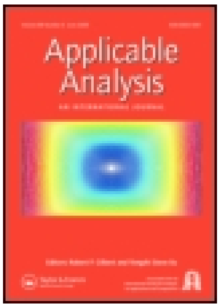

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The Double Layer Potential Operator over Polyhedral Domains I: Solvability in Weighted Sobolev Spaces

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Dedicated to Professor Erhard Meister on the occasion of his 60th birthday

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Abstract We consider the integral equation $(\lambda - K)u = f$, where K is the double layer (harmonic) potential operator on the boundary of a bounded polyhedron in $\mathbb{R}^3$ and λ, $|\lambda| \geq 1$, is a complex constant. We study the mapping properties of $\lambda - K$ in weighted Sobolev spaces, applying Mellin transformation techniques directly to the integral equation.

KEY WORDS: Double layer, potential theory.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded Lipschitz domain. This means that its boundary $\partial\Omega$ is locally the graph of a Lipschitz function. The harmonic double layer potential operator on $\partial\Omega$ is defined by

$$\begin{aligned} Ku(x) : &= \frac{1}{2\pi} \int_{\partial\Omega} u(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \, d\sigma(y) \\ &= \frac{1}{2\pi} \int_{\partial\Omega} \frac{(x-y) \cdot n_y}{|x-y|^3} u(y) d\sigma(y) , \quad x \in \partial\Omega , \end{aligned} \tag{1.1}$$

where $d\sigma$ is the surface measure on $\partial\Omega$, n the outward pointing normal vector to $\partial\Omega$, and the dot denotes the scalar product in $\mathbb{R}^3$. We consider the second kind integral equation on $\partial\Omega$,

$$(\lambda - K)u(x) = f(x), \quad x \in \partial\Omega , \tag{1.2}$$

where λ is a constant satisfying $|\lambda| \geq 1$. For $\lambda = 1$, Eq. (1.2) occurs in the indirect boundary integral equation method for the Dirichlet problem, whereas the case $\lambda \neq 1$ corresponds to certain transmission problems for the Laplace equation; see e.g. [10].

The main aim of the present paper is to establish solvability results for Eq. (1.2) in weighted Sobolev spaces when Ω is, in addition, a polyhedron, i.e. $\partial\Omega$ is the union of a finite number of plane polygons. The results can be used to derive quasi–optimal error estimates for certain boundary element methods; this will be done in the second part of this paper. Applying the theory of elliptic boundary value problems in domains with corners and edges developed e.g. in [11], Maz'ya [9] (cf. also the survey [10]) proved solvability in weighted Hölder spaces for the more general boundary integral equations of elasticity on polyhedral boundaries. Our approach is based on a direct application of Mellin transformation techniques to the integral equation (1.2); see also Rempel [15] who studied the adjoint of the double layer potential in certain classical Sobolev spaces on $\partial\Omega$, and Schmitz [16] who considered a class of integral operators containing the single layer potential over a polyhedral domain.

In order to establish Fredholmness and invertibility of the operator $\lambda - K$ defined in (1.2), we localize it, near each corner, to an integral operator on the boundary Γ of an infinite polyhedral cone with vertex at the origin 0. The intersection γ of Γ with the two–dimensional unit sphere S^2 is then a spherical polygon consisting of arcs of great circles. After introducing spherical coordinates $x = r\omega$, $y = r'\omega'$ $(r = dist(x,0),\, \omega \in S^2)$, the double layer potential on Γ can be written

$$\begin{aligned} Ku(r\omega) &= \frac{1}{2\pi} \int\limits_{R^+\times\gamma} \frac{n_{\omega'} \cdot \omega r r'}{|r\omega - r'\omega'|^3}\, u(r'\omega') d\omega' dr' \\ &= \frac{1}{2\pi} \int\limits_{R^+\times\gamma} \frac{n_{\omega'} \cdot \omega r/r'}{|(r/r')\omega - \omega'|^3}\, u(r'\omega') d\omega' \,\frac{dr'}{r'}\,, x \in \Gamma\,, \end{aligned} \tag{1.3}$$

where $d\omega'$ denotes the arc element on γ. Thus (1.3) can be regarded as a Mellin convolution operator (with respect to the radial variables) which has the operator kernel

$$k(t)\cdot = \frac{1}{2\pi} \int\limits_{\gamma} \frac{n_{\omega'} \cdot \omega t}{t\omega - \omega'|^3} \cdot d\omega' = \frac{1}{2\pi} \int\limits_{\gamma} \frac{n_{\omega'} \cdot \omega t}{(t^2 - 2t\omega\cdot\omega' + 1)^{3/2}} \cdot d\omega'\,, \tag{1.4}$$

acting on functions defined on γ. The operator–valued Mellin symbol of K is then defined by the Mellin transform of the kernel:

$$\mathcal{A}(z)\cdot = \int\limits_0^\infty t^{z-1} k(t) \cdot dt \quad z \in \mathbf{C}\,. \tag{1.5}$$

In Section 2 we collect the properties of the operator function (1.5) which is analytic in the strip $-1 < Re\, z < 2$ and takes values in weighted L^2 spaces on γ. In particular, the kernel of $\mathcal{A}(z)$ in $L^2(\gamma)$ is trivial if $0 \le Re\, z \le 1$; see Rempel [15] who reduced this to the investigation of an eigenvalue problem for the Laplace–Beltrami operator on S^2, and also Rathsfeld [14] for the case $Re\, z = 0$. In Appendix B we present a new proof of this, which relies entirely on integral equation methods. A key ingredient in the proof is a certain (non–standard) strong ellipticity result for the double layer potential on Lipschitz surfaces which yields, as a by–product, another approach to Verchota's theorem [18]; see Appendix A.

In Section 3 the results on the operator function (1.5) are used to prove the invertibility of the operators $\lambda - K$, $|\lambda| \geq 1$, in weighted Sobolev spaces over the infinite polyhedral cone. Our final results concerning the solvability of Eq. (1.2) on a closed polyhedral surface $\partial\Omega$ are contained in Section 4. They imply, in particular, that the Fredholm radius of (1.1) in those weighted Sobolev spaces on $\partial\Omega$ is larger than one. Note that precise formulas for the Fredholm radius of K in the case of domains with edges but without vertices were given by Grachev and Maz'ya [5]; see also [10]. Similar (but more complicated) formulas can also be derived in the presence of vertices (V.G. Maz'ya, personal communication, February 1990).

2. THE OPERATOR FUNCTION $\mathcal{A}(z)$

Let Γ be the boundary of a simply connected infinite polyhedral Lipschitz cone with vertex 0. The faces of Γ are (open) plane sectors, say $F_j (j = 1, \ldots, J)$, and the edges of Γ are denoted by f_j. The spherical polygon $\gamma = \Gamma \cap S^2$ then consists of the (open) arcs $\gamma_j = S^2 \cap F_j$ of great circles and the corner points $e_j = f_j \cap S^2$, $j = 1, \ldots, J$. Let Σ be the set of singularities of the cone, i.e. $\Sigma = \{f_1, \ldots, f_J\} \cup \{0\}$.

We further introduce the set $C^\infty_{pw}(\gamma)$ of all continuous functions on γ which are C^∞ on each closed arc $\overline{\gamma}_j$. In the sequel functions $u(\omega)$ on γ are identified with (periodic) functions $u(s)$ on the real axis, where $s, 0 \leq s \leq l$, denotes the arc length on γ. We choose a function $q \in C^\infty_{pw}(\gamma)$ such that $q(s) = |s - s_j|$ near each corner e_j of γ and $q(s) \neq 0$, $s \in [0, l) \backslash \{s_1, \ldots, s_J\}$, where e_j is parametrized by s_j. Thus q may be considered as the regularized geodesic distance on S^2 to the set of corner points of γ. Finally, after introducing the weighted L^2 spaces $L^2_\rho(\gamma) = q^\rho L^2(\gamma)$, $\rho \in \mathbb{R}$, with norm $||q^{-\rho} v; L^2(\gamma)||$ and the corresponding operator norm $||\cdot; \mathcal{L}(L^2_\rho(\gamma))||$ on these spaces, we can formulate the main properties of the operator function $\mathcal{A}(z)$ defined in (1.5).

Theorem 2.1. *Let $\rho \in [0, 1/2)$, and let $\epsilon > 0$ be sufficiently small.*

(i) $\mathcal{A}(z) \in \mathcal{L}(L^2_\rho(\gamma))$ is an analytic operator function in the strip $-1 < Re\, z < 2$.

(ii) On each closed strip $-1 + \epsilon \leq Re\, z \leq 2 - \epsilon$, the representation $\mathcal{A}(z) = \mathcal{A}_o(z) + \mathcal{A}_1(z)$ holds, where $||\mathcal{A}_o(z); \mathcal{L}(L^2_\rho(\gamma))|| \leq 1 - \delta$ for some $\delta \in (0,1)$ and all $z, \mathcal{A}_1(z)$ is compact on $L^2_\rho(\gamma)$ for any z and $||\mathcal{A}_1(z) : \mathcal{L}(L^2_\rho(\gamma))|| \to 0$ as $|Im\, z| \to \infty$.

(iii) For any $i, k \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$, $v \in C^\infty_{pw}(\gamma)$ and each closed strip $-1 + \epsilon \leq Re\, z \leq 2 - \epsilon$,

the estimate

$$\sum_{1\leq j\leq J} ||q^k z^k q^{i-\rho}(d/ds)^i \mathcal{A}(z)v; L^2(\gamma_j)|| \leq c||v; L^2_\rho(\gamma)|| \tag{2.1}$$

holds, where c does not depend on u and on z from that strip.

(iv) The kernel of $\lambda - \mathcal{A}(z)$ *in* $L^2(\gamma)$ *is trivial if* $|\lambda| \geq 1$ *and* $0 \leq Re\, z \leq 1$.

A simple consequence of (ii) is the following

Corollary 2.2. *Let* $\rho \in [0, 1/2)$, $|\lambda| \geq 1$ *and* $-1 < Re\, z < 2$. *Then* $\lambda - \mathcal{A}(z)$ *is a Fredholm operator on* $L^2_\rho(\gamma)$ *with index* 0. *Moreover, on each closed strip* $-1 + \epsilon \leq Re\, z \leq 2 - \epsilon$, $(\lambda - \mathcal{A}(z))^{-1} \in \mathcal{L}(L^2_\rho(\gamma))$ *exists and is uniformly bounded whenever* $|Im\, z|$ *is large enough, and there is only a finite number of z's lying in that strip such that the kernel of* $\lambda - \mathcal{A}(z)$ *is non-trivial.*

Proof of (i)–(iii): For $v \in C^\infty_{pw}(\gamma)$, consider the expression (cf. (1.4), (1.5))

$$\mathcal{A}(z)v(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}^+}\int_\gamma t^{z-1}\frac{n_{\omega'}\cdot \omega t}{(t^2 - 2t\omega\cdot\omega' + 1)^{3/2}}v(\omega')d\omega' dt\,, \tag{2.2}$$

where $-1 < Re\, z < 2$ and $\omega \in \gamma\backslash\{e_1, \dots, e_J\}$; it is pointwise well-defined since $n_{\omega'}\cdot\omega = 0$ whenever ω' and ω belong to the same arc γ_j of γ.

Step 1. For each corner e_j of γ, choose a function $\varphi_j \in C^\infty_{pw}(\gamma)$ such that $0 \leq \varphi_j \leq 1$ on γ, $\varphi_j = 1$ in a small neighborhood of e_j and $\varphi_j = 0$ outside a somewhat larger neighborhood not containing e_i if $i \neq j$. We set

$$\mathcal{A}_o(z) = \sum_{1\leq j\leq J}\varphi_j\mathcal{A}(z)\varphi_j,\ \mathcal{A}_1(z) = \mathcal{A}(z) - \mathcal{A}_o(z)\,. \tag{2.3}$$

The second operator takes the form

$$\mathcal{A}_1(z)\cdot = \int_{\mathbb{R}^+\times\gamma} t^{z-1}k(t,\omega,\omega')\cdot d\omega' dt\,,$$

where the kernel function k is continuous on $[0,\infty)\times\gamma\times\gamma$, C^∞ on each set $[0,\infty)\times\overline{\gamma}_j\times\overline{\gamma}_l$ and satisfies there the estimates

$$|(\partial/\partial s)^i k(t,s,s')| \leq c_i(1+t)^{-3}t\,,\ i \in \mathbb{N}_o \tag{2.4}$$

uniformly with respect to s and s'. Thus the expressions on the left-hand side of (2.4) multiplied by t^{z-1} are integrable on $\mathbb{R}^+$ uniformly with respect to s, s' and $-1+\epsilon \leq Re\, z \leq 2-\epsilon$ ($\epsilon > 0$). This implies easily that $\mathcal{A}_1(z) \in \mathcal{L}(L^2(\gamma), H^1(\gamma))$ is an analytic operator function in the strip $-1 < Re\, z < 2$, H^1 denoting the classical Sobolev space, and that $||\mathcal{A}_1(z); \mathcal{L}(L^2(\gamma), H^1(\gamma))||$ tends to zero as $|Im\, z| \to \infty$ and $-1+\epsilon \leq Re\, z \leq 2-\epsilon$. In view of the compact embedding $H^1(\gamma) \hookrightarrow L^2_\rho(\gamma)$ for $\rho < 1/2$, the desired properties of $\mathcal{A}_1(z)$ are established. Furthermore, the estimates (2.4) imply (iii) for $k = 0$ and $\mathcal{A}(z)$ replaced by $\mathcal{A}_1(z)$.

Step 2. We now consider the operator $\mathcal{A}_o(z)$ in a neighborhood of $e_1 = \overline{\gamma}_1 \cap \overline{\gamma}_2$, for example. Let χ_i be the characteristic function of $\overline{\gamma}_i$. Putting $\varphi' = \chi_1\varphi_1$, $\varphi'' = \chi_2\varphi_1$ and $a = \omega \cdot \omega'$, $b = n_{\omega'} \cdot \omega$, the operator $B(z) = \varphi''\mathcal{A}(z)\varphi'$ takes the form

$$B(z)v(s) = \int_0^\infty \frac{t^z}{2\pi}\varphi''(s)\int_0^{s_o} \frac{b}{(t^2-2at+1)^{3/2}}\varphi'(s')v(s')ds'dt\,,\ s \in (0,s_o)\,, \quad (2.5)$$

where

$$n_{\omega'} \cdot \omega = -\sin\alpha\sin s,\ \omega \cdot \omega' = \cos s\cos s' + \cos\alpha\sin s\sin s'\,. \quad (2.6)$$

Here $\omega \in \gamma_2$, $\omega' \in \gamma_1$ are parametrized by the arc lengths measured from the corner point e_1 of γ, and α denotes the interior angle of this spherical polygon at e_1. To verify the relations (2.6), we choose Cartesian coordinates (x_1, x_2, x_3) and assume without loss of generality that e_1 coincides with the north pole of S^2 and that γ_1 lies in the $x_1 - x_3$-plane. Then

$$n_{\omega'} = (0,-1,0),\ \omega' = (\sin s', 0, \cos s'),\ \omega = (\sin s\cos\alpha, \sin s\sin\alpha, \cos s)$$

which implies (2.6).

We first estimate the operator norm of (2.5) on the weighted L^2 space $s^\rho L^2(0,s_o)$ if s_o is sufficiently small and $Re\, z = 0$. From (2.6) we obtain

$$|B(i\xi)v(s)| \le \frac{1}{2\pi}\int_0^{s_o} \frac{|b|}{1-a}|v(s')|ds',\ \xi \in I\!R,\ s \in (0,s_o)\,,$$

where we have used the integral (cf. e.g. [4], 380.003)

$$\int_0^\infty \frac{dt}{(t^2-2at+t)^{3/2}} = \frac{1}{1-a}\,,\ a < 1.$$

Using (2.6) and the substitution

$$\sigma = \tan(s/2), \sigma' = \tan(s'/2), ds' = 2d\sigma'/(1+\sigma'^2)\,,$$

we arrive at

$$|B(i\xi)v(\sigma)| \le \frac{1}{\pi}\int_0^{\sigma_o} \frac{\sigma|\sin\alpha|}{\sigma^2+\sigma'^2-2\sigma\sigma'\cos\alpha}|v(\sigma')|d\sigma'\,, \quad (2.7)$$
$$\xi \in I\!R,\ \sigma \in (0,\sigma_o),\ \sigma_o = \tan(s_o/2)\,.$$

The right–hand side of (2.7) can be interpreted as a (scalar) Mellin convolution operator on the interval $(0,\sigma_o)$ having the kernel

$$\frac{1}{\pi}\frac{t|\sin\alpha|}{1+t^2-2t\cos\alpha}\,.$$

The norm of this operator on the weighted L^2 space $\sigma^\rho L^2(0,\sigma_o)$ is equal to (cf. e.g. [1])

$$|\sin(1/2-\rho)(\pi-\alpha)/\sin(1/2-\rho)\pi|\,,$$

and this quantity is strictly smaller than one if

$$0 \le \rho < 1/2 + 1/(1 + |1 - \alpha/\pi|)\,. \tag{2.8}$$

Since the measure ds' is equivalent to $2d\sigma'$ for small s', we have

$$||B(i\xi); \mathcal{L}(L^2_\rho(\gamma))|| \le 1 - \delta\,,\ \xi \in I\!R\,,$$

for some $\delta \in (0,1)$ if ρ satisfies (2.8) and supp φ_1 is small enough.

Consider now an arbitrary $\nu = Re\, z \in [-1+\epsilon, 2-\epsilon]$, where $\epsilon > 0$ is fixed, and let $\xi = Im\, z$. Then, for $s \in (0, s_o)$,

$$\begin{aligned} |[B(z) - B(i\xi)]v(s)| &\le c \int_0^\infty |t^\nu - 1| \int_0^{s_o} \frac{|b|}{(t^2 - 2at + 1)^{3/2}} |v(s')| ds' dt \\ &= : I_1 + I_2 + I_3\,, \end{aligned} \tag{2.9}$$

where the terms I_1, I_2 and I_3 correspond to the integration with respect to t over the intervals $(1-\eta, 1+\eta)$, $(0, 1-\eta)$ and $(1+\eta, \infty)$, $\eta > 0$, respectively. Assume that ρ satisfies (2.8). Choosing η sufficiently small, $|t^\nu - 1|$ can be made as small as desired for all $t \in (1-\eta, 1+\eta)$. Therefore, repeating the above estimates applied to $B(i\xi)$, we observe that the norm of I_1 in $\mathcal{L}(L^2\rho(\gamma))$ tends to zero as $\eta \to 0$ uniformly with respect to z belonging to the strip $-1+\epsilon \le Re\, z \le 2-\epsilon$. Furthermore, on that strip the L^2_ρ operator norm of the terms I_2 and I_3 can be made as small as we like if supp φ_1 is chosen small enough. This follows easily from the fact that the kernels of those integral operators do not have singularities and satisfy appropriate estimates as $t \to 0$ and $t \to \infty$. Consequently,

$$||B(z); \mathcal{L}(L^2_\rho(\gamma))|| \le 1 - \delta,\ -1 + \epsilon \le Re\, z \le 2 - \epsilon$$

with some $\delta \in (0,1)$ if supp φ_1 is sufficiently small and (2.8) holds. The operators $\varphi' \mathcal{A}(z) \varphi''$ can be treated analogously. Together with the considerations in Step 1, this establishes (ii).

Moreover, splitting the integral over t as in (2.9), for any z_o, $-1 < Re\, z_o < 2$, one obtains easily that in the norm of $\mathcal{L}(L^2_\rho(\gamma))$

$$\frac{B(z) - B(z_o)}{z - z_o} \cdot \to \int_0^\infty \frac{t^{z_o} ln t}{2\pi} \int_0^{s_o} \varphi'' \frac{b}{(t^2 - 2at + 1)^{3/2}} \varphi' \cdot ds' dt$$

as $z \to z_o$. Thus $B(z)$ is an analytic operator function in the strip $-1 < Re\, z < 2$, which finishes the proof of (i).

Step 3. Finally, we sketch the proof of (iii). Consider the operator $B(z)$ defined in (2.5). For $s \in (0, s_o)$, we have

$$s(\partial/\partial s) \frac{b}{(t^2 - 2at + 1)^{3/2}} = A + BC$$

where $A = s(\partial/\partial s) b/(t^2 - 2at + 1)^{3/2}$, $B = b/(t^2 - 2at + 1)^{3/2}$ and $C = 3ts(\partial/\partial s)a/(t^2 - 2at + 1)$. Using the relations (2.6), we observe that

$$|A| \leq const\ |B| \leq const\ |b|/(t^2 - 2at + 1)^{3/2}$$

$$|C| \leq const\ ts|(\partial/\partial s)a|/\{(t-1)^2 + 2t(1-a)\} \leq const\ s|(\partial/\partial s)a|/(1-a)$$

$$\leq \quad const\ s(s+s')/(s^2 + s'^2) \leq const$$

for $s, s' \in (0, s_o)$ and $t \in (0, \infty)$. Thus the estimates of Step 2 can be applied to $|s(d/ds)B(z)v(s)|$, and together with the properties of $\mathcal{A}_1(z)$ established in Step 1, we obtain (2.1) for $k = 0$ and $i = 1$. Repeating the above considerations, one gets (iii) for $k = 0$.

To verify (iii) for $k \in \mathbb{N}$, we note that the operator function $qs\mathcal{A}(z)$ corresponds to the Mellin transform of the (operator) kernel $-qt(\partial/\partial t)k(t)\cdot$ with k given by (1.4). We have

$$t(\partial/\partial t)\frac{b}{(t^2 - 2at + 1)^{3/2}} = \frac{b}{(t^2 - 2at + 1)^{3/2}}(1 + D),$$

where $D = 3t(t-a)/(t^2 - 2at + 1)$. Note that

$$\begin{aligned} |qD| &\leq const\ \{(1-a)t + |q|\,|t-1|t\}/\{(t-1)^2 + 2t(1-a)\} \\ &\leq const\ + const\ |q|/(1-a)^{1/2} \leq const \end{aligned}$$

for any $\omega, \omega' \in \gamma \backslash \{e_1, \dots, e_J\}$ and $t \in \mathbb{R}^+$. Similarly, the quantities $q^{i+1}(\partial/\partial s)^i D$ can be shown to be bounded for any $i \in \mathbb{N}$. Therefore, the above considerations can be applied to the operator $qz\mathcal{A}(z)$ instead of $\mathcal{A}(z)$, which gives (iii) for $k = 1$. The operators $q^i z^i \mathcal{A}(z)$, $i \geq 2$, can be treated in an analogous manner so that (iii) is established. $\square$

The proof of (iv) is postponed to Appendix B.

3. SOLVABILITY IN THE CASE OF AN INFINITE CONE

We retain the basic notation of the preceding section and introduce two scales of weighted Sobolev spaces on the infinite tangent cone Γ. Let $C^\infty_{pw}(\Gamma)$ be the set of all continuous functions on Γ with compact support which are C^∞ on each closed face $\overline{F}_j$ of Γ. In the sequel functions $u(r\omega)$ on Γ are identified with functions $u(r,s)$, $s \in [0, l)$. For $k \in \mathbb{N}_o$, $\rho \in [0, 1/2)$ and $\mu \in [0, 1)$, let $X^k_{\rho,\mu}(\Gamma)$ be the completion of $C^\infty_{pw}(\Gamma)$ in the norm

$$||u; X^k_{\rho,\mu}(\Gamma)|| = \sum_{1 \leq i+l \leq k} \sum_{1 \leq j \leq J} ||r^{-\mu}(qr)^l(\partial/\partial r)^l q^{i-\rho}(\partial/\partial s)^i u; L^2(F_j)||\,.$$

Note that $X^k_{\rho,\mu}$ can be considered as the space of traces of functions in certain weighted Sobolev spaces on domains with singularities, which were introduced by Maz'ya and Plamenevskiĭ [12]. For the special case $\rho = \mu$, one easily obtains the equivalent description

$$\begin{aligned} X^k_\rho(\Gamma) := X^k_{\rho,\rho}(\Gamma) = \{u \in \kappa^\rho L^2(\Gamma) : \kappa^{|\alpha|-\rho} D^\alpha_x u \in L^2(F_j), \\ 1 \leq |\alpha| \leq k,\ 1 \leq j \leq J\}\,, \end{aligned}$$

where $\kappa(x)$ denotes the regularized distance of x to the singularity set Σ of Γ. (Note that κ is equivalent to qr near Σ; cf. also [11].) In order to obtain optimal error estimates for certain boundary element methods applied to Eq. (1.2) which will be the subject of the second part of this paper, we need another scale of weighted Sobolev spaces. For $k \in \mathbb{N}_o$, $\rho \in [0,1/2)$ and $\mu \in [0,1)$, denote by $Y^k_{\rho,\mu}(\Gamma)$ the completion of $C^\infty_{pw}(\Gamma)$ in the norm

$$||u; Y^k_{\rho,\mu}(\Gamma)|| = \sum_{0\le i+l\le k} \sum_{1\le j\le J} ||r^{l-\mu}(\partial/\partial r)^l q^{i-\rho}(\partial/\partial s)^i u; L^2(F_j)||$$

and set $Y^k_\rho(\Gamma) := Y^k_{\rho,\rho}(\Gamma)$.

Let K be the double layer potential defined in (1.3). We are now in a position to state our solvability results.

Theorem 3.1. *Let $0 \le \rho < 1/2$, $0 \le \mu < 1$ and $|\lambda| \ge 1$.*

(i) The operator $\lambda - K \in \mathcal{L}(X^o_{\rho,\mu}(\Gamma))$ is invertible.

(ii) For any $k \in \mathbb{N}$, $K \in \mathcal{L}(X^o_{\rho,\mu}(\Gamma),\ X^k_{\rho,\mu}(\Gamma))$.

(iii) For any $k \in \mathbb{N}_o$, $\lambda - K \in \mathcal{L}(Y^k_{\rho,\mu}(\Gamma))$ is invertible.

As a direct consequence of (i) and (ii) one obtains

Corollary 3.2. *Under the assumptions of Theorem 3.1, $\lambda - K \in \mathcal{L}(X^k_{\rho,\mu}(\Gamma))$ is invertible for any $k \in \mathbb{N}$.*

Proof of (i)–(iii). (i): Note that $X^o_{\rho,\mu}(\Gamma)$ is simply the weighted L^2 space $r^{\mu-1/2}L^2(\mathbb{R}^+) \otimes L^2_\rho(\gamma)$. Let

$$\tilde{u}\,(z,s) = Mu(z,s) := \int_0^\infty r^{z-1}u(r,s)dr$$

be the partial Mellin transform of u with respect to the radial variable. It is a standard fact (cf. e.g. [6]) that M is an isomorphism of $X^o_{\rho,\mu}(\Gamma)$ onto the weighted L^2 space

$$\tilde{X}^o_{\rho,\mu} := L^2(\{Re\, z = 1-\mu\}) \otimes L^2_\rho(\gamma)\,.$$

Now Corollary 2.2 and Theorem 2.1 (iv) imply that $(\lambda - \mathcal{A}(z))^{-1} \in \mathcal{L}(L^2_\rho(\gamma))$ is uniformly bounded if $0 \le Re\, z \le 1$. Therefore, the mapping $\tilde{u} \to (\lambda - \mathcal{A}(z))\tilde{u}$ is an isomorphism of $\tilde{X}^o_{\rho,\mu}$ onto itself, which proves (i).

(ii): Since the image of $X^k_{\rho,\mu}(\Gamma)$ under the Mellin transform can be characterized by

$$\left\{\tilde{u} \in \tilde{X}^o_{\rho,\mu}:\ z^l q^{l+i}(\partial/\partial s)^i\, \tilde{u} \in \tilde{X}^o_{\rho,\mu},\ i+l \le k\right\}, \tag{3.1}$$

Theorem 2.1 (iii) implies the assertion. (In (3.1) the derivatives have to be understood piecewise on each arc γ_j.)

(iii): The Mellin image of $Y^k_{\rho,\mu}(\Gamma)$ is obviously the space

$$\overset{\sim k}{Y}_{\rho,\mu}= \left\{\tilde{u}\in\overset{\sim o}{X}_{\rho,\mu}: z^l q^i(\partial/\partial s)^i\, \tilde{u}\in\overset{\sim o}{X}_{\rho,\mu},\ i+l\leq k\right\}$$

endowed with the canonical norm, where the derivatives have to be interpreted as in (3.1). Now we observe that the operators K and $(r\partial/\partial r)^i$, $i\in I\!N$, commute (at least on $C^\infty_{pw}(\Gamma)$) so that Theorem 2.1 (iii) for $k=0$ implies the continuity of K on $Y^k_{\rho,\mu}(\Gamma)$. Furthermore, consider the equations

$$(\lambda-\mathcal{A}(z))z^l\,\tilde{u}= z^l\,\tilde{f},\ l=0,\dots,k\ , \tag{3.2}$$

where $f\in\overset{\sim k}{Y}_{\rho,\mu}$. For $l=0$, (3.2) possesses a unique solution $u\in\overset{\sim o}{X}_{\rho,\mu}$ by virtue of (i). Moreover, since $z^l\,\tilde{f}\in\overset{\sim o}{X}_{\rho,\mu}$, we obtain $z^l\,\tilde{u}\in\overset{\sim o}{X}_{\rho,\mu}$, $1\leq l\leq k$, by applying (i) to Eq. (3.2) for $l\geq 1$. Finally, using the estimates (2.1) with $k=0$ and $v(s)=z^l\,\tilde{u}\,(z,s)$, where z is fixed, we get

$$z^l q^i(\partial/\partial s)^i\,\tilde{u}\in\overset{\sim o}{X}_{\rho,\mu},\ l+i\leq k\ ,$$

which finishes the proof of (iii). □

4. SOLVABILITY IN THE CASE OF THE BOUNDED DOMAIN

Let $\partial\Omega$ be the boundary of a simply connected bounded polyhedron $\Omega\subset I\!R^3$ having the vertices E_i, $i=1,\dots,I$, and the plane faces F_j, $j=1,\dots,J$. It is assumed, in addition, that $\partial\Omega$ is locally the graph of a Lipschitz function. Let $C^\infty_{pw}(\partial\Omega)$ be the set of all continuous functions $\partial\Omega$ which are C^∞ on each set $\overline{F}_j$, and choose a partition of unity $\{\varphi_i,\ i=1,\dots,I\}$ on $\partial\Omega$ such that $\varphi_i\in C^\infty_{pw}(\partial\Omega)$, $\varphi=1$ in a neighborhood of E_i and $\varphi_i=0$ outside a somewhat larger neighborhood not containing E_j if $j\neq i$.

By Γ_i we denote the (infinite) tangent cone to $\partial\Omega$ at the corner E_i. Using the notation of the preceding section, we define the weighted Sobolev spaces $X^k_\rho(\partial\Omega)$ and $Y^k_\rho(\partial\Omega)$, $k\in I\!N_o$, $0\leq\rho<1/2$. Let $X^k_\rho(\partial\Omega)$ be the completion of $C^\infty_{pw}(\partial\Omega)$ in the norm

$$||u;X^k_\rho(\partial\Omega)||=\sum_{1\leq i\leq I}||\varphi_i u;\ X^k_\rho(\Gamma_i)||\ , \tag{4.1}$$

and $Y^k_\rho(\partial\Omega)$ is defined analogously. Let K be the double layer potential on $\partial\Omega$; cf. (1.1). We are now ready to state our main result.

Theorem 4.1. *Let $0\leq\rho<1/2$, $k\in I\!N_o$ and $|\lambda|\geq 1$.*

(i) For $\lambda\neq -1$, the operators $\lambda-K\in\mathcal{L}(X^k_\rho(\partial\Omega))$ and $\lambda-K\in\mathcal{L}(Y^k_\rho(\partial\Omega))$ are invertible.

(ii) For $\lambda=-1$, the above operators are Fredholm with index 0.

This theorem obviously has the following

Corollary 4.2 (Smoothness of solutions). *Under the assumptions of Theorem 4.1, $u \in L^2(\partial\Omega)$ and $(\lambda - K)u \in X^k_\rho(\partial\Omega)$ imply that $u \in X^k_\rho(\partial\Omega)$. The same result is valid for the Y-spaces.*

Proof of (i) and (ii). Step 1: We first show that $A = \lambda - K$ is a semi–Fredholm operator with finite dimensional kernel if $|\lambda| \geq 1$. Let $A_i = \lambda - K_i$, where K_i is the double layer potential on the tangent cone Γ_i, and consider the representation

$$A = \sum_{1 \leq i \leq I} \psi_i A_i \varphi_i + A'$$

with functions $\psi_i \in C^\infty_{pw}(\partial\Omega)$ having a somewhat larger support than the φ_i's. Let α be an arbitrary multiindex. It is easy to check that the kernels of the integral operators $D^\alpha_x A'$ are C^∞ on each set $\overline{F}_j \times \overline{F}_l$, $j, l = 1, \ldots, J$, whereas the kernel of A' is continuous on $\partial\Omega \times \partial\Omega$. Therefore, A' is a continuous mapping of the Sobolev space $H^{-1}(\partial\Omega)$ into $X^k_\rho(\partial\Omega)$ and $Y^k_\rho(\partial\Omega)$ for all $k \in \mathbb{N}_o$. Here we have used the fact that the functions which belong piecewise to the (classical) Sobolev space of order $k + 1$ on each face F_j embed continuously into those weighted spaces. Similarly, the integral operators $(1 - \psi_i)A_i\varphi_i$ map $H^{-1}(\partial\Omega)$ continuously into $X^k_\rho(\Gamma_i)$ and $Y^k_\rho(\Gamma_i)$, since the corresponding kernels have an appropriate decay at infinity.

Together with the estimates

$$||A_i\varphi_i u; X^k_\rho(\Gamma_i)|| \geq c||\varphi_i u; X^k_\rho(\Gamma_i)||,\ u \in X^k_\rho(\partial\Omega),\ i = 1, \ldots, I$$

which follow from Theorem 3.1 applied to A_i, the above considerations imply the inequality

$$||Au; X^k_\rho(\partial\Omega)|| \geq c||u; X^k_\rho(\partial\Omega)|| - c_1||u; H^{-1}(\partial\Omega)||,\ u \in X^k_\rho(\partial\Omega)$$

with positive constants c, c_1. In view of the compact embedding $L^2(\partial\Omega) \hookrightarrow H^{-1}(\partial\Omega)$, this proves that $A \in \mathcal{L}(X^k_\rho(\partial\Omega))$ is semi–Fredholm with finite dimensional kernel. The same is of course true for the Y–spaces.

Step 2: Since $\lambda - K$ is invertible on the corresponding spaces if $|\lambda|$ is sufficiently large and the index of semi–Fredholm operators is a homotopy invariant, $\lambda - K$ is Fredholm with index 0 for all $|\lambda| \geq 1$. It remains to prove that the kernel of $\lambda - K$ in $L^2(\partial\Omega)$ is trivial if $\lambda \neq -1$. This is a consequence of Corollary A.4 below. Note that the kernel of $1 + K$ is the one–dimensional space spanned by the constants; cf. e.g. [18]. □

Remark 4.3. A more careful study of the operator function $\mathcal{A}(z)$ shows that our solvability results may be extended to weighted Sobolev spaces defined for a larger range of the indices ρ and μ. Let Γ be a tangent cone as in Section 2 and set $\rho^* = 1/2 + 1/\lambda$, where λ is the maximum of $1 + |1 - \alpha_j/\pi|$ taken over all interior angles α_j of the spherical polygon

$\gamma = S^2 \cap \Gamma$, cp. with (2.8). For $0 \le \rho < \rho^*$, $0 \le \mu < 2$ and $k \in I\!N$, we define the spaces $X^k_{\rho,\mu}(\Gamma)$, $Y^k_{\rho,\mu}(\Gamma)$ as the completion of $C^\infty_{pw}(\Gamma)$ in the norm

$$||u; L^2(\Gamma)|| \quad + \sum_{1\le i+l\le k} ||r^{-\mu}(1+r)^\mu (qr)^l (\partial/\partial r)^l q^{i-\rho} (\partial/\partial s)^i u; L^2(\Gamma)||\,,$$

$$||u; L^2(\Gamma)|| \quad + \sum_{1\le l\le k} ||r^{-\mu}(1+r)^\mu r^l (\partial/\partial r)^l u; L^2(\Gamma)||$$
$$+ \sum_{1\le i, i+l\le k} ||r^{-\mu}(1+r)^\mu r^l (\partial/\partial r)^l q^{i-\rho} (\partial/\partial s)^i u;\, L^2(\Gamma)||\,,$$

respectively, where the derivatives are to be interpreted piecewise on all faces of Γ. If one still introduces the corresponding weighted Sobolev spaces $X^k_\rho(\partial\Omega)$ and $Y^k_\rho(\partial\Omega)$ by using (4.1), then Theorems 3.1, 4.1 and Corollaries 3.2, 4.2 generalize to all indices $\rho \in [0, \rho_o)$, $\mu \in [0, \mu_o)$ and $k \in I\!N$ with some $\rho_o \in (1, 3/2)$ and $\mu_o \in (1, 2]$. This covers, in particular, the solvability in the classical Sobolev space $H^1(\partial\Omega)$ which corresponds to the choice $\rho = k = 1$.

APPENDIX A: ANOTHER APPROACH TO VERCHOTA'S THEOREM

Let $\Omega \subset I\!R^3$ be a simply connected bounded Lipschitz domain with boundary $\partial\Omega$; for properties of Lipschitz domains we refer to [13]. We first introduce some tools; see [2] where a nice exposition of this material can be found.

Let $H^s(\Omega)$, $s \ge 0$, be the usual Sobolev spaces on Ω, and denote by $H^s(\partial\Omega)$, $|s| \le 1$, the Sobolev spaces on the Lipschitz manifold $\partial\Omega$. For $0 \le s \le 1$, $H^{-s}(\partial\Omega)$ is the dual space of $H^s(\partial\Omega)$ with respect to the bilinear form

$$\langle u, v\rangle = \int_{\partial\Omega} uv\,d\sigma\,.$$

With the Laplace operator Δ we associate the standard bilinear form

$$\Phi(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx,\; u, v \in H^1(\Omega)\,.$$

Let t_o be the trace operator $t_o u = u|_{\partial\Omega}$, $u \in C^\infty(\overline{\Omega})$, which by Gagliardo's trace lemma extends to a continuous surjective map $t_o : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ with bounded right inverse t_o^{-1}. Let further $t_1 u = \partial_\nu u|_{\partial\Omega}$, $u \in C^\infty(\overline{\Omega})$, where $\partial_\nu = n \cdot \nabla$ and n denotes the exterior unit normal to $\partial\Omega$. Consider the space $H^1_\Delta(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ endowed with the canonical norm. t_1 extends to a continuous map of $H^1_\Delta(\Omega)$ into $H^{-1/2}(\partial\Omega)$ by setting

$$\langle t_1 u, v\rangle = \Phi(u, t_0^{-1} v) + \int_\Omega t_o^{-1} v \Delta u\, dx,\; v \in H^{1/2}(\partial\Omega)\,; \tag{A.1}$$

note that $C^\infty(\overline{\Omega})$ is dense in $H^1_\Delta(\Omega)$. (A.1) is justified by the Green formula

$$\int_\Omega v\Delta u\, dx = -\Phi(u, v) + \langle t_1 u, t_o v\rangle,\; u, v \in C^\infty(\overline{\Omega}) \tag{A.2}$$

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which also holds for any $u, v \in H^1_\Delta(\Omega)$. Let

$$Wv(x) := \frac{1}{2\pi} \int_{\partial\Omega} \frac{v(y)}{|x-y|}\, d\sigma(y),\ x \in \mathbb{R}^3 \backslash \partial\Omega \tag{A.3}$$

be the single layer potential which is a continuous mapping of $H^{-1/2}(\partial\Omega)$ into $H^1_{loc}(\mathbb{R}^3)$. By K we denote again the double layer potential on $\partial\Omega$ (see (1.1)), and V stands for the single layer potential operator

$$Vv(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{v(y)}{|x-y|}\, d\sigma(y),\ x \in \partial\Omega \tag{A.4}$$

on the surface $\partial\Omega$. Note that, for any Lipschitz function on $\partial\Omega$, the integrals in (1.1) and (A.4) are to be understood as a Cauchy principal value; cf. [18]. Furthermore, $V \in \mathcal{L}(H^{-1/2-\beta}(\partial\Omega), H^{1/2-\beta}(\partial\Omega))$ and $K \in \mathcal{L}(H^{1/2-\beta}(\partial\Omega))$, $\beta \in (-1/2, 1/2)$; the proof of this does not rely on the deep theorem of Calderón. (See [2]; the endpoint result for $\beta = \pm 1/2$ can be found in [18].) Finally, we need the relations

$$t_o Wv = Vv,\ t_1 Wv = (1+K')v,\ v \in H^{-1/2}(\partial\Omega)\,, \tag{A.5}$$

where K' denotes the adjoint of K with respect to the scalar product $\langle u, \overline{v}\rangle$ in $L^2(\partial\Omega)$.

Now we can state two results on the strong ellipticity of our boundary integral operators. The first one is due to Costabel [2]; see also [3]. In the sequel $c, c_1, \ldots$ denote generic positive constants.

Proposition A.1. *For any $v \in H^{-1/2}(\partial\Omega)$, we have*

$$(i) \qquad \langle Vv, \overline{v}\rangle \geq c\|v; H^{-1/2}(\partial\Omega)\|^2, \tag{A.6}$$

$$(ii) \qquad \langle (1 \pm K')v, V\overline{v}\rangle \geq c\|v; H^{-1/2}(\partial\Omega)\|^2 - c_1 \left|\int_{\partial\Omega} Vv\, d\sigma\right|^2. \tag{A.7}$$

Proof. (i): See [2] and [3] (Bemerkung 3.9) where the case of general elliptic differential operators of second order was considered.

(ii): Consider the operator $1 + K'$, and let $v \in H^{-1/2}(\partial\Omega)$. Then (A.2) and (A.5) imply the relation

$$\Phi(Wv, W\overline{v}) = \langle t_1 Wv, t_o W\overline{v}\rangle = \langle (1+K')v, V\overline{v}\rangle\,. \tag{A.8}$$

On the other hand, from (i) and Gagliardo's trace lemma,

$$\|v; H^{-1/2}(\partial\Omega)\|^2 \leq c\|Vv; H^{1/2}(\partial\Omega)\|^2 \leq c\|Wv; H^1(\Omega)\|^2\,. \tag{A.9}$$

The last quantity can be dominated in the form

$$\|Wv; H^1(\Omega)\|^2 \leq c\left\{\Phi(Wv, W\overline{v}) + \left|\int_{\partial\Omega} Vv d\sigma\right|^2\right\}\,, \tag{A.10}$$

which follows by a standard compactness argument (and arguing by contradiction). Combining the estimates (A.8) to (A.10), we obtain (A.7) for $1 + K'$. Using the corresponding versions of (A.2) and (A.5) for the exterior domain $\Omega^c = \mathbb{R}^3 \backslash \overline{\Omega}$, one gets the estimate (A.7) in case of the minus sign. □

Corollary A.2. *For any* $|\lambda| \geq 1$, *the operators* $\lambda - K' \in \mathcal{L}(H^{-1/2}(\partial\Omega))$ *and* $\lambda - K \in \mathcal{L}(H^{1/2}(\partial\Omega))$ *are Fredholm with index* 0.

Proof. (A.6) and (A.7) imply that

$$\begin{aligned} |\langle(\lambda - K')v, V\overline{v}\rangle| &\geq c\langle v, V\overline{v}\rangle - c_1 |\int_{\partial\Omega} Vv\, d\sigma|^2 \\ &\geq c||v; H^{-1/2}(\partial\Omega)||^2 - c_1||v; H^{-1/2-\beta}(\partial\Omega)||^2 \end{aligned} \tag{A.11}$$

for any $v \in H^{-1/2}(\partial\Omega)$ and some $\beta \in (0, 1/2)$. Consequently,

$$||(\lambda - K')v; H^{-1/2}(\partial\Omega)|| \geq c||v; H^{-1/2}(\partial\Omega)|| - c_1||v; H^{-1/2-\beta}(\partial\Omega)||$$

so that $\lambda - K' \in \mathcal{L}(H^{-1/2}(\partial\Omega))$ is semi-Fredholm with finite dimensional kernel in view of the compact embedding $H^{-1/2} \hookrightarrow H^{-1/2-\beta}$. Since $\lambda - K'$ is clearly invertible for sufficiently large $|\lambda|$, one obtains the first assertion of the corollary. The second follows by duality. □

Remark A.3. Since V is self-adjoint with respect to the L^2 scalar product, it follows from (A.6) that $V \in \mathcal{L}(H^{-1/2}(\partial\Omega)), H^{1/2}(\partial\Omega))$ is an isomorphism, and by introducing the scalar product $\langle u, V\overline{v}\rangle$ one obtains an equivalent norm on $H^{-1/2}(\partial\Omega)$. Introducing the corresponding operator norm on $\mathcal{L}(H^{-1/2}(\partial\Omega))$, (A.7) implies that the essential norm of K' is smaller than one. Therefore, using the corresponding dual norm on $H^{1/2}(\partial\Omega)$, we observe that the essential norm of K is smaller than one. Unfortunately, this given no information about the essential norm of the double layer potential on $L^2(\partial\Omega)$. Apparently there is no result in the literature concerning that question, even in the case of a convex polyhedral domain Ω. This is in contrast to the situation when the operator K is considered in the space of continuous functions on $\partial\Omega$; cf. [7], [8], [10].

Corollary A.4. *For any* λ *satisfying* $|\lambda| \geq 1$ *and* $\lambda \neq -1$, *the operator* $\lambda - K'$*is invertible on* $H^{-1/2}(\partial\Omega)$.

Proof. We proceed as in [10] (Chap. 1, Thm. 12) where the case of a domain with smooth boundary has been treated. Assume that $\lambda \neq -1$ and that $v \neq 0$ belongs to the kernel of $\lambda - K'$ in $H^{-1/2}(\partial\Omega)$. Consider the function $w = Wv$ defined in (A.3). Note that $w \in H^1_{loc}(\mathbb{R}^3)$ and $\nabla w \in L^2(\mathbb{R}^3)$. Applying Green's formula, the relations (A.5) and their corresponding versions for the exterior domain Ω^c (cf. [2], [18]) as well, we get

$$\int_\Omega |\nabla w|^2 dx = \langle(1 + K')v, V\overline{v}\rangle, \int_{\Omega^c} |\nabla w|^2 dx = \langle(1 - K')v, V\overline{v}\rangle\,. \tag{A.12}$$

Let $\lambda = a + ib$, $a, b \in \mathbb{R}$. Together with $\langle(\lambda - K')v, V\overline{v}\rangle = 0$, (A.12) then yields

$$b\int_{\mathbb{R}^3} |\nabla w|^2 dx = 0, \; (a-1)\int_{\Omega} |\nabla w|^2 dx + (1+a)\int_{\Omega^c} |\nabla w|^2 dx = 0 \; . \tag{A.13}$$

On the other hand, $\int_{\Omega^c} |\nabla w|^2 dx \neq 0$, since otherwise $w = 0$ in Ω^c would imply that $Vv = w|_{\partial\Omega} = 0$, hence $v = 0$ which is a contradiction. By the first equality of (A.13), we now have $b = 0$ and $|a| \geq 1$, $a \neq -1$, whereas the second relation leads to $a \neq 1$ and $(a+1)/(a-1) \leq 0$. This is again a contradiction, which proves the result. □

Remark A.5. It follows from Corollaries A.2 and A.4 that $1 - K \in \mathcal{L}(H^{1/2}(\partial\Omega))$ is invertible. Verchota's theorem [18] says that this operator is invertible on $H^s(\partial\Omega)$, $0 \leq s \leq 1$. To prove this, it is now sufficient to apply the following smoothness result: If $u \in L^2(\partial\Omega)$ and $(1 - K)u \in H^1(\partial\Omega)$, then $u \in H^1(\partial\Omega)$. The latter, however, is a consequence of a result of Nečas [13]; see [2].

Finally, we emphasize that all results of this section are of course valid for Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The case $n = 3$ was only considered for notational convenience.

APPENDIX B: ON THE INVERTIBILITY OF $\mathcal{A}(z)$

Let Γ be such an infinite tangent cone as in Sections 2 and 3, and let K' be the adjoint of the double layer potential K (cf. (1.3)) with respect to the scalar product in $L^2(\Gamma)$. Define the single layer potential V by (A.4) with $\partial\Omega$ replaced by Γ. To prove Theorem 2.1 (iv), we need some auxiliary results. The first one is a version of Proposition A.1 for the unbounded Lipschitz surface Γ.

Proposition B.1. *Let $|\lambda| \geq 1$. Then, for any $u \in C^\infty_{pw}(\Gamma)$,*

$$\left|\int_\Gamma (\lambda - K')uV\overline{u}\,d\sigma\right| \geq c\int_\Gamma uV\overline{u}\,d\sigma \geq c_1||u; H^{-1/2}(\Gamma)||^2 . \tag{B.1}$$

Proof. Consider the bounded closed Lipschitz surface Γ_o composed of Γ and S^2. By Proposition A.1, we have

$$\int_{\Gamma_o} uV_o\overline{u}\,d\sigma \geq c||u; H^{-1/2}(\Gamma_o)||^2 , \tag{B.2}$$

$$\left|\int_{\Gamma_o} (\lambda - K_o')uV_o\overline{u}\,d\sigma\right| \geq c\int_{\Gamma_o} uV_o\overline{u}\,d\sigma - c_1||u; H^{-1/2-\beta}(\Gamma_o)||^2 \tag{B.3}$$

for all $u \in H^{-1/2}(\Gamma_o)$ and some sufficiently small $\beta > 0$, where V_o resp. K_o' denote the single layer potential resp. the L^2 adjoint of the double layer potential K_o on Γ_0. To deduce (B.1), we now use the following dilation argument.

For $u \in C^\infty_{pw}(\Gamma)$ and $R > 0$, we set $u_R(x) = u(x/R)$. Then

$$\int_\Gamma uv\,d\sigma = R^{-2}\int_\Gamma u_R v_R\,d\sigma , \tag{B.4}$$

$$(Vu)_R = R^{-1}Vu_R,\ (K'u)_R = K'u_R\,. \tag{B.5}$$

Furthermore, since the norm in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ is given by

$$||u; H^{1/2}(\Gamma)||^2 = \int_\Gamma |u|^2 d\sigma + |u|^2_{1/2}\,,$$

$$|u|^2_{1/2} := \int_{\Gamma\times\Gamma} \frac{|u(x)-u(y)|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y)\,;$$

$$||u, H^{-1/2}(\Gamma)|| = \sup_{v\neq 0} |\int_\Gamma uv\, d\sigma|/||v; H^{1/2}(\Gamma)||$$

respectively, we have

$$||u; H^{1/2}(\Gamma)||^2 = R^{-2}\int_\Gamma |u_R|^2 d\sigma + R^{-1}|u_R|^2_{1/2}\,, \tag{B.6}$$

$$||u_R; H^{-1/2}(\Gamma)|| \geq R^{3/2}||u; H^{-1/2}(\Gamma)||\,. \tag{B.7}$$

Applying (B.2) to u_R for sufficiently small R, one obtains

$$\int_\Gamma u_R V\overline{u}_R\, d\sigma = \int_{\Gamma_o} u_R V_o \overline{u}_R\, d\sigma \geq c||u_R; H^{-1/2}(\Gamma)||^2\,, \tag{B.8}$$

and by virtue of (B.4), (B.5) and (B.7), this implies the second inequality of (B.1).

Moreover, for any $u \in C^\infty_{pw}(\Gamma)$ satisfying supp $u \subset \{x \in \Gamma : |x| < 1/2\}$, we can write

$$Vu = \varphi V_o u + Tu,\ K'u = \varphi K'_o u + T'u \tag{B.9}$$

where $\varphi \in C^\infty_{pw}(\Gamma)$ vanishes for $|x| > 1$, $\varphi = 1$ in some neighborhood of supp u, and the integral operators T, T' extend to continuous mappings of $H^{-1}(\Gamma_o)$ into $H^1(\Gamma)$. Therefore, for those u we get from (B.3)

$$|\int_\Gamma (\lambda - K')uV\overline{u}\, d\sigma| \geq c\int_\Gamma uV\overline{u}\, d\sigma - c_1||u; H^{-1/2-\beta}(\Gamma_o)||^2\,. \tag{B.10}$$

For any $u \in C^\infty_{pw}(\Gamma)$, $u \neq 0$, we now set

$$v_R = R^{-3/2}u_R / \int_\Gamma uV\overline{u}\, d\sigma\,.$$

Then $\int_\Gamma v_R V \overline{v}_R d\sigma = 1$ and $||v_R; H^{-1/2}(\Gamma_o)|| \leq c_2$, $R > 0$ (cf. (B.4), (B.5) and (B.9)), hence $||v_R; H^{-1/2-\beta}(\Gamma_o)|| \to 0$ as $R \to \infty$ because the diameter of supp v_R tends to 0. Setting $u = v_R$ in (B.10), we thus obtain

$$|\int_\Gamma (\lambda - K')v_R V\overline{v}_R\, d\sigma| \geq c/2$$

for sufficiently small R, hence (B.5) implies the first estimate of (B.1). □

Proposition B.2. *For any $|\lambda| \geq 1$, the operator $\lambda - K \in \mathcal{L}(H^{1/2}(\Gamma))$ is invertible.*

Proof. We first verify that K is a continuous operator on $H^{1/2}(\Gamma)$. In analogy to (B.9), $Ku = \varphi K_o u + Tu$ for any $u \in C^\infty_{pw}(\Gamma)$ satisfying supp $u \subset \{x \in \Gamma : |x| < 1/2\}$, where φ and T have the same properties as there. Consequently, for those u, $||Ku; H^{1/2}(\Gamma)|| \leq c||u; H^{1/2}(\Gamma)||$ because of the continuity of K_o on $H^{1/2}(\Gamma_o)$. For an arbitrary element $u \in C^\infty_{pw}(\Gamma)$, we again consider the function u_R for sufficiently small $R > 0$. Then

$$||Ku_R; L^2(\Gamma)||^2 + |Ku_R|^2_{1/2} \leq c\{||u_R; L^2(\Gamma)||^2 + |u_R|^2_{1/2}\}$$

and by the homogeneity of K and (B.6),

$$R^2||Ku; L^2(\Gamma)||^2 + R|Ku|^2_{1/2} \leq c\{R^2||u; L^2(\Gamma)||^2 + R|u|^2_{1/2}\},$$

hence $|Ku|^2_{1/2} \leq c\{|u|^2_{1/2} + ||u; L^2(\Gamma)||^2\}$. Together with the continuity of K on $L^2(\Gamma)$, the last estimate proves the assertion.

We next consider the symmetric operator

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),\ D(V) = C^\infty_{pw}(\Gamma).$$

Then its closure $\overline{V}$ is self–adjoint (with respect to the L^2 scalar product), and its domain of definition is given by

$$D(\overline{V}) = \{v \in H^{-1/2}(\Gamma) : Vv \in H^{1/2}(\Gamma)\}.$$

Indeed, for any $\varphi \in C^\infty_{pw}(\Gamma)$ and $\psi \in C^\infty_{pw}(\Gamma)$ with a somewhat larger support, $\varphi V \psi \in \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ and $\varphi V(1-\psi) \in \mathcal{L}(H^{-1}(\Gamma), H^1(\Gamma))$. Thus V is a continuous map of $H^{-1/2}(\Gamma)$ into $H^{1/2}_{loc}(\Gamma)$. Moreover, if $u \in D(\overline{V})$ and the sequence $\{u_n\} \subset C^\infty_{pw}(\Gamma)$ converges to the element u in $H^{-1/2}(\Gamma)$, then $\varphi V u_n \to \varphi V u$ (in $H^{1/2}(\Gamma)$) for any $\varphi \in C^\infty_{pw}(\Gamma)$, hence $Vu_n \to Vu$.

Now, by the second estimate of (B.1), $\overline{V}$ is an isomorphism of $D(\overline{V})$ onto $H^{1/2}(\Gamma)$, where the Banach space $D(\overline{V})$ is equipped with the graph norm $||u; H^{1/2}(\Gamma)|| + ||Vu; H^{1/2}(\Gamma)||$.

Finally, we consider the operator $(\lambda - K)\overline{V} \in \mathcal{L}(D(\overline{V}), H^{1/2}(\Gamma))$ which is the closure of

$$(\lambda - K)V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),\ D((\lambda - K)V) = C^\infty_{pw}(\Gamma).$$

By Proposition B.1, $(\lambda - K)\overline{V}$ is semi–Fredholm with trivial kernel for any $|\lambda \geq 1$. Since this operator is obviously invertible for sufficiently large $|\lambda|$, we obtain its invertibility for all $|\lambda| \geq 1$. This finishes the proof of the proposition. ◫

Proof of Theorem 2.1 (iv). By Theorem 2.1 (i), Corollary 2.2 and a general result on analytic Fredholm operator functions (see e.g. [17]), the operator function $\mathcal{B}(z) := (\lambda - \mathcal{A}(z))^{-1}$ is analytic in the strip $-\epsilon < Re\, z < 1 + \epsilon$ (for some sufficiently small $\epsilon > 0$) and possesses only a finite number of poles, say $z_o, \ldots, z_m$, there. Moreover in a neighborhood of such a pole, say z_o, $\mathcal{B}(z)$ always takes the form

$$\mathcal{B}(z) = \mathcal{A}'(z) + B_r(z - z_o)^{-r} + \ldots + B_1(z - z_o)^{-1} \tag{B.11}$$

with an analytic operator function $\mathcal{A}'(z)$, some $r \in I\!N$ and finite dimensional operators $B_j \in \mathcal{L}(L^2(\gamma))$. Furthermore, $\mathcal{B}(z) \in \mathcal{L}(L^2(\gamma))$ is uniformly bounded on $\{z \in \mathbf{C} : -\epsilon \leq Re\, z \leq 1 + \epsilon\} \backslash U$ for any open neighborhood U of the set $\{z_o, \ldots, z_m\}$.

It suffices to prove that $\mathcal{B}(z)$ has no poles on the strip $1/2 \leq Re\, z \leq 1$. Indeed, the operator kernel (1.4) satisfies $k(t) = t^{-1}k(t^{-1})$, hence $\mathcal{A}(z) = \mathcal{A}(1 - z)$ for $0 \leq Re\, z \leq 1$, and $\lambda - \mathcal{A}(z) \in \mathcal{L}(L^2(\gamma))$ is then invertible for those z. Suppose on the contrary that $\mathcal{B}$ takes the form (B.11) with $B_r \neq 0$ near z_o, where $\nu = Re\, z_o \in [1/2, 1]$. We choose a function $\varphi(s) \in H^1(\gamma)$ such that $B_r\varphi \neq 0$ and set

$$g(z, s) = \varphi(s)(z - z_o)^{r-1}(z - 2)^{-p} \prod_{1 \leq j \leq m} (z - z_j)^{r_j},$$

where $p \in I\!N$ is sufficiently large and z_j are the other poles of $\mathcal{B}(z)$ in $-\epsilon < Re\, z < 1 + \epsilon$ having the orders r_j. Then, for any fixed s, g is analytic in the strip $-\epsilon < Re\, z < 1 + \epsilon$. Moreover, g is square integrable with weight $(1 + |z|)^2$ uniformly (with respect to s) on each line $Re\, z = \mu, \mu \in [-\epsilon, 1 + \epsilon]$. Therefore the inverse Mellin transform

$$f(r, s) = \frac{1}{2\pi i} \int\limits_{Re\, z = \mu} r^{-z} g(z, s)dz, \ \mu \in [-\epsilon, 1 + \epsilon] \tag{B.12}$$

of g is an element of $H^1(\Gamma)$. (Note that $H^1(\Gamma)$ is the completion of $C^\infty_{pw}(\Gamma)$ in the norm $||u|| + ||(\partial/\partial r)u|| + ||r^{-1}(\partial/\partial s)u||$, where $|| \cdot ||$ denotes the norm in $L^2(\Gamma)$.)

Consider now the function $\tilde{u}\,(z, s) = \mathcal{B}(z)g(z, s)$ which is analytic in $\{z \in \mathbf{C} : -\epsilon < Re\, z < 1 + \epsilon,\ z \neq z_o\}$ for almost all fixed s and square integrable on $\{Re\, z = \mu\} \times \gamma$ for all $\mu \in [-\epsilon, 1 + \epsilon] \backslash \{\nu\}$. However, $\tilde{u}$ is not square integrable on $\{Re\, z = \nu\} \times \gamma$ since

$$|| \tilde{u}\,(z, \cdot); L^2(\gamma)|| \geq c||B_r\varphi; L^2(\gamma)||\, |z - z_o|^{-1},\ z \neq z_o,\ Re\, z = \nu$$

in a neighborhood of z_o; cf. (B.11). On the other hand, by Proposition B.2 there exists a unique solution $u \in H^{1/2}(\Gamma)$ of the equation $(\lambda - K)u = f$ with f given by (B.12). Then $u \in L^2(\Gamma) \cap r^{1/2}L^2(\Gamma)$ (cf. e.g. [6]), hence $\tilde{u} = Mu$ is square integrable on $\{Re\, z = \mu\} \times \gamma$ for all $\mu \in [0, 1/2]$. This contradiction finishes the proof. □

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