Journal of Mathematical Analysis and Applications **262,** 688–699 (2001)
doi:10.1006/jmaa.2001.7599, available online at http://www.idealibrary.com on IDEAL®

A Multilevel Method for Solving Operator Equations[1]

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Submitted by J. Diestel

Received April 26, 1999

We develop a multilevel method suitable for solving operator equations. This method combines the multiresolution structure of the spaces used to solve the operator equation with a Gauss–Seidel strategy to solve the associated matrix equations. We prove that this multilevel scheme has an optimal order of convergence and provide an application of it to the solution of second kind integral equations. © 2001 Academic Press

Key Words: wavelets; multilevel methods; multiresolutions; a Gauss–Seidel strategy; operator equations.

[1] This work is partially supported by the National Science Foundation under Grant DMS-9504780.

[2] Supported in part by NNSF of China and the Advanced Research Foundation of Zhongshan University.

0022-247X/01 $35.00

1. INTRODUCTION

Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are separable Banach spaces and $\mathcal{A}$: $\mathbb{X} \to \mathbb{Y}$ is a bounded linear operator. We consider the operator equation

$$\mathcal{A}u = f, \tag{1.1}$$

where $f \in \mathbb{Y}$ is given and $u \in \mathbb{X}$ is the unique solution of (1.1) to be determined. A general method to solve Eq. (1.1) requires a sequence of spaces $\mathbb{X}_n$, $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$, such that

$$\overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{X}_n} = \mathbb{X}$$

and a sequence of projections $\mathcal{P}_n$: $\mathbb{Y} \to \mathbb{Y}_n$ with $\mathbb{Y}_n := \mathcal{P}_n\mathbb{Y}$ such that

$$\overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{Y}_n} = \mathbb{Y}.$$

With this setup, we find a $u_n \in \mathbb{X}_n$ satisfying the equation

$$\mathcal{P}_n\mathcal{A}u_n = \mathcal{P}_n f,$$

provided that it exists.

An effective numerical method of this type produces an approximate solution u_{n+1} (fine level solution) which is more accurate than u_n (coarse level solution). The multilevel method to be described here allows us to first solve the operator equation at a coarse level and then add to it correction terms to form an approximation to the solution at all subsequent finer levels. Using this method, we can avoid solving the operator equation at finer levels. Instead, we add successive correction terms to the solution computed at some coarse level. Computationally, this requires solving linear systems of a smaller size than those which determine the finer level approximants.

This paper is organized into four sections. In Section 2, we shall describe in operator theoretic terms the multilevel method and then devote Section 3 to its error analysis. We will show that it produces an approximate solution which gives an optimal order of convergence. In Section 4, we will discuss in detail an application of this scheme to solving second kind integral equations.

2. THE MULTILEVEL METHOD

A multilevel method for solving Eq. (1.1) requires two sequences of finite dimensional subspaces $X := \{\mathbb{X}_n : n \in \mathbb{N}_0\}$ and $Y := \{\mathbb{Y}_n : n \in \mathbb{N}_0\}$, to be chosen such that

$$\mathbb{X}_n \subseteq \mathbb{X}_{n+1}, \qquad n \in \mathbb{N}_0, \qquad \overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{X}_n} = \mathbb{X}, \tag{2.1}$$

and

$$\mathbb{Y}_n \subseteq \mathbb{Y}_{n+1}, \qquad n \in \mathbb{N}_0, \qquad \overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{Y}_n} = \mathbb{Y}. \tag{2.2}$$

It is the nesting of the spaces in (2.1) and (2.2) that distinguishes a multilevel scheme from the general case described in the Introduction. This condition implies that there exist subspaces $\mathbb{W}_n$ and $\mathbb{Z}_n$ of $\mathbb{X}_{n+1}$ and $\mathbb{Y}_{n+1}$, respectively, such that

$$\mathbb{X}_{n+1} = \mathbb{X}_n \oplus \mathbb{W}_n, \qquad \mathbb{Y}_{n+1} = \mathbb{Y}_n \oplus \mathbb{Z}_n,\ n \in \mathbb{N}_0, \tag{2.3}$$

where $\oplus$ denotes the direct sum. We further require for each $n \in \mathbb{N}_0$ that the subspaces $\mathbb{X}_n$ and $\mathbb{Y}_n$ have the same dimension, that is,

$$\dim \mathbb{X}_n = \dim \mathbb{Y}_n := d_n, \qquad n \in \mathbb{N}_0.$$

Associated with the decomposition $\mathbb{Y}_{n+1} = \mathbb{Y}_n \oplus \mathbb{Z}_n$, there is a linear projection $\mathscr{P}_n$: $\mathbb{Y} \to \mathbb{Y}_n$. We require that the operator

$$\mathscr{Q}_n := \mathscr{P}_{n+1} - \mathscr{P}_n, \qquad n \in \mathbb{N}_0$$

have the property that $\mathscr{Q}_n \mathbb{Y}_{n+1} = \mathbb{Z}_n$. In general, $\mathscr{Q}_n$ is not a projection. In fact this is the case if and only if

$$\mathscr{P}_n \mathscr{Q}_n = 0, \qquad n \in \mathbb{N}_0.$$

Note that even if $\mathscr{Q}_n$ is not a linear projection, it follows that $\mathscr{Q}_n \mathscr{P}_n = 0$, $n \in \mathbb{N}_0$.

For an operator $\mathscr{B}$: $\mathbb{X} \to \mathbb{Y}$ and a subspace $\mathbb{S} \subseteq \mathbb{X}$ we use $\mathscr{B}|_{\mathbb{S}}$ to denote the restriction of the operator $\mathscr{B}$ on the subspace $\mathbb{S}$. For each $n \in \mathbb{N}_0$, we introduce four operators:

$$\mathscr{A}_n := \mathscr{P}_n \mathscr{A}|_{\mathbb{X}_n}\colon\ \mathbb{X}_n \to \mathbb{Y}_n,$$
$$\mathscr{B}_n := \mathscr{P}_n \mathscr{A}|_{\mathbb{W}_n}\colon\ \mathbb{W}_n \to \mathbb{Y}_n,$$
$$\mathscr{C}_n := \mathscr{Q}_n \mathscr{A}|_{\mathbb{X}_n}\colon\ \mathbb{X}_n \to \mathbb{Z}_n,$$

and

$$\mathscr{D}_n := \mathscr{Q}_n \mathscr{A}|_{\mathbb{W}_n}\colon\ \mathbb{W}_n \to \mathbb{Z}_n.$$

Our final requirement is the existence of a positive integer N and positive constants α and δ such that

$$\|\mathscr{A}_n^{-1}\| \le \alpha, \qquad \|\mathscr{D}_n^{-1}\| \le \delta,\ n \ge N. \tag{2.4}$$

With this setup, we call $u_n \in \mathbb{X}_n$ satisfying the equation

$$\mathcal{A}_n u_n = \mathcal{P}_n f \tag{2.5}$$

the nth level solution of Eq. (1.1). The idea of the multilevel method is to obtain an *approximation* of the $(n+1)$st level solution from the nth solution and a *correction* from $\mathbb{W}_n$. We now describe the method in detail. To this end, we write $u_{n+1} = u_{n,0} + v_{n,0}$, where $u_{n,0} \in \mathbb{X}_n$ and $v_{n,0} \in \mathbb{W}_n$, and observe that

$$\begin{aligned}\mathcal{A}_{n+1}u_{n+1} &= (\mathcal{P}_n\mathcal{A} + \mathcal{Q}_n\mathcal{A})(u_{n,0} + v_{n,0}) \\ &= \mathcal{P}_n\mathcal{A}u_{n,0} + \mathcal{P}_n\mathcal{A}v_{n,0} + \mathcal{Q}_n\mathcal{A}u_{n,0} + \mathcal{Q}_n\mathcal{A}v_{n,0} \\ &= \mathcal{A}_n u_{n,0} + \mathcal{B}_n v_{n,0} + \mathcal{C}_n u_{n,0} + \mathcal{D}_n v_{n,0}.\end{aligned}$$

We prefer rewriting this equation in the following form. For $g_1 \in \mathbb{X}_n$ (or $g_1 \in \mathbb{Y}_n$) and $g_2 \in \mathbb{W}_n$ (or $g_2 \in \mathbb{Z}_n$), we identify the column vector $[g_1, g_2]^T$ in $\mathbb{X}_n \times \mathbb{W}_n$ with the vector $g_1 + g_2$ in $\mathbb{X}_n \oplus \mathbb{W}_n$. Therefore, we can express Eq. (2.5) at the $(n+1)$st level as

$$\begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & \mathcal{D}_n \end{bmatrix}\begin{bmatrix} u_{n,0} \\ v_{n,0} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_n f \\ \mathcal{Q}_n f \end{bmatrix}, \tag{2.6}$$

where the operator $\mathcal{A}_{n+1}$ is identified as

$$\begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n \\ \mathcal{C}_n & \mathcal{D}_n \end{bmatrix}. \tag{2.7}$$

This notation leads us to an iterative method to solve Eq. (2.6) in the spirit of Gauss–Seidel iteration; cf. [3, pp. 506–509]. First, we decompose the matrix (2.7) into the sum of an upper triangular matrix

$$\mathcal{U}_n := \begin{bmatrix} \mathcal{A}_n & \mathcal{B}_n \\ 0 & \mathcal{D}_n \end{bmatrix}$$

and a lower triangular matrix

$$\mathcal{L}_n := \begin{bmatrix} 0 & 0 \\ \mathcal{C}_n & 0 \end{bmatrix}.$$

Next, we define $u^1_{n,0} \in \mathbb{X}_n$, $v^1_{n,0} \in \mathbb{W}_n$ by the equation

$$\mathcal{U}_n \begin{bmatrix} u^1_{n,0} \\ v^1_{n,0} \end{bmatrix} + \mathcal{L}_n \begin{bmatrix} u_n \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{P}_n f \\ \mathcal{Q}_n f \end{bmatrix}. \tag{2.8}$$

Note that our hypotheses on $\mathcal{A}_n$ and $\mathcal{D}_n$ ensure for $n \geq N$ that this equation uniquely determines $u^1_{n,0} \in \mathbb{X}_n$ and $v^1_{n,0} \in \mathbb{W}_n$. This leads us to a *one step predictor-correction* method to compute u_{n+1} approximately. Specifically, we choose $u_{n,1} = u^1_{n,0} + v^1_{n,0}$ as our approximation to u_{n+1}. The advantage of this method is that a linear system of size d_{n+1} corresponding to Eq. (2.6) is approximately solved by solving two linear systems, one of size d_n and the other of size $d_{n+1} - d_n$, corresponding to the two equations of (2.8), respectively. In practice this approach saves computational effort. In general, our $(i+1)$st step predictor-correction method to compute $u_{n,\,i+1}$, an approximation to u_{n+i+1}, is defined by first setting

$$u_{n,\,0} = u_n,$$

then computing $u^1_{n,\,i} \in \mathbb{X}_{n+i}$ and $v^1_{n,\,i} \in \mathbb{W}_{n+i}$ from the equation

$$\mathscr{U}_{n+i}\begin{bmatrix} u^1_{n,\,i} \\ v^1_{n,\,i} \end{bmatrix} + \mathscr{L}_{n+i}\begin{bmatrix} u_{n,\,i} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathscr{P}_{n+i}f \\ \mathscr{Q}_{n+i}f \end{bmatrix}, \tag{2.9}$$

to obtain

$$u_{n,\,i+1} = u^1_{n,i} + v^1_{n,i}, \qquad i \in \mathbb{N}_0.$$

We call $u_{n,\,i+1}$ the $(i+1)$st multilevel solution of Eq. (1.1). Practical applications of this algorithm requires bases for $\mathbb{X}_0$, $\mathbb{Y}_0$, $\mathbb{W}_i$, and $\mathbb{Z}_i$, $i \in \mathbb{N}_0$, to be specified. They must be carefully chosen relative to properties of the operator $\mathscr{A}$ to obtain an effective numerical procedure. As this issue is not the focus of this paper, instead, we turn to the theoretically important issue of estimating the order of convergence of the multilevel algorithm.

3. ERROR ANALYSIS OF THE MULTILEVEL METHOD

The goal of this section is to analyze the convergence of the multilevel method. We shall demonstrate that the multilevel solution $u_{n,\,i}$ approximates the exact solution u of operator equation (1.1) at least as well as u_{n+i}.

We begin with an estimate for the difference between $u_{n,\,i+1}$ and u_{n+i+1}. For this purpose, we recall that when there exist a positive integer N and a positive constant α such that for $n \geq N$ there holds for all $x \in \mathbb{X}_n$ the estimate

$$\alpha \|\mathscr{A}_n x\|_{\mathbb{Y}} \geq \|x\|_{\mathbb{X}}, \tag{3.1}$$

the approximate solution u_n of Eq. (2.5) exists. Moreover, if there exists a positive constant p such that for $n \geq N$ we have that $\|\mathscr{P}_n\| \leq p$ then

$$\|u - u_n\|_{\mathbb{X}} \leq \rho E_n, \tag{3.2}$$

where

$$E_n := \min_{v \in \mathbb{X}_n} \|u - v\|_{\mathbb{X}}$$

and

$$\rho := 1 + p\alpha\|\mathscr{A}\|;$$

cf. [2].

Our first observation establishes a useful equation for $u_{n,\,i+1}$.

LEMMA 3.1. *The* $(i+1)$*st level multilevel solution* $u_{n,\,i+1}$ *of Eq.* (1.1) *satisfies the operator equation*

$$\mathscr{A}_{n+i+1}u_{n,\,i+1} = \mathscr{P}_{n+i+1}f + \mathscr{C}_{n+i}e_{n,\,i}, \tag{3.3}$$

where $e_{n,\,i} := u^1_{n,\,i} - u_{n,\,i}$.

Proof. Recalling that $u_{n,\,i+1}$ is identified with $[u^1_{n,\,i}, v^1_{n,\,i}]^T$ and $\mathscr{A}_{n+i+1}$ with

$$\begin{bmatrix} \mathscr{A}_{n+i} & \mathscr{B}_{n+i} \\ \mathscr{C}_{n+i} & \mathscr{D}_{n+i} \end{bmatrix},$$

we obtain that $\mathscr{A}_{n+i+1}u_{n,i+1}$ is identified with

$$\left[\mathscr{A}_{n+i}u^1_{n,\,i} + \mathscr{B}_{n+i}v^1_{n,\,i}, \mathscr{C}_{n+i}u^1_{n,\,i} + \mathscr{D}_{n+i}v^1_{n,\,i}\right]^T.$$

Therefore using the first equation of (2.9) we have that

$$\mathscr{A}_{n+i+1}u_{n,\,i+1} = \mathscr{P}_{n+i}f + \mathscr{C}_{n+i}u^1_{n,\,i} + \mathscr{D}_{n+i}v^1_{n,\,i}.$$

Since $e_{n,i} = u^1_{n,\,i} - u_{n,\,i}$, the second equation in (2.9) yields the formula

$$\mathscr{A}_{n+i+1}u_{n,\,i+1} = \mathscr{P}_{n+i}f + \mathscr{Q}_{n+i}f + \mathscr{C}_{n+i}e_{n,\,i}$$

which completes the proof. ∎

We are now ready to estimate the difference between $u_{n,\,i+1}$ and u_{n+i+1}.

LEMMA 3.2. *Suppose that inequality* (3.1) *holds. Then for* $n \geq N$ *and* $i \in \mathbb{N}_0$ *there hold the equation*

$$u_{n,\,i+1} - u_{n+i+1} = \mathscr{A}^{-1}_{n+i+1}\mathscr{C}_{n+i}e_{n,\,i} \tag{3.4}$$

and the estimate

$$\|u_{n,\,i+1} - u_{n+i+1}\|_{\mathbb{X}} \leq \alpha\|\mathscr{C}_{n+i}\|\,\|e_{n,\,i}\|_{\mathbb{X}}.$$

Proof. By Lemma 3.1, $u_{n,i+1}$ satisfies Eq. (3.3). Using the equation

$$\mathscr{A}_{n+i+1}u_{n+i+1} = \mathscr{P}_{n+i+1}f$$

yields relation

$$\mathscr{A}_{n+i+1}(u_{n+i,1} - u_{n+i+1}) = \mathscr{C}_{n+i}e_{n,i}. \tag{3.5}$$

By the assumption there exists a positive integer N and a positive constant α such that for $n \geq N$ the inverse $\mathscr{A}^{-1}_{n+i+1}$ exists and satisfies the bound

$$\|\mathscr{A}^{-1}_{n+i+1}\| \leq \alpha.$$

Thus, the result of this lemma follows from (3.5). ∎

Lemma 3.2 leads us to estimate the norm of $e_{n,i}$. To this end, we assume that condition (2.4) is satisfied and for each $n \in \mathbb{N}_0$, we introduce operators $\mathscr{G}_n$: $\mathbb{X} \to \mathbb{X}_n$

$$\mathscr{G}_n := \mathscr{A}_n^{-1}\mathscr{B}_n\mathscr{D}_n^{-1}\mathscr{Q}_n\mathscr{A}$$

and $\mathscr{H}_n$: $\mathbb{X}_{n-1} \to \mathbb{X}_n$

$$\mathscr{H}_n := (\mathscr{G}_n - \mathscr{I})\mathscr{A}_n^{-1}\mathscr{C}_{n-1}.$$

LEMMA 3.3. *For $n \geq N$, let $e_{n,-1} = 0$ and $e_n := u_n - u$ for $n \geq N$. Then for $n \geq N$ and $i \in \mathbb{N}_0$ we have that*

$$e_{n,i} = \mathscr{H}_{n+i}e_{n,i-1} + \mathscr{G}_{n+i}e_{n+i} \tag{3.6}$$

and

$$e_{n,i} = \mathscr{G}_{n+i}e_{n+i} + \sum_{j=0}^{i-1}\mathscr{H}_{n+i} \cdots \mathscr{H}_{n+j+1}\mathscr{G}_{n+j}e_{n+j}. \tag{3.7}$$

Proof. By the definition of $e_{n,i}$, we have that

$$e_{n,i} = u^1_{n,i} - u_{n+i} + u_{n+i} - u_{n,i}.$$

We first derive a formula for $u^1_{n,i} - u_{n+i}$. To this end, we use Eqs. (2.5) and (2.9) to conclude that

$$\mathscr{A}_{n+i}(u^1_{n,i} - u_{n+i}) = -\mathscr{B}_{n+i}v^1_{n,i}. \tag{3.8}$$

Note that again by Eq. (2.9) we have that

$$v^1_{n,i} = \mathscr{D}^{-1}_{n+i}(\mathscr{Q}_{n+i}f - \mathscr{C}_{n+i}u_{n,i}) = \mathscr{D}^{-1}_{n+i}\mathscr{Q}_{n+i}\mathscr{A}(u - u_{n,i}). \tag{3.9}$$

Upon substituting (3.9) into (3.8) and solving for $u^1_{n,i} - u_{n+i}$, we obtain that

$$u^1_{n,i} - u_{n+i} = \mathscr{G}_{n+i}(u_{n,i} - u_{n+i}) + \mathscr{G}_{n+i}e_{n+i}.$$

This equation implies that

$$e_{n,i} = (\mathcal{I} - \mathcal{G}_{n+i})(u_{n+i} - u_{n,i}) + \mathcal{G}_{n+i} e_{n+i}.$$

Using Eq. (3.4) in Lemma 3.2 yields the recursive formula (3.6).

We next prove Eq. (3.7) by induction on i. The case when $i = 1$ follows directly from (3.6) and the assumption that $e_{n,-1} = 0$. We assume that (3.7) holds for $i = m$ and prove that it holds for $i = m + 1$. By using (3.6) with $i = m + 1$, we have that

$$e_{n,m+1} = \mathcal{H}_{n+m+1} e_{n,m} + \mathcal{G}_{n+m+1} e_{n+m+1}.$$

Substituting Eq. (3.7) with $i = m$ into the equation above yields the formula

$$\begin{aligned} e_{n,m+1} = {} & \mathcal{H}_{n+m+1} \mathcal{G}_{n+m} e_{n+m} + \mathcal{H}_{n+m+1} \\ & \times \sum_{j=0}^{m-1} \mathcal{H}_{n+m} \cdots \mathcal{H}_{n+j+1} \mathcal{G}_{n+j} e_{n+j} + \mathcal{G}_{n+m+1} e_{n+m+1} \end{aligned}$$

which simplifies to Eq. (3.7) with $i = m + 1$. This advances the induction hypothesis and proves the lemma. ∎

In the next lemma, we provide an upper bound for the norm of $e_{n,i}$.

LEMMA 3.4. *Suppose that there exists a positive integer N such that for all $n \geq N$*

$$\|\mathcal{G}_n\| \leq \frac{1}{2}$$

and γ_n, $n \geq N$ is a sequence such that for $n \geq N$

$$\|\mathcal{C}_n\| \leq \frac{1}{3\alpha} \frac{\gamma_{n+1}}{\gamma_n}$$

and $\gamma_n \geq E_n$. Then for all $n \geq N$ and $i \in \mathbb{N}_0$ there holds the estimate

$$\|e_{n,i}\|_{\mathbb{X}} \leq \rho \gamma_{n+i}.$$

Proof. For each $n \geq N$ we have that

$$\|\mathcal{H}_n\| \leq \alpha(1 + \|\mathcal{G}_n\|)\|\mathcal{C}_{n-1}\| \leq \frac{\gamma_n}{2\gamma_{n-1}}.$$

Therefore, it follows from Lemma 3.3 that

$$\begin{aligned} \|e_{n,i}\|_{\mathbb{X}} & \leq \rho \|\mathcal{G}_{n+i}\| \gamma_{n+i} + \rho \sum_{j=0}^{i-1} \|\mathcal{H}_{n+1}\| \cdots \|\mathcal{H}_{n+j+1}\| \, \|\mathcal{G}_{n+j}\| \gamma_{n+j} \\ & \leq \frac{\rho}{2} \gamma_{n+i} + \left(\sum_{j=0}^{i-1} \frac{\rho}{2^{i-j+1}} \right) \gamma_{n+i} \\ & \leq \rho \gamma_{n+i}. \end{aligned}$$

∎

The next result summarizes our error bound for the multilevel algorithm.

THEOREM 3.5. *Suppose that that exists a positive integer N such that for $n \geq N$*

$$\|\mathcal{A}_n^{-1}\| \leq \alpha, \qquad \|\mathcal{D}_n^{-1}\| \leq \delta, \tag{3.10}$$

and the hypotheses of Lemma 3.4 *are satisfied. Then for all $i \in \mathbb{N}_0$ and $n \geq N$ there holds the error estimate*

$$\|u - u_{n,\,i+1}\|_{\mathbb{X}} \leq \frac{4}{3}\rho\gamma_{n+i+1}.$$

Proof. By Lemmas 3.2 and 3.4, and the hypotheses of this theorem we conclude that

$$\begin{aligned} \|u_{n,\,i+1} - u_{n+i+1}\|_{\mathbb{X}} &\leq \alpha\|\mathcal{C}_{n+i}\|\,\|e_{n,\,i}\|_{\mathbb{X}} \\ &\leq \alpha\frac{\rho}{3\alpha}\frac{\gamma_{n+i+1}}{\gamma_{n+i}}\gamma_{n+i} \\ &= \frac{\rho}{3}\gamma_{n+i+1}. \end{aligned}$$

Therefore, the triangle inequality ensures that

$$\|u - u_{n,\,i+1}\|_{\mathbb{X}} \leq \frac{4}{3}\rho\gamma_{n+i+1}.$$

∎

Next, we consider an application of Theorem 3.5. To this end, we recall that when the solution u of Eq. (1.1) is known to be in some Besov space, typical estimates from Approximation Theory yield error bounds of the type $E_n \leq \gamma_n$ where γ_n has the property that there exists a positive constant c such that $\gamma_{n+1}\gamma_n^{-1} \geq c$ for all $n \in \mathbb{N}$. Specifically, $\gamma_n = Cn^{-s}$ where C depends on u and $s > 0$ specifies its regularity. With this comment in mind we are led to the following corollary.

COROLLARY 3.6. *Suppose that condition* (3.10) *holds. If there exists a positive constant c such that*

$$\frac{\gamma_{n+1}}{\gamma_n} \geq c, \qquad n \geq N.$$

and

$$\lim_{n\to\infty}\|\mathcal{B}_n\| = \lim_{n\to\infty}\|\mathcal{C}_n\| = 0,$$

then there exists a positive constant c_0 such that for $n \geq N$ and $i \in \mathbb{N}_0$,

$$\|u - u_{n,\,i+1}\|_{\mathbb{X}} \leq c_0\gamma_{n+i+1}.$$

Proof. Since $\lim_{n\to\infty} \|\mathscr{B}_n\| = 0$ and

$$\|\mathscr{G}_n\| \leq 2\alpha\delta p\|\mathscr{A}\| \, \|\mathscr{B}_n\|,$$

we conclude that $\lim_{n\to\infty} \|\mathscr{G}_n\| = 0$. Hence, there exists a positive integer N such that for $n \geq N$ there follows $\|\mathscr{G}_n\| \leq \frac{1}{2}$. On the other hand, by our hypotheses we have for $n \in \mathbb{N}_0$ that

$$\frac{1}{3\alpha}\frac{\gamma_{n+1}}{\gamma_n} \geq \frac{c}{3\alpha}.$$

The assumption that $\lim_{n\to\infty} \|\mathscr{C}_n\| = 0$ ensures that there exists a positive integer N such that for $n \geq N$,

$$\|\mathscr{C}_n\| \leq \frac{1}{3\alpha}\frac{\gamma_{n+1}}{\gamma_n}.$$

Thus, this corollary follows directly from Theorem 3.5. ∎

4. APPLICATIONS TO INTEGRAL EQUATIONS

In this section, we illustrate the general results provided in Section 3 with some examples. For this purpose, we choose to work in a Hilbert space $\mathbb{X}$ and consider the equation

$$u - \mathscr{K}u = f, \tag{4.1}$$

where $f \in \mathbb{X}$ and $\mathscr{K}\colon \mathbb{X} \to \mathbb{X}$ is a compact linear operator such that one is not an eignevalue of $\mathscr{K}$. Thus, the operator $\mathscr{A}$ in Eq. (1.1) has the special form $\mathscr{A} := \mathscr{I} - \mathscr{K}$. We restrict our discussion to the case that $\mathbb{X}_n = \mathbb{Y}_n$, $\mathbb{W}_n = \mathbb{Z}_n$ for $n \in \mathbb{N}_0$ and choose $\mathscr{P}_n$ to be the orthogonal projection of $\mathbb{X}$ onto $\mathbb{X}_n$. As before, we also require that (2.1) holds. A typical circumstance where this setup arises is the well-known Galerkin method for solving second kind integral equations.

We have the following proposition.

PROPOSITION 4.1. *Suppose that $\mathscr{K}$ is a compact operator and $\mathscr{P}_n$ is the orthogonal projection as described above. Then there exist a positive integer N and positive constants α and δ such that for all $n \geq N$, the inverse operators $\mathscr{A}_n^{-1}\colon \mathbb{X}_n \to \mathbb{X}_n$ and $\mathscr{D}_n^{-1}\colon \mathbb{W}_n \to \mathbb{W}_n$ exist and satisfy the bounds*

$$\|\mathscr{A}_n^{-1}\| \leq \alpha, \qquad \|\mathscr{D}_n^{-1}\| \leq \delta.$$

Moreover, we have that

$$\lim_{n\to\infty} \|\mathscr{B}_n\| = \lim_{n\to\infty} \|\mathscr{C}_n\| = 0.$$

Proof. We conclude from the density of the subspaces $\mathbb{X}_n$, $n \in \mathbb{N}_0$ in $\mathbb{X}$ that for all $f \in \mathbb{X}$

$$\lim_{n\to\infty} \|\mathcal{P}_n f - f\| = 0.$$

This ensures that there exists a positive integer N such that whenever $n \ge N$, Eq. (2.5) has a unique solution $u_n \in \mathbb{X}_n$. In fact, since $\mathcal{K}$ is a compact operator, the uniform boundedness principle implies that

$$\lim_{n\to\infty} \|\mathcal{P}_n\mathcal{K} - \mathcal{K}\| = 0; \tag{4.2}$$

cf. [1]. Therefore, for any $\lambda \in (0, 1)$ there exists a positive integer N such that for all $n \ge N$

$$\|(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\| \le (1-\lambda)^{-1}\|(\mathcal{I} - \mathcal{K})^{-1}\|.$$

Thus, the bound on $\|\mathcal{A}_n^{-1}\|$ holds with $\alpha := (1-\lambda)^{-1}\|(\mathcal{I} - \mathcal{K})^{-1}\|$.

To show the result concerning $\mathcal{D}_n^{-1}$, we observe that the inequality

$$\|\mathcal{Q}_n\mathcal{K}\| \le \|\mathcal{P}_{n+1}\mathcal{K} - \mathcal{K}\| + \|\mathcal{P}_n\mathcal{K} - \mathcal{K}\|$$

and Eq. (4.2) imply that

$$\lim_{n\to\infty} \|\mathcal{Q}_n\mathcal{K}\| = 0$$

which confirms that there exist positive constant δ and positive integer N such that for $n \ge N$, $(\mathcal{I} - \mathcal{Q}_n\mathcal{K})^{-1}$ exists and $\|(\mathcal{I} - \mathcal{Q}_n\mathcal{K})^{-1}\| \le \delta$.

To prove the remaining claims, we first estimate the norm of $\mathcal{C}_n$. It follows from the definition of $\mathcal{C}_n$ that

$$\begin{aligned}
\|\mathcal{C}_n\| &\le \sup_{x\in\mathbb{X}\setminus\{0\}} \frac{\|\mathcal{Q}_n\mathcal{K}\mathcal{P}_n x\|}{\|x\|} \\
&= \sup_{x\in\mathbb{X}\setminus\{0\}} \frac{\|\mathcal{Q}_n\mathcal{K}(\mathcal{P}_n x)\|}{\|\mathcal{P}_n x\|}\frac{\|\mathcal{P}_n x\|}{\|x\|} \\
&\le \|\mathcal{Q}_n\mathcal{K}\|.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \|\mathcal{C}_n\| = 0$ because $\lim_{n\to\infty} \|\mathcal{Q}_n\mathcal{K}\| = 0$. By our hypothesis we also have that

$$\lim_{n\to\infty} \|\mathcal{Q}_n\mathcal{K}^*\| = 0.$$

Moreover, as above, we conclude that $\|\mathcal{B}_n^*\| \le \|\mathcal{Q}_n\mathcal{K}^*\|$, from which we conclude that

$$\lim_{n\to\infty} \|\mathcal{B}_n\| = \lim_{n\to\infty} \|\mathcal{B}_n^*\| = 0.$$

This completes the proof. ∎

Proposition 4.1 allows us to satisfy all the requirements of Theorem 3.5. We give two examples of it. The first depends on *having available* the spectral decomposition of $\mathscr{K}$ which sometimes occurs in applications (albeit rarely). Thus, we suppose that $1 \notin \{\lambda_n : n \in \mathbb{N}_0\}$ such that

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \to 0, \qquad n \to \infty,$$

$\{v_n\colon\ n \in \mathbb{N}_0\}$ is a complete orthonormal basis of $\mathbb{X}$, and

$$\mathscr{K} = \sum_{n\in\mathbb{N}_0} \lambda_n v_n \otimes v_n.$$

We choose

$$\mathbb{X}_n := \operatorname{span}\{v_j : j \in Z_n\}, \qquad n \in \mathbb{N}_0$$

and let $\mathscr{P}_n$ be the orthogonal projection from $\mathbb{X}$ onto $\mathbb{X}_n$. Then the multilevel method gives us for $n, i \in \mathbb{N}_0$ that $u_{n,i} = u_{n+i}$.

In our final example, we choose $\mathbb{X} = L^2[0,1]$ and $\mathscr{K}$ to be an integral operator with kernel $K \in L^2([0,1]^2)$. We choose $\mathbb{X}_n$ to be the piecewise polynomial space of degree $\ell - 1$ with knots at $j/2^n$, $j \in Z_{2^n+1}$. Thus, $d_n = \ell 2^n$ and this sequence of spaces satisfies all required conditions. We choose $\mathbb{W}_n$ to be the wavelet spaces; see [4] for constructions of the wavelet spaces. Let $u \in L^2[0,1]$ be the unique solution of Eq. (4.1). In addition, if $f \in H^r[0,1]$ and $K \in H^{r-1}([0,1]^2)$ for some $1 \leq r \leq \ell$, then the solution u is in $H^r[0,1]$; cf. [1]. Hence, there exists a positive constant c such that

$$E_n \leq c2^{-rn}\|u\|_{H^r[0,1]}, \qquad n \in \mathbb{N}_0.$$

This fact allows us to apply Theorem 3.5 and Proposition 4.1 with $\gamma_n = c2^{-rn}\|u\|_{H^r[0,1]}$ to conclude that there is a positive integer N and a positive constant c_0 such that the multilevel approximants $u_{n,i}$ for this example satisfy for $n \geq N$ the optimal error bound

$$\|u - u_{n,i}\|_{L^2[0,1]} \leq c_0 2^{-r(n+i)}\|u\|_{H^r[0,1]}.$$

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