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Research Statement

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Since my undergraduate studies, *string theory* has fascinated me as a potential unified theory of nature. A key aspect of string theory is its vast array of solutions, known as the *string landscape*. My research focuses on identifying string theory solutions consistent with observed physics or proving their absence. To achieve this, I rely on geometric methods, particularly toric and algebraic geometry. I specialize in constructive and enumerative approaches, using computational tools to identify optimal string theory solutions. A notable contribution is my work on the computer algebra system `OSCAR` (https://www.oscar-system.org/), where I have added or modified over 160,000 lines of code.

My research spans three areas: physics, focusing on *F-theory* (a set of string theory solutions encoded in elliptic fibrations); mathematics, with emphasis on toric and algebraic geometry, and connections to combinatorics, graph theory, and number theory; and computer science, particularly open-source computer algebra systems, data science, and machine learning.

INTRODUCTION TO STRING THEORY

Black holes are extremely massive objects where gravitational forces dominate the surrounding region. However, near their centers, where spacetime is highly curved and the scale is incredibly small, quantum effects become significant. These interactions between quantum mechanics and gravity remain one of the major conceptual challenges in theoretical physics. Motivated by these phenomena, the physics community has long sought a unified description of the four fundamental forces: electromagnetism, the weak and strong nuclear interactions, and gravity. [1–4]. The *standard model of particle physics* provides a unified framework for the first three forces [5, 6], but gravity's integration remains elusive [7]. *String theory* offers a potential solution, proposing that elementary particles are small vibrating strings [8–11].

A consistent quantum description of strings requires a 10-dimensional spacetime $\mathcal{S}$ [8–11], differing from our familiar 4 dimensions (time plus three spatial directions). This is reconciled through compactification, with $\mathcal{S}$ often expressed as $\mathcal{E} \times \mathcal{M}$, where $\mathcal{E}$ is the observable 4-dimensional spacetime and $\mathcal{M}$ a compact 6-dimensional Calabi-Yau manifold. Experimental endeavors have yet to yield evidence for extra spacetime dimensions, leading to the presumption that $\mathcal{M}$ is exceptionally minuscule. The geometry of $\mathcal{M}$ significantly impacts the physics observed in $\mathcal{E}$, and identifying geometries $\mathcal{M}$ to match string theory's predictions with observations remains a critical challenge.

String theory is governed by a functional, where sections of vector bundles represent elementary particles. Minimizing this functional leads to differential equations that dictate particle dynamics. For consistency, each particle must have a "partner," a concept known as supersymmetry. Though central to the theory, supersymmetry remains experimentally unconfirmed. It is worth recalling that the Higgs boson [12, 13] took fifty years to discover, and *gravitational waves* required a century. I remain optimistic that supersymmetry will eventually be validated, e.g. through supersymmetry breaking mechanisms [14, 15].

String phenomenology aims to reconcile string theory with our experimentally observed physics. This involves identifying suitable geometries $\mathcal{M}$ or proving their absence. Enormous efforts in this direction have been undertaken. Broadly, these research efforts diverge into two categories: matching string theory with cosmological

observations and aligning it with particle accelerator results. My research focuses on the latter.
String theory can be defined by five different, yet equivalent, actions [8,9,16]. Early efforts in string phenomenology focused on the $E_8 \times E_8$ heterotic string [17–24] and eventually expanded to type IIA and IIB string theory [25–32]. For these solutions, one assume that a parameter – the *interaction strength* – is small enough to ensure convergence of Taylor series, which are used to approximate solutions to the differential equations resulting from minimizing the action. These solutions are collected termed the perturbative sector of string theory. The first perturbative realization of the minimally supersymmetric standard model (MSSM) – a minimal extension of the empirically-backed standard model of particle physics via supersymmetry – exists in [20, 21] with further insights in [33, 34]. However, many other perturbative models introduce unobserved exotic particles. A prevailing challenge, as noted in [22–24] and detailed in [35], is that some observed particle interactions are omitted or differ significantly from experiments like those at CERN.
Alongside ongoing studies on (perturbative) heterotic line bundle standard models [22–24, 35], significant attention has been given to *F-theory* in probing strongly coupled IIB string theory. F-theory adeptly bridges the gap between geometry and physics [36–38], enriched by techniques from algebraic geometry. Due to its geometric consistency, F-theory supports all particle interactions needed for consistency. Pioneering studies in this realm encompass references such as [39–51]. A milestone in this domain is the discovery of the *Quadrillion F-theory standard models (QSMs)* [52], renowned for their physically appealing properties (global consistency, gauge coupling unification, and absence of chiral exotics). The QSMs stand out as the largest known class (more than 10^{15} geometries) of F-theory standard model solutions with these attributes.

MY PAST CONTRIBUTIONS

Higgs pairs are a vital component in our understanding of particle physics, as underpinned by the Nobel Prize awarded to Higgs and Englert [12, 13]. For instance, a *single* Higgs pair is imperative for the MSSM. To check for alignment among F-theory solutions and the MSSM, it is thus imprudent to compute the total number of Higgs pairs. This leads to the study of vector-like spectra. Those spectra are encode in line bundle cohomology on smooth, irreducible curves of high genus, which among others introduces challenges due to the curve's continuous Picard group. My engagement with vector-like spectra in F-theory is not only rooted in the dire need to investigate this property of F-theory solutions, but also inspired by the rich mathematical tapestry of this topic, which invites the application of modern mathematics and computational tools. My explorations encompass cohomologies of coherent sheaves on toric varieties [44, 48, 50], Freyd categories [53], and machine learning enhancements [54]. More recently, I focused on the F-theory QSMs [55–59], for which cohomologies of root bundles encode the vector-like spectra. Significant progress towards the Brill-Noether theory of root bundles on nodal curves has been made, which serves as upper bounds for the vector-like spectra of the QSMs.
Navigating the computational challenges in F-theory can be arduous. This slows progress, limits the exploration of complex geometries, and presents a steep entry barrier for newcomers. In collaboration with my colleague A. Turner (Virginia Tech, USA), I have initiated a program to enrich the OSCAR computer algebra system [60, 61] with tools specifically designed for F-theory applications [62]. As of now, I have added and modified more than 160,000 lines of code, with significant contributions to its toric geometry functionality [63, 64]. `OSCAR` uses the modern `Julia` programming language, a significant advantage over earlier attempts [65–67]. A notable feature of the `FTheoryTools` is its capability to effortlessly extract and modify geometric constructions from existing literature. Another key feature are the advanced singularity resolution techniques – F-theory is inherently defined on a singular space, but properties of the underlying physics are easiest extracted from a resolution of this singular space. Advancing these developments and establishing `FTheoryTools` as a recognized computational tool in the F-theory community is one of my main future research goals.

FTheoryTools – TACKLING F-THEORY'S COMP. CHALLENGES

To fully appreciate the computational challenges in F-theory, a brief introduction in order. Recall that type IIB string theory, just as the other five perturbative string theories, are defined by a functional, which governs functions – or rather sections of vector bundles – that represent elementary particles. Among those functions – often termed *fields* in physics lingo – the dilaton field ϕ and the RR gauge potentials C_0, C_8 are of ample importance. The dilaton field's significance stems from its linkage to the strength of string interactions [8–11]. Interestingly, C_8's equations of motion (the differential equation for C_8 resulting from minimizing the type IIB action) rule out any non-trivial solutions on a smooth space $\mathcal{M}$. The application of an involution σ to $\mathcal{M}$, i.e. $\mathcal{M} \to \mathcal{B} := \mathcal{M}/\sigma$, leads to a space $\mathcal{B}$ which allows for non-trivial solutions to the C_8 equations of motion. Spaces akin to $\mathcal{B}$ serve as the foundation for type IIB orientifold theories [14, 68, 69]. In such orientifold theories, the dilaton field ϕ can manifest singularities, implying an infinite/undefined value at specific loci, leading to strong string interactions. Such strength contradicts the essential premise of weak interactions in perturbation theory. This necessitates a transition from the perturbative type IIB supergravity description to a non-perturbative framework [70], namely *F-theory* [36]. To this end, the dilaton ϕ and RR gauge field C_0 merge into the axio-dilaton τ:

$$\tau \colon \mathcal{E} \times \mathcal{B} \to \mathbb{C}\,,\ x \mapsto C_0\,(x) + ie^{-\phi(x)}\,.$$

Due to *Lorentz invariance*, τ is constant on $\mathcal{E}$ and a section of a holomorphic $\mathrm{SL}\,(2, \mathbb{Z})$ line bundle over $\mathcal{B}$ [37, 38]. For any $x \in \mathcal{B}$, we may thus understand $\tau(x)$ as the complex structure modulus of an elliptic curve and an elliptically fibered 4-fold $\pi \colon Y_4 \twoheadrightarrow \mathcal{B}$ with fibre $\mathbb{C}_{1,\tau(x)}$ can serve as "book-keeping" device of the axio-dilaton. The geometry of Y_4 enforces consistency in that it ensures that the equation of motion for C_8 has a solution and leads to the encoded axio-dilaton field τ. In addition, Y_4 encodes much information about the underlying physics [71–73]. Non-trivial physics necessitates a singular Y_4. In need for better alternatives, it is common to crepantly resolve Y_4 [74]. My past contributions to F-theory assume that, up to $\mathbb{Q}$-factorial terminal singularities, at least one such crepant resolution $\widehat{Y}_4$ does exist. Furthermore, I assume that $\widehat{Y}_4$ admits at least one smooth section. For F-theory without section, see for instance [75–78].

Goals and Features of `FTheoryTools`

In F-theory setups, the initial geometric challenge is the crepant resolution of Y_4. A comprehensive algorithm is still missing, especially in determining $\mathbb{Q}$-factorial terminal singularities. Typically, we apply the entire toolkit to singularities, presuming absence of a crepant resolution when those standard methods fail. Hence, incorporating state-of-the-art resolution routines, for instance including the weighted blowups explored in [79], is paramount. An equally significant feature is a database to automatically utilize established literature constructions, including the set of known resolutions.

Upon resolution, the ensuing step involves examining the given geometry using topological tools, notably through the application of pushforward formulae [47, 80]. This technique facilitates the translation of intersection theory computations from the resolved 4-fold $\widehat{Y}_4$ to the base $\mathcal{B}$, often simplifying the calculations and revealing patterns. For instance, it shows that specific physical quantities solely depend on a base intersection number, as seen in F-theory QSMs which hinge on the triple intersection number of the anticanonical class of the base 3-fold. Enhancing the `FTheoryTools` with intersection theory and topological intersection numbers, and venturing beyond the toric regime, presents collaborative prospects.

Incorporating prevalent F-theory methodologies into the `FTheoryTools` offers an excellent avenue for students to delve into advanced research, interact with relevant (computer) geometries, and contribute to our database. Although concise, these interactions hold significant potential for deeper exploration. For instance, similar to [54], this database may be used for machine learning and data science explorations [81]. A case in point is to probe a theory of Brill-Noether numbers or, if unattainable, to investigate a then promising cryptosystem. I will elaborate on this avenue in larger detail below.

The singularities of Y_4 determine a non-abelian (gauge) group. To make contact with the observed physical laws, it is advantageous to augment this group with abelian group factors, as this allows to enforce so-called selection rules. These abelian factors originate from the torsion-free subgroup of the Mordell–Weil group of $\widehat{Y}_4$, which represents the group of infinite-order rational sections of the fibration, governed by elliptic curve addition. Consequently, the rank of the abelian part of the gauge algebra aligns with the Mordell–Weil rank. It is noteworthy that the torsional part of the Mordell–Weil rank relates to the gauge group's global structure in the physical theory. Similarly, exploring F-theory on elliptic fibrations with multi-sections can be pursued, the study of which leads to the Weil–Châtalet group and discrete factors thereof [77].

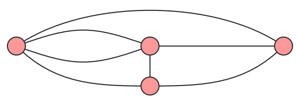
To explore alignment of F-theory solutions with empirical observations in particle physics, it is critical to understand the chiral spectrum [39, 41–43, 45–47, 80, 82]. It remains to implement algorithms into `FTheoryTools`, which facilitate this task, e.g. by use of the renowned `cohomCalg`-algorithm [83–87]. In extending towards vector-like spectra, which count the number of Higgs pairs in a given F-theory solution, foundational investigations into Deligne cohomology and root bundles are necessary.

FROM F-THEORY QSMS TO BRILL-NOETHER NUMBERS AND BACK

The F-theory QSMs [52] provide 10^{15} solutions apt for the standard model of particle physics. My investigation into their vector-like spectra – a crucial ingredient to compare these solutions to experimental findings from particle accelerators – directed me to root bundles on nodal curves. Before we explain how this topic arises from the physics, I wish to first provide a noval introduction to Brill-Noether numbers and subsequently elaborate on the significance of these numbers for F-theory.

Brill-Noether Numbers – A Noval Introduction

Root bundles generalize spin bundles. We recall that there are 2^{2g} spin bundles on a smooth, irreducible genus g curve, each of which corresponds to a divisor class D with $2 \cdot D = K_C$, where K_C is the canonical bundle. For root bundles, we consider divisors D with $r \cdot D = E$, $r \in \mathbb{Z}_{\geq 2}$, and E not necessarily the canonical bundle, yielding r^{2g} roots, if existent. Nodal curves $C^\bullet$ bring two nuances: they typically have multiple irreducible components due to nodal singularities, and their root bundles can be enumerated as limit roots [88] (cf. [89–91]). Here is the dual graph (vertices are irreducible components and edges nodal singularities) of a nodal curve for which all irreducible components are $\mathbb{P}^1$s:



(1)

Our objective is to enumerate limit roots $P^\bullet$ with $12 P^\bullet = 12 K_{C^\bullet}$ and discern their global section count. As we shall motivate below, this count of global sections filters F-theory solutions potentially aligned with experimental findings. Here, there are 12^8 limit roots, of which 12^4 roots have $h^0(C^\bullet, P^\bullet) = 4$ and the rest three global sections:

Roots Count	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
12^8	$12^4 \cdot (12^4 - 1)$	0	12^4	0

(2)

Although the number of global sections for some roots $P^\bullet$ cannot be uniquely determined, an optimal lower bound can be computed, as shown in every second column of the previous table for more complex examples. This leads to the partition:

$$12^8 = 12^4 \cdot \left(12^4 - 1\right) + 0 + 12^4 + 0\,. \tag{3}$$

The order of the summands is crucial, and summands can reappear. Our technology recently culminated in [58], computing in a certain sense an optimal partition. This optimal partition is likely to bear a deeper meaning and definitely carries a striking resemblance to the Brill-Noether theory for line bundles on smooth, irreducible curves [92]. This leads me to dub the summands *Brill-Noether numbers*. These numbers can be linked to the results in the works [93, 94], which showed that about half of the spin bundles on a genus g curve have an even number of global sections, though little is known beyond this.

Spin Bundle: A Critical Piece in F-theory's Puzzle

Vector-like spectra are essential in analyzing the matter fields in string theory solutions. Those matter fields include the Higgs pair [12, 13], a salient component highlighted by the Nobel Prize awarded to Higgs and Englert. The matter fields manifest as section of line bundles over *matter curves* $C_{\mathbf{R}} \subset \mathcal{B}$ [95–97]. The matter curves and the group representations of the matter fields are in turn determined by Y_4's singularities [73]. The sections, which represent the matter fields, are determined by gauge fields. The latter are described by the Deligne cohomology $H^4_D\left(Y_4, \mathbb{Z}\left(2\right)\right)$ [98–104]. The results in [44] imply that these gauge fields induce a line bundle $\mathcal{L}_{\mathbf{R}}$ on $C_{\mathbf{R}}$. Classical results then lead to the formulae [105, 106]:

- $\mathcal{N} = 1$ chiral multiplets are elements of $H^0\left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.
- $\mathcal{N} = 1$ anti-chiral multiplets are elements of $H^1\left(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}\right)$.[1]

On a curve of genus g, the selection of the appropriate spin bundle $\sqrt{K_{C_{\mathbf{R}}}}$ from 2^{2g} possibilities significantly influences the number of sections of $L_{\mathbf{R}} = \mathcal{L}_{\mathbf{R}} \otimes \sqrt{K_{C_{\mathbf{R}}}}$. Identifying the correct spin bundles is essential but complex. For instance, [96] shows that the Freed-Witten anomaly cancellation [107] requires spin^c-structures on gauge surfaces $S \subset B_3$ in F-theory GUTs. Recent advances, such as [59], underscore the necessity of addressing this question in future research endeavors.

Brill-Noether Numbers: Upper Bound to Vector-Like Spectra of F-theory QSMs

The pioneering work on F-theory QSMs [52] do not investigate the vector-like spectra of these F-theory solutions. Amidst the intricacies of the spin bundle, we deduced a pivotal constraint: the line bundle $L_{\mathbf{R}}$ is a specific root bundle $P_{\mathbf{R}}$ [55]. On the quark-doublet curve $C_{(\mathbf{3},\mathbf{2})_{1/6}}$, this root bundle satisfies $P_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 2\overline{K}_{\mathcal{B}}^3} = K_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes(6+\overline{K}_{\mathcal{B}}^3)}$, where $\overline{K}_{\mathcal{B}}^3 \in \{6, 10, 18, 30\}$ is the triple self-intersection number of the F-theory base's anticanonical class. A limited number of all solutions to this constraint will arise from an apt choice of the spin bundle. Instead of pinpointing this correct choice, we took a statistical approach: First, enumerate all solutions to the root bundle constraint, then count the global sections of all roots, and finally determined probable F-theory geometries devoid of absence of exotic vector-like quark-doublets. Still, enumerating all roots and working out their global section count is close to impossible for smooth, irreducible curves of genus $g > 2$. Indeed, for the quark-doublet curve $2g = \overline{K}_{\mathcal{B}}^3 + 2$. Fortunately, Brill-Noether numbers offer a practical upper bound:

1. Deform $C_{(\mathbf{3},\mathbf{2})_{1/6}}$ into $C^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}$, which is shared across various geometries [52] due to their origin from toric K3-surfaces desingularizations [56], cf. [108–110]. A computer scan of the majority of the 10^{15} QSM geometries ensues [52, 111].
2. Employ techniques from [88] (cf. [89–91]) to list all *limit root* $P^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}$ with our software [112].
3. Compute the global sections of each limit root $P^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}$ with the techniques developed in [55–57], which recently culminated in an optimal approach [58]. This leads to the Brill-Noether numbers introduced above.
4. If $h^0(C^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}, P^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}) > \chi(P_{(\mathbf{3},\mathbf{2})_{1/6}})$, the number of global sections may decrease as we smooth $C^\bullet_{(\mathbf{3},\mathbf{2})_{1/6}}$ into $C_{(\mathbf{3},\mathbf{2})_{1/6}}$. Hence, the Brill-Noether numbers serve as upper bound to the desired numbers on $C_{(\mathbf{3},\mathbf{2})_{1/6}}$.

Queston 1: Towards a Theory of Brill-Noether Numbers

Even though the computation of the Brill-Noether numbers are resource-intensive, we currently use them as upper bound to the F-theory QSMs' vector-like spectra. A deeper understanding of the link between a nodal curve and these numbers is desired. One may posit if the Brill-Noether numbers can be inferred from the nodal curve's dual graph and the root bundle constraint. Employing both machine learning tools and analytic/algebraic insights can be beneficial in this pursuit, mimicking efforts in [54]. If the systematics are revealed, it opens doors for analogous analyses on the intricate Higgs curve of the F-theory QSMs. To illustrate the Higgs curve's complexity, consider a base $\mathcal{B}$ with $\overline{K}_{\mathcal{B}}^3 = 6$. Then, $g(C_{(\mathbf{3},\mathbf{2})_{1/6}}) = 4$, while $g(C_{(\mathbf{1},\mathbf{2})_{-1/2}}) = 28$. Not only are there many more limit roots to be enumerated, but also the dual graph is more complicated.

Question 2: An F-Theory-Inspired Cryptosystem?

For a given integer partition, can we identify a graph and a root bundle constraint, such that the ensuing Brill-Noether numbers match exactly with the initial partition? The inverse of this question is feasible with our existing methods, but computationally extremely demanding. Attempting the direct approach seems daunting. This disparity raises the possibility of unveiling a new cryptosystem: A promising avenue for future studies.

[1] Supersymmetric theories only contain chiral superfields. "Anti-chiral" refers to fields in the charge conjugate representation $\overline{\mathbf{R}}$.

Question 3: Jumps Meet Yukawa Interactions

A Higgs pair [12, 13] is present in an F-theory solution exactly if

$$h^0\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, P_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = h^1\left(C_{(\mathbf{1},\mathbf{2})_{-1/2}}, P_{(\mathbf{1},\mathbf{2})_{-1/2}}\right) = 1\,. \tag{4}$$

However, as $h^0(C^\bullet_{(\mathbf{1},\mathbf{2})_{1/6}}, P^\bullet_{(\mathbf{1},\mathbf{2})_{1/6}}) > \chi(P^\bullet_{(\mathbf{1},\mathbf{2})_{1/6}})$, the counts on $C^\bullet_{(\mathbf{1},\mathbf{2})_{1/6}}$ are merely upper bounds and jumping phenomena may occur, just as in classical Brill-Noether theory [92, 113]. To refine the analysis, we must understand the differences between the global section counts on $C^\bullet_{(\mathbf{1},\mathbf{2})_{-1/2}}$ and $C_{(\mathbf{1},\mathbf{2})_{-1/2}}$. The physics expects interactions at the nodes of $C^\bullet_{(\mathbf{1},\mathbf{2})_{-1/2}}$, which may be summarized in a so-called mass matrix M, whose rank is expected to be the difference in global section counts. The challenge lies in computing M [114, 115] and aligning with e.g. the mathematics of *limit linear series* [116–118].

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