Iterative Substructuring Methods: Algorithms and Theory for Elliptic Problems in the Plane

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Abstract Finite element problems can often be divided into subproblems corresponding to subregions into which the region is partitioned or from which it was originally assembled. In this paper, iterative methods are developed in which the interactions across the curves or surfaces which divide the region are handled by a conjugate gradient method. Recent progress on iterative substructuring methods for elliptic problems in two dimensions is discussed with particular attention to the case of many subregions. A connection between these methods and certain multigrid algorithms is also explored.

1. Introduction Iterative substructuring methods are domain decomposition methods with which the often very large linear systems of algebraic equations, which arise when elliptic problems are discretized by finite differences or finite elements, are solved with the aid of solvers for the same or similar equations defined on subregions. The subregions, also called substructures, do not overlap. The interaction between the substructures, to enforce various continuity requirements, is handled by an iterative method, typically of conjugate gradient type.

In this paper, we will assume that the finite dimensional subproblems are solved exactly, but we note that the use of certain inexact solvers are well understood; cf. Bramble, Pasciak and Schatz [6] , Dryja , Proskurowski and Widlund [15] and Section 2 of this paper.

The new results reported here have recently been obtained in joint work with Maksymilian Dryja, who in his report to this conference focuses on our results for three dimensional problems; see Dryja [12] . In this paper, only problems in two dimensions are discussed . We consider finite element approximations of self adjoint, linear, elliptic problems formulated in the traditional variational form as minimization problems. In developing the theory, we need an extension theorem for finite element functions on regions

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with Lipschitz boundaries, similar to well known results on Sobolev spaces; cf. Nečas [19] or Stein [21]. We know how to prove such a result for general geometric configurations and triangulations, but only for conforming finite elements; cf. Widlund [24]. Therefore, for the time being, we confine our study to finite element spaces which are subspaces of the Sobolev space appropriate for the variational form of the elliptic equation being considered.

Good iterative substructuring algorithms lead to improved methods already for uniprocessor systems and two substructures. In the context of parallel computing, we are of course primarily concerned with problems with many substructures. In our discussion, two themes emerges: When specialized to the case of two substructures the algorithms should be optimal, i.e. their rates of convergence should be bounded independently of the size of the problem. Secondly, we should also assure that, in each step of the algorithm, there is enough global transportation of information. These issues are discussed in Section 2, where we review two basic optimal algorithms for the case of two substructures; cf. Bjørstad and Widlund [3,4], Bramble, Pasciak and Schatz [6], Dryja [9,10] or Widlund [23,24]. We also consider the case of many substructures and give a lower bound on the rate of convergence of iterative substructuring methods for which, in each iteration step, information is exchanged only between neighboring substructures.

In Section 3, we collect a number of results on Sobolev and finite element spaces, which are fundamental for the proofs of our results as well as for an understanding of the algorithms and their structure. Some of these tools do not have counterparts in the continuous case. The central technical result is given in Lemma 3.2. In Section 4, we introduce a number of algorithms and establish, by using the tools of Section 3, that they are almost optimal. To simplify the presentation, we develop these methods and the theory primarily for second order problems, but as demonstrated by a discussion of a biharmonic problem, our techniques are also applicable to higher order equations.

We note that the first algorithm for a large number of substructures and theoretical results of this kind are due to Bramble, Pasciak and Schatz [7]. We present a different derivation of their algorithm. Our proofs of the almost optimality of their algorithm and others are, in our opinion, shorter and add new insight.

For a further discussion of the literature cf. Bjørstad and Widlund[4], where various algorithms for the case of two substructures are discussed in detail. The interest in iterative substructuring algorithms has grown rapidly recently. A number of papers have appeared since that paper was given its final form; cf. Widlund [25], where further information on the literature is given. A more detailed discussion of the results presented in this paper will be given in Dryja, Proskurowski and Widlund [16].

We will not present any numerical results in this paper. It deserves to be mentioned, however, that in our experience 7-10 iteration are sufficient to decrease the error by a factor 10^4 for many problems.

2. Basic algorithms and a lower bound. Let Ω be a Lipschitz region. For an important family of variational problems, we partition boundary of the region into the union of Γ_D and Γ_N on which essential (Dirichlet) and natural (Neumann) boundary conditions are imposed. A linear, self adjoint, elliptic problem of order 2ℓ is given in variational form as

$$a_\Omega(u,v) = f(v), \ \forall\ v \in V_0(\Omega) .$$

The solution u and, in the case of higher order equations, $\ell - 1$ of its normal derivatives take on given values on Γ_D. The space $V_0(\Omega)$ is the subspace of $V(\Omega)$, the Sobolev space $H^\ell(\Omega)$, or $(H^\ell(\Omega))^d$, with zero data on Γ_D. The linear functional $f(v)$ is obtained from the right hand side of the elliptic equation and the natural boundary conditions defined on Γ_N by using a Green's formula. We assume that the bilinear form $a_\Omega(u,v)$ satisfies the following standard conditions,

$$\begin{aligned} a_\Omega(u,v) &= a_\Omega(v,u) , \\ c\|u\|^2_{V_0} &\le a(u,u) , \\ |a_\Omega(u,v)| &\le C\ \|u\|_{V_0}\|v\|_{V_0} , \end{aligned} \tag{2.1}$$

where c and C are positive constants. It follows immediately that the discrete problem obtained by a Galerkin procedure using a conforming finite element space $V^h \subset V$, has a positive definite, symmetric stiffness matrix K. We also note that we can treat a number of problems with so called intermediary boundary conditions in a similar way.

Associated with the finite dimensional space V^h is a triangulation of Ω, which can be quite general, except that we insist that it is shape regular, i.e. there is a uniform bound on h_K/ρ_K. Here h_K is the diameter and ρ_K the radius of the largest inscribed circle of the element K. No essential complications arise if we allow some of the triangles to be curved as in isoparametric methods; cf. Bernardi [2] and Ciarlet [8]. We could also develop the same theory for quadrilateral elements.

To be able to discuss the case of two substructures, we partition Ω into two nonoverlapping subregions Ω_1 and Ω_2. The curve which dissects the region is $\Gamma_3 = \bar{\Omega}_1 \cap \bar{\Omega}_2$, the intersection of the closures of Ω_1 and Ω_2, and the boundaries of Ω, Ω_1 and Ω_2 are given as $\Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_3$, respectively. We always assume that Γ_3 follows element boundaries of the triangulation. We use basis functions in V^h which satisfies the standard conditions, e.g. they vanish outside the triangles with which their respective degrees of freedom are associated; cf. Ciarlet [8]. The elements of K are obtained as $k_{ij} = a_\Omega(\phi_i,\phi_j)$, where ϕ_i and ϕ_j are basis functions. This leads to zero blocks in the stiffness matrix and a linear system of equations corresponding to the Galerkin equation which is of the form,

$$Kx = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2.2}$$

The matrix K_{11} represents the couplings between pairs of degrees of freedom associated with the open set Ω_1 as well as the intersection of Γ_N and $\bar{\Omega}_1$, K_{13} the couplings between pairs with one element in this set and the other associated with Γ_3, etc. The right hand sides are obtained from $f(\phi_j)$ and the Dirichlet and Neumann data.

When considering iterative substructuring algorithms, we assume that it is acceptable to solve discrete problems on the subregions, with appropriate boundary conditions on their

boundaries. We can therefore concentrate on the case when b_1 and b_2 are zero, since we can reduce system (2.2) to such a form in a preliminary step. We call the finite element functions corresponding to such special right hand sides piecewise, discrete harmonic functions. By block Gaussian elimination, the system (2.2) then reduces to

$$Sx_3 = (K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23})x_3 = b_3 . \tag{2.3}$$

The so called Schur complement S is given by $S = S^{(1)} + S^{(2)}$, where $S^{(1)} = K_{33}^{(1)} - K_{13}^T K_{11}^{-1} K_{13}$, is the the Schur complement associated with the problem ,

$$K^{(1)}x^{(1)} = \begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_3 \end{pmatrix} . \tag{2.4}$$

Equation (2.4) is the finite element approximation of the elliptic problem restricted to Ω_1 with a natural boundary condition added on Γ_3. The elements of $K_{33}^{(1)}$ are $a_{\Omega_1}(\phi_i, \phi_j)$, where ϕ_i and ϕ_j are basis functions associated with degrees of freedom on Γ_3. The matrix K_{33} in (2.2) is thus the sum of $K_{33}^{(1)}$ and $K_{33}^{(2)}$, the contributions from Ω_1 and Ω_2 respectively. It is also clear that K can be derived easily from $K^{(1)}$ and $K^{(2)}$.

While this discussion might seem somewhat special, since it refers to a problem cut in two, matrices of a general structure as in (2.2) appear very frequently when two subproblems are merged. It is of course easy to extend the discussion to more than two subproblems. In industrial finite element work a multi-level substructuring approach is often adopted in which, at successive levels, groups of increasingly large substructures are merged; cf. Bell, Hatlestad, Hansteen and Araldsen [1].

As shown, cf. Bjørstad and Widlund [4] and Widlund [24], the condition number of S goes to infinity when the problem size increases and a preconditioner for S should therefore be found. Not surprisingly, $S^{(1)}$ serves well in a conjugate gradient algorithm which solves equation (2.3). The use of this preconditioner involves the solution of the Neumann problem (2.4), with a particular choice of c_3, followed by a Dirichlet problem on Ω_2; see discussion below for a matrix representation of these operations. As always, the rate of convergence is determined by the stationary values of a generalized Rayleigh quotient

$$x_3^T S x_3 / x_3^T S^{(1)} x_3 . \tag{2.5}$$

In particular, the method is optimal if the values of the Rayleigh quotient lie in a fixed interval. We denote the corresponding condition number by $\kappa(S(S^{(1)})^{-1})$.

It is important to note that this Rayleigh quotient is the ratio of the strain energy $x^T K x$ which, by a simple calculation, is equal to $x_3^T S x_3$ for a piecewise, discrete harmonic function represented by the vector x, and $x_3^T S^{(1)} x_3$ the strain energy attributable to the subregion Ω_1. A lower bound is thus obtained immediately but an upper bound requires an extension theorem; see Widlund [24] for a discussion and a proof of such a theorem for general, conforming finite element spaces.

Remark. The method just introduced could be called the Neumann-Dirichlet algorithm. A Dirichlet-Neumann preconditioner can also be introduced. To simplify our notations, we only

consider a second order problem. We first solve a problem in Ω_1 with a Dirichlet condition on Γ_3. The function is then continued to Ω_2 by solving a Neumann problem, while assuring that the jump in the normal derivative across Γ_3 takes on the right value. Finally the residual is computed as the difference between the function values on Γ_3. The iteration operator for this algorithm is $(S^{(2)})^{-1}S$ and there is thus a close relationship between these methods.

For a discussion of a number of non-optimal algorithms; see Bjørstad and Widlund [4]. Among these are the rather natural algorithms which use the values of u or $\partial u/\partial n$ on Γ_3 as the basic unknown in the iteration and proceed simultaneously to compute the solution in the two subregions with the Dirchlet or Neumann boundary values.

Before turning to our other basic algorithm, we write down the first preconditioner in matrix form. A vector x, corresponding to a piecewise, discrete harmonic function, is constructed by first solving equation (2.4). Then values of the degrees of freedom in Ω_2 are found by solving the equation

$$K_{22}\, x_2 = -K_{23}\, x_3 \,. \tag{2.6}$$

We can represent the operations corresponding to (2.4) and (2.6) in terms of a linear system of equations,

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \tag{2.7}$$

After a permutation, this matrix resembles a block Gauss-Seidel operator, but we note that one of its diagonal blocks differs in an important way from the one which appear in that method. The splitting we are considering here is of course based on problems defined on substructures.

The matrix in (2.7) is not symmetric, but as can be shown by block Gaussian elimination, the restriction of this operator to the subspace of vectors, which represent functions which are discrete harmonic in Ω_2, is symmetric. By introducing an additional factor, we obtain a preconditioner which, by a simple calculation, is symmetric on the entire space,

$$\hat{K} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & K_{23}^T K_{22}^{-1} & I \end{pmatrix} \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & 0 & K_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & \hat{K}_{33} \end{pmatrix} \tag{2.8}$$

where $\hat{K}_{33} = K_{33}^{(1)} + K_{23}^T K_{22}^{-1} K_{23}$. We note that the first factor in the factored form of $\hat{K}$ represents an operation which is trivial in the case when the second subvector of the right hand side vanishes.

When this method is used, an additional Dirichlet problem, on one of the subregions, has to be solved in each iteration but the advantage is that exact solvers for the subproblems no longer are required. Thus if we wish to solve a linear system of equations of the form $\tilde{K}x = b$, where

$$c\, x^T K x \leq x^T \tilde{K} x \leq C x^T K x,$$

then we can use the matrix given in (2.8) as a preconditioner. It is easy to see that

$$\kappa(\hat{K}^{-1}\tilde{K}) \leq \kappa(\hat{K}^{-1}K)\,\kappa(K^{-1}\tilde{K}) \leq \kappa(\hat{K}^{-1}K)\,C/c\ .$$

The condition number of $\hat{K}^{-1}K$ is equal to the largest eigenvalue of $S^{(1)^{-1}}S$. This can be seen by using the same block Cholesky factors to turn K and $\hat{K}$ into block diagonal matrices. The appropriate block lower factor is

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_{13}^T K_{11}^{-1} & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & K_{23}^T K_{22}^{-1} & I \end{pmatrix}$$

The resulting block diagonal matrices are

$$\begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & S^{(1)} \end{pmatrix}$$

Thus, the spectra of $\hat{K}^{-1}K$ and $S^{(1)^{-1}}S$ differ only by a highly multiple eigenvalue which is located at one. The result now follows by observing that all the eigenvalues of $S^{(1)^{-1}}S$ are larger than one. We note that this study of preconditioners and inexact solvers requires tools of linear algebra only.

Similar constructions are possible for other preconditioners. This idea was in fact first introduced in Bramble, Pasciak and Schatz [6] for the case of two subregions and the second standard preconditioner, which we will now introduce. To simplify our notations, we will confine our discussion to the case of a second order problem with Dirichlet boundary conditions on $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

A discrete harmonic function is uniquely defined by its trace, i.e. its boundary values, and it is therefore natural to construct a preconditioner of S by using an appropriate norm of a function defined on the boundary. It is well known that in the continuous case the $H^{1/2}(\partial\Omega)$ norm of the boundary values is equivalent to the $H^1(\Omega)$ norm of the function which is the harmonic extension of these boundary values. This result can be extended to the discrete case by using the trace theorem and the extension theorem previously mentioned; cf. Widlund [24]. The $H^{1/2}(\partial\Omega)$ norm can be defined in terms of the square root of the differential operator $-(d/ds)^2$. In the case at hand, the only non-trivial part of the trace is confined to Γ_3. The corresponding preconditioner, denoted by J in Bjørstad and Widlund [4], is essentially the square root of the finite element approximation of this differential operator on Γ_3, using zero Dirchlet values at the end points of Γ_3 and the restriction of the finite element space V^h to Γ_3. The related space, $H_{00}^{1/2}(\Gamma_3)$, will be further discussed in Section 3.

It deserves to be noted that in simple cases the operator J can be represented with the aid of fast Fourier transforms; cf. Bjørstad and Widlund [4] , Bramble, Pasciak and Schatz [6] or Dryja [9].

We conclude this section by introducing the case of many substructures. Let Ω_i be a substructure of Ω and let $K^{(i)}$ and $x^{(i)T} K^{(i)} x^{(i)}$ be the stiffness matrix and the strain energy corresponding to Ω_i. We denote by Γ_{ij} the curve that is the intersection of $\overline{\Omega}_i$ and $\overline{\Omega}_j$, the closure of two neighboring substructures. Without seriously decreasing the generality of our arguments, we will assume that the substructures are triangular. We thus have two levels of triangulations of Ω, in terms of the substructures and the finite element triangulation. We will refer to these as the coarse and fine triangulations. Our estimates of the rate of convergence will be given in terms of log(H/h), where H and h are the diameters of a substructure and an element, respectively. Many of our arguments will be local; one substructure and its next neighbors will be examined at a time. We can think of this logarithmic quantity as a measure of the maximum number of refinement steps required when going from any substructure to its finest refinement level typically by repeatedly cutting triangles in four; cf. Yserentant [26].

It is well known that for elliptic problems, a residual of modestly low frequency will result in an error which cannot be neglected in any part of the region, even if the residual differs from zero only in one part of the region. Therefore any iterative method in which information is exchanged only between neighboring substructures must necessarily, for certain initial errors, require a number of steps which is at least equal to the diameter of the dual graph corresponding to the partioning of the region into substructures. We recall that the dual graph corresponding to a triangulation is constructed by introducing a vertex for each triangle and an edge between any pair of nodes which represents triangles which have an edge in common. The diameter of the graph is the maximum distance between pairs of nodes, where the distance is measured by the length of the shortest path between nodes. In a case where the diameter of the region is one, this diameter is typically on the order of 1/H. It is now easy to show, by an argument of contradiction and a well known estimate of the rate of convergence of conjugate gradient algorithm, that the condition number of the iteration operator must be at least on the order of $1/H^2$. It will be shown, in Section 4, that it is easy to obtain algorithms, for which a slightly larger upper bound is valid, by modifying some of the good algorithms.

3.Sobolev spaces and other tools. In this section, we will develop the technical tools required to prove the almost optimality of the algorithms to be introduced in Section 4. Some of our tools can be found in the books by Grisvard [17] and Nečas [19], but there are also some results for finite element spaces that do not have continuous counterparts. The two books just mentioned consider Sobolev spaces and related elliptic problems on non-smooth domains, normally Lipschitz regions. For plane problems the curvilinear polygons considered in Grisvard's book is a natural family of regions. These are regions with corners with smooth curves connecting the corners. To simplify our discussion, we will assume that all substructures are triangular and shape regular. We will also consider a special class of regions with boundaries which, according to the definitions adopted in these books, are not even continuous.

In most instances, we will work with Sobolev semi-norms and norms defined on an individual substructure or part or all of its boundary. Without restricting the generality of our theory, we will principally consider the case of second order equations and Sobolev spaces

appropriate for that case. The norms contain certain scale factors which are obtained from the standard norms defined on a region with unit diameter by a simple change of variables. We have

$$|u|^2_{H^1(\Omega_i)} = \sum_{i=1}^{2} \int_{\Omega_i} |\frac{\partial u}{\partial x_i}|^2 \, dx \tag{3.1}$$

and

$$\|u\|^2_{H^1(\Omega_i)} = |u|^2_{H^1(\Omega_i)} + (1/diam(\Omega_i))^2 \, \|u\|^2_{L^2(\Omega_i)} .$$

The space $H^1(\Omega_i)$ has a well-known subspace $H^1_0(\Omega_i)$ defined as the closure of $C^\infty_0(\Omega_i)$ with respect to the $H^1(\Omega_i)$ norm. This is a true subspace of $H^1(\Omega_i)$ and it is also the maximal subspace of $H^1(\Omega_i)$ for which the extension by zero to the complement of Ω_i defines a bounded operator into $H^1(R^2)$. The space of traces of $H^1(\Omega_i)$ is $H^{1/2}(\partial\Omega_i)$ where, for an arbitrary curve Γ, the corresponding semi-norm and norm are given by

$$|u|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma |u(x(s)) - u(x(s'))|^2/|x(s) - x(s')|^2 \, ds \, ds' \tag{3.2a}$$

and

$$\|u\|^2_{H^{1/2}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + (1/d(\Gamma))\|u\|^2_{L^2(\Gamma)} . \tag{3.2b}$$

We note that we have a scale factor $d(\Gamma)$, which is the diameter of the curve Γ. The closure of $C^\infty_0(\Gamma)$ in the $H^{1/2}(\Gamma)$ norm is denoted by $H^{1/2}_0(\Gamma)$, a space which turns out to be equal to $H^{1/2}(\Gamma)$, while the maximal subspace of $H^{1/2}(\Gamma)$ for which the extension by zero, to a curve $C\Gamma$ which is the complement of Γ, defines a bounded operator into $H^{1/2}(\Gamma \cup C\Gamma)$ is known as $H^{1/2}_{00}(\Gamma)$. This space, previously briefly mentioned in Section 2, is defined in terms of a norm given by

$$\|u\|^2_{H^{1/2}_{00}(\Gamma)} = |u|^2_{H^{1/2}(\Gamma)} + \int_0^\ell \{|u(x(s))|^2/s + |u(x(s))|^2/(\ell - s)\} ds \ . \tag{3.3}$$

Here $\ell = \ell(\Gamma)$ is the length of the curve.

It is shown in Grisvard [17], that the last two terms of this norm cannot be bounded in terms of $\|u\|^2_{H^{1/2}(\Gamma)}$. Therefore $H^{1/2}_{00}(\Gamma)$ is a true subspace of $H^{1/2}(\Gamma)$. The origin of the extra terms in the norm is explained in Grisvard [17] and Nečas [19]. What is required is a straightforward estimate of the $H^{1/2}(\Gamma \cup C\Gamma)$ norm, as defined in (3.2), of a function defined on Γ and extended by zero to $C\Gamma$. We will carry out a quite similar computation in the proof of Lemma 3.2.

It is well known that the norm of the quotient space $H^\ell(\Omega_i)/P_{\ell-1}$, i.e.

$$\inf_{p \in P_{\ell-1}} \|u + p\|_{H^\ell(\Omega_i)} \tag{3.4}$$

is equivalent to the $H^\ell(\Omega_i)$ semi-norm given for $\ell = 1$ in formula (3.1). Here P_k is the space of polynomials of degree k. We note that this is a standard tool used in error analysis for finite elements; cf. Ciarlet [8].

We note, that $H_{00}^{\frac{1}{2}}(\Gamma)$ can also be defined by using the K-method of interpolation. Thus,

$$H_{00}^{\frac{1}{2}}(\Gamma) = [H_0^1(\Gamma), L_2(\Gamma)]_{\frac{1}{2}} ,$$

cf. Lions and Magenes [18, p. 66].

The following lemma plays an important role in the theory.

Lemma 3.1. Let $u_h \in V^h(\Omega_i)$ be a finite element function defined on a substructure of diameter H and let the diameter of the smallest element be h. Then there exists a constant C, independent of h and H, such that

$$\|u_h\|^2_{L^\infty(\Omega_i)} \leq C(1 + log(H/h))\|u_h\|^2_{H^{\frac{1}{2}}(\Omega_i)} .$$

This result is often given for piecewise linear finite elements only, but it holds for much more general cases too; cf. Bramble [5], Bramble, Pasciak and Schatz [7], Oganesyan and Rukhovets [20], Thomée [22] or Yserentant [26].

As we already know, from our discussion in Section 2, it is important to be able to estimate the strain energy associated with a discrete harmonic function on a substructure in terms of the strain energy of its neighbors. Consider a union of four triangles where the middle triangle Ω_i shares an edge with each of its three neighbors. Let u_h be a continuous finite element function which is discrete harmonic in Ω_i. We note that the union of the three outer triangles is not a Lipschitz region; cf. Grisvard [17] or Nečas [19] for a detailed definition of Lipschitz boundaries. The following result is essentially an extension theorem. It does not have an obvious continuous counterpart.

Lemma 3.2 The contribution to the strain energy from the triangle Ω_i of the piecewise discrete harmonic function u_h, measured by the square of the $H^1(\Omega_i)$ semi-norm, exceeds the sum of the strain energies of the three neighboring triangles by at most a factor $C(1+log(H/h))^2$.

Proof: We first observe that

$$|u_h|_{H^{\frac{1}{2}}(\Omega_i)} \leq C \ |u_h|_{H^{\frac{1}{2}}(\partial\Omega_i)} .$$

This result follows, for the individual Lipschitz region Ω_i, from the extension theorem discussed above; cf. Bjørstad and Widlund [4] or Widlund [24] for details. We now use the definition of the trace norm and find that, since $\partial\Omega_i = U\Gamma_{ij}$,

$$|u_h|^2_{H^{\frac{1}{2}}(\partial\Omega_i)} = \sum_j |u_h|^2_{H^{\frac{1}{2}}(\Gamma_{ij})} + \sum_{j\neq k} \int_{\Gamma_{ij}} \int_{\Gamma_{ik}} |u_h(x(s)) - u_h(x(s'))|^2/|x(s) - x(s')|^2 ds ds' .$$

Using the trace theorem , the first terms can be bounded by the strain energies of the neighboring triangles. We now consider one of the terms with $j \neq k$. By using some elementary inequalities , it can be estimated by

$$\int_0^{\ell_{ij}} \int_{-\ell_{ik}}^{0} |u_h(x(s)) - u_h(x(s'))|^2/(s-s')^2 ds ds' \leq$$

$$\leq 2\int_0^{\ell_{ij}}\int_{-\ell_{ik}}^{0} |u_h(x(s)) - u_h(x(0))|^2/(s-s')^2\, dsds' + 2\int_0^{\ell_{ij}}\int_{-\ell_{ik}}^{0} |u_h(x(s')) - u_h(x(0))|^2/(s-s')^2 dsds'$$

$$\leq 2\int_0^{\ell_{ij}} |u_h(x(s)) - u_h(x(0))|^2/s ds + 2\int_{-\ell_{ik}}^{0} |u_h(x(s')) - u_h(x(0))|^2/s' ds' .$$

Divide each resulting integral into two and use Lemma 3.1. Here h_{ij} is the distance to the next mesh point.

$$\int_0^{\ell_{ij}} |u_h(x(s)) - u_h(x(0))|^2/s\, ds = \int_0^{h_{ij}} |u_h(x(s)) - u_h(x(0))|^2/s\, ds$$

$$+ \int_{h_{ij}}^{\ell_{ij}} |u_h(x(s)) - u_h(x(0))|^2/s\, ds \leq$$

$$\leq C(1+\log(H/h))\|u_h(x(\cdot)) - u_h(x(0))\|^2_{L^\infty(\Omega_j)}$$

$$= C(1+\log(H/h))\|u_h(x(\cdot)) + p_0 - (u_h(x(0)) + p_0)\|^2_{L^\infty(\Omega_j)}$$

$$\leq C(1+\log(H/h))^2\|u_h + p_0\|^2_{H^1(\Omega_j)} .$$

The proof of the lemma is now completed by a simple quotient space argument.

We will also need to study certain finite element functions that vanish at the vertices of the substructures.

Lemma 3.3. Let $w_h = u_h - I_H u_h$, where $I_H u_h$ is the piecewise linear interpolant of u_h on the coarse triangulation. Then the following estimates hold

$$\sum_j \|w_h\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C(1+\log(H/h))^2 |u_h|^2_{H^1(\Omega_j)}$$

and

$$|w_h|^2_{H^1(\Omega_j)} \leq C\sum_j \|w_h\|^2_{H^{1/2}_{00}(\Gamma_{ij})}$$

Proof: Since w_h does not change if a constant is added to u_h, the left hand side of the first inequality defines a semi norm. The proof of the inequality involves virtually the same computation as that of Lemma 3.2 and no details are therefore given. We next use the extension theorem, as in the proof of Lemma 3.2. This is possible because w_h is a discrete, harmonic function. We write w_h as the sum of three discrete, harmonic functions each with nontrivial boundary values on only one edge Γij. Since an element of $H^{1/2}_{00}$ can be extended by zero, the proof is concluded by using the triangle inequality.

We have mostly worked with semi-norms in this section. Results such as Lemma 3.2 can be extended to the case when the energy is measured in terms of $|u_h|^2_{H^1(\Omega_j)} + \|u_h\|^2_{L^2(\Omega_j)}$, by proving a variant for the scaled norms introduced in (3.2) and then taking a convex combination of the resulting inequality and that of Lemma 3.2.

4. Some almost optimal algorithms for problems with many substructures. In this section we will develop a number of algorithms, and prove the following result.

Main Theorem The condition number of every algorithm developed in this section is bounded by $C(1+log(H/h))^q$, where $q=2$ or 3.

We begin by considering a direct extension of the Neumann- Dirichlet algorithm introduce in Section 2. We assume that we can color the substructures by two colors, red and black. This means that no pair of elements which have a common edge have the same color. Such a coloring is of course not always possible.

Denote the union of the red and black substructures by $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively. We now apply the Neumann-Dirichlet algorithm introduced in Section 2 to these sets. When we solve a Neumann problem on $\Omega^{(1)}$, we determine values of the degrees of freedom at the edges of all the substructures including the vertices of the substructures. It is therefore easy to see that information will be transported globally in this step , which involves a region from which roughly every second substructure has been eliminated. Once this Neumann problem has been solved, Dirichlet problems are solved on all the black substructures using the traces of the solution of the Neumann problem as boundary data. These Dirichlet problems are independent of each other and they can be solved simultaneously by using different processors, if they are available.

As in the simple case of two substructures, discussed in Section 2, the rate of convergence is determined by,

$$x_3^T S x_3 / x_3^T S^{(1)} x_3 \ .$$

The proof of the almost optimality follows by applying Lemma 3.2 , estimating the strain energy of each red substructure in terms of its black neighbors, and by adding the results.

There are several economical ways of implementing this algorithm. We first assume that the Neumann problem on each substructure is positive definite. The linear system corresponding to the Neumann problem on $\Omega^{(1)}$ has about half as many variables as the original problem on Ω . However, because of the very special form of $\Omega^{(1)}$, we can decouple the local problems in an inexpensive preprocessing step. We rewrite the Neumann problem on $\Omega^{(1)}$ as,

$$inf \sum_{red\ \Omega_i} \{a_{\Omega_i}(u_h,u_h) - 2\int_{\Omega_i} f u_h dx - 2\int_{\partial\Omega_i} g_N u_h\, ds\} \ ,$$

subject to u_h being continuous at all vertices of the coarse mesh.

We account for the constraints by introducing one Lagrange multiplier for each unknown at the vertices of the substructures. When we write out the resulting linear equations to determine the values of the nodal values of the finite element function and the Lagrange multipliers, we obtain an indefinite symmetric system. The coefficient matrix has a large principal minor which is the direct sum of matrices which correspond to finite element approximations of Neumann problems on the individual red substructures. Each column corresponding to a Lagrange multiplier has only two non-zeros. We can therefore proceed to compute the Schur complement corresponding to the Lagrange multipliers. This Schur

complement is a sparse matrix, corresponding to a seven point stencil in a standard case, and can be factored into its Cholesky factors at an acceptable expense unless the number of substructures is very large. Once this work has been carried out, an iteration step involves the use of these factors in a back solving step. In addition, we also need to solve a problem on each substructure. Details, including operation counts, will be given in Dryja, Proskurowski and Widlund [16].

As an alternative, we can solve the large linear system associated with $\Omega^{(1)}$ by block Gaussian elimination. It is easy to see that we get a quite similar algorithm if we order the unknowns corresponding to the vertices of the substructures last. As in the other method, we have a very large principal minor which is the direct sum of matrices corresponding to individual red substructures but with Dirichlet conditions at the vertices of the substructures. When using this method, we need only assume that $a_{\Omega^{(1)}}(u_h,u_h)$ is positive definite.

We next consider a family of methods which includes the one developed by Bramble, Pasciak and Schatz [7] and a special multigrid method due to Yserentant [26]. Inspired by Yserentant, we partition the finite element functions into low and high frequency components.

$$u_h = I_H u_h + (u_h - I_H u_h) . \tag{4.1}$$

Here I_H is the interpolation operator associated with piecewise linear continuous elements on the coarse mesh defined by the substructures. Since the second term in (4.1) vanishes at all vertices of the coarse triangulation, it represents higher frequencies; the pitch of a drum increases substantially if held fixed at a number of points. The underlying finite element problem can be defined in terms of a more complicated finite element space.

Remark. By choosing the simplest coarse mesh interpolant, the resulting algorithm will be as simple as possible. By examining our proof of Lemma 3.3, it becomes clear how to extend to a more general I_H .

We assume that u_h is a piecewise, discrete harmonic function. It is easy to show that $I_H u_h$ and $u_h - I_H u_h$ belong to the same class. A preliminary preconditioner is introduced by the bilinear form

$$a_\Omega(I_H u_h, I_H v_h) + a_\Omega (u_h - I_H u_h, v_h - I_H v_h) . \tag{4.2}$$

The resulting linear system of equations is block diagonal if we work with the standard basis functions for the piecewise linear functions on the coarse mesh and a complementary space built by the standard basis functions of V^h. We note that for the basis functions of the first kind the second term in (4.2) vanishes while the first vanishes for those of the second kind. The first block corresponds to the original problem solved on the coarse mesh. If this system is solved directly, then it provides the necessary quite modest amount of global transportation of information. The second term should be changed since, by itself, it gives rise to a linear system which is almost as difficult as the original problem.

As we have noted the first term in (4.2) assures us of global transportation of information. If we would replace the matrix representing the first term in (4.2) by a diagonal matrix, destroying this feature, the condition number of the resulting iteration matrix would

increase by a factor $1/H^2$. Technically, the use of a method where the interaction between the substructures is local only would make it impossible to remove terms of the form $1/H^2\|u\|_{L^2(\Omega_j)}$ in our estimates.

Before we replace the second term of (4.2), it is important to note that for $u_h = v_h$ the expression in (4.2) is bounded from below;

$$\tfrac{1}{2}a_\Omega(u_h,u_h) \leq a_\Omega(I_H u_h, I_H u_h) + a_\Omega(u_h - I_H u_h, u_h - I_H u_h) . \tag{4.3}$$

This follows from the triangle inequality. An upper bound for the first term is given by

$$|I_H u_h|^2_{H^1(\Omega_j)} \leq C(1+\log(H/h))\ |u_h|^2_{H^1(\Omega_j)} , \tag{4.4}$$

which is obtained by using Lemma 3.1, an elementary calculation and the quotient space argument of the previous section. By using the triangle inequality, the second term of the right hand side of (4.3) is bounded similarly. We can therefore conclude that the method based on the preconditioner defined by (4.2) has a condition number on the order of (1+ log(H/h)).

We will now explore several different ways of replacing the second term in the quadratic form (4.3). To arrive at the algorithm of Bramble, Pasciak and Schatz [7], we replace this second term by terms related to the trace of the function, resulting in a preconditioner defined by,

$$a_\Omega(I_H u_h, I_H v_h) + \sum (u_h - I_H u_h, v_h - I_H v_h)_{H_{00}^{1/2}(\Gamma_{ij})} \tag{4.5}$$

A proof that the condition number of the resulting algorithms is bounded by $C(1 + \log(H/h))^2$ follows directly from (4.3), (4.4) and Lemma 3.3.

In order to derive Yserentant's algorithm , we recursively split the second term of (4.1),

$$u_h = I_0 u_h + (I_1 u_h - I_0 u_h) + \cdots + (I_j u_h - I_{j-1} u_h) = I_0 u_h + u_{1,h} + \cdots + u_{j,h} .$$

where

$$I_0 u_h = I_H u_h$$

$$I_1 u_h = I_{H/2} u_h$$

$$\cdots$$

and finally

$$I_j u_h = u_h .$$

The preconditioner is now obtained by replacing the second term in (4.2) by

$$\sum_{k=1}^{j} \sum_{x \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} u_{k,h}(x)\ v_{k,h}(x)$$

Here $\mathcal{N}_k$ is the set of mesh points obtained after k successive refinements of the original mesh defined by the substructures. Thus $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ is the set of mesh points introduced in the kth step of the refinement. For a discussion of the triangulations for which this argument applies

and a proof that this algorithm has a condition number on the order of $C(1+log(H/h))^2$, see Yserentant [26].

A hybrid algorithm can also be derived. Combining the ideas of the two previous algorithms, we obtain a preconditioner of the following form,

$$(I_0 u_h, I_0 v_h)_{H^{1/2}(\Omega)} + \sum_{k=1}^{\ell} \sum_{x \in N_k \setminus N_{k-1}} u_{k,h}(x) \cdot v_{k,h}(x) \tag{4.6}$$

$$+ \sum (u_h - I_\ell u_h, v_h - I_\ell v_h)_{H_{00}^{1/2}(\Gamma_{ij}^\ell)} .$$

Here the curves Γ_{ij}^ℓ correspond to a division of Ω into finer substructures. It presents no further difficulties to establish that the condition number of this algorithm is on the order of $(1+log(H/h))^2$. This algorithm might have certain advantages over the two from which it originates. It might thus be preferable to decrease the size of the substructures, if the resulting linear systems of equations are solved by a method with a work estimate that grows faster than linear in the number of unknowns, especially if this can be done without increasing the cost of the coarse mesh computation. Compared with multi-grid algorithms, domain decomposition algorithms might also offer advantages in that the granularity of the computational tasks is coarser.

It is also natural to replace the second term in (4.2) by the quadratic form of the Neumann-Dirichlet algorithm. Since by Lemma 3.2

$$a_{\Omega^{(1)}}(w_h, w_h) \leq a_\Omega(w_h, w_h) \leq C(1 + log(H/h))^2 a_{\Omega^{(1)}}(w_h, w_h),$$

it is appropriate to introduce a scale factor. The resulting preconditioner is

$$a_\Omega(I_H u_h, I_H v_h) + (1 + \log(H/h))\, a_{\Omega^{(1)}}(u_h - I_H u_h, v_h - I_H v_h) \tag{4.7}$$

It is easy to show, by using (4.3), (4.4) and the estimate just given, that the iteration operator has a condition number which is bounded by $C(1+log(H/h))^3$. In spite of the fact that it has a worse bound, this algorithm is quite interesting since it does not require the use of the J operator introduced in Section 2 and used in the algorithm introduced by Bramble, Pasciak and Schatz [7]. Compared with the Neumann-Dirichlet algorithm discussed earlier in this section it has the advantage that the problems on the substructures, represented by the second term of (4.7) are uncoupled. This follows from the fact that $u_h - I_H u_h$ vanishes at the vertices of the coarse mesh. Consideration such as these are even more important in the three dimensional case, where edges with a substantial number of points correspond to the vertices of a plane problem . We note that the three dimensional case differs considerably from the two dimensional problems considered here. Thus Lemma 3.1 is far from true in that case; cf. the discussion in Dryja [12].

We conclude this paper by outlining how these algorithms can be extended to the biharmonic Dirichlet problem. We first note that, in this case, the Dirichlet conditions essentially amount to providing values of the components of the gradient ∇u_h on the boundary. The variational formulation is given in the Sobolev space $H^2(\Omega)$ and by using the trace theorem $\nabla u_h \in (H^{1/2}(\partial\Omega))^2$. For biharmonic functions, we have the following inequality,

$$|u|_{H^2(\Omega)} \leq C|\nabla u|_{(H^{\frac{1}{2}}(\partial\Omega))^2}.$$

A similar bound follows, for any conforming finite element method, from the extension theorem given in Widlund [24]. The extension of the Neumann-Dirichlet algorithm and a proof of its almost optimality is immediate. Here we impose the continuity of u_h and its gradient at the vertices of the red substructures.

When turning to the other algorithms, we must understand what properties are required of the coarse mesh interpolant $I_H u_h$. We find that $\nabla I_H u_h$ should be bounded by ∇u_h in the $L^\infty(\Omega_i)$ norm, and that $I_H(ax + by + c) = ax + by + c$ for all values of a, b and c. We can then estimate the H^2 norm of $I_H u_h$ by using Lemma 3.1. An example of a method which satisfies these requirements is given by Bell's triangle, which has 18 degrees of freedom ; cf. Ciarlet [8], and the following construction. Compute $I_H u_h$ by using the same finite element method on the coarse mesh, defined by the substructures. The values of u_h and ∇u_h are interpolated while we set all the second derivatives of $I_H u_h$ equal to zero. Certain bounds for the nodal basis functions have to be established in order to prove the L^∞ estimate. This is done in a computation resembling that in the study of Argyris' triangle given in Ciarlet [8]. The proof otherwise proceeds just as in the second order case, using a variant of Lemma 3.3 etc..

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