wird $\varrho = \varrho_1 = \nu = m|_{[0,1]}$,

$$k(z) = \begin{cases} 2 & \text{für} \quad z \in \mathfrak{a}, \\ 1 & \text{für} \quad z \in [0,1]-\mathfrak{a}, \end{cases} \qquad \# \Psi^{-1}\{z\} = 2 \text{ f.a. } z \in [0,1]$$

(also gleich der Hellinger–Hahnschen Vielfachheitsfunktion von $M_\Psi$).

4. Zum Abschluß wird noch kurz das Beispiel D aus [1] diskutiert: Mit Hilfe eines stetigen singulären Wahrscheinlichkeitsmaßes $\sigma$ mit Träger $[0,1]$ und der dadurch definierten Funktion $F(x) = \sigma([0,x))$ wird dort eine stetige Funktion $\varphi\colon [0,1] \to \boldsymbol{R}$ eingeführt, so daß für den Operator $M_\varphi$ auf $L^2([0,1], m)$ gilt: $k(z) = 1$ für $\nu$-f.a. $z \in [0,1]$, $\#\varphi^{-1}\{z\} = 2$ f.a. $z \in [0,1]$; $\nu = \frac{1}{2}(m+\sigma)$. Die Überlegungen von Abschnitt 3–5 kann man hier mit $E_1 = [0, \frac{1}{2}]$, $E_2 = (\frac{1}{2}, 1]$ und $E_3 = \ldots = E_\infty =$ leere Menge durchführen. Man findet $\varrho_1 = \frac{1}{2}\cdot m$, $\varrho_2 = \frac{1}{2}\sigma$. Sei $S \subset [0,1]$ eine Lebesgue-Nullmenge, auf der $\sigma$ konzentiert ist. Dann ist $\frac{d\varrho_1}{d\varrho} = \chi_{[0,1]-S}$, $\frac{d\varrho_2}{d\varrho} = \chi_S$; $\mathfrak{d}_1 = [0,1]-S$, $\mathfrak{d}_2 = S$. Satz 1 bestätigt, daß $k(z) = 1$ für $\nu$-f.a. $z \in [0,1]$. Die im Beweis von Satz 4 auftretende Ausnahmenullmenge ist hier

$$N_0 = [0, \tfrac{1}{2}] \cap \varphi^{-1}(S) \cup (\tfrac{1}{2}, 1] \cap \varphi^{-1}([0,1]-S) = \tfrac{1}{2}\cdot S \cup \big(1 - \tfrac{1}{2}F([0,1]-S)\big);$$

es ist $\varphi(N_0) = S \cup ([0,1]-S) = [0,1]$.

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# On the Poisson integral for Lipschitz and $C^1$-domains

by

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**Abstract.** Let $D$ be a Lipschitz domain and let $Hf$ denote the solution of the Dirichlet problem with boundary values $f$. In this note we prove that if $f \in L^p(\sigma)$, where $\sigma$ is the surface measure of $\partial D$ and $2 \leqslant p \leqslant \infty$, then the nontangential maximal function associated to $Hf$ is also in $L^p$. Here the exponent 2 is sharp. However, if $D$ is assumed to be a $C^1$-domain, then the results hold for $1 < p \leqslant \infty$. The method we use is to characterize the corresponding Carleson measures.

**1. Introduction.** Let $D$ be a bounded Lipschitz domain and denote by $\delta$ the surface measure of $\partial D$. If $E \subset \partial D$ and $P \in D$, we denote by $\omega(P, E)$ the harmonic measure of $E$ evaluated at $P$, and if $f$ is integrable with respect to the harmonic measure, we denote by

$$Hf(P) = \int_{\partial D} f(Q)\,\omega(P, dQ)$$

the Poisson integral of $f$. For the basic properties of $\omega$ we refer to Helms [5], Chapter 8. We recall that if $D$ is the unit ball and $1 < p < \infty$, then

$$\int_{\partial D} \sup_{0<r<1} |Hf(rQ)|^p\, d\sigma(Q) \leqslant C_p \int_{\partial D} |f|^p\, d\sigma.$$

In this article we shall study the analogues of this result for Lipschitz domains. It turns out that the analogue holds if $2 \leqslant p < \infty$ but not always if $1 < p < 2$. However, if we assume that $D$ is a $C^1$-domain, then we can extend the result to $1 < p < \infty$. We shall formulate these results in Section 3, but we can give the following characterization of the corresponding Carleson measures.

THEOREM 1. *Let $D \subset \boldsymbol{R}^n$, $n \geqslant 3$, be a Lipschitz domain and let $2 \leqslant p < \infty$. If $\mu$ is a positive measure on $D$, then the following conditions are equivalent:*

(i) *There is a constant $M$ such that for all $P \in \partial D$ and all $r > 0$ we have*

$$\mu\{Q \in D\colon |Q-P| < r\} \leqslant Mr^{n-1}. \tag{1.1}$$

(ii) *There is a constant $K$ such that for all $f \in L^p(\sigma)$ we have*

$$\int_D |Hf|^p\, d\mu \leqslant K \int_{\partial D} |f|^p\, d\sigma. \tag{1.2}$$

*If condition* (1.1) *holds, then the constant $K$ may be chosen to depend only on $p$, $M$ and $D$.*

*If $D$ is assumed to be a $C^1$-domain, then the above result holds when $1 < p < \infty$.*

For domains with smooth boundaries the result of Theorem 1 is contained in Hörmander [8]. We would like to point out that it has been proved in Dahlberg [3] that $L^2(\sigma) \subset L^1(\omega(P, \cdot))$ and also observed that if $1 < p < 2$ is given, then there is a Lipschitz domain such that $L^p(\sigma) \not\subset L^1(\omega(P, \cdot))$. This explains the restriction $p \geqslant 2$ in the first part of the theorem.

We remark that our results continue to hold in the case when $n = 2$, but to avoid the minor complications due to the logarithmic singularity of the Green functions we have restricted ourselves to the case $n \geqslant 3$.

**2. Technical preliminaries.** We start by recalling that a function $\varphi$ is called a *Lipschitz function* if

$$|\varphi(x) - \varphi(x_1)| \leqslant M|x - x_1|. \tag{2.1}$$

The smallest possible $M$ such that (2.1) holds is called the *Lipschitz constant of* $\varphi$, which we denote by $\Lambda(\varphi)$. We say that a bounded domain $D$ is a *Lipschitz domain with Lipschitz constant less than $M$* if $\partial D$ can be covered by right circular cylinders whose bases have positive distance from $\partial D$ and corresponding to each cylinder $L$ there is a coordinate system $(x, y)$ with $x \in \boldsymbol{R}^{n-1}$, $y \in \boldsymbol{R}$, with the $y$-axis parallell to the axis of $L$ and a Lipschitz function $\varphi$ with $\Lambda(\varphi) < M$ such that

$$L \cap D = \{(x, y): \varphi(x) < y\} \cap L \text{ and } L \cap \partial D = \{(x, y): y = \varphi(x)\} \cap L.$$

The greatest lower bound of all possible $M$ is called the *Lipschitz constant of $D$*. If in addition the function $\varphi$ can be taken to be $C^1$-functions, we say that $D$ is a *$C^1$-domain*. With this terminology it is easily verified that a $C^1$-domain has Lipschitz constant 0. If $D$ is a Lipschitz domain, we shall denote by $\sigma$ the surface measure of $\partial D$. If $P \in \partial D$ and $r > 0$, we put $\Delta(P, r) = \partial D \cap B(P, r)$, where $B(P, r) = \{Q: |Q - P| < r\}$. Since Lipschitz functions are differentiable almost everywhere with respect to Lebesgue measure (Stein [11], p. 250), it follows that for all points $Q$ on $\partial D$ outside a set of vanishing $\sigma$-measure there is an inward unit normal, which we denote by $n_Q$.

THEOREM A (Dahlberg [3]). *Let $D \subset \boldsymbol{R}^n$, $n \geqslant 3$, be a Lipschitz domain and denote by $G$ the Green function of $D$. Let $P \in D$ and put $g = G(P, \cdot)$. Then there exists a set $E \subset \partial D$ such that $\sigma(E) = 0$ and for all $Q \in \partial D - E$ the limit $\lim_{t \downarrow 0} (\partial/\partial n_Q) g(Q + t n_Q)$ exists. If we denote this limit by $(\partial/\partial n) g(Q)$, then the following holds:*

(a) *If $Q \in \partial D - E$, then $0 < (\partial/\partial n) g(Q) < \infty$.*



(b) *Let $\sigma_n$ denote the surface measure of $\{P \in \boldsymbol{R}^n: |P| = 1\}$ and define $\gamma_n$ by $\gamma_n^{-1} = \sigma_n(n-2)$. If $F \subset \partial D$, then*

$$\omega(P, F) = \gamma_n \int_F (\partial/\partial n) g(Q)\, d\sigma(Q).$$

(c) *There is a number $C > 0$ such that for all $P' \in \partial D$ and all $r \in (0, 1)$ we have*

$$\sigma(\Delta(P', r)) \int_{\Delta(P', r)} [(\partial/\partial n) g(Q)]^2 d\sigma(Q) \leqslant C \int_{\Delta(P', r)} [(\partial/\partial n) g(Q)]^2 d\sigma(Q).$$

We will need to compare positive harmonic functions which vanish on a part of the boundary. The following result was stated in Kemper [9]. For a proof see Dahlberg [3].

THEOREM B. *Let $D$ be a Lipschitz domain. Suppose $V$ is an open set such that $V \cap \partial D \neq \emptyset$. Suppose $W$ is a domain such that $W \subset D$ and $\overline{W} \subset V$. Let $P_0 \in W$. Then there is a constant $C > 0$ such that if $u$ and $v$ are non-negative harmonic functions in $D$ which vanish on $V \cap \partial D$ and satisfy $u(P_0) \leqslant v(P_0)$, then $u(P) \leqslant C v(P)$ for all $P \in W$.*

We shall be working with a class of domains which we shall now describe. For $m > 0$ let $L(m)$ be the class of Lipschitz functions $\varphi$ such that $\varphi(0) = 0$, $\Lambda(\varphi) < m$ and the support of $\varphi$ is contained in $\{x \in \boldsymbol{R}^{n-1}: |x| < 1\}$. If $\varphi \in L(m)$, we put

$$D(\varphi, m) = \{(x, y): \varphi(x) < y < Am, |x| < 10\},$$
$$S(\varphi) = \{(x, \varphi(x)): |x| \leqslant 1\}.$$

We suppose that $A$ is chosen so large that if $m \geqslant 1$, then $D(\varphi, m)$ is star-shaped with respect to $P_m(0, \frac{1}{2}mA)$ for all $\varphi \in L(m)$ and $P_m \in \Gamma + P$ for all $P \in \{(x, \varphi(x)): |x| \leqslant 10\}$, when $\Gamma = \{(x, y): |x| < 2my\}$. Also if $A$ is chosen large enough, we have the following result from Dahlberg [3], Lemma 1:

LEMMA 2.1. *Let $m \geqslant 1$ and $\varphi \in L(m)$. Let $G$ be the Green function of $D(\varphi, m)$. Then there is a number $C$ such that for all $Q \in S(\varphi)$ and all $r \in (0, 1)$ we have*

$$C^{-1} G(Q + (0, r), P_m) \leqslant \omega(P_m, \Delta(Q, r)) \leqslant C G(Q + (0, r), P_m).$$

We will make use of the following consequence from Hunt and Wheeden [7], p. 512:

LEMMA 2.2. *Let $m \geqslant 1$ and $\varphi \in L(m)$. Then there is a number $C$ with the following property: If $u$ is positive and harmonic in $D(\varphi, m)$ and vanishes on $\partial D(\varphi, m) - \Delta(Q, r)$ for some $Q \in S(\varphi)$ and $r \in (0, 1)$, then*

$$u(P_m) \leqslant C u(Q + (0, r))\, \omega(P_m, \Delta(Q, r)).$$

We have the following estimate of the harmonic measure for sets of the form $\Delta(P, r)$ (see Hunt and Wheeden [7], Lemma 2.1):

LEMMA 2.3. *Let $D$ be a Lipschitz domain. Then there is a number $\delta > 0$ such that if $P \in \partial D$ and $r > 0$, then $\omega(Q, \Delta(P, r)) \geqslant \delta$ whenever $Q \in D \cap B(P, \delta r)$.*

**3. The main result.** We start with the following preliminary version of Theorem 1.

LEMMA 3.1. *Let $\varphi$ be a Lipschitz function such that $\varphi(0) = 0$. Suppose the positive numbers $A$ and $B$ have been chosen such that $A > \sup\{|\varphi(x)|: |x| \leqslant 10B\}$ and the domain $D = \{(x, y): \varphi(x) < y < 10A,\ |x| < 10B\}$ is starshaped with respect to a point $P_0 = (0, A_1)$. Let $\mu$ be a positive measure on $D$ such that $\mu(D \cap B(P, r)) \leqslant Mr^{n-1}$ whenever $P \in \{(x, \varphi(x)): |x| \leqslant 10B\}$. Let $k$ be the density of $\omega(P_0, \cdot)$ with respect to $\sigma$. Suppose there is a $q \in (1, \infty)$ and an $\varepsilon > 0$ such that if $P \in \{(x, \varphi(x)): |x| \leqslant 4B\}$ and $0 < r < \varepsilon$, then*

$$\Big(r^{1-n} \int_{\Delta(P,r)} k^q\, d\sigma\Big)^{1/q} \leqslant L r^{1-n} \int_{\Delta(P,r)} k\, d\sigma.$$

*Then there is a number $K$, which can be taken to depend only on $D$, $P_0$, $L$ and $q$ such that if $f \in L^p(S^*, \sigma)$, where $S^* = \{(x, \varphi(x)): |x| \leqslant B\}$ and $p = q(q-1)^{-1}$, then*

$$\mu\{P \in D^*: |Hf(P)| > s\} \leqslant K s^{-p} \int_{S^*} |f|^p\, d\sigma.$$

*Here $D^* = \{(x, y): |x| < 2B,\ \varphi(x) < y < 2A\}$.*

Proof. Let $f^+ = \max(f, 0)$, $f^- = f^+ - f$. Then $f = f^+ - f^-$. Since $\mu\{P \in D^*: |Hf(P)| > s\} \leqslant \mu\{P \in D^*: Hf^+ > s/2\} + \mu\{P \in D^*: Hf^- > s/2\}$, it follows that it is sufficient to prove the lemma in the case $f \geqslant 0$. Put $S_m = \{(x, \varphi(x)): |x| \leqslant mB\}$ and $D_m = \{(x, y): |x| \leqslant mB,\ \varphi(x) < y < mA\}$. Let $\gamma = \{(x, y): a|x| < y,\ 0 < y < h\}$, where $a$ and $h$ have been chosen so small that $\gamma(P) = \gamma + P \in D_3$ whenever $P \in S_2$. We assume from now on that $f \geqslant 0$. Then we have from Hunt and Wheeden [6], Lemma 4, that if $P \in S_3$, then

$$\sup\{Hf(Q): Q \in \gamma(P)\} \leqslant Cf^*(P), \tag{3.1}$$

where $f^*(P) = \sup_{r>0} \big(\int_{\Delta(P,r)} fk\, d\sigma\big)\big(\omega(P_0, \Delta(P, r))\big)^{-1}$. Let $V = \{P \in D^*: P \in \gamma(Q)$ for some $Q \in S_2\}$ and put $U = D^* - V$. Since $\overline{U} \subset D$, it follows from Harnack's inequality that there is a constant $C$ such that if $P \in U$, then $Hf(P) \leqslant CHf(P_0) \leqslant C\|f\|_p$, where $\|f\|_p = (\int_{\partial D} |f|^p\, d\sigma)^{1/p}$. Hence

$$\mu\{P \in U: Hf(P) > s\} \leqslant C_1 s^{-p} \|f\|_p^p \tag{3.2}$$

for a suitable choice of $C_1$. If $P = (x, y) \in D$, we put $P^* = (x, \varphi(x))$.



From (3.1) follows the existence of a positive number $\delta > 0$ such that if $P = (x, y) \in V$ and $Hf(P) > s$, then $f^*(Q) > \delta s$ whenever $|Q - P^*| < \delta(y - \varphi(x))$. We now define $R(P) = \{(x', y'): |x - x'| < t(y - \varphi(x)),\ |y - y'| < 2(y - \varphi(x))\}$, where $t$ has been chosen so small that if $Q \in R(P) \cap S_3$, then $|Q - P^*| < \delta(y - \varphi(x))$. Therefore we have from (3.1) that if $P \in V$ and $Hf(P) > s$, then $f^*(Q) > \delta s$ whenever $Q \in R(P) \cap S_3$. Suppose $F \subset \{P \in V: Hf(P) > s\}$ is compact. Then there exists finitely many $P_j \in F$ such that $F \subset \bigcup R(P_j)$. Moreover, we may assume that no point $P$ belongs to more than $2^n$ of the sets $R(P_j)$ (see Stein and Weiss [12], p. 54). Hence $\mu(F) \leqslant \sum \mu(R(P_j)) \leqslant C\sigma\{Q \in S_3: f^*(Q) > \delta s\}$. Since $F$ was arbitrary, it follows that

$$\mu(V) \leqslant C\sigma\{Q \in S_3: f^*(Q) > \delta s\}. \tag{3.3}$$

We now see from (3.2) and (3.3) that the lemma follows if we can show that $\sigma\{Q \in S_3: f^*(Q) > s\} \leqslant Cs^{-p}\|f\|_p^p$. To this end we note that if $0 < r < \varepsilon$ and $Q \in S_3$, then

$$\int_{\Delta(Q,r)} fk\, d\sigma \leqslant r^{n-1}\Big(r^{1-n} \int_{\Delta(Q,r)} f^p\, d\sigma\Big)^{1/p}\Big(r^{1-n} \int_{\Delta(Q,r)} [k^q\, d\sigma\Big)^{1/q}.$$

From our condition on $k$ follows now that

$$\sup_{0<r<\varepsilon} \big(\omega(P_0, \Delta(Q, r))\big)^{-1} \int_{\Delta(Q,r)} fk\, d\sigma \leqslant C(Mf^p)^{1/p}(Q), \tag{3.4}$$

where $Mf(Q) = \sup_{0<r} r^{1-n} \int_{\Delta(Q,r)} f\, d\sigma$. From Lemma 2.3 follows the existence of a number $c = c(\varepsilon, D)$ such that $\omega(P_0, \Delta(Q, \varepsilon)) \geqslant c$. If $r > \varepsilon$ and $Q \in S_3$, then $\big(\omega(P_0, \Delta(Q, r))\big)^{-1} \int_{\Delta(Q,r)} fk\, d\sigma \leqslant C\|f\|_p^p$, which together with (3.4) implies that $f^*(Q) \leqslant C(Mf^p)^{1/p} + C\|f\|_p$. From the ordinary maximal inequality (see Stein [11], Chapter 1) follows now that $\sigma\{Q \in S_3: f^*(Q) > s\} \leqslant Cs^{-p}\|f\|_p^p$ which proves the lemma.

Let $D$ be a Lipschitz domain and let $1 < q < \infty$. We say that a function $f \in L^q(\sigma)$ *satisfies condition* $\mathrm{B}_q$ if there is a constant $B_q$ such that for all $P \in \partial D$ and all $r \in (0, 1)$

$$\Big[\big(\sigma(\Delta(P, r))\big)^{-1} \int_{\Delta(P,r)} f^q\, d\sigma\Big]^{1/q} \leqslant B_q\big(\sigma(\Delta(P, r))\big)^{-1} \int_{\Delta(P,r)} f\, d\sigma.$$

LEMMA 3.2. *Let $D \subset \mathbf{R}^n$, $n \geqslant 3$, be a Lipschitz domain and let $q \in (1, \infty)$. Suppose there is a point $P_0 \in D$ such that $k$ satisfies condition $\mathrm{B}_q$, where $k$ is the density of $\omega(P_0, \cdot)$ with respect to $\sigma$. Suppose $\mu$ is a positive measure on $D$ such that $\mu(B(P, r) \cap D) \leqslant Mr^{n-1}$ for all $P \in \partial D$ and all $r > 0$. Suppose $p < s < \infty$, where $p = q(q-1)^{-1}$. Then there is a constant $K$, which can be taken to depend only on $D$, $k$, $s$ and $M$ such that*

$$\int_D |Hf|^s\, d\mu \leqslant K \int_{\partial D} |f|^s\, d\sigma.$$

Proof. As in the proof of Lemma 3.1 it is sufficient to treat the case $f \geqslant 0$. From the definition of a Lipschitz domain follows the existence of finitely many domains $D_0, D_1, \ldots, D_N$ with the following properties. First, $D_i \subset D$ for all $i$ and $\bar{D}_0 \subset D$. Secondly, each of the domains $D_1, \ldots, D_N$ is congruent to a domain of the type considered in Lemma 3.1 and to each $i \geqslant 1$ there is a right circular cylinder $L_i$ whose bases are on positive distance from $\partial D$ and $L_i \cap D = D_i$, $L_i \cap \partial D = L_i \cap \partial D_i$. At last, the domains can be chosen so that $\bigcup_1^N S_i^* = \partial D$ and $D_0 \cup \left(\bigcup_1^N D_i^*\right) = D$.

Suppose now that $f$ has its support on $S_i^*$ and $f \geqslant 0$. From Theorem B follows the existence of a constant $C$ such that $Hf(P) \leqslant CHf(P_0) \leqslant C\|f\|_p$ for all $P \in D - D_1^*$. Hence

$$\int_D (Hf)^s d\mu \leqslant \int_{D_i^*} (Hf)^s d\mu + C\|f\|_p^p. \tag{3.5}$$

Let $G$ and $G_i$ be the Green functions of $D$ and $D_i$, respectively. Let $k_i$ denote the density of the harmonic measure of $D_i$ evaluated at $P_i$. If $V$ is an open set such that $\bar{V} \subset L_i$ and $P_i \notin \bar{V}$, then it follows from Theorem B that

$$C^{-1}G(P_0, Q) \leqslant G_i(P_i, Q) \leqslant CG(P_0, Q) \quad \text{for all } Q \in V. \tag{3.6}$$

From Theorem A and (3.6) follows now that to each compact set $K \subset L_i \cap \cap \partial D$ there is a number $C$ such that if $Q \in K$, then $C^{-1}k(Q) \leqslant k_i(Q) \leqslant Ck(Q)$. Hence we have from Lemma 3.1 that $\mu\{P \in D_i^*: H_i f > s\} \leqslant Cs^{-p}\|f\|_p^p$ when $H_i$ denotes the Poisson integral with respect to $D_i$. The argument leading to (3.5) gives that $Hf\,|\,D_i \leqslant H_i f + C\|f\|_p$. Hence $\mu\{P \in D_i^*: Hf > s\} \leqslant Cs^{-p}\|f\|_p^p$ which taken together with the Marcinkiewicz interpolation theorem (Stein [11], p. 272) and (3.5) gives that $\int_D (Hf)^s d\mu \leqslant K_i\|f\|_p^p$. Since any $f \in L^p(\sigma)$ can be written as $\sum_{i=1}^N h_i f$, where $h_i$ is the characteristic function of $S_i^*$, the lemma is proved.

We shall next show that if $D$ is a $C^1$-domain, then the density of the harmonic measure satisfies condition $\mathrm{B}_q$ for all $q \in (1, \infty)$. We start with the following fact. Suppose $u$ is harmonic in a domain $D \subset \boldsymbol{R}^n$ and $2 < q < \infty$. Then the function

$$F_q(\nabla u) = \left(|\nabla_x u|^2 + t\left((\partial/\partial y)u\right)^2\right)^{q/2} - q|\nabla_x u|^2\left(|\nabla_x u|^2 + t\left((\partial/\partial y)u\right)^2\right)^{(q-2)/2} \tag{3.7}$$

is superharmonic in $D$ (see Küran [10]), where $t = (q-1)^{-1}(n-1)^{-1}$ and $|\nabla_x u|^2 = \sum_{i=1}^{n-1}\left((\partial/\partial x_i)u\right)^2$.

Let $\delta > 0$ be chosen so that for all $\varphi \in L(1)$ we have $B(P_1, 5\delta) \subset D(\varphi, 1)$ and put $D^*(\varphi) = D(\varphi, 1) - \overline{B(P_1, \delta)}$.



LEMMA 3.3. *Let $n \geqslant 3$ and suppose $0 < m < (n-1)^{-1/2}$. For $\varphi \in L(m)$ let $\gamma$ denote the harmonic measure of $\partial D^*(\varphi) - \{(x, \varphi(x)): |x| \leqslant 10\}$. Suppose $2 < q < 1 + m^{-1}(n-1)^{-1/2}$. Then there are positive numbers $A$ and $B$, which can be taken to depend only on $m$ and $q$, such that*

$$F_q(\nabla g) + A\gamma \geqslant 2B\left((\partial/\partial y)g\right)^q \quad in \quad D^*(\varphi),$$

*where $g$ is the Green function of $D(\varphi, 1)$ with pole at $P_1$.*

Proof. Put $L = \partial D^*(\varphi) - \{(x, \varphi(x)): |x| \leqslant 10\}$. Since $0 \leqslant g(P) \leqslant |P - P_1|^{2-n}$, it follows from the standard Schauder estimates that there are numbers $A_1 > 0$ and $B_1 > 0$, which can be taken to depend only on $q$, such that $F_q(\nabla g) \geqslant -A_1$ and $|\nabla g| \leqslant B_1$ on $L$, where $\nabla g$ denotes the gradient of $g$. We now assume $\varphi \in C^\infty(\boldsymbol{R}^{n-1}) \cap L(m)$. Then $g$ is smooth up to the boundary of $D^*(\varphi)$ and, in particular, the tangential part of the gradient vanishes on $L_1 = \{(x, \varphi(x)): |x| < 10\}$. Hence, if $P \in L_1$, then $|\nabla_x g(P)| \leqslant m|(\partial/\partial y)g(P)|$. From (3.7) follows now that there is a number $B = B(q, m)$ such that $F_q(\nabla g) \geqslant 2B|(\partial/\partial y)g(P)|^q$ on $L_1$. Hence we can now choose a number $A = A(q, m) > 0$ such that $F_q(\nabla g) - 2B|(\partial/\partial y)g|^q + A \geqslant 0$ on $L$. From (3.7) we have that $v = F_q(\nabla u) - 2B|(\partial/\partial y)g|^q + A\gamma$ is superharmonic in $D^*(\varphi)$ and has boundary values $\geqslant 0$ on $\partial D^*(\varphi)$ which gives the lemma in the case when $\varphi \in C^\infty(\boldsymbol{R}^{n-1}) \cap L(m)$.

If $\varphi \in L(m)$ and not assumed to be $C^\infty$, we can find $\varphi_i \in C^\infty \cap L(m)$ such that $\varphi_i \geqslant \varphi$ and $\varphi_i \to \varphi$ uniformly. Let $g_i$ denote the Green function of $D(\varphi_i, 1)$ with pole at $P_1$ and let $\gamma_i$ denote the harmonic measure of $D^*(\varphi_i) - \{(x, \varphi_i(x)): |x| \leqslant 10\}$. Then it follows from the arguments given in Helms [5], p. 89, that $g_i \to g$ uniformly on compact subsets of $D(\varphi, 1) - \{P_1\}$ and $\gamma_i \to \gamma$ uniformly on compact subsets of $D^*(\varphi)$. Hence the lemma follows from the previous case.

We shall need the following simple consequence from Theorems A and B.

LEMMA 3.4. *Let $D_1$ and $D_2$ be two Lipschitz domains in $\boldsymbol{R}^n$, $n \geqslant 3$. Let $P_i \in D_i$, $i = 1, 2$, and denote by $k_i$ the density of the harmonic measure of $D_i$ evaluated at $P_i$. Suppose there is an open set $V$ such that $V \cap D_1 = V \cap D_2$ and $V \cap \partial D_1 = U \cap \partial D_2 \neq \emptyset$. Then to each compact set $F \subset V \cap \partial D_1$ there is a number $C > 0$ such that $C^{-1}k_1(P) \leqslant k_2(P) \leqslant Ck_1(P)$ a.e. on $F$.*

Proof. Let $G_i$ denote the Green function of $D_i$, $i = 1, 2$. From Theorem B follows the existence of a neighbourhood $W$ of $F$ such that $W \subset V$ and $C^{-1}G_1(P_1, Q) \leqslant G_2(P_2, Q) \leqslant CG_1(P_1, Q)$ for all $Q \in W$. From Theorem A follows that $k_i$ is a.e. given as the normal derivative of $G_i(P_i, \cdot)$ from which the lemma follows.

For $m > 0$ let $A(m) = \{(x, y): |x| < my,\ |x|^2 + y^2 = 1\}$ and put $e = (0, 1) \in A(m)$. Let $\delta$ denote the Beltrami operator of $S^{n-1} = \{P \in \boldsymbol{R}^n: |P| = 1\}$ and let $\lambda(m)$ be the first eigenvalue of $\delta f + \lambda f = 0$, $f = 0$ on the boundary of $A(m)$. Let $\varphi_m$ be the corresponding eigenfunction, normalized by $\varphi_m(e) = 1$. Let $\alpha(m)$ denote the positive root of the equation $t(t+n-2) = \lambda(m)$. It follows from Courant–Hilbert [2], p. 321, that $\alpha(m)$ is a continuous increasing function of $m$ and

$$\lim_{m\to 0} \alpha(m) = \alpha(0) = 1. \tag{3.8}$$

The fact which is important for us is that the function

$$h_m(P) = |P|^{\alpha(m)}\varphi_m(P|P|^{-1}) \tag{3.9}$$

is non-negative and harmonic in $\Gamma(m) = \{(x, y)\colon |x| < my\}$ and vanishes on the boundary of $\Gamma(m)$.

LEMMA 3.5. *Let $m, \varphi$ and $q$ be as in Lemma 3.3. Suppose in addition that $(q-1)\alpha(m) \leqslant q$. Denote by $k$ the density of the harmonic measure of $D(\varphi, 1)$ evaluated at $P_1$. Then there is a number $C > 0$ such that for all $P \in S(\varphi)$ and all $r \in (0, 1)$ we have*

$$\Big(r^{1-n} \int_{A(P',r)} k^{q+1}\, d\sigma\Big)^{1/(q+1)} \leqslant Cr^{1-n} \int_{A(P',r)} k\, d\sigma.$$

Proof. Define $v(P) = F_q(\nabla g) + (A+1)\gamma - B|(\partial/\partial y)g|^q$, where $A$ and $B$ are as in Lemma 3.3. Then $v$ is non-negative and superharmonic in $D^*(\varphi)$ and $v \geqslant 1$ on $L$, where $L$ is as in the proof of Lemma 3.3. Moreover, $v \geqslant B|(\partial/\partial y)g|^q$ in $D^*(\varphi)$. For $0 < t < 1$ let $v_t(P) = v(tP + (1-t)P_1)$. If $t$ is sufficiently near 1, then $v_t$ is non-negative and superharmonic in $D_1(\varphi) = D(\varphi, 1) - \overline{B(P_1, 2\delta)}$. Fix a point $P_2 \in D_1(\varphi)$ and denote by $k_1$ the density of the harmonic measure of $D_1(\varphi)$ evaluated at $P_2$. Since $v_t$ is superharmonic and continuous in $D_1(\varphi)$, we have

$$v_t(P_2) \geqslant \int_{\partial D_1(\varphi)} v_t(P) k_1(P)\, d\sigma(P). \tag{3.10}$$

We now observe that from Lemma 2.1 follows that $\nabla g$ is non-tangentially bounded a.e. on $S(\varphi)$, which means that $\nabla g$ has non-tangential limits a.e. on $S(\varphi)$. Also it follows that $\lim_{t\to 1} |\nabla g(P_t)| \geqslant C^{-1}k(P)$ a.e. on $S(\varphi)$. From Lemma 3.4 follows that there is a constant $C > 0$ such that $k_1(P) \geqslant C^{-1}k(P)$ a.e. on $S(\varphi)$. Hence we find from (3.8) and Fatou's lemma that $Cv(P_2) \geqslant \int_{S(\varphi)} (k(P))^{q+1} d\sigma(P)$. This means $k \in L^{q+1}(S(\varphi), \sigma)$. However, since the support of $\varphi$ is contained $m\{x \in \boldsymbol{R}^{n-1}\colon |x| < 1\}$, it follows that

$$\int_{L_1} k^{q+1}\, d\sigma < \infty, \tag{3.11}$$

where $L_1 = \{(x, \varphi(x))\colon |x| < 10\}$. Let $P' \in S(\varphi)$ and $0 < r < 1$ and put $u(P) = \int_{A(P',r)} |\partial/\partial y(g(x))|^q \omega(P, dQ)$. From (3.11) follows that there is a constant $C$ such that $u(P) \leqslant C$ on $L$. Hence $u \leqslant Cv$ in $D^*(\varphi)$ for a suitable constant $C$. From Lemma 2.2 follows the existence of a constant $C$ such that



$$u(P_1) \leqslant Cu(P' + (0, r))\,\omega(P_1, A(P', r)). \tag{3.12}$$

We now observe

$$\begin{aligned} u(P' + (0, r)) &\leqslant C(F_q(\nabla g) + A\gamma)(P' + (0, r)) \\ &\leqslant C(|\nabla g|^q + \gamma)(P' + (0, r)). \end{aligned} \tag{3.13}$$

Let $P_r = P' + (0, r)$. Then there is a number $\varepsilon > 0$ such that $B(P_r, \varepsilon r) \subset D(\varphi) - (P_1)$ for all $r \in (0, 1)$. Since $g$ is positive and harmonic in $B(P_r, \varepsilon r)$, we have that there is a number $C$, depending only on $\varepsilon$ and $n$, such that $|\nabla g(P_r)| \leqslant Cr^{-1}g(P_r)$. Let $\Gamma = \Gamma(m) = \{(x, y)\colon |x| < my\}$ and for $P \in S(\varphi)$ define $\Gamma(P) = \Gamma + P$. Fix an $r_0 > 0$ such that $B(P, r_0) \cap \Gamma(P) \subset D(\varphi, 1)$ for all $P \in S(\varphi)$. Then there is a number $C > 0$ such that $g(Q) \geqslant C$, $Q \in \bigcup_{P\in S(\varphi)} \{Q\colon Q \in \Gamma(P), |Q - P| = r_0\}$. If $s$ denotes the harmonic measure $\{P \in \Gamma\colon |P| = r_0\}$ with respect to $\Gamma \cap B(0, r_0)$, then $g(Q) \geqslant Cs(Q - P)$ for all $Q \in B(P, r_0) \cap \Gamma(P)$. From Theorem B follows that there is a number $C > 0$ such that $s(Q) \geqslant ch_m(Q)$ for all $Q \in B(0, \frac{1}{2}r_0) \cap \Gamma$, where $h_m$ is defined by (3.9). Hence $g(P_r) \geqslant Cr^{\alpha(m)} > 0$ for all $r$, $0 < r < 1$. From Theorem B and the assumption on $q$ follows the existence of a number $C > 0$ such that $\gamma(P_r) \leqslant Cg(P_r) \leqslant C(r^{-1}g(P_r))^q$.

From (3.13) follows now that $u(P_r) \leqslant C(r^{-1}g(P_r))^q$ and using Lemma 2.1 we find $u(P_r) \leqslant C(r^{1-n}\omega(P_1, A(P', r)))^q$. Considering (3.10) we now have

$$r^{1-n}u(P_1) \leqslant C(r^{1-n}\omega(P_1, A(P', r)))^{q+1}.$$

From Lemma (2.1) follows now that $\lim_{r\to 1} (\partial/\partial y)g(P + (0, r)) \geqslant C^{-1}k(P)$ a.e. on $\{(x, \varphi(x))\colon |x| < 10\}$. Hence $u(P_1) \geqslant C^{-q} \int_{A(P',r)} k^{q+1} d\sigma$ from which the lemma follows.

Proof of Theorem 1. Let $D \subset \boldsymbol{R}^n$ be a Lipschitz domain; then it follows from Lemma 2.3 that (ii) implies (i). If $k$ denotes the density of the harmonic measure evaluated at some point $P$, then $k$ satisfies a $\mathrm{B}_2$-condition (Theorem A). Hence it follows from Coifman–Fefferman [1] that $k$ satisfies a $\mathrm{B}_q$-condition for some $q > 2$. Lemma 3.2 shows now that (i) implies (ii) in this case.

If $D$ is a $C^1$-domain, then it follows from (3.8), Lemma 3.4 and Lemma 3.5 that $k$ satisfies a $\mathrm{B}_q$-condition for all $q \in (1, \infty)$. As above, considering Lemma 3.2 completes the proof.

We shall now turn to some consequences of Theorem 1. Let $D$ be a Lipschitz domain and let $\Gamma$ be a given open bounded circular cone with vertex at 0. We say that a compact set $F \subset \partial D$ is *$\Gamma$-regular* if there is an open right circular cylinder $L$ with axis parallell to the axis of $\Gamma$ and a coordinate system $(x, y)$ with $x \in \boldsymbol{R}^{n-1}$, $y \in \boldsymbol{R}$, with the $y$-axis parallell to the axis of $\Gamma$ such that $L\cap D = \{(x, y)\colon \varphi(x) < y\}\cap L$ and $L\cap\partial D = \{(x, y)\colon y = \varphi(x)\}\cap L$ for some Lipschitz function $\varphi$. Moreover, we require that $L\cap\partial D\supset F$ and there exists an open circular cone $\Gamma'$ with vertex at 0 such that $\bar\Gamma-\{0\}\subset\Gamma'$ and $\Gamma'+P\subset D$ for all $P\in\partial D\cap L$.

THEOREM 2. *Let $D\subset \boldsymbol{R}^n$, $n\geqslant 3$, be a Lipschitz domain and assume that $F\subset\partial D$ is a compact set which is $\Gamma$-regular for some open circular cone $\Gamma$ with axis along $e\in s^{n-1}$. Let $2\leqslant p<\infty$ and put for $f\in L^p(\sigma)$ $f^*(P) = \sup\{|Hf(Q)|\colon Q\in\Gamma+P\}$. Then*

$$\int_F (f^*)^p d\sigma \leqslant C\int_{\partial D}|f|^p d\sigma \tag{3.14}$$

*and* $\lim_{\delta\to 0}\int_F |Hf(P+\delta e)-Hf(P)|\,d\sigma(P) = 0$.

*If, in addition, $D$ is a $C^1$-domain, then the above results hold for all $p\in(1,\infty)$.*

Proof. To prove the theorem it is sufficient to prove (3.14) for the appropriate range of $p$. It is sufficient to prove (3.14) for the case when $f\geqslant 0$. From our assumptions of $\Gamma$ and Harnack's inequality it follows $f^*(P)\leqslant Cf^{**}(P)$, where $f^{**}(P) = \sup_{0<t<h} Hf(P+te)$, where $h$ is the height of $\Gamma$. If $s$ is a non-negative function with $0\leqslant s\leqslant h$, we define $F^* = \{P+s(P)e\colon P\in F\}$ and let $T\colon F\to F$ be defined by $\varphi(P) = P+s(P)$. Define now the positive measure $\mu$ on $F^*$ by $\mu(M) = \sigma(T^{-1}(M))$ for $M\subset F^*$. Then $\mu$ is a measure such that $\mu(B(P,r)\cap F)\leqslant Cr^{n-1}$ for all $P\in F^*$, where $C$ is independent of $s$. From Theorem 1 follows now that $\int_{F^*}(Hf)^p d\mu\leqslant K\int_{\partial D} f^p d\sigma$, where $K$ is independent of $s$. By a suitable choice of $s$ we can arrange that $Hf(T(P))\geqslant \frac{1}{2}f^{**}(P)$ for all $P\in F$, which proves the theorem.

We shall now show a converse of Theorem 2.

THEOREM 3. *Let $D\subset\boldsymbol{R}^n$, $n\geqslant 3$, be a Lipschitz domain and suppose $V\subset\partial D$ is a relatively open set such that $\bar V$ is $\Gamma$-regular for some open circular cone $\Gamma$. Let $2\leqslant p\leqslant\infty$ and put $V_\varepsilon = \{P+\varepsilon e\colon P\in V\}$, where $e$ is the axis of $\Gamma$. Let $u$ be harmonic in $D$ and suppose that $\limsup_{\varepsilon\to 0}\left(\int_{V_\varepsilon}|u|^p d\sigma\right)^{1/p}<\infty$. Then there is a function $f\in L^p(\sigma)$ such that $\lim_{P\to Q}(u(P)-Hf(P)) = 0$ for all $Q\in V$.*

*If, in addition, $D$ is assumed to be a $C^1$-domain, then the above results also hold when $1<p<2$.*

Proof. Define $f_\varepsilon$ by $f_\varepsilon(Q) = u(Q+\varepsilon e)$ if $Q\in V$ and zero otherwise. Then there is a number $M>0$ such that if $0<\varepsilon<\varepsilon_0$, then $\|f_\varepsilon\|_p\leqslant M$. Pick a sequence $\varepsilon_j\to 0$ such that $f_{\varepsilon_j}$ converges weakly to $f\in L^p(\sigma)$ as $j\to\infty$.

Fix a point $Q\in V$ and let $L$ be a right circular cylinder with its basis at a positive distance from $\partial D$ and its axis parallell to $e$ such that $Q\in L\cap\partial D\subset\overline{L\cap\partial D}\subset V$ and $\overline{L\cap D}-\partial D\subset D$. Define $u_\varepsilon(P) = |u(P+\varepsilon e)-Hf_\varepsilon(P)|$ if $P\in L\cap D$ and zero otherwise. Then there is a number $\varepsilon'>0$ such that $u_\varepsilon$ is subharmonic in $L$ whenever $0<\varepsilon<\varepsilon'$. Also, it follows from Theorem 2 that if $\varepsilon'$ is chosen sufficiently small, then there is a number $\delta_0$ such that

$$\Big(\int_V \big(u_\varepsilon(P+\delta e)^p d\sigma\big)\Big)^{1/p}\leqslant C,\qquad 0<\delta<\delta_0,\ 0<\varepsilon<\varepsilon'. \tag{3.15}$$

Let $R$ be an open set such that $\overline{L\cap D}-\partial D\subset R\subset\bar R-\partial D\subset\{P+se\colon P\in V,\ 0<s<\delta_0/2\}$. Hence we have from (3.15) that $\int_R u_\varepsilon dP\leqslant \mathrm{Const}\times\left(\int_R u_\varepsilon^p d\sigma\right)^{1/p}\leqslant\mathrm{Const}$ if $0<\varepsilon<\varepsilon'$. Also, there is a number $t>0$ such that if $P\in L\cap D$, then $B(P, td(P))\subset R$, where $d(P)$ denotes the distance from $P$ to $\partial D$. Since $u_\varepsilon$ is subharmonic, we find

$$u_\varepsilon(P)\leqslant Cd(P)^{-n}\int_{B(P,td(P))}u_\varepsilon dP\leqslant Cd(P)^{-n}\int_R u_\varepsilon dP\leqslant Kd(P)^{-n}.$$

Define $F(P) = Kd(P)^{-n}$ if $P\in L\cap D$, $F(P) = +\infty$ if $P\in L\cap\partial D$ and zero otherwise. Then $F$ is upper semicontinous in $L$, $\int_L(\log^+F)^q dP<\infty$ for all $q>0$ and $u_\varepsilon\leqslant F$ whenever $0<\varepsilon<\varepsilon'$. Now it is known that if a non-negative function $F$ is upper semicontinuous in a domain $\Omega\subset\boldsymbol{R}^n$ and $(\log^+F)^{n-1+\varepsilon}$ is locally integrable in $\Omega$ for some $\varepsilon>0$, then $S(F)$ contains a largest element $v$, where $S(F) = \{u\colon u$ is subharmonic in $\Omega$ and $u\leqslant F\}$ (see Domar [4], Theorem 2). Hence there is a function $v$ subharmonic in $L$, vanishing outside $L\cap\bar D$ such that if $0<\varepsilon<\varepsilon'$, then $u_\varepsilon\leqslant v$ in $L$. Hence $|u(P)-Hf(P)| = \lim_{j\to\infty}u_{\varepsilon_j}\leqslant v(P)$. It follows from the Wiener criterion (Helms [5], p. 220) that $L-\bar D$ is not thin (Helms [5], p. 209) at $Q$. Hence $v(Q) = \limsup_{P\to Q, P\in L-\bar D} v(P) = 0$, which implies that $\limsup_{P\to Q}|u(P)-Hf(P)|\leqslant\limsup_{P\to Q}v(P) = 0$. Since $Q$ was arbitrary, Theorem 3 follows.

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# On weakly* conditionally compact dynamical systems

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1. Let $(X, \varphi)$ be a topological dynamical system, i.e. $X$ is a compact metric space and $\varphi\colon X \to X$ is a continuous mapping. Denote by $C(X)$ the space of all continuous, real (or complex) valued functions on $X$, and let $U_\varphi$ be an operator defined as follows: $U_\varphi f = f\circ\varphi$, $f \in C(X)$.

A sequence $(f_n)$ of elements of a Banach space $E$ is said to be *weakly* conditionally compact* if for every sequence of positive integers $(n_k)$ there is a subsequence $(n_{k_i})$ such that for every linear continuous functional $\Phi \in E^*$ the sequence of scalars $(\Phi(f_{n_{k_i}}))$ is convergent. In the case of $E = C(X)$ it means that the sequence $(f_{n_{k_i}}(x))$ is pointwise convergent (not necessarily to a continuous function).

If for every sequence $(n_k)$ there exists a subsequence $(n_{k_i})$ and an element $f \in E$ such that $(f_{n_{k_i}})$ is weakly convergent to $f$, then the sequence $(f_n)$ is said to be weakly conditionally compact.

Definition. A system $(X, \varphi)$ is said to be *weakly* [weakly] conditionally compact* if for every $f \in C(X)$ the sequence $(U^n f)$ is weakly* [weakly] conditionally compact. For brevity, we shall call these systems w*cc [wcc] systems.

The aim of the paper is to study some spectral properties, the strict ergodicity (under some additional assumptions) and the sequence entropy of w*cc systems.

In view of Rosenthal's theorem [8] for every $f \in C(X)$ there are two possibilities:

(1) The sequence $(U^n f)$ contains a subsequence $(U^{n_k} f)$ such that for some $c > 0$ and for every sequence of numbers (real or complex) $a_0, \ldots, a_{m-1}$ the following inequality holds:

$$\sup_{x\in X}\Big|\sum_{k=0}^{m-1} a_k U^{n_k} f(x)\Big| \geqslant c \sum_{k=0}^{m-1} |a_k|. \tag{1}$$

(2) The sequence $(U^n f)$ is w*cc.