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The *hp*-Version of the Boundary Element Method in $\mathbb{R}^3$ The Basic Approximation Results

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This paper deals with the basic approximation properties of the *h–p* version of the boundary element method (BEM) in $\mathbb{R}^3$. We extend the results on the exponential convergence of the *h–p* version of the boundary element method on geometric meshes from problems in polygonal domains to problems in polyhedral domains. In 2D elliptic boundary value problems the solutions have only corner singularities whereas in 3D problems they contain additional edge and corner-edge singularities. The solutions of the corresponding boundary integral equations inherit those singularities. The detailed investigations in our analysis take care of the various types of those singularities. While edge singularities can be analysed using standard one-dimensional approximation results the corner-edge singularities demand a new analysis.

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1. Introduction

The solutions of three-dimensional elliptic boundary value problems in polyhedral domains or on screens have special singular forms at corners and edges. When those problems are converted via the direct method into boundary integral equations then the solutions of the latter inherit those edge and corner singularities. The singularities disturb the convergence of numerical schemes e.g. finite element or boundary element methods. Numerical schemes converge as fast as the solution to be approximated by the chosen trial functions due to the quasioptimality of the Galerkin method, and thus the rate of convergence is determined by the regularity of the solution. Even for smooth data, the regularity is reduced by edge and corner singularities, and therefore the convergence of these procedures is much slower than in case of smooth domains.

In 2D-boundary value problems we have only corner singularities (see [2]). From [13–15] which are based on [7] it is known that the solutions of three-dimensional elliptic boundary value problems also contain edge and corner-edge singularities. While edge singularities can be analysed using one-dimensional approximation

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results (see [9]) the corner-edge singularities demand further investigation. Therefore we have defined countably normed spaces which contain not only the well-known corner singularities but also the edge singularities and even more the corner-edge singularities. Because we have not only point singularities all approximation results of [8, 9] have to be extended and generalized. First results for the *h–p* version of the finite element method for 3D problems have been announced in [3]. However those results are not suitable for the *h–p* version of boundary element method considered here. From Lemmas 3.1 and 3.2 it can be seen that the singularities which are described in [13–15] are contained in our countably normed spaces. Here we restrict ourselves to square integrable singularities. From [13–15] it is known that for opening angles $\omega < 2\pi$ of the polyhedral, i.e. we have no screen problem, the singularities have this property. The case of screen problems for the Laplacian is analysed in [10] and for the Helmholtz operator in [11].

In this paper it will be shown that under proper assumptions usually satisfied in practice, the *h–p* version with a geometric mesh has exponential rate of convergence with respect to the number of degrees of freedom. The countably normed spaces introduced in section 2 are motivated by the form of the corner-edge singularities described in [14].

In section 2 we define the mesh and the appropriate countably normed spaces. For the proof of our main result (Theorem 3.1), the exponential convergence of the *h–p* version of the BEM for 3D problems, we define an equivalent countably normed space on a reference element using a cartesian co-ordinate system. In section 3 we present the approximation theorems which are proved in the appendix to derive the exponentially fast convergence of the *hp*-version for functions belonging to the countably normed space. In section 4 we describe some numerical results. In the Appendix the approximation problem at the corner and the edges is reduced by Lemma A.2 to a 1-dim problem which is solved in Lemma A.3 for the corner and in Theorem A.1 for the edges. In Theorem A.2 the approximation problem is treated for the elements inside the reference element. In the following C denotes a generic constant.

2. Notation and preliminaries

Let Γ be the boundary of a simply connected bounded polyhedron Ω in $\mathbb{R}^3$. We first introduce appropriate countably normed spaces $\mathscr{B}^l_\beta(\Gamma)$ $(0 < \beta < 1)$. Let F be a face of Ω having the corner points $e_1, \ldots, e_m$. For each i, f_i denotes the edge of F connecting e_i with e_{i+1}, where the periodicity convention $e_{m+1} = e_1$, $f_{m+1} = f_1$ is adopted. Consider a covering of F by neighbourhoods U_i $(i = 1, \ldots, m)$ of e_i not containing e_j, $j \neq i$, and introduce polar co-ordinates (r_i, Θ_i) for origin e_i in U_i such that $r_i = \operatorname{dist}(x, e_i)$ and the edges f_{i-1} and f_i are given by $\Theta_i = \omega_i$ and $\Theta_i = 0$ respectively. For $0 < \beta < 1$, let

$$\begin{aligned} \mathscr{B}^l_\beta(F) = \{u \mid u \in H^{l-1}(F)\colon \exists C, d > 0 \text{ independent of } k \text{ such that} \\ \|r_i^{\beta+\alpha_r-l}(\Theta_i(\omega_i - \Theta_i))^{(\beta+\alpha_\Theta-l)_+}(\partial/\partial r_i)^{\alpha_r}(\partial/\partial\Theta_i)^{\alpha_\Theta} u\|_{L^2(U_i)} \\ \leqslant C d^{\alpha_r+\alpha_\Theta-l}(\alpha_r + \alpha_\Theta - l)!, \\ \alpha_r + \alpha_\Theta = k = l, \quad l+1, \ldots, i = 1, \ldots, m\}, \end{aligned} \tag{2.1}$$

$$\mathscr{B}^l_\beta(\Gamma) = \{u \mid u \in H^{l-1}(\Gamma),\ u \in \mathscr{B}^l_\beta(F) \text{ for all faces } F \text{ of } \Gamma\}.$$

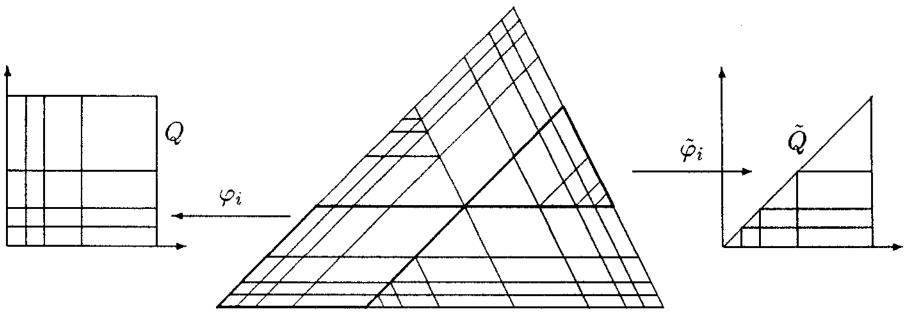

Fig. 1. Geometric mesh with $\sigma = 0.5$ on a polyhedral face.

with $(a)_+ := \max(a, 0)$. If we would like to emphasize the dependence on the constants C, d we will write $\mathscr{B}^l_\beta(F) = \mathscr{B}^l_{\beta,C,d}(F)$, etc.

Now we define the geometric mesh on the faces of the polyhedron. We assume that the face F is a triangle. This is no loss of generality because every polygonal domain can be decomposed into triangles. We divide this triangle into three parallelograms and three triangles where each parallelogram lies in a corner of the face F and each triangle lies at an edge of F apart of the corners. By linear transformations φ_i we can transform the parallelograms on to the reference square $Q = [0,1]^2$ such that the vertices of the face F are transformed to $(0,0)$. The triangles can be transformed by a linear transformation $\tilde{\varphi}_i$ on to the reference triangle $\tilde{Q} = \{(x,y) \in Q \,|\, y \leqslant x\}$ such that the corner point of the triangle in the interior of the face F is transformed to $(1,1)$ of the reference triangle. By Definition 2.1 the geometric mesh and appropriate spline spaces are defined on the reference element Q. Analogously the geometric mesh can be defined on the reference triangle $\tilde{Q}$ (see Fig. 1). Via the transformations φ_i^{-1}, $\tilde{\varphi}_i^{-1}$ the geometric mesh is also defined on the faces of the polyhedron. The approximation on the reference square is the more interesting case because it handles the corner-edge singularities. Therefore we deal with the following only with the approximation on the reference square.

Definition 2.1 (Geometric mesh). *Let $Q = [0,1] \times [0,1]$. For $0 < \sigma < 1$ we use the partition Q^n_σ of Q into n^2 subsquares R_{kl}*

$$R_{kl} = [x_{k-1}, x_k] \times [x_{l-1}, x_l] \quad (k, l = 1, \dots, n), \qquad Q = \bigcup_{k,l=1}^{n} R_{kl} \tag{2.2}$$

where

$$x_0 = 0, \qquad x_k = \sigma^{n-k}, \quad k = 1, \dots, n. \tag{2.3}$$

With Q^n_σ we associate a degree vector $p = (p_1, \dots, p_n)$ and define $S^p(Q^n_\sigma)$ as the vector space of all piecewise polynomials $v(x,y)$ on Q having degree p_k in x and p_l in y on $[x_{k-1}, x_k] \times [x_{l-1}, x_l]$, $k, l = 1, \dots, n$, i.e. $v|_{[x_{k-1},x_k]\times[x_{l-1},x_l]} \in P_{p_k,p_l}(R_{kl})$. For the difference $h_k = x_k - x_{k-1}$ we have

$$h_k = x_k - x_{k-1} = x_{k-1}\left(\frac{1}{\sigma} - 1\right) \leqslant x\left(\frac{1}{\sigma} - 1\right) = x\lambda,$$

$$\forall x \in [x_{k-1}, x_k], \; 2 \leqslant k \leqslant n, \quad \lambda = \frac{1}{\sigma} - 1 \tag{2.4}$$

Now we define the countably normed space on the reference element Q using Cartesian co-ordinates which we need to prove the approximation results.

Definition 2.2 (Weighted Sobolev spaces $H_\beta^{m,l}(Q)$). *Let β a real number with $0 < \beta < 1$. The weight function $\Phi_{\beta,\alpha,l} = \Phi_{\beta,\alpha,l}(x, y)$ is for $\alpha = (\alpha_1, \alpha_2)$ and an integer $l \geqslant 1$ defined by*

$$\begin{aligned}\Phi_{\beta,\alpha,l} = x^\beta & \sum_{\gamma_1 = \max(\alpha_1 - l, 0)}^{\min(\alpha_1 - 1, \alpha_1 + \alpha_2 - l)} x^{\gamma_1} y^{\alpha_1 + \alpha_2 - l - \gamma_1} \\ + y^\beta & \sum_{\gamma_2 = \max(\alpha_2 - l, 0)}^{\min(\alpha_2 - 1, \alpha_1 + \alpha_2 - l)} x^{\alpha_1 + \alpha_2 - l - \gamma_2} y^{\gamma_2}.\end{aligned} \tag{2.5}$$

Let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} = \partial_x^{\alpha_1} \partial_y^{\alpha_2}.$$

The weighted Sobolev spaces are defined for integers m and l, $m \geqslant l \geqslant 1$, by

$$H_\beta^{m,l}(Q) = \{u : u \in H^{l-1}(Q) \text{ for } l > 0, \|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(Q)} < \infty \text{ for } l \leqslant |\alpha| \leqslant m\}, \tag{2.6}$$

with the norm

$$\|u\|^2_{H_\beta^{m,l}(Q)} = \|u\|^2_{H^{l-1}(Q)} + \sum_{k=|\alpha|=l}^{m} \int_Q |D^\alpha u(x)|^2 \Phi^2_{\beta,\alpha,l}(x) \, \mathrm{d}x \tag{2.7}$$

and the seminorm

$$|u|^2_{H_\beta^{m,l}(Q)} = \sum_{k=|\alpha|=l}^{m} \int_Q |D^\alpha u(x)|^2 \Phi^2_{\beta,\alpha,l}(x) \, \mathrm{d}x. \tag{2.8}$$

Definition 2.3 (Countably normed spaces $B_\beta^l(Q)$). *The countably normed spaces for $l \geqslant 1$ are defined by*

$$\begin{aligned}B_\beta^l(Q) = \{u : u \in H_\beta^{k,l}(Q) \forall k \geqslant l, \|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(Q)} \leqslant C d^{k-l}(k-l)! \\ \text{for } |\alpha| = k = l, l+1, \ldots, \text{ with } C \geqslant 1, d \geqslant 1 \text{ independent of } k\}.\end{aligned} \tag{2.9}$$

If we would like to emphasize the dependence on the constants C, d we will write $B_\beta^l(Q) = B_{\beta,C,d}^l(Q)$, etc.

Theorem 2.1 (Maischak [12]). *Let Q be the reference element and let φ be the linear transformation from a parallelogram, which lies in a corner of the face F, to the reference element Q. Then*

$$u \circ \varphi^{-1} \in B^1_{\beta,C,d}(Q) \tag{2.10}$$

if and only if

$$u \in \mathscr{B}^1_{\beta,\tilde{C},\tilde{d}}(\varphi(Q)) \tag{2.11}$$

where C, d (resp. $\tilde{C}, \tilde{d}$) are the constants in the definition of $B^1_\beta(Q)$ (resp. $\mathscr{B}^1_\beta$ $(\varphi(Q))$.

Corollary 2.1. *For $l = 1$ we have the following weight function from* (2.5):

$$\Phi_{\beta,(\alpha_1,\alpha_2),1} = \begin{cases} x^{\beta+\alpha_1-1}, & \alpha_1 \geqslant 1, \alpha_2 = 0, \\ x^{\beta+\alpha_1-1}y^{\alpha_2} + x^{\alpha_1}y^{\beta+\alpha_2-1}, & \alpha_1 \geqslant 1, \alpha_2 \geqslant 1, \\ y^{\beta+\alpha_2-1}, & \alpha_1 = 0, \alpha_2 \geqslant 1. \end{cases} \tag{2.12}$$

The weighted Sobolev spaces $H_\beta^{k,l}(Q)$ are defined only for positive integers $k \geqslant l > 0$. For the proof of the approximation theorems we need also the weighted Sobolev spaces $H_\beta^{s,l}(Q)$ of a non-integral s which are defined as interpolation spaces by the K-method (see [5]):

$$(H_\beta^{k,l}(Q), H_\beta^{k+1,l}(Q))_{\theta,\infty} = H_\beta^{k+\theta,l}(Q), \quad 0 < \theta < 1.$$

It can be shown [9] that if $u \in B^l_{\beta,C,d}(Q)$, then for any $k \geqslant l$

$$|u|_{H_\beta^{k+\theta,l}(Q)} \leqslant Cd^{k+\theta-l}\Gamma(k+\theta-l+1) \quad 0 < \theta < 1. \tag{2.13}$$

3. Proof of the exponentially fast convergence

Theorem 3.1. *Let $u(x) \in B_\beta^1(Q)$ with $0 < \beta < 1$. Then there is a spline function $\phi(x,y) \in S^p(Q_\sigma^n)$ and constants $C, b > 0$ independent of N, but depending on σ, μ, β such that*

$$\|u(x,y) - \phi(x,y)\|_{L^2(Q)} \leqslant Ce^{-b\sqrt[4]{N}} \tag{3.1}$$

with $p = (p_1, \ldots, p_n)$, $p_k = [\mu(k-1)]$ for $\mu > 0$ and $N = \dim S^p(Q_\sigma^n)$. Moreover $\phi(x,y)$ can be chosen to be the L^2-projection of u on $S^p(Q_\sigma^n)$ (Fig. 2).

Proof. Let $\phi(x,y)$ be the L^2-projection of $u(x,y)$ on $S^p(Q_\sigma^n)$. Then the restriction of ϕ to the corner element R_{11}, $\phi|_{R_{11}}$, is a constant polynomial. Therefore we can apply Lemma A.3 of the Appendix to the corner element of Q and we get

$$\|u - \phi\|^2_{L^2(R_{11})} \leqslant Ch_1^{2(1-\beta)}|u|^2_{H_\beta^{1,1}(Q)} \tag{3.2}$$

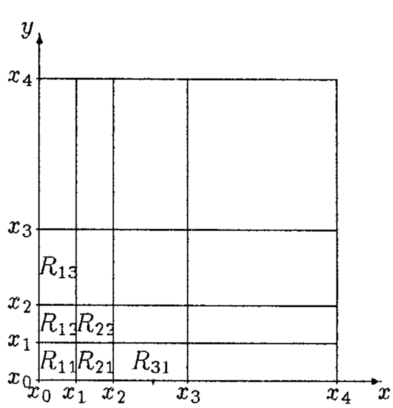

Fig. 2. Geometric mesh with $\sigma = 0.5$ on the reference element.

with $h_1 = \sigma^{n-1}$, Due to Theorem A.1 we get for the elements R_{k1} ($k \geqslant 2$) at the x-edge of Q

$$\|u-\phi\|^2_{L^2(R_{k1})} \leqslant C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} + C x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+1,1}_\beta(Q)} \tag{3.3}$$

with $\lambda = (1/\sigma) - 1 > 0$ due to (2.4). Analogously we get by Theorem A.1 for the elements R_{1l} ($l \geqslant 2$) at the y-edge of Q

$$\|u-\phi\|^2_{L^2(R_{1l})} \leqslant C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} + C x_{l-1}^{2(1-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 3)} \left(\frac{\lambda}{2}\right)^{2(s_l+1)} |u|^2_{H^{s_l+1,1}_\beta(Q)}. \tag{3.4}$$

Due to Theorem A.2 (analogous to [8, Theorem 4.1]) we obtain for the inner elements R_{kl} ($2 \leqslant k, l \leqslant n$):

$$\|u-\phi\|^2_{L^2(R_{kl})} \leqslant C \left\{ x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+1,1}_\beta(Q)} + x_{l-1}^{2(1-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 3)} \left(\frac{\lambda}{2}\right)^{2(s_l+1)} |u|^2_{H^{s_l+1,1}_\beta(Q)} \right\}. \tag{3.5}$$

Combining (3.2)–(3.5) we have

$$\begin{aligned}
&\|u-\phi\|^2_{L^2(Q)} \\
&= \|u-\phi\|^2_{L^2(R_{11})} + \sum_{k=2}^n \|u-\phi\|^2_{L^2(R_{k1})} + \sum_{l=2}^n \|u-\phi\|^2_{L^2(R_{11})} \\
&\quad + \sum_{k,l=2}^n \|u-\phi\|^2_{L^2(R_{kl})} \leqslant C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} \\
&\quad + \sum_{k=2}^n \left(C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} + C x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+1,1}_\beta(Q)} \right) \\
&\quad + \sum_{l=2}^n \left(C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} + C x_{l-1}^{2(1-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 3)} \left(\frac{\lambda}{2}\right)^{2(s_l+1)} |u|^2_{H^{s_l+1,1}_\beta(Q)} \right) \\
&\quad + \sum_{k,l=2}^n C \left\{ x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+1,1}_\beta(Q)} \right. \\
&\qquad \left. + x_{l-1}^{2(1-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 3)} \left(\frac{\lambda}{2}\right)^{2(s_l+1)} |u|^2_{H^{s_l+1,1}_\beta(Q)} \right\} \\
&= (2n-1) C h_1^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} \\
&\quad + 2nC \sum_{k=2}^n x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+1,1}_\beta(Q)}.
\end{aligned}$$

Due to $u \in B^1_\beta(Q)$ we have $u \in H^{s_k+1,1}_\beta(Q)$ with $|u|_{H^{s_k+1,1}_\beta(Q)} \leqslant C d^{s_k}\Gamma(s_k+1)$ (see (2.13)). Therefore we get

$$\|u-\phi\|^2_{L^2(Q)} \leqslant (2n-1)Ch_1^{2(1-\beta)}|u|^2_{H^{1,1}_\beta(Q)} + 2nC\sum_{k=2}^{n} x_{k-1}^{2(1-\beta)}\frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+3)}\left(\frac{\lambda}{2}\right)^{2(s_k+1)}(d^{s_k}\Gamma(s_k+1))^2. \tag{3.6}$$

With

$$F(\alpha,d) = \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}}\left(\frac{\alpha d}{2}\right)^{2\alpha} \tag{3.7}$$

we have for $s_k = \alpha p_k$ (see [8, 9, 12]):

$$\frac{\Gamma(p_k-s_k+1)}{\Gamma(p_k+s_k+3)}\left(\frac{\lambda}{2}\right)^{2(s_k+1)}(d^{s_k}\Gamma(s_k+1))^2 \leqslant CF\left(\alpha,\frac{\lambda d}{2}\right)^{p_k}(p_k+1)^{-1}. \tag{3.8}$$

Thus we obtain with $x_k = \sigma^{n-k}$ $(k = 1, \ldots, n)$

$$\begin{aligned}\|u-\phi\|^2_{L^2(Q)} &\leqslant (2n-1)Ch_1^{2(1-\beta)}|u|^2_{H^{1,1}_\beta(Q)} + 2nC\sum_{k=2}^{n} F\left(\alpha,\frac{\lambda d}{2}\right)^{p_k}(p_k+1)^{-1}x_{k-1}^{2(1-\beta)}\\ &\leqslant (2n-1)C\sigma^{2(1-\beta)(n-1)} + 2nC\sum_{k=2}^{n} F\left(\alpha,\frac{\lambda d}{2}\right)^{p_k}(p_k+1)^{-1}\sigma^{2(1-\beta)(n-k+1)}\\ &\leqslant 2nC\sigma^{2(1-\beta)(n-1)}\left(1+\sum_{k=2}^{n} F\left(\alpha,\frac{\lambda d}{2}\right)^{p_k}(p_k+1)^{-1}\sigma^{2(1-\beta)(-k+2)}\right).\end{aligned} \tag{3.9}$$

From [8, 9, 12] we know that the series in (3.9) is bound for a parameter α depending on σ, μ. Therefore we have

$$\|u-\phi\|^2_{L^2(Q)} \leqslant Cn\sigma^{2(1-\beta)(n-1)} = Ce^{\log n}e^{2(1-\beta)(n-1)\log\sigma} \leqslant Ce^{-2bn}$$

for $n \geqslant n_0$ with a fixed integer n_0 and with $b = -((\log n_0)/n_0 + (1-\beta)\log\sigma) > 0$. For the number of degrees of freedom $N = \dim S^p(Q^n_\sigma)$ we have

$$N = \left(\sum_{i=0}^{n-1}(1+[\mu i])\right)^2 \leqslant \left(n+\mu\sum_{i=0}^{n-1} i\right)^2 = \left(n+\mu\frac{n(n-1)}{2}\right)^2 \leqslant Cn^4. \tag{3.10}$$

Finally we get

$$\|u-\phi\|_{L^2(Q)} \leqslant Ce^{-b\sqrt[4]{N}} \qquad \square \tag{3.11}$$

Now we want to show that singularities (which typically appear in solutions of boundary integral equations on polyhedral domains Ω [13–15]) are contained in the countably normed spaces considered here. The arising singularities have the forms (3.14), (3.18) near the corners of the faces of Ω. Therefore we need countably normed spaces which describe singularities in the angular variable Θ.

Definition 3.1 (Countably normed spaces $B^l_\beta([0,\omega])$). *The countably normed spaces $B^l_\beta([0,\omega])$ for $l \geqslant 1$ are defined by*

$$B^l_{\beta,C,d}([0,\omega]) = \{u : u \in H^{l-1}([0,\omega]), \|(\Theta(\omega-\Theta))^{\beta+\alpha-l}\partial^\alpha_\Theta u\|_{L^2([0,\omega])} \leqslant C d^{\alpha-l}(\alpha-l)! \text{ for } \alpha = l, l+1, \dots, \text{ with } C \geqslant 1,\ d \geqslant 1 \text{ independent of } \alpha\}. \tag{3.12}$$

Lemma 3.1 (Corner-edge singularities). *Let $0 < \beta < 1$. Let $(a_k)_k$ be a sequence of real numbers. Let $(\lambda_k)_k$ be a sequence of real numbers with $\lambda_{k+1} - \lambda_k \geqslant 0$ for all $k \geqslant 1$ and $\lambda_1 > l - \beta - 1$. Let $w_k \in B^l_{\beta,C_k,d_k}([0,\omega])$ with $d_k \leqslant \bar{d} < \infty$ for all $k \geqslant 1$ and $\|w_k\|_{H^{l-1}([0,\omega])} \leqslant C_k$. Let*

$$\limsup_{k\to\infty} \frac{C_{k+1}|a_{k+1}|}{C_k|a_k|} 2^{|\lambda_{k+1}|-|\lambda_k|} R^{\lambda_{k+1}-\lambda_k} \leqslant q < 1. \tag{3.13}$$

Then we have

$$\sum_{k=1}^\infty a_k r^{\lambda_k} w_k(\Theta) \in \mathscr{B}^l_{\beta,C,d}(S^R_\omega) \tag{3.14}$$

with the sector $S^R_\omega = \{(r,\Theta) | 0 < r < R,\ 0 < \Theta < \omega\}$.

Note: $\mathscr{B}^l_{\beta,C,d}(S^R_\omega)$ is the restriction of $\mathscr{B}^l_\beta(F)$ (see (2.1)) to the sector S^R_ω on the face F.

Proof. For $\alpha_\Theta \geqslant l$ we have with

$$|\partial^{\alpha_r}_r r^\lambda| = \left|\alpha_r! \binom{\lambda}{\alpha_r} r^{\lambda-\alpha_r}\right| \leqslant \alpha_r!\, 2^{\alpha_r+|\lambda|} r^{\lambda-\alpha_r}$$

$$\begin{aligned}
&\|r^{\beta+\alpha_r-l}(\Theta(\omega-\Theta))^{\beta+\alpha_\Theta-l}\partial^{\alpha_r}_r\partial^{\alpha_\Theta}_\Theta \sum_{k=1}^\infty a_k r^{\lambda_k} w_k(\Theta)\|_{L^2(S^R_\omega)} \\
&\leqslant \sum_{k=1}^\infty |a_k|\, \|r^{\beta+\alpha_r-l}\alpha_r!\, 2^{\alpha_r+|\lambda_k|} r^{\lambda_k-\alpha_r}(\Theta(\omega-\Theta))^{\beta+\alpha_\Theta-l}\partial^{\alpha_\Theta}_\Theta w_k(\Theta)\|_{L^2(S^R_\omega)} \\
&\leqslant \alpha_r! 2^{\alpha_r} \sum_{k=1}^\infty |a_k|\, \|r^{1/2} r^{\beta+\lambda_k-l} 2^{|\lambda_k|} C_k d_k^{\alpha_\Theta-l}(\alpha_\Theta-l)!\,\|_{L^2([0,R])} \\
&\leqslant \alpha_r!(\alpha_\Theta-l)!\, 2^{\alpha_r}\bar{d}^{\alpha_\Theta-l} \sum_{k=1}^\infty |a_k| C_k 2^{|\lambda_k|}\, \|r^{\beta+\lambda_k-l+1/2}\|_{L^2([0,R])} \\
&\leqslant (\alpha_r+\alpha_\Theta-l)!\,(\max(2,\bar{d}))^{\alpha_r+\alpha_\Theta-l} \sum_{k=1}^\infty |a_k| C_k 2^{|\lambda_k|} \frac{R^{\beta+\lambda_k-l+1}}{\sqrt{(2(\beta+\lambda_k-l+1))}}
\end{aligned} \tag{3.15}$$

Then the convergence of the series follows by the quotient criterion due to (3.13). Similarly we get the analogous result for $0 \leqslant \alpha_\Theta < l$ and also

$$\sum_{k=1}^\infty a_k r^{\lambda_k} w_k(\Theta) \in H^{l-1}(S^R_\omega). \tag{3.16}$$

Hence the assertion (3.14) follows. □

Lemma 3.2 (Edge singularities). *Let $0 < \beta < 1$. Let $(a_k)_k$ be a sequence of real numbers. Let $(\lambda_k)_k$ be a sequence of real numbers with $\lambda_{k+1} - \lambda_k \geqslant 0$ for all $k \geqslant 1$ and $\lambda_1 > l - \beta - \frac{1}{2}$. Let*

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} 2^{1.5|\lambda_{k+1}|-1.5|\lambda_k|} R^{\lambda_{k+1}-\lambda_k} \leqslant q < 1. \tag{3.17}$$

Let ϱ be the distance to the edge defined by $\Theta = 0$, i.e. $\varrho = r\sin\Theta$. Then we have

$$\sum_{k=1}^{\infty} a_k \varrho^{\lambda_k} \in \mathscr{B}^l_{\beta,C,d}(S^R_\omega) \tag{3.18}$$

with $S^R_\omega = \{(r,\Theta) | 0 < r < R,\ 0 < \Theta < \omega\}$.

Proof. It can be shown [12] that

$$|\partial^\alpha_\Theta \sin^\lambda \Theta| \leqslant C\alpha! 4^\alpha 2^{0.5|\lambda|} |\sin^{\lambda-\alpha}\Theta|. \tag{3.19}$$

For $\omega \in (0,\pi)$ we have

$$\begin{aligned}
&(\Theta(\omega-\Theta))^{\beta+\alpha-l}\\
&\quad\leqslant \max_{\Theta\in[0,\omega]} \left(\frac{\Theta(\omega-\Theta)}{\sin\Theta\sin(\omega-\Theta)}\right)^{\beta+\alpha-1} (\sin\Theta\sin(\omega-\Theta))^{\beta+\alpha-l}\\
&\quad\leqslant \left(\frac{\omega}{\sin\omega}\right)^{2(\beta+\alpha-l)} (\sin\Theta\sin(\omega-\Theta))^{\beta+\alpha-l}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\|(\Theta(\omega-\Theta))^{\beta+\alpha-l}\partial^\alpha_\Theta \sin^\lambda\Theta\|_{L^2([0,\omega])}\\
&\quad\leqslant C\alpha! 4^\alpha 2^{0.5|\lambda|} \|(\Theta(\omega-\Theta))^{\beta+\alpha-l}\sin^{\lambda-\alpha}\Theta\|_{L^2([0,\omega])}\\
&\quad\leqslant C\alpha! 4^\alpha 2^{0.5|\lambda|} (\omega/\sin\omega)^{2(\beta+\alpha-l)} \|(\sin\Theta\sin(\omega-\Theta))^{\beta+\alpha-l}\sin^{\lambda-\alpha}\Theta\|_{L^2([0,\omega])}\\
&\quad\leqslant C\alpha! 4^\alpha 2^{0.5|\lambda|} (\omega/\sin\omega)^{2(\beta+\alpha-l)} \|(\sin(\omega-\Theta))^{\beta+\alpha-l}\sin^{\beta+\lambda-l}\Theta\|_{L^2([0,\omega])}\\
&\quad\leqslant C\alpha! 4^\alpha 2^{0.5|\lambda|} (\omega/\sin\omega)^{2(\beta+\alpha-l)} \|\sin^{\beta+\lambda-l}\Theta\|_{L^2([0,\omega])}.
\end{aligned}$$

The integral exists for $\lambda > l - \beta - \frac{1}{2}$. Therefore we have

$$\sin^{\lambda_k}\Theta \in B^l_{\beta,C_k,d_k}([0,\omega]) \tag{3.20}$$

with $d_k = \bar{d} = 4(\omega/\sin\omega)^2$ and $C_k = 2^{0.5|\lambda_k|}$. Using Lemma 3.1 we get the desired result (3.18) from

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{C_{k+1}|a_{k+1}|}{C_k|a_k|} 2^{|\lambda_{k+1}|-|\lambda_k|} R^{\lambda_{k+1}-\lambda_k}\\
&\quad= \limsup_{k\to\infty} \frac{2^{0.5|\lambda_{k+1}|}|a_{k+1}|}{2^{0.5|\lambda_k|}|a_k|} 2^{|\lambda_{k+1}|-|\lambda_k|} R^{\lambda_{k+1}-\lambda_k}\\
&\quad= \limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} 2^{1.5|\lambda_{k+1}|-1.5|\lambda_k|} R^{\lambda_{k+1}-\lambda_k} \leqslant q < 1. \quad \square
\end{aligned}$$

4. Numerical results

Here we present numerical results (performed on the CDC 990 and Siemens S400/40 at the RRZN, University of Hannover) for the Dirichlet problem of the Laplacian on a screen. We consider the square plate G with sidelength 2. We choose the function 1 as the right-hand side, i.e. $f = 1$. For the Dirichlet problem we look at the capacitance C of the plate G, which is given by

$$C = \frac{1}{4\pi} \int_G \Psi \, \mathrm{d}s, \tag{4.1}$$

where the charge density Ψ solves the integral equation

$$V\Psi = \frac{1}{4\pi} \int_G \frac{1}{|x-y|} \Psi(y) \, \mathrm{d}s_y = f. \tag{4.2}$$

In [1] a numerical value of 0.733576 in case of the square plate is given for the capacitance C. One obtains an approximation C_N to the capacitance C by inserting the boundary element Galerkin solution Ψ_N for (4.2) into (4.1). Because of the special right-hand side of $f = 1$ there hold the inequalities

$$c_1 \|\Psi - \Psi_N\|^2_{\tilde{H}^{-1/2}(G)} \leqslant |C - C_N| \leqslant c_2 \|\Psi - \Psi_N\|^2_{\tilde{H}^{-1/2}(G)} \tag{4.3}$$

for constants $c_1, c_2 > 0$ and therefore the error estimates

$$|C - C_N| \leqslant c_3 \begin{cases} h^{1-\varepsilon} \\ p^{-2+2\varepsilon} \end{cases} \tag{4.4}$$

for the pure h- and p-versions (see Lemma 1.6 in [1] for the h-version and [16] for the p-version).

In Table 1 we present the experimental convergence rates α_j for the error $|C - C_N|$, which underline the theoretical estimates (4.4). The α_j's are computed by

$$\alpha_j = 2 \log \frac{|C - C_{N_{j-1}}|}{|C - C_{N_j}|} \Bigg/ \log \frac{N_j}{N_{j-1}}. \tag{4.5}$$

Table 1. Convergence rates for the Dirichlet problem on the square

h-version, $p = 0$			p-version, 4 elements			hp-version, $\sigma = 0.17$		
N	$\|C - C_N\|$	α	N	$\|C - C_N\|$	α	N	$\|C - C_N\|$	α
4	0.06090		16	0.02630		16	0.013478	
9	0.04229	0.899	36	0.01370	1.608	64	0.002714	2.312
16	0.03362	0.797	64	0.00846	1.674	144	0.000671	3.445
25	0.02773	0.862	100	0.00573	1.746	324	0.000136	3.940
36	0.02364	0.877	144	0.00414	1.786	576	0.000036	4.581
49	0.02060	0.892	196	0.00313	1.816	1024	0.000008	5.335
64	0.01826	0.903	256	0.00245	1.838	1600	0.000003	4.279
81	0.01640	0.911	324	0.00197	1.856			
100	0.01489	0.918	400	0.00162	1.870			
121	0.01364	0.923	484	0.00135	1.881			
	Theoretically: 1			Theoretically: 2				

Due to Theorem 3.1 we expect for the hp-version with geometric mesh numerically an exponentially fast convergence. In Fig. 3 we compare the errors for the different versions on the square plate: the error curves are linear for the pure h- and p-versions and curved downward for the hp-version with geometric mesh. The convergence of the latter is indeed exponential as Fig. 4 indicates where we obtain nearly straight lines by plotting the logarithmic errors in capacitance against the fourth root of the number of unknowns.

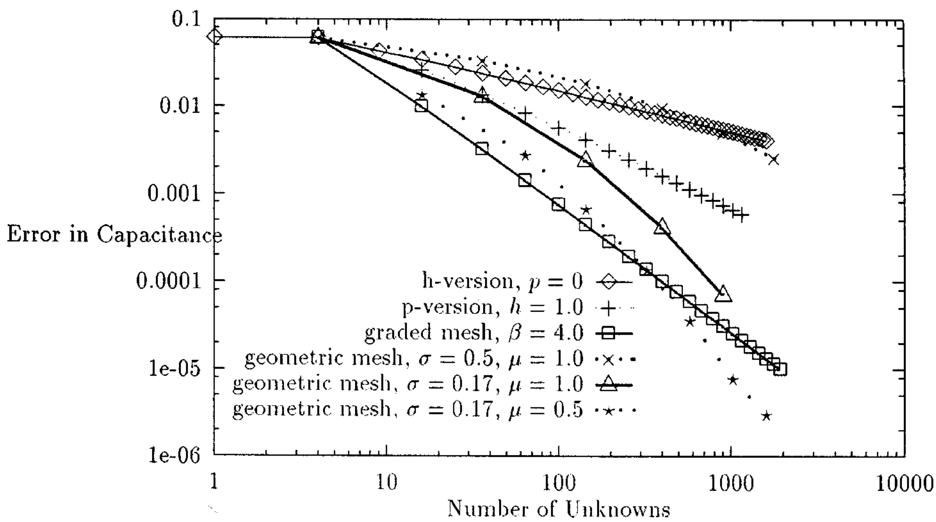

Fig. 3. Dirichlet problem on the square plate in $\mathbb{R}^3$.

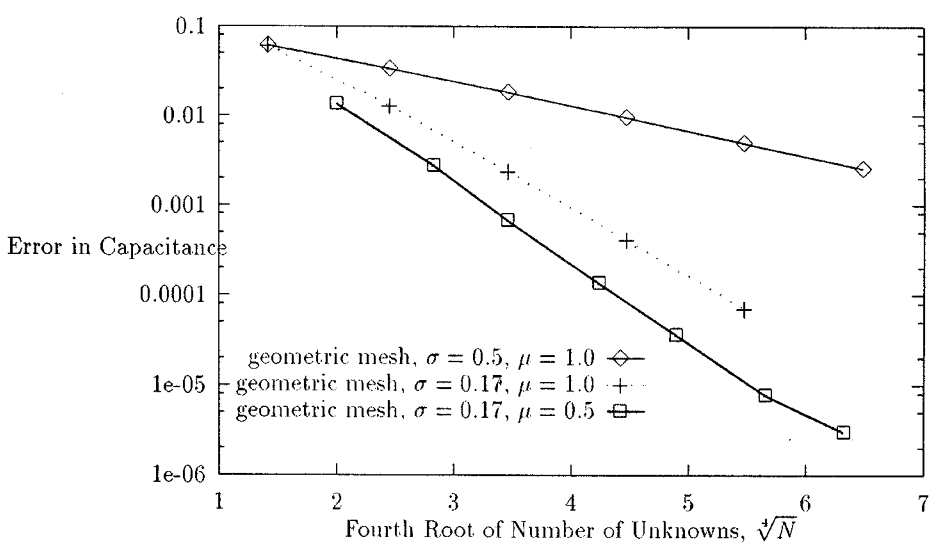

Fig. 4. Dirichlet problem on the square plate in $\mathbb{R}^3$.

Appendix: Elementary approximation results

Here we prove the approximation results needed in the proof of Theorem 3.1.

Lemma A.1 (Babuska *et al.* [4, *Lemma* 4.2]). *One has the inequality*

$$\int_0^1 t^{\alpha-2}[z(t)-a]^2\,\mathrm{d}t \leqslant C(\alpha)\int_0^1 t^{\alpha}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t, \quad \alpha \neq 1 \tag{A.1}$$

where $a = z(0)$ *if* $\alpha < 1$, $a = z(1)$ *if* $\alpha > 1$.

Lemma A.2. *Let* $u \in H^{1,1}_{\beta}(Q)$ *for* $0 < \beta < 1$, *then with*

$$\bar{u}(x) := \frac{1}{h}\int_0^h u(x,y)\,\mathrm{d}y$$

there holds

$$\|u(x,y)-\bar{u}(x)\|_{L^2([0,1]\times[0,h])} \leqslant Ch^{1-\beta}|u|_{H^{1,1}_{\beta}(Q)}. \tag{A.2}$$

Note: From (2.12) and (2.8) we have

$$|u|^2_{H^{1,1}_{\beta}(Q)} = \|x^{\beta}\partial_x u\|^2_{L^2(Q)} + \|y^{\beta}\partial_y u\|^2_{L^2(Q)}. \tag{A.3}$$

Proof. Using Cauchy–Schwarz inequality and Lemma A.1 we have for $\frac{1}{2} < \beta < 1$

$$\begin{aligned}
&\int_0^1\int_0^h (u(x,y)-\bar{u}(x))^2\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{h^2}\int_0^1\int_0^h\left(\int_0^h (u(x,y)-u(x,z))\,\mathrm{d}z\right)^2\mathrm{d}y\,\mathrm{d}x\\
&\leqslant \frac{1}{h}\int_0^1\left\{\int_0^h\int_0^y (u(x,y)-u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y + \int_0^h\int_y^h (u(x,y)-u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y\right\}\mathrm{d}x\\
&= \frac{1}{h}\int_0^1\left\{\int_0^h\int_0^y (u(x,y)-u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y + \int_0^h\int_0^z (u(x,y)-u(x,z))^2\,\mathrm{d}y\,\mathrm{d}z\right\}\mathrm{d}x\\
&= 2\frac{1}{h}\int_0^1\int_0^h\int_0^y (u(x,y)-u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant 2\frac{1}{h}\int_0^1\int_0^h y^{2-2\beta}\int_0^y z^{2\beta-2}\,(u(x,y)-u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant C\frac{1}{h}\int_0^1\int_0^h y^{2-2\beta}\int_0^y z^{2\beta}\,(\partial_z u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant C\frac{1}{h}\int_0^1\int_0^h h^{2-2\beta}\int_0^h z^{2\beta}\,(\partial_z u(x,z))^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
&= C\frac{1}{h}hh^{2-2\beta}\int_0^1\int_0^h z^{2\beta}\,(\partial_z u(x,z))^2\,\mathrm{d}z\,\mathrm{d}x\\
&\leqslant Ch^{2-2\beta}|u|^2_{H^{1,1}_{\beta}(Q)}. \quad \text{(For } 0<\beta\leqslant \tfrac{1}{2} \text{ see [12])} \quad \square
\end{aligned}$$

Lemma A.3. *Let $u \in H^{1,1}_\beta(Q)$ with $0 < \beta < 1$, then we have for the L^2 projection of $u(x,y)$ on the piecewise constant polynomials on $[0,h] \times [0,h]$*

$$\left\| u - \frac{1}{h^2} \int_0^h \int_0^h u(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^2([0,h]^2)} \leqslant C h^{1-\beta} |u|_{H^{1,1}_\beta(Q)}. \tag{A.4}$$

Proof. Let

$$\bar{u}(x) := \frac{1}{h} \int_0^h u(x,y)\,\mathrm{d}y. \tag{A.5}$$

By Lemma A.2 we have

$$\int_0^h \int_0^h (u(x,y) - \bar{u}(x))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C h^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)}. \tag{A.6}$$

Furthermore application of Lemma A.1 yields for $\frac{1}{2} < \beta < 1$

$$\begin{aligned}
&\int_0^h \int_0^h (\bar{u}(x) - \bar{u}(h))^2\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^h \int_0^h \left(\frac{1}{h} \int_0^h (u(x,z) - u(h,z))\mathrm{d}z \right)^2 \mathrm{d}y\,\mathrm{d}x \\
&\leqslant \int_0^h \int_0^h \frac{1}{h^2}\, h \int_0^h (u(x,z) - u(h,z))^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant h^{2-2\beta} \int_0^h \int_0^h x^{2\beta-2} (u(x,z) - u(h,z))^2\,\mathrm{d}z\,\mathrm{d}x \\
&\leqslant C h^{2-2\beta} \int_0^h \int_0^h x^{2\beta} (\partial_x u(x,z))^2\,\mathrm{d}z\,\mathrm{d}x \\
&\leqslant C h^{2-2\beta} |u|_{H^{1,1}_\beta(Q)}.
\end{aligned} \tag{A.7}$$

Similarly there holds

$$\begin{aligned}
&\int_0^h \int_0^h \left(\bar{u}(h) - \frac{1}{h^2} \int_0^h \int_0^h u(z,w)\mathrm{d}w\,\mathrm{d}z \right)^2 \mathrm{d}y\,\mathrm{d}x \\
&= h^2 \left(\frac{1}{h} \int_0^h u(h,y)\,\mathrm{d}y - \frac{1}{h^2} \int_0^h \int_0^h u(x,y)\,\mathrm{d}y\,\mathrm{d}x \right)^2 \\
&= h^2 \left(\frac{1}{h^2} \int_0^h \int_0^h (u(h,y) - u\,(x,y)) \right)^2 \mathrm{d}y\,\mathrm{d}x \\
&\leqslant \int_0^h \int_0^h (u(h,y) - u(x,y))^2\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant h^{2-2\beta} \int_0^h \int_0^h x^{2\beta-2} (u(h,y) - u(x,y))^2\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant C h^{2-2\beta} \int_0^h \int_0^h x^{2\beta} (\partial_x u(x,y))^2\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant C h^{2-2\beta} |u|^2_{H^{1,1}_\beta(Q)}.
\end{aligned} \tag{A.8}$$

Thus using (A.6)–(A.8) we obtain (A.4). (For $0 < \beta \leqslant \frac{1}{2}$ see [12]). $\square$

Lemma A.4 (Heuer [9, *Lemma* 2.3]). *Let $J = (a, b)$ with $h = b - a$. If $u \in H^{k+1}(J)$, $k \geqslant 0$, then there is a polynomial $\phi(x) = \sum_{0 \leqslant i \leqslant k} d_i x^i$, such that for any integers $0 \leqslant s \leqslant k$*

$$\|u - \phi\|^2_{L^2(J)} \leqslant C \frac{(k-s)!}{(k+s+2)!} \left(\frac{h}{2}\right)^{2(s+1)} \left\| \frac{\partial^{s+1} u}{\partial x^{s+1}} \right\|^2_{L^2(J)} \tag{A.9}$$

where C is independent of k. Moreover ϕ can be chosen to be the L^2-projection of u on this polynomial space.

Theorem A.1. *Let $Q_1 = (a, b) \times (0, h_2) \subseteq Q = [0, 1] \times [0, 1]$ with $h_1 = b - a \leqslant \lambda_1 a$, $\lambda_1 \geqslant 0$. If $u \in H^{k+1,1}_\beta(Q)$ with the weight function $\Phi_{\beta,\alpha,1}$ of (2.12) then there exists a polynomial $\phi(x, y) = \sum_{0 \leqslant i \leqslant k_1} d_i x^i$ on Q_1 with $0 \leqslant k_1 \leqslant k$ such that*

$$\|u - \phi\|^2_{L^2(Q_1)} \leqslant C h_2^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)} + C a^{2(1-\beta)} \frac{\Gamma(k_1 - s_1 + 1)}{\Gamma(k_1 + s_1 + 3)} \left(\frac{\lambda_1}{2}\right)^{2(s_1+1)} |u|^2_{H^{s_1+1,1}_\beta(Q)} \tag{A.10}$$

where s_1 is any real number $0 \leqslant s_1 \leqslant k_1$ and $H^{s_1+1,1}_\beta(Q)$ is the interpolation space $(H^{\tilde{k}_1+0,1}_\beta(Q), H^{\tilde{k}_1+1,1}_\beta(Q))_{\Theta_1,\infty}$ for some integer $\tilde{k}_1 = s_1 + 1 - \Theta_1 \leqslant k_1$, $0 \leqslant \Theta_1 \leqslant 1$ and C is independent of k.

Proof. By Lemma A.2 we have

$$\bar{u}_{h_2} := \frac{1}{h_2} \int_0^{h_2} u(x, y)\, \mathrm{d}y$$

$$\|u(x, y) - \bar{u}_{h_2}(x)\|^2_{L^2(Q_1)} \leqslant \|u(x, y) - \bar{u}_{h_2}(x)\|^2_{L^2([0,1]\times[0,h_2])} \leqslant C h_2^{2(1-\beta)} |u|^2_{H^{1,1}_\beta(Q)}.$$

By Lemma A.4 for $\bar{u}_{h_2}(x)$ there is a polynomial $\phi(x) = \sum_{0 \leqslant i \leqslant k_1} d_i x^i$, such that for all integers $0 \leqslant \tilde{k}_1 \leqslant k_1$

$$\|\bar{u}_{h_2} - \phi\|^2_{L^2(a,b)} \leqslant C \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2)!} \left(\frac{h_2}{2}\right)^{2(\tilde{k}_1+1)} \left\| \frac{\partial^{\tilde{k}_1+1} \bar{u}_{h_2}}{\partial x^{\tilde{k}_1+1}} \right\|^2_{L^2(a,b)}.$$

Therefore we get

$$\begin{aligned}
\|\bar{u}_{h_2} - \phi\|^2_{L^2(Q_1)} &= \int_0^{h_2} \|\bar{u}_{h_2}(x) - \phi(x)\|^2_{L^2(a,b)}\, \mathrm{d}y \\
&\leqslant C \int_0^{h_2} \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2)!} \left(\frac{h_2}{2}\right)^{2(\tilde{k}_1+1)} \left\| \frac{\partial^{\tilde{k}_1+1} \bar{u}_{h_2}}{\partial x^{\tilde{k}_1+1}} \right\|^2_{L^2(a,b)} \mathrm{d}y \\
&= C \frac{(k_1 - \tilde{k}_1)!}{(k_1 + \tilde{k}_1 + 2)!} \left(\frac{h_2}{2}\right)^{2(\tilde{k}_1+1)} \left\| \frac{\partial^{\tilde{k}_1+1} \bar{u}_{h_2}}{\partial x^{\tilde{k}_1+1}} \right\|^2_{L^2(Q_1)}.
\end{aligned}$$

Using Cauchy–Schwarz inequality we have

$$\begin{aligned}
\left\| \frac{\partial^{\tilde{k}_1+1} \bar{u}_{h_2}}{\partial x^{\tilde{k}_1+1}} \right\|^2_{L^2(Q_1)} &= \int_a^b \int_0^{h_2} \left(\frac{\partial^{\tilde{k}_1+1}}{\partial x^{\tilde{k}_1+1}} \frac{1}{h_2} \int_0^{h_2} u(x, z)\, \mathrm{d}z \right)^2 \mathrm{d}y\, \mathrm{d}x \\
&\leqslant \int_a^b \int_0^{h_2} \left(\frac{\partial^{\tilde{k}_1+1}}{\partial x^{\tilde{k}_1+1}} u(x, z) \right)^2 \mathrm{d}z\, \mathrm{d}x \\
&= \left\| \frac{\partial^{\tilde{k}_1+1} u}{\partial x^{\tilde{k}_1+1}} \right\|^2_{L^2(Q_1)}.
\end{aligned}$$

With $h_1 \leqslant \lambda_1 a \leqslant \lambda_1 x$ for $x \in (a,b)$ and $\Phi_{\beta,(\tilde{k}_1+1,0),1}(x,y) = x^{\beta+\tilde{k}_1}$ (2.12) it follows

$$\begin{aligned}
\|\bar{u}_{h_2} - \phi\|^2_{L^2(Q_1)} &\leqslant C\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{h_1}{2}\right)^{2(\tilde{k}_1+1)}\left\|\frac{\partial^{\tilde{k}_1+1}u}{\partial x^{\tilde{k}_1+1}}\right\|^2_{L^2(Q_1)} \\
&= C\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{\lambda_1}{2}\right)^{2(\tilde{k}_1+1)} a^{2(1-\beta+\beta+\tilde{k}_1)}\left\|\frac{\partial^{\tilde{k}_1+1}u}{\partial x^{\tilde{k}_1+1}}\right\|^2_{L^2(Q_1)} \\
&\leqslant Ca^{2(1-\beta)}\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{\lambda_1}{2}\right)^{2(\tilde{k}_1+1)}\left\|x^{\beta+\tilde{k}_1}\frac{\partial^{\tilde{k}_1+1}u}{\partial x^{\tilde{k}_1+1}}\right\|^2_{L^2(Q_1)} \\
&\leqslant Ca^{2(1-\beta)}\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{\lambda_1}{2}\right)^{2(\tilde{k}_1+1)}\left\|\Phi_{\beta,(\tilde{k}_1+1,0),1}(x,y)\frac{\partial^{\tilde{k}_1+1}u}{\partial x^{\tilde{k}_1+1}}\right\|^2_{L^2(Q_1)} \\
&\leqslant Ca^{2(1-\beta)}\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{\lambda_1}{2}\right)^{2(\tilde{k}_1+1)} |u|^2_{H_\beta^{\tilde{k}_1+1,1}(Q)}\,. \qquad \text{(A.11)}
\end{aligned}$$

Let $Tu = \bar{u}_{h_2} - \phi$. Then T is an operator: $H_\beta^{\tilde{k}_1+1,1}(Q) \to L^2(Q_1)$ and the norm of the operator is bounded

$$\|T\|^2_{H_\beta^{\tilde{k}_1+1,1}(Q),\,L^2(Q_1)} \leqslant Ca^{2(1-\beta)}\,\frac{(k_1-\tilde{k}_1)!}{(k_1+\tilde{k}_1+2)!}\left(\frac{\lambda_1}{2}\right)^{2(\tilde{k}_1+1)}.$$

By interpolation with the K-method [5] (see also [8, 12]) T is also a linear and continuous operator: $H_\beta^{s_1+1,1}(Q) \to L^2(Q_1)$ and we have

$$\|T\|^2_{H_\beta^{s_1+1,1}(Q),\,L^2(Q_1)} \leqslant Ca^{2(1-\beta)}\left(\frac{\lambda_1}{2}\right)^{2(s_1+1)}\frac{\Gamma(k_1-s_1+1)}{\Gamma(k_1+s_1+3)}. \qquad \text{(A.12)}$$

Therefore we finally get (A.10). $\square$

By a modification of [8, Lemma 4.3] we obtain the following result.

Lemma A.5. *Let $\Omega = (a,b)\times(c,d)$ with $h_1 = b-a$ and $h_2 = d-c$. If $u \in H^{k+1}(\Omega)$, $k = \max(k_1,k_2)$, $k_1,\ k_2 \geqslant 0$, then there is a polynomial*

$$\phi(x,y) = \sum_{\substack{0\leqslant i\leqslant k_1\\ 0\leqslant j\leqslant k_2}} d_{i,j}x^i y^j,$$

such that for any integers $0 \leqslant s_i \leqslant k_i$, $i = 1,2$

$$\begin{aligned}
\|u-\phi\|^2_{L^2(\Omega)} \leqslant 2\Bigg\{&\frac{(k_1-s_1)!}{(k_1+s_1+2)!}\left(\frac{h_1}{2}\right)^{2(s_1+1)}\left\|\frac{\partial^{s_1+1}u}{\partial x^{s_1+1}}\right\|^2_{L^2(\Omega)} \\
&+\frac{(k_2-s_2)!}{(k_2+s_2+2)!}\left(\frac{h_2}{2}\right)^{2(s_2+1)}\left\|\frac{\partial^{s_2+1}u}{\partial y^{s_2+1}}\right\|^2_{L^2(\Omega)}\Bigg\} \qquad \text{(A.13)}
\end{aligned}$$

Moreover ϕ can be chosen to be the L^2-projection of u on this polynomial space.

Theorem A.2. *Let* $Q_1 = (a_1, b_1) \times (a_2, b_2) \subset\subset Q = [0,1] \times [0,1]$ *with* $h_1 = b_1 - a_1 \leqslant \lambda_1 a_1$, $h_2 = b_2 - a_2 \leqslant \lambda_2 a_2$, $\lambda_i \geqslant 0$ $(i = 1,2)$. *If* $u \in H_\beta^{k+1,1}(\Omega)$ *with* $\Phi_{\beta,\alpha,1}(x,y)$ *of* (2.12) *then there exists a polynomial*

$$\phi(x,y) = \sum_{\substack{0 \leqslant i \leqslant k_1 \\ 0 \leqslant j \leqslant k_2}} d_{i,j} x^i y^j,$$

on Q_1 *with* $0 \leqslant k_1, k_2 \leqslant k$ *such that*

$$\|u - \phi\|^2_{L^2(Q_1)} \leqslant C \sum_{j=1}^{2} a_j^{2(1-\beta)} \frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3)} \left(\frac{\lambda_j}{2}\right)^{2(s_j+1)} |u|^2_{H_\beta^{s_j+1,1}(Q)} \tag{A.14}$$

where s_j *is any real number* $0 \leqslant s_j \leqslant k_j$ $(j = 1,2)$ *and* $H_\beta^{s_j+1,1}(Q)$ *is the interpolation space* $(H_\beta^{\tilde{k}_j+0,1}(Q), H_\beta^{\tilde{k}_j+1,1}(Q))_{\Theta_j,\infty}$ *for some integer* $\tilde{k}_j = s_j + 1 - \Theta_j \leqslant k_j$, $0 \leqslant \Theta_j \leqslant 1$ *and* C *is independent of* k.

Proof. Estimating (A.13) as in (A.11) we obtain (A.14). For details see [12].

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