Preconditioning tests

Using the square root of the weighted Laplace operator

Let $\Delta_\omega = (\omega(x)\partial x)^2$ where $\omega(x) = \sqrt{1-x^2}$, that is, for any twice differentiable function u, $\Delta_\omega u = (\omega(x)u'(x))'$. Note that for $\omega(x) = 1$, Δ_ω is the usual Laplace-Beltrami operator on the segment. When Γ is an infinite straight line, using a Fourier decomposition, one can show that the trace of the single-layer potential is the square-root of $-\Delta$. In fact, for the open segment, a similar result holds. One can check that Δ_ω and S_ω commute, and that the Chebyshev polynomials T_n are a common basis of eigenvectors for those two operators. To approximate the eigenvectors of an operator A with respect to the scalar product

$$(u, v) \to \int_\Gamma \frac{uv}{\omega}$$

we write that for any v, the n-th eigenvector ϕ^n, associated to the eigenvalue λ_n satisfies

$$\int_\Gamma \frac{(A\phi^n)v}{\omega} = \lambda_n \int_\Gamma \frac{\phi^n v}{\omega}$$

In the discrete setting, we find ϕ_h^n such that for all v_h,

$$\int_\Gamma \frac{(A\phi_h^n)v_h}{\omega} = \lambda_n \int_\Gamma \frac{\phi_h^n v_h}{\omega}$$

That is, if we denote by $[A]_\omega$ the Galerkine matrix of A for the scalar product defined previously,

$$[A]_\omega \Phi_n = \lambda_n [I]_\omega \Phi_n$$

This is a generalized eigenvalue / eigen vectors problem. Here S_ω and $-\Delta_\omega$ are self adjoint positive, $[I_\omega]$ is positive definite. Therefore, the exact and approximate eigenvalues are necessarily real and positive.

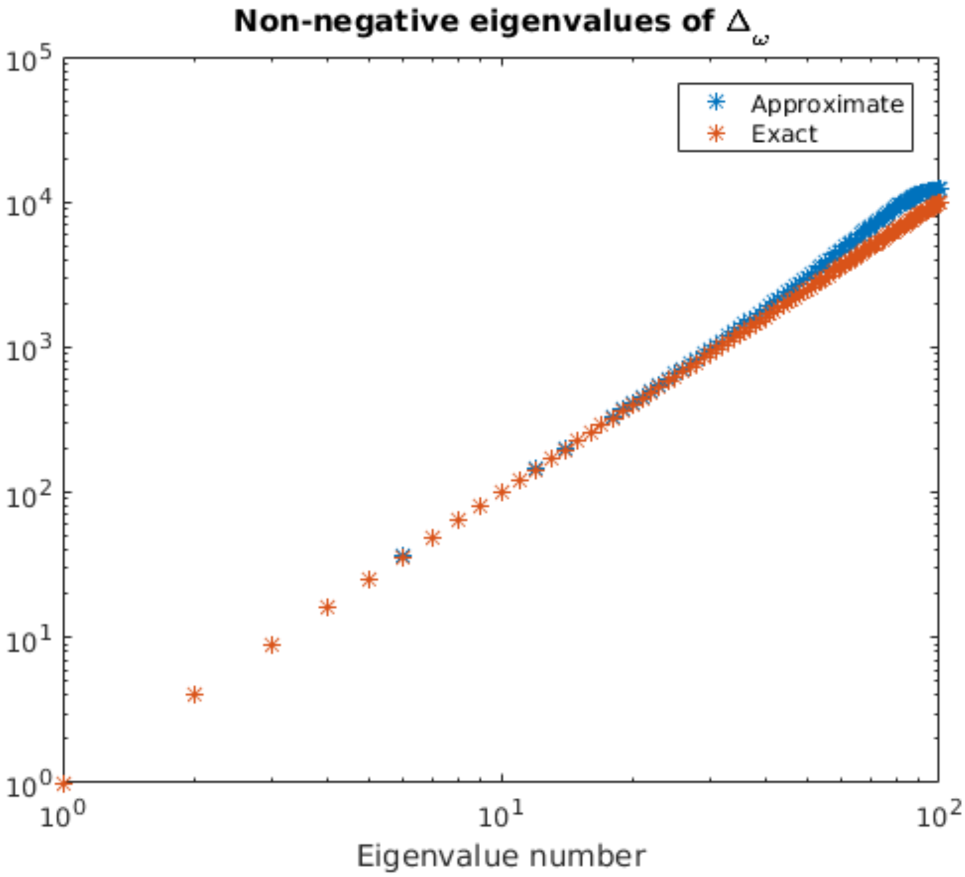
Eigenvalues of the weighted Laplace operator

For $-\Delta_\omega$, one has for all n.

$$-\Delta_\omega T_n = n^2 T_n$$

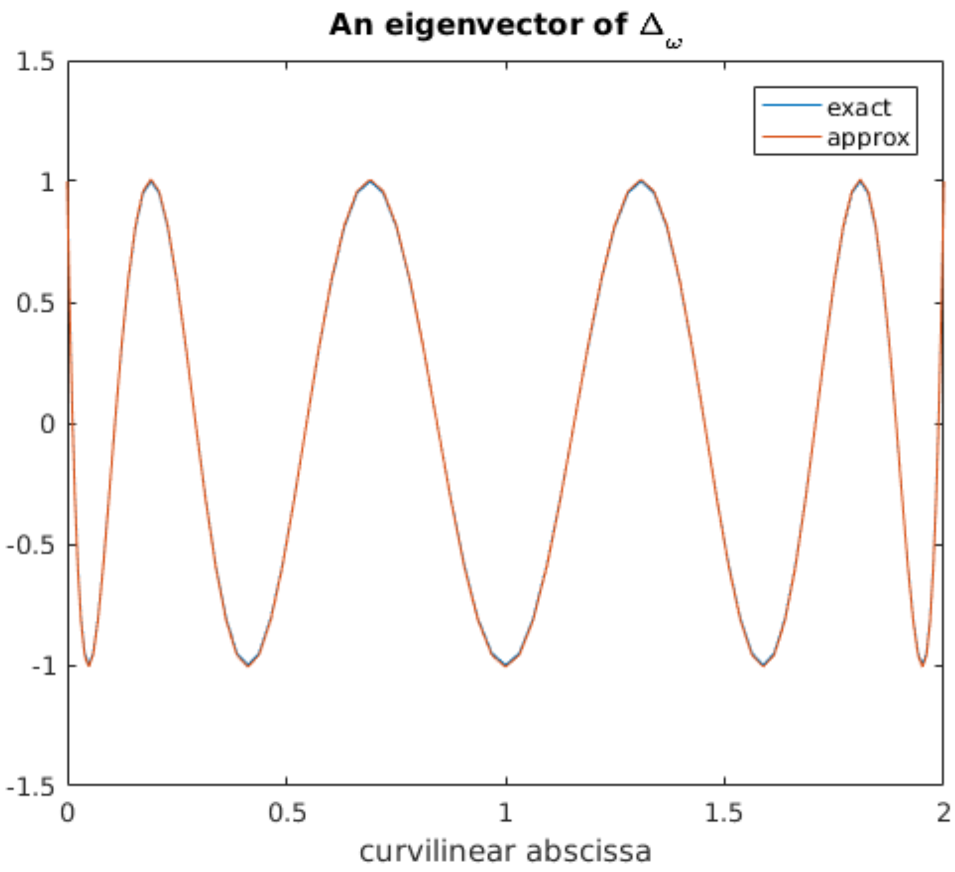
```
clear all
close all
clc;
segment = unitSegment;
N = 100;
repartition = @cos;
bounds = [-pi,0];
mesh = MeshCurve(segment,N,repartition,bounds);
Vh = weightedFEspace(mesh,'P1','1/sqrt(1-t^2)',5);
M = full(Vh.Mass);
Wh =  weightedFEspace(mesh,'P1','sqrt(1-t^2)',5);
dM = full(Wh.dMass);
```

```
[P1,D1] = eig(dM,M);
[eigenVals_dM,I] = sort(diag(D1));
figure;
loglog(1:Vh.ndof-1,sort(eigenVals_dM(2:end)),'*');
hold on
loglog(1:Vh.ndof-1,(1:Vh.ndof-1).^2,'*')
title('Non-negative eigenvalues of \Delta_{\omega}')
xlabel('Eigenvalue number')
legend({'Approximate','Exact'});
```



We can also check that the first eigenvectors are close to the Chebyshev polynomials. For $n = 10$, for example :

```
X = Vh.dofCoords;
s = Vh.mesh.sVertices;
n = 10;
Tn = R2toRfunc.Tn(n);
figure;
plot(s,Tn(X));
hold on;
plot(s,P1(:,I(n+1))/P1(end,I(n+1)))
title('An eigenvector of \Delta_\omega')
legend({'exact','approx'});
xlabel('curvilinear abscissa')
```



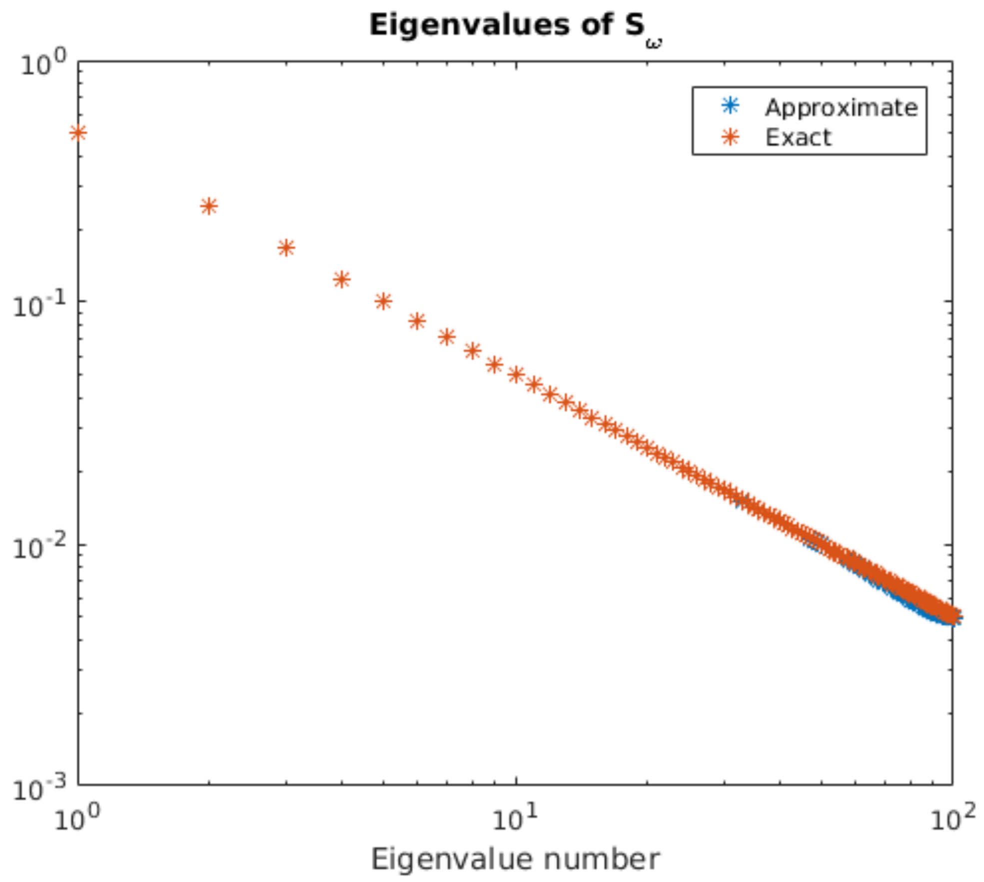
Eigenvalues of the weighted Laplace operator

For S_ω. One has

$$S_\omega T_n = s_n T_n$$

where $s_n = \frac{1}{2n}$ if $n > 1$, and $s_0 = \frac{\ln(2)}{2}$.

```
Somega = singleLayer(0,Vh,[],{'full',true});
Sgalerk = Somega.galerkine(Vh,'U');
% The full option computes the matrix without SBD compression.
[P2,D2] = eig(full(Sgalerk),M);
[eigenVals_S,I] = sort(diag(D2),'descend');
eigenVals_S(1:2) = eigenVals_S(2:-1:1);
I(1:2) = I(2:-1:1);
figure;
loglog(0:Vh.ndof-1,eigenVals_S,'*');
hold on
s_n = 1./(2*(1:(Vh.ndof-1))); s_0 = log(2)/2;
s_n = [s_0, s_n];
loglog(0:Vh.ndof-1,s_n,'*')
title('Eigenvalues of S_{\omega}')
xlabel('Eigenvalue number')
legend({'Approximate','Exact'});
```



Application to preconditionning

Based on the values of the eigenvalues of the two operator, it follows that

$$(S_\omega)^{-1} = 2\sqrt{-\Delta_\omega} + \frac{1}{s_0}T_0^*$$

where $(T_n^*)_n$ is the orthogonal family of projector defined by

$$T_n^* u = \frac{\langle u, T_n \rangle}{\langle T_n, T_n \rangle} T_n$$

Moreover, here the square root is taken in the following sense : for any operator A of the form

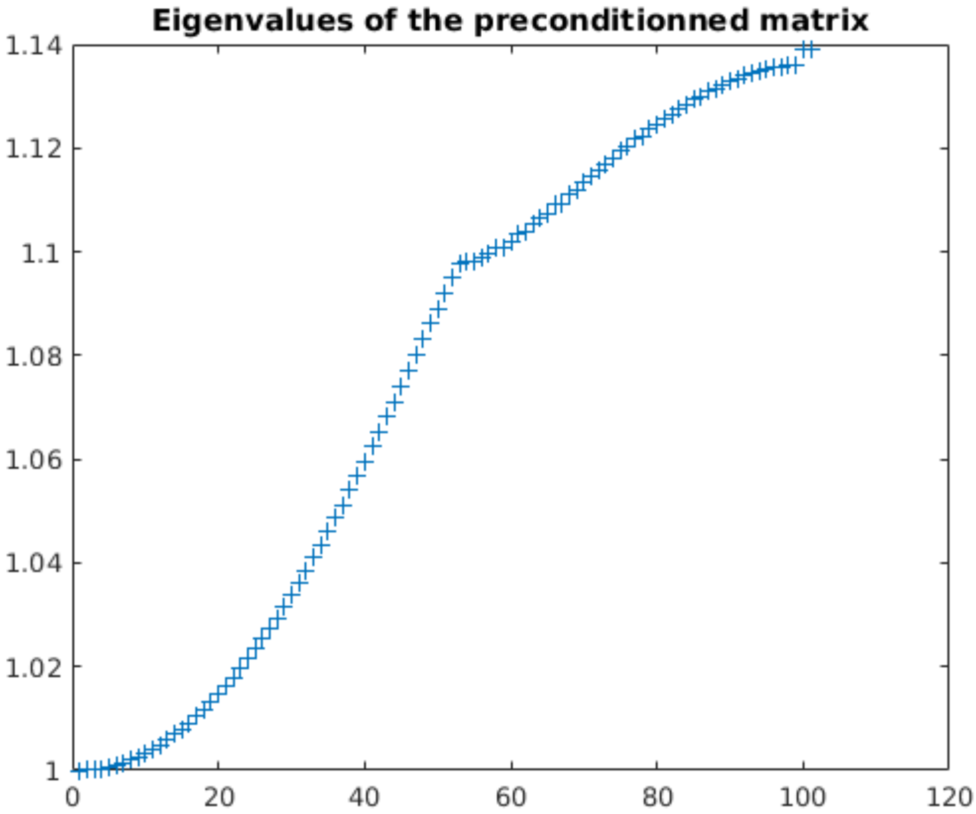
$$A = \sum_{n=0}^{+\infty} a_n T_n^*$$

$$\sqrt{A}u = \sum_{n=0}^{+\infty} \sqrt{a_n} T_n^*$$

We exploit this fact to build a preconditionner for S_ω. One method is to compute the squre root of the operator $-\Delta_\omega$. This can be done exactly with the following code.

```
sqrtdM = M*P1*sqrt(D1)*P1^(-1); % sqrtdM is as M*sqrtm(M^(-1)*dM);
```

```
T0_scal_phi = Vh.phi'*Vh.W;
T0_star_galerk = T0_scal_phi*T0_scal_phi'/sum(Vh.W);
Prec_galerk = M^(-1)*(2*sqrtdM + 1/s_0*T0_star_galerk)*M^(-1);
figure
plot(sort(eig(Prec_galerk*full(Sgalerk))),'+');
title('Eigenvalues of the preconditionned matrix')
```



However, if the number of unknowns gets large, it is not tractable to perform an eigenvalue/ eigenvector decomposition of the matrix *dM*. Instead, one can use a method to get a fast approximation of the matrix-vector product

```
N = 100;
mesh = mesh.remesh(N);
Wh = weightedFEspace(mesh,'P1','sqrt(1-t^2)',5);
dM = Wh.dMass.concretePart;
Vh = weightedFEspace(mesh,'P1','1/sqrt(1-t^2)',5);
M = Vh.Mass.concretePart;
a_factor = 15; % This has been balanced to get at the same time fast MV product
% and not too long an assembling. A big a_factor is in favor of the $S_0$
% preconditioner since more local interactions are stored.
Somega = singleLayer(0,Vh,[],{'a_factor',a_factor});
Sgalerk = Somega.galerkine(Vh,'U');
T0_scal_phi = Vh.phi'*Vh.W;
T0_star_galerk = T0_scal_phi*T0_scal_phi'*(1/sum(Vh.W));
```

```
u0 = R2toRfunc(@(Z)(sin(5*Z(:,1))));
l = Somega.Vh.secondMember(u0);
t0 = tic;
disp('No preconditioner')
[~,~,~,~,resvec0]  = variationalSol(Sgalerk,l,20,1e-8,N);
fprintf('\nGmres returned a solution in %s iteration\n',num2str(length(resvec0)));
t0 = toc(t0);
fprintf('t = %s s\n\n',num2str(t0))
t1 = tic;

disp('preconditionned by SBD local matrix')
Prec1 = Sgalerk.concretePart;
[~,~,~,~,resvec1]  = variationalSol(Sgalerk,l,20,1e-8,N,Prec1);
fprintf('Gmres returned a solution in %s iteration\n',num2str(length(resvec1)));
t1 = toc(t1);
fprintf('t = %s s \n\n',num2str(t1))
disp('preconditionned by Square root of weighted Laplace operator, à la volée')
t2 = tic;
Prec2 = @(u)(M\TrefethenSqrt(4*dM,3,M\u,M,4,4.5*Vh.ndof^2));
[~,~,~,~,resvec2]  = variationalSol(Sgalerk,l,[],1e-8,N,Prec2);
fprintf('\nGmres returned a solution in %s iteration\n',num2str(length(resvec2)));
t2 = toc(t2);
fprintf('t = %s s \n\n',num2str(t2))

disp('preconditionned by Square root of weighted Laplace operator, fully assembled')
t3 = tic;
Minv = M^(-1);
Prec3 = Minv*TrefethenSqrt(4*dM,3,[],M,4,4.5*Vh.ndof^2)*Minv;
t3bis = tic;
[~,~,~,~,resvec3]  = variationalSol(Sgalerk,l,[],1e-8,N,@(u)(Prec3*u));
fprintf('\nGmres returned a solution in %s iteration\n',num2str(length(resvec3)));
t3bis = toc(t3bis);
t3 = toc(t3);
fprintf('t = %s s assemble, %s krylov \n\n',num2str(t3),num2str(t3bis))

figure;
semilogy(0:length(resvec0)-1,resvec0/norm(full(l)));
hold on
semilogy(0:length(resvec1)-1,resvec1/norm(Prec1\full(l)));
hold on
semilogy(0:length(resvec2)-1,resvec2/norm(Prec2(full(l))));
legend({['No precond : ' num2str(t0) 's'],...
    ['$S_0$ : ' num2str(t1) 's'],...
    ['$\sqrt{-\Delta_{\omega}}$ : ' num2str(t2) 's']},...
    'interpreter','latex');
xlabel('Iteration count');
ylabel('relative residual');
title('Gmres history, comparing preconditionners')
```

```
*********************************************
SBD package : launching the radial decomposition
Warning: Condition number too high, restarting with a =
1.677051e-01
```

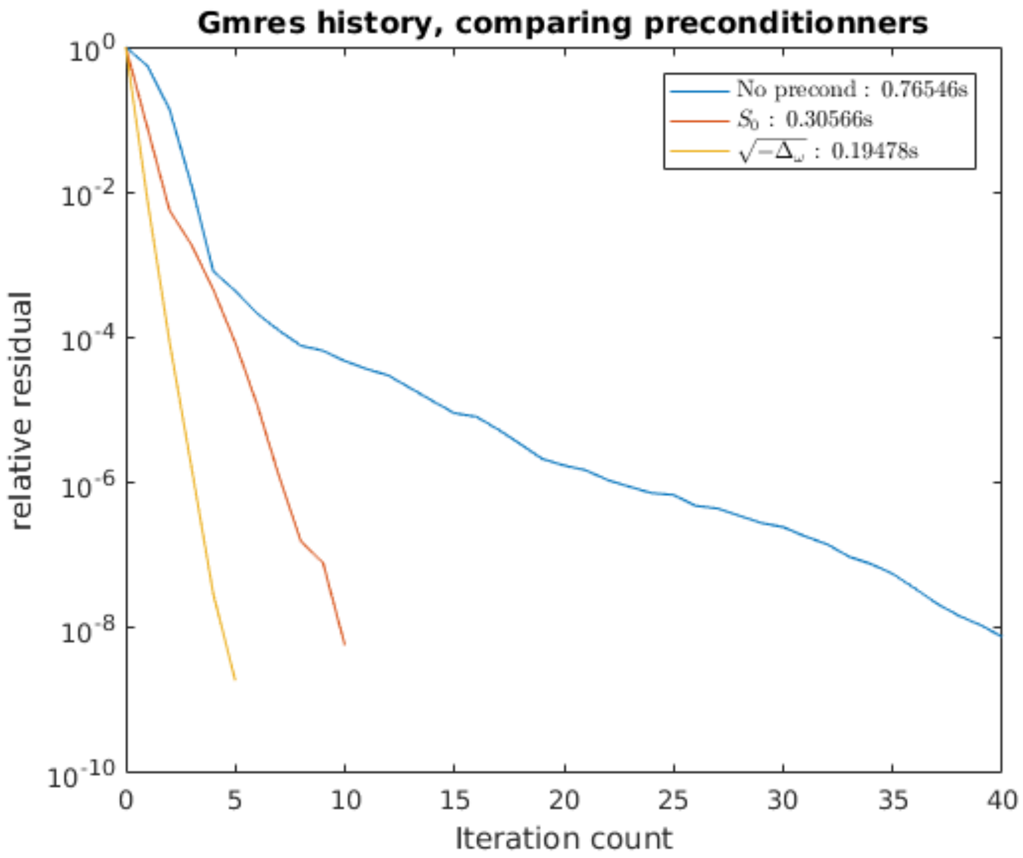
```
Bessel decomposition successfully computed. Number of terms : 35
Radial quadrature of 35 components computed in 0.057588 seconds
Done

NUFFT for local correction : *
No preconditioner
******************************************
Gmres returned a solution in 41 iteration
t = 0.76546 s

preconditionned by SBD local matrix
************Gmres returned a solution in 11 iteration
t = 0.30566 s

preconditionned by Square root of weighted Laplace operator, à la volée
*******
Gmres returned a solution in 6 iteration
t = 0.19478 s

preconditionned by Square root of weighted Laplace operator, fully assembled
*******
Gmres returned a solution in 6 iteration
t = 0.13292 s assemble, 0.12337 krylov
```



Alternative preconditionner

The relation between the eigenvalues of S_ω and Δ_ω also implies that

$$S_\omega^{-1} = -4 S_\omega \Delta_\omega + \frac{1}{s_0} T_0^*$$

so that the matrix

$$[I]_\omega^{-1}[S_\omega]^{-1}]_\omega [I]_\omega^{-1}[\Delta_\omega]_\omega [I]_\omega^{-1}$$

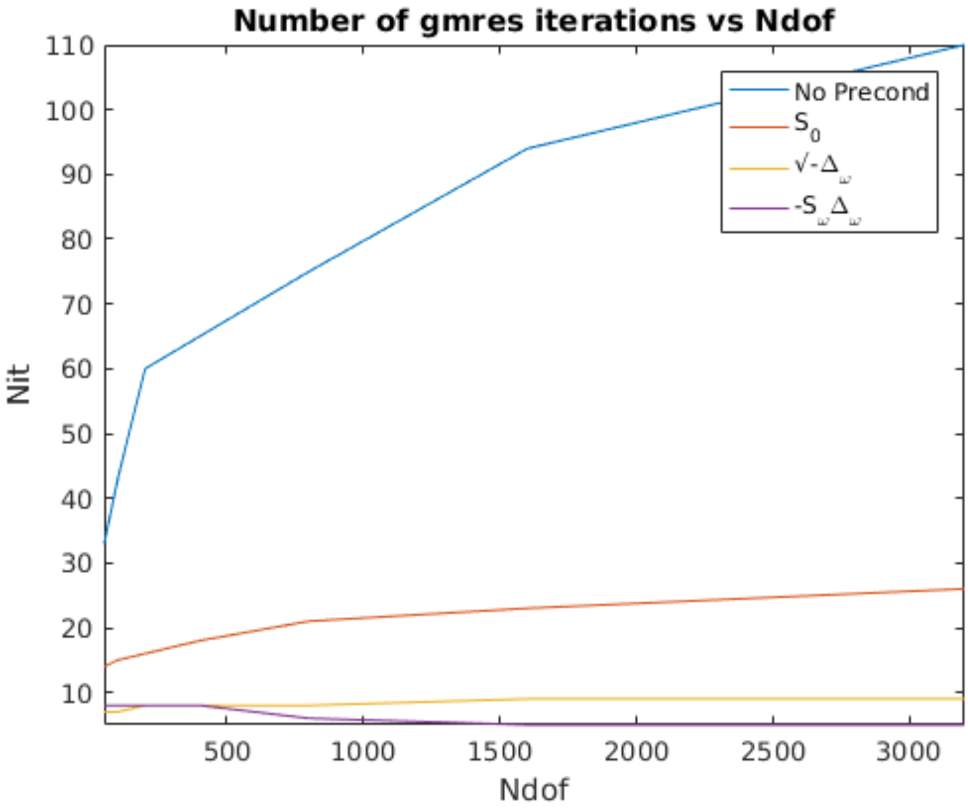
should provide an efficient preconditioner for S_ω. We perform once again the comparison between those methods

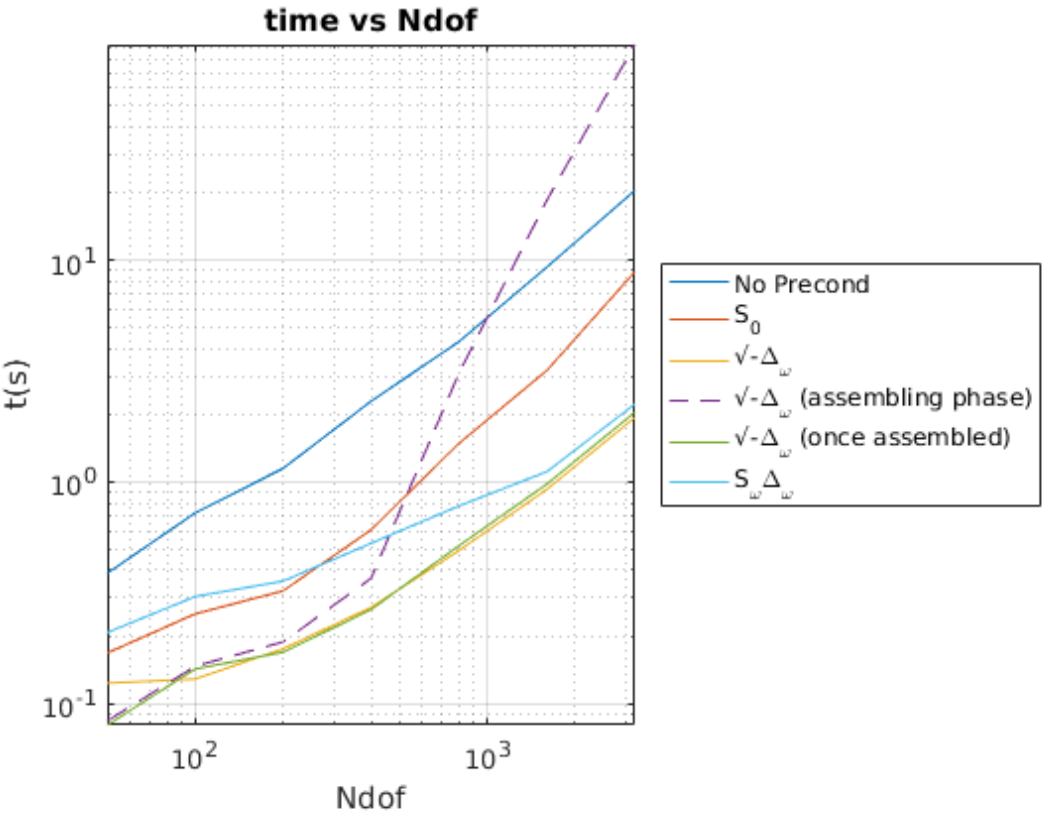
Asymptotic behavior

We now run the previous test for several values of N and show the evolution of iteration number, time and preconditioner assembling for the full square root method.

```
run('asymptoticPrecondSqrtSegment.m');
figure;
plot(ns,nit0,'DisplayName','No Precond')
hold on
plot(ns,nit1,'DisplayName','S_0')
plot(ns,nit2,'DisplayName','\surd{-\Delta_{\omega}}')
plot(ns,nit4,'DisplayName','-S_{\omega}\Delta_{\omega}')
legend show
axis tight
title('Number of gmres iterations vs Ndof');
ylabel('Nit')
xlabel('Ndof')
figure;
loglog(ns,t0_save,'DisplayName','No Precond')
hold on
loglog(ns,t1_save,'DisplayName','S_0')
loglog(ns,t2_save,'DisplayName','\surd{-\Delta_{\omega}}')
loglog(ns,t3_save,'--','DisplayName','\surd{-\Delta_{\omega}} (assembling phase)')
loglog(ns,t3bis_save,'DisplayName','\surd{-\Delta_{\omega}} (once assembled)')
loglog(ns,t4_save,'DisplayName','S_{\omega}\Delta_{\omega}')
grid on;
legend show;
set(legend,'location','eastoutside')
axis tight
title('time vs Ndof')
xlabel('Ndof');
ylabel('t(s)');
```

```
Warning: Condition number too high, restarting with a =
2.371708e-01
Warning: Condition number too high, restarting with a =
1.677051e-01
```





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