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Research Statement



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From my undergraduate days, string theory captivated me as a foremost candidate for a unified theory of nature. Central to string theory is its vast *string landscape*, with myriads of solution. Despite advances, we are far from uncovering string theory's full implications. My research aims to discern solutions consistent with observed physics, presenting exciting opportunities to unveil hitherto unknown facets of the universe. To this end, I employ geometric logic, especially in toric and algebraic geometry. My strength lies in constructive and enumerative techniques, harnessing computational tools to pinpoint optimal string theory solutions. Notably, I have added/modified 142,000+ lines of code in the open-source computer algebra system OSCAR.

My research trifurcates into: physics, emphasizing F-theory's geometric framework for string theory; mathematics, delving into toric and algebraic geometry as well as touching on combinatorics, graph theory, and number theory; and computer science, focusing on open-source computer algebra systems like OSCAR but also interests in machine learning and data science applications.

The following text is tailored for a broad readership. Experts may skip the introduction to string and F-theory.

INTRODUCTION TO STRING THEORY

Physics has long pursued the unification of the four fundamental forces: electromagnetism, weak and strong nuclear interactions, and gravity [1–4]. While the standard model effectively unifies the first three forces [5–7], gravity's integration remains elusive by use of perturbative quantum field theories [8]. This challenge has spotlighted string theory as a potential solution [9–12].

At the heart of string theory lies the proposition that elementary particles are not point-like but instead resemble strings in shape. Intriguingly, a consistent quantum representation of these strings mandates a 10-dimensional spacetime \mathcal{S} [9–12], a deviation from our conventional 4-dimensional (time plus three spatial directions) observation of the world surrounding us. The disparity is bridged through compactification, often illustrating the 10-dimensional spacetime as $\mathcal{S} = \mathcal{E} \times \mathcal{M}_6$, with \mathcal{E} representing our everyday-observed 4-dimensional spacetime, and \mathcal{M}_6 being a 6-dimensional compact Calabi-Yau manifold. Experimental endeavors have yet to yield evidence for extra spacetime dimensions, leading to the presumption that \mathcal{M}_6 is exceptionally minuscule. The configuration of \mathcal{M}_6 profoundly affects the physics we discern in \mathcal{E} according to string theory. Pinpointing an \mathcal{M}_6 that ensures a seamless match between string theory's predictions and experimental findings remains a pressing concern.

String theory is fundamentally defined by its action: a functional dependent on fields that denote elementary particles. From a mathematical perspective, these fields can be conceptualized as sections of vector bundles. Minimizing this action produces differential equations, the solutions of which dictate the dynamics of the elementary particles represented by these fields. As the intricacies are unpacked, an additional consistency criterion surfaces: string theory demands a particular symmetry among its elementary particles, termed supersymmetry. Despite its theoretical prominence, this symmetry has yet to be substantiated through experimentation. It's worth noting that the discovery of the Higgs boson [13, 14] took fifty years, while the detection of *gravitational waves* spanned an entire century. My optimism persists in believing that supersymmetry will eventually gain empirical validation, potentially indirectly via supersymmetry breaking mechanisms [15, 16], .

String phenomenology aims to reconcile string theory with our experimentally observed physics. This involves

identifying suitable geometries \mathcal{M}_6 and solutions to the equations of motion obtained from minimizing the action functional. Enormous efforts in this direction have been undertaken. Broadly, these research efforts diverge into two categories: matching string theory with cosmological observations and aligning it with particle accelerator results. My research focus is on the latter.

String theory has five equivalent formulations [9,10,17], leading to identical physical theories from varied action functionals on spaces \mathcal{M}_6 . Early efforts focused on the $E_8 \times E_8$ heterotic string [18–25] and eventually expanded to include intersecting brane scenarios in type IIA and IIB string theory [26–33]. Said solutions to string theory solve the differential equations of motion by use of Taylor expansion. Physicist justify this strategy by saying that there is a small parameter in said Taylor series – the so-called interaction strength – which is believed to ensure convergence of said series. Put yet differently, said solutions to string theory explore the perturbative sector. The first perturbative realization of the minimally supersymmetric standard model (MSSM) – a minimal extension of the empirically-backed standard model of particle physics via supersymmetry – exists in [21, 22] with further insights in [34, 35]. However, many other perturbative models introduce unobserved exotic particles. A prevailing challenge, as noted in [23–25] and detailed in [36], is that some observed particle interactions are omitted or differ significantly from experiments like those at CERN.

Alongside ongoing studies on (perturbative) heterotic line bundle standard models, significant attention has been given to the potential of *F-theory* in probing strongly coupled IIB *string theory*. F-theory adeptly bridges the gap between geometry and physics [37–39], enriched by techniques from algebraic geometry. This framework inherently supports the required particle interactions due to its geometric consistency. By the same principle, its solutions are always globally consistent, which is not a given for perturbative string theory explorations. Pioneering studies in this realm encompass references such as [?, 40–51]. A milestone in this domain is the discovery of the *Quadrillion F-theory standard models (QSMs)* [52], renowned for their physically appealing properties (global consistency, gauge coupling unification, and absence of chiral exotics). The QSMs stand out as the largest known class (more than 10^{15} constructions) of F-theory standard model solutions with these attributes.

PAST CONTRIBUTIONS

Higgs pairs are a vital component in our understanding of particle physics, as underpinned by the Nobel Prize awarded to Higgs and Englert in 2013 [13, 14]. For instance, a *single* Higgs pair is imperative for the MSSM. To check for alignment among F-theory solutions and the MSSM, it is thus imprudent to compute the total number of Higgs pairs. This leads to the study of vector-like spectra. My engagement with this topic is not only rooted in the dire need to investigate this property of F-theory solutions, but also by the rich mathematical tapestry of this topic, which provides avenues to apply modern mathematics and computational tools in physics. My explorations encompass cohomologies of coherent sheaves on toric varieties [44, 48, 50], Freyd categories [53], and machine learning enhancements [54]. Recently, I focused on the F-theory QSMs [55–58]. While F-theory adeptly resolves consistency issues in perturbative string theory, computing the vector-like spectra, and thereby the number of Higgs pairs, is more demanding compared to the heterotic string theory. In the latter, Higgs pairs are encoded in line bundles on a Calabi-Yau 3-fold, often assumed to be a toric space for practicality, simplifying the computation of line bundle cohomologies. Conversely, F-theory centers on line bundles on smooth, irreducible curves, introducing complexities due to the curve's continuous Picard group. My QSM program – encapsulated in a recent review [59] and remaining a major focus in my future research – provides significant arithmetic approaches towards the Brill-Noether theory of root bundles on nodal curves and employ the ensuing Brill-Noether numbers as upper bounds for the vector-like spectra of the F-theory QSMs.

Navigating the geometric computations in F-theory can be arduous, which slows progress, limits the exploration of complex geometries (e.g., those requiring techniques beyond those suitable for toric varieties), and presents a steep entry barrier for newcomers. In collaboration with A. Turner from the University of Pennsylvania, I have enriched the OSCAR computer algebra system, introducing tools specifically designed for F-theory applications. A notable feature of the upcoming FTheoryTools is its capability to effortlessly extract and modify geometric constructions from existing literature. It will also incorporate advanced singularity resolution techniques, since the physics is most easiest extracted from the resolved space. Advancing these developments, e.g. by facilitating topological properties of non-toric-spaces as well as establishing FTheoryTools as a recognized computational tool in F-theory encapsulates my second research focus in the future.

RESEARCH FOCUS I:

FTheoryTools – TACKLING F-THEORY’S COMP. CHALLENGES

Navigating F-theory’s geometric computations is challenging, slowing advancements. To streamline this, I am making major contributions to the development of FTheoryTools within the OSCAR computer algebra system [60, 61]. As of now, I have added and modified over 142,000 lines in OSCAR, with significant contributions to its toric geometry functionality [62]. Additionally, I sought to modernize my earlier work, the *ToricVarieties_project* [63], which is part of [64, 65] and written in the gap programming language [66]. In contrast, OSCAR uses the modern Julia programming language. A brief overview of F-theory’s geometric intricacies ensues.

F-Theory in a Nutshell

Type IIB *string theory* hinges on a supergravity action governed by fields which, for mathematicians, are sections of particular vector bundles. Notably, the scalar dilaton field ϕ (with $\phi(x) \in \mathbb{C}$ for every $x \in \mathcal{M}_6$) and the RR gauge potentials C_0 and C_8 are of primary focus. The dilaton field’s significance stems from its linkage to the strength of string interactions in 10-dimensional spacetime $\mathcal{S} = \mathcal{E} \times \mathcal{M}_6$ [9–12]. Interestingly, C_8 ’s equations of motion allow only trivial solutions on a smooth \mathcal{M}_6 . The application of an involution σ to \mathcal{M}_6 , i.e. $\mathcal{M}_6 \rightarrow \mathcal{B}_6 := \mathcal{M}_6/\sigma$, leads to a space \mathcal{B}_6 which allows non-trivial solutions to the C_8 equations of motion. In physics lingo, the involution introduces orientifold $O7$ planes, which are represented by the fixed points of σ [9, 10]. Spaces akin to \mathcal{B}_6 serve as the foundation for type IIB orientifold theories [15, 67, 68]. It turns out that in such orientifold theories, the dilaton field ϕ can manifest singularities, implying an infinite value at specific loci and leading to strong string interactions. Such strength contradicts the essential premise of weak interactions in perturbation theory. This necessitates a transition from the perturbative type IIB supergravity description to a non-perturbative framework [69], namely *F-theory* [37]. To this end, the dilaton ϕ and RR gauge field C_0 merge into the axio-dilaton τ :

$$\tau: \mathcal{E} \times \mathcal{B}_6 \rightarrow \mathbb{C}, \quad x^\mu \mapsto C_0(x^\mu) + ie^{-\phi(x^\mu)}.$$

Due to *Lorentz invariance*, τ is constant on \mathcal{E} and a section of a holomorphic $\mathrm{SL}(2, \mathbb{Z})$ line bundle over \mathcal{B}_6 [38, 39]. We understand the value of τ at $x^\mu \in \mathcal{B}_6$ as the complex structure modulus of an elliptic curve. Consequently, an elliptically fibered 4-fold $\pi: Y_4 \rightarrow \mathcal{B}_6$ with fibre $\mathbb{C}_{1, \tau(x^\mu)}$ serves as “book-keeping” device of the axio-dilaton. At the same time, the geometry of Y_4 enforces consistency in that it ensures that the equation of motion for C_8 has a solution and leads to the encoded axio-dilaton field τ .

The geometry of Y_4 encodes much of the physics, as detailed in [70–73]. Non-trivial physics necessitates a singular Y_4 . In need for better alternatives, it is common to try to crepantly resolve Y_4 [74]. Most of my past contributions to F-theory assume that, up to \mathbb{Q} -factorial terminal singularities, at least one such crepant resolution \hat{Y}_4 exist in the form of a sequence of blowups. Furthermore, I assume that \hat{Y}_4 admits at least one smooth section.¹

Goals and Features of FTheoryTools

In F-theory setups, the initial geometric challenge is the crepant resolution of singular Y_4 . A comprehensive algorithm is still elusive, especially in determining \mathbb{Q} -factorial terminal singularities. Typically, we apply the entire F-theory toolkit to singularities, presuming non-resolvability when standard methods fail. Hence, incorporating state-of-the-art resolution routines, for instance including the weighted blowups explored in [79], is paramount. An equally significant feature is a database to automatically utilize established literature constructions, including the set of known resolutions.

Upon resolution, the ensuing step involves examining the given geometry using topological tools, notably through the application of pushforward formulae. This technique facilitates the translation of intersection theory computations from the resolved 4-fold \hat{Y}_4 to the base, often simplifying the calculations and revealing patterns. For instance, it shows that specific physical quantities solely depend on a base intersection number, as seen in F-theory QSMs which hinge on the triple intersection number of the anticanonical class of the base 3-fold. Enhancing the

¹Certainly, F-theory has been studied in more general contexts, for instance without section [75–78].

FTheoryTools with intersection theory and topological intersection numbers, and venturing beyond the toric regime, presents collaborative prospects.

Incorporating prevalent F-theory methodologies into the FTheoryTools offers an excellent avenue for students to delve into advanced research, interact with relevant (computer) geometries, and contribute to the literature constructions database. Although these efforts are concise—fitting, they hold potential for deeper exploration. Notably, this database not only furthers the study of F-theory geometries but also initiates explorations into machine learning and data science, reflecting my interdisciplinary spirit. A case in point involves probing a theory of Brill-Noether numbers or, if unattainable, investigating a related F-theory inspired cryptosystem. These concepts form part of my secondary research focus, elaborated towards the end of this proposal, and hint at potential fruitful collaborations.

In F-theory studies, exploring beyond topology is vital. For example, the singularities of Y_4 determine a non-abelian (gauge) group. It is advantageous to augment this group with abelian group factors for purposes such as enforcing selection rules. These abelian factors originate from the torsion-free subgroup of the Mordell–Weil group of \hat{Y}_4 , which represents the group of infinite-order rational sections of the fibration, governed by elliptic curve addition. Consequently, the rank of the abelian part of the gauge algebra aligns with the Mordell–Weil rank. It is noteworthy that the torsional part of the Mordell–Weil rank relates to the gauge group’s global structure in the physical theory. Similarly, exploring F-theory on elliptic fibrations with multi-sections can be pursued, the study of which leads to the Weil–Châtelet group and discrete factors (cf. [77] and references therein).

My secondary research aim explores the alignment of F-theory solutions with empirical observations in particle physics. Central to this endeavor is the G_4 -flux, an element of $H^{2,2}(\hat{Y}_4, \mathbb{Z})$, which encodes the count of matter particle families in a specified compactification. Initially, the chiral spectrum, founded on topological computations as outlined in [80–88], provides insight. Advancing beyond, I aim to ascertain whether an F-theory solution encompasses zero, one, or multiple instances of the notable Higgs boson - a pivotal step in connecting F-theory solutions with observed particle physics. Presently, FTheoryTools has limited capacity for such advanced inquiries. However, support for the renowned `cohomCalc`-algorithm [89–93], along with the vanishing sets from `cohomCalc` in [50], lays a promising groundwork in OSCAR. Further explorations include Deligne cohomology and root bundles and necessitate foundational investigations, marking the core of my second research focus.

I am excited by the numerous applications of FTheoryTools, from uncovering crepant resolutions to enabling previously unmanageable geometries, and streamlining standard processes for novices and experts.

RESEARCH FOCUS 2:

FROM F-THEORY QSMS TO BRILL-NOETHER NUMBERS AND BACK

Brill-Noether Numbers – A Noval Introduction

The *F-theory QSMs* [52] provide 10^{15} solutions apt for the *standard model* of particle physics. My investigation into their vector-like spectra – a crucial ingredient to compare these solutions to experimental findings from particle accelerators – directed me to root bundles on nodal curves. Before we explain how this topic arises from the physics, I wish to provide a noval introduction to Brill-Noether numbers.

Root bundles generalize spin bundles. We recall that there are 2^{2g} spin bundles on a smooth, irreducible genus g curve, each of which corresponds to a divisor class D with $2 \cdot D = K_C$, where K_C is the canonical bundle. For root bundles, we consider divisors D with $r \cdot D = E$, $r \in \mathbb{Z}_{\geq 2}$, and E not necessarily the canonical bundle, yielding r^{2g} roots, if existent. Nodal curves C^\bullet bring two nuances: they typically have multiple irreducible components due to nodal singularities, and their root bundles can be enumerated as limit roots [94] (see also [95–97]). Here is the dual graph (vertices are irreducible components and edges nodal singularities) of a nodal curve for which all irreducible components are \mathbb{P}^1 s:



Our objective is to enumerate limit roots P^\bullet that satisfy $12P^\bullet = 12K_{C^\bullet}$ and discern their global section count. As

we shall motivate below, this count of global sections filters F-theory solutions potentially aligned with experimental findings. For the given case, we have 12^8 limit roots. Our techniques reveal that 12^4 roots have $h^0(C^\bullet, P^\bullet) = 4$, whereas the rest possess precisely three global sections:

Roots Count	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
12^8	$12^4 \cdot (12^4 - 1)$	0	12^4	0

(2)

While a unique determination of the number of global sections is elusive for some roots P^\bullet , an optimal lower bound can be computed, as would be reflected in every 2nd column of the prior table for a more complicated example. This gives rise to the partition:

$$12^8 = 12^4 \cdot (12^4 - 1) + 0 + 12^4 + 0. \quad (3)$$

The order of the summands is crucial, and summands can reappear. Our technology recently culminated in [58], computing in a certain sense an optimal partition. This optimal partition is likely to bear a deeper meaning and definitely carries a striking resemblance to the Brill-Noether theory for line bundles on smooth, irreducible curves [98]. This leads me to dub the summands *Brill-Noether numbers*. It is worth recalling the historic works [99,100], showing that roughly half of the spin bundles on a genus g curve possess an even numbers of global sections. Beyond this, knowledge is limited. The following subsections will explain the significance of Brill-Noether-numbers for F-theory QSMs, our current understanding, and tasks for future studies.

Spin Bundle: A Critical Piece in F-theory's Puzzle

Vector-like spectra are essential in analyzing string theory solutions and elucidating the number of Higgs pairs in a theory [?, 13, 14] – a salient component highlighted by the Nobel Prize awarded to Higgs and Englert. For instance, a single Higgs pair is imperative for the MSSM. My engagement with this topic, also driven by its rich mathematical tapestry, was initiated in [44]. The relevant matter fields, arising from strings between D7-branes in the perturbative type II string theory, localized in F-theory on *matter curves* $C_{\mathbf{R}} \subset \mathcal{B}_6$ [101–103]. The gauge groups in F-theory, including the group representations of the matter fields, are determined by Y_4 's geometry [70,71]. The gauge fields correspond to the Deligne cohomology $H_D^4(Y_4, \mathbb{Z}(2))$ [104–110]. The results in [44] imply that these gauge fields induce a line bundle $\mathcal{L}_{\mathbf{R}}$ on $C_{\mathbf{R}}$. Classical results guide our computation of massless matter [111, 112]:

- $\mathcal{N} = 1$ chiral multiplets: $H^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}})$.
- $\mathcal{N} = 1$ anti-chiral multiplets: $H^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}})^2$.

On a curve of genus g , the selection of the appropriate spin bundle $\sqrt{K_{C_{\mathbf{R}}}}$ from 2^{2g} possibilities significantly influences the number of sections of $L_{\mathbf{R}} = \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}$ and the Freed-Witten anomaly cancelation [113]. As highlighted in [102], the anomaly cancelation requires spin^c-structures on gauge surfaces $S \subset B_3$ in F-theory GUTs. Identifying the correct spin bundles is essential but complex. While traditionally underserved, recent advances underscore the need to address this question. I aspire to explore this area in subsequent research endeavors.

Brill-Noether Numbers: Upper Bound to Vector-Like Spectra of F-theory QSMs

The pioneering work on F-theory QSMs [52] concentrated on the G_4 -flux, which governs the chiral index [40–42, 108, 114–119] but leaves the *F-theory* gauge field A undetermined. Amidst the intricacies of the spin bundle (cf. section 4.2), we deduced a pivotal constraint: the line bundle $L_{\mathbf{R}}$ is a specific root bundle $P_{\mathbf{R}}$ [55]. Prior investigations centered on the quark-doublet curve $C_{(3,2)_{1/6}}$ of genus $g = \frac{\overline{K}_{B_6}^3 + 2}{2}$, where $\overline{K}_{B_6}^3$ denotes the triple intersection of the F-theory base's anticanonical class $\overline{K}_{B_6}^3 \in \{6, 10, 18, 30\}$. The root bundle constraint is

$$P_{(3,2)_{1/6}}^{\otimes 2\overline{K}_{B_6}^3} = K_{(3,2)_{1/6}}^{\otimes (6+\overline{K}_{B_6}^3)}. \quad (4)$$

²A *supersymmetric field theory* only contains chiral fields. We count chiral superfields in the charge conjugate representation $\overline{\mathbf{R}}$. The wording "anti-chiral" is inspired from low energy physics.

A limited number of these solutions likely arise from an apt choice of the spin bundle, yet its correct identification remains elusive. Instead of pinpointing this, our prior research took a statistical approach: We enumerated all solutions to eq. (4), counted all roots with exactly three global sections, and determined probable F-theory geometries devoid of absence of exotic vector-like quark-doublets. Still, enumerating all roots and distinguishing their global sections is formidable on smooth, irreducible curves. Fortunately, the Brill-Noether numbers offer a practical upper bound. Our methodology ensues:

1. Deforming $C_{(3,2)_{1/6}}$ into $C_{(3,2)_{1/6}}^\bullet$, which is shared across various geometries [52] due to their origin from toric K_3 -surfaces desingularizations [120–122], see [56] for a detailed explanation. This key observation facilitated a computer scan of the majority of the 10^{15} F-theory QSM geometries [52, 123].
2. Employing techniques from [94] (cf. [95–97]), we list all *limit root* $P_{(3,2)_{1/6}}^\bullet$ with our software [124].
3. Computing the global sections of each *limit root* $P_{(3,2)_{1/6}}^\bullet$ with the techniques developed in [55–57], which recently culminated in an optimal approach [58]. The interested reader may wish to consult [59] for a summary of this program. This step leads to the Brill-Noether numbers introduced above.
4. Lastly, we bridge the number of global sections between all limit roots and the roots on the smooth curve $C_{(3,2)_{1/6}}$. The number of global sections may decrease if $h^0(C_{(3,2)_{1/6}}^\bullet, P_{(3,2)_{1/6}}^\bullet) > \chi(P_{(3,2)_{1/6}})$. Hence, the Brill-Noether numbers serve as upper bound to the desired statistics on $C_{(3,2)_{1/6}}$.

Question 1: Towards a Theory of Brill-Noether Numbers

Even though the computation of the Brill-Noether numbers are resource-intensive, we currently use them as upper bound to the F-theory QSMs' vector-like spectra. A deeper understanding of the link between a nodal curve and these numbers is desired. One may posit if the Brill-Noether numbers can be inferred from the nodal curve's dual graph and the root bundle constraint. Employing both *machine learning tools* and *analytic/algebraic insights* can be beneficial in this pursuit, mimicking efforts in [54]. If the systematics are revealed, it opens doors for analogous analyses on the intricate Higgs curve of the F-theory QSMs. To illustrate the Higgs curve's complexity, consider a base \mathcal{B}_6 with $\overline{K}_{\mathcal{B}_6}^3 = 6$. Then, $g(C_{(3,2)_{1/6}}) = 4$, while $g(C_{(1,2)_{-1/2}}) = 28$. The complexity not only manifests in a much larger number of limit roots to be enumerated, but also in a much more complicated dual graph.

Question 2: An F-Theory-Inspired Cryptosystem?

Recently, the idea of a cryptosystem based on Brill-Noether numbers was sparked: For a given integer partition, can we identify a graph and a root bundle constraint, such that the ensuing Brill-Noether numbers match exactly with the initial partition? The inverse of this question is feasible with our existing methods, but this is already computationally taxing. Attempting the direct approach seems, at the very least, daunting. This disparity raises the possibility of unveiling a new cryptosystem: A promising avenue for future studies.

Question 3: Jumps Meet Yukawa Interactions

Computing the Brill-Noether numbers for the nodal Higgs curve is imperative. Yet, linking these numbers to the vector-like spectra on the smooth, irreducible Higgs curve introduces challenges. Specifically, a drop in the number of global sections might occur if $h^0(C_{(1,2)_{1/6}}^\bullet, P_{(1,2)_{1/6}}^\bullet) > \chi(P_{(1,2)_{1/6}}^\bullet)$. Realizing a Higgs pair [13, 14] indeed necessitates such a non-minimal number of sections:

$$h^0(C_{(1,2)_{-1/2}}, P_{(1,2)_{-1/2}}) = h^1(C_{(1,2)_{-1/2}}, P_{(1,2)_{-1/2}}) = 1. \quad (5)$$

Changes in \widehat{Y}_4 's complex structure can alter $P_{\mathbf{R}}$ and $C_{\mathbf{R}}$, potentially causing h^0 and h^1 jumps, as explained by Brill-Noether theory [98, 125]. To refine our analysis [55] towards a single Higgs pair, understanding the cohomology differences between the limit root line bundles on $C_{(1,2)_{-1/2}}^\bullet$ and $C_{(1,2)_{-1/2}}$ is crucial. Physically, interactions are anticipated at the nodes of $C_{(1,2)_{-1/2}}^\bullet$, leading to a mass matrix M , whose rank is expected to be difference in vector-like spectra between $C_{(1,2)_{-1/2}}^\bullet$ and $C_{(1,2)_{-1/2}}$. The challenge lies in computing M [126, 127] and aligning the predicted jump to mathematical concepts like Brill-Noether jumps [98] or *limit linear series* [128–130].

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