Definition 1. *Let the M-transform be defined by*

$$\mathcal{M}\phi(\xi) := \int_0^{+\infty} \cos(2\pi\sqrt{\xi u})\frac{\phi(u)}{\sqrt{u}}du$$

Definition 2. *For any function ϕ defined on $\mathbb{R}^+$, let C the operator defined by*

$$C\phi(t) = \phi(t^2), \quad t \in \mathbb{R}$$

For any function even function ϕ defined on $\mathbb{R}$, $C^{-1}\phi$ is a function defined on $\mathbb{R}^+$ by

$$C^{-1}\phi(u) = \phi(\sqrt{u})$$

Definition 3. *We note $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ the space of ϕ such that $C\phi \in \mathcal{S}(\mathbb{R})$. Let $S_p(\mathbb{R})$ the subspace of real even functions that belong to the Schwartz class.*

Proposition 1. $C\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right) = \mathcal{S}_p(\mathbb{R})$

Proposition 2. *If $\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, then*

$$\sqrt{x}\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$$

For all polynomial P,

$$C^{-1}P(dx^2)C\phi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$$

Where dx is the differentiation operator.

Proposition 3. *The operator $\mathcal{M} : \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right) \longrightarrow \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ is an involution, and can be rewritten as*

$$\mathcal{M} = C^{-1}\mathcal{F}C$$

where $\mathcal{F}$ is the Fourier transform defined on $\mathcal{S}_p(\mathbb{R})$

$$\mathcal{F}u(\xi) = \int_{-\infty}^{+\infty} e^{-i2\pi x\xi}u(x)dx$$

or equivalently

$$\mathcal{F}u(\xi) = \int_0^{+\infty} 2\cos(2\pi\xi x)u(x)dx$$

$\mathcal{F}$ is self-adjoint on $S_p(\mathbb{R})$.

Definition 4. *For ψ and ϕ in $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, we define the duality product*

$$\langle\phi, \psi\rangle_\omega = \int_0^{+\infty} \frac{\phi(x)\psi(x)}{\sqrt{x}}$$

Proposition 4.

$$\langle \phi, \psi \rangle_\omega = \langle C\phi, C\psi \rangle$$

Proposition 5. *For any ϕ, $\psi \in \mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$, one has*

$$\langle \mathcal{M}\phi, \psi \rangle_\omega = \langle \phi, \mathcal{M}\psi \rangle_\omega$$

Definition 5. *Let Δ_ω the operator defined on $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ by*

$$\Delta_\omega \phi(x) = 2\sqrt{x}\left(2\sqrt{x}\phi'(x)\right)'$$

If we call Δ the usual Laplace operator defined on $S_p(\mathbb{R})$, we have

Proposition 6.

$$\Delta_\omega \phi = C^{-1} \Delta C \phi$$

Corollary 1. *Δ_ω maps $\mathcal{S}\left(\sqrt{\mathbb{R}^+}\right)$ on itself.*

Corollary 2. $\langle \Delta_\omega \phi, \psi \rangle_\omega = \langle \phi, \Delta_\omega \psi \rangle_\omega$

Proposition 7. *One has, for all $\xi \in \mathbb{R}^+$*

$$\mathcal{M}(\Delta_\omega \phi) = -\xi \mathcal{M}\phi$$

Definition 6. *For $s \in \mathbb{R}$, we define $\mathcal{M}^s(\mathbb{R})$ as*

$$f \in \mathcal{M}^s(\mathbb{R}) \iff \int_0^{+\infty} \frac{(1+\xi)^s}{\sqrt{\xi}} \left|\mathcal{M}f\right|^2(\xi) < +\infty$$

Proposition 8.

$$f \in \mathcal{M}^s(\mathbb{R}^+) \iff Cf \in H^s(\mathbb{R})$$

and we have

$$\|f\|_{\mathcal{M}^s} = \|Cf\|_{H^s}$$

Definition 7. *For $s = 0$, we note $L^2_\omega = \mathcal{M}^0(\mathbb{R}^+)$. It is a Hilbert space with the scalar product corresponding to $\langle \cdot, \cdot \rangle_\omega$ defined earlier.*

Proof. To prove that L^2_ω is complete, it suffices to show that any Cauchy sequence f_n has a limit in L^2_ω. Obviously, $g_n := \dfrac{f_n}{x^{1/4}}$ is a Cauchy sequence in $L^2(\mathbb{R}^+)$, so it admits a limit $g_\infty \in L^2(\mathbb{R}^+)$. Then $f_\infty := x^{1/4} g_\infty$ belongs to L^2_ω and we have

$$\|f_n - f_\infty\|_{L^2_\omega} = \|g_n - g_\infty\|_{L^2}$$

which ensures $f_n \to f_\infty$ in L^2_ω. □

Proposition 9. *$\mathcal{M}^s(\mathbb{R}^+)$ is a closed subspace of L^2_ω. It is also a Hilbert space for the scalar product defined by*

$$\langle u, v \rangle_{\omega,s} := \int_0^{+\infty} \frac{(1+\xi)^s}{\sqrt{\xi}} \mathcal{M}u(\xi)\mathcal{M}v(\xi)$$

Proof. Let u_n a Cauchy sequence in $\mathcal{M}^s(\mathbb{R}^+)$. Then, by the same arguments as above,

$$v_n := \frac{(1+\xi)^{s/2}\mathcal{M}u_n(\xi)}{\xi^{1/4}}$$

has a limit v_∞ in L^2, and

$$u_\infty = \mathcal{M}\left(\frac{\xi^{1/4}}{(1+\xi)^{s/2}}v_\infty\right)$$

is the limit of u_n in $\mathcal{M}^s(\mathbb{R}^+)$. $\square$

Proposition 10. *The injection*