Square root preconditioners for the Helmholtz integral equation

Martin Averseng*

December 12, 2018

Abstract

We introduce new analytical tools to study the first-kind integral equations on open curves. Those tools are applied to analyze two new preconditioners and study the convergence orders of a Galerkin method on weighted L^2 spaces.

Introduction

For the resolution of the Helmholtz scattering problem, one of the most popular strategy is to use of integral equations, as it reduces drastically the size of the problem. The unknowns of the equation are functions on the boundary of the scatterer instead of the whole space outside this scatterer. With this method, one is eventually led to solve large and dense linear systems. Since direct method are often too expensive in time and memory in this case, iterative methods like GMRES [22] are generally used. With this method, the number of operations is $N = N_{\text{iter}}N_{\text{mat}}$, where N_{iter} is the number of iterations and N_{mat} is the complexity of a matrix-vector products. Several compression and acceleration, methods have emerged, such as FMM (see [8, 21] and references therein), the Hierarchical Matrix [6], or more recently, the Sparse Cardinal Sine Decomposition [1] and the Efficient Bessel Decomposition [4], adressing the problem of reducing the matrix-vector product complexity.

To reduce the number of iterations, the main approach is preconditioning, which basically consists in finding an approximate inverse of the matrix of the linear system. Algebraical preconditioners exist, such as SPAI [10], but in many cases, they may be unsufficient to capture the physics underlying the linear system problem. An alternative approach, referred as analytical preconditioning, is to build a preconditioner at the continuous level. Say we solve an equation of the form

$$\mathcal{K}u = v$$

where $\mathcal{K}$ is an operator on a Hilbert space, then an analytical preconditioner $\mathcal{L}$ is an operator such that $\mathcal{L}\mathcal{K}$ is a compact perturbation of the identity. In this case, a numerical approximation of u can be obtained through the resolution of a linear system involving discrete approximations of $\mathcal{K}$ and $\mathcal{L}$, with a condition number independent of the mesh size [24].

*CMAP, Ecole polytechnique, Route de Saclay, 91128 Palaiseau Cedex.

Many approaches exist to produce an analytical preconditioner, among which the Calderon preconditioners [?, ?]. In the case of smooth closed curves, pseudo-differential theory provides a systematic way to design such analytical preconditioners [2, 15] références de François. Roughly speaking, one computes the principal symbol $k(\xi)$ of the operator $\mathcal{K}$ and chooses an operator $\mathcal{L}$ with principal symbol $\frac{1}{k(\xi)}$. For the single-layer potential on a smooth closed curve, for example, this leads to a preconditioner of the form

$$\mathcal{L} = \sqrt{-\partial_\tau^2 - k^2} \tag{1}$$

where ∂_τ is the tangential derivative on the curve. This method only works for smooth scatterers though, since pseudo-differential calculus is only well-defined on smooth manifolds. Nevertheless, in [5], efficient preconditioners in a very similar form as in (1) have been introduced for weighted versions of the single layer potential and the hypersingular operator. For the weighted single layer potential, for example, the preconditioner has the form

$$\mathcal{L}_\omega = \sqrt{-(\omega\partial_\tau)^2 - k^2\omega^2} \tag{2}$$

where ω is a simple weight function defined on the curve. For $\omega \equiv 1$, this reduces to the previous preconditioner. This proximity suggests that pseudo-differential tools could be extended to arbitrary curves with a weight accounting for the singularities. The present work is an effort in this direction. We define two scales of spaces, T^s and U^s, with some interlacing properties, that replace the scale of Sobolev spaces H^s for smooth curves. In the scale T^s, we define a class of operators S^p enjoying some of the properties of pseudo-differential operators on smooth manifolds. Those tools are then applied to prove the theorems announced in [5].

The paper is organized as follows. We start by introducing the new analytical tools in section 1. We define the families of spaces T^s and U^s and establish some properties of these spaces. We then introduce the class S^p of operators on T^s. In **??**, we apply these tools to study the new preconditioners for the Helmholtz scattering problem outside an open curve. Finally in **??**, we describe a Galerkin scheme with piecewise polynomial functions on a weighted L^2 to solve the scattering problem. We establish the optimal convergence rates for this setting.

1 Analytical setting

In this section, we will use extensively the properties of Chebyshev polynomials of first and second kinds, respectively given by

$$T_n(x) = \cos(n \arccos(x)),$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos(x))}{\sqrt{1-x^2}}$$

for $x \in [-1, 1]$ Inclure une ref. Let ω the operator $u(x) \mapsto \omega(x)u(x)$ with $\omega(x) = \sqrt{1-x^2}$ and let ∂_x the derivation operator. The Chebyshev polynomials satisfy the ordinary differential equations

$$(1-x^2)T_n'' - xT_n' + n^2T_n = 0 \text{ and } (1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0$$

which can be rewritten under the form

$$
\begin{aligned}
(\omega\partial_x)^2 T_n &= -n^2 T_n\,, && (3)\\
(\partial_x\omega)^2 U_n &= -(n+1)^2 U_n\,. && (4)
\end{aligned}
$$

(Notice that by $(\partial_x\omega)f$ we mean $(\omega f)'$.) As we shall see, the preceding equations are crucial in our analysis.

1.1 Spaces T^s and U^s

1.1.1 Definitions

Both T_n and U_n are polynomials of degree n, and form orthogonal families respectively of the Hilbert spaces

$$L^2_{\frac{1}{\omega}} := \left\{ u \in L^1_{\mathrm{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}}dx < +\infty \right\}$$

and

$$L^2_{\omega} := \left\{ u \in L^1_{\mathrm{loc}}(-1,1) \;\middle|\; \int_{-1}^{1} f^2(x)\sqrt{1-x^2}dx < +\infty \right\}.$$

We denote by $\langle\cdot,\cdot\rangle_{\frac{1}{\omega}}$ and $\langle\cdot,\cdot\rangle_{\omega}$ the inner products in $L^2_{\frac{1}{\omega}}$ and L^2_{ω} respectively. The Chebyshev polynomials satisfy

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } m = n = 0 \\ \pi/2 \text{ otherwise} \end{cases} \tag{5}$$

and

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi/2 \text{ otherwise} \end{cases} \tag{6}$$

which provides us with the so-called Fourier-Chebyshev decomposition. Any $u \in L^2_{\frac{1}{\omega}}$ can be decomposed through the first kind Chebyshev series

$$u(x) = \sum_{n=0}^{+\infty} \hat{u}_n T_n(x) \tag{7}$$

where the Fourier-Chebyshev coefficients $\hat{u}_n$ are given by

$$\hat{u}_n := \begin{cases} \dfrac{2}{\pi}\displaystyle\int_{-1}^{1} \frac{u(x)T_n(x)}{\sqrt{1-x^2}}dx & \text{if } n \neq 0\,, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \frac{u(x)}{\sqrt{1-x^2}}dx & \text{otherwise,} \end{cases}$$

and satisfy the Parseval equality

$$\int_{-1}^{1} \frac{u^2(x)}{\sqrt{1-x^2}}dx = \frac{\pi\hat{u}_0^2}{2} + \pi\sum_{n=1}^{+\infty} \hat{u}_n^2\,.$$

When u is furthermore a smooth function, the series (7) converges uniformly to u. Similarly, any function $v \in L^2_\omega$ can be decomposed along the U_n as

$$v(x) = \sum_{n=0}^{+\infty} \check{v}_n U_n(x)$$

where the coefficients $\check{v}_n$ are given by

$$\check{v}_n := \frac{2}{\pi} \int_{-1}^{1} v(x) U_n(x) \sqrt{1-x^2} dx$$

with the Parseval identity

$$\int_{-1}^{1} v^2(x)\sqrt{1-x^2}dx = \frac{\pi}{2} \sum_{n=0}^{+\infty} \check{v}_n^2 \, .$$

The preceding analysis can be generalized to define Sobolev-like spaces.

Definition 1. *For all $s \geq 0$, we may define*

$$T^s = \left\{ u \in L^2_{\frac{1}{\omega}} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^s \hat{u}_n^2 < +\infty \right\}.$$

This is a Hilbert space for the scalar product

$$\langle u, v \rangle_{T^s} = \frac{\pi}{2} \hat{u}_0 \hat{v}_0 + \pi \sum_{n=1}^{+\infty} (1+n^2)^s \hat{u}_n \hat{v}_n.$$

We also define a semi-norm

$$|u|_{T^s} := \sum_{n=1}^{+\infty} n^{2s} \left| \hat{u}_n \right|^2 .$$

We denote by T^∞ the Fréchet space $T^\infty := \cap_{s\in\mathbb{R}} T^s$, and by $T^{-\infty}$ the set of continuous linear forms on T^∞. For $l \in T^{-\infty}$, we note $\hat{l}_n = l(T_n)$, so that for $u \in T^\infty$,

$$l(u) = \frac{\pi}{2} \hat{l}_0 \hat{u}_0 + \pi \sum_{n=1}^{+\infty} \hat{l}_n \hat{u}_n \, .$$

We choose to identify the dual of $L^2_{\frac{1}{\omega}}$ to itself using the previous bilinear form. With this identification, any element of T^s with $s \geq 0$ can also be seen as an element of $T^{-\infty}$. Furthermore, the space T^{-s} can be defined for all $s \geq 0$ as

$$T^{-s} = \left\{ u \in T^{-\infty} \;\middle|\; \sum_{n=0}^{+\infty} (1+n^2)^{-s} \hat{u}_n^2 < \infty \right\}.$$

Using the former identification T^{-s} becomes the dual of T^s. For $s < t$, the inclusion $T^s \subset T^t$ is compact.

Remark 1. *The spaces T^n correspond, up to a variable change, to the spaces H_e^n defined in [3, 7, 29, 30] among other works, that is, the restriction of the usual Sobolev space H^n to even periodic functions, as stated in Lemma 6. In what follows, $H_p^s(0,T)$ denotes the space of T-periodic functions in $H^s(\mathbb{R})$, and $H_e^s(0,T)$ is the set of even functions if $H_p^s(0,T)$.*

In a similar fashion, we define the following spaces:

Definition 2. *For all $s \geq 0$, we set*

$$U^s = \left\{ u \in L^2_\omega \,\middle|\, \sum_{n=0}^{+\infty} (1+n^2)^s \check{u}_n^2 \right\}.$$

We extend as before the definition to negative indices by setting U^{-s} to be the dual of U^s for $s \geq 0$, this time with respect to the duality $\langle \cdot, \cdot \rangle_\omega$.

1.1.2 Basic properties

Obviously, for any real s, if $u \in T^s$ the sequence of polynomials

$$S_N(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x)$$

converges to u in T^s. The same assertion holds for $u \in U^s$ when T_n is replaced by U_n. Therefore

Lemma 1. *$C^\infty([-1,1])$ is dense in T^s and U^s for all $s \in \mathbb{R}$.*

The polynomials T_n and U_n are connected by the following formulas:

$$\forall n \geq 2, \quad T_n(x) = \frac{1}{2}\left(U_n - U_{n-2}\right), \tag{8}$$

$$\forall n \in \mathbb{N}, \quad U_{2n} = 2\sum_{j=0}^{n} T_{2j} - 1, \quad U_{2n+1} = 2\sum_{j=0}^{n} T_{2j+1}. \tag{9}$$

We deduce the following inclusions:

Lemma 2. *For all real s, $T^s \subset U^s$ and for all $s > 1/2$, $U^s \subset T^{s-1}$.*

Before starting the proof, we introduce the Cesàro operator C defined on $l^2(\mathbb{N}^*)$ by

$$(Cu)_n = \frac{1}{n}\sum_{k=1}^{n} u_k\,.$$

As is well-known, this is a linear continuous operator on $l^2(\mathbb{N}^*)$. Its adjoint

$$(C^* u)_n = \sum_{k=n}^{+\infty} \frac{u_k}{k}\,,$$

is therefore also continuous on $l^2(\mathbb{N}^*)$. In other words, for all $u \in l^2(\mathbb{N})$,

$$\sum_{n=1}^{+\infty} \left(\sum_{k=n}^{+\infty} \frac{u_k}{k} \right)^2 \leq C \sum_{k=1}^{+\infty} u_k^2\,.$$

Proof. The first property is immediate from (8). When $u \in U^s$ for $s > 1/2$, the series $\sum |\check{u}_n|$ is converging, allowing to identify u to a function in $T^{-\infty}$, with, in view of (9),

$$\hat{u}_0 = 2\sum_{n=0}^{+\infty} \check{u}_{2n}, \quad \hat{u}_j = 2\sum_{n=0}^{+\infty} \check{u}_{j+2n} \text{ for } j \geq 1.$$

Since $u \in U^s$, the sequence $\left((1+n^2)^{s/2}\,|\check{u}|\right)_{n\geq 1}$ is in $l^2(\mathbb{N}^*)$. Thus, using the continuity of the adjoint of the Cesàro operator mentioned previously, the sequence $r_n := \left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}\,|\check{u}_k|\right)_{n\geq 0}$ is in $l^2(\mathbb{N})$. But

$$\begin{aligned}
\|u\|_{T^{s-1}}^2 &= \sum_{n=0}^{+\infty}(1+n^2)^{s-1}\,|\hat{u}_n|^2 \\
&\leq 4\sum_{n=0}^{+\infty}(1+n^2)^{s-1}\left(\sum_{k=n}^{+\infty}|\check{u}_k|\right)^2 \\
&\leq 4\sum_{n=0}^{+\infty}\left(\sum_{k=n}^{+\infty}(1+k^2)^{\frac{s-1}{2}}\,|\check{u}_k|\right)^2. \\
&= 4\,\|r_n\|_{l^2}^2.
\end{aligned}$$

□

One immediate consequence is that $T^\infty = U^\infty$. Moreover, we have the following result:

Lemma 3.
$$T^\infty = C^\infty([-1,1]).$$

Proof. If $u \in C^\infty([-1,1])$, then we can obtain by induction using integration by parts and (3), that for any $k \in \mathbb{N}$

$$\hat{u}_n = \frac{(-1)^k}{n^{2k}}\int_{-1}^{1}\frac{(\omega\partial_x)^{2k}u(x)T_n(x)}{\omega(x)}dx.$$

Noting that $(\omega\partial_x)^2 = (1-x^2)\partial_x^2 - x\partial_x$, the function $(\omega\partial_x)^{2k}u$ is C^∞, and since $\|T_n\|_\infty = 1$, the integral is bounded independently of n. Thus, the coefficients $\hat{u}_n$ have a fast decay, proving that $C^\infty([-1,1]) \subset T^\infty$.

To prove the converse inclusion, let $u \in T^\infty$. Then, one has

$$u(x) = \sum_{n=0}\hat{u}_nT_n(x)$$

where the series is normally converging. This ensures $T^\infty \subset C^0([-1,1])$. Now, let $u \in T^\infty$, it suffices to show that $u' \in T^\infty$ and apply an induction argument. Applying term by term differentiation, we obtain

$$u'(x) = \sum_{n=1}^{+\infty} nu_nU_{n-1}(x).$$

Therefore, u' is in $U^\infty = T^\infty$. This proves the result. □

Remark 2. *For $s \leq \frac{1}{2}$, the functions of U^s cannot be identified to functions in $T^{-\infty}$. Indeed, let assume that this is the case. Then, there must exist a map I continuous from $U^{\frac{1}{2}}$ to $T^{-\infty}$ with the property*

$$\forall u \in U^{\infty}, \quad Iu = u.$$

Now, let us consider for example the function u defined by $\check{u}_n = \frac{1}{n\ln(n)}$. Note that u is in $U^{1/2}$. Let $u_N = \sum_{n=0}^{N} \check{u_n} U_n$. This is a sequence of elements of U^∞ converging to u in $U^{1/2}$. By continuity of I, and since $Iu_N = u_N$, the sequence $(\langle u_N, T_0\rangle_{\frac{1}{\omega}})_{N\in\mathbb{N}}$ must converge with limit $\langle Iu, T_0\rangle$. This is not the case since

$$\langle u_N, T_0\rangle_{\frac{1}{\omega}} = \sum_{n=0}^{N} \check{u}_n \langle U_n, T_0\rangle_{\frac{1}{\omega}} = \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2k\ln(2k)}$$

is diverging.

Two natural derivation operators arise in our context, that give another link between T^s and U^s. They are given by the identities

$$\partial_x T_n = nU_{n-1}\,, \tag{10}$$

$$\omega\partial_x\omega U_n = -(n+1)T_{n+1}\,. \tag{11}$$

The first one is obtained for example from the trigonometric definition of T_n. This combined with $-(\omega\partial_x)^2 T_n = n^2 T_n$ gives the second identity.

Definition 3. *For all real s, the operator ∂_x can be extended into a continuous map from T^{s+1} to U^s defined as*

$$\forall v \in U^{\infty}, \quad \langle \partial_x u, v\rangle_{\omega} := -\langle u, \omega\partial_x\omega v\rangle_{\frac{1}{\omega}}\ .$$

In a similar fashion, the operator $\omega\partial_x\omega$ can be extended into a continuous map from U^{s+1} to T^s defined as

$$\forall v \in T^{\infty}, \quad \langle \omega\partial_x\omega u, v\rangle_{\frac{1}{\omega}} := -\langle u, \partial_x v\rangle_{\omega}\,.$$

Proof. Using the identities (10) and (11), one can check that the formulas indeed extend the usual definition of the two operators for smooth functions. We now show that the map ∂_x extended this way is continuous from T^{s+1} to U^s. The definition

$$\forall v \in U^{\infty}, \langle \partial_x u, v\rangle := -\langle u, \omega\partial_x\omega v\rangle$$

gives a sense to $\partial_x u$ for all u in $T^{-\infty}$, as a duality $T^{-\infty} \times T^{\infty}$ product, because if $v \in U^{\infty}(= C^{\infty})$, then $\omega\partial_x\omega v = (1-x^2)v' - xv$ also lies in $C^{\infty}(= T^{\infty})$. It remains to check the announced continuity. Letting $w = \partial_x u$, we have, by definition, for all n

$$\check{w}_n = \langle w, U_n\rangle_{\omega} = -\langle u, \omega\partial_x\omega U_n\rangle_{\frac{1}{\omega}} = n\langle u, T_{n+1}\rangle_{\frac{1}{\omega}} = n\hat{u}_{n+1}$$

Obviously, this implies the announced continuity with

$$\|w\|_{U^s} \leq \|u\|_{T^{s+1}}\ .$$

The properties of $\omega\partial_x\omega$ on T^s are established in a similar way. ∎

The operator ∂_x is not continuous from T^s to T^{s-1}. However, the following result holds:

Corollary 1. *The operator ∂_x is continuous from T^{s+2} to T^s for all $s > -1/2$ and from U^{s+2} to U^s for all $s > -3/2$.*

Proof. For the first case we use ∂_x is continuous from T^{s+2} to U^{s+1} and then the identity is continuous from U^{s+1} to T^s. For the second, we use the same arguments in the reverse order. □

Lemma 4. *For all $\varepsilon > 0$, if $u \in T^{\frac{1}{2}+\varepsilon}$, then u is continuous and there exists a constant C such that for all $x \in [-1, 1]$,*

$$|u(x)| \leq C \, \|u\|_{T^{1/2+\varepsilon}} .$$

Similarly, if $u \in U^{3/2+\varepsilon}$, then u is continuous and

$$|u(x)| \leq C \, \|u\|_{U^{3/2+\varepsilon}} .$$

Proof. Using triangular inequality,

$$|u(x)| \leq \sum_{n=0}^{+\infty} |\hat{u}_n|$$

since for all n, $\|T_n\|_{L^\infty} = 1$. Cauchy-Schwarz's inequality then yields

$$|u(x)| \leq \sqrt{\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^{\frac{1}{2}+\varepsilon}}} \, \|u\|_{T^{\frac{1}{2}+\varepsilon}} .$$

For the second statement, we use the inclusion $U^s \subset T^{s-1}$ valid for $s > 1/2$, as established in Lemma 2. □

1.1.3 Characterization of T^n and U^n.

In this section, we provide a characterization of the spaces T^s and U^s in terms of L^2 norms of the derivatives.

Lemma 5. *The operator $\omega\partial_x$ has a continuous extension from T^1 to T^0. Similarly, the operator $\partial_x\omega$ has a continuous extension from U^1 to U^0.*

Proof. Obviously, the operator ω maps $L^2_\omega = U^0$ to $L^2_{\frac{1}{\omega}} = T^0$. This is in fact a bijective isometry with inverse $\frac{1}{\omega}$. Since ∂_x is continuous from T^1 to U^0, we have the announced continuity of $\omega\partial_x$. For the second part, we write

$$\partial_x\omega = \frac{1}{\omega}\left(\omega\partial_x\omega\right).$$

Where $\omega\partial_x\omega$ is continuous from U^1 to T^0, and the multiplication by $\frac{1}{\omega}$ is continuous from T^0 to U^0. □

We can now state the main result of this paragraph. For a function u defined on $[-1, 1]$, we denote by Cu the function defined on $[0, \pi]$ by

$$Cu(\theta) = u(\cos(\theta))$$

and by Su the function defined as

$$Su(\theta) := \sin(\theta)Cu(\theta).$$

Lemma 6. *A function u belongs to the space T^n if and only if $u = \tilde{u} \circ \arccos$ for some even function $\tilde{u} \in H_p^n(-\pi, \pi)$. In this case, $Cu = \tilde{u}$ and*

$$\|u\|_{T_n} \sim \|Cu\|_{H^n} \text{ and } |u|_{T^n} \sim |Cu|_{H^n}.$$

Similarly, u belongs to the space U^n if and only if $u = \frac{1}{\sqrt{1-x^2}}\tilde{u} \circ \arccos$ for some odd function $\tilde{u}$ in $H^n(-\pi, \pi)$. In this case, $Su = \tilde{u}$ and

$$\|u\|_{U_n} \sim |Su|_{H^n}.$$

Moreover, if $u \in T^n$, then $(\omega\partial_x)^n u$ is in $L^2_{\frac{1}{\omega}}$ and

$$|u|^2_{T^n} \sim \int_{-1}^{1} \frac{((\omega\partial_x)^n u)^2}{\omega}.$$

Similarly, if $u \in U^n$, then $(\partial_x\omega)^n u \in L^2_\omega$ and

$$\|u\|_{U_n} = \int_{-1}^{1} \omega((\partial_x\omega)^n u)^2.$$

Proof. The first two equivalences stem from the fact that

$$\hat{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Cu(\theta)\cos(n\theta), \quad \check{u}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} Su(\theta)\sin((n+1)\theta)d\theta,$$

which can be verified by using the change of variables $x = \cos\theta$ in the definitions of $\hat{u}_n$ and $\check{u}_n$. Now, let us show that if $u \in T^n$, then $(\omega\partial_x)^n$ is in $L^2_{\frac{1}{\omega}}$. The operator $(\omega\partial_x)^2$ is continuous from T^s to T^{s-2} for all real s which implies the result if n is even. If n is odd, say $n = 2k+1$, we write $(\omega\partial_x)((\omega\partial_x)^2)^k$, and conclude using Lemma 5. The same kind of proof also shows that if $u \in U^n$, $(\partial_x\omega)^n u \in L^2_\omega$. The rest of the proof can be performed by computing the quantities for functions in $C^\infty([-1,1])$, performing integrations by parts and concluding with the density of T^∞ in T^s and U^s. □

1.2 Periodic Pseudo-differential operators

On the family of spaces $H_p^s(0,T)$, a class of periodic pseudo-differential operators has been introduced in [28], with symbolic calculus. A periodic pseudo-differential operator on $H_p^s(0,T)$ of order p is an operator of the form

$$A : u \in H_p^s(0,T) \mapsto \sum_{n\in\mathbb{Z}} \sigma_A(t,n)\hat{u}_n e^{\frac{2in\pi t}{T}}.$$

for a symbol $\sigma_A \in C^\infty(\mathbb{T}_T \times \mathbb{R})$ satisfying

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \quad \left|\partial_t^j \partial_\xi^k \sigma_A(t,\xi)\right| \leq C_{j,k}(1+|\xi|)^{p-k}.$$

Here, $\hat{u}_n = \frac{1}{T}\int_0^T u(t)e^{-i\frac{2n\pi t}{T}}dt$ are the usual Fourier coefficients of u and $\partial_t := \frac{1}{iT}\frac{\partial}{\partial t}$, $\partial_\xi := \frac{1}{iT}\frac{\partial}{\partial \xi}$. The class of symbols that satisfy this condition is denoted by

Σ^p, and $\Sigma^{-\infty} := \Cup_{p\in\mathbb{Z}}\Sigma^p$. The operator corresponding to a symbol σ is denoted by $Op(\sigma)$ and thus, the set of periodic pseudo-differential operators is $Op(\Sigma^p)$.

The symbol is not unique but is determined uniquely at the integer values of ξ by

$$\sigma_A(t,n) = e^{-\frac{2in\pi t}{T}} A(e^{\frac{2in\pi t}{T}})\,.$$

As shown in [28], $A \in Op(\Sigma^p)$ if and only if those values satisfy

$$\forall j,k \in \mathbb{N}, \exists C_{j,k} > 0 : \left|\partial_t^j \Delta_n^k \sigma_A(t,n)\right| \leq C_{j,k}(1+|n|)^{p-k}\,.$$

An operator in $Op(\Sigma^p)$ maps continuously $H_p^s(0,T)$ to H_p^{s+p} for all $s \in \mathbb{R}$. The composition of two operators in $Op(\sigma^p)$ and $Op(\sigma^q)$ gives rise to an operator in $Op(\sigma^{p+q})$. If two symbols a and b in $\Sigma^{-\infty}$ satisfy $a-b \in \Sigma^p$, we write $a = b+\Sigma^p$ and if $A = Op(a)$ and $B = Op(b)$, we write $A = B + R_p$. Let $a \in \Sigma^{-\infty}$. If there exists a sequence of reals $(p_j)_{j\in\mathbb{N}}$ such that $p_j < p_{j+1}$ and a sequence of symbols $a_j \in \Sigma^{p_j}$ such that for all N,

$$a = \sum_{i=0}^{N} a_i + \Sigma^{p_j+1}\,,$$

we then write

$$a = \sum_{i=0}^{+\infty} a_j\,.$$

This is called an asymptotic expansion of the symbol a. Consider an integral operator K of the form

$$K : u \mapsto \frac{1}{T}\int_0^T a(t,s)\kappa(t-s)u(s)ds.$$

where a is T-periodic and C^∞ in both arguments and κ is a T-periodic distribution. Assume that the Fourier coefficients $\hat{\kappa}(n)$ of κ can be prolonged to a function $\hat{\kappa}(\xi)$ on $\mathbb{R}$ such that

$$\forall k \in \mathbb{N}, \exists C_k > 0 : \quad \left|\partial_\xi^k \hat{\kappa}(\xi)\right| \leq C_k(1+|\xi|)^{\alpha-k}\,.$$

for some α. Then K is in $Op(\Sigma^\alpha)$ with a symbol satisfying the asymptotic expansion

$$\sigma_K(\xi,t) = \sum_{j=0}^{+\infty} \frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j \hat{\kappa}(\xi)\partial_s^j a(t,s)_{|s=t}\,.$$

1.2.1 Classes of operators

We shall now introduce a class of pseudo-differential operators on the open segment. Essentially, they are, up to a change of variables, the periodic pseudo-differential operators on the torus $\mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$ that preserve the space of 1-periodic even functions, that is, that map $H_e^n(0,1)$ to itself. A particular property of these pseudo-differential operators is that their symbol is not defined uniquely. We show that a symbolic calculus is still available within this restricted class.

Definition 4. *Let $p \in \mathbb{R}$. If $A : T^{\infty} \to T^{-\infty}$ can be extended into a continuous operator from T^s to T^{s+p} for any $s \in \mathbb{R}$, we shall say that it is of order p in the scale T^s. When an operator is of order p for all $p \in \mathbb{N}$, we call it a smoothing operator.*

An operator $A : U^{\infty} \to U^{-\infty}$ which maps continuously U^s to U^{s+p} for all real s is said to be of order p in the scale U^s. When the family (U^s or T^s) is clear from the context, we simply say that A is of order p.

Definition 5. *Let A and B two operators of order s in the scale T^s. When the operator $A - B$ is of order $p \in (0, +\infty]$, we shall write*

$$A = B + T_p.$$

When the scale is U^s instead of T^s, we write

$$A = B + U_p.$$

To ease the computations, we define $T_n = T_{|n|}$ for $n \in \mathbb{Z}$.

Definition 6. *An operator $A \in \mathcal{L}(T^{-\infty})$ belongs to the class $S^{-\infty}$ if there exists a "discrete symbol" $\sigma_A : \mathbb{N} \times \mathbb{Z}$ of A such that, for all $n \in \mathbb{N}$,*

$$AT_n = \sum_{i \in \mathbb{Z}} \sigma_A(n,i) T_{n-i} \,,$$

with furthermore the property that there exists an integer N for which

$$|i| \geq N \implies \forall n \in \mathbb{N}, \quad \sigma_A(n,i) = 0 \,.$$

A single operator A admits infinitely many discrete symbols as shown by the following example. Given n_0, let σ the symbol defined by

$$\sigma(n,i) = \begin{cases} 0 & \text{if } n \neq n_0, i \neq 0 \text{ or } i \neq 2n_0 \\ 1 & \text{if } n = n_0 \text{ and } i = 0 \\ -1 & \text{if } n = n_0 \text{ and } i = 2n_0 \,. \end{cases}$$

Then, σ is a non-trivial discrete symbol of the null operator. Nevertheless, the next result ensures that two symbols of the same operator agree for n large enough.

Lemma 7. *Let σ a discrete symbol of the null operator. Then there exists an integer N such that*

$$n \geq N \implies \forall i \in \mathbb{Z}, \quad \sigma(n,i) = 0 \,.$$

Proof. By definition of $S^{-\infty}$, there exists an integer N such that

$$|i| \geq N \implies \forall n \in \mathbb{N}, \quad \sigma_A(n,i) = 0 \,.$$

Let $n \geq N$, we can write

$$0 = \sum_{i=-N+1}^{N-1} \sigma(n,i) T_{n-i} \,.$$

Two Chebyshev polynomials of the same order do not arise in the summation since $\forall i \in [-N+1, N-1]$, $n-i \geq 1$. Orthogonal projection against each T_{n-i} thus yields

$$\forall i \in [-N+1, N-1], \quad \sigma(n,i) = 0\,,$$

and by definition of N,

$$\forall i \in \mathbb{Z}, \quad \sigma(n,i) = 0\,.$$

This has been shown for all $n \geq N$ so the claim is proved. □

An alternative definition is given as follows

Definition 7. *An operator A belongs to $S^{-\infty}$ if there exists two functions $a_1, a_2 : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ and an integer N such that for all n, $a_1(n,\cdot)$ and $a_2(n,\cdot)$ are polynomials of order at most N, and such that, for $n \geq 1$,*

$$AT_n(x) = a_1(n,x)T_n(x) + a_2(n,x)\omega^2 U_{n-1}(x),$$

while

$$AT_0(x) = a_1(0,x)T_0\,.$$

We say that (a_1, a_2) is a couple of semi-discrete symbols of A.

Let us prove the equivalence betwwen the two definitions.

Proof. We prove that the first definition implies the second one. The calculations can also be made in the reverse order to prove the converse. Let $A \in S^{-\infty}$ and let σ_A a discrete symbol of A. For $(n,i) \in \mathbb{N} \times \mathbb{Z}$, let

$$f_A(n,i) = \frac{\sigma_A(n,i) + \sigma_A(n,-i)}{2}\,,$$

$$g_A(n,i) = \frac{\sigma_A(n,i) - \sigma_A(n,-i)}{2}\,.$$

We then have

$$\begin{aligned} AT_n = f_A(n,0)T_n &+ \sum_{i=1}^{+\infty} f_A(n,i)\left(T_{n+i} + T_{n-i}\right) \\ &+ \sum_{i=1}^{+\infty} g_A(n,i)\left(T_{n-i} - T_{n+i}\right)\,. \end{aligned}$$

We use the trigonometric identities

$$T_i T_n = \frac{T_{n+i} + T_{n-i}}{2}\,.$$

$$\omega^2 U_{i-1} U_{n-1} = \frac{T_{n-i} - T_{n+i}}{2}\,,$$

to get

$$AT_n = a_1(n,x)T_n + \omega^2 a_2(n,x)U_{n-1}$$

with

$$a_1(n,x) = f_A(n,0) + 2\sum_{i=1}^{+\infty} f_A(n,i)T_i(x)\,,$$

$$a_2(n,x) = 2\sum_{i=1}^{+\infty} g_A(n,i)U_{i-1}(x)\,.$$

There exists N such that for each n, $f_A(n,i)$ and $g_A(n,i)$ are null for $|i| \geq N$. This implies that $a_1(n,\cdot)$ and $a_2(n,\cdot)$ are polynomials of degree at most N. $\square$

As before, there are non-trivial couple of semi-discrete symbols for the null operator. For example, let a_1 and a_2 be defined as follows. Fix n_0 and take $a_1(x,n) = a_2(x,n) = 0$ for $n \neq n_0$, while

$$a_1(n_0,x) = -\omega^2 U_{n_0-1}(x)\,,$$
$$a_2(n_0,x) = T_{n_0}(x)\,.$$

Nevertheless, two couple of semi-discrete symbols of the same operator agree for n large enough:

Lemma 8. *Let (a_1,a_2) a couple of semi-discrete symbols for the null operator. Then there exists N such that $n \geq N$ implies $a_1(n,x) = a_2(n,x) = 0$.*

Proof. By definition, there exists an integer N such that for all n, $a_1(n,\cdot)$ is a polynomial of order less than N. Fix $n \geq N$. We have

$$0 = a_1(n,x)T_n(x) + a_2(n,x)\omega^2 U_{n-1}(x)$$

and let (z_i) the $(n+1)$ distinct roots of $\omega^2 U_{n-1}$. Note that $T_n(z_i)$ is non-zero for all i, thus, for

$$\forall 1 \leq i \leq n+1, \quad a_1(n,z_i) = 0\,.$$

Thus, $a_1(n,\cdot)$ has $n+1$ distinct roots while being a polynomial of order less than n. Therefore, $a_1(n,\cdot) = 0$. This in turn implies that $a_2(n,\cdot) = 0$. $\square$

As $a_1(n,\cdot)$ and $a_2(n,\cdot)$ are required to be polynomials, it is in fact possible to ensure uniqueness of the couple by Euclidean division. We may for example ask that $a_1(n,x)$ has a degree less than n. If there exists an admissible couple, then we can change a_1 such that this condition be satisfied by writing

$$a_1(n,\cdot) = b_1(n,\cdot) + Q(x)\omega^2 U_{n-1}(x)$$

where $b_1(n,\cdot)$ has a degree less than n. Then, letting $b_2(n,x) = a_2(n,x) + Q(x)$, we have exhibited such a couple. Of course, with this condition, the null operator only has trivial semi-discrete symbols, by the same arguments as in the proof of the previous lemma. In practice, this uniqueness condition does no seem to bear any special interest.

Notice that the action of A on a function u is given in terms of a couple of semi-discrete symbols (a_1,a_2) by

$$Au(x) = \sum_{n=0}^{+\infty} a_1(n,x)\hat{u}_n T_n(x) + \omega^2 \sum_{n=1}^{+\infty} a_2(n,x)\hat{u}_n U_{n-1}(x)\,.$$

Letting $x = \cos(\theta)$ we have

$$\begin{aligned} Au(\cos(\theta)) = &\sum_{n=0}^{+\infty} a_1(n,\cos(\theta))\hat{u}_n \cos(n\theta) \\ &+ \sum_{n=1}^{+\infty} \sin(\theta) a_2(n,\cos(\theta))\hat{u}_n \sin(n\theta)\,. \end{aligned}$$

Let $\tilde{A}$ the operator defined on $H_e^n(0,1)$ by

$$\tilde{A}f(t) = Au(\cos(2\pi t))$$

where $f(\theta) = u(\cos(2\pi t))$. The Fourier coefficients of f defined by

$$\hat{f}_n = \int_0^1 e^{-2i\pi t\cdot n} f(t)dt$$

satisfy $\hat{f}_n = \frac{\hat{u}_n}{2}$ for $n \neq 0$, and $\hat{f}_0 = \hat{u}_0$. Thus

$$\begin{aligned}\tilde{A}f(t) = a(0,x)\hat{f}_0 + & 2\sum_{n=1}^{+\infty} a_1(n, \cos(2\pi t))\hat{f}_n \cos(2n\pi t) \\ & +2\sum_{n=1}^{+\infty} \sin(2\pi t) a_2(n, \cos(2\pi t))\hat{f}_n \sin(2n\pi t)\,.\end{aligned}.$$

This coincides with a discrete pseudo-differential operator, as defined in [28], with a symbol σ_A equal, for $n \neq 0$, to

$$\sigma_A(n,t) = a_1(|n|, \cos(2\pi t)) + i\mathrm{sign}(n)\sin(2\pi t) a_2(|n|, \cos(2\pi t))\,.$$

and for $n = 0$ to

$$\sigma_A(0,t) = a_1(0, \cos(2\pi t))\,.$$

Contrary to [28], the resulting symbol σ_A is restricted to the class of trigonometric polynomials of bounded degree in the second variable as would result here of our hypothesis for a_1 and a_2. We need this assumption in order to have a form of uniqueness in our symbols, but consequently, this prevents us from directly applying the results of the previous work. We thus need to prove that our class is still an algebra.

Definition 8. *For any real p, an operator A belongs to the class S^p if one of its discrete symbols σ_A satisfies:*

$$\forall (\alpha, i) \in \mathbb{N} \times \mathbb{Z}\,, \exists C_{i,k} > 0 : \forall k \in \mathbb{Z}, \quad |\Delta_k^\alpha \sigma_A(k,i)| \leq C_{\alpha,i}(1+k)^{-p-\alpha}\,. \quad (12)$$

Here Δ_k is the difference operator in the variable k (denoted simply by Δ when there is only one variable), defined by

$$\Delta_k a(k,i) = a(k+1,i) - a(k,i),$$

and Δ_k^α is the α-th iterate of Δ_k.

By Lemma 7, if $A \in S^p$, then any symbol of A satisfies the previous condition. We state an equivalent definition using this time a couple of semi-discrete symbols.

Lemma 9. *The following conditions are equivalent:*

- $A \in S^p$
- *There exists a couple of semi-discrete symbols (a_1, a_2) of A such that, for $i = 1, 2$,*

$$\forall (\alpha, x) \in \mathbb{N}\,, \exists C_\alpha > 0 : \forall (k,x) \in \mathbb{Z}\times[-1,1], \quad |\Delta_k^\alpha a_i(k,x)| \leq C_\alpha(1+k)^{-p-\alpha}\,. \quad (13)$$

If one of those conditions hold, then the condition (13) *holds for any couple of semi-discrete symbols of A.*

Proof. Given a discrete symbol σ_A of A, A admits the couple of semi-discrete symbols (a_1, a_2) given by

$$a_1(n,x) = f_A(n,0) + 2\sum_{i=1}^{+\infty} f_A(n,i)T_i(x)\,,$$

$$a_2(n,x) = 2\sum_{i=1}^{+\infty} g_A(n,i)U_{i-1}(x)\,.$$

with f_A and g_A defined as

$$f_A(n,i) = \frac{\sigma_A(n,i)+\sigma_A(n,-i)}{2}\,,$$

$$g_A(n,i) = \frac{\sigma_A(n,i)-\sigma_A(n,-i)}{2}\,.$$

We have by linearity

$$\Delta_k^\alpha a_1(k,x) = \Delta_k^\alpha \sigma_A(n,0) + \sum_{i=1}^{+\infty}\left(\Delta_k^\alpha \sigma_A(n,i) + \Delta_k^\alpha \sigma_A(n,-i)\right)T_i(x)\,.$$

We now apply condition (12) to each term in the sum and with triangular inequality,

$$|\Delta_n^\alpha a_1(k,x)| \le C_\alpha (1+k)^{-p-\alpha}$$

with $C_\alpha = \sum_{i=0}^{+\infty} C_{\alpha,i}$, where $C_{\alpha,i}$ are the constant from condition (12) (recall the sum is actually finite). The same kind of computations gives a similar estimate for a_2 and the direct implication is proved. Conversely, let (a_1, a_2) a couple of semi-discrete symbols of A satisfying the condition (13). Then, reversing the computations, A admits the discrete symbol σ_A defined for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ by

$$\sigma_A(n,i) = f_A(n,|i|) + \operatorname{sign}(i) g_A(n,|i|) \tag{14}$$

with, for $i \in \mathbb{N}$, (by orthonormal projection),

$$f_A(n,i) = \begin{cases} \frac{1}{2\|T_i\|^2_{\frac{1}{\omega}}} \langle a_1(n,x), T_i\rangle_{\frac{1}{\omega}} & \text{if } i \neq 0 \\ \frac{1}{\|T_0\|^2_{\frac{1}{\omega}}} \langle a_1(n,x), T_0\rangle_{\frac{1}{\omega}} & \text{if } i = 0\,. \end{cases}$$

and

$$g_A(n,i) = \begin{cases} \frac{1}{2\|U_{i-1}\|^2_{\omega}} \langle a_2(n,x), U_{i-1}\rangle_{\omega} & \text{if } i \neq 0 \\ 0 & \text{if } i = 0\,. \end{cases}$$

Fix $i \in \mathbb{N}^*$. We have for all k, by linearity,

$$\Delta_k^\alpha f_A(k,i) = \frac{1}{2\,\|T_i\|^2_{\frac{1}{\omega}}} \langle \Delta_k^\alpha a_1(k,x), T_i\rangle_{\frac{1}{\omega}}\,.$$

Using Cauchy-Schwarz inequality and assumption (13) for a_1, we get

$$|\Delta_k^\alpha f_A(k,i)| \leq C_i C_\alpha (1+k)^{-p-\alpha}\,,$$

with $C_i = \frac{1}{\|T_i\|_{\frac{1}{\omega}}}$. Similar estimates can be shown in the same way for $i = 0$ and also for g_A. We finally combine them with (14) to conclude for the reverse implication. The last statement is obvious in view of Lemma 8. □

An interpolation procedure gives the following result:

Lemma 10. *The following propositions are equivalent:*

- $A \in S^p$
- *There exists a couple of functions $(\nu_1(\xi,x),\nu_2(\xi,x))$ defined for $(\xi,x) \in \mathbb{R}^+ \times [-1,1]$, and an integer N such that a_1 and a_2 are C^∞ in ξ and $(\nu_{1|\mathbb{N}\times[-1,1]},\nu_{2|\mathbb{N}\times[-1,1]})$ is a couple of symbols for A, with furthermore the property*

$$\forall\alpha, \exists C_\alpha > 0 : \forall(\xi,x) \in \mathbb{R}+\times[-1,1], \quad \left|\partial_\xi^\alpha \nu_i(x,\xi)\right| \leq C_\alpha(1+\xi)^{-p-\alpha}\,.$$

We call (ν_1,ν_2) a couple of continuous symbols of A.

Proof. Utiliser la procédure d'interpolation du papier PDO discrets. □

1.2.2 Symbols that are rational functions of n.

Il va peut-être falloir enlever cette section à cause de l'équivalence du lemme précédent.

Given a discrete symbol $\sigma(n,i)$, is not straightforward to check that the condition (12) holds for this symbol. We now show that this is the case when $\sigma(n,i)$ is a rational function of n. We will repeatedly make use the following forms of Peetre's inequality: for any real a,b and s,

$$(1+|a+b|)^s \leq (1+|a|)^{|s|}\,(1+|b|)^s \tag{15}$$

and

$$\left(1+|a+b|^2\right)^s \leq 2^{|s|}\left(1+|a|^2\right)^{|s|}\left(1+|b|^2\right)^s\,. \tag{16}$$

Lemma 11. *Let f a C^α function on $[k,k+\alpha]$. Then for all k,*

$$\Delta^\alpha f(k) = \int_{x_1=k}^{k+1}\int_{x_2=x_1}^{x_1+1}\cdots\int_{x_\alpha=x_{\alpha-1}}^{x_{\alpha-1}+1} f^{(\alpha)}(x_\alpha)dx_1dx_2\cdots dx_\alpha \tag{17}$$

Proof. We show by induction that for all $1 \leq \beta \leq \alpha$,

$$\Delta^\alpha f(x) = \int_x^{x+1}\int_{x_1}^{x_1+1}\int_{x_{\beta-1}}^{x_{\beta-1}+1}\Delta^{\alpha-\beta}f^{(\beta)}(x_\beta)dx_1dx_2\cdots dx_\beta. \tag{18}$$

For $\beta = 1$, we write

$$\Delta^\alpha f(x) = \Delta^{\alpha-1}f(x+1) - \Delta^{\alpha-1}f(x),$$

therefore,

$$\Delta^\alpha f(x) = \int_x^{x+1} \frac{d}{dx_1} \left(\Delta^{\alpha-1} f\right) dx_1.$$

This proves the property for $\beta = 1$. Let $1 \leq \beta < \alpha$ and assume that (18) holds for this β. Then we write

$$\Delta^{\alpha-\beta} f^\beta(x_\beta) = \int_{x_\beta}^{x_\beta+1} \frac{d}{dx_\beta} \Delta^{\alpha-\beta} f^{(\beta)}(x_{\beta+1}) dx_{\beta+1}.$$

Of course, Δ and $\frac{d}{dx}$ commute, thus

$$\Delta^{\alpha-\beta} f^\beta(x_\beta) = \int_{x_\beta}^{x_\beta+1} \Delta^{\alpha-\beta} f^{(\beta+1)}(x_{\beta+1}) dx_{\beta+1}.$$

Replacing in (18), this proves the heredity of the property. Finally, taking $\beta = \alpha$ in (18) gives the announced result. ☐

Corollary 2. *If f is a rational function of degree p which poles are contained in $\mathbb{C} \setminus \mathbb{N}$, then there exists a constant C_α such that for all $k \in \mathbb{N}$,*

$$|\Delta^\alpha f(k)| \leq C_\alpha (1+k)^{p-\alpha}\,.$$

Proof. Fix a rational fraction F of degree p, with poles in $\mathbb{C} \setminus \mathbb{N}$. F is of the form

$$F = P + R$$

where P is zero if $p < 0$ and a polynomial of degree p otherwise, and R is a finite linear combination of quantities of the form

$$Q_i(X) = \frac{1}{(X - x_i)^q}$$

with $x_i \in \mathbb{C} \setminus \mathbb{N}$ and $q \geq -p$. The claimed result is an easy consequence of the following two fact:

- If the polynomial P is of degree $p \geq 0$, there holds

$$|\Delta^\alpha P|\,(k) \leq C(1+k)^{p-\alpha},$$

- For the terms $Q_i(X) = \frac{1}{(X-x_i)^q}$, there holds

$$|\Delta^\alpha Q_i|\,(k) \leq C(1+k)^{-q-\alpha}\,. \tag{19}$$

We first treat the polynomial case. If $\alpha > p$, we have $P^{(\alpha)} = 0$, and thus, by Lemma 11, $\Delta^\alpha P(k) = 0$, and the result is obvious. If $\alpha \leq p$, there exists a constant C such that

$$\left|P^{(\alpha)}(x)\right| \leq C(1+x)^{p-\alpha}.$$

We inject this in (17). In the domain of integration, $x_\alpha \leq k + \alpha$, thus

$$|\Delta^q P|\,(k) \leq C(1+(k+\alpha))^{p-\alpha} \leq C(1+\alpha)^{p-\alpha}(1+k)^{p-\alpha}$$

by Peetre's inequality (15). This proves the claim for the polynomial term.

For Q_i, we write:

$$Q_i^{(\alpha)}(x) = \frac{(-1)^k p(p+1)\cdots(p+\alpha)}{(x-x_i)^{p+\alpha}}.$$

For $x \geq \max(1, 2|x_i|)$, using Peetre's inequaliy (15), we get

$$\left|Q^{(\alpha)}(x)\right| \leq \frac{p(p+1)\cdots(p+\alpha)}{(1+x)^{p+\alpha}}.$$

We then proceed with the same arguments as for P and conclude that (19) holds with some constant C_1 for all $k \geq k_0$ where k_0 is an integer greater than $\max(1, 2|x_i|)$. Then (19) holds for all k with

$$C = \max(C_1, \Delta^\alpha Q_i(0), \cdots, \Delta^\alpha Q_i(k_0)).$$

□

1.2.3 Algebraic properties of S^p

Lemma 12. *If $A \in S^p$, then A is of order p in the scale T^s.*

Proof. Since the sum in the condition (i) is finite, by linearity, showing that the operator A is of order p amounts to proving that the operator A_i defined by

$$\forall k \in \mathbb{Z}, \quad A_i T_k = a(k,i) T_{k-i}$$

is of order p. We treat the case $i > 0$, the opposite case being analogous. Let $u \in T^s$ for some s, there holds

$$A_i u = \sum_{k=0}^{+\infty} a(i+k,i)\hat{u}_{k+i} T_k + \sum_{k=0}^{i} a(i-k,i)\hat{u}_{i-k} T_k.$$

Let Vu and Ru respectively the two terms of the rhs. Obviously, R is a smoothing operator. Now, for all $k \in \mathbb{N}$ let

$$\hat{v}_k := a(i+k,i)\hat{u}_{i+k}.$$

Applying Peetre's inequality (16),

$$(1+k^2)^{n+s} |\hat{v}_k|^2 \leq 2^{|p+s|} \left(1+i^2\right)^{|p+s|} \left(1+(i+k)^2\right)^{p+s} |a(k+i,i)|^2 |\hat{u}_{k+i}|^2.$$

Condition (ii) with $\alpha = 0$ yields

$$|a(k+i,i)|^2 \leq C\left(1+(k+i)\right)^{-2p} \leq 2^{|p|} C \left(1+(k+i)^2\right)^{-p}.$$

Therefore, $\|Vu\|_{T^{s+p}} \leq C(1+i)^{|n+s|} \|u\|_{T^s}$ which shows that A is of order p. □

Lemma 13. *If $A \in S^p$, $B \in S^q$, then AB is in S^{p+q},*

Proof. A symbol of AB is given by

$$c(k,i) = \sum_{j=-\infty}^{+\infty} a(k-j, i-j) b(k,j).$$

This formula is obtained writing the expression of ABT_n using a symbol of A and B and using the identity $T_iT_j = T_{i+j} + T_{i-j}$. Let N_a such that $|i| \geq N_a \implies a(k,i) = 0$ and let N_b defined in a similar way. Then, it is easy to check that $|i| \geq N_a + N_b \implies c(k,i) = 0$. It remains to check the requirement (ii). Since the sum defining c only has a finite number of non-zero terms, we just have to show that for any $j \in \mathbb{Z}$, the function $c_j(k,i) := a(k-j, i-j)b(k,j)$ satisfies

$$\forall \alpha \in \mathbb{N}, \forall i, k, \in \mathbb{Z}, \quad |\Delta_k^\alpha c_j(k,i)| \leq C_{i,j,\alpha}(1+k^2)^{p+q-\alpha}.$$

The announced result then follows by linearity. To prove this, one can check by induction that for any $\alpha \in \mathbb{N}$, $\Delta^\alpha c_j(k,i)$ is of the form

$$\Delta_k^\alpha c_j(k,i) = \sum_{l=1}^{L} \lambda_l \Delta_k^{\beta_l} a(k_{l,1}, i-j) \Delta_k^{\alpha-\beta_l} b(k_{l,2}, j)$$

for some coefficients λ_l, where L is a finite number, and where, for all l, $\beta_l \leq \alpha$ while $k_{l,1}$ and $k_{l,2}$ respectively lie in the interval $[k-j, k-j+\beta_l]$ and $[k, k+\alpha-\beta_l]$. Let us fix $l \in [1, L]$. We have

$$\left| \Delta_k^{\beta_l} a(k_{l,1}, i-j) \right| \leq C_{i,j,\alpha,l}(1+k_1)^{p-\beta_l}$$

and by Peetre's inequality

$$(1+k_1)^{p-\beta_l} \leq C(1+k)^{p-\beta}(1+\alpha+|j|)^{|p-\beta_l|}.$$

The same arguments applied to b lead to

$$\left| \Delta_k^{\beta} a(k_1, i-j) \Delta_k^{\alpha-\beta} b(k_2, j) \right| \leq C_{i,j,\alpha,l}(1+k)^{p+q}$$

which implies our claim. $\square$

Theorem 1. *If $A \in S^p$ and $B \in S^q$, then $AB - BA$ is in S^{p+q+1}.*

Proof. A symbol of $C = AB - BA$ is given by

$$\sigma_C(k,i) = \sum_{j=-\infty}^{+\infty} a(k-j, i-j)b(k,j) - \sum_{j=-\infty}^{+\infty} b(k-j, i-j)a(k,j).$$

In the second sum, we change the index to $j' = i - j$, yielding

$$\begin{aligned}
\sigma_C(k,i) &= \sum_{j=-\infty}^{+\infty} a(k-j, i-j)b(k,j) - \sum_{j=-\infty}^{+\infty} b(k-i+j', j')a(k, i-j'). \\
&= \sum_{j=-\infty}^{+\infty} \left[a(k-j, i-j) - a(k, i-j)\right] b(k,j) \\
&\quad - \sum_{j=-\infty}^{+\infty} a(k, i-j) \left[b(k-i+j') - b(k,j)\right].
\end{aligned}$$

Let us consider one of the terms of the first sum when j is positive. We can write

$$[a(k-j,i-j)-a(k,i-j)]\,b(k,j)=-\sum_{l=0}^{j-1}\Delta_k a(k-j+l+1,i-j)b(k,j).$$

The estimation (ii) required for the symbol σ_C can be established for this individual term from the same considerations as in the proof of the previous result. The other terms are treated in an analogous way. $\square$

1.2.4 Symbolic calculus

For a couple of (semi-discrete or continuous) symbols a_1, a_2, let H the operator defined by

$$H(a_1,a_2)=(-\omega\partial_x\omega a_2,\partial_x a_1)$$

and let the multiplication of couples of symbols be defined by

$$(a_1,a_2)\times(b_1,b_2)=\left(a_1b_1-\omega^2a_2b_2,a_1b_2+a_2b_1\right).$$

Let $\nu=(\nu_1,\nu_2)$ a couple of continuous symbols of an operator in S^p. For all $\alpha\in\mathbb{N}$, we write $\partial_n^\alpha\nu:=(\partial_\xi^\alpha\nu_1,\partial_\xi^\alpha\nu_2)$. This is of course a couple of continuous symbols for an operator in $S^{p+\alpha}$.

Lemma 14. *Let (a_1,a_2) and (b_1,b_2) two couple of symbols of two operators $A\in S^p$ and $B\in S^q$. Then $H(a_1,a_2)$ is a couple of symbols for an operator in S^p and $(a_1,a_2)\times(b_1,b_2)$ is a couple of symbols for an operator in S^{p+q}.*

Preuve pédestre de ça. Definition of Asymptotic expansion

Theorem 2. *Let $A\in S^p$ and $B\in S^q$ with respctive couples of continuous symbols $a=(a_1,a_2)$ and $b=(b_1,b_2)$. Then a couple of continuous symbols $c=(c_1,c_2)$ of $C=AB$ is given by*

$$c=\sum_{\alpha=0}^{+\infty}\frac{1}{\alpha!}\partial_\xi^\alpha a\times H^k(b)$$

Proof. Appliquer les résultats connus pour les opérateurs discrets, ou simplement reproduire la preuve qui est tout simple ici. $\square$

Lemma 15. *If P is a polynomial, the multiplication by P defines an operator of S^0.*

Proof. For all n, we have $xT_n=\frac{T_{n+1}+T_{n-1}}{2}$, thus x is in the class S^0. By Lemma 13, the same is true for x^n for any n, and by linearity, for any polynomial. $\square$

The next two lemmas enlarge the class of operators for which we can assess the order. Lemma 17 allows us to treat a rather general class of operators and will allow us to evaluate the order of the remainders in Taylor expansions of the kernels.

Lemma 16. *If ψ is a C^∞ function on $[-1,1]$, then the operator*

$$u(x) \mapsto \psi(x)u(x)$$

is of order 0*, and for any $s \in \mathbb{R}$,*

$$\|\psi u\|_{T^s} \leq C 2^{|s|/2} \|u\|_{T^s} \|\psi\|_{T^{|s|+1}} .$$

where C is independent of ψ and s.

Proof. Let $u \in T^s$, we rewrite u as

$$u = \sum_{n=-\infty}^{+\infty} u'_n T_n$$

where for $n < 0$ we define $T_n = T_{|n|}$, and with

$$u'_n = \begin{cases} u_0 & \text{if } n = 0 \\ \frac{u_{|n|}}{2} & \text{otherwise.} \end{cases}$$

We apply the same idea to ψ, and using $T_m T_n = T_{m+n} + T_{m-n}$,

$$\psi u = \sum_{m,n} u'_n \psi'_m (T_{m+n} + T_{m-n}) = \sum_m \left(\sum_n u'_n (\psi'_{n+m} + \psi'_{n-m}) \right) T_m$$

that is,

$$\psi u = 2 \sum_{m,n} u'_m \psi'_{m-n} T_n$$

Using Peetre's inequality, we have

$$(1+n^2)^{s/2} |(\psi u)_n| \leq 2^{|s|/2+1} \sum_m (1+m^2)^{s/2} |u'_m| (1+|n-m|^2)^{|s/2|} \left|\psi'_{n-m}\right|$$

and by Young's inequality with $r = 2, p = 2, q = 1$,

$$\|\psi u\|_s^2 \leq 2^{|s|+2} \|u\|_s^2 \sum_{m=-\infty}^{+\infty} (1+m^2)^{|s|/2} |\psi'_m|$$

The last sum is finite because $\psi \in T^\infty$ and

$$\sum_{m=-\infty}^{\infty} (1+m^2)^{|s|/2} |\psi'_m| \leq \left(\sum_{m=-\infty}^{+\infty} \frac{1}{1+m^2} \right) \sum_{m=-\infty}^{+\infty} (1+m^2)^{|s|+1} |\psi'|_m^2 .$$

□

Lemma 17. *Let G an integral operator with kernel g, that is*

$$G : u \mapsto \int_{-1}^{1} \frac{g(x,y)u(y)}{\omega(y)} dy .$$

We assume that G is of order p. Let $r(x,y)$ a C^∞ function. Then the operator

$$K : \int_{-1}^{1} \frac{g(x,y)r(x,y)u(y)}{\omega(y)} dy$$

is of order p.

Proof. Since r is in C^∞, one can show that r admits the following expression:

$$r(x,y) = \sum_{m,n} r_{m,n} T_m(x) T_n(y) \tag{20}$$

Moreover, the regularity of R ensures $r_{m,n}$ satisfies for all $s,t \in \mathbb{R}$,

$$\sum_{m,n} (1+m^2)^s (1+n^2)^t \left| r_{m,n} \right|^2 < +\infty.$$

To prove this property, one can for example apply the operator $(\omega \partial_x)^2$ repeatedly in the two variables. The resulting function is C^∞, and in particular, square integrable on $[0,1] \times [0,1]$. We then write the Parseval's identity and the result follows. We can write

$$Ku = \sum_{m,n} r_{m,n} T_m G T_n u$$

where for each m,n, the operator $T_m G T_n$ is defined by

$$T_m G T_n u(x) = T_m(x) \int_{-1}^{1} \frac{G(x,y) T_n(y) u(y)}{\omega(y)} dy.$$

Fix $s \in \mathbb{R}$, this operator is in $L(T^s, T^{s+p})$ by the previous lemma, with

$$\|T_m G T_n\|_{T^s \to T^{s+p}} \leq \|G\|_{T^s \to T^{s+p}} 2^{|s|+|s+p|} (1+n^2)^{|s|+1} (1+m^2)^{|s+p|+1}.$$

thus, the series in (20) is normally convergent in $L(T^s, T^{s+p})$, which proves the claim. ☐

As a consequence, since the operator G with kernel $g \equiv 1$ is a smoothing operator, we have the following result:

Corollary 3. *Let $r \in C^\infty([-1,1]^2)$. Then*

$$u \mapsto \int_{-1}^{1} \frac{r(x,y) u(y)}{\omega(y)} dy$$

is a smoothing operator.

2 Preconditioners for the Helmholtz scattering problem

In this section, we apply the analytical tools introduced in the previous section to the study of the Helmholtz scattering problems. The two main results are **??** and **??**. We start by introducing the notations.

2.1 The scattering problem for an open curve

Let Γ be a smooth non-intersecting open curve in $\mathbb{R}^2$, and let $k \geq 0$ the wave number. We seek a solution to the two problems

$$-\Delta u_i - k^2 u_i = 0, \text{ in } \mathbb{R}^2 \setminus \Gamma, \quad i = 1,2 \tag{21}$$

with the following additional conditions

- Dirichlet or Neumann boundary conditions, respectively

$$u_1 = u_D, \text{ and } \frac{\partial u_2}{\partial n} = u_N \text{ on } \Gamma \tag{22}$$

- Suitable decay at infinity, given for $k > 0$ by the Sommerfeld condition

$$\frac{\partial u}{\partial r} - iku = o\left(\frac{1}{\sqrt{r}}\right) \tag{23}$$

with $r = |x|$ for $x \in \mathbb{R}^2$.

When $k = 0$, the radiation condition must be replaced by an appropriate decay of u and ∇u at infinity, see for example [26, 27], or [18, Chap. 7] Vérifier le chapitre et la page. In the preceding equation n stands for a smooth unit normal vector to Γ. Existence and uniqueness results are available for those problems, but the solutions fail to be regular even with smooth data u_D and u_N. More precisely, let $\lambda = \left[\frac{\partial u_1}{\partial n}\right]_\Gamma$ and $\mu = [u_2]_\Gamma$ where $[\cdot]_\Gamma$ refers to the jump of a quantity across Γ, we have the following result.

Theorem 3. *(see e.g. [19,26,27]) Assume $u_D \in H^{1/2}(\Gamma)$, and $u_N \in H^{-1/2}(\Gamma)$. Then problems (21,22,23) both possess a unique solution $u_i \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$, which is of class C^∞ outside Γ. Near the edges of the screen Γ, λ is unbounded:*

$$\lambda(x) = O\left(\frac{1}{\sqrt{d(x, \partial\Gamma)}}\right).$$

while μ satisfies

$$\mu(x) = C\sqrt{d(x, \partial\Gamma)} + \psi$$

where $\psi \in \tilde{H}^{3/2}(\Gamma)$.

For the definition of Sobolev spaces on smooth open curves, we follow [18] by considering any smooth closed curve $\tilde{\Gamma}$ containing Γ, and defining

$$H^s(\Gamma) = \left\{ U_{|\Gamma} \;\middle|\; U \in H^s(\tilde{\Gamma}) \right\}.$$

Obviously, this definition does not depend on the particular choice of the closed curve $\tilde{\Gamma}$ containing Γ. Moreover,

$$\tilde{H}^s(\Gamma) = \left\{ u \in H^s(\Gamma) \;\middle|\; \tilde{u} \in H^s(\tilde{\Gamma}) \right\}$$

where $\tilde{u}$ denotes the extension by zero of u on $\tilde{\Gamma}$.

Single-layer potential We define the single-layer potential by

$$\mathcal{S}_k\lambda(x) = \int_\Gamma G_k(x-y)\lambda(y)d\sigma(y) \tag{24}$$

where G_k is the Green's function

$$\begin{cases} G_0(z) = -\dfrac{1}{2\pi}\ln|z|, & \text{if } k = 0, \\ G_k(z) = \dfrac{i}{4}H_0(k|z|), & \text{if } k > 0, \end{cases} \tag{25}$$

for $x \in \mathbb{R}^2 \setminus \Gamma$. Here H_0 is the Hankel function of the first kind. For $k > 0$, the solution u_1 to the Dirichlet problem admits the representation

$$u_1 = \mathcal{S}_k \lambda \tag{26}$$

where $\lambda \in \tilde{H}^{-1/2}(\Gamma)$ is the jump of the normal derivative of u_1 across Γ and is the unique solution to

$$S_k \lambda = u_D\,. \tag{27}$$

Here, $S_k := \gamma \mathcal{S}_k$ where γ is the trace operator on Γ. The operator S_k maps continuously $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$. When $k = 0$, the computation of u_1 also involves the resolution of (27) but some subtleties arise in the representation of u_1 (26). On this topic, see [26, Theorem 1.4].

Double-layer and hypersingular potentials Similarly, we introduce the double layer potential $\mathcal{D}_k$ by

$$\mathcal{D}_k \mu(x) = \int_\Gamma n(y) \cdot \nabla G_k(x-y) \mu(y) d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \Gamma$$

for any smooth function μ defined on Γ. The normal derivative of $\mathcal{D}_k \mu$ is continuous across Γ, allowing us to define the hypersingular operator $N_k = \frac{\partial}{\partial_n} \mathcal{D}_k$. This operator admits the following representation for $x \in \Gamma$

$$N_k \mu(x) = \lim_{\varepsilon \to 0^+} \int_\Gamma n(y) \cdot \nabla G(x + \varepsilon n(x) - y) \mu(y) d\sigma(y). \tag{28}$$

The kernel of this operator has a non-integrable singularity, but numerical calculations are made possible by the following formula, valid for smooth functions μ and ν that vanish at the extremities of Γ:

$$\begin{aligned} \langle N_k \mu, \nu \rangle = \int_{\Gamma \times \Gamma} & G_k(x-y) \mu'(x) \nu'(y) \\ & - k^2 G_k(x-y) \mu(x) \nu(y) n(x) \cdot n(y) d\sigma(x) d\sigma(y)\,. \end{aligned} \tag{29}$$

It is also known that N_k maps $\tilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ continuously, and that the solution u_2 to the Neumann problem can be written as

$$u_2 = \mathcal{D}_k \mu \tag{30}$$

where $\mu \in \tilde{H}^{1/2}(\Gamma)$ is the jump of u_2 across Γ and is the unique solution to

$$N_k \mu = u_N\,. \tag{31}$$

Weighted layer potentials. Theorem 3 shows that even if u_D and u_N are smooth, the solutions λ and μ to the corresponding integral equations have singularities. For this reason, we consider the following weighted operators. Let $\omega_\Gamma(r(x)) := \frac{|\Gamma|}{2} \omega(x)$ where $|\Gamma|$ is the length of Γ, $\omega(x) = \sqrt{1-x^2}$ as in the previous section, and $r : [-1,1] \to \Gamma$ is a smooth parametrisation. We define $S_{k,\omega_\Gamma} := S_k \frac{1}{\omega_\Gamma}$ and $N_{k,\omega_\Gamma} := N_k \omega_\Gamma$. The operator $S_{k,\omega}$ reads

$$S_{k,\omega_\Gamma} \alpha(x) = \int_\Gamma \frac{G_k(x-y) \alpha(y)}{\omega_\Gamma(y)} dy\,.$$

As for the hypersingular operator, the identity (29) can be rewritten equivalently

$$\begin{aligned}\langle N_{k,\omega_\Gamma}\beta,\beta'\rangle_\omega = &\langle S_{k,\omega_\Gamma}\left(\omega_\Gamma\partial_\tau\omega_\Gamma\right)\beta,\left(\omega_\Gamma\partial_\tau\omega_\Gamma\right)\beta'\rangle_{\frac{1}{\omega}}\\ &- k^2\left\langle S_{k,\omega_\Gamma}\omega_\Gamma^2\beta,\omega_\Gamma^2\beta'\right\rangle_{\frac{1}{\omega}}\end{aligned} \tag{32}$$

Solving the integral equations (27) and (31), is equivalent to solving

$$\begin{aligned}S_{k,\omega_\Gamma}\alpha &= u_D\\ N_{k,\omega_\Gamma}\beta &= u_N\end{aligned}$$

and letting $\lambda = \frac{\alpha}{\omega_\Gamma}$, $\mu = \omega_\Gamma u_N$. When $k = 0$, and Γ is the flat segment, that is $r \equiv 1$, we simply write S_ω and N_ω. The weighted integral operators appear in many related works such as [7,13,14]. From [7], we know that the operators S_{k,ω_Γ} and N_{k,ω_Γ} map smooth functions to smooth functions, more precisely, they define bicontinuous maps T^s to T^{s+1} and T^{s+1} to T^s respectively. Moreover, $N_{k,\omega_\Gamma}S_{k,\omega_\Gamma}$ has its spectrum concentrated around $\frac{1}{4}$. This motivates the use of the pair $S_{k,\omega_\Gamma}, N_{k,\omega_\Gamma}$ as mutual preconditioners, in close analogy to the well-known Calderon relations for smooth closed curves Inclure citation.

2.2 Operators S_ω and N_ω on the flat segment

When the wavenumber is equal to 0 and the curve Γ is the flat segment $(-1,1)\times 0$, then $\partial_\tau = \partial_x$ and $\omega_\Gamma = \omega$, and the operators S_ω and N_ω become very simple to analyze in T^s and U^s, because they are diagonal in the basis $(T_n)_n$ and $(U_n)_n$ respectively, with explicit eigenvalues.

Single layer potential The operator S_ω takes the form

$$S_\omega\alpha(x) = \int_{-1}^{1}\frac{\ln|x-y|\,\alpha(y)}{\sqrt{1-y^2}}dy\,.$$

We have the following explicit formulas, stated in [5], but also at the core of many other related works, for example [7,13,14]. A proof can be found in [16]. There holds

$$S_\omega T_n = \sigma_n T_n \tag{33}$$

where

$$\sigma_n = \begin{cases}\dfrac{\ln(2)}{2} & \text{if } n = 0\\ \dfrac{1}{2n} & \text{otherwise.}\end{cases}.$$

In particular, Corollary 2, S_ω is in the class S^1. As a consequence, S_ω maps T^∞ to itself, so the image of a smooth function is a smooth function. We can also deduce the following characterization of $T^{-1/2}$ and $T^{1/2}$, also obtained independently in [13] Ou bien [12], vérifier et citer le thm.

Lemma 18. *We have $T^{-1/2} = \omega\tilde{H}^{-1/2}(-1,1)$ and for all $u \in \tilde{H}^{-1/2}(-1,1)$,*

$$\|u\|_{\tilde{H}^{-1/2}} \sim \|\omega u\|_{T^{-1/2}}\,.$$

Moreover, $T^{1/2} = H^{1/2}(-1,1)$ and

$$\|u\|_{T^{1/2}} = \|u\|_{H^{1/2}}$$

Proof. Since the logarithmic capacity of the segment is $\frac{1}{4}$, the (unweighted) single-layer operator S is positive and bounded from below on $\tilde{H}^{-1/2}(-1,1)$, (see [18] chap. 8). Therefore the norm on $\tilde{H}^{-1/2}(-1,1)$ is equivalent to

$$\|u\|_{\tilde{H}^{-1/2}} \sim \sqrt{\langle Su, u\rangle}.$$

On the other hand, the explicit expression (33) imply that if $\alpha \in T^{-1/2}$

$$\|\alpha\|_{T^{-1/2}} \sim \sqrt{\langle S_\omega \alpha, \alpha\rangle_{\frac{1}{\omega}}}.$$

It remains to notice that, since $\alpha = \omega u$, $\langle S_\omega \alpha, \alpha\rangle_{\frac{1}{\omega}} = \langle Su, u\rangle$. This proves the first result. For the second result, we know that, $(H^{1/2}(-1,1))' = \tilde{H}^{-1/2}(-1,1)$ (taking the dual with respect to the usual L^2 duality, [17] chap. 3), and therefore

$$\|u\|_{H^{\frac{1}{2}}} = \sup_{v\neq 0} \frac{\langle u, v\rangle}{\|v\|_{\tilde{H}^{-\frac{1}{2}}}}.$$

According to the previous result, for all $v \in \tilde{H}^{-\frac{1}{2}}$, the function $\alpha = \omega v$ is in $T^{-1/2}$, and $\|v\|_{\tilde{H}^{-1/2}} \sim \|\alpha\|_{T^{-1/2}}$, while $\langle u, v\rangle = \langle u, \alpha\rangle_\omega$. Thus

$$\|u\|_{H^{1/2}} \sim \sup_{\alpha\neq 0} \frac{\langle u, \alpha\rangle_\omega}{\|\alpha\|_{T^{-1/2}}}$$

The last quantity is the $T^{1/2}$ norm of u since $T^{1/2}$ is identified to the dual of $T^{-1/2}$ for $\langle\cdot,\cdot\rangle_\omega$.

□

The main consequence for our purpose is the possibility to derive an explicit inverse of S_ω as the square root of a local operator. Recall that

$$-(\omega\partial_x)^2 T_n = n^2 T_n$$

the operator $-(\omega\partial_x)^2$ is in S^{-2} and

$$-(\omega\partial_x)^2 S_\omega^2 = \frac{I_d}{4} + T_\infty.$$

This shows that $\sqrt{-(\omega\partial_x)^2}$ and S_ω can be thought as inverse operators (modulo smoothing operators) and that $\sqrt{-(\omega\partial_x)^2}$ can thus be used as an efficient preconditioner for S_ω.

Hypersingular operator For $k = 0$ and when $\Gamma = (-1,1)\times\{0\}$, the identity (32) becomes

$$\langle N_\omega \beta, \beta'\rangle_\omega = \langle S_\omega(\omega\partial_x\omega)\beta, (\omega\partial_x\omega)\beta'\rangle_{\frac{1}{\omega}}$$

Noticing that $(\omega\partial_x\omega)U_n = -(n+1)T_{n+1}$, we have for all $n \neq m$

$$\langle N_\omega U_n, U_m\rangle_\omega = 0\,,$$

so $N_\omega U_n = \nu_n U_n$ with

$$\nu_n \|U_n\|_\omega^2 = (n+1)^2 \sigma_{n+1} \|T_{n+1}\|_{\frac{1}{\omega}}^2\,,$$

that is, $\nu_n = \frac{(n+1)}{2}$. so N_ω is of order -1 in the scale U^s. In particular, N_ω maps smooth functions to smooth functions. As before, a characterization of U^s for $s = \pm\frac{1}{2}$ from the previous formula:

Lemma 19. *The following identities hold,*

$$U^{1/2} = \frac{1}{\omega}\tilde{H}^{1/2}(-1,1),$$

$$U^{-1/2} = H^{-1/2}(-1,1),$$

with

$$\|\omega u\|_{\tilde{H}^{1/2}} \sim \|u\|_{U^{1/2}}, \quad \|u\|_{U^{-1/2}} \sim \|u\|_{H^{-1/2}}.$$

Proof. It suffices to remark that

$$\|\omega u\|_{\tilde{H}^{1/2}} \sim \sqrt{\langle N\omega u, \omega u\rangle} = \sqrt{\langle N_\omega u, u\rangle_\omega} \sim \|u\|_{U^{1/2}}\ .$$

The second equality follows from the same calculations that were done in Lemma **??**, as well as the norm equivalence. $\square$

Here again, we can express the inverse of N_ω in the form of the square root of a local operator. Recall that

$$-(\partial_x\omega)^2 U_n = -(n+1)^2 U_n$$

the operator $-(\partial_x\omega)^2$ is of order -2 in the scale U^s and

$$N_\omega^2 = -\frac{1}{4}(\partial_x\omega)^2.$$

In what follows, we show that those simple identities can be generalized to non-zero wavenumber k and arbitrary smooth and non-intersecting open curve Γ.

2.3 Weighted single-layer operator on the flat segment for $k \neq 0$

In this section and the next, Γ is the flat segment $(-1,1)\times\{0\}$. The general case is treated in **??**. We first focus on the weighted single-layer operator problem with non-zero frequency, and establish the following result, announced in [5].

Theorem 4. *$S_{k,\omega}$ is of order 1 in the scale T^s, and*

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega}^2 = \frac{I_d}{4} + T_4.$$

Remark 3. *The previous result also implies that*

$$-(\omega\partial_x)^2 S_{k,\omega}^2 = \frac{I_d}{4} + R$$

where R is of order 2. This is also a compact perturbation of the identity. We have $R = k^2\omega^2 S_{k,\omega}^2 + T_4$. Thus, the term $k^2\omega^2 S_{k,\omega}^2$ is the leading first order correction accounting for the wavenumber. The inclusion of this term leads to a drastic reduction of the number of GMRES iterations in numerical applications, as demonstrated in [5].

The perturbation analysis is based on the following expansion for the Hankel:

$$H_0(z) = \frac{-1}{2\pi} \ln|z| J_0(z) + F_1(z^2)$$

where J_0 is the Bessel function of first kind and order 0 and where F_1 is analytic. Using the power series definition of J_0, this gives

$$\begin{aligned}\frac{i}{4} H_0(k\,|x-y|) = &\frac{-1}{2\pi} \ln|x-y| \\ &+ \frac{1}{2\pi}\frac{k^2}{4}(x-y)^2 \ln|x-y| \\ &+ (x-y)^4 \ln|x-y| F_2(x,y) + F_3(x,y)\end{aligned} \tag{34}$$

where F_2 and F_3 are C^∞. Let us study the operators O_n defined for $n \geq 1$ as

$$O_n : \alpha \mapsto -\frac{1}{2\pi} \int_{-1}^{1} (x-y)^{n-1} \ln|x-y| \frac{\alpha(y)}{\omega(y)}\,.$$

Lemma 20. *For every n, O_n is in the class S^n.*

Proof. This can be shown by a simple induction. O_1 is just S_ω, which is indeed in S^1. Let $n \geq 2$, and assume $O_{n-1} \in S^{n-1}$. We have

$$O_n = xO_{n-1} - O_{n-1}x.$$

As shown in Lemma 15, the multiplication by x defines an operator of S^0. By assumption, O_{n-1} is in S^{n-1}, thus Theorem 1 implies that $O_n \in S^n$, which concludes the proof. □

We define a new operator y defined for $n \geq 1$ by

$$yT_n = \frac{T_{n+1} - T_{n-1}}{2}\,.$$

and $yT_0 = 0$. It is easy to check that y is in S^0 and $y^2 = -\omega^2 + T_\infty$. Moreover, y commutes with the multiplication by x, and the adjoint of y (in the $L^2_{\frac{1}{\omega}}$ duality) is $-y$. Since, for $n \geq 0$, $(x+y)T_n = T_{n+1}$ and $(x-y)T_n = T_{n-1}$, we see that any operator in $A \in S^p$ can be expressed as

$$Au = \sum_{n=0}^{+\infty} a(x,y,n)\hat{u}_n T_n(x)$$

where for each n, $(x,y) \mapsto a(x,y,n)$ is a polynomial in x and y. A déplacer probablement. We show an intermediary result:

Lemma 21. *For all $n \geq 0$, there exists an operator $R_{n+3} \in S^{n+3}$ such that*

$$xS_\omega^n - S_\omega^n x = 2nyS_\omega^{n+1} - 2n(n+1)xS_\omega^{n+2} + R_{n+3}\,. \tag{35}$$

Proof. We must show that

$$R_{n+2} := xS_\omega^n - S_\omega^n x - 2nyS_\omega^{k+1} + 2n(n+1)xS_\omega^{n+2}$$

belongs to the class S^{n+3} for all $n \in \mathbb{N}$. For $n = 0$ this is obvious. Let us fix $n \geq 1$. We check the three requirements of **??**. Using $xT_k = \frac{T_{k+1}+T_{k-1}}{2}$ and $S_\omega T_k = \sigma_k T_k$, we have

$$R_{n+2}T_k = a(k,-1)T_{k-1} + a(k,1)T_{k+1}$$

with

$$\begin{aligned} a(k,1) &= \frac{\sigma_k^n - \sigma_{k+1}^n - 2n\sigma_k^{n+1} + 2n(n+1)\sigma_k^{n+2}}{2} \\ a(k,-1) &= \frac{\sigma_k^n - \sigma_{k-1}^n + 2n\sigma_k^{n+1} + 2n(n+1)\sigma_k^{n+2}}{2} \end{aligned}$$

The symbol a thus satisfies the requirements (i) and (iii). It remains to show the estimate (ii). We do this for $a(k,1)$, the other case being similar. Of course, it suffices to establish the estimate for $k \geq 1$. In this case, we have $\sigma_k = \frac{1}{2k}$, thus

$$a(k,1) = g(k+1) - g(k) - g'(k) - \frac{g''(k)}{2}$$

where

$$g(x) = -\frac{1}{2 \times (2x)^n}\,.$$

Applying Δ_k^α, on both sides and using the commutation of $\frac{d}{dx}$ and Δ_k, we obtain

$$a(k,1) = \Delta_k^\alpha g(k+1) - \Delta_k^\alpha g(k) - (\Delta_k^\alpha g)'(k) - \frac{(\Delta_k^\alpha g)''(k)}{2}\,.$$

This can be rewritten using Taylor's formula

$$a(k,1) = \int_k^{k+1} \frac{(k+1-\xi)^2}{2} \left(\Delta_k^\alpha g^{(3)}\right)(\xi)d\xi\,.$$

Using Lemma 11 and the explicit derivatives of g, for $\xi \geq 1$, there holds

$$\left|\left(\Delta_k^\alpha g^{(3)}\right)(\xi)\right| \leq \frac{C}{(1+\xi)^{k+3+\alpha}}$$

and thus, for $k \geq 1$,

$$|\Delta_k^\alpha a(k,1)| \leq \frac{C}{(1+n)^{n+\alpha+3}}\,,$$

as needed. $\square$

With the same method, we obtain:

Lemma 22. *For all $n \in \mathbb{N}$,*

$$yS_\omega^n - S_\omega^n y = 2nxS_\omega^{n+1} - 2n(n+1)yS_\omega^{n+2} + R_n + 2\,.$$

with $R_{n+2} \in S^{n+2}$.

Lemma 23. *For all $n \geq 1$, the operator O_n satisfies*

$$O_n = 2^{n-1}(n-1)!y^{n-1}S_\omega^n + 2^n(n-1)n!xy^{n-1}S_\omega^{n+1} + R_{n+2} \tag{36}$$

where $R_{n+2} \in S^{n+2}$.

Proof. We show this by induction. For $n = 1$, the formula is obvious, with $R_{n+2} = 0$. Assume that the formula is true for $n \geq 1$. Then by definition,

$$O_{n+1} = xO_n - O_n x$$

and using the commutation of x and y:

$$\begin{aligned} O_{n+1} =& 2^{n-1}(n-1)! y^{n-1} \left(x S_\omega^n - S_\omega^n x \right) \\ & + 2^n (n-1) n! x y^{n-1} \left(x S_\omega^{n+1} - S_\omega^{n+1} x \right) \\ & + \left(x R_{n+2} - R_{n+2} x \right) . \end{aligned}$$

The operator on the last line is in S^{n+3} by Theorem 1. By Lemma 21, there exists an operator $R_{n+3} \in S^{n+3}$ such that

$$\begin{aligned} O_{n+1} = \quad & 2^{n-1}(n-1)! y^{n-1} \left(2ny S_\omega^{n+1} - 2n(n+1) x S_\omega^{n+2} \right) \\ & + 2^n (n-1) n! x y^{n-1} \left(2(n+1) y S_\omega^{n+2} \right) \\ & + R_{n+3} . \end{aligned}$$

And we obtain the expected formula for O_{n+1}. □

Using the notation introduced in Definition 5, we have the following result:

Lemma 24. *The operator $S_{k,\omega}$ admits the following expansion*

$$S_{k,\omega} = S_\omega - \frac{k^2}{4} O_3 + T_5.$$

Proof. From equation (34), it suffices to show that the operator

$$R_5 : \alpha \mapsto \int_{-1}^{1} (x-y)^4 \ln |x-y| F_2(x,y) \frac{\alpha(y)}{\omega(y)}$$

is of order 5. Since O_5 is of order 5, this is true in view of Lemma 17. □

In particular, the operator $S_{k,\omega}$ is well defined on $T^{-\infty}$, and is of order 1.

Lemma 25. *There holds*

$$S_\omega (\omega \partial_x)^2 O_3 + O_3 (\omega \partial_x)^2 S_\omega = 4 S_\omega \omega^2 S_\omega + T_4.$$

Proof. We have $S_\omega \in S^1$, $O_3 \in S^3$ and $(\omega \partial_x)^2 \in S^{-2}$. □

Theorem 5. *There holds*

$$\left[-(\omega \partial_x)^2 - k^2 \omega^2 \right] S_{k,\omega}^2 = \frac{I_d}{4} + T_4.$$

Proof. Using the expansion of Lemma 24, we can write

$$\begin{aligned} -S_{k,\omega} (\omega \partial_x)^2 S_{k,\omega} \quad = \quad & -S_\omega (\omega \partial_x)^2 S_\omega \\ & + \frac{k^2}{4} \left(S_\omega (\omega \partial_x)^2 O_3 + O_3 (\omega \partial_x)^2 S_\omega \right) + T_4 \end{aligned}$$

By **??**, the first term is $\frac{Id}{4} + T_\infty$ and by Lemma 25 the second term is $k^2\omega^2 + T_4$ Finally, using Lemma 24, on can check that

$$S_\omega \omega^2 S_\omega = S_{k,\omega}\omega^2 S_{k,\omega} + T_4$$

We have thus proved

$$-S_{k,\omega}(\omega\partial_x)^2 S_{k,\omega} = \frac{I_d}{4} + k^2 S_{k,\omega}\omega^2 S_{k,\omega} + T_4.$$

If we substract the term $k^2 S_{k,\omega}\omega^2 S_{k,\omega}$ of each side, and use the first commutation proved in **??**, we finally get

$$\left[-(\omega\partial_x)^2 - k^2\omega^2\right] S_{k,\omega}^2 = \frac{I_d}{4} + T_4,$$

and the result is proved. □

Recall that $\lambda_{n,k}^2$ are the eigenvalues of $-(\omega\partial_x) - k^2\omega^2$. Let $s_{n,k}$ the eigenvalues of $S_{k,\omega}$ on the basis of Mathieu cosines, that is

$$S_{k,\omega} T_n^k = s_{n,k} T_n^k.$$

The previous theorem has the following consequence:

Corollary 4. *One has*

$$s_{n,k}\lambda_{n,k} = \frac{1}{4} + r_{n,k}$$

where $r_{n,k}$ satisfies

$$\sum_{n=0}^{+\infty}(1+n^2)^4 \left|r_{n,k}\right|^2 < +\infty$$

The results of this section prompt us to use $\sqrt{-(\omega\partial_x)^2 - k^2\omega^2}$ as a preconditioner for $S_{k,\omega}$. Problème d'inversibilité possible pour certaines valeurs de k. Je n'arrive pas à l'écarter.

2.4 Neumann problem

Similarly, if we define $N_{k,\omega} := N_k\omega$, we have

Theorem 6.

$$N_{k,\omega}^2 = \left[-(\partial_x\omega)^2 - k^2\omega^2\right] + U_2.$$

This result suggests $\left[-(\partial_x\omega)^2 - k^2\omega^2\right]^{-1/2}$ as a candidate preconditioner for $N_{k,\omega}$. Problème d'inversibilité idem. The proof is reported to section 5.

2.5 Non-flat arc

In the more general case of a C^∞ non-intersecting open curve Γ and non-zero frequency k, the results of the previous sections can be extended using again compact perturbations arguments. Essentially, in the decomposition Equation 34, x and y must be replaced by $r(x)$ and $r(y)$, where the function r is a smooth, constant-speed parametrisation of Γ defined on $[-1,1]$ and satisfying $|r(x) - r(y)|^2 = \frac{|\Gamma|^2}{4}|x-y|^2 + |x-y|^4 G(x,y)$ where $|\Gamma|$ is the length of Γ and G is a C^∞ function on $[-1,1]$. Letting $\omega_\Gamma(x) = |\Gamma|\,\omega(x)$, ∂_τ the tangential derivative on Γ and $S_{k,\omega_\Gamma} := S_k\frac{1}{\omega_\Gamma}$, we have:

Theorem 7.

$$S_{k,\omega_\Gamma}\left(-(\omega_\Gamma\partial_\tau)^2 - k^2\omega_\Gamma^2\right)S_{k,\omega_\Gamma} = \frac{I_d}{4} + T_4$$

Theorem 8.

$$N_{k,\omega_\Gamma}^2 = -(\partial_\tau\omega_\Gamma)^2 - k^2\omega_\Gamma^2 + U_2.$$

where R_2 is of order 2 in the scale U^s.

The proofs of those facts are omitted.

3 Galerkin analysis

In this section, we describe and analyze the Galerkin scheme used to solve the integral equations in this work. To keep matters simple, we focus on equations (**??**) and (**??**) on the flat strip. The results extend to the general case using standard arguments in the theory of boundary element methods. Standard discretization on a uniform mesh with piecewise polynomial trial functions leads to very poor rates of convergences (see for example [23, Chap. 4,] and subsequent remark). Several methods have been developed to remedy this problem. One can for example enrich the trial space with special singular functions, refine the mesh near the segment tips, (h-BEM) or increase the polynomial order in the trial space. The combination of the last two methods, known as h-p BEM, can achieve an exponential rate of convergence with respect to the dimension of the trial space, see [20] and references therein. Spectral methods, involving trigonometric polynomials have also been analyzed for example [7], and some results exist for piecewise linear functions in the colocation setting [9].

Here, we describe a simple Galerkin scheme using piecewise affine functions on an adapted mesh, that is both stable and easy to implement. Our analysis shows that the usual rates of convergence one would obtain with smooth closed boundary with smooth solution, are recovered thanks to this new analytic setting. The orders of convergence are stated in Theorem 9 and Theorem 10.

In what follows, we introduce a discretization of the segment $[-1, 1]$ as $-1 = x_0 < x_1 < \cdots < x_N = 1$, and let $\theta_i := \arccos(x_i)$. We define the parameter h of the discretization as

$$h := \min_{i=0\cdots N-1} |\theta_{i+1} - \theta_i|\,.$$

In practice, one should use a mesh for which $|\theta_i - \theta_{i+1}|$ is constant. This turns out to be analog to a graded mesh with the grading parameter set to 2, that is, near the edge, the width of the $i-th$ interval is approximately $(ih)^2$. In comparison, in the h-BEM method with $p = 1$ polynomial order, this would only lead to a convergence rate in $O(h)$ (cf. [20, Theorem 1.3]).

3.0.1 Dirichlet problem

In this section, we present the method to compute a numerical approximation of the solution λ of (**??**). To achieve it, we use a variational formulation of (**??**) to compute an approximation α_h of α, and set $\lambda_h = \frac{\alpha_h}{\omega}$. Let V_h the Galerkin space of (discontinuous) piecewise affine functions with breakpoints at x_i. Let α_h the unique solution in V_h to

$$\langle S_\omega\alpha_h, \alpha_h'\rangle_{\frac{1}{\omega}} = -\langle u_D, \alpha_h'\rangle_{\frac{1}{\omega}}\,, \quad \forall\alpha_h' \in V_h.$$

We shall prove the following result:

Theorem 9. *If the data u_D is in T^{s+1} for some $-1/2 \leq s \leq 2$, then there holds:*

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \leq Ch^{s+1/2} \|u_D\|_{T^{s+1}}.$$

In particular, when u_D is smooth, it belongs to T^∞ so the rate of convergence is $h^{5/2}$. We start by proving an equivalent of Céa's lemma:

Lemma 26. *There exists a constant C such that*

$$\|\alpha - \alpha_h\|_{T^{-1/2}} \leq C \inf_{\alpha'_h \in V_h} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

Proof. In view of the properties of S_ω stated in **??**, we have the equivalent norm

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha_h \rangle.$$

Since $\langle S_\omega \alpha, \alpha'_h \rangle = \langle S_\omega \alpha_h, \alpha'_h \rangle = -\langle u_D, \alpha'_h \rangle$ for all $\alpha'_h \in V_h$, we deduce

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq \langle S_\omega(\alpha - \alpha_h), \alpha - \alpha'_h \rangle, \quad \forall \alpha'_h \in V_N.$$

By duality

$$\|\alpha - \alpha_h\|^2_{T^{-1/2}} \leq C \|S_\omega(\alpha - \alpha_h)\|_{T^{1/2}} \|\alpha - \alpha'_h\|_{T^{-1/2}}$$

which gives the desired result after using the continuity of S_ω from $T^{-1/2}$ to $T^{1/2}$. □

From this we can derive the rate of convergence for α_h to the true solution α. We use the $L^2_{\frac{1}{\omega}}$ orthonormal projection $\mathbb{P}_h$ on V_h, which satisfies the following properties:

Lemma 27. *For any function u,*

$$\|(\mathrm{I} - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq C \|u\|_{L^2_{\frac{1}{\omega}}},$$

$$\|(\mathrm{I} - \mathbb{P}_h)u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T_2}.$$

The proof requires the following well-known result:

Lemma 28. *Let $\tilde{u}$ in the Sobolev space $H^2(\theta_1, \theta_2)$,such that $\tilde{u}(\theta_1) = \tilde{u}(\theta_2) = 0$. Then there exists a constant C independent of θ_1 and θ_2 such that*

$$\int_{\theta_1}^{\theta_2} \tilde{u}(\theta)^2 \leq C(\theta_1 - \theta_2)^4 \int_{\theta_1}^{\theta_2} \tilde{u}''(\theta)^2 d\theta$$

Proof. The first inequality is obvious since $\mathbb{P}_h$ is an orthonormal projection. For the second inequality, we first write, since the orthogonal projection minimizes the $L^2_{\frac{1}{\omega}}$ norm,

$$\|I - \mathbb{P}_h u\|_{L^2_{\frac{1}{\omega}}} \leq \|I - I_h u\|_{L^2_{\frac{1}{\omega}}}, \tag{37}$$

where $I_h u$ is the piecewise affine (continuous) function that matches the values of u at the breakpoints x_i. By Lemma 6, on each interval $[x_i, x_{i+1}]$, the function $\tilde{u}(\theta) := u(\cos(\theta))$ is in the Sobolev space $H^2(\theta_i, \theta_{i+1})$ so we can apply Lemma 28:

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} = \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I_h u})^2 \leq (\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (\tilde{u} - \tilde{I_h u})''^2.$$

This gives

$$\int_{x_i}^{x_{i+1}} \frac{(u - I_h u)^2}{\omega} \leq 2h^4 \left(\int_{\theta_i}^{\theta_{i+1}} \tilde{u}''^2 + \int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 \right). \tag{38}$$

Before continuing, we need to establish the following result

Lemma 29. *There holds*

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 \leq C \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega}$$

Proof. The expression of $I_h u$ is given by

$$\tilde{I_h u}(\theta) = u(x_i) + \frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} (\cos(\theta) - \cos(\theta_i)),$$

thus

$$\int_{\theta_i}^{\theta_{i+1}} \tilde{I_h u}''^2 = \left(\frac{u(x_i) - u(x_{i+1})}{\cos(\theta_{i+1}) - \cos(\theta_i)} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 d\theta.$$

We can rewrite

$$(u(x_{i+1}) - u(x_i))^2 = \left(\int_{x_i}^{x_{i+1}} u'(t) dt \right)^2,$$

and apply Cauchy-Schwarz's inequality and the variable change $t = \cos(\theta)$ to find

$$(\tilde{u}(\theta_{i+1}) - \tilde{u}(\theta_i))^2 \leq \int_{x_i}^{x_{i+1}} \frac{u'^2}{\omega} \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2 d\theta.$$

To conclude, it remains to notice that the quantity

$$\frac{\int_{\theta_i}^{\theta_{i+1}} \cos(\theta)^2 \int_{\theta_i}^{\theta_{i+1}} \sin(\theta)^2}{(\cos(\theta_{i+1}) - \cos(\theta_i))^2}$$

is bounded uniformly in (θ_i, θ_{i+1}). Indeed, since cos is injective on $[0, \pi]$, the only problematic case is the limit when $\theta_i = \theta_{i+1}$. It is easy to check that this limit is $\cos(\theta_i)^2$, which is indeed uniformly bounded in θ_i. □

We can now conclude the proof of Lemma 27. Summing all inequalities (38) for $i = 0, \cdots N + 1$, we get

$$\|u - I_h u\|^2_{L^2_{\frac{1}{\omega}}} \leq Ch^4 \left(\|u\|^2_{T^2} + \|u'\|^2_{T_0} \right).$$

By Corollary 1, the operator ∂_x is continuous from T^2 to T^0 which gives

$$\|u - I_h u\|_{L^2_{\frac{1}{\omega}}} \leq Ch^2 \|u\|_{T^2}.$$

Thanks to (37), this concludes the proof. □

We obtain the following corollary by interpolation:

Corollary 5. *The operator* $\mathrm{I} - \mathbb{P}_N$ *is continuous from* $L^2_{\frac{1}{\omega}}$ *to* T^s *for* $0 \leq s \leq 2$ *with*

$$\|(\mathrm{I} - \mathbb{P}_N)u\|_{L^2_{\frac{1}{\omega}}} \leq ch^s \|u\|_{T^s}.$$

We can now prove Theorem 9:

Proof. First, using Lemma 18, one has

$$\|\lambda - \lambda_h\|_{\tilde{H}^{-1/2}} \sim \|\alpha - \alpha_h\|_{T^{-1/2}}.$$

Moreover, if u_D is in T^{s+1}, then $\alpha = S_\omega^{-1} u_D$ is in T^s and $\|\alpha\|_{T^s} \sim \|u_D\|_{T^{s+1}}$. By the analog of Céa's lemma, Lemma 26, it suffices to show that

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq Ch^{s+1/2} \|\alpha\|_{T^s}.$$

For this, we write

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_h \alpha, \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}$$

and since $\mathbb{P}_h$ is an orthonormal projection on $L^2_{\frac{1}{\omega}}$,

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} = \inf_{\eta \in T^{1/2}, \eta \neq 0} \frac{(\alpha - \mathbb{P}_N \alpha, \eta - \mathbb{P}_h \eta)_{\frac{1}{\omega}}}{\|\eta\|_{T^{1/2}}}.$$

Using Cauchy-Schwarz's inequality and Corollary 5 ($s = \frac{1}{2}$),

$$\|\alpha - \mathbb{P}_h \alpha\|_{T^{-1/2}} \leq \frac{h^s \|\alpha\|_{T^s} h^{1/2} \|\eta\|_{T^{1/2}}}{\|\eta\|_{T^{1/2}}} = h^{s+\frac{1}{2}} \|\alpha\|_{T^s}.$$

□

3.0.2 Neumann problem

We now turn to the numerical resolution of (**??**). We use a variational form for equation (**??**), and solve it using a Galerkin method with continuous piecewise affine functions. We introduce W_h the space of continuous piecewise affine functions with breakpoints at x_i, and we denote by β_h the unique solution in W_h to the variational equation:

$$\langle N_\omega \beta_h, \beta'_h \rangle_\omega = \langle u_N, \beta'_h \rangle_\omega, \quad \forall \beta'_h \in W_h. \tag{39}$$

Then, $\mu_h = \omega \beta_h$ is the proposed approximation for μ. We shall prove the following:

Theorem 10. *If* $u_N \in U^{s-1}$, *for some* $\frac{1}{2} \leq s \leq 2$, *there holds*

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq Ch^{s-\frac{1}{2}} \|u_N\|_{U^{s-1}}.$$

Like before, we start with an analog of Céa's lemma:

Lemma 30. *There exists a constant C such that*

$$\|\beta - \beta_h\|_{U^{1/2}} \leq C \inf_{\beta'_h \in W_h} \|\beta - \beta'_h\|_{U^{1/2}}$$

In a similar fashion as in the previous section, it is possible to show the following continuity properties of the interpolation operator I_h:

Lemma 31. *There holds*

$$\|u - I_h u\|_{L^2_\omega} \leq Ch^2 \|u\|_{U^2}$$

and

$$\|u - I_h u\|_{U^1} \leq Ch \|u\|_{U^2}$$

Proof. We only show the first estimation, the method of proof for the second being similar. Using again Lemma 28 on each segment $[x_i, x_{i+1}]$, one can write

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \omega(u - I_h u)^2 &\leq C(\theta_{i+1} - \theta_i)^4 \int_{\theta_i}^{\theta_{i+1}} (Vu - VI_h u)''^2 \\ &\leq Ch^4 \left(2 \int_{\theta_i}^{\theta_{i+1}} Vu''^2 + 2 \int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \right) \end{aligned}$$

where we recall that for any function u, Vu is defined as

$$Vu(\theta) = \sin(\theta) u(\cos(\theta)).$$

Before continuing, we need to establish the following estimate:

Lemma 32.

$$\int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \left(\|u\|_{U_2}^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 + \int_{x_i}^{x_{i+1}} \omega(\partial_x u)^2 \right)$$

Proof. Using the expression of I_h, one can write

$$\begin{aligned} \int_{\theta_i}^{\theta_{i+1}} (VI_h u)''^2 \leq C \Bigg(&|u(x_i)|^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2 \\ &+ \left(\frac{u(x_{i+1}) - u(x_i)}{\cos\theta_{i+1} - \cos\theta_i} \right)^2 \int_{\theta_i}^{\theta_{i+1}} \sin^2(1 + \cos^2) \Bigg) \end{aligned} \tag{40}$$

We can estimate the first term, thanks to Lemma 4:

$$|u(x_i)| \leq C \|u\|_{U^2},$$

while for the second term, the numerator of is estimated as follows:

$$\begin{aligned}(u(x_{i+1})-u(x_i))^2 &= \left(\int_{x_i}^{x_{i+1}}\partial_x u\right)^2\\ &\leq \int_{x_i}^{x_{i+1}}\omega(\partial_x u)^2\int_{x_i}^{x_{i+1}}\frac{1}{\omega}\\ &= |\theta_{i+1}-\theta_i|\int_{x_i}^{x_{i+1}}\omega(\partial_x u)^2.\end{aligned}$$

to conclude, it remains to observe that the quantity

$$\frac{|(\theta_{i+1}-\theta_i)|\int_{\theta_i}^{\theta_{i+1}}\sin^2(1+\cos^2)}{(\cos(\theta_i)-\cos(\theta_{i+1}))^2}$$

is bounded by a constant independent of θ_i and θ_{i+1}. Indeed, in the limit $\theta_{i+1}\to\theta_i$, the fraction has the value $1+\cos^2(\theta_i)$ □

We now plug the estimate Lemma 32 in (40), and sum over i:

$$\|u-I_h u\|_{L^2_\omega}^2 \leq Ch^4(\|u\|_{U^2}^2+\|u'\|_{L^2_\omega}^2).$$

This implies the claim once we use the continuity of ∂_x from U^2 to U^0, cf. Corollary 1. □

We can now prove Theorem 10

Proof. Let us denote by Π_h the Galerkin projection operator defined by $\beta\mapsto\beta_h$. Since it is an orthogonal projection on W_h with respect to the scalar product $(\beta,\beta') := \langle N_\omega\beta,\beta'\rangle$, it is continuous from $U^{1/2}$ to itself, so we have for any u in $U^{1/2}$.

$$\|(I-\Pi_h)u\|_{U^{1/2}} \leq C\|u\|_{U^{1/2}}.$$

We are now going to show the estimate

$$\|(I-\Pi_h)u\|_{U^{1/2}} \leq Ch^{3/2}\|u\|_{U^2}.$$

By the analog of Céa's lemma Lemma 30, one has

$$\|(I-\Pi_h)u\|_{U^{1/2}} \leq \|(I-I_h)u\|_{U^{1/2}}.$$

By interpolation, this norm satisfies

$$\|(I-I_h)u\|_{U^{1/2}} \leq C\sqrt{\|(I-I_h)u\|_{U^0}}\sqrt{\|(I-I_h)u\|_{U^1}},$$

which yields, applying Lemma 31,

$$\|(I-I_h)u\|_{U^{1/2}} \leq Ch^{3/2}\|u\|_{U^2}.$$

By interpolation, for all $s\in[1/2,2]$, we get

$$\|(I-\Pi_h)u\|_{U^{1/2}} \leq Ch^{s-1/2}\|u\|_{U^s}.$$

In view of Lemma 19, we have $\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \sim \|(I - \Pi_h)\beta\|_{U^{1/2}}$. In addition, since N_ω is a continuous bijection from U^{s+1} to U^s for all s, there holds

$$\|\beta\|_{U^s} = \left\|N_\omega^{-1} u_N\right\|_{U^s} = \|u_N\|_{U^{s-1}} .$$

Consequently,

$$\|\mu - \mu_h\|_{\tilde{H}^{1/2}} \leq C \|(I - \Pi_h)\beta\|_{U^{1/2}} \leq Ch^{s-1/2} \|\beta\|_{U^s} \leq Ch^{s-1/2} \|u_N\|_{U^{s-1}} .$$

□

4 Conclusion

5 Proof of Theorem 6

From equation (29), we can deduce the following formula for the weighted operator:

$$N_{k,\omega} = -\partial_x S_{k,\omega} \omega \partial_x \omega - k^2 S_{k,\omega} \omega^2 \tag{41}$$

If we define $L_n := -\partial_x O_{n+2} \omega \partial_x \omega$, then using the mapping properties of ∂_x and $\omega \partial_x \omega$ given by Definition 3, and since, by Lemma 20, O_{n+2} is of order $n + 2$ in the scale T^s, we deduce that L_n is of order n in the scale U^s. The expansion obtained for the weighted single-layer operator in Lemma 24 yields the following expansion for $N_{k,\omega}$.

Lemma 33.

$$N_{k,\omega} = N_\omega + k^2 \left(-\frac{L_1}{4} - S_\omega \omega^2 \right) + U_3$$

As a consequence, $N_{k,\omega}$ is an operator of order -1 in the scale U^s. Using equation (41), we have the following expression:

$$N_{k,\omega}^2 = N_\omega^2 - k^2 \left(\frac{L_1 N_\omega + N_\omega L_1}{4} + N_\omega S_\omega \omega^2 + S_\omega \omega^2 N_\omega \right) + U_2.$$

We have proved in By definition, $L_1 = -\partial_x O_3 \omega \partial_x \omega$, while $N_\omega = -\partial_x S_\omega \omega \partial_x \omega$, thus

$$L_1 N_\omega = \partial_x (O_3 (\omega \partial_x)^2 S_\omega) \omega \partial_x \omega.$$

Moreover,

$$N_\omega L_1 = \partial_x (S_\omega (\omega \partial_x)^2 O_3) \omega \partial_x \omega.$$

Adding these two inequalities and using Lemma 25, we get

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = \partial_x (S_\omega \omega^2 S_\omega) \omega \partial_x \omega + U_2.$$

Here again, we use the formula $\partial_x S_\omega \omega^2 = S_\omega \omega \partial_x \omega$, which yields

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = S_\omega \omega \partial_x \omega \partial_x S_\omega \omega^2 = \left(-\frac{I_d}{4} + T_\infty \right) \omega^2.$$

Since ω^2 is continuous from U^s to T^s by **??** and using the injections $T^s \subset U^s$, any operator of the form $R\omega^2$ is smoothing in the scale U^s as soon as R is smoothing in the scale T^s. Therefore,

$$\frac{L_1 N_\omega + N_\omega L_1}{4} = -\frac{\omega^2}{4} + U_\infty.$$

Moreover, we have

$$\begin{aligned} S_\omega \omega^2 N_\omega &= -S_\omega \omega^2 \partial_x S_\omega \omega \partial_x \omega \\ &= -S_\omega \omega^2 \partial_x^2 S_\omega \omega^2 \end{aligned}$$

using again **??**. Since $\omega^2 \partial_x^2 = (\omega \partial_x)^2 + x\partial_x$, we get

$$S_\omega \omega^2 N_\omega = \frac{\omega^2}{4} - S_\omega x \partial_x S_\omega \omega^2 + U_\infty$$

Futhermore,

$$N_\omega S_\omega \omega^2 = -\partial_x S_\omega \omega \partial_x \omega S_\omega \omega^2.$$

We use $\omega \partial_x \omega = \omega^2 \partial_x - x$:

$$\begin{aligned} N_\omega S_\omega \omega^2 &= -\partial_x S_\omega \omega^2 \partial_x S_\omega \omega^2 + \partial_x S_\omega x S_\omega \omega^2 \\ &= \frac{\omega^2}{4} + \partial_x S_\omega x S_\omega \omega^2 \end{aligned}$$

Thus,

$$S_\omega \omega^2 N_\omega + N_\omega S_\omega \omega^2 = \frac{\omega^2}{2} + \left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2\right) + U_\infty.$$

We are done if we prove that the operator in parenthesis is of order 2 in the scale U^s. For this, we may compute the action of each one of them on U_n. Using the various identities at our disposal, we obtain on the one hand for $n \geq 2$

$$\partial_x S_\omega x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8(n+2)} + \frac{U_n + U_{n-2}}{8n(n+2)}.$$

and on the other hand for $n > 0$

$$S_\omega x \partial_x S_\omega \omega^2 U_n = -\frac{T_{n+2}}{8(n+2)} - \frac{T_n}{8n}.$$

After substracting, this gives the rather surprising identity identity for $n \geq 2$

$$\left(\partial_x S_\omega x S_\omega \omega^2 - S_\omega x \partial_x S_\omega \omega^2\right) U_n = \frac{U_n}{4n(n+2)}$$

which of course proves our claim.

6 Suggestion de découpage

J'y ai un tout petit peu réfléchi :

- Les analyses pseudo-diffs des espaces T^s, bien qu'intéressantes, sont trop longues et ne se justifient pas vraiment dans le simple but de faire une méthode numérique.

- La méthode de Galerkine est bien analysée et nouvelle (à ma connaissance) mais n'est pas vraiment essentielle pour le message.

Je pense qu'on pourrait envisager 3 articles. Un très concis sur la méthode numérique en elle-même. Utiliser le minimum d'info pour k=0, donner les inverses exacts, prouver la commutation des opérateurs pour k non nul, puis balancer les préconditionneurs, et mettre les figures.

Un article un peu à part sur la méthode de Galerkine, et tous les aspects numériques (bcp moins d'impact)

Un article (peut-être juste sur arxiv ?) sur les espaces T^s et U^s, qui donne toutes les justifications théoriques. (une sorte de version étendue de cet article.)

References

[1] François Alouges and Matthieu Aussal. The sparse cardinal sine decomposition and its application for fast numerical convolution. *Numerical Algorithms*, 70(2):427–448, 2015.

[2] Xavier Antoine and Marion Darbas. Generalized combined field integral equations for the iterative solution of the three-dimensional helmholtz equation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 41(1):147–167, 2007.

[3] Kendall E Atkinson and Ian H Sloan. The numerical solution of first-kind logarithmic-kernel integral equations on smooth open arcs. *mathematics of computation*, 56(193):119–139, 1991.

[4] Martin Averseng. Fast discrete convolution in r 2. *arXiv preprint :*.

[5] Martin Averseng. New preconditioners for the first kind integral equations on open curves. *arXiv preprint :*.

[6] S. Börm, L. Grasedyck, and W. Hackbusch. Introduction to hierarchical matrices with applications. *Engineering analysis with boundary elements*, 27(5):405–422, 2003.

[7] Oscar P Bruno and Stéphane K Lintner. Second-kind integral solvers for te and tm problems of diffraction by open arcs. *Radio Science*, 47(6), 2012.

[8] R. Coifman, V. Rokhlin, and S. Wandzura. The fast multipole method for the wave equation: A pedestrian prescription. *IEEE Antennas and Propagation Magazine*, 35(3):7–12, 1993.

[9] Martin Costabel, Vince J Ervin, and Ernst P Stephan. On the convergence of collocation methods for symm's integral equation on open curves. *Mathematics of computation*, 51(183):167–179, 1988.

[10] Marcus J Grote and Thomas Huckle. Parallel preconditioning with sparse approximate inverses. *SIAM Journal on Scientific Computing*, 18(3):838–853, 1997.

[11] Nicholas Hale, Nicholas J Higham, and Lloyd N Trefethen. Computing aˆα,\log(a), and related matrix functions by contour integrals. *SIAM Journal on Numerical Analysis*, 46(5):2505–2523, 2008.

[12] C Jerez-Hanckes and J-C Nédélec. Boundary hybrid galerkin method for elliptic and wave propagation problems in r3 over planar structures. *Integral Methods in Science and Engineering, Volume 2*, pages 203–212, 2010.

[13] Carlos Jerez-Hanckes and Jean-Claude Nédélec. Explicit variational forms for the inverses of integral logarithmic operators over an interval. *SIAM Journal on Mathematical Analysis*, 44(4):2666–2694, 2012.

[14] Shidong Jiang and Vladimir Rokhlin. Second kind integral equations for the classical potential theory on open surfaces ii. *Journal of Computational Physics*, 195(1):1–16, 2004.

[15] David P Levadoux. Proposition de préconditionneurs pseudo-différentiels pour l'équation cfie de l'électromagnétisme. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(1):147–155, 2005.

[16] John C Mason and David C Handscomb. *Chebyshev polynomials*. CRC Press, 2002.

[17] W McLean. A spectral galerkin method for a boundary integral equation. *Mathematics of computation*, 47(176):597–607, 1986.

[18] William Charles Hector McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge university press, 2000.

[19] Lars Mönch. On the numerical solution of the direct scattering problem for an open sound-hard arc. *Journal of computational and applied mathematics*, 71(2):343–356, 1996.

[20] FV Postell and Ernst P Stephan. On the h-, p-and hp versions of the boundary element method—numerical results. *Computer Methods in Applied Mechanics and Engineering*, 83(1):69–89, 1990.

[21] V. Rokhlin. Diagonal forms of translation operators for the helmholtz equation in three dimensions. *Applied and Computational Harmonic Analysis*, 1(1):82–93, 1993.

[22] Youcef Saad and Martin H Schultz. Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM Journal on scientific and statistical computing*, 7(3):856–869, 1986.

[23] Stefan A Sauter and Christoph Schwab. Boundary element methods. *Boundary Element Methods*, pages 183–287, 2011.

[24] O Steinbach and WL Wendland. Efficient preconditioners for boundary element methods and their use in domain decomposition methods. In *Domain Decomposition Methods in Sciences and Engineering: 8th International Conference*, pages 3–18, 1995.

[25] Olaf Steinbach and Wolfgang L Wendland. The construction of some efficient preconditioners in the boundary element method. *Advances in Computational Mathematics*, 9(1-2):191–216, 1998.

[26] Ernst P Stephan and Wolfgang L Wendland. An augmented galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. *Applicable Analysis*, 18(3):183–219, 1984.

[27] Ernst P Stephan and Wolfgang L Wendland. A hypersingular boundary integral method for two-dimensional screen and crack problems. *Archive for Rational Mechanics and Analysis*, 112(4):363–390, 1990.

[28] V Thrunen and Gennadi Vainikko. On symbol analysis of periodic pseudodifferential operators. *ZEITSCHRIFT FUR ANALYSIS UND IHRE ANWENDUNGEN*, 17:9–22, 1998.

[29] Yeli Yan, Ian H Sloan, et al. *On integral equations of the first kind with logarithmic kernels.* University of NSW, 1988.

[30] Yi Yan. Cosine change of variable for symm's integral equation on open arcs. *IMA Journal of Numerical Analysis*, 10(4):521–535, 1990.