1 Gaussian quadrature adapted to a weight

We look to design a quadrature formula adapted to the computation of integrals of the form

$$I = \int_{[A,B]} \frac{f(Y)}{\omega(Y)} dY$$

where f is a smooth function and ω is a singular weight of the form $\omega(Y) = d(A,Y)^\alpha$. For this we use a Gaussian quadrature adapted to the weight. The problem is reduced to finding nodes (x_k) and weights w_k such that

$$\int_0^1 \frac{f(y)}{y^\alpha} dy \approx \sum_{k=1}^{n} f(x_k) w_k \, .$$

The method for the latter is to find a family of orthogonal polynomials p_q such that

$$\int_0^1 \frac{p_q(x) p_r(x)}{x^\alpha} = 0$$

with $\deg(p_q) = q$. Then, the quadrature nodes are the roots of p_n in $[0,1]$ and the weights are obtained numerically using the Golub-Welsch algorithm.

Remark: One can replace the integrals appearing above by principal values or finite part integrals. For $\alpha \notin \mathbb{Z}$, this gives well-defined nodes and weights (x_k, w_k), which are in general complex (x_k do no longer lie in $[0,1]$). This behavior is known (thesis of kutt).

2 Multiply singular integrals

2.1 Weakly singular kernel

The weighted integral operators involve the computation of singular integrals of the type

$$I = \int_{[A,B]} \frac{\ln|X-Y|}{\omega(Y)} dY$$

where $[A,B]$ is an element of the mesh (in 2D), $|X|$ represents the Euclidean norm of X and $\omega(Y) = d(A,Y)^\alpha$ for some α. Let $X = d\vec{n} + x\vec{u}$ where $\vec{n}$ is a unit normal vector to $[A,B]$ and $u = \frac{\overrightarrow{AB}}{|B-A|}$. Let $g(x) = \ln\sqrt{d^2 + x^2}$. Then we have

$$I = \int_0^{|B-A|} \frac{g(x-y)}{y^\alpha} dy \, .$$

It is easier to compute

$$J = \int_0^{|B-A|} \frac{g\left(\frac{k(x)-k(y)}{k'(x)}\right)}{y^\alpha} dy$$

where k is chosen so that $k'(y) = \frac{1}{y^\alpha}$, that is $k(y) = \dfrac{y^{1-\alpha}}{1-\alpha}$. Then we have simply have simply

$$J = \left[k'(x) G\left(\frac{[k(x) - k(y)]}{k'(x)} \right) \right]_0^{|B-A|}$$

where $G(x) = -x + x \ln \sqrt{d^2 + x^2} + d \arctan \frac{x}{d}$ is a primitive of g. When $d = 0$, one should put instead $G(x) = -x + x \ln |x|$ which is the limit of the previous expression when $d \to 0$. The first integral I can be seen as an approximation of J replacing the term $k(x) - k(y)$ by its first order Taylor expansion. In fact the remainder $J - I$ is of the form

$$J - I = \int_0^{|B-A|} \frac{f(y)}{y^\alpha} dy$$

where $f(y) = \ln |\frac{k(x)-k(y)}{k'(x)(x-y)}|$. When $x \neq 0$, this is a smooth function of y. When $x = 0$, f is not well defined, but the initial integral has the form

$$I = \int_0^{|B-A|} \frac{\ln \sqrt{d^2 + y^2}}{y^\alpha} dy \,.$$

When $d \neq 0$, this is already of the form $I = \int_0^{|B-A|} \frac{f(y)}{y^\alpha}$ where f is a smooth function. When $d = 0$, then letting $u = y^{1-\alpha}$, we have explicitly

$$I = \int_0^{|B-A|^{1-\alpha}} \ln |u| du = [G(u)]_0^{|B-A|^{1-\alpha}}$$

with $G(u) = u \ln |u| - u$.

We also need to compute integrals of the form

$$J = \int_{[A,B]} \frac{(Y - X) \ln |X - Y|}{\omega(Y)} \,.$$

Let n be the normal vector to $[A, B]$. Then

$$J \cdot n = \int_{[A,B]} \frac{d \ln |X - Y|}{\omega(Y)}$$

where $d = (X - A) \cdot n$. This is regularized with the same method as above, though it is expected that the error is small because when $|X - Y|$ is small, then d is small too. The tangential component of J has the form

$$J \cdot t = \frac{1}{2} \int_0^b y^{-\alpha} (x - y) \ln(d^2 + (x - y)^2) \,.$$

One approach to approximate this when $d = 0$ could be to write

$$J \cdot t = x \int_0^b y^{-\alpha} \ln |x - y| - \int_0^b y^{-\beta} \ln |x - y|$$

each term is computed as before. Note that in the second term, $\beta = \alpha - 1$ is smaller than α so we expect the regularization to perform better.

2.2 Principal value

We now show how to deal with integrals of the form

$$J = \mathrm{PV} \int_0^b \frac{y^{-\alpha}}{(x - y)} dy$$

where the PV abbreviation denotes the fact that the integral is understood in the sense of the Cauchy principal value, that is

$$J = \lim_{\varepsilon \to 0^+} \int_0^{x-\varepsilon} + \int_{x+\varepsilon}^{b} \frac{y^{-\alpha}}{(y-x)} dy \,.$$

This limit exists indeed as we can see by adding and subtracting to J the term

$$x^{-\alpha} \mathrm{PV} \int_0^b \frac{1}{(y-x)} = x^{-\alpha} \ln \frac{|b-x|}{x} \,,$$

which leads to the following expression of J:

$$J = x^{-\alpha} \ln \frac{|b-x|}{x} + \int_0^b \frac{y^{-\alpha} - x^{-\alpha}}{(y-x)}$$

Note that there exists a relation between this integral and the previous one. By formally integrating by parts, we find

$$J = \int_0^b \alpha y^{-\alpha-1} \ln |x-y| + b^{-\alpha} \ln |b-x|$$