{V}_2 & V_1 \\ -\widetilde{V}_2^d & \frac{1}{2} + \widetilde{V}_2^* \end{bmatrix}.$$

Similarly, the second Calderòn projector is

$$\mathscr{C}_2 = \begin{bmatrix} \frac{1}{2} + \widetilde{V}_2 & -V_1 \\ \widetilde{V}_2^d & \frac{1}{2} - \widetilde{V}_2^* \end{bmatrix}.$$

Thus, we indeed have

$$\mathscr{C}_1 + \mathscr{C}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

proved in Proposition 4.1. From the projection property $\mathscr{C}_1^2 = \mathscr{C}_1$ (or, identically, $\mathscr{C}_2^2 = \mathscr{C}_2$) in Theorem 4.6,

$$\mathscr{C}_1^2 = \begin{bmatrix} \frac{1}{4} - \widetilde{V}_2 + (\widetilde{V}_2)^2 - V_1\widetilde{V}_2^d & V_1 - \widetilde{V}_2V_1 + V_1\widetilde{V}_2^* \\ -\widetilde{V}_2^d + \widetilde{V}_2^d\widetilde{V}_2 - \widetilde{V}_2^*\widetilde{V}_2^d & -\widetilde{V}_2^d V_1 + \frac{1}{4} + \widetilde{V}_2^* + (\widetilde{V}_2^*)^2 \end{bmatrix}$$

$$= \mathscr{C}_1 = \begin{bmatrix} \frac{1}{2} - \widetilde{V}_2 & V_1 \\ -\widetilde{V}_2^d & \frac{1}{2} + \widetilde{V}_2^* \end{bmatrix},$$

we obtain the following simple relations between these layer potential operators:

$$-\frac{1}{4} + (\widetilde{V}_2)^2 - V_1\widetilde{V}_2^d = 0, \tag{4.126}$$

$$V_1\widetilde{V}_2^* - \widetilde{V}_2V_1 = 0, \tag{4.127}$$

$$\widetilde{V}_2^d\widetilde{V}_2 - \widetilde{V}_2^*\widetilde{V}_2^d = 0, \tag{4.128}$$

$$-\frac{1}{4} + (\widetilde{V}_2^*)^2 - \widetilde{V}_2^d V_1 = 0. \tag{4.129}$$

Note that (4.128) and (4.129) are respectively the adjoint relations of (4.127) and (4.126). □

4.6 Fredholm operators

Let V and H be Hilbert spaces such that

$$V \subset H, \quad V \quad \text{is dense in} \quad H, \tag{4.130}$$

and the injection is continuous. The *dual space* of all continuous linear functionals l on V satisfying

$$l: \; V \to \mathbb{C}, \quad |l(v)| \leqslant M\|v\|_V \; \text{ for some } M > 0, \; \forall\, v \in V,$$

is V'. We write

$$l(v) = (v, l)_{V \times V'}.$$

Assume that $l_0 \in V'$ also satisfies

$$|l_0(v)| \leqslant M_0\|v\|_H \; \text{ for some } M_0 > 0, \forall\, v \in H.$$

Then, by the closed graph theorem (see Appendix A.3(ii) or [161]), l_0 can be uniquely extended as a bounded linear functional to all of H. We still call it l_0. Then $l_0 \in H'$ and we have

$$(v, l_0)_{V \times V'} = (v, l_0)_{H \times H'}. \tag{4.131}$$

We can identify H' with H through the Riesz representation theorem (see Appendix A.4). For any $l \in H \equiv H'$, through the pivotal relation (4.131), we define

$$l(v) \equiv (v, l)_{H \times H'},$$

so

$$|l(v)| \leqslant M_0\|v\|_H \leqslant M_1\|v\|_V \; \text{ for some } M_0, M_1 > 0, \forall\, v \in V,$$

by the dense imbedding of V in H. Therefore $l \in V'$ and (algebraically)

$$V \subset H (\equiv H') \subset V'. \tag{4.132}$$

It is not difficult to verify that the injection from H into V' is also dense and continuous, so the inclusions in (4.132) are also topological.

Proposition 4.2. *Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X, Y)$. Then*

$$[\mathscr{R}(A)]^{\perp} = \mathscr{N}(A^*). \tag{4.133}$$

Proof. Let $y^* \in \mathscr{N}(A^*)$ and $y \in \mathscr{R}(A)$. Then $y = Ax$ for some $x \in X$. We have

$$(y,y^*)_{Y\times Y'} = (Ax,y^*) = (x,A^*y^*) = 0,$$

so $\mathscr{N}(A^*) \subset [\mathscr{R}(A)]^{\perp}$.
Now let $y^* \in [\mathscr{R}(A)]^{\perp}$. Then

$$(Ax,y^*) = 0 \quad \forall x \in X.$$

So

$$(x,A^*y^*) = 0 \qquad \forall x \in X.$$

Hence $[\mathscr{R}(A)]^{\perp} \subset \mathscr{N}(A^*)$. □

The dual of Proposition 4.2 is

Proposition 4.3. *Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X,Y)$. Let $\mathscr{R}(A)$ be closed. Then*

$$\mathscr{R}(A^*) = [\mathscr{N}(A)]^{\perp}. \tag{4.134}$$

Proof. Let $x^* \in \mathscr{R}(A^*)$. Then $x^* = A^*y^*$ for some $y^* \in Y'$. Let $x \in \mathscr{N}(A)$. Then

$$(x,x^*)_{X\times X'} = (x,A^*y^*) = (Ax,y^*) = 0.$$

Thus $x^* \in [\mathscr{N}(A)]^{\perp}$. It follows that $\mathscr{R}(A^*) \subset [\mathscr{N}(A)]^{\perp}$.
To prove the reverse inclusion, more work is required. For each $x \in X$, let $[x] = x + \mathscr{N}(A)$, the equivalence class containing x in the quotient space (see Appendix A.6) $\overline{X} \equiv X/\mathscr{N}(A)$. Then A induces a 1–1 mapping

$$\overline{A}:\ X/\mathscr{N}(A) \longrightarrow \mathscr{R}(A), \qquad \overline{A}[x] = Ax.$$

This 1–1 mapping has an inverse $\overline{A}^{-1} \in \mathscr{L}(\mathscr{R}(A),\overline{X})$ because $\mathscr{R}(A)$ is closed (and therefore is a Banach space with the subspace topology of Y). Therefore there is a constant $M > 0$ such that for any $y \in \mathscr{R}(A)$ such that $y = Ax$ for some $x \in X$,

$$\|[x]\|_{\overline{X}} \leqslant M\|y\|_Y \quad \forall x \in X. \tag{4.135}$$

Now let $x^* \in [\mathscr{N}(A)]^{\perp}$. For $y \in \mathscr{R}(A)$ and $x \in X$ satisfying $Ax = y$, define

$$f(y) = (\xi,x^*)_{X\times X'} \quad \forall \xi \in [x]. \tag{4.136}$$

Then for any $a_1,a_2 \in \mathbb{C}$, $y_1,y_2 \in \mathscr{R}(A)$, we have

$$\begin{aligned} f(a_1y_1 + a_2y_2) &= (a_1\xi_1 + a_2\xi_2 + \mathscr{N}(A),x^*) \\ &= (a_1\xi_1 + a_2\xi_2,x^*) \\ &= a_1f(y_1) + a_2f(y_2), \end{aligned}$$

for any $x_1, x_2 \in X$ such that $y_1 = Ax_1, y_2 = Ax_2$, and $\xi_1 \in [x_1]$, $\xi_2 \in [x_2]$. Therefore f is well defined and linear. f is bounded because

$$|f(y)| = |([x], x^*)| \leqslant \|x^*\|_{X'} \|[x]\|_{\overline{X}} \leqslant M \|x^*\|_{X'} \|y\|_Y,$$
$$\text{for } x \text{ such that } Ax = y, \text{by (4.135) and (4.136).}$$

We now invoke the closed graph theorem (see Appendix A.3(ii)). Extend f to a linear functional $F \in Y^*$. From

$$\begin{aligned}(x, A^*F^*) &= (Ax, F^*)\\ &= (y, F^*)\\ &= (x, x^*) \quad \forall x \in X,\end{aligned}$$

it follows that $A^*F^* = x^*$, and thus $\mathscr{R}(A^*) \supset [\mathscr{N}(A)]^{\perp}$. □

Definition 4.4. Let X and Y be two Banach spaces. Let A be a bounded linear operator from X into Y, with $\mathscr{R}(A)$ closed in Y. We define the *cokernel* of A to be

$$\text{Coker}\, A = \{y^* \in Y' \mid (Ax, y^*) = 0 \; \forall x \in X\}.$$

we say that A is a *Fredholm operator* if the dimension of its null space (or kernel) and the codimension of its range (i.e., dim Coker A) are both finite. For a Fredholm operator A, the number

$$\text{Ind}\, A = \dim \mathscr{N}(A) - \text{ codim}\, \mathscr{R}(A).$$

is called the *index* of A. □

Theorem 4.7. *Let X and Y be reflexive Banach spaces, and let $A \in \mathscr{L}(X,Y)$. If A is Fredholm then so is A^*, and* $\text{Ind}\, A = -\text{ Ind}\, A^*$.

Proof. By Propositions 4.2 and 4.3 , because of reflexivity and $A = A^{**}$, we have the symmetric relations

$$\dim \mathscr{N}(A^*) = \text{codim}\, \mathscr{R}(A), \tag{4.137}$$

$$\dim \mathscr{N}(A) = \text{codim}\, \mathscr{R}(A^*). \tag{4.138}$$

$\mathscr{R}(A^*)$ is closed in X' because $\mathscr{R}(A)$ is closed in X due to the closed range theorem of Banach [191, p. 205]. □

Corollary 4.1. *Let H be a Hilbert space, and suppose $A \in \mathscr{L}(H)$ is self-adjoint, i.e., $A = A^*$. If A is Fredholm then* $\text{Ind}\, A = 0$.

Definition 4.5. Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X,Y)$. We say that A is *of finite rank* if $\dim \mathscr{R}(A)$ is finite. □

Proposition 4.4. *Suppose that X and Y are reflexive Banach spaces and that $A \in \mathscr{L}(X,Y)$. Then the following two properties are equivalent:*

(i) *A is Fredholm;*

(ii) *There is $B \in \mathscr{L}(Y,X)$ such that $AB - I_Y$ and $BA - I_X$ both have finite rank.*

If the above hold then the operator B in (ii) is also Fredholm. We can choose B such that

$$A \circ B \circ A = A, \qquad B \circ A \circ B = B. \tag{4.139}$$

Proof. First assume that (i) holds. Let $d = \operatorname{codim} \mathscr{R}(A)$. Then by (4.133) we can select a basis $f_1^*, f_2^*, \ldots, f_d^*$ in $\mathscr{N}(A^*)$, and d vectors $f_1, f_2, \ldots, f_d$ in Y, such that $(f_k, f_j^*) = \delta_{kj}$. Define

$$P\colon\ Y \to Y$$
$$Py = y - \sum_{j=1}^{d} (y, f_j^*) f_j.$$

Then $(Py, f_j^*) = 0$ for all $y \in Y$ and $j = 1, 2, \ldots, d$. Therefore the range of P is contained in $A(X)$. The restriction of P to $A(X)$ is the identity. P is a continuous projection (see Appendix A.7) of Y onto $A(X)$.

Let X_0 be a closed subspace of X such that X is the direct sum (see Appendix A.7) of X_0 and $\mathscr{N}(A)\colon\ X = X_0 \oplus \mathscr{N}(A)$. We restrict A to X_0 by defining

$$A_0 : X_0 \longrightarrow A(X),$$
$$A_0(x_0) = Ax \quad \text{if } x = x_0 + n \text{ for some } n \in \mathscr{N}(A).$$

Then A_0 is a continuous linear isomorphism, so its inverse $A_0^{-1}\colon\ A(X) \to X$ exists. Now define

$$B = A_0^{-1} \circ P.$$

The range of B is X_0, and it can easily be checked that $A \circ B = P$. The operator $I_Y - P$ $(= I_Y - A \circ B)$ is a continuous projection of Y into span $\{f_1, f_2, \ldots, f_d\}$ and thus has finite rank.

Similarly, $PA = A$, so $BA = A_0^{-1}A$ is a continuous projection of X onto X_0 whose kernel in $\mathscr{N}(A)$. Hence $I_X - BA$ is a continuous projection of X onto $\mathscr{N}(A)$; it also has finite rank.

It is easy to check that (4.139) holds.

Suppose that (ii) holds. By (4.139), since $\mathscr{N}(A)$ is contained in the range of $I_X - BA$, it is finite-dimensional. Let $Y_0 = \mathscr{R}(I_Y - AB)$. From (4.139), it is clear that

$$Y_0 + A(X) = Y.$$

Therefore $\operatorname{codim} A(X) \leqslant \dim Y_0$. The fact that $A(X)$ is closed in Y is obvious from (4.139).

□

Theorem 4.8. *Let X and Y be reflexive Banach spaces. Then $A \in \mathscr{L}(X,Y)$ is Fredholm if and only if there exists $B \in \mathscr{L}(Y,X)$ such that both $I_Y - AB$ and $I_X - BA$ are compact operators.*

Proof. The "only if" part follows directly from Proposition 4.4 because every linear operator with finite rank is compact.

Now consider the "if" part. Define $K_X = I_X - BA$ and $K_Y = I_Y - AB$. Then K_X and K_Y are compact operators on X and Y respectively. We have

$$BA = I_X - K_X.$$

On $\mathscr{N}(I_X - K_X)$,

$$x = K_X x, \quad \text{for } x \in \mathscr{N}(I_X - K_X),$$

i.e., the identity operator is a compact operator on $\mathscr{N}(I_X - K_X)$. This is possible when and only when $\mathscr{N}(I_X - K_X)$ is finite-dimensional, implying $\dim \mathscr{N}(BA) < \infty$. Since

$$\mathscr{N}(A) \subset \mathscr{N}(BA),$$

so $\mathscr{N}(A)$ has finite dimension.

On the other hand, K_Y being compact implies that K_Y^* is compact [191, p. 282], so

$$B^*A^* = (AB)^* = (I_Y - K_Y)^* = I_{Y^*} - K_Y^*,$$

with K_Y^* being compact. We again have $\dim \mathscr{N}[(AB)^*] < \infty$, giving

$$\begin{aligned} &\mathscr{N}(A^*) \subset \mathscr{N}(B^*A^*) = \mathscr{N}[(AB)^*], \\ &\dim \mathscr{N}(A^*) = \dim[\mathscr{R}(A)]^{\perp} \leqslant \dim \mathscr{N}[(AB)^*] < \infty. \end{aligned}$$

So $\mathscr{R}(A)$ has finite codimension. Now

$$Y \supset \mathscr{R}(A) \supset \mathscr{R}(AB) = [\mathscr{R}(I_Y - K_Y)].$$

But $\mathscr{R}(I_Y - K_Y)$ is closed in Y (see Exercise 4.2), whose algebraic complement in Y is finite dimensional. Therefore $\mathscr{R}(A)$ is also closed in Y. Hence A is Fredholm. □

Exercise 4.2. Let X be a Banach space, and let $K \in \mathscr{L}(X)$ be compact. Show that $\mathscr{R}(I-K)$ is closed in X. □

Exercise 4.3.

(i) Let X_j and Y_j be Banach spaces and let $A_j:\ X_j \to Y_j$ be bounded linear with closed range, for $j = 1,2$. Define

$$A_1 \oplus A_2:\ X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2$$

by

$$(A_1 \oplus A_2)(x_1 \oplus x_2) = A_1x_1 \oplus A_2x_2.$$

Let A_1 and A_2 be Fredholm. Show that $A_1 \oplus A_2$ is also Fredholm, and

$$\operatorname{Ind}(A_1 \oplus A_2) = \operatorname{Ind}A_1 + \operatorname{Ind}A_2.$$

(ii) Let

$$X \xrightarrow{A} Y \xrightarrow{B} Z,$$

where X, Y and Z are Banach spaces and A, B are Fredholm. Then $B \circ A$ is also Fredholm, and

$$\operatorname{Ind}(B \circ A) = \operatorname{Ind}A + \operatorname{Ind}B.$$ □

The following theorem and its corollaries contain further properties of the index of Fredholm operators. Their proofs are not difficult, and are based upon the above discussion. See also [179, Chap. II, § 2.1], for example.

Theorem 4.9. *Let X and Y be Banach spaces. If $A \in \mathscr{L}(X,Y)$ is Fredholm and $K \in \mathscr{L}(X,Y)$ is compact, then $A+K$ is Fredholm. Furthermore,* $\operatorname{Ind}(A+K) = \operatorname{Ind}A$. □

Corollary 4.2. *Let X be a Banach space and let $K \in \mathscr{L}(X)$ be compact. Then*

$$\operatorname{Ind}(I_X + K) = 0.$$ □

Corollary 4.3. *Let X and Y be Banach spaces. The following two properties of an operator $A \in \mathscr{L}(X,Y)$ are equivalent.*

(i) *A is Fredholm and* $\operatorname{Ind}A = 0$*;*
(ii) *There is a compact operator $K \in \mathscr{L}(X,Y)$ such that $A+K$ is invertible.* □

Now we are able to present the following theorems, which will be important in the study of boundary element methods.

Let X and Y be Banach spaces and let $A\colon\ X \to Y$ be Fredholm. Assume that

$$m = \dim \mathscr{N}(A) = \operatorname{codim} \mathscr{R}(A) \in \mathbb{N}, \tag{4.140}$$

so $\operatorname{Ind} A = 0$. We want to augment A to become an operator $\widetilde{A}\colon\ \mathbb{R}^m \oplus X \to \mathbb{R}^m \oplus Y$ that is *invertible*. The augmented operator $\widetilde{A}$ then takes the form

$$\widetilde{A} = \begin{bmatrix} R & T \\ S & A \end{bmatrix}\colon\ \mathbb{R}^m \oplus X \to \mathbb{R}^m \oplus Y, \tag{4.141}$$

where

$$R\colon \mathbb{R}^m \to \mathbb{R}^m,\ \ S\colon \mathbb{R}^m \to Y,\ \ T\colon\ X \to \mathbb{R}^m. \tag{4.142}$$

Obviously, R is an $m \times m$ constant matrix. From (4.142), we also see that S and T are both operators of finite rank, admitting representations

$$S = \sum_{j=1}^{m} \sigma_j \otimes e_j, \quad T = \sum_{j=1}^{m} e_j \otimes \tau_j^*, \tag{4.143}$$

where

$$\{\sigma_j \mid 1 \leqslant j \leqslant m\} \subset Y,\ \ \{\tau_j^* \mid 1 \leqslant j \leqslant m\} \subset X^*,$$
$$e_j = (0,0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^m,\ \text{the } j\text{th unit vector},\ \ 1 \leqslant j \leqslant m,$$

and

$$\left.\begin{aligned} S\vec{a} &= \sum_{j=1}^{m} a_j \sigma_j, && \text{for } \vec{a} = (a_1,\ldots,a_m) \in \mathbb{R}^m, \\ T\vec{f} &= \vec{\alpha} = (\alpha_1,\ldots,\alpha_m), && \text{with } \alpha_j = (f_j,\tau_j^*)_{X\times X^*},\ \ 1 \leqslant j \leqslant m. \end{aligned}\right\} \tag{4.144}$$

We state and prove the following.

Theorem 4.10 ([40]). *Let X and Y be Banach spaces, and let $A \in \mathscr{L}(X,Y)$ be a Fredholm operator with index zero, with m given by (4.140). Let $\{k_j \mid 1 \leqslant j \leqslant m\}$ and $\{k_j^* \mid 1 \leqslant j \leqslant m\}$ be, respectively, the bases of $\mathscr{N}(A)$ and $\mathscr{N}(A^*)$. Then an augmented operator $\widetilde{A}$ as given by (4.141), with S and T represented by (4.143), is invertible if and only if*

$$\det[\langle k_i,\tau_j^*\rangle]_{1\leqslant i,j\leqslant m} \neq 0, \quad \det[\langle \sigma_i,k_j^*\rangle]_{1\leqslant i,j\leqslant m} \neq 0. \tag{4.145}$$

Proof. First note that

$$\widetilde{A} = \begin{bmatrix} R & T \\ S & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \equiv \widetilde{\mathscr{A}_1} + \widetilde{\mathscr{A}_2},$$

and that $\widetilde{A}_1$ is of finite rank, and thus compact. We see by Theorem 4.9 that $\widetilde{A}$ is Fredholm, with index

$$\begin{aligned}
\operatorname{Ind}\widetilde{A} &= \operatorname{Ind}\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \\
&= \dim\left\{\mathscr{N}\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right\} - \dim\left\{\mathbb{R}^m \oplus Y \Big/ \mathscr{R}\left(\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right)\right\} \\
&= m + \dim \mathscr{N}(A) - m + \dim[Y/\mathscr{R}(A)] \\
&= \dim \mathscr{N}(A) - \dim[Y/\mathscr{R}(A)] \\
&= \operatorname{Ind} A = 0.
\end{aligned}$$

By Theorem 4.9, $\widetilde{A}$ is invertible if and only if $\mathscr{N}(\widetilde{A}) = \{0\}$. Let $(\vec{a}, f) \in \mathbb{R}^m \oplus X$ be such that $\widetilde{A}(\begin{bmatrix} \vec{a} \\ f \end{bmatrix}) = 0$. Thus

$$\left.\begin{aligned} R\vec{a} + Tf &= 0 \in \mathbb{R}^m, \\ S\vec{a} + Af &= 0 \ \in Y. \end{aligned}\right\} \tag{4.146}$$

We get

$$\begin{aligned}
0 &= \langle S\vec{a} + Af, k_j^* \rangle_{Y \times Y^*} \\
&= \sum_{i=1}^m a_i \langle \sigma_i, k_j^* \rangle + \langle f, A^* k_j^* \rangle \\
&= \sum_{i=1}^m a_i \langle \sigma_i, k_j^* \rangle \quad \text{for } 1 \leqslant j \leqslant m, \text{ by (4.144)}.
\end{aligned} \tag{4.147}$$

If the second relation in (4.145) is satisfied then $\vec{a} = (a_1, \ldots, a_m) = 0$. Thus (4.146) gives

$$Tf = 0 \text{ in } \mathbb{R}^m, \ \ Af = 0 \text{ in } Y. \tag{4.148}$$

So $f \in \mathscr{N}(A)$, implying

$$f = \sum_{i=1}^m c_i k_i \in X, \quad c_i \in \mathbb{R}. \tag{4.149}$$

Substituting (4.149) into the first equation of (4.148), we get

$$0 = Tf = \sum_{i=1}^m c_i \sum_{j=1}^m \langle k_i, \tau_j^* \rangle e_j.$$

Therefore

$$\sum_{i=1}^m c_i \langle k_i, \tau_j^* \rangle = 0, \quad \text{for } j = 1, 2, \ldots, m. \tag{4.150}$$

By the first condition in (4.145), we get $c_i = 0$ for $1 \leqslant i \leqslant m$. Therefore $f = 0$ in X. Therefore, if (4.145) holds, $\mathscr{N}(A) = \{0\}$.
To show that (4.145) is also necessary for $\mathscr{N}(A) = \{0\}$, we set, separately, $\vec{a} = 0$ and $f = 0$ in (4.146) and conclude that (4.150) and (4.147) can admit only trivial solutions $c_i = 0$ and $a_i = 0$ for $1 \leqslant i \leqslant m$. Hence (4.145) must hold. □

We now show that the augmented system will always provide a solution to the original problem, if $\widetilde{A}$ is invertible. The original equation is

$$Au = g, \tag{4.151}$$

where the m compatibility conditions

$$\langle g, k_j^* \rangle_{Y \times Y^*} = 0, \qquad 1 \leqslant j \leqslant m, \tag{4.152}$$

are satisfied. Let us solve an augmented equation

$$\widetilde{A} \begin{bmatrix} \vec{a} \\ f \end{bmatrix} = \begin{bmatrix} \vec{a} \\ g \end{bmatrix} \in \mathbb{R}^m \oplus Y \tag{4.153}$$

for any given $\vec{\alpha} \in \mathbb{R}^m$. If $\widetilde{A}$ is invertible, we get a unique solution $(\vec{a}, f) \in \mathbb{R}^m \oplus X$, satisfying

$$\left.\begin{aligned} R\vec{a} + Tf &= \vec{\alpha}, \\ S\vec{a} + Af &= g. \end{aligned}\right\} \tag{4.154}$$

From the second equation above, using (4.152), we get

$$\begin{aligned} &\langle S\vec{a}, k_j^* \rangle + \langle Af, k_j^* \rangle = \langle g, k_j^* \rangle, \quad 1 \leqslant j \leqslant m, \\ &\sum_{i=1}^{m} a_i \langle \sigma_i, k_j^* \rangle + 0 = 0, \quad 1 \leqslant j \leqslant m, \\ &\qquad\qquad\qquad \text{by (4.152), (4.144) and } A^* k_j^* = 0. \end{aligned}$$

By (4.145), we have

$$\vec{a} = (a_1, \ldots, a_m) = 0.$$

Hence the second equation in (4.154) gives

$$Af = g.$$

Thus, the second component f in the solution of the augmented system (4.153) can always be used to serve as the solution u in (4.151). The extra m equations will always be consistent with the original equation (4.151).

Remark 4.3. The presence of R (cf. (4.141)) in $\widetilde{A}$ is immaterial in the statement of Theorem 4.10; therefore it is convenient to just set $R = 0$. □

Theorem 4.10 will be applied to various types of BVP in the next section.

The following two theorems are fundamental in the analysis of boundary integral equations, where the compact differentiable manifold M is just $\partial\Omega$, the boundary of some bounded open set Ω in $\mathbb{R}^N$.

Theorem 4.11. *Let M be a compact differentiable manifold and let $p(x,\xi) \in S^m$ be elliptic of order m. Then*

$$P\colon\ H^s(M) \to H^{s-m}(M), \tag{4.155}$$

is Fredholm for any $s \in \mathbb{R}$.

Proof. By applying Theorem 4.1, and a partition of unity on M, we find that the mapping P in (4.155) is linear continuous. Since P is elliptic, by Theorem 4.4, we can let $q(x,\xi) = \sigma(Q) \in S^{-m}$ be such that

$$Q\colon\ H^{s-m}(M) \to H^s(M)$$

and $I - QP$ is regularizing, i.e., Q is a parametrix of P. So

$$I - QP\colon {}^s(M) \to C^\infty(M) \subset H^s(M). \tag{4.156}$$

But $C^\infty(M) = \cap_{\tau \geqslant s} H^\tau(M)$, and the injection $H^\tau(M) \to H^s(M)$ is compact if $\tau > s$, by Theorem 2.5. So $I - QP \in \mathscr{L}(H^s(M))$ is compact. By Theorem 4.8, P is Fredholm. □

Corollary 4.4. *Under the assumptions of Theorem 4.11, we have*

$$\mathscr{N}(P) \subset C^\infty(M).$$

Proof. Let $u \in \mathscr{N}(P)$. By (4.156), we have

$$(I - QP)u = Ru, \ \text{ where } R\colon\ H^s(M) \to C^\infty(M) \ \text{ is infinitely smoothing.}$$

Since $Pu = 0$, we get

$$u = Ru \in C^\infty(M). \tag{4.157}$$

□

4.7 Applications to BIE of elliptic BVP with Neumann boundary conditions

An elliptic partial differential operator satisfies the Gårding inequality (4.59), which leads to a Fredholm opeorator when the boundary conditions are also properly posed. In our subsequent applications, this Fredholm operator has zero index. Therefore m orthogonality (or compatibility) conditions must be satisfied by the data, and the solutions have m degrees of freedom—they are nonunique. In order to fix these m degrees of freedom, m accessory linear conditions are usually prescribed. This leads to an augmented system like (4.141). In this section, we apply the theory of § 4.6, particularly Theorem 4.10, to elliptic BVP whose boundary conditions involve higher-order derivatives, i.e., Neumann-type boundary conditions. We will use Theorem 4.10 to determine when the augmented linear system is uniquely solvable.

The main advantages of the invertibility of a Fredholm operator by augmentation are the following.

(i) *It produces a unique solution* satisfying the given PDE, boundary data and the m accessory conditions. Further, our characterization of the augmented operator indicates what types of accessory conditions may be used for solving a boundary value problem uniquely.

(ii) *It bypasses the verification of the compatibility conditions*. When the compatibility conditions are satisfied, the augmented system will automatically provide a solution to the original system. This is particularly useful in the numerical solutions for such BVP, because numerically the discretized compatibility conditions usually do not correspond to the compatibility conditions for the matrix equation obtained by discretizing the PDE and boundary conditions.

We mention two applications: one to the Neumann boundary value problem of the Laplacian, and the other to the traction value problem of linear elastostatics. The plate equation will also be discussed in Example 4.7.

Throughout (except in Example 4.7) we let Ω be a simply connected bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We want to solve the BVP by boundary integral equations.

Example 4.4 (The Neumann boundary value problem of the Laplace equation). Consider the following problem:

$$\left.\begin{array}{l} \Delta u(x) = 0 \quad \text{on} \quad \Omega, \\ \dfrac{\partial u(x)}{\partial n} = g(x) \in H^r(\partial\Omega), r \in \mathbb{R}, \\ (n \equiv \text{unit exterior normal on } \partial\Omega). \end{array}\right\} \tag{4.158}$$

From elliptic regularity (cf. § 6.7), we know there exists a solution $u \in H^{r+3/2}(\Omega)$ satisfying (4.158), and, by (6.157),

$$\inf \|u\|_{H^{r+3/2}(\Omega)} \leqslant C\|g\|_{H^r(\partial\Omega)}$$

for some $C > 0$ independent of g, with the infimum taken over all such u satisfying (4.158), if and only if the compatibility condition

$$\int_{\partial\Omega} g(x)\, d\sigma = 0 \tag{4.159}$$

is satisfied. We note that in the above, the integral should be interpreted as a duality pairing $\langle 1, g\rangle$ in $H^{-r}(\partial\Omega) \times H^r(\partial\Omega)$ if $r < 0$. From now on throughout this section, we will assume such an interpretation for all similar situations without further mention. □

Let

$$E(x,\xi) = \frac{1}{4\pi}\frac{1}{|x-\xi|}, \quad x, \xi \in \mathbb{R}^3, \tag{4.160}$$

be the fundamental solution of the Laplacian satisfying

$$\Delta_\xi E(x,\xi) = -\delta(x-\xi), \quad \Delta_\xi \equiv \sum_{i=1}^{3} \frac{\partial^2}{\partial \xi_i^2},$$

The simple-layer potential solution for (4.158) is based on the *ansatz* (see [72, 95] and Theorem 6.20)

$$u(x) = \int_{\partial\Omega} E(x,\xi) f(\xi)\, d\sigma_\xi, \quad x \in \overline{\Omega}. \tag{4.161}$$

Taking the normal derivation, from the jump property of the boundary layer potentials (see Theorem 6.3), we get

$$\frac{\partial u(x)}{\partial n} = \frac{1}{2} f(x) + \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_x} f(\xi)\, d\sigma_\xi, \quad x \in \partial\Omega, \tag{4.162}$$

where the integral on the RHS of (4.162) is interpreted as a weakly singular integral for $f \in C^\infty(\partial\Omega)$. Define boundary integral operators

$$(\mathscr{L}_1 f)(x) = \int_{\partial\Omega} E(x,\xi) f(\xi)\, d\sigma_\xi, \tag{4.163}$$

$$(\mathscr{L}_2 f)(x) = \frac{1}{2} f(x) + \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_x} f(\xi)\, d\sigma_\xi. \tag{4.164}$$

According to the theory in §4.4 and §6.8, it is known that the following hold:

(1) $\mathscr{L}_1$ is a strongly elliptic pseudodifferential operator of order -1, mapping continuously

$$\mathscr{L}_1 : \ H^r(\partial\Omega) \to H^{r+1}(\partial\Omega). \tag{4.165}$$

$\mathscr{L}_1$ is positive definite and invertible, satisfying

$$\langle \mathscr{L}_1 f, f\rangle_{H^0(\partial\Omega)} = \int_{\partial\Omega} |\nabla V(f)|^2\, d\sigma \geqslant \gamma \|\mathscr{L}_1 f\|^2_{H^{1/2}(\partial\Omega)},$$
$$\gamma > 0, \ \forall f \in H^0(\partial\Omega), \tag{4.166}$$

where

$$V(f)(x) \equiv \int_{\partial\Omega} E(x-\xi) f(\xi)\, d\sigma_\xi, \quad x \in \mathbb{R}^3,$$

$$C_r^{-1} \|f\|_{H^r(\partial\Omega)} \leqslant \|\mathscr{L}_1 f\|_{H^{r+1}(\partial\Omega)} \tag{4.167}$$

$$\leqslant C_r \|f\|_{H^r(\partial\Omega)}, \tag{4.168}$$

$$C_r > 0 \text{ independent of } f \in H^r(\partial\Omega).$$

(2) $\mathscr{L}_2$ is a Fredholm operator with zero index on $H^r(\partial\Omega), r \in \mathbb{R}$. It is a pseudodifferential operator of order 0, with principal symbol equal to $\frac{1}{2}$. The integral operator on the RHS of (4.164) is a compact operator on $H^r(\partial\Omega)$. Further,

$$\mathscr{N}(\mathscr{L}_2) = \text{span}\{k \in C^\infty(\partial\Omega) \mid \mathscr{L}_1 k = 1 \text{ on } \partial\Omega\} \tag{4.169}$$

$$\text{Coker } \mathscr{L}_2 = \mathscr{N}(\mathscr{L}_2^*) = \text{span}\{1 \text{ on } \partial\Omega\}, \tag{4.170}$$

$$(\mathscr{L}_2^* f)(x) = \frac{1}{2} f(x) + \int_{\partial\Omega} \frac{\partial}{\partial n_\xi} E(x,\xi) f(\xi)\, d\sigma_\xi, \quad x \in \partial\Omega, \tag{4.171}$$

$$\mathscr{L}_2^* : H^{-r}(\partial\Omega) \to H^{-r}(\partial\Omega),\ r \in \mathbb{R},$$

is also Fredholm with zero index.

To solve the BVP (4.158) subject to (4.159), we use the simple-layer solution (4.161). In order to fix the solution of (4.158) and (4.159) uniquely, we consider appending an extra condition such as

$$\int_{\partial\Omega} x_j^n u(x)\, d\sigma_x = \alpha \in \mathbb{R}, \ \ j = 1, 2 \text{ or } 3 \text{ (exclusively)} \tag{4.172}$$

$$n \text{ is a given nonnegative integer,}$$

or

$$u(x^{(0)}) = \alpha \in \mathbb{R}, \quad x^{(0)} \in \partial\Omega \text{ given.} \tag{4.173}$$

Condition (4.172) means that the nth moment of the solution u in the x_j-direction on $\partial\Omega$ must be equal to a specified number α. (Note that only one pair (j,n) is allowed here.) Condition (4.173) means that the solution takes a specified value α at a given boundary point $x^{(0)}$.

Let us amend the boundary integral equation

$$\frac{\partial u(x)}{\partial n} = g(x) = \frac{1}{2} f(x) + \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_x} f(\xi)\, d\xi \tag{4.174}$$

$$= (\mathscr{L}_2 f)(x), \quad x \in \partial\Omega,$$

by (4.172) or (4.173). We treat these two cases separately.

(a) *Extra condition (4.172):* $\int_{\partial\Omega} x_j^n u(x)\, d\sigma_x = \alpha$.

We first note that, corresponding to the simple-layer representation (4.161), (4.173) gives

$$\begin{aligned}\int_{\partial\Omega} x_j^n \int_{\partial\Omega} E(x,\xi) f(\xi)\, d\sigma_\xi\, d\sigma_x &= \int_{\partial\Omega} \left[\int_{\partial\Omega} x_j^n E(x,\xi)\, d\sigma_x\right] f(\xi)\, d\sigma_\xi \\ &\equiv \int_{\partial\Omega} \tau^*(\xi) f(\xi)\, d\sigma\ (= \langle f, \tau^* \rangle).\end{aligned} \tag{4.175}$$

From the smoothing property of the simple-layer potential (4.74), we see that $\tau^* \in C^\infty(\partial\Omega)$. Let $k \in \mathscr{N}(\mathscr{L}_1)$ be its basis element satisfying

$$\int_{\partial\Omega} E(x,\xi) k(\xi)\, d\sigma_\xi = 1 \quad \forall x \in \partial\Omega,\ \text{cf. (4.169)}. \tag{4.176}$$

We let the augmented system be

$$\begin{bmatrix} 0 & \tau^* \\ 1 & \mathscr{L}_2 \end{bmatrix} \begin{bmatrix} a \\ f \end{bmatrix} = \begin{bmatrix} \alpha \\ g \end{bmatrix} \quad \text{in} \quad \mathbb{R} \oplus H^r(\partial\Omega). \tag{4.177}$$

i.e.,

$$\left.\begin{aligned} &\langle f, \tau^* \rangle = \int_{\partial\Omega} f(\xi) \tau^*(\xi)\, d\sigma = \alpha, \\ &a + (\mathscr{L}_2 f)(x) = g(x). \end{aligned}\right\}$$

According to Theorem 4.10, (4.177) is invertible if and only if

$$\langle k, \tau^* \rangle \neq 0, \quad \langle 1, k^* \rangle \neq 0. \tag{4.178}$$

The second condition above is always satisfied, because by (4.170), $k^* \equiv 1$ and

$$\begin{aligned}\langle 1, k^* \rangle_{H^r(\partial\Omega)\times H^{-r}(\partial\Omega)} &= \langle 1, k^* \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} 1 \cdot 1\, d\sigma \\ &= \text{area of } \partial\Omega \\ &> 0.\end{aligned}$$

The first condition in (4.145) requires that

$$\begin{aligned} 0 \neq \langle k, \tau^* \rangle_{H^r(\partial\Omega)\times H^{-r}(\partial\Omega)} &= \langle k, \tau^* \rangle_{L^2(\partial\Omega)} \\ &= \int_{\partial\Omega} k(\xi) \left[\int_{\partial\Omega} x_j^n E(x,\xi)\, d\sigma_x\right] d\sigma_\xi \\ &= \int_{\partial\Omega} x_j^n \left[\int_{\partial\Omega} E(x,\xi) k(\xi)\, d\sigma_\xi\right] d\sigma_x \\ &= \int_{\partial\Omega} x_j^n\, d\sigma_x \qquad (\text{cf.}(4.176)).\end{aligned} \tag{4.179}$$

It is strictly positive if $n = 0, 2, 4, \ldots$, but can vanish if n is odd (just let Ω be the unit ball and use the spherical coordinate representation for x_1, x_2 or x_3). Therefore, the augmented system (4.177) is invertible if and only if

$$\int_{\partial\Omega} x_j^n \, d\sigma_x \neq 0 \quad \text{for given } (j, n).$$

It is obvious that the invertibility of the augmented system

$$\widetilde{A} = \begin{bmatrix} R & \tau^* \\ S & \mathscr{L}_2 \end{bmatrix} \tag{4.180}$$

in the present case of (4.177) will remain unchanged even if R and S in (4.180) are chosen differently from (4.175), because every harmonic function (with sufficiently regular boundary data on $\partial\Omega$) on $\Omega \subset \mathbb{R}^3$ is representable by a simple-layer potential, as a consequence of the invertibility of the operator $\mathscr{L}_1$.

Therefore, we conclude the following.

Corollary 4.5. *The Neumann BVP*

$$\left.\begin{aligned} &\Delta u(x) = 0 \quad \textit{on } \Omega \\ &\frac{\partial u(x)}{\partial n} = g(x) \in H^r(\partial\Omega), \quad r \in \mathbb{R}, \end{aligned}\right\} \tag{4.181}$$

with the compatibility condition $\int_{\partial\Omega} g \, d\sigma = 0$ *has a* unique *solution* $u \in H^{r+3/2}(\Omega)$ *also satisfying the accessory condition*

$$\int_{\partial\Omega} x_j^n u(x) \, d\sigma_x = \alpha, \quad j = 1, 2 \textit{ or } 3, \tag{4.182}$$

with $n =$ *a nonnegative integer and* $\alpha \in \mathbb{R}$ *both given, if and only if*

$$\int_{\partial\Omega} x_j^n \, d\sigma_x \neq 0. \tag{4.183}$$

□

Furthermore, the augmented boundary integral equation system (4.177) based on the simple-layer potential (4.161) corresponding to (4.181) and (4.182) has a unique solution $(a, f) = (0, \widetilde{f}) \in \mathbb{R} \times H^r(\partial\Omega)$ satisfying

$$\langle \widetilde{f}, \tau^* \rangle = \alpha = \int_{\partial\Omega} x_j^n u(x) \, d\sigma_x,$$

where τ^* is defined by (4.175).

(b) *Extra condition (4.173)*: $u(x^{(0)}) = \alpha, x^{(0)} \in \partial\Omega$

First, note that u is defined pointwise at $x^{(0)} \in \partial\Omega$, provided that $g \in H^r(\partial\Omega)$ with $r > \frac{1}{2}$ and, thus, $u \in H^{3/2+r}(\Omega)$. Also, its trace satisfies

$$u|_{\partial\Omega} \in H^{3/2+r-1/2}(\partial\Omega) = H^{1+r}(\partial\Omega) \subset H^{1+\varepsilon}(\partial\Omega) \subset C^0(\partial\Omega),$$
$$\text{for } r > 1/2, \text{ and for } \varepsilon:\ 0 < \varepsilon \leqslant \frac{1}{2},$$

since $\partial\Omega$ is two-dimensional (by the Sobolev imbedding theorem; cf. Theorem 2.2). Corresponding to the simple-layer representation (4.161), (4.173) implies that

$$\int_{\partial\Omega} E(x^{(0)},\xi) f(\xi)\, d\sigma_\xi = \alpha. \tag{4.184}$$

So we let

$$\int_{\partial\Omega} E(x^{(0)},\xi) f(\xi)\, d\sigma_\xi \equiv \int_{\partial\Omega} f(\xi)\tau^*(\xi)\, d\sigma = \langle f, \tau^* \rangle_{L^2}. \tag{4.185}$$

Note that the distribution $\tau^* \in L^{2-\varepsilon_1}(\partial\Omega) \cap H^{-\varepsilon_2}(\partial\Omega)$ for all $\varepsilon_1,\ \varepsilon_2 > 0$.
Again, we let the augmented system be (4.177), where $\tau^* = E(x^{(0)},\cdot)$ is used in (4.177) instead. The system (4.177) is invertible if and only if

$$\langle k, \tau^* \rangle_{H^r(\partial\Omega)\times H^{-r}(\partial\Omega)} = \langle k, \tau^* \rangle_{L^2(\partial\Omega)} \neq 0,$$

i.e.,

$$\int_{\partial\Omega} E(x^{(0)},\xi) k(\xi)\, d\sigma \neq 0. \tag{4.186}$$

But we know that k satisfies (4.176). So $1 =$ LHS of (4.186) $\neq 0$ is always satisfied. We conclude the following.

Corollary 4.6. *The Neumann BVP*

$$\left.\begin{aligned} &\Delta u(x) = 0 \\ &\frac{\partial u(x)}{\partial n} = g(x) \in H^r(\partial\Omega),\ r > \tfrac{1}{2}, \end{aligned}\right\} \tag{4.187}$$

with the compatibility condition $\int_{\partial\Omega} g\, d\sigma = 0$ *has a* unique *solution* $u \in H^{r+3/2}(\Omega)$ *also satisfying the accessory condition*

$$u(x^{(0)}) = \alpha, \quad \textit{with } x^{(0)} \in \partial\Omega \textit{ and } \alpha \in \mathbb{R} \textit{ given.} \tag{4.188}$$

□

Furthermore, the augmented boundary integral equation system (4.177) based on the simple-layer potential (4.161) corresponding to (4.187) and (4.188) has a unique solution $(a, f) = (0, \widetilde{f}) \in \mathbb{R} \times H^r(\partial\Omega)$ satisfying

$$\langle \widetilde{f}, \tau^* \rangle = \alpha = u(x^{(0)}),$$

where τ^* is defined by (4.185).

Example 4.5. (The traction boundary value problem in three-dimensional linear elastostatics (Chapter 9)). Consider

$$\left\{ \begin{array}{l} \mu\Delta\vec{u}(x) + (\lambda+\mu)\nabla[\nabla\cdot\vec{u}(x)] = 0 \text{ on } \Omega, \\ \vec{\tau}(\vec{u})(x) = \vec{g}(x) \in [H^r(\partial\Omega)]^3, \qquad r \in \mathbb{R}, \end{array} \right\} \tag{4.189}$$

where $\vec{u}(x) = (u_1(x), u_2(x), u_3(x))$ represents the displacement field of a homogeneous isotropic Hookean solid at $x \in \Omega$; $\lambda,\ \mu > 0$ are the Lamé constants; $\vec{\tau}(\vec{u})(x)$ represents the traction vector given by

$$\vec{\tau}(\vec{u})(x) = \lambda[\nabla\cdot\vec{u}(x)]n(x) + 2\mu\frac{\partial\vec{u}(x)}{\partial n} + \mu\vec{n}(x)\times[\nabla\times\vec{u}(x)] \tag{4.190}$$

at $x \in \partial\Omega$. The BVP (4.189) behaves very similarly to the Neumann BVP in Example 4.4, except that now it is a *system* and more compatibility conditions must be satisfied: (4.189) has a solution $\vec{u}$ if and only if

$$\int_{\partial\Omega} \vec{g}(x)\cdot\vec{m}_j(x)\,d\sigma = 0, \quad j = 1, 2, \ldots, 6, \tag{4.191}$$

are satisfied, where $\vec{m}_j(x)$ represents the j-th column vectors of the rigid-body motion matrix

$$M(x) = \begin{bmatrix} 1 & 0 & 0 & -x_2 & 0 & x_3 \\ 0 & 1 & 0 & x_1 & -x_3 & 0 \\ 0 & 0 & 1 & 0 & x_2 & -x_1 \end{bmatrix}. \tag{4.192}$$

(The first three column vectors represent translations along the three coordinate directions, while the last three represent angular rotations.) □

Again, let us solve (4.189) by the simple-layer potential method. Write

$$\vec{u}(x) = \int_{\partial\Omega} E(x,\xi)\vec{f}(\xi)\,d\sigma_\xi, \quad x \in \overline{\Omega}, \tag{4.193}$$

where the fundamental 3×3 matrix solution

$$E(x,\xi) = \frac{\lambda+3\mu}{8\pi(\lambda+2\mu)}\left[\frac{1}{|x-\xi|}I_3 + \frac{\lambda+\mu}{\lambda+3\mu}\frac{1}{|x-\xi|^3}(x-\xi)(x-\xi)^{\mathrm{Tr}}\right] \tag{4.194}$$

satisfies

$$\mu\Delta_\xi E(x,\xi) + (\lambda+\mu)\nabla_\xi[\nabla_\xi\cdot E(x,\xi)] = -\delta(x-\xi)I_3,$$

with I_3 being the 3×3 identity matrix. Taking the traction vector of the simple-layer displacement field (4.193), similarly to (4.162), we get

$$\begin{aligned} \vec{\tau}(\vec{u})(x) &= \frac{1}{2}\vec{f}(x) + \int_{\partial\Omega} \vec{\tau}_x(E)(x,\xi)\,\vec{f}(\xi)\,d\sigma_\xi \\ &= \vec{g}(x), \quad x \in \partial\Omega. \end{aligned} \tag{4.195}$$

where $\overrightarrow{\tau}_x(E)(x,\xi)$ is given by (9.99). Again, we define boundary integral operators

$$\mathscr{L}_1(\overrightarrow{f})(x) = \int_{\partial\Omega} E(x,\xi)\overrightarrow{f}(\xi)\,d\sigma_\xi, \qquad x \in \partial\Omega, \tag{4.196}$$

$$\mathscr{L}_2(\overrightarrow{f})(x) = \frac{1}{2}\overrightarrow{f}(x) + \int_{\partial\Omega} \overrightarrow{\tau}_x(E)(x,\xi)\overrightarrow{f}(\xi)\,d\sigma_\xi, \quad x \in \partial\Omega. \tag{4.197}$$

The singularity of the kernel $E(x,\xi)$ is essentially similar to that of the kernel (4.160) for the Laplacian. Using the same ideas in Example 4.4, it is known (see [98]) the following.

(1) $\mathscr{L}_1$ is a strongly pseudodifferential operator of order -1 mapping continuously from

$$\mathscr{L}_1:\ [H^r(\partial\Omega)]^3 \longrightarrow \left[H^{r+1}(\partial\Omega)\right]^3, \quad r \in \mathbb{R}. \tag{4.198}$$

$\mathscr{L}_1$ is positive definite and invertible, satisfying

$$\begin{aligned}\langle \mathscr{L}_1\overrightarrow{f},\overrightarrow{f}\rangle_{[H^0(\partial\Omega)]^3} &= \int_{\mathbb{R}^3}\left\{\lambda[\nabla\cdot\overrightarrow{V}(\overrightarrow{f})]^2 + \frac{\mu}{2}\sum_{j,k=1}^{3}\left[\frac{\partial}{\partial x_j}V_k(\overrightarrow{f}) + \frac{\partial}{\partial x_k}V_j(\overrightarrow{f})\right]^2\right\}dx \\ &\geqslant \gamma\|\mathscr{L}_1\overrightarrow{f}\|^2_{[H^{1/2}(\partial\Omega)]^3}, \quad \gamma > 0,\ \forall\,\overrightarrow{f} \in [H^0(\partial\Omega)]^3,\end{aligned} \tag{4.199}$$

where

$$\overrightarrow{V}(\overrightarrow{f})(x) \equiv \int_{\partial\Omega} E(x,\xi)\overrightarrow{f}(\xi)\,d\sigma_\xi, \quad x \in \mathbb{R}^3, \tag{4.200}$$

and

$$\begin{aligned}c_r^{-1}\|\overrightarrow{f}\|_{[H^r(\partial\Omega)]^3} &\leqslant \|\mathscr{L}_1\overrightarrow{f}\|_{[H^{r+1}(\partial\Omega)]^3} \\ &\leqslant c_r\|\overrightarrow{f}\|_{[H^r(\partial\Omega)]^3},\ c_r > 0 \text{ independent of } \overrightarrow{f}.\end{aligned} \tag{4.201}$$

(2) $\mathscr{L}_2$ is Fredholm with zero index on $[H^r(\partial\Omega)]^3, r \in \mathbb{R}$. $\mathscr{L}_2$ is a pseudodifferential operator whose principal matrix symbol at each point $x \in \partial\Omega$ is similar to a symmetric symbol matrix with a diagonal part $\frac{1}{2}I_3$, with some nonzero off-diagonal entries (cf. [98, p. 49]). The integral operator on the RHS of (4.164) represents a Cauchy principal value. The overall principal matrix symbol of $\mathscr{L}_2$ is strongly elliptic of order 0 at each $x \in \partial\Omega$, and therefore Fredholm with zero index on $[H^r(\partial\Omega)]^3$ by Corollary 4.2. Further,

$$\mathscr{N}(\mathscr{L}_2) = \operatorname{span}\left\{\overrightarrow{k}_j \in [C^\infty(\partial\Omega)]^3 \mid \mathscr{L}_1\overrightarrow{k}_j = \overrightarrow{m}_j \text{ on } \partial\Omega, 1 \leqslant j \leqslant 6\right\} \tag{4.202}$$

$$\operatorname{Coker}\mathscr{L}_2 = \mathscr{N}(\mathscr{L}_2^*) = \operatorname{span}\{\overrightarrow{m}_j \text{ on } \partial\Omega \mid 1 \leqslant j \leqslant 6\}, \tag{4.203}$$

$$(\mathscr{L}_2^* f)(x) = \frac{1}{2}\overrightarrow{f}(x) + \int_{\partial\Omega}\overrightarrow{\tau}_\xi(E)(x,\xi)\overrightarrow{f}(\xi)\,d\sigma_\xi, \quad x \in \partial\Omega.$$

$\mathscr{L}_2^*$ is also a pseudodifferential operator of order 0, with principal matrix symbol at each point $x \in \partial\Omega$ similar to a symmetric symbol matrix whose diagonal part is $\frac{1}{2}I_3$, and some of whose off-diagonal entries are nonzero, and with an overall principal matrix symbol strongly elliptic of order 0. Thus $\mathscr{L}_2^*$ is also a Fredholm operator with zero index on $[H^r(\partial\Omega)]^3$. In order to uniquely solve the solution of (4.189) satisfying (4.191), we need to *fix six additional constants*. We may consider, e.g.,

$$\int_{\partial\Omega} \vec{u}(x)\cdot\vec{m}_j(x)\,d\sigma = \alpha_j, \quad j = 1,2,\ldots,6, \tag{4.204}$$

or

$$\vec{u}(x^{(1)}) = \vec{\beta}^{(1)}, \; \vec{u}(x^{(2)}) = \vec{\beta}^{(2)} \tag{4.205}$$
$$\text{where } x^{(1)}, x^{(2)} \in \overline{\Omega}, \vec{\beta}^{(1)}, \vec{\beta}^{(2)} \in \mathbb{R}^3 \text{ are given,}$$

or six additional restrictive conditions on the simple layer density $\vec{f}$:

$$\int_{\partial\Omega} \vec{m}_j(x)\cdot\vec{f}(x)\,d\sigma = \alpha_j \in \mathbb{R}, \tag{4.206}$$
$$1 \leqslant j \leqslant 6, \; \alpha_j\text{s are given.}$$

Any of (4.204)–(4.206) would provide six extra conditions to possibly help obtain a unique solution $\vec{u}(x)$. Note that (4.205) is analogous to (4.173), but we now use two points $x^{(1)}$ and $x^{(2)}$ (providing six data) instead.

We use the simple-layer solution (4.193). We treat cases (4.204), (4.205) and (4.206) separately below.

(a) *Six extra conditions (4.204):* $\int_{\partial\Omega} \vec{u}(x)\cdot\vec{m}_j(x)\,d\sigma = \alpha_j, j = 1,2,\ldots,6.$

Note that, corresponding to (4.193), conditions (4.204) imply

$$\begin{aligned}\int_{\partial\Omega}\int_{\partial\Omega}[E(x,\xi)\vec{f}(\xi)]\cdot\vec{m}_j(x)\,d\sigma_\xi\,d\sigma_x = \alpha_j&\\ &= \int_{\partial\Omega}\vec{f}(x)\cdot\left[\int_{\partial\Omega}E(x,\xi)\vec{m}_j(\xi)\,d\sigma_\xi\right]d\sigma_x\\ &\equiv \int_{\partial\Omega}\vec{f}(x)\cdot\vec{\tau}_j^*(x)\,d\sigma_x\\ &= \langle\vec{f},\vec{\tau}_j^*\rangle_{[L^2(\partial\Omega)]^3}, \;\; \tau_j^*\in C^\infty(\partial\Omega).\end{aligned} \tag{4.207}$$

From (4.195), (4.197) and (4.207), we form an augmented system by

$$\widetilde{A}\begin{bmatrix}\vec{a}\\ \vec{f}\end{bmatrix} \equiv \begin{bmatrix}0 & \sum_{j=1}^{6} e_j\otimes\vec{\tau}_j^*\\ \sum_{j=1}^{6}\vec{m}_j\otimes e_j & \mathscr{L}_2\end{bmatrix}\begin{bmatrix}\vec{a}\\ \vec{f}\end{bmatrix} \tag{4.208}$$
$$= \begin{bmatrix}\vec{\alpha}\\ \vec{g}\end{bmatrix}, \;\text{ in } \mathbb{R}^6\oplus[H^r(\partial\Omega)]^3.$$

Then, $\widetilde{A}$ is invertible if and only if

$$\det[\langle \vec{k}_i, \vec{\tau}^*_j\rangle] \neq 0, \quad \det[\langle \vec{m}_i, k^*_j\rangle] \neq 0. \tag{4.209}$$

We have

$$\begin{aligned}\langle \vec{k}_i, \vec{\tau}^*_j\rangle &= \int_{\partial\Omega} \vec{k}_i(x)\cdot\int_{\partial\Omega} E(x,\xi)\vec{m}_j(\xi)\,d\sigma_\xi\,d\sigma_x\\ &= \int_{\partial\Omega} \vec{m}_j(x)\cdot\int_{\partial\Omega} E(x,\xi)\,\vec{k}_i(\xi)\,d\sigma_\xi\,d\sigma_x\\ &= \int_{\partial\Omega} \vec{m}_j(x)\cdot\vec{m}_i(x)\,d\sigma.\end{aligned} \tag{4.210}$$

Because $\vec{m}_i(x), i = 1,\ldots,6$, are linearly independent functions in $L^2(\partial\Omega)$, we see that the first condition in (4.145) is always satisfied. The second condition is also satisfied, because by (4.203), $\langle \vec{m}_i, \vec{k}^*_j\rangle = \langle \vec{m}_i, \vec{m}_j\rangle$, which form entries of an invertible matrix because $\vec{m}_i, 1 \leqslant i \leqslant 6$, are linearly independent in $[L^2(\Omega)]^3$.
Therefore we conclude the following.

Corollary 4.7. *The traction BVP*

$$\left.\begin{array}{ll}\mu\Delta\vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) = 0 & on\ \Omega\\ \vec{\tau}(\vec{u})(x) = \vec{g}(x) \in [H^r(\partial\Omega)]^3, & r\in\mathbb{R},\end{array}\right\} \tag{4.211}$$

with compatibility conditions $\int_{\partial\Omega}\vec{g}\cdot\vec{m}_j\,d\sigma = 0,\ 1\leqslant j\leqslant 6$, *has a unique solution* $\vec{u} \in [H^{r+3/2}(\Omega)]^3$ *if six accessory conditions*

$$\int_{\partial\Omega}\vec{u}(x)\cdot\vec{m}_j(x)\,d\sigma = \alpha_j, \tag{4.212}$$

$$j = 1,2,\ldots,6, \textit{ with } \alpha_j \in \mathbb{R} \textit{ given,}$$

are specified. □

Furthermore, the augmented boundary integral equation system (4.208) based on the simple-layer potential (4.193) corresponding to (4.211) and (4.212) has a unique solution $(\vec{a}, \vec{f}) = (0, \vec{\widetilde{f}}) \in \mathbb{R}^6 \oplus [H^r(\partial\Omega)]^3$ satisfying

$$\left\langle \vec{\widetilde{f}}, \sum_{j=1}^{6} e_j \otimes \tau^*_j \right\rangle = \vec{\alpha} = \left[\int_{\partial\Omega}\vec{u}(x)\cdot\vec{m}_j(x)\,d\sigma\right]_{j=1}^{6},$$

where $\tau^*_j,\ 1\leqslant j\leqslant 6$, are defined by (4.207).

(b) *Six extra conditions (4.205):* $\vec{u}(x^{(1)}) = \vec{\beta}^{(1)}$ and $\vec{u}(x^{(2)}) = \vec{\beta}^{(2)}$

As in Example 4.4, case (b), in order for $\vec{u}$ to be defined pointwise at $x^{(1)}, x^{(2)} \in \partial\Omega$, we require the traction data $\vec{g}$ in (4.188) to satisfy

$$\vec{g} \in [H^r(\partial\Omega)]^3, \quad r > \frac{1}{2}.$$

Then $\vec{u} \in [H^{r+3/2}(\Omega)]^3$, and by the trace theorem and the Sobolev imbedding theorem (Theorem 2.2),

$$\vec{u}\,|_{\partial\Omega} \in [H^{1+r}(\partial\Omega)]^3 \subset [H^{1+\varepsilon}(\partial\Omega)]^3 \subset [C^0(\partial\Omega)]^3,$$
$$\text{for} \quad r > \frac{1}{2}, \; 0 < \varepsilon \leqslant \frac{1}{2}.$$

Denote $\vec{\alpha} = (\vec{\beta}^{(1)}, \vec{\beta}^{(2)}) \in \mathbb{R}^6$. The conditions

$$\vec{u}(x^{(i)}) = \vec{\beta}^{(i)}, \quad i = 1, 2,$$

require that

$$\int_{\partial\Omega} E(x^{(i)}, \xi)\vec{f}(\xi)\, d\sigma_\xi = \vec{\beta}^{(i)},$$

so we define τ_j^* by

$$\langle \vec{f}, \vec{\tau}_j^* \rangle = \int_{\partial\Omega} \vec{f}(\xi) \cdot \vec{\tau}_j^*(\xi)\, d\sigma$$

where

$$\left.\begin{aligned} &\vec{\tau}_j^* \quad \text{is the } j\text{th row vector of } E(x^{(1)}, \xi), \\ &\qquad\qquad \text{transposed for } j = 1, 2, 3; \\ &\vec{\tau}_j^* \text{ is the } (j-3)\text{th row vector of } E(x^{(2)}, \xi), \\ &\qquad\qquad \text{transposed for } j = 4, 5, 6. \end{aligned}\right\} \tag{4.213}$$

Similarly, we now form the augmented system (4.208), where $\vec{\tau}_j^*$ defined by (4.213) are used instead in (4.208). $\vec{\tau}_j^*$ here has the same type of singularity as the $\vec{\tau}^*$ in (4.185), so $\vec{\tau}_j^* \in [L^{2-\varepsilon_1}(\partial\Omega)]^3 \cap [H^{-\varepsilon_2}(\partial\Omega)]^3$ for any $\varepsilon_1, \varepsilon_2 > 0$. We now check the two conditions in (4.145). The second condition in (4.145) is easily verified as before, so we need only check the first condition in (4.145). We have

$$\begin{aligned} \langle \vec{k}_i, \vec{\tau}_j^* \rangle &= \int_{\partial\Omega} \vec{k}_i(\xi) \cdot \vec{E}_j(x^{(1)}, \xi)\, d\sigma_\xi \\ &= \text{the } j\text{th component of } \vec{m}_i(x^{(1)}), \quad 1 \leqslant j \leqslant 3, \\ \langle \vec{k}_i, \vec{\tau}_j^* \rangle &= \int_{\partial\Omega} \vec{k}_i(\xi) \cdot \vec{E}_{j-3}(x^{(2)}, \xi)\, d\sigma_\xi \\ &= \text{the } (j-3)\text{th component of } \vec{m}_i(x^{(2)}), \quad 4 \leqslant j \leqslant 6, \end{aligned}$$

where $\vec{E}_l(x, \xi)$ denotes the lth row vector of $E(x, \xi)$, transposed. Therefore the matrix of $\langle \vec{k}_i, \vec{\tau}_j^* \rangle$ is assembled, yielding

$$\det \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -x_2^{(1)} & 0 & x_3^{(1)} & -x_2^{(2)} & 0 & x_3^{(2)} \\ x_1^{(1)} & -x_3^{(1)} & 0 & x_1^{(2)} & -x_3^{(2)} & 0 \\ 0 & x_2^{(1)} & -x_1^{(1)} & 0 & x_2^{(2)} & -x_1^{(2)} \end{bmatrix} = 0. \tag{4.214}$$

So the first condition of (4.145) is violated. Because every solution of linear elastostatics is representable by the simple-layer potential (4.193), we see that the six extra conditions (4.205) will not yield an augmented system that is invertible. We conclude the following.

Corollary 4.8. *Consider the traction BVP*

$$\left.\begin{aligned}&\mu\Delta\vec{u}+(\lambda+\mu\nabla)(\nabla\vec{u})=0,\\&\vec{\tau}(\vec{u})=\vec{g}\in[H^r(\partial\Omega)]^3,\quad r>\tfrac{1}{2},\end{aligned}\right\}\tag{4.215}$$

satisfying the compatibility conditions $\int_{\partial\Omega}\vec{g}(x)\cdot\vec{m}_j(x)\,d\sigma=0$ *for* $j=1,2,\ldots,6$. *Then adding the six extra accessory conditions*

$$\vec{u}(x^{(1)})=\vec{\beta}^{(1)},\ \ \vec{u}(x^{(2)})=\vec{\beta}^{(2)},\tag{4.216}$$

where $x^{(1)},\ x^{(2)}\in\overline{\Omega}$ *and* $\vec{\beta}^{(1)},\ \vec{\beta}^{(2)}\in\mathbb{R}^3$ *are given, will not yield a unique solution* $\vec{u}$ *satisfying (4.215) and (4.216).* □

Physically, Corollary 4.1 implies that *holding the displacements of two points of a (3D) solid fixed will not yield a unique equilibrium configuration when a traction force is applied to the surface of the solid.*

(c) *Six extra conditions (4.206)*: $\int_{\partial\Omega}\vec{f}(x)\cdot\vec{m}_j(x)\,d\sigma=\alpha_j,\ 1\leqslant j\leqslant 6$

Obviously, we choose

$$\vec{\tau}^{*}_j(x)=\vec{m}_j(x),\qquad j=1,2,\ldots,6,\tag{4.217}$$

in (4.208).

The second condition in (4.209) is fulfilled in the same way as before. To check the first condition in (4.209), we note that

$$\langle\vec{k}_i,\vec{\tau}^{*}_j\rangle=\langle\mathscr{L}_1\vec{m}_i,\vec{m}_j\rangle,\qquad 1\leqslant i,j\leqslant 6.\tag{4.218}$$

Because $\mathscr{L}_1$ is positive definite (cf. (4.199)) and invertible (cf. (4.201)), the matrix with entries (4.218) is positive definite. So

$$\det[\langle\vec{k}_i,\vec{\tau}^{*}_j\rangle]>0.$$

Thus Theorem 4.10 is satisfied. We state the following.

Corollary 4.9. *The augmented boundary integral equation system*

$$\begin{bmatrix}0&\sum\limits_{j=1}^{6}e_j\otimes\vec{m}_j\\\sum\limits_{j=1}^{6}\vec{m}_j\otimes e_j&\mathscr{L}_2\end{bmatrix}\begin{bmatrix}\vec{a}\\\vec{f}\end{bmatrix}=\begin{bmatrix}\vec{\alpha}\\\vec{g}\end{bmatrix}\tag{4.219}$$

corresponding to the simple-layer potential (4.193) for the traction BVP (4.189), where the simple-layer density $\vec{f}$ *is subject to the constraints (4.206), is uniquely solvable with solution*

$$(\vec{a}, \vec{f}) = (0, \vec{\tilde{f}}) \in \mathbb{R}^m \oplus [H^r(\partial\Omega)]^3. \qquad \square$$

Remark 4.4.

(1) Example 4.4 can be immediately generalized to the Neumann problem in an N-dimensional space, $N \geqslant 3$.

(2) Both Examples 4.4 and 4.5, after suitable modification, also work for a two-dimensional domain Ω. In the 2 D case, because of the logarithmic potential, the operators $\mathscr{L}_1$ in (4.163) and (4.196) may not be invertible, and the arguments are slightly more complicated. $\square$

Example 4.6. Corollaries 4.5–4.9 above are most useful when the boundary integral equation approach is used for these boundary value problems. We now show *a shortcut to directly determine whether a given set of accessory conditions ensures the uniqueness of the solution*. Consider the Neumann BVP (4.181) subject to (4.182):

$$\int_{\partial\Omega} x_j^n u(x)\, d\sigma = \alpha, \quad \text{with } \alpha \in \mathbb{R} \text{ given.} \tag{4.220}$$

$\square$

First let $u_0(x)$ be any particular solution to (4.181). (u_0 exists if the compatibility condition $\int_{\partial\Omega} g(x)\, d\sigma = 0$ is satisfied.) Write the general solution as

$$u(x) = u_0(x) + c, \tag{4.221}$$

where the constant $c \in \mathbb{R}$ is yet to be determined. Substituting (4.221) into (4.220), we get

$$\int_{\partial\Omega} x_j^n [u_0(x) + c]\, d\sigma = \alpha.$$

So

$$c \int_{\partial\Omega} x_j^n\, d\sigma = \alpha - \int_{\partial\Omega} x_j^n u_0(x)\, d\sigma,$$

where the RHS above is known. Thus c is uniquely determinable if and only if

$$\int_{\partial\Omega} x_j^n\, d\sigma \neq 0.$$

This is exactly condition (4.183) in Corollary 4.5.

When the Neumann BVP (4.181) is subject to (4.173),

$$u(x^{(0)}) = \alpha, \quad \text{with } x^{(0)} \in \partial\Omega \text{ and } \alpha \in \mathbb{R} \text{ given,} \tag{4.222}$$

let us again substitute (4.221) into (4.222), and see that the constant c is always uniquely determinable:

$$c = \alpha - u_0(x^{(0)}).$$

Note that $u_0(x)$ is pointwise-defined at $x^{(0)} \in \partial\Omega$ once $g \in H^r(\partial\Omega)$ with $r > 0$. So uniqueness of solution u is always assured, just as Corollary 4.6 states.

Next, we consider the elastostatic traction BVP (4.189), where the traction data $\vec{g}$ satisfies the compatibility conditions (4.191). Let the six accessory conditions be imposed as (4.204). Let $\vec{u}^{\,0}$ be a particular solution of (4.189). Then any general solution $\vec{u}$ can be represented as

$$\vec{u}\,(x) = \vec{u}^{\,0}(x) + \sum_{i=1}^{6} c_i \vec{m}_i(x), \quad x \in \Omega. \tag{4.223}$$

Substituting (4.223) into (4.204), we get

$$\sum_{i=1}^{6} c_i \int_{\partial\Omega} \vec{m}_i(x) \cdot \vec{m}_j(x)\, d\sigma = \alpha_j - \int_{\partial\Omega} \vec{u}^{\,0}(x) \cdot \vec{m}_j(x)\, d\sigma, \quad j = 1, 2, \ldots, 6. \tag{4.224}$$

Because of the linear independence of $\{\vec{m}_i \mid 1 \leqslant i \leqslant 6\}$ in $[L^2(\partial\Omega)]^3$, c_i, $1 \leqslant i \leqslant 6$, are uniquely solvable. Therefore $\vec{u}\,(x)$ is uniquely given by (4.223), which agrees with what is stated in Corollary 4.7.

If instead, the six constraints (4.205) are imposed,

$$\vec{u}\,(x^{(1)}) = \vec{\beta}^{\,(1)}, \;\; \vec{u}\,(x^{(2)}) = \vec{\beta}^{\,(2)}, \tag{4.225}$$
$$\text{with } x^{(1)}, x^{(2)} \in \overline{\Omega} \text{ and } \vec{\beta}^{\,(1)}, \vec{\beta}^{\,(2)} \in \mathbb{R}^3 \text{ given,}$$

we substitute (4.223) into (4.225), getting

$$\begin{bmatrix} \vec{m}_1(x^{(1)})\; \vec{m}_2(x^{(1)}) \;\ldots\; \vec{m}_6(x^{(1)}) \\ \vec{m}_1(x^{(2)})\; \vec{m}_2(x^{(2)}) \;\ldots\; \vec{m}_6(x^{(2)}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_6 \end{bmatrix} = \begin{bmatrix} \vec{\beta}^{\,(1)} - \vec{u}^{\,0}(x^{(1)}) \\ \vec{\beta}^{\,(2)} - \vec{u}^{\,0}(x^{(2)}) \end{bmatrix} \in \mathbb{R}^6. \tag{4.226}$$

The determinant of the matrix on the LHS of (4.226) is zero for the same reason as (4.214). Therefore $c_1, \ldots, c_6$ are not uniquely determinable, and the solution lacks uniqueness, just as stated in Corollary 4.8.

The uniqueness of solution in Corollary 4.9 is *not obtainable* by this shortcut argument, because the six accessory conditions are formulated in terms of the simple-layer density $\vec{f}$ rather than the solution $\vec{u}$. (Recall that the simple-layer density $\vec{f}$ still has the physical meaning of being twice the jump discontinuity of the traction field across the layer surface.) □

Example 4.7. In the same spirit of Example 4.6, we now look at a fourth-order (biharmonic) elastostatic Kirchhoff pure bending plate model (Chapter 8):

$$\left.\begin{aligned} &\Delta^2 u(x) = 0, \qquad \Delta^2 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^2, \\ &\qquad\qquad\qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \\ &B_1 u(x) = g_1(x),\ x \in \partial\Omega, \\ &B_2 u(x) = g_2(x),\ x \in \partial\Omega, \end{aligned}\right\} \tag{4.227}$$

where

$$B_1 u \equiv \frac{\partial}{\partial n}\Delta u + (1-\nu)\frac{\partial}{\partial s}\left[(n_1^2 - n_2^2)\frac{\partial^2 u}{\partial x_1 \partial x_2} - n_1 n_2\left(\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right)\right],$$

$$B_2 u \equiv \nu\Delta u + (1-\nu)\left(n_1^2\frac{\partial^2 u}{\partial x_1^2} + n_2^2\frac{\partial^2 u}{\partial x_2^2} + 2n_1 n_2\frac{\partial^2 u}{\partial x_1 \partial x_2}\right),$$

are, respectively, the shear and bending moment operators on $\partial\Omega$; $n = (n_1, n_2) = (n_1(x), n_2(x))$ and $s = (-n_2, n_1)$ are respectively the unit exterior normal vector and the unit counterclockwise tangent vector on $\partial\Omega$; $\partial/\partial s$ denotes the counterclockwise tangential derivative; $\nu\colon\ 0 < \nu < \frac{1}{2}$ denotes the Poisson ratio; and $u(x)$ is the vertical displacement of the thin plate. □

The variational formulation, Sobolev space setting and ellipticity of the BVP (4.227) can be found in §§ 8.1–8.3. Assume that the boundary data g_1 and g_2 in (4.227) are sufficiently regular. It is not hard to verify that (4.227) has a solution if and only if the three compatibility conditions

$$\int_{\partial\Omega} [g_1(x)\phi_1(x) + g_2(x)\phi_2(x)]\, d\sigma = 0 \tag{4.228}$$

are satisfied, where

$$\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \in \left\{ \begin{bmatrix}1 \\ 0\end{bmatrix}, \begin{bmatrix}x_1 \\ -n_1(x)\end{bmatrix}, \begin{bmatrix}x_2 \\ -n_2(x)\end{bmatrix} \,\middle|\, x \in \partial\Omega \right\}.$$

Once (4.228) is met, the solution has three degrees of freedom:

$$\begin{gathered} u(x) = u_0(x) + a_0 + a_1 x_1 + a_2 x_2, \\ a_0,\ a_1,\ a_2 \in \mathbb{R},\ \ x = (x_1, x_2) \in \Omega, \end{gathered} \tag{4.229}$$

where u_0 is some particular solution to (4.227).

In order to fix the three constants in (4.229), we may impose one of the two sets of commonly used accessory conditions in the following forms: either

$$u(x^{(0)}) = \alpha_0, \quad \frac{\partial u(x^{(0)})}{\partial n} = \alpha_1, \quad \frac{\partial u(x^{(0)})}{\partial s} = \alpha_2, \tag{4.230}$$
$$\text{with } x^{(0)} \in \partial\Omega \text{ and } \alpha_j \in \mathbb{R},\ 0 \leqslant j \leqslant 2 \text{ given,}$$

or

$$u(x^{(1)}) = \beta_1, \quad u(x^{(2)}) = \beta_2, \quad u(x^{(3)}) = \beta_3, \tag{4.231}$$
$$\text{with } x^{(1)},\ x^{(2)},\ x^{(3)} \in \overline{\Omega} \text{ and } \beta_j \in \mathbb{R},\ 1 \leqslant j \leqslant 3, \text{ given.}$$

We see that (4.231) is analogous to (4.173) or (4.205) treated earlier. We study (4.230) and (4.231) separately below:

(a) *Three extra conditions (4.230)*

First of all, to ensure that the solution u and first-order derivatives are pointwise well defined in (4.230), it is sufficient that

$$g_1 \in H^{-3/2+\varepsilon_1}(\partial\Omega),\ g_2 \in H^{-1/2+\varepsilon_2}(\partial\Omega),\ \ \forall \varepsilon_1,\ \varepsilon_2 > 0. \tag{4.232}$$

(Thus $u \in H^{2+\varepsilon}(\Omega)$, $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, must be interpreted as a weak solution.) We substitute the general solution (4.229) into (4.230), yielding

$$\left.\begin{aligned} a_0 + a_1 x_1^{(0)} + a_2 x_2^{(0)} &= \alpha_0 - u_0(x^{(0)}), \\ a_1 n_1(x^{(0)}) + a_2 n_2(x^{(0)}) &= \alpha_1 - \frac{\partial}{\partial n} u_0(x^{(0)}), \\ -a_1 n_2(x^{(0)}) + a_2 n_1(x^{(0)}) &= \alpha_2 - \frac{\partial}{\partial s} u_0(x^{(0)}), \\ \text{with } x^{(0)} &= (x_1^{(0)}, x_2^{(0)}) \in \partial\Omega. \end{aligned}\right\} \tag{4.233}$$

Since the determinant satisfies

$$\det \begin{bmatrix} 1 & x_1^{(0)} & x_2^{(0)} \\ 0 & n_1(x^{(0)}) & n_2(x^{(0)}) \\ 0 & -n_2(x^{(0)}) & n_1(x^{(0)}) \end{bmatrix} = 1,$$

a_0, a_1 and a_2 in (4.229) are uniquely solvable. Therefore *uniqueness of the solution $u(x)$ for (4.227) is always assured.*

(b) *Three extra conditions (4.231)*

First, assume regularities

$$g_1 \in H^{-5/2+\varepsilon_1}(\partial\Omega),\ \ g \in H^{-3/2+\varepsilon_2}(\partial\Omega), \quad \varepsilon_1, \varepsilon_2 > 0$$

to assure that $u \in H^{1+\varepsilon}(\Omega)$, $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, with trace $u \in H^{1/2+\varepsilon}(\partial\Omega) \subset C^0(\partial\Omega)$. Next, substitute the general solution (4.229) into (4.231), yielding

$$\left\{\begin{aligned} a_0 + a_1 x_1^{(1)} + a_2 x_2^{(1)} &= \beta_1 - u_0(x^{(1)}),\\ a_0 + a_1 x_1^{(2)} + a_2 x_2^{(2)} &= \beta_2 - u_0(x^{(2)}),\\ a_0 + a_1 x_1^{(3)} + a_2 x_2^{(3)} &= \beta_3 - u_0(x^{(3)}),\\ \text{with } x^{(j)} &= (x_1^{(j)}, x_2^{(j)}) \in \overline{\Omega},\quad 1 \leqslant j \leqslant 3. \end{aligned}\right\} \tag{4.234}$$

The determinant of the coefficient matrix on the LHS of (4.234), evaluated,

$$\begin{aligned} &\det \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ 1 & x_1^{(3)} & x_2^{(3)} \end{bmatrix} \\ &= \left(x_1^{(2)} - x_1^{(1)}\right)\left(x_2^{(3)} - x_2^{(1)}\right) - \left(x_1^{(3)} - x_1^{(1)}\right)\left(x_2^{(2)} - x_2^{(1)}\right) \neq 0, \end{aligned}$$

if and only if $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are not colinear. Therefore, we see that *the three extra displacement data (4.231) assure uniqueness of the solution of the plate BVP (4.227) if and only if the three locations $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are chosen to be non colinear*. Physically, this implies that *holding (the displacements of) three points fixed on a plate subject to boundary shear and bending will yield a unique equilibrium configuration if and only if the three points are not colinear*. □

Note that the conclusions in Example 4.7 are also obtainable by a boundary integral equation method.