Research statement
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By upbringing, I am a physicist, searching for solutions to **string theory**, which resemble the physics of our everyday experience. This is an involved geometric task and my research relies heavily on **algebraic geometry**, in particular **constructive and enumerative methods**. The resulting computations can often only be handled by modern **computer implementations**. Therefore, I have made major contributions to computer algebra systems, most recently the **OSCAR Computer Algebra System**.

Outline Section 1 provides an introduction to *string theory* and *F-theory*, which experts may skip. In section 2, I formulate a criterion to tell if solutions to *F-theory* contain certain exotic particles – *vector-like exotics* – which have not been observed in experiments. This leads to the quest for line bundles on certain curves subject to physical constraints, in particular their sheaf cohomologies. Section 3 summarizes my past contributions towards finding *F-theory* solutions without *vector-like exotics* and section 4 my current efforts.

1 Motivation

1.1 Why *string theory*?

It is experimentally established that four "rules" are necessary to describe all phenomena of our physical world: the *electromagnetic, weak and strong interaction* [1–3] and *gravity* [4]. The first three admit a unified description in the *standard model of particle physics* [5–7]. *General relativity* follows different rules and cannot be unified with the *standard model of particle physics* by means of perturbative quantum field theory [8]. The search for a unified description of *quantum gravity* has lead to a remarkable candidate for a theory of *quantum gravity*, namely *superstring theory* [9–12].

For reasons of consistency, *superstring theory* requires a 10 (real) dimensional spacetime $\mathcal{S}$ and *supersymmetry* [9–12]. This is in contrast to our everyday experience of 4 spacetime directions. To account for this mismatch, one performs a compactification. We write

$$\mathcal{S} = \mathcal{E} \times_w \mathcal{M}_6 \,,$$

where "w" reminds us that this can in general be a warp-product and $\mathcal{E}$ represents the 4 (real) dimensional spacetime of our everyday experience. The so-called *internal space* $\mathcal{M}_6$ is, in the simplest case, a *compact Calabi-Yau manifold* of real dimension 6.[1]

Experiments have not found evidence for additional spacetime dimension. For this reason, we take $\mathcal{M}_6$ sufficiently small. Similarly, *supersymmetry* has not been found in experiments – at least not yet. While recent experiments put a lot of stresses on

[1] In absence of fluxes, a necessary condition for conformal invariance of quantized *superstring theory* is $0 = \alpha' R_{\mu\nu} + \frac{1}{2}(\alpha')^2 R_{\mu\lambda\rho\sigma} R_{\nu}^{\lambda\rho\sigma} + \ldots$ [13], where $R_{\mu\lambda\rho\sigma}$ is the *Riemann* tensor of the metric $g_{\mu\nu}$ on $\mathcal{S}$. For the special case of a direct product and "small" α', i.e. in the SUGRA limit of *string theory*, this implies a Ricci flat $\mathcal{M}_6$. This is equivalent to the *Calabi-Yau* condition $c_1(T_{\mathcal{M}_6}) = 0$ [14].

supersymmetric theories, we must not forget that the detection of the Higgs boson took a half-century, and that of *gravitational waves* an entire century. Therefore, I remain optimistic that *supersymmetry* can eventually be established at high energy scales and is broken at low energies. An exposition of mechanism for *supersymmetry* breaking in the context of my research can be found e.g. in [15, 16].

String phenomenology goes by the assumption that *superstring theory* is the correct unification of *gravity* and the *standard model of particle physics*. The goal is to find solutions to the string equations of motion such that upon compactification and *supersymmetry* breaking, only the observed 4-dimensional physics remains. To tell if a compactification represents the experimentally observed physics, among others the exact massless spectrum of this compactification must be computed.[2] This is what my research focuses on.

1.2 Why *F-theory*?

Superstring theory in 10 dimensions can be phrased in five consistent formulations [9,10,13]. We focus on so-called type IIB *superstring theory* for which the following are important:

- Dilaton ϕ: A scalar field (massless bosonic string excitation) on the spacetime $\mathcal{S}$.
- D7-branes: 8 (real) dimensional, dynamic objects in the spacetime $\mathcal{S}$, which are electrically charged under the 8-form RR gauge potential C_8.

The dilaton ϕ is of ample importance since $\phi_0 := \lim_{x \to \infty} \phi(x)$ fixes the string coupling far away from D-branes via $g_s = e^{\phi_0}$ [9–12]. D7-branes are crucial since they realize $\mathrm{U}(N)$ gauge theories [15]. The consistency of the field equation for the C_8 gauge potential requires that the C_8-charges of all D7-branes in a type IIB compactification cancel. Therefore, non-trivial and stable compactifications with D7-branes must contain orientifold planes $O7$, which are negatively charged under C_8 [9, 10]. To achieve this, the internal space $\mathcal{M}_6$ must be modified by the operation of an involution σ via

$$\mathcal{M}_6 \to \mathcal{B}_6 := \mathcal{M}_6 / \sigma \, .$$

Superstring theory on $\mathcal{B}_6$ is obtained by removing the orientation of the strings. To this end, one focuses on string excitations which are invariant under the operation of the parity operator Ω and $(-1)^{F_L}$ [3] This defines the *orientifold theory* on $\mathcal{B}_6$ [15, 17, 18].

While type IIB orentifold theories with D7-branes are phenomenologically desirable, the dilaton ϕ then develops a profile which challenges the perturbative descriptions and often requires non-perturbative techniques [19]. This can be seen as the origin of *F-theory* – the exact description of strongly coupled type IIB *superstring theory* [20]. The key is to combine the dilaton $\phi \colon \mathcal{E} \times_w \mathcal{B}_6 \to \mathbb{R}$ with the 0-form valued RR gauge field $C_0 \colon \mathcal{E} \times_w \mathcal{B}_6 \to \mathbb{R}$ to the axio-dilaton τ:

$$\tau \colon \mathcal{E} \times_w \mathcal{B}_6 \to \mathbb{C} \, , \; x^\mu \mapsto C_0(x^\mu) + i e^{-\phi(x^\mu)} \, .$$

[2] Those massless states are rendered massive by spontaneous symmetry breaking via the Higgs mechanism.
[3] F_L is the number operator for left-handed, fermionic string excitations.

By *Lorentz invariance*, τ is constant on $\mathcal{E}$ and the section of a holomorphic $\mathrm{SL}(2,\mathbb{Z})$ line bundle over $\mathcal{B}_6$ [21,22]. We interpret the value of τ at $x^\mu \in \mathcal{B}_6$ as the complex structure modulus of an elliptic curve $\mathbb{C}_{1,\tau(x^\mu)}$. A geometry which keeps track of the axio-dilaton profile is thus obtained by "attaching" the elliptic curve $\mathbb{C}_{1,\tau(x^\mu)}$ at $x^\mu \in \mathcal{B}_6$. This leads to the study of elliptically fibered 4-folds $\pi\colon Y_4 \twoheadrightarrow \mathcal{B}_6$ and fibre $\mathbb{C}_{1,\tau(x^\mu)} := \pi^{-1}(x^\mu)$. The geometry of Y_4 contains information about the physics. The phenomenological implications are summarized e.g. in [23–26]. For non-trivial physics, Y_4 must be singular. Since the analysis of smooth geometries is typically simpler, it is standard to resolve the singularities [27], most commonly by a crepant blowup resolution. For simplicity, I will not make a strict distinction between the singular Y_4 and its resolved cousin $\widehat{Y}_4$.

An elliptic curve is topologically isomorphic to a torus surface with a marked point [28]. Hence, in considering an elliptically fibered 4-fold Y_4, there exists for each $x^\mu \in \mathcal{B}_6$ a marked point $P(x^\mu) \in \pi^{-1}(x^\mu) = \mathbb{C}_{1,\tau(x^\mu)}$ and we can consider $s\colon \mathcal{B}_6 \to Y_4\ ,\ x^\mu \mapsto P(x^\mu)$. For *F-theory* applications, we demand that this map is (at least) continuous, i.e. Y_4 admits a (continuous) section s. While our applications focus on *F-theory with a section*, it is worth mentioning that *F-theory without section* has been described e.g. in [29–32].

1.3 Why *F-theory standard models*?

Enormous efforts have been undertaken to demonstrate the particle spectrum of the *standard model* from *string theory*. The earliest studies focus on the $E_8 \times E_8$ heterotic string [33–40] and were later extended by intersecting branes models [41–48]. While these compactifications realize the gauge sector and chiral spectrum of the *standard model*, they are limited to the perturbative regime in the string coupling. Typically, they also suffer from vector-like exotics. The first globally consistent, perturbative MSSM constructions are [36,37] (see [49,50] for more details).

As *F-theory* describes strongly coupled IIB *superstring theory* and provides a coherent approach to analyze the relations among the geometry of Y_4 and the 4-dimensional physics [20–22], efforts were undertaken to realize the *standard model* in *F-theory* [51–59]. These works have recently led to the discovery of the largest, currently-known, class of one quadrillion globally consistent *F-theory standard models (QSMs)* with gauge coupling unification and no chiral exotics [60]. I aim towards finding even better solutions and analyze the exact matter content to tell if unobserved vector-like particles are predicted.

2 Towards *F-theory vacua* without vector-like exotics

2.1 Counting exact massless spectra in *F-theory*

In [61], the study of massless spectra in *F-theory* was initiated. The matter fields relevant for our study localize along so-called *matter curves* $C_{\mathbf{R}} \subset \mathcal{B}_6$ [62–64].[4] Mathematically, $C_{\mathbf{R}}$

[4] The matter fields relevant for our study are, in type IIB language, the strings stretching between two D7-branes. Other matter fields stem from open strings on the same D7-brane and closed strings. In quantum field theory, particles are specified by a representation of a group G [65–68]. In *F-theory*, the representation $\mathbf{R}$ of particles localized on $C_{\mathbf{R}}$ is determined by the geometry of Y_4 [23–26].

is a complex curve/Riemann surface [27]. The number of matter fields on $C_{\mathbf{R}}$ is specified by gauge information. In part, this information is given by the 4-form valued gauge flux $G_4 \in \Omega^4(Y_4)$, which is induced by *F-theory*/M-theory duality from the M-Theorie 3-form valued gauge potential C_3 [23–26]. G_4 is a "field strength" (curvature) and can therefore not provide the complete gauge information (connection). In [69, 70], *F-theory* gauge fields were identified with elements of the Deligne cohomology $H^4_D(Y_4, \mathbb{Z}(2))$[5], which can be understood via the short exact sequence [75, 76]

$$0 \to J^2(Y_4) \to H^4_D(Y_4, \mathbb{Z}(2)) \xrightarrow{\hat{c}_2} H^{2,2}_{\mathbb{Z}}(Y_4) \to 0\,.$$

For $A \in H^4_D(Y_4, \mathbb{Z}(2))$, we interpret $\hat{c}_2(A)$ as G_4-flux and the kernel of $\hat{c}_2$ as the C_3-flat configurations.[6] We argue in [61] that A leads to a line bundle $L_{\mathbf{R}}$ on the matter curve $C_{\mathbf{R}}$. In analogy with classical results on type IIB *superstring theory* [79–85], we then propose to count the massless matter on $C_{\mathbf{R}}$ as follows:

- $\mathcal{N} = 1$ chiral multiplets are given by elements of $H^0\left(C_{\mathbf{R}}, L_R \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.
- $\mathcal{N} = 1$ anti-chiral multiplets are given by elements of $H^1\left(C_{\mathbf{R}}, L_R \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.[7]

$\sqrt{K_{C_{\mathbf{R}}}}$ is the spin bundle on $C_{\mathbf{R}}$ which is fixed by the holomorphic embedding $\iota\colon C_{\mathbf{R}} \hookrightarrow Y_4$ in a $\mathcal{N} = 1$ supersymmetric configuration. In the following, I write $\mathcal{L}_{\mathbf{R}} = L_R \otimes \sqrt{K_{C_{\mathbf{R}}}}$.

2.2 Absence of vector-like exotic particles

Experiments have found 3 types of leptons[8] [86]. To include them in an *F-theory* vacuum, there must be a matter curve $C_{\mathbf{R}}$ which realizes the representation $(\mathbf{1}, \mathbf{2})_{-1/2}$ of the leptons under the *standard model gauge group* $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_{\mathrm{Y}}$. In addition, the *F-theory* gauge field $A \in \mathrm{H}^4_D(Y_4, \mathbb{Z}(2))$ must induce $\mathcal{L}_{(\mathbf{1},\mathbf{2})_{-1/2}} \in \mathrm{Pic}(C_{(\mathbf{1},\mathbf{2})_{-1/2}})$ with

$$h^0\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, \mathcal{L}_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = 3 + a\,, \qquad h^1\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, \mathcal{L}_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = a\,.$$

The chiral index – a topological invariant – is given by the Riemann-Roch theorem

$$\chi = \int_{C_{\mathbf{R}}} c_1(L_{\mathbf{R}}) = h^0(C_{\mathbf{R}}, L_{\mathbf{R}}) - h^1(C_{\mathbf{R}}, L_{\mathbf{R}})\,.$$

This famous result cannot predict the number a above. If $a \neq 0$, there is a non-trivial pair – known as *vector-like pair* – consisting of an element of $h^0(C_{\mathbf{R}}, L_{\mathbf{R}})$ and $h^1(C_{\mathbf{R}}, L_{\mathbf{R}})$. *Vector-like pairs* have not been observed for leptons [86] and we must satisfy $a = 0$.

Variations of the complex structure of Y_4 change $\mathcal{L}_{\mathbf{R}}$, $C_{\mathbf{R}}$ and, as is well-known e.g. in Brill-Noether theory [87] (see [88] for a more modern exposition and [89] for an earlier

[5] An equivalent approach can be found in [71–74].
[6] Equivalently, different gauge potentials in $H^4_D(Y_4, \mathbb{Z}(2))$ differ by their Wilson lines [77, 78].
[7] A *supersymmetric field theory* only contains chiral fields. What we count are chiral superfields in the charge conjugate representation $\overline{\mathbf{R}}$. The terminology "anti-chiral" is inspired from low energy physics.
[8] These include the electron, which drives all our electric devices.

application of Brill-Noether theory in *F-theory*), can lead to jumps in $h^i\left(C_{\mathbf{R}}, L_{\mathbf{R}}\right)$. This means that one could find $h^i(C_{\mathbf{R}}, L_{\mathbf{R}}) = (3,0)$ for generic and $(4,1)$ for special complex structure of Y_4. For leptons the generic outcome is desired. In contrast, a non-generic configuration is necessary for the Higgs field – the particle that gives mass to all other matter particles [90–92] and lead to the *Nobel Prize in Physics 2013*: A matter curve C_{Higgs} and $\mathcal{L}_{\text{Higgs}} \in \text{Pic}(C_{\text{Higgs}})$ – induced by the above *F-theory* gauge field – with

$$h^0\left(C_{\text{Higgs}}, \mathcal{L}_{\text{Higgs}}\right) = h^1\left(C_{\text{Higgs}}, \mathcal{L}_{\text{Higgs}}\right) = 1\,.$$

By extending this task to all particles in the *standard model*, we arrive at the quest for *F-theory* realizations of *minimally supersymmetric standard models* (*F-theory* MSSMs).

3 Contributions towards *F-theory* MSSMs

3.1 F-theory and coherent sheaf cohomologies

In [61], we argued that for special gauge fields A the line bundle $\mathcal{L}_R$ is the restriction of a bundle defined on a simple (typically toric) ambient space. While arithmetically favorable, this is not a generic feature of *F-theory* vacua. Consider for example *F-theory* SU(5) GUT models [63, 64, 93–96]. The gauge group SU(5) is realized on a D7-brane $\mathcal{E} \times \Sigma$[9] and must be broken by a hypercharge flux. The latter is given by a holomorphic $\mathrm{U}(1)$ line bundle $\mathcal{L}_Y$ on $\mathcal{E} \times \Sigma$. The absence of exotic massive particles requires[10] $c_1\left(\mathcal{L}_Y\right) \in H^{1,1}\left(\Sigma\right) - \iota^* H^{1,1}\left(\mathcal{B}_6\right)$. Consequently, the physics in $\mathcal{E}$ observes the breaking $\mathrm{SU}\left(5\right) \to \mathrm{SU}\left(3\right)_C \times \mathrm{SU}\left(2\right)_L \times \mathrm{U}\left(1\right)_Y$ iff $\mathcal{L}_Y^\Sigma = \mathcal{L}_Y|_\Sigma$ is not pullback of a line bundle on $\mathcal{B}_6$. While our study in [97] focused on exactly this type of application, it could also show that in the presence of *matter surface fluxes* the massless spectra are typically encoded in non-pullback line bundles. Consequently, the computation of cohomologies of pullback line bundles – e.g. via the famous *cohomCalg* algorithm [98–102] – is insufficient to determine the massless spectra of most *F-theory* vacua. This makes the quest for *F-theory standard models* without vector-like exotics very challenging.

We can extend every line bundle $\mathcal{L}_{\mathbf{R}}$ by "zero" outside of $C_{\mathbf{R}}$ and obtain a coherent sheaf. On toric varieties, the cohomologies of coherent sheaves can be computed [103, 104]. In [105], I put forward an algorithm for such computations. It combines *cohomCalg* [98–102] with [106]. This algorithm was developed in collaboration with *Prof. Dr. Mohamed Barakat* at the *university of Siegen*, whose previous works establish a constructive approach to coherent sheaves on toric spaces is achieved by *Serre Quotient categories* [107–111]. The new algorithm is implemented in the `gap`-package [112]. It is written in the language of *categorical programming* (`CAP`) [113] and models coherent sheaves as objects of *Freyd categories* [114, 115]. The latter were first implemented in `CAP` in [116–118].

This new algorithm in [112] employs *cohomCalg* [98–102] to computed vanishing sets, which refine the semigroup K_{sat} introduced in [119] and used in [120] (see also [121] for

[9] $\mathcal{E}$ is the spacetime of our everyday experience and $\Sigma \subset \mathcal{B}_6$ the complex surface wrapped in $\mathcal{B}_6$.

[10] $\mathcal{L}_Y$ defines an embedding $\mathfrak{u}\left(1\right)_Y \hookrightarrow \mathfrak{su}\left(5\right)$. The commutant of $\mathfrak{u}\left(1\right)_Y$ does not obtain masses from the *Dirac-Born-Infeld*-action. A mass for the $\mathrm{U}\left(1\right)_Y$ gauge boson can be induced from the *Chern-Simons* action. An explicit study shows that this mass vanishes iff $c_1\left(\mathcal{L}_Y\right) \in H^{1,1}\left(\Sigma\right) - \iota^* H^{1,1}\left(\mathcal{B}_6\right)$.

more background). In [97, 105], we made use of these implementations and analyzed zero modes in *F-Theory* GUTs. While our results could not satisfy the experimental requirements, we gave the first examples of jumps in massless spectra of *F-theory vacua*.[11]

3.2 Promonoidal structures in Freyd categories

As outlined above, the computations necessary to determine massless spectra in *F-theory* are very challenging. The motivation for [123] was motivated by the desire for further improvements for the algorithm in [112]. The latter models a coherent sheaf $\mathcal{F}$ on a toric space X_Σ by an object F in the Freyd category $\mathbf{A}(S)$ [114, 115] associated to the Cox ring S of X_Σ. The sheaf cohomology $H^0(X_\Sigma, \mathcal{F})$ is computed by $H^0(X_\Sigma, \mathcal{F}) \cong \underline{\mathrm{Hom}}_S(I, F)_0$ where I is a suitable other object of $\mathbf{A}(S)$, $\underline{\mathrm{Hom}}_S$ denotes the so-called *Internal-Hom* of this category and the subscript 0 indicates the truncation to degree 0. The *Internal-Hom* is part of the *monoidal structure* of $\mathbf{A}(S)$. Consequently, the desire for quick determination of sheaf cohomologies sparks an interest in *monoidal structures* in *Freyd categories*.

In [123] we show that such *monoidal structures* derive from *pro-monoidal structures*. This insight allows us to formulate a constructive approach to tensor products in the category of finitely presented functors. We argue that our construction is equivalent to the restriction of the so-called *Day convolution of arbitrary functors* to finitely-presented functors [124, 125]. The software packages [116–118, 126] were subsequently remodeled and are now available in [127]. Accordingly, also [112] has been completely remodeled.

3.3 Systematics of jumps in exact massless spectra of *F-theory*

The geometry of Y_4 and choice of gauge field A determine the matter curves $C_{\mathbf{R}}$ and line bundles $L_{\mathbf{R}}$. As we vary the complex structure Y_4, it is expected that jumps in the massless spectra occur. Such jumps are crucial to match the experimental data – e.g. to accommodate the Higgs field at non-generic complex structure – we conducted a systematic study of such jumps in [128]. To simplify the computations, our study was inspired by an *F-theory* SU(5)-GUT model. We considered a D7 GUT-brane wrapped on $\mathcal{E} \times \Sigma$, where the complex surface $\Sigma \subset \mathcal{B}_6$ was chosen as a del Pezzo surface $\Sigma = dP_3$. We further focused on a gauge field $A \in H^4_D(Y_4, \mathbb{Z}(2))$ such that $L_{\mathbf{R}} \in \mathrm{Pic}(C_{\mathbf{R}})$ on the hypersurface matter curve $C_{\mathbf{R}} \subseteq dP_3$ was the restriction of a line bundle on dP_3 to $C_{\mathbf{R}}$.

This setup could not satisfy the experimental constraints, but is computationally rather simple. By employing the algorithms in [112], we therefore managed to generate a large database [129], which we then analyzed with decision trees – an interpretable machine learning tool. We found that the majority of jumps could be predicted from factorization of the curve $C_{\mathbf{R}}$. For example, we notice that once a $\mathbb{P}^1$ factors-off, there is a good chance that the vector-like spectrum increases. Along this rational, we engineered an additional vector-like pair in a geometry which is – with the current technology – arithmetically almost inaccessible. Our database also contains jumps for which the curve remains smooth

[11]The study in [97, 105] was facilitated by modelling *F-theory* gauge potentials with elements of the Chow group $\mathrm{CH}^2(Y_4, \mathbb{Z})$. While putting the finishing touches to [97], we found that (local) anomalies in *F-Theory* follow similar rules and can be understood from Chow groups of Y_4 [122].

and irreducible. Those are Brill-Noether jumps [87] – the line bundle divisor moves into a fairly non-generic configuration so that it admits additional sections.

A Koszul resolution of $L_{\mathbf{R}}$ allowed us to analytically determine the cohomologies as the (co)kernel of a complex-structure valued matrix [128]. By analyzing this matrix, we stratified the parameter space of $C_{\mathbf{R}}$ with respect to the massless spectra and verified that jumps arise from a combination of curve factorization and Brill-Noether jumps.

3.4 (Limit) Roots in the F-theory QSMs

Recently, one quadrillion (10^{15}) *F-theory* vacua were found which have many attractive physical features for *standard models* (e.g. gauge coupling unification and no chiral exotics) [60] [12] The vector-like spectra for these compactifications were not identified in [60]. Since only the G_4-flux was specified – it is sufficient to determine the chiral index [51–53, 74, 94, 130–134] but does not uniquely determine the *F-theory* gauge field A – we only found the fairly non-trivial and necessary constraint that $L_{\mathbf{R}}$ is a root bundle [135]. For the geometries in [60], the triple intersection number $\overline{K}_{\mathcal{B}_6}{}^3$ of the anticanonical class of $\mathcal{B}_6$ satisfies $\overline{K}_{\mathcal{B}_6}{}^3 \in \{6, 10, 18, 30\}$ and there are five matter curves:[13]

curve	genus	root bundle constraint	h^i constraint
$C_{(\mathbf{3},\mathbf{2})_{1/6}}$	$\frac{\overline{K}_{\mathcal{B}_6}{}^3+2}{2}$	$P_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes\left(6+\overline{K}_{\mathcal{B}_6}{}^3\right)}$	$(3,0)$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}}$	$\frac{9\overline{K}_{\mathcal{B}_6}{}^3+2}{2}$	$P_{(\mathbf{1},\mathbf{2})_{-1/2}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\mathbf{1},\mathbf{2})_{-1/2}}^{\otimes\left(4+\overline{K}_{\mathcal{B}_6}{}^3\right)} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30\cdot Y_1)$	$(3,0)\oplus(1,1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}$	$\frac{\overline{K}_{\mathcal{B}_6}{}^3+2}{2}$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}^{\otimes\left(6+\overline{K}_{\mathcal{B}_6}{}^3\right)}$	$(3,0)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}$	$\frac{9\overline{K}_{\mathcal{B}_6}{}^3+2}{2}$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}^{\otimes\left(4+\overline{K}_{\mathcal{B}_6}{}^3\right)} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30\cdot Y_3)$	$(3,0)$
$C_{(\mathbf{1},\mathbf{1})_{1}}$	$\frac{\overline{K}_{\mathcal{B}_6}{}^3+2}{2}$	$P_{(\mathbf{1},\mathbf{1})_{1}}^{\otimes 2\overline{K}_{\mathcal{B}_6}{}^3} = K_{(\mathbf{1},\mathbf{1})_{1}}^{\otimes\left(6+\overline{K}_{\mathcal{B}_6}{}^3\right)}$	$(3,0)$

For a specific choice of $\mathcal{B}_6$, we argue in [135] that $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ admits $2\overline{K}_{\mathcal{B}_6}{}^3$th root $P_{(\mathbf{3},\mathbf{2})_{1/6}}$ of the $(6+\overline{K}_{\mathcal{B}_6}{}^3)$-th power of the canonical bundle of $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ with cohomologies $(3,0)$:

1. We deform $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ into a canonical nodal curve $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$.[14]
2. An expilcit construction proves the existence of a so-called *limit root* $P^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ on $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ with cohomologies $(3,0)$ [141] (see also [142–144]).
3. The number of global sections cannot increase along $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}} \to C_{(\mathbf{3},\mathbf{2})_{1/6}}$ by upper semicontinuity. It cannot decrease neither since it is already minimal on $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$. Hence, the root $P_{(\mathbf{3},\mathbf{2})_{1/6}}$ on $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ with cohomologies $(3,0)$ exists.

[12] See [52, 56, 58, 59] for earlier works in this *F-theory* program.
[13] The Higgs and lepton matter curves coincide in the models in [60].
[14] $\mathcal{B}_6$ is obtained from the desingularization of a toric K3-surface [136] (see also [104, 137]). By classical results such as [138, 139], the Picard lattice of said K3-surface is closely related to $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ [140].

This last step fails for $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$ since we need a *non-minimal* number of global sections to realize the Higgs field. This sparks part of my current research efforts.

3.5 Enumeration of limit root bundles

The techniques in [141] allow constructing limit roots on nodal curves by solving certain combinatorics tasks. To gain deeper insights into the root bundle constraints outlined above, we have thus conducted a systematic study in [140]. A simple estimation shows that even in the simplest geometries, we expect about 10^8 root bundles. Indeed, the resulting combinatorics tasks are huge. To complete them in a finite time, we have therefore developed `C++`-code (available as part of [145]) which – after several performance optimizations and imposing simplifying assumptions – was able to complete this task.

A tremendous simplification arises because, the curve $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$ is identical for many of the geometries in [60], which are obtained from desingularizations of toric K3-surfaces [136,137]. It then turns out the Picard lattice of the K3-surface in question is closely related to the dual graph of $C^{\bullet}_{(\mathbf{3},\mathbf{2})_{1/6}}$. This allowed us to scan over the majority over these quadrillion geometries [60, 146]. Our computations find the largest fraction of root bundles $P_{(\mathbf{3},\mathbf{2})_{1/6}}$ with cohomologies $(3, 0)$ for a particular family of spaces $\mathcal{B}_6$, namely the toric varieties obtained from fine regular star triangulations of the 8-th 3-dimensional reflexive polytope Δ°_8 in the Kreuzer-Skarke list [137].[15] At the current state-of-affairs, these spaces are the most promising base spaces $\mathcal{B}_6$ for *F-theory MSSMs*.

4 Current research efforts

My current and – to the extent that I can anticipate – future research efforts will focus on further elucidating the quest for *F-theory standard models* without vector-like exotics. I believe that we made a lot of progress in this direction over the last years, and I hope to find more insights in the foreseeable future. My current efforts are as follows.

Equidimensional deformations A key towards extending our work [135] to exactly one Higgs field is to understand by how much the cohomologies of limit root line bundles on $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$ differ from those on $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$. From the physics, we expect interactions among fields to happen in the nodes of $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$. This effect is accounted for by the complex structure valued mass matrix M. The physics predicts that the rank $\mathrm{rk}(M)$ tells the difference between the vector-like spectra on $C^{\bullet}_{(\mathbf{1},\mathbf{2})_{-1/2}}$ and $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$. The key challenges are to compute the mass matrix M as pioneered in [148] and recently extended in [149], and then to relate the physics prediction to the corresponding mathematics, e.g. Brill-Noether jumps [87] or *limit linear series* [150–152].

[15] Star means that every simplex in the triangulation contains the origin, fine that every lattice point of Δ° is used as ray generator and regular implies that the resulting variety is projective [103, 147].

Counting limit root bundles on the Higgs curve The genus of $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ is much smaller than $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$ for the *F-theory QSMs* [60]. For example, for bases $\mathcal{B}_6$ with $\overline{K}_{\mathcal{B}_6}{}^3 = 6$, it holds $g(C_{(\mathbf{3},\mathbf{2})_{1/6}}) = 4 < 28 = g(C_{(\mathbf{1},\mathbf{2})_{-1/2}})$. Consequently, we have to analyze 12^8 limit roots on $C^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}$ and 12^{56} on $C^\bullet_{(\mathbf{1},\mathbf{2})_{-1/2}}$ (cf. section 3.4). Clearly, the combinatorics task on $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$ is huge compared to $C_{(\mathbf{3},\mathbf{2})_{1/6}}$. This is the reason why, for simplicity, our analysis in [140] counted limit root line bundles on curves $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ with $g = 4, 6$. Still, the enumeration of limit root bundles on $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$ is crucial to gain further insights in the accommodation of the Higgs field in the *F-theory QSMs* [60]. Therefore, we have refined and extended our algorithm. Significant computational efforts have led to partial results in tandem with our findings in [140]. We are currently investigating further extensions of our computational powers to gain further insights.

Brill-Noether theory for roots on nodal curves Under simplifying assumptions, one can order nodal curves via their dual graphs from "simple" to "involved". Our algorithms in [145] approximate their Brill-Noether theory in that they find a lower bound to the number $n(h^0)$ of root bundle with h^0 global sections.[16] Generally speaking, these computations require a lot of computational resources. Inspired by applications to the F-theory QSMs [60], one may wonder if $n(h^0)$ can be predicted/approximated from the dual graph of the nodal curve in question. It would be interesting to study this question – in spirit similar to [128] – driven both by insights from *machine learning tools* and *analytic/algebraic insights*.

OSCAR Computer Algebra System I have developed software for the computation of vector-like spectra since around 2015. My main focus was on the *ToricVarieties_project* [145], which by now consists of more than 40,000 lines of code. I have also contributed to [111, 113, 155]. According to *github*, I have added/modified almost 1,000,000 lines of code. For comparison, the code for this LaTeX-document consists of less than 300 lines of code.

While [145] is a nice software collection, improvements are necessary. This project was mainly written in the programming language `gap` [156], which is – measured in time-scales of computer sciences – somewhat antique by now. For this reason, I have contributed to the computer algebra system `Oscar` [155] which uses the modern language of `Julia`. I am one of the main contributors on *toric geometry* and hope to refactor the entire functionality of the [145]. Thereby, this code should be modernized and subsequently available to a much broader audience.

[16]By classical results, roughly half of the spin bundles on a smooth curve have even and the other half odd number of global sections [153, 154]. We are trying to refine this classical statement.

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