Link between Sobolev norms after cosine change

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1 Notations and preliminary results

In this note, when u refers to a function defined on $\Gamma_x = (-1,1)$, we will denote by α the function such that

$$u = \frac{\alpha}{\omega}, \quad \omega(x) = \sqrt{1-x^2}, \tag{1}$$

and by $\tilde{\alpha}(\theta) = \alpha(\cos\theta)$. Let $\Gamma_\theta = (0,\pi)$. Let H^s stand for the usual Sobolev space of order s. For $\Gamma \subset \tilde{\Gamma}$, where $\tilde{\Gamma}$ is any closed Lipschitz curve, recall that $\tilde{H}^s(\Gamma)$ is defined by

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\tilde{\Gamma}) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}, \tilde{u}(x) = \left\{ \begin{array}{ll} u(x) & x \in \Gamma \\ 0 & x \in \tilde{\Gamma} \setminus \overline{\Gamma} \end{array} \right.$$

We denote by $\|\cdot\|_s$ the H^s norm. Also, recall that for integer s, the norms $\tilde{H}^s$ and H^s are equivalent. Given that functions in $H^1(\mathbb{R})$ are continuous, the elements of $\tilde{H}^1(-1,1)$ must vanish at $x = -1, 1$, so we have simply

$$\tilde{H}^1(\Gamma_x) = H^1_0(\Gamma_x)$$

with equivalent norms.

We first state the following lemma which is a particular case of a weighted Hardy inequality (see for example the introduction of [1]).

Lemma 1.1. *Let $\alpha \in C_0^\infty(\Gamma_x)$. There holds*

$$\int_{\Gamma_x} \frac{\alpha^2(x)}{\omega^3(x)} \leq \int_{\Gamma_x} \alpha'^2(x)\omega(x)dx$$

Remark 1.1. Observe that after cosine change of variable, this result is equivalent to

$$\int_0^\pi \frac{\tilde{\alpha}^2(\theta)}{\sin^2\theta} d\theta \leq \int_0^\pi \tilde{\alpha}'^2(\theta)d\theta,$$

which, taking into account $\sin\theta \underset{\theta\to 0}{\sim} \theta$, is under the form of a classical Hardy inequality.

We also introduce S and N the usual single layer operator and hypersingular operator on Γ_x. The kernel of S is chosen so that it is positive definite and bounded below on $\tilde{H}^{-\frac{1}{2}}(\Gamma_x)$. For example, one can choose

$$Su(x) = -\frac{1}{2\pi}\int_{-1}^1 \ln|x-y|u(y)dy, \quad x \in \Gamma \tag{2}$$

In this case, we have

Lemma 1.2.

1. *For any $u \in \tilde{H}^{-\frac{1}{2}}(\Gamma_x)$, one has*

$$\|u\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_x)} \sim \sqrt{\langle Su, u\rangle}_{H^{\frac{1}{2}}(\Gamma_x), \tilde{H}^{-\frac{1}{2}}(\Gamma_x)}$$

2. For any $u \in \tilde{H}^{\frac{1}{2}}(\Gamma_x)$, one has

$$\|u\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_x)} \sim \sqrt{\langle Nu, u\rangle}_{H^{-\frac{1}{2}}(\Gamma_x),\tilde{H}^{\frac{1}{2}}(\Gamma_x)}$$

By $a \sim b$, we imply that there exist two constants c and C such that $ca \leq b \leq Ca$.

As was shown in [2], we have the following result: (the proof will be reproduced here for convenience)

Proposition 1.1. *Let $x = \cos\theta$, we have the identity*

$$Su(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right| \tilde{\alpha}(\theta')d\theta' \tag{3}$$

Proof. In (2), do the variable change

$$x = \cos\theta, y = \cos\theta', \frac{-dy}{\omega} = d\theta' \tag{4}$$

leading to

$$Su(x) = -\frac{1}{2\pi}\int_{0}^{\pi} \ln\left|\cos\theta - \cos\theta'\right| \tilde{\alpha}(\theta)d\theta$$

The result is then obtained with the help of the formula $\cos\theta - \cos\theta' = -2\sin\frac{\theta-\theta'}{2}\sin\frac{\theta+\theta'}{2}$. $\square$

Remark 1.2. This shows that, in the variable θ, the single layer potential is actually a convolution by the kernel $A(\theta) = -\frac{1}{2\pi}\ln\left|\sqrt{2}\sin\frac{\theta}{2}\right|$.

We easily deduce the following formula

Lemma 1.3. *For smooth u and v in $C_0^\infty(\Gamma_x)$,*

$$\langle Su, v\rangle = \frac{-1}{4\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \ln\left|\sqrt{2}\sin\frac{\theta-\theta'}{2}\right| \tilde{\alpha}(\theta)\tilde{\beta}(\theta')d\theta d\theta' \tag{5}$$

2 Norm estimates

Theorem 2.1. *We have the following four estimates :*

(i) $\|u\|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_x)} \leq C\,\|\tilde{\alpha}\|_{-1/2}$

(ii) $\|\sqrt{\omega}u\|_{L^2(\Gamma)} \leq C\,\|\tilde{\alpha}\|_0$

(iii) $\|\omega u\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_x)} \leq C\,\|\tilde{\alpha}\|_{1/2}$

(iv) $\left\|\omega^{\frac{3}{2}}u\right\|_1 \leq C\,\|\tilde{\alpha}\|_1$

Proof. **Proof of (ii):**
By the change of variables $t = \cos\theta$, we can see that

$$\int_{-1}^{1}\frac{\alpha^2(x)}{\omega(x)}dx = \int_0^\pi \tilde{\alpha}^2(\theta)d\theta$$

Proof of (iv):
The same change of variables also yields

$$\int_{-1}^{1}\omega\alpha'(x)^2dx = \int_0^\pi \tilde{\alpha'}^2(\theta)d\theta$$

Morever, observe that

$$\begin{aligned}\left(\omega^{\frac{3}{2}}u\right)' &= (\sqrt{\omega}\alpha)' \\ &= -\frac{x\alpha}{2\omega^{\frac{3}{2}}} + \alpha'\sqrt{\omega}.\end{aligned}$$

The second term has its L^2 norm controlled by the H^1 norm of $\tilde{\alpha}$. It remains to show that this also holds for the first one that is,

$$\int_{-1}^{1}\frac{\alpha^2}{\omega^3} \leq C\,\|\tilde{\alpha}\|_1\,,$$

which is a simple consequence of Lemma 1.1

Proof of (i):

Since $\tilde{\alpha}$ can be extended as an even $2\pi-$*periodic* function, its Sobolev norm of order s can be expressed as

$$\|\tilde{\alpha}\|_s^2 = |\alpha_0|^2 + \sum_{n=1}^{+\infty}|\alpha_n|^2\,n^{2s},$$

where α_k are the usual Fourier coefficients. Simple calculations show that

$$\|\tilde{\alpha}\|_{-\frac{1}{2}} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\tilde{\alpha}(\theta)\tilde{\alpha}(\theta')\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos\left(n\left(\theta-\theta'\right)\right)}{n}\right)$$

We aim to compute the function G in parenthesis:

$$G(\theta) = \frac{1}{2} + \sum_{n=1}^{+\infty}\frac{\cos n\theta}{n}$$

To achieve this, we will need the following well-known property of Chebyshev's polynomials $T_n(x)$.

Lemma 2.1. *For any $t \in (-1,1)$,*

$$\sum_{n=0}^{+\infty}t^n T_n(x) = \frac{1-tx}{1-2tx+t^2}$$

Integrating in t and taking the value at $t=1$ leads to the following identity:

$$\sum_{n=1}^{+\infty}\frac{T_n(x)}{n} = -\ln\sqrt{2-2x}$$

Therefore, taking $x = \cos\theta$, we find:

$$\begin{aligned}G(\theta) &= \frac{1}{2} - \ln\sqrt{2-2\cos\theta} \\ &= -\ln\left[2e^{-1/2}\sin\frac{|\theta|}{2}\right]\end{aligned}$$

By Proposition 1.1, we see that there exists a constant C such that

$$\|\tilde{\alpha}\|_{-1/2} = C\,|\alpha_0|^2 + \sqrt{\langle Su, u\rangle} \tag{6}$$

which implies the first inequality. $\square$

References

[1] David Eric Edmunds and Ritva Hurri-Syrjänen. Weighted hardy inequalities. *Journal of mathematical analysis and applications*, 310(2):424–435, 2005.

[2] Yeli Yan, Ian H Sloan, et al. *On integral equations of the first kind with logarithmic kernels.* University of NSW, 1988.