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COLLOCATION WITH CHEBYSHEV POLYNOMIALS FOR SYMM'S INTEGRAL EQUATION ON AN INTERVAL

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Abstract

A collocation method for Symm's integral equation on an interval (a first-kind integral equation with logarithmic kernel), in which the basis functions are Chebyshev polynomials multiplied by an appropriate singular function and the collocation points are Chebyshev points, is analysed. The novel feature lies in the analysis, which introduces Sobolev norms that respect the singularity structure of the exact solution at the ends of the interval. The rate of convergence is shown to be faster than any negative power of n, the degree of the polynomial space, if the driving term is smooth.

1. Introduction

Symm's integral equation [14] on an interval

$$-\frac{1}{\pi}\int_a^b \log|x-y|\,v(y)\,dy = g(x), \qquad x\in[a,b] \tag{1.1}$$

for $b-a\neq 4$ and g suitably smooth, has a unique solution with endpoint singularities of the form $(x-a)^{-1/2}(b-x)^{-1/2}$ (see [6]).

The collocation method for Symm's equation to be considered here, based on Chebyshev polynomials, is probably the easiest method of obtaining a numerical solution. It correctly represents the endpoint singularities of the exact solution, and yields faster-than-polynomial convergence if g is smooth.

We do not claim that the method is new. The new element lies in the

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analysis: in the present work we establish rates of convergence in suitable Sobolev spaces, by means of an analysis similar to that used by Saranen, Arnold and Wendland [9, 1] for spline collocation on smooth *closed* curves. One point of interest lies in the definition of the Sobolev norms, here defined in such a way as to respect the singularity structure of the exact solution.

Recently Costabel, Ervin and Stephan [4] proved in weighted Sobolev spaces the convergence of the collocation method for Symm's integral equation for open curves with piecewise linear trial functions which are constant near the endpoints.

The Galerkin method with piecewise polynomial test and trial functions for Symm's integral equation on an interval has been analysed by Stephan and Wendland in [13]. There higher convergence rates, compared with the traditional Galerkin method, have been obtained by augmenting the test and trial spaces by special singular elements which imitate the behaviour of the exact solution at the endpoints of the interval. The effect of graded meshes on the convergence rate of the standard Galerkin error has been analysed by several authors, including Bourlard, Nicaise, and Paquet [3], von Petersdorff [7], Yan and Sloan [15]. All the above mentioned papers deal with the h-version of the Galerkin method where the degree p of the elements is fixed, usually at a low value, typically $p = 0, 1, 2$, and the accuracy is achieved by refining the mesh. The p-version, which fixes the mesh and achieves the accuracy by increasing the degree p of the elements, has been analysed by Stephan and Suri in [12] for Symm's integral equation on an open curve. There they show that the p-version of the Galerkin boundary element method has twice the rate of convergence of the usual h-version with uniform mesh. Meanwhile, convergence results have also been derived for the $h-p$ version of the boundary element method, which is a combination of the standard h-version and the p-version (see Stephan [11], and Guo, von Petersdorff, Stephan [5]). If a geometric mesh refinement towards the endpoints of the interval is used together with suitably chosen piecewise polynomial test and trial functions, then the convergence of the Galerkin error of the h-version is exponential (see [11]). For numerical experiments, see [5].

A method similar to the present method is obtained as a special case of one recently proposed by Atkinson and Sloan [2] for Symm's integral equation on a smooth open curve. The latter is a fully discrete method based on first making the variable transformation of Yan and Sloan [16] (see below), and then applying a discrete Galerkin method. If the curve becomes straight then the method is close to the present method. The analysis, however, is quite different.

Initially we consider the special case $a = -1$, $b = +1$, deferring to the last section the almost trivial modifications that are needed for general intervals.

Thus the equation we consider is

$$Vv(x) := -\frac{1}{\pi}\int_{-1}^{1} \log|x-y|v(y)\,dy = g(x), \qquad x \in [-1, 1]. \tag{1.2}$$

The method and the analysis that follows are based on the following special property of the operator V (see [8]):

$$V(\omega T_j) = \begin{cases} \log 2 & \text{if } j = 0, \\ \frac{1}{j}T_j & \text{if } j \geq 1, \end{cases} \tag{1.3}$$

where

$$\omega(x) = (1-x^2)^{-1/2}, \qquad x \in (-1, 1), \tag{1.4}$$

and T_j is the Chebyshev polynomial of the first kind of degree j, defined by

$$T_j(\cos\theta) = \cos(j\theta), \qquad j \geq 0. \tag{1.5}$$

One way of presenting the material of this paper would be to make the explicit change of variable

$$x = \cos\theta \tag{1.6}$$

in (1.2), as in [16], and to replace the Cheybshev polynomials throughout by cosine polynomials. We choose to work in terms of the original variable x, but the alternative view will often emerge.

2. The collocation method

In the light of the property (1.3), it is natural to approximate the solution v of (1.2) by ω times a polynomial of degree $\leq n-1$. As collocation points we use

$$x_j = x_j^{(n)} = \cos\frac{2j-1}{2n}\pi, \qquad j = 1, \dots, n, \tag{2.1}$$

the n zeros of $T_n(x)$. Thus the method, in principle, is: find $v_h \in \omega\mathbb{P}_{n-1}$ such that

$$Vv_h(x_j) = g(x_j), \qquad j = 1, \dots, n. \tag{2.2}$$

Here $\mathbb{P}_{n-1}$ denotes the space of polynomials of degree $\leq n-1$ restricted to $[-1, 1]$. In practice one writes

$$v_h = \omega\left(\frac{1}{2}a_0 + \sum_{k=1}^{n-1} a_k T_k\right), \tag{2.3}$$

so that, with the aid of (1.3),

$$Vv_h(x) = \frac{\log 2}{2}a_0 + \sum_{k=1}^{n-1}\frac{a_k}{k}T_k(x). \tag{2.4}$$

Then (2.2) represents a set of linear equations for $a_0, a_1, \ldots, a_{n-1}$, with easily computed matrix elements.

In fact, however, one may even obtain explicit expressions for $a_0, \ldots, a_{n-1}$ by exploiting the discrete orthogonality property of the Chebyshev polynomials: define

$$\langle u, w\rangle = \int_{-1}^{1} u(x)w(x)\omega(x)\,dx, \tag{2.5}$$

an inner product incorporating the weight ω, and

$$\langle u, w\rangle_n = \frac{\pi}{n}\sum_{k=1}^{n} u(x_k)w(x_k), \tag{2.6}$$

a corresponding discrete inner product obtained by using the Gauss-Chebyshev quadrature rule, with x_k as in (2.1). Then because the n-point Gauss-Chebyshev quadrature rule is exact for all polynomials of degree $\leq 2n-1$, we have, for $j+k \leq 2n-1$ (and therefore in particular for $j, k = 0, 1, \ldots, n-1$), the discrete orthogonality property

$$\langle T_k, T_j\rangle_n = \langle T_k, T_j\rangle = \delta_{kj}\cdot\begin{cases} \pi & \text{if } k=j=0, \\ \frac{\pi}{2} & \text{if } k=j\neq 0, \end{cases} \tag{2.7}$$

with $\delta_{kj}=0$ for $k \neq j$ and $\delta_{kj}=1$ for $k=j$. It then follows from (2.2), (2.4) and (2.7) that the coefficients in (2.3) are given explicitly by

$$a_0 = \frac{2}{\pi\log 2}\langle g, 1\rangle_n, \tag{2.8}$$

$$a_k = \frac{2}{\pi}k\langle g, T_k\rangle_n, \qquad k = 1, \ldots, n-1. \tag{2.9}$$

In particular, if $g \in \mathbb{P}_{n-1}$ then the method yields the exact answer, i.e., $v_h = v$.

3. The convergence result

It is useful to begin by defining some norms by which the error may be described. We start by writing the solution of (1.2) as $v = \omega u$. It is well known from the theory of orthogonal polynomials that if

$$\|u\|_{\tilde{L}_2} := \langle u, u\rangle^{1/2} = \left(\int_{-1}^{1} |u(x)|^2\omega(x)\,dx\right)^{1/2} < \infty,$$

then u has a Chebyshev polynomial expansion

$$u = \frac{1}{2}\tilde{u}(0) + \sum_{k=1}^{\infty}\tilde{u}(k)T_k,$$

where

$$\tilde{u}(k) = \frac{2}{\pi}\langle u, T_k\rangle, \qquad k = 0, 1, \dots, \tag{3.1}$$

and the expansion converges in the sense that

$$\|u\|_{\widetilde{L}_2} = \left(\frac{\pi}{2}\right)^{1/2}\left[\frac{1}{2}|\tilde{u}(0)|^2 + \sum_{k=1}^{\infty}|\tilde{u}(k)|^2\right]^{1/2}. \tag{3.2}$$

Guided by the latter expression, we may define Sobolev-type norms for arbitrary $s \in \mathbb{R}$ by

$$\|u\|_{\widetilde{H}^s} := \left(\frac{\pi}{2}\right)^{1/2}\left[\frac{1}{2}|\tilde{u}(0)|^2 + \sum_{k=1}^{\infty}k^{2s}|\tilde{u}(k)|^2\right]^{1/2}, \tag{3.3}$$

and define $\widetilde{H}^s$ to be the closure of the set of all polynomials with respect to this norm. Roughly speaking, $u \in \widetilde{H}^s$ if $U(\theta) := u(\cos\theta)$ has s square-integrable derivatives. More precisely, we may write U as a Fourier cosine series,

$$U(\theta) = \frac{1}{2}\widehat{U}(0) + \sum_{k=1}^{\infty}\widehat{U}(k)\cos k\theta,$$

where

$$\begin{aligned}\widehat{U}(k) &= \frac{2}{\pi}\int_0^{\pi}\cos k\theta U(\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi}\cos k\theta u(\cos\theta)\,d\theta \\ &= \frac{2}{\pi}\int_{-1}^{1}T_k(x)u(x)\omega(x)\,dx = \frac{2}{\pi}\langle u, T_k\rangle = \tilde{u}(k).\end{aligned}$$

Then the usual Sobolev norm of U is

$$\|U\|_{H^s}^2 := \frac{\pi}{2}\left[\frac{1}{2}|\widehat{U}(0)|^2 + \sum_{k=1}^{\infty}k^{2s}|\widehat{U}(k)|^2\right] = \|u\|_{\widetilde{H}^s}^2.$$

The definition (3.3) is actually a very convenient way to define the norm of u, if we think about the application: the potential at a point (x_1, x_2) off the slit $(-1, 1)\times\{0\}$ is, by definition,

$$\begin{aligned}\phi(x_1, x_2) &= -\frac{1}{\pi}\int_{-1}^{1}\log|(x_1, x_2) - (x, 0)|v(x)\,dx \\ &= -\frac{1}{\pi}\int_{-1}^{1}\log|(x_1, x_2) - (x, 0)|u(x)\omega(x)\,dx \\ &= -\frac{1}{\pi}\int_{0}^{\pi}\log|(x_1, x_2) - (\cos\theta, 0)|u(\cos\theta)\,d\theta \\ &= -\frac{1}{\pi}\int_{0}^{\pi}\log|(x_1, x_2) - (\cos\theta, 0)|U(\theta)\,d\theta.\end{aligned}$$

With $z(\theta) := -\frac{1}{\pi}\log|(x_1, x_2) - (\cos\theta, 0)|$, a smooth function, we see that ϕ is just the L_2 inner product of U and z on the interval $(0, \pi)$. Hence, by the standard duality property of the H^s and H^{-s} norms,

$$|\phi(x_1, x_2)| \leq \|U\|_{H^{-s}}\|z\|_{H^s} = \|u\|_{\widetilde{H}^{-s}}\|z\|_{H^s}.$$

Similarly, if an approximate potential ϕ_h is defined by

$$\phi_h(x_1, x_2) := -\frac{1}{\pi}\int_{-1}^{1} \log|(x_1, x_2) - (x, 0)|v_h(x)\,dx$$

where $v_h = \omega u_h$ is the solution of (2.2), then we have

$$|\phi_h(x_1, x_2) - \phi(x_1, x_2)| \leq \|u_h - u\|_{\widetilde{H}^{-s}}\|z\|_{H^s}. \tag{3.4}$$

Thus the error in the potential inherits all the negative norm convergence of $u_h - u$ in the sense of the norm (3.3).

Defining analogous norms directly in terms of v, where $v = \omega u$, we may say, correspondingly, that if

$$\|v\|_{\overline{L}_2} := \left[\int_{-1}^{1} |v(x)|^2(1 - x^2)^{1/2}\,dx\right]^{1/2} = \|u\|_{\widetilde{L}_2} < \infty,$$

then v has an expansion of the form

$$v = \omega\left[\frac{1}{2}\overline{v}(0) + \sum_{k=1}^{\infty}\overline{v}(k)T_k\right],$$

where

$$\overline{v}(k) := \frac{2}{\pi}(v, T_k) := \frac{2}{\pi}\int_{-1}^{1} v(x)T_k(x)\,dx = \tilde{u}(k), \tag{3.5}$$

and

$$\|v\|_{\overline{L}_2} = \left(\frac{\pi}{2}\right)^{1/2}\left[\frac{1}{2}|\overline{v}(0)|^2 + \sum_{k=1}^{\infty}|\overline{v}(k)|^2\right]^{1/2}. \tag{3.6}$$

Similarly, a Sobolev-type norm for arbitrary $s \in \mathbb{R}$ is defined by

$$\|v\|_{\overline{H}^s} = \left(\frac{\pi}{2}\right)^{1/2}\left[\frac{1}{2}|\overline{v}(0)|^2 + \sum_{k=1}^{\infty}k^{2s}|\overline{v}(k)|^2\right]^{1/2}. \tag{3.7}$$

Hence

$$\|v\|_{\overline{H}^s} = \|u\|_{\widetilde{H}^s}. \tag{3.8}$$

Armed with these definitions, the existence and regularity properties of solutions of (1.2) are easily stated:

Theorem 1. *For arbitrary* $t \in \mathbb{R}$, *the operator* V *defined by* (1.2) *is an isomorphism from* $\overline{H}^t$ *to* $\widetilde{H}^{t+1}$.

Proof. This property, established in Yan and Sloan [16] by using the change of variable $x = \cos\theta$, follows readily from (1.3), (3.1), (3.3), (3.5) and (3.7).

There holds the following convergence result for the solution of the collocation scheme (2.2).

Theorem 2. *Let* V *be as in* (1.2) *and assume* $g \in \widetilde{H}^{t+1}$. *Then for any* $n \in \mathbb{Z}^+$ *there exists a solution* $v_h \in \omega\mathbb{P}_{n-1}$ *of* (2.2). *Moreover, if* $t > -\frac{1}{2}$ *and* $t \geq s$ *then for the solution* $v \in \overline{H}^t$ *of* (1.2) *there holds*

$$\|v_h - v\|_{\overline{H}^s} \leq cn^{-\min(t-s,\,t+1)}\|g\|_{\widetilde{H}^{t+1}}, \tag{3.9}$$

where the constant c *is independent of* n.

The above estimates lead to fast convergence of the approximate potential ϕ_h towards the exact potential ϕ at points x away from the slit: since

$$\|u_h - u\|_{\widetilde{H}^s} = \|v_h - v\|_{\overline{H}^s}, \tag{3.10}$$

where $v = \omega u$ solves (1.2) and $v_h = \omega u_h$ solves (2.2), it follows from (3.4) and (3.9) that

$$|\phi_h(x) - \phi(x)| \leq c(x)n^{-\min(t-s,\,t+1)}\|g\|_{\widetilde{H}^{t+1}}.$$

4. Proof of Theorem 2

The approximation scheme (2.2) is equivalent to

$$\langle Vv_h, T_j\rangle_n = \langle Vv, T_j\rangle_n, \qquad j = 0, \dots, n-1. \tag{4.1}$$

Now, from 91.3) and (3.1), we have, with $v = \omega u$,

$$Vv = \frac{1}{2}\log 2\ \tilde{u}(0) + \sum_{k=1}^{\infty}\frac{\tilde{u}(k)}{k}T_k.$$

Thus with the aid of (2.7)

$$\langle Vv, T_j\rangle_n = \begin{cases} \frac{\pi}{2}\log 2\ \tilde{u}(0) + \sum_{k=2n}^{\infty}\frac{\tilde{u}(k)}{k}\langle T_k, T_0\rangle_n, & j = 0, \\ \frac{\pi}{2}\frac{\tilde{u}(j)}{j} + \sum_{k=2n-j}^{\infty}\frac{\tilde{u}(k)}{k}\langle T_k, T_j\rangle_n, & j = 1, \dots, n-1. \end{cases} \tag{4.2}$$

A similar result can be written for $\langle Vv_h, T_j\rangle_n$—with the difference that, because $u_h = v_h/\omega$ is a polynomial of degree $\le n-1$, $\tilde{u}_h(k) = 0$ for $k \ge n$. Thus

$$\langle Vv_h, T_j\rangle_n = \begin{cases} \frac{\pi}{2}\log 2\, \tilde{u}_h(0), & j = 0, \\ \frac{\pi}{2}\frac{\tilde{u}_h(j)}{j}, & j = 1, \ldots, n-1. \end{cases} \tag{4.3}$$

On solving the defining equation (4.1), we obtain

$$\tilde{u}_h(j) = \begin{cases} \tilde{u}(0) + \frac{2}{\pi \log 2}\sum_{k=2n}^{\infty} \frac{\tilde{u}(k)}{k}\langle T_k, T_0\rangle_n, & j = 0, \\ \tilde{u}(j) + \frac{2}{\pi} j \sum_{k=2n-j}^{\infty} \frac{\tilde{u}(k)}{k}\langle T_k, T_j\rangle_n, & j = 1, \ldots, n-1. \end{cases} \tag{4.4}$$

Next we estimate the error $u_h - u$ in the $\widetilde{H}^s$ norm. Since $\tilde{u}_h(k) = 0$ for $k \ge n$ we obtain

$$\begin{aligned} \|u_h - u\|_{\widetilde{H}^s}^2 &= \frac{\pi}{4}|\tilde{u}_h(0) - \tilde{u}(0)|^2 + \frac{\pi}{2}\sum_{k=1}^{\infty} k^{2s}|\tilde{u}_h(k) - \tilde{u}(k)|^2 \\ &= \frac{\pi}{4}|\tilde{u}_h(0) - \tilde{u}(0)|^2 + \frac{\pi}{2}\sum_{j=1}^{n-1} j^{2s}|\tilde{u}_h(j) - \tilde{u}(j)|^2 + \frac{\pi}{2}\sum_{k=n}^{\infty} k^{2s}|\tilde{u}(k)|^2. \end{aligned} \tag{4.5}$$

With the aid of (4.4) we can estimate the first term as follows.

$$\begin{aligned} |\tilde{u}_h(0) - \tilde{u}(0)|^2 &= c\left|\sum_{k=2n}^{\infty} \frac{\tilde{u}(k)}{k}\langle T_k, T_0\rangle_n\right|^2 \\ &= c\left|\sum_{k=2n}^{\infty} \frac{\langle T_k, T_0\rangle_n}{k^{t+1}} k^t \tilde{u}(k)\right|^2 \\ &\le c\left(\sum_{l=2n}^{\infty} \frac{\langle T_l, T_0\rangle_n^2}{l^{2t+2}}\right)\sum_{k=2n}^{\infty} k^{2t}|\tilde{u}(k)|^2 \\ &\le c\left(\sum_{l=2n}^{\infty} \frac{\langle T_l, T_0\rangle_n^2}{l^{2t+2}}\right)\|u\|_{\widetilde{H}^t}^2. \end{aligned}$$

We want to show that the first factor of this is $\le c/n^{2t+2}$. Note that it is not enough to use $|\langle T_l, T_0\rangle_n| \le c$, since that would lead to a bound that is only of order $O(n^{-2t-1})$. Here we need the property that for $j, k \in \{0, 1, \ldots, n-1\}$ and $a \in \mathbb{Z}^+$,

$$\langle T_{2na+k}, T_j\rangle_n = \langle T_{2na-k}, T_j\rangle_n = (-1)^a\langle T_k, T_j\rangle_n = (-1)^a\delta_{kj}\langle T_j, T_j\rangle_n,$$

which is obvious from the corresponding expressions in terms of cosine functions.

Using these results one finds

$$\langle T_l, T_0\rangle_n = 0 \quad \text{unless } l \text{ is a multiple of } 2n,$$

$$|\langle T_{2na}, T_0\rangle_n| = \pi, \qquad a \in \mathbb{Z}^+.$$

Hence

$$\sum_{l=2n}^{\infty} \frac{\langle T_l, T_0\rangle_n^2}{l^{2t+2}} = \pi^2 \sum_{a=1}^{\infty} \frac{1}{(2na)^{2t+2}} = \frac{c}{n^{2t+2}} \sum_{a=1}^{\infty} \frac{1}{a^{2t+2}} = \frac{c}{n^{2t+2}},$$

provided $t > -\frac{1}{2}$. Thus

$$|\tilde{u}_h(0) - \tilde{u}(0)|^2 \le c n^{-2t-2} \|u\|_{\tilde{H}^t}^2. \tag{4.6}$$

Next, we estimate the third term in (4.5),

$$\begin{aligned} \sum_{k=n}^{\infty} k^{2s} |\tilde{u}(k)|^2 &= \sum_{k=n}^{\infty} k^{2s-2t} k^{2t} |\tilde{u}(k)|^2 \\ &\le n^{2s-2t} \sum_{k=n}^{\infty} k^{2t} |\tilde{u}(k)|^2 \le n^{2s-2t} \|u\|_{\tilde{H}^t}^2, \end{aligned} \tag{4.7}$$

provided $t \ge s$. Finally, we consider the second term in (4.5). From (4.4) we have

$$\sum_{j=1}^{n-1} j^{2s} |\tilde{u}_h(j) - \tilde{u}(j)|^2 = c \sum_{j=1}^{n-1} j^{2s+2} \left(\sum_{k=2n-j}^{\infty} \frac{\tilde{u}(k)}{k} \langle T_k, T_j\rangle_n \right)^2.$$

Now we set $k = 2an \pm l$, with $a \in \mathbb{Z}^+$ and $0 \le l \le n$. Then the discrete orthogonality property (2.7) yields, for $1 \le j \le n-1$,

$$\langle T_k, T_j\rangle_n = (-1)^a \langle T_l, T_j\rangle_n = \begin{cases} 0 & \text{if } l \ne j, \\ \frac{\pi}{2}(-1)^a & \text{if } l = j. \end{cases}$$

Thus for $1 \le j \le n-1$,

$$\sum_{k=2n-j}^{\infty} \frac{\tilde{u}(k)}{k} \langle T_k, T_j\rangle_n = \frac{\pi}{2} \sum_{a=1}^{\infty} (-1)^a \left[\frac{\tilde{u}(2an-j)}{2an-j} + \frac{\tilde{u}(2an+j)}{2an+j} \right].$$

The inequality $(a+b)^2 \le 2(a^2+b^2)$ then gives

$$\left(\sum_{k=2n-j}^{\infty} \frac{\tilde{u}(k)}{k} \langle T_k, T_j\rangle_n \right)^2 \le c \left[\left(\sum_{a=1}^{\infty} \frac{|\tilde{u}(2an-j)|}{2an-j} \right)^2 + \left(\sum_{a=1}^{\infty} \frac{|\tilde{u}(2an+j)|}{2an+j} \right)^2 \right].$$

Now

$$\left(\sum_{a=1}^{\infty}\frac{|\tilde{u}(2an\mp j)|}{2an\mp j}\right)^2 = \left(\sum_{a=1}^{\infty}\frac{1}{(2an\mp j)^{t+1}}(2an\mp j)^t|\tilde{u}(2an\mp j)|\right)^2$$
$$\leq \left(\sum_{b=1}^{\infty}\frac{1}{(2bn\mp j)^{2t+2}}\right)\sum_{a=1}^{\infty}(2an\mp j)^{2t}|\tilde{u}(2an\mp j)|^2.$$

But for $1\leq j\leq n-1$,

$$\sum_{b=1}^{\infty}\frac{1}{(2bn\mp j)^{2t+2}} = \frac{1}{(2n)^{2t+2}}\sum_{b=1}^{\infty}\frac{1}{\left(b\mp\frac{j}{2n}\right)^{2t+2}}$$
$$\leq \frac{2^{-2t-2}}{n^{2t+2}}\sum_{b=1}^{\infty}\frac{1}{\left(b-\frac{1}{2}\right)^{2t+2}} = cn^{-2t-2}, \quad \text{for } t>-\frac{1}{2}.$$

Thus

$$\left(\sum_{a=1}^{\infty}\frac{|\tilde{u}(2an\mp j)|}{2an\mp j}\right)^2 \leq cn^{-2t-2}\sum_{a=1}^{\infty}(2an\mp j)^{2t}|\tilde{u}(2an\mp j)|^2.$$

Working back, we have

$$\left(\sum_{k=2n-j}^{\infty}\frac{\tilde{u}(k)}{k}\langle T_k, T_j\rangle_n\right)^2$$
$$\leq cn^{-2t-2}\sum_{a=1}^{\infty}\left[(2an-j)^{2t}|\tilde{u}(2an-j)|^2 + (2an+j)^{2t}|\tilde{u}(2an+j)|^2\right],$$

and therefore

$$\sum_{j=1}^{n-1} j^{2s}|\tilde{u}_h(j)-\tilde{u}(j)|^2$$
$$\leq c\sum_{j=1}^{n-1} j^{2s+2}n^{-2t-2}\sum_{a=1}^{\infty}\left[(2an-j)^{2t}|\tilde{u}(2an-j)|^2+(2an+j)^{2t}|\tilde{u}(2an+j)|^2\right].$$

If $s\geq -1$, we use $j^{2s+2}\leq n^{2s+2}$. If $s\leq -1$, we use $j^{2s+2}\leq 1$. Thus we get

$$\sum_{j=1}^{n-1} j^{2s}|\tilde{u}_h(j)-\tilde{u}(j)|^2 \leq cn^{-2\min(t-s,\,t+1)}$$
$$\times\sum_{j=1}^{n-1}\sum_{a=1}^{\infty}[(2an-j)^{2t}|\tilde{u}(2an-j)|^2 + (2an+j)^{2t}|\tilde{u}(2an+j)|^2] \tag{4.8}$$
$$\leq cn^{-2\min(t-s,\,t+1)}\|u\|^2_{\tilde{H}^t}.$$

Putting the three terms (4.6), (4.7), (4.8) together, we get from 94.5), for $u \in \widetilde{H}^t$,

$$\|u_h - u\|_{\widetilde{H}^s} \leq cn^{-\min(t-s,t+1)}\|u\|_{\widetilde{H}^t}, \tag{4.9}$$

provided $t > -\frac{1}{2}$ and $t \geq s$. To complete the argument, from Theorem 1 there exists a $c > 0$ such that

$$\|u\|_{\widetilde{H}^t} = \|v\|_{\overline{H}^t} \leq c\|g\|_{\widetilde{H}^{t+1}}. \tag{4.10}$$

On combining (3.8), (4.9) and (4.10) we obtain the required estimate (3.9). □

5. Modification for general intervals $[a, b]$

With the change of variables

$$x = \frac{b+a}{2} + \frac{b-a}{2}t, \qquad y = \frac{b+a}{2} + \frac{b-a}{2}s,$$

(1.1), the equation for the general interval $[a, b]$, can be written as

$$-\frac{1}{\pi}\int_{-1}^{1} \log|t-s|w(s)\,ds = f(t) + \frac{1}{\pi}\log\frac{b-a}{2}\int_{-1}^{1} w(s)\,ds, \qquad t \in [-1, 1], \tag{5.1}$$

where

$$w(t) = v(x)\frac{b-a}{2} \quad \text{and} \quad f(t) = g(x).$$

Thus the solution for the general interval $[a, b]$ may be found by superposition of the solution of (1.2) with right-hand side f and the solution of the same equation with a constant right-hand side. Explicitly, we write

$$w = w_1 + w_0,$$

where w_1 satisfies

$$-\frac{1}{\pi}\int_{-1}^{1} \log|t-s|w_1(s)\,ds = f(t), \qquad t \in [-1, 1], \tag{5.2}$$

which can be solved approximately by the method of this paper, and w_0 satisfies

$$\begin{aligned} -\frac{1}{\pi}\int_{-1}^{1} \log|t-s|w_0(s)\,ds &= \frac{1}{\pi}\log\frac{b-a}{2}\int_{-1}^{1} w(s)\,ds \\ &= \frac{1}{\pi}\log\frac{b-a}{2}[(w_1, 1) + (w_0, 1)], \end{aligned}$$

where we have used again $w = w_1 + w_0$ and the inner product defined in (3.5). Because the right-hand side is constant, the latter equation has the solution (from the $j=0$ case of (1.3))

$$w_0(s)=\omega(s)\frac{\log\frac{b-a}{2}}{\pi\log 2}[(w_1,1)+(w_0,1)],\qquad s\in[-1,1].$$

On integrating from -1 to 1 and solving for $(w_0, 1)$, we obtain

$$w_0(s)=\omega(s)\frac{\log\frac{b-a}{2}(w_1,1)}{\pi\left(\log 2-\log\frac{b-a}{2}\right)}. \tag{5.3}$$

Note that this fails if $b-a=4$; as it should, because the logarithmic capacity of an interval of length 4 is 1, making (1.2) not uniquely solvable. (For a discussion see [10].)

Finally, the collocation method applied to (5.2) approximates w_1 by an expression of the form

$$w_{1,h}=\omega\left(\frac{1}{2}a_0+\sum_{k=1}^{n-1}a_kT_k\right),$$

which leads to $(w_{1,h}, 1)=\frac{\pi}{2}a_0$. Thus w_0 is naturally approximated by

$$w_{0,h}(s):=\omega(s)\frac{\frac{1}{2}\log\frac{b-a}{2}a_0}{\log 2-\log\frac{b-a}{2}}, \tag{5.4}$$

and in turn v is approximated by

$$v_h(x)=\frac{2}{b-a}(w_{1,h}(t)+w_{0,h}(t)).$$

The error estimate in Theorem 2 then holds without alteration if the definitions of the norms are extended in the obvious way.

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References

[1] D. N. Arnold and W. L. Wendland, "The convergence of spline collocation for strongly elliptic equations on curves", *Numer. Math.* **47** (1985) 317–341.

[2] K. E. Atkinson and I. H. Sloan, "The numerical solution of first kind logarithmic-kernel integral equations on smooth open arcs," *Math. Comp.* **56** (1991) 119–139.

[3] M. Bourlard, S. Nicaise and L. Paquet, "An adapted boundary element method for the Dirichlet problem in polygonal domains", *Preprint, Math. Department, State University, Mons, Belgium* (1988).
[4] M. Costabel, V. J. Ervin and E. P. Stephan, "On the convergence of collocation methods for Symm's integral equation on open curves", *Math. Comp.* **51** (1988) 167–179.
[5] B. Guo, T. von Petersdorff and E. P. Stephan, "An *hp* version for BEM for plane mixed boundary value problems." in *Proc. Conference BEM-11, 1989, Cambridge USA* (ed. C. A. Brebbia) (1989).
[6] K. Jörgens, *Lineare Integral Operatoren*, (Teubner-Verlag, Stuttgart, 1970).
[7] T. von Petersdorff, "Elasticity problems in polyhedra–Singularities and approximations with boundary elements", Dissertation, *TH Darmstadt* (1989).
[8] J. B. Reade, "Asymptotic behavior of eigenvalues of certain integral equations", *Proc. Edin. Math. Soc.* **22** (1979) 137–144.
[9] J. Saranen and W. L. Wendland, "On the asymptotic convergence of collocation methods with spline functions of even degree", *Math. Comp.* **45** (1985) 91–108.
[10] I. H. Sloan and A. Spence, "The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory". *IMA Journal of Numerical Analysis* **8** (1988) 105–122.
[11] E. P. Stephan, "The *hp* version of the Galerkin boundary element method for the integral equations on polygons and open arcs". In *Proc. Conference BEM-10, Southhampton, U.K.* (ed. C. A. Brebbia) (1988).
[12] E. P. Stephan and M. Suri, "On the convergence of the *p*-version of some boundary element Galerkin methods", *Math. Comp.* **52** (1989) 31–48.
[13] E. P. Stephan and W. L. Wendland, "An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems", *Applicable Analysis* **18** (1984) 183–219.
[14] G. T. Symm, "Integral equation methods in potential theory. II", *Proc. Roy. Soc.* (*London*) **A275** (1963) 33–46.
[15] Y. Yan and I. H. Sloan, "Mesh grading for integral equations of the first kind with logarithmic kernel". *SIAM J. Numer. Anal.,* **26** (1989) 574–587.
[16] Y. Yan and I. H. Sloan, "On integral equations of the first kind with logarithmic kernels", *J. Integral Equations and Applications*, **1** (1988) 517–548.