JOURNAL OF FUNCTIONAL ANALYSIS **130**, 161–219 (1995)

The Inhomogeneous Dirichlet Problem in Lipschitz Domains

DAVID JERISON

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

CARLOS E. KENIG

University of Chicago, Chicago, Illinois 60637

Communicated by the Editors

Received October 29, 1993

INTRODUCTION

The purpose of this paper is to study the inhomogenous Dirichlet problem

$$\begin{aligned} \Delta u &= f \qquad \text{on } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{0.1}$$

with data in Sobolev spaces of domains Ω in $\mathbf{R}^n$ with Lipschitz boundary. We obtain a complete description of all Sobolev spaces for which estimates hold, and, in the process, we obtain the best possible estimates in Besov spaces as well. Lions and Magenes [20] made a systematic study of this and other elliptic boundary value problems for smooth domains and L^2 Sobolev spaces. Grisvard [13] treats various related problems for domains with minimal smoothness, mostly in the L^2 context. The present paper can be viewed as a continuation of that study. Unlike the case of smoother domains in which Green's function inherits exactly the degree of differentiability of the boundary, in C^1 and Lipschitz domains Green's function need not have a bounded derivative. The limitations on the exponent p and the order of differentiability α of function spaces L^p_α for which estimates are true can be quite subtle. For example, it is easy to see that for every $p > 3$

Research supported in part by NSF Grants DMS-8451770, DMS-9106507, and DMS-9200908.

0022-1236/95 $6.00

and $n \geqslant 3$ there is a Lipschitz domain and an infinitely differentiable right hand side f for which ∇u cannot belong to $L^p(\Omega)$. We show here that sharp estimates do succeed in $L^3(\Omega)$. These estimates are applied to give operator bounds on fractional powers of the solution operator to the Dirichlet problem. We obtain characterizations of harmonic functions in a given Sobolev space that are of independent interest. Finally, we construct counterexamples in borderline cases involving careful dyadic Fourier decomposition.

In the case of smooth domains, estimates for the inhomogeneous problem are essentially equivalent to estimates in the homogeneous Dirichlet problem

$$\begin{aligned} \Delta v &= 0 \quad \text{on } \Omega, \\ v &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{0.2}$$

For certain exceptional function spaces this is no longer true for Lipschitz domains. The customary approach of reducing the inhomogeneous problem to the homogeneous one can fail. A great deal is already known about the homogeneous Dirichlet problem in Lipschitz domains, but new estimates are needed to complete the study of the inhomogeneous problem. In addition to proving these new estimates for the homogeneous problem, we construct some nontrivial counterexamples to show that we have the best possible estimates.

For $1 \leqslant p \leqslant \infty$ and $-\infty < \alpha < \infty$ the Sobolev space L^p_α is defined by

$$L^p_\alpha = \{(I - \Delta)^{-\alpha/2} g : g \in L^p(\mathbf{R}^n)\},$$

with norm

$$\|f\|_{L^p_\alpha} = \|(I - \Delta)^{\alpha/2} f\|_{L^p(\mathbf{R}^n)}.$$

For $\alpha \geqslant 0$, define the Sobolev space $L^p_\alpha(\Omega)$ as the space of restrictions of functions in L^p_α to Ω. It is well known that for $\alpha > 1/p$, the restriction mapping $\operatorname{Tr} u = u|_{\partial\Omega}$, defined initially on $C^\infty(\bar{\Omega})$, extends to a continuous mapping $\operatorname{Tr}: L^p_\alpha(\Omega) \to L^p(\partial\Omega)$. Thus the boundary condition $u = 0$ on $\partial\Omega$ is interpreted as $\operatorname{Tr} u = 0$. In the range $\alpha \leqslant 1/p$, the mapping Tr is no longer continuous, and uniqueness in the homogeneous problem fails. For instance, the function $u \equiv 1$ belongs to the closure of $C^\infty_0(\Omega)$ in the $L^p_\alpha(\Omega)$ norm. Thus, we will restrict our attention to the range $\alpha > 1/p$.

For $\alpha > 0$, $1 < p < \infty$, $1/p' + 1/p = 1$, define $L^{p'}_{\alpha,0}(\Omega)$ as the space of functions in $L^{p'}_\alpha$ supported in $\bar{\Omega}$, and define $L^p_{-\alpha}(\Omega)$ as the space dual to $L^{p'}_{\alpha,0}(\Omega)$. The inhomogeneous Dirichlet problem can be solved in smooth domains with estimates in any Sobolev space.

THEOREM 0.3. *Let $1 < p < \infty$ and $\alpha > 1/p$. If Ω is a bounded domain with C^∞ boundary, then for every $f \in L^p_{\alpha-2}(\Omega)$ there is a unique solution $u \in L^p_\alpha(\Omega)$ to the inhomogeneous Dirichlet problem* (0.1). *Moreover,*

$$\|u\|_{L^p_\alpha(\Omega)} \leqslant C \|f\|_{L^p_{\alpha-2}(\Omega)} \qquad (1 < p < \infty)$$

for all $f \in L^p_{\alpha-2}(\Omega)$.

Theorem 0.3 is a consequence of the Calderón–Zygmund theory of singular integrals and boundary layer potential (or pseudodifferential operator) techniques. Denote Green's function by $G(x, y)$. G is the Schwartz kernel of the solution operator $\mathbf{G}$ for (0.1),

$$\mathbf{G}f(x) = u(x) = \int_\Omega G(x, y) f(y)\, dy.$$

The bounds on $\mathbf{G}$ as an operator from $L^p_{-1}(\Omega)$ to $L^p_1(\Omega)$ are the same as estimates on $L^p(\Omega)$ for the singular integral operator with kernel $\nabla_x \nabla_y G(x, y)$. The bounds on $\mathbf{G}$ from $L^p(\Omega)$ to $L^p_2(\Omega)$ are the same as estimates on $L^p(\Omega)$ for the singular integral operator with kernel $\nabla_x^2 G(x, y)$.

On Lipschitz domains the following results have been known for some time.

THEOREM A (Negative Results).

1. *If $n \geqslant 3$, then for any $p > 3$ there is a Lipschitz domain Ω and $f \in C^\infty(\bar{\Omega})$ such that the solution u to the inhomogeneous Dirichlet problem* (0.1) *does not belong to $L^p_1(\Omega)$.* (*For $n = 2$, the result is valid for any $p > 4$.*)

2. *For any $p > 1$ there is a Lipschitz domain Ω and $f \in C^\infty(\bar{\Omega})$ such that the solution u to the inhomogeneous Dirichlet problem* (0.1) *does not belong to $L^p_2(\Omega)$.*

3. *For any $\alpha > 3/2$ there is a Lipschitz domain Ω and $f \in C^\infty(\bar{\Omega})$ such that the solution u to the inhomogeneous Dirichlet problem* (0.1) *does not belong to $L^2_\alpha(\Omega)$.*

Parts 1 and 3 of Theorem A come from easy examples on cones. Part 2 is an example due to Dahlberg [9].

THEOREM B (Positive Results). *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$.*

1. *If $n \geqslant 3$, there exists an exponent $p_0 > 3$ such that the fractional integral operator*

$$f \mapsto \nabla u(x) = \int_\Omega \nabla_x G(x, y) f(y)\, dy$$

is bounded from $L^q(\Omega)$ *to* $L^p(\Omega)$ $(1/p = 1/q - 1/n,\ 1 < p < p_0)$. *(If the domain is* C^1, *then* $p_0 = \infty$.)

2. *If* $f \in L^2(\Omega)$, *then* $u \in L^2_{3/2}(\Omega)$.

Part 1 of Theorem **B** is due to Dahlberg [9]. Part 2 follows from estimates in [14] for the homogeneous Dirichlet problem. Here is the proof. Extend f to be zero outside Ω and let $w = f * N$, the convolution with the Newtonian potential $N(x) = c_n |x|^{2-n}$. Then $w \in L^2_2(\Omega)$. It is easy to show that $\operatorname{Tr} \nabla w$ belongs to $L^2(\partial\Omega)$ and hence $g = \operatorname{Tr} w$ belongs to $L^2_1(\partial\Omega)$. The result of [14] is that the harmonic function v satisfying $\Delta v = 0$ in Ω and $v = g$ on $\partial\Omega$ satisfies $v \in L^2_{3/2}(\Omega)$. Hence $u = w - v$ belongs to $L^2_{3/2}(\Omega)$ and satisfies $\Delta u = f$ in Ω and $\operatorname{Tr} u = 0$.

One motivation for the more systematic study undertaken here is as follows. Nečas [21] posed the question of solving the equations

$$\Delta u = \operatorname{div} \vec{f} \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

on Lipschitz domains, with estimates

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C\, \|\vec{f}\|_{L^p(\Omega)}.$$

(He asked this question in particular for the case $n = 3$ and $p = 3$.) This estimate is the same as that showing that **G** is a bounded operator from $L^p_{-1}(\Omega)$ to $L^p_1(\Omega)$. Since we already know from Theorem B that **G** is bounded from $L^q(\Omega)$ to L^p_1 for certain p, this might seem to be a routine exercise. But it is not. In fact, the corresponding extension of Theorem B(2) is false!

THEOREM 0.4. *There is a* C^1 *domain* Ω *and* $f \in L^2_{-1/2}(\Omega)$ *such that the solution* u *to the inhomogeneous Dirichlet problem* (0.1) *does not belong to* $L^2_{3/2}(\Omega)$.

Another motivation is that the estimates we obtain here lead to estimates in the Neumann problem and to estimates for fractional powers of the Laplace operator. In a future paper we plan to treat the case of Neumann boundary conditions.

For the purposes of this introduction, let us highlight the special case $\alpha = 1$ and the special case $p = 2$.

THEOREM 0.5. *Let* Ω *be a bounded Lipschitz domain in* $\mathbf{R}^n$.

(a) *There is an exponent* p_1 *with* $p_1 > 4$ *when* $n = 2$ *and* $p_1 > 3$ *when* $n \geqslant 3$ *such that if* $p_1' < p < p_1$, *then the inhomogeneous Dirichlet problem* (0.1) *has a unique solution* $u \in L^p_1(\Omega)$ *and*

$$\|u\|_{L^p_1(\Omega)} \leqslant C\, \|f\|_{L^p_{-1}(\Omega)}$$

for all $f \in L^p_{-1}(\Omega)$.

(b) *Suppose that* $\frac{1}{2} < \alpha < \frac{3}{2}$. *Then the inhomogeneous Dirichlet problem* (0.1) *has a unique solution* $u \in L^2_\alpha(\Omega)$ *and*

$$\|u\|_{L^2_\alpha(\Omega)} \leqslant C \|f\|_{L^2_{\alpha-2}(\Omega)}$$

for every $f \in L^2_{\alpha-2}(\Omega)$.

(Here and elsewhere p' denotes the dual exponent to p: $1/p + 1/p' = 1$.)

Theorem 0.5(a) gives a positive answer to the question posed by Nečas. Part (b) shows that the incompatibility, illustrated in Theorem 0.4, between the homogeneous and inhomogeneous problems is confined to the endpoint $\alpha = \frac{3}{2}$.

Let us see how the general strategy of reducing the inhomogeneous problem to the homogeneous one proceeds. In the case of Theorem 0.5(b), we are given $f \in L^2_{\alpha-2}(\Omega)$. Define f outside Ω so that f has compact support and $f \in L^2_{\alpha-2}(\mathbf{R}^n)$. Let $w = f * N$, the convolution with the Newtonian potential $N(x) = c_n |x|^{2-n}$. Then $\Delta w = f$ in Ω and $w \in L^2_\alpha(\Omega)$. Define $g = \operatorname{Tr} w$, the trace of w on $\partial\Omega$. The well-known restriction theorem for Sobolev spaces says that for $\alpha > 1/2$ and $\partial\Omega$ smooth, Tr is bounded from $L^2_\alpha(\Omega)$ to $L^2_{\alpha-1/2}(\partial\Omega)$. In the case of smooth boundary, standard estimates for the homogeneous Dirichlet problem say that if

$$\Delta v = 0 \quad \text{on } \Omega, \qquad v = g \quad \text{on } \partial\Omega,$$

and $g \in L^2_{\alpha-1/2}(\partial\Omega)$, then $v \in L^2_\alpha(\Omega)$. It follows that $u = w - v$ is the solution to the inhomogeneous Dirichlet problem (0.1) and $u \in L^2_\alpha(\Omega)$. In the case of Lipschitz domains, the regularity estimate $v \in L^2_\alpha(\Omega)$ is valid in the range $\frac{1}{2} \leqslant \alpha \leqslant \frac{3}{2}$ [14,16]. The trace theorem is also valid in the range $\frac{1}{2} < \alpha < \frac{3}{2}$ (See [19]). What goes wrong for $\alpha = \frac{3}{2}$ in the Lipschitz and C^1 cases is that the trace Tr, when applied to $L^2_{3/2}(\Omega)$, has a range larger than the usual range $L^2_1(\partial\Omega)$. (See Section 3 for definitions of function spaces associated to the boundary.) Not only does the method of proof fail, but the counterexample for restriction leads to the counterexample to regularity for the inhomogeneous Dirichlet problem of Theorem 0.4. This is where the homogeneous and inhomogeneous problems take separate paths.

The organization of the paper is as follows. In the first section we state the main results. In the second section and third sections, we introduce the function spaces that we will need and examine extension and restriction properties. In the fourth section we show that certain Besov and Sobolev norms are equivalent when restricted to the subspace of harmonic functions. This step is crucial to the proof of Sobolev estimates and is the reason why Besov spaces are introduced. In the fifth section we prove all the necessary estimates in the homogeneous problem. In the sixth section we deduce the main estimates in the inhomogeneous problem and present some fairly subtle counterexamples. In the final section we apply our estimates to derive bounds on the square root and other fractional powers of the Laplace operator.

The emphasis in all of what follows is on estimates that are different, or require different proofs, from the smooth case.

Many of the results described here were announced in [18]. We thank Martin Costabel for questioning a claim in an earlier paper [16; further results in 3, p. 62]. There we claimed that the trace of the space $L^2_{3/2}(\Omega)$ on $\partial\Omega$ is the space of functions $L^2_1(\partial\Omega)$ with one tangential derivative in $L^2(\partial\Omega)$. We want to correct that mistake here. The estimates for the homogeneous problem described in [16] are valid as stated, but the identification of the space $L^2_1(\partial\Omega)$ with the trace space of $L^2_{3/2}(\Omega)$ is not. We also thank Guy David, who showed us the example, given below, that proves that this trace theorem is false. As we have indicated above, the failure of the trace theorem does not affect estimates in the homogeneous problem starting with data in $L^2_1(\partial\Omega)$, but it does necessitate the extra work described here to reach best possible results in the inhomogeneous problem.

1. Main Results

Our main positive result is as follows.

THEOREM 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geqslant 3$. There exists ε, $0 < \varepsilon \leqslant 1$, depending only on the Lipschitz constant of Ω such that for every $f \in L^p_{\alpha-2}$ there is a unique solution $u \in L^p_\alpha(\Omega)$ to the inhomogeneous Dirichlet problem*

$$\Delta u = f \quad \text{in } \Omega; \qquad \operatorname{Tr} u = 0 \quad \text{on } \partial\Omega$$

provided one of the following holds:

$$\text{(a)} \quad p_0 < p < p_0' \quad \text{and} \quad \frac{1}{p} < \alpha < 1 + \frac{1}{p}$$

$$\text{(b)} \quad 1 < p \leqslant p_0 \quad \text{and} \quad \frac{3}{p} - 1 - \varepsilon < \alpha < 1 + \frac{1}{p}$$

$$\text{(c)} \quad p_0' \leqslant p < \infty \quad \text{and} \quad \frac{1}{p} < \alpha < \frac{3}{p} + \varepsilon$$

where $1/p_0 = 1/2 + \varepsilon/2$ and $1/p_0' = 1/2 - \varepsilon/2$. Moreover, we have the estimate

$$\|u\|_{L^p_\alpha(\Omega)} \leqslant C\, \|f\|_{L^p_{\alpha-2}}$$

for all $f \in L^p_{\alpha-2}$. When the domain is C^1 the exponent p_0 may be taken to be 1.

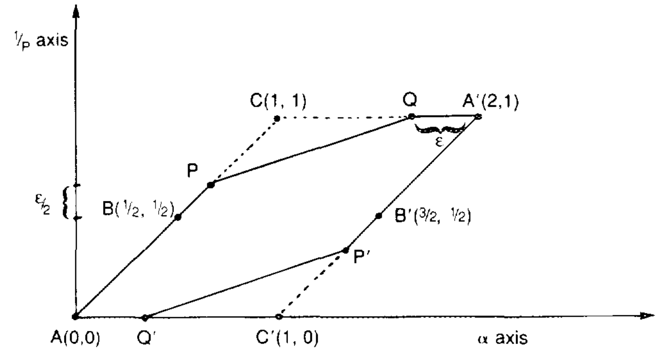

FIGURE 1

The collection of α and p for which Theorem 3 holds is best understood as the open hexagon in $(\alpha, 1/p)$-space $APQA'P'Q'$ with corners $A=(0,0)$, $P=(1/p_0, 1/p_0)=(1/2+\varepsilon/2, 1/2+\varepsilon/2)$, $Q=(2-\varepsilon, 1)$, $A'=(2,1)$, $P'=(2-1/p_0, 1/p_0')$, and $Q'=(\varepsilon, 0)$. There is a symmetry under the transformation $(\alpha, 1/p) \mapsto (2-\alpha, 1/p')$ which comes from the symmetry of Green's function (Fig. 1).

Also of interest are the points $B=(1/2, 1/2)$, $B'=(3/2, 1/2)$, $C=(1,1)$, and $C'=(1,0)$. Theorem 0.5(a) corresponds to the intersection of the hexagon with the segment from C to C'. Theorem 0.5(b) is the segment from B to B'. The simplest way to describe the full theorem is that it says that the region for which estimates hold contains the open parallelogram $ABA'B'$ widened by an ε along the segments BA' and AB'. In the C^1 case, estimates hold in the full parallelogram $ACA'C'$.

The complementary negative result is as follows.

THEOREM 1.2. (a) *For $n \geqslant 3$, and any $p>2$ and any $\alpha>3/p$ there exists a Lipschitz domain Ω in $\mathbf{R}^n$ and $f \in C^\infty(\bar{\Omega})$ for which the solution u to (0.1) does not belong to $L^p_\alpha(\Omega)$.*

(b) *For any $n \geqslant 2$, there exists a C^1 domain Ω in $\mathbf{R}^n$ and $f \in C^\infty(\bar{\Omega})$ for which the second derivative $\nabla^2 u$ of the solution u to (0.1) does not belong to $L^1(\Omega)$.*

(c) *For any $n \geqslant 2$, there exists a C^1 domain Ω in $\mathbf{R}^n$ and $f \in L^p_{-1+1/p}(\Omega)$ for which the solution u to (0.1) does not belong to $L^p_{1+1/p}(\Omega)$ for all p, $1<p<\infty$.*

Theorem 1.2 shows that Theorem 1.1 is sharp, including all endpoint cases. Theorem 1.2(a) $(n \geqslant 3)$ says that regularity can fail for any point below the segment AB'. Duality shows that the solution may fail to exist for f in the negative Sobolev spaces corresponding to any point above the opposite side, BA'. Part (c) says that even for C^1 domains regularity fails on the segment $C'A'$. In particular, it fails at the midpoint B' as stated in Theorem 0.4. Duality shows that existence fails along the segment AC, which is, coincidentally, the segment at which uniqueness fails even in the smooth case. Part (b) treats the endpoint A'. It improves on the counterexample of Theorem A(2) due to Dahlberg.

The two-dimensional result is as follows.

THEOREM 1.3. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. There exists ε, $0 < \varepsilon \leqslant \frac{1}{2}$, depending only on the Lipschitz constant of Ω such that for every $f \in L^p_{\alpha-2}$ there is a unique solution $u \in L^p_\alpha(\Omega)$ to the inhomogeneous Dirichlet problem provided one of the following holds:*

$$\text{(a)}\quad p_0 < p < p_0' \qquad \text{and} \qquad \frac{1}{p} < \alpha < 1 + \frac{1}{p}$$

$$\text{(b)}\quad 1 < p \leqslant p_0 \qquad \text{and} \qquad \frac{2}{p} - \frac{1}{2} - \varepsilon < \alpha < 1 + \frac{1}{p}$$

$$\text{(c)}\quad p_0' \leqslant p < \infty \qquad \text{and} \qquad \frac{1}{p} < \alpha < \frac{2}{p} + \frac{1}{2} + \varepsilon,$$

where $1/p_0 = 1/2 + \varepsilon$ and $1/p_0' = 1/2 - \varepsilon$. Moreover, we have the estimate

$$\|u\|_{L^p_\alpha(\Omega)} \leqslant C\,\|f\|_{L^p_{\alpha-2}}$$

for all $f \in L^p_{\alpha-2}$. When the domain is C^1 the exponent p_0 may be taken to be 1.

The negative result not covered by Theorem 1.2 is as follows.

PROPOSITION 1.4. *For any $p > 2$ and any $\alpha > 2/p + 1/2$ there exists a Lipschitz domain Ω in $\mathbf{R}^2$ and $f \in C^\infty(\bar{\Omega})$ for which the solution u to* (0.1) *does not belong to $L^p_\alpha(\Omega)$.*

Estimates in the homogeneous Dirichlet problem are stated in Theorems 5.1 and 5.18. Estimates for the square root of the Laplace operator are as follows. Let A denote the square root of the Laplace operator with Dirichlet boundary conditions. (See Section 7 for definitions and Theorem 7.5 for a more detailed statement.)

THEOREM 1.5. *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain.* (a) *If $n \geqslant 3$, then there exists $q_0 > 3$, depending on the Lipschitz constant of Ω such that if $1 < p < q_0$, $f \in \mathscr{D}(A)$ and $Af \in L^p(\Omega)$, then*

$$\int_\Omega |\nabla f|^p \leqslant C \int_\Omega |Af|^p.$$

If $n=2$, then the exponent q_0 can be chosen greater than 4. If Ω is C^1, then q_0 can be chosen equal to infinity.

(b) *The lower bounds on the exponent q_0 in part* (a) *are best possible.*

(c) *Let $n \geqslant 2$ and $1 < p < \infty$. If $f \in \mathscr{D}(A) \cap L_1^p(\Omega)$, then*

$$\int_\Omega |Af|^p \leqslant C \int_\Omega |\nabla f|^p.$$

2. Sobolev and Besov Spaces on Lipschitz Domains

Let $R_\Omega f$ denote the restriction of function f from $\mathbf{R}^n$ to Ω.

Definition 2.1. For $1 \leqslant p \leqslant \infty$ and $\alpha \geqslant 0$ we define $L_\alpha^p(\Omega) = R_\Omega L_\alpha^p$ with the usual quotient norm

$$\|f\|_{L_\alpha^p(\Omega)} = \inf\{\|g\|_{L_\alpha^p(\mathbf{R}^n)} : R_\Omega g = f\}.$$

For each non-negative integer k, define

$$W_k^p(\Omega) = \left\{f\colon \frac{\partial^\beta f}{\partial x^\beta} \in L^p(\Omega), |\beta| \leqslant k\right\},$$

so that the $W_k^p(\Omega)$ norm is equivalent to

$$\|f\|_{k,p} = \left(\sum_{|\beta| \leqslant k} \int_\Omega \left|\frac{\partial^\beta f}{\partial x^\beta}\right|^p dx\right)^{1/p}.$$

Calderón's extension theorem [3] says that if Ω has a Lipschitz boundary, then there is a bounded linear extension operator E_k from $W_k^p(\Omega)$ to $W_k^p(\mathbf{R}^n)$, that is, $R_\Omega \circ E_k$ is the identity operator on $W_k^p(\Omega)$. (See 3.10 for the explicit form of E_1.) It follows from standard Calderón–Zygmund estimates for singular integrals that $L_k^p(\mathbf{R}^n) = W_k^p(\mathbf{R}^n)$, for $1 < p < \infty$, and hence

$$L_k^p(\Omega) = W_k^p(\Omega) \tag{2.2}$$

for all non-negative integers k and all p, $1 < p < \infty$.

Next, recall the extension operator defined by Stein [22]. (The key feature of this extension operator is that it is linear and extends Sobolev spaces for more than one integer at a time. Hence it is amenable to interpolation.)

THEOREM 2.3. *For any bounded Lipschitz domain Ω, there is a bounded linear extension operator E: $W_k^p(\Omega) \to W_k^p$ for every non-negative integer k and every p, $1 \leqslant p \leqslant \infty$.*

Recall interpolation by the complex method of the spaces L_α^p:

$$[L_{\alpha_0}^{p_0}, L_{\alpha_1}^{p_1}]_\theta = L_\alpha^p, \tag{2.4}$$

provided $1 < p_0, p_1 < \infty$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. (See [2, Theorem 6.4.5].) We say that a family of spaces like X_α^p is a *complex interpolation scale* if an assertion like (2.4) for the parameters p and α is true.

PROPOSITION 2.4. *E extends to a bounded linear operator from $L_\alpha^p(\Omega)$ to L_α^p for all $\alpha \geqslant 0$ and all p, $1 < p < \infty$. Moreover, $L_\alpha^p(\Omega)$ is a complex interpolation scale for $\alpha \geqslant 0$ and $1 < p < \infty$.*

Proof. By complex interpolation, Stein's extension operator, E, is a bounded linear mapping from $[L^p(\Omega), W_k^p(\Omega)]_\theta$ to $L_{\theta k}^p$. It follows that $[L^p(\Omega), W_k^p(\Omega)]_\theta$ is a subset of $L_{\theta k}^p(\Omega)$. On the other hand, the restriction mapping, R_Ω, maps L_α^p to $L_\alpha^p(\Omega)$ for all α, by definition. Recall that for $1 < p < \infty$, $L_{\theta k}^p = [L^p, W_k^p]_\theta$. By complex interpolation, R_Ω is a bounded operator from $L_{\theta k}^p$ to $[L^p(\Omega), W_k^p(\Omega)]_\theta$. Hence, $L_{\theta k}^p(\Omega)$ is a subset of $[L^p(\Omega), W_k^p(\Omega)]_\theta$. In all

$$[L^p(\Omega), W_k^p(\Omega)]_\theta = L_{\theta k}^p(\Omega). \tag{2.5}$$

The proposition now follows from the reiteration theorem for complex interpolation [2, Theorem 4.6.1].

DEFINITION 2.6. Let $-\infty < \alpha < \infty$ and $1 < p < \infty$. Define $L_{\alpha,0}^p(\Omega)$ as the space of all distributions $f \in L_\alpha^p(\mathbf{R}^n)$ with support in $\bar{\Omega}$. The norm is

$$\|f\|_{L_{\alpha,0}^p(\Omega)} = \|f\|_{L_\alpha^p(\mathbf{R}^n)}$$

REMARK 2.7. *The space $C_0^\infty(\Omega)$ is dense in $L_{\alpha,0}^p(\Omega)$ for all values of α and p.*

Proof. It suffices to consider a distribution ψ supported in a small neighborhood U of a boundary point. After suitable translation and rotation the boundary point is the origin and the boundary is represented as the graph of a Lipschitz function ϕ,

$$\partial\Omega \cap U = \{x = (x', x_n) : x_n > \phi(x'), x' \in \mathbf{R}^{n-1}\} \cap U.$$

If ψ has small support, then the translate $\psi_t(x) = \psi(x + (0, t))$ for small $t > 0$ vanishes in a neighborhood of $\partial\Omega$. Convolution with an approximate identity shows that ψ_t can be approximated by functions in $C_0^\infty(\Omega)$. Moreover, it is well known that as t tends to zero ψ_t tends to ψ in L_α^p. This proves the remark.

DEFINITION 2.8. Let $0 < \alpha < \infty$, $1 < p < \infty$, and $1/p' + 1/p = 1$. The space $L_{-\alpha}^p(\Omega)$ is defined as the space of linear functionals on $C_0^\infty(\Omega)$ with norm

$$\|g\|_{L_{-\alpha}^p(\Omega)} = \sup\{|g(f)| : f \in C_0^\infty(\Omega), \|f\|_{L_\alpha^{p'}} \leqslant 1\}.$$

PROPOSITION 2.9. *For all* $\alpha \geqslant 0$ *and all* $p, 1 < p < \infty$, $1/p + 1/p' = 1$, $L_{-\alpha}^p(\Omega)$ *is the dual space to* $L_{\alpha,0}^{p'}(\Omega)$. $L_{-\alpha,0}^{p'}(\Omega)$ *is the dual space to* $L_\alpha^p(\Omega)$. *Moreover,* $C^\infty(\bar{\Omega})$ *is dense in* $L_\alpha^p(\Omega)$ *for* $-\infty < \alpha < \infty$.

Proof. The first assertion is an easy consequence of Remark 2.7. The second assertion is not much more difficult. Given a distribution $f \in L_{-\alpha,0}^{p'}(\Omega)$, define a linear functional on $L_\alpha^p(\Omega)$ by $u \mapsto f(u_1)$ where $u_1 \in L_\alpha^p$ is any extension of u satisfying $R_\Omega u_1 = u$. This functional is well-defined because if u_2 is another extension of u, then $u_1 - u_2 \in L_{\alpha,0}^p(\Omega')$, where $\Omega' = \mathbf{R}^n \backslash \bar{\Omega}$. Remark 2.7 (for the region Ω') implies that $u_1 - u_2$ is the limit in L_α^p norm by functions in $C_0^\infty(\Omega')$. But f annihilates such functions, so taking the limit, $f(u_1 - u_2) = 0$. Thus there is a natural mapping $L_{-\alpha,0}^{p'}(\Omega) \to (L_\alpha^p(\Omega))^*$. Its inverse is given by composition with R_Ω. $C^\infty(\bar{\Omega})$ is dense in $L_\alpha^p(\Omega)$ for $\alpha \geqslant 0$ because of density in all of $\mathbf{R}^n$. The space $L_{\alpha,0}^{p'}(\Omega)$ is a closed subspace of the uniformly convex space $L_{\alpha,0}^{p'}(\mathbf{R}^n)$ and hence reflexive. It follows that $X = L_{-\alpha}^p(\Omega)$ is reflexive. If $C^\infty(\bar{\Omega})$ were not dense in X, then by the Hahn–Banach theorem, there would be a nonzero function $f \in X^* = L_{\alpha,0}^{p'}(\Omega)$ such that $\int f\phi = 0$ for every $\phi \in C^\infty(\bar{\Omega})$. This is a contradiction.

COROLLARY 2.10. $L_{-\alpha,0}^p(\Omega)$ *is a complex interpolation scale for* $\alpha > 0$ *and* $1 < p < \infty$.

This follows because the dual of an interpolation scale is an interpolation scale [2].

PROPOSITION 2.11. $L_{\alpha,0}^p(\Omega)$ *is a complex interpolation scale for* $\alpha \geqslant 0$ *and* $1 < p < \infty$.

Proof. We begin with a general abstract lemma, due to Lions and Magenes [20, Vol. I], which will also be useful in Section 4.

LEMMA 2.12. *Suppose that* $X_i, Y_i, Z_i, i = 0, 1$, *are Banach spaces. Suppose that* X_0 *and* X_1 *are contained in a larger Banach space in such a* *way that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0 and Z_1. Suppose that $Y_i \subset Z_i$ and that there is a continuous linear mapping $D\colon X_i \to Z_i$. Define Banach spaces*

$$X_i(D) = \{u \in X_i : Du \in Y_i\}$$

with the graph norm. Suppose that there exist continuous linear mappings $G\colon Z_i \to X_i$ and $K\colon Z_i \to Y_i$ such that $D \circ Gf = f + Kf$ for all $f \in Z_i$. Then we can define D by complex interpolation as a mapping from $[X_0, X_1]_\alpha$ to $[Z_0, Z_1]_\alpha$ and the natural inclusion $[Y_0, Y_1]_\alpha \subset [Z_0, Z_1]_\alpha$, $0 < \alpha < 1$. It follows that

$$[X_0(D), X_1(D)]_\alpha = \{f \in [X_0, X_1]_\alpha : Df \in [Y_0, Y_1]_\alpha\}.$$

Let $\Omega' = \mathbf{R}^n \setminus \bar{\Omega}$. Consider $X_0 = L^p_{\alpha_0}$, $X_1 = L^p_{\alpha_1}$, $Z_0 = L^p_{\alpha_0}(\Omega')$, and $Z_1 = L^p_{\alpha_1}(\Omega')$. Let Y_i be the zero subspace of Z_i, and let $D\colon X_i \to Z_i$ be given by $D = R_{\Omega'}$. If G denotes Stein's extension operator from $L^p_\alpha(\Omega')$ to L^p_α, then $D \circ G$ is the identity on Z_i. Note that $L^p_{\alpha_i, 0}(\Omega) = \{f \in X_i\colon Df = 0\}$. Thus the proposition follows from Lemma 2.12.

Let us introduce Besov spaces. For $0 < \alpha < 1$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant q < \infty$, define $B^{p,q}_\alpha$ as the space of all functions f in L^p such that the norm

$$\|f\|_{L^p} + \left(\int_{\mathbf{R}^n} \frac{\|f(x+t) - f(x)\|^q_{L^p}}{|t|^{n+\alpha q}} \, dt \right)^{1/q}$$

is finite. In the case $q = \infty$, the L^q norm in t is replaced by the supremum norm:

$$\|f\|_{L^p} + \sup_{t \neq 0} \frac{\|f(x+t) - f(x)\|_{L^p}}{|t|^\alpha}.$$

In particular, for $1 \leqslant p < \infty$, we have

$$B^p_\alpha \equiv B^{p,p}_\alpha = \left\{ f \in L^p(\mathbf{R}^n) : \iint \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy < \infty \right\}.$$

When $p = \infty$, $q = \infty$, the Besov space is also known as a Hölder class. For $0 < \alpha < 1$, denote

$$B^\infty_\alpha \equiv B^{\infty,\infty}_\alpha = \{f \in L^\infty(\mathbf{R}^n) : |f(x) - f(y)| \leqslant C\,|x-y|^\alpha\}.$$

For $0 < \alpha < 2$ there is a definition involving second differences, and there are many other definitions, valid for all real values of α (see [2,19].) We define $B^{p,q}_\alpha$ as the space of distributions for which the norm

$$\|f\|_{B^{p,q}_\alpha} = \|(I - \Delta)^{(\alpha - 1/2)/2} f\|_{B^{p,q}_{1/2}}$$

is finite. The fact that this definition is equivalent to the one given above for $0 < \alpha < 1$ and the equivalence with many other definitions can be found in [2, Theorem 6.2.7; 19]. Singular integral operator bounds imply that for all real α and $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$,

$$f \in B_\alpha^{p,q} \quad \text{if and only if} \quad f \in B_{\alpha-1}^{p,q} \quad \text{and} \quad \nabla f \in B_{\alpha-1}^{p,q}. \tag{2.13}$$

(See [2, Exercise 30, p. 167, or Theorem 6.2.5].) Similarly, we have the corresponding result for $1 < p < \infty$ and all real α,

$$f \in L_\alpha^p \quad \text{if and only if} \quad f \in L_{\alpha-1}^p \quad \text{and} \quad \nabla f \in L_{\alpha-1}^p \tag{2.14}$$

(see [2]).

Interpolation by the real and complex methods gives

$$[B_{\alpha_0}^{p_0,q}, B_{\alpha_1}^{p_1,q}]_\theta = B_\alpha^{p,q}, \qquad [B_{\alpha_0}^{p,q_1}, B_{\alpha_1}^{p,q_2}]_{\theta,q} = B_\alpha^{p,q} \tag{2.15}$$

provided $1 \leqslant p_1, p_2, q_1, q_2, q \leqslant \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\alpha_0 \neq \alpha_1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. The main connection between Sobolev and Besov spaces is the real interpolation result

$$[L^p, W_k^p]_{\alpha,q} = B_{\alpha k}^{p,q} \qquad \text{for} \quad 1 \leqslant p \leqslant \infty \quad \text{and} \quad 0 < \alpha < 1, \tag{2.16}$$

where k is any positive integer. See [1, p. 341].

Define the space $B_\alpha^{p,q}(\Omega)$ as the set of $R_\Omega f$ with $f \in B_\alpha^{p,q}$, with the usual quotient norm.

PROPOSITION 2.17. (a) *$[L^p(\Omega), W_k^p(\Omega)]_{\alpha,q} = B_{\alpha k}^{p,q}(\Omega)$ for $1 \leqslant p, q \leqslant \infty$ and $0 < \alpha < 1$, k a positive integer.*

(b) *Stein's extension operator, E, is a bounded operator from $B_\alpha^{p,q}(\Omega)$ to $B_\alpha^{p,q}$, for all $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$, and $[B_{\alpha_0}^{p,q_1}(\Omega), B_{\alpha_1}^{p,q_2}(\Omega)]_{\theta,q} = B_\alpha^{p,q}(\Omega)$, provided $1 \leqslant p_1, p_2, q_1, q_2, q \leqslant \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\alpha_0 \neq \alpha_1$, $\alpha_0 > 0$, $\alpha_1 > 0$, and $1/p = (1-\theta)/p_0 + \theta/p_1$.*

Proof. The argument leading to (2.5) with real interpolation instead of complex interpolation yields (a). Part (b) then follows from real interpolation and the reiteration theorem for real interpolation [2, Theorem 3.5.3].

PROPOSITION 2.18. (a) *Suppose that $\alpha > 0$ and $1 < p < \infty$. Then $f \in L_{\alpha+1}^p(\Omega)$ if and only if $f \in L^p(\Omega)$ and $\nabla f \in L_\alpha^p(\Omega)$.*

(b) *Suppose that $\alpha > 0$, $1 \leqslant p, q \leqslant \infty$. Then $f \in B_{\alpha+1}^{p,q}(\Omega)$ if and only if $f \in L^p(\Omega)$ and $\nabla f \in B_\alpha^{p,q}(\Omega)$.*

Proof. In both (a) and (b) the "only if" assertion is trivial. We recall the formula for Stein's extension operator. Suppose that the domain Ω is given by the region above the graph of a Lipschitz function ϕ, i.e.,

$$\Omega = \{x = (x', x_n) : x_n > \phi(x'), x' \in \mathbf{R}^{n-1}\}.$$

Following [22], there exists a function $\delta^*(x)$ which is an infinitely differentiable function in Ω comparable to the distance to the boundary and satisfying

$$\delta^*(x', x_n) \geqslant 2(x_n - \phi(x')); \qquad |(\partial/\partial x)^\beta \delta^*(x)| \leqslant C^\beta \delta^*(x)^{1-|\beta|}$$

for every multi-index β. There is also a continuous function ψ defined on $[1, \infty)$ satisfying $\psi(\lambda) = O(\lambda^{-N})$ for every N and

$$\int_1^\infty \psi(\lambda)\, d\lambda = 1; \qquad \int_1^\infty \lambda^k \psi(\lambda)\, d\lambda = 0, \quad \text{for } k = 1, 2, ...$$

The extension operator E is given by

$$Ef(x) = \int_1^\infty f(x', x_n + \lambda\delta^*(x))\, \psi(\lambda)\, d\lambda.$$

Note that

$$\partial_j Ef = E(\partial_j f) + T_j(\partial_n f) \tag{1.19}$$

where $\partial_j = \partial/\partial x_j$ and

$$T_j f(x) = \int_1^\infty f(x', x_n + \lambda\delta^*(x))\, \lambda\partial_j\delta^*(x)\, \psi(\lambda)\, d\lambda.$$

The exact same argument [22, p. 187] showing that E is a bounded operator from $W_k^p(\Omega)$ to W_k^p shows that the same is true for T_j, $j = 1, 2, ..., n$. It follows from complex interpolation and Proposition 2.4 that both E and T_j are bounded from $L_\alpha^p(\Omega)$ to L_α^p. Next, if $f \in L^p(\Omega)$ and $\partial_j f \in L_\alpha^p(\Omega)$, then formula (1.19) shows that Ef belongs to $L_{\alpha+1}^p$. This proves (a) in the case of a region above a graph. The general case follows using a suitable partition of unity. The proof of (b) is similar, using real interpolation and Proposition 2.17(b) in place of complex interpolation.

3. Boundary Values of Sobolev and Besov Functions

Consider a region Ω above the graph of a Lipschitz function ϕ. For $0 < s \leqslant 1$, define the space $B_s^p(\partial\Omega)$ as the space of functions

$f(x, \phi(x)) = g(x)$ where $g \in B_s^p(\mathbf{R}^{n-1})$. This definition can be extended to boundaries of all bounded Lipschitz domains in an obvious way using a partition of unity. For $0 \leqslant s \leqslant 1$, $L_s^p(\partial\Omega)$ is defined in a similar way. We will also make use of the tangential component of the gradient $\nabla_T f$ for functions $f \in L_1^p(\partial\Omega)$. This may be defined almost everywhere as $\nabla_T f(x, \phi(x)) = \nabla_x f(x, \phi(x))$.

THEOREM 3.1. *Suppose that* $1/p < \alpha < 1 + 1/p$ *and* $s = \alpha - 1/p$. *If* $1 < p < \infty$, *then the mapping* Tr, *initially defined on* $C^\infty(\bar{\Omega})$ *as the restriction to* $\partial\Omega$, *extends to a bounded linear operator from* $L_\alpha^p(\Omega)$ *to* $B_s^p(\partial\Omega)$. *If* $1 \leqslant p \leqslant \infty$, *then the mapping* Tr *extends to a bounded linear operator from* $B_\alpha^p(\Omega)$ *to* $B_s^p(\partial\Omega)$. *Moreover, there is a linear operator* $\mathscr{E}$ *that maps* $B_s^p(\partial\Omega)$ *to* $L_\alpha^p \cap B_\alpha^p$ *for all* s, $0 < s < 1$, *and all* p, $1 < p < \infty$. *In addition,* $\mathscr{E}$ *maps* $B_s^p(\partial\Omega)$ *to* B_α^p *for* $p = 1$ *and* $p = \infty$, *and* $\mathrm{Tr} \circ \mathscr{E}$ *is the identity operator on* $B_s^p(\partial\Omega)$.

Theorem 3.1 is a special case of a theorem of Jonsson and Wallin [19]. For the case of smooth domain see [22, p. 193]. The extension operator $\mathscr{E}$ of [19] can be defined as

$$\mathscr{E}f(x) = \sum_Q c_Q \phi_Q$$

where the sum is taken over cubes Q of a Whitney decomposition of the complement of $\partial\Omega$ in $\mathbf{R}^n$, the functions ϕ_Q form a partition of unity subordinate to the Whitney decomposition, and the coefficients c_Q are obtained as averages of f over "cubes" in the boundary near Q of size comparable to Q.

There is a substantial change in the character of the trace operator on the boundary of a Lipschitz domain in the case $\alpha = 1 + 1/p$. Jonsson and Wallin have characterized the trace of L_α^p for all values of α. But when $\alpha \geqslant 1 + 1/p$, the function spaces on the boundary look very different and the extension operator is not the same as the one above. For example, in the case $s = \alpha - 1/p = 1$, the function space which is the image under the trace operator Tr of $B_{1+1/p}^p$ and of $L_{1+1/p}^p$ is the space of functions f such that there exist first degree polynomials $P_Q(x)$ satisfying

$$\sum_Q \int_{Q^* \cap \partial\Omega} |f(x) - P_Q(x)|^p \, d\sigma(x) \, d(Q)^{-p} < \infty$$

and

$$\sum_Q \sum_{Q'} \int_{Q^* \cap \partial\Omega} |\nabla P_Q(x) - \nabla P_{Q'}(x)|^p \, d\sigma(x) < \infty,$$

where $d(Q)$ is the diameter of Q, the first sum is over cubes Q in the Whitney decomposition and the second is over cubes Q' of size comparable to Q, which are near Q. The cube Q^* is a cube with the same center as Q and diameter a multiple of $d(Q)$, in such a way that the sets $Q^* \cap \partial\Omega$ for cubes of each dyadic diameter cover $\partial\Omega$.

Even in the case of a C^1 domain, this trace space does not coincide with one of the obvious definitions of the corresponding space in the smooth case. In the smooth case, it is well known that the trace of $L^p_{1+1/p}$ to $\partial\Omega$ is the space $B^p_1(\partial\Omega)$. In particular, when $1 < p \leqslant 2$, $B^p_1(\partial\Omega) \subset L^p_1(\partial\Omega)$. However, the following example, due to Guy David, shows that in the Lipschitz case functions in the trace space of $L^p_{1+1/p}$ need not have a tangential derivative in $L^p(\partial\Omega)$. Consider a periodic sawtooth region, Ω_ε, with period 2ε and with slopes alternating between 1 and -1. In other words, define

$$\Omega_\varepsilon = \{(x, y) \in \mathbf{R}^2 \colon y > \varepsilon\phi(x/\varepsilon)\},$$

where

$$\phi(x) = x \quad \text{for } 0 \leqslant x \leqslant 1; \qquad \phi(x) = 2 - x \quad \text{for } 1 \leqslant x \leqslant 2;$$

and

$$\phi(x+2) = \phi(x).$$

It is well-known that there are unbounded functions in $L^p_{1/p}(\mathbf{R})$. Thus we can choose $f \in L^p_{1+1/p}(\mathbf{R})$ such that $f'(y)$ tends to infinity as y tends to 0, e.g., $f(y) = y(\log 10/|y|)^\beta$ for suitable β. Let $g(x, y) = \theta(x, y) f(y)$ where θ is a smooth function equal to 1 in B_2 and compactly supported in B_3.

In the region where $\theta = 1$, the tangential derivative of g equals $(\pm 1/\sqrt{2}) f'(y)$. Therefore, the L^p norm of the tangential derivative of g on $\partial\Omega$ is at least $|f'(\varepsilon)|$ which tends to infinity as ε tends to zero. Thus there can be no estimate on the trace operator dependent only on the Lipschitz constant.

Let us modify David's example slightly to construct a C^1 domain on which the smooth trace estimate fails. The domain constructed here will be used again in Section 7.

PROPOSITION 3.2. *Let $1 < p \leqslant 2$. There is a C^1 domain $\Omega \subset \mathbf{R}^2$ and a function $g \in L^p_{1+1/p}(\mathbf{R}^2)$ whose trace on $\partial\Omega$ does not have a tangential derivative in $L^p(\partial\Omega)$.*

(An example of this sort is of no interest in the case $2 < p < \infty$ because in that range B^p_1 is not contained in L^p_1. In other words, if $2 < p < \infty$, the conclusion holds in every smooth domain.)

Proof. Let f and g be as above. Define a_j and b_j so that $a_j = jb_j$ and b_j tends to zero so fast that $(1/j) f'(2b_j) \geqslant 2^j$ for sufficiently large j. Let ϕ denote a smooth function $C_0^\infty(-1, 1)$ such that $0 \leqslant \phi \leqslant 2$, $\phi(x) = 1 + x$ for $-1/2 < x < 1/2$, and $\max |\phi'| \leqslant 2$. Consider $I^j = [2^{-j-1}, 2^{-j}]$, $j = 3, 4, ...,$ and N_j disjoint subintervals $I_k^j(*)$, $k = 1, ..., N_j$, of equal length $2a_j$, whose union has measure at least half the measure of I^j. (Thus N_j must satisfy $2^{-j-2} \leqslant 2a_j N_j \leqslant 2^{-j-1}$.) Let x_k^j be the center of $I_k^j(*)$, and define I_k^j as the interval of length a_j centered at x_k^j.

Consider

$$\rho(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{N_j} b_j \phi((x - x_k^j)/a_j)$$

It is easy to see that because b_j/a_j tends to zero, ρ is of class C^1. Furthermore,

$$\rho'(x) = b_j/a_j = 1/j \qquad \text{for all} \quad x \in I_k^j$$

Let D be the unbounded domain

$$D = \{(x, y) : y > \rho(x)\}$$

The tangential derivative $\nabla_T g$ can be written as

$$\nabla_T g(x, \rho(x)) = \frac{1}{\sqrt{1 + \rho'(x)^2}} \frac{d}{dx} g(x, \rho(x)) = \frac{1}{\sqrt{1 + \rho'(x)^2}} \rho'(x) f'(\rho(x))$$

for all $(x, \rho(x))$ belonging to the set where $\theta = 1$. It follows that for j sufficiently large,

$$\nabla_T g(x, \rho(x)) \approx (1/j) f'(2b_j) \geqslant 2^j \qquad \text{for all} \quad x \in I_k^j.$$

The length of union over k of the intervals I_k^j is greater than a constant times 2^{-j}, so even the L^1 norm of the tangential derivative is divergent.

Proposition 3.3. *Suppose that $1/p < \alpha < 1 + 1/p$, $1 < p < \infty$, and Ω is a bounded Lipschitz domain in $\mathbf{R}^n$. Then $L^p_{\alpha,0}(\Omega)$ is the space of all functions in $L^p_\alpha(\Omega)$ whose boundary trace is* 0.

Proof. Recall Strichartz's characterization of $L^p_\alpha(\mathbf{R}^n)$ [25].

Theorem 3.4. *Let $1 < p < \infty$, $0 < \alpha < 1$. Denote*

$$(S_\alpha f)(x) = \left(\int_0^\infty \left(\int_{|y| < 1} |f(x + ry) - f(x)| \, dy \right)^2 \frac{dr}{r^{1+2\alpha}} \right)^{1/2}.$$

A function f belongs to L^p_α if and only if f belongs to L^p and $S_\alpha f$ belongs to L^p. In addition,

$$\|f\|_{L^p_\alpha} \text{ is comparable to } \|f\|_{L^p} + \|S_\alpha f\|_{L^p}.$$

An easy consequence of Strichartz's theorem is the following.

PROPOSITION 3.5. *Let Ω be a bounded Lipschitz domain. Suppose that $1 < p < \infty$ and $0 \leqslant \alpha < 1/p$. Then*

$$\|\chi_\Omega f\|_{L^p_\alpha} \leqslant C\, \|f\|_{L^p_\alpha}.$$

Proof. The result can be localized to a neighborhood of a boundary point. Furthermore, since a bi-Lipschitz change of variables preserves L^p and W^p_1, it also preserves L^p_α for all α, $0 < \alpha < 1$. Thus we can reduce matters to checking this property in the case in which Ω is the half-space $\{x : x_n > 0\}$. Here, we follow Strichartz. Denote by T_k the Fourier multiplier

$$\widehat{(T_k f)}(\xi) = |\xi_k|^\alpha \hat{f}(\xi).$$

The Marcinkiewicz multiplier theorem [22] implies that the L^p_α norm is comparable to

$$\|f\|_{L^p} + \|T_1 f\|_{L^p} + \cdots + \|T_n f\|_{L^p}.$$

When $k < n$, $T_k(\chi_\Omega f) = \chi_\Omega T_k f$ and L^p estimates follow immediately. When $k = n$, the estimate will follow from Fubini's theorem and the corresponding estimates in the case of one variable. In the one-variable case, an easy calculation shows that

$$S_\alpha(fg)(x) \leqslant \|g\|_{L^\infty} S_\alpha f(x) + |f(x)|\, S_\alpha g(x).$$

If $g = \chi_{(0,\infty)}$, then $S_\alpha g(x) \leqslant C\,|x|^{-\alpha}$. Thus

$$S_\alpha(\chi_{(0,\infty)} f)(x) \leqslant S_\alpha f(x) + C\,|f(x)|\,|x|^{-\alpha} \tag{3.6}$$

The fractional integral estimate says $f \in L^q$ for $1/q = 1/p - \alpha$, and hence, by Hölder's inequality, $|f|\,|x|^{-\alpha} \in L^p$ for $\alpha < 1/p$. This implies

$$\|\chi_{(0,\infty)} f\|_{L^p_\alpha(\mathbf{R})} \leqslant C\, \|f\|_{L^p_\alpha(\mathbf{R})}.$$

In other words, the inequality of Proposition 3.5 holds in the one-variable case. Thus, by Fubini's theorem,

$$\|T_n(\chi_\Omega f)\|_{L^p(\mathbf{R}^n)} \leqslant C(\|T_n f\|_{L^p(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)}).$$

This concludes the proof.

LEMMA 3.7. *If* $1/p < \alpha < 1 + 1/p$, *then*

$$\int_0^\infty |f(x) - f(0)|^p \, x^{-\alpha p} \, dx \leqslant C \, \|f\|^p_{L^p_\alpha(\mathbf{R})}.$$

Proof. Because f belongs to $L^p_\alpha(\mathbf{R})$ it can be written as

$$f(x) = \int_{-\infty}^\infty K(x-y) \, g(y) \, dy$$

for some $g \in L^p(\mathbf{R})$ and K satisfying $|K(x)| \leqslant |x|^{\alpha-1}$ and $|K'(x)| \leqslant |x|^{\alpha-2}$. Then

$$f(x) - f(0) = \int_{-\infty}^\infty (K(x-y) - K(-y)) \, g(y) \, dy$$

and $|K(x-y) - K(-y)| \leqslant Cx \, |y|^{\alpha-2}$ for $|y| > 2x > 0$ and $|K(x-y) - K(-y)| \leqslant C(|x-y|^{\alpha-1} + |y|^{\alpha-1})$ for $2x \leqslant |y|$. The estimate in Lemma 3.7 then follows from applications of Hardy's inequality [22, Appendix A.4]. The details are left to the reader.

Next, we state an immediate consequence of the inequality (3.6).

LEMMA 3.8. *If* $1/p < \alpha < 1 + 1/p$, *then*

$$\|\chi_{(0,\infty)} f\|_{L^p_\alpha(\mathbf{R})} \leqslant C \left[\|f\|_{L^p_\alpha(\mathbf{R})} + \left(\int_0^\infty |f(x)|^p \, x^{-\alpha p} \, dx \right)^{1/p} \right].$$

Suppose that $1 < p < \infty$, $1/p < \alpha < 1 + 1/p$; we now prove Proposition 3.3 by showing that if f belongs to $L^p_\alpha(\Omega)$ and $\operatorname{Tr} f = 0$, then

$$\|\chi_\Omega f\|_{L^p_\alpha} \leqslant C \, \|f\|_{L^p_\alpha(\Omega)}. \tag{3.9}$$

The case of one variable ($n = 1$) follows from Lemmas 3.7 and 3.8. A bi-Lipschitz change of variables reduces (3.9) to the case of a half-space when $1/p < \alpha \leqslant 1$. In the case of a half-space, estimate (3.9) can be reduced to the one-variable case using the multipliers T_i in the same way as in the proof of Proposition 3.5. In order to prove (3.9) in the case $1 < \alpha < 1 + 1/p$, it suffices to consider the region above a Lipschitz graph. Thus, let $\Omega = \{x : x_n > \phi(x)\}$ where ϕ is a Lipschitz function. Calderón's extension operator [3] is given by

$$E_1 f(x) = \int K(x-y) \frac{x-y}{|x-y|} \cdot \chi_\Omega(y) \, \nabla f(y) \, dy \tag{3.10}$$

with K supported in a cone pointing up (inside Ω), K homogeneous of degree $1-n$, $K \in C^{\infty}(\mathbf{R}^n \backslash \{0\})$, and

$$\int_{|x|=1} K(x)\, d\sigma(x) = 1.$$

One can calculate that $E_1 f(x) = f(x)$ for $x \in \Omega$ and if $\operatorname{Tr} f = 0$ (on $\partial\Omega$), then $E_1 f(x) = 0$ for all $x \in \mathbf{R}^n \backslash \bar{\Omega}$. On the other hand, $f \in L^p_\alpha$ implies $\nabla f \in L^p_{\alpha-1}$. Now since $\alpha - 1 < 1/p$, Proposition 3.5 implies that $\chi_\Omega \nabla f \in L^p_{\alpha-1}$. Since $\nabla K(x)$ is the kernel of a singular integral operator, convolution with $K_1(x) = K(x)\, x/|x|$ is smoothing of order 1. Hence $\chi_\Omega f = K_1 * (\chi_\Omega \nabla f)$ belongs to L^p_α, with the desired norm estimate.

Proposition 3.3 and Remark 2.7 imply the following.

COROLLARY 3.11. *Let $1/p < \alpha < 1 + 1/p$. Then $C_0^\infty(\Omega)$ is dense in the space of functions of $L^p_\alpha(\Omega)$ with boundary trace 0.*

PROPOSITION 3.12. *Suppose that $1/p < \alpha < 1 + 1/p$, $1 \leqslant p < \infty$, and Ω is a bounded Lipschitz domain. Then $C_0^\infty(\Omega)$ is dense in the space of all functions in $B^p_\alpha(\Omega)$ whose boundary trace is 0.*

Proof. Consider a function $f \in B^p_\alpha$ with $\operatorname{Tr} f = 0$ on $\partial\Omega$. Then there exist C^∞ functions f_j which tend to f in B^p_α. It follows from the continuity of the trace operator that $\operatorname{Tr} f_j$ tends to zero in $B^p_{\alpha-1/p}(\partial\Omega)$. Consider the Jansson–Wallin extension operator $\mathscr{E}$ acting on these functions: $g_j = \mathscr{E}(\operatorname{Tr} f_j)$. Then g_j tends to zero in B^p_α. It follows that $f_j - g_j$ tends to f in B^p_α and that $\operatorname{Tr}(f_j - g_j) = 0$. For each fixed j, $f_j \in L^q_s$ for all s and hence, by Theorem 3.1, $g_j \in L^q_s$ for all $s < 1 + 1/q$. Fix values of q and s so that $qs > \alpha p$ and $s < 1 + 1/q$. The density result we just proved for Sobolev spaces implies that there is a function $h_j \in C_0^\infty(\Omega)$ such that $\|h_j - (f_j - g_j)\|_{L^q_s(\Omega)} \leqslant 1/j$. Since the L^q_s norm is larger than the B^p_α norm, h_j tends to f in $B^p_\alpha(\Omega)$.

Lemma 3.8 suggests the conjecture that for any Lipschitz domain Ω, if $\delta(x)$ denotes $\operatorname{dist}(x, \partial\Omega)$, then

$$L^p_{1/p,0}(\Omega) = \left\{ f \in L^p_{1/p}(\Omega) : \int_\Omega |f(x)|^p\, \delta(x)^{-1}\, dx < \infty \right\}. \tag{3.13}$$

We expect a similar characterization for $L^p_{k+1/p,0}(\Omega)$ for every integer k.

The reader may be surprised to learn that the function space $L^p_{1/p,0}(\Omega)$ is not the closure of $C_0^\infty(\Omega)$ in $L^p_{1/p}(\Omega)$. It is not hard to show that $C_0^\infty(\Omega)$ is dense in $L^p_\alpha(\Omega)$ for $0 \leqslant \alpha \leqslant 1/p$. The defining requirement for the function space $L^p_{1/p,0}(\Omega)$ that the extension vanish identically outside Ω is a significant extra assumption on the extension in the borderline case $\alpha = 1/p$. The norms on the spaces $L^p_{1/p}(\Omega)$ and $L^p_{1/p,0}(\Omega)$ are not equivalent. We refer to Lions and Magenes [20] for the case $p=2$. We will not pursue characterizations of these function spaces here since they are not needed for our treatment of the Dirichlet problem.

4. Harmonic Sobolev Spaces

Recall that $\delta(x)$ is the distance from x to $\partial\Omega$. Define $\nabla^k u$ as the vector of all kth order derivatives of a function u.

THEOREM 4.1. *Suppose that u is a harmonic function in Ω. Let $0<\alpha<1$, let k be a nonnegative integer, and let $1 \leqslant p \leqslant \infty$. Then the following are equivalent*:

(a) *u belongs to $B^p_{k+\alpha}(\Omega)$*,

(b) *$\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$.*

THEOREM 4.2. *Suppose that u is a harmonic function in Ω. Let $0 \leqslant \alpha \leqslant 1$, let k be a nonnegative integer, and let $1<p<\infty$. Then the following are equivalent*:

(a) *u belongs to $L^p_{k+\alpha}(\Omega)$*,

(b) *$\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$.*

Proposition 2.18 shows that in Theorems 4.1 and 4.2 it suffices to prove the case $k=0$. Henceforth, let $k=0$.

Proof that (b) *Implies* (a) *in Theorem* 4.1. Recall that Proposition 2.17 says

$$B^p_\alpha(\Omega) = [L^p(\Omega), W^p_1(\Omega)]_{\alpha,p}.$$

We make use of the norm for the interpolation space on the right-hand side given in terms of traces [20, 2]. Thus, the $B^p_\alpha(\Omega)$ norm of a function u is comparable to the infimum over all functions $f\colon [0,\infty) \mapsto L^p(\Omega) + W^p_1(\Omega)$ with $f(0)=u$ of

$$\left(\int_0^\infty \|t^{1-\alpha}f(t)\|^p_{W^p_1(\Omega)}\, t^{-1}\,dt\right)^{1/p} + \left(\int_0^\infty \|t^{1-\alpha}f'(t)\|^p_{L^p(\Omega)}\, t^{-1}\,dt\right)^{1/p}.$$

Since u belongs to W^p_1 for any $O \subset\subset \Omega$, interior estimates are straightforward, and only estimates in a small, but fixed, neighborhood of the boundary are required. There is a constant $r>0$ depending only on the domain such that if B is the ball of radius r centered at a boundary point, then, after a suitable rigid motion, the boundary point is the origin and

$$B \cap \Omega = \{(x', y) \in B \colon y > \phi(x')\} \qquad \text{and} \qquad \phi(0) = 0,$$

where $|\nabla\phi| \leqslant M$. Moreover, $\delta(x)$ is comparable to $y - \phi(x')$ for $x = (x', y) \in B$. Choose $\eta \in C_0^\infty(B)$ such that $\eta(x) = 1$ on the concentric ball of radius $r/2$ and $\theta \in C_0^\infty((-r, r))$ such that $\theta(y) = 1$ for $|y| < r/2$. Define $g(t) = \eta(x)\, u(x', y + t)\, \theta(t)$. Then $g(0) = \eta u$, and

$$\int_0^\infty \|t^{1-\alpha} g(t)\|_{L^p(\Omega)}^p\, t^{-1}\, dt \leqslant \int_0^r \int_{B\cap\Omega} |u(x, y)|^p\, dx\, dy t^{p-\alpha p-1}\, dt \leqslant C\, \|u\|_{L^p(\Omega)}^p,$$

because $p - \alpha p - 1 > -1$. The first derivative in x and t of $g(t)$ is a sum of terms for which the derivative falls on one of the cut-off functions η or θ which are all controlled by the preceding inequality plus one other term:

$$\int_0^\infty \int_{B\cap\Omega} |t^{1-\alpha}\theta(t)\, \eta(x)\, \nabla u(x + (0, t))|^p\, dx\, t^{-1}\, dt.$$

Note that for $x \in B \cap \Omega$, $t \leqslant C\,\delta(x + (0, t))$. Thus we can change variables and exchange the order of integration to show that this integral is less than

$$\int_{B\cap\Omega} |\nabla u(x)|^p \int_0^{C\,\delta(x)} t^{p-\alpha p-1}\, dt\, dx,$$

which is less than

$$C \int_{B\cap\Omega} |\delta(x)^{1-\alpha}\, \nabla u(x)|^p\, dx.$$

Proof that (a) *Implies* (b) *in Theorem* 4.1. Suppose that $u \in B_\alpha^p(\Omega)$. Then

$$\int_\Omega \int_\Omega \frac{|u(x) - u(z)|^p}{|x - z|^{n + p\alpha}}\, dx\, dz < \infty.$$

Because u is harmonic,

$$\begin{aligned}
|\nabla u(x)| &\leqslant \frac{C}{\delta(x)^{n+1}} \int_{B(x,\,\delta(x)/2)} |u(z) - u(x)|\, dz \\
&\leqslant C\,\delta(x)^{-1} \left(\frac{1}{\delta(x)^n} \int_{B(x,\,\delta(x)/2)} |u(z) - u(x)|^p\, dz\right)^{1/p} \\
&\leqslant C\,\delta(x)^{\alpha-1} \left(\int_{B(x,\,\delta(x)/2)} \frac{|u(z) - u(x)|^p}{|x - z|^{n+\alpha p}}\, dz\right)^{1/p}
\end{aligned}$$

and hence

$$\int_{\Omega} |\delta(x)^{1-\alpha} \nabla u(x)|^p \, dx < \infty.$$

This concludes the proof of Theorem 4.1.

Proof that (a) *Implies* (b) *in Theorem* 4.2. The theorem is obvious for $\alpha = 1$. Moreover, for $\alpha = 0$ we can deduce the equivalence by observing that for a cube Q in Ω whose double Q^* is also contained in Ω, and whose distance to $\partial\Omega$ is comparable to the diameter of Q,

$$\int_{Q} |\delta \nabla u|^p \, dx \leqslant \int_{Q^*} |u|^p \, dx$$

for any harmonic function in Ω. The equivalence then follows from summing over a disjoint covering of Ω by such cubes (a Whitney covering). Next we wish to deduce the result for all intermediate values of α from complex interpolation. We need to show that

$$[\mathscr{H} \cap L^p(\Omega), \mathscr{H} \cap W_1^p(\Omega)]_\alpha = \mathscr{H} \cap L_\alpha^p(\Omega), \tag{4.3}$$

where $\mathscr{H}$ is the space of harmonic functions in Ω.

We apply Lemma 2.12 with $X_0 = L^p(\Omega)$, $X_1 = L_\alpha^p(\Omega)$. Choose a cube Q containing $\bar{\Omega}$. For $0 \leqslant \alpha \leqslant 1$, define $Z_\alpha = L_{\alpha-2,0}^p(Q)$ and

$$Y_\alpha = \{ f \in Z_\alpha \colon f(\psi) = 0 \text{ for all } \psi \in C_0^\infty(\Omega) \} = L_{\alpha-2,0}^p(Q \backslash \bar{\Omega}).$$

The operator D is given by $D = \Delta \circ E$, where E is Stein's extension operator, $E\colon X_0 \to L^p(\mathbf{R}^n)$ and $E\colon X_1 \to L_1^p(\mathbf{R}^n)$. We may assume that E maps to functions supported in Q, so that D maps X_i to Z_i, $i = 0, 1$. The operator G is given by $G = R \circ N$, where R is the restriction map from functions on $\mathbf{R}^n$ to functions on Ω and N is convolution with the Newtonian potential—the inverse operator to Δ. For a distribution f in Z_0 we calculate $DGf - f$ for $\psi \in C_0^\infty(\Omega)$ by

$$(DGf - f)(\psi) = \int (EGf)\, \Delta\psi - f(\psi) = \int (Nf)\, \Delta\psi - f(\psi) = 0.$$

Therefore, the operator K defined by $Kf = DGf - f$ maps X_i to Y_i, and the hypothesis of Lemma 2.12 is satisfied. We also have

$$\mathscr{H} \cap L_\alpha^p(\Omega) = \{ f \in L_\alpha^p(\Omega) \colon Df \in Y_\alpha \},$$

so the Lemma 2.12 implies

$$[\mathscr{H} \cap L^p(\Omega), \mathscr{H} \cap W_1^p(\Omega)]_\alpha = \{ f \in L_\alpha^p(\Omega) \colon Df \in [Y_0, Y_1]_\alpha \}$$

Corollary 2.10 implies $[Y_0, Y_1]_\alpha = Y_\alpha$, which concludes the proof of formula (4.3).

The mapping $u \mapsto \nabla u$ is continuous from $\mathscr{H} \cap L^p(\Omega)$ to $L^p(\Omega, \delta(x)^p\, dx)$ and from $\mathscr{H} \cap L_1^p(\Omega)$ to $L^p(\Omega)$. The theorem of Stein and Weiss [24] says that

$$[L^p(w_0(x)\, dx), L^p(w_1(x)\, dx)]_\theta = L^p(w(x)\, dx),$$

where $w = w_0^{1-\theta} w_1^\theta$. Therefore, by complex interpolation the mapping $u \mapsto \nabla u$ is continuous from $\mathscr{H} \cap L_\theta^p(\Omega)$ to $L^p(\Omega, \delta(x)^{(1-\theta)p}\, dx)$.

Proof that (b) *Implies* (a) *in Theorem* 4.2. When $p \leqslant 2$, $B_\alpha^p(\Omega) \subset L_\alpha^p(\Omega)$, so the assertion is a special case of Theorem 4.1. Thus we assume that $p > 2$. The proof that follows uses some deep facts about harmonic functions on Lipschitz domains. Another less difficult proof, valid in all cases except $\alpha = 1/p$, is presented following this one.

Let $D = \{x \in \mathbf{R}^n : r < \phi(x/|x|)\}$ for some Lipschitz function $\phi > 0$ on the unit sphere. Thus D is star-shaped with respect to the origin. Every Lipschitz domain is covered by a finite collection (of translates) of such star-shaped domains. It is easy to see that it suffices to prove that (b) implies (a) separately for each such domain. Furthermore, the value of the harmonic function u at a single interior point of D is dominated by a multiple of the $L^p(D)$ norm of u, so we can subtract this constant and restrict to the space

$$\mathscr{H}_0 = \{u : \Delta u = 0 \text{ in } D \text{ and } u(0) = 0\}.$$

For $\operatorname{Re} z > 0$ define the operator $T_z : \mathscr{H}_0 \to \mathscr{H}_0$ by

$$T_z u(x) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} u(e^{-t}x)\, dt.$$

Note that the integral is absolutely convergent for all $x \in D$ because $u(e^{-t}x) \leqslant Ce^{-t}$. The operator T_z is a fractional integral operator based on dilation from the origin, with the special property that it preserves harmonic functions and is analytic in z. (We use starshaped domains instead of multiplication by a smooth cut-off function because the cut-off procedure is incompatible with the limitations of regularity theory in Lipschitz domains: the first derivative of Green's function need not belong to L^p for large p.)

Proposition 4.4. *The operator T_z can be extended by continuity to be defined for* $\operatorname{Re} z = 0$. *Moreover,*

(a) *If* $\operatorname{Re} z_1 \geqslant 0$ *and* $\operatorname{Re} z_2 \geqslant 0$, *then* $T_{z_1} \circ T_{z_2} = T_{z_1+z_2}$.

(b) *If* $0 \leqslant \alpha = \operatorname{Re} z \leqslant 1$, *then* T_z *is a bounded operator from* $L^p(D) \cap \mathscr{H}_0$ *to* $L^p_\alpha(D) \cap \mathscr{H}_0$.

Proof. Let $f(s) = u(e^{-s}x)$. Then

$$\begin{aligned}(T_z u)(e^{-s}x) &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} u(e^{-t-s}x)\, dt \\ &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} f(s+t)\, dt \\ &= \int_{-\infty}^\infty f(s-t)\, k_z(t)\, dt\end{aligned}$$

where

$$k_z(t) = \frac{1}{\Gamma(z)} (-t)^{z-1} \quad \text{if} \quad t<0 \qquad \text{and} \qquad k_z(t) = 0 \quad \text{if} \quad t>0.$$

One can calculate by contour integration that the Fourier transform of k_z is given by

$$\hat{k}_z(\xi) = (\xi - i0)^{-z}\, e^{\pi i z/2}. \tag{4.5}$$

It follows that the operator T_z can be defined by continuity in z for $\operatorname{Re} z = 0$. Thus T_{iy} is a bounded Fourier multiplier operator along each ray to the origin with the bound

$$\int_0^\infty |T_{iy} u(e^{-t}x)|^p\, dt \leqslant C e^{C|y|} \int_0^\infty |u(e^{-t}x)|^p\, dt.$$

Integrating over $x \in \partial D$ we find

$$\int_D |T_{iy} u|^p\, |x|^{-n}\, dx \leqslant C e^{c|y|} \int_D |u|^p\, |x|^{-n}\, dx$$

for $1 < p < \infty$. Since $u \in \mathscr{H}_0$ we also have

$$|u(x)|^p \leqslant C\, |x|^p \int_D |u(x')|^p\, dx'$$

and hence

$$\int_D |T_{iy} u|^p\, |x|^{-n}\, dx \leqslant C e^{c|y|} \int_D |u|^p\, dx.$$

In particular, $T_{i\gamma}$ is a bounded operator on $L^p(D) \cap \mathscr{H}_0$. The composition law follows from direct calculation using Euler's formula for the beta integral or the formula for the Fourier transform (4.5). This proves Proposition 4.4(a).

Recall that (4.3) says that $L^p_\alpha(D) \cap \mathscr{H}$ is a complex interpolation scale. It is an easy exercise to use Lemma 2.12 to deduce from (4.3) that $L^p_\alpha(D) \cap \mathscr{H}_0$ is a complex interpolation scale as well. To prove (b) we interpolate between $\operatorname{Re} z = 0$ where we have already proved the estimate and $\operatorname{Re} z = 1$. The estimate for the case $\operatorname{Re} z = 1$ is proved as follows. Because $T_{1+i\gamma} = T_1 \circ T_{i\gamma}$, it suffices to prove that T_1 is bounded from $L^p(D) \cap \mathscr{H}_0$ to $L^p_\alpha(D) \cap \mathscr{H}_0$. In other words, we need only confirm estimate (b) in the case $z = 1$. This assertion can be restated as follows.

LEMMA 4.6. *Let $1 \leqslant p < \infty$. Let $u \in \mathscr{H}_0$ and let $v = T_1 u$. Then*

$$\int_D |\nabla v(x)|^p \, dx \leqslant C \int_D |u(x)|^p \, dx.$$

Proof. One can calculate that

$$x \cdot \nabla v(x) = u(x). \tag{4.7}$$

Let $R_{ij} = x_i \partial_{x_j}$. Then

$$\begin{aligned}\int_D |R_{ij} v(x)|^p \, dx &= \int_D \left| \int_{1/2}^1 \frac{d}{ds} R_{ij} v(sx) \, ds + R_{ij} v(x/2) \right|^p dx \\ &= \int_D \left| \int_{1/2}^1 \frac{1}{s} R_{ij} u(sx) \, ds + R_{ij} v(x/2) \right|^p dx.\end{aligned}$$

The term $R_{ij} v(x/2)$ is an interior term, which is easy to bound. To handle the main term recall that if u is a harmonic function in a ball B^* of radius r and B is the concentric ball of half the radius, then

$$\sup_B |\nabla u|^p \leqslant C r^{-n-1} \int_{B^*} |u(x)|^p \, dx. \tag{4.8}$$

It follows that

$$\begin{aligned}&\int_D \left| \int_{1/2}^1 \frac{1}{s} R_{ij} u(sx) \, ds \right|^p dx \\ &\quad \leqslant C \int_D \left| \int_{1/2}^1 \delta(sx)^{-n-1} \int_{B(sx,\, \delta(sx)/2)} |u(z)| \, dz \, ds \right|^p dx. \end{aligned} \tag{4.9}$$

The set of $z \in B(sx, \delta(sx)/2)$ for some s between $\frac{1}{2}$ and 1 is contained in the set $\Gamma(x) = \{z : \delta(x) + |x - z| \leqslant C\,\delta(z)\}$ for a suitable C. Let x^* be the dilate of x on the boundary of D. $\Gamma(x)$ is the portion of the nontangential cone above x^* whose distance from ∂D is at least $\delta(x)$. Using Fubini's theorem and a dyadic partition into sets where the integrand is roughly constant $(\delta(z) \approx 2^k\,\delta(x))$, we obtain

$$\int_{1/2}^{1} \delta(sx)^{-n-1}\,\chi_{B(sx,\,\delta(sx))}(z)\,ds \leqslant C\,\delta(z)^{-n}\,\chi_{\Gamma(x)}(z).$$

It follows that (4.9) is majorized by

$$\int_D \left| \int_{\Gamma(x)} \delta(z)^{-n}\,|u(z)|\,dz \right|^p dx.$$

Finally, we need to show

$$\int_D \left| \int_{\Gamma(x)} \delta(z)^{-n}\,|u(z)|\,dz \right|^p dx \leqslant C \int_D |u(x)|^p\,dx. \tag{4.10}$$

Introduce the measure $d\mu(x) = \delta(x)^{-\varepsilon}\,dx$ and $w(x) = |u(x)|\,\delta(x)^{\varepsilon/p}$. Then (4.10) is the same as

$$\int_D \left(\int_D K(x, z)\,w(z)\,d\mu(z) \right)^{1/p} d\mu(x) \leqslant C \int_D w(z)^p\,d\mu(z)$$

with $K(x, z) = \chi_{\Gamma(x)}\,\delta(x)^{\varepsilon/p}\,\delta(z)^{-n+\varepsilon/p'}$ and $1/p' + 1/p = 1$. It is easy to check that if $0 < \varepsilon < 1$,

$$\sup_{x \in D} \int_D K(x, z)\,d\mu(z) = \sup_x \delta(x)^{\varepsilon/p} \int_{\Gamma(x)} \delta(z)^{-n-\varepsilon/p}\,dz \leqslant C \tag{4.11}$$

and

$$\sup_{z \in D} \int_D K(x, z)\,d\mu(x) = \sup_z \int_{\Gamma(z)} \delta(x)^{-\varepsilon/p'}\,dx\,\delta(z)^{-n+\varepsilon/p'} \leqslant C. \tag{4.12}$$

This proves (4.10) and hence Lemma 4.6 and Proposition 4.4.

LEMMA 4.13. *If $1 \leqslant p < \infty$, $\alpha > 0$, and $v \in \mathscr{H}_0$, then*

$$\|T_\alpha v\|_{L^p(D)} \leqslant C\,\|\delta^\alpha v\|_{L^p(D)}.$$

Proof. Let $L(t) = t^{1/p}(1-t)_+^{\alpha-1}$. Then for $\alpha > 0$ and $p < \infty$, we have

$$\int_0^\infty L(t)\,t^{-1}\,dt < \infty.$$

Hence, for $1 \leqslant p \leqslant \infty$,

$$\int_0^\infty \left| \int_0^\infty L(t/s)\, g(s)\, s^{-1}\, ds \right|^p t^{-1}\, dt \leqslant C \int_0^\infty |g(t)|^p\, t^{-1}\, dt \tag{4.14}$$

Let $f(t) = t^{-\alpha - 1/p} g(t)$ and

$$(K_\alpha f)(t) = \int_0^\infty (t-s)_+^{\alpha-1} f(s)\, ds.$$

Then inequality (4.14) is equivalent to

$$\int_0^\infty |K_\alpha f(t)|^p\, dt \leqslant C \int_0^\infty |t^\alpha f(t)|^p\, dt.$$

Finally, with the notation $f(s) = v(e^{-s}x)$, $(T_\alpha v)(e^{-s}x) = (K_\alpha f)(s)$, and so it follows that

$$\int_D |T_\alpha v(x)|^p\, |x|^{-n}\, dx \leqslant C \int_D |\delta(x)^\alpha\, v(x)|^p\, |x|^{-n}\, dx.$$

As before, routine interior estimates imply

$$\int_D |\delta(x)^\alpha\, v(x)|^p\, |x|^{-n}\, dx \leqslant \int_D |\delta(x)^\alpha\, v(x)|^p\, dx$$

and Lemma 4.13 follows.

We can now combine the lemmas to prove that (b) implies (a) in Theorem 4.2. Given a harmonic function u such that $u(0) = 0$ and $\delta^{1-\alpha} |\nabla u| + |u| \in L^p(D)$, let $w = x \cdot \nabla u$. Then $w \in \mathscr{H}_0$, and Lemma 4.13 implies $T_{1-\alpha} w \in L^p(D) \cap \mathscr{H}_0$. Hence, by Lemma 4.4, $T_\alpha(T_{1-\alpha} w) \in L^p_\alpha(D)$. But $T_\alpha \circ T_{1-\alpha} = T_1$ and $T_1 w = u$, so we are done.

The proof that (b) implies (a) in Theorem 4.2 just given was somewhat complicated. Moreover, it seems natural that the implication should not depend on whether u is harmonic. We have only been able to show this when $\alpha \neq 1/p$. We present the proof below. The case $\alpha \neq 1/p$ is the only case that will be needed in the sequel.

PROPOSITION 4.15. *Let Ω be a bounded Lipschitz domain. Let $0 \leqslant \alpha \leqslant 1$, let k be a nonnegative integer, and let $1 < p < \infty$. In the case $p > 2$, assume further that $\alpha \neq 1/p$. Then*

$$\|u\|_{L^p_{k+\alpha}(\Omega)} \leqslant C \left\| \delta^{1-\alpha} |\nabla^{k+1} u| + |\nabla^k u| + |u| \right\|_{L^p(\Omega)}$$

Proof. As usual we can assume $k=0$. Make a bi-Lipschitz change of variables and localize to reduce to the case in which the function u has compact support in $\bar{\Omega}$ and the domain Ω is the upper half-space. Note that the implication that (b) implies (a) in Theorem 4.1 did not depend on the fact that u is harmonic. Thus as before the case $p \leqslant 2$ is already proved. Hence we can make the further assumption that $\alpha \neq 1/p$.

Denote

$$\widehat{(\Lambda^z f)}(\xi) = |\xi|^z \hat{f}(\xi).$$

LEMMA 4.16. *For every $g \in L^p_{1,0}(\Omega)$, $0 \leqslant \alpha \leqslant 1$,*

$$\|\Lambda^\alpha V(y^{\alpha-1}g)\|_{L^p} + \|y^{-1}g\|_{L^p(\Omega)} \leqslant C\, \|\nabla g\|_{L^p(\Omega)}$$

where V denotes the even extension of a function from the upper half-space to $\mathbf{R}^n$.

Proof. It suffices to take $g \in C_0^\infty(\Omega)$. Then $g(x, 0) = 0$, and Hardy's inequality implies

$$\int |y^{-1}g(x, y)|^p\, dy = \int_0^\infty \left| y^{-1} \int_0^y (\partial/\partial s)\, g(x, s)\, ds \right|^p dy$$

$$\leqslant C \int_0^\infty |(\partial/\partial y)\, g(x, y)|^p\, dy,$$

provided $p > 1$. In other words, we have

$$\|y^{-1}Vg\|_{L^p} \leqslant C\, \|\nabla g\|_{L^p(\Omega)}.$$

We can now apply Stein's interpolation theorem for analytic families of operators that the mapping $g \mapsto \Lambda^z V(y^{z-1}g)$ as an operator from $L^p_{1,0}(\Omega)$ to L^p. For $z = i\gamma$ we have

$$\|\Lambda^{i\gamma} V(y^{i\gamma-1}g)\|_{L^p} \leqslant C\, \|V(y^{i\gamma-1}g)\|_{L^p} \leqslant C\, \|\nabla g\|_{L^p(\Omega)}.$$

For $z = 1 + i\gamma$ we have

$$\|\Lambda^{1+i\gamma} V(y^{i\gamma}g)\|_{L^p} \leqslant C\, \|\nabla V(y^{i\gamma}g)\|_{L^p} \leqslant C(\|y^{-1}g\|_{L^p(\Omega)} + \|\nabla g\|_{L^p(\Omega)}).$$

It follows that the operator is bounded for all values of z with real part between 0 and 1. In particular, it is bounded for $z = \alpha$, which is the assertion in Lemma 4.16.

We first treat the case $\alpha < 1/p$. Consider a function u satisfying $y^{1-\alpha}|\nabla u| + |u| \in L^p(\Omega)$ with support in $0 \leqslant y \leqslant 1$. Because $\alpha < 1/p$, Hardy's inequality implies

$$\int_0^\infty |y^{-\alpha}u(x, y)|^p\, dy = \int_0^\infty \left| y^{-\alpha} \int_y^\infty (\partial/\partial s)\, u(x, s)\, ds \right|^p dy$$

$$\leqslant C \int_0^\infty |y^{1-\alpha}(\partial/\partial y)\, u(x, y)|^p\, dy$$

Therefore, the function $g = y^{1-\alpha}u$ satisfies $\nabla g \in L^p(\Omega)$. Furthermore,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\varepsilon^{2\varepsilon} |g(x, y)|^p\, dx\, dy = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\varepsilon^{2\varepsilon} y^{(1-\alpha)p} |u(x, y)|^p\, dx\, dy = 0,$$

because $u \in L^p$ and $(1-\alpha)\,p - 1 > 0$. It follows that $\operatorname{Tr} g = 0$, i.e., $g \in L^p_{1,0}(\Omega)$. Lemma 4.16 implies $\Lambda^\alpha V u \in L^p$, and, consequently, $u \in L^p_\alpha(\Omega)$.

Next, consider the case $\alpha > 1/p$. We showed in the proof of Theorem 4.1 that $u \in B^p_\alpha(\Omega)$. It follows from the trace theorem (Theorem 3.1) that $f(x) = u(x, 0)$ belongs to $B^p_{\alpha - 1/p}(\mathbf{R}^{n-1})$. Define $f(x, y) = f * \phi_y(x)$, where $\phi_y(x) = y^{-n+1}\phi(x/y)$, $\phi \in C_0^\infty(\mathbf{R}^{n-1})$, and $\int \phi = 1$. Let $w = u - f$. It is well known that $f \in L^p_\alpha(\Omega)$ [22, p. 193] and also that $y^{1-\alpha}\nabla f \in L^p(\Omega)$ [22, p. 150–154]. Therefore, $y^{1-\alpha}\nabla w \in L^p(\Omega)$ and it suffices to prove that $w \in L^p_\alpha(\Omega)$. But $\operatorname{Tr} w = 0$, so we can write

$$w(x, y) = \int_0^y (\partial/\partial s)\, w(x, s)\, ds$$

for almost every x. Since $\alpha > 1/p$, it follows from Hardy's inequality that

$$\int_0^\infty |y^{-\alpha}w(x, y)|^p\, dy = \int_0^\infty \left| y^{-\alpha} \int_0^y (\partial/\partial s)\, w(x, s)\, ds \right|^p dy$$

$$\leqslant C \int_0^\infty |y^{1-\alpha}(\partial/\partial y)\, w(x, y)|^p\, dy.$$

Therefore, as in the previous case, $g = y^{1-\alpha}w \in L^p_{1,0}(\Omega)$ and Lemma 4.16 implies that $w = y^{\alpha-1}g \in L^p_\alpha(\Omega)$.

We do not know whether Proposition 4.15 is true or false in the exceptional case $\alpha = 1/p$ and $p > 2$ $(n \geqslant 2)$, despite the fact that this is just a question about function spaces on the upper half-space.

5. The Homogeneous Problem

Our goal is to prove the following.

THEOREM 5.1. *Consider ε such that $0 < \varepsilon \leqslant 1$. Define p_0 and p_0' by $1/p_0 = (1+\varepsilon)/2$ and $1/p_0' = (1-\varepsilon)/2$. Let s and p be numbers satisfying one of the following*:

(a) $p_0 < p < p_0'$ *and* $0 < s < 1$.

(b) $1 < p \leqslant p_0$ *and* $2/p - 1 - \varepsilon < s < 1$.

(c) $p_0' \leqslant p < \infty$ *and* $0 < s < 2/p + \varepsilon$.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$ for some $n \geqslant 3$. There exists ε depending only on the Lipschitz constant of Ω such that for every $g \in B_s^p(\partial\Omega)$ there exists a unique harmonic function v such that $\operatorname{Tr} v = g$ and $v \in L_{s+1/p}^p(\Omega)$. Also, the solution v belongs to $B_{s+1/p}^p(\Omega)$. If Ω is a C^1 domain, then we can take $p_0 = 1$ ($\varepsilon = 1$).

This theorem is best understood by looking at a picture in $(s, 1/p)$-space. In Fig. 2, the points that are labelled are

$$A_1 = (0, 0), \quad A_2 = (\varepsilon, 0), \quad B_1 = (0, 1/2), \quad B_2 = (0, 1/p_0),$$
$$C_3 = (0, 1), \quad C_2 = (1-\varepsilon, 1), \quad C_1 = (1, 1),$$
$$D_1 = (1, 1/2), \quad D_2 = (1, 1/p_0'), \quad D_3 = (1, 0).$$

The theorem says that estimates are valid for the interior of the polygon $A_1B_2C_2C_1D_2A_2$. In the C^1 case, the figure is the interior of the unit square $A_1C_3C_1D_3$. Consider the parallelogram P given by $A_1B_1C_1D_1$. The region described by (a), (b), and (c) above is the subset of the unit square of points within some small positive distance of the parallelogram P.

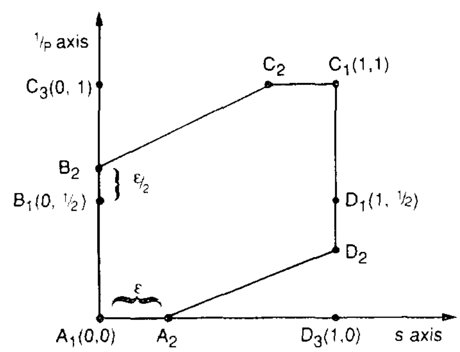

FIGURE 2

We now review known estimates. They include estimates that correspond to several edges of the region in which we want to prove the theorem. We will then prove new estimates on the segment C_2C_1 and the segment C_1D_2.

We begin with the classical estimate of Kellogg. (For a proof, see [15], for example.)

THEOREM 5.2. *Let Ω be a bounded Lipschitz domain. There exists $\alpha_0 > 0$ such that for all α, $0 < \alpha < \alpha_0$, and all $g \in B_\alpha^\infty(\partial\Omega)$, the solution v to $\Delta v = 0$ in Ω and $v = g$ on $\partial\Omega$ satisfies*

$$\|v\|_{B_\alpha^\infty(\Omega)} + \sup_\Omega \delta^{1-\alpha} |\nabla v| \leqslant \|g\|_{B_\alpha^\infty(\partial\Omega)}.$$

When $n = 2$, then we can take $\alpha_0 > \frac{1}{2}$. Moreover, in every dimension, if Ω is of class C^1 we can take $\alpha_0 = 1$.

The next two results, the fundamental theorems of Dahlberg [7, 8], concern the edge A_1B_2, although the function spaces are not the Sobolev or Besov spaces. Denote by $\delta(x)$ the distance from $x \in \Omega$ to $\partial\Omega$. Denote by $\Gamma(z)$ a nontangential cone with vertex at z, that is,

$$\Gamma(z) = \{x \in \mathbf{R}^n \colon |x - z| < C\,\delta(x)\},$$

for a suitable constant $C > 1$. The nontangential maximal function v^* of a function v is defined by

$$v^*(z) = \sup_{x \in \Gamma(z)} |v(x)|.$$

THEOREM 5.3 [7]. *There exists $p_0 < 2$ such that for all p, $p_0 < p \leqslant \infty$, and for all $g \in L^p(\partial\Omega)$ there exists a unique v such that $\Delta v = 0$ in Ω, v tends nontangentially to g almost everywhere on $\partial\Omega$, and v satisfies*

$$\|v^*\|_{L^p(\partial\Omega)} \leqslant C\, \|g\|_{L^p(\partial\Omega)}.$$

When Ω is of class C^1, then we can take $p_0 = 1$.

Next, define the area integral of v by

$$S(v)(z) = \left(\int_{\Gamma(z)} \delta(x)^{2-n}\, |\nabla u(x)|^2\, dx\right)^{1/2}.$$

THEOREM 5.4 [8]. *If v is a harmonic function in a Lipschitz domain Ω and $0 < p < \infty$, then there is a constant depending only on the Lipschitz constant of Ω such that*

$$\|S(v)\|_{L^p(\partial\Omega)} \leqslant C\, \|v^*\|_{L^p(\partial\Omega)}.$$

If, in addition, $u(x^0) = 0$ *for some fixed point* x^0 *in* Ω, *then*

$$\|v^*\|_{L^p(\partial\Omega)} \leq C \, \|S(v)\|_{L^p(\partial\Omega)}.$$

We single out for special attention the case of the point B_1.

COROLLARY 5.5. *Suppose that* v *is a harmonic function in* Ω. *Then the following are equivalent*:

(a) $v^* \in L^2(\partial\Omega)$

(b) $v \in L^2_{1/2}(\Omega)$

(c) $S(v) \in L^2(\partial\Omega)$, *or equivalently*,

$$\int_\Omega \delta(x) \, |\nabla v(x)|^2 \, dx < \infty,$$

and in each case there exists a function $g \in L^2(\partial\Omega)$ *such that* v *tends to* g *nontangentially, a.e. on* $\partial\Omega$.

The equivalence of (a) and (c) is Theorem 5.4. The equivalence of (b) and (c) is given by Theorems 4.1 and 4.2. (Recall that $L^2_{1/2}(\Omega) = B^2_{1/2}(\Omega)$; see [22, p. 155].)

Next, we have a result on the segment $C_1 D_2$. Define the space $L^p_1(\partial\Omega)$ as the space of functions f for which $f(x', \phi(x')) = g(x')$ for some $g \in L^p_1(\mathbf{R}^{n-1})$, when $\partial\Omega$ is given locally as the graph of a Lipschitz function ϕ.

THEOREM 5.6. *There exists* $q > 2$, *such that for* $g \in L^p_1(\partial\Omega)$ *and* $1 < p < q$, *there exists* v *such that* $\Delta v = 0$ *in* Ω, *and* v *tends nontangentially to* g *almost everywhere on* Ω *and*

$$\|v^*\|_{L^p(\partial\Omega)} + \|(\nabla v)^*\|_{L^p(\partial\Omega)} \leq C \, \|g\|_{L^p_1(\partial\Omega)}.$$

When Ω *is of class* C^1 *we can take* $q = \infty$.

This theorem was proved in [10] for C^1 domains, in [14] for Lipschitz domains when $p = 2$, and in general in [26].

In particular, a result at D_1 is as follows.

COROLLARY 5.7. *If* v *is a harmonic function in* Ω *then the following are equivalent.*

(a) $(\nabla v)^* \in L^2(\partial\Omega)$

(b) $v \in L^2_{3/2}(\Omega)$

(c) $S(\nabla v) \in L^2(\partial\Omega)$, *or equivalently*,

$$\int_\Omega \delta(x)\, |\nabla^2 v(x)|^2\, dx < \infty.$$

When (a), (b), *and* (c) *are satisfied, there exists a function* $g \in L_1^2(\partial\Omega)$ *such that* v *tends to* g *nontangentially, a.e. on* $\partial\Omega$, *and* $\mathrm{Tr}\, v = g$.

This corollary follows from Corollary 5.5 and the fact that ∇v is harmonic.

The new estimate associated with the segment $C_1 C_2$ is as follows.

THEOREM 5.8. *There exists* $\varepsilon > 0$ *depending on* Ω *such that, for all* α, $1 - \varepsilon < \alpha < 1$, *and all* $g \in B_\alpha^1(\partial\Omega)$, *there is a unique solution* v *to the homogeneous Dirichlet problem*

$$\begin{aligned} \Delta v &= 0 \qquad \text{on } \Omega, \\ \mathrm{Tr}\, v &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

satisfying

$$\int_\Omega \delta^{1-\alpha}\, |\nabla^2 v| + |\nabla v| + |v| \leqslant C\, \|g\|_{B_\alpha^1(\partial\Omega)}.$$

In particular, $\nabla v \in B_\alpha^1(\Omega)$, *i.e.*, $v \in B_{1+\alpha}^1(\Omega)$.

Proof. We use the atomic characterization of $B_\alpha^1(\partial\Omega)$ [11]. An $(\alpha, 1)$-atom is a function a on $\partial\Omega$ satisfying

$$|a| \leqslant r^{\alpha+1-n}, \qquad |\nabla_T a| \leqslant r^{\alpha-n}, \qquad \text{and} \qquad \operatorname{supp} a \subset D$$

for some "surface ball" $D = B(z, r) \cap \partial\Omega$ of size r. Every $g \in B_\alpha^1(\partial\Omega)$ has the form $g = \sum s_k a_k$ where each function a_k is an atom and $\sum |s_k| \leqslant C\, \|g\|_{B_\alpha^1(\partial\Omega)}$. It follows that to prove the theorem we need only check it in the case $g = a$, an atom. Since the domain Ω is bounded, it suffices to consider only $r < r_0$ for some fixed constant r_0, $0 < r_0 < 1$. For notational simplicity we suppose that the surface ball D on which a is supported has center at $z = 0$, the origin. Choose $M > 1$ and denote

$$U(\rho) = \{(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} \colon |x| < \rho \text{ and } |y| < 2M\rho\}.$$

We can suppose that in a neighborhood $U = U_{(r_1)}$ of 0, with $r_1 > 100 r_0$, the boundary $\partial\Omega$ is given by the graph of a Lipschitz function:

$$U \cap \Omega = U \cap \{(x, y) \colon y > \phi(x)\}$$

with $|\nabla\phi| \leqslant M$ and $\phi(0) = 0$. Harmonic measure ω^x is defined as the measure on $\partial\Omega$ such that

$$u(x) = \int_{\partial\Omega} f\, d\omega^x$$

for every harmonic function u with continuous boundary values f.

LEMMA 5.9. *There exists $\varepsilon > 0$ depending only on the Lipschitz constant of Ω such that if $x \in \Omega$ and $|x| \geqslant Mr$, then*

$$\omega^x(D) \leqslant C(r/|x|)^{\varepsilon + n - 2}.$$

Moreover, if $n = 2$, we can take $\varepsilon > \frac{1}{2}$. For every $n \geqslant 2$, if $\partial\Omega$ is of class C^1 then we can take ε arbitrarily close to 1.

Proof. This lemma is proved by comparison with the harmonic measure of an infinite cone.

LEMMA 5.10. *Let ρ satisfy $4r \leqslant \rho \leqslant r_1/16$. Then there exists ρ' such that $2\rho < \rho' < 4\rho$ and*

$$\int_{\Omega \cap \partial U(\rho')} |\nabla v|^2\, d\sigma \leqslant C\rho^{n-3} \max_{\Omega \cap (U(8\rho)\setminus U(\rho))} v^2 \leqslant C\rho^{n-3} r^{2(\alpha+1-n)} (r/\rho)^{2(\varepsilon+n-2)}.$$

Proof. Since v vanishes on $(U(8\rho)\setminus U(\rho)) \cap \partial\Omega$ it follows from Cacciopoli's inequality that

$$\int_{U(4\rho)\cap\Omega\setminus U(2\rho)} |\nabla v|^2 \leqslant C\rho^{-2} \int_{U(8\rho)\cap\Omega\setminus U(\rho)} v^2 \leqslant C\rho^{n-2} \max_{\Omega \cap (U(8\rho)\setminus U(\rho))} v^2.$$

By Fubini's theorem there is at least one number ρ', $2\rho < \rho' < 4\rho$, for which

$$\int_{\Omega \cap \partial U(\rho')} |\nabla v|^2 \leqslant C\rho^{-1} \int_{\Omega \cap (U(4\rho)\setminus U(2\rho))} |\nabla v|^2.$$

The first inequality of Lemma 5.10 follows. Finally, Lemma 5.9 says that if $x \in U(8\rho) \cap \Omega\setminus U(\rho)$, then

$$|v(x)| = \left| \int_{\partial\Omega} a\, d\omega^x \right| \leqslant C(r/\rho)^{\varepsilon+n-2} \max |a| \leqslant Cr^{\alpha+1-n}(r/\rho)^{\varepsilon+n-2}.$$

This implies the second inequality of Lemma 5.10.

Let N be the largest integer for which $2^{N+3}r < r_1$. Applying Lemma 5.10 with $\rho = 2^{k-1}r$, we find that for every integer k such that $3 \leqslant k \leqslant N$, there exists ρ_k such that $2^k r < \rho_k < 2^{k+1} r$ and

$$\int_{\Omega \cap \partial U(\rho_k)} |\nabla v|^2 \, d\sigma \leqslant C(2^k r)^{n-3} \, r^{2(\alpha+1-n)} (2^{-k})^{2(\varepsilon+n-2)}. \tag{5.11}$$

Define

$$R_k = \Omega \cap (U(\rho_{k+3}) \backslash U(\rho_k)) \quad \text{for} \quad k \geqslant 3 \quad \text{and} \quad R_2 = U(\rho_5) \cap \Omega$$

$$\delta_k(x) = \operatorname{dist}(x, \partial R_k) \quad \text{for} \quad x \in R_k; \quad \delta_k(x) = 0 \quad \text{for} \quad x \in \mathbf{R}^n \backslash R_k.$$

Let ∇_T denote the tangential component of the gradient on ∂R_k. For $k \geqslant 3$, v and $\nabla_T v$ vanish on $(\partial R_k) \cap \partial \Omega$. Also, $|\nabla_T v| \leqslant |\nabla v|$, so that (5.11) implies

$$\int_{\partial R_k} |\nabla_T v|^2 \, d\sigma \leqslant C(2^k r)^{n-3} \, r^{2(\alpha+1-n)} (2^{-k})^{2(\varepsilon+n-2)}. \tag{5.12}$$

For $k = 2$,

$$\int_{\partial R_2 \cap \partial \Omega} |\nabla_T v|^2 \, d\sigma = \int |\nabla_T a|^2 \, d\sigma \leqslant C r^{2\alpha - n - 1}.$$

This is the same as the bound given by (5.12) for the other portion of the boundary, $\Omega \cap \partial R_2$. Therefore, (5.12) is valid for $k = 2$ as well. Theorem 5.4 with $p = 2$ and Theorem 5.6 imply

$$\int_{R_k} \delta_k \, |\nabla^2 v|^2 \leqslant C \int_{\partial R_k} |\nabla_T v|^2 \, d\sigma.$$

(Note that this inequality is dilation invariant, so that R_k can be rescaled to unit size.) It follows from Schwarz's inequality that

$$\begin{aligned} \int_{R_k} \delta_k^{1-\alpha} \, |\nabla^2 v| &\leqslant \left(\int_{R_k} \delta_k \, |\nabla^2 v|^2 \right)^{1/2} \left(\int_{R_k} \delta_k^{1-2\alpha} \right)^{1/2} \\ &\leqslant C(2^k r)^{(n-2\alpha+1)/2} \left(\int_{\partial R_k} |\nabla_T v|^2 \, d\sigma \right)^{1/2}. \end{aligned} \tag{5.13}$$

Finally, (5.13) and (5.12) yield

$$\sum_{k=2}^{N} \int_{R_k} \delta_k^{1-\alpha} \, |\nabla^2 v| \leqslant C \sum_{k=2}^{N} (2^k)^{-\varepsilon+1-\alpha} \leqslant C'$$

because $-\varepsilon+1-\alpha<0$. The overlap of the sets R_k imply that $C\sum\delta_k\geqslant\delta$, so we have

$$\int_\Omega \delta^{1-\alpha}|\nabla^2 v|\leqslant C$$

whenever the harmonic function v has boundary value equal to an atom. The bound in Lemma 5.9 implies

$$\int_{R_k}|v|\leqslant Cr^{\alpha+1-n}(2^{-k})^{\varepsilon+n-2}\operatorname{vol}R_k.$$

Hence,

$$\int_\Omega|v|\leqslant\sum_{k=2}^N\int_{R_k}|v|\leqslant Cr^{\alpha+1}\sum_{k=2}^N(2^k)^{2-\varepsilon}=Cr^{\alpha-1+\varepsilon}<C,$$

because 2^N is comparable to $1/r$, $2-\varepsilon>0$, and $\alpha-1+\varepsilon>0$. For $k\geqslant 3$ we can estimate $|\nabla v|$ as in Lemma 5.10 by Cacciopoli's inequality

$$\begin{aligned}\left(\int_{R_k}|\nabla v|^2\right)^{1/2}&\leqslant C(2^kr)^{-1}r^{\alpha+1-n}(2^{-k})^{\varepsilon+n-2}(\operatorname{vol}R_k)^{1/2}\\&=Cr^{\alpha-n/2}2^{k(1-n/2-\varepsilon)}.\end{aligned}\tag{5.14}$$

In the case $k=2$, Theorem 5.6 implies

$$\int_{R_2}|\nabla v|^2\leqslant Cr\int_{\partial\Omega}((\nabla v)^*)^2\leqslant Cr\int_{\partial\Omega}|\nabla_T a|^2\leqslant Cr^{2\alpha-n}.$$

This is the same estimate as in (5.14) for larger k. Thus by Schwarz's inequality we have

$$\begin{aligned}\int_\Omega|\nabla v|&\leqslant C\sum_{k=2}^N\int_{R_k}|\nabla v|\leqslant C\sum_{k=2}^N\left(\int_{R_k}|\nabla v|^2\right)^{1/2}(2^kr)^{n/2}\\&\leqslant Cr^\alpha\sum_{k=2}^N(2^k)^{1-\varepsilon}\leqslant Cr^{\alpha-1+\varepsilon}<C.\end{aligned}$$

The final two inequalities are true because 2^N is comparable to $1/r$, $1-\varepsilon>0$, and $\alpha-1+\varepsilon>0$. This concludes the proof of existence and regularity in Theorem 5.8.

To prove uniqueness, it suffices to consider a function $v\in B^1_{1+\alpha}(\Omega)$ such that $\Delta v=0$ in Ω and $\operatorname{Tr}v=0$ on $\partial\Omega$. Proposition 3.12 says that there exists a sequence of functions $\phi_j\in C_0^\infty(\Omega)$ such that ϕ_j tends to v in $B^1_{1+\alpha}(\Omega)$. Choose smooth domains Ω_j such that $\bar{\Omega}_j\subset\Omega$, $\phi_j\in C_0^\infty(\Omega_j)$, and $\partial\Omega_j$ is uniformly Lipschitz and tends to $\partial\Omega$. If Tr_j denotes the restriction to $\partial\Omega_j$, then as $j \to \infty$,

$$\|\mathrm{Tr}_j v\|_{B^1_\alpha(\partial\Omega_j)} = \|\mathrm{Tr}_j(\phi_j - v)\|_{B^1_\alpha(\partial\Omega_j)} \leqslant \|\phi_j - v\|_{B^1_{1+\alpha}(\Omega)} \to 0.$$

The regularity estimate just proved says that

$$\|v\|_{B^1_{1+\alpha}(\Omega_j)} \leqslant C\,\|\mathrm{Tr}_j v\|_{B^1_\alpha(\partial\Omega_j)}.$$

Since the right-hand side tends to zero, $v \equiv 0$.

Next we derive estimates for the vertical segments A_1B_2 and C_1D_2. In contrast to the estimates for the horizontal segments C_1C_2 and A_1A_2, the boundary data in these estimates are in a function space that is smaller than the trace space. Nevertheless, these estimates will be sufficient for interpolation.

THEOREM 5.15. *There exists $p_0 < 2$ depending on Ω such that*

(a) *if $2 \leqslant p < \infty$ and $g \in L^p(\partial\Omega)$, then there exists a unique harmonic function $v \in L^p_{1/p}(\Omega)$ such that v converges nontangentially to g and $v^* \in L^p(\partial\Omega)$. For $p_0 < p \leqslant 2$ and $g \in L^p(\partial\Omega)$, there exists a unique harmonic function $v \in B^{p,2}_{1/p}(\Omega)$ such that v converges nontangentially to g and $v^* \in L^p(\partial\Omega)$.*

(b) *If $2 \leqslant p < p_0'$ and $g \in L^p_1(\partial\Omega)$, then there exists a unique harmonic function $v \in L^p_{1+1/p}(\Omega)$ with $\mathrm{Tr}\, v = g$. Moreover, v converges nontangentially to g. If $1 < p \leqslant 2$ and $g \in L^p_1(\partial\Omega)$, then there exists a unique harmonic function $v \in B^{p,2}_{1+1/p}(\Omega)$ such that $\mathrm{Tr}\, v = g$.*

(c) *If Ω is of class C^1, then we can take $p_0 = 1$.*

Proof. Uniqueness in part (a) is contained in the uniqueness statement of Theorem 5.3. For uniqueness in part (b), we consider a harmonic function v with trace 0 in the class $L^p_{1+1/p}(\Omega)$ or $B^{p,2}_{1+1/p}(\Omega)$. In either case, $v \in L^p_\alpha(\Omega)$ for $\alpha < 1 + 1/p$ and $v \equiv 0$ by Proposition 5.17 below.

For the proof of part (a), consider first the case $p \geqslant 2$. Theorem 5.4 and Corollary 5.5 imply that, for $g \in L^2(\partial\Omega)$, the solution v of the homogeneous Dirichlet problem ($\Delta v = 0$ in Ω and v tends to g nontangentially almost everywhere on $\partial\Omega$) satisfies $v \in L^2_{1/2}(\Omega)$. The maximum principle (and absolute continuity of harmonic measure with respect to surface measure on the boundary) implies that if $g \in L^\infty(\partial\Omega)$ then $v \in L^\infty(\Omega)$. Denote by E Stein's extension operator from functions on Ω to functions on $\mathbf{R}^n$. Let Λ^z denote the fractional integral operator as in Lemma 4.16. The mapping $g \mapsto \Lambda^{z/2}Ev$ for $\mathrm{Re}\, z = 0$ maps $L^\infty(\partial\Omega) \to BMO(\mathbf{R}^n)$. For $\mathrm{Re}\, z = 1$, it maps $L^2(\partial\Omega) \to L^2(\mathbf{R}^n)$. Therefore, by complex interpolation, when $z = 2/p$ it maps $L^p(\partial\Omega) \to L^p(\mathbf{R}^n)$, which proves that if $g \in L^p(\partial\Omega)$, then $v \in L^p_{1/p}(\Omega)$.

Next, we turn to the case $p_0 < p < 2$. Recall that the space $B_\alpha^{p,2}$ is a real interpolation space between L^p and W_1^p. As in the proof of Theorem 4.1, let $x = (x', y)$ and define $f(t) = \eta(x)\, v(x', y+t)\, \theta(t)$. We need to prove that

$$\int_0^\infty \|t^{1-\alpha} f(t)\|^2_{W_1^p(\Omega)}\, t^{-1}\, dt + \int_0^\infty \|t^{1-\alpha} f'(t)\|^2_{L^p(\Omega)}\, t^{-1}\, dt \leqslant C\, \|g\|^2_{L^p(\partial\Omega)}.$$

The left-hand side is dominated by the expression

$$\int_0^{r_1} t^{2-2/p} \left(\int_{|x'| < r_1} \int_t^{r_1} |\nabla v(x', \phi(x') + s)|^p\, dx'\, ds \right)^{2/p} \frac{dt}{t} \tag{5.16}$$

and the similar, but lower order, expression with ∇v replaced by v. Formula (5.16) equals

$$\int_0^{r_1} \left(\int_t^{r_1} (t/s)^{p-1} (s^p h(s))\, \frac{ds}{s} \right)^{2/p} \frac{dt}{t}$$

where

$$h(s) = \int_{|x'| < r_1} |\nabla v(x', \phi(x') + s)|^p\, dx'.$$

Now we can use Hardy's inequality (valid because $1 < p \leqslant 2$), and Minkowski's inequality for the exponent $2/p > 1$ to dominate 5.16 by

$$\begin{aligned} C \int_0^{r_1} (y^p h(y))^{2/p} \frac{dy}{y} &= C \int_0^{r_1} \left(\int_{|x'| < r_1} |\nabla v(x', \phi(x') + y)|^p\, dx' \right)^{2/p} y\, dy \\ &\leqslant C \left(\int_{|x'| < r_1} \left(\int_0^{r_1} y\, |\nabla v(x', \phi(x') + y)|^2\, y\, dy \right)^{p/2} dx' \right)^{2/p} \\ &\leqslant C \left(\int_{U \cap \partial\Omega} S(v)^p \right)^{2/p} \leqslant C\, \|g\|^2_{L^p(\partial\Omega)}. \end{aligned}$$

This controls the main term. The term with ∇v replaced by v is easily controlled using $v^* \in L^p(\partial\Omega)$ and $2 - 2/p > 0$.

For the proof of part (b) consider a function $g \in L_1^p(\partial\Omega)$ with $1 \leqslant p < p_0'$. Then by Theorem 5.6, $(\nabla v)^* \in L^p(\partial\Omega)$ and the proof of part (a) applied to ∇v in place of v implies that $\nabla v \in L^p_{1/p}(\Omega)$, for $2 \leqslant p < p_0'$ and $\nabla v \in B^{p,2}_{1/p}(\Omega)$ for $1 < p \leqslant 2$. Hence by Proposition 2.18 the solution v belongs to $L^p_{1+1/p}(\Omega)$ for $2 \leqslant p < p_0'$ and v belongs to $B^{p,2}_{1+1/p}(\Omega)$ for $1 < p \leqslant 2$. The fact that $\operatorname{Tr} v = g$ is obvious if $g \in C^\infty(\partial\Omega)$ and follows in case (b) by continuity of the trace operator and solution operator $g \mapsto v$.

Part (c) follows from the corresponding statement in Theorem 5.3.

Now we will use interpolation to prove Theorem 5.1. We begin with part 5.1(a). Recall that for $0<s<1$,

$$[L^p(\partial\Omega), W_1^p(\partial\Omega)]_{s,p} = B_s^p(\partial\Omega), \qquad [B_{1/p}^{p,2}(\Omega), B_{1+1/p}^{p,2}(\Omega)]_{s,p} = B_{s+1/p}^p(\Omega),$$

and

$$[L_{1/p}^p(\Omega), L_{1+1/p}^p(\Omega)]_{s,p} = B_{s+1/p}^p(\Omega).$$

Thus if $g \in B_s^p(\partial\Omega)$ and $p_0 < p < p_0'$, then by Theorem 5.15 and real interpolation, $v \in B_{s+1/p}^p(\Omega)$. It follows from Theorems 4.1 and 4.2 that $v \in L_{s+1/p}^p(\Omega)$, provided $0<s<1$ and $s+1/p \neq 1$. For the exceptional value $s+1/p=1$, we obtain the result by complex interpolation:

$$[B_{s_0}^p(\partial\Omega), B_{s_1}^p(\partial\Omega)]_\mu = B_s^p(\partial\Omega)$$

and

$$[L_{s_0+1/p}^p(\Omega), L_{s_1+1/p}^p(\Omega)]_\mu = L_{s+1/p}^p(\Omega),$$

with $0<s_0, s_1<1$ and $s=(1-\mu)s_0+\mu s_1$. Thus, for $0<\alpha<1$ and every $g \in B_s^p(\partial\Omega)$, we have found a harmonic function $v \in L_{\alpha+1/p}^p(\Omega)$ such that v tends to g nontangentially almost everywhere on $\partial\Omega$. The fact that $\operatorname{Tr} v = g$ follows from continuity of the trace operator.

The fact that $g \in B_s^p(\partial\Omega)$ implies $v \in B_{s+1/p}^p(\Omega)$ in the range specified by 5.1(b) follows by complex interpolation from the estimates of Theorems 5.1(a) and 5.8. Similarly, in the range of 5.1(c) it follows from complex interpolation of Theorems 5.1(a) and 5.2. Since v is harmonic, Theorems 4.1 and 4.2 imply that $v \in B_{s+1/p}^p(\Omega)$ implies $v \in L_{s+1/p}^p(\Omega)$, except in the case $s+1/p=1$. But then this case is treated by complex interpolation of the operator $g \mapsto v$ as in the proof of Theorem 5.1(a).

Finally, we prove uniqueness.

PROPOSITION 5.17. *Let Ω be a bounded Lipschitz domain and suppose that v satisfies $\Delta v = 0$ in Ω, $\operatorname{Tr} v = 0$ on $\partial\Omega$, and $v \in L_\alpha^p(\Omega)$ (or, equivalently, $v \in B_\alpha^p(\Omega)$) for some $\alpha > 1/p$. Then v is identically zero.*

Proof. This is a repetition of the uniqueness argument in Theorem 5.8. Since $v \in L_\alpha^p(\Omega)$, Proposition 3.11 says that there exists a sequence of functions $\phi_j \in C_0^\infty(\Omega)$ such that ϕ_j tends to v in $L_\alpha^p(\Omega)$. Choose smooth domains Ω_j such that $\bar{\Omega}_j \subset \Omega$, $\phi_j \in C_0^\infty(\Omega_j)$, and $\partial\Omega_j$ is uniformly Lipschitz and tends to $\partial\Omega$. If Tr_j denotes the restriction to $\partial\Omega_j$, then as $j \to \infty$,

$$\|\operatorname{Tr}_j v\|_{B_{\alpha-1/p}^p(\partial\Omega_j)} = \|\operatorname{Tr}_j(\phi_j - v)\|_{B_{\alpha-1/p}^p(\partial\Omega_j)} \leqslant C\,\|\phi_j - v\|_{L_\alpha^p(\Omega)} \to 0.$$

The regularity estimate just proved says that

$$\|v\|_{L^p_\alpha(\Omega_j)} \leqslant C\,\|\mathrm{Tr}_j v\|_{B^p_{\alpha-1/p}(\partial\Omega_j)}.$$

Since the right-hand side tends to zero, $v \equiv 0$.

6. The Inhomogeneous Problem and Counterexamples

Theorems 1.1 and 1.3 are straightforward consequences of the corresponding results in the homogeneous problem, Theorems 5.1 and 5.17, coupled with the results in Sections 2 and 3.

The counterexample of Theorem 1.2(a) is given by the complement of a slender cone. Similarly, the example in Proposition 1.4 is given by the complement of a narrow sector.

The example in Theorem 1.2(b) is somewhat more complicated. Recall that we wish to show that there exists a bounded C^1 domain D in $\mathbf{R}^2$ and a function u such that $\Delta u \in C^\infty(\bar{D})$, $u = 0$ on ∂D, but $\nabla^2 u \notin L^1(D)$.

Lemma 6.1. *There is a continuous function h supported on the unit interval, $[0, 1]$, satisfying*

(a) $\max |h| < \pi/4$.

(b) *If $h(x, y) = h * P_y(x)$ is the Poisson integral of h, then*

$$\int_0^1 \int_0^1 |\nabla h(x, y)|\, dx\, dy = \infty.$$

For a proof of Lemma 6.1, see [12, Chap. 6, Theorem 5.2 and Exercises 9a, 14b]. Let $v(x, y)$ be the conjugate harmonic function of h. Then

$$\log \Phi'(x+iy) = v(x, y) - ih(x, y)$$

is analytic in the upper half-plane. Let $\Phi(\mathbf{R}^2_+) = \Omega$. It is well known that Φ is a conformal mapping of $\mathbf{R}^2_+$ onto Ω. Ω is the region above the graph of a Lipschitz function and $|\arg \Phi'| = |h| < \pi/4$ implies that the Lipschitz constant is less than 1. Let Ψ denote the inverse of Φ. It is also well known that Φ and Ψ are continuous up to the boundary. We will consistently identify (x, y) with $x + iy$ in the sequel. The tangent vector to $\partial\Omega$ at z can be written in complex notation as $e^{-ih(\Psi(z))}$. This is a continuous function of z, so $\partial\Omega$ is C^1. Let $Q = \Phi([0, 1] \times [0, 1])$. $\partial\Omega$ is a straight line at all points z where $h(\Psi(z)) = 0$, i.e., outside Q. Consider the function $w = \mathrm{Im}\, \Psi$

on Ω. Then $w=0$ on $\partial\Omega$ and w is C^∞ up to the boundary, $\partial\Omega$, in the complement of Q. The Cauchy–Riemann equations imply

$$|\nabla^2 w| = |\Psi''| \qquad \text{and} \qquad |\nabla h| = |(\log \Phi')'|.$$

Moreover, $\Psi' \circ \Phi = 1/\Phi'$ implies $(\Psi'' \circ \Phi) = -\Phi''/(\Phi')^3$. Hence

$$\begin{aligned}\int_Q |\nabla^2 w| = \int_Q |\Psi''| &= \int_0^1 \int_0^1 |\Psi'' \circ \Phi(x+iy)|\,|\Phi'(x+iy)|^2\,dx\,dy \\ &= \int_0^1 \int_0^1 |\Phi''/\Phi'|\,dx\,dy = \int_0^1 \int_0^1 |(\log \Phi')'|\,dx\,dy \\ &= \int_0^1 \int_0^1 |\nabla h|\,dx\,dy = \infty.\end{aligned}$$

Choose a disk $B \supset Q$. Let $D = \Omega \cap B$. Let $u = \phi w$, where $\phi \in C_0^\infty(B)$ and $\phi = 1$ in a neighborhood of Q. Then $\Delta u \in C^\infty(\bar{D})$, $u=0$ on ∂D, but $|\nabla^2 u| \notin L^1(D)$.

Finally, we wish to prove Theorem 1.2(c). This counterexample requires quite a bit of work in the case $p>2$. With the notations of Proposition 3.2, let a_j and b_j be given by

$$b_j = e^{-2^{2^j}} \qquad \text{and} \qquad a_j = j b_j.$$

Recall that ρ is C^1 and

$$\rho'(x) = b_j/a_j = 1/j \qquad \text{for all} \quad x \in I_k^j.$$

D is the unbounded domain

$$D = \{(x, y) \colon y > \rho(x)\}.$$

Let Ω be a domain in $D \cap B_5$, where B_5 is the ball around the origin of radius 5, and such that $\partial\Omega \cap \partial D = B_3 \cap \partial D$. (In other words, Ω shares the interesting portion of its boundary with D.) We also assume that $\partial\Omega$ is infinitely differentiable except at the origin, where it coincides with ∂D. Let θ be a smooth function equal to 1 in B_2 and compactly supported in B_3.

Remark 6.2. *Let $g(x, y) = \theta(x, y)\, f(y)$, with θ a cut-off function as above and $f(y) = y(\log 10/|y|)^\beta$. If $\beta < 1 - 1/p$, then g belongs to $L^p_{1+1/p}(\mathbf{R}^2)$.*

Proof. If $f'(y)$ is cut-off in a neighborhood of 0, so that it can be viewed as an even periodic function of period 2π, then [27, Theorem 2.15, p. 188] shows that its Fourier series has the form

$$\sum_{n=0}^{\infty} n^{-1} b(n) \cos nx$$

for $b(n) \sim (\log 10n)^{\beta - 1}$. Fractional differentiation of order $1/p$ gives

$$\sum_{n=0}^{\infty} n^{1/p - 1} b(n) \cos ny.$$

It then follows from [27, Theorem 2.6, p. 187] that this cosine series is asymptotic as y tends to 0 to $y^{-1/p} b(1/y)$. This last expression belongs to L^p near $y = 0$ if and only if $(\beta - 1)\, p < -1$, which is the same as $\beta < 1 - 1/p$.

Another proof of the remark can be given making use of Strichartz's characterization of L^p_α using the operator S_α described in the Appendix. One can calculate

$$S_{1/p}(f')(y) \leqslant C(\log 10/|y|)^{\beta - 1} \, |y|^{-1/p},$$

which belongs to $L^p([0, 1])$ provided $\beta < 1 - 1/p$.

We now show that the harmonic function u in Ω with boundary values equal to g on $\partial\Omega$ does not belong to $L^p_{1+1/p}(\Omega)$.

The case $p = 2$ has a particularly simple proof. Since $g \in L^2_{3/2}(\Omega)$, $\Delta g \in L^2_{-1/2}(\Omega)$. Recall that $\operatorname{Tr} g \notin L^2_1(\partial\Omega)$, by Proposition 3.2. Now suppose that the solution to $\Delta u = \Delta g$ in Ω is such that $\operatorname{Tr} u = 0$ on $\partial\Omega$ satisfies $u \in L^2_{3/2}(\Omega)$. We derive a contradiction. Let $v = u - g$, then v is harmonic and $v \in L^2_{3/2}(\Omega)$ implies, by Corollary 5.7, $\nabla v \in L^2(\partial\Omega)$. But $u = 0$ on $\partial\Omega$, so $\operatorname{Tr} g \in L^2_1(\partial\Omega)$. This contradicts Proposition 3.2.

Here is the sketch of a proof in the case $1 < p < 2$ along the same lines. We solve $\Delta u = \Delta g$ in Ω with $\operatorname{Tr} u = 0$ and suppose that $u \in L^p_{1+1/p}(\Omega)$. Then $v = u - g \in L^p_{1+1/p}(\Omega)$ and, by Theorem 4.2, $\delta^{1-1/p} \, |\nabla^2 v| \in L^p(\Omega)$. But because $p \leqslant 2$, one can prove

$$\|S(\nabla v)\|_{L^p(d\sigma)} \leqslant C \, \|\delta^{1-1/p} \, \nabla^2 v\|_{L^p(\Omega)}.$$

This follows from Fubini's theorem and the fact that the L^2 average of $\nabla^2 v$ on a Whitney cube is dominated by the L^p average on a slightly larger cube. Finally, by Theorem 5.4,

$$\|\nabla v\|_{L^p(d\sigma)} \leqslant C \, \|S(\nabla v)\|_{L^p(d\sigma)}$$

for all p, $1 < p < \infty$. So we obtain, as before in the case L^2, that $g \in L^p_1(\partial\Omega)$, contradicting Proposition 3.2.

A proof of the type just described cannot work in the case $p > 2$ because the trace space for $L^p_{1+1/p}(\Omega)$ is larger than $L^p_1(\partial\Omega)$ even in the smooth case. The following more elaborate argument is valid for all p, $1 < p < \infty$.

Denote the Poisson kernel by

$$P_t(s) = \frac{1}{\pi} \frac{t}{s^2 + t^2}.$$

The harmonic extension v to the upper half-plane of a function f on the real line is given by

$$v(s, t) = f * P_t(s)$$

and the s derivative of v is given by the formula

$$(\partial/\partial s)\, v(s, t) = (1/\pi t)\, f * \psi_t(s),$$

where

$$\psi(s) = \frac{s}{(s^2+1)^2}; \qquad \psi_t(s) = t^{-1}\psi(s/t) = st^2/(s^2 + t^2)^2.$$

LEMMA 6.3. *Let N be an integer and $N\varepsilon = 1$. For $k = 1, \ldots, N$, consider the intervals $I_k = [(k-1)\,\varepsilon, k\varepsilon]$ and functions f_k supported in I_k satisfying*

$$\int_{I_k} f_k(s)\, ds = \varepsilon, \qquad |f_k(s)| \leqslant C, \qquad \text{and} \qquad |f_k'(s)| \leqslant C/\varepsilon.$$

Define

$$f(s) = \sum_{k=1}^{N} f_k(s).$$

Then

$$|\psi_t * f(s)| \leqslant C\left(\frac{\varepsilon}{t} + \frac{(|s|+1)\, t^2}{(s^2+t^2)((s-1)^2+t^2)}\right)$$

$$|\psi_t * f(s)| \leqslant C\frac{(|s|+1)\, t^2}{(s^2+t^2)^2} \qquad \text{if} \quad |s| > 2 \quad \text{or} \quad t > 1$$

$$|\psi_t * f(s)| \leqslant Ct/\varepsilon.$$

Proof.

$$\psi_t * f(s) = \sum_{k=1}^{N} \int_{(k-1)\varepsilon}^{k\varepsilon} \psi_t(s-s')\, f_k(s')\, ds'$$
$$= \sum_{k=1}^{N} \int_{(k-1)\varepsilon}^{k\varepsilon} \psi_t(s-s')(f_k(s')-1)\, ds' + \int_0^1 \psi_t(s-s')\, ds'.$$

Moreover,

$$\int_0^1 \psi_t(s-s')\, ds' = \left.\frac{t^2}{(s-s')^2+t^2}\right|_{s'=0}^{s'=1} = \frac{(2s-1)\,t^2}{(s^2+t^2)((s-1)^2+t^2)}.$$

Let F_k be defined by $F_k((k-1)\varepsilon)=0$ and $F_k'(s)=f_k(s)-1$. Then $F_k(k\varepsilon)=0$, so that

$$\sum_{k=1}^{N} \int_{(k-1)\varepsilon}^{k\varepsilon} \psi_t(s-s')(f_k(s')-1)\, ds' = \sum_{k=1}^{N} \int_{(k-1)\varepsilon}^{k\varepsilon} \psi_t'(s-s')\, F_k(s')\, ds'.$$

Furthermore, $|F_k(s')| \leqslant C\varepsilon$ and

$$\int_0^1 |\psi_t'(s-s')|\, ds' \leqslant \int_{-\infty}^{\infty} |\psi_t'(s')|\, ds' \leqslant C/t,$$

so that the first inequality follows. For the second inequality, we have

$$|\psi_t * f(s)| \leqslant C \int_0^1 |\psi_t(s-s')|\, ds' \leqslant C \int_0^1 \frac{(|s|+1)\,t^2}{(s^2+t^2)^2}\, ds'$$

whenever $|s|>2$ or $t>1$. For the third inequality, note that $|f'(s)| \leqslant C/\varepsilon$, so that $|f(s')-f(s)| \leqslant (C/\varepsilon)\,|s-s'|$. Since ψ has integral zero,

$$|\psi_t * f(s)| = \left|\int \psi_t(s-s')(f(s')-f(s))\, ds'\right|$$
$$\leqslant (C/\varepsilon) \int |\psi_t(s-s')|\, |s-s'|\, ds' \leqslant Ct/\varepsilon.$$

Consider a weight function $w(s)$ on the real line. We will be interested in the case in which w is the derivative of a conformal mapping from the upper half-plane to the domain D above. Denote

$$w_t(s) = \frac{1}{2t} \int_{s-t}^{s+t} w(s')\, ds',$$

that is, $w_t(s)$ is the average of w on the interval of length $2t$ centered at s.

LEMMA 6.4. *Let $1 < p < \infty$. Let f be defined as in Lemma 6.3 and let v be the harmonic extension of f to the upper half-plane. Let w be a non-negative function on the real line in the Muckenhoupt class A_q, for all $q > 1$. (The definition of the Muckenhoupt class can be found in* [4].) *Then*

$$\int_0^\infty \int_{-\infty}^\infty t^{p-1} |\nabla v(s,t)|^p w_t(s)\, ds\, dt \leqslant C \int_{-1}^1 w(s)\, ds.$$

Proof. It suffices to prove the estimates for $(\partial/\partial s)\, v$, rather than the full gradient. This is because $(\partial/\partial t)\, v(s,t)$ is the Hilbert transform in the s variable of $(\partial/\partial s)\, v(s,t)$ for each fixed t. Moreover, $w_t(s)$ belongs to the Muckenhoupt class A_p, uniformly for all t. Therefore, by the theorem of Hunt *et al.* [4],

$$\int_{-\infty}^\infty |(\partial/\partial t)\, v(s,t)|^p w_t(s)\, ds \leqslant C \int_{-\infty}^\infty |(\partial/\partial s)\, v(s,t)|^p w_t(s)\, ds.$$

Thus the bounds for $(\partial/\partial s)\, v$ are valid for $(\partial/\partial t)\, v$ as well.

Split the region of integration into three parts

$$R_1 = \{(s,t) \colon |s| < 2, \varepsilon < t < 1\},$$
$$R_2 = \{(s,t) \colon |s| > 2 \text{ or } t > 1\},$$
$$R_3 = \{(s,t) \colon |s| < 2, 0 < t < \varepsilon\}.$$

Recall that $t(\partial/\partial s)\, v = (1/\pi)\, f * \psi_t(s)$. By the first estimate in Lemma 6.3,

$$\begin{aligned}
&\int_{R_1} t^{p-1} |(\partial/\partial s)\, v(s,t)|^p w_t(s)\, ds\, dt \\
&\quad = \pi^{-p} \int_{R_1} t^{-1}(|f * \psi_t(s)|^p w_t(s)\, ds\, dt \\
&\quad \leqslant C \int_{R_1} t^{-1} \left(\frac{\varepsilon}{t} + \frac{(|s|+1)\, t^2}{(s^2+t^2)((s-1)^2+t^2)}\right)^p w_t(s)\, ds\, dt.
\end{aligned}$$

This is majorized by the sum of

$$C \int_{-2}^2 \int_\varepsilon^1 t^{-1} \left|\frac{\varepsilon}{t}\right|^p w_t(s)\, ds\, dt \leqslant C \int_{-4}^4 w(s)\, ds \varepsilon^p \int_\varepsilon^1 t^{-p-1}\, dt \leqslant C \int_{-4}^4 w(s)\, ds,$$

and

$$C \iint_{R_1} t^{-1} \left(\frac{(|s|+1)\, t^2}{(s^2+t^2)((s-1)^2+t^2)}\right)^p w_t(s)\, ds\, dt.$$

It suffices to estimate the latter integral for s near 0, say $|s| < \frac{1}{2}$. (For s near 1 the estimate is similar.) This we estimate by

$$\int_\varepsilon^1 \int_{-1/2}^{1/2} t^{-1} \left(\frac{t^2}{(s^2+t^2)}\right)^p w_t(s)\, ds\, dt.$$

Recall the definition of w_t. If we exchange the order of integration with respect to s' and s, and rename the variable s' as s, we find that the integral is majorized by

$$C \int_\varepsilon^1 \int_{-2}^{2} t^{-1} \left(\frac{t^2}{(s^2+t^2)}\right)^p w(s)\, ds\, dt.$$

Now perform the integration in t first to find that the integral is majorized by

$$C \int_{-2}^{2} (1+\log(1/s))\, w(s)\, ds \leqslant C \int_{-2}^{2} w(s)\, ds.$$

The last inequality follows from the fact that the A_q property for any q implies that w satisfies a reverse Hölder inequality. See [4]. Furthermore, an A_p weight also satisfies the doubling property; in particular, the integral of w over $[-4, 4]$ is majorized by a multiple of the integral over $[-1, 1]$, so the estimate on R_1 is completed.

Next, by the second estimate in Lemma 6.3,

$$\int_{R_2} t^{p-1} |(\partial/\partial s)\, v(s, t)|^p\, w_t(s)\, ds\, dt C \leqslant \int\int_{R_2} t^{-1} \left(\frac{(|s|+1)\, t^2}{(s^2+t^2)^2}\right)^p w_t(s)\, ds\, dt.$$

In the region $t > 1$ we use the fact that for $p > 1$ there exists $\delta > 0$ (e.g., $\delta = (p-1)/2$) such that

$$\left(\frac{(|s|+1)\, t^2}{(s^2+t^2)^2}\right)^p \leqslant \frac{1}{t^\delta (1+|s|)^{1+\delta}}.$$

Arguing as before, we can replace w_t by w. Next, if we perform the integration in t first, we find that the integral is majorized by

$$\int_1^\infty t^{-1-\delta}\, dt \int_{-\infty}^{\infty} \frac{1}{(1+|s|)^{1+\delta}}\, w(s)\, ds.$$

Recall that the theorem of Muckenhoupt says that the mapping $f \to Mf$, where Mf is the maximal function of f, is bounded on $L^{1+\delta}$ for every weight w in the class $A_{1+\delta}$. (See [4].) In the case in which the function f is the characteristic function of the interval $[-1, 1]$, its maximal function, Mf, is larger than a constant times $1/(1+|s|)$. Hence we have

$$\int_{-\infty}^{\infty} \frac{1}{(1+|s|)^{1+\delta}} w(s)\, ds \leqslant C \int_{-1}^{1} w(s)\, ds.$$

Therefore we have the desired bound for our integral in the range $t > 1$.

Next, we consider the remaining portion of R_2, $0 < t < 1$ and $|s| > 2$. We use the bound

$$\left(\frac{(|s|+1)\, t^2}{(s^2+t^2)^2}\right)^p \leqslant C\, |s|^{-3}\, t^2.$$

Our integral is bounded by

$$\int_{|s|>2} \int_0^1 |s|^{-3p}\, w_t(s)\, t^{2p-1}\, dt\, ds \leqslant C \int_{|s|>1} |s|^{-3p}\, w(s)\, ds$$

which is easily seen to be bounded by the integral of w over $[-1, 1]$.

For the third region we have, using the doubling property for w,

$$\begin{aligned}
&\int_{R_3} t^{p-1}\, |(\partial/\partial s)\, v(s,t)|^p\, w_t(s)\, dt\, ds \\
&\quad \leqslant \int_{|s|<2} \int_0^{\varepsilon} t^{-1} \left(\frac{t}{\varepsilon}\right)^p w_t(s)\, dt\, ds \\
&\quad \leqslant C \int_{|s|<3} w(s)\, ds \leqslant C \int_{-1}^{1} w(s)\, ds.
\end{aligned}$$

LEMMA 6.5. *Let* $1 < p < \infty$. *Let* J_k, $k = 1, \ldots, M$, *be a sequence of adjacent intervals of equal length* h. *Let* J *be the interval of length* ε_1 *which is the union of these intervals. Let* f *be a function supported in* J *satisfying*

$$f = \sum_{k=1}^{M} f_k; \qquad \int_{J_k} f(s)\, ds = \varepsilon_1; \qquad |f(s)| \leqslant C;$$

$$|f'(s)| \leqslant C/\varepsilon_1; \qquad \operatorname{supp} f_k \subset J_k.$$

Let v *be the harmonic extension of* f *to the upper half-plane. Let* w *be a nonnegative function on the real line in the Muckenhoupt class* A_q, *for all* $q > 1$. *Then*

$$\int_0^{\infty} \int_{-\infty}^{\infty} t^{p-1}\, |\nabla v(s,t)|^p\, w_t(s)\, ds\, dt \leqslant C \int_J w(s)\, ds.$$

This lemma is just a rescaled version of Lemma 6.4. If the length of J is r, then we apply Lemma 6.4 to the function $f(rs)$ and the weight function $w(rs)$. Because $w(rs)$ satisfies the Muckenhoupt class estimates with the same constants as $w(s)$, the inequality in the conclusion has a constant independent of the length of J.

Now assume that u belongs to $L^p_{1+1/p}(\Omega)$. It follows from Theorem 4.2 that u satisfies

$$\int_\Omega \delta^{p-1} |\nabla^2(u)|^p \, dx \, dy < \infty,$$

where $\delta(x, y)$ is the distance from (x, y) to the boundary of Ω. Let ϕ be a smooth cut-off function ϕ compactly supported in the ball of radius 4 and equal to 1 on the ball of radius 2. Then the function ϕu is infinitely differentiable in a neighborhood of the set where $\nabla\phi$ is nonzero. Let Φ denote the conformal mapping from the upper half-plane to D which takes the origin to the origin and the point i (or $(0, 1)$) to itself. Let Ψ denote the inverse mapping to Φ. We identify variables in $\mathbf{R}^2$ with complex variables by $z = x + iy$ and $\zeta = s + it$ and put $z = \Phi(\zeta)$ and $G = (G_1, G_2)$ defined by

$$G_1(s, t) = (\partial/\partial x)(\phi u)(\Phi(s + it)); \qquad G_2(s, t) = (\partial/\partial y)(\phi u)(\Phi(s + it)).$$

The Jacobian matrix of Φ is equivalent in norm to multiplication by the complex number Φ'. Therefore,

$$|\nabla_{(s,t)} G| \leqslant C \, |\Phi'(s + it)| \, (|\nabla^2\phi| \, |u| + |\nabla\phi| \, |\nabla u| + |\phi| \, |\nabla^2 u|).$$

We also have that $|\Phi'(s + it)|$ is comparable to $\delta_D(\Phi(s + it))/t$, where δ_D is the distance to the boundary in D rather than Ω. (This is the distortion lemma, valid even more generally for quasicircles. See, for example, [17].) Changing variables by $x + iy = \Phi(s + it)$, we have $dx\, dy = |\Phi'(s + it)|^2 \, ds\, dt$ and

$$\int_0^\infty \int_{-\infty}^\infty t^{p-1} \, |\nabla G|^2 \, |\Phi'(s + it)| \, ds\, dt < \infty.$$

Denote $w(s) = |\Phi'(s)|$. Then $|\Phi'(s + it)|$ is comparable to $w_t(s)$. (See [17].) So our estimate can be rewritten as

$$\int_0^\infty \int_{-\infty}^\infty t^{p-1} \, |\nabla G|^2 \, w_t(s) \, ds\, dt < \infty. \tag{6.6}$$

Define $J^j_k = \Psi(I^j_k)$ and $J^j = \Psi(I^j)$. Let r_j denote the length of J^j. We have

$$\int_{J^j} w(s) \, ds = |I^j| = 2^{-j-1}.$$

REMARK 6.7. *There is an absolute constant C such that*

$$(1/C)\,|J_k^j| \leqslant |J_{k'}^j| \leqslant C\,|J_k^j|$$

for all $k, k' = 1, ..., N_j$.

Proof. Dahlberg's comparison principle can be phrased as follows,

LEMMA 6.8. *Let D_1 be a Lipschitz domain in $\mathbf{R}^2$. Let I be an interval in ∂D_1 of length r. Let X be a point of D_1 at a distance at least $2r$ from I and let A be a point at a distance r from I and also at a distance greater than r/C_1 from ∂D_1. Let $G_X(Y)$ be Green's function for D_1 with pole at X. Let ω^X be harmonic measure at X for D_1, i.e., $d\omega^X(Q) = \partial_n u G_X(Q)\, d\sigma(Q)$, where $\partial_n u$ is the normal derivative of G_X. Then there is a constant C depending only on the Lipschitz constant of D (and on C_1) such that*

$$(1/C)\,\omega^X(I) \leqslant G_X(A) \leqslant C\omega^X(I).$$

LEMMA 6.9. *Let $Q(r) = \{(x, y): |x| < r, |y| < r\}$. Suppose that v_1 and v_2 are two non-negative harmonic functions in $Q(r) \cap D$ which vanish on $\partial D \cap Q(r)$ and satisfy $v_1(0, r/2) = v_2(0, r/2)$. Then*

$$(1/C)\,v_1(x, y) \leqslant v_2(x, y) \leqslant Cv_1(x, y)$$

for all $(x, y) \in D$ such that $|x| < (9/10)\,r$ and $|y| < (9/10)\,r$. (The constant C depends only on the Lipschitz constant of the boundary.)

Let G^j denote Green's function for D with pole at $X^j = (0, 2^{-j})$. Let ω^j denote harmonic measure for ∂D with pole at $(0, 2^{-j})$. Recall that $\partial D \cap Q(2^{-j})$ is contained in the horizontal strip $T(\varepsilon) = \{(x, y): |y| < \varepsilon\}$, with $\varepsilon = 2^{-j+2}/N_j$. Consider the semicircle of radius 2^{-j+2}

$$S = \{(x, y): x^2 + y^2 < 2^{-2j+4} \text{ and } y > 0\}.$$

The region $S + (0, \varepsilon)$ is contained in D and the region $D \cup (S - (0, \varepsilon))$ contains D_1. By Lemma 6.8 and the maximum principle, the value of Green's function with pole at $X = (0, 2^{-j})$ for these two regions is comparable to the value of Green's function for S with pole at X, provided we evaluate Green's function at a point $(x, 2\varepsilon)$ with $|x| < 2^{-j+1}$. Moreover, they are comparable, in turn, to G^j. Now Lemma 6.9 says that the values of Green's function at a distance comparable to ε from the boundary are comparable to harmonic measure of intervals of length ε. Since an explicit calculation shows the harmonic measures of any two intervals of equal length in $\{(x, 0) \in \partial S: |x| < 2^{-j+1}\}$ are comparable, we can now conclude that $\omega^j(I_k^j)$ and $\omega(I_{k'}^j)$ are comparable for all k and k'. Next, Harnack's inequality implies that the values of G^j on the circle of radius 2^{-j-1} about

X^j are comparable to each other. It then follows from Lemmas 6.8 and 6.9 that there is a factor $c_j = G^j(0, 2^{-j-1})/G^1(0, 2^{-j-1})$ such that $\omega^j(I_k^j)$ is comparable to $c_j \omega^1(I_k^j)$ for all k. Similarly, we can compare G^1 to the imaginary part of Ψ to see that $|J_k^j|$ is comparable to $\omega^1(I_k^j)$. This concludes the proof of the remark.

We are now ready to derive the contradiction to the estimate (6.6). The tangential derivative of $g(x, y)$ is

$$\begin{aligned}(\partial/\partial x + p'(x)\,\partial/\partial y)\, g(x, y) &= \left(\frac{1}{j}\right)(\partial/\partial y)\left[y \left(\log \frac{10}{y}\right)^{\beta}\right] \\ &= \left(\frac{1}{j}\right)\left(\left(\log \frac{10}{y}\right)^{\beta} - \beta\left(\log \frac{10}{y}\right)^{\beta-1}\right)\end{aligned}$$

for $x \in I_k^j$. We also have $b_j/2 \leqslant p(x) \leqslant (3/2)\, b_j$ for $x \in I_k^j$. Therefore, the tangential derivative of $g(x, y)$ is comparable to

$$(-\log b_j)^{\beta} \approx 2^{\beta 2^j}.$$

Thus

$$G_1(s) + (1/j)\, G_2(s) \approx 2^{\beta 2^j}$$

for all $s \in J_k^j$. Denote $\varepsilon_1 = \min_k |J_k^j|$. There is ε_2 comparable to ε_1 such that if we subdivide J^j into intervals E_ℓ, $\ell = 1, \ldots, N$, of exactly equal length ε_2, there is an interval $J(\ell) = J_k^j$, for some k, in the middle third of each E_ℓ. Define the function f to be a smooth function supported on the intervals $J(\ell)$ and satisfying the conditions of Lemma 6.5 for the interval J^j and ε_2. The function f can be assumed to be equal to 1 on the middle third of $J(\ell)$ and non-negative everywhere on $J(\ell)$. The formula

$$\begin{aligned}\int f(s)\, G_1(s)\, ds &= 2 \int_0^{\infty} \int_{-\infty}^{\infty} t\, \nabla G_1(s, t) \cdot \nabla v(s, t)\, ds\, dt \\ &\quad + \int_0^{\infty} \int_{-\infty}^{\infty} (\Delta G_1(s, t))\, t v(s, t)\, ds\, dt\end{aligned}$$

follows from integration by parts and the fact that v is harmonic. The integral on the left is easily seen to be convergent because G_1 is smooth on the support of f. The integral on the right is majorized as follows. The term involving ΔG_1 is zero except where the cut-off function ϕ is not constant. This set is far from the support of f, so the functions involved are infinitely differentiable there and this is a trivial error term. The main term is majorized by

$$\int_0^\infty \int_{-\infty}^\infty t\,|\nabla G_1(s,t)|\,|\nabla v(s,t)|\,ds\,dt$$

$$\leqslant \left(\int_0^\infty \int_{-\infty}^\infty (t^{1/p'}\,|\nabla G_1|)^p\,w_t\,ds\,dt\right)^{1/p}$$

$$\times\left(\int_0^\infty \int_{-\infty}^\infty (t^{1/p}\,|\nabla v|)^p\,w_t^{-p'/p}\,ds\,dt\right)^{1/p'}$$

(with $1/p+1/p'=1$). If w belongs to every weight class A_q, then so does the dual weight $w'=w^{-p'/p}$. Moreover, $w_t^{-p'/p}$ is comparable to w'_t. (See, for example, [4].)

Lemma 6.5 implies that

$$\left(\int_0^\infty \int_{-\infty}^\infty (t^{1-1/p'}\,|\nabla v|)^p\,w_t^{-p'/p}\,ds\,dt\right)^{1/p'} \leqslant C\int_{J^j} w'_t(s)\,ds.$$

We also have the crude estimate

$$\int_{J^j} w'_t(s)\,ds \leqslant \int_0^1 w'_t(s)\,ds \leqslant C.$$

Therefore, using the estimate (6.6) for G_1 we have

$$\left|\int f(s)\,G_1(s)\,ds\right| \leqslant C.$$

We also have the similar estimate for G_2, and, in particular,

$$\left|\int (G_1(s)+(1/j)\,G_2(s))\,f(s)\,ds\right| \leqslant C.$$

But we can compute the order of magnitude of this integral explicitly because f is supported where the value of $G_1(s)+(1/j)\,G_2(s)$ is known. Indeed, since f equals 1 on a fixed positive fraction of J^j, we have

$$\int (G_1(s)+(1/j)\,G_2(s))\,f(s)\,ds \approx 2^{\beta 2^j}\,|J^j|.$$

Finally, the doubling property for harmonic measure implies that there is a constant $\delta>0$ such that

$$|J^j|/|J^1| \geqslant (|I^j|/|I^1|)^\delta \geqslant 2^{-\delta j}.$$

Thus,

$$2^{\beta 2^j - \delta j} \leqslant C$$

for all j, which is a contradiction.

7. The Square Root of the Laplacian

Recall that $L^2_{1,0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the L^2_1 norm. The Laplace operator on Ω with Dirichlet boundary conditions has a complete set of eigenfunctions $\phi_1, \phi_2, \ldots$ in $L^2_{1,0}(\Omega)$, satisfying $-\Delta\phi_k = \lambda_k \phi_k$, for a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$. After normalization, these eigenfunctions form an orthonormal basis for $L^2(\Omega)$. Define $A = \sqrt{-\Delta}$ as the unbounded operator on $L^2(\Omega)$ such that $A\phi_k = \sqrt{\lambda_k}\,\phi_k$. The purpose of this section is to compare the L^p norm of Af and ∇f.

Before formulating our theorem, we present a few well-known technical remarks. The domain, $\mathscr{D}(A)$, of A is given by

$$\mathscr{D}(A) = \left\{ f \in L^2(\Omega) : f = \sum_{k=1}^{\infty} a_k \phi_k,\ \sum_{k=1}^{\infty} \lambda_k a_k^2 < \infty \right\}.$$

Remark 7.1. *There is a constant $c > 0$ depending on Ω such that*

$$ck^{2/n} \leqslant \lambda_k \leqslant c^{-1} k^{2/n}.$$

This follows from the min–max comparison principle, comparing the domain Ω to a cube containing Ω and to a cube contained in Ω. (See [5, Courant–Hilbert].)

Remark 7.2. *If $f \in C_0^\infty(\Omega)$, then*

$$f = \sum_{k=1}^{\infty} a_k \phi_k$$

with $|a_k| \leqslant C_M \lambda_k^{-M}$ for every M, and

$$-\Delta f = \sum_{k=1}^{\infty} \lambda_k a_k \phi_k = A^2 f.$$

Proof.

$$a_k = \int f\phi_k = -\int f \lambda_k^{-1}\, \Delta\phi_k = -\lambda_k^{-1} \int \Delta f \phi_k.$$

This integration by parts can be repeated any number of times, proving that $|a_k|$ is rapidly decreasing. To prove the second claim, note that $-\Delta f = \sum_k b_k \phi_k$, and b_k is calculated by integration by parts:

$$b_k = \int (-\Delta f)\, \phi_k = \int f(-\Delta \phi_k) = \lambda_k a_k.$$

REMARK 7.3. *$\mathscr{D}(A) = L^2_{1,0}(\Omega)$ and A is an isomorphism from $L^2_{1,0}(\Omega)$* to $L^2(\Omega)$.

Proof. If $f \in C_0^\infty(\Omega)$, then by Remark 7.2 and an integration by parts,

$$\int (Af)^2 = \sum_k \lambda_k a_k^2 = -\int (\Delta f)\, f = \int |\nabla f|^2.$$

In other words,

$$\|Af\|^2_{L^2(\Omega)} = \sum_k \lambda_k a_k^2 = \|\nabla f\|^2_{L^2(\Omega)}. \tag{7.4}$$

Hence, $\mathscr{D}(A) = L^2_{1,0}(\Omega)$ and equality (7.4) holds for all $f \in \mathscr{D}(A)$.

THEOREM 7.5. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$.*

(a) *If $n \geq 3$, then there exists $q_0 > 3$, depending only on dimension and the Lipschitz constant of Ω such that if $1 < p < q_0$, $f \in \mathscr{D}(A)$ and $Af \in L^p(\Omega)$, then $f \in L^p_{1,0}(\Omega)$ and*

$$\int_\Omega |\nabla f|^p \leq C \int_\Omega |Af|^p.$$

Thus the operator A^{-1}, initially defined on $L^2(\Omega)$, extends to a bounded linear operator from $L^p(\Omega)$ to $L^p_{1,0}(\Omega)$. If $n = 2$, then the exponent q_0 can be chosen greater than 4. If Ω is C^1, then q_0 can be chosen equal to infinity.

(b) *For all $n \geq 2$, $1 < p < \infty$, if $f \in \mathscr{D}(A) \cap L^p_1(\Omega)$, then*

$$\int_\Omega |Af|^p \leq C \int_\Omega |\nabla f|^p.$$

Thus the operator A extends to a bounded linear operator from $L^p_{1,0}(\Omega)$ to $L^p(\Omega)$.

(c) *The assertions in part* (a) *are sharp in the sense that when $n \geq 3$, given any $q_0 > 3$, there is a Lipschitz domain for which the inequality fails for $p = q_0$. When $q_0 > 4$ there is a Lipschitz domain in $\mathbf{R}^2$ for which the inequality fails for $p = q_0$.*

The main idea is to develop interpolation results for the analytic family of operators A^z defined by $A^z\phi_k = \lambda_k^{z/2}\phi_k$.

LEMMA 7.6. *If $f \in C_0^\infty(\Omega)$, then $A^z f \in L^\infty(\Omega)$ and $A^{z_1}A^{z_2}f = A^{z_1+z_2}f$ for all complex numbers z, z_1, and z_2.*

Proof. We claim that

$$|\phi_k(x)| \leqslant C\lambda_k^m \tag{7.7}$$

for any constant $m > n/4$. To see this, note that by the maximum principle, Green's function $G(x, y)$ for Ω satisfies $|G(x, y)| \leqslant C|x-y|^{2-n}$ (or a logarithm in the case $n = 2$). Denote

$$Gf(x) = \int_\Omega G(x, y) f(y)\, dy.$$

G is a bounded operator from $L^p(\Omega)$ to $L^q(\Omega)$ provided $1/q \geqslant 1/p - 2/n$ and $q < \infty$. It is also a bounded operator from $L^p(\Omega)$ to $L^\infty(\Omega)$ provided $p > n/2$. It follows that G^m is a bounded operator from $L^2(\Omega)$ to $L^\infty(\Omega)$. On the other hand, $\phi_k = \pm\lambda_k^m G^m \phi_k$, which proves (7.7). The assertion that $A^z f$ is bounded follows from (7.7) and Remark 7.2. The composition formula now follows by computing the coefficients of the expansion of both sides.

The basic ingredients of the proof of Theorem 7.5 are bounds on the operators $A^{i\gamma}$, A^{-2}, and the L^2 bound on A of Remark 7.3.

LEMMA 7.8. *Let γ be a real number. The operator $A^{i\gamma}$, defined initially as a bounded operator on $L^2(\Omega)$, extends to a bounded operator on $L^p(\Omega)$ for all p, $1 < p < \infty$, with operator norm bounded by a constant times $e^{c|\gamma|}$.*

This theorem follows from the Littlewood–Paley theory of Stein. See [23] and [6].

Our main result, Theorem 1.1, implies that there is an $\varepsilon > 0$ depending only on the Lipschitz constant of Ω such that if $1/p_0 = 1/2 + \varepsilon/2$,

$$A^{-2}\colon L^p_{\alpha-2}(\Omega) \to L^p_{\alpha,0}(\Omega) \tag{7.9}$$

provided

$$1 < p \leqslant p_0 \qquad \text{and} \qquad 3/p - 1 - \varepsilon < \alpha < 1 + 1/p$$

or

$$p_0 < p < p_0' \qquad \text{and} \qquad 1/p < \alpha < 1 + 1/p$$

or

$$p_0' \leqslant p < \infty \qquad \text{and} \qquad 1/p < \alpha < 3/p + \varepsilon.$$

Here, and in the future, we will consider operators A and A^z as defined initially on $C_0^\infty(\Omega)$. They will be said to be defined on larger function spaces if they can be extended (uniquely) as bounded linear operators on these larger spaces. The range will always be specified. In particular, we will no longer restrict A to $\mathscr{D}(A)$. By Remark 7.3 and duality we have isomorphisms

$$A: L^2_{1,0}(\Omega) \to L^2(\Omega) \qquad A: L^2(\Omega) \to L^2_{-1}(\Omega). \tag{7.10}$$

By Stein's complex interpolation theorem [24], for all $0 \leqslant s \leqslant 1$, there is an isomorphism

$$A^{s+i\gamma}: L^2(\Omega) \to L^2_{-s}(\Omega). \tag{7.11}$$

Composition with A^{-2} using Lemma 7.6 gives an isomorphism

$$A^{-3/2+\delta+i\gamma}: L^2(\Omega) \to L^2_{3/2-\delta,0}(\Omega) \tag{7.12}$$

for $0 < \delta \leqslant 1/2$. Using Stein's interpolation with endpoints (7.12) and Lemma 7.8, we obtain an isomorphism

$$A^{-1+i\gamma}: L^p(\Omega) \to L^p_{1,0}(\Omega) \tag{7.13}$$

for $3/2 < p < 3$.

Next, we wish to extend estimate (7.13) to values of p greater than 3 and less than 3/2. Interpolation of (7.13) with Lemma 7.8 gives an isomorphism

$$A^{-s+i\gamma}: L^p(\Omega) \to L^p_{s,0}(\Omega) \tag{7.14}$$

for $0 \leqslant s \leqslant 1$, $s/3 < 1/p < 1 - s/3$. In other words, there is an isomorphism

$$A^{s-i\gamma}: L^p_{s,0}(\Omega) \to L^p(\Omega). \tag{7.14'}$$

By duality, there is an isomorphism

$$A^{s-i\gamma}: L^p(\Omega) \to L^p_{-s}(\Omega) \tag{7.15}$$

provided $0 \leqslant s \leqslant 1$ and $s/3 < 1/p < 1 - s/3$. (Note that the range of $1/p$ is the same as the range of the dual exponent.) Let $p = 3$ in 7.15, then for $1 < \alpha = 2 - s < 1 + \varepsilon$, one can compose with A^{-2} to obtain

$$A^{-\alpha-i\gamma}: L^3(\Omega) \to L^3_{\alpha,0}(\Omega) \tag{7.16}$$

Interpolation of Lemma 7.8 with (7.16) for values of α between 1 and $1+\varepsilon$ yields the main bound on operator A^{-1},

$$A^{-1}: L^p(\Omega) \to L^p_{1,0}(\Omega) \tag{7.17}$$

for all p, $3 \leqslant p < 3(1+\varepsilon)$. Thus, when combined with 7.13, we see that 7.17 is valid for $3/2 < p < 3(1+\varepsilon)$.

For $1 < p \leqslant 2$ and $3/p - 1 < \alpha < 1 + 1/p$, A^{-2} is bounded from $L^p_{\alpha-2}(\Omega)$ to $L^p_{\alpha,0}(\Omega)$. If $\alpha = 2 - s$, then $3/p - 1 < \alpha$ is equivalent to $1/p < 1 - s/3$, and since $p \leqslant 2$ we have in addition that $s/3 < 1/p$. Thus we can compose 7.15 with A^{-2} to obtain

$$A^{-\alpha - i\gamma}: L^p(\Omega) \to L^p_{\alpha,0}(\Omega). \tag{7.18}$$

The range of α is nonempty and contains values greater than 1. (Indeed, as p approaches 1 the range of α is near 2.) Thus interpolation with Lemma 7.8 gives (7.17) in the range $1 < p \leqslant 2$. This completes the proof of part (a) of Theorem 7.5, when $n \geqslant 3$. The case $n = 2$ is similar, using the improved bounds on A^{-2} given by Theorem 1.3.

To prove Theorem 7.5(b) note that Remark 7.2 implies that for all $f \in C_0^\infty(\Omega)$ and all p, $1 < p < \infty$,

$$\|A^2 f\|_{L^p(\Omega)} \leqslant C \, \|f\|_{L^p_{2,0}(\Omega)}.$$

Use Lemma 7.6 to compose with $A^{i\gamma}$ to obtain

$$\|A^{2+i\gamma} f\|_{L^p(\Omega)} \leqslant C \, \|f\|_{L^p_{2,0}(\Omega)}$$

with a constant C that increases exponentially with γ. Since $C_0^\infty(\Omega)$ is dense in $L^p_{2,0}(\Omega)$ (Remark 2.7), the operator and estimate can be extended to all $f \in L^p_{2,0}(\Omega)$. Finally, interpolation with Lemma 7.8 implies Theorem 7.5(b). (Recall that $L^p_{\alpha,0}(\Omega)$ was identified in Proposition 2.11 as a complex interpolation scale.)

Next, we need to prove part (c) of Theorem 7.5. Suppose that $q > 3$ and

$$\int_\Omega |\nabla f|^q \leqslant C \int_\Omega |Af|^q$$

for all $f \in C_0^\infty(\Omega)$. It follows that A^{-1} is a bounded operator from $L^q(\Omega)$ to $L^q_{1,0}(\Omega)$. Let $1/q + 1/q' = 1$. Then $q' < 2$, so Theorem 7.5(a) implies that A^{-1} is a bounded operator from $L^{q'}(\Omega)$ to $L^{q'}_{1,0}(\Omega)$. By duality, A^{-1} is bounded from $L^q_{-1}(\Omega)$ to $L^q(\Omega)$. Hence A^{-2} is bounded from $L^q_{-1}(\Omega)$ to $L^q_{1,0}(\Omega)$. But if $f \in C_0^\infty(\Omega)$, then $u = A^{-2} f$ solves $\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$. So well known counterexamples involving cones in $\mathbf{R}^n$ for $n \geqslant 3$ show that for any $q > 3$, there exists a Lipschitz domain Ω and $f \in C_0^\infty(\Omega)$ for which $\nabla u \notin L^q(\Omega)$. (Similarly, in $\mathbf{R}^2$ there is a counterexample if $q > 4$.)

Finally, let us remark that the argument just given shows that bounds on A^{-1} imply the sharp regularity estimates of $\Delta u = f$ when $f \in L^q_{-1}(\Omega)$. Thus Theorem 7.5(a) implies the result stated in the Introduction as Theorem 0.5(a). Dahlberg's estimate, stated as Theorem B(1) in the Introduction, also follows easily from this estimate of A^{-1}. In fact, if $1 < p < q < q_0$ and $1/q = 1/p - 1/n$, then by Theorem 7.5(a) and the Sobolev lemma A^{-1} is bounded from $L^p(\Omega)$ to $L^p_{1,0}(\Omega) \subset L^q(\Omega)$. Applying Theorem 7.5(a) again we see that A^{-1} is bounded from $L^q(\Omega)$ to $L^q_{1,0}(\Omega)$, and hence A^{-2} is bounded from $L^p(\Omega)$ to $L^q_{1,0}(\Omega)$, which is Theorem B(1).

References

1. C. Bennett and R. C. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.
2. J. Bergh and J. Lofström, "Interpolation Spaces, An Introduction," Springer-Verlag, Berlin, 1976.
3. A. P. Calderon, Lebesgue spaces of differentiable functions and distributions, *Proc. Sympos. Pure Math.* **4** (1961), 33–49.
4. R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 214–250.
5. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. I, Interscience, New York, 1953.
6. M. Cowling, Harmonic analysis on semigroups, *Ann. Math.* **117** (1983), 267–283.
7. B. E. J. Dahlberg, Estimates for harmonic measure, *Arch. Rational Mech. Anal.* **65** (1977), 275–283.
8. B. E. J. Dahlberg, Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for harmonic functions in a Lipschitz domain, *Studia Math.* **67** (1980), 297–314.
9. B. E. J. Dahlberg, L^q estimates for Green potentials in Lipschitz domains, *Math. Scand* **44** (1979), 149–170.
10. E. B. Fabes, M. Jodeit, Jr., and N. Rivière, Potential techniques for boundary value problems on C^1 domains, *Acta Math.* **141** (1978), 165–186.
11. M. Frazier and B. Jawerth, Decomposition of Besov spaces, *Indiana Univ. Math. J.* **34** (1985), 777–799.
12. J. B. Garnett, "Bounded Analytic Functions," Academic Press, San Diego, 1981.
13. P. Grisvard, "Elliptic Problems in Nonsmooth Domains," Pitman, Boston, 1985.
14. D. Jerison and C. E. Kenig, The Neumann problem in Lipschitz domains, *Bull. Amer. Math. Soc.* **4** (1981), 203-207.
15. D. Jerison and C. E. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), 80–147.
16. D. Jerison and C. E. Kenig, Boundary value problems on Lipschitz domains, *in* "Studies in Partial Differential Equations" (W. Littman, Ed.), MAA Studies in Math., Vol. 23, pp. 1–68, 1982.
17. D. Jerison and C. E. Kenig, Hardy spaces, A_∞, and singular integrals on chord-arc domains, *Math. Scand.* **50** (1982), 221-247.
18. D. Jerison and C. E. Kenig, The functional calculus for the Laplacian on Lipschitz domains, *in* "Journées Equations aux dérivées partielles," Saint-Jean-de-Monts, Ecole Polytechnique, Paris, 1989.

19. A. JONSSON AND H. WALLIN, "Function Spaces on Subsets of $\mathbf{R}^n$," Harwood Academic, New York, 1984.
20. J.-L. LIONS AND E. MAGENES, Problèmes aux limites non homogènes et applications I, II, Dunod, Paris, 1968 (Engl. transl. Springer-Verlag, 1972).
21. J. NEČAS, personal communication.
22. E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
23. E. M. STEIN, Topics in harmonic analysis related to the Littlewood-Paley theory, *in* "Annals of Math. Studies," Vol. 63, Princeton Univ. Press, Princeton, NJ, 1970.
24. E. M. STEIN AND G. WEISS, "Introduction to Fourier Analysis in Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
25. R. S. STRICHARTZ, Multipliers on fractional Sobolev spaces, *J. Math. Mech.* **16** (1967), 1031–1060.
26. G. VERCHOTA, Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, *J. Funct. Anal.* **59** (1984), 572–611.
27. A. ZYGMUND, "Trigonometric Series," Cambridge Univ. Press, Cambridge, 1959.