The weighted integral operators involve the computation of singular integrals of the type

$$I = \int_{[A,B]} \frac{\ln|X-Y|}{\omega(Y)} dY$$

where $[A, B]$ is an element of the mesh (in 2D), $|X|$ represents the Euclidean norm of X and $\omega(Y) = d(A, Y)^\alpha$ for some α. Let $X = d\vec{n} + x\vec{u}$ where $\vec{n}$ is a unit normal vector to $[A, B]$ and $u = \frac{\overrightarrow{AB}}{|B-A|}$. Let $g(x) = \ln\sqrt{d^2 + x^2}$. Then we have

$$I = \int_0^{|B-A|} \frac{g(x-y)}{y^\alpha} dy\,.$$

It is easier to compute

$$J = \int_0^{|B-A|} \frac{g\left(\frac{k(x)-k(y)}{k'(x)}\right)}{y^\alpha} dy$$

where k is chosen so that $k'(y) = \frac{1}{y^\alpha}$, that is $k(y) = \dfrac{y^{1-\alpha}}{1-\alpha}$. Then we have simply have simply

$$J = \left[k'(x)G\left(k'(x)[k(x)-k(y)]\right)\right]_0^{|B-A|}$$

where $G(x) = -x + x\ln\sqrt{d^2+x^2} + d\arctan\frac{x}{d}$ is a primitive of g. When $d = 0$, one should put instead $G(x) = -x + x\ln|x|$ which is the limit of the previous expression when $d \to 0$. The first integral I can be seen as an approximation of J replacing the term $k(x) - k(y)$ by its first order Taylor expansion. In fact the remainder $J - I$ is of the form

$$J - I = \int_0^{|B-A|} \frac{f(y)}{y^\alpha} dy$$

where $f(y) = \ln|\frac{k(x)-k(y)}{k'(x)(x-y)}|$. When $x \neq 0$, this is a smooth function of y. When $x = 0$, f is not well defined, but the initial integral has the form

$$I = \int_0^{|B-A|} \ln\frac{\sqrt{d^2+y^2}}{y^\alpha} dy\,.$$

When $d \neq 0$, this is already of the form $I = \int_0^{|B-A|} \frac{f(y)}{y^\alpha}$ where f is a smooth function. When $d = 0$, then letting $u = y^{1-\alpha}$, we have explicitly

$$I = \int_0^{|B-A|^{1-\alpha}} \ln|u|du = [G(u)]_0^{|B-A|^{1-\alpha}}$$

with $G(u) = u\ln|u| - u$.