Department of Physics and Astronomy
University of Heidelberg

Cohomologies Of Holomorphic Pullback Line Bundles In Smooth And Compact Normal Toric Varieties

Master Thesis in Physics
submitted by

Martin Bies

2014

This Master Thesis has been carried out by
Martin Bies
at the Institut für Theoretische Physik
under the supervision of
Apl. Professor Dr. T. Weigand

Cohomologies Of Holomorphic Pullback Line Bundles In Smooth And Compact Normal Toric Varieties

Both in type IIB orientifold compactifications with D3- and D7-branes as well as in F-theory model building the need to compute the spectrum of localised zero modes arises. This task can be formulated as the computation of cohomologies of holomorphic pullback line bundles. In this thesis we present the Koszul spectral sequence and establish it as the optimal way to compute these cohomology groups. In particular we present a *Mathematica* notebook that computes a first approximation of the Koszul spectral sequence. Extending this notebook to compute the entire spectral sequence, to improve the used algorithms and to apply it to model building is reserved for future work.

Kohomologien Holomorpher Rückzugsgeradenbündel In Glatten Und Kompakten Normalen Torischen Varietäten

Sowohl in Type IIB-Orientifold Kompaktifizierungen mit D3- und D7-Branen als auch im F-Theory Modelbau besteht die Notwendigkeit das Spektrum der lokalisierten Nullmoden zu berechnen. Diese Aufgabe kann formuliert werden als die Berechnung der Kohomologiegruppen von holomorphen Rückzugsgeradenbündeln. In dieser Arbeit präsentieren wir die Koszul'sche Spektralsequenz und etablieren diese als den optimalen Zugang zur Berechnung dieser Kohomologiegruppen. Insbesondere stellen wir ein *Mathematica* Notebook vor, welches eine erste Approximation der Koszul'schen Spektralsequenz berechnet. Die Erweiterung dieser Notebooks zur Bestimmung der gesammten Spektralsequenz, die Verbesserung der verwendeten Algorithm und die Anwendung des Notebooks im Modelbau sind für zukünftige Arbeiten vorgesehen.

Contents

Part I.

A Motivation From Physics

1. Why Cohomologies Of Pullback Line Bundles?

1.1. From String Theory To F-Theory

Superstring theory is one of the most promising candidates for a unified description of the standard model and gravity [1, 2, 3, 4]. In taking superstring theory seriously, one approach to obtain the standard model from perturbative string theory is to consider heterotic string theory models [5]. Another well understood perturbative string theory corner is the perturbative type IIA string theory. There one can consider the setup of intersecting D6-branes in type IIA superstring theory [6, 7, 8, 9]. In these constructions the standard model gauge group $SU(3) \times SU(2) \times U(1)$ can be implemented very easily and in the past years many such models have been constructed [10, 11, 12]. However, to date we are still missing the perfect model describing our universe from such a construction.
Yet another well understood perturbative string theory corner is type IIB string theory. The latter is very similar to the case of intersecting D6-branes in type IIA in that the standard model gauge group can be easily implemented in constructions with D3 and D7 branes on the internal Calabi-Yau space X_3 [8, 9, 13]. We denote the associated orientifold by B_3, so that B_3 is not a Calabi-Yau manifold in general. Whilst the gauge group picture is very similar in the type IIA and type IIB construction, there is also a crucial difference - the perturbative description generically breaks down in the type IIB situation [14]. Consequently one has to turn to non-perturbative techniques to describe such models. F-theory provides such a means in that the varying axio-dilaton from the type IIB picture is identified with the complex structure modulus of an elliptic curve [15, 16, 17, 18]. This elliptic curve is then fibred over the three complex dimensional internal space B_3. Thereby the internal space X_3 is replaced by the elliptically fibred 4-fold Y_4 in F-theory.
In conclusion, turning to type IIB for model buidling naturally leads to the study of elliptic fibrations. A general feature of such fibrations is that they become singular whenever the fibred elliptic curve degenerates. On a first glance these singularities might look worrying and one might be tempted to restrict to F-theory models without singularities to begin with. Surprisingly, the picture is precisely opposite. The singularities of the elliptically fibred internal 4-fold Y_4 encode the gauge degrees of freedom [14]. So focusing on non-singular F-theory models is not an option for model building. In addition, the singularity structures allow to implement more general gauge groups than available in type IIB constructions [19, 20, 21, 22]. For example E_8 gauge the-

ories can be achieved in this way.
F-theory thus allows to handle non-perturbative type IIB string dynamics and extends the accessible gauge groups. In particular one can therefore use F-theory for GUT model building [21, 19, 20, 22, 23]. Thereby one makes use of the fact that the singularities of the elliptically fibred 4-fold Y_4 are in one-to-one correspondance with the gauge groups along the 7-branes in the type IIB setup, the localised matter fields at the 7-brane intersection curves as well as the Yukawa interactions between these localised modes [24, 25, 21, 19, 20]. This makes F-theory an interesting setup for GUT model building.
In most of the F-theory model building, the ellitpically fibred 4-fold $Y_4 \overset{\pi}{\to} B_3$ is required to admit at least one global section in order to ensure that we can identify B_3 with a suitable subset of Y_4 [1]. This restricts the elliptic fibration to be of Weierstrass form up to birational equivalence [27, 28]. Hence the elliptic fibre can be described as hypersurface in $\mathbb{CP}_{231}$ parametrised by two complex parameters f, g

$$\mathcal{C}\left(f,g\right) \coloneqq \left\{ [x,y,z] \in \mathbb{CP}_{2,3,1} \, , \; y^2 - x^3 - fxz^4 - gz^6 = 0 \right\} \tag{1.1}$$

Therefore f and g describe the shape of the elliptic curve. They are related to the complex strucutre modulus of $\mathcal{C}\left(f,g\right)$ via a special modular function, termed the j-invariant [28, 29]

$$j\left(\tau\right) = \frac{4 \cdot \left(24 f\right)^3}{\Delta\left(f,g\right)} \tag{1.2}$$

In the elliptic fibration, f and g are promoted to global sections of $\overline{K}_{B_3}^{\otimes 4}$ and $\overline{K}_{B_3}^{\otimes 6}$, so that they become functions of the coordinates u_i of the base space B_3. The singularities of the elliptic fibration are then found to lie over the following set

$$\Delta \coloneqq \left\{ u \in B_3 \, , \; 27 g^2\left(u\right) + 4 f^3\left(u\right) = 0 \right\} \subset B_3 \tag{1.3}$$

To analyse this singular locus further one has to resolve it. For this reason we require that a smooth resolution $\widehat{Y}_4$ exists in which the singular fibres are replaced by suitable chains of $\mathbb{CP}^1$s. Over the irreducible components Δ_i of Δ, the intersection structure of the $\mathbb{CP}^1$s correspond to an affine Dynkin diagram of an affine Lie algebra $\mathfrak{g}_i$. Vector multiplets propagating on Δ_i and transforming in the adjoint representation of the associated Lie group G_i finally encode the gauge and matter degrees of freedom [14, 30].
Recall that we required B_3 to be non-Calabi-Yau. The reason behind this can be seen as follows. First of all it can be shown that for an elliptic fibration $Y_4 \overset{\pi}{\to} B_3$ it holds [31]

$$c_1\left(T_{Y_4}\right) \cong \pi^* \left[c_1\left(T_{B_3}\right) - \sum_i \frac{\delta_i}{12} \left[\Gamma_i\right] \right] \tag{1.4}$$

In this expression the Poincaré dual of the irreducible components Δ_i of the singularity locus are denoted by $[\Gamma_i]$ and the vanishing order of Δ along Δ_i is denoted by

[1] In [26] F-theory on elliptically fibred 4-folds without such a section is discussed.

δ_i. Consequently Y_4 is a Calabi-Yau manifold precisely if

$$\sum_i \delta_i \left[\Gamma_i\right] = 12 c_1 \left(T_{B_3}\right) \tag{1.5}$$

This strongly resembles Tadpole cancellation conditions in type IIB-theory. This relation can be made more precise [14] and leads to the conclusion that the tadpole cancellation conditions from the type IIB picture are reflected in a well-defined geometry in F-theory. In particular note that if B_3 was a Calabi-Yau manifold, i.e. $c_1\left(T_{B_3}\right) = 0$, we would find from Y_4 being a Calabi-Yau manifold as well, that Δ must not vanish to any order on any irreducible component of B_3. This in turn would lead only to trivial gauge dynamics as mentioned above. As the latter is not of interest for model building, the case that B_3 is a Calabi-Yau manifold is to be excluded.
In addition to these gauge symmetries we can enforce additional $U(1)$ gauge symmetries. To achieve this let us recall that the elliptic fibration $Y_4 \xrightarrow{\pi} B_3$ is required to have one global section, so that we can identify B_3 with a suitable subset of Y_4. Suppose now that we specialise the Weierstrass form of the elliptic fibre further, such as to ensure the existence of a second global section. Then this additional section gives rise to a new divisor class of the elliptic fibration. Poincaré duality relates this class to a $(1,1)$-form w. The duality between F- and M-theory [30] finally allows to expand the 3-form potential C_3 according to $C_3 = A \wedge w + \ldots$. In this expression A is identified as the gauge potential of the new $U(1)$ gauge symmetry [17, 18, 32]. Such constructions have been studied extensively in F-theory model building [33, 34, 35, 36].
The interest in such constructions is based on the study of the G_4-flux associated to the M-theory 3-form potential C_3. This G_4-flux should be treated at the same footing as the field strength in gauge theories. The field strength however does not provide all data about the gauge theory, rather the gauge field does so. Consequently the 'gauge field' for the G_4-flux in the F-theory construction is what one should look for. As proposed in [37] this object should be identified with an algebraic cycle $A \in Z^2(Y_4)$. An interesting long term objective is to investigate the validity of this proposal by applying it to the F-theory model building. To this end the ability to compute cohomologies of pullback line bundles is needed as we will point out momentarily.

1.2. Gauge Data In Type IIB Orientifold Compactifications

Before we explain this reasoning further, let us first recall how gauge data is specified in type IIB orientifold compactifications with stacks of $D7$-branes. We denote the holomorphic 4-cycles in X_3 that are wrapped by the D7-branes by D_i. As pointed out in [38], the gauge data for such a setup is specified by a family $\{\mathcal{F}_i\}$ of derived and bounded coherent sheaves on the holomorphic 4-cycles D_i. Note also that the existence of charged open string zero modes with non-zero vacuum expectation value between the stacks on D_i and D_j is reflected in non-trivial sheaf homomorphisms

$\alpha_{ij}: \mathcal{F}_i \to \mathcal{F}_j$. To make contact with the Abelian $U(1)$ gauge symmetries described above, let us simplify this setup by making the following three assumptions.

- $\alpha_{ij} \equiv 0$. This means that all charged open string zero modes between the D7-branes are massless.

- We restrict to Abelian gauge data.

Given these simplifications one would think that the gauge data is now specified by holomorphic line bundles $\mathcal{L}_i$ on the holomorphic 4-cycles. However in [39] it was shown that this data is incomplete. To illustrate this let us for a moment consider two D7-brane stacks D_1, D_2 and holomorphic line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on the respective holomorphic 4-cycles. We assume that D_1 and D_2 intersect. Then on $D_1 \cap D_2$ we can consider the holomorphic line bundles $\mathcal{L}_1|_{D_1 \cap D_2}$ and $\mathcal{L}_2|_{D_1 \cap D_2}$. The enhanced gauge theory on the intersection locus would naively be described by $\mathcal{L}_1^\vee|_{D_1 \cap D_2} \otimes \mathcal{L}_2|_{D_1 \cap D_2}$. This however is non-generic and to be more general one would specify how the two line bundles glue along the intersection $D_1 \cap D_2$. This data is provided by a gluing morphism which is a meromorphic map between $\mathcal{L}_1|_{D_1 \cap D_2}$ and $\mathcal{L}_2|_{D_1 \cap D_2}$. Here we prefer to avoid this subtlety. For this reason we assume that all gluing morphisms vanish and consequently the above-presented naive picture does apply. For setups with non-trivial gluing morphisms the interested reader is refered to [40, 41].
In summary, given the above simplifications, the gauge data in a type IIB orientifold compactification with stacks of D7-branes along holomorphic 4-cycles D_i in the compactification space X_3 is given by a family $\{\mathcal{L}_i\}$ of holomorphic line bundles on the 4-cycles D_i. Had we worked with coherent sheaves instead, the localised zero modes at the intersections $C_{ab} = D_a \cap D_b$ would be described in the language of Ext groups [42]. Still the above assumptions pay off a second time, in that this description simplifies to that of cohomology classes of line bundles. By setting $L_{ab} = \mathcal{L}_a^\vee|_{C_{ab}} \otimes \mathcal{L}_b|_{C_{ab}}$ we thus find that massless open strings stretched between D_a and D_b are counted by

$$H^i\left(C_{ab}, L_{ab} \otimes \sqrt{K_{C_{ab}}}\right) \quad i = 0, 1 \tag{1.6}$$

We require C_{ab} to be a compact and connected Riemann surface. As proven in [43] the spin structures on a compact complex manifold correspond bijectively to the isomorphism classes of holomorphic line bundles S with $S^{\otimes 2} \cong K_{C_{ab}}$ with $K_{C_{ab}}$ the canonical bundle of C_{ab}. By use of this non-trivial statement, we can shematically state that $\sqrt{K_{C_{ab}}}$ is a spin bundle on C_{ab}. Unfortunately however, it was also proven in [43] that on a compact and connected Riemann surface of genus g there are 2^{2g} inequivalent spin structures. This we exemplify in section C.4 where we describe the 4 different spin structures on $\mathbb{C}_{1,\tau}$.
In conclusion we have to choose a spin bundle S on C_{ab}. A canonical choice is made by use of the embedding of $C_{ab} \hookrightarrow D_a$ [19, pp. 58]. By means of the adjunction formula [44] it holds

$$K_{C_{ab}} = K_{D_a}|_{C_{ab}} \otimes N_{C_{ab}/D_a} \tag{1.7}$$

In addition we have since X_3 is a Calabi-Yau manifold

$$K_{D_a}|_{C_{ab}} = \left(K_{X_3} \otimes \mathcal{O}_{X_3}(D_a)\right)|_{C_{ab}} = \mathcal{O}_{X_3}(D_a)|_{C_{ab}} \qquad N_{C_{ab}/D_a} = \mathcal{O}_{X_3}(D_b)|_{C_{ab}} \quad (1.8)$$

The Freed-Witten-quantisation condition now ensure that the bundle

$$\widetilde{L}_{ab} = \mathcal{O}^{\vee}_{X_3}(D_a) \otimes \mathcal{O}_{X_3}(D_b) \otimes \mathcal{O}_{X_3}\left(\frac{1}{2}\left[D_a + D_b\right]\right) \quad (1.9)$$

is well defined. In conclusion, massless strings between D_a and D_b are counted by $H^i\left(C_{ab}, \widetilde{L}_{ab}|_{C_{ab}}\right)$, i.e. the cohomology classes of a pullback line bundle.
In the mathematics literature there are many very elegant theorems describing cohomologies of holomorphic line bundles. For a holomorphic line bundle $\mathcal{L}$ on a compact and connected Riemann surface M_g of genus g, it holds [44]

$$h^0(M_g, \mathcal{L}) - h^1(M_g, \mathcal{L}) = \deg(\mathcal{L}) - 1 + g = \int_{M_g} c_1(\mathcal{L}) \quad (1.10)$$

The first equality is known as the theorem of Riemann and Roch. Apart from Koidaira vanishing and Serré duality [44] [45] the above result is the most important result and can be used to constrain the chiral index of $\mathcal{L}$. This result also shows that the chiral index is only sensitive to the first Chern class of $\mathcal{L}$. In particular note that the chiral index cannot tells us the dimension of the individual cohomology groups $H^i(\mathcal{L})$ but only their difference, as we see from the second equality in Equation 1.10. Still these dimensions count the massless zero modes between the stacks D_a and D_b and are what the model builder is really looking for. We conclude that index theorems are not enough to determine the spectrum at the intersection locus C_{ab}. Rather additional techniques need to be used.

1.3. Gauge Data In Global Tate-Models In F-theory

We mentioned already that the singularity structure of $Y_4 \xrightarrow{\pi} B_3$ encodes the gauge data in an F-theory model. Therefore one is interested in studying this singularity structure in detail. Such an analysis starts locally. Let us be more precise here an state that this means that for a point $p \in B_3$ there exists an open neighbourhood $p \in U \subset B_3$ such that for every $\widetilde{p} \in U$ application of the Tate algorithm allows to express the Weierstrass polynomial at $\widetilde{p}$ as the so-called Tate polynomial [24]

$$P_W(x,y,z,\widetilde{p}) = x^3 - y^2 + xyza_1(\widetilde{p}) + x^2z^2a_2(\widetilde{p}) + yz^3a_3(\widetilde{p}) + xz^4a_4(\widetilde{p}) + z^6a_6(\widetilde{p}) \quad (1.11)$$

Note that in this expression $a_i \in \overline{K}^{\otimes i}_{B_3}(U)$. So over U the elliptically fibred 4-fold Y_4 is obtained by fibering the elliptic curves

$$\mathcal{C}(\widetilde{p}) = \{[x,y,z] \in \mathbb{CP}_{231}\ ,\ P_W(x,y,z,\widetilde{p}) = 0\} \quad (1.12)$$

over U. The advantage of the Tate polynomial is, that from it $\Delta \cap U$ can be easily read off. A summary of the so-obtained local singularity classification can be found in [24, Table 2].
A special class of F-theory models are the so-called global Tate-models. In those models on describes the fibration $Y_4 \overset{\pi}{\to} B_3$ not only locally, but globally by a Tate polynomial. Then the local classification of singularities gives the global singularity classification and one can easily read of the gauge degrees of freedom in the model.
Let us exemplify this construction on an $SU(5) \times U(1)_X$-model along the hypesurface

$$S = \{p \in B_3\ ,\ w(p) = 0\} \subset B_3 \tag{1.13}$$

where w is a holomorphic function. To obtain such a model, which is also refered to as a $U(1)$-restricted model, we now take the sections $a_i \in \overline{K}_{B_3}^{\otimes i}$ to have the following form [32, 46]

$$a_2 = a_{2,1} \cdot w, \quad a_3 = a_{3,2} \cdot w, \quad a_4 = a_{4,3} \cdot w, \quad a_6 \equiv 0 \tag{1.14}$$

and require that $a_{2,1}$, $a_{3,2}$ and $a_{4,3}$ are not divisiable by w in the ring $\mathcal{O}_{B_3}(B_3)$. Given this setup we consider the following curves.

- $C_{10} \coloneqq \{p \in B_3\ ,\ w(p) = a_1(p) = 0\}$
- $C_{\overline{5}m} \coloneqq \{p \in B_3\ ,\ w(p) = a_{3,2}(p) = 0\}$
- $C_{5H} \coloneqq \{p \in B_3\ ,\ w(p) = a_{4,3}(p) \cdot a_1(p) - a_{3,2}(p)\, a_{2,1}(p) = 0\}$

Due to the gauge enhancement new matter transforming in the $\mathbf{10}$, $\overline{\mathbf{5}}$ and $\mathbf{5}$ representation appear localised on the curves C_{10}, $C_{\overline{5}m}$ and C_{5H} [14], which are at the same time charged under the additional $U(1)_X$ symmetry. The above representations of $SU(5)$ can in turn be split into the following representations of $SU(3) \times SU(2)$

- $\mathbf{10} \to (\mathbf{3}, \mathbf{2}) + (\overline{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})$
- $\overline{\mathbf{5}} \to (\overline{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$
- $\mathbf{5} \to (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$

Those of course quality for interpretation in terms of standard model particles. The only exception to this statement is the exotic $(\mathbf{3}, \mathbf{1})$. To rule this one out, suitable fluxes have to be used [30]. We will not discuss this any further but rather focus on the additional $U(1)_X$ symmetry.
This $U(1)_X$-symmetry corresponds to an additional divisor in the elliptic fibration. Poincaré duality relates this additional divisor to a $(1,1)$ form w, so that one can expand the M-theory 3-form potential C_3 according to $C_3 = A \wedge w + \ldots$. The A in this expression is then identified as the gauge potential of the additional $U(1)_X$-symmetry [17, 18, 32]. This $U(1)_X$ symmetry can be described by a holomorphic line bundle $\mathcal{L}$ on the GUT [37] and is oftentimes referred to as a G_4-flux.
The particles on the C_{10}-curve carry charge -1, those on the $C_{\overline{5}m}$-curve charge $+3$ and

the ones on the C_{5H}-curve charge +2 under the $U(1)_X$-symmetry [47]. Consequently the cohomologies of $\mathcal{L}^\vee|_{C_{10}} \otimes \sqrt{K_{C_{10}}}$ count the number of matter states localised on the C_{10}-curve. Similarly the cohomologies of $\mathcal{L}^{\otimes 3}|_{C_{\overline{5}m}} \otimes \sqrt{K_{C_{\overline{5}m}}}$ count the matter states along $C_{\overline{5}m}$ and the cohomologies of $\mathcal{L}^{\otimes 2}|_{C_{5H}} \otimes \sqrt{K_{C_{5H}}}$ the states along C_{5H}. Consequently this brings us back to computing pullback cohomologies.

2. Why Toric Varieties?

In [48] Witten introduced in the context of heterotic string theory a gauged linear sigma model whose vacuum configurations form toric varieties. This is the most prominent appearance of toric varieties in physics. For a nice exposition of this material the interested reader is refered to [49].

Also type II superstring theories can make use of toric varieties once fluxes are introduced. Then namely one wishes to stabilize the moduli of the theory. To this end one has to use complex spaces with $SU(3)$ structure, and toric varieties give a nice means to construct such spaces as outlined in [50].

For our applications however the motivation stemms from the fact that for model building a lot of computations need to be handled. For example intersection products have to be computed. On toric varieties the intersection theory is so well-understood, that it was even implemented in *Sage* [51] [52].

Before we mention yet another attractive feature about toric varieties let us recall that an analytic subvariety of $\mathbb{C}^n$ is a subset of $\mathbb{C}^n$ that is *locally* cut out by the vanishing of a finite number of polynomials. In particular the number of polynomials that locally cut out the subvariety need not be constant. In case it is, one terms such a variety a *pure-dimensional variety.* Note that analytic varieties are very general and play a crucial role in the local theory of complex spaces, which naively can be thought of as complex manifolds with singularites. In particular note that all submanifolds of $\mathbb{C}^n$ are smooth analytic varieties. So handeling all submanifolds of $\mathbb{C}^n$ requires the ability to handle all smooth analytic subvarieties, which for most practial purposes is not the case.

Now let us look at $\mathbb{CP}^n$. We pick homogeneous polynomials $Q_1, \ldots, Q_n$ and then define the associated *algebraic variety* as the set

$$C = \{p \in \mathbb{CP}^n \, , \; Q_1(p) = \cdots = Q_n(p) = 0\} \tag{2.1}$$

Chow's theorem now tells us that in fact any analytic subvariety of $\mathbb{CP}^n$ is an algebraic subvariety. This is a major simplification and should be contrasted to the case of $\mathbb{C}^n$. And this result even generalises to simplicial toric varieties, which includes smooth and compact toric varieties, i.e. those toric varieties that we will focus on during the major part of this thesis [52]. So we conclude the following.

Let X_Σ a smooth and compact normal toric variety and $C \subset X_\Sigma$ a manifold. Then there exist a finite number of homogeneous polynomials $Q_1, \ldots, Q_n$ on X_Σ such that

$$C = \{p \in X_\Sigma \, , \; Q_1(p) = \cdots = Q_n(p) = 0\} \tag{2.2}$$

This major simplification means that we can even in pratice handle every submanifold of X_Σ [1].
Yet another simplification for the case of smooth and compact normal toric varieties is that an isomorphism class of line bundles is uniquely specified by its first Chern class. Physically speaking this means that a $U(1)$ gauge theory on X_Σ is uniquely specified by its field strength. But recall that the first Chern class is only determined up to $\mathcal{C}^\infty$-isomorphisms [44]. This freedom allows us to trade $\mathcal{C}^\infty$-line bundles for holomorphic line bundles. The latter in turn are much easier to handle than their smooth counterparts. This in essence follows from the observation that a holomorphic function is much easier to handle than a smooth function. In particular this means that the computation of cohomology classes simplies considerably. The latter has been used to implement the computation of cohomology classes of holomorphic line bundles on X_Σ in the *cohomCalg* algorithm as well as applications thereof [53, 54, 55, 56].
The ability to handle submanifolds, intersection products and cohomology classes of holomorphic line bundle are all imporant in the model building. As all of those simplify enourmously on toric varieties, we focus on such spaces.

[1] More details on this statement are given in section 8.2.

3. Outline Of This Thesis

3.1. The Content

In this thesis we expect that the reader is familiar with the notion of holomorphic line bundles, the first Chern class of a holomorphic line bundle, sheaves and cohomology of sheaves. In addition basic knowledge about toric varieties is required. For convience of the reader, we give a discussion of all this material in the appendix.
In Appendix A we discuss the notion of a holomorphic line bundle, its first Chern class as well as its divisor. To exemplify these notions, we discuss holomorphic line bundles on special manifolds - in Appendix B on compact and connected Riemann surfaces and in Appendix C on complex tori $\mathbb{C}^n/\Lambda$. For the latter we make use of the Appell-Humbert theorem, which classifies holomorphic line bundles on complex tori. The complex 2-torus $\mathbb{C}_{1,\tau}$ is a compact and connected Riemann surface which is a special complex torus at the same time. Thus we can use the combined power of both approaches to discuss holomorphic line bundles on $\mathbb{C}_{1,\tau}$ in a very detailed fashion. Finally we introduce the topic of toric varieties in Appendix D.
In the actual thesis we then describe the techniques needed to compute the cohomologies of pullback line bundles. We restrict on the study of the situation that a holomorphic line bundle $\mathcal{L}$ is given on a smooth and compact normal toric variety X_Σ and then pulled back onto an algebraic submanifold $C \subset X_\Sigma$. Note however that the general techniques can be applied to far more general geometries.
We first discuss the notion of holomorphic line bundles on smooth and compact normal toric varieties in Part II. This discussion includes computing the cohomology classes of such line bundles by two means - first we present the classical way of computing line bundle cohomology on toric varieties via chamber counting in section 6.3 and subsequently the modern and fast *cohomCalg* algorithm in section 6.6.
After that we present the sheaf exact Koszul sequence, which gives us a means of computing the cohomologies of $\mathcal{L}|_C$ from cohomologies of line bundles on X_Σ and direct sums thereof. In particular we discuss how this sequence is evaluated by means of the exactness properties in Part III and point out that exactness is in general not even enough to only determine the dimension of the cohomology classes of $\mathcal{L}|_C$.
Whilst the evaluation of the Koszul spectral sequence by means of exactness properties has been implemented in the *Koszul extension of cohomCalg* [57] [54] and an algorithm for the evaluation of the generic mappings in direct product of $\mathbb{CP}^n$ has been formulated in [58], a computer implementation that evaluates the Koszul spectral sequence by use of the actual mappings and is applicable beyond direct products of $\mathbb{CP}^n$ is so-far missing. Here we aim to make one step towards closing this gap.

To this end we discuss the Koszul spectral sequence in detail in Part IV. The technologies presented in this chapter are then implemented in a *Mathematica* notebook whose source code is displayed in Appendix E. Details on the implemented functionality is given in chapter 15.
We mention that this notebook is so far only able to compute the mappings in the E_1-sheet of the Koszul spectral sequence. Therefore this solves only those cases that the spectral sequence converges on the E_2-sheet. Hence we present a computer implementation which solves the study of pullback cohomologies onto hypersurfaces completely. Whilst in fortunate cases this analysis can be enough to study pullback cohomologies onto higher codimension loci, there are cases in which this is not the case. Thus for those cases a full computer implementation is still missing. Extending the functionality of the notebook to cover also these cases fully is reserved for future work.
Finally we put our notebook to a use in a model building teaser in chapter 16. Studying more models by use of our notebook is also reserved to future work.

3.2. How To Read This Thesis

The core of this thesis is to investigate the following purely mathematical question.

> Given a smooth and compact normal toric variety X_Σ, $\mathcal{L}$ a holomorphic line bundle on X_Σ and an algebraic submanifold $C \subset X_\Sigma$ defined as the common zero locus of a finite number of polynomials $Q_1, \ldots, Q_n$ in the homogeneous coordinates of X_Σ, how does one compute the pullback cohomologies $\left.\mathcal{L}\right|_C$?

For this reason we aim at a rigorous mathematical investigation of this question. Whilst such a rigorous exposition of the material is certainly in favour of a technical reader, the non-technical reader might find it hard to work through this thesis in a timely fashion. To bridge this gap we give summaries at the beginning of all parts and chapters. These summaries give a brief and less-technical presentation of the material covered in the respective parts and chapters. That said we point out that there are essentially two ways to read this thesis.

1. The non-technical reader is advised to read the summaries.

2. The technically interested reader however should read everything.

The author hopes that this kind of exposition of the material allows many people to profit from this thesis.

3.3. Convention

Our main source of reference for toric varieties is [52]. There $\mathrm{Cl}\,(X_\Sigma)$ is used to denote the divisor class group. We follow this notation whenever we deal with toric varieties. However, when we present material about holomorphic line bundles on Riemann surfaces and complex tori in the appendix we use $\mathrm{Div}\,(X)$ to denote the divisor class group on these spaces. The latter is in agreement with the convention in [44], which is one of our main sources of reference for general background on holomorphic line bundles.

This convention can be confusing because $\mathrm{Div}\,(X_\Sigma)$ is used for the Weil divisors on a toric variety, whilst $\mathrm{Div}\,(X)$ is the divisor class group on X. To emphasise this difference we reserve the symbol X_Σ for the rest of this thesis to denote the normal toric variety associated to the fan Σ. We hope that it will then always be clear from the context, in what meaning we use these symbols.

Part II.

Cohomology Of Holomorphic Line Bundles On Smooth And Compact Normal Toric Varieties

4. Summary

Let us recall that our goal is to answer the following question.

> Given a holomorphic line bundle $\mathcal{L}$ on a smooth and compact normal toric variety X_Σ and an algebraic submanifold $C \subset X_\Sigma$ given as the common zero locus of a finite number of polynomials $Q_1, \ldots, Q_n$ in the homogeneous coordinates of X_Σ, how does one compute the cohomology classes of $\mathcal{L}|_C$?

This we will achieve by considering the so-called Koszul spectral sequence. This spectral sequence we will present in chapter 12. For the time-being we suffice it to state that it relates the cohomologies of certain holomorphic line bundles on X_Σ with the cohomologies of $\mathcal{L}|_C$. Consequently our first step is to understand the cohomologies of holomorphic line bundles on X_Σ. This is what we do in this part of the thesis.

The task of understanding these cohomologies we divide into two tasks. First of all we want to have a nice description for all holomorphic line bundles on X_Σ. In smooth normal toric varieties such a nice description is given by the divisor class of a holomorphic line bundle. This statement we establish in chapter 5. Secondly, given this description we want to study the cohomology classes of the corresponding line bundle. It turns out that if we restrict X_Σ to be compact also, the cohomology classes are finite dimensional vector spaces whose bases can be expressed as certain rationoms, that is quotients of suitable monomials in the homogeneous coordinates of X_Σ. The latter we describe in chapter 6.

Let us mention that our main source of reference for toric varieties is [52]. In particular most of the theorems, lemmas, . . . that we present on toric varieties are taken from there. Whilst we omit the proofs of those statements here in order to give a brief but precise exposition of the necessary material, the interested reader is refered to [52] for the proofs and more details.

5. Divisors On Toric Varieties

5.1. Summary

In this chapter we discuss divisors and establish them as suitable description of holomorphic line bundles. Therefore we first discuss divisors in section 5.2. A careful treatment needs to differ prime, Weil and Cartier divisors. Let us first give the rough picture of prime and Weil divisors.

- A prime divisor of a toric variety X is a special codimension one subvariety of X_Σ.
- Let $D_1, \ldots, D_n$ prime divisors and $a_1, \ldots, a_n \in \mathbb{Z}$, then $D = \sum_{i=1}^{n} a_i D_i$ is a Weil divisor. The set of all Weil divisors we denote by $\operatorname{Div}(X_\Sigma)$.

On any toric variety one can consider the rational functions $\mathbb{C}(X)$. Given a prime divisor $D \subset X_\Sigma$ one would now like to discuss the vanishing order of $f \in \mathbb{C}^*(X)$ along D. As it turns out this is not possible in just any toric variety, rather the toric variety has to be well behaved. This well-behavedness is phrased mathematically as the condition that the local coordinate ring of X is normal [52].
As we point out in Appendix D a normal toric variety X is biholomorphically equivalent to the toric variety of a fan Σ. So to give an example of a non-normal toric variety we have to consider a toric variety that does not stemm from a fan. A standard example of a non-normal but irreducible toric variety is the cuspidal cubic curve

$$C := V\left(x^3 - y^2\right) = \left\{(x,y) \in \mathbb{C}^2\ ,\ x^3 - y^2 = 0\right\} \tag{5.1}$$

It can be shown that a curve is non-singular precisely if its coordinate ring is normal. As it turns out C is singular at the origin.
Whilst for curves singularity and normality are linked quantites, this is not true for higher dimensional toric varieties. To shed more intuition on the term normal let us cite from [59] that normality is equivalent to $R_1 + S_2$ where

- R_1 means that the singular locus has at least codimension two.
- S_2 is the so-called 'extension property' stating, that every function defined on an open set whose complement is of codimension at least two, extends to the entire variety.

As special case, we obtain the above-mentioned statement for curves.
To cut things short, we want to avoid such subteties and hence focus on normal toric

varieties X_Σ. In particular this allows for the following additional divisor constructions.

- Let $f \in \mathbb{C}(X_\Sigma)$ a rational function on X_Σ. Then the vanishing order of f along prime divisors $D_i \subset X_\Sigma$ can be discussed. It turns out that there are only finitely many prime divisors along which the vanishing order a_i of f is non-zero. We then set $\mathrm{div}(f) = \sum_i a_i D_i$ and denote the collection of these divisors by $\mathrm{Div}_0(X_\Sigma)$.
- A Weil divisor $D = \sum_i a_i D_i$ which is locally the divisor of a rational function is a Cartier divisor. Those divisors form $\mathrm{CDiv}(X_\Sigma)$.

From the above we find $\mathrm{Div}_0(X_\Sigma) \subset \mathrm{CDiv}(X_\Sigma) \subset \mathrm{Div}(X_\Sigma)$. This allows to consider

- The class group $\mathrm{Cl}(X_\Sigma) = \mathrm{Div}(X_\Sigma)/\mathrm{Div}_0(X_\Sigma)$. Oftentimes the class group is refered to as the divisor class group.
- The Picard group $\mathrm{Pic}(X_\Sigma) = \mathrm{CDiv}(X_\Sigma)/\mathrm{Div}_0(X_\Sigma)$. This group describes holomorphic line bundles on X_Σ [44].

So the task that we have set for is to find a map $\mathrm{Cl}(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$ which is bijective. On a smooth and normal toric variety X_Σ this can indeed be achieved by the following mapping. Let $D \in \mathrm{Div}(X_\Sigma)$. Then D gives rise to a divisor class and also defines an element $\mathcal{O}_{X_\Sigma}(D) \in \mathrm{Pic}(X_\Sigma)$ given by

$$(\mathcal{O}_{X_\Sigma}(D))(U) := \left\{ f \in \mathbb{C}(X_\Sigma)^* \ , \ \mathrm{div}(f)|_U + D \geq 0 \right\} \cup \{0\} \tag{5.2}$$

Note that the above really defines a sheaf, so that for a complete understanding of the above equation one has to know the sheaf theoretic picture of line bundles. This construction we describe in Appendix A.
This finally reduces the task of finding a nice description of holomorphic line bundles on a smooth and normal toric variety X_Σ to finding a nice description of the class group $\mathrm{Cl}(X_\Sigma)$. This we explain in section 5.3. To formulate the central result let us mention that given a fan Σ we can consider its ray generators $\Sigma(1)$. Moreover note that the proper definition of X_Σ includes the definition of an action of an algebraic torus $T \cong \mathbb{C}^k$ onto X_Σ. As it turns out there are canonical prime divisors associated to the ray generators $\rho \in \Sigma(1)$ which are torus invariant. These torus invariant Weil divisors form the group

$$\mathrm{Div}_T(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \subset \mathrm{Div}(X_\Sigma) \tag{5.3}$$

Now the central result is that the following sequence is exact

$$0 \to \mathbb{Z}^n \xrightarrow{\alpha} \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0 \tag{5.4}$$

In this expression n is the dimension of the fan Σ. Note that exactness implies

$$\mathrm{Cl}(X_\Sigma) \cong \mathrm{coker}(\alpha) = \mathrm{Div}_T(X_\Sigma)/\mathrm{im}(\alpha) \tag{5.5}$$

This gives us a way of computing the class group of X_Σ and thereby, given that X_Σ is smooth, even the Picard group. The reader might find it instructive to see the above in action. To this end we give a few examples at the end of section 5.3 including the famous result

$$\mathbb{Z} \cong \mathrm{Cl}(\mathbb{CP}^n) \cong \mathrm{Pic}(\mathbb{CP}^n) \tag{5.6}$$

Finally we conclude this chapter by presenting for completeness in section 5.4 that in analogy to Equation 5.3 there also exists an exact sequence involving the Picard group. This sequence could therefore be used to compute the Picard group directly. However, as we focus on smooth normal toric varieties in this thesis, we find it more comfortable to use the sequence in Equation 5.3 for the computation of the class group and then to use $\mathrm{Cl}(X_\Sigma) \cong \mathrm{Pic}(X_\Sigma)$.

5.2. Prime, Weil and Cartier Divisors

5.2.1. Prime Divisors

Definition 5.2.1 (Prime Divisor)**:**
Let X an irreducible affine, projective or abstract variety. An irreducible subvariety $D \subset X$ of codimension 1 is a prime divisor in X.

Remark:
Recall that any toric variety is by definition irreducible. Therefore the above gives the notion of prime divisors on toric varieties.

Lemma 5.2.1:
Let X a toric variety with coordinate ring $\mathbb{C}[X]$. Then there is the following one-to-one correspondance.

$$\boxed{\text{prime divisors of } X \Leftrightarrow \text{prime ideals } I \subset \mathbb{C}[X] \text{ of codimension } 1}$$

Remark:
The notion of dimensionality of rings and ideals is given by the Krull dimension [60].

5.2.2. Weil Divisors

Definition 5.2.2 (Weil Divisor)**:**
Let X a toric variety. Then we define the following.

- $\mathrm{Div}(X)$ is the Abelian group generated by the prime divisors on X over $\mathbb{Z}$.
- The elements of $\mathrm{Div}(X)$ are termed Weil divisors.

Example 5.2.1:
$\mathbb{C}^n$ is a toric variety and we find prime divisors by

$$D_i = V(X_i) = \{p \in \mathbb{C}^n \, , \, x_i = 0\} \tag{5.7}$$

since $\langle x_i \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ is a prime ideal of codimension 1. Thus $D := \sum_{i=1}^n a_i D_i$ with $a_i \in \mathbb{Z}$ is a Weil divisor.

Definition 5.2.3 (Effective Weil Divisor)**:**
Let X a toric variety and let

$$D = \sum_{i=1}^{n} a_i D_i \in \mathrm{Div}(X) \tag{5.8}$$

a Weil divisor. Then D is effective precisely if $a_i \geq 0$ for all i. We denote an effective divisor by $D \geq 0$.

Definition 5.2.4 (Support Of A Weil Divisor)**:**
Let X a toric variety and $D = \sum_{i=1}^n a_i D_i \in \mathrm{Div}(X)$ a Weil divisor. Then we define the support of D as

$$\mathrm{Supp}(D) := \bigcup_{a_i \neq 0} D_i \tag{5.9}$$

5.2.3. The divisor Of A Rational Function

Lemma 5.2.2:
Let X_Σ a normal toric variety with field of rational functions $\mathbb{C}(X_\Sigma)$ and $D \subset X_\Sigma$ a prime divisor. Under these conditions the following holds true:

- There exists a discrete valuation

$$\nu_D : \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}\,,\ f \mapsto \text{vanishing degree of } f \text{ along } D \tag{5.10}$$

- It holds $\nu_D(f) \neq 0$ only for finitely many prime divisors on X_Σ.

Remark:
Note that in the above statement X_Σ is required to be normal. Otherwise such a discrete valuation need not exist. This is the reason why we restrict to normal toric varieties.

Definition 5.2.5 (Principal Divisor)**:**
Let X_Σ a normal toric variety and $f \in \mathbb{C}(X_\Sigma)^*$. Then

- $\mathrm{div}(f) := \sum \nu_D(f) \cdot D \in \mathrm{Div}(X_\Sigma)$ where the sum runs over all prime divisors D of X_Σ.
- $\mathrm{Div}_0(X_\Sigma) := \left\{ \mathrm{div}(f)\,,\ f \in \mathbb{C}(X_\Sigma)^* \right\}$ - this is the set of principal divisors on X_Σ.
- Let $D, E \in \mathrm{Div}(X_\Sigma)$. Then we define a relation

$$D \sim E \iff \exists f \in \mathbb{C}(X_\Sigma)^*\,,\ D = E + \mathrm{div}(f) \tag{5.11}$$

  This we term linear equivalence.

Remark:

- $\mathrm{Div}_0(X_\Sigma) \subset \mathrm{Div}(X_\Sigma)$ a subgroup.
- The relation $D \sim E$ is an equivalence relation.

5.2.4. Cartier Divisors

Definition 5.2.6 (Cartier Divisor)**:**
Let X_Σ a normal toric variety and $D \in \mathrm{Div}(X_\Sigma)$. D is a Cartier divisor precisely if f is locally principal.

Remark:
This means that X_Σ has an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ such that there exist functions $\{f_i \in \mathbb{C}^*(X_\Sigma)(U_i)\}$ with the property

$$D|_{U_i} = \mathrm{div}(f_i)|_{U_i} \ \ \forall i \in I \tag{5.12}$$

In particular we then term $\{U_i, f_i\}_{i\in I}$ the local data of D.

Note:

- The Cartier divisors form an Abelian group $\mathrm{CDiv}(X_\Sigma)$ which is a subgroup of the Weil divisors.
- Every principal divisor is a Cartier divisor.

Consequence:
This implies $\mathrm{Div}_0(X_\Sigma) \subset \mathrm{CDiv}(X_\Sigma) \subset \mathrm{Div}(X_\Sigma)$ where the inclusions mean subgroup inclusions. Consequently we can make the following definition.

Definition 5.2.7 (Divisor Classes)**:**
Let X a normal toric variety. Then we define

- $\mathrm{Cl}(X_\Sigma) \coloneqq \mathrm{Div}(X_\Sigma)/\mathrm{Div}_0(X_\Sigma)$ - the class group.
- $\mathrm{Pic}(X_\Sigma) \coloneqq \mathrm{CDiv}(X_\Sigma)/\mathrm{Div}_0(X_\Sigma)$ - the Picard group.

Consequence:
There is a canonical inclusion $\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Cl}(X_\Sigma)$. Recall that $\mathrm{Pic}(X_\Sigma)$ is the set of all equivalence classes of holomorphic line bundles on X_Σ. Therefore the canonical inclusion $\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Cl}(X_\Sigma)$ states that one can always associate to a holomorphic line bundle a divisor class. For our purposes however we want exactly the converse - namely we want to associate to every divisor class a holomorphic line bundle. To achieve this we need *smoothness* as the following lemma shows.

Lemma 5.2.3:
Let X_Σ a smooth normal toric variety. Then every Weil divisor on X_Σ is a Cartier divisor.

Example 5.2.2:
Since $\mathbb{CP}^n$ is smooth and normal, we thus find $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathrm{Cl}(\mathbb{CP}^n)$.

5.2.5. Computing Divisor Classes

Note:
Let X_Σ a normal toric variety and $U \subset X_\Sigma$ open and non-empty. Then

$$\cdot|_U : \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(U)\ ,\ [D] \mapsto [D|_U] \tag{5.13}$$

is a well-defined mapping.

Theorem 5.2.1:
Let X_Σ a normal toric variety, $U \subset X_\Sigma$ open and non-empty. Moreover let $D_1, \dots, D_s$ the irreducible components of $X_\Sigma - U$. Note that those are prime divisors. Then the following sequence is exact.

$$\bigoplus_{j=1}^{s} \mathbb{Z}D_j \to \mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(U) \to 0 \tag{5.14}$$

Example 5.2.3:
Recall that $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$. With the topology of the Riemann sphere, the set $\{\infty\} \subset \mathbb{CP}^1$ is open and an irreducible subvariety of $\mathbb{CP}^1$. Consequently we have the exact sequence

$$\mathbb{Z}\{\infty\} \xrightarrow{f} \mathrm{Cl}\left(\mathbb{CP}^1\right) \to \mathrm{Cl}(\mathbb{C}^n) \to 0 \tag{5.15}$$

But $\mathrm{Cl}(\mathbb{C}^n) = 0$ since $\mathbb{C}^n$ is a UFD. Thus we have the exact sequence

$$\mathbb{Z}\{\infty\} \xrightarrow{f} \mathrm{Cl}\left(\mathbb{CP}^1\right) \to 0 \to 0 \tag{5.16}$$

So by exactness the map f is surjective. But note that f is explicitely given by

$$f(a\{\infty\}) \mapsto [a \cdot \{\infty\}] \tag{5.17}$$

This shows that f is injective also. To see this let $g \in \mathbb{C}\left(\mathbb{CP}^1\right)$ with

$$a \cdot \{\infty\} = \mathrm{div}(g) \Rightarrow \mathrm{div}(g)|_{\mathbb{C}} = 0 \Rightarrow g \text{ is constant} \Rightarrow a = 0 \tag{5.18}$$

where we used that g is continous. Consequently $\mathrm{Cl}\left(\mathbb{CP}^1\right) \cong \mathbb{Z}$.

5.2.6. Sheaves Of $\mathcal{O}_{\mathbf{X}}$-Modules On Toric Varieties

Remark:
Let X_Σ a normal toric variety and $U \subset X_\Sigma$ open. Then the sheaf $\mathcal{O}_{X_\Sigma}$ is defined by

$$\mathcal{O}_{X_\Sigma}(U) = \left\{ f \in \mathbb{C}(X_\Sigma)^*\ ,\ \mathrm{div}(f)|_U \geq 0 \right\} \cup \{0\} \tag{5.19}$$

Remark (Generalisation)**:**
Let X_Σ a normal toric variety and $D \in \mathrm{Div}(X_\Sigma)$. Then we can define a sheaf $\mathcal{O}_{X_\Sigma}(D)$ by

$$(\mathcal{O}_{X_\Sigma}(D))(U) = \left\{ f \in \mathbb{C}(X_\Sigma)^*\ ,\ \mathrm{div}(f)|_U + D \geq 0 \right\} \cup \{0\} \tag{5.20}$$

for $U \subset X$ open.

Note:
This definition coincides with the sheaf-theoretic notion of a holomorphic line bundle associated to a divisor D that we give in section A.6 in more general setups.

Lemma 5.2.4:
Let X_Σ a normal toric variety and $D \in \mathrm{Div}(X_\Sigma)$ a Weil divisor. Then $\mathcal{O}_{X_\Sigma}(D)$ is a coherent sheaf of $\mathcal{O}_{X_\Sigma}$-modules. If in addition $D \in \mathrm{CDiv}(X_\Sigma)$, then $\mathcal{O}_{X_\Sigma}(D)$ is even invertible.

Lemma 5.2.5:
Let X_Σ a normal toric variety and $D, E \in \mathrm{Div}(X_\Sigma)$. Then $\mathcal{O}_{X_\Sigma}(D) \cong \mathcal{O}_{X_\Sigma}(E)$ precisely if $D \sim E$.

5.3. Computing The Class Group

5.3.1. The Divisor Of A Character

Remark:

- Recall that a toric variety X contains an algebraic torus $(\mathbb{C}^*)^n$ as Zariski open subset. We will denote this torus by T. In particular note that the self-action of T extends to a T-action on X. This then also gives the notion of T-invariant subsets of X.
- For normal toric varieties X_Σ, there is the cone-orbit correspondance
$$\text{k-dim cones } \sigma \in \Sigma \Leftrightarrow (n-k)\text{-dimensional } T\text{-orbits in } X_\Sigma \tag{5.21}$$
- Let $\rho \in \Sigma(1)$ a ray of a fan Σ. Then by the above there exists a codimension one orbit $O(\rho) \subset X_\Sigma$.
- More information no characters and 1-parameters is presented in subsection D.2.2.

Lemma 5.3.1:
Let X_Σ a normal toric variety, $\rho \in \Sigma(1)$ and $O(\rho)$ the associated codimension one orbit. Then $\overline{O(\rho)}$ is a T-invariant prime divisor on X_Σ.

Lemma 5.3.2:
Let X_Σ a normal toric variety with fan Σ in $\mathbb{R}^n$ and $D_\rho \in \mathrm{Div}(X_\Sigma)$ for $\rho \in \Sigma(1)$. We denote the primitive element of ρ by $u_\rho \in \mathbb{Z}^n$. Now let $m \in \mathbb{Z}^n$. Then it holds
$$\nu_{D_\rho}(\chi^m) = \langle m, u_\rho \rangle \tag{5.22}$$

Theorem 5.3.1:
The character χ^m is a rational function on a smooth and compact normal toric variety X_Σ and its divisor is given by
$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle \cdot D_\rho \tag{5.23}$$

5.3.2. Exact Sequences For the Class Group

Definition 5.3.1:
The set of torus invariant Weil divisors on X_Σ forms a group given by

$$\mathrm{Div}_T(X_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho \subset \mathrm{Div}(X_\Sigma) \tag{5.24}$$

Theorem 5.3.2:
Let X_Σ the normal toric variety of the fan Σ in $\mathbb{R}^n$. Then the following holds true.

- The following sequence is exact

$$\begin{aligned} \mathbb{Z}^n &\to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0 \\ m &\mapsto \mathrm{div}(\chi^m) \\ &\sum_{\rho\in\Sigma(1)} a_\rho D_\rho \mapsto \left[\sum_{\rho\in\Sigma(1)} a_\rho D_\rho\right] \end{aligned} \tag{5.25}$$

- If and only if $\{u_\rho\ ,\ \rho\in\Sigma(1)\}$ is a basis of $\mathbb{R}^n$ (i.e. precisely if X_Σ has no torus factors), then even the following sequence is exact

$$\begin{aligned} 0 \to \mathbb{Z}^n &\to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Cl}(X_\Sigma) \to 0 \\ m &\mapsto \mathrm{div}(\chi^m) \\ &\sum_{\rho\in\Sigma(1)} a_\rho D_\rho \mapsto \left[\sum_{\rho\in\Sigma(1)} a_\rho D_\rho\right] \end{aligned} \tag{5.26}$$

Consequence:

- $\mathrm{Cl}(X_\Sigma)$ is a finitely generated Abelian group.
- As any smooth and compact normal toric variety X_Σ has no torus factors, for such toric varieties the second sequence is always exact.

5.3.3. Examples

Remark:

- The rays of Σ are listed as $\rho_1,\dots,\rho_r$ and the corresponding ray generators are denoted by $u_1,\dots,u_r\in\mathbb{Z}^n$. In particular we have

$$u_i = \left(\langle e_1,u_i\rangle,\dots,\langle e_n,u_i\rangle\right)^T \tag{5.27}$$

- Consequently we can represent the map $A\colon\mathbb{Z}^n\to\mathrm{Div}_T(X_\Sigma)$ as

$$A\colon\mathbb{Z}^n\to\mathbb{Z}^r\ ,\ \begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} \mapsto \begin{pmatrix} \langle e_1,u_1\rangle & \dots & \langle e_n,u_1\rangle\\ \vdots & \ddots & \vdots\\ \langle e_1,u_r\rangle & \dots & \langle e_n,u_r\rangle\end{pmatrix}\cdot\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} \tag{5.28}$$

Then by noting that all the maps in the above exact sequence can be considered as vector space homomorphism, one easily deduces

$$\mathrm{Cl}\left(X_{\Sigma}\right) \cong \mathrm{coker}\left(A\right) = \mathbb{Z}^r/\mathrm{im}\left(A\right) \tag{5.29}$$

- Whenever we want to think in terms of divisors, we set D_i to be the torus-invariant prime divisor associated to ρ_i via the cone-orbit correspondance. [1]

Example 5.3.1:
Consider the fan Σ in $\mathbb{R}^2$ of $\mathrm{Bl}_0\left(\mathbb{C}^2\right)$. This fan Σ has ray generators

$$u_0 = e_1 + e_2, \quad u_1 = e_1, \quad u_2 = e_2 \tag{5.30}$$

Then one easily finds

$$0 = \left[\mathrm{div}\left(\chi^{e_1}\right)\right] = \left[D_1\right] + \left[D_0\right], \qquad 0 = \left[\mathrm{div}\left(\chi^{e_2}\right)\right] = \left[D_2\right] + \left[D_0\right] \tag{5.31}$$

Thus $\mathrm{Cl}\left(X_{\Sigma}\right)$ is generated by $\left[D_0\right]$ which gives us

$$\mathrm{Cl}\left(X_{\Sigma}\right) \cong \mathbb{Z} \tag{5.32}$$

Alternatively one can evaluate the image of the map $A\colon \mathbb{Z}^2 \to \mathrm{Div}_T\left(X_{\Sigma}\right) \cong \mathbb{Z}^3$ which in this case is given by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{5.33}$$

Then it follows easily that $\mathrm{im}\left(A\right) \cong \mathbb{Z}^2$ and thus

$$\mathrm{Cl}\left(X_{\Sigma}\right) \cong \mathbb{Z}^3/\mathbb{Z}^2 \cong \mathbb{Z} \tag{5.34}$$

Example 5.3.2 (Complex Projective Space)**:**
$\mathbb{CP}^n$ has a fan Σ in $\mathbb{R}^n$ with ray generators

$$u_0 = -e_1 - \cdots - e_n, \quad u_1 = e_1, \quad \ldots, u_n = e_n \tag{5.35}$$

Now we pursue the above mentioned two approaches.

1. The map $A\colon \mathbb{Z}^n \to \mathrm{Div}_T\left(\mathbb{CP}^n\right) \cong \mathbb{Z}^{n+1}$ is given by the matrix

$$A = \begin{pmatrix} -1 & -1 & \ldots & -1 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \tag{5.36}$$

   Its image is easily found to be $\mathbb{Z}^n$, so that

$$\mathrm{Cl}\left(\mathbb{CP}^n\right) \cong \mathbb{Z}^{n+1}/\mathbb{Z}^n \cong \mathbb{Z} \tag{5.37}$$

[1] By means of the homogenisation of the smooth and compact normal toric variety X_{Σ} one can identify those divisors D_i with the sets $\{x_i = 0\} \subset X_{\Sigma}$.

2. From $0 = [\mathrm{div}\,(\chi^{e_i})]$ it can be found
$$D_0 \sim D_i \ \forall 1 \leq i \leq n \tag{5.38}$$
Thus $\mathrm{Cl}\,(\mathbb{CP}^n)$ is generated by $[D_0]$ which gives us the above result.

5.4. Computing The Picard Group

Note:
Every Cartier divisor $D \in \mathrm{CDiv}\,(X_\Sigma)$ is a Weil divisor. Thus there exist $a_i \in \mathbb{Z}$ such that $D \sim \sum_{\rho\in\Sigma(1)} a_\rho \cdot D_\rho$. Those coefficients a_i might however be very special. This motivates the following definition.

Definition 5.4.1:
Let X_Σ the normal toric variety associated to the fan Σ in $\mathbb{R}^n$. Then we denote by $\mathrm{CDiv}_T\,(X_\Sigma) \subset \mathrm{Div}_T\,(X_\Sigma)$ the Abelian group of T-invariant Cartier divisors.

Remark:
$\mathrm{div}\,(\chi^m) \in \mathrm{CDiv}_T\,(X_\Sigma)$ for all $m \in \mathbb{Z}^n$.

Theorem 5.4.1:
Let X_Σ the normal toric variety associated to the fan Σ in $\mathbb{R}^n$. Then the following sequence is exact
$$\mathbb{Z}^n \to \mathrm{CDiv}_T\,(X_\Sigma) \to \mathrm{Pic}\,(X_\Sigma) \to 0 \tag{5.39}$$
where $m \in \mathbb{Z}^n \mapsto \mathrm{div}\,(\chi^m)$.
Given that X_Σ has no torus factors, even the following sequence is exact
$$0 \to \mathbb{Z}^n \to \mathrm{CDiv}_T\,(X_\Sigma) \to \mathrm{Pic}\,(X_\Sigma) \to 0 \tag{5.40}$$

Note:
In order to evalute the above sequence and thereby compute $\mathrm{Pic}\,(X_\Sigma)$ one needs to understand $\mathrm{CDiv}_T\,(X_\Sigma)$ first. But recall that we will eventually focus on smooth and compact normal toric varieties. Then the following lemma gives us a means to compute the Picard group.

Lemma 5.4.1:
Let X_Σ be the normal toric variety of a fan Σ. Then the following three statements are equivalent.

- Every Weil divisor on X_Σ is a Cartier divisor.
- $\mathrm{Pic}\,(X_\Sigma) = \mathrm{Cl}\,(X_\Sigma)$.
- X_Σ is smooth.

Consequence:
We computed the class group for $\mathbb{CP}^n$ in the previous section to be isomorphic to $\mathbb{Z}$. Since $\mathbb{CP}^n$ is smooth, we thus conclude
$$\mathrm{Pic}\,(\mathbb{CP}^n) \cong \mathbb{Z} \tag{5.41}$$

6. Line Bundle Cohomology On Toric Varieties

6.1. Summary

In this chapter we turn to computing cohomologies of holomorphic line bundles on toric varieties. However before we do so, let us briefly recall the description that we found for holomorphic line bundles in the previous chapter.
First of all recall that we learned that for 'well-behavedness' we should restrict to normal toric varieties X_Σ. Consequently we can consider the ray generators $\Sigma(1)$ of the fan Σ. Via the so-called cone-orbit-correspondance [52] a ray generator $\rho \in \Sigma(1)$ is then one-to-one to a torus-invariant prime divisor $D_\rho \in \mathrm{Div}(X_\Sigma)$. Now let

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho in \mathrm{Div}_T(X_\Sigma) \tag{6.1}$$

Then this divisor induces a holomorphic line bundle $\mathcal{O}_{X_\Sigma}(D)$ which is defined sheaf-theoretically as

$$(\mathcal{O}_{X_\Sigma}(D))(U) = \{f \in \mathbb{C}^*(X_\Sigma)\ ,\ \mathrm{div}(f) + D \geq 0\} \cup \{0\} \tag{6.2}$$

for $U \subset X_\Sigma$ open. Now assume that $\widetilde{D} \in \mathrm{Div}_T(X_\Sigma)$ is a divisor which differs from D via and element in the image of

$$f\colon \mathbb{Z}^n \to \mathrm{Div}_T(X_\Sigma) \tag{6.3}$$

Then by the result from the previous chapter $\mathcal{O}_{X_\Sigma}(D) \cong \mathcal{O}_{X_\Sigma}(\widetilde{D})$ and we should consequently identify these two line bundles to form a unique element in $\mathrm{Pic}(X_\Sigma)$. Also we should identify D and $\widetilde{D}$ and work with the associated divisor class $[D] \in \mathrm{Cl}(X_\Sigma)$. Given that X_Σ is smooth, we found that every holomorphic lien bundle can be described as $\mathcal{O}_{X_\Sigma}(D)$ for a suitable divisor class $[D] \in \mathrm{Cl}(X_\Sigma)$. This is the notation that we will use for holomorphic line bundles.
That said we point out in section 6.2 that the global sections of the sheaf $\mathcal{O}_{X_\Sigma}(D)$ are described by

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m = \bigoplus_{m\in\mathbb{Z}^n\cap P_D} \mathbb{C}\cdot\chi^m \tag{6.4}$$

In this expression χ^m is a so-called character - a special rational function on X_Σ. Details on these functions we give in Appendix D. P_D is the so-called polyhedron of the divisor D given by

$$P_D = \{m \in \mathbb{R}^n\ ,\ \langle m, u_\rho\rangle \geq -a_\rho\ ,\ \forall \rho \in \Sigma(1)\} \tag{6.5}$$

Note that by the second equality in Equation 6.4 we can compute the dimension of $\Gamma\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)\right)$ by counting integral points in P_D. It turns out that this polyhedron is bounded given that X_Σ is compact. This leads to the following crucial insight.

Let X_Σ a compact normal toric variety and $\mathcal{F}$ a coherent sheaf on X_Σ. Then $H^i\left(X_\Sigma, \mathcal{F}\right)$ are finite dimensional vector spaces.

This observation leads on to the classical approach in computing line bundle cohomology on smooth and compact normal toric varieties X_Σ, which goes by the name *chamber counting*. We present this approach in section 6.3 and give examples on how to apply this approach in examples on the del Pezzo surfaces dP_1 and dP_3 in section 6.4 and section 6.5 respectively.
Whilst we will make use of this approach towards the end of this thesis, in what follows first this approach is far less important than the *cohomCalg* algorithm. So in a first reading, the non-technical reader might want to skip reading about chamber counting completely. For this reason we decide to avoid giving more details on this construction in this summary, but rather state the crucial result from the *cohomCalg* algorithm which we present in section 6.6.
To state this result let us briefly remind us that a smooth and compact normal toric variety X_Σ can be expressed conveniently as

$$X_\Sigma \cong \left(\mathbb{C}^r - Z\right) / \left(\mathbb{C}^*\right)^a \tag{6.6}$$

which is termed the *homogenisation* of X_Σ. We give details on this construction in section D.5. This construction identifes equivalence classes of the coordinates of $\mathbb{C}^r$ (minus the exceptional set Z) as the homogeneous coordinates of X_Σ. Those we will denote by $x_1, \ldots, x_r$ in the rest of this thesis. Given this setup, the *cohomCalg* algorithm allows to find a basis of the cohomology classes for a given holomorphic line bundle $\mathcal{L}$ on X_Σ by quotients of monomials in the homogeneous coordinates of X_Σ [54].
Let us exemplify this statement on a del Pezzo 1 surface dP_1. Note that dP_1 is a smooth and compact normal toric variety. We present the toric data of this variety in subsection 6.4.1. In particular note that

$$dP_1 = \left(\mathbb{C}^4 - \langle x_1 x_2, x_3 x_4 \rangle\right) / \left(\mathbb{C}^*\right)^2 \tag{6.7}$$

with the torus action described in Table 6.1. So in particular there are four homogeneous coordinates. To each of these homogeneous coordinates one can associate a torus invariant prime divisor which we denote by D_i. These divisors generate the group $\mathrm{Div}_T\left(X_\Sigma\right)$ over $\mathbb{Z}$. As we learned previously, to describe the Picard group, we have to divide out this group by the image of the map

$$f \colon \mathbb{Z}^2 \to \mathrm{Div}_T\left(X_\Sigma\right) \tag{6.8}$$

By these means we point out in subsection 6.4.2 that

$$\mathrm{Pic}\left(dP_1\right) \cong \mathbb{Z}\left[D_1\right] \oplus \mathbb{Z}\left[D_4\right] \tag{6.9}$$

This now estabilshes the notation for holomorphic line bundles on dP_1. For example we can consider

$$\mathcal{L} = \mathcal{O}_{dP_1}\left(5\left[D_1\right] - 2\left[D_4\right]\right) \equiv \mathcal{O}_{dP_1}\left(5, -2\right) \tag{6.10}$$

From the *cohomCalg* algorithm one can now compute the cohomology classes of this line bundle. As we point out in section 6.6 these turn out to be as follows

- $H^0\left(dP_1, \mathcal{L}\right) = \{0\}$
- $H^1\left(dP_1, \mathcal{L}\right) = \left\{\frac{a_1 x_1^6}{x_3 x_4} + \frac{a_2 x_1^5 x_2}{x_3 x_4} + \frac{a_3 x_1^4 x_2^2}{x_3 x_4} + \frac{a_4 x_1^3 x_2^3}{x_3 x_4} + \frac{a_5 x_1^2 x_2^4}{x_3 x_4} + \frac{a_6 x_1 x_2^5}{x_3 x_4} + \frac{a_7 x_2^6}{x_3 x_4}\ ,\ a_i \in \mathbb{C}\right\}$
- $H^2\left(dP_1, \mathcal{L}\right) = \{0\}$

So bases of the cohomology classes are indeed quotients of monomials in the homogeneous coordinates of dP_1 as mentioned before.
We conclude this chapter by pointing out that on smooth and compact normal toric varieties, the elements of $\operatorname{Pic}\left(X_\Sigma\right)$ can be described in yet another way. It turns out that $\mathcal{L} \in \operatorname{Pic}\left(X_\Sigma\right)$ is uniquely described by its first Chern class. Physically specaking this means that every $U\left(1\right)$ gauge theory on a smooth and compact normal toric variety is uniquely specified by its field strength. We give the details on this observation in section 6.7.

6.2. Global Sections Of The Sheaf Of A Torus-Invariant Divisor

Remark:
Let X_Σ a normal toric variety and $D = \sum\limits_{\rho \in \Sigma(1)} a_\rho D_\rho \in \operatorname{Div}_T\left(X_\Sigma\right)$. Then D defines a sheaf $\mathcal{O}_{X_\Sigma}\left(D\right)$ defined sheaf-theoretically via

$$\left(\mathcal{O}_{X_\Sigma}\left(D\right)\right)\left(U\right) = \left\{f \in \mathbb{C}\left(X_\Sigma\right)^*\ ,\ \operatorname{div}\left(f\right)|_U + D \geq 0\right\} \cup \{0\} \tag{6.11}$$

for $U \subset X_\Sigma$ open.

Lemma 6.2.1 (Global Sections I)**:**
The global sections of the above sheaf $\mathcal{O}_{X_\Sigma}\left(D\right)$ are given by

$$\Gamma\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(D\right)\right) = \bigoplus_{\operatorname{div}(\chi^m) + D \geq 0} \mathbb{C} \cdot \chi^m \tag{6.12}$$

Note:
Let Σ a fan in $\mathbb{R}^n$, $D = \sum\limits_{\rho \in \Sigma(1)} a_\rho D_\rho \in \operatorname{Div}_T\left(X_\Sigma\right)$ and $m \in \mathbb{Z}^n$. Then it holds

$$\begin{aligned} \operatorname{div}\left(\chi^m\right) + D \geq 0 \iff & \sum_{\rho \in \Sigma(1)} \left\langle m, u_\rho\right\rangle D_\rho + \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \geq 0 \\ \iff & \left\langle m, u_\rho \right\rangle \geq -a_\rho\ \forall \rho \in \Sigma\left(1\right) \end{aligned} \tag{6.13}$$

Definition 6.2.1:
Let X_Σ the toric variety of the fan Σ in $\mathbb{R}^n$. Then consider $D \in \mathrm{Div}_T(X_\Sigma)$. For this divisor we define

$$P_D := \{m \in \mathbb{R}^n\ ,\ \langle m, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \tag{6.14}$$

We term P_D the polyhedron of the divisor D.

Note:
P_D need not be bounded. This we illustrate momentarily.

Consequence (Global Sections II)**:**
Let X_Σ the normal toric variety of the fan Σ in $\mathbb{R}^n$ and consider $D \in \mathrm{Div}_T(X_\Sigma)$. Then it holds

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap \mathbb{Z}^n} \mathbb{C} \cdot \chi^m \tag{6.15}$$

where $P_D \subset \mathbb{R}^n$ is the polyhedron of the divisor D.

Example 6.2.1:
Consider $\mathrm{Bl}_0(\mathbb{C}^2)$ which is given by a fan Σ with ray generators

$$u_0 = e_1 + e_2, \qquad u_1 = e_1, \qquad u_2 = e_2 \tag{6.16}$$

We wish to consider $D = D_0 + D_1 + D_2 \in \mathrm{Div}_T(X_\Sigma)$ and its associated sheaf. For this sheaf we have by the above results

$$\Gamma(X_\Sigma, \mathcal{O}_\Sigma(D)) = \bigoplus_{m \in P_D \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \tag{6.17}$$

An easy calculation yields

$$P_D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2\ ,\ x \geq -1, y \geq -1, x + y \geq -1 \right\} \tag{6.18}$$

which is not limited. Consequently $\dim_{\mathbb{C}}(\Gamma(X_\Sigma, \mathcal{O}_\Sigma(D))) = |P_D \cap \mathbb{Z}^2|$ is infinite. This is because $\mathrm{Bl}_0(\mathbb{C}^2)$ is not compact, as the following lemma shows.

Lemma 6.2.2:
Let X_Σ a compact normal toric variety, i.e. the fan Σ is complete. Then the following holds true.

- $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}) = \mathbb{C}$ since the only morphisms $X_\Sigma \to \mathbb{C}$ are the constant ones.
- P_D is bounded for any $D \in \mathrm{Div}_T(X_\Sigma)$ and $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is a finite-dimensional complex vector space.
- For any coherent sheaf $\mathcal{F}$ on X_Σ, the space $\Gamma(X_\Sigma, \mathcal{F})$ is a finite dimensional vector space.

Consequence:
From the first bullet point it follows that every compact normal toric variety is connected.

6.3. Line Bundle Cohomology Via Chamber Counting

Remark (The Affine Open Cover)**:**
Let X_Σ a compact and smooth normal toric variety. In the following we will always cover X_Σ by the affine open cover. This cover is given by

$$\mathcal{U} = \{U_\sigma\}_{\sigma \in \Sigma_{\max}} \tag{6.19}$$

Lemma 6.3.1:
Let X_Σ a smooth and compact normal toric variety. Then for any $p \in \mathbb{N}_{\geq 0}$ and any $D \in \mathrm{Div}_T(X_\Sigma)$ it holds

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \check{H}^p(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) \tag{6.20}$$

Consequence:
We can replace the harder task of computing sheaf cohomology, by the much easier task of computing Čech cohomology.

Lemma 6.3.2:
Let X_Σ a smooth and compact normal toric variety. Let us consider $D \in \mathrm{Div}_T(X_\Sigma)$. Then the Čech cochain groups (with respect to the affine open cover $\mathcal{U}$) of the sheaf $\mathcal{O}_{X_\Sigma}(D)$ are given by

$$\check{C}^p(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{(i_0,\ldots,i_p)\in[l]_p} H^0\left(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}_{X_\Sigma}(D)\right) \tag{6.21}$$

If we set $\sigma_\gamma := \sigma_{i_0} \cap \cdots \cap \sigma_{i_p} \in \Sigma$ for $\gamma = (i_0, \ldots, i_p) \in [l]_p$, then we can rewrite this relation more easily as

$$\check{C}^p(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma\in[l]_p} H^0\left(U_{\sigma_\gamma}, \mathcal{O}_{X_\Sigma}(D)\right) \tag{6.22}$$

Remark:
Since U_σ is a normal toric variety for every $\sigma \in \mathcal{U}$, we can apply the results from section 6.2. If we thus consider $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho \in \mathrm{Div}_T(X_\Sigma)$, then this observation gives us

$$H^0(U_\sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D(U_\sigma)} \mathbb{C}\cdot\chi^m \tag{6.23}$$

where

$$P_D(U_\sigma) = \{m \in \mathbb{R}^n\ ,\ \langle m, u_\rho\rangle \geq -a_\rho\ \forall \rho \in \sigma(1)\} \tag{6.24}$$

Consequence:
We can thus conclude

$$\check{C}^p(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma\in[l]_p} \left(\bigoplus_{m\in P_D(U_\gamma)\cap\mathbb{Z}^n} \mathbb{C}\cdot\chi^m \right) \tag{6.25}$$

Remark:
Since we require that X_Σ is compact, both sums are finite and we can change their order. Thereby however, computing the Čech cochain groups turns into counting points on the lattice $\mathbb{Z}^n$, where each point constitutes its Laurent monomial as basis element to the Čech cochain. This computation can be organised by splitting the lattice $\mathbb{Z}^n$ into regions, called chambers, such that all points in one chamber contribute to only one cochain group. After this 'chamber counting' is done, one can finally evaluate the Čech complex and thereby determine the sheaf cohomology groups. We will illustrate this approach by computing line bundle cohomology on dP_1 and dP_3. Before we give these examples, let us mention that the above description is the classical way to consider line bundle cohomology on toric varieties. More details on this picture can be found in [52]. Finally we mention that the chamber counting approach has been computerised. Details on this can be found in [61].

6.4. Line Bundle Cohomology On dP$_1$ Via Chamber Counting

6.4.1. The Toric Data Of dP$_1$

Remark:
In the remainder of this thesis we will specify a normal toric variety X_Σ by its homogenisation. To this end we describe the group $G \subset (\mathbb{C}^*)^r$ explicitely. From this group action we can then determine the ray generators of the fan of Σ. These ray generators we then triangulate with the program *Sage* [51] in order to determine all Stanley-Reisner-ideals $I_{SR} \subset \mathbb{C}[x_1, \ldots, x_n]$ with Alexander dual irrelevant ideals B_{X_Σ}, such that we obtain a smooth and compact toric variety

$$X_\Sigma \cong (\mathbb{C}^n - V(B_{X_\Sigma}))/G \tag{6.26}$$

This strategy we also exemplify in this particular example.

Note:
In Table 6.1 we specify a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$. This gives us the group G that we want to consider. Note that written explicitely we have

$$G = \left\{(\mu, \mu, \mu + \nu, \nu) \in \mathbb{C}^4\ ,\ \mu, \nu \in \mathbb{C}^*\right\} \cong (\mathbb{C}^*)^2 \tag{6.27}$$

Consequence:
Given the group G, we consider a fan Σ in $\mathbb{R}^N$ with

$$N = \#\text{ of homogenous coordinates} - \#\ \mathbb{C}^*\text{ operations} = 4 - 2 = 2 \tag{6.28}$$

We then obtain the ray generators of Σ as follows.

1. Assign to each homogeneous coordinate x_i a vector $u_i \in \mathbb{Z}^N = \mathbb{Z}^2$.

homogeneous coordinates	Q_1	Q_2
x_1	1	0
x_2	1	0
x_3	1	1
x_4	0	1

Table 6.1.: $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ which for a suitably chosen fan gives a del Pezzo 1-surface dP_1.

2. Next impose that these vectors satisfy the following constraints

$$u_1 + u_2 + u_3 = 0, \qquad u_3 + u_4 = 0 \tag{6.29}$$

   Those conditions originate from the $(\mathbb{C}^*)^2$-action in Table 6.1.

3. These equations can be solved by

$$u_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{6.30}$$

   Note that the solution is not unique.

Remark:
The justification for this procedure originates from the definition of the group G as given in subsection D.5.1.

Consequence:
Given the above data we can triangulate with *Sage* [51] to obtain the Stanley-Reisner-Ideals I_{SR} that yield a smooth and compact normal toric variety X_Σ. In this case there exists a unique such ideal, namely

$$I_{\text{SR}}(X_\Sigma) = \langle x_1x_2, x_3x_4\rangle \subset \mathbb{C}[x_1, x_2, x_3, x_4] \tag{6.31}$$

6.4.2. The Picard Group

Remark:
Recall that the divisors $D_i := \{p \in dP_1\ ,\ x_i = 0\} \subset dP_1$ are torus-invariant prime divisors that generate $\text{Div}_T(dP_1)$ over $\mathbb{Z}$.

Note:
Since dP_1 is smooth we have the short exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{f} \text{Div}_T(dP_1) \to \text{Cl}(dP_1) \to 0 \tag{6.32}$$

In this particular situation the map f takes the form

$$f\colon \mathbb{Z}^2 \to \mathbb{Z}^4\,,\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6.33}$$

This implies $\mathrm{im}\,(f) \cong \mathbb{Z}^2$. By exactness of the above sequence we thus learn

$$\mathrm{Cl}\,(dP_1) \cong \mathbb{Z}^4 / \mathrm{im}\,(f) \cong \mathbb{Z}^2 \tag{6.34}$$

Remark:
Alternatively, one concludes from

- $[0] = [\mathrm{div}\,(\chi^{e_1})] = -[D_1] + [D_2]$
- $[0] = [\mathrm{div}\,(\chi^{e_2})] = -[D_1] + [D_3] - [D_4]$

that $\mathrm{Cl}\,(dP_1) \cong \mathbb{Z}\,[D_1] \oplus \mathbb{Z}\,[D_4] \cong \mathbb{Z}^2$.

Consequence:
Since dP_1 is a smooth normal toric variety it holds

$$\mathrm{Pic}\,(dP_1) = \mathrm{Cl}\,(dP_1) \cong \mathbb{Z}\,[D_1] \oplus \mathbb{Z}\,[D_4] \tag{6.35}$$

Remark:
For a condensed notation we write $\mathcal{O}_{dP_1}(m,n)$ for the isomorphism class of holomorphic line bundles associated to the divisor class $m\,[D_1] \oplus n\,[D_4] \in \mathrm{Cl}\,(X_\Sigma)$.

6.4.3. Example: Cohomologies Of $\mathcal{O}_{\mathbf{dP_1}}(\mathbf{5},\mathbf{-2})$

Remark (The Fan)**:**
To begin the chamber counting let us first recall what the fan of dP_1 looks like. Therefore we display this fan in Figure 6.1. Note in particular that the affine open cover of dP_1 does consist of four different open sets. In fact the number of these open sets is a measure for the effort that it takes to compute the Čech cohomology groups via the chamber counting approach.

Note:
We now want to compute the cohomologies of the holomorphic line bundles associated to the divisor class

$$[D] = \sum_{i=1}^{4} a_i\,[D_i] = 5\,[D_1] + 0\,[D_2] + 0\,[D_3] + (-2)\,[D_4] \in \mathrm{Cl}\,(dP_1) \tag{6.36}$$

Consequently we have $a_1 = 5$, $a_2 = a_3 = 0$ and $a_4 = -2$.

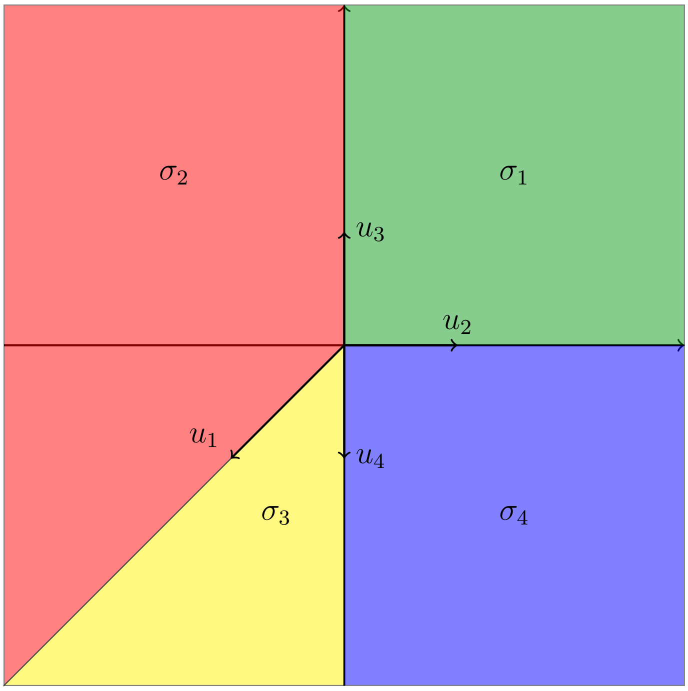


Figure 6.1.: The fan of a del Pezzo 1 surface dP_1.

Construction 6.4.1 (Chamber Construction)**:**
Recall that

$$H^0\left(U_\sigma, \mathcal{O}_{X_\Sigma}(D)\right) = \bigoplus_{m \in P_D(U_\sigma) \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \tag{6.37}$$

where

$$P_D\left(U_\sigma\right) = \left\{ m \in \mathbb{R}^2 \;,\; \langle m, u_\rho \rangle \geq -a_\rho \;\forall \rho \in \sigma(1) \right\} \tag{6.38}$$

Hence we need an efficient way to compute the number of points in the polytopes $P_D\left(U_\sigma\right)$. To this end one introduces the following sets

$$L_\rho = \left\{ m \in \mathbb{R}^2 \;,\; \langle m, u_\rho \rangle + a_\rho = 0 \right\} \tag{6.39}$$

In the current dP_1-situation these sets form lines because the fan Σ lies in $\mathbb{R}^2$. In general though these sets are affine planes.
The lines L_i separate the fan Σ of the dP_1 into disjoint sets - the so-called *chambers*. We illustrate the chambers in Figure 6.2. It is then immediately clear, that there are only two compact chambers, namely R_{+++-} and R_{+-+-}. One can argue from compactness of dP_1 that for the calculation of the Čech cohomology groups on dP_1 only contributions from compact chambers need to be taken into account. Consequently we can focus on Laurent monomials stemming from R_{+++-} and R_{+-+-}.

Consequence:
Given the chamber decomposition in Figure 6.2, it is not too hard to compute the

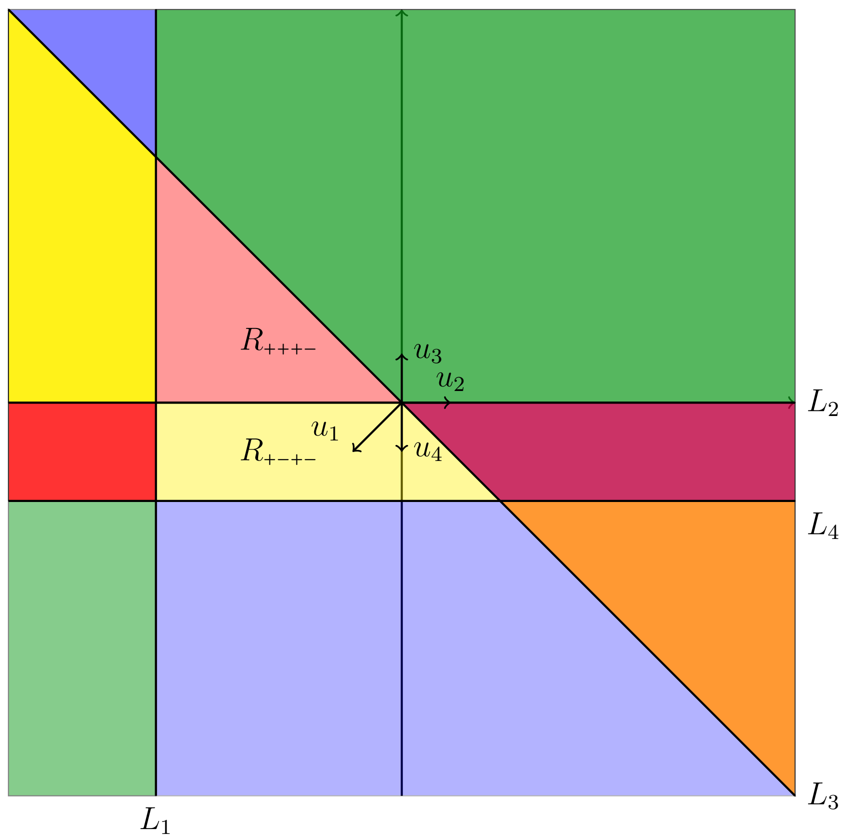


Figure 6.2.: The chambers for the computation of the cohomologies of $\mathcal{O}_{dP_1}(5,-2)$.

Čech cochains. Therefore we only state the results and leave it to the interested reader to confirm these results. We define $R \coloneqq R_{+++-} \cup R_{+-+-}$. Then the result reads

$$\check{C}^0\left(\mathcal{U}, \mathcal{O}_{dP_1}(5,-2)\right) = \begin{pmatrix} \bigoplus\limits_{m \in R_{+++-}} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R_{+++-}} \mathbb{C} \cdot \chi^m \\ 0 \\ 0 \end{pmatrix} \tag{6.40}$$

$$\check{C}^1\left(\mathcal{U}, \mathcal{O}_{dP_1}(5,-2)\right) = \begin{pmatrix} \bigoplus\limits_{m \in R_{+++-}} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ 0 \end{pmatrix} \tag{6.41}$$

$$\check{C}^2\left(\mathcal{U}, \mathcal{O}_{dP_1}(5,-2)\right) = \begin{pmatrix} \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \\ \bigoplus\limits_{m \in R} \mathbb{C} \cdot \chi^m \end{pmatrix} \tag{6.42}$$

$$\check{C}^3\left(\mathcal{U}, \mathcal{O}_{dP_1}(5,-2)\right) = \left(\bigoplus_{m \in R} \mathbb{C} \cdot \chi^m \right) \tag{6.43}$$

Note:
To compute the Čech cohomologies of the holomorphic line bundle $\mathcal{L} = \mathcal{O}_{dP_1}(5,-2)$ we now consider the Čech complex given by

$$0 \to \check{C}^0(\mathcal{U},\mathcal{L}) \xrightarrow{f_0} \check{C}^1(\mathcal{U},\mathcal{L}) \xrightarrow{f_1} \check{C}^2(\mathcal{U},\mathcal{L}) \xrightarrow{f_2} \check{C}^3(\mathcal{U},\mathcal{L}) \to 0 \to 0 \to \dots \tag{6.44}$$

where the mappings f_i are given by the following matrices

$$\begin{aligned} M_{f_0} &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \\ M_{f_1} &= \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \\ M_{f_2} &= \begin{pmatrix} -1 & 1 & -1 & 1 \end{pmatrix} \end{aligned} \tag{6.45}$$

Consequence:
Given the previous results, it is now an easy task to compute the cohomology of the above Čech complex. Again we leave it to the interested reader to confirm the following results.

- $\check{H}^0(\mathcal{U},\mathcal{L}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \cong \{0\}$

- $\check{H}^1(\mathcal{U},\mathcal{L}) = \left\{ \begin{pmatrix} 0 \\ \sum\limits_{m \in R_{+-+-}} b_m \cdot \chi^m \\ \sum\limits_{m \in R_{+-+-}} b_m \cdot \chi^m \\ \sum\limits_{m \in R_{+-+-}} b_m \cdot \chi^m \\ \sum\limits_{m \in R_{+-+-}} b_m \cdot \chi^m \\ 0 \end{pmatrix}, \; b_m \in \mathbb{C} \right\} \cong \left\{ \sum\limits_{m \in R_{+-+-}} b_m \cdot \chi^m \;, \; b_m \in \mathbb{C} \right\}$

homogeneous coordinates	Q_1	Q_2	Q_3	Q_4
x_1	1	0	0	1
x_2	1	0	1	0
x_3	1	1	0	0
x_4	0	1	0	0
x_5	0	0	1	0
x_6	0	0	0	1

Table 6.2.: $(\mathbb{C}^*)^4$-action on $\mathbb{C}^6$ which for a suitably chosen fan gives a del Pezzo 3-surface dP_3.

- $\check{H}^2(\mathcal{U},\mathcal{L}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \cong \{0\}$
- $\check{H}^3(\mathcal{U},\mathcal{L}) = \{0\}$

Note:
By working out the Laurent monomials in the chamber R_{+-+-} explicitely one finds that $\check{H}^1(\mathcal{U},\mathcal{L})$ is the following linear span over $\mathbb{C}$

$$\check{H}^1(\mathcal{U},\mathcal{L}) = \left\langle \frac{x_1^6}{x_3x_4}, \frac{x_2x_1^5}{x_3x_4}, \frac{x_2^2x_1^4}{x_3x_4}, \frac{x_2^3x_1^3}{x_3x_4}, \frac{x_2^4x_1^2}{x_3x_4}, \frac{x_2^5x_1}{x_3x_4}, \frac{x_2^6}{x_3x_4} \right\rangle \tag{6.46}$$

6.5. Line Bundle Cohomology on dP$_3$ Via Chamber Counting

6.5.1. The Toric Data

Note:
As discussed in the dP_1 example we specify dP_3 via the $(\mathbb{C}^*)^4$-action displayed in Table 6.2 and its Stanley-Reisner ideal

$$I_{\mathrm{SR}} = \langle x_1x_2, x_1x_3, x_1x_6, x_2x_3, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_5x_6 \rangle \tag{6.47}$$

Consequence:
The ray generators are easily obtained from Table 6.2. This gives

$$u_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6.48}$$

$$u_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad u_5 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad u_6 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.49}$$

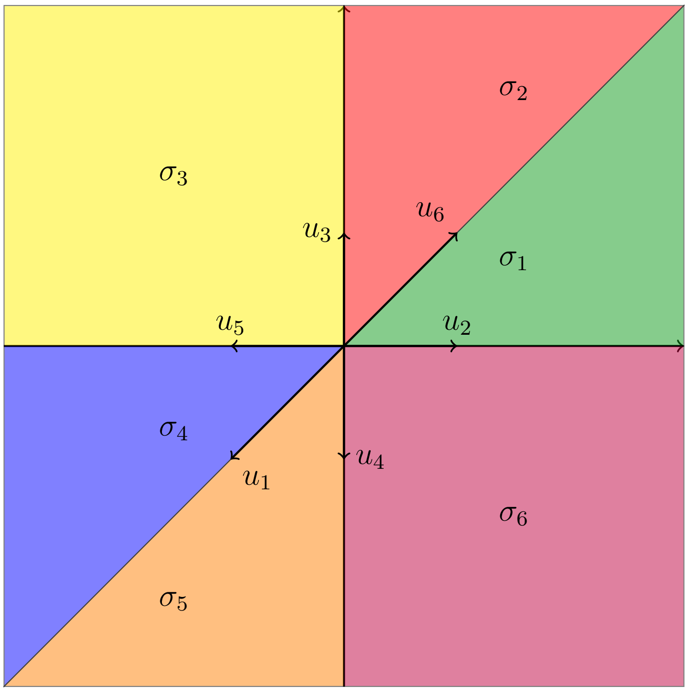


Figure 6.3.: The fan of a del Pezzo 3 surface dP_3.

Together with the Stanley-Reisner-Ideal, this enables us to plot the fan Σ in $\mathbb{R}^2$ of dP_3. This fan we depicture in Figure 6.3.

6.5.2. The Picard Group

Note:
It is readily verified that

$$\operatorname{Pic}(dP_3) \cong \mathbb{Z}[D_1] \oplus \mathbb{Z}[D_2] \oplus \mathbb{Z}[D_3] \oplus [D_4] \tag{6.50}$$

We agree on writing $\mathcal{O}_{dP_3}(a,b,c,d)$ for the isomophism class of holomorphic line bundles associated to the divisor class $a[D_1] + b[D_2] + c[D_3] + d[D_4] \in \mathrm{Cl}(dP_3)$.

6.5.3. Example: Cohomologies of $\mathcal{O}_{\mathbf{dP_3}}(-1,-1,-1,0)$

Note:
Our goal is to compute the Čech cohomology groups of $\mathcal{O}_{dP_3}(-1,-1,-1,0)$. Let us emphasise that now the affine open cover consists of 6 affine toric varieties. Therefore this calculation is more elaborate than for a dP_1.

Construction 6.5.1 (Chamber Decomposition)**:**
To obtain the chamber decomposition we again introduce the sets

$$L_i := \left\{ m \in \mathbb{R}^2 \,,\ \langle m, u_i \rangle + a_i = 0 \right\} \tag{6.51}$$

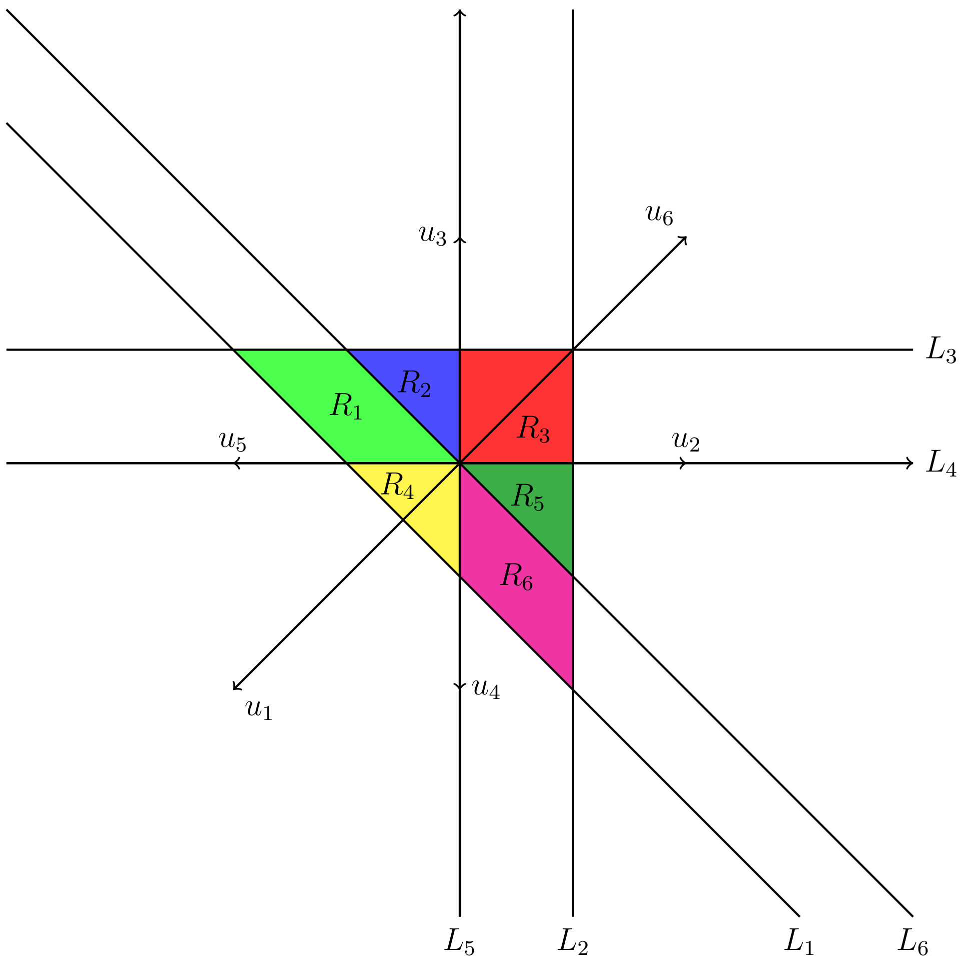


Figure 6.4.: The chambers for the computation of the cohomologies of $\mathcal{O}_{dP_3}(-1,-1,-1,0)$. Note that only the compact chambers are coloured.

These sets separate $\mathbb{R}^2$ into the chambers. We picture the chamber decomposition in Figure 6.4. Note that we only colour the compact chambers.

Consequence:
From the chamber decomposition one easily computes the Čech cochains. Let us set $R := R_1 \cup R_2 \cup \cdots \cup R_6$ and agree that in the following equations R_i is to mean the $\mathbb{C}$ vector space spanned by all Laurent monomials in R_i. With this agreement the Čech cochains for $\mathcal{L} = \mathcal{O}_{dP_3}(-1,-1,-1,0)$ can be written as follows

- $\check{C}^0(\mathcal{U},\mathcal{L}) = (0,0,0,0,0,0)^T$
- $\check{C}^1(\mathcal{U},\mathcal{L}) = (R_2 \cup R_3 \cup R_5, R, R, R, 0, 0, R, R, R, R_1 \cup R_2 \cup R_4, R, R, 0, R, R_4 \cup R_5 \cup R_6)^T$
- $\check{C}^2(\mathcal{U},\mathcal{L}) = \underbrace{(R, R, \ldots, R)^T}_{20 \text{ times}}$

- $\check{C}^3(\mathcal{U},\mathcal{L}) = \underbrace{(R,R,\ldots,R)^T}_{15 \text{ times}}$

- $\check{C}^4(\mathcal{U},\mathcal{L}) = (R,R,R,R)^T$

- $\check{C}^5(\mathcal{U},\mathcal{L}) = (R)$

- $\check{C}^p(\mathcal{U},\mathcal{L}) = 0$ for $p \geq 6$

Note:
The Čech complex for $\mathcal{L}$ is now given by

$$0 \to \check{C}^0(\mathcal{U},\mathcal{L}) \overset{f_0}{\to} \check{C}^1(\mathcal{U},\mathcal{L}) \overset{f_1}{\to} \check{C}^2(\mathcal{U},\mathcal{L}) \overset{f_2}{\to} \check{C}^3(\mathcal{U},\mathcal{L}) \overset{f_3}{\to} \ldots \tag{6.52}$$

The mapping matrices are dictated by the alternating property of the Čech differential maps. Explicitely one has

$$M_{f_0} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{6.53}$$

and

$$M_{f_1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix} \tag{6.54}$$

Consequence:
One finds that $\check{H}^0(\mathcal{U}, \mathcal{L}) = \check{H}^2(\mathcal{U}, \mathcal{L}) = \check{H}^3(\mathcal{U}, \mathcal{L}) = \{0\}$ whilst

$$\check{H}^1(\mathcal{U}, \mathcal{L}) = \left(\bigoplus_{m \in R_4 \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \right) \cdot \boldsymbol{v}_1 \oplus \left(\bigoplus_{m \in R_2 \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \right) \cdot \boldsymbol{v}_2 \tag{6.55}$$

where

$$\begin{aligned} \boldsymbol{v}_1 &= (0, 0, -1, -1, 0, 0, -1, -1, 0, -1, -1, 0, 0, 1, 1)^T \\ \boldsymbol{v}_2 &= (-1, -1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0)^T \end{aligned} \tag{6.56}$$

But since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly indepedent over $\mathbb{C}$ we can thus conclude

$$\check{H}^1(\mathcal{U}, \mathcal{L}) \cong \left(\bigoplus_{m \in R_4 \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \right) \oplus \left(\bigoplus_{m \in R_2 \cap \mathbb{Z}^2} \mathbb{C} \cdot \chi^m \right) \tag{6.57}$$

Remark:
We now come to a crucial point - we have to determine $R_4 \cap \mathbb{Z}^2$ and $R_2 \cap \mathbb{Z}^2$. To this end we have to think carefully about the boundaries of R_2 and R_4 in order to determine which lattice points are contained in R_2 and R_4 respectively.
To this end first recall the definition of the polytopes $P_D(U_\sigma)$ as

$$P_D(U_\sigma) = \left\{ m \in \mathbb{Z}^2 \, , \, \langle m, u_\rho \rangle \geq -a_\rho \; \forall \rho \in \sigma(1) \right\} \tag{6.58}$$

where we emphasise the inequality. This inequality needs to be taken care of - in particular when one defines the chamber decomposition. This is also the reason why the chambers in Figure 6.2 got termed R_{+++-} and R_{+-+-} respectively.
This notation is made such that a plus indicates that the corresponding line is contained in the chamber, whilst a minus sign signals that it is not. So for example the line L_1 is part of R_{+++-} as well as R_{+-+-} since there is a plus at position one in both cases. Conversely L_2 is only part of R_{+++-} but not of R_{+-+-}.
If one goes through the same analysis in the current calculation on dP_3 one finds that

$$R_2 \cap \mathbb{Z}^2 = R_4 \cap \mathbb{Z}^2 = \{\mathbf{0}\} \tag{6.59}$$

which corresponds to the Laurent monomial $\frac{1}{x_1 x_2 x_3}$.

Consequence:
We thus conclude that $\check{H}^0(\mathcal{U}, \mathcal{L}) = \check{H}^2(\mathcal{U}, \mathcal{L}) = \check{H}^3(\mathcal{U}, \mathcal{L}) = \{0\}$ whilst

$$\check{H}^1(\mathcal{U}, \mathcal{L}) \cong \left\{ \frac{a_1}{x_1 x_2 x_3} \ , \ a_1 \in \mathbb{C} \right\} \oplus \left\{ \frac{a_2}{x_1 x_2 x_3} \ , \ a_2 \in \mathbb{C} \right\} \tag{6.60}$$

Note:
In this case, there is only a unique Laurent monomial contributing to $\check{H}^1(\mathcal{U}, \mathcal{L})$. Still the dimension of this cohomology group is two. Such multiplicities are thus important and need to be taken care of. We will come back to this important observation momentarily, when discussing a faster means to compute cohomology groups of holomorphic line bundles on smooth and compact normal toric varieties, the *cohomCalg* algorithm.

6.6. Line Bundle Cohomology Via *cohomCalg*

6.6.1. Introduction

For the computation of sheaf cohomology of holomorphic line bundles on smooth and compact normal toric varieties there exists a fast algorithm that was orginally proposed in [54] and subsequently proven in [62] and [63]. This algorithm goes by the name *cohomCalg*. Its applications are outlined in [55] and [56]. Therefore we will only briefly state the algorithm and subsequently exemplify its power and speed by working out the cohomology calculation from the preceeding two examples with the *cohomCalg* algorithm. Still, let us emphasis that the chamber counting algorithm will be anything but useless towards the end of this thesis.

6.6.2. The Algorithm

- Let $I_{\mathrm{SR}} = \left\{ S_1, \ldots, S_{|\mathrm{SR}|} \right\}$ the Stanley-Reisner ideal of X_Σ and $P(I_{\mathrm{SR}})$ its power set (i.e. the set of all subsets of I_{SR}). Then

$$P(I_{\mathrm{SR}}) = \bigcup_{k=0}^{|I_{\mathrm{SR}}|} P_k(I_{\mathrm{SR}}) \tag{6.61}$$

where $P_k(I_{\mathrm{SR}})$ is the set of all subsets of I_{SR} which are composed of exactly k subsets of I_{SR}.

- Now let $A = \{\alpha_1, \dots, \alpha_k\} \subset \{1, \dots, |I_{\mathrm{SR}}|\}$ and consider

$$P_A^k = \{S_{\alpha_1}, \dots, S_{\alpha_k}\} \in P_k(I_{\mathrm{SR}}) \tag{6.62}$$

  Then define

  - $Q_A^k := \bigcup_{i=1}^{k} S_{\alpha_i}$, the set of all homogeneous coordinates in P_A^k.
  - $N_A^k := |Q_A^k| - k$, the so-called 'c-degree'.

- For fixed k define

$$Q^k := \bigcup_A Q_A^k \tag{6.63}$$

  where A runs over all subsets of $\{1, \dots, |\mathrm{SR}|\}$ of lenght k.

- In order to determine to which cohomology group a given set Q_A^k does contribute, one follows the following steps:
  1. Let $c^i(Q_A^k)$ be the number of times that Q_A^k does appear with c-degree i in the sets Q^k.
  2. Then consider the complex [1]

$$\dots \to 0 \to c^0\left(Q_A^k\right) \to c^1\left(Q_A^k\right) \to \dots \to c^d\left(Q_A^k\right) \to \dots \tag{6.64}$$

     and denote the dimension of its cohomology class at position i by $h^i(Q_A^k)$.
  3. For every $h^i(Q_A^k) \neq 0$ the set Q_A^k gives a contribution to $H^i(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ via

$$h^i\left(Q_A^k\right) \cdot \frac{T(\boldsymbol{x})}{\prod y_i \cdot W(\boldsymbol{y})} \tag{6.65}$$

     with $\boldsymbol{y} \in Q_A^k$ and $\boldsymbol{x} \in H - Q_A^k$, where T, W are monomials of degrees such that the above rationom does match the degrees of D under the $(\mathbb{C}^*)^a$-torus action on X_Σ.

6.6.3. Example: Cohomologies Of $\mathcal{O}_{\mathrm{dP}_1}(5, -2)$ On dP_1

Data For *cohomCalg*

We prepare the application of the *cohomCalg* algorithm by defining the generators of the Stanley-Reisner-Ideal as

$$S_1 := x_1 x_2, \qquad S_2 := x_3 x_4 \tag{6.66}$$

[1]The meaning and proper definition of these complexes are given in the proofs of the *cohomCalg* algorithm, i.e. in [62] and [63]. We just mention that it is these intermediate cohomologies that take care of multiplicities, such as the one encountered in the calculation of $H^i(dP_3, \mathcal{O}_{dP_3}(-1,-1,-1,0))$ where the single rationom $\frac{1}{x_1 x_2 x_3}$ gave rise to a 2-complex dimensional cohomology class.

Consequently it holds

$$P\left(I_{\text{SR}}\left(X_{\Sigma}\right)\right) = \left\{\varnothing, \left\{S_1\right\}, \left\{S_2\right\}, \left\{S_1, S_2\right\}\right\} \tag{6.67}$$

In particular we get

- $P_0^0 = \{\varnothing\}$ with $Q_0^0 = \varnothing$
- $P_1^1 = \{S_1\}$ with $Q_1^1 = \{x_1, x_2\}$
- $P_2^1 = \{S_2\}$ with $Q_2^1 = \{x_3, x_4\}$
- $P_{12}^2 = \{S_1, S_2\}$ with $Q_{12}^2 = \{x_1, x_2, x_3, x_4\}$

The c-degrees are

$$N_0^0 = 0 - 0 = 0, \quad N_1^1 = 2 - 1 = 1, \quad N_2^1 = 2 - 1 = 1, \quad N_{12}^2 = 4 - 2 = 2 \tag{6.68}$$

$\mathbf{H^0\left(dP_1, \mathcal{O}_{dP_1}\left(5, -2\right)\right)}$:

The only P_A^i with c-degree 0 is P_0^0. Thus only P_0^0 does contribute to $H^0\left(dP_1, \mathcal{O}\left(5, -2\right)\right)$. Consequently only monomials

$$T\left(x_1, x_2, x_3, x_4\right) = x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}, \quad a_i \geq 0 \tag{6.69}$$

with charge $(5, -2)$ contribute. However, inspection of Table 6.1 shows that there are no monomials with negative charge under Q_2. Thus we conclude

$$H^0\left(dP_1, \mathcal{O}_{dP_1}\left(5, -2\right)\right) = \{0\} \tag{6.70}$$

$\mathbf{H^1\left(dP_1, \mathcal{O}_{dP_1}\left(5, -2\right)\right)}$:

Now we have to consider the P_A^i with c-degree 1. There are two, namely

$$P_1^1 = \{S_1\} = \{x_1 x_2\}, \quad P_2^1 = \{S_2\} = \{x_3 x_4\} \tag{6.71}$$

Consequently there are in principle two rationom contributions to this cohomology group, namely

$$R_{(1)} = \frac{x_3^{a_3} x_4^{a_4}}{x_1^{a_1+1} x_2^{a_2+1}}, \qquad R_{(2)} = \frac{x_1^{a_1} x_2^{a_2}}{x_3^{a_3+1} x_4^{a_4+1}} \tag{6.72}$$

where $a_i \in \mathbb{Z}_{\geq 0}$. Next we impose that these rationoms have to carry charge $(5, -2)$ under the action of $\left(\mathbb{C}^*\right)^2$ as given in Table 6.1. This requirement rules out $R_{(1)}$ and leaves us with

$$\check{H}^1\left(\mathcal{U}, \mathcal{L}\right) = \left\{b_1 \frac{x_1^6}{x_3 x_4} + b_2 \frac{x_2 x_1^5}{x_3 x_4} + b_3 \frac{x_2^2 x_1^4}{x_3 x_4} + b_4 \frac{x_2^3 x_1^3}{x_3 x_4} + b_5 \frac{x_2^4 x_1^2}{x_3 x_4} + b_6 \frac{x_2^5 x_1}{x_3 x_4} + b_7 \frac{x_2^6}{x_3 x_4}\right\} \tag{6.73}$$

In concluding this we used that $S_1 \cap S_2 = \varnothing$ implies that all intermediate cohomologies are trivial.

$\mathbf{H^2}\left(\mathbf{dP_1}, \mathcal{O}\left(\mathbf{5}, \mathbf{-2}\right)\right)$:

In this case only P_{12}^2 has c-degree 2. Imposing that the corresponding rationoms be of charge $(5, -2)$ rules all of them out. Consequently we have

$$H^2\left(dP_1, \mathcal{O}_{dP_1}\left(5, -2\right)\right) = \{0\} \tag{6.74}$$

6.6.4. Example - Cohomologies Of $\mathcal{O}_{\mathbf{dP_3}}\left(-1, -1, -1, 0\right)$ On $\mathbf{dP_3}$

This example is worked out in detail in [54]. Hence we can just quote that indeed one reproduces the result that we obtained from the chamber counting approach. Note that the multiplicity which we encountered from $R_2 \cap R_4 = \{0\}$ is in this computation reflected in the non-trivial intermediate cohomologies stemming from the fact that in the Stanley-Reisner-ideal of dP_3 variables x_i appear multiple times. Note in particular that a deep insight into this interplay has been obtained from the two proofs of the cohomCalg algorithm in [62] and [63].

6.6.5. Basis Of The Cohomology Groups

Note:
Let us assume that we consider a smooth and compact normal toric variety X_Σ and considered a line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}\left(D\right)$ on X_Σ for some $D \in \mathrm{Div}_T\left(X_\Sigma\right)$. Let us further assume that the above procedure gave us that $H^0\left(X_\Sigma, \mathcal{L}\right)$ was spanned by the rationoms R_1, R_2, but with mutliplicities 1 and 2 say. Then we could represent this cohomology class as follows

$$H^0\left(X_\Sigma, \mathcal{L}\right) = \{\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_2\ ,\ \alpha_i \in \mathbb{C}\} \cong \mathbb{C}^3 \tag{6.75}$$

The different coefficients $\alpha_2, \alpha_3 \in \mathbb{C}$ for R_2 indicate that this vector space does contain the direct sum of two spaces spanned by R_2.

Consequence:
Let X_Σ a smooth and compact normal toric variety. Then the *cohomCalg* algorithm allows for the calculation of a basis of the cohomology classes of all holomorphic line bundles $\mathcal{O}_{X_\Sigma}\left(D\right)$ for $D \in \mathrm{Cl}\left(X_\Sigma\right)$.

6.6.6. Computer Implementation

Comment:
The existing *cohomCalg* software [2] does not return a basis of the cohomology groups to the user. Rather it either returns the dimension of the cohomology classes or via the option 'integrated' an output styled for use in *Mathematica*. The latter can be used in order to compute a basis of the cohomology groups in *Mathematica*. Therefore

[2]as well as its Koszul extension, that we discuss in the next chapter.

we have written a *Mathematica* notebook that performs this task for us. The source code of an extended version of this notebook is given in Appendix E and explained in chapter 15.

6.7. The First Chern Class Of Holomorphic Line Bundles

Remark:
In this section we present the *Demazure vanishing theorem*, from which we conclude that holomorphic line bundles on a smooth and compact normal toric variety are uniquely specified by their first Chern class. To this end we follow closely to [52].

Definition 6.7.1 (Convex Support)**:**
A fan $\Sigma \subset \mathbb{R}^n$ such that

- $|\Sigma| \subset \mathbb{R}^n$ is convex
- $\dim_{\mathbb{R}} (|\Sigma|) = n$

is said to have *convex support*.

Consequence:
Any complete fan has convex support.

Definition 6.7.2:
Let X_Σ a normal toric variety and $D \in \mathrm{Div}(X_\Sigma)$ a Weil divisor. Then D is a $\mathbb{Q}$-Cartier divisor precisely if there exists $a \in \mathbb{Z}$ such that $a \cdot D$ is a Cartier divisor.

Example 6.7.1:
On any normal toric variety X_Σ, the trivial divisor is the divisor of a constant and non-zero function and thus a Cartier divisor. In particular this implies that the trivial divisor is a $\mathbb{Q}$-Cartier divisor.

Remark:
The following definition makes use of the intersection product which we do not introduce here. For details the interested reader is refered to [52].

Definition 6.7.3 (Nef Divisor)**:**
Let X_Σ a normal toric variety and D a $\mathbb{Q}$-Cartier divisor on X_Σ. Then D is nef precisely if for every irreducible complete curve $C \subset X_\Sigma$ it holds $D \cdot C \geq 0$.

Remark:
Nef is short for *numerically efficient*.

Lemma 6.7.1 (A Criterion For Nef)**:**
Let X_Σ a normal toric variety such that Σ has convex support. Moreover let D a $\mathbb{Q}$-Cartier divisor on X_Σ. Then the following are equivalent.

- D is nef.
- There exists $a \in \mathbb{Z}_{>0}$ such that the holomorphic line bundle $\mathcal{O}_{X_\Sigma}(a \cdot D)$ is generated by its global sections, i.e. is flabby.

Consequence:
Let X_Σ a smooth and compact normal toric variety. Then Σ is complete and hence has convex support. Moreover any compact normal toric variety is connected, i.e.

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(0)) \cong \mathbb{C} \tag{6.76}$$

and all holomorphic functions on X_Σ are constant functions. Consequently $\mathcal{O}_{X_\Sigma}(0)$ is generated by its global sections. Together with the above lemma, this shows that the trivial divisor on a smooth and compact normal toric variety is nef.

Theorem 6.7.1 (Demazure vanishing)**:**
Let X_Σ a normal toric variety such that $|\Sigma|$ is convex and $D \in \mathrm{Div}(X_\Sigma)$ a $\mathbb{Q}$-Cartier nef divisor. Then

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = 0 \quad \forall p > 0 \tag{6.77}$$

Consequence:
Let us consider a smooth and compact normal toric variety X_Σ. Then Σ is complete, which shows that $|\Sigma|$ is convex. We already argued that the trivial divisor $D = 0$ on such a toric variety is a $\mathbb{Q}$-Cartier nef divisor. Consequently we find from Demazure vanishing

$$H^p(X_\Sigma, \mathcal{O}_{X_\Sigma}) = 0 \quad \forall p > 0 \tag{6.78}$$

The structure sheaf of a smooth and compact normal toric variety X_Σ is hence acyclic.

Consequence:
Let us now consider on a smooth and compact toric variety X_Σ the sheaf exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_\Sigma} \to \mathcal{O}^*_{X_\Sigma} \to 0 \tag{6.79}$$

The associated long exact sequence in sheaf cohomology contains the following part

$$\cdots \to H^1(X_\Sigma, \mathbb{Z}_{X_\Sigma}) \to H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}) \to H^1\left(X_\Sigma, \mathcal{O}^*_{X_\Sigma}\right) \to H^2(X_\Sigma, \mathbb{Z}_{X_\Sigma}) \to \dots \tag{6.80}$$

The preceeding result thus shows that all holomorphic line bundles on a smooth and compact normal toric variety are uniquely specified by their first Chern class. We thus conclude.

A $U(1)$ gauge theory on a smooth and compact normal toric variety X_Σ is uniquely specified by the curvature 2-form or equivalently its field strength.

Remark:

- This situation is to be contrasted to complex torus $\mathbb{C}_{1,\tau}$, where the structure sheaf is not acyclic and consequently, as we point out in Appendix C, holomorphic line bundles are *not* uniquely determined by their first Chern class.

- A different means to find this result is to use that a smooth and compact normal toric variety is simply connected. As pointed out in [37] this already implies that holomorphic line bundles are uniquely specified by their first Chern class.

- Finally note that on a smooth and compact normal toric variety X_Σ, we have an isomorphism $\mathrm{Cl}\left(X_\Sigma\right) \cong \mathrm{Pic}\left(X_\Sigma\right)$, so that a holomorphic line bundle is on the one hand uniquely specified by its first Chern class and on the other hand by its associated divisor. There are thus two means by which one can specify a holomorphic line bundle on X_Σ and one can also identify them with each other. Our prefered picture in this thesis will be to specify a holomorphic line bundle on X_Σ by its divisor class.

Part III.

Cohomology Of Holomorphic Pullback Line Bundles On Algebraic Submanifolds Of Smooth And Compact Normal Toric Varieties Via Exactness Properties

7. Summary

In Part II we learned the following.

- How to describe a holomorphic line bundle $\mathcal{L}$ on a smooth and compact normal toric variety X_Σ by a divisor class $D \in \mathrm{Cl}(X_\Sigma)$.
- How to compute the cohomology classes of $\mathcal{L}$.
- That we can think of the cohomology classes of $\mathcal{L}$ as spanned over the complex numbers by quotients of monomials in the homogeneous coordinates of X_Σ. Let us also mention that such quotients are refered to as *rationoms*.

In this part we now want to consider submanifolds of a smooth and compact normal toric variety X_Σ. A non-trivial statement is, that any such manifold $C \subset X_\Sigma$ is an algebraic submanifold, i.e. of the form

$$C = \{p \in X_\Sigma \, , \; P_1(p) = \cdots = P_n(p) = 0\} \tag{7.1}$$

for finitely many homogeneous polynomials $P_1, \ldots, P_n$ in the homogeneous coordinates of X_Σ. This can intuitively be thought of as the generalisation of Chow's theorem [64] to toric varieties. We give details on this statement in section 8.2.
Note that the set C as defined above is not guaranteed to be smooth. Rather one has to pick $P_1, \ldots, P_n$ such that the so-called variety C is smooth. Then C is an algebraic submanifold. Let us briefly exemplify that the smoothness condition is indeed necessary. First consider the algebraic variety given by

$$C_1 = \left\{(x_1, x_2) \in \mathbb{C}^2 \, , \; x_1 = 0\right\} \tag{7.2}$$

This variety C_1 is indeed smooth. However the algebraic variety C_2 given by

$$C_2 = \left\{(x_1, x_2) \in \mathbb{C}^2 \, , \; x_1 x_2 = 0\right\} \tag{7.3}$$

is singular at the origin.
Given a smooth and compact normal toric variety X_Σ, a holomorphic line bundle $\mathcal{L}$ on X_Σ and an algebraic submanifold C of X_Σ one can consider the line bundle $\mathcal{L}|_C$. Sheaf theoretically this line bundle is obtained from the restriction of the sections of $\mathcal{L}$ onto C. Our final task is to compute the cohomologies of $\mathcal{L}|_C$ from cohomologies of suitable line bundles on the ambient space X_Σ.
The technology used to perform this calculation is the so-called Koszul sequence. This sequence we present in chapter 8. A simplified means to evaluate this sequence is by

use of its exactness property. Unter fortunate circumstances this allows to compute the dimension of the cohomology classes of $\mathcal{L}|_C$. Still this is not true generally. We will explain how one evaluates the Koszul sequence by exactness properties in chapter 9 and also give examples in which exactness is not enough to compute the cohomologies of $\mathcal{L}|_C$.

8. The Koszul Resolution

8.1. Summary

For briefness, let us exemplify the Koszul sequence for the case that C is of codimension 3, as this is the setup that we will encounter in applications to physics most of the time. So let us consider the following setup.

- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ for $D \in \mathrm{Cl}(X_\Sigma)$.
- $\widetilde{s}_1 \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_1))$ a non-trivial polynomial. [1]
- $\widetilde{s}_2 \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_2))$ a non-trivial polynomial.
- $\widetilde{s}_3 \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_3))$ a non-trivial polynomial.

Then we consider

$$C \coloneqq \{p \in X_\Sigma\ ,\ \widetilde{s}_1(p) = \widetilde{s}_2(p) = \widetilde{s}_3(p) = 0\} \tag{8.1}$$

and require that

- C is smooth
- $C \subset X_\Sigma$ has codimension 3

In this setup the Koszul sequence is then given by

$$0 \to \mathcal{L}' \xrightarrow{\alpha} \mathcal{V}_2 \xrightarrow{\beta} \mathcal{V}_1 \xrightarrow{\gamma} \mathcal{L} \xrightarrow{r_C} \mathcal{L}|_C \to 0 \tag{8.2}$$

where

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(D - S_1 - S_2 - S_3)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}(D - S_2 - S_3) \oplus \mathcal{O}_{X_\Sigma}(D - S_1 - S_3) \oplus \mathcal{O}_{X_\Sigma}(D - S_1 - S_2)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(D - S_1) \oplus \mathcal{O}_{X_\Sigma}(D - S_2) \oplus \mathcal{O}_{X_\Sigma}(D - S_3)$

[1] For such a non-trivial polynomial to exist, the divisor class $S_1 \in \mathrm{Cl}(X_\Sigma)$ has to be effective, i.e. all integer coefficients must be non-negative.

and the mappings are induced from the following matrices

$$\alpha = \begin{pmatrix} \widetilde{s}_1 \\ -\widetilde{s}_2 \\ \widetilde{s}_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix}, \quad \gamma = (\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) \tag{8.3}$$

Note that r_C is the ordinary restriction onto C. We give details on this construction in section 8.4 and section 8.5.
The crucial point about the Koszul sequence is that this sequence is exact. This means [2]

$$\ker(\beta) = \operatorname{im}(\alpha), \quad \ker(\gamma) = \operatorname{im}(\beta), \quad \ldots \tag{8.4}$$

To compute the cohomologies of $\mathcal{L}|_C$ from the exactness of the Koszul sequence one now applies the following three-step procedure.

1. First we apply the so-called splitting principle. This tells us that instead of the long exact Koszul sequence we can consider the following collection of short exact sequences.
   - $0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_2 \to \mathcal{I}_1 \to 0$
   - $0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0$
   - $0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_C \to 0$

   In these expressions the sheaves $\mathcal{I}_1$, $\mathcal{I}_2$ are quotients of direct sums of holomorphic line bundles. The general theory of sheaves tells us that those are in general no longer vector bundles but coherent sheaves. In particular e.g. the mapping $\mathcal{V}_2 \to \mathcal{I}_1$ is involved. We give more details on this in section A.1 and section 8.6.

2. Every short exact sequence of sheaves induces a long exact sequence in the cohomologies of the appearing sheaves. For example the short exact sequence $0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_2 \to \mathcal{I}_1 \to 0$ induces the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{L}') \arrow[r, "\alpha^0"] & H^0(X,\mathcal{V}_2) \arrow[r, "\beta^0"] & H^0(X,\mathcal{I}_1) \arrow[dll, "\delta^0"] \\
& H^1(X,\mathcal{L}') \arrow[r, "\alpha^1"] & H^1(X,\mathcal{V}_2) \arrow[r, "\beta^1"] & H^1(X,\mathcal{I}_1) \arrow[dll, "\delta^1"] \\
& H^2(X,\mathcal{L}') \arrow[r, "\alpha^2"] & H^2(X,\mathcal{V}_2) \arrow[r, "\beta^2"] & H^2(X,\mathcal{I}_1) \arrow[dll, dashed] \\
& H^n(X,\mathcal{L}') \arrow[r, "\alpha^n"] & H^n(X,\mathcal{V}_2) \arrow[r, "\beta^n"] & H^n(X,\mathcal{I}_1)
\end{tikzcd}$$

[2]Let us mention that we actually we mean sheaf exactness. Then e.g. $\operatorname{im}(\alpha)$ must be read as the sheaf image. This construction is non-trivial and we give details in section A.1.

In particular note that the maps α^i are induced from the map α and β^i is induced from $\beta\colon \mathcal{V}_2 \to \mathcal{I}_1$. We mentioned already that the mapping β is involved. Consequently alo the induced maps β^i are involved. Consequently we prefer not to touch them just yet but will come back to this in Part IV.

3. Under fortunate circumstances one can compute from exactness properties of the above long exact sequence in cohomologies, the cohomologies of $\mathcal{I}_1$ from knowledge about the cohomologies of $\mathcal{L}'$ and $\mathcal{V}_2$. The cohomologies of the latter two can always be computed from the *cohomCalg* algorithm. So let us assume that indeed we were able to compute the cohomologies of $\mathcal{I}_1$ from exactness only. Then we would plug these cohomology classes into the long exact sequence of cohomologies induced from the second short exact sequence

$$0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0 \tag{8.5}$$

and thereby, under fortunate circumstances, compute the cohomologies of $\mathcal{I}_2$ from exactness only. Playing the same game with the third short exact sequence

$$0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{8.6}$$

then allows us to deduce the cohomologies of $\mathcal{L}|_C$ from exactness.

Let us emphasis though that exactness is in general not enough to deduce from a long exact sequence in cohomologies and knowledge about the cohomologies of two of the three sheaves the cohomologies of the third sheaf. This however can be done if the mappings in the long exact sequence of cohomologies are known. This we will explain in Part IV.

8.2. Submanifolds Of Smooth And Compact Normal Toric Varieties

Remark:
Recall that we are interested in smooth and compact normal toric varieties X_Σ. For such toric varieties we always have

$$X_\Sigma \cong \left(\mathbb{C}^r - V\left(B_{X_\Sigma}\right)\right) / \left(\mathbb{C}^*\right)^a \tag{8.7}$$

where $B_{X_\Sigma} \subset \mathbb{C}\left[x_1, \ldots, x_r\right]$ is the irrelevant ideal. Note also that $\mathbb{C}\left[x_1, \ldots, x_r\right]$ together with the grading induced by the $\left(\mathbb{C}^*\right)^a$ is the total coordinate ring S of X_Σ.

Theorem 8.2.1 (Hilbert Basis Theorem [65])**:**
Let R a commutative, Noetherian ring. Then $R\left[x_1, \ldots, x_n\right]$ is also Noetherian.

Consequence:
The coordinate ring S is a Noetherian ring. So every ideal $I \subset S$ is finitely generated.

Definition 8.2.1 (Radical Ideal)**:**
Let $I = \langle p_1, \ldots, p_n \rangle \subset S$ an ideal. Then one defines the radical of I as

$$\sqrt{I} := \{ Q \in S \ , \ Q^n \in I \text{ for suitable } n \in \mathbb{N}_{>0} \} \tag{8.8}$$

The ideal I is a radical ideal precisely if $I = \sqrt{I}$.

Example 8.2.1:
Let $\langle x^2 \rangle \subset \mathbb{C}[x]$ an ideal. Then $\sqrt{I} = \langle x \rangle$.

Definition 8.2.2 (Homogeneous Ideal)**:**
Recall that the coordinate ring S is graded from the $(\mathbb{C}^*)^a$ action defining X_Σ. Consequently we can require $f \in S$ to be homogeneous. An ideal

$$I = \langle f_1, \ldots, f_m \rangle \subset S \tag{8.9}$$

is a homogeneous ideal precisely if $f_1, \ldots, f_m$ are all homogeneous.

Note:
In the above definition $f_1, \ldots, f_m \in S$ need not have the same homogeneous degrees, but each of the polynomials f_i has to be a homogeneous polynomial by itself.

Theorem 8.2.2 (Subvarieties of X_Σ)**:**
Let X_Σ a smooth and compact toric variety. Then the following two are one-to-one.

- Closed subvarieties of X_Σ.
- Radical homogeneous ideals $I \subseteq B_{X_\Sigma} \subseteq S$.

Note:
The above theorem holds true even if X_Σ is only simplicial. Note that 'radical' can intuitiely be though of as 'minimal'. For example we can consider the variety

$$V\left(\left\langle x_1^2 \right\rangle\right) = \left\{ (x_1, x_2) \in \mathbb{C}^2 \ , \ x_1^2 = 0 \right\} \tag{8.10}$$

Set theoretically

$$V\left(\langle x_1 \rangle\right) = \left\{ (x_1, x_2) \in \mathbb{C}^2 \ , \ x_1 = 0 \right\} = V\left(\left\langle x_1^2 \right\rangle\right) \tag{8.11}$$

Hence taking x_1 as defining polynomial is 'better' than x_1^2 or even x_1^3. This is also in favour of computation that we will perform later - then namely reducing the degree of a possibly very long polynomial means to reduce the computational effort considerably.
Still in principle we can ignore the issue of I being radical and just take a homogeneous ideal. Then the following major implication arises.

Consequence:
Every closed variety of a smooth and compact normal toric variety is an algebraic variety. So in particular all analytic submanifolds of X_Σ are obtained from a finitely generated ideal $I \subset S$.

Definition 8.2.3 (Dimension Of An Algebraic Subvariety):
Let X_Σ a smooth and compact normal toric variety and $C \subset X_\Sigma$ an algebraic subvariety defined by $I \subset S$ a homogeneous ideal. Then

$$\mathbb{C}_C := \mathbb{C}\left[X_\sigma\right]/I \tag{8.12}$$

is the coordinate ring of C. The dimension of C is the Krull dimension of $\mathbb{C}_V$.

Remark:
The definition of Krull dimension for rings coincides with the geometric definition [60]. However the Krull dimension can be easier implemented in computer algebra software. For example the computation of dimensionality in *Sage* [51] relies on this definition. In particular one can use *Sage* to check that a submanifold is indeed of the desired codimension.

8.3. The Notion Of A Pullback Line Bundle

Definition 8.3.1 (Pullback $\mathcal{O}$-Modul):
Let X_Σ a smooth and compact normal toric variety and $C \subset X_\Sigma$ an algebraic submanifold. Then we have a canonical inclusion map

$$\iota: C \to X_\Sigma \tag{8.13}$$

Let $\mathcal{F}$ an $\mathcal{O}_{X_\Sigma}$-modul and $U \subset C$ open with the induced topology. Then by definition of the induced topology there exists $V \subset X_\Sigma$ open in X_Σ such that $U = V \cap C$. Now we define an $\mathcal{O}_C$-modul $\mathcal{F}|_C$ via

$$\left(\mathcal{F}|_C\right)(U) = \left\{g|_C \ , \ g \in \mathcal{F}(V)\right\} = \mathcal{F}(V)|_C \tag{8.14}$$

This we term the pullback $\mathcal{O}_C$-modul of $\mathcal{F}$.

Remark:

- We make use of the smoothness of C in that we consider $U \subset C$ open. In case that C was a singular space, one would consider analytic subsets instead.
- Holomorphic line bundles and direct sums thereof can be pulled back according to the above definition, since a holomorphic line bundle is a locally free $\mathcal{O}_{X_\Sigma}$-modules, i.e. a special $\mathcal{O}_{X_\Sigma}$-module.

8.4. The Koszul Complex

Definition 8.4.1 (Koszul Complex):
Let R a commutative ring, $n \in \mathbb{N}_{>0}$ and $x_1, \ldots, x_n \in R$. We want to define a complex $K_n^\bullet(x_1, \ldots, x_n)$ from this. To this end set

- $K_p := 0$ for $p < 0$ and $p > n$.
- $K_0 := R$
- For $1 \leq p \leq n$ set
$$K_p := \bigoplus_{1 \leq i_1 < \cdots < i_p \leq n} R \cdot e_{i_1} \wedge \cdots \wedge e_{i_p} \tag{8.15}$$
so that K_p is the free module of rank $\binom{n}{p}$ with basis $\left\{e_{i_1} \wedge e_{i_p}\ ,\ 1 \leq i_1 < \cdots < i_p \leq n\right\}$ over R.

Now we define differential maps $\partial \colon K_p \to K_{p-1}$ by setting

- $\partial\left(e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_r} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p}$
- for $p = 1$ set $\partial\left(e_i\right) = x_i$.

Claim:
This construction defines a complex.

Proof
From the definition of $K_p^\bullet$ is follows immediately that we have defined a complex. ∎

Example 8.4.1:

- The complex $K_1^\bullet$ is given by
$$\ldots 0 \leftarrow 0 \leftarrow R \xleftarrow{x_1} R \leftarrow 0 \leftarrow 0 \leftarrow \ldots \tag{8.16}$$
- The complex $K_2^\bullet$ is given by
$$\ldots 0 \leftarrow 0 \leftarrow R \xleftarrow{\beta} R^2 \xleftarrow{\alpha} R \leftarrow 0 \leftarrow 0 \leftarrow \ldots \tag{8.17}$$
where $\alpha = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$ and $\beta = (x_1, x_2)$.
- The complex $K_3^\bullet$ is given by
$$\ldots 0 \leftarrow 0 \leftarrow R \xleftarrow{\gamma} R^3 \xleftarrow{\beta} R^3 \xleftarrow{\alpha} R \leftarrow 0 \leftarrow 0 \leftarrow \ldots \tag{8.18}$$
where the maps are given by
$$\alpha = \begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}, \quad \gamma = (x_1, x_2, x_3) \tag{8.19}$$

Note:
$K_p^\bullet(x_1, \ldots, x_n)$ need not be exact. This is because the elements $x_1, \ldots, x_n$ are so-far just some elements in the commutative ring R. This we change now.

Definition 8.4.2 (Weak Regular Sequence)**:**
Let R a commutative ring and $x_1, \ldots, x_n \in R$. These elements form a weak regular sequence precisely if the following holds true.

- x_1 is not a zero-divisor in R.
- x_2 is not a zero-divisor in $R/(x_1)$.
- x_3 is not a zero-divisor in $R/(x_1, x_2)$.
- $\ldots$

Theorem 8.4.1:
Let R a commutative ring and $x_1, \ldots, x_p \in R$. We set $I := (x_1, \ldots, x_p) \subset R$ and assume that $x_1, \ldots, x_p$ form a weak regular sequence. Then

$$0 \to K_p^\bullet(x_1, \ldots, x_p) \to R/I \to 0 \tag{8.20}$$

is exact.

Proof
A proof can be found in [60]. ∎

Comment:
Theorem 8.4.1 is the local version of the Koszul resolution, to which we turn next. In particular note that the condition of a weak regular sequence then translates into the local condition that $C \subset X_\Sigma$ has codimension p.

8.5. The Koszul Resolution

Claim (Koszul Resolution)**:**
Let X_Σ a smooth compact toric variety. Let us consider effective divisor classes $S_1, \ldots, S_n \in \mathrm{Cl}(X_\Sigma)$ and non-trivial holomorphic sections

$$\widetilde{s}_i \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_i)) \tag{8.21}$$

such that

$$C := \{p \in X_\Sigma\ ,\ \widetilde{s}_1(p) = \cdots = \widetilde{s}_n(p) = 0\} \subset X_\Sigma \tag{8.22}$$

We is an algebraic submanifold of X_Σ of codimension n.
Given this setup consider the holomorphic line bundle $\mathcal{O}_{X_\Sigma}(D)$ on X_Σ described by its divisor class $D \in \mathrm{Cl}(X_\Sigma)$. From section 8.3 we know that this setup allows us to consider the pullback line bundle $\mathcal{O}_{X_\Sigma}(D)|_C$.
Given such a setup, we claim that the following sequence is sheaf exact

$$0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{O}_{X_\Sigma}(D) \xrightarrow{\mathrm{r}} \mathcal{O}_{X_\Sigma}(D)|_C \to 0 \tag{8.23}$$

The sheaves appearing in this sequence are defined as

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}\left(D - \sum_{i=1}^{n} S_i\right)$
- $\mathcal{V}_k = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} \mathcal{O}_{X_\Sigma}\left(D - \sum_{j=1}^{k} S_{i_j}\right)$

The last map in Equation 8.23 is the restriction map to C. All other mappings are induced from the Koszul complex $K_n^\bullet$ over the ring $\mathcal{O}_{X_\Sigma} = \mathbb{C}\left(X_\Sigma\right)$.

Example 8.5.1:
Before we prove the above statement let us give an example of a Koszul resolution. To this end let us consider three effective divisor classes $S_1, S_2, S_3 \in \mathrm{Cl}\left(X_\Sigma\right)$. Given such divisor classes we pick three non-trivial global holomorphic sections

$$\widetilde{s}_1 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_1\right)\right), \quad \widetilde{s}_2 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_2\right)\right), \quad \widetilde{s}_3 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_3\right)\right) \tag{8.24}$$

We assume that their common zero locus

$$C := \{p \in X_\Sigma \, , \, \widetilde{s}_1(p) = \widetilde{s}_2(p) = \widetilde{s}_3(p) = 0\} \tag{8.25}$$

is an algebraic submanifold of X_Σ of codimension 3. Then according to the above statement we should consider the sequence

$$0 \to \mathcal{L}' \xrightarrow{\alpha} \mathcal{V}_2 \xrightarrow{\beta} \mathcal{V}_1 \xrightarrow{\gamma} \mathcal{O}_{X_\Sigma}(D) \xrightarrow{\mathrm{r}} \mathcal{O}_{X_\Sigma}(D)|_C \to 0 \tag{8.26}$$

for a line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ of interest. The other sheaves in that sequence are given by

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_2 - S_3\right)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}\left(D - S_2 - S_3\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_3\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_2\right)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}\left(D - S_1\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_2\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_3\right)$

The sheaf homomorphisms in this sequence are induced by the following matrices of global sections

$$\alpha = \begin{pmatrix} \widetilde{s}_1 \\ -\widetilde{s}_2 \\ \widetilde{s}_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix}, \quad \gamma = \left(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3\right) \tag{8.27}$$

This is to be compared to Equation 8.19, as these maps are induced from the Koszul complex $K_3^\bullet$.

Proof
For the proof of sheaf exactness of the Koszul resolution we recall two important facts.

- Sheaf exactness is a local property.
- Vector bundles of rank r are locally free $\mathcal{O}_{X_\Sigma}$-modules of rank r.

Given a point $p \in X_\Sigma$ and a sufficiently small open neighbourhood $p \in U \subset X_\Sigma$ one consequently has to prove exactness of the following sequence

$$0 \to \mathcal{O}_{X_\Sigma}(U) \to \mathcal{O}_{X_\Sigma}(U)^{\oplus \binom{n}{n-1}} \to \cdots \to \mathcal{O}_{X_\Sigma}(U) \to \left(\mathcal{O}_{X_\Sigma}(U)|_C \right)(U) \to 0 \qquad (8.28)$$

We are therefore looking at the Koszul complex $K_n^\bullet(\widetilde{s}_1, \ldots, \widetilde{s}_n)$ over the ring $\mathcal{O}_{X_\Sigma}(X_\Sigma)$. The condition that C has codimension n translates into the condition that the germs of $\widetilde{s}_1, \ldots, \widetilde{s}_n \in \mathcal{O}_{X_\Sigma}(X_\Sigma)$ form a weak regular sequence in the local coordinate ring at p. The statement now follows from Theorem 8.4.1. ■

Remark:

- The proof does not make use of the smoothness of C. This is correct, since the statement holds more generally for algebraic subvarieties of X_Σ. Then however, the definition of a pullback line bundle must be refined. We briefly mentioned this in section 8.3. Let us therefore mention that the smoothness condition is supplemented with a look towards applications in F-theory model building.
- More information on the Koszul resolution can be found in standard textbooks such as [44] or [66]. Note however that usually only the Koszul complex is treated in detail. The Koszul resolution as we presented it above and will use it during the rest of this thesis, is just a simple consequence from the technology of Koszul complexes.

8.6. Splitting Principle Applied

Lemma 8.6.1:
Any long exact sequence of Abelian groups can be split into a number of short exact sequences.

Proof
We assume that the long exact sequence of Abelian groups is of the form

$$0 \to A_1 \to A_2 \to A_3 \to \ldots \qquad (8.29)$$

Then introduce an Abelian group I_1 such that

$$0 \to A_1 \to A_2 \to I_1 \to 0 \qquad (8.30)$$

is a short exact sequence. Subsequently introduce an Abelian group I_2 such that

$$0 \to I_1 \to A_3 \to I_2 \to 0 \qquad (8.31)$$

is a short exact sequence.
It is clear that the abelian groups I_i do exist and that by following the above procedure one obtains a number of short exact sequences which give the same information as the long exact sequence that we started with. ■

Consequence:
The above generalises to sheaf exact sequences and in particular allows to split the Koszul resolution by introducing auxilliary sheaves $\mathcal{I}_1, \ldots, \mathcal{I}_{N-1}$ into a number of short sheaf exact sequences

- $0 \to \mathcal{L}' \to \mathcal{V}_{N-1} \to \mathcal{I}_1 \to 0$
- $0 \to \mathcal{I}_1 \to \mathcal{V}_{N-2} \to \mathcal{I}_2 \to 0$
- $0 \to \mathcal{I}_2 \to \mathcal{V}_{N-3} \to \mathcal{I}_{N-3} \to 0$
- $\ldots$
- $0 \to \mathcal{I}_{N-2} \to \mathcal{V}_1 \to \mathcal{I}_{N-1} \to 0$
- $0 \to \mathcal{I}_{N-1} \to \mathcal{L} \to \mathcal{L}|_C \to 0$

Note:
The fundamental theorem of homological algebra states that any short exact sequence of complexes of Abelian groups gives rise to a long exact sequence in cohomology of these three complexes in the short exact sequence. Via the Godement resolution of any sheaf this theorem implies that also any short exact sequence of sheaves gives rise to a long exact sequence in sheaf cohomologies.
By use of this fact the above splitting of the Koszul resolution can be used to compute from knowledge of the cohomology groups of $\mathcal{L}'$, $\mathcal{V}_k$ and $\mathcal{L}$ the cohomologies of $\mathcal{L}|_C$. Under good circumstances [3] one can even deduce the dimensions of the cohomologies of $\mathcal{L}|_C$ from exactness of the long exact cohomology sequences alone, without having to know anything about the maps involved.
The computation of the dimension of the cohomologies of $\mathcal{L}|_C$ based on the use of exactness alone has been implemented in the Koszul extension of *cohomCalg* [53]. In the next chapter we will give an example of the above-mentioned good circumstances, but will also give two other examples where these good circumstances do not appear. The remainder of this thesis will then focus on determining the cohomology groups of $\mathcal{L}|_C$ by means of the sheaf homomorphisms involved.

[3] This means that a sufficiently high number of the ambient space cohomology groups are trivial. An example of this is presented in the next chapter.

9. Limits Of The Koszul-Extension Of cohomCalg

9.1. Summary

In chapter 8 we presented the Koszul sequence and pointed out that it relates the desired pullback cohomologies to certain ambient space cohomologies. We also mentioned that exactness can be used to constraint the pullback cohomologies easily, but not necessarily uniquely. In this chapter we explain in detail how this works.
To this end we present simple consequences from the exactness of sequences in section 9.2. The most important result is as follows. Let

$$A_1 \to A_2 \to \cdots \to A_n \tag{9.1}$$

a long exact sequence of finite dimensional vector spaces of dimensions a_1, a_2, ..., a_n respectively. Then it holds

$$a_1 - a_2 + a_3 - a_4 + \cdots \pm a_n = 0 \tag{9.2}$$

This knowledge we put to a use in section 9.3, where we start the discussion of an example that will guide us through the remainder of this thesis. For this example we specify a smooth and compact normal toric variety in subsection 9.3.1. Here we mention briefly that X_Σ is of the form

$$X_\Sigma = \left(\mathbb{C}^8 - Z\right) / \left(\mathbb{C}^*\right)^4 \tag{9.3}$$

and that divisor classes can canonically be identified with elements in $\mathbb{Z}^4$. For this reason we can consider the effective divisor classes [1]

- $S_{B_3} = (3, 2, 1, 1)$
- $S_{\text{GUT}} = (1, 1, 0, 0)$
- $S_{10} = (2, 1, 2, 1)$
- $S_{\overline{5}m} = (4, 1, 6, 3)$
- $S_{5H} = (7, 2, 10, 5)$

[1]The terminology for these divisor classes is motivated from an F-theory application that we will present at the very end of this thesis.

Now consider non-trivial global sections $\widetilde{s}_i \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_i))$ in order to define

$$C_{10} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{10}}}(p) = 0\} \tag{9.4}$$

$$C_{\overline{5}m} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{5m}}}(p) = 0\} \tag{9.5}$$

$$C_{5H} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{5H}}}(p) = 0\} \tag{9.6}$$

All three varieties are required to be smooth and of codimension 3 in X_Σ. In addition it should be pointed out that the actual form of the polynomials $\widetilde{s}_i$ shape the curves C_i. This implies that the coefficients of the polynomials $\widetilde{s}_i$ can be identified as a redundant description of the complex structure of the curves C_i. In particular one should expect that for $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$ the cohomologies of $\mathcal{L}|_{C_i}$ do depend on the complex structure of the curve C_i via the coefficients in the defining polynomials of C_i.
This indeed will turn out to be true. In contrast however, evaluating the Koszul sequence by means of exactness only, does not require this information. So one should not be surprised that exactness alone is in general not enough to compute the cohomologies of $\mathcal{L}|_{C_i}$. This we illustrate in the above setup. As we point out in subsection 9.3.5 the cohomologies of $\mathcal{L}|_{C_{10}}$ turn out to be indepedent of the actual description of C_{10}, whilst for $C_{\overline{5}m}$ and C_{5H} this does not hold true as shown in subsection 9.3.6 and subsection 9.3.7 respectively. In the latter two cases this dependence is reflected in an unknown integer valued constant appearing in the dimensions of the cohomologies of $\mathcal{L}|_{C_{\overline{5}m}}$ and $\mathcal{L}|_{C_{5H}}$.
This problem is of course known to the authors of the *cohomCalg* algorithm and its *Koszul extension* which automises the evaluation of the Koszul spectral sequence by means of exactness only [53, pp. 20]. In particular it is mentioned there, that so-far no functionality is implemented to compute this complex structure dependence. Therefore our final goal is to make one step towards closing this gap.
However, before we do so, let us mention that sometimes bounds on the dimension of the cohomology groups are all that a model builder needs to rule out a certain model. For this reason we present in section 9.4 how exact sequence technology allows for such simple bounds. The results presented there are of the following type. Assume that $A_1 \to X \to A_2 \to 0$ is an exact sequence of finite dimensional vector spaces. Then it follows

$$\dim(X) \leq \dim(A_1) + \dim(A_2) \tag{9.7}$$

We exemplify the use of these kinds of inequalities in section 9.5 where we estimate the cohomology classes on $C_{\overline{5}m}$ and C_{5H}. The so-obtained estimates are summarised in Table 9.3.

9.2. Exact Sequence Technology - Part I

9.2.1. The Basics

Remark (Exact Sequence)**:**
Let $(A_i)_{i\in\mathbb{Z}}$ be a family of finite dimensional $\mathbb{K}$-vector spaces and $(f_i\colon A_i \to A_{i+1})_{i\in\mathbb{Z}}$ a

family of vector space homomorphisms such that $f_{i+1} \circ f_i = 0$ for all $i \in \mathbb{Z}$. Then the complex

$$\cdots \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \ldots \tag{9.8}$$

is exact precisely if

$$\mathrm{im}\left(A_{i-1} \to A_i\right) = \ker\left(A_i \to A_{i+1}\right) \quad \forall i \in \mathbb{Z} \tag{9.9}$$

Remark (Dimension Formula For Vector Space Homomorphisms)**:**
Let V, W finite dimensional $\mathbb{K}$-vector spaces and $\varphi \colon V \to W$ a vector space homomorphism. Then it holds

$$\dim_{\mathbb{K}}\left(\mathrm{im}\left(\varphi\right)\right) = \dim_{\mathbb{K}}\left(V\right) - \dim_{\mathbb{K}}\left(\mathrm{Ker}\left(\varphi\right)\right) \tag{9.10}$$

9.2.2. Simple Consequences

Claim:
Let $n \geq 3$. Then consider the exact sequence $A_1 \to A_2 \to A_3 \to \cdots \to A_n$ of $\mathbb{K}$-vector spaces of dimensions a_1, a_2, ..., a_n over $\mathbb{K}$. Then it holds

$$\dim_{\mathbb{K}}\left(\ker\left(A_{n-1} \to A_n\right)\right) = a_{n-2} - a_{n-3} + a_{n-4} - a_{n-5} \cdots \pm a_1 \mp \dim_{\mathbb{K}}\left(\ker\left(A_1 \to A_2\right)\right) \tag{9.11}$$

Proof

- We begin by analysing the exactness at position A_{n-1}. There we find

$$\begin{aligned} \dim_{\mathbb{K}}\left(\ker\left(A_{n-1} \to A_n\right)\right) &= \dim_{\mathbb{K}}\left(\mathrm{im}\left(A_{n-2} \to A_{n-1}\right)\right) \\ &= a_{n-2} - \dim_{\mathbb{K}}\left(\ker\left(A_{n-2} \to A_{n-1}\right)\right) \end{aligned} \tag{9.12}$$

- Similarly one finds at position A_k

$$\begin{aligned} \dim_{\mathbb{K}}\left(\ker\left(A_k \to A_{k+1}\right)\right) &= \dim_{\mathbb{K}}\left(\mathrm{im}\left(A_{k-1} \to A_k\right)\right) \\ &= a_{k-1} - \dim_{\mathbb{K}}\left(\ker\left(A_{k-1} \to A_k\right)\right) \end{aligned} \tag{9.13}$$

An induction by the sequence index now yields the claim. ■

Consequence:
Be $n \geq 2$ and $0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$ an exact sequence of $\mathbb{K}$-vector spaces of dimensions a_1, a_2, ..., a_n over $\mathbb{K}$. Then it holds

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} - a_{n-4} \cdots \pm a_1 \tag{9.14}$$

Proof
The two zeros limiting the sequence imply

- $\dim_{\mathbb{K}}\left(\ker\left(A_1 \to A_2\right)\right) = 0$

	Q_1	Q_2	Q_3	Q_4
x_1	1	1	0	0
x_2	0	0	1	0
x_3	0	0	0	1
x_4	1	0	0	0
x_5	1	1	0	0
x_6	1	0	1	1
x_7	1	0	1	0
x_8	0	1	0	0

Table 9.1.: Action of the algebraic torus $(\mathbb{C}^*)^4$ onto $\mathbb{C}^8$. This is part of the defining data of the toric ambient space X_Σ in the exhaustive example.

- $\dim_{\mathbb{K}}(\ker(A_{n-1} \to A_n)) = a_{n-1} - a_n$

The statement now follows from the preceeding one. ∎

Consequence:

- Let $0 \to A \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then $A = \{0\}$.
- Let $0 \to A_1 \to A_2 \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then $\dim_{\mathbb{K}}(A_1) = \dim_{\mathbb{K}}(A_2)$.
- Let $0 \to A_1 \to A_2 \to A_3 \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then it holds $\dim_{\mathbb{K}}(A_3) = \dim_{\mathbb{K}}(A_2) - \dim_{\mathbb{K}}(A_1)$.

9.3. An Exhaustive Example

9.3.1. Toric Ambient Space And Stanley-Reisner-Ideal

We start from an action of the algebraic torus $(\mathbb{C}^*)^4$ onto $\mathbb{C}^8$ as outlined in table Table 9.1. From these one finds the ray generators of the fan Σ to be

$$u_1 = e_1, \quad u_2 = e_2, \quad u_3 = e_3, \quad u_4 = e_4, \quad u_5 = -e_1 + e_2 - e_4 \tag{9.15}$$

$$u_6 = -e_3, \quad u_7 = e_3 - e_2, \quad u_8 = -e_2 + e_4 \tag{9.16}$$

We note that any fan with these ray generators is smooth. It remains to require in addition that the fan Σ be complete, so that X_Σ is compact. From this condition, the computer program *Sage* [51] is able to find two triangulations[2] with the following Stanley-Reisner ideals.

$$I_{\mathrm{SR}}^{(a)} = \langle x_2x_7, x_2x_8, x_3x_6, x_3x_8, x_1x_4x_5, x_1x_5x_8, x_4x_6x_7 \rangle \tag{9.17}$$

[2]We described in subsection D.5.2 how this is done.

$$I_{\mathrm{SR}}^{(b)} = \langle x_2x_7, x_2x_8, x_3x_6, x_4x_7, x_1x_4x_5, x_1x_5x_8 \rangle \tag{9.18}$$

9.3.2. Divisor Classes

One easily confirms

$$\mathrm{Cl}\left(X_\Sigma\right) = \mathbb{Z}\left[D_1\right] \oplus \mathbb{Z}\left[D_2\right] \oplus \mathbb{Z}\left[D_3\right] \oplus \mathbb{Z}\left[D_4\right] \cong \mathbb{Z}^4 \tag{9.19}$$

Therefore we agree on the following notation for a divisor class.

$$\left(a_1, a_2, a_3, a_4\right) \equiv a_1\left[D_1\right] + a_2\left[D_2\right] + a_3\left[D_3\right] + a_4\left[D_4\right] \tag{9.20}$$

9.3.3. Pullback Setup

We consider the following effective divisor classes

- $S_{B_3} = (3,2,1,1)$
- $S_{\mathrm{GUT}} = (1,1,0,0)$
- $S_{10} = (2,1,2,1)$
- $S_{\overline{5}m} = (4,1,6,3)$
- $S_{5H} = (7,2,10,5)$

Given non-trivial holomorphic sections $\widetilde{s}_i$ in the associated holomorphic line bundles, we wish to consider the following algebraic subvarieties

$$C_{10} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{10}}}(p) = 0\} \tag{9.21}$$
$$C_{\overline{5}m} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{5m}}}(p) = 0\} \tag{9.22}$$
$$C_{5H} = \{p \in X_\Sigma\ ,\ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = \widetilde{s_{D_{5H}}}(p) = 0\} \tag{9.23}$$

Those are subject to being smooth and of codimension 3 in X_Σ. Finally we consider the line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$. Our task is to compute the cohomologies of $\mathcal{L}|_{C_{10}}$, $\mathcal{L}|_{C_{\overline{5}m}}$ and $\mathcal{L}|_{C_{5H}}$.

9.3.4. Ambient Space Cohomology Dependence On The Stanley-Reisner-Ideal

To compute the pullback cohomologies one uses the Koszul resolution. Therefore the first objects to compute are the cohomology classes of all the bundles on X_Σ that appear in the Koszul resolution. For the time being all that we need are the dimensions of these cohomology classes, and those turn out to be independent of I_{SR} in this particular setup.

Whilst this indepence of the dimensions of the cohomology classes from the Stanley-Reisner-ideal is expected, it should be mentioned that representations for the basis of these cohomology classes will in general depend on the Stanley-Reisner-ideal, as follows since the *cohomCalg* algorithm is heavily dependend on I_{SR}.

9.3.5. Pullback to C_{10}

Remark (First Exact Sequence)**:**
The first exact sequence from the splitting gives the following long exact sequence in cohomology.

$\mathcal{O}_{X_\Sigma}(-4,-3,-1,-1)$	$\mathcal{O}_{X_\Sigma}(-2,-2,1,0) \oplus \mathcal{O}_{X_\Sigma}(-3,-2,-1,-1) \oplus \mathcal{O}_{X_\Sigma}(-1,-1,0,0)$	$\mathcal{I}_1$
0	0	A_1
0	0	A_2
0	1	A_3
0	0	A_4
0	0	A_5

where $\mathcal{I}_1$ is an auxillary sheaf. From the exact sequence technology developed in section 9.2, it follows immediately

$$h^i\left(X_\Sigma, \mathcal{I}_1\right) = (0,0,1,0,0) \tag{9.24}$$

Remark (Second Exact Sequence)**:**
The second exact sequence relates the auxillary sheaf $\mathcal{I}_1$ with the auxillary sheaf $\mathcal{I}_2$ and takes the form

$\mathcal{I}_1$	$\mathcal{O}_{X_\Sigma}(-1,-1,1,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(0,0,0,0)$	$\mathcal{I}_2$
0	4	A_6
0	0	A_7
1	0	A_8
0	0	A_9
0	0	A_{10}

From this we conclude

$$h^i\left(X_\Sigma, \mathcal{I}_2\right) = (4,1,0,0,0) \tag{9.25}$$

Remark (Third Exact Sequence)**:**
The third exact sequence now relates $\mathcal{I}_2$ to the cohomology of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{10}}$ which we intent to compute. This sequence looks like

$\mathcal{I}_2$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)\|_{C_{10}}$
4	11	A_{11}
1	0	A_{12}
0	0	A_{13}
0	0	A_{14}
0	0	A_{15}

Consequently we find

$$h^i\left(C_{10}, \mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{10}}\right) = (8,0,0,0,0) \tag{9.26}$$

Note:
In this particular situation, the exact sequence technology as presented in section 9.2 was enough to determine the dimensions of the cohomology groups of the pullback line bundle uniquely. This will not be true in the following two examples.

9.3.6. Pullback to $C_{\overline{5}m}$

Remark (First Exact Sequence)**:**
The first exact sequence in the splitting of the Koszul resolution gives the following long exact sequence in cohomology.

$\mathcal{O}_{X_\Sigma}(-6,-3,-5,-3)$	$\mathcal{O}_{X_\Sigma}(-2,-2,0,0) \oplus \mathcal{O}_{X_\Sigma}(-5,-2,-5,-3) \oplus \mathcal{O}_{X_\Sigma}(-3,-1,-4,-2)$	$\mathcal{I}_1$
0	0	A_1
0	0	A_2
0	2	A_3
0	0	A_4
3	0	A_5

where $\mathcal{I}_1$ is an auxillary sheaf. The exactness of this sequence implies

$$h^i(X_\Sigma, \mathcal{I}_1) = (0,0,2,3,0) \tag{9.27}$$

Remark (Second Exact Sequence)**:**
The second exact sequence looks like

$\mathcal{I}_1$	$\mathcal{O}_{X_\Sigma}(-1,-1,0,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(-2,0,-4,-2)$	$\mathcal{I}_2$
0	3	A_6
0	0	A_7
2	4	A_8
3	0	A_9
0	0	A_{10}

From this we conclude

$$h^i(X_\Sigma, \mathcal{I}_2) = (3, A_8 - 5, A_8, 0, 0) \tag{9.28}$$

Remark (Third Exact Sequence)**:**
The third exact sequence finally takes the following shape

$\mathcal{I}_2$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)\|_{C_{\overline{5}m}}$
3	11	A_{11}
$-5 + A_8$	0	A_{12}
A_8	0	0
0	0	0
0	0	0

Exact sequence technology now implies

$$h^i\left(C_{\overline{5}m}, \mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{\overline{5}m}}\right) = (3 + A_{12}, A_{12}, 0, 0, 0) \tag{9.29}$$

leaving us with an unconstrained, nonnegative, integer-valued constant A_{12}.

9.3.7. Pullback to $\mathbf{C_{5H}}$

Remark (First Exact Sequence)**:**
The first exact sequence is given as

$\mathcal{O}_{X_\Sigma}(-9,-4,-9,-5)$	$\mathcal{O}_{X_\Sigma}(-2,-2,0,0) \oplus \mathcal{O}_{X_\Sigma}(-8,-3,-9,-5) \oplus \mathcal{O}_{X_\Sigma}(-6,-2,-8,-4)$	$\mathcal{I}_1$
0	0	A'_1
0	0	A'_2
0	17	A'_3
0	0	A'_4
34	10	A'_5

This implies

$$h^i(X_\Sigma, \mathcal{I}_1) = (0, 0, 17, 24 + A'_5, A'_5) \tag{9.30}$$

Remark (Second Exact Sequence)**:**
The second exact sequence then looks like

$\mathcal{I}_1$	$\mathcal{O}_{X_\Sigma}(-1,-1,0,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(-5,-1,-8,-4)$	$\mathcal{I}_2$
0	3	A'_6
0	0	A'_7
17	30	A'_8
$24 + A_5$	0	A'_9
A_5	0	A'_{10}

Exactness properties now yield

$$h^i(X_\Sigma, \mathcal{I}_2) = (3, -37 + A'_8 - A'_9, A'_8, A'_9, 0) \tag{9.31}$$

Remark (Third Exact Sequence)**:**
The third exact sequence finally looks like

$\mathcal{I}_2$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)$	$\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{5H}}$
3	11	A'_{11}
$-37 + A'_8 - A'_9$	0	A'_{12}
A'_8	0	0
A'_9	0	0
0	0	0

From exactness properties we conclude

$$h^i\left(C_{5H}, \mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{5H}}\right) = (A'_{12} - 29, A'_{12}, 0, 0, 0) \tag{9.32}$$

where again $A'_{12} \in \mathbb{N}_{\geq 29}$ is left unconstrained.

	C_{10}	$C_{\overline{5}m}$	C_{5H}
$h^0\left(C_i, \mathcal{L}\|_{C_i}\right)$	8	$3 + A_{12}$	$A'_{12} - 29$
$h^1\left(C_i, \mathcal{L}\|_{C_i}\right)$	0	A_{12}	A'_{12}
$h^2\left(C_i, \mathcal{L}\|_{C_i}\right)$	0	0	0
$h^3\left(C_i, \mathcal{L}\|_{C_i}\right)$	0	0	0
$h^4\left(C_i, \mathcal{L}\|_{C_i}\right)$	0	0	0

Table 9.2.: Cohomology groups of $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$ pulled back onto C_{10}, $C_{\overline{5}m}$ and C_{5H}. Note that $A_{12} \in \mathbb{N}_{\geq 0}$ and $A'_{12} \in \mathbb{N}_{\geq 29}$ are left unconstrained.

9.3.8. Summary On The Cohomologies From Exactness

The dimensions of the pullback cohomologies as determined above, are summarised in Table 9.2.

9.4. Exact Sequence Technology - Part II

Lemma 9.4.1:
Let $A_1 \to X \to A_2 \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then it holds

$$\dim_{\mathbb{K}}(X) \leq \dim_{\mathbb{K}}(A_1) + \dim_{\mathbb{K}}(A_2) \tag{9.33}$$

Proof
From exactness at A_2 it follows

$$\operatorname{im}(X \to A_2) = \ker(A_2 \to 0) = A_2 \tag{9.34}$$

Consequently the dimension formula for vector space homomorphisms implies

$$\dim_{\mathbb{K}}(X) - \dim_{\mathbb{K}}(\ker(X \to A_2)) = \dim_{\mathbb{K}}(A_2) \tag{9.35}$$

Exactness at X then implies

$$\dim_{\mathbb{K}}(X) - \dim_{\mathbb{K}}(\operatorname{im}(A_1 \to X)) = \dim_{\mathbb{K}}(A_2) \tag{9.36}$$

Finally applying the dimension formula a second time gives

$$\dim_{\mathbb{K}}(X) - \dim_{\mathbb{K}}(A_1) + \dim_{\mathbb{K}}(\ker(A_1 \to X)) = \dim_{\mathbb{K}}(A_2) \tag{9.37}$$

This gives the inequality that we are looking for by noting that $\dim_{\mathbb{K}}(\ker(A_1 \to X))$ is always non-negative. ∎

Consequence:
Let $A \to X \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then it holds $\dim_{\mathbb{K}}(X) \leq \dim_{\mathbb{K}}(A)$.

Lemma 9.4.2:
Let $A_0 \to A_1 \to X \to A_2 \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then it holds

$$\dim_{\mathbb{K}}(X) \geq \dim_{\mathbb{K}}(A_2) + \dim_{\mathbb{K}}(A_1) - \dim_{\mathbb{K}}(A_0) \tag{9.38}$$

Proof
Following the strategy used to prove the first lemma, one easily finds

$$\dim_{\mathbb{K}}(X) = \dim_{\mathbb{K}}(\ker(A_0 \to A_1)) + \dim_{\mathbb{K}}(A_2) + \dim_{\mathbb{K}}(A_1) - \dim_{\mathbb{K}}(A_0) \tag{9.39}$$

From this the claim follows immediately. ∎

Consequence:
Let $A_0 \to A_1 \to X \to 0$ an exact sequence of finite dimensional $\mathbb{K}$-vector spaces. Then it holds

$$\dim_{\mathbb{K}}(X) \geq \dim_{\mathbb{K}}(A_1) - \dim_{\mathbb{K}}(A_0) \tag{9.40}$$

9.5. An Exhaustive Example - Bounds On The Cohomology Groups

9.5.1. Pullback To C_{10} - Bounds

The cohomologies on C_{10} were uniquely determined by exactness. Consequently we are already done in this case.

9.5.2. Pullback To $C_{\overline{5}m}$ - Bounds

Remark:
Since $A_{12} = A_8$ we are looking for bounds to A_8.

Claim:
It holds $5 \leq A_8 \leq 7$.

Proof
We have the exact sequence $2 \to 4 \to A_8 \to 3 \to 0$ from which the two bounds follow. ∎

Consequence:
We have thus found $5 \leq A_{12} \leq 7$.

9.5.3. Pullback To C_{5H} - Bounds

Remark:
We first recall $A'_{12} = A'_8$. Thus our first task is to find a bound for A'_8.

	$h^0(C_i, \mathcal{L}\|_C)$	$h^1(C_i, \mathcal{L}\|_C)$	Parameter range
C_{10}	8	0	0
$C_{\overline{5}m}$	$3 + A_{12}$	A_{12}	$5 \leq A_{12} \leq 7$
C_{5H}	$A'_{12} - 29$	A'_{12}	$37 \leq A'_{12} \leq 64$

Table 9.3.: Bounds on the parameters which describe the dimensions of the cohomology groups of the pullback line bundle on the matter curves $C_{\overline{5}m}$ and C_{5H}.

Claim:
It holds $37 + A'_5 \leq A'_8 \leq 54 + A'_5$.

Proof
The exact sequence $17 \to 30 \to A'_8 \to 24 + A'_5 \to 0$ implies both bounds. ∎

Consequence:
We consequently need to find bounds for A'_5 next.

Claim:
It holds $0 \leq A'_5 \leq 10$.

Proof
The exact sequence $34 \to 10 \to A'_5 \to 0$ implies

$$-24 \leq A'_5 \leq 10 \tag{9.41}$$

But since A'_5 is the dimension of a vector space we know $A'_5 \geq 0$. Thus we obtain the statement. ∎

Consequence:
It holds $37 \leq A'_{12} \leq 64$.

9.5.4. Summary

We have thus found the cohomologies as listed in Table 9.3.

Part IV.

Cohomology Of Holomorphic Pullback Line Bundles On Algebraic Submanifolds Of Smooth And Compact Normal Toric Varieties Via The Koszul Spectral Sequence

10. Summary

In this chapter we present the true story behind computing pullback cohomologies. To start off this part, we continue in chapter 11 the analysis of the exhaustive example given in chapter 9. This time however, we use the actual mappings in the Koszul sequence to compute the pullback cohomologies. Thereby we can in these examples identify the dependence of the pullback cohomologies $\mathcal{L}|_{C_i}$ on the complex structure of the curves C_i reflected in the coefficients of their defining polynomials. In particular we point out that the exhaustive example turns out to be an example of the simpler kind. This is reflected in the fact that when we compute the pullback cohomologies $\mathcal{L}|_{C_i}$ there are several maps that are not directly induced from the Koszul sequence, but whose details we need not know for the computations in this particular example. Let us term these maps 'mysterious maps' for the time-being.
The general story is unfortunately more involved than this exhaustive example shows on a first glance. To see this we introduce the Koszul spectral sequence in chapter 12. General theory about sheaves tells us that the Koszul spectral sequence allows for a very efficient and well-organised way of computing pullback cohomologies. In particular it clearifies the origin of the 'mysterious maps'.
Thus the construction of the 'mysterious maps' is in general the real problem in computing pullback cohomologies. The general strategy for the construction of these maps is of course well-known in the mathematics literature. To illustrate this abstract construction, we give an example of the construction of a special such 'mysterous map' called the 'Knight's move' in chapter 13. In this chapter we also give a hint towards a simplified construction of this 'mysterious map'. This hint involves the use of the chamber counting algorithm and the *cohomCalg* algorithm for the calculation of line bundle cohomology on the toric ambient space. In particular we will point out that the chamber counting encodes more information than *cohomCalg*. This additional information in turn is needed to perform the abstract construction of the 'mysterious maps'. Unfortunately obtaining this additional information comes at a high cost - the computer implementation of the chamber counting algorithm in [61] is much slower than *cohomCalg* [57].
Consequently one might ask if there is a faster way for the construction of these 'mysterious' maps. We turn towards this question in chapter 14. In particular we point out that the answer is affirmative given that the smooth and compact normal toric ambient space X_Σ is a so-called generalised Flag variety. Whilst it is well-known that $\mathbb{CP}^n$ does indeed fall into that category it is not known to the author if this holds true for a general smooth and compact normal toric variety. Therefore a proof or disproof of this statement would be very interesting to know of. This however is currently beyond the abilities of the author.

A first approximation of the results from the Koszul spectral sequence is independent of the construction of the 'mysterious' maps. In the language of spectral sequences this is the evaluation of the E_1-sheet. This task has been computerised in a *Mathematica* notebook. We give the source code of this notebook in Appendix E and present a brief manual of this notebook in chapter 15.
We conclude this thesis by putting this notebook to a use in a model building teaser in chapter 16. There we present an $SU(5) \times U(1)_X$ toy-model in F-theory with a special choice of G_4-flux. Counting the number of zero modes along the curves C_{10}, $C_{\overline{5}m}$ and C_{5H} which are charged under this G_4-flux is then performed by computing certain pullback cohomology classes. The latter we use our *Mathematica* notebook for. Let us mention that we do not perform any global checks in this toy-model. So existence of a smooth resolution $\widehat{Y}_4$, tadpole cancellation etc. are not checked, as this toy-model is presented only to demonstrate how our *Mathematica* notebook can be used in future model building work.

11. Leray Property And Induced Cohomology Maps

11.1. Summary

In this chapter we continue the analysis of the exhaustive example presented in chapter 9. In constrast to using exactness properties only, we will now make use of the actual mappings in the Koszul sequence. For a codimension 3 algebraic submanifold C of a smooth and compact normal toric variety X_Σ this sequence is given by

$$0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_2 \overset{\beta}{\to} \mathcal{V}_1 \overset{\gamma}{\to} \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{11.1}$$

We can split this long exact sequence by the so-called splitting-principle into three short exact sequences

- $0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_2 \to \mathcal{I}_1 \to 0$
- $0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0$
- $0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_C \to 0$

This we presented back in section 8.6. Let us focus on the first of the three short exact sequences $0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_2 \to \mathcal{I}_1 \to 0$. This sequence induces the following long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{L}') \arrow[r, "\alpha^0"] & H^0(X,\mathcal{V}_2) \arrow[r] & H^0(X,\mathcal{I}_1) \arrow[dll, "\delta^0"] \\
 & H^1(X,\mathcal{L}') \arrow[r, "\alpha^1"] & H^1(X,\mathcal{V}_2) \arrow[r] & H^1(X,\mathcal{I}_1) \arrow[dll, "\delta^1"] \\
 & H^2(X,\mathcal{L}') \arrow[r, "\alpha^2"] & H^2(X,\mathcal{V}_2) \arrow[r] & H^2(X,\mathcal{I}_1) \arrow[dll, dashed] \\
 & H^n(X,\mathcal{L}') \arrow[r, "\alpha^n"] & H^n(X,\mathcal{V}_2) \arrow[r] & H^n(X,\mathcal{I}_1)
\end{tikzcd}$$

Now two questions are immediate.

- How exactly does the map α induce the maps α^i?

- Given that we know the cohomology groups of $\mathcal{L}'$ and $\mathcal{V}_2$ and even know the mappings α^i, how do we then compute the cohomologies of $\mathcal{I}_1$?

The answer to the first question we give in section 11.2. There we point out that on a smooth and compact normal toric variety there exists a particularly nice open cover such that computing Čech cohomology from this open cover does actually give sheaf cohomology. This open cover is the so-called *affine open cover* $\mathcal{U}$. Consequently we can think about the maps between the sheaf cohomology groups as maps between Čech cohomology groups, i.e. collections of Čech cocycles. But Čech cocycles are equivalence classes of Čech cochains, and Čech cochains in turn only collections of functions on the open sets of X_Σ that form the affine open cover $\mathcal{U}$.
That said let us focus on the situation of a sheaf homomorphism $\alpha\colon \mathcal{O}_{X_\Sigma}(D) \to \mathcal{O}_{X_\Sigma}(D')$ with $D' \geq D$. Then this homomorphism is induced from a global section $\widetilde{s} \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D'-D))$. By the above-presented observation about the affine open cover $\mathcal{U}$, we can think of Čech cochains in $\mathcal{O}_{X_\Sigma}(D)$ to be mapped to Čech cochains in $\mathcal{O}_{X_\Sigma}(D')$ simply by multiplication with $\widetilde{s}$. This observation we can express in the following commutative diagram

$$\begin{tikzcd} \check{C}^0(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) \arrow[r, "\delta"] \arrow[d, "\cdot\widetilde{s}"] & \check{C}^1(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) \arrow[d, "\cdot\widetilde{s}"] \\ \check{C}^0(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D')) \arrow[r, "\delta"] & \check{C}^1(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D')) \end{tikzcd}$$

From this it follows that the maps α^i are just the canonically induced mappings of quotient spaces, i.e. for example

$$\alpha^0\colon H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \to H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D'))\ ,\ x = [X] \mapsto [\widetilde{s}\cdot X] \equiv y \tag{11.2}$$

where $X \in \check{C}^0(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D))$ and $\widetilde{s}\cdot X \in \check{C}^0(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D'))$. We should mention that this construction preserves closure, and that therefore this construction answers the first question.
The answer to the second question is given in section 11.3 and can be stated in the equation

$$H^i(X_\Sigma, \mathcal{I}_1) \cong \operatorname{coker}\left(\alpha^i\right) \oplus \ker\left(\alpha^{i+1}\right) \tag{11.3}$$

By use of these two results it is actually possible to compute the example from chapter 9 completely. This is because the 'mysterious' maps are all trivial in these examples. To illustrate this we point out in the detailed computations outlined in section 11.3, section 11.4 and section 11.5 all 'mysterious maps'. The results from the computations in this chapter are summarised in Table 11.6 and should be compared to the results in Table 9.3 which are obtained only from exactness considerations.
Finally we would like to point out that we discuss the meaning of a 'generic pullback setup' in detail in subsection 11.3.7. This will establish our terminology throughout this thesis.

11.2. Natural Leray Cover Of Toric Varieties And Consequences

11.2.1. The Affine Open Cover

Remark:
Let X_Σ a smooth and compact normal toric variety. Then the affine open cover $\mathcal{U} = \{U_\sigma\}_{\sigma \in \Sigma_{\max}}$ is a Leray cover and thus allows to compute sheaf cohomology from Čech cohomology.

Note:
We heavily used that fact when introducing the classical view of sheaf cohomology on toric varieties in section 6.3.

11.2.2. Induced Maps On Čech Cocycles - Codimension One

Note:
Let X_Σ a smooth and compact normal toric variety. In addition let $S \in \mathrm{Cl}(X_\Sigma)$ an effective divisor class and $\widetilde{s} \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S))$ a global holomorphic section of the associated holomorphic line bundle $\mathcal{O}_{X_\Sigma}(S)$ such that

$$C := \{p \in X_\Sigma \ , \ \widetilde{s}(p) = 0\} \tag{11.4}$$

is an algebraic submanifold with codimension 1 in X_Σ. Then for any $D \in \mathrm{Cl}(X_\Sigma)$ we have the Koszul resolution as given by

$$0 \to \mathcal{O}_{X_\Sigma}(D - S) \overset{\otimes \widetilde{s}}{\to} \mathcal{O}_{X_\Sigma}(D) \overset{\mathrm{r}}{\to} \mathcal{O}_{X_\Sigma}(D)|_C \to 0 \tag{11.5}$$

Recall that the maps in this short exact sequence are sheaf homomorphisms. In particular we have for any open $U \subset X_\Sigma$ a homomorphism of $\mathcal{O}_{X_\Sigma}(U)$-modules

$$\varphi_U \colon (\mathcal{O}_{X_\Sigma}(D - S))(U) \to (\mathcal{O}_{X_\Sigma}(D))(U) \tag{11.6}$$

given by multiplication with $\widetilde{s}|_U$.

Consequence:
Assume that the collection $C = (f_\sigma \in \mathcal{O}_{X_\Sigma}(D - S)(U_\sigma))_{\sigma \in \Sigma_{\max}}$ is a Čech 0-cochain in the sheaf $\mathcal{O}_{X_\Sigma}(D - S)$. Then the following holds true.

- We can use the maps φ_{U_σ} to map the Čech 0-cochain C to a Čech 0-cochain C' in the sheaf $\mathcal{O}_{X_\Sigma}(D)$.
- C is closed with respect to the Čech differential precisely if C' is closed.

Proof
The image of $C = \left(f_\sigma \in \left(\mathcal{O}_{X_\Sigma}(D-S)\right)(U_\sigma)\right)_{\sigma\in\Sigma_{\max}}$ by the map induced from the sheaf homomorphism $\otimes\widetilde{s}$ is

$$C' := \left(\widetilde{s}|_{U_\sigma} \cdot f_\sigma \in \left(\mathcal{O}_{X_\Sigma}(D)\right)(U_\sigma)\right)_{\sigma\in\Sigma_{\max}} \tag{11.7}$$

which indeed forms a Čech 0-cochain in the sheaf $\mathcal{O}_{X_\Sigma}(D)$.
Next we note that $\widetilde{s} \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(S)\right)$. By means of the homogenisation of X_Σ one justifies that $\widetilde{s}$ is a polynomial in the homogeneous coordinates of X_Σ with multidegree S with respect to the defining scaling relations of X_Σ. Recall also that the affine open cover is the union of the following open affine subsets of X_Σ

$$U_i := \{p \in X_\Sigma\ ,\ x_i \neq 0\} \tag{11.8}$$

Consequently $\widetilde{s}|_{U_\sigma} = \widetilde{s}$ for any $\sigma \in \Sigma_{\max}$. This implies that the cocycle condition for C' is just the one for C multiplied by $\widetilde{s}$. But $\widetilde{s}$ is non-trivial, as otherwise its zero locus cannot form a codimension 1 submanifold of X_Σ. Consequently C is closed with respect to the Čech differential precisely if C' is closed. ∎

Comment:
This argument easily generalises to Čech p-cocycles with $p \geq 1$.

Remark:
Čech cohomology classes are sets of equivalence classes of Čech cochains. *cohomCalg* picks certain Čech cochains to represent such a class and thus to form a basis of the cohomology class of interest. By the above means we have maps that take Čech cochains in one sheaf to Čech cochains in another sheaf and respect closure. Thus these maps induce maps on the Čech cohomology classes.
Note however that a choice of basis is not unique. In particular it can happen that we use a Čech cochain C_1 in the domain and map it to a Čech cochain C_2 in the target, of which only part is expressable by means of the Čech cochains that *cohomCalg* chose as target space basis. The crucial insight is then, that the part of C_2 not expressable in the target space basis is zero with respect to the equvialence relations in the target space. Therefore one can focus on the part of the image cochain C_2 that can be expressed in terms of the target space basis and use this part to draw conclusions on the maps of Čech cocycles. We will encounter such situations momentarily.

11.2.3. Induced Maps On Čech Cocycles - Arbitary Codimension

Remark:
All maps in the Koszul resolution are sheaf homomorphism. That said, the above strategy immediately generalises to the situation of arbitary codimension smooth subvarieties of a smooth and compact normal toric variety.

11.3. An Exhaustive Example Continued - Pullback To C_{10}

Remark:
In chapter 9 we found that the dimensions of the cohomology groups of $\mathcal{L}|_{C_{10}}$ could be deduced from exactness alone. Nevertheless we decide to use this example in order to demonstrate the applicability of the techniques introduced in section 11.2.

11.3.1. The Defining Polynomials of C_{10}

Recall that we have to choose holomorphic sections

$$\widetilde{s}_1 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{B_3}\right)\right), \quad \widetilde{s}_2 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{\text{GUT}}\right)\right), \quad \widetilde{s}_3 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{10}\right)\right) \tag{11.9}$$

to define C_{10} as their common zero locus, subject to the condition that C_{10} is smooth and of codimension 3. By recalling that the above divisor classes are given by

$$S_{B_3} = \mathcal{O}_{X_\Sigma}\left(3,2,1,1\right), \quad S_{\text{GUT}} = \mathcal{O}_{X_\Sigma}\left(1,1,0,0\right), \quad S_{10} = \mathcal{O}_{X_\Sigma}\left(2,1,2,1\right) \tag{11.10}$$

one computes from the toric data of X_Σ as given in chapter 9 a basis of these cohomology groups by means of the *cohomCalg* algorithm. This was automatised in the *Mathematica* notebook whose sourcecode we display in Appendix E and discuss the implemented functionality in chapter 15. Hence we can simply quote the results.

$$\begin{aligned}
\widetilde{s}_1 &= C_{18}x_1^2x_2x_3x_4 + C_{14}x_1x_2x_3x_4x_5 + C_7x_2x_3x_4x_5^2 + C_{16}x_1^2x_6 + C_{10}x_1x_5x_6 \\
&\quad + C_1x_5^2x_6 + C_{17}x_1^2x_3x_7 + C_{12}x_1x_3x_5x_7 + C_4x_3x_5^2x_7 + C_{15}x_1x_2x_3x_4^2x_8 \\
&\quad + C_8x_2x_3x_4^2x_5x_8 + C_{11}x_1x_4x_6x_8 + C_2x_4x_5x_6x_8 + C_{13}x_1x_3x_4x_7x_8 \\
&\quad + C_5x_3x_4x_5x_7x_8 + C_9x_2x_3x_4^3x_8^2 + C_3x_4^2x_6x_8^2 + C_6x_3x_4^2x_7x_8^2 \\
\widetilde{s}_2 &= C_{21}x_1 + C_{19}x_5 + C_{20}x_4x_8 \\
\widetilde{s}_3 &= C_{32}x_1x_2^2x_3x_4 + C_{28}x_2^2x_3x_4x_5 + C_{30}x_1x_2x_6 + C_{24}x_2x_5x_6 + C_{31}x_1x_2x_3x_7 \\
&\quad + C_{26}x_2x_3x_5x_7 + C_{29}x_2^2x_3x_4^2x_8 + C_{25}x_2x_4x_6x_8 + C_{27}x_2x_3x_4x_7x_8 \\
&\quad + C_{22}x_6x_7x_8 + C_{23}x_3x_7^2x_8
\end{aligned} \tag{11.11}$$

The algebraic subvariety C_{10} is now given by

$$C_{10} = \left\{p \in X_\Sigma\ ,\ \widetilde{s}_1\left(p\right) = \widetilde{s}_2\left(p\right) = \widetilde{s}_3\left(p\right) = 0\right\} \tag{11.12}$$

The 32-parameters C_i that appear in the sections $\widetilde{s}_i$ are thus identified as a redundant description of the complex structure of C_{10}. A priori any of these parameters can take any value in $\mathbb{C}$. However, recall that they are subject to the condition that the algebraic subvariety C_{10} is smooth and of codimension 3.

Space	Basis	Dimension
P_1	$\begin{pmatrix} 0 \\ 0 \\ A_1 \frac{x_7}{x_1 x_4 x_5} \end{pmatrix}$	1
P_2	$\begin{pmatrix} 0 \\ A_2 x_2 x_6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ A_3 x_2 x_3 x_7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ A_4 x_2^2 x_3 x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ A_5 \end{pmatrix}$	4
P_3	$A_{16} x_1 x_2^2 x_3 x_4 + A_{12} x_2^2 x_3 x_4 x_5 + A_{14} x_1 x_2 x_6 + A_8 x_2 x_5 x_6 + A_{15} x_1 x_2 x_3 x_7 + A_{10} x_2 x_3 x_5 x_7 + A_{13} x_2^2 x_3 x_4^2 x_8 + A_9 x_2 x_4 x_6 x_8 + A_{11} x_2 x_3 x_4 x_7 x_8 + A_6 x_6 x_7 x_8 + A_7 x_3 x_7^2 x_8$	11

Table 11.1.: Non-trivial ambient space cohomologies in the computation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{10}}$, their bases and dimensions.

11.3.2. The Ambient Space Cohomologies

For this particular situation the Koszul resolution is given by

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_{C_{10}} \to 0 \tag{11.13}$$

where

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(-4,-3,-1,-1)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}(-1,-1,0,0) \oplus \mathcal{O}_{X_\Sigma}(-3,-2,-1,-1) \oplus \mathcal{O}_{X_\Sigma}(-2,-2,1,0)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(-1,-1,1,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(0,0,0,0)$
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$

The cohomologies of these bundles are easily computed by our *Mathematica* notebook from Appendix E. We list the non-trivial cohomology classes in Table 11.1 and use them to neatly organise the ambient space cohomologies in Table 11.2.

11.3.3. Computation Of The First Short Exact Sequence

Remark:
Recall that the first short exact sequence resulting from the splitting of the Koszul resolution looks like

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0 \tag{11.14}$$

with the auxillary sheaf $\mathcal{I}_1$. Our first task is therefore to learn how to compute the cohomologies of $\mathcal{I}_1$ from the cohomologies of $\mathcal{L}'$, $\mathcal{V}_2$ and knowledge about the map $\mathcal{L}' \to \mathcal{V}_2$.

$\mathcal{L}'$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$	P_1	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	P_2	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_3	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 11.2.: Ambient space cohomologies in the computation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{10}}$.

Claim:
Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ a short exact sequence of coherent sheaves on a complex and compact manifold X. Then the finiteness theorem tells us that the cohomology groups of these sheaves are finite dimensional vector spaces. Moreover we know that there exists a long exact sequence in these cohomologies which looks as follows

$$\begin{aligned}
0 \longrightarrow H^0(X,\mathcal{F}_1) &\xrightarrow{\alpha^0} H^0(X,\mathcal{F}_2) \xrightarrow{\beta^0} H^0(X,\mathcal{F}_3) \xrightarrow{\delta^0} \\
H^1(X,\mathcal{F}_1) &\xrightarrow{\alpha^1} H^1(X,\mathcal{F}_2) \xrightarrow{\beta^1} H^1(X,\mathcal{F}_3) \xrightarrow{\delta^1} \\
H^2(X,\mathcal{F}_1) &\xrightarrow{\alpha^2} H^2(X,\mathcal{F}_2) \xrightarrow{\beta^2} H^2(X,\mathcal{F}_3) \dashrightarrow \\
&\cdots \\
H^n(X,\mathcal{F}_1) &\xrightarrow{\alpha^n} H^n(X,\mathcal{F}_2) \xrightarrow{\beta^n} H^n(X,\mathcal{F}_3)
\end{aligned}$$

Given this situation we claim the following.

$$H^i\left(X,\mathcal{F}_3\right) \cong \operatorname{coker}\left(\alpha^i\right) \oplus \ker\left(\alpha^{i+1}\right) \tag{11.15}$$

Proof
For simplicity we perform the proof for $i = 0$. We thus consider the finite dimensional complex vector space $H^0\left(X,\mathcal{F}_3\right)$. By means of the vector space homomorphism

$$\delta^0 \colon H^0\left(X,\mathcal{F}_3\right) \to H^1\left(X,\mathcal{F}_1\right) \tag{11.16}$$

we have the natural isomorphism

$$H^0\left(X,\mathcal{F}_3\right) \cong \ker\left(\delta^0\right) \oplus \operatorname{im}\left(\delta^0\right) \tag{11.17}$$

By means of exactness we have $\ker\left(\delta^0\right) \cong \operatorname{im}\left(\beta^0\right)$. From the natural ismorphism

$$H^0\left(X,\mathcal{F}_2\right) \cong \ker\left(\beta^0\right) \oplus \operatorname{im}\left(\beta^0\right) \tag{11.18}$$

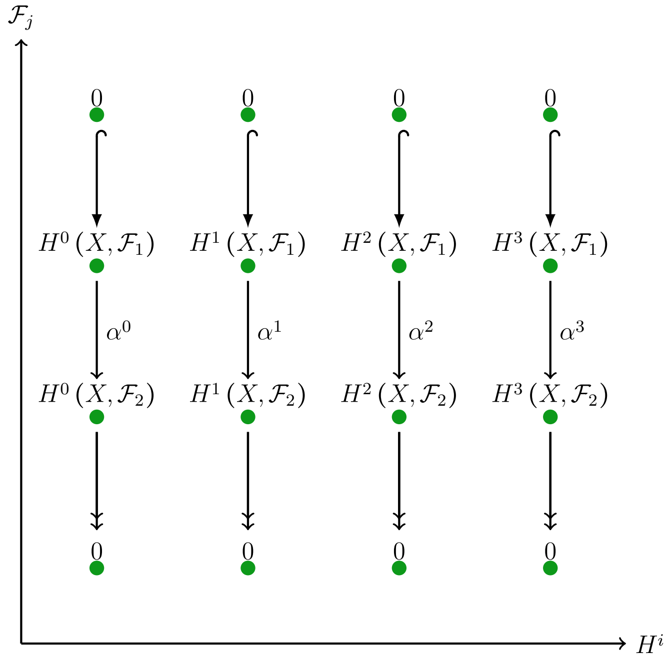


Figure 11.1.: Given a short exact sequence $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, the cohomologies of $\mathcal{F}_3$ can be computed from the cohomologies of $\mathcal{F}_1$ and $\mathcal{F}_2$ and the mappings on their cohomologies as induced from the sheaf homomorphism α. To this end one computes the cohomologies of the vertical complexes. Denoting the resulting cohomology class at position (i, j) by E^{ij} one obtains $H^i(X, \mathcal{F}_3) \cong E^{i,0} \oplus E^{i+1,1}$.

we consequently find

$$H^0(X, \mathcal{F}_3) \cong \operatorname{coker}\left(\alpha^0\right) \oplus \operatorname{im}\left(\delta^0\right) \tag{11.19}$$

By using the exactness property at $H^1(X, \mathcal{F}_1)$ we finally obtain the result. ■

Consequence:
This result enables us to compute the cohomologies of the sheaf $\mathcal{I}_1$ as defined by the short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0 \tag{11.20}$$

from knowledge of the cohomology classes of $\mathcal{L}'$ and $\mathcal{V}_2$ and the mapping between these classes. We depicture the result of this consideration in Figure 11.1.

Consequence:
For the calculation of the cohomologies of the auxillary sheaf $\mathcal{I}_1$ as defined by the short exact sequence

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0 \tag{11.21}$$

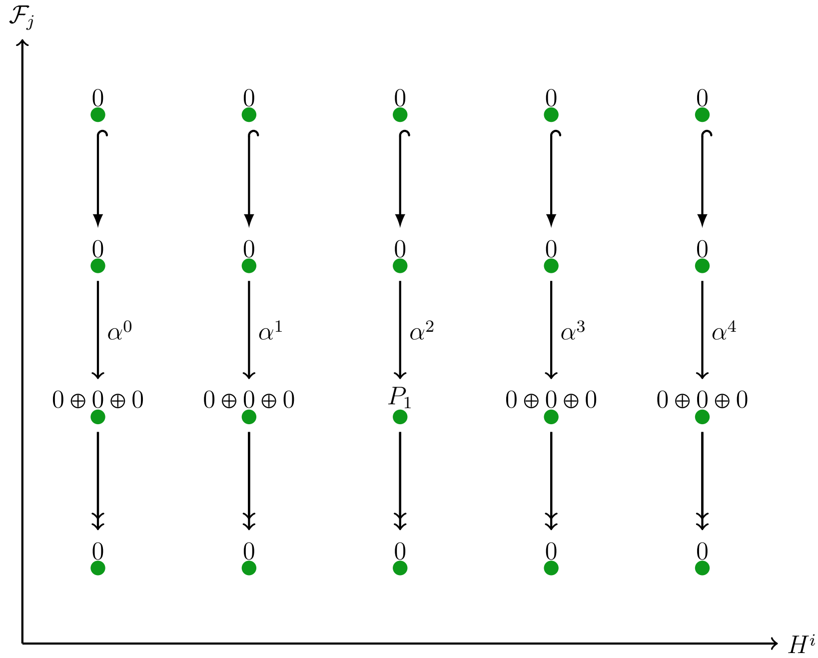


Figure 11.2.: Sheet describing the first short exact sequence $0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0$ for the computation of line bundle cohomology on C_{10}.

we have to consider the sheet as given in Figure 11.2. Note that from the Koszul complex we have

$$\alpha^i \colon H^i\left(X_\Sigma, \mathcal{L}'\right) \to H^i\left(X_\Sigma, \mathcal{V}_2\right)\ ,\ [t] \mapsto \left[\begin{pmatrix} \widetilde{s}_1 \\ -\widetilde{s}_2 \\ \widetilde{s}_3 \end{pmatrix} \cdot t\right] \tag{11.22}$$

Consequently it follows

$$H^i\left(X_\Sigma, \mathcal{I}_1\right) = \left(0 \oplus 0 \oplus 0, 0 \oplus 0 \oplus 0, P_1, 0 \oplus 0 \oplus 0, 0 \oplus 0 \oplus 0\right) \tag{11.23}$$

where $P_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ A_1 \cdot \frac{x_7}{x_1 x_4 x_5} \end{pmatrix}\ ,\ A_1 \in \mathbb{C} \right\}$.

11.3.4. Computation Of The Second Short Exact Sequence

Remark:
We now have to compute the cohomologies of the auxillary sheaf $\mathcal{I}_2$ defined by the short exact sequence

$$0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0 \tag{11.24}$$

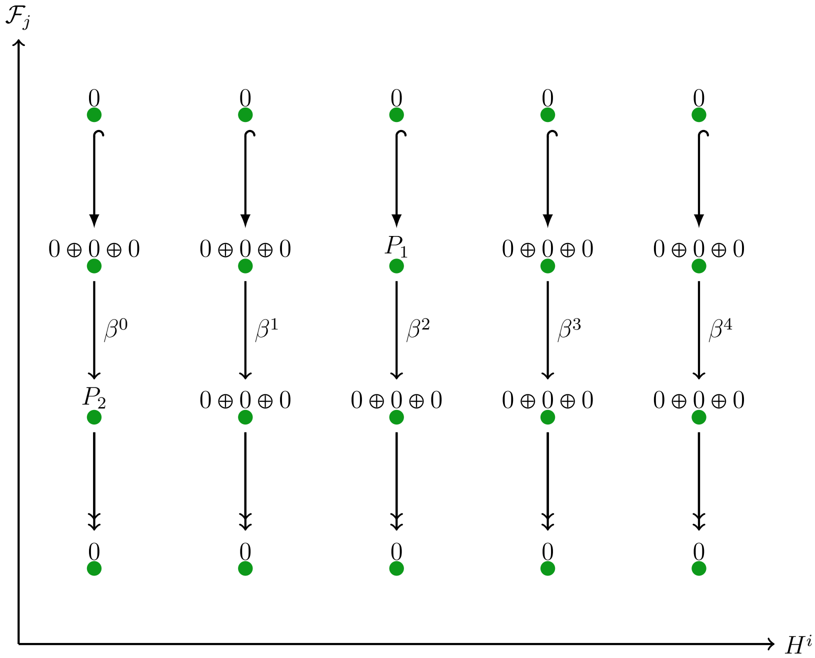


Figure 11.3.: Sheet describing the second short exact sequence $0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0$ for the computation of line bundle cohomology on C_{10}.

With the result from the above calculation we can organise the corresponding sheet as outlined in Figure 11.3. Note also that we found a canonical isomorphism

$$H^i(X_\Sigma, \mathcal{I}_1) \cong H^i(X_\Sigma, \mathcal{V}_2) \tag{11.25}$$

Consequently the maps β^i are just the ones induced from the map in the Koszul sequence, that is we have

$$\beta^i\colon H^i(X_\Sigma, \mathcal{I}_1) \to H^i(X_\Sigma, \mathcal{V}_2)\ , \quad \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.26}$$

Comment:
Let us look at the map β^2 in more detail. Recall that we have

$$\beta^2\colon P_1 \to 0 \oplus 0 \oplus 0\ , \quad \left[\begin{pmatrix} 0 \\ 0 \\ \frac{A_1 x_7}{x_1 x_4 x_5} \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \frac{A_1 x_7}{x_1 x_4 x_5} \end{pmatrix}\right] \tag{11.27}$$

On a first glance this might look worrying, since in general

$$\boldsymbol{v} = \begin{pmatrix} -\widetilde{s}_2 \cdot A_1 \cdot \frac{x_7}{x_1 x_4 x_5} \\ \widetilde{s}_1 \cdot A_1 \cdot \frac{x_7}{x_1 x_4 x_5} \\ 0 \end{pmatrix} \notin 0 \oplus 0 \oplus 0 \tag{11.28}$$

Whilst this holds true, the crucial insight is that we use Čech cochains to represent equivalence classes of Čech cochains, namely Čech cocycles, and induce maps of Čech cocycles from maps of Čech cochains. As well-definedness of β^2 follows from the well-definedness of the sheaf homomorphism

$$\beta\colon \mathcal{V}_2 \to \mathcal{V}_1 \tag{11.29}$$

as outlined in section 11.2, we must thus conclude that the non-zero vector $\boldsymbol{v}$ describes the same equivalence class of Čech cochains in $H^2(X_\Sigma, \mathcal{V}_1)$ as does the cochain $0 \oplus 0 \oplus 0$. In particular we note that any element in $H^2(X_\Sigma, \mathcal{I}_1)$ is mapped to $0 \oplus 0 \oplus 0$, so that

$$\ker\left(\beta^2\right) = H^2(X_\Sigma, \mathcal{I}_1), \quad \operatorname{coker}\left(\beta^2\right) \cong 0 \oplus 0 \oplus 0 \tag{11.30}$$

Consequence:
The cohomologies of $\mathcal{I}_2$ are now easily obtained as

- $H^0(X_\Sigma, \mathcal{I}_2) \cong P_2$
- $H^1(X_\Sigma, \mathcal{I}_2) \cong P_1$
- $H^2(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0$
- $H^3(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0$
- $H^4(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0$

11.3.5. Computation Of The Third Short Exact Sequence

Remark:
Finally, we can compute the cohomologies of $\mathcal{L}|_{C_{10}}$. Recall that this is achieved via the third short exact sequence which takes the form

$$0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{10}} \to 0 \tag{11.31}$$

With the cohomologies for $\mathcal{I}_2$ we can organise the calculation in the sheet given in Figure 11.4. Note that

$$H^0(X_\Sigma, \mathcal{I}_2) \cong H^0(X_\Sigma, \mathcal{V}_1) \tag{11.32}$$

Consequently the map γ^0 is just given by the one induced from the Koszul complex, i.e. we have

$$\gamma^0\colon H^0(X_\Sigma, \mathcal{I}_2) \to H^0(X_\Sigma, \mathcal{L})\ , \quad \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.33}$$

This also holds true for γ^2, γ^3 and γ^4. Note however that the story is different for γ^1. This is because we have

$$H^1(X_\Sigma, \mathcal{I}_2) \cong H^1(X_\Sigma, \mathcal{V}_2) \tag{11.34}$$

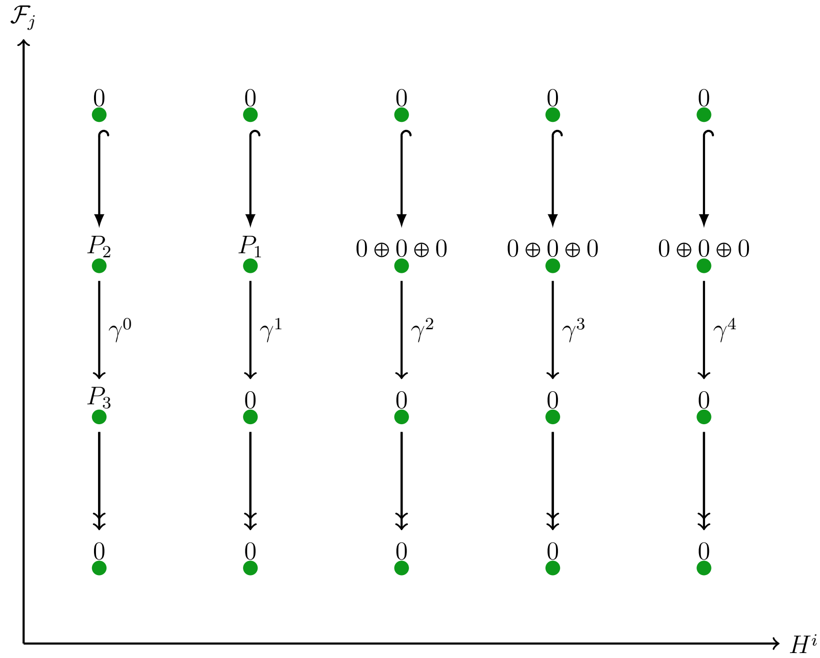


Figure 11.4.: Sheet describing the third short exact sequence $0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{10}} \to 0$ for the computation of line bundle cohomology on C_{10}.

and the Koszul resolution does not include a sheaf homomorphism $\varphi\colon \mathcal{V}_2 \to \mathcal{L}$. This is the first 'mysterious' map that we got to see so far. For the application in this and the following examples however, we will not need to know the details of these 'mysterious maps'. Therefore we decide to remain silent on the details of these maps now, but will discuss them in section 12.3.

Consequence:
From the sheet in Figure 11.4 we learn that

$$H^0\left(C_{10}, \mathcal{L}|_{C_{10}}\right) \cong \operatorname{coker}\left(\gamma^0\right) \oplus P_1 \tag{11.35}$$

whilst all higher cohomology groups are trivial.

Remark (γ^0 As Matrix)**:**
Recall that

$$P_2 = \left\{ \left(\begin{array}{c} 0 \\ A_2 x_2 x_6 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ A_3 x_2 x_3 x_7 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ A_4 x_2^2 x_3 x_4 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \\ A_5 \end{array} \right),\ A_i \in \mathbb{C} \right\} \tag{11.36}$$

From this it is easily found that the map γ^0 can be represented by the following matrix

$$M_{\gamma^0} = \begin{pmatrix} 0 & 0 & C_{21} & C_{32} \\ 0 & 0 & C_{19} & C_{28} \\ C_{21} & 0 & 0 & C_{30} \\ C_{19} & 0 & 0 & C_{24} \\ 0 & C_{21} & 0 & C_{31} \\ 0 & C_{19} & 0 & C_{26} \\ 0 & 0 & C_{20} & C_{29} \\ C_{20} & 0 & 0 & C_{25} \\ 0 & C_{20} & 0 & C_{27} \\ 0 & 0 & 0 & C_{22} \\ 0 & 0 & 0 & C_{23} \end{pmatrix} \tag{11.37}$$

Comment:
From the above we see how the parameters $C_i \in \mathbb{C}$ influence the image of γ^0 and thereby also $H^0\left(C_{10}, \mathcal{L}|_{C_{10}}\right)$.
Let us recall that from chapter 9 we know $h^0\left(C_{10}, \mathcal{L}|_{C_{10}}\right) = 8$. Thus a detailed study of the allowed values for the parameters C_i, subject to the conditions that C_{10} is smooth and of codimension 3, must yield

$$\mathrm{im}\left(M_{\gamma^0}\right) \cong \mathbb{C}^4 \tag{11.38}$$

Whilst we do not perform this analysis in detail, we give an example for a smooth C_{10} locus and subsequently introduce the notion of the generic C_{10}-curve.

11.3.6. A Smooth Example

Construction 11.3.1:
We now want to give at least one smooth example. To this end we replace the coefficients C_i by pseudo-random numbers between 0 and 1. The program *Sage* [51] is then used to check if the so-defined C_{10} curve is smooth and has the correct

codimension [1]. An example of polynomials that define such a C_{10} locus is as follows.

$$\begin{aligned}
\widetilde{s}_1 &= 0.419639901494 x_2 x_3 x_4^3 x_8^2 + 0.616192161696 x_1 x_2 x_3 x_4^2 x_8 + 0.324512347891 x_2 x_3 x_4^2 x_5 x_8 \\
&+ 0.516449982403 x_3 x_4^2 x_7 x_8^2 + 0.689070716759 x_1^2 x_2 x_3 x_4 + 0.288808226706 x_1 x_2 x_3 x_4 x_5 \\
&+ 0.208510506734 x_2 x_3 x_4 x_5^2 + 0.684379783437 x_1 x_3 x_4 x_7 x_8 + 0.37391574191 x_3 x_4 x_5 x_7 x_8 \\
&+ 0.188034542702 x_4^2 x_6 x_8^2 + 0.095326756162 x_1^2 x_3 x_7 + 0.811677985296 x_1 x_3 x_5 x_7 \\
&+ 0.818928548878 x_3 x_5^2 x_7 + 0.492366395328 x_1 x_4 x_6 x_8 + 0.678754309531 x_4 x_5 x_6 x_8 \\
&+ 0.5482483103870 x_1^2 x_6 + 0.228988834049 x_1 x_5 x_6 + 0.177375013782 x_5^2 x_6 \\
\widetilde{s}_2 &= 0.97133169913 x_4 x_8 + 0.5800783043480 x_1 + 0.0708936406458 x_5 \\
\widetilde{s}_3 &= 0.451943628152 x_2^2 x_3 x_4^2 x_8 + 0.0944105997866 x_1 x_2^2 x_3 x_4 + 0.888785313706 x_2^2 x_3 x_4 x_5 \\
&+ 0.448344364944 x_2 x_3 x_4 x_7 x_8 + 0.0498775090408 x_1 x_2 x_3 x_7 + 0.452990860452 x_2 x_3 x_5 x_7 \\
&+ 0.0275749267842 x_2 x_4 x_6 x_8 + 0.833986475345 x_3 x_7^2 x_8 + 0.604239004892 x_1 x_2 x_6 \\
&+ 0.216791670544 x_2 x_5 x_6 + 0.648429504954 x_6 x_7 x_8
\end{aligned} \tag{11.39}$$

Consequence:
It is not too hard to check that with the above values for the parameters C_i, the matrix M_{γ^0} indeed has a four-dimensional kernel. So we find in this smooth example $h^0\left(C_{10}, \mathcal{L}|_{C_{10}}\right) = 8$ as expected.

11.3.7. The Generic Pullback Setup

Remark (The Word Generic)**:**
The word *generic* appears very often in algebraic geometry. Thus it is highly adequate to explain its precise meaning. To this end we quote [44, pp. 20-21].

> "We should mention here a piece of terminology that is pervasive in algebraic geometry: the word *generic*. When we are dealing with a family of objects parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement that "a (or the) generic member of the family has a certain property" means exactly that "the set of objects in the family that do not have that property is contained in a subvariety of strictly smaller dimension"."

Note:
In our situation, the subvariety C_{10} is parametrised by points in a certain analytic subvariety of $\mathbb{C}^{32}$. Note in particular that not every point in $\mathbb{C}^{32}$ corresponds to an allowed algebraic subvariety C_{10}. This is easily seen as e.g. the origin is not allowed - it corresponds to the situation where all sections are trivial.
Still in order to demonstrate the meaning of 'generic' let us for a moment assume

[1] This is achieved via introducing C_{10} as algebraic subscheme of X_Σ and then checking its smoothness and dimension.

that in fact all points in $\mathbb{C}^{32}$ were allowed. Then points in

$$G := \left\{ (C_1, \ldots, C_{32}) \in \mathbb{C}^{32} \;,\; C_i \neq C_j \forall i \neq j \right\} \tag{11.40}$$

were generic in the above sense and one would indeed obtain $\mathrm{im}\left(M_{\gamma^0}\right) \cong \mathbb{C}^4$ as expected.
Yet the situation is much more involved since the points in $\mathbb{C}^{32}$ that correspond to well-defined pullback setups form proper subsets of $\mathbb{C}^{32}$. Therefore the above naive guess of 'generic' does not apply in general.

Remark:
For simplicity we will ignore this issue in the remainder of this thesis and agree on the following simplified convention.

> 'Generic pullback setup': We replace the coefficients C_i by their index i. Thereby kernel and image of the mapping matrices can be easily computed. We then term any smooth pullback setup of the correct codimensionality, such that for any involved mapping matrix the kernel is of the same dimension as computed by the replacements $C_i \to i$, a generic pullback setup.

Consequence:
In the generic pullback setup, we have $\ker\left(M_{\gamma^0}\right) \cong \mathbb{C}^4$.

Note:
Whilst it is very tempting to guess that the above terminology of 'generic' agrees with the one given in [44], the author does not know how to proof or disproof this assertion.

11.4. An Exhaustive Example Continued - Pullback To $\mathbf{C_{\overline{5}m}}$

11.4.1. The Defining Polynomials of $\mathbf{C_{\overline{5}m}}$

Recall that we have to choose holomorphic sections

$$\widetilde{s}_1 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{B_3}\right)\right), \quad \widetilde{s}_2 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{\mathrm{GUT}}\right)\right), \quad \widetilde{s}_3 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{\overline{5}m}\right)\right) \tag{11.41}$$

to define $C_{\overline{5}m}$ as their common zero locus. By recalling that

$$S_{B_3} = \mathcal{O}_{X_\Sigma}(3,2,1,1), \quad S_{\mathrm{GUT}} = \mathcal{O}_{X_\Sigma}(1,1,0,0), \quad S_{\overline{5}m} = \mathcal{O}_{X_\Sigma}(4,1,6,3) \tag{11.42}$$

one computes from the toric data of X_Σ a basis of these cohomology groups by means of our *Mathematica* notebook displayed in Appendix E. This gives

$$\begin{aligned}
\widetilde{s}_1 &= C_{18}x_1^2x_2x_3x_4 + C_{14}x_1x_2x_3x_4x_5 + C_7x_2x_3x_4x_5^2 + C_{16}x_1^2x_6 + C_{10}x_1x_5x_6 \\
&+ C_1x_5^2x_6 + C_{17}x_1^2x_3x_7 + C_{12}x_1x_3x_5x_7 + C_4x_3x_5^2x_7 + C_{15}x_1x_2x_3x_4^2x_8 \\
&+ C_8x_2x_3x_4^2x_5x_8 + C_{11}x_1x_4x_6x_8 + C_2x_4x_5x_6x_8 + C_{13}x_1x_3x_4x_7x_8 \\
&+ C_5x_3x_4x_5x_7x_8 + C_9x_2x_3x_4^3x_8^2 + C_3x_4^2x_6x_8^2 + C_6x_3x_4^2x_7x_8^2 \\
\widetilde{s}_2 &= C_{21}x_1 + C_{19}x_5 + C_{20}x_4x_8 \\
\widetilde{s}_3 &= C_{22}x_2^2x_6^3x_7x_8 + C_{23}x_2^2x_3x_6^2x_7^2x_8 + C_{24}x_2^2x_3^2x_6x_7^3x_8 + C_{25}x_2^2x_3^3x_7^4x_8 \\
&+ C_{26}x_2^3x_5x_6^3 + C_{27}x_2^3x_4x_6^3x_8 + C_{28}x_2^3x_3x_5x_6^2x_7 + C_{29}x_2^3x_3x_4x_6^2x_7x_8 \\
&+ C_{30}x_2^3x_3^2x_5x_6x_7^2 + C_{31}x_2^3x_3^2x_4x_6x_7^2x_8 + C_{32}x_2^3x_3^3x_5x_7^3 + C_{33}x_2^3x_3^3x_4x_7^3x_8 \\
&+ C_{34}x_2^4x_3x_4x_5x_6^2 + C_{35}x_2^4x_3x_4^2x_6^2x_8 + C_{36}x_2^4x_3^2x_4x_5x_6x_7 + C_{37}x_2^4x_3^2x_4^2x_6x_7x_8 \\
&+ C_{38}x_2^4x_3^3x_4x_5x_7^2 + C_{39}x_2^4x_3^3x_4^2x_7^2x_8 + C_{40}x_2^5x_3^2x_4^2x_5x_6 + C_{41}x_2^5x_3^2x_4^3x_6x_8 \\
&+ C_{42}x_2^5x_3^3x_4^2x_5x_7 + C_{43}x_2^5x_3^3x_4^3x_7x_8 + C_{44}x_2^6x_3^3x_4^3x_5 + C_{45}x_2^6x_3^3x_4^4x_8 \\
&+ C_{46}x_1x_2^3x_6^3 + C_{47}x_1x_2^3x_3x_6^2x_7 + C_{48}x_1x_2^3x_3^2x_6x_7^2 + C_{49}x_1x_2^3x_3^3x_7^3 \\
&+ C_{50}x_1x_2^4x_3x_4x_6^2 + C_{51}x_1x_2^4x_3^2x_4x_6x_7 + C_{52}x_1x_2^4x_3^3x_4x_7^2 + C_{53}x_1x_2^5x_3^2x_4^2x_6 \\
&+ C_{54}x_1x_2^5x_3^3x_4^2x_7 + C_{55}x_1x_2^6x_3^3x_4^3
\end{aligned} \tag{11.43}$$

The $C_{\overline{5}m}$-curve is now given by

$$C_{\overline{5}m} = \{p \in X_\Sigma\ ,\ \widetilde{s}_1(p) = \widetilde{s}_2(p) = \widetilde{s}_3(p) = 0\} \tag{11.44}$$

The 54 parameters C_i, that appear in the sections $\widetilde{s}_i$ and which give a redundant description of the complex structure of $C_{\overline{5}m}$, are subject to the condition that their common zero locus is a smooth subvariety of codimension 3 in X_Σ.

11.4.2. The Ambient Space Cohomologies

For this particular situation the Koszul resolution is given by

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_{C_{\overline{5}m}} \to 0 \tag{11.45}$$

where

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(-6,-3,-5,-3)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}(-3,-1,-4,-2) \oplus \mathcal{O}_{X_\Sigma}(-5,-2,-5,-3) \oplus \mathcal{O}_{X_\Sigma}(-2,-2,1,0)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(-1,-1,1,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(-2,0,-4,-2)$
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$

The cohomologies of these bundles are easily computed by our *Mathematica* notebook given in Appendix E. We list the non-trivial cohomology classes in Table 11.3 and use them to neatly represent the ambient space cohomologies in Table 11.4.

Space	Basis	Dimension
P_1	$\frac{A_1}{x_1 x_2^2 x_3 x_4 x_5 x_6^2 x_7 x_8} + \frac{A_2}{x_1 x_2^2 x_3^2 x_4 x_5 x_6 x_7^2 x_8} + \frac{A_3}{x_1 x_2^3 x_3^2 x_4^2 x_5 x_6 x_7 x_8}$	3
P_2	$\begin{pmatrix} 0 \\ 0 \\ \frac{A_4 x_7}{x_1 x_4 x_5} \end{pmatrix}, \begin{pmatrix} \frac{A_5}{x_2 x_3 x_6 x_7^2 x_8} \\ 0 \\ 0 \end{pmatrix}$	2
P_3	$\begin{pmatrix} 0 \\ A_6 x_2 x_6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ A_7 x_2 x_3 x_7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ A_8 x_2^2 x_3 x_4 \\ 0 \end{pmatrix}$	3
P_4	$\begin{pmatrix} 0 \\ 0 \\ \frac{A_9 x_4}{x_2 x_3 x_6 x_7^2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{A_{10}}{x_2^2 x_3 x_6 x_7} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{A_{11} x_5}{x_2 x_3 x_6 x_7^2 x_8} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{A_{12} x_1}{x_2 x_3 x_6 x_7^2 x_8} \end{pmatrix}$	4
P_5	$A_{13} x_6 x_7 x_8 + A_{14} x_3 x_7^2 x_8 + A_{15} x_2 x_5 x_6 + A_{16} x_2 x_4 x_6 x_8 + A_{17} x_2 x_3 x_5 x_7 + A_{18} x_2 x_3 x_4 x_7 x_8 + A_{19} x_2^2 x_3 x_4 x_5 + A_{20} x_2^2 x_3 x_4^2 x_8 + A_{21} x_1 x_2 x_6 + A_{22} x_1 x_2 x_3 x_7 + A_{23} x_1 x_2^2 x_3 x_4$	11

Table 11.3.: Non-trivial ambient space cohomologies in the computation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{\overline{5}m}}$, their bases and dimensions.

11.4.3. Computation Of The First Short Exact Sequence

Remark:
Recall that the first short exact sequence that results from the splitting of the Koszul sequence takes the form

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0 \tag{11.46}$$

with the auxillary sheaf $\mathcal{I}_1$. In order to compute the cohomologies of $\mathcal{I}_1$ we consider the sheet given in Figure 11.5 and recall that

$$\alpha^i \colon H^i(X_\Sigma, \mathcal{L}') \to H^i(X_\Sigma, \mathcal{V}_2)\ ,\ [t] \mapsto \left[\begin{pmatrix} \widetilde{s}_1 \\ -\widetilde{s}_2 \\ \widetilde{s}_3 \end{pmatrix} \cdot t \right] \tag{11.47}$$

From this it is readily confirmed that

- $H^0(X_\Sigma, \mathcal{I}_1) \cong 0 \oplus 0 \oplus 0$
- $H^1(X_\Sigma, \mathcal{I}_1) \cong 0 \oplus 0 \oplus 0$
- $H^2(X_\Sigma, \mathcal{I}_1) \cong P_2 \cong H^2(X_\Sigma, \mathcal{V}_2)$
- $H^3(X_\Sigma, \mathcal{I}_1) \cong P_1 \cong H^3(X_\Sigma, \mathcal{L}')$
- $H^4(X_\Sigma, \mathcal{I}_1) \cong 0 \oplus 0 \oplus 0$

$\mathcal{L}'$	0	0	0	0	P_1
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$	P_2	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	P_3	$0\oplus 0\oplus 0$	P_4	$0\oplus 0\oplus 0$	$0\oplus 0\oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_5	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 11.4.: Ambient space cohomologies in the computation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{\overline{5}m}}$.

11.4.4. Computation Of The Second Short Exact Sequence

Remark:
We now have to compute the cohomologies of the auxillary sheaf $\mathcal{I}_2$ defined by the short exact sequence

$$0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0 \tag{11.48}$$

With the result from the above computation we can organise the corresponding sheet as outlined in Figure 11.6. Note that we have a canonical isomorphism

$$H^i(X_\Sigma, \mathcal{I}_1) \cong H^i(X_\Sigma, \mathcal{V}_2) \tag{11.49}$$

for $i = 0,1,2,4$, so that for these values of i the maps β^i are just the ones induced from the map in the Koszul sequence. Consequently we have for $i = 0,1,2,4$

$$\beta^i\colon H^i(X_\Sigma, \mathcal{I}_1) \to H^i(X_\Sigma, \mathcal{V}_1)\ ,\ \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.50}$$

For β^3 however the story is different since we have

$$H^3(X_\Sigma, \mathcal{I}_1) \cong H^4(X_\Sigma, \mathcal{L}') \tag{11.51}$$

and there is no sheaf homomorphism $\varphi\colon \mathcal{L}' \to \mathcal{V}_1$ in the Koszul resolution. This again is one of those 'mysterious maps'. However, in the same fashion as we were able to perform the calculation for C_{10} we will not need the details of this map here. In particular it should be clear from Figure 11.6, that we have

- $H^0(X_\Sigma, \mathcal{I}_2) \cong P_3 \cong H^0(X_\Sigma, \mathcal{V}_1)$
- $H^1(X_\Sigma, \mathcal{I}_2) \cong \ker(\beta^2)$
- $H^2(X_\Sigma, \mathcal{I}_2) \cong \operatorname{coker}(\beta^2) \oplus P_1$
- $H^3(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0 \cong H^3(X_\Sigma, \mathcal{V}_1)$

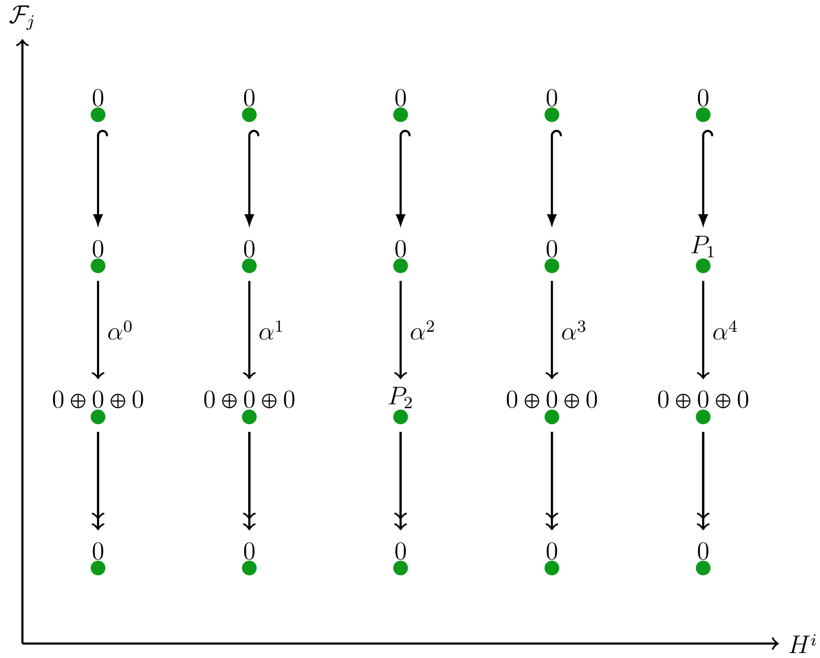


Figure 11.5.: Sheet describing the first short exact sequence $0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0$ for the computation of line bundle cohomology on $C_{\overline{5}m}$.

- $H^4(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0 \cong H^4(X_\Sigma, \mathcal{V}_1)$

Comment:
Let us take a closer look at the map β^2. Recall that we have

$$\begin{gathered}\beta^2\colon P_2 \to P_4 \\ \left[\begin{pmatrix} \frac{A_5}{x_2x_3x_6x_7^2x_8} \\ 0 \\ \frac{A_4x_7}{x_1x_4x_5} \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{A_5}{x_2x_3x_6x_7^2x_8} \\ 0 \\ \frac{A_4x_7}{x_1x_4x_5} \end{pmatrix}\right]\end{gathered} \tag{11.52}$$

On a first glance this might look worrying, since for non-trivial sections $\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3$ the first two entries will not vanish in general. This however is necessary for the map β^2 to map into P_4, as can be seen from Table 11.3. Recall however that we are mapping equivalence classes of Čech cochains by means of maps of Čech cochains. Thus we conclude that the first two entries are zero in the codomain with respect to the equivalence relations in it. Therefore we have

$$\operatorname{im}\left(\beta^2\right) = 0 \oplus 0 \oplus \left\{\frac{A_5C_{20}x_4}{x_2x_3x_6x_7^2} + \frac{A_5C_{21}x_1}{x_2x_3x_6x_7^2x_8} + \frac{A_5C_{19}x_5}{x_2x_3x_6x_7^2x_8}\,,\ A_5 \in \mathbb{C}\right\} \tag{11.53}$$

Note that the mapping depends on the parameters C_{19}, C_{20} and C_{21}. In terms of

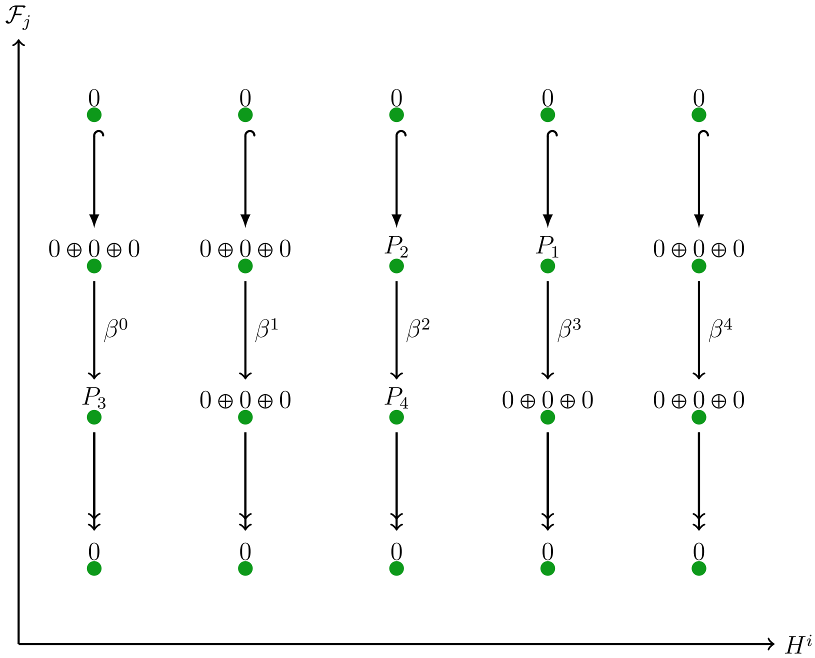


Figure 11.6.: Sheet describing the second short exact sequence $0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0$ for the computation of line bundle cohomology on $C_{\overline{5}m}$.

matrices we can express this the map β^2 as

$$M_{\beta^2} = \begin{pmatrix} 0 & 0 \\ C_{20} & 0 \\ C_{21} & 0 \\ C_{19} & 0 \end{pmatrix} \tag{11.54}$$

11.4.5. Computation Of The Third Short Exact Sequence

Remark:
Finally, we can compute the cohomologies of $\mathcal{L}|_{C_{\overline{5}m}}$. Recall that this is achieved via the short exact sequence

$$0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{\overline{5}m}} \to 0 \tag{11.55}$$

With the cohomologies for $\mathcal{I}_2$ we can organise the calculation in the sheet given in Figure 11.7. Note that

$$H^0(X_\Sigma, \mathcal{I}_2) \cong H^0(X_\Sigma, \mathcal{V}_1) \tag{11.56}$$

Consequently the map γ^0 is just given by the one induced from the Koszul complex. Thus we have

$$\gamma^0 \colon H^0(X_\Sigma, \mathcal{I}_2) \to H^0(X_\Sigma, \mathcal{L})\ , \quad \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.57}$$

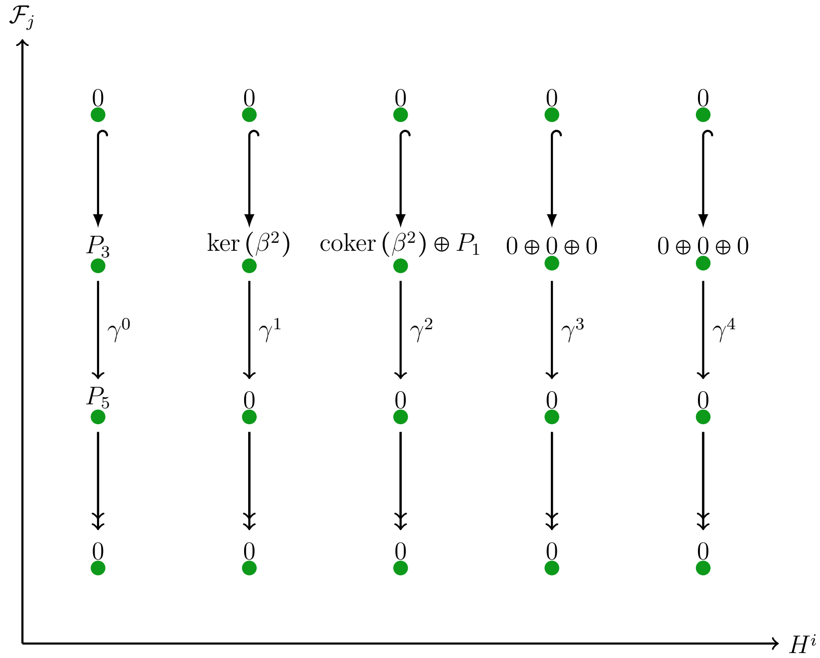


Figure 11.7.: Sheet describing the third short exact sequence $0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{\overline{5}m}} \to 0$ for the computation of line bundle cohomology on $C_{\overline{5}m}$.

This does not hold true for γ^1 and γ^2, but is again the case for the higher maps γ^3 and γ^4. So the maps γ^1 and γ^2 are again identified as 'mysterious maps'.

Remark:
The map γ^0 is easily evaluated to have the following image

$$\begin{aligned} \operatorname{im}\left(\gamma^0\right) = \{ & A_8 C_{21} x_1 x_2^2 x_3 x_4 + A_8 C_{19} x_2^2 x_3 x_4 x_5 + A_6 C_{21} x_1 x_2 x_6 + A_6 C_{19} x_2 x_5 x_6 \\ & + A_7 C_{21} x_1 x_2 x_3 x_7 + A_7 C_{19} x_2 x_3 x_5 x_7 + A_8 C_{20} x_2^2 x_3 x_4^2 x_8 \\ & + A_6 C_{20} x_2 x_4 x_6 x_8 + A_7 C_{20} x_2 x_3 x_4 x_7 x_8 \ , \ A_i \in \mathbb{C} \} \end{aligned} \tag{11.58}$$

Hence the image does depend on the paramters C_{19}, C_{20} and C_{21}, just as does the map β^2. One can also express γ^0 as matrix. We leave it to the interested reader to confirm that this matrix takes the following form.

$$M_{\gamma^0} = \begin{pmatrix} 0 & 0 & C_{21} & C_{19} & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{21} & C_{19} & 0 & 0 & C_{20} & 0 & 0 \\ C_{21} & C_{19} & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{11.59}$$

Consequence:
Having computed that matrix we can now formulate our result as follows.

- $H^0\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) \cong \operatorname{coker}(\gamma^0) \oplus \ker(\beta^2)$

- $H^1\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) \cong \operatorname{coker}\left(\beta^2\right) \oplus P_1$
- All higher cohomology groups are trivial.

Note that this implies

$$\begin{aligned} h^0\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) &= 13 - \dim_{\mathbb{C}}\left(\operatorname{im}\left(\beta^2\right)\right) - \dim_{\mathbb{C}}\left(\operatorname{im}\left(\gamma^0\right)\right) \\ h^1\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) &= 7 - \dim_{\mathbb{C}}\left(\operatorname{im}\left(\beta^2\right)\right) \end{aligned} \tag{11.60}$$

Consequently we have just found

$$\chi\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) = 6 - \dim_{\mathbb{C}}\left(\operatorname{im}\left(\gamma^0\right)\right) \tag{11.61}$$

11.4.6. Analysis Of $\mathbf{C_{19}}$, $\mathbf{C_{20}}$ and $\mathbf{C_{21}}$

Remark:
In principle we would now have to analyse under what conditions the common zero locus of the three sections $\widetilde{s}_1$, $\widetilde{s}_2$ and $\widetilde{s}_3$ is a smooth algebraic subvariety of codimension 3. Given the size of the polynomials $\widetilde{s}_i$, this however is a laborious task. Therefore we decide to take a different and minimalist approach, where we make use of the fact that at least one of the parameters C_{19}, C_{20}, C_{21} must be non-zero to ensure that $C_{\overline{5}m}$ can be of codimension 3.

Consequence:
From the fact that at least one of the parameters C_{19}, C_{20}, C_{21} is non-zero one concludes

$$\dim_{\mathbb{C}}\left(\operatorname{im}\left(\gamma^0\right)\right) = 3, \qquad \dim_{\mathbb{C}}\left(\operatorname{im}\left(\beta^2\right)\right) = 1 \tag{11.62}$$

This in turn suffices to deduce

$$h^0\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) = 9, \qquad h^1\left(C_{\overline{5}m}, \mathcal{L}|_{C_{\overline{5}m}}\right) = 6 \tag{11.63}$$

whilst the higher cohomology classes are trivial.

Note:
This result is in agreement with the one obtained from exactness considerations only. The latter is listed in Table 9.3.

11.5. An Exhaustive Example Continued - Pullback To $\mathbf{C_{5H}}$

11.5.1. The Defining Polynomials of $\mathbf{C_{5H}}$

Recall that we have to choose holomorphic sections

$$\widetilde{s}_1 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{B_3}\right)\right), \quad \widetilde{s}_2 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{\mathrm{GUT}}\right)\right), \quad \widetilde{s}_3 \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_{5H}\right)\right) \tag{11.64}$$

to define C_{5H} as their common zero locus. By recalling that

$$S_{B_3} = \mathcal{O}_{X_\Sigma}(3,2,1,1), \quad S_{\mathrm{GUT}} = \mathcal{O}_{X_\Sigma}(1,1,0,0), \quad S_{10} = \mathcal{O}_{X_\Sigma}(7,2,10,5) \tag{11.65}$$

one computes from the toric data of X_Σ a basis of these cohomology groups by means of our *Mathematica* notebook as displayed in Appendix E. This gives

$$\begin{aligned}
\widetilde{s}_1 = {} & C_{18}x_1^2x_2x_3x_4 + C_{14}x_1x_2x_3x_4x_5 + C_7x_2x_3x_4x_5^2 + C_{16}x_1^2x_6 + C_{10}x_1x_5x_6 \\
& + C_1x_5^2x_6 + C_{17}x_1^2x_3x_7 + C_{12}x_1x_3x_5x_7 + C_4x_3x_5^2x_7 + C_{15}x_1x_2x_3x_4^2x_8 \\
& + C_8x_2x_3x_4^2x_5x_8 + C_{11}x_1x_4x_6x_8 + C_2x_4x_5x_6x_8 + C_{13}x_1x_3x_4x_7x_8 \\
& + C_5x_3x_4x_5x_7x_8 + C_9x_2x_3x_4^3x_8^2 + C_3x_4^2x_6x_8^2 + C_6x_3x_4^2x_7x_8^2 \\
\widetilde{s}_2 = {} & C_{21}x_1 + C_{19}x_5 + C_{20}x_4x_8 \\
\widetilde{s}_3 = {} & C_{171}x_1^2x_3^5x_4^5x_2^{10} + C_{100}x_3^5x_4^5x_5^2x_2^{10} + C_{102}x_3^5x_4^7x_8^2x_2^{10} + C_{149}x_1x_3^5x_4^5x5x_2^{10} \\
& + C_{150}x_1x_3^5x_4^6x_8x_2^{10} + C_{101}x_3^5x_4^6x_5x_8x_2^{10} + C_{96}x_3^4x_4^6x_6x_8^2x_2^9 + C_{99}x3^5x_4^6x_7x_8^2x_2^9 \\
& + C_{169}x_1^2x_3^4x_4^4x_6x_2^9 + C_{94}x_3^4x_4^4x_5^2x_6x_2^9 + C_{145}x_1x_3^4x_4^4x_5x_6x_2^9 + C_{170}x_1^2x_3^5x_4^4x_7x_2^9 \\
& + C_{97}x_3^5x_4^4x_5^2x_7x_2^9 + C_{147}x_1x_3^5x_4^4x_5x_7x_2^9 + C_{146}x_1x_3^4x_4^5x_6x_8x_2^9 \\
& + C_{95}x_3^4x_4^5x_5x_6x_8x_2^9 + C_{148}x_1x_3^5x_4^5x_7x_8x_2^9 + C_{98}x_3^5x_4^5x_5x_7x_8x_2^9 \\
& + C_{166}x_1^2x_3^3x_4^3x_6^2x_2^8 + C_{85}x_3^3x_4^3x_5^2x_6^2x_2^8 + C_{168}x_1^2x_3^5x_4^3x_7^2x_2^8 + C_{91}x_3^5x_4^3x_5^2x_7^2x_2^8 \\
& + C_{143}x_1x_3^5x_4^3x_5x_7^2x_2^8 + C_{87}x_3^3x_4^5x_6^2x_8^2x_2^8 + C_{93}x_3^5x_4^5x_7^2x_8^2x_2^8 + C_{90}x_3^4x_4^5x_6x_7x_8^2x_2^8 \\
& + C_{167}x_1^2x_3^4x_4^3x_6x_7x_2^8 + C_{88}x_3^4x_4^3x_5^2x_6x_7x_2^8 + C_{141}x_1x_3^4x_4^3x_5x_6x_7x_2^8 \\
& + C_{140}x_1x_3^3x_4^4x_6^2x_8x_2^8 + C_{86}x_3^3x_4^4x_5x_6^2x_8x_2^8 + C_{144}x_1x_3^5x_4^4x_7^2x_8x_2^8 \\
& + C_{92}x_3^5x_4^4x_5x_7^2x_8x_2^8 + C_{142}x_1x_3^4x_4^4x_6x_7x_8x_2^8 + C_{89}x_3^4x_4^4x_5x_6x_7x_8x_2^8 \\
& + C_{162}x_1^2x_3^2x_4^2x_6^3x_2^7 + C_{73}x_3^2x_4^2x_5^2x_6^3x_2^7 + C_{131}x_1x_3^2x_4^2x_5x_6^3x_2^7 + C_{165}x_1^2x_3^5x_4^2x_7^3x_2^7 \\
& + C_{82}x_3^5x_4^2x_5^2x_7^3x_2^7 + C_{137}x_1x_3^5x_4^2x_5x_7^3x_2^7 + C_{164}x_1^2x_3^4x_4^2x_6x_7^2x_2^7 \\
& + C_{79}x_3^4x_4^2x_5^2x_6x_7^2x_2^7 + C_{135}x_1x_3^4x_4^2x_5x_6x_7^2x_2^7 + C_{75}x_3^2x_4^4x_6^3x_8^2x_2^7 \\
& + C_{84}x_3^5x_4^4x_7^3x_8^2x_2^7 + C_{81}x_3^4x_4^4x_6x_7^2x_8^2x_2^7 + C_{78}x_3^3x_4^4x_6^2x_7x_8^2x_2^7 \\
& + C_{163}x_1^2x_3^3x_4^2x_6^2x_7x_2^7 + C_{76}x_3^3x_4^2x_5^2x_6^2x_7x_2^7 + C_{133}x_1x_3^3x_4^2x_5x_6^2x_7x_2^7 \\
& + C_{74}x_3^2x_4^3x_5x_6^3x_8x_2^7 + C_{138}x_1x_3^5x_4^3x_7^3x_8x_2^7 + C_{83}x_3^5x_4^3x_5x_7^3x_8x_2^7 \\
& + C_{136}x_1x_3^4x_4^3x_6x_7^2x_8x_2^7 + C_{80}x_3^4x_4^3x_5x_6x_7^2x_8x_2^7 + C_{134}x_1x_3^3x_4^3x_6^2x_7x_8x_2^7 \\
& + C_{77}x_3^3x_4^3x_5x_6^2x_7x_8x_2^7 + C_{58}x_3x_4x_5^2x_6^4x_2^6 + C_{157}x_1^2x_3x_4x_6^4x_2^6 \\
& + C_{121}x_1x_3x_4x_5x_6^4x_2^6 + C_{70}x_3^5x_4x_5^2x_7^4x_2^6 + C_{161}x_1^2x_3^5x_4x_7^4x_2^6 \\
& + C_{129}x_1x_3^5x_4x_5x_7^4x_2^6 + C_{67}x_3^4x_4x_5^2x_6x_7^3x_2^6 + C_{160}x_1^2x_3^4x_4x_6x_7^3x_2^6 \\
& + C_{127}x_1x_3^4x_4x_5x_6x_7^3x_2^6 + C_{64}x_3^3x_4x_5^2x_6^2x_7^2x_2^6 + C_{159}x_1^2x_3^3x_4x_6^2x_7^2x_2^6 \\
& + C_{125}x_1x_3^3x_4x_5x_6^2x_7^2x_2^6 + C_{60}x_3x_4^3x_6^4x_8^2x_2^6 + C_{72}x_3^5x_4^3x_7^4x_8^2x_2^6 \\
& + C_{69}x_3^4x_4^3x_6x_7^3x_8^2x_2^6 + C_{66}x_3^3x_4^3x_6^2x_7^2x_8^2x_2^6 + C_{63}x_3^2x_4^3x_6^3x_7x_8^2x_2^6 \\
& + C_{61}x_3^2x_4x_5^2x_6^3x_7x_2^6 + C_{158}x_1^2x_3^2x_4x_6^3x_7x_2^6 + C_{123}x_1x_3^2x_4x_5x_6^3x_7x_2^6 \\
& + C_{122}x_1x_3x_4^2x_6^4x_8x_2^6 + C_{59}x_3x_4^2x_5x_6^4x_8x_2^6 + C_{130}x_1x_3^5x_4^2x_7^4x_8x_2^6 \\
& + C_{71}x_3^5x_4^2x_5x_7^4x_8x_2^6 + C_{128}x_1x_3^4x_4^2x_6x_7^3x_8x_2^6 + C_{68}x_3^4x_4^2x_5x_6x_7^3x_8x_2^6
\end{aligned}$$

$$
\begin{aligned}
&+ C_{126}x_1x_3^3x_4^2x_6^2x_7^2x_8x_2^6 + C_{65}x_3^3x_4^2x_5x_6^2x_7^2x_8x_2^6 + C_{124}x_1x_3^2x_4^2x_6^3x_7x_8x_2^6 \\
&+ C_{62}x_3^2x_4^2x_5x_6^3x_7x_8x_2^6 + C_{151}x_1^2x_6^5x_2^5 + C_{40}x_5^2x_6^5x_2^5 + C_{109}x_1x_5x_6^5x_2^5 \\
&+ C_{156}x_1^2x_3^5x_7^5x_2^5 + C_{55}x_3^5x_5^2x_7^5x_2^5 + C_{119}x_1x_3^5x_5x_7^5x_2^5 + C_{132}x_1x_3^2x_4^3x_6^3x_8x_2^7 \\
&+ C_{155}x_1^2x_3^4x_6x_7^4x_2^5 + C_{52}x_3^4x_5^2x_6x_7^4x_2^5 + C_{117}x_1x_3^4x_5x_6x_7^4x_2^5 \\
&+ C_{154}x_1^2x_3^3x_6^2x_7^3x_2^5 + C_{49}x_3^3x_5^2x_6^2x_7^3x_2^5 + C_{115}x_1x_3^3x_5x_6^2x_7^3x_2^5 \\
&+ C_{153}x_1^2x_3^2x_6^3x_7^2x_2^5 + C_{46}x_3^2x_5^2x_6^3x_7^2x_2^5 + C_{113}x_1x_3^2x_5x_6^3x_7^2x_2^5 + C_{42}x_4^2x_6^5x_8^2x_2^5 \\
&+ C_{57}x_3^5x_4^2x_7^5x_8^2x_2^5 + C_{54}x_3^4x_4^2x_6x_7^4x_8^2x_2^5 + C_{51}x_3^3x_4^2x_6^2x_7^3x_8^2x_2^5 \\
&+ C_{48}x_3^2x_4^2x_6^3x_7^2x_8^2x_2^5 + C_{45}x_3x_4^2x_6^4x_7x_8^2x_2^5 + C_{43}x_3x_5^2x_6^4x_7x_2^5 \\
&+ C_{152}x_1^2x_3x_6^4x_7x_2^5 + C_{111}x_1x_3x_5x_6^4x_7x_2^5 + C_{110}x_1x_4x_6^5x_8x_2^5 \\
&+ C_{41}x_4x_5x_6^5x_8x_2^5 + C_{120}x_1x_3^5x_4x_7^5x_8x_2^5 + C_{56}x_3^5x_4x_5x_7^5x_8x_2^5 \\
&+ C_{118}x_1x_3^4x_4x_6x_7^4x_8x_2^5 + C_{53}x_3^4x_4x_5x_6x_7^4x_8x_2^5 + C_{116}x_1x_3^3x_4x_6^2x_7^3x_8x_2^5 \\
&+ C_{50}x_3^3x_4x_5x_6^2x_7^3x_8x_2^5 + C_{114}x_1x_3^2x_4x_6^3x_7^2x_8x_2^5 + C_{47}x_3^2x_4x_5x_6^3x_7^2x_8x_2^5 \\
&+ C_{112}x_1x_3x_4x_6^4x_7x_8x_2^5 + C_{44}x_3x_4x_5x_6^4x_7x_8x_2^5 + C_{39}x_3^5x_4x_7^6x_8^2x_2^4 \\
&+ C_{37}x_3^4x_4x_6x_7^5x_8^2x_2^4 + C_{35}x_3^3x_4x_6^2x_7^4x_8^2x_2^4 + C_{33}x_3^2x_4x_6^3x_7^3x_8^2x_2^4 \\
&+ C_{31}x_3x_4x_6^4x_7^2x_8^2x_2^4 + C_{29}x_4x_6^5x_7x_8^2x_2^4 + C_{108}x_1x_3^5x_7^6x_8x_2^4 + C_{38}x_3^5x_5x_7^6x_8x_2^4 \\
&+ C_{107}x_1x_3^4x_6x_7^5x_8x_2^4 + C_{36}x_3^4x_5x_6x_7^5x_8x_2^4 + C_{106}x_1x_3^3x_6^2x_7^4x_8x_2^4 \\
&+ C_{34}x_3^3x_5x_6^2x_7^4x_8x_2^4 + C_{105}x_1x_3^2x_6^3x_7^3x_8x_2^4 + C_{32}x_3^2x_5x_6^3x_7^3x_8x_2^4 \\
&+ C_{104}x_1x_3x_6^4x_7^2x_8x_2^4 + C_{30}x_3x_5x_6^4x_7^2x_8x_2^4 + C_{103}x_1x_6^5x_7x_8x_2^4 \\
&+ C_{28}x_5x_6^5x_7x_8x_2^4 + C_{27}x_3^5x_7^7x_8^2x_2^3 + C_{26}x_3^4x_6x_7^6x_8^2x_2^3 + C_{25}x_3^3x_6^2x_7^5x_8^2x_2^3 \\
&+ C_{24}x_3^2x_6^3x_7^4x_8^2x_2^3 + C_{23}x_3x_6^4x_7^3x_8^2x_2^3 + C_{22}x_6^5x_7^2x_8^2x_2^3 + C_{139}x_1x_3^3x_4^3x_5x_6^2x_2^8
\end{aligned}
$$

The C_{5H}-curve is now given by

$$C_{5H} = \{p \in X_\Sigma \;,\; \widetilde{s}_1(p) = \widetilde{s}_2(p) = \widetilde{s}_3(p) = 0\} \tag{11.66}$$

The 177-parameters C_i that appear in the sections $\widetilde{s}_i$ give a redundant description of the complex structure of C_{5H}. They are subject to the condition that C_{5H} be a smooth subvariety of codimension 3 in X_Σ.

11.5.2. The Ambient Space Cohomologies

For this particular situation the Koszul resolution is given by

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_{C_{5H}} \to 0 \tag{11.67}$$

where

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(-9,-4,-9,-5)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}(-6,-2,-8,-4) \oplus \mathcal{O}_{X_\Sigma}(-8,-3,-9,-5) \oplus \mathcal{O}_{X_\Sigma}(-2,-2,1,0)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(-1,-1,1,0) \oplus \mathcal{O}_{X_\Sigma}(1,0,2,1) \oplus \mathcal{O}_{X_\Sigma}(-5,-1,-8,-4)$
- $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$

$\mathcal{L}'$	0	0	0	0	P_1
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	P_2	$0 \oplus 0 \oplus 0$	P_3
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	P_4	$0 \oplus 0 \oplus 0$	P_5	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_6	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 11.5.: Ambient space cohomologies in the computation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)|_{C_{5H}}$.

The cohomologies of these bundles are easily computed by our *Mathematica* notebook which is outlined in Appendix E. We give bases of the non-trivial ambient space cohomology groups in Table 11.7, which are then used to display all ambient space cohomologies in Table 11.5.

11.5.3. Computation Of The First Short Exact Sequence

Remark:
The first short exact sequence that results from splitting of the Koszul resolution is given by

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0 \tag{11.68}$$

with the auxillary sheaf $\mathcal{I}_1$. The corresponding sheet is given in Figure 11.8. Note that the only difficulty in determining the cohomologies of $\mathcal{I}_1$ is the map α^4.

Note:
Recall that the map α^4 is given by

$$\alpha^4\colon H^4(X_\Sigma, \mathcal{L}') \to H^4(X_\Sigma, \mathcal{V}_2), [t] \mapsto \left[\begin{pmatrix} \widetilde{s}_1 \\ -\widetilde{s}_2 \\ \widetilde{s}_3 \end{pmatrix} \cdot t\right] \tag{11.69}$$

By choosing an appropriate basis of domain and codomain the map α^4 can be represented by the matrix in Figure 11.9. Note that this matrix depends on the parameters C_{19}, C_{20} and C_{21} which will thus determine image and kernel of the map α^4.

Consequence:
We conclude from Figure 11.8 that the cohomologies of the auxillary sheaf $\mathcal{I}_1$ are given as

- $H^0(X_\Sigma, \mathcal{I}_1) = 0 \oplus 0 \oplus 0 \cong H^0(X_\Sigma, \mathcal{V}_2)$
- $H^1(X_\Sigma, \mathcal{I}_1) = 0 \oplus 0 \oplus 0 \cong H^1(X_\Sigma, \mathcal{V}_2)$

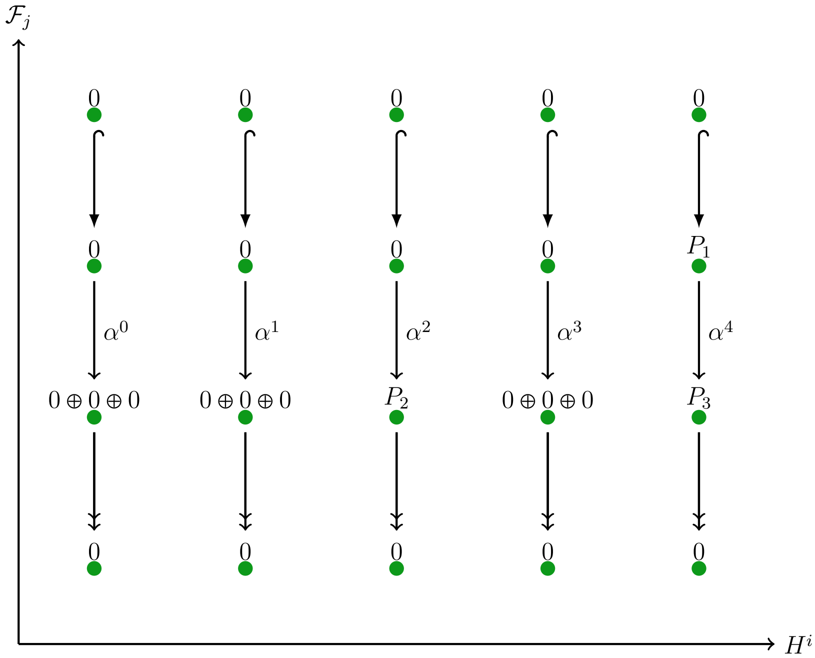


Figure 11.8.: Sheet describing the first short exact sequence $0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{I}_1 \to 0$ for the computation of line bundle cohomology on C_{5H}.

- $H^2(X_\Sigma, \mathcal{I}_1) = P_2 \cong H^2(X_\Sigma, \mathcal{V}_2)$
- $H^3(X_\Sigma, \mathcal{I}_1) \cong \ker(\alpha^4)$
- $H^4(X_\Sigma, \mathcal{I}_1) \cong \mathrm{coker}(\alpha^4)$

11.5.4. Computation Of The Second Short Exact Sequence

Remark:
We now have to compute the cohomologies of the auxillary sheaf $\mathcal{I}_2$ defined by the short exact sequence

$$0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0 \tag{11.70}$$

With the result from the above calculation we can organise the corresponding sheet as outlined in Figure 11.11. Recall that we have a canonical isomorphism

$$H^i(X_\Sigma, \mathcal{I}_1) \cong H^i(X_\Sigma, \mathcal{V}_2) \tag{11.71}$$

for $i = 0, 1, 2$. So β^3 and β^4 are identified as 'mysterious maps'. Note however that the only non-trivial map is β^2. This map in turn is given by

$$\beta^2 \colon H^2(X_\Sigma, \mathcal{I}_1) \to H^2(X_\Sigma, \mathcal{V}_2)\ , \quad \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & -\widetilde{s}_3 & -\widetilde{s}_2 \\ -\widetilde{s}_3 & 0 & \widetilde{s}_1 \\ \widetilde{s}_2 & \widetilde{s}_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.72}$$

$$
M_{\alpha^4} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} \\
0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21}
\end{pmatrix}^T
$$

Figure 11.9.: A matrix representing the map α^4 that appears in the calculation of the first short exact sequence of the pullback cohomology of $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$ onto C_{5H}.

A simple but cumbersome calculation reveals that β^2 can be represented by the matrix given in Figure 11.10.

Consequence:
The cohomologies of $\mathcal{I}_2$ are thus obtained as

- $H^0(X_\Sigma, \mathcal{I}_2) \cong P_4 \cong H^0(X_\Sigma, \mathcal{V}_1)$
- $H^1(X_\Sigma, \mathcal{I}_2) \cong \ker(\beta^2)$
- $H^2(X_\Sigma, \mathcal{I}_2) \cong \operatorname{coker}(\beta^2) \oplus \ker(\alpha^4)$
- $H^3(X_\Sigma, \mathcal{I}_2) \cong \operatorname{coker}(\alpha^4)$
- $H^4(X_\Sigma, \mathcal{I}_2) \cong 0 \oplus 0 \oplus 0$

11.5.5. Computation Of The Third Short Exact Sequence

Remark:
Finally we can compute the cohomologies of $\mathcal{L}|_{C_{5H}}$. Recall that this is achieved via

$$
M_{\beta^2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & C_{16} & C_{17} \\
0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & C_{10} & C_{12} \\
0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & C_{1} & C_{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & C_{16} & C_{17} & 0 \\
0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & C_{10} & C_{12} & 0 \\
0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{1} & C_{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & C_{16} & C_{17} & 0 & 0 \\
C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & C_{10} & C_{12} & 0 & 0 \\
C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{1} & C_{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & C_{11} & C_{13} \\
0 & 0 & 0 & 0 & C_{20} & C_{19} & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & C_{2} & C_{5} \\
0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & C_{11} & C_{13} & 0 \\
0 & 0 & C_{20} & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{2} & C_{5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & C_{11} & C_{13} & 0 & 0 \\
C_{20} & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{2} & C_{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{3} & C_{6} \\
0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{3} & C_{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{3} & C_{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Figure 11.10.: The matrix representing the map β^2 that appears in the computation of the second short exact sequence of the pullback cohomologies of $\mathcal{L} = \mathcal{O}_{X_\Sigma}(2,1,2,1)$ onto C_{5H}.

the third short exact sequence

$$0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{5H}} \to 0 \tag{11.73}$$

With the cohomologies for $\mathcal{I}_2$ we can organise the computation on the sheet given in Figure 11.12. Note that

$$H^0(X_\Sigma, \mathcal{I}_2) \cong H^0(X_\Sigma, \mathcal{V}_1) \tag{11.74}$$

Consequently the map γ^0 is just given by the one induced from the Koszul complex. Thus we have

$$\gamma^0 \colon H^0(X_\Sigma, \mathcal{I}_2) \to H^0(X_\Sigma, \mathcal{L}) \ , \ \left[\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \mapsto \left[(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}\right] \tag{11.75}$$

This does not hold for γ^1, γ^2, γ^3 since these maps again fall into the category of 'mysterious maps'. However the map γ^4 is induced from the Koszul complex.

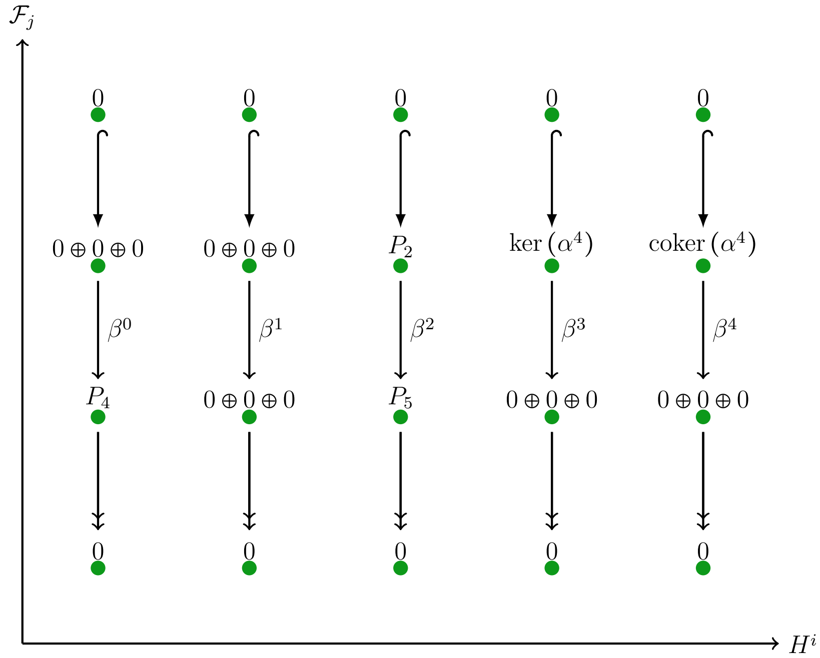


Figure 11.11.: Sheet describing the second short exact sequence $0 \to \mathcal{I}_1 \to \mathcal{V}_1 \to \mathcal{I}_2 \to 0$ for the computation of line bundle cohomology on C_{5H}.

Note:
A simple calculation reveals that the map γ^0 can be represented by the following matrix

$$M_{\gamma^0} = \begin{pmatrix} 0 & 0 & C_{21} \\ 0 & C_{21} & 0 \\ C_{21} & 0 & 0 \\ 0 & 0 & C_{20} \\ 0 & 0 & C_{19} \\ 0 & C_{20} & 0 \\ 0 & C_{19} & 0 \\ C_{20} & 0 & 0 \\ C_{19} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11.76}$$

Consequence:
From the above we now conclude that the cohomologies of $\mathcal{L}|_{C_{5H}}$ are given by

- $H^0\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = \mathrm{coker}\left(\gamma^0\right) \oplus \ker\left(\beta^2\right)$
- $H^1\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = \mathrm{coker}\left(\beta^2\right) \oplus \ker\left(\alpha^4\right)$
- $H^2\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = \mathrm{coker}\left(\alpha^4\right)$

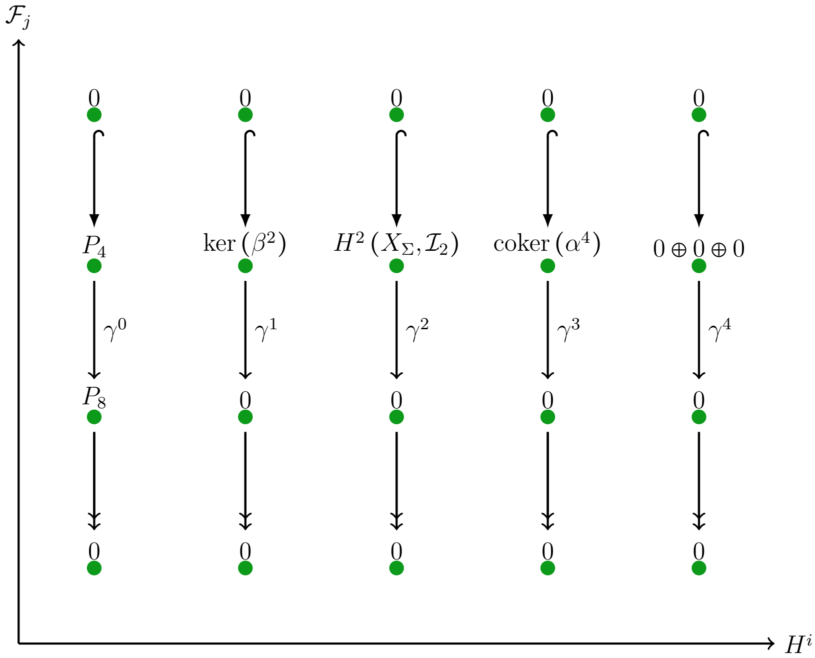


Figure 11.12.: Sheet for the third short exact sequence $0 \to \mathcal{I}_2 \to \mathcal{L} \to \mathcal{L}|_{C_{5H}} \to 0$ for the computation of line bundle cohomology on C_{5H}. For better readability we have set $H^2(X_\Sigma, \mathcal{I}_2) \cong \mathrm{coker}(\beta^2) \oplus \ker(\alpha^4)$.

- $H^3\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$
- $H^4\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$

Remark:
In principle the story ends here, and one has to go back to the sections $\widetilde{s}_i$. The condition that their common zero locus has to be a smooth subvariety of codimension 3 in X_Σ places constraints on the parameters C_i. Having chosen a set of such parameters one can then use the above analysis to determine the cohomologies of the pullback line bundle $\mathcal{L}|_{C_{5H}}$.
Unfortunately it is a cumbersome exercise to determine which values of the parameters C_i meet this requirements, in particular given the length of the polynomials $\widetilde{s}_i$. Therefore we will take a minimalistic approach in which we make use of the fact that all three sections $\widetilde{s}_i$ must be non-trivial to allow C_{5H} to be of codimension 3. This alone will already allow us to improve the bounds for the dimensions of the cohomology groups of $\mathcal{L}|_{C_{5H}}$ that we obtained in chapter 9.

11.5.6. Analysis Of The Parameters $\mathbf{C_i}$

Remark:
Recall that at least one of the parameters C_{19}, C_{20}, C_{21} is non-zero in order to ensure

that $\widetilde{s}_2$ is non-trivial. This implies

$$\operatorname{im}\left(\alpha^4\right) \cong \mathbb{C}^{10}, \qquad \ker\left(\alpha^4\right) \cong \mathbb{C}^{24} \tag{11.77}$$

In particular this implies that $H^2\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$ as expected from the finiteness theorem. In addition the fact that at least one of the parameters C_{19}, C_{20}, C_{21} is non-zero does imply

$$\operatorname{im}\left(\gamma^0\right) \cong \mathbb{C}^3 \tag{11.78}$$

Consequence:
From this observation alone we conclude

- $h^0\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 25 - \operatorname{im}\left(\beta^2\right)$
- $h^1\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 54 - \operatorname{im}\left(\beta^2\right)$
- $h^2\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = h^3\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = h^4\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$

Note:
Having at least one of the parameters C_{19}, C_{20}, C_{21} non-zero gives [2]

$$12 \leq \dim_{\mathbb{C}}\left(\operatorname{im}\left(\beta^2\right)\right) \leq 16 \tag{11.79}$$

This then gives $1 \leq \dim_{\mathbb{C}}\left(\ker\left(\beta^2\right)\right) \leq 5$ and therefore implies bounds for the dimension of the pullback cohomology groups according to

- $h^0\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 8 + \ker\left(\beta^2\right)$
- $h^1\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 37 + \ker\left(\beta^2\right)$
- $h^2\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$
- $h^3\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$
- $h^4\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 0$

This result improves the bounds that we computed back in chapter 9.

Note:
For the generic pullback setup we find that M_{β^2} has one-dimensional kernel. Thus

$$h^0\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 9, \qquad h^0\left(C_{5H}, \mathcal{L}|_{C_{5H}}\right) = 38 \tag{11.80}$$

11.6. Collection Of Results

We summarise the results from the calculations performed in this chapter in Table 11.6. They should be compared to Table 9.3, which lists the results obtained by use of exactness properties only.

[2] Linear algebra tells us that row rank and column rank of a matrix are identical.

	$h^0(C_i, \mathcal{L}\|_P)$	$h^1(C_i, \mathcal{L}\|_P)$	Parameter range
C_{10} 'general'	8	0	0
C_{10} 'generic'	8	0	0
$C_{\overline{5}m}$ 'general'	9	6	0
$C_{\overline{5}m}$ 'generic'	9	6	0
C_{5H} 'general'	$8 + A$	$37 + A$	$1 \leq A \leq 5$
C_{5H} 'generic'	9	38	0

Table 11.6.: The pullback cohomology dimensions in the exhaustive example as computed by evaluation of the appearing maps. Note that $A = \dim_{\mathbb{C}}(\ker(\beta^2))$. These results should be compared to the results obtained from exactness alone. The latter are summarised in Table 9.3.

Space	Basis	Dimension
P_1	$\frac{A_4}{x_1x_2^3x_3^4x_4x_5x_6x_7^5x_8^2} + \frac{A_3}{x_1x_2^3x_3^3x_4x_5x_6^2x_7^4x_8^2} + \frac{A_{12}}{x_1x_2^4x_3^4x_4^2x_5x_6x_7^4x_8^2} + \frac{A_2}{x_1x_2^3x_3^2x_4x_5x_6^3x_7^3x_8^2} + \frac{A_{10}}{x_1x_2^4x_3^3x_4^2x_5x_6^2x_7^3x_8^2} + \frac{A_{18}}{x_1x_2^5x_3^4x_4^3x_5x_6x_7^3x_8^2} + \frac{A_1}{x_1x_2^3x_3x_4x_5x_6^4x_7^2x_8^2} + \frac{A_8}{x_1x_2^4x_3^2x_4^2x_5x_6^3x_7^2x_8^2} + \frac{A_{16}}{x_1x_2^5x_3^3x_4^3x_5x_6^2x_7^2x_8^2} + \frac{A_{22}}{x_1x_2^6x_3^4x_4^4x_5x_6x_7^3x_8^2} + \frac{A_6}{x_1x_2^4x_3x_4^2x_5x_6^4x_7x_8^2} + \frac{A_{14}}{x_1x_2^5x_3^2x_4^3x_5x_6^3x_7x_8^2} + \frac{A_{20}}{x_1x_2^6x_3^3x_4^4x_5x_6^2x_7x_8^2} + \frac{A_{24}}{x_1x_2^7x_3^4x_4^5x_5x_6x_7x_8^2} + \frac{A_{11}}{x_1x_2^4x_3^4x_4x_5^2x_6x_7^4x_8} + \frac{A_{28}}{x_1^2x_2^4x_3^4x_4x_5x_6x_7^4x_8} + \frac{A_9}{x_1x_2^4x_3^3x_5^2x_6^2x_7^3x_8} + \frac{A_{27}}{x_1^2x_2^4x_3^3x_4x_5x_6^2x_7^3x_8} + \frac{A_{17}}{x_1x_2^5x_3^4x_4^2x_5^2x_6x_7^3x_8} + \frac{A_{31}}{x_1^2x_2^5x_3^4x_4^2x_5x_6x_7^3x_8} + \frac{A_7}{x_1x_2^4x_3^2x_4x_5^2x_6^3x_7^2x_8} + \frac{A_{26}}{x_1^2x_2^4x_3^2x_4x_5x_6^3x_7^2x_8} + \frac{A_{15}}{x_1x_2^5x_3^3x_4^2x_5^2x_6^2x_7^2x_8} + \frac{A_{30}}{x_1^2x_2^5x_3^3x_4^2x_5x_6^2x_7^2x_8} + \frac{A_{21}}{x_1x_2^6x_3^4x_4^3x_5^2x_6x_7^2x_8} + \frac{A_{33}}{x_1^2x_2^6x_3^4x_4^3x_5x_6x_7^2x_8} + \frac{A_5}{x_1x_2^4x_3x_4x_6^2x_6^4x_7x_8} + \frac{A_{25}}{x_1^2x_2^4x_3x_4x_5x_6^4x_7x_8} + \frac{A_{13}}{x_1x_2^5x_3^2x_4^2x_5^2x_6^3x_7x_8} + \frac{A_{29}}{x_1^2x_2^5x_3^2x_4^2x_5x_6^3x_7x_8} + \frac{A_{19}}{x_1x_2^6x_3^3x_4^3x_5^2x_6^2x_7x_8} + \frac{A_{32}}{x_1^2x_2^6x_3^3x_4^3x_5x_6^2x_7x_8} + \frac{A_{23}}{x_1x_2^7x_3^4x_4^4x_5^2x_6x_7x_8} + \frac{A_{34}}{x_1^2x_2^7x_3^4x_5x_6x_7x_8}$	34

P_2	$\begin{pmatrix} 0 \\ 0 \\ \frac{A_{35}x_7}{x_1x_4x_5} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{39}}{x_2x_3^4x_6x_7^7x_8^3} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{38}}{x_2x_3^3x_6^2x_7^6x_8^3} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{37}}{x_2x_3^2x_6^3x_7^5x_8^3} \\ 0 \end{pmatrix} +$ $\begin{pmatrix} 0 \\ \frac{A_{36}}{x_2x_3x_6^4x_7^4x_8^3} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{61}x_1}{x_2x_3^3x_6x_7^6x_8^3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{54}x_5}{x_2x_3^3x_6x_7^6x_8^3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{60}x_1}{x_2x_3^2x_6^2x_7^5x_8^3} \\ 0 \\ 0 \end{pmatrix} +$ $\begin{pmatrix} \frac{A_{52}x_5}{x_2x_3^2x_6^2x_7^5x_8^3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{59}x_1}{x_2x_3x_6^3x_7^4x_8^3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{50}x_5}{x_2x_3x_6^3x_7^4x_8^3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{55}x_4}{x_2x_3^3x_6x_7^6x_8^2} \\ 0 \\ 0 \end{pmatrix} +$ $\begin{pmatrix} \frac{A_{53}x_4}{x_2x_3^2x_6^2x_7^5x_8^2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{58}}{x_2^2x_3^3x_6x_7^5x_8^2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{51}x_4}{x_2x_3x_6^3x_7^4x_8^2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{A_{57}}{x_2^2x_3^2x_6^2x_7^4x_8^2} \\ 0 \\ 0 \end{pmatrix} +$ $\begin{pmatrix} \frac{A_{56}}{x_2^2x_3x_6^3x_7^3x_8^2} \\ 0 \\ 0 \end{pmatrix}$	17
P_3	$\begin{pmatrix} 0 \\ \frac{A_{43}}{x_1x_2^4x_3^4x_4x_5x_6x_7^4x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{42}}{x_1x_2^4x_3^3x_4x_5x_6^2x_7^3x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{46}}{x_1x_2^5x_3^4x_4^2x_5x_6x_7^3x_8} \\ 0 \end{pmatrix} +$ $\begin{pmatrix} 0 \\ \frac{A_{41}}{x_1x_2^4x_3^2x_4x_5x_6^3x_7^2x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{45}}{x_1x_2^5x_3^3x_4^2x_5x_6^2x_7^2x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{48}}{x_1x_2^6x_3^4x_4^3x_5x_6x_7^2x_8} \\ 0 \end{pmatrix} +$ $\begin{pmatrix} 0 \\ \frac{A_{40}}{x_1x_2^4x_3x_4x_5x_6^4x_7x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{44}}{x_1x_2^5x_3^2x_4^2x_5x_6^3x_7x_8} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{A_{47}}{x_1x_2^6x_3^3x_4^3x_5x_6^2x_7x_8} \\ 0 \end{pmatrix} +$ $\begin{pmatrix} 0 \\ \frac{A_{49}}{x_1x_2^7x_3^4x_4^4x_5x_6x_7x_8} \\ 0 \end{pmatrix}$	10
P_4	$\begin{pmatrix} 0 \\ A_{64}x_2^2x_3x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ A_{62}x_2x_6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ A_{63}x_2x_3x_7 \\ 0 \end{pmatrix}$	3

P_5	$\begin{pmatrix}0\\0\\\frac{A_{94}x_1^2}{x_2x_3^3x_6x_7^6x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{87}x_1x_5}{x_2x_3^3x_6x_7^6x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{71}x_5^2}{x_2x_3^3x_6x_7^6x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{93}x_1^2}{x_2x_3^2x_6^2x_7^5x_8^3}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{85}x_1x_5}{x_2x_3^2x_6^2x_7^5x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{68}x_5^2}{x_2x_3^2x_6^2x_7^5x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{92}x_1^2}{x_2x_3x_6^3x_7^4x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{83}x_1x_5}{x_2x_3x_6^3x_7^4x_8^3}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{65}x_5^2}{x_2x_3x_6^3x_7^4x_8^3}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{88}x_1x_4}{x_2x_3^3x_6x_7^6x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{72}x_4x_5}{x_2x_3^3x_6x_7^6x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{86}x_1x_4}{x_2x_3^2x_6^2x_7^5x_8^2}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{69}x_4x_5}{x_2x_3^2x_6^2x{-}7^5x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{91}x_1}{x_2^2x_3^3x_6x_7^5x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{78}x_5}{x_2^2x_3^3x_6x_7^5x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{84}x_1x_4}{x_2x_3x_6^3x_7^4x_8^2}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{66}x_4x_5}{x_2x_3x_6^3x_7^4x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{90}x_1}{x_2^2x_3^2x_6^2x_7^4x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{76}x_5}{x_2^2x_3^2x_6^2x_7^4x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{89}x_1}{x_2^2x_3x_6^3x_7^3x_8^2}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{74}x_5}{x_2^2x_3x_6^3x_7^3x_8^2}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{73}x_4^2}{x_2x_3^3x_6x_7^6x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{70}x_4^2}{x_2x_3^2x_6^2x_7^5x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{79}x_4}{x_2^2x_3^3x_6x_7^5x_8}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{67}x_4^2}{x_2x_3x_6^3x_7^4x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{77}x_4}{x_2^2x_3^2x_6^2x_7^4x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{82}}{x_2^3x_3^3x_6x_7^4x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{75}x_4}{x_2^2x_3x_6^3x_7^3x_8}\end{pmatrix}+$ $\begin{pmatrix}0\\0\\\frac{A_{81}}{x_2^3x_3^2x_6^2x_7^3x_8}\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{A_{80}}{x_2^3x_3x_6^3x_7^2x_8}\end{pmatrix}$	30
P_6	$A_{105}x_1x_2^2x_3x_4 + A_{101}x_2^2x_3x_4x_5 + A_{103}x_1x_2x_6 + A_{97}x_2x_5x_6 + A_{104}x_1x_2x_3x_7 + A_{99}x_2x_3x_5x_7 + A_{102}x_2^2x_3x_4^2x_8 + A_{98}x_2x_4x_6x_8 + A_{100}x_2x_3x_4x_7x_8 + A_{95}x_6x_7x_8 + A_{96}x_3x_7^2x_8$	11

Table 11.7.: Non-trivial ambient space cohomologies in the calculation of the cohomologies of $\mathcal{O}_{X_\Sigma}(2,1,2,1)\big|_{C_{5H}}$.

12. The Koszul Spectral Sequence

12.1. Summary

In this chapter we wish to introduce the subject of spectral sequences briefly and then turn to the Koszul spectral sequence, which allows us to reorganise the above computations of pullback cohomology. Unfortunately the topic of spectral sequences is very technical. For this reason the following chapter is probably the most technical chapter in the entire thesis. The technical reader might want to consult in addition [58], references therein as well as [52, 44, 66].
In section 12.2 we introduction spectral sequences briefly. The intuitive picture is as follows.

- Think about a paper stack. On each page we draw the lattice $\mathbb{Z}^2$. This we illustrate in $Figure$ 12.2.
- At page E_0 we place at each lattice point (p, q) an Abelian group $E_0^{p,q}$. In addition we specify a number of horizontal complexes as pictured in Figure 12.3.
- We now compute the cohomologies of these horizontal complexes. These cohomology groups are quotients of Abelian groups. We denote the corresponding quotient at the lattice point (p, q) by $C_0^{p,q}$. Subsequently we place precisely this cohomology group at the lattice point (p, q) on the sheet E_1.

⇒ Roughly speaking the sheet E_1 is the cohomology of the sheet E_0.

- As a next step we specify complexes on the sheet E_1. These are indicated in Figure 12.3. We again compute their cohomologies and place these cohomology groups on the sheet E_2.
- We specify complexes on the sheet E_2 and compute cohomology from those complexes ...

In particular it can happen that starting from the sheet E_p all complexes are trivial. Then the cohomology groups become stable and the sheets E_p, E_{p+1}, ... are identical as long as the Abelian groups on them are concerned. These Abelian groups one denotes by $E_\infty^{p,q}$ and says that the spectral sequence converges on the sheet E_p. It is precisely such a 'limit sheet' that will tell us about the pullback cohomologies - namely the 'limit sheet' of the Koszul spectral sequence.
The Koszul spectral sequence is a special spectral sequence that we introduce in section 12.3. The crucial point in introducing this spectral sequence is to make use of the affine open cover $\mathcal{U}$ of the smooth and compact normal toric variety X_Σ which allows to compute sheaf cohomology on X_Σ from Čech cohomology. Recall that we

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^0"] & \check{C}^1(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^1"] & \check{C}^2(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^2"] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^0"] & \check{C}^1(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^1"] & \check{C}^2(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^2"] & \dots \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \dots \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Figure 12.1.: Double complex from Koszul sequence.

used this already in section 11.1 and section 11.2 where we pointed out that the maps in the Koszul sequence induce mappings of the Čech cocycles and thereby also mappings of the Čech cochains. Consequently the Koszul sequence

$$0 \to \mathcal{L}' \overset{\alpha}{\to} \mathcal{V}_{n-1} \overset{\beta}{\to} \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{12.1}$$

allows to consider the commutative double complex given in Figure 12.1 where the maps α^i, β^i, ... are induced from the corresponding mappings in the original Koszul sequence.

The main result from this chapter is that this double complex induces a spectral sequence, in that it systematically induces the complexes of the higher sheets, and that the so-obtained Koszul spectral sequence allows for a systematic computation of the pullback cohomologies. We give details in section 12.3 and recommend that also the non-technical reader has a look into this chapter. In particular we indicate how the above double complex induces a spectral sequence. Unfortunately the induction of the complexes is involved. For this reason section 12.3 only sketches this construction and we devote the entire chapter chapter 13 on a detailed discussion of this construction.

We conclude this chapter with example applications of the Koszul spectral sequence. To this end we work out a simple computation on $\mathbb{CP}^4$ from the spectral sequence perspective in section 12.4 and have a final look at the exhaustive example from chapter 9 and chapter 11 in section 12.5.

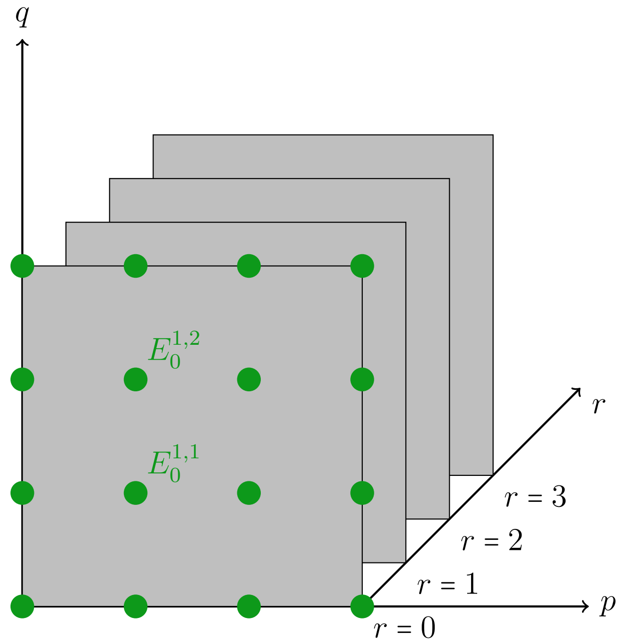


Figure 12.2.: Organisation of the Abelian groups in a spectral sequence.

12.2. Brief Introduction To Spectral Sequences

Definition 12.2.1 (Cohomology Spectral Sequence)**:**
A (cohomology) spectral sequence is a collection of Abelian groups $(E_r^{p,q})_{r,p,q \in \mathbb{Z}}$ and group homomorphisms $(d_r^{p,q})_{p,q,r \in \mathbb{Z}}$ such that

- r labels the sheets, that is we can think of the collection of groups $(E_r^{p,q})$ as organised in a paper stack. This we picture in Figure 12.2.
- in the r-th sheet there are group homomorphisms

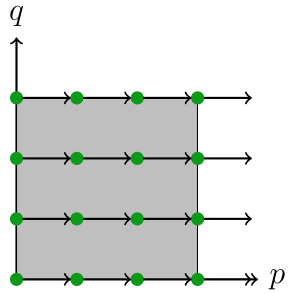
$$d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+1-r,q-r} \tag{12.2}$$

  We give pictures of this situation for the sheets E_0, E_1 and E_2 in Figure 12.3. In particular note that the r-th sheet thus splits up into a collection of complexes.
- The $(r+1)$-th sheet is given by the cohomology groups of the complexes $(E_r^{p,q}, d_r^{p,q})$ in the r-th sheet, i.e.

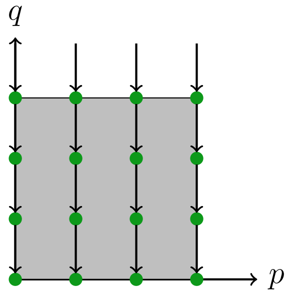
$$E_{r+1}^{p,q} := \frac{\ker\left(d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+1-r,q-r}\right)}{\operatorname{im}\left(d_r \colon E^{p+r-1,q+r} \to E_r^{p,q}\right)} \tag{12.3}$$

Note:

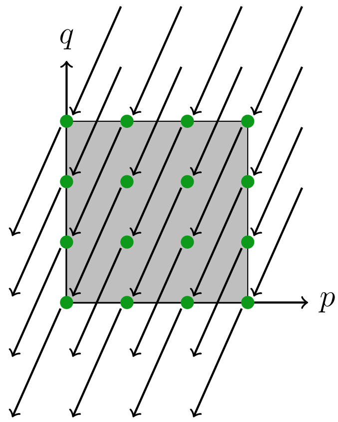
- For our purposes the group $E_r^{p,q}$ carry more structure than just being an Abelian group - they are $\mathcal{O}_{X_\Sigma}$-modules. The maps $d_r^{p,q}$ consequently are homomorphisms of $\mathcal{O}_{X_\Sigma}$-modules.



(a) Sheet E_0



(b) Sheet E_1



(c) Sheet E_2

Figure 12.3.: Splitting of the sheets E_0, E_1 and E_2 into complexes.

- A first quadrant cohomology spectral sequences is a spectral sequence for which $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$.
- If the minimum value for r is 1, then we talk about an E_1 spectral sequence. Similarly if $r \geq 2$ holds, we term the corresponding cohomology spectral sequence an E_2 spectral sequence.

In the remainder of this thesis we will only consider first quadrant E_0 cohomology spectral sequences.

Definition 12.2.2 (Convergence)**:**
A first quadrant cohomology spectral sequence E_0 converges at sheet R precisely if all the differential maps $d_r^{p,q}$ with $r \geq R$ vanish identically. We then use the notation

$$E_\infty^{p,q} := E_R^{p,q} \tag{12.4}$$

Definition 12.2.3 (Convergence)**:**
A first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to a sequence of Abelian groups H^k $(k \geq 0)$ precisely if there exists a filtration

$$0 = F^{k+1}H^k \subseteq F^k H^k \subseteq F^{k-1}H^k \subseteq \cdots \subseteq F^1 H^k \subseteq F^0 H^k = H^k \tag{12.5}$$

of H^k by subgroups such that

$$E_\infty^{p,q} \simeq \frac{F^q H^{p+q}}{F^{p+1}H^{p+q}} \tag{12.6}$$

Remark:
We will now introduce a special spectral sequence, the Koszul spectral sequence. This spectral sequence enables us to organise the computation of pullback cohomologies in a very efficient way. This reorganisation of the pullback computation will in particular shed light on the origin of the 'mysterious maps'.

12.3. The Koszul Spectral Sequence

Remark:
Let X_Σ a smooth and compact normal toric variety. Moreover let $S_1, \ldots, S_n \in \mathrm{Cl}(X_\Sigma)$ effective divisor classes. We then pick non-trivial holomorphic sections

$$\widetilde{s}_i \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_i)\right) \tag{12.7}$$

such that their common zero locus

$$C := \{p \in X_\Sigma \ , \ \widetilde{s}_1(p) = \cdots = s_n(p) = 0\} \tag{12.8}$$

is a smooth algebraic subvariety of codimension n in X_Σ. Finally consider $D \in \mathrm{Cl}(X_\Sigma)$ and the associated holomorphic line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$. Given all that structure we know that the Koszul resolution

$$0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{12.9}$$

is a sheaf exact sequence. By means of the Leray property of the affine open cover of X_Σ we can then induce maps on the Čech cochains of the vector bundles appearing in the Koszul resolution. This gives rise to the commutative double complex as given in Figure 12.4. Note in particular that the columns of this double complex are exact, as they are stemming from the exact Koszul resolution.

Note:
It is possible to remove the last row from the commutative double complex in Figure 12.4. The resulting double complex is still commutative, but exactness of the columns is lost.

Theorem 12.3.1:
The double complex given in Figure 12.5 gives rise to a convergent, first quadrant E_0-spectral sequence with the property

$$H^i\left(C, \mathcal{L}|_C\right) \cong \bigoplus_{m=0}^{\infty} E_\infty{}^{i+m,m} \tag{12.10}$$

We term this particular spectral sequence the *Koszul spectral sequence.*

Proof
We first illustrate how one constructes a spectral sequence from the double complex in Figure 12.5.

- Sheet E_0:
  The finite dimensional vector spaces that we put in this sheet are the ones that appear in Figure 12.5. The differential maps $d_0^{p,q}$ are the horizontal Čech coboundaries.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^0"] & \check{C}^1(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^1"] & \check{C}^2(\mathcal{U}, \mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^2"] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^0"] & \check{C}^1(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^1"] & \check{C}^2(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^2"] & \dots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U}, \mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U}, \mathcal{L}|_C) \arrow[r, "\delta"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 12.4.: The commutative double complex induced from the Koszul resolution $0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0$ by use to the Leray property of the affine open cover $\mathcal{U}$ of the smooth and compact normal toric variety X_Σ.

- Sheet E_1:
  The sheet E_1 is obtained by computing the cohomology of the horizontal complexes in the sheet E_0. By use of the Leray property of the affine open cover $\mathcal{U}$ of the smooth and compact normal toric variety X_Σ, the sheet E_1 consequently looks as outlined in Figure 12.6. Note that the horizontal maps disappeared as we computed the cohomology of the corresponding complexes. In addition note that the maps α^i, β^i, ... that appear in the sheet E_0 give rise to maps between the cohomology groups of the horizontal E_0-complexes. For ease of notation we will not use different labels for these maps.

- Sheet E_2:
  The vector spaces in the sheet E_2 are obtained from computing the cohomologies in the sheet E_1. Whilst on a first glance there are no maps between the so-obtained quotients of vector spaces, a more throughout analysis reveals that this is not true.
  To see this consider $x \in E_2^{p,q}$. As x represents an equivalence class of elements in $E_1^{p,q}$ one can find a representative $\widetilde{x} \in E_1^{p,q}$ of x. This $\widetilde{x}$ in turn represents an equivalence class of elements in $E_0^{p,q}$. We thus conclude that one can represent $x \in E_2^{p,q}$ by very special elements $\widetilde{\widetilde{x}} \in E_0^{p,q}$.
  In the sheet $E_0^{p,q}$ however, we have the vertical and horizontal differential maps.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^0"] & \check{C}^1(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^1"] & \check{C}^2(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^2"] & \cdots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^0"] & \check{C}^1(\mathcal{U},\mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^1"] & \check{C}^2(\mathcal{U},\mathcal{V}_{n-1}) \arrow[r, "\delta"] \arrow[d, "\beta^2"] & \cdots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 12.5.: The commutative double complex induced from the Koszul resolution $0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0$ which gives rise to a convergent spectral sequence - the Koszul spectral sequence. The limit of that spectral sequence allows us to determine the cohomologies of $\mathcal{L}|_C$.

By use of the special properties of $\overset{\approx}{x}$, which we decide to remain silent on for now, a simple diagram chase reveals that it is possible to assign to $\overset{\approx}{x}$ a very special element $\overset{\approx}{y} \in E_0^{p-1,q-2}$. This element $\overset{\approx}{y} \in E_0^{p-1,p-2}$ in turn, has precisely the special properties to represent an element $y \in E_0^{p-1,p-2}$. This constructs the 'Knight's move' [1].

- Higher sheets:
  The construction of the differential maps in the higher sheets follows the same principle.
  1. Represent an element $x \in E_r^{p,q}$ by very special elements $\overset{\approx}{x} \in E_0^{p,q}$.
  2. Make a diagram chase in the sheet E_0 to associate to $\overset{\approx}{x}$ a new element $\overset{\approx}{y} \in E_0^{p+1-r,q-r}$ which has the same special properties as $\overset{\approx}{x}$.
  3. The special properties of $\overset{\approx}{y}$ mean that it represents an element $y \in E_0^{p+1-r,q-r}$.
  4. Finally one defines $d_r^{p,q}(x) := y$.

Consequently, the above forms a first quadrant E_0 spectral sequence. It is clear that this sequence converges as the first quadrant does only contain a rectangle of finite size in which the $\mathcal{O}_{X_\Sigma}$-modules are non-trivial.
For the proof that this spectral sequence does converge to $\mathcal{O}_{X_\Sigma}$-modules $E_\infty^{p,q}$ with the property $H^i(C, \mathcal{L}|_C) \cong \bigoplus_{m=0}^{\infty} E_\infty{}^{i+m,m}$ we refer the interested reader to [67]. ∎

[1] We give more details on this construction in chapter 13.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
H^0(\mathcal{U}, \mathcal{L}') \arrow[d, "\alpha^0"] & H^1(\mathcal{U}, \mathcal{L}') \arrow[d, "\alpha^1"] & H^2(\mathcal{U}, \mathcal{L}') \arrow[d, "\alpha^2"] & \ldots \\
H^0(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[d, "\beta^0"] & H^1(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[d, "\beta^1"] & H^2(\mathcal{U}, \mathcal{V}_{n-1}) \arrow[d, "\beta^2"] & \ldots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
H^0(\mathcal{U}, \mathcal{L}) \arrow[d] & H^1(\mathcal{U}, \mathcal{L}) \arrow[d] & H^2(\mathcal{U}, \mathcal{L}) \arrow[d] & \ldots \\
0 & 0 & 0 &
\end{tikzcd}$$

Figure 12.6.: Commuting double complex induced from the Koszul resolution $0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0$. This complex gives rise to a convergent spectral sequence. The limit of that spectral sequence allows us to determine the cohomologies of $\mathcal{L}|_C$.

Remark:
We give more details on the construction of the maps d_r with $r \geq 2$ later. For example we discuss the Knight's move in detail in chapter 13 and formulate a proposal of a simplified construction of these maps in chapter 14. Note that the maps d_r with $r \geq 2$ are the 'mysterious maps'.

12.4. An Example On Complex Projective Space

12.4.1. Some Facts About Complex Projective Space

Remark:
We want to illustrate the application of the Koszul spectral sequence with a particularly simple example. In order to do so we consider line bundles on the smooth and compact normal toric variety $\mathbb{CP}^4$. Let us therefore recall a few facts about this toric variety first.

- We have $\Sigma(1) = \{u_1, u_2, u_3, u_4, u_5\}$ where the ray generators $u_i \in \mathbb{Z}^4$ are given by

$$u_1 = e_1, \quad u_2 = e_2, \quad u_3 = e_3, \quad u_4 = e_4, \quad u_5 = -e_1 - e_2 - e_3 - e_4 \tag{12.11}$$

  In this situation it is straight-forward to extend $\Sigma(1)$ to a fan Σ, which is complete and smooth, showing that $\mathbb{CP}^4$ is indeed a compact and smooth normal toric variety. From this fan the Stanley-Reisner ideal is obtained as

$$I_{\mathrm{SR}} = \langle x_1 x_2 x_3 x_4 x_5 \rangle \subset \mathbb{C}\left[x_1, x_2, x_3, x_4, x_5\right] \tag{12.12}$$

- Via the cone-orbit-correspondance there is a unique torus invariant prime divisor associated to each ray generator. These we denote D_1, D_2, D_3, D_4 and D_5 respectively.
- By recalling that the characters χ^m are meromorphic functions and thus have a principal divisor, one easily confirms that $[D_5]$ generates the class group of $\mathbb{CP}^4$ over $\mathbb{Z}$. Since $\mathbb{CP}^4$ is smooth this even gives

$$\operatorname{Pic}\left(\mathbb{CP}^4\right) \cong \mathbb{Z}\left[D_5\right] \tag{12.13}$$

  It is thus meaningful to refer to a holomorphic line bundle on $\mathbb{CP}^4$ by $\mathcal{O}_{\mathbb{CP}^4}(m)$ with $m \in \mathbb{Z}$.

12.4.2. Setup

Let us now consider the holomorphic line bundles

$$S_1 = \mathcal{O}_{\mathbb{CP}^4}(1), \qquad S_2 = \mathcal{O}_{\mathbb{P}^4}(1), \qquad S_3 = \mathcal{O}_{\mathbb{P}^4}(1) \tag{12.14}$$

which all have effective divisor classes. It then easily follows from application of *cohomCalg* that the most general global holomorphic sections of these bundles are given by

- $\widetilde{s}_1 = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 x_4 + C_5 x_5$
- $\widetilde{s}_2 = C_6 x_1 + C_7 x_2 + C_8 x_3 + C_9 x_4 + C_{10} x_5$
- $\widetilde{s}_3 = C_{11} x_1 + C_{12} x_2 + C_{13} x_3 + C_{14} x_4 + C_{15} x_5$

where $c_i \in \mathbb{C}$ are arbitary complex numbers. Now we set

$$C := \left\{p \in \mathbb{CP}^4 \ , \ \widetilde{s}_1(p) = \widetilde{s}_2(p) = \widetilde{s}_3(p) = 0\right\} \tag{12.15}$$

and require that this is a smooth submanifold of codimension 3 in $\mathbb{CP}^4$. We now ask for the cohomology of the holomorphic line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^4}(1)$ pulled back onto C.

12.4.3. The Koszul Sequence And Koszul Spectral Sequence

In this particular situation the Koszul sequence becomes

$$0 \to \mathcal{L}' \to \mathcal{V}_2 \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{12.16}$$

where

- $\mathcal{L}' = \mathcal{O}_{\mathbb{CP}^4}(-2)$
- $\mathcal{V}_2 = \mathcal{O}_{\mathbb{CP}^4}(-1) \oplus \mathcal{O}_{\mathbb{CP}^4}(-1) \oplus \mathcal{O}_{\mathbb{CP}^4}(-1)$

$\mathcal{L}'$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	P_1	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$	$0 \oplus 0 \oplus 0$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_2	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 12.1.: Ambient space cohomologies in the computation of an example application of the Koszul spectral sequence in $\mathbb{CP}^4$.

- $\mathcal{V}_1 = \mathcal{O}_{\mathbb{CP}^4}(0) \oplus \mathcal{O}_{\mathbb{CP}^4}(0) \mathcal{O}_{\mathbb{CP}^4}(0)$
- $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^4}(1)$

By means of our *Mathematica* notebook, which is given in Appendix E, the cohomologies of the above line bundles are easily calculated. It turns out that only the following cohomology groups are non-trivial.

- $H^0\left(\mathbb{CP}^4, \mathcal{O}_{\mathbb{CP}^4}(0)\right) = \{a\ ,\ a \in \mathbb{C}\} \cong \mathbb{C}$

$\Rightarrow\ H^0\left(\mathbb{CP}^4, \mathcal{V}_1\right) \cong P_1 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\ ,\ a_i \in \mathbb{C} \right\} \cong \mathbb{C}^3$

- $P_2 := H^0\left(\mathbb{CP}^4, \mathcal{L}\right) = \left\{ \sum\limits_{i=1}^{5} a_i x_i\ ,\ a_i \in \mathbb{C} \right\} \cong \mathbb{C}^5$

From this the first sheet of the Koszul spectral sequence is easily obtained. We present this sheet in Table 12.1. This immediately shows that the spectral sequence degenerates at sheet E_2, so that all we have to do is to compute the map

$$\gamma^0 \colon P_1 \to P_4 \tag{12.17}$$

with the result

$$H^i\left(C, \mathcal{L}|_C\right) = \left(\operatorname{coker}\left(\gamma^0\right), 0, 0, 0, 0\right) \tag{12.18}$$

The map γ^0 is easily found to be represented by the matrix

$$M_{\gamma^0} = \begin{pmatrix} c_1 & c_6 & c_{11} \\ c_2 & c_7 & c_{12} \\ c_3 & c_8 & c_{13} \\ c_4 & c_9 & c_{14} \\ c_5 & c_{10} & c_{15} \end{pmatrix} \tag{12.19}$$

12.5. An Exhaustive Example - Revisited

Remark:
In chapter 9 and chapter 11 we considered a very special set of examples. We now want to complete these examples by rephrasing them in terms of spectral sequences. This then shows that these examples are of the simpler kind as they do not require knowledge about the 'mysterious' differential maps d_r with $r \geq 2$. It were those maps, that we termed 'mysterious' and decided to remain silent on back in chapter 11.

Note:
The sheet E_1 for the three computations is given in Table 11.2, Table 11.4 and Table 11.5 respectively. From the spectral sequence perspective one then concludes that for these computations

- indeed only knowledge about the d_1 differential maps is required.
- the results from the computations in chapter 11 are reproduced by the use of the Koszul spectral sequence.

13. The Knight's Move

13.1. Summary

In section 12.3 we did not give the full details on the higher maps $d_r^{p,q}$ with $r \geq 2$ that appear in the Koszul spectral sequence. To remedy this we want to use this chapter to discuss the Knight's move $d_2^{p,q}$ [1] in detail. We give the general construction of the Knight's move in section 13.2 and hope that this exposition is also accessible for the non-technical reader.
Whilst this general construction works flawless from the theoretical point of view, it is computationally very elaborate. This is because the general construction requires knowledge about the the Čech cochains displayed in the sheet E_0 of the Koszul spectral sequence. Unfortunately this information is not accessible from the *cohomCalg* algorithm but only via the chamber counting approach. The chamber counting in turn is computationally very elaborate. In particular it is much slower than *cohomCalg*.
To illustrate this point we give an example in section 13.3 which makes use of the chamber counting approach to compute the E_0-sheet. Subsequently we apply the general strategy from section 13.2 in order to construct the Knight's move.
After going through this example it should be clear to the reader that a computerisation of the general strategy from section 13.2 is very time-consuming both while programming the algorithm as well as while running the program. For this reason one can ask the following question.

> Suppose we use *cohomCalg* to compute the E_1-sheet directly - is there a way to construct the maps d_r with $r \geq 2$ without going back to the E_0-sheet in the Koszul spectral sequence?

We give a hint towards the answer of this question in the example in section 13.3. In this example we have to construct a Knight's move

$$\alpha^0_{(2)} \colon P_1 \to P_2 \tag{13.1}$$

where

- $P_1 = \left\{ A_1 \cdot \frac{x_4}{x_5 x_6} + A_2 \cdot \frac{x_3}{x_5 x_6} \ , \ A_i \in \mathbb{C} \right\} \cong \mathbb{C}^2$
- $P_2 = \{ A_3 x_2 x_4 + A_4 x_2 x_3 + A_5 x_1 x_4 + A_6 x_1 x_3 \ , \ A_i \in \mathbb{C} \} \cong \mathbb{C}^4$

[1] In chess the Knight performs precisely the move of the maps $d_2^{p,q}$. For that reason the d_2-maps are known as the 'Knight's move'.

Let us set

$$a, b \in \{1, 2\}\,, \quad i, j \in \{3, 4\}\,, \quad A, B \in \{5, 6\} \tag{13.2}$$

Then we can describe the above spaces roughly as

$$P_1 = S_i \epsilon_{AB} \cdot \frac{x_i}{x_A x_B}\,, \qquad P_2 = T_i U_a \cdot x_i x_a \tag{13.3}$$

Consequently both P_1 and P_2 have tensor properties. Consequently the map $\alpha^0_{(2)}$ should respect those (anti)-symmetrisation properties. Inducing the map $\alpha^0_{(2)}$ from the global sections $\widetilde{s}_i$ in the Koszul sequence and in addition asking for respect of the tensor structures of the spaces P_1 and P_2 then leads to proposing that polynomially

$$\alpha^0_{(2)} = x_1 x_5 x_6 \left[C_4 C_5 - C_3 C_6\right] + x_2 x_5 x_6 \left[C_2 C_5 - C_1 C_6\right] \tag{13.4}$$

We will prove in section 13.4 that in this particular example the proposal is indeed correct. Motivated from this example we will have much more to say about this simplified construction in chapter 14.

13.2. General Strategy

Note (The sheets)**:**
We will need to look at the sheets E_0, E_1 and E_2 simultaneously to understand the construction of the maps d_2. To this end we picture these sheets in Figure 13.1, Figure 13.2 and Figure 13.3.

Construction 13.2.1:
We now wish to map $x \in \ker\left(\widetilde{\beta}^2\right)/\mathrm{im}\left(\widetilde{\alpha}^2\right)$ to an element $z \in \ker\left(\widetilde{\epsilon}^1\right)/\mathrm{im}\left(\widetilde{\gamma}^1\right)$.

1. Note that $x \in \ker\left(\widetilde{\beta}^2\right)/\mathrm{im}\left(\widetilde{\alpha}^2\right)$ can be represented by $x' \in H^2\left(X_\Sigma, \mathcal{V}_3\right)$ which has the property

$$\widetilde{\beta}^2\left(x'\right) = 0 \tag{13.5}$$

But note that $H^2\left(X_\Sigma, \mathcal{V}_3\right) = \check{H}^2\left(\mathcal{U}, \mathcal{V}_3\right)$ where $\mathcal{U}$ is the affine open cover of X_Σ. Consequently $H^2\left(X_\Sigma, \mathcal{V}_3\right)$ is a quotient space also. For that reason a general element in $H^2\left(X_\Sigma, \mathcal{V}_3\right)$ can be expressed by an element in $\check{C}^2\left(\mathcal{U}, \mathcal{V}_3\right)$ which lies in the kernel of the subsequent Čech differential. Consequently it is not hard to see that the special element $x' \in H^2\left(X_\Sigma, \mathcal{V}_3\right)$ can be represented by $x'' \in \check{C}^2\left(\mathcal{U}, \mathcal{V}_3\right)$ which has the following two properties

   - $\delta\left(x''\right) = 0$
   - $\exists y'' \in \check{C}^1\left(\mathcal{U}, \mathcal{V}_2\right) : \delta\left(y''\right) = \beta^2\left(x''\right)$

   Let us mention that $y'' \in \check{C}^1\left(\mathcal{U}, \mathcal{V}_2\right)$ is unique only up to addition of $\delta\left(w\right)$ for $w \in \check{C}^0\left(\mathcal{U}, \mathcal{V}_2\right)$.

2. Now we set

$$z'' := \gamma^1\left(y''\right) \in \check{C}^1\left(\mathcal{U}, \mathcal{V}_1\right) \tag{13.6}$$

   Note that z'' is unqiue only up to addition of $\gamma^1 \circ \delta\left(w\right)$ for $w \in \check{C}^0\left(\mathcal{U}, \mathcal{V}_2\right)$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^0"] & \check{C}^1(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^1"] & \check{C}^2(\mathcal{U},\mathcal{L}') \arrow[r, "\delta"] \arrow[d, "\alpha^2"] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{V}_3) \arrow[r, "\delta"] \arrow[d, "\beta^0"] & \check{C}^1(\mathcal{U},\mathcal{V}_3) \arrow[r, "\delta"] \arrow[d, "\beta^1"] & x'' \in \check{C}^2(\mathcal{U},\mathcal{V}_3) \arrow[r, "\delta"] \arrow[d, "\beta^2"] & \dots \\
0 \arrow[r, "\delta"] & w \in \check{C}^0(\mathcal{U},\mathcal{V}_2) \arrow[r, "\delta"] \arrow[d, "\gamma^0"] & y'' \in \check{C}^1(\mathcal{U},\mathcal{V}_2) \arrow[r, "\delta"] \arrow[d, "\gamma^1"] & \check{C}^2(\mathcal{U},\mathcal{V}_2) \arrow[r, "\delta"] \arrow[d, "\gamma^2"] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{V}_1) \arrow[r, "\delta"] \arrow[d, "\epsilon^0"] & z'' \in \check{C}^1(\mathcal{U},\mathcal{V}_1) \arrow[r, "\delta"] \arrow[d, "\epsilon^1"] & \check{C}^2(\mathcal{U},\mathcal{V}_1) \arrow[r, "\delta"] \arrow[d, "\epsilon^2"] & \dots \\
0 \arrow[r, "\delta"] & \check{C}^0(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{L}) \arrow[r, "\delta"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 13.1.: The E_0-sheet of the Koszul spectral sequence for a codimension 4 locus. Note that this diagramm is commutative.

3. We now note two important properties of z''. In order to do so, we make use of the commutativity of the sheet E_0.
   - $\delta(z'') = \delta \circ \gamma^1(y'') = \gamma^2 \circ \delta(y'') = \gamma^2 \circ \beta^2(x'') = 0$.
   - $\epsilon^1(z'') = \epsilon^1 \circ \gamma^1(y'') = 0$. But since $0 \in \check{C}^0(\mathcal{U},\mathcal{L})$ we thus have

$$\delta(0) = \epsilon^1(z'') \tag{13.7}$$

   These two properties now show, that z'' gives rise to an element $z \in \ker(\widetilde{\gamma}^1)/\mathrm{im}(\widetilde{\beta}^1)$. It is not too hard to verify that z is well-defined, i.e. independent of changing z'' by elements $\gamma^1 \circ \delta(w)$.

We conclude that we have just constructed a map

$$d_2 \colon \ker(\widetilde{\beta}^2)/\mathrm{im}(\widetilde{\alpha}^2) \to \ker(\widetilde{\epsilon}^1)/\mathrm{im}(\widetilde{\gamma}^1) \tag{13.8}$$

This is the Knight's move.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
H^0(X_\Sigma, \mathcal{L}') \arrow[d, "\widetilde{\alpha}^0"] & H^1(X_\Sigma, \mathcal{L}') \arrow[d, "\widetilde{\alpha}^1"] & H^2(X_\Sigma, \mathcal{L}') \arrow[d, "\widetilde{\alpha}^2"] & \ldots \\
H^0(X_\Sigma, \mathcal{V}_3) \arrow[d, "\widetilde{\beta}^0"] & H^1(X_\Sigma, \mathcal{V}_3) \arrow[d, "\widetilde{\beta}^1"] & x' \in H^2(X_\Sigma, \mathcal{V}_3) \arrow[d, "\widetilde{\beta}^2"] & \ldots \\
H^0(X_\Sigma, \mathcal{V}_2) \arrow[d, "\widetilde{\gamma}^0"] & H^1(X_\Sigma, \mathcal{V}_2) \arrow[d, "\widetilde{\gamma}^1"] & H^2(X_\Sigma, \mathcal{V}_2) \arrow[d, "\widetilde{\gamma}^2"] & \ldots \\
H^0(X_\Sigma, \mathcal{V}_1) \arrow[d, "\widetilde{\epsilon}^0"] & H^1(X_\Sigma, \mathcal{V}_1) \arrow[d, "\widetilde{\epsilon}^1"] & H^2(X_\Sigma, \mathcal{V}_1) \arrow[d, "\widetilde{\epsilon}^2"] & \ldots \\
H^0(X_\Sigma, \mathcal{L}) \arrow[d] & H^1(X_\Sigma, \mathcal{L}) \arrow[d] & H^2(X_\Sigma, \mathcal{L}) \arrow[d] & \ldots \\
0 & 0 & 0 &
\end{tikzcd}$$

Figure 13.2.: The E_1-sheet of the Koszul spectral sequence for a codimension 4 locus.

13.3. An Example With A Simple Knight's Move And A Proposal For A Simplified Construction

13.3.1. An Example With $\mathbf{d_1}$-Maps Only

Ambient Space

Let us consider $\widetilde{X}_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1$ as ambient space. Its toric data is given in Table 13.1.

vertices	homogeneous coordinates	Q_1	Q_2	divisor class
$(1,0)$	x_1	1	0	H
$(-1,0)$	x_2	1	0	H
$(0,1)$	x_3	0	1	H'
$(0,-1)$	x_4	0	1	H'

Table 13.1.: Toric data of $\widetilde{X}_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1$. Note that the Stanley-Reisner ideal of this space is $I_{\widetilde{X}_\Sigma} = \langle x_1 x_2, x_3 x_4 \rangle$.

$$
\begin{tikzcd}
0 & 0 & 0 & \\
\ker\left(\widetilde{\alpha}^0\right) & \ker\left(\widetilde{\alpha}^1\right) & \ker\left(\widetilde{\alpha}^2\right) & \dots \\
\ker\left(\widetilde{\beta}^0\right)/\mathrm{im}\left(\widetilde{\alpha}^0\right) & \ker\left(\widetilde{\beta}^1\right)/\mathrm{im}\left(\widetilde{\alpha}^1\right) & x \in \ker\left(\widetilde{\beta}^2\right)/\mathrm{im}\left(\widetilde{\alpha}^2\right) \arrow[ddl, dashed] & \dots \\
\ker\left(\widetilde{\gamma}^0\right)/\mathrm{im}\left(\widetilde{\beta}^0\right) & \ker\left(\widetilde{\gamma}^1\right)/\mathrm{im}\left(\widetilde{\beta}^1\right) & \ker\left(\widetilde{\gamma}^2\right)/\mathrm{im}\left(\widetilde{\beta}^2\right) & \dots \\
\ker\left(\widetilde{\epsilon}^0\right)/\mathrm{im}\left(\widetilde{\beta}^0\right) & z \in \ker\left(\widetilde{\epsilon}^1\right)/\mathrm{im}\left(\widetilde{\beta}^1\right) & \ker\left(\widetilde{\epsilon}^2\right)/\mathrm{im}\left(\widetilde{\beta}^2\right) & \dots \\
H^0\left(X_\Sigma, \mathcal{L}\right)/\mathrm{im}\left(\widetilde{\epsilon}^0\right) & H^1\left(X_\Sigma, \mathcal{L}\right)/\mathrm{im}\left(\widetilde{\epsilon}^1\right) & H^2\left(X_\Sigma, \mathcal{L}\right)/\mathrm{im}\left(\widetilde{\epsilon}^2\right) & \dots \\
0 & 0 & 0 &
\end{tikzcd}
$$

Figure 13.3.: The E_2-sheet of the Koszul spectral sequence for a codimension 4 locus. The Knight's move is given by a dashed line.

Pullback Setup

Now let us consider the effective divisor class $S = (1,0) \in \mathrm{Cl}\left(\mathbb{CP}^1 \times \mathbb{CP}^1\right)$. The most general element $\widetilde{s} \in H^0\left(\widetilde{X}_\Sigma, \mathcal{O}_{\widetilde{X}_\Sigma}(S)\right)$ is given by

$$\widetilde{s} = \widetilde{C}_2 x_1 + \widetilde{C}_1 x_2 \tag{13.9}$$

where $\widetilde{C}_1, \widetilde{C}_2 \in \mathbb{C}$. Then we consider the algebraic variety

$$\widetilde{C} := \left\{ p \in \widetilde{X}_\Sigma \, , \; \widetilde{s}(p) = 0 \right\} \tag{13.10}$$

Finally we require that $\widetilde{C}_1$, $\widetilde{C}_2$ are chosen such that $\widetilde{C}$ is an algebraic submanifold of codimension 1 in $\widetilde{X}_\Sigma$.
Given this setup we want to compute the cohomologies of $\mathcal{L}|_{\widetilde{C}}$ for $\mathcal{L} = \mathcal{O}_{X_\Sigma}(1,1)$.

The E_1-Sheet

In this particular situation the Koszul sequence becomes

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}|_{\widetilde{C}} \to 0 \tag{13.11}$$

where $\mathcal{L}' = \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(0,1)$. By use of the *cohomCalg* algorithm one easily computes the ambient space cohomologies. We list the non-trivial ambient space cohomologies

Space	Basis	Dimension
$P_1 = H^0\left(\widetilde{X}_\Sigma, \mathcal{O}_{\widetilde{X}_\Sigma}(0,1)\right)$	$A_1 \cdot x_4 + A_2 \cdot x_3$	2
$P_2 = H^0\left(\widetilde{X}_\Sigma, \mathcal{O}_{\widetilde{X}_\Sigma}(1,1)\right)$	$A_3 \cdot x_2 x_4 + A_4 \cdot x_2 x_3 + A_5 x_1 x_4 + A_6 x_1 x_3$	4

Table 13.2.: Non-trivial ambient space cohomologies in the simple Knight's move example on $\widetilde{X}_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1$.

$\mathcal{L}'$	P_1	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_2	0	0
	H^0	H^1	H^2

Table 13.3.: The E_1-sheet of the Koszul spectral sequence in the simple Knight's move example on $\widetilde{X}_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1$.

in Table 13.2.
Thereby we can neatly organise the ambient space cohomologies in the E_1-sheet of the Koszul spectral sequence which is displayed in Table 13.3. Note that this spectral sequence does converge on the E_2-sheet. By means of the by now familiar construction of the d_1-maps we can represent the map α^0 by the following matrix

$$M_{\alpha^0} = \begin{pmatrix} \widetilde{C}_1 & 0 \\ 0 & \widetilde{C}_1 \\ \widetilde{C}_2 & 0 \\ 0 & \widetilde{C}_2 \end{pmatrix} \tag{13.12}$$

Consequently $H^0\left(\widetilde{C}, \mathcal{L}|_{\widetilde{C}}\right) \cong \operatorname{coker}\left(M_{\alpha^0}\right)$ whilst all higher cohomology classes vanish.

13.3.2. An Example With A Single $\mathbf{d_2}$-Map And A Proposal For This Map

Ambient Space

Now we consider the different ambient space $X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Its toric data is given in Table 13.4.

Pullback Setup

Now let us consider the effective divisor classes

$$S_1 = (1,0,1)\,, \quad S_2 = (0,0,1) \tag{13.13}$$

ray generators	homogeneous coordinates	Q_1	Q_2	Q_3	divisor class
$(1,0,0)$	x_1	1	0	0	H
$(-1,0,0)$	x_2	1	0	0	H
$(0,1,0)$	x_3	0	1	0	H'
$(0,-1,0)$	x_4	0	1	0	H'
$(0,0,1)$	x_5	0	0	1	H''
$(0,0,-1)$	x_6	0	0	1	H''

Table 13.4.: Toric data of $X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Note that the Stanley-Reisner ideal of this space is $I_{X_\Sigma} = \langle x_1 x_2, x_3 x_4, x_5 x_6 \rangle$.

The most general global sections of the associated holomorphic line bundles are given by

- $\widetilde{s}_1 = C_4 x_1 x_5 + C_2 x_2 x_5 + C_3 x_1 x_6 + C_1 x_2 x_6$
- $\widetilde{s}_2 = C_6 x_5 + C_5 x_6$

where the parameters $C_i \in \mathbb{C}$ are subject to the conditions that

$$C := \{p \in X_\Sigma\ ,\ \widetilde{s}_1(p) = \widetilde{s}_2(p) = 0\} \tag{13.14}$$

is an algebraic submanifold of codimension 2 in X_Σ.
Given this setup we wish to compute the cohomologies of $\mathcal{L}|_C$ for $\mathcal{L} = \mathcal{O}_{X_\Sigma}(1,1,0)$.

Equivalent Setups

It should be apparent that this situation is very similar to that given in subsection 13.3.1. In particular it is simple to verify that we have a canonical biholomorphism $\widetilde{C} \cong C$ precisely if

$$\widetilde{C}_1 = \epsilon\left(C_2 C_5 - C_1 C_6\right), \quad \widetilde{C}_2 = \epsilon\left(C_4 C_5 - C_3 C_6\right) \tag{13.15}$$

where $\epsilon = \pm 1$. We will use this relation momentarily.

The E_1-Sheet

For this example the Koszul sequence is given by

$$0 \to \mathcal{L}' \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{13.16}$$

where

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(0,1,-2)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(0,1,-1) \oplus \mathcal{O}_{X_\Sigma}(1,1,-1)$

Space	Basis	Dimension
$P_1 = H^1(X_\Sigma, \mathcal{L}')$	$A_1 \cdot \frac{x_4}{x_5 x_6} + A_2 \cdot \frac{x_3}{x_5 x_6}$	2
$P_2 = H^0(X_\Sigma, \mathcal{L})$	$A_3 \cdot x_2 x_4 + A_4 \cdot x_2 x_3 + A_5 x_1 x_4 + A_6 x_1 x_3$	4

Table 13.5.: Non-trivial ambient space cohomologies in the simple Knight's move example on $X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

$\mathcal{L}'$	0	P_1	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}$	P_2	0	0	0
	H^0	H^1	H^2	H^3

Table 13.6.: Ambient space cohomologies in the simple example on $X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

We use the *cohomCalg* algorithm to compute the ambient space cohomologies and list the non-trivial ones in Table 13.5. Thereby we can neatly organise the ambient space cohomologies in the E_1-sheet of the Koszul spectral sequence. This sheet we display in Table 13.6. Note that this spectral sequence does converge on the E_3-sheet and that given the d_2-map

$$\alpha^0_{(2)} : P_1 \to P_2 \tag{13.17}$$

we have $H^0(C, \mathcal{L}|_C) \cong \operatorname{coker}\left(\alpha^0_{(2)}\right)$ whilst all higher cohomology classes do vanish.

Proposal For The $\mathbf{d_2}$-Map

We propose that the d_2-map $\alpha^0_{(2)}$ is polynomially represented as

$$\alpha^0_{(2)} = x_1 x_5 x_6 \left[C_4 C_5 - C_3 C_6\right] + x_2 x_5 x_6 \left[C_2 C_5 - C_1 C_6\right] \tag{13.18}$$

Equivalently it can be given by the following matrix

$$M_{\alpha^0_{(2)}} = \begin{pmatrix} C_2 C_5 - C_1 C_6 & 0 \\ 0 & C_2 C_5 - C_1 C_6 \\ C_4 C_5 - C_3 C_6 & 0 \\ 0 & C_4 C_5 - C_3 C_6 \end{pmatrix} \tag{13.19}$$

Consistency Check

Recall that $C \cong \widetilde{C}$ precisely if

$$\widetilde{C}_1 = \epsilon \left(C_2 C_5 - C_1 C_6\right), \quad \widetilde{C}_2 = \epsilon \left(C_4 C_5 - C_3 C_6\right) \tag{13.20}$$

with $\epsilon = \pm 1$. Then however we have

$$M_{\alpha^0_{(2)}} = \epsilon \begin{pmatrix} \widetilde{C}_1 & 0 \\ 0 & \widetilde{C}_1 \\ \widetilde{C}_2 & 0 \\ 0 & \widetilde{C}_2 \end{pmatrix} \tag{13.21}$$

which is up to a sign just the matrix M_{α^0} from subsection 13.3.1. Hence the cohomology classes computed from the two setups are precisely the same. Therefore the above proposal passes this consistency check.
Whilst this simple consistency check is passed, we want to prove that this is indeed the correct map to consider. To this end we will apply the general strategy of constructing the map $\alpha^0_{(2)}$ to this particular situation. The detailed analysis presented in section 13.4 then finally proves the given proposal.

13.4. Proof Of The Proposal For A Simplified Construction Of The Knight's Move In The Preceeding Example

Remark:
Recall that the general construction of the Knight's move makes use of the Čech cochains. To prove the propsal for the d_2 in section 13.3 we will therefore need to compute the Čech cochains and subsequently follow the general construction outlined in section 13.2. This is what we need the chamber counting for - it allows for the explicit computation of the Čech cochains and therefore gives more information than the faster *cohomCalg* algorithm alone.

13.4.1. Strategy Of The Proof

Our strategy in this proof is as follows.

1. Construct the maximal cones in the fan Σ of X_Σ in order to obtain the affine open cover $\mathcal{U}$.
2. Compute $\check{C}^i(\mathcal{U}, \mathcal{L}')$ for $0 \leq i \leq 1$ where $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(0, 1, -2)$.
3. Compute $\check{C}^0(\mathcal{U}, \mathcal{V})$ for $\mathcal{V} = \mathcal{O}_{X_\Sigma}(0, 1, -1) \oplus \mathcal{O}_{X_\Sigma}(1, 1, -1)$.
4. Represent the general element in $H^1(X_\Sigma, \mathcal{L}')$ by a special element $y \in \check{C}^1(X_\Sigma, \mathcal{L}')$.
5. Find the corresponding element $x \in \check{C}^0(\mathcal{U}, \mathcal{V})$ subject to the condition

$$\delta(x) = \alpha^1(y) \tag{13.22}$$

   where δ is the Čech coboundary and $\alpha^1 : \check{C}^1(\mathcal{U}, \mathcal{L}') \to \check{C}^1(\mathcal{U}, \mathcal{V})$ the d_1-map given by $(\widetilde{s}_1, -\widetilde{s}_2)^T$ with

- $\widetilde{s}_1 = C_4 x_1 x_5 + C_2 x_2 x_5 + C_3 x_1 x_6 + C_1 x_2 x_6$
- $\widetilde{s}_2 = C_6 x_5 + C_5 x_6$

6. Subsequently map the element x via the d_1-map β^0 to $z \in \check{C}^0(\mathcal{U}, \mathcal{L})$. As argued in section 13.2 z will then give rise to an element in $H^0(X_\Sigma, \mathcal{L})$. For this reason it will give us the d_2-map $\alpha^0_{(2)}$ that we are looking for.

13.4.2. The Affine Open Cover

Recall that the toric data of X_Σ is given in Table 13.4. In particular we have

$$I_{\mathrm{SR}} = \langle x_1 x_2, x_3 x_4, x_5 x_6 \rangle \subset \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6] \tag{13.23}$$

and that the ray generators in $\mathbb{Z}^3$ are given by

$$n_1 = -n_2 = e_1, \quad n_3 = -n_4 = e_2, \quad n_5 = -n_6 = e_3 \tag{13.24}$$

From this we conclude that the maximal cones in Σ are the eight octans in $\mathbb{R}^3$ given by

- $\sigma_1 = \langle n_1, n_3, n_5 \rangle$
- $\sigma_2 = \langle n_1, n_3, n_6 \rangle$
- $\sigma_3 = \langle n_1, n_4, n_5 \rangle$
- $\sigma_4 = \langle n_1, n_4, n_6 \rangle$
- $\sigma_5 = \langle n_2, n_3, n_5 \rangle$
- $\sigma_6 = \langle n_2, n_3, n_6 \rangle$
- $\sigma_7 = \langle n_2, n_4, n_5 \rangle$
- $\sigma_8 = \langle n_2, n_4, n_6 \rangle$

Via the cone-orbit-correspondance the open cover $\mathcal{U}$ is now given by

$$\mathcal{U} = \{U_{\sigma_1}, \ldots, U_{\sigma_8}\} \tag{13.25}$$

13.4.3. Čech Cochains For $\mathcal{L}'$

We consider $D = \sum\limits_{\rho \in \Sigma(1)} a_\rho D_\rho$ with

$$a_1 = a_2 = a_4 = a_6 = 0, \quad a_3 = 1, \quad a_5 = -2 \tag{13.26}$$

Next we introduce the following planes in order to determine the compact chambers in $\mathbb{R}^3$ that we need to take into account in the computation of the Čech cochains.

- $E_1 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_1 \rangle = -a_1\} = \left\{ \begin{pmatrix} 0 \\ m_2 \\ m_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$

- $E_2 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_2 \rangle = -a_2\} = \left\{ \begin{pmatrix} 0 \\ m_2 \\ m_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$
- $E_3 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_3 \rangle = -a_3\} = \left\{ \begin{pmatrix} m_1 \\ -1 \\ m_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$
- $E_4 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_4 \rangle = -a_4\} = \left\{ \begin{pmatrix} m_1 \\ 0 \\ m_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$
- $E_5 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_5 \rangle = -a_5\} = \left\{ \begin{pmatrix} m_1 \\ m_2 \\ 2 \end{pmatrix} \in \mathbb{R}^3 \right\}$
- $E_6 = \{m \in \mathbb{R}^3\ ,\ \langle m, n_6 \rangle = -a_6\} = \left\{ \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \right\}$

These planes separate $\mathbb{R}^3$ such that there is only a single compact chamber, namely

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 \\ m_2 \\ m_3 \end{pmatrix} \in \mathbb{R}^3\ ,\ -1 \le m_2 \le 0,\ 0 \le m_3 \le 2 \right\} \tag{13.27}$$

Since X_Σ is compact it suffices to consider only Laurent monomials that stem from $\mathcal{C}$.

A simple but tedious computation now determines the Čech cochains. Therefore we only state the results and leave it to the interested reader to confirm these results. Let us first introduce the following three spaces

- $S_0 = \left\{ \alpha_1 \frac{x_3}{x_5^2} + \alpha_2 \frac{x_4}{x_5^2}\ ,\ \alpha_i \in \mathbb{C} \right\}$
- $S_1 = \left\{ \alpha_1 \frac{x_3}{x_5 x_6} + \alpha_2 \frac{x_4}{x_5 x_6}\ ,\ \alpha_i \in \mathbb{C} \right\}$
- $S_2 = \left\{ \alpha_1 \frac{x_3}{x_6^2} + \alpha_2 \frac{x_4}{x_6^2}\ ,\ \beta_i \in \mathbb{C} \right\}$

and note that $\mathcal{C} = S_0 \oplus S_1 \oplus S_2$ in the meaning that the union of S_0, S_1 and S_2 gives all Laurent monomials stemming from integral points in $\mathcal{C}$. In an abuse of notion, we can then write

$$\begin{aligned} \check{C}^0(\mathcal{U}, \mathcal{L}') &= (S_2, S_0, S_2, S_0, S_2, S_0, S_2, S_0) \\ \check{C}^1(\mathcal{U}, \mathcal{L}') &= (C, C, C, S_2, C, S_2, C, C, C, C, S_0, C, S_0, \\ &\qquad C, S_2, C, S_2, C, C, S_0, C, S_0, C, S_2, C, C, S_0, C) \end{aligned} \tag{13.28}$$

13.4.4. Čech Cohomology For $\mathcal{L}'$

The mappings in the Čech complex

$$0 \to \check{C}^0(\mathcal{U}, \mathcal{L}') \overset{\delta_0}{\to} \check{C}^1(\mathcal{U}, \mathcal{L}') \overset{\delta_1}{\to} \dots \tag{13.29}$$

are given by huge matrices which we display in Figure 13.4 and Figure 13.5. From these mapping matrices M_{δ_0} and M_{δ_1} it is not too hard to verify that $\check{H}^0(\mathcal{U}, \mathcal{L}')$ is trivial, whilst $\check{H}^1(\mathcal{U}, \mathcal{L})$ is spanned over $\mathbb{C}$ by the vectors

$$v_1 = \begin{pmatrix} \alpha \\ 0 \\ \alpha \\ 0 \\ \alpha \\ 0 \\ \alpha \\ -\alpha \\ 0 \\ -\alpha \\ 0 \\ -\alpha \\ 0 \\ \alpha \\ 0 \\ \alpha \\ 0 \\ \alpha \\ -\alpha \\ 0 \\ -\alpha \\ 0 \\ \alpha \\ 0 \\ \alpha \\ -\alpha \\ 0 \\ \alpha \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \beta \\ 0 \\ \beta \\ 0 \\ \beta \\ 0 \\ \beta \\ -\beta \\ 0 \\ -\beta \\ 0 \\ -\beta \\ 0 \\ \beta \\ 0 \\ \beta \\ 0 \\ \beta \\ -\beta \\ 0 \\ -\beta \\ 0 \\ \beta \\ 0 \\ \beta \\ -\beta \\ 0 \\ \beta \end{pmatrix} \tag{13.30}$$

where $\alpha = \frac{x_3}{x_5 x_6}$ and $\beta = \frac{x_4}{x_5 x_6}$.

13.4.5. Čech Cochains For $\mathbf{D = 1 \cdot [D_3] + (-1) \cdot [D_5]}$

For this divisor only a single compact chamber does exist. This chamber is given by

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 \\ m_2 \\ m_3 \end{pmatrix}, \ -1 \leq m_2 \leq 0, \ 0 \leq m_3 \leq 1 \right\} \tag{13.31}$$

Let us define

- $\widetilde{S}_1 = \left\{ \alpha_1 \frac{x_4}{x_6} + \alpha_2 \frac{x_3}{x_6} \ , \ \alpha_i \in \mathbb{C} \right\}$

- $\widetilde{\widetilde{S}}_2 = \left\{ \alpha_1 \frac{x_4}{x_5} + \alpha_2 \frac{x_3}{x_5} \ , \ \alpha_i \in \mathbb{C} \right\}$

Then a simple but tedious computation reveals

$$\check{C}^0 \left(\mathcal{U}, \mathcal{O}_{X_\Sigma} (0,1,-1) \right) \cong \widetilde{\widetilde{S}}_1 \oplus \widetilde{\widetilde{S}}_2 \oplus \widetilde{\widetilde{S}}_1 \oplus \widetilde{\widetilde{S}}_2 \oplus \widetilde{\widetilde{S}}_1 \oplus \widetilde{\widetilde{S}}_2 \oplus \widetilde{\widetilde{S}}_1 \oplus \widetilde{\widetilde{S}}_2 \tag{13.32}$$

In particular one concludes from this result that $\check{H}^0 \left(\mathcal{U}, \mathcal{O}_{X_\Sigma} (0,1,-1) \right) = 0$.

13.4.6. Čech Cochains For $\mathbf{D} = \mathbf{1} \cdot [\mathbf{D_1}] + \mathbf{1} \cdot [\mathbf{D_3}] + (-\mathbf{1}) \cdot [\mathbf{D_5}]$

For this divisor there is again only a single compact chamber. It is given by

$$\mathcal{C} = \left\{ \left(\begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \right) \ , \ -1 \leq m_i \leq 0 \right\} \tag{13.33}$$

We define

- $\widetilde{\widetilde{\widetilde{S}}}_1 = \left\{ \alpha_1 \frac{x_2 x_4}{x_6} + \alpha_2 \frac{x_2 x_3}{x_6} + \alpha_3 \frac{x_1 x_4}{x_6} + \alpha_4 \frac{x_1 x_3}{x_6} \ , \ \alpha_i \in \mathbb{C} \right\}$
- $\widetilde{\widetilde{\widetilde{S}}}_2 = \left\{ \alpha_1 \frac{x_2 x_4}{x_5} + \alpha_2 \frac{x_2 x_3}{x_5} + \alpha_3 \frac{x_1 x_4}{x_5} + \alpha_4 \frac{x_1 x_3}{x_5} \ , \ \alpha_i \in \mathbb{C} \right\}$

Then one confirmes by a tedious calculation

$$\check{C}^0 \left(\mathcal{U}, \mathcal{O}_{X_\Sigma} (0,1,-1) \right) \cong \widetilde{\widetilde{\widetilde{S}}}_1 \oplus \widetilde{\widetilde{\widetilde{S}}}_2 \oplus \widetilde{\widetilde{\widetilde{S}}}_1 \oplus \widetilde{\widetilde{\widetilde{S}}}_2 \oplus \widetilde{\widetilde{\widetilde{S}}}_1 \oplus \widetilde{\widetilde{\widetilde{S}}}_2 \oplus \widetilde{\widetilde{\widetilde{S}}}_1 \oplus \widetilde{\widetilde{\widetilde{S}}}_2 \tag{13.34}$$

13.4.7. Čech Cochains For $\mathcal{V}$

It holds $\check{C}^0 \left(\mathcal{U}, \mathcal{V} \right) = \check{C}^0 \left(\mathcal{U}, \mathcal{O}_{X_\Sigma} (1,1,-1) \right) \oplus \check{C}^0 \left(\mathcal{U}, \mathcal{O}_{X_\Sigma} (0,1,-1) \right)$. Thus the results from subsection 13.4.5 and subsection 13.4.6 determine the Čech cochains of $\mathcal{V}$.

13.4.8. The Knight's Move

Let us recall that our aim is to construct the map

$$\alpha^0_{(2)} \colon H^1 \left(X_\Sigma, \mathcal{L}' \right) \to H^0 \left(X_\Sigma, \mathcal{L} \right) \tag{13.35}$$

To this end we start with an element in $H^1 \left(X_\Sigma, \mathcal{L}' \right)$. We found in subsection 13.4.4 that this space is spanned over $\mathbb{C}$ by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. So let us first take care of $_1$ and take care of $_2$ later.
We can consider $_1$ as an element in $\check{C}^1 \left(\mathcal{U}, \mathcal{L}' \right)$. Our first task is therefore to find an element $y \in \check{C}^0 \left(\mathcal{U}, \mathcal{V} \right)$ such that

$$\delta \left(y \right) = \alpha^1 \left(\boldsymbol{v}_1 \right) \tag{13.36}$$

To this end let us define

- $\check{x}_1 = -\frac{C_6 x_3}{x_6}$

- $\check{x}_2 = +\frac{C_5 x_3}{x_5}$
- $\check{\check{x}}_1 = \frac{C_4 x_1 x_3}{x_6} + \frac{C_2 x_2 x_3}{x_6}$
- $\check{\check{x}}_2 = -\frac{C_3 x_1 x_3}{x_5} - \frac{C_1 x_2 x_3}{x_5}$

Then it holds

$$\begin{aligned} \widetilde{x}_1 &:= (\check{x}_1, \check{x}_2, \check{x}_1, \check{x}_2, \check{x}_1, \check{x}_2, \check{x}_1, \check{x}_2)^T \in \check{C}^0 (\mathcal{U}, \mathcal{O}_{X_\Sigma} (1,1,-1)) \\ \widetilde{x}_2 &:= \left(\check{\check{x}}_1, \check{\check{x}}_2, \check{\check{x}}_1, \check{\check{x}}_2, \check{\check{x}}_1, \check{\check{x}}_2, \check{\check{x}}_1, \check{\check{x}}_2\right)^T \in \check{C}^0 (\mathcal{U}, \mathcal{O}_{X_\Sigma} (0,1,-1)) \end{aligned} \tag{13.37}$$

Consequently $y := (\widetilde{x}_1, \widetilde{x}_2)^T \in \check{C}^0 (\mathcal{U}, \mathcal{V})$. And indeed one readily confirms

$$\delta\left(\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix}\right) = \begin{pmatrix} \widetilde{s}_1 \cdot v_1 \\ -\widetilde{s}_2 \cdot v_1 \end{pmatrix} \tag{13.38}$$

The next step is consequently to map $y \in \check{C}^0 (\mathcal{U}, \mathcal{V})$ by means of the map β^1 to an element $z \in \check{C}^0 (\mathcal{U}, \mathcal{L})$. Recall that β^1 is given by $(\widetilde{s}_1, \widetilde{s}_2)$. Thus we have to consider

$$z = \widetilde{s}_1 \widetilde{x}_1 + \widetilde{s}_2 \widetilde{x}_2 \tag{13.39}$$

But z is canonically identified with $\widetilde{s}_1 \check{x}_1 + \widetilde{s}_2 \check{\check{x}}_1$ and an easy computation shows

$$\widetilde{s}_1 \check{x}_1 + \widetilde{s}_2 \check{\check{x}}_1 = x_1 x_3 [C_4 C_5 - C_3 C_6] + x_2 x_3 [C_2 C_5 - C_1 C_6] \tag{13.40}$$

Indeed one can show that z gives rise to an element in $H^0 (X_\Sigma, \mathcal{L})$. This is left for the interested reader.

In conclusion we have done the following.

- We started with $\frac{x_3}{x_5 x_6} \in H^1 (X_\Sigma, \mathcal{L}')$. This rationom is canonically identified with $\boldsymbol{v}_1$.
- Then we mapped $\boldsymbol{v}_1$ to an element in $z \in H^0 (X_\Sigma, \mathcal{L})$ which can canonically be identified with

$$z = x_1 x_3 [C_4 C_5 - C_3 C_6] + x_2 x_3 [C_2 C_5 - C_1 C_6] \tag{13.41}$$

We can thus write

$$z = (x_1 x_5 x_6 [C_4 C_5 - C_3 C_6] + x_2 x_5 x_6 [C_2 C_5 - C_1 C_6]) \cdot \frac{x_3}{x_5 x_6} \tag{13.42}$$

Finally we have to repeat the above analysis for $\boldsymbol{v}_2$. We leave it to the interested reader to perform this computation in detail and thereby to confirm that $\frac{x_4}{x_5 x_6}$ gets mapped to

$$z' = (x_1 x_5 x_6 [C_4 C_5 - C_3 C_6] + x_2 x_5 x_6 [C_2 C_5 - C_1 C_6]) \cdot \frac{x_4}{x_5 x_6} \tag{13.43}$$

In conclusion we can polynomially represent the map $\alpha^0_{(2)}$ by

$$\alpha^0_{(2)} = x_1 x_5 x_6 [C_4 C_5 - C_3 C_6] + x_2 x_5 x_6 [C_2 C_5 - C_1 C_6] \tag{13.44}$$

This is precisely the proposal given in section 13.3.

$$M_{\delta_0} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Figure 13.4.: Computing the Čech complex for an open cover $\mathcal{U}$ with $|\mathcal{U}| = 8$, the Čech coboundary $\delta_0 \colon \check{C}^0(\mathcal{U}, \cdot) \to \check{C}^1(\mathcal{U}, \cdot)$ is given by the above 8 x 28 matrix.

$$M_{\delta_1} = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}$$

Figure 13.5.: Computing the Čech complex for an open cover $\mathcal{U}$ with $|\mathcal{U}| = 8$, the Čech coboundary $\delta_1 \colon \check{C}^1(\mathcal{U}, \cdot) \to \check{C}^2(\mathcal{U}, \cdot)$ is given by the above 28 x 56 matrix.

14. Simplified Construction Of Higher d-Maps

14.1. Summary

In chapter 13 we learned the general construction of the maps d_r with $r \geq 2$ in the Koszul spectral sequence. We also gave a proposal for a simplified Knight's move construction in section 13.3 and proved this proposal subsequently in section 13.4. Our goal in this chapter is to analyse the structure underlying this proposal further. In particular this analysis will lead to formulate a proposal for simplified construction of all maps d_r with $r \geq 2$.
We begin our analysis by introducing the notion of a generlised Flag variety in section 14.2. The most prominent example of a Flag variety is $\mathbb{CP}^n$. The property that makes $\mathbb{CP}^n$ a flag variety is that one can prove [68]

$$\mathbb{CP}^n \cong U(n+1) / (U(1) \times U(n)) \tag{14.1}$$

from which a transitive $U(1) \times U(n)$-action on $\mathbb{CP}^n$ is induced. As a consequence of this, the theorem of Bott-Borel-Weil implies that the cohomology groups of holomorphic line bundles on $\mathbb{CP}^n$ are classified by representations of the Lie group $U(1) \times U(n)$ [68].
By means of the Künneth formula this generalises to direct products of $\mathbb{CP}^n$. In particular line bundle cohomology on direct products of $\mathbb{CP}^n$ is again labeled by representations of certain Lie groups. This observation was heavily exploited in [58] and following works. In particular it is those tensor properties that where present in the cohomology groups in section 13.3 and lead to the proposal for the simplified construction of the Knight's move $\alpha^0_{(2)}$ in section 13.3.
Consequently we might ask if also for more general smooth and compact normal toric varieties the line bundle cohomology groups are labeled by representations of certain Lie groups and thus come equipped with a certain tensor structure. This would for example be the case if we knew that every smooth and compact normal toric variety indeed formed a generalised Flag varieties.
Proving or disproving his assertion is currently beyond the abilities of the author. Still, all line bundle cohomology groups presented in this thesis have rationom bases which naturally lead to such tensor structures in the cohomology groups. As we only looked at a finite number of cohomology groups in this thesis, this of course is no proof of the general principle but is enough to formulate the following two propositions.

- Every smooth and compact normal toric variety X_Σ is a generalised Flag variety.
- The higher d_r-maps are the (anti)-symmetrised partitions of the Koszul complex.

The construction of the (anti)-symmetrised partitions of the Koszul complex we describe in detail in section 14.3. Proving or disproving these statements is left for future work.

14.2. Generalised Flag Varieties And Toric Varieties

14.2.1. Homogeneous Spaces

Definition 14.2.1 (G-Space)**:**
Let G a topological group and X a nonempty topological space. Suppose that we have a continuous map $\varphi: G \to \mathrm{Aut}(X)$. Then (X, φ) is a G-space.

Example 14.2.1:
Let us consider the group $\mathbb{Z}_2$ equipped with the *trivial topology.* This gives $\mathbb{Z}_2$ the structure of a topological group. Now consider the map

$$\varphi: \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{R}) \tag{14.2}$$

given by

- $\varphi(0) = (\mathbb{R} \to \mathbb{R}\ ,\ x \mapsto 0)$
- $\varphi(1) = (\mathbb{R} \to \mathbb{R}\ ,\ x \mapsto 1)$

Then this map is continuous and thus gives us a $\mathbb{Z}_2$-space $(\mathbb{R}, \varphi)$.

Definition 14.2.2 (Homogeneous G-Space)**:**
A G-space (X, φ) is a homogeneous space precisely if G acts transitively, i.e. the following property is satisfied

$$\forall x, y \in X\ \exists g \in G: (\varphi(g))(x) = y \tag{14.3}$$

Example 14.2.2:
The above $\mathbb{Z}_2$-space $(\mathbb{R}, \varphi)$ is not a homogeneous $\mathbb{Z}_2$-space since for any $g \in \mathbb{Z}_2$ it holds

$$2 \notin (\varphi(g))(\mathbb{R}) \tag{14.4}$$

Note however that we can build a homogeneous $\mathbb{Z}_2$-space by defining $\varphi': \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{R})$ as follows

- $\varphi'(0) = (\mathbb{R} \to \mathbb{R}\ ,\ x \mapsto 0)$
- $\varphi'(1) = (\mathbb{R} \to \mathbb{R}\ ,\ x \mapsto x)$

14.2.2. Generalised Flag Varieties

Definition 14.2.3 (Generalised Flag Variety)**:**
Simply connected, compact, complex, homogeneous G-spaces are termed generalised Flag varieties or C-spaces.

Remark:
It can be shown [69] that every generalised Flag variety is homeomorphic to a torus fibration over a certain product of quotient spaces $G_{(C)}/H$ where $G_{(C)}$ is a compact and simple Lie group, whilst H is a regular semi-simple Lie group. More details can also be found in [68].

Note:
The Bott-Borel-Weil theorem implies that holomorphic line bundles over a Flag variety of the form $\mathbb{F} = G_{(C)}/H$ are labeled by representations of H [68] [70]. By means of the Künneth formula this generalises to holomorphic line bundles over products of Flag varieties.

14.2.3. Toric Varieties As Generalised Flag Varieties

Remark:
Let X_Σ a smooth and compact normal toric variety. Then it turns out [52] that X_Σ is already simply connected. Moreover it is a complex manifold. Thus if we knew that X_Σ was for some group action $G \to \mathrm{Aut}(X_\Sigma)$ a homogeneous G-space, then we could conclude that X_Σ was a generalised Flag variety. Unfortunately the action of the algebraic torus $(\mathbb{C}^*)^a$ does in general not act transitively, but rather there exist various torus orbits on general smooth and compact normal toric variety [52]. In particular on $\mathbb{CP}^n$ there exist multiple such torus orbits. This is easily seen from the cone-orbit-correspondance and the fact that one can associate to every ray generator a torus-invariant prime divisor. This observation shows that we have to consider a different group action. Let us stay with $\mathbb{CP}^n$ to illustrate this.

Construction 14.2.1:
Let us define the following $U(n+1)$ group action on $\mathbb{CP}^n$

$$\rho_{U(n+1)}: U(n+1) \to \mathrm{Aut}(\mathbb{CP}^n)\ ,\ A \mapsto (\mathbb{CP}^n \to \mathbb{CP}^n\ ,\ x = [x_0, \ldots, x_n] \mapsto [A \cdot (x_0, \ldots, x_n)]) \tag{14.5}$$

Similarly one can define a $U(1) \times U(n)$ group action on $\mathbb{CP}^n$ if we represent elements in $U(1) \times U(n)$ by block matrices of the form

$$A = \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & B \end{pmatrix} \in U(1) \times U(n) \tag{14.6}$$

where $B \in U(n)$. Given these group actions one can prove [68] that the isomorphism

$$\mathbb{CP}^n \cong U(n+1) / (U(1) \times U(n)) \tag{14.7}$$

induces a transitive group action of $U(1) \times U(n)$ on $\mathbb{CP}^n$. This establishes $\mathbb{CP}^n$ as Flag variety.

ray generators	homogeneous coordinates	Q_1	Q_2	divisor class
$(1,0)$	x_1	1	0	H
$(0,1)$	x_2	1	1	$H+E$
$(-1,-1)$	x_3	1	0	H
$(0,-1)$	x_4	0	1	E

Table 14.1.: Toric data of a del Pezzo 1-surface dP_1. The Stanley-Reisner ideal is $I_{\mathrm{SR}} = \langle x_1x_3, x_2x_4\rangle$.

Consequence:
Holomorphic line bundles on $\mathbb{CP}^n$ are hence labeled by representation of the regular semi-simple Lie group $U(1)\times U(n)$. This relationsship was heavily exploited in [58].

Example 14.2.3:
Let us consider $\mathbb{CP}^1$ and the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^1}(2)$ on $\mathbb{CP}^1$. Then it follows from *cohomCalg*

$$H^0\left(\mathbb{CP}^1,\mathcal{L}\right) = \left\{\alpha_1x_1^2+\alpha_2x_1x_2+\alpha_3x_2^2\ ,\ \alpha_i\in\mathbb{C}\right\}\cong\mathbb{C}^3 \tag{14.8}$$

whilst the higher cohomology classes are all trivial. Now let $i,j\in\{1,2\}$. Then we can describe $H^0\left(\mathbb{CP}^1,\mathcal{L}\right)$ by a symmetric tensor S_{ij}, i.e.

$$H^0\left(\mathbb{CP}^1,\mathcal{L}\right) = \{S_{ij}x_ix_j\} \tag{14.9}$$

Note:
Let us now look at a del Pezzo 1-surface dP_1. Its toric data is given in Table 14.1. Subsequently let us consider the line bundle $\mathcal{L} = \mathcal{O}_{dP_1}(5,-2)$. We learned back in section 6.4

$$H^0(X_\Sigma,\mathcal{L}) = \left\{\frac{\alpha_1x_3^6+\alpha_2x_1x_3^5+\alpha_3x_1^2x_3^4+\alpha_4x_1^3x_3^3+\alpha_5x_1^4x_3^2+\alpha_6x_1^5x_3+\alpha_7x_1^6}{x_2x_4}\ ,\ \alpha_i\in\mathbb{C}\right\} \tag{14.10}$$

We now set $i_1,\ldots,i_6\in\{1,3\}$ and $J,K\in\{2,4\}$. Then we observe

$$S_{i_1\ldots i_6}\cdot\epsilon_{JK}\cdot\frac{x_{i_1}\cdot\cdots\cdot x_{i_6}}{x_Jx_K} \tag{14.11}$$

where S is a symmetric tensor and ϵ the totally antisymmetric tensor in two dimensions.
This looks as if the cohomology groups of this holomorphic line bundle on dP_1 were also classified by a tensor representation of some possible regular and semi-simple Lie group. In fact this observation generalises to all line bundle cohomologies on smooth and compact normal toric varieties that the author got to see to date. This observation therefore motivates the following proposition.

Proposition:
Every smooth and compact normal toric variety X_Σ is a generalised flag variety.

Remark:
The author would like to emphasize that the above is a proposition and that it is currently beyond the knowledge of the author to proof or disprove it.

14.3. (Anti-)Symmetrised Partitions Of The Koszul Complex

Remark:
We have described in chapter 13 the general construction of the maps d_r for $r \geq 2$ in the Koszul spectral sequence. This procedure involves knowledge about the Čech cochains as these form the E_0-sheet of the Koszul spectral sequence. Unfortunately their computation takes much longer than the application of *cohomCalg*. For this reason it is tempting to compute the sheaf cohomology groups via *cohomCalg* and thereby to start the computation at the E_1-sheet instead.
This speed advantage comes at the cost that it is impossible to construct the maps d_r with $r \geq 2$ via tracing them back onto the E_0-sheet as presented out in section 13.2. For this reason we intent to give a proposition for a different way of constructing these maps. We start with a few examples and fomulate the general proposition afterwards.

Note (Codimension 1)**:**
Recall that for the computation of pullback cohomologies on a hypersurface, the Koszul spectral sequence converges on the E_2-sheet. Thus only knowledge about the d_1-maps is needed in this situation. Consequently the computation of pullback cohomologies on codimension 2 algebraic subvarieties is the simplest situation with non-trivial d_2-maps.

Example 14.3.1 (Codimension 2)**:**
Let X_Σ a smooth and compact normal toric variety. Then consider two effective divisor classes $S_1, S_2 \in \mathrm{Cl}(X_\Sigma)$ and pick holomorphic sections $\widetilde{s}_i \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(S_i))$ such that

$$C := \{p \in X_\Sigma \ , \ \widetilde{s}_1(p) = \widetilde{s}_2(p) = 0\} \subset X_\Sigma \tag{14.12}$$

is an algebraic submanifold of codimension 2 in X_Σ. Now pick a holomorphic line bundle $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$ on X_Σ for $D \in \mathrm{Cl}(X_\Sigma)$.
Given this information we know that the E_1-sheet of the Koszul spectral sequence looks as illustrated in Figure 14.1. Also recall

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}(D - S_1 - S_2)$
- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(D - S_2) \oplus \mathcal{O}_{X_\Sigma}(D - S_1)$

The Koszul resolution gives us maps

- $\alpha^i_{(1)} \colon H^i(X_\Sigma, \mathcal{L}') \to H^i(X_\Sigma, \mathcal{V}_1)$
- $\beta^i_{(1)} \colon H^i(X_\Sigma, \mathcal{V}_1) \to H^i(X_\Sigma, \mathcal{L})$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
H^0\left(X_\Sigma, \mathcal{L}'\right) \arrow[d, "\alpha^0_{(1)}"] & H^1\left(X_\Sigma, \mathcal{L}'\right) \arrow[d, "\alpha^1_{(1)}"] \arrow[ddl, dashed, "\alpha^0_{(2)}"'] & H^2\left(X_\Sigma, \mathcal{L}'\right) \arrow[d, "\alpha^2_{(1)}"] \arrow[ddl, dashed, "\alpha^1_{(2)}"'] & \cdots \\
H^0\left(X_\Sigma, \mathcal{V}_1\right) \arrow[d, "\beta^0_{(1)}"] & H^1\left(X_\Sigma, \mathcal{V}_1\right) \arrow[d, "\beta^1_{(1)}"] & H^2\left(X_\Sigma, \mathcal{V}_1\right) \arrow[d, "\beta^2_{(1)}"] & \cdots \\
H^0\left(X_\Sigma, \mathcal{L}\right) \arrow[d] & H^1\left(X_\Sigma, \mathcal{L}\right) \arrow[d] & H^2\left(X_\Sigma, \mathcal{L}\right) \arrow[d] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}$$

Figure 14.1.: The sheet E_1 of the Koszul spectral sequence for a codimension 2 locus. The Knight's moves $\alpha^i_{(2)}$ to be constructed on the E_2-sheet are indicated by dashed green lines.

We now wish to construct the d_2-mappings $\alpha^i_{(2)}$. A natural guess is to consider the mappings

$$\widetilde{\alpha}^i_{(2)} \colon H^{i+1}\left(X_\Sigma, \mathcal{L}'\right) \to H^i\left(X_\Sigma, \mathcal{L}\right) \tag{14.13}$$

and then to induce from these maps the Knight's moves on the sheet E_2. Therefore our task has turned into constructing the maps $\widetilde{\alpha}^i_{(2)}$. A naive guess for these maps in turn would be to construct them from the global section valued matrix

$$M_{\widetilde{\alpha}^i_{(2)}} = \left(\widetilde{s}_1 \cdot \widetilde{s}_2\right) \tag{14.14}$$

We will come back to turning this naive guess into an educated guess momentarily.

Example 14.3.2 (Codimension 3)**:**
We consider precisely the same situation as in the preceeding example, except that we consider a codimension 3 algebraic variety $C \subset X_\Sigma$. Hence we consider three effective divisor classes $S_1, S_2, S_3 \in \mathrm{Cl}\left(X_\Sigma\right)$ and holomorphic sections $\widetilde{s}_i \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(S_i\right)\right)$ such that

$$C \coloneqq \left\{p \in X_\Sigma \text{ , } \widetilde{s}_1\left(p\right) = \widetilde{s}_2\left(p\right) = 0\right\} \subset X_\Sigma \tag{14.15}$$

is an algebraic submanifold of codimension 3 of X_Σ. Then the E_1-sheet of the Koszul spectral sequence looks as illustrated in Figure 14.2. Recall in particular that we have

- $\mathcal{L}' = \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_2 - S_3\right)$
- $\mathcal{V}_2 = \mathcal{O}_{X_\Sigma}\left(D - S_2 - S_3\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_3\right) \oplus \mathcal{O}_{X_\Sigma}\left(D - S_1 - S_2\right)$

- $\mathcal{V}_1 = \mathcal{O}_{X_\Sigma}(D - S_1) \oplus \mathcal{O}_{X_\Sigma}(D - S_2) \oplus \mathcal{O}_{X_\Sigma}(D - S_3)$

The Koszul resolution gives us maps

- $\alpha^i_{(1)} \colon H^i(X_\Sigma, \mathcal{L}') \to H^i(X_\Sigma, \mathcal{V}_2)$
- $\beta^i_{(1)} \colon H^i(X_\Sigma, \mathcal{V}_2) \to H^i(X_\Sigma, \mathcal{V}_1)$
- $\gamma^i_{(1)} \colon H^i(X_\Sigma, \mathcal{V}_1) \to H^i(X_\Sigma, \mathcal{L})$

We now wish to construct the d_2-mappings $\alpha^i_{(2)}$, $\beta^i_{(2)}$ and the d_3-mappings $\alpha^i_{(3)}$. Again we consider the maps

- $\widetilde{\alpha}^i_{(2)} \colon H^{i+1}(X_\Sigma, \mathcal{L}') \to H^i(X_\Sigma, \mathcal{V}_1)$
- $\widetilde{\beta}^i_{(2)} \colon H^{i+1}(X_\Sigma, \mathcal{V}_2) \to H^i(X_\Sigma, \mathcal{L})$
- $\widetilde{\alpha}^i_{(3)} \colon H^{i+2}(X_\Sigma, \mathcal{L}') \to H^i(X_\Sigma, \mathcal{L})$

instead and induce the higher d_r-maps from these level E_1-mappings. The task of constructing the mappings $\widetilde{\alpha}^i_{(2)}$, $\widetilde{\beta}^i_{(2)}$ and $\widetilde{\alpha}^i_{(3)}$ in turn can naively be done by inducing these mappings from the following global section valued matrices.

$$M_{\widetilde{\alpha}_{(2)}} = \begin{pmatrix} \widetilde{s}_2 \widetilde{s}_3 \\ \widetilde{s}_1 \widetilde{s}_3 \\ \widetilde{s}_1 \widetilde{s}_2 \end{pmatrix}, \quad M_{\widetilde{\beta}_{(2)}} = (\widetilde{s}_2 \widetilde{s}_3, \widetilde{s}_1 \widetilde{s}_3, \widetilde{s}_1 \widetilde{s}_2), \quad M_{\widetilde{\alpha}_{(3)}} = (\widetilde{s}_1 \widetilde{s}_2 \widetilde{s}_3) \tag{14.16}$$

We now turn this naive guess into an educated guess via the following proposition.

Proposition (Alternative Construction Of Higher d-Maps)**:**
The maps d_r with $r \geq 2$ in the Koszul spectral sequence are the, up to an overall minus sign, unique maps which have the following properties.

1. They are induced from mappings on the E_1-sheet given by a matrix, whose entries are products, but no inverses, of the global sections $\widetilde{s}_i$ and which in addition are (anti)-symmetrised such as to respect the tensor structure of the corresponding cohomology groups on the E_1-sheet.
2. The mappings split the E_r-sheet of the Koszul spectral sequence into complexes, i.e. subsequent application of two neighbouring d_r-maps gives the trivial mapping.

Definition 14.3.1 ((Anti)-Symmetrised Partitions Of The Koszul Complex)**:**
We term the maps in the preceeding proposition the *(anti)-symmetrised partitions of the Koszul complex.*

Example 14.3.3 (Application Of The Proposal)**:**
We illustrate this proposition on the Knight's move example presented in subsection 13.3.2. In this example all d_1-maps are trivial and we have $\widetilde{\alpha}^0_{(2)} = \alpha^0_{(2)}$. Hence the Knight's move to be constructed is just the map

$$\alpha^0_{(2)} \colon H^1(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{L}') \to H^0(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{L}) \tag{14.17}$$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
H^0(X_\Sigma, \mathcal{L}') \arrow[d, "\alpha^0_{(1)}"] & H^1(X_\Sigma, \mathcal{L}') \arrow[d, "\alpha^1_{(1)}"] \arrow[ddl, dashed, "\alpha^0_{(2)}"'] & H^2(X_\Sigma, \mathcal{L}') \arrow[d, "\alpha^2_{(1)}"] \arrow[ddl, dashed, "\alpha^1_{(2)}"'] \arrow[dddll, dashed, red] & \cdots \\
H^0(X_\Sigma, \mathcal{V}_2) \arrow[d, "\beta^0_{(1)}"] & H^1(X_\Sigma, \mathcal{V}_2) \arrow[d, "\beta^1_{(1)}"] \arrow[ddl, dashed, "\beta^0_{(2)}"'] & H^2(X_\Sigma, \mathcal{V}_2) \arrow[d, "\beta^2_{(1)}"] \arrow[ddl, dashed, "\beta^1_{(2)}"'] & \cdots \\
H^0(X_\Sigma, \mathcal{V}_1) \arrow[d, "\gamma^0_{(1)}"] & H^1(X_\Sigma, \mathcal{V}_1) \arrow[d, "\gamma^1_{(1)}"] & H^2(X_\Sigma, \mathcal{V}_1) \arrow[d, "\gamma^2_{(1)}"] & \cdots \\
H^0(X_\Sigma, \mathcal{L}) \arrow[d] & H^1(X_\Sigma, \mathcal{L}) \arrow[d] & H^2(X_\Sigma, \mathcal{L}) \arrow[d] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}$$

Figure 14.2.: The sheet E_1 of the Koszul spectral sequence for a codimension 3 locus. The red dashed arrow indicates the d_3-map $\alpha^0_{(3)}$, whilst $\alpha^i_{(2)}$ and $\beta^i_{(2)}$ are the Knight's moves on the E_2-sheet.

To this end we first consider the global section valued matrix

$$M^{\text{naive}}_{\alpha^0_{(2)}} = (\widetilde{s}_1 \cdot \widetilde{s}_2) \tag{14.18}$$

Secondly we have to make sure that all entries of this matrix respect the tensor structure of the corresponding cohomology groups. To this end let us recall that

- $P_1 = H^1(X_\Sigma, \mathcal{L}') = \left\{ A_1 \cdot \frac{x_4}{x_5 x_6} + A_2 \cdot \frac{x_3}{x_5 x_6} \, , \; A_i \in \mathbb{C} \right\}$
- $P_2 = H^0(X_\Sigma, \mathcal{L}) = \{ A_3 \cdot x_2 x_4 + A_4 \cdot x_2 x_3 + A_5 x_1 x_4 + A_6 x_1 x_3 \, , \; A_i \in \mathbb{C} \}$

Let us now use the following indices

$$\alpha \in \{1, 2\} \, , \quad a \in \{3, 4\} \, , \quad A \in \{5, 6\} \tag{14.19}$$

Then we can describe the spaces P_1 and P_2 via the following tensor structure

$$P_1 = S_a \epsilon_{AB} \cdot \frac{x_a}{x_A x_B} \, , \qquad P_2 = S_a T_\alpha \cdot x_a x_\alpha \tag{14.20}$$

Consequently a mapping $P_1 \to P_2$ has to cancel the antisymmetrisation ϵ_{AB}. This is achieved by antisymmetrising $\widetilde{s}_1 \widetilde{s}_2$ with respect to the variables x_5 and x_6. Let us perform this in detail. We first recall

$$\widetilde{s}_1 = (C_4 x_1 + C_2 x_2) \, x_5 + (C_3 x_1 + C_1 x_2) \, x_6 \, , \quad \widetilde{s}_2 = C_6 x_5 + C_5 x_6 \tag{14.21}$$

Then one obtains

$$\widetilde{s}_1 \widetilde{s}_2 = (C_4 x_1 + C_2 x_2)\, C_6 x_5^2 + (C_4 x_1 + C_2 x_2)\, C_5 x_5 x_6 \tag{14.22}$$
$$+ (C_3 x_1 + C_1 x_2)\, C_6 x_5 x_6 + (C_3 x_1 + C_1 x_2)\, C_5 x_6^2 \tag{14.23}$$

Antisymmetrisation with respect to x_5 and x_6 rules out the two terms with x_5^2 and x_6^2. Hence we conclude that we should consider up to an overall minus sign the matrix

$$M^{\text{final}}_{\alpha^0_{(2)}} = ((C_4 x_1 + C_2 x_2)\, C_5 x_5 x_6 - (C_3 x_1 + C_1 x_2)\, C_6 x_5 x_6) \tag{14.24}$$
$$= (x_1 x_5 x_6 \,[C_4 C_5 - C_3 C_6] + x_2 x_5 x_6 \,[C_2 C_5 - C_1 C_6]) \tag{14.25}$$

This matrix now induces a map $\alpha^0_{(2)}$ which satisfies the first bullet point in the proposition of (anti-)symmetrised partitions of the Koszul complex. In addition it is clear that we obtain a complex by adding the trivial mappings leading to $H^1(X_\Sigma, \mathcal{L}')$ and away from $H^0(X_\Sigma, \mathcal{L})$ respectively. Thus this is up to an overall minus sign the unique map described in the proposition.

Comparison with section 13.4 shows that this is indeed the correct Knight's move mapping.

15. Computation Of The E_1-Sheet With Mathematica

15.1. Summary

In this chapter we summarise the commands that are implemented in the *Mathematica* notebook given in Appendix E. This notebook allows for the computation of the E_1-sheet of the Koszul spectral sequence. It is based on the *cohomCalg* algorithm [57]. Therefore the *C++* program *cohomCalg* need to be downloaded from `http://wwwth.mpp.mpg.de/members/bjurke/cohomcalg/`. The path of the executable file is needed as input in the *Mathematica* notebook. Given that the notebook is run on a *Windows* system and *cohomCalg.exe* is placed in the same folder as the *Mathematica* notebook, one has to set

```
cohomCalgExecutable="cohomcalg.exe"
```

at the very top of the notebook. Subsequently the first two large code blocks have to be executed. Thereafter all implemented functions can be used.

15.2. Collection Of Implemented Commands

15.2.1. Toric Variety Input

To enter a toric variety the following information is required.

- The homogeneous coordinates. In this notebook they are always taken as x_i.
- The scaling relations.
- The Stanley-Reisner ideal.

For example the following command implements $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

```
In[1]:=CP2CP1CP1 = {
    (*Coordinates*){x1, x2, x3, x4, x5, x6, x7},
    (*Stanley Reisner*)Map[Variables[#] &, {x1*x2*x3, x4*x5, x6*x7}],
```

```
(*Equivalence Relations*){{1, 0, 0}, {1, 0, 0}, {1, 0, 0}, {0, 1,
  0}, {0, 1, 0}, {0, 0, 1}, {0, 0, 1}}
};
```

It is expected that the input gives a smooth and compact normal toric variety. No check on this is performed in the notebook. Consequently the user has to make sure this is the case.

15.2.2. Holomorphic Line Bundles And Divisor Classes

In smooth and compact normal toric varieties both holomorphic line bundles as well as divisor classes can be specified by an integer valued vector of appropriate length. Therefore both are entered as such integer valued vectors into the *Mathematica* notebook.

15.2.3. Ambient Space Cohomology Computations

Let us compute the cohomologies of the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^2\times\mathbb{CP}^1\times\mathbb{CP}^1}(1,0,0)$. To this end we use that we have already implemented $\mathbb{CP}^2\times\mathbb{CP}^1\times\mathbb{CP}^1$ as CP2CP1CP1. Therefore the following command computes a basis of the cohomology classes.

```
In[2]:=GetBasisOfLineBundleCohomology[CP2CP1CP1, {1, 0, 0}]
Out[2]={{x3, x2, x1}, {}, {}, {}, {}}
```

This result says

$$H^0\left(\mathbb{CP}^2\times\mathbb{CP}^1\times\mathbb{CP}^1,\mathcal{L}\right) = \{A_1x_1 + A_2x_2 + A_3x_3\ ,\ A_i\in\mathbb{C}\}\cong\mathbb{C}^3 \tag{15.1}$$

and all higher cohomology classes are trivial.
Similarly we can compute the cohomologies of a direct sum of holomorphic line bundles.

```
In[3]:=GetBasisOfVectorBundleCohomology[CP2CP1CP1, {{0,0,1},{0,1,0}}]
Out[3]={{{{{x7}, {0}}, {{x6}, {0}}, {{0}, {x5}}, {{0}, {x4}}}, {}}, {{{{0},
        {0}}}, {}}, {{{{0}, {0}}}, {}}, {{{{0}, {0}}}, {}}, {{{{0}, {0}}},
        {}}}
```

This means that on $X_\Sigma = \mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ the cohomologies of the holomorphic vector bundle $\mathcal{V} = \mathcal{O}_{X_\Sigma}(0,0,1) \oplus \mathcal{O}_{X_\Sigma}(0,1,0)$ are as follows. We have

$$H^0(X_\Sigma, \mathcal{V}) = \left\{ A_1 \begin{pmatrix} x_6 \\ 0 \end{pmatrix} + A_2 \begin{pmatrix} x_7 \\ 0 \end{pmatrix} + A_3 \begin{pmatrix} 0 \\ x_4 \end{pmatrix} + A_4 \begin{pmatrix} 0 \\ x_5 \end{pmatrix},\ A_i \in \mathbb{C} \right\} \cong \mathbb{C}^5 \quad (15.2)$$

and all higher cohomology classes are trivial.

15.2.4. Computation Of Sheet E_1

Let us now turn to the computation of the E_1-sheet of the Koszul spectral sequence. Let us compute the cohomologies of $\mathcal{L} = \mathcal{O}_{X_\Sigma}(1,1,0)$ on the algebraic submanifold C of codimension 3 in $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ which in divisor language is the complete intersection

$$C = (1,0,0) \cap (0,1,0) \cap (1,0,1) \quad (15.3)$$

In a first approach we compute only the ambient space cohomologies. This suffices to plot the E_1-sheet. This task is performed by the following command.

```
In[4]:=DrawFirstSheetWithMaps[CP2CP1CP1, {{1,0,0}, {0,1,0}, {1,0,1}}, {1,1,0},
        {{"none"}}]
```

The output of this command is given in Figure 15.1. From this we see that this spectral sequence does indeed converge on the sheet E_2. In addition only the map $P_2 \to P_3$ has to be computed to obtain the pullback cohomology classes. So we repeat the same computation, but this time we compute the map $\gamma^0\colon P_2 \to P_3$. This is achieved via the following command.

```
In[5]:=DrawFirstSheetWithMaps[CP2CP1CP1, {{1,0,0}, {0,1,0}, {1,0,1}}, {1,1,0},
        {{3,1}}]
```

From the output in Figure 15.2 we conclude that the map $\gamma^0\colon P_1 \to P_2$ can be represented by the matrix

$$M_{\gamma^0} = \begin{pmatrix} C_1 & 0 & C_4 & 0 & 0 \\ 0 & C_1 & C_5 & 0 & 0 \\ C_2 & 0 & 0 & C_4 & 0 \\ 0 & C_2 & 0 & C_5 & 0 \\ C_3 & 0 & 0 & 0 & C_4 \\ 0 & C_3 & 0 & 0 & C_5 \end{pmatrix} \quad (15.4)$$

This is all the information needed to analyse pullback cohomology dependence on the parameters $C_i \in \mathbb{C}$ which give a redundant description of the complex structure of C. In particular we have

$$H^0\left(C, \mathcal{L}|_C\right) \cong \ker\left(M_{\gamma^0}\right), \qquad H^i\left(C, \mathcal{L}|_C\right) = 0 \quad i \geq 1 \tag{15.5}$$

Let us mention that the command

```
In[6]:=DrawFirstSheetWithMaps[CP2CP1CP1, {{1,0,0}, {0,1,0}, {1,0,1}}, {1,1,0},
    {{3,1},{2,1}}]
```

computes the maps at position $(3,1)$ and $(2,1)$. Therefore the above command can be used to compute several mapping matrices in one run. However one can also compute all maps in one run by use of.

```
In[7]:=DrawFirstSheetWithMaps[CP2CP1CP1, {{1,0,0}, {0,1,0}, {1,0,1}}, {1,1,0},
    {{"all"}}]
```

It should be mentioned that most of the computational time is used for computing the mapping matrices. The needed time for determining these matrices grows fast as the dimensions of the ambient space cohomologies increases. Therefore we advise the user to apply the following two step procedure.

1. First compute no mapping matrices at all, but only the cohomology groups in the E_1-sheet by use of the option "none".
2. Secondly identify the maps needed for the analysis of the E_2-sheet and compute those maps only.

15.2.5. Application To Model Building

In chapter 16 we give a simple application of the technology of pullback cohomology computations to model building. Whilst we give a detailed description of the models there, let us mention that we need the following input for this model.

- A smooth and compact normal toric variety X_Σ.
- $D_{B_3} \in \mathrm{Cl}\left(X_\Sigma\right)$.
- $D_{\mathrm{GUT}} \in \mathrm{Cl}\left(X_\Sigma\right)$.
- A quasi-divisor-class D with $2D \in \mathrm{Cl}\left(X_\Sigma\right)$.

So for example for $X_\Sigma = \mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ we could have $D_{B_3} = (1,2,1)$, $D_{\mathrm{GUT}} = (1,0,1)$ and $D = \left(-\frac{7}{2}, 0, \frac{9}{2}\right)$. Then the following command computes the model for us.

In[8]:=Model[CP2CP1CP1, {1, 2, 1}, {1, 0, 1}, {−7/2, 0, 9/2}]

We display the output in Figure 15.3, Figure 15.4, Figure 15.5, Figure 15.6 and Figure 15.7.

```
In[105]:= DrawFirstSheetWithMaps [CP2CP1CP1 , {{1, 0, 0}, {0, 1, 0}, {1, 0, 1}}, {1, 1, 0}, {{"none"}}]
```

Computation started

Section	Explicit form	Charges
s1	C3 x1 + C2 x2 + C1 x3	{1, 0, 0}
s2	C5 x4 + C4 x5	{0, 1, 0}
s3	C11 x1 x6 + C9 x2 x6 + C7 x3 x6 + C10 x1 x7 + C8 x2 x7 + C6 x3 x7	{1, 0, 1}

Global sections defining the complete intersection subvariety.

Space	Basis	Equivalence Relations	Naive Dimension
P1	$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$	{ }	1
P2	$\left\{\begin{pmatrix}x5\\0\\0\end{pmatrix},\begin{pmatrix}x4\\0\\0\end{pmatrix},\begin{pmatrix}0\\x3\\0\end{pmatrix},\begin{pmatrix}0\\x2\\0\end{pmatrix},\begin{pmatrix}0\\x1\\0\end{pmatrix}\right\}$	{ }	5
P3	{(x3 x5), (x3 x4), (x2 x5), (x2 x4), (x1 x5), (x1 x4)}	{ }	6

Rationom spaces in the first sheet of the Koszul spectral sequence.

$\mathcal{O}$(-1,0,-1)	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	
↓ $\begin{pmatrix}s3\\-s2\\s1\end{pmatrix}$ ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	
$\mathcal{O}$(0,0,0)⊕$\mathcal{O}$(-1,1,-1)⊕$\mathcal{O}$(0,0,-1)	P1	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	
↓ $\begin{pmatrix}-s2 & -s3 & 0\\ s1 & 0 & -s3\\ 0 & s1 & s2\end{pmatrix}$ ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	
$\mathcal{O}$(0,1,0)⊕$\mathcal{O}$(1,0,0)⊕$\mathcal{O}$(0,1,-1)	P2	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\},\{\}\right\}$	
↓ (s1 s2 s3) ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	↓ not computed ↓	
$\mathcal{O}$(1,1,0)	P3	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	
	H^0	H^1	H^2	H^3	H^4	

First sheet of the Koszul exact sequence and the maps therein.

Out[105]= Computation finished after 0.7710441 seconds.

Figure 15.1.: Computation of E_1-sheet by *Mathematica* notebook - map computations are supressed.

```
In[129]:= DrawFirstSheetWithMaps [CP2CP1CP1 , {{1, 0, 0}, {0, 1, 0}, {1, 0, 1}}, {1, 1, 0}, {{3, 1}}]

Computation started
```

Section	Explicit form	Charges
s1	C3 x1 + C2 x2 + C1 x3	{1, 0, 0}
s2	C5 x4 + C4 x5	{0, 1, 0}
s3	C11 x1 x6 + C9 x2 x6 + C7 x3 x6 + C10 x1 x7 + C8 x2 x7 + C6 x3 x7	{1, 0, 1}

Global sections defining the complete intersection subvariety.

Space	Basis	Equivalence Relations	Naive Dimension
P1	$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$	{}	1
P2	$\left\{\begin{pmatrix}x5\\0\\0\end{pmatrix}, \begin{pmatrix}x4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\x3\\0\end{pmatrix}, \begin{pmatrix}0\\x2\\0\end{pmatrix}, \begin{pmatrix}0\\x1\\0\end{pmatrix}\right\}$	{}	5
P3	{(x3 x5), (x3 x4), (x2 x5), (x2 x4), (x1 x5), (x1 x4)}	{}	6

Rationom spaces in the first sheet of the Koszul spectral sequence.

$\mathcal{O}$(-1,0,-1)	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}
$\begin{pmatrix}s3\\-s2\\s1\end{pmatrix}$	not computed	not computed	not computed	not computed	not computed
$\mathcal{O}$(0,0,0)⊕$\mathcal{O}$(-1,1,-1)⊕$\mathcal{O}$(0,0,-1)	P1	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$
$\begin{pmatrix}-s2 & -s3 & 0\\ s1 & 0 & -s3\\ 0 & s1 & s2\end{pmatrix}$	not computed	not computed	not computed	not computed	not computed
$\mathcal{O}$(0,1,0)⊕$\mathcal{O}$(1,0,0)⊕$\mathcal{O}$(0,1,-1)	P2	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$	$\left\{\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, \{\}\right\}$
(s1 s2 s3)	$\begin{pmatrix}C1 & 0 & C4 & 0 & 0\\ 0 & C1 & C5 & 0 & 0\\ C2 & 0 & 0 & C4 & 0\\ 0 & C2 & 0 & C5 & 0\\ C3 & 0 & 0 & 0 & C4\\ 0 & C3 & 0 & 0 & C5\end{pmatrix}$	not computed	not computed	not computed	not computed
$\mathcal{O}$(1,1,0)	P3	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}	{{(0)}, {}}
	H^0	H^1	H^2	H^3	H^4

First sheet of the Koszul exact sequence and the maps therein.

```
Out[129]= Computation finished after 0.8600012 seconds.
```

Figure 15.2.: Computation of E_1-sheet by *Mathematica* notebook - only the important map at position $(3, 1)$ was computed.

```
In[131]:= Model[CP2CP1CP1, {1, 2, 1}, {1, 0, 1}, {-7/2, 0, 9/2}]
Started computation.
```

Name	Charge	Random Section
Base	{1, 2, 1}	$0.809356 x1 x4^{2} x6 + 0.146078 x2 x4^{2} x6 + 0.0591562 x3 x4^{2} x6 + 0.376335 x1 x4 x5 x6 + 0.637865 x2 x4 x5 x6 + 0.784765 x3 x4 x5 x6 + 0.43033 x1 x5^{2} x6 + 0.188448 x2 x5^{2} x6 + 0.439943 x3 x5^{2} x6 + 0.903981 x1 x4^{2} x7 + 0.781722 x2 x4^{2} x7 + 0.232888 x3 x4^{2} x7 + 0.975963 x1 x4 x5 x7 + 0.46887 x2 x4 x5 x7 + 0.346646 x3 x4 x5 x7 + 0.823916 x1 x5^{2} x7 + 0.319159 x2 x5^{2} x7 + 0.0863057 x3 x5^{2} x7$
GUT	{1, 0, 1}	$0.0550177 x1 x6 + 0.609207 x2 x6 + 0.94568 x3 x6 + 0.682536 x1 x7 + 0.822599 x2 x7 + 0.232181 x3 x7$
a1	{2, 0, 1}	$0.965737 x1^{2} x6 + 0.399804 x1 x2 x6 + 0.269027 x2^{2} x6 + 0.846127 x1 x3 x6 + 0.422912 x2 x3 x6 + 0.157133 x3^{2} x6 + 0.555685 x1^{2} x7 + 0.561758 x1 x2 x7 + 0.192047 x2^{2} x7 + 0.513682 x1 x3 x7 + 0.983103 x2 x3 x7 + 0.625067 x3^{2} x7$
a21	{3, 0, 1}	$0.529199 x1^{3} x6 + 0.838528 x1^{2} x2 x6 + 0.840043 x1 x2^{2} x6 + 0.445524 x2^{3} x6 + 0.948063 x1^{2} x3 x6 + 0.0943822 x1 x2 x3 x6 + 0.330786 x2^{2} x3 x6 + 0.453929 x1 x3^{2} x6 + 0.737271 x2 x3^{2} x6 + 0.460944 x3^{3} x6 + 0.456702 x1^{3} x7 + 0.657483 x1^{2} x2 x7 + 0.372096 x1 x2^{2} x7 + 0.270854 x2^{3} x7 + 0.306919 x1^{2} x3 x7 + 0.647227 x1 x2 x3 x7 + 0.180077 x2^{2} x3 x7 + 0.036952 x1 x3^{2} x7 + 0.653654 x2 x3^{2} x7 + 0.326238 x3^{3} x7$
a32	{4, 0, 1}	$0.718925 x1^{4} x6 + 0.258641 x1^{3} x2 x6 + 0.831174 x1^{2} x2^{2} x6 + 0.0748824 x1 x2^{3} x6 + 0.104985 x2^{4} x6 + 0.60516 x1^{3} x3 x6 + 0.813322 x1^{2} x2 x3 x6 + 0.807712 x1 x2^{2} x3 x6 + 0.978169 x2^{3} x3 x6 + 0.142021 x1^{2} x3^{2} x6 + 0.212022 x1 x2 x3^{2} x6 + 0.517164 x2^{2} x3^{2} x6 + 0.5311 x1 x3^{3} x6 + 0.0907438 x2 x3^{3} x6 + 0.461829 x3^{4} x6 + 0.821614 x1^{4} x7 + 0.804168 x1^{3} x2 x7 + 0.569678 x1^{2} x2^{2} x7 + 0.963772 x1 x2^{3} x7 + 0.77205 x2^{4} x7 + 0.274347 x1^{3} x3 x7 + 0.468654 x1^{2} x2 x3 x7 + 0.274181 x1 x2^{2} x3 x7 + 0.607986 x2^{3} x3 x7 + 0.468464 x1^{2} x3^{2} x7 + 0.387115 x1 x2 x3^{2} x7 + 0.246592 x2^{2} x3^{2} x7 + 0.145521 x1 x3^{3} x7 + 0.931167 x2 x3^{3} x7 + 0.640857 x3^{4} x7$
a43	{5, 0, 1}	$0.710384 x1^{5} x6 + 0.916609 x1^{4} x2 x6 + 0.756759 x1^{3} x2^{2} x6 + 0.559207 x1^{2} x2^{3} x6 + 0.576848 x1 x2^{4} x6 + 0.371201 x2^{5} x6 + 0.823283 x1^{4} x3 x6 + 0.133729 x1^{3} x2 x3 x6 + 0.760824 x1^{2} x2^{2} x3 x6 + 0.220988 x1 x2^{3} x3 x6 + 0.396195 x2^{4} x3 x6 + 0.815669 x1^{3} x3^{2} x6 + 0.0762325 x1^{2} x2 x3^{2} x6 + 0.386434 x1 x2^{2} x3^{2} x6 + 0.846821 x2^{3} x3^{2} x6 + 0.74492 x1^{2} x3^{3} x6 + 0.455711 x1 x2 x3^{3} x6 + 0.513031 x2^{2} x3^{3} x6 + 0.224662 x1 x3^{4} x6 + 0.162035 x2 x3^{4} x6 + 0.320198 x3^{5} x6 + 0.93546 x1^{5} x7 + 0.718881 x1^{4} x2 x7 + 0.0369824 x1^{3} x2^{2} x7 + 0.789456 x1^{2} x2^{3} x7 + 0.0807137 x1 x2^{4} x7 + 0.739937 x2^{5} x7 + 0.523926 x1^{4} x3 x7 + 0.26974 x1^{3} x2 x3 x7 + 0.0353793 x1^{2} x2^{2} x3 x7 + 0.918838 x1 x2^{3} x3 x7 + 0.441546 x2^{4} x3 x7 + 0.0820188 x1^{3} x3^{2} x7 + 0.404392 x1^{2} x2 x3^{2} x7 + 0.550655 x1 x2^{2} x3^{2} x7 + 0.191122 x2^{3} x3^{2} x7 + 0.611362 x1^{2} x3^{3} x7 + 0.997466 x1 x2 x3^{3} x7 + 0.720157 x2^{2} x3^{3} x7 + 0.0434576 x1 x3^{4} x7 + 0.536702 x2 x3^{4} x7 + 0.993361 x3^{5} x7$
C10	{2, 0, 1}	$0.965737 x1^{2} x6 + 0.399804 x1 x2 x6 + 0.269027 x2^{2} x6 + 0.846127 x1 x3 x6 + 0.422912 x2 x3 x6 + 0.157133 x3^{2} x6 + 0.555685 x1^{2} x7 + 0.561758 x1 x2 x7 + 0.192047 x2^{2} x7 + 0.513682 x1 x3 x7 + 0.983103 x2 x3 x7 + 0.625067 x3^{2} x7$
C5m	{4, 0, 1}	$0.718925 x1^{4} x6 + 0.258641 x1^{3} x2 x6 + 0.831174 x1^{2} x2^{2} x6 + 0.0748824 x1 x2^{3} x6 + 0.104985 x2^{4} x6 + 0.60516 x1^{3} x3 x6 + 0.813322 x1^{2} x2 x3 x6 + 0.807712 x1 x2^{2} x3 x6 + 0.978169 x2^{3} x3 x6 + 0.142021 x1^{2} x3^{2} x6 + 0.212022 x1 x2 x3^{2} x6 + 0.517164 x2^{2} x3^{2} x6 + 0.5311 x1 x3^{3} x6 + 0.0907438 x2 x3^{3} x6 + 0.461829 x3^{4} x6 + 0.821614 x1^{4} x7 + 0.804168 x1^{3} x2 x7 + 0.569678 x1^{2} x2^{2} x7 + 0.963772 x1 x2^{3} x7 + 0.77205 x2^{4} x7 + 0.274347 x1^{3} x3 x7 + 0.468654 x1^{2} x2 x3 x7 + 0.274181 x1 x2^{2} x3 x7 + 0.607986 x2^{3} x3 x7 + 0.468464 x1^{2} x3^{2} x7 + 0.387115 x1 x2 x3^{2} x7 + 0.246592 x2^{2} x3^{2} x7 + 0.145521 x1 x3^{3} x7 + 0.931167 x2 x3^{3} x7 + 0.640857 x3^{4} x7$
C5H	{7, 0, 2}	$-0.30559 x1^{7} x6^{2} - 0.429506 x1^{6} x2 x6^{2} - 0.0277435 x1^{5} x2^{2} x6^{2} + 0.184965 x1^{4} x2^{3} x6^{2} - 0.0524425 x1^{3} x2^{4} x6^{2} - 0.218305 x1^{2} x2^{5} x6^{2} - 0.182041 x1 x2^{6} x6^{2} - 0.0530896 x2^{7} x6^{2} - 0.394313 x1^{6} x3 x6^{2} - 0.283381 x1^{5} x2 x3 x6^{2} + 0.630356 x1^{4} x2^{2} x3 x6^{2} + 1.036 x1^{3} x2^{3} x3 x6^{2} + 0.842395 x1^{2} x2^{4} x3 x6^{2} + 0.440347 x1 x2^{5} x3 x6^{2} + 0.206953 x2^{6} x3 x6^{2} - 0.62072 x1^{5} x3^{2} x6^{2} + 0.701849 x1^{4} x2 x3^{2} x6^{2} + 0.739129 x1^{3} x2^{2} x3^{2} x6^{2} + 1.00504 x1^{2} x2^{3} x3^{2} x6^{2} + 0.0294108 x1 x2^{4} x3^{2} x6^{2} + 0.177674 x2^{5} x3^{2} x6^{2} - 0.517135 x1^{4} x3^{3} x6^{2} + 0.473963 x1^{3} x2 x3^{3} x6^{2} + 1.07212 x1^{2} x2^{2} x3^{3} x6^{2} + 0.263449 x1 x2^{3} x3^{3} x6^{2} + 0.422661 x2^{4} x3^{3} x6^{2} + 0.115898 x1^{3} x3^{4} x6^{2} + 0.140353 x1^{2} x2 x3^{4} x6^{2} + 0.522825 x1 x2^{2} x3^{4} x6^{2} + 0.674321 x2^{3} x3^{4} x6^{2} + 0.128017 x1^{2} x3^{5} x6^{2} + 0.142336 x1 x2 x3^{5} x6^{2} + 0.22277 x2^{2} x3^{5} x6^{2} + 0.148215 x1 x3^{6} x6^{2} + 0.221444 x2 x3^{6} x6^{2} + 0.162563 x3^{7} x6^{2} - 0.535026 x1^{7} x6 x7 - 0.271346 x1^{6} x2 x6 x7 + 0.836493 x1^{5} x2^{2} x6 x7 + 1.01847 x1^{4} x2^{3} x6 x7 + 1.34652 x1^{3} x2^{4} x6 x7 + 0.435864 x1^{2} x2^{5} x6 x7 + 0.500435 x1 x2^{6} x6 x7 + 0.102052 x2^{7} x6 x7 - 0.698735 x1^{6} x3 x6 x7 - 0.547846 x1^{5} x2 x3 x6 x7 + 0.772938 x1^{4} x2^{2} x3 x6 x7 + 1.08812 x1^{3} x2^{3} x3 x6 x7 + 0.812676 x1^{2} x2^{4} x3 x6 x7 - 0.417988 x1 x2^{5} x3 x6 x7 - 0.0626312 x2^{6} x3 x6 x7 - 0.831582 x1^{5} x3^{2} x6 x7 + 0.411994 x1^{4} x2 x3^{2} x6 x7 + 1.33165 x1^{3} x2^{2} x3^{2} x6 x7 +$

Figure 15.3.: Computation of the model presented in chapter 16 - page 1.

		$0.985085\,x1^2\,x2^3\,x3^2\,x6\,x7 + 0.339257\,x1\,x2^4\,x3^2\,x6\,x7 + 0.126454\,x2^5\,x3^2\,x6\,x7 -$ $0.522241\,x1^4\,x3^3\,x6\,x7 - 0.353962\,x1^3\,x2\,x3^3\,x6\,x7 + 0.137949\,x1^2\,x2^2\,x3^3\,x6\,x7 +$ $0.63582\,x1\,x2^3\,x3^3\,x6\,x7 + 0.569253\,x2^4\,x3^3\,x6\,x7 - 0.196679\,x1^3\,x3^4\,x6\,x7 +$ $0.280202\,x1^2\,x2\,x3^4\,x6\,x7 - 0.485726\,x1\,x2^2\,x3^4\,x6\,x7 + 0.310395\,x2^3\,x3^4\,x6\,x7 -$ $0.753854\,x1^2\,x3^5\,x6\,x7 - 0.307724\,x1\,x2\,x3^5\,x6\,x7 + 0.174523\,x2^2\,x3^5\,x6\,x7 - 0.603937\,x1\,x3^6\,x6\,x7 +$ $0.312672\,x2\,x3^6\,x6\,x7 + 0.0898299\,x3^7\,x6\,x7 - 0.144589\,x1^7\,x7^2 - 0.0175113\,x1^6\,x2\,x7^2 +$ $0.490579\,x1^5\,x2^2\,x7^2 + 0.738952\,x1^4\,x2^3\,x7^2 + 0.92061\,x1^3\,x2^4\,x7^2 + 0.412399\,x1^2\,x2^5\,x7^2 +$ $0.117152\,x1\,x2^6\,x7^2 + 0.06701\,x2^7\,x7^2 - 0.394204\,x1^6\,x3\,x7^2 - 0.560141\,x1^5\,x2\,x3\,x7^2 +$ $0.381174\,x1^4\,x2^2\,x3\,x7^2 + 0.491803\,x1^3\,x2^3\,x3\,x7^2 + 0.00612297\,x1^2\,x2^4\,x3\,x7^2 +$ $0.089792\,x1\,x2^5\,x3\,x7^2 - 0.508529\,x2^6\,x3\,x7^2 - 0.570922\,x1^5\,x3^2\,x7^2 - 0.000798621\,x1^4\,x2\,x3^2\,x7^2 +$ $0.669647\,x1^3\,x2^2\,x3^2\,x7^2 - 0.203938\,x1^2\,x2^3\,x3^2\,x7^2 - 0.0957055\,x1\,x2^4\,x3^2\,x7^2 -$ $0.252372\,x2^5\,x3^2\,x7^2 - 0.220926\,x1^4\,x3^3\,x7^2 + 0.0472718\,x1^3\,x2\,x3^3\,x7^2 - 0.201172\,x1^2\,x2^2\,x3^3\,x7^2 -$ $0.678661\,x1\,x2^3\,x3^3\,x7^2 + 0.343708\,x2^4\,x3^3\,x7^2 + 0.0546948\,x1^3\,x3^4\,x7^2 - 0.414094\,x1^2\,x2\,x3^4\,x7^2 -$ $0.785641\,x1\,x2^2\,x3^4\,x7^2 - 0.229731\,x2^3\,x3^4\,x7^2 - 0.601563\,x1^2\,x3^5\,x7^2 - 0.829329\,x1\,x2\,x3^5\,x7^2 -$ $0.36404\,x2^2\,x3^5\,x7^2 - 0.46628\,x1\,x3^6\,x7^2 - 0.58937\,x2\,x3^6\,x7^2 - 0.411845\,x3^7\,x7^2$
L1	{4, 0, -4}	-
L2	{-9, 0, 14}	-
L3	{-4, 0, 10}	-

Global sections defining the complete intersection subvariety.

Start computation on C10 curve

Computation started

$\mathcal{O}(0,-2,-7)$	0	0	6	0	0
$\downarrow$ $\begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix}$ $\downarrow$	Ker = 0	Ker = 0	Ker = 0	Ker = 0	Ker = 0
$\mathcal{O}(2,-2,-6)\oplus\mathcal{O}(1,-2,-6)\oplus\mathcal{O}(1,0,-6)$	0	15	45	0	0
$\downarrow$ $\begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix}$ $\downarrow$	Ker = 0	Ker = 0	Ker = 6	Ker = 0	Ker = 0
$\mathcal{O}(3,-2,-5)\oplus\mathcal{O}(3,0,-5)\oplus\mathcal{O}(2,0,-5)$	0	64	40	0	0
$\downarrow$ $\begin{pmatrix} s1 & s2 & s3 \end{pmatrix}$ $\downarrow$	Ker = 0	Ker = 19	Ker = 40	Ker = 0	Ker = 0
$\mathcal{O}(4,0,-4)$	0	45	0	0	0
	H^0	H^1	H^2	H^3	H^4

First sheet of the Koszul exact sequence and the maps therein.

Figure 15.4.: Computation of the model presented in chapter 16 - page 2.

$\mathcal{O}(0,-2,-7)$	0	0	0	0	0
$\downarrow \begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix} \downarrow$					
$\mathcal{O}(2,-2,-6)\oplus\mathcal{O}(1,-2,-6)\oplus\mathcal{O}(1,0,-6)$	0	0	0	0	0
$\downarrow \begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix} \downarrow$					
$\mathcal{O}(3,-2,-5)\oplus\mathcal{O}(3,0,-5)\oplus\mathcal{O}(2,0,-5)$	0	4	1	0	0
$\downarrow (\,s1 \;\; s2 \;\; s3\,) \downarrow$					
$\mathcal{O}(4,0,-4)$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Second sheet of the Koszul exact sequence and the maps therein.

```
Computation finished after 2.9328052 seconds.
Start computation on C5m curve
Computation started
```

$\mathcal{O}(-15,-2,11)$	0	0	0	1092	0
$\downarrow \begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix} \downarrow$	↓ Ker = 0 ↓	↓ Ker = 0 ↓	↓ Ker = 0 ↓	↓ Ker = 0 ↓	↓ Ker = 0 ↓
$\mathcal{O}(-11,-2,12)\oplus\mathcal{O}(-14,-2,12)\oplus\mathcal{O}(-14,0,12)$	0	0	1014	1599	0
$\downarrow \begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix} \downarrow$	↓ Ker = 0 ↓	↓ Ker = 0 ↓	↓ Ker = 6 ↓	↓ Ker = 1095 ↓	↓ Ker = 0 ↓
$\mathcal{O}(-10,-2,13)\oplus\mathcal{O}(-10,0,13)\oplus\mathcal{O}(-13,0,13)$	0	0	1428	504	0
$\downarrow (\,s1 \;\; s2 \;\; s3\,) \downarrow$	↓ Ker = 0 ↓	↓ Ker = 0 ↓	↓ Ker = 1008 ↓	↓ Ker = 504 ↓	↓ Ker = 0 ↓
$\mathcal{O}(-9,0,14)$	0	0	420	0	0
	H^0	H^1	H^2	H^3	H^4

First sheet of the Koszul exact sequence and the maps therein.

Figure 15.5.: Computation of the model presented in chapter 16 - page 3.

$\mathcal{O}(-15,-2,11)$	0	0	0	0	0
$\downarrow \begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix} \downarrow$					
$\mathcal{O}(-11,-2,12)\oplus\mathcal{O}(-14,-2,12)\oplus\mathcal{O}(-14,0,12)$	0	0	6	3	0
$\downarrow \begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix} \downarrow$					
$\mathcal{O}(-10,-2,13)\oplus\mathcal{O}(-10,0,13)\oplus\mathcal{O}(-13,0,13)$	0	0	0	0	0
$\downarrow (\, s1 \;\; s2 \;\; s3 \,) \downarrow$					
$\mathcal{O}(-9,0,14)$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Second sheet of the Koszul exact sequence and the maps therein.

```
Computation finished after 1044.4199890 seconds.
Start computation on C5H curve
Computation started
```

$\mathcal{O}(-13,-2,6)$	0	0	0	462	0
$\downarrow \begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix} \downarrow$	Ker = 0	Ker = 0	Ker = 0	Ker = 0	Ker = 0
$\mathcal{O}(-6,-2,8)\oplus\mathcal{O}(-12,-2,7)\oplus\mathcal{O}(-12,0,7)$	0	0	440	530	0
$\downarrow \begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix} \downarrow$	Ker = 0	Ker = 0	Ker = 9	Ker = 470	Ker = 0
$\mathcal{O}(-5,-2,9)\oplus\mathcal{O}(-5,0,9)\oplus\mathcal{O}(-11,0,8)$	0	0	465	60	0
$\downarrow (\, s1 \;\; s2 \;\; s3 \,) \downarrow$	Ker = 0	Ker = 0	Ker = 432	Ker = 60	Ker = 0
$\mathcal{O}(-4,0,10)$	0	0	33	0	0
	H^0	H^1	H^2	H^3	H^4

First sheet of the Koszul exact sequence and the maps therein.

Figure 15.6.: Computation of the model presented in chapter 16 - page 4.

$\mathcal{O}(-13,-2,6)$	0	0	0	0	0
$\downarrow \begin{pmatrix} s3 \\ -s2 \\ s1 \end{pmatrix} \downarrow$					
$\mathcal{O}(-6,-2,8)\oplus\mathcal{O}(-12,-2,7)\oplus\mathcal{O}(-12,0,7)$	0	0	9	8	0
$\downarrow \begin{pmatrix} -s2 & -s3 & 0 \\ s1 & 0 & -s3 \\ 0 & s1 & s2 \end{pmatrix} \downarrow$					
$\mathcal{O}(-5,-2,9)\oplus\mathcal{O}(-5,0,9)\oplus\mathcal{O}(-11,0,8)$	0	0	1	0	0
$\downarrow \begin{pmatrix} s1 & s2 & s3 \end{pmatrix} \downarrow$					
$\mathcal{O}(-4,0,10)$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Second sheet of the Koszul exact sequence and the maps therein.

```
Computation finished after 181.1066774 seconds.
Out[131]= Finished the computation after 1229.1178728 seconds.
```

Figure 15.7.: Computation of the model presented in chapter 16 - page 5.

16. Application To Model Building - A Teaser

16.1. Summary

In section 1.3 we discussed a global Tate model with $SU(5) \times U(1)_X$ gauge symmetry. This type of F-theory model can be used for GUT-model building. Recall in particular that we pointed out that pullback cohomologies of a certain line bundle count the number of the states that experiments would observe as electrons, quarks etc. To conclude this thesis we therefore decide to put the developed technology about the computation of pullback cohomologies to a use in a model building teaser.
In section 16.2 we give a cooking recipe for this model. In particular we mention that a pseudo-random representative for such a model can be generated by our *Mathematica* notebook. The implemented functionality is described in subsection 15.2.5. We run a scan over roughly $45.8 \cdot 10^6$ parameter values that describe these kinds of models in the toric ambient space $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Of those parameter values only 20 give distinct candidate models. We give the details on this scan in section 16.3. Finally we exemplify the needed cohomology computations on the 10-th model that the scan found. This analysis is presented in section 16.4.
We have to mention that the ability to resolve the singularity structure is crucial in F-theory model building. As our intention is to exemplify the application of the computational methods on pullback cohomologies, we do not check if such a resolution does exist. Similarly we do not check Tadpole cancellations. In fact we focus entirely on the cohomological aspects of the model and leave further checks to the future. In this sense the presented models are toy-models only.

16.2. Setup For $\mathrm{SU}(5) \times \mathrm{U}(1)_{\mathrm{X}}$-Models With Line Bundle G_4-Flux

Construction 16.2.1 (Pick A Geometric Backbone, ...)**:**

1. Pick a smooth and compact normal toric varietyX_Σ of complex dimension 4. This toric variety we specify by means of a homogenisation, i.e.

$$X_\Sigma \cong \left(\mathbb{C}^r - Z\right) / \left(\left(\mathbb{C}^*\right)^a\right) \tag{16.1}$$

The $(\mathbb{C}^*)^a$ action is given by an $a \times r$ integer-valued matrix M. If we sum the entries in each row of M we obtain a vector

$$T \in \mathbb{Z}^a \tag{16.2}$$

In particular we can consider the divisor class

$$D_T := T^t \in \mathrm{Cl}(X_\Sigma) \tag{16.3}$$

2. Next consider an effective divisor $D_{B_3} \in \mathrm{Cl}(X_\Sigma)$ and $\widetilde{s}_{B_3} \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_{B_3}))$ a holomorphic section. Then we define

$$B_3 := \{p \in X_\Sigma \, , \, \widetilde{s}_{B_3}(p) = 0\} \tag{16.4}$$

We require that B_3 is an algebraic submanifold of codimension 1 in X_Σ. B_3 is termed the 'base'. Subsequently we consider the divisor class

$$\overline{K}_{B_3} = T^t - D_{B_3} \in \mathrm{Cl}(X_\Sigma) \tag{16.5}$$

which gives rise to an isomorphism class of holomorphic line bundles

$$\mathcal{L}_{\overline{K}_{B_3}} = \mathcal{O}_{X_\Sigma}\left(\overline{K}_{B_3}\right) \in \mathrm{Pic}(X_\Sigma) \tag{16.6}$$

We claim that $\mathcal{L}_{\overline{K}_{B_3}}\Big|_{B_3}$ is isomorphic to the anticanonical bundle of B_3.
The proof of this statements rests on the adjunction formula [44] which states that in this particular situation

$$K_{B_3} = K_{X_\Sigma}|_{B_3} \otimes N_{B_3/X_\Sigma}\big|_{B_3} \tag{16.7}$$

General theory of toric varieties implies $K_{X_\Sigma} = \mathcal{O}_{X_\Sigma}(-T^t)$ [1]. Moreover we have $N_{B_3/X_\Sigma}\big|_{B_3} = \mathcal{O}_{X_\Sigma}(D_{B_3})|_{B_3}$. Consequently

$$K_{B_3} = \mathcal{O}_{X_\Sigma}(-T^t + D_{B_3})|_{B_3} = \mathcal{L}_{\overline{K}_{B_3}}\Big|_{B_3} \tag{16.8}$$

which proves the claim. In particular this justifies our notation.

Construction 16.2.2 (. . . Form A Global Tate Model, . . .)**:**
In this geometry we now construct a global Tate model.

1. We require that the divisor class $\overline{K}_{B_3}$ is effective. This enables us to consider for $i \in \{1, 2, 3, 4, 6\}$

$$\widetilde{a}_i \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(i\overline{K}_{B_3}\right)\right) \tag{16.9}$$

Note that this implies

$$a_i := \widetilde{a}_i|_{B_3} \in H^0\left(B_3, \mathcal{L}^{\otimes i}_{\overline{K}_{B_3}}\Big|_{B_3}\right) \tag{16.10}$$

[1] The more familiar case is the canonical bundle of complex projective space. For this situation it holds $K_{\mathbb{CP}^n} = \mathcal{O}_{\mathbb{CP}^n}(-(n+1))$.

2. Next let $[x,y,z] \in \mathbb{CP}_{2,3,1}$ the homogeneous coordinates and $p \in B_3$. Then we set

$$\mathcal{C}_p := \{[x,y,z] \in \mathbb{CP}_{2,3,1} \, , \; P(x,y,z,p) = 0\} \tag{16.11}$$

where

$$\begin{aligned} P(x,y,z,p) := & x^2 - y^2 + xyza_1(p) + x^2z^2a_2(p) + yz^3a_3(p) \\ & + xz^4a_4(p) + z^6a_6(p) \end{aligned} \tag{16.12}$$

is the Tate polynomial. Thereby we have obtained a family of elliptic curves labeled by points $p \in B_3$.

3. Finally we set

$$Y_4 := \bigcup_{p \in B_3} C_p \tag{16.13}$$

so that we can define the holomorphic projection map

$$\pi \colon Y_4 \to B_3 \tag{16.14}$$

with the property $\pi^{-1}(p) = \mathcal{C}_p$ for any $p \in B_3$.

Let us mention two things here. First of all note that the above elliptic fibration is a special form of the Weierstrass form, which we described in section 1.1. This special form of elliptic fibrations is called a global Tate form elliptic fibration. Secondly we highlight that the motivation to study such a global Tate model is that the singularity structure of Y_4 is much easier worked out than in the Weierstrass model. This has been described in detail in [14]. It is this simplifcation that allows to easily work out an $SU(5) \times U(1)_X$ gauge symmetry.

Construction 16.2.3 (... Shed Some SU (5) In This Model, ...)**:**

1. As a next step consider an effective divisor $D_{\mathrm{GUT}} \in \mathrm{Cl}(X_\Sigma)$ and a non-trivial holomorphic section $\widetilde{s}_{\mathrm{GUT}} \in H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_{\mathrm{GUT}}))$. Then consider

$$\mathrm{GUT} := \{p \in X_\Sigma \, , \; \widetilde{s}_{B_3}(p) = \widetilde{s}_{\mathrm{GUT}}(p) = 0\} \subset B_3 \tag{16.15}$$

We require that this is an algebraic submanifold of codimension 2 in X_Σ. This manifold we term 'the GUT'. Moreover we set

$$w \colon B_3 \to \mathbb{C} \, , \; p \mapsto \widetilde{s}_{\mathrm{GUT}}(p) \tag{16.16}$$

2. As pointed out in [14] for an $SU(5)$ singularity structure along the GUT, we have to require that the holomorphic functions a_2, a_3, a_4 and a_6 factor according to

$$a_2 = a_{2,1}w, \quad a_3 = a_{3,2}w^2, \quad a_4 = a_{4,3}w^3, \quad a_6 = a_{6,5}w^5 \tag{16.17}$$

where $a_{i,j} := \widetilde{a}_{i,j}|_{B_3}$ for

$$\widetilde{a}_{i,j} \in H^0\left(X_\Sigma, \mathcal{O}_{X_\Sigma}\left(i \cdot \overline{K}_{B_3} - j \cdot D_{\mathrm{GUT}}\right)\right) \tag{16.18}$$

such that all $a_{i,j}$ are not diviable by w in the ring $\mathcal{O}_{B_3}(B_3)$.

3. Finally we consider the following codimension 3 curves of X_Σ.
   - $C_{10} := \{p \in X_\Sigma \ , \ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\text{GUT}}(p) = \widetilde{a}_1(p) = 0\}$
   - $C_{\overline{5}m} := \{p \in X_\Sigma \ , \ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\text{GUT}}(p) = \widetilde{a}_{3,2}(p) = 0\}$
   - $C_{5H} := \{p \in X_\Sigma \ , \ \widetilde{s}_{B_3}(p) = \widetilde{s}_{\text{GUT}}(p) = \widetilde{a}_{3,2}(p) \cdot \widetilde{a}_{2,1}(p) - \widetilde{a}_{4,3}(p) \cdot \widetilde{a}_1(p) = 0\}$

   Note that they are all required to be smooth and of codimension 3 in X_Σ. We term these complex curves the 10-curve, the $\overline{5}m$-curve and the $5H$-curve respectively.
4. The homogeneity of $\widetilde{a}_1$, $\widetilde{a}_{3,2}$ and $\widetilde{a}_{3,2}\widetilde{a}_{2,1} - \widetilde{a}_{4,3}\widetilde{a}_1$ canonically induce divisor classes on X_Σ, which we term $D_{C_{10}}$, $D_{C_{\overline{5}m}}$ and $D_{C_{5H}}$.

Construction 16.2.4 (...Restrict To An $SU(5) \times U(1)_X$ Model, ...)**:**

1. We restrict the form of the elliptic fibration further, so as to enlarge the gauge group.
2. Enlarging $SU(5)$ to $SU(5) \times U(1)_X$ is most easily achieved by requiring $a_6 \equiv 0$. This type of model is known as a $U(1)$ *restricted model*. More details on this type of model can be found in [46].

Construction 16.2.5 (...And Finally Add G_4-Flux To It)**:**

1. Last but not least, we add G_4-flux to the model. Hence this is the point where a special form of G_4-flux kicks in. We mentioned that in the above type of model one can pick holomorphic line bundles on the GUT [37] to form special G_4-fluxes. Here we will be even more special, in that we choose a holomorphic line bundle on X_Σ. Via pullback onto the GUT this gives us a special holomorphic line bundle on the GUT and therefore a very special G_4-flux.
   So let $D \in \text{Cl}(X_\Sigma)$ [2]. The associated holomorphic line bundle $\widetilde{\mathcal{L}} = \mathcal{O}_{X_\Sigma}(D)$ then plays the role of a G_4-flux. Note that in [47] the line bundle $\widetilde{\mathcal{L}}$ is refered to as F_X.
2. Next let us compute the canonical bundle of C_{10}. To this end we use the adjunction formula twice. First we realise

$$K_{C_{10}} = K_{\text{GUT}}|_{C_{10}} \otimes N_{C_{10}/\text{GUT}}\big|_{C_{10}} \tag{16.19}$$

   with $N_{C_{10}/\text{GUT}}\big|_{C_{10}} = \mathcal{O}_{X_\Sigma}(D_{C_{10}})|_{C_{10}}$. Secondly we have again from the adjunction formula

$$K_{\text{GUT}} = K_{B_3}|_{\text{GUT}} \otimes N_{\text{GUT}/B_3}\big|_{\text{GUT}} \tag{16.20}$$

   By using $K_{B_3} = \mathcal{O}_{X_\Sigma}(-T^t + D_{B_3})|_{B_3}$ we then find

$$K_{\text{GUT}} = \mathcal{O}_{X_\Sigma}(D_{\text{GUT}} + D_{B_3} - T^t)|_{\text{GUT}} \tag{16.21}$$

   From $K_{B_3} = D_{B_3} - T^t = -\overline{K}_{B_3}$ we finally conclude

$$K_{C_{10}} = \mathcal{O}_{X_\Sigma}\left(D_{\text{GUT}} - \overline{K}_{B_3} + D_{C_{10}}\right)\big|_{C_{10}} \tag{16.22}$$

[2] This divisor class should not be effective, as otherwise supersymmetry constraints are hard to satsify.

3. Motivated by this finding we focus on the following three formal [3] divisor classes
   - $D_1 = -1D + \frac{1}{2}\left(D_{C_{10}} + D_{\mathrm{GUT}} - \overline{K}_{B_3}\right)$
   - $D_2 = 3D + \frac{1}{2}\left(D_{C_{\overline{5}m}} + D_{\mathrm{GUT}} - \overline{K}_{B_3}\right)$
   - $D_3 = 2D + \frac{1}{2}\left(D_{C_{5H}} + D_{\mathrm{GUT}} - \overline{K}_{B_3}\right)$

   Note that the second part $\frac{1}{2}(\dots)$ resembles in analogy to the type IIB-picture a formal spin divisor. This observation is based on the fact that on a connected and compact Riemann surface M_g of genus g the spin structures are one-to-one with holomorphic line bundles S with the property $S^{\otimes 2} = K_{M_g}$ as proven in [43].
4. As a next step we implement the Freed-Witten quantisation condition. In this particular context this requires that D_1, D_2 and D_3 are divisor classes, i.e. canonically identified with integer valued vectors of length a.
5. Given Freed-Witten quantisation, we can thus consider the holomorphic line bundles $\widetilde{\mathcal{L}}_i = \mathcal{O}_{X_\Sigma}(D_i)$ on X_Σ. We are then interested in the cohomologies of the following three holomorphic line bundles.
   - $\mathcal{L}_1 := \widetilde{\mathcal{L}}_1\big|_{C_{10}}$
   - $\mathcal{L}_2 := \widetilde{\mathcal{L}}_2\big|_{C_{\overline{5}m}}$
   - $\mathcal{L}_3 := \widetilde{\mathcal{L}}_3\big|_{C_{5H}}$

   Note that it is this final step that needs the computation of cohomologies of pullback line bundles. It is hence the ability to perform this task that most of the work in this thesis focused on.
6. Finally we mention that for model-building purposes we are mostly interested in situations, such that the cohomologies are according to one of the following two cases.
   - Case 1:
     $h^0(C_{10}, \mathcal{L}_1) = 3$ and $h^1(C_{10}, \mathcal{L}_1) = 0$
     $h^0(C_{\overline{5}m}, \mathcal{L}_2) = 3$ and $h^1(C_{\overline{5}m}, \mathcal{L}_2) = 0$
     $h^0(C_{5H}, \mathcal{L}_3) = 1$ and $h^1(C_{5H}, \mathcal{L}_3) = 1$
   - Case 2:
     $h^0(C_{10}, \mathcal{L}_1) = 0$ and $h^1(C_{10}, \mathcal{L}_1) = 3$
     $h^0(C_{\overline{5}m}, \mathcal{L}_2) = 0$ and $h^1(C_{\overline{5}m}, \mathcal{L}_2) = 3$
     $h^0(C_{5H}, \mathcal{L}_3) = 1$ and $h^1(C_{5H}, \mathcal{L}_3) = 1$

16.3. A Scan On $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$

Note:
A particularly simply ambient space to handle is the smooth and compact normal

[3]This means that a priori they are half-integer valued vectors of length a. So up to $\frac{1}{2}$ we can always identify them with a divisor class in X_Σ. Hence they are not quite canonically isomorphic to divisor classes of X_Σ.

toric variety $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Recall in particular that $\mathbb{CP}^n$ is a Flag variety. Hence we know that the cohomology classes of holomorphic line bundles on $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ come equipped with natural tensor structures that we can use to construct the maps d_r with $r \geq 2$ in the Koszul spectral sequence, as we described in chapter 14. Let us therefore exemplify the $SU(5) \times U(1)_X$-models presented in section 16.2 on this particular toric ambient space.

Consequence:
To check for promising models on this ambient space, we should scan over *all* possible setups in $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ and pick the models of interest to us. Our strategy will be as follows.

Construction 16.3.1 (Finding Candidate Models On $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$)**:**
To find candidate models on $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ we scan over effective divisor classes D_{B_3}, D_{GUT}. Then we check if the divisor classes $\overline{K}_{B_3}$, $2\overline{K}_{B_3} - D_{\mathrm{GUT}}$, $3\overline{K}_{B_3} - 2D_{\mathrm{GUT}}$, $4\overline{K}_{B_3} - 3D_{\mathrm{GUT}}$, $D_{C_{10}}$, $D_{C_{\overline{5}m}}$ and $D_{C_{5H}}$ are all effective as well. If this is found to be true, we scan over G_4-fluxes $\widetilde{\mathcal{L}} = \mathcal{O}_{X_\Sigma}(D)$ and check if the divisor classes D_1, D_2 and D_3 are well-defined, that is if the Freed-Witten quantisation is satisfied. Given that this check is also passed we use the Koszul extension of the *cohomCalg* algorithm to compute the chiral index of the holomorphic line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ [4]. In case we find

$$|\chi(\mathcal{L}_1)| = 3 \ \wedge \ |\chi(\mathcal{L}_2)| = 3 \ \wedge \ \chi(\mathcal{L}_3) = 0 \tag{16.23}$$

this setup is a candidate model.

Note (Candidate Models on $\mathbb{CP}_2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$)**:**
We perform an explicit scan over the following parameter ranges.

- $D_{B_3} = (\alpha, \beta, \delta)$ with $0 \leq \alpha \leq 3$ and $0 \leq \beta, \delta \leq 2$.
- $D_{\mathrm{GUT}} = (\mu, \nu, \gamma)$ with $0 \leq \mu \leq 3$ and $0 \leq \nu, \gamma \leq 2$.
- $D = \frac{1}{2}(a, b, c)$ with $-20 \leq a \leq 0$ and $-20 \leq b, c \leq 20$.

This scan therefore checks roughly $45.8 \cdot 10^6$ configurations. It yields a number of *only* 20 disctinct candidate-models which we list in Table 16.1.

Remark:
Of course on each of these candidate models more checks need to be performed - we should check that all subvarieties are smooth and of the correct codimension and we should also compute the pullback cohomologies by use of the pullback-cohomology-computation-technology developed in this thesis.
The codimension checks are always easily performed in *Sage* [51]. Unfortunately however the polynomials defining the curves $\mathcal{C}_i$ grow large very fast. Even if we choose numerical prefactors to replace the complex valued coefficients C_i that encode

[4] As this task can be phrased as the calculation of certain intersection numbers, the performance of the scan can be increased by replacing the use of the Koszul extension of *cohomCalg* by use of some software that can compute intersection numbers in toric varieties, such as *Sage* [51].

a redundant description of the complex structure of $\mathcal{C}_i$, *Sage* [51] has to work very hard and long to check for smoothness. Therefore we were so-far only able to perform the smoothness checks on B_3, GUT and C_{10} in a timely fashion, whilst $C_{\overline{5}m}$ and C_{5H} usually take many days to be checked.
Similarly also the computation of the pullback cohomologies proves very hard in practice. This is for essentially two reasons.

- The cohomology classes on the E_1-sheet grow rapidely. In many of the candidate models that we will present, those cohomology groups reach dimensions of thousands to hundreds of thousands. Even though the performance of our *Mathematica* notebook was increased by a factor of 100, it still takes days to work out only the E_1-sheet for such big cohomology groups.
- The Koszul spectral sequences are to be worked out for codimension 3 loci. These spectral sequences do not converge on the E_1-sheet but the E_3-sheet. Under fortunate circumstances of course it can happen that the sequences converges on the E_1-sheet, but in general this is not the case. As we have not yet implemented a construction of the maps d_r with $r \geq 2$ into our notebook such computations need to be done by hand. This however is impractial even for cohomology classes of dimensions of 50 - 100, not to mention hundreds, thousands or hundreds of thousands. Therefore the current state of our *Mathematica* notebook allows only to find estimates for the cohomology classes from the computation of the E_1-sheet.

It is left for future work to overcome these limitations.

Note:
It is readily checked, that model #15 and #17 have $h^0(C_{5H}, \mathcal{L}_3) \geq 9$, whilst for model #16 we have $h^0(C_{5H}, \mathcal{L}_3) \geq 13$. Let us be very restrictive and not allow for any exotics. Then by these simple means we have just ruled out 3 of the 20 models. The analysis of the remaining 17 models is harder though. We leave this analysis for future work and decide to only give a baby-version of the analysis needed. To this end we have a closer look onto a special representative of model #10.

16.4. A Special Form Of Model #10

Note:
We now investigate the model #10 for a special set of sections $\widetilde{s}_{B_3}$, $\widetilde{s}_{GUT}$, $\widetilde{a}_i$, $\widetilde{a}_2$, $\widetilde{a}_3$ and $\widetilde{a}_4$. To this end we let *Mathematica* generate pseudo-random numbers for us, plug those into the corresponding polynomials and compute the pullback cohomologies without checking the smoothness conditions. This is what the function *Model* presented in subsection 15.2.5 does for us. The source code of this function can be found in Appendix E.

#	D_{B_3}	D_{GUT}	$2D$	$D_{C_{10}}$	$D_{\mathcal{L}_1}$	$D_{\bar{5}m}$	$D_{\mathcal{L}_2}$	D_{5H}	$D_{\mathcal{L}_3}$	Ruled out?
1	$(1,1,2)$	$(1,1,0)$	$(-17,-1,20)$	$(2,1,0)$	$(9,1,-10)$	$(4,1,0)$	$(-24,-1,30)$	$(7,2,0)$	$(-14,0,20)$	
2	$(1,1,2)$	$(1,1,0)$	$(-15,1,16)$	$(2,1,0)$	$(8,0,-8)$	$(4,1,0)$	$(-21,2,24)$	$(7,2,0)$	$(-12,2,16)$	
3	$(1,1,0)$	$(1,0,2)$	$(-15,14,14)$	$(2,1,2)$	$(8,-7,-6)$	$(4,3,2)$	$(-21,22,22)$	$(7,5,4)$	$(-12,16,16)$	
4	$(1,1,2)$	$(1,1,0)$	$(-13,3,12)$	$(2,1,0)$	$(7,-1,-6)$	$(4,1,0)$	$(-18,5,18)$	$(7,2,0)$	$(-10,4,12)$	
5	$(1,1,2)$	$(1,1,0)$	$(-11,5,8)$	$(2,1,0)$	$(6,-2,-4)$	$(4,1,0)$	$(-15,8,12)$	$(7,2,0)$	$(-8,6,8)$	
6	$(1,1,0)$	$(1,0,2)$	$(-9,2,18)$	$(2,1,2)$	$(5,-1,-8)$	$(4,3,2)$	$(-12,4,28)$	$(7,5,4)$	$(-6,4,20)$	
7	$(1,2,1)$	$(1,0,1)$	$(-9,4,7)$	$(2,0,1)$	$(5,-2,-3)$	$(4,0,1)$	$(-12,6,11)$	$(7,0,2)$	$(-6,4,8)$	
8	$(1,0,1)$	$(1,2,0)$	$(-9,6,10)$	$(2,2,1)$	$(5,-2,-5)$	$(4,2,3)$	$(-12,10,16)$	$(7,4,5)$	$(-6,8,12)$	
9	$(1,0,1)$	$(1,2,1)$	$(-7,-10,19)$	$(2,2,1)$	$(4,6,-9)$	$(4,2,1)$	$(-9,-14,29)$	$(7,4,2)$	$(-4,-8,20)$	
10	$(1,2,1)$	$(1,0,1)$	$(-7,0,9)$	$(2,0,1)$	$(4,0,-4)$	$(4,0,1)$	$(-9,0,14)$	$(7,0,2)$	$(-4,0,10)$	
11	$(1,2,1)$	$(1,0,1)$	$(-5,-4,11)$	$(2,0,1)$	$(3,2,-5)$	$(4,0,1)$	$(-6,-6,17)$	$(7,0,2)$	$(-2,-4,12)$	
12	$(1,0,1)$	$(1,2,1)$	$(-5,-2,9)$	$(2,2,1)$	$(3,2,-4)$	$(4,2,1)$	$(-6,-2,14)$	$(7,4,2)$	$(-2,0,10)$	
13	$(1,1,2)$	$(1,1,0)$	$(-3,-19,20)$	$(2,1,0)$	$(2,10,-10)$	$(4,1,0)$	$(-3,-28,30)$	$(7,2,0)$	$(0,-18,20)$	
14	$(1,2,1)$	$(1,0,1)$	$(-3,-8,13)$	$(2,0,1)$	$(2,4,-6)$	$(4,0,1)$	$(-3,-12,20)$	$(7,0,2)$	$(0,-8,14)$	
15	$(1,0,1)$	$(1,2,0)$	$(-3,-2,6)$	$(2,2,1)$	$(2,2,-3)$	$(4,2,3)$	$(-3,-2,10)$	$(7,4,5)$	$(0,0,8)$	Yes
16	$(1,1,0)$	$(1,0,2)$	$(-3,-2,10)$	$(2,1,2)$	$(2,1,-4)$	$(4,3,2)$	$(-3,-2,16)$	$(7,5,4)$	$(0,0,12)$	Yes
17	$(1,1,0)$	$(1,1,2)$	$(-3,-1,6)$	$(2,1,2)$	$(2,1,-2)$	$(4,1,2)$	$(-3,-1,10)$	$(7,2,4)$	$(0,0,8)$	Yes
18	$(1,1,2)$	$(1,1,0)$	$(-1,-17,16)$	$(2,1,0)$	$(1,9,-8)$	$(4,1,0)$	$(0,-25,24)$	$(7,2,0)$	$(2,-16,16)$	
19	$(1,2,1)$	$(1,0,1)$	$(-1,-12,15)$	$(2,0,1)$	$(1,6,-7)$	$(4,0,1)$	$(0,-18,23)$	$(7,0,2)$	$(2,-12,16)$	
20	$(1,1,0)$	$(1,1,2)$	$(-1,-11,14)$	$(2,1,2)$	$(1,6,-6)$	$(4,1,2)$	$(0,-16,22)$	$(7,2,4)$	$(2,-10,16)$	

Table 16.1.: Models with correct chiralities as found by our scan on $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ within the specified parameter ranges.

Construction 16.4.1 (Base)**:**
We pick the following function.

$$
\begin{aligned}
\widetilde{s}_{B_3} = {} & 0.989028379852613 x_1 x_4^2 x_6 + 0.734032360376816 x_2 x_4^2 x_6 \\
& + 0.938375320175355 x_3 x_4^2 x_6 + 0.339317919191924 x_1 x_4 x_5 x_6 \\
& + 0.649822960090401 x_2 x_4 x_5 x_6 + 0.582373999640213 x_3 x_4 x_5 x_6 \\
& + 0.353911797094248 x_1 x_5^2 x_6 + 0.160360451666156 x_2 x_5^2 x_6 \\
& + 0.267511914379556 x_3 x_5^2 x_6 + 0.110625148626521 x_1 x_4^2 x_7 \\
& + 0.117119296015655 x_2 x_4^2 x_7 + 0.0897198604905047 x_3 x_4^2 x_7 \\
& + 0.820490527048718 x_1 x_4 x_5 x_7 + 0.487051129121177 x_2 x_4 x_5 x_7 \\
& + 0.392781180105981 x_3 x_4 x_5 x_7 + 0.839333450337773 x_1 x_5^2 x_7 \\
& + 0.753182202403554 x_2 x_5^2 x_7 + 0.519756066830027 x_3 x_5^2 x_7
\end{aligned}
\tag{16.24}
$$

It is checked via *Sage* [51] that the base is smooth and of the correct codimension.

Construction 16.4.2 (GUT)**:**
We pick

$$
\begin{aligned}
\widetilde{s}_{\mathrm{GUT}} = {} & 0.2417425434195 95 x_1 x_6 + 0.801228849161860 x_2 x_6 \\
& + 0.432085830206161 x_3 x_6 + 0.300089241240703 x_1 x_7 \\
& + 0.286717978324859 x_2 x_7 + 0.807949402715996 x_3 x_7
\end{aligned}
\tag{16.25}
$$

Sage [51] confirmed that the GUT is smooth and of the correct codimension.

Construction 16.4.3 (The elliptic fibration)**:**
For the elliptic fibration we pick the following functions.

$$
\begin{aligned}
a_1 = {} & 0.665801789799261 x_1^2 x_6 + 0.669664904120052 x_1 x_2 x_6 \\
& + 0.318435395893956 x_2^2 x_6 + 0.829533654639629 x_1 x_3 x_6 \\
& + 0.865654510031526 x_2 x_3 x_6 + 0.0341480184712135 x_3^2 x_6 \\
& + 0.580463865504795 x_1^2 x_7 + 0.500320022518483 x_1 x_2 x_7 \\
& + 0.208509205382034 x_2^2 x_7 + 0.441200276585968 x_1 x_3 x_7 \\
& + 0.0552895324802994 x_2 x_3 x_7 + 0.969710856811344 x_3^2 x_7 \\
a_{2,1} = {} & 0.788002465577774 x_1^3 x_6 + 0.439069680539317 x_1^2 x_2 x_6 \\
& + 0.137986051013561 x_1 x_2^2 x_6 + 0.874619184264689 x_2^3 x_6 \\
& + 0.562300489147410 x_1^2 x_3 x_6 + 0.329839199646780 x_1 x_2 x_3 x_6 \\
& + 0.801050951715531 x_2^2 x_3 x_6 + 0.523939295616758 x_1 x_3^2 x_6 \\
& + 0.203937616308565 x_2 x_3^2 x_6 + 0.502034849368105 x_3^3 x_6 \\
& + 0.298737666994317 x_1^3 x_7 + 0.496643608788407 x_1^2 x_2 x_7 \\
& + 0.912400586812497 x_1 x_2^2 x_7 + 0.852814225518237 x_2^3 x_7 \\
& + 0.248915975392212 x_1^2 x_3 x_7 + 0.865092973825429 x_1 x_2 x_3 x_7
\end{aligned}
$$

$$
\begin{aligned}
&+ 0.690062985617382 x_2^2 x_3 x_7 + 0.897542003372793 x_1 x_3^2 x_7 \\
&+ 0.0822689553271077 x_2 x_3^2 x_7 + 0.484256802362263 x_3^3 x_7 \\
a_{3,2} &= 0.686598371466560 x_1^4 x_6 + 0.736877098213456 x_1^3 x_2 x_6 \\
&+ 0.519887608005832 x_1^2 x_2^2 x_6 + 0.631121503597466 x_1 x_2^3 x_6 \\
&+ 0.523916095921866 x_2^4 x_6 + 0.540171387853479 x_1^3 x_3 x_6 \\
&+ 0.774177382238416 x_1^2 x_2 x_3 x_6 + 0.478051853456718 x_1 x_2^2 x_3 x_6 \\
&+ 0.489079773624477 x_2^3 x_3 x_6 + 0.966896920963908 x_1^2 x_3^2 x_6 \\
&+ 0.337380994053327 x_1 x_2 x_3^2 x_6 + 0.487462632081411 x_2^2 x_3^2 x_6 \\
&+ 0.578893438222033 x_1 x_3^3 x_6 + 0.175482030248971 x_2 x_3^3 x_6 \\
&+ 0.0719383254616413 x_3^4 x_6 + 0.382073262023434 x_1^4 x_7 \\
&+ 0.692613178728784 x_1^3 x_2 x_7 + 0.373915048972292 x_1^2 x_2^2 x_7 \\
&+ 0.593037827943180 x_1 x_2^3 x_7 + 0.0579722711581433 x_2^4 x_7 \\
&+ 0.257991537789341 x_1^3 x_3 x_7 + 0.960658931792121 x_1^2 x_2 x_3 x_7 \\
&+ 0.452926172891729 x_1 x_2^2 x_3 x_7 + 0.267573462371527 x_2^3 x_3 x_7 \\
&+ 0.566377266632345 x_1^2 x_3^2 x_7 + 0.0828467155289336 x_1 x_2 x_3^2 x_7 \\
&+ 0.865574210986096 x_2^2 x_3^2 x_7 + 0.544972262424058 x_1 x_3^3 x_7 \\
&+ 0.0372063843633103 x_2 x_3^3 x_7 + 0.203286716867415 x_3^4 x_7 \\
a_{4,3} &= 0.200390298762100 x_1^5 x_6 + 0.686140096451894 x_1^4 x_2 x_6 \\
&+ 0.739728976539899 x_1^3 x_2^2 x_6 + 0.803247432595487 x_1^2 x_2^3 x_6 \\
&+ 0.504703388119129 x_1 x_2^4 x_6 + 0.557865317402868 x_2^5 x_6 \\
&+ 0.225701312612143 x_1^4 x_3 x_6 + 0.0464734863833506 x_1^3 x_2 x_3 x_6 \\
&+ 0.259250756947604 x_1^2 x_2^2 x_3 x_6 + 0.858641538134234 x_1 x_2^3 x_3 x_6 \\
&+ 0.0748948037831674 x_2^4 x_3 x_6 + 0.416652723219461 x_1^3 x_3^2 x_6 \\
&+ 0.971655475179566 x_1^2 x_2 x_3^2 x_6 + 0.965112632872295 x_1 x_2^2 x_3^2 x_6 \\
&+ 0.874558662460118 x_2^3 x_3^2 x_6 + 0.890657311506406 x_1^2 x_3^3 x_6 \\
&+ 0.706627228635256 x_1 x_2 x_3^3 x_6 + 0.131047290866268 x_2^2 x_3^3 x_6 \\
&+ 0.208293062208944 x_1 x_3^4 x_6 + 0.576212693410161 x_2 x_3^4 x_6 \\
&+ 0.967416113084352 x_3^5 x_6 + 0.190152053193513 x_1^5 x_7 \\
&+ 0.991630829448392 x_1^4 x_2 x_7 + 0.352306555543267 x_1^3 x_2^2 x_7 \\
&+ 0.993527895225648 x_1^2 x_2^3 x_7 + 0.788179133456002 x_1 x_2^4 x_7 \\
&+ 0.685392463899916 x_2^5 x_7 + 0.958791956684563 x_1^4 x_3 x_7 \\
&+ 0.680943123824964 x_1^3 x_2 x_3 x_7 + 0.540208674830397 x_1^2 x_2^2 x_3 x_7 \\
&+ 0.124729049492497 x_1 x_2^3 x_3 x_7 + 0.200958170773620 x_2^4 x_3 x_7 \\
&+ 0.356803465628049 x_1^3 x_3^2 x_7 + 0.0958530600359767 x_1^2 x_2 x_3^2 x_7 \\
&+ 0.819356258009017 x_1 x_2^2 x_3^2 x_7 + 0.268023672600283 x_2^3 x_3^2 x_7 \\
&+ 0.595208783195890 x_1^2 x_3^3 x_7 + 0.777925884306606 x_1 x_2 x_3^3 x_7
\end{aligned}
$$

$\mathcal{L}'$	0	0	6	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	15	45	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	64	40	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	45	0	0	0
	H^0	H^1	H^2	H^3	H^4

$\mathcal{L}'$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	4	1	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 16.2.: The E_1- and E_2-sheet in the calculation of $h^i\left(C_{10}, \mathcal{L}_1\right)$ in model #10 on the toric ambient space $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

$$
\begin{aligned}
&+ 0.736419813783850 x_2^2 x_3^3 x_7 + 0.862244928722523 x_1 x_3^4 x_7 \\
&+ 0.794027013298842 x_2 x_3^4 x_7 + 0.996627270641613 x_3^5 x_7 \\
a_6 &= 0
\end{aligned}
$$

Sage [51] verifies that all curves C_{10}, $C_{\overline{5}m}$ and C_{5H} are of the correct codimension. In addition we checked that C_{10} is smooth. The corresponding smoothness checks for $C_{\overline{5}m}$ and C_{5H} were cancelled after two days of running *Sage* [51] on a *Windows* 7 system with i7 quad-core processor in a virtual machine equipped with 3GB RAM and 1 CPU.

Consequence:
The above-presented sections describe a special representative geometry of model #10. We can thus proceed by computing the cohomologies of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. To this end we use our *Mathematica* notebook as presented in Appendix E to compute the E_2-sheet. We display the E_1-sheets as well as the generic E_2-sheets in Table 16.2, Table 16.3 and Table 16.4. From this we conclude

- $h^0\left(C_{10}, \widetilde{\mathcal{L}}_1\right) = 4$ and $h^1\left(C_{10}, \widetilde{\mathcal{L}}_1\right) = 1$
- $h^0\left(C_{\overline{5}m}, \widetilde{\mathcal{L}}_2\right) = 6$ and $h^1\left(C_{\overline{5}m}, \widetilde{\mathcal{L}}_2\right) = 3$
- $h^0\left(C_{5H}, \widetilde{\mathcal{L}}_3\right) = 9$ and $h^1\left(C_{5H}, \widetilde{\mathcal{L}}_3\right) = 9$

So this special representative of model #10 is ruled out if we allow for no exotics.

Remark:
A question that we have to leave open here is the following.

> Is it possible to choose other sections such that the cohomology groups on C_{10}, $C_{\overline{5}m}$ and C_{5H} are tuned to the desired values?

Let us mention though, that in principle this question is answered by the technology presented here. Answering it only hinges on sufficient computational power.

$\mathcal{L}'$	0	0	0	1092	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	0	1014	1599	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	0	1428	504	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	0	420	0	0
	H^0	H^1	H^2	H^3	H^4

$\mathcal{L}'$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	0	6	3	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 16.3.: The E_1- and E_2-sheet in the computation of $h^i\left(C_{\overline{5}m}, \mathcal{L}_2\right)$ in model #10 on the toric ambient space $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

$\mathcal{L}'$	0	0	0	462	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	0	440	530	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	0	465	60	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	0	33	0	0
	H^0	H^1	H^2	H^3	H^4

$\mathcal{L}'$	0	0	0	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_2$	0	0	9	8	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{V}_1$	0	0	1	0	0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{L}_1$	0	0	0	0	0
	H^0	H^1	H^2	H^3	H^4

Table 16.4.: The E_1- and E_2-sheet in the calculation of $h^i\left(C_{5H}, \mathcal{L}_3\right)$ in model #10 on the toric ambient space $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

Part V.

Conclusion, Outlook And Appendix

17. Conclusion And Outlook

Conclusions And Outlook

In type IIB and F-theory model building one wishes to compute the spectrum of massless string zero modes. This task leads under simplified assumptions to the computation of cohomology groups of pullback line bundles. In many applications to physics, the geometry is build from toric varieties. This is because toric varieties are mathematically well-understood and easy to handle. In particular one can write computer programs that compute intersection numbers, Chern classes, indices, ... A prominent example of such a software is *Sage* [51]. Consequently computing the spectrum in many model building applications boils down to understanding the answer to the following question.

> Given a smooth and compact normal toric variety X, a holomorphic line bundle $\mathcal{L}$ on X and $C \subset X$ a submanifold. How does one compute the cohomologies of $\mathcal{L}|_C$?

This is the question that we adressed in this thesis. Let us briefly recall the answer.

1. For every holomorphic line bundle $\mathcal{L}$ on X_Σ there exists a divisor class $D \in \mathrm{Cl}(X_\Sigma)$ with $\mathcal{L} = \mathcal{O}_{X_\Sigma}(D)$.
2. Every analytic submanifold $C \subset X_\Sigma$ turned out to be even an algebraic submanifold, i.e. cut out by a finite number of polynomials $Q_1, \ldots, Q_n$.
3. The cohomologies of $\mathcal{L}|_C$ are then related to the cohomologies of certain line bundles on X_Σ via the sheaf exact Koszul sequence

$$0 \to \mathcal{L}' \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_1 \to \mathcal{L} \to \mathcal{L}|_C \to 0 \tag{17.1}$$

   The ambient space cohomologies can be computed via the fast *cohomCalg* algorithm. Then a novel approach to computing the pullback cohomologies from this sequence is the use of exactness. This is implemented in [53]. We pointed out in chapter 9 that in general however, exactness is not enough to determine the pullback cohomologies.

4. The core of this thesis was then to go beyond these exactness calculations. To this end we presented in Part IV the Koszul spectral sequence and established it as the optimal approach to computing pullback cohomologies. In particular we present all the information that is needed to perform these computations in principle.

For practical applications in the model building, this principle knowledge is not enough. It must be possible to handle this computation even when the cohomology classes are hugh and one wishes to obtain the result in a timely fashion.
In a first attempt to achieve this goal we present a *Mathematica* notebook that enables us to compute the E_1-sheet of the Koszul spectral sequence. The source code is displayed in Appendix E and we present a brief manual in chapter 15.
Whilst this notebook enables us to compute a first approximation of the pullback cohomology groups it is not yet a full practical answer to the given question. Rather it needs to be extended such that is can also compute the higher sheets in the Koszul spectral sequence. For most model building applications it would even be sufficient to implement this functionality only up to the E_3-sheet. Unfortunately this extension faces the following two problems.

- On the one hand we could use the standard construction in order to obtain the d_r-maps with $r \geq 2$. This construction was described in section 13.2. In particular we pointed out that for this approach to work we need the information about the E_0-sheet. This information unfortunately is not accessible from the fast *cohomCalg* algorithm but needs to make use of the chamber counting.
  In conclusion this way of constructing the maps is certainly possible to implement in a computer. The calculation that we presented in section 13.4 should however motivate that implementing this functionality will take quite some time. In addition, based on the fact that the performance of the chamber counting algorithm presented in[61] performs slower than *cohomCalg*, one should expect that the so- obtained program would perform slowly too.
- In principle a slow program is not a problem, in practice however it is. So one would hope to find a faster algorithm. A hint towards such a faster construction is given in the mathematics literature where it reads that the maps d_r with $r \geq 2$ are 'natural'. But of course 'naturalness' is not accessible for a computer and one has to look more carefully into the construction to determine what 'natural' is suppose to mean.
  Given that we work on a toric variety X_Σ which is a generalised Flag variety at the same time, we know from the Bott-Borel-Weil theorem that the cohomology classes of holomorphic line bundles are labeled by representations of certain Lie groups. Intuitively this means that the cohomology groups come equipped with a tensor structure. In this context 'natural' then means, among others, that these tensor structures are to be respected.
  In [58] model building on direct products of $\mathbb{CP}^n$ have been performed. Such toric varieties are known to form generalised Flag varieties. Therefore a 'natural' construction for the maps d_r with $r \geq 2$ arises from the demand to respect the tensor structures.
  For model building however, a freedom in choice of ambient toric variety is desirable. Therefore one can ask the following question.

Is a smooth and compact normal toric variety X_Σ a generalised Flag variety?

> Our proposition is that the answer to this question is affirmative and we formulate a proposition for a simplified construction of the maps d_r with $r \geq 2$ in chapter 14.

Let us emphasise that these are propositions. Turning these propositions into solid statements is left for future work. In particular extending the *Mathematica* notebook to give a full practical answer to the computation of pullback cohomologies is left for future work.
Whilst the *Mathematica* notebook is to date not yet complete in the above sense, it can be used to go one step beyond the Koszul extension of *cohomCalg* [53]. We exemplified such an application in chapter 16. A particularly interesting application would be to exploit the proposal to understand the G_4-flux in F-theory from Chow groups and Deligne cohomology [37]. Such applications are reserved for future work.

Acknowledgements

The author thanks all members of the *String Theory and Physics Beyond the Standard Model* research group at Heidelberg university for the many useful discussions. Special thanks go to Christoph Mayrhofer for his expertise in toric geometry and *Sage* [51], Stefan Sjörs for discussions of holomorphic line bundle on $\mathbb{C}_{1,\tau}$, Lara Brian Anderson for a discussion of the higher differential maps in the Koszul spectral sequence and Eberhard Freitag, whose lectures on *Riemann surfaces* and *complex spaces* proved a perfect supplement to the topic of my thesis.
Finally I would like to thank my supervisor Timo Weigand for his endless patience, the many discussions, encouragements and helpful advices.

A. Line Bundles, Divisors And Chern Classes

A.1. (Pre-) Sheaves And Sheaf Cohomology

A.1.1. Introduction

We now give a brief introduction to the topic of sheaves. For alternative brief introductions consult [67, 38]. For a more careful treatment the interested reader is refered to [44, 66, 45, 71].

A.1.2. Presheaves

Definition A.1.1 (Presheaf)**:**
A presheaf F of Abelian groups on a topological space M is a map which assigns to every open subset $U \subset X$ an Abelian group $F(U)$ and to every pair U, V of open subsets of M with the property $V \subset U$ a group homomorphism

$$r_V^U : F(U) \to F(V) \tag{A.1}$$

such that

- $r_U^U = \mathrm{id}$ for all $U \subset M$ open
- for any three open subsets $W \subset V \subset U$ of M it holds $r_W^U = r_W^V \circ r_V^U$

Remark:
The notion of a presheaf can be defined on a given category with values in a second category. Whilst this point of view if far more general than the one given in the above definition, for our purposes though, it will suffice to work with the above definition.

Example A.1.1:
Consider a topolgical space M. Then define for $U \subset M$ open

$$F(U) := \{f : U \to \mathbb{C} \,,\ f \text{ is constant}\} \tag{A.2}$$

Any such set $F(U)$ forms an Abelian group. As restriction maps we pick the ordinary restriction of functions. Then this structure forms a presheaf.

Definition A.1.2 (Presheaf Homomorphsm)**:**
Let M a topological space and F, G presheaves on M. Then we define that a presheaf homomorphism $f{:}\, F \to G$ is a collection

$$\{f_U{:}\, F(U) \to G(U)\ ,\ U \subset M \text{ open }\} \tag{A.3}$$

of group homomorphisms, such that for any two $V \subset U \subset M$ open, the following diagram commutes.

$$\begin{tikzcd} F(U) \arrow[r, "f_U"] \arrow[d, "(r_F)^U_V"'] & G(U) \arrow[d, "(r_G)^U_V"] \\ F(V) \arrow[r, "f_V"] & G(V) \end{tikzcd}$$

Note:
There are natural notions of kernel, image and cokernel of a presheaf homomorphism. All of them are presheaves. In particular this allows to define the notion of a complex of presheaves.

Lemma A.1.1:
A sequence $0 \to F \to G \to H \to 0$ of presheaves F, G, H on a topological space M is presheaf exact precisely if for any $U \subset M$ open, the induced sequence

$$0 \to F(U) \to G(U) \to H(U) \to 0 \tag{A.4}$$

is exact.

A.1.3. Sheaves

Definition A.1.3 (Sheaf)**:**
A sheaf F of Abelian groups on a topological space M is a presheaf of Abelian groups, which satisfies in addition the following three conditions.

(S1) Let $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of $U \subset M$ open and $s, t \in F(U)$ with the property

$$s|_{U_i} = t|_{U_i} \qquad \forall i \in I \tag{A.5}$$

Then it holds $s = t$.

(S2) Let $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of $U \subset M$ open and $\{s_i \in F(U_i)\}_{i\in I}$ a family with the property

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \forall i, j \in I \tag{A.6}$$

Then there exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all $i \in I$.

(S3) $F(\varnothing)$ is the zero group.

Example A.1.2:
We first give an example of a presheaf that is not a sheaf. To this end let M a topological space which is not connected. Then the constant functions on M with the ordinary restriction of functions form a presheaf. However they do not form a sheaf. This can be seen as follows.
For simplicity let us assume $M = U \cup V$ with $U \cap V = \varnothing$ but U, V both connected. Then we consider the functions

$$s_0 \colon U \to \mathbb{C}\,,\ z \mapsto 0, \qquad s_1 \colon V \to \mathbb{C}\,,\ z \mapsto 1 \tag{A.7}$$

Consequently $s_0 \in F(U)$ and $s_1 \in F(V)$. This setup now satisfies the requirements for (S2) since $U \cap V = \varnothing$. However there does not exist a constant function

$$s \colon M \to \mathbb{C}\,,\ z \mapsto c, \qquad c \in \mathbb{C} \text{ constant} \tag{A.8}$$

with the property

$$s|_{U_0} = s_0, \qquad s|_{U_1} = s_1 \tag{A.9}$$

since $0 \neq 1$. Thus (S2) is not satisfied and the constant functions on M do not form a sheaf.

Remark:
If we replace the constant functions in the preceeding example by the *locally* constant functions with ordinary restriction of functions, then this structure does form a sheaf on M. This is a manifestation of the more general rule

Sheafifying means to make properties local.

Consequently the continous, the differentiable, the smooth and the holomorphic functions on M with ordinary restriction of functions do form sheaves on M as all these properties are local properties.

Notation:
Let M a complex manifold. Then $\mathcal{O}_M$ is the sheaf of holomorphic functions, $\mathcal{O}_M^*$ the sheaf of non-vanishing holomorphic functions and $\mathcal{M}_M^*$ the sheaf of meromorphic function on M which are not identically zero on any connected component of M. [1]

Definition A.1.4 (Sheaf Homomorphism)**:**
Let M a topological space and F, G two sheaves on M. Then a sheaf homomorphism

$$f \colon F \to G \tag{A.10}$$

is a homomorphism of presheaves.

[1] The symbol $\mathcal{O}$ is used to represent holomorphic functions in order to honour the great achievements of the japanese mathematician Kiyoshi Oka in complex analysis.

Note:
Let M a topological space, F, G two presheaves on M and $f: F \to G$ a presheaf homomorphism. Then there are natural notions of kernel, image and cokernel of f and all of them are presheaves. Given that F is a sheaf it turns out that $\ker(f)$ is a sheaf also. However, even if F and G are both sheaves, then the presheaf image need not be a sheaf.

Example A.1.3:
To illustrate this fact let us consider $M = \mathbb{C}$ and the sheaf homomorphism

$$\mathcal{O} \to \mathcal{O}^* \;,\; f \mapsto \exp(2\pi i f) \tag{A.11}$$

Let us then consider the open subset

$$U := \{z \in \mathbb{C} \;,\; |z| > 1\} \tag{A.12}$$

which we cover by

$$U_1 := \left\{z \in U \;,\; \mathrm{Re}(z) < \frac{1}{2}\right\}, \quad U_2 := \left\{z \in U \;,\; \mathrm{Re}(z) > -\frac{1}{2}\right\} \tag{A.13}$$

This is illustrated in Figure A.1. Now we consider the functions

$$f_1: U_1 \to \mathbb{C}^* \;,\; z \mapsto \frac{1}{z}, \quad f_2: U_2 \to \mathbb{C}^* \;,\; z \mapsto \frac{1}{z}, \tag{A.14}$$

Note that both functions agree on $U_1 \cap U_2$ and that both admit a holomorphic logarithm since U_1, U_2 are both simply connected. Therefore f_1, f_2 are in the image of the above sheaf homomorphism.
If the presheaf image was a sheaf, it would satisfy property (S2), i.e. since f_1, f_2 are local sections in the presheaf image there would exist a function $f: U \to \mathbb{C}^*$ with the properties

- $f|U_i = f_i$
- f admits a holomorphic logarithm

The first requirement implies $f: U \to \mathbb{C}^* \;,\; z \mapsto \frac{1}{z}$. This function however is known not to admit a holomorphic logarithm since U is not simply connected.

Lemma A.1.2 (Generated Sheaf)**:**
Let M a topological space and F a presheaf on M. Then there exists a smallest sheaf $\widehat{F}$ on M which contains F as a subpresheaf. $\widehat{F}$ is known as the *generated sheaf.*

Consequence:
One thus defines that for two sheaves F, G on a topological space M and a sheaf homomorphism $f: F \to G$ one has

- The sheaf image of f is the generated sheaf $\widehat{f(F)}$.

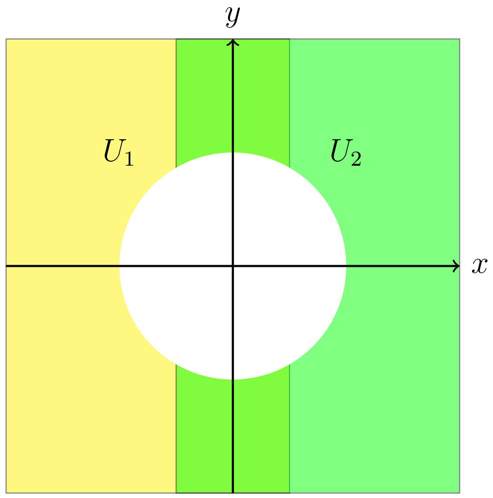


Figure A.1.: Subsets $U_1, U_2 \subset \mathbb{C}$ used to illustrate that the presheaf image need not be a sheaf in general.

- The sheaf cokernel of f is the generated sheaf $\widehat{\mathrm{coker}\,(f)}$.

Note:
This notion now allows for the defintion of complexes of sheaves.

Lemma A.1.3:
A sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ of sheaves on a topological space M is sheaf exact precisely if induced sequence of stalks

$$0 \to (F_1)_p \to (F_2)_p \to (F_3)_p \to 0 \tag{A.15}$$

is exact at every point $p \in M$.

Remark:
The above lemma is quite important in that it shows that sheaf exactness is a local property.

Sheaf exactness can be checked stalkwise.

A.1.4. Sheaf Cohomology

Lemma A.1.4 (Godement Resolution)**:**
Let F a sheaf of Abelian groups on a topological space M. Then F admits a Godement resolution, that is there exist a sheaf exact sequence

$$0 \to F \to F^{(0)} \to F^{(1)} \to F^{(2)} \to \dots \tag{A.16}$$

where the sheaves $F^{(i)}$ are all flabby and $F^{(0)}$ is the Godement sheaf of F. This resolution is known as the *Godement resolution* or the *canonical flabby resolution*.

Definition A.1.5 (Sheaf Cohomology):
By application of the global section functor Γ one can obtain a long sequence from the Godement resolution

$$0 \to \Gamma F^{(0)} \to \Gamma F^{(1)} \to \Gamma F^{(2)} \to \dots \tag{A.17}$$

which is no longer exact. One now defines the cohomology of the sheaf F as the cohomology of this long sequence and denotes it by $H^i(M, F)$.

Remark:
It holds $H^0(M, F) \cong \Gamma F = F(M)$.

Definition A.1.6 (Acyclic Sheaf):
A sheaf F of Abelian groups on a topological space M is acyclic precisely if

$$H^n(M, F) = 0 \quad \forall n \geq 0 \tag{A.18}$$

Definition A.1.7 (Acyclic Resolution):
Let F a sheaf of Abelian groups on a topological space M. An acyclic resolution of F is a long sheaf exact sequence

$$0 \to F \to F_0 \to F_1 \to F_2 \to \dots \tag{A.19}$$

where the sheaves F_i are acyclic.

Remark:
The cohomology of a sheaf F of Abelian groups on a topological space M can be computed from any acyclic resolution of F.

Note:
In general the computation of the cohomology groups of a sheaf is involved. However, under certain nice conditions, one can compute sheaf cohomology from Čech cohomology. Let us emphasise though, that Čech cohomology and sheaf cohomology are by no means the same, but completely different structures that happen to give the same answer under special circumstances.

A.2. Čech Cohomology

A.2.1. Introduction

In this section we will introduce the notion of Čech cohomology. For alternative brief introductions to the topic we refer the intersted reader to [67] and [38]. More careful treatments are to be found in [44], [66], [45] and [71].

A.2.2. Čech Cohomology - Definitions

Definition A.2.1 (Čech Cochain)**:**
Let M a complex manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open cover of M. We consider the sheaf $\mathcal{O}_M^*$ on M and introduce the notion of Čech cohomology for that sheaf. To this end we make the following definitions.

- $U_{\alpha_0,\dots,\alpha_p} := U_{\alpha_0}\cap\dots\cap U_{\alpha_p}$. This set is open by the definition of topology since all sets U_α are open.
- For $p\in\mathbb{N}\cup\{0\}$ we define $\mathcal{I}^p := \{(\alpha_0,\dots,\alpha_p)\in I^{p+1}\ ,\ \alpha_i\neq\alpha_j\forall i,j\in\{0,\dots,p\}\}$.
- Now we set
$$\check{\check{C}}^p(\mathcal{U},\mathcal{O}_M^*) := \prod_{(\alpha_0,\dots,\alpha_p)\in\mathcal{I}^p}\mathcal{O}_M^*\left(U_{\alpha_0,\dots,\alpha_p}\right) \tag{A.20}$$
An element in $\check{\check{C}}^p(\mathcal{U},\mathcal{O}_M^*)$ is thus a family $\left(s_{\alpha_0,\dots,\alpha_p}\in\mathcal{O}_M^*\left(U_{\alpha_0,\dots,\alpha_p}\right)\right)_{(\alpha_0,\dots,\alpha_p)\in\mathcal{I}^p}$. Still we want to impose the additional requirement of total antisymmetry, by which we mean here
$$s_{\alpha_0,\dots,\alpha_i,\dots,\alpha_j,\dots,\alpha_p} = \frac{1}{s_{\alpha_0,\dots,\alpha_j,\dots,\alpha_i,\dots,\alpha_p}} \tag{A.21}$$
We denote the so-obtained families by $\check{C}^p(\mathcal{U},\mathcal{O}_M^*)$.
- If the indexing set I is finite, then we define $\check{C}^p(\mathcal{U},\mathcal{O}_M^*)$ to be the trivial group for $p\geq|I|$.

Example A.2.1:
Let M a complex manifold with open cover $\mathcal{U} = \{U_0\}$. Then we have

- $\check{C}^0(\mathcal{U},\mathcal{O}_M^*) = \mathcal{O}_M^*(U_0)$ since the total antisymmetry is trivially satisfied in this case.
- $\check{C}^p(\mathcal{U},\mathcal{O}_M^*)$ is the trivial group for $p\geq 1$.

Example A.2.2:
Let M a complex manifold with open cover $\mathcal{U} = \{U_0, U_1\}$. Then we have

- $\check{C}^0(\mathcal{U},\mathcal{O}_M^*) = \mathcal{O}_M^*(U_0)\times\mathcal{O}_M^*(U_1)$. Thus to describe an element $x\in\check{C}^0(\mathcal{U},\mathcal{O}_M^*)$ we pick two holomorphic functions
$$f_0\colon U_0\to\mathbb{C}^*, \qquad f_1\colon U_1\to\mathbb{C}^* \tag{A.22}$$
and obtain $x = (f_0, f_1)$. Note that $x = x_\alpha$ has just one index, namely α, and that this index can take two different values, namely 0, 1. The antisymmetrisation condition that we impose does not operate on these different values that α can take but on different indices. Since x has just a single index, namely α, the antisymmetrisation condition is still trivial.

- $\check{\tilde{C}}^1(\mathcal{U}, \mathcal{O}_M^*) = \mathcal{O}_M^*(U_0 \cap U_1) \times \mathcal{O}_M^*(U_1 \cap U_0)$. Following the above procedure we describe an element $y = y_{\alpha,\beta} \in \check{\tilde{C}}^1(\mathcal{U}, \mathcal{O}_M^*)$ by picking holomorphic functions

$$f, g: U_0 \cap U_1 \to \mathbb{C}^* \tag{A.23}$$

so that $y_{01} = f$ and $y_{10} = g$. To finally obtain and element in $\check{C}^1(\mathcal{U}, \mathcal{O}_M^*)$ we impose in addition the total antisymmetry, i.e. we enforce

$$y_{\alpha\beta} = \frac{1}{y_{\beta\alpha}} \tag{A.24}$$

Thus we have to require $y_{00} = y_{11} = 0$. In addition we have $y_{01} = \frac{1}{y_{10}}$, or equivalently $f = \frac{1}{g}$. Consequently we can write

$$y = (y_{00}, y_{01}, y_{10}, y_{11}) = \left(0, f, \frac{1}{f}, 0\right) \in \check{C}^1(\mathcal{U}, \mathcal{O}_M^*) \tag{A.25}$$

for $f \in \mathcal{O}_M^*(U_0 \cap U_1)$, or in a more condensed notation

$$y = \left(f, \frac{1}{f}\right) \in \check{C}^1(\mathcal{U}, \mathcal{O}_M^*) \tag{A.26}$$

- All higher Čech cochain groups are trivial.

Definition A.2.2 (Boundary Operator)**:**
We wish to define a map

$$\delta_p: \check{C}^p(\mathcal{U}, \mathcal{O}_M^*) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{O}_M^*) \tag{A.27}$$

To this end recall that an element $a \in \check{C}^p(\mathcal{U}, \mathcal{O}_M^*)$ is a tuple with one entry $a_{\alpha_0,\ldots,\alpha_p}$ for each p-tuple $(\alpha_0, \ldots, \alpha_p) \in \mathcal{I}^p$ and the total antisymmetry imposed. Also recall that

$$a_{\alpha_0,\ldots,\alpha_p} \in \mathcal{O}_M^*\left(U_{\alpha_0,\ldots,\alpha_p}\right) \tag{A.28}$$

Then we define

$$\delta_p\left[\left(a_{\alpha_0,\ldots,\alpha_p}\right)_{(\alpha_0,\ldots,\alpha_p)\in\mathcal{I}^{p+1}}\right] := \left(\prod_{k=0}^{p+1} a^{(-1)^k}_{\alpha_0,\ldots,\widehat{\alpha}_k,\ldots,\alpha_{p+1}}\Big|_{U_{\alpha_0,\ldots,\alpha_{p+1}}}\right)_{(\alpha_0,\ldots,\alpha_{p+1})\in\mathcal{I}^{p+2}} \tag{A.29}$$

This family forms an element of $\check{C}^{p+1}(\mathcal{U}, \mathcal{O}_M^*)$ as is readily checked.

Remark:
We will supress the index p on δ_p and simply write δ instead.

Example A.2.3:
Let us consider a complex manifold M.

- Let $\mathcal{U} = \{U_0\}$ an open cover of M. Then by our earlier findings we know $a = (f_0) \in \check{C}^0(\mathcal{U}, \mathcal{O}_M^*)$ for $f_0 \in \mathcal{O}_M^*(U_0)$. Then

$$\delta a = \left(a_{\alpha_1} \cdot a_{\alpha_0}^{-1}\Big|_{U_{\alpha_0,\alpha_1}}, \, a_{\alpha_0} \cdot a_{\alpha_1}^{-1}\Big|_{U_{\alpha_0,\alpha_1}} \right)_{(\alpha_0,\alpha_1)\in\mathcal{I}^2} \tag{A.30}$$

  We see that the antisymmetry is taken care of automatically. Now note that since $I = \{0\}$ we must have $\alpha_0 = \alpha_1 = 0$. So

$$\delta a = (1, 1) \tag{A.31}$$

  which is the identity in the multiplicative group of pairs of nowhere vanishing holomorphic functions. Thus δa is trivial. This matches our earlier finding that for $\mathcal{U} = \{U_0\}$ the Čech cochains $\check{C}^p(\mathcal{U}, \mathcal{O}_M^*)$ are trivial for $p \geq 1$.

- Now let us consider another open cover of M, namely $\mathcal{U} = \{U_0, U_1\}$. In this case we have

$$a = (f_0, f_1) \in \check{C}^0(\mathcal{U}, \mathcal{O}_M^*) \tag{A.32}$$

  where $f_i \in \mathcal{O}_M^*(U_i)$. Following the above logic we then find

$$(\delta a) = \left(\frac{f_1}{f_0}\Big|_{U_{0,1}}, \frac{f_0}{f_1}\Big|_{U_{0,1}} \right) \tag{A.33}$$

  In general this is non-trivial as opposed to the case $\mathcal{U} = \{U_0\}$. Still the higher Čech cochain-groups are trivial again.

Remark:

- Note that $\delta^2 = 0$, as is readily checked.
- By means of the coboundary maps δ we obtain the *Čech complex* for the open cover $\mathcal{U}$ given by

$$0 \to \check{C}^0(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \dots \tag{A.34}$$

Definition A.2.3 (Cocycles and Coboundaries)**:**
Let M a complex manifold and $\mathcal{U}$ an open cover of M. Then we define

- The cocycles are given by [2]

$$Z^k(\mathcal{U}, \mathcal{O}_M^*) := \left\{ a \in \check{C}^k(\mathcal{U}, \mathcal{O}_M^*) \; ; \; \delta a = 1 \right\} \tag{A.35}$$

- The coboundaries are given by

$$B^k(\mathcal{U}, \mathcal{O}_M^*) := \left\{ a \in \check{C}^k(\mathcal{U}, \mathcal{O}_M^*) \; ; \; \exists b \in \check{C}^{k-1}(\mathcal{U}, \mathcal{O}_M^*) \text{ such that } a = \delta b \right\} \tag{A.36}$$

Remark:
We define that $B^0(\mathcal{U}, \mathcal{O}_M^*)$ is trivial.

[2]Recall that $\mathcal{O}_M^*$ is a multiplicative group with identity 1.

Example A.2.4:
For $\mathcal{U} = \{U_0, U_1\}$ we have by our ealier considerations

- $Z^1(\mathcal{U}, \mathcal{O}_M^*) = \check{C}^1(\mathcal{U}, \mathcal{O}_M^*)$
- $B^1(\mathcal{U}, \mathcal{O}_M^*) = \left\{ \left(\frac{f_1}{f_0}\Big|_{U_{0,1}}, \frac{f_0}{f_1}\Big|_{U_{0,1}} \right), f_i \in \mathcal{O}_M^*(U_i) \right\}$

Definition A.2.4 (Čech Cohomology Group)**:**
We define the Čech Cohomology groups as the following quotient groups

$$\check{H}^k(\mathcal{U}, \mathcal{O}_M^*) := Z^k(\mathcal{U}, \mathcal{O}_M^*) / B^k(\mathcal{U}, \mathcal{O}_M^*) \tag{A.37}$$

or equivalently as the cohomologies of the Čech complex

$$0 \to \check{C}^0(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \check{C}^1(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \check{C}^2(\mathcal{U}, \mathcal{O}_M^*) \xrightarrow{\delta} \dots \tag{A.38}$$

Note:
The Čech cohomology groups are coset spaces, i.e. objects differing by a coboundary are considered the same, which in addition carries the structure of an Abelian group.

Example A.2.5:
Let $\mathcal{U} = \{U_0, U_1\}$. Then we have

- $\check{H}^0(\mathcal{U}, \mathcal{O}_M^*) = Z^0(\mathcal{U}, \mathcal{O}_M^*)$ since $B^0(\mathcal{U}, \mathcal{O}_M^*)$ is trivial.
- $\check{H}^1(\mathcal{U}, \mathcal{O}_M^*) = \left\{ \left[\left(f_{01}, \frac{1}{f_{01}} \right) \right], f_{01} \in \mathcal{O}_M^*(U_0 \cap U_1) \right\}$ where $\left(f_{01}, \frac{1}{f_{01}} \right) \sim \left(g_{01}, \frac{1}{g_{01}} \right)$ iff there exist $f_0 \in \mathcal{O}_M^*(U_0)$ and $f_1 \in \mathcal{O}_M^*(U_1)$ such that

$$g_{01}(z) = \frac{f_0(z)}{f_1(z)} \cdot f_{01}(z) \quad \forall z \in U_{01} \tag{A.39}$$

Remark:
Although we have introduced Čech cohomology for the sheaf $\mathcal{O}_M^*$ only, it should be clear that this construction generalises to any sheaf of Abelian groups on a topological space M.

Definition A.2.5 (Leray Cover)**:**
Let M a complex manifold and $\mathcal{U}$ an open cover of M. The open cover $\mathcal{U}$ is a Leray cover precisely if for any sheaf F on M it holds

$$H^i(M, F) \cong \check{H}^i(\mathcal{U}, \mathcal{F}) \tag{A.40}$$

Remark:
We will make use of the fact that the affine open cover of a smooth and compact normal toric variety is known to be a Leray cover in the later parts of this thesis. In the first part we will one make moderate use of such covers.

A.3. Holomorphic Line Bundles Cohomologically And The Picard Group

A.3.1. Holomorphic Line Bundles Cohomologically

Definition A.3.1:
Let M a complex manifold and $\mathcal{U}$ an open cover of M. Then we introduce the following terminology.

- The elements of $H^1(M, \mathcal{O}_M^*)$ are holomorphic line bundles.
- The elements of $\check{H}^1(\mathcal{U}, \mathcal{O}_M^*)$ are holomorphic coordinate line bundles with respect to the open cover $\mathcal{U}$.

Consequence:
Thus a holomorphic coordinate line bundle with respect to an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a complex manifold M can be represented by a family $G = \{g_{\alpha\beta} \in \mathcal{O}_M^*(U_{\alpha\beta})\}_{\alpha,\beta \in I}$ such that

- $g_{\alpha\beta}(z) = \frac{1}{g_{\beta\alpha}(z)}$ for all $z \in U_{\alpha\beta}$
- $g_{\alpha\beta}(z) g_{\beta\gamma}(z) = g_{\alpha\gamma}(z)$ for all $z \in U_{\alpha\beta\gamma}$

This family however is not unique, since it only has to represent an element of $\check{H}^1(\mathcal{U}, \mathcal{O}_M^*)$. So if we pick a family $S = \{f_\alpha \in \mathcal{O}_M^*(U_\alpha)\}_{\alpha \in I}$ and construct from it

$$G' = \left\{ g'_{\alpha\beta} = \left. \frac{f_\beta}{f_\alpha} \right|_{U_{\alpha\beta}} \cdot g_{\alpha\beta} \in \mathcal{O}_M^*(U_{\alpha\beta}) \right\}_{\alpha,\beta \in I} \tag{A.41}$$

then G' represents the same coordinate line bundle.

Definition A.3.2 (Equivalent Holomorphic Coordinate Line Bundles)**:**
Let M a complex manifold. We consider two open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{V} = \{V_\mu\}_{\mu \in J}$ for I, J suitable indexing sets. Then consider two holomorphic coordinate line bundles $\mathcal{L}$ and $\mathcal{L}'$ with respect to $\mathcal{U}$ and $\mathcal{V}$. Let them be represented by the families

$$G = \{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}_{\alpha,\beta \in I}, \qquad H = \{h_{\mu\nu} \in \mathcal{O}^*(V_\alpha \cap V_\beta)\}_{\mu,\nu \in J} \tag{A.42}$$

Now we construct a new open cover of M as

$$\mathcal{W} := \{U_\alpha \cap V_\mu\}_{\alpha,\mu} := \{W_\kappa\}_{\kappa \in I \times J} \tag{A.43}$$

In particular we can restrict the functions $g_{\alpha\beta}$ and $h_{\mu\nu}$ to intersections $W_\kappa \cap W_\lambda$. This gives

$$\widetilde{G} = \{\widetilde{g}_{\kappa,\lambda} \in \mathcal{O}^*(W_\kappa \cap W_\lambda)\}_{\kappa,\lambda \in I \times J}, \qquad \widetilde{H} := \{h_{\kappa\lambda} \in \mathcal{O}^*(W_\kappa \cap W_\lambda)\}_{\kappa,\lambda \in I \times J} \tag{A.44}$$

These families now represent holomorphic coordinate line bundles with respect to the same open cover $\mathcal{W}$.
We now define that $\mathcal{L}$ and $\mathcal{L}'$ are equivalent holomorphic coordinate line bundles precisely if the holomorphic coordinate line bundles represented by $\widetilde{G}$ and $\widetilde{H}$ respectively, are equivalent in the sense that they lie in the same class in $\check{H}^1(\mathcal{W}, \mathcal{O}_M^*)$.

Remark:
This definition is independent of the chosen representative families G, H for $\mathcal{L}$ and $\mathcal{L}'$ respectively.

Note:
Given a complex manifold M with Leray cover $\mathcal{U}$. Then we can represent a holomorphic line bundle by a holomorphic coordinate line bundle with respect to the Leray cover $\mathcal{U}$.

A.3.2. The Picard Group

Note:
Let M a complex manifold. The holomorphic line bundles on M are the elements of $H^1(X, \mathcal{O}_M^*)$. Given that $\mathcal{U}$ is a Leray cover we can represent holomorphic line bundles by holomorphic coordinate line bundles since we then have an isomorphism

$$H^1(X, \mathcal{O}_M^*) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_M^*) \tag{A.45}$$

Remark:
$H^1(\mathcal{U}, \mathcal{O}_M^*)$ corresponds to the coordinate line bundles of the open cover $\mathcal{U}$ of the complex manifold M. We can represent the elements of $H^1(\mathcal{U}, \mathcal{O}_M^*)$ by a Čech cochain. In particular we can define a group action between Čech cochains as

- $\{g_{\alpha\beta}\} + \{g'_{\alpha\beta}\} := \{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$
- $\{g_{\alpha\beta}\}^{-1} = \{g_{\alpha\beta}^{-1}\}$

Thereby $H^1(\mathcal{U}, \mathcal{O}_M^*)$ becomes an abelian group.

Note:
One can show that this group action carries over to $H^1(M, \mathcal{O}_M^*)$ giving it a structure of a group - the so-called Picard group. It is denoted as

$$\mathrm{Pic}(M) := H^1(M, \mathcal{O}^*) \tag{A.46}$$

Let us mention that the above mentioned group operation of Čech cochains become tensor product and dual bundle construction at the level of sheaf cohomology.

A.4. Holomorphic Line Bundles Topologically

Comment:
Not only can one define holomorphic line bundles, but also continous, differential, smooth line bundles. The latter are in principle accesible via the topological definition that we will give momentarily. However as we are not interested in just any line bundles but holomorphic line bundles, we give the definition of a holomorphic line bundle from a topological perspective. To obtain e.g. a smooth line bundle one has to replace the word holomorphic everywhere in the following definition by smooth.

Definition A.4.1 (Representative Of A Holomorphic Coordinate Line Bundle)**:**
Let M a complex manifold. Then a representative of a holomorphic coordinate bundle $\mathcal{L}$ is a collection of data $(L, \pi, \mathcal{U} = \{U_\alpha\}_{\alpha\in I}, \{\varphi_\alpha\}_{\alpha\in I})$ as follows

- L is a complex manifold with $\dim_{\mathbb{C}}(L) = \dim_{\mathbb{C}}(M) + 1$.
- The map $\pi: L \to M$ is holomorphic and satisfies

$$\pi^{-1}(p) \cong \mathbb{C}, \qquad \forall p \in M \tag{A.47}$$

- $\mathcal{U}$ is an open cover of M.
- The maps $\varphi_\alpha: U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha)$ are biholomorphisms.

Remark (Transition Functions)**:**
Let us consider a representative of a holomorphic coordinate line bundle on a complex manifold M given by the data $(L, \pi, \mathcal{U} = \{U_\alpha\}_{\alpha\in I}, \{\varphi_\alpha\}_{\alpha\in I})$. Recall that the maps

$$\varphi_\alpha: U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha) \tag{A.48}$$

are biholomorphisms. We consider now $p \in U_\alpha \cap U_\beta$ and define the maps

$$\varphi_\alpha|_{\{p\}\times\mathbb{C}}: \{p\} \times \mathbb{C} \to \pi^{-1}(p) \cong \mathbb{C}, \qquad \varphi_\beta|_{\{p\}\times\mathbb{C}}: \{p\} \times \mathbb{C} \to \pi^{-1}(p) \cong \mathbb{C} \tag{A.49}$$

These maps are invertible since φ_α and φ_β are biholomorphisms. Consequently we can consider the map

$$\varphi_\beta^{-1}|_{\{p\}\times\mathbb{C}} \circ \varphi_\alpha|_{\{p\}\times\mathbb{C}}: \{p\} \times \mathbb{C} \to \{p\} \times \mathbb{C}\ ,\ (p, x) \mapsto (p, g_{\alpha\beta}(p) \cdot x) \tag{A.50}$$

where $g_{\alpha\beta}(p) \in \mathbb{C}^*$.
We can repeat this construction for every $p \in U_\alpha \cap U_\beta$. Thereby we obtain a map $g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$. The collection of all these functions we term the transition functions for the holomorphic coordinate line bundle $(L, \pi, \mathcal{U} = \{U_\alpha\}_{\alpha\in I}, \{\varphi_\alpha\}_{\alpha\in I})$.

Consequence (Properties Of The Transition Functions)**:**
By construction, the transition functions have the following properties.

- $g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$
- $g_{\alpha\beta}(p) \cdot g_{\beta\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta$
- $g_{\alpha\beta}(p)\, g_{\beta\gamma}(p)\, g_{\gamma\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$

Construction A.4.1 (Minimal Defining Data)**:**
Let M a complex manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open cover of M. Moreover let us consider the family $G = \{g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)\}_{\alpha,\beta\in I}$ such that

- $g_{\alpha\beta}(p)\, g_{\beta\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta$
- $g_{\alpha\beta}(p)\, g_{\beta\gamma}(p)\, g_{\gamma\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$

We claim that from this data we can reconstruct a representative of a holomorphic coordinate line bundle in the sense presented above. We outline the steps to achieve this.

1. Construct the space $S \coloneqq \bigcup_{\alpha \in I} U_\alpha \times \mathbb{C}$.
2. Now define an equivalence relation on S as follows. For $(p,x) \in U_\alpha \times \mathbb{C}$ and $(q,y) \in U_\beta \times \mathbb{C}$ define

$$(p,x) \sim (q,y) \Leftrightarrow p = q \text{ and } x = g_{\alpha\beta}(p) \cdot y \tag{A.51}$$

3. Now construct $L \coloneqq S/\sim$. Note that S carries the product topology and L the quotient topology. In particular both S and L become complex manifolds.
4. Now we define the projection map as

$$\pi: L \to M \ , \ [(p,x)] \mapsto p \tag{A.52}$$

5. Finally we construct the trivialisations φ_α as

$$\varphi_\alpha: U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha) \ , \ (p,x) \mapsto [(p,x)] \tag{A.53}$$

Definition A.4.2 (Equivalence Of Representants)**:**
Let M a complex manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover of M. We consider the families $G = \{g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)\}_{\alpha,\beta \in I}$ and $G' = \{g'_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)\}_{\alpha,\beta \in I}$. We assume that

- $g_{\alpha\beta}(p)\, g_{\beta\alpha}(p) = g'_{\alpha\beta}(p)\, g'_{\beta\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta$
- $g_{\alpha\beta}(p)\, g_{\beta\gamma}(p)\, g_{\gamma\alpha}(p) = g'_{\alpha\beta}(p)\, g'_{\beta\gamma}(p)\, g'_{\gamma\alpha}(p) = 1$ for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$

From the above we know that the families G and G' thus describe a representative of possibly different holomorphic coordinate line bundles. We define that G and G' are equivalent iff there exists a family $S = \{f_\alpha \in \mathcal{O}_M^*(U_\alpha)\}_{\alpha \in I}$ such that for all $p \in U_\alpha \cap U_\beta$ it holds

$$g_{\alpha\beta}(p) = \left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta}(p)\, g'_{\alpha\beta}(p) \tag{A.54}$$

Definition A.4.3 (Holomorphic Coordinate Line Bundle)**:**
Let M a complex manifold. An equivalence class of representants of holomorphic coordinate line bundles according to the above definition is a holomorphic coordinate line bundle.

Remark:
From this it is now apparent that we can also say that a holomorphic coordinate line bundle on a complex manifold M with respect to an open cover $\mathcal{U}$ is an element of $\check{H}^1(\mathcal{U}, \mathcal{O}_M^*)$, just as we defined in section A.3. Thereby we made the connection between the topological and cohomological perspective of holomorphic line bundles.

A.5. Divisors, Holomorphic Line Bundles And Chern Classes

A.5.1. Divisors On Complex Manifolds

Definition A.5.1 (Analytic Variety):
Let M a complex manifold. A subset $V \subset M$ is an analytic variety precisely if for every $p \in V$ there exists $p \in U_p \subset M$ open and a finite number of holomorphic functions $f_1, \ldots, f_{n_p} \in \mathcal{O}_M (U(p))$ such that

$$V \cap U_p = \left\{ z \in U_p \, , \; f_1(z) = \cdots = f_{n_p}(z) = 0 \right\} \tag{A.55}$$

Remark (Analytic Hypersurface):
An analytic variety $V \subset M$ of a complex manifold is termed an analytic hypersurface, if it is locally given by the vanishing of a single holomorphic function.

Note:
An analytic variety is not a manifold as it can have singular loci. For an example of this consider the manifold $M = \mathbb{C}^2$ and consider the analytic hypersurface

$$V := \left\{ (z_1, z_2) \in \mathbb{C}^2 \, , \; z_1 \cdot z_2 = 0 \right\} \tag{A.56}$$

It is not hard to verify that V has a singularity at $(z_1, z_2) = (0, 0)$. [3]

Definition A.5.2 (Irreducibility):
An alaytic variety $V \subset M$ of a complex manifold is irreducible precisely if it cannot be written as the union $V = V_1 \cup V_2$ of two analytic varieties V_1, V_2 such that neither $V_1 = V$ nor $V_2 = V$.

Note:
Here are a few facts about analytic varieties $V \subset M$ of a complex manifold M.

- There exists $U \subset M$ open with $V \subset U$ such that V is closed in the induced topology on U.
- V has only a finite number of connected components $V_1^*, \ldots, V_m^*$ and it holds

$$V = \bigcup_{i=1}^{m} \overline{V_i^*} \tag{A.57}$$

- V is irreducible precisely if its smooth locus is connected.

Consequence:
Any analytic hypersurface $V \subset M$ of a complex manifold can be written as the union

$$V = \bigcup_{i=1}^{m} \overline{V_i^*} \tag{A.58}$$

[3] For this reason analytic varieties play a crucial role in the construction of complex spaces, which generalise the notion of complex manifolds to 'complex manifolds with singularities'.

where the V_i^* are the connected components of V. Moreover the sets $\overline{V_i^*}$ are irreducible, that is we have a decomposition of an analytic hypersurface in its irreducible components.

Definition A.5.3 (Divisor)**:**
Let M a complex manifold. A divisor D on M is a locally finite formal linear combination

$$D = \sum_{i \in I} a_i V_i, \qquad a_i \in \mathbb{Z} \tag{A.59}$$

of irreducible analytic hypersurfaces V_i of M.

Remark:
Locally finite in the above definition means, that for any $p \in M$ there exists an open neighbourhood $p \in U_p \subset M$ such that $U_p \cap V_i \neq \varnothing$ only for finitely many $i \in I$.

Consequence:
If M is a compact complex manifold, then any divisor on M is of the form $D = \sum_{i=1}^{N} a_i V_i$ with $a_i \in \mathbb{Z}$.

Remark (Divisor Group)**:**
Divisors on M naturally form a group under addition induced by $(\mathbb{Z}, +)$. This group is termed the divisor group on M and is denoted $\mathrm{Div}(M)$.

A.5.2. Holomorphic And Meromorphic Functions On Irreducible Analytic Hypersurfaces

Definition A.5.4 (Vanishing Order)**:**
Let M a complex manifold and $V \subset M$ an irreducible analytic hypersurface. Consider $p \in V$. Then there exists an open neighbourhood $p \in U_p \subset M$ such that $V \cap U_p$ is the zero locus of $f_p \in \mathcal{O}_M(U_p)$.
Now consider $g \in \mathcal{O}_M(M)$. Then there exists $a \in \mathbb{Z}$ and $h \in \mathcal{O}_M^*(U_p)$ such that

$$g|_{U_p} = f^a \cdot h \tag{A.60}$$

One can thus define

$$\mathrm{ord}_{V,p}(g) := \max\left\{a \in \mathbb{Z}\,,\ g|_{U_p} = f^a \cdot h\right\} \tag{A.61}$$

Consequence:
It can be shown that $\mathrm{ord}_{V,p}(g)$ is independent of p. The proof strongly relies on the fact that V is taken as irreducible here, as this implies that the smooth locus V^* is connected, as we pointed out earlier. We leave the details to [44] and just note that it thus makes sense to write $\mathrm{ord}_V(g)$. Moreover it holds for any two $g_1, g_2 \in \mathcal{O}_M(M)$

$$\mathrm{ord}_V(g_1 \cdot g_2) = \mathrm{ord}_V(g_1) + \mathrm{ord}_V(g_2) \tag{A.62}$$

Remark:
Let M a complex manifold and $f \in \mathcal{M}_M^* (M)$. Then the poles and zeros of f form a discrete subset of M.

Definition A.5.5 (Vanishing Order II)**:**
Let M a complex manifold and $f \in \mathcal{M}_M^* (M)$, that is f is a meromorphic function on M that does not vanish identically. Let $V \subset M$ an irreducible analytic hypersurface. We consider $p \in V$. In particular there exists an open neighbourhood $p \in U_p \subset M$ such that

$$f|_{U_p} = \frac{g}{h} \tag{A.63}$$

with $g, h \in \mathcal{O}_M (U_p)$ relatively prime in $\mathcal{O}_M (U_p)$. In particular we can define

$$\operatorname{ord}_V (f) := \operatorname{ord}_V (g) - \operatorname{ord}_V (h) \tag{A.64}$$

Definition A.5.6 (Principal Divisor)**:**
Let M a complex manifold and $f \in \mathcal{M}_M^* (M)$. Then we define

$$(f) := \sum_V \operatorname{ord}_V (f) \cdot V \in \operatorname{Div} (M) \tag{A.65}$$

Divisors of this form are termed principal divisors.

Remark:

- The principal divisors (f) can uniquely be split as $(f) = (f)_0 - (f)_\infty$ such that $(f)_0$ is the divisor of the zeros of f and $(f)_\infty$ the divisor of its poles.
- In the definition of (f), the sum runs over all irreducible analytic hypersurfaces $V \subset M$. [4]

A.5.3. Divisors Sheaf-Theoretically

Remark:
Let M a complex manifold and $\mathcal{U}$ an open cover of M. Let F a sheaf of abelian groups on M. Then there always exists a canonical isomorphism

$$H^0 (M, F) \cong \check{H}^0 (\mathcal{U}, F) \tag{A.66}$$

We will make use of this in the following prove.

Claim:
Let M a complex manifold. Then there is a one-to-one correspondance between divisor $D \in \operatorname{Div} (M)$ and global section of the quotient sheaf $\mathcal{M}_M^* / \mathcal{O}_M^*$.

[4] The branch of mathematical logic teaches to only allow sets as indices. We are thus lead to question if the collection of all irreducible analytic hypersurfaces does form a set for an arbitary complex manifold. This in fact is not clear, but rather leads to the concept of categories which we will not touch here. However, let us note that the situation is much better when M is compact and that this will be the situation that we will focus on.

Proof

(⇒) Let $D \in \operatorname{Div}(M)$ be given by

$$D = \sum_{i\in I} a_i V_i, \quad a_i \in \mathbb{Z} \tag{A.67}$$

We pick an open cover $\{U_\alpha\}_{\alpha\in I}$ of M such that every V_i appearing in D has a locally defining function $g_{i\alpha} \in \mathcal{O}_M(U_\alpha)$. Then set

$$f_\alpha = \prod_i (g_{i\alpha})^{a_i} \in \mathcal{M}_M^*(U_\alpha) \tag{A.68}$$

from which one obtains a global section of $\mathcal{M}_M^*/\mathcal{O}_M^*$ according to the preceeding remark.

(⇐) Let $f \in (\mathcal{M}_M^*/\mathcal{O}_M^*)(M)$. Then, again by the preceeding remark, we can pick an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M and represent f by a family $\{f_\alpha \in \mathcal{M}_M^*(U_\alpha)\}_{\alpha\in I}$ such that on all intersections $U_{\alpha\beta}$ it holds

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta) \tag{A.69}$$

Then for any irreducible analytic hypersurface $V \subset U_\alpha \cap U_\beta$ we also have

$$\operatorname{ord}_V(f_\alpha) = \operatorname{ord}_V(f_\beta) \tag{A.70}$$

Thus we can associate to f the divisor

$$D = \sum_V \operatorname{ord}_V(f_\alpha) \cdot V \tag{A.71}$$

where for each irreducible analytic hypersurface V that the sum runs over, we choose $\alpha \in I$ such that $V \cap U_\alpha \neq \varnothing$ and thus $\operatorname{ord}_V(f_\alpha)$ is defined.

■

Consequence:
Let M a complex manifold. Then $\operatorname{Div}(M) \cong H^0(M, \mathcal{M}_M^*/\mathcal{O}_M^*)$.

A.5.4. Divisors And Holomorphic Line Bundles

Remark:
Let M a complex manifold. Then we consider the sheaves $\mathcal{O}_M^*$ and $\mathcal{M}_M^*$. Then the canonical inclusion

$$\mathcal{O}_M^* \hookrightarrow \mathcal{M}_M^* \tag{A.72}$$

is a sheaf homomorphism. By considering the quotient sheaf $\mathcal{M}_M^*/\mathcal{O}_M^*$ we then obtain a short sheaf exact sequence

$$0 \to \mathcal{O}_M^* \to \mathcal{M}_M^* \to \mathcal{M}_M^*/\mathcal{O}_M^* \to 0 \tag{A.73}$$

The associated long exact sequence in sheaf cohomology then induces a map

$$[\cdot]: H^0\left(M, \mathcal{M}_M^*/\mathcal{O}_M^*\right) \to H^1\left(M, \mathcal{O}_M^*\right) \tag{A.74}$$

by which we can associate to a divisor a holomorphic line bundle.

Note:
Whilst this might look quite abstract at this stage, we will make this map quite explicit when we discuss holomorphic line bundles from the sheaf-theoretical perspective in section A.6.

Definition A.5.7:
Let M a complex manifold and $D \in \mathrm{Div}(M) = H^0\left(M, \mathcal{M}_M^*/\mathcal{O}_M^*\right)$. Then we term $[D] \in H^1\left(M, \mathcal{O}_M^*\right)$ the line bundle associated to the divisor D.

Remark:
Recall that for M a complex manifold, both $\mathrm{Div}(M)$ and $\mathrm{Pic}(M)$ carry the structure of an Abelian group. In fact this structure is respected by $[\cdot]$, i.e. the sheaf homomorphism $[\cdot]$ is also a group homomorphism. In more concrete terms this means that for any two $D, D' \in \mathrm{Div}(M)$ it holds

$$[D + D'] = [D] \otimes [D'] \tag{A.75}$$

Moreover the inverse holomorphic line bundle associated to $[D]$ is $[-D]$. This inverse bundle in addition happens to be the dual bundle in the case of holomorphic line bundles.

Definition A.5.8:
A holomorphic line bundle $L \in H^1\left(M, \mathcal{O}_M^*\right)$ is trivial iff in any associated holomorphic coordinate line bundle the transition functions can be taken to be one, i.e. $g_{\alpha\beta}(p) = 1$ for all $p \in U_\alpha \cap U_\beta$.

Claim:
Let M a complex manifold. Then there is a one-to-one correspondance between trivial holomorphic line bundles and principal divisors on M.

Proof

- Let D a principal divisor on M, that is $D = (f)$ for $f \in \mathcal{M}_M^*(M)$. We pick as local defining data of D the restrictions of the function f, so have $\{f|U_\alpha\}$ for an open cover $\mathcal{U} = \{U_\alpha\}$ of M. In particular this implies that on $U_\alpha \cap U_\beta$ it holds

  $$g_{\alpha\beta} = \left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha \cap U_\beta} = 1 \tag{A.76}$$

  Since however, this holds for any open cover $\mathcal{U}$, the holomorphic line bundle $[D]$ is trivial.

- Conversely let $[D]$ a trivial holomorphic line bundle. Then we pick an open cover $\mathcal{U}$ of M and choose as representative of D a family $\{f_\alpha \in \mathcal{M}_M^*(U_\alpha)\}_{\alpha\in I}$ with the property

$$\left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha\cap U_\beta} = g_{\alpha\beta} \tag{A.77}$$

We also know that an equivalent holomorphic line bundle is obtained by picking a family $\{h_\alpha \in \mathcal{O}_M^*(U_\alpha)\}_{\alpha\in I}$ and setting

$$g'_{\alpha\beta} = \left.\frac{h_\alpha}{h_\beta}\right|_{U_\alpha\cap U_\beta} \cdot g_{\alpha\beta} \tag{A.78}$$

But since $[D]$ is trivial, we can pick the functions h_α such that

$$g'_{\alpha\beta} = 1 \tag{A.79}$$

This then implies

$$\left.\frac{f_\alpha}{f_\beta}\right|_{U_\alpha\cap U_\beta} = g_{\alpha\beta} = \left.\frac{h_\beta}{h_\alpha}\right|_{U_\alpha\cap U_\beta} \Rightarrow \; f_\alpha\cdot h_\alpha|_{U_\alpha\cap U_\beta} = f_\beta\cdot h_\beta|_{U_\alpha\cap U_\beta} \tag{A.80}$$

But since $\mathcal{M}_M^*$ is a sheaf, there now exists by property (S2) a meromorphic function $g\in\mathcal{M}_M^*(M)$ with

$$g|_{U_\alpha} = f_\alpha\cdot h_\alpha \tag{A.81}$$

Consequently $D = (g)$. Since this argument holds true for any representative of D, the statement follows.

■

Definition A.5.9:
Let M a complex manifold and $D, D' \in \mathrm{Div}(M)$. Then we define the relation

$$D \sim D' \iff \exists f\in\mathcal{M}_M^*(M): D = D' + (f) \tag{A.82}$$

Remark:
An important question is under which conditions divisor classes and holomorphic line bundles are in one-to-one correspondance. To this end we state the following theorem [72].

Theorem A.5.1:
Let M a projective algebraic manifold. Then the sequence

$$0\to H^0(M,\mathcal{M}_M^*)\to H^0(M,\mathcal{M}_M^*/\mathcal{O}^*)\to H^1(M,\mathcal{O}_M^*)\to 0 \tag{A.83}$$

is exact.

Consequence:
Exactness implies that the map

$$[\cdot]: H^0(M, \mathcal{M}^*_M/\mathcal{O}^*) \to H^1(M, \mathcal{O}^*_M) \tag{A.84}$$

is surjective. So any holomorphic line bundle $\mathcal{L}$ is of the form

$$\mathcal{L} = [D] \tag{A.85}$$

for a suitable $D \in \mathrm{Div}(M)$. But since a holomorphic line bundle stemming from a divisor is trivial precisely if that divisor is a principal divisor we learn that the map $[\cdot]$ is also injective. So we conclude that on any projective algebraic manifold divisor classes and holomorphic line bundles are in a one-to-one relationship.

Remark:
This statement also holds true for compact Riemann surfaces [45].

A.5.5. The Chern Class Of A Holomorphic Line Bundle

Remark:
Let M a compact and complex manifold of complex dimension n. Then the sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0 \tag{A.86}$$

is sheaf-exact [5]. This sequence induces a long exact sequence in sheaf cohomology, thereby giving us the connecting homomorphism

$$\delta: H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \tag{A.87}$$

Definition A.5.10 (Chern Class Of Line Bundle)**:**
Let M a compact and complex manifold. Then we know already

$$\mathrm{Pic}(M) = H^1(M, \mathcal{O}^*) \tag{A.88}$$

Consequently a line bundle $\mathcal{L}$ is an element of $H^1(M, \mathcal{O}^*)$. This allows for the definition

$$c_1(\mathcal{L}) := \delta(\mathcal{L}) \in H^2(M, \mathbb{Z}) \tag{A.89}$$

We term this the *first Chern class* of the line bundle $\mathcal{L}$.

Note:
There is a natural map $H^2(M, \mathbb{Z}) \to H^2_{\mathrm{D.R.}}(M)$, so that in an abuse of notation one sometimes writes $c_1(\mathcal{L}) \in H^2_{\mathrm{D.R.}}(M)$.

Remark:
It holds

- $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$
- $c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L})$

[5] Recall that the existence of a holomorphic logarithm is locally secured, whilst global obstructions can hinder the existence. This however is not a problem for sheaf exactness, since the latter is a local property.

A.5.6. Determining Holomorphic Line Bundles By Their Chern Class

Remark:
It is also possible to consider smooth line bundles. To this end let us introduce the following two sheaves

- $\mathcal{C}_M$ is the sheaf of $\mathcal{C}^\infty$ functions on M.
- $\mathcal{C}_M^*$ is the sheaf of $\mathcal{C}^\infty$ functions on M which do not vanish.

Then a smooth line bundle on M is given by an element in $H^1(M, \mathcal{C}_M^*)$.

Note:
Let M a compact complex manifold. Then there is a sheaf exact sequence

$$0 \to \mathbb{Z} \to \mathcal{C}_M \to \mathcal{C}_M^* \tag{A.90}$$

which allows us to define the Chern class of smooth line bundles in just the same fashion as we did for holomorphic line bundles.

Consequence:
Let M a compact complex manifold. Then there is a commutative diagram

$$\begin{tikzcd}
H^1(M, \mathcal{C}_M) \arrow[r] & H^1(M, \mathcal{C}_M^*) \arrow[r, "\delta"] & H^2(M, \mathbb{Z}) \\
H^1(M, \mathcal{O}_M) \arrow[r] \arrow[u, "\iota"] & H^1(M, \mathcal{O}_M^*) \arrow[r, "\delta"] \arrow[u, "\iota"'] & H^2(M, \mathbb{Z}) \arrow[u]
\end{tikzcd}$$

where both rows are exact and the map in the last column is an isomorphism. Note that $H^1(M, \mathcal{C}_M) = 0$ since $\mathcal{C}_M$ is flabby. This implies

$$\boxed{\text{Holomorphic line bundles are determined by their Chern class up to } \mathcal{C}^\infty \text{ isomorphism.}}$$

Lemma A.5.1:
On the complex projective space $\mathbb{CP}^n$, the sequence

$$H^1(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}) \to H^1(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}^*) \to H^2(\mathbb{CP}^n, \mathbb{Z}) \tag{A.91}$$

is exact and it holds $H^1(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}) = 0$ since $\mathcal{O}_{\mathbb{CP}^n}$ is flabby. From this one concludes that every holomorphic line bundle on $\mathbb{CP}^n$ is uniquely determined by its Chern class, that is we have

$$\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z} \tag{A.92}$$

Note:
The above is just one example of a more general statement. We will later find that on smooth and compact normal toric varieties, every holomorphic line bundle is uniquely determined by its Chern class. This we will heavily use.

A.6. Holomorphic Line Bundles Sheaf-Theoretically

A.6.1. Definitions

Definition A.6.1 ($\mathcal{O}_M$-Module)**:**
Let M a complex manifold. Then we have the notion of holomorphic functions on M, which gives us the sheaf $\mathcal{O}_M$ of holomorphic functions on M. An $\mathcal{O}_M$-module $\mathcal{V}$ on M is a sheaf of Abelian groups, such that

- for each $U \subset M$ open, the Abelian group $\mathcal{V}(U)$ carries the structure of an $\mathcal{O}_M(U)$-module.
- for $V \subset U \subset M$ any two open subsets of M the restriction maps of the sheaf $\mathcal{V}$ are compatible with the module-structure, i.e. are module-homomorphisms.

Definition A.6.2 (Finitely Generated Free $\mathcal{O}_M$-Modules)**:**
Let M a complex manifold. An $\mathcal{O}_X$-module $\mathcal{V}$ on M is finitely generated free iff there exists $n \in \mathbb{N}$ such that $\mathcal{V}$ is isomorphic to $\mathcal{O}_M^n$.

Definition A.6.3 (Holomorphic Vector Bundle)**:**
A holomorphic vector bundle $\mathcal{V}$ on a complex manifold M is a locally free $\mathcal{O}_M$-module, i.e. an $\mathcal{O}_M$-module such that every point admits an open neighbourhood U such that $\mathcal{V}|_U$ is isomorphic to $(\mathcal{O}_M|_U)^n$ for some integer $n \geq 0$.

Remark:
The integer n need not be the same for all $U \subset M$ open. For example this can happen if M is not connected.

Definition A.6.4 (Rank Of A Vector Bundle)**:**
Let M a complex manifold and $\mathcal{V}$ a holomorphic vector bundle on M. If there exists $N \in \mathcal{M}$ such that $\mathcal{V}$ is at every point $p \in M$ locally isomorphic to $\mathcal{O}_M^N$, then N is termed the rank of $\mathcal{V}$.

Definition A.6.5 (Holomorphic Line Bundle)**:**
Let M a complex manifold. A holomorphic vector bundle $\mathcal{V}$ on M of rank 1 is a holomorphic line bundle.

A.6.2. Associating A Holomorphic Line Bundle To A Divisor

Let M a compact Riemann surface, i.e. a complex manifold of complex dimension one which is compact. Then divisors on M are finite sums of points

$$D = \sum_{a \in M} D(a) \cdot [a] \tag{A.93}$$

One can then define for every $U \subset M$ open

$$\mathcal{L}_D(U) := \left\{ f : U \to \overline{\mathbb{C}} \text{ meromorphic },\ (f) \geq -D \right\} \tag{A.94}$$

It is not too hard to see that this forms a holomorphic line bundle in the sense defined in the previous subsection [45]. This exemplifies the map

$$[\cdot] : H^0(M, \mathcal{M}_M^*/\mathcal{O}_M^*) \to H^1(M, \mathcal{O}_M^*) \tag{A.95}$$

discussed in subsection A.5.4. Let us note two important properties of this mapping.

- $H^0(M, \mathcal{L}_D) \cong \left\{ f : M \to \overline{\mathbb{C}} \text{ meromorphic },\ (f) \geq -D \right\}$
- All holomorphic line bundles on compact Riemann surfaces can be obtained in this way, since there is an isomorphism from the divisors classes on M to the Picard group of M.

A.6.3. Making Connection With The Cohomological And Topological Picture Of Holomorphic Line Bundles

Remark:
Let M a complex manifold and $\mathcal{L}$ a holomorphic line bundle on M. Consider in addition an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M. In particular we can consider the sets $\mathcal{L}(U_\alpha)$ of local holomorphic sections of $\mathcal{L}$.
It is not hard to verify that there exist functions $g_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$ with the property that the maps

$$\varphi_{\alpha\beta} : \mathcal{L}(U_\alpha)|_{U_{\alpha\beta}} \to \mathcal{L}(U_\beta)|_{U_{\alpha\beta}}\ ,\ f \mapsto g_{\alpha\beta} \cdot f \tag{A.96}$$

are isomorphisms.
The collection $\{g_{\alpha\beta} \in \mathcal{O}_M^*(U_{\alpha\beta})\}_{\alpha,\beta \in I}$ does then form the transition function of $\mathcal{L}$.

Consequence:
The sheaf-theoretic picture of line bundles is completely equivalent to the cohomological and topological picture. We can thus fluently switch between them.

B. Line Bundles On Compact Riemann Surfaces

We will now familiarise ourselves with the concepts introduced in Appendix A. To keep matters as simple as possible we pick the simplest complex manifolds, namely compact and connected complex manifolds of complex dimension one. These are known as compact Riemann surfaces.

In this chapter we summarise classical results about holomorphic line bundles on compact Riemann surfaces. As this is a well-known topic there are many good references. Historical references include [73] and [74]. Of the many available textbooks we would like to highlight [75], [76] and [45] for information on vector bundles on compact Riemann surfaces. Background on Riemann surfaces is given in [77]. In addition there are many good lecture notes available on the topic of vector bundles on Riemann surfaces. Of the many we point the interested reader to [78] for a discussion of vector bundles on Riemann surfaces and to [79] for more background on Riemann surfaces.

Finally we mention that a topological classification of compact, connected Riemann surfaces is given by their genus $g \in \mathbb{N}_{\geq 0}$. A proof of this can be found in [77]. Throughout this chapter M_g represents a compact, connected Riemann surface of genus g.

B.1. General Facts And Theorems

Theorem B.1.1 (Finiteness Theorem)**:**
Let $\mathcal{L}$ a holomorphic line bundle on M_g. Then it holds

- $H^i(M_g, \mathcal{L}) = 0$ for $i \geq 2$
- $H^0(M_g, \mathcal{L})$ and $H^1(M_g, \mathcal{L})$ are finite-dimensional complex vector spaces

Remark (Convention)**:**
Consequently we can introduce for a holomorphic line bundle $\mathcal{L}$ on M_g the notion

$$h^i(M_g, \mathcal{L}) := \dim_{\mathbb{C}}\left(H^i(M_g, \mathcal{L})\right) \tag{B.1}$$

Theorem B.1.2 (Divisor Classes And Holomorphic Line Bundles)**:**
The divisor class group $\mathrm{Div}(M_g)$ is isomorphic to $\mathrm{Pic}(M_g) = H^1\left(M_g, \mathcal{O}^*_{M_g}\right)$. There is thus a one-to-one relation between divisor classes and isomorphism classes of holomorphic line bundles on M_g.

Definition B.1.1 (Degree Of A Holomorphic Line Bundle)**:**
Let $\mathcal{L}$ a holomorphic line bundle on M_g. Then we know from the above isomorphism that there exists $D \in \mathrm{Div}(M_g)$ such that $\mathcal{L} \cong \mathcal{L}_D$. We then define

$$\deg(\mathcal{L}) := \deg(D) \tag{B.2}$$

Remark (Welldefinedness)**:**
The divisor $D \in \mathrm{Div}(M_g)$ in the above definition is only defined up to linear equivalence, that is up to the addition of a principal divisor. But the degree of a principal divisor vanishes. Consequently $\deg(\mathcal{L})$ is well-defined.

Theorem B.1.3 (Riemann-Roch)**:**
For a holomorphic line bundle $\mathcal{L}$ over M_g it holds

$$h^0(M_g, \mathcal{L}) - h^1(M_g, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g \tag{B.3}$$

Remark:
This statement holds more generally for coherent sheaves. Note also that the Euler-Characteristic of $\mathcal{L}$ is defined via

$$\chi(\mathcal{L}) := h^0(M_g, \mathcal{L}) - h^1(M_g, \mathcal{L}) \tag{B.4}$$

In the physics literature this quantity is often refered to as the *chiral index*.

Theorem B.1.4:
All compact Riemann surfaces of genus g admits a non-trivial meromorphic 1-form ω.

Remark:
One can define the divisor for a meromorphic 1-form in a natural way.

Definition B.1.2 (The Canonical Line Bundle)**:**
A holomorphic line bundle on M_g is termed canonical precisely if there exists a non-trivial meromorphic 1-form ω on M_g such that $\mathcal{L} \cong \mathcal{L}_{D_\omega}$. We denote the canonical bundle on M_g by $\mathcal{K}$.

Note:
If M_g admits more than one non-trivial meromorphic 1-form, then their divisors are linearly equivalent. Thus the canonical line bundle is well-defined.

Theorem B.1.5 (Duality Theorem)**:**
Let $\mathcal{L}$ a holomorphic line bundle on M_g, then it holds

$$h^1(M_g, \mathcal{L}) = h^0(M_g, \mathcal{K} \otimes \mathcal{L}^\vee) \tag{B.5}$$

Remark:
This is a special form of Serre duality.

Theorem B.1.6:
Let $\mathcal{L} \cong \mathcal{L}_D$ a holomorphic line bundle on M_g with curvature form Ω and $D \in \mathrm{Div}(M_g)$. Then it holds [44]

$$\frac{i}{2\pi} \int_{M_g} \Omega = \deg(D) = \deg(\mathcal{L}) \tag{B.6}$$

B.2. Simple Consequences

Remark:
By means of the duality theorem, we can rephrase the Riemann-Roch theorem for a holomorphic line bundle $\mathcal{L}$ on M_g as

$$h^0(M_g, \mathcal{L}) - h^0(M_g, \mathcal{K} \otimes \mathcal{L}^\vee) = \deg(\mathcal{L}) + 1 - g \tag{B.7}$$

Claim:
It holds $h^0(M_g, \mathcal{K}) = g$.

Proof

- Let $D \in \mathrm{Div}(M_g)$ the trivial divisor. We note that this is the divisor of a non-zero constant function on M_g. But since the non- zero constant functions are trivially meromorphic, we learn that the divisor D is a principal divisor. The holomorphic line bundle L_D is consequently the trivial holomorphic line bundle $\mathcal{L}_0$ over M_g.
- Let us apply the Riemann-Roch theorem to the trivial bundle $\mathcal{L}_0$ on M_g. From the above we conclude

$$h^0(M_g, \mathcal{L}_0) - h^0(M_g, \mathcal{K}) = \deg(\mathcal{L}_0) + 1 - g = 1 - g \tag{B.8}$$

  where we used that the trivial holomorphic line bundle $\mathcal{L}_0$ has the degree of the trivial divisor, which is zero.
- We can describe $H^0(M_g, \mathcal{L}_0)$ explicitely as

$$H^0(M_g, \mathcal{L}_0) = \left\{ f : M_g \to \overline{\mathbb{C}} \text{ meromorphic }, \ (f) \geq 0 \right\} \tag{B.9}$$

  But since M_g is compact it follows from the maximum modulus principle that these are the constant functions and the constant functions only [1]. Thus

$$H^0(M_g, \mathcal{L}_0) \cong \mathbb{C} \tag{B.10}$$

Putting all the pieces together we find $h^0(M_g, \mathcal{K}) = g$. ∎

Claim:
It holds $\deg(\mathcal{K}) = 2g - 2$.

Proof
Let us apply the Riemann-Roch theorem to $\mathcal{K}$. Then we find

$$h^0(M_g, \mathcal{K}) - h^0(M_g, \mathcal{L}_0) = \deg(\mathcal{K}) - g + 1 \tag{B.11}$$

From the previous result it follows immediately $g - 1 = \deg(\mathcal{K}) - g + 1$ which concludes the proof. ∎

[1] By convention the function $f \equiv 0$ satisfies $(f) \geq D$ for any divisor $D \in \mathrm{Div}(M_g)$.

Claim:
Let $\mathcal{L}$ a holomorphic line bundle with $\deg(\mathcal{L}) < 0$, then $h^0(M_g, \mathcal{L}) = 0$.

Proof
Let $D \in \mathrm{Div}(M_g)$ be the divisor of $\mathcal{L}$. Then we know

$$H^0(M_g, \mathcal{L}) = \left\{ f\colon M_g \to \overline{\mathbb{C}} \text{ meromorphic },\ (f) \geq -D \right\} \tag{B.12}$$

But $\deg(D) < 0$. We are thus looking for meromorphic functions on M_g with positive degree. But this can only apply to functions that are holomorphic on the entire M_g, since all other meromorphic functions have a divisor of vanshing degree. Yet holomorphic functions on all of M_g are constant by the maximum modulus principle. Finally, since $\deg(f) > 0$ they must have at least one zero of order one. But then these functions are identically zero on all of M_g. Thus

$$H^0(M_g, \mathcal{L}) = \left\{ f\colon M_g \to \overline{\mathbb{C}} ,\ z \mapsto 0 \right\} \tag{B.13}$$

This concludes the proof. ∎

Theorem B.2.1 (Vanishing Theorem of Kodaira)**:**
Let $\mathcal{L}$ a holomorphic line bundle on M_g with $\deg(\mathcal{L}) \geq 2g - 1$. Then $h^1(M_g, \mathcal{L}) = 0$.

Proof
By the duality theorem

$$h^1(M_g, \mathcal{L}) = h^0(M_g, \mathcal{K} \otimes \mathcal{L}^\vee) \tag{B.14}$$

But $\deg(\mathcal{K} \otimes \mathcal{L}^\vee) = \deg(\mathcal{K}) - \deg(\mathcal{L}) = 2g - 2 - \deg(\mathcal{L}) < 0$. The statement now follows from the preceeding one. ∎

Claim:
Let $\mathcal{L}$ a holomorphic line bundle on M_g with $\deg(\mathcal{L}) = 0$. Then it holds

- $h^0(M_g, \mathcal{L}) = 1 \iff \mathcal{L} \cong \mathcal{L}_0$.
- $h^0(M_g, \mathcal{L}) = 0 \iff \mathcal{L} \ncong \mathcal{L}_0$.

Proof
We first pick $D \in \mathrm{Div}(M_g)$ such that $\mathcal{L} \cong \mathcal{L}_D$.

- Let us first consider the case of $\mathcal{L} \cong \mathcal{L}_0$. But it holds

  $$\mathcal{L} \cong \mathcal{L}_0 \ \Leftrightarrow \exists f \in \mathcal{M}^*_{M_g}(M_g) : D = (f) \tag{B.15}$$

  Consequently this is the case iff there exists $f \in \mathcal{M}^*_{M_g}(M_g)$ such that

  $$H^0(M_g, \mathcal{L}) = \left\{ g \in \mathcal{M}^*_{M_g}(M_g) \ ,\ (g) \geq (f) \right\} \tag{B.16}$$

  Now it is easy to see that $g = f \cdot h$ with $h \in \mathcal{O}_{M_g}(M_g)$ which in addition gives an isomorphism. So we conclude

  $$\mathcal{L} \cong \mathcal{L}_0 \ \Leftrightarrow H^0(M_g, \mathcal{L}) \cong \mathcal{O}_{M_g}(M_g) \cong \mathbb{C} \tag{B.17}$$

  where in the last step the maximum modulus principle was applied.

- Now we consider the case of $\mathcal{L} \ncong \mathcal{L}_0$. But we know

$$\mathcal{L} \ncong \mathcal{L}_0 \;\Leftrightarrow \nexists\, f \in \mathcal{M}^*_{M_g}\left(M_g\right) : D = (f) \tag{B.18}$$

  Consequently D is not linearly equivalent to any principal divisor. Then we recall

$$H^0\left(M_g, \mathcal{L}\right) = \left\{g \in \mathcal{M}^*_{M_g}\left(M_g\right)\ ,\ (g) \geq -D\right\} \tag{B.19}$$

  Now it is easy to see that since D is not the divisor of meromorphic function, all elements in H^0 are functions that are holomorphic on all of M_g. But since M_g is compact, these functions have to be constant. Since D is not a principal divisor (and thus not linearly equivalent to the trivial divisor), D includes at least one pole. This forces all elements in $H^0\left(M_g, \mathcal{L}\right)$ to have at least one zero. Putting everything together we conclude

$$\mathcal{L} \ncong \mathcal{L}_0 \;\Leftrightarrow h^0\left(M_g, \mathcal{L}\right) = 0 \tag{B.20}$$

This concludes the proof. ■

B.3. Spin Bundles

Remark:
The first two Stiefel-Whitney classes vanish on compact and connected Riemann surfaces. Thus compact and connected Riemann surfaces are orientable (as is any complex manifold) and in addition admit spin structures [80].

Definition B.3.1 (Spin Divisor And Spin Bundle)**:**
Let $D_{\mathcal{K}}$ the canonical divisor on M_g. Then we define

- $D \in \mathrm{Div}\left(M_g\right)$ is a spin divisor precisely if $2D = D_{\mathcal{K}}$.
- A holomorphic line bundle $\mathcal{L}$ on M_g with $\mathcal{L} \cong \mathcal{L}_D$ is a spin bundle precisely if $D \in \mathrm{Div}\left(M_g\right)$ a linearly equivalent to a spin divisor.

Remark:
Michael Atiyah proved in [43] that on M_g there exist 2^{2g} linearly independent spin divisors. Consequently there are 2^{2g} different spin-bundles on M_g. The choice of a spin-structure on M_g is thus far from unique. Moreover he pointed out that

- $2^{g-1} \cdot \left(2^g - 1\right)$ spin structures have at least one non-trivial section.
- $2^{g-1} \cdot \left(2^g + 1\right)$ spin structures either have no or at least two (linearly independent) non-trivial sections.

The same statements were also proven by David Mumford in [81].

Note (Open Question)**:**
Let $\mathcal{L}$ a holomorphic line bundle on M_g that describes a bosonic theory. To add fermionic degrees of freedom to that theory one considers the bundle $\mathcal{L} \otimes \mathcal{S}$ with a spin bundle $\mathcal{S}$ chosen from the 2^{2g} possible ones. Subsequently one makes the following identification

- $H^0(M_g, \mathcal{L} \otimes \mathcal{S})$ are the chiral fields in the theory.
- $H^1(M_g, \mathcal{L} \otimes \mathcal{S})$ are the anti-chiral fields in the theory.

Minimal information on the chiral and antichiral fields in the theory is the number of their generators. Thus we are asking for knowledge about $h^0(M_g, \mathcal{L} \otimes \mathcal{S})$ and $h^1(M_g, \mathcal{L} \otimes \mathcal{S})$. By the vanishing theorems presented in the previous section we can deduce the answer from the Riemann-Roch theorem for the following cases.

1. $\deg(L \otimes S) < 0$:
   This implies $h^0(M_g, \mathcal{L} \otimes \mathcal{S}) = 0$ and $h^1(M_g, \mathcal{L} \otimes \mathcal{S}) = g - 1 - \deg\ (\mathcal{L} \otimes \mathcal{S})$
2. $\deg(L \otimes S) \geq 2g - 1$:
   This implies $h^0(M_g, \mathcal{L} \otimes \mathcal{S}) = \deg(\mathcal{L} \otimes \mathcal{S}) - g + 1$ and $h^1(M_g, \mathcal{L} \otimes \mathcal{S}) = 0$

This leaves unanswered the cases $0 \leq \deg(\mathcal{L} \otimes \mathcal{S}) < 2g-1$. Making use of $\deg(\mathcal{S}) = g-1$ we find that the holomorphic line bundles $\mathcal{L}$ to be studied in more detail are the ones that satisfy

$$1 - g \leq \deg(\mathcal{L}) < g \tag{B.21}$$

In fact, determining h^0 and h^1 for these bundles and for arbitary genus g is very hard and to this day remains an open question in mathematical research. What we can do easily however, is to close this gap for $g = 0$ and $g = 1$. This we will outline in the preceeding sections.

Remark:
Before we do this, we would like to say one more word on the ambiguity of choosing the spin bundle. In many situation there are additional constraints on the spin bundle to be satisfied and those constraints can under certain circumstances fix the spin bundle uniquely. In fact in any good physical situation they should do so. An example of such constraints and how they fix the spin bundle can be found in [19, pp. 58].

B.4. Line Bundles For g = 0 And g = 1

Example B.4.1 ($g = 0$: The Riemann Sphere)**:**
Let us consider a holomorphic line bundle $\mathcal{L}$ on a compact Riemann surface of genus $g = 0$. To this end let $\deg(\mathcal{L}) = m$. Then an easy analysis shows

$$h^0(M_g, \mathcal{L}) = \begin{cases} m+1 & m \geq 0 \\ 0 & m < 0 \end{cases}, \qquad h^1(M_g, \mathcal{L}) = \begin{cases} 0 & m \geq 0 \\ -(m+1) & m < 0 \end{cases} \tag{B.22}$$

Note in particular that the left-open cases as mentioned in section B.3 cannot occure in the case $g = 0$. We thus conclude that on the Riemann sphere one can only have either non-trivial holomorphic or non-trivial antiholomorphic sections, but never both.

Example B.4.2 ($g = 1$: The Complex 2-Torus)**:**
In this case one finds from the preceeding results

$$h^0\left(M_g, \mathcal{L}\right) = \begin{cases} m & m > 0 \\ 1 & m = 0 \text{ and } \mathcal{L} \cong \mathcal{L}_0 \\ 0 & m = 0 \text{ and } \mathcal{L} \ncong \mathcal{L}_0 \\ 0 & m < 0 \end{cases}, \quad h^1\left(M_g, \mathcal{L}\right) = \begin{cases} 0 & m > 0 \\ 1 & m = 0 \text{ and } L \cong \mathcal{L}_0 \\ 0 & m = 0 \text{ and } L \ncong \mathcal{L}_0 \\ -m & m < 0 \end{cases} \tag{B.23}$$

where $m = \deg(\mathcal{L})$. Note that the case $m = 0$ corresponds to the left-open cases mentioned in section B.3. With the results on degree zero bundles, we are however able to solve this case. From the above result we read off that on the 2-torus one can in fact have both chiral and antichiral fields - namely for the trivial bundle.

Remark:
For any divisor D of degree 0 which is not the trivial divisor it holds $h^0\left(M_1, \mathcal{L}\right) = h^1\left(M_1, \mathcal{L}\right)$. Still if D is the trivial divisor both dimensions jump by one. Therefore the above example shows that the dimension of the cohomology classes can jump as we vary the divisor D.

B.5. Comparing Holomorphic Line Bundles Of Different Degree

Remark:
Our task is now to compare holomorphic line bundles of different degree on compact, connected Riemann surfaces. We will make use of the following definition.

Definition B.5.1:
Let M_g a compact connected Riemann surface of genus g. Then we define for $d \in \mathbb{Z}$

$$\mathrm{Div}_d\left(M_g\right) := \left\{D \in \mathrm{Div}\left(M_g\right)\,,\ \deg\left(D\right) = d\right\} \tag{B.24}$$

Claim:
Be $\widetilde{D} \in \mathrm{Div}_d\left(M_g\right)$ arbitary but fixed. Then the following map

$$\varphi\colon \mathrm{Div}_0\left(M_g\right) \to \mathrm{Div}_d\left(M_g\right)\,,\ D \mapsto D + \widetilde{D} \tag{B.25}$$

is an isomorphism.

Proof

- Be $D' \in \mathrm{Div}_d\left(M_g\right)$ a divisor. Then we set

$$D_0 := D' - \widetilde{D} \in \mathrm{Div}_0\left(M_g\right) \tag{B.26}$$

  Thus $D' = \widetilde{D} + D_0 \in \mathrm{im}\left(\varphi\right)$. Hence φ is surjective.

- Let $D', D'' \in D_0(M_g)$ two different divisors. Then we consider

$$\varphi(D') = D' + \widetilde{D}, \quad \varphi(D'') = D'' + \widetilde{D} \tag{B.27}$$

  But from $D' \neq D''$ it follows $\varphi(D') \neq \varphi(D'')$, proving that φ is injective.

Consequently φ is an isomorphism. ■

Consequence:
The map φ induces an isomorphism

$$\widetilde{\varphi}\colon \mathrm{Pic}_0(M_g) \to \mathrm{Pic}_d(M_g) \tag{B.28}$$

We thus conclude that there are as many equivalence classes of holomorphic line bundles of degree d_1 as there are equivalence classes of holomorphic line bundles of degree d_2 on any compact and connected Riemann surface M_g of genus g.

C. The Appell-Humbert Theorem

In this chapter we will be interested in compact, connected and complex Lie groups X of complex dimension d, i.e. a compact connected complex manifold of dimension d equipped with a group structure such that the maps

$$X \times X \to X \ , \ (x, y) \mapsto x \cdot y \qquad X \to X \ , \ x \mapsto x^{-1} \tag{C.1}$$

are holomorphic. Throughout this chapter, X will represent such a structure.
The Appell-Humbert theorem gives a classification of all holomorphic line bundles on such spaces. In particular the complex 2-torus is therefore covered. After introducing the Appell-Humbert theorem in its general form we will specialise to the case of a complex 2-torus to get a better feeling for the interplay between divisors and holomorphic line bundles.
Finally we mention that this theorem dates back to works of Appell [82] and Humbert [83] around 1890 and was later generalised by Lefschetz in 1921 [84]. A particularly nice exposition of this material can be found in [85] to which we will refer for several proofs.

C.1. Classification Of Holomorphic Line Bundles On Complex Tori

C.1.1. Complex Tori

Definition C.1.1 (Lattice)**:**
Let V a complex vector space of complex dimension d. Then any basis $\mathcal{B}$ of V over $\mathbb{R}$ is made up of $2d$ elements, i.e.

$$\mathcal{B} = \{v_1, \dots, v_{2d}\} \tag{C.2}$$

Via the Abelian group structure of V upon addition, the elements of $\mathcal{B}$ generate a group

$$\Lambda_{\mathcal{B}} := \left\{ \sum_{i=1}^{2d} a_i \cdot v_i \ , \ a_i \in \mathbb{Z} \right\} \tag{C.3}$$

We term any such group $\Lambda_{\mathcal{B}}$ a lattice in V (of full rank).

Remark:
Let V a complex vector space of complex dimension d. Then any lattice $\Lambda_{\mathcal{B}}$ in V satisfies $\Lambda \cong \mathbb{Z}^{2d}$, since a lattice is a free and finitely generated Abelian group.

Remark (The Exponential Map)**:**
Let G a connected matrix Lie group and its associated Lie algebra. Consider the exponential map

$$\mathfrak{g} \to G \,,\; X \mapsto \exp X \tag{C.4}$$

Then first recall the following fact.

$\boxed{\text{For every } A \in G \text{ there exists a finite number of elements in the Lie algebra } X_1, \ldots, X_m \in \mathfrak{g} \text{ such that } A = \exp X_1 \cdot \cdots \cdot \exp X_m.}$

In particular note that in general for a connected matrix Lie group, the exponential map is not surjective. To see this consider the non-commutative and simply connected matrix Lie group $\mathrm{SL}(2, \mathbb{C})$. Then we have

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \tag{C.5}$$

Still for every $X \in \mathfrak{sl}(2, \mathbb{C})$ it holds $\exp X \neq A$. This is pointed out in [86], page 49. More information on the exponential map can also be found in [86], from where we quote the following two important facts.

1. If G is a commutative and connected matrix Lie group, then the exponential map is surjective, i.e. for every $A \in G$ there exists $X \in \mathfrak{g}$ such that $A = \exp X$.
2. If G is a compact and connected matrix Lie group, then the exponential map is surjective.
3. If G is in a simply connected and nilpotent matrix Lie group, then the exponential map is a homeomorphism.

Theorem C.1.1:
Let X a compact, connected and complex Lie group of complex dimension d and $V := T_e(X)$. Then the following holds true.

- X is commutative. As a consequence of the above remark we thus know that the exponential map
$$\pi \colon V \to X \tag{C.6}$$
is a surjective group homomorphism.
- Moreover $\ker(\pi) = \Lambda \subset V$ is a lattice (of full rank). Consequently we have by means of the exponential map
$$X \cong V/\Lambda \cong \mathbb{C}^d/\Lambda \tag{C.7}$$

Proof
The proof can be found in [85]. ∎

Definition C.1.2 (Complex Torus)**:**
Let V a complex vector space of dimension d and $\Lambda \subset V$ a lattice (of full rank). Any complex manifold X which is biholomorphic to V/Λ is termed a complex torus.

Consequence:
Any compact, connected and complex Lie group X of dimension d is a complex torus with $X \cong \mathbb{C}^d/\Lambda$.

Remark:
Recall the following two facts.

- $H^p\left(\mathbb{C}^d, \mathcal{O}_{\mathbb{C}^d}\right) = 0$ for $p \geq 1$.
- $H^p\left(\mathbb{C}^d, \mathcal{O}^*_{\mathbb{C}^d}\right) = 0$ for $p \geq 0$.

Consequence:
In Appendix A we identified the elements of $H^1\left(\mathbb{C}^d, \mathcal{O}^*_{\mathbb{C}^d}\right)$ with holomorphic line bundles on $\mathbb{C}^d$. Since this cohomology group is trivial, we conclude that all holomorphic line bundles on $\mathbb{C}^d$ are trivial.
A different means to justify this result is to use the homotopy axiom [80] - $\mathbb{C}^d$ can be contracted to a point. This property is termed *point-homotopic*. The homotopy-axiom states that on a point-homotopic topological space, every smooth line bundle is trivial. Applied to $\mathbb{C}^d$ we obtain, as special case, that all holomorphic line bundle on $\mathbb{C}^d$ are trivial.

Remark:
Let X a compact, connected and complex Lie group of dimension d. Then we can consider the surjective homomorphism

$$\pi\colon V \cong \mathbb{C}^d \to X \cong \mathbb{C}^d/\Lambda \tag{C.8}$$

Now let $\mathcal{L}$ a holomorphic line bundle on X. The pullback line bundle $\pi^*\left(\mathcal{L}\right)$ is a line bundle on V. From the above observation and $V \cong \mathbb{C}^d$ we conclude, that this bundle is trivial, i.e.

$$\pi^*\left(\mathcal{L}\right) \cong V \times \mathbb{C} \tag{C.9}$$

The important consequence is now, that π induces an isomorphism which allows us to represent every holomorphic line bundle on X as a space of the form $\left(V \times \mathbb{C}\right)/\widehat{\Lambda}$ with a so-called lift action $\widehat{\Lambda}$ of the lattice $\Lambda \subset V$. Understanding these lift-actions is therefore equivalent to understanding all holomorphic line bundles on X. To this end we first give a brief introduction to the subject of group cohomology.

C.1.2. Disgression - Group Cohomology

Remark:
So far we have introduced sheaf cohomology and sheaf-valued Čech cohomology. In the next subsection we will however need to make use of a different cohomology, the so-called group cohomology. Group cohomology can be approached in a manner very similar to Čech cohomology. For that reason we will introduce it in this fashion. For more details see [87].

Definition C.1.3 (G-Module)**:**
A G-Module is an abelian group M together with a action of the group G on M, such that every element of G acts as an automorphism of M.

Example C.1.1:
We construct the $\mathbb{Z}$-module $\mathbb{R}$. To this end we define a map

$$\rho: \mathbb{Z} \to \mathrm{Aut}(\mathbb{R}) \ , \ n \mapsto [\rho_n: \mathbb{R} \to \mathbb{R} \ , \ x \mapsto x + n] \tag{C.10}$$

Then $(\mathbb{R}, \rho)$ does form a $\mathbb{Z}$-module.

Remark:
The notion of a module is often introduced to mean ring module. This notion we use when we state that a holomorphic line bundle on X is locally free $\mathcal{O}_X$ module. Still it is possible to generalise this notion to mean group module, as we just did. Note in particular that a field module is a vector space.

Definition C.1.4 (Chains)**:**
Let G a group and M a G-module. Then we define for $n \in \mathbb{N}_{\geq 0}$ the cochains as

$$C^n(G, M) = \{f: G^n \to M\} \tag{C.11}$$

Note:
The definition of group cochains is much simpler than the notion of a Čech cochain.

Definition C.1.5 (Boundary Operator)**:**
We now define the boundary operator $d^n: C^n(G, M) \to C^{n+1}(G, M)$ as follows. Let $\varphi \in C^n(G, M)$, i.e. $\varphi: G^n \to M$. Then $(d^n \varphi): G^{n+1} \to M$ is defined by

$$\begin{aligned}(d^n \varphi)(g_1, \ldots, g_{n+1}) = & g_1 \circ \varphi(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_{i-1}, g_i \circ g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} \varphi(g_1, \ldots, g_n)\end{aligned} \tag{C.12}$$

Remark:

- The term $g_1 \circ \varphi(g_2, \ldots, g_{n+1})$ is determined by the action of the group G onto the G-module M.
- It is easily checked that $d^{n+1} \circ d^n = 0$. Consequently we have a complex

$$0 \to C^0(G, M) \xrightarrow{d^0} C^1(G, M) \xrightarrow{d^1} C^2(G, M) \xrightarrow{d^2} \ldots \tag{C.13}$$

Definition C.1.6 (Cocycles And Coboundaries)**:**
Let G a group and M a G-module. For $n \in \mathbb{N}_{\geq 0}$ we define

- $Z^n(G, M) = \ker(d^n)$ to be the n-th cocycle group.

- $B^n(G,M) = \operatorname{im}(d^{n-1})$ to be the n-th coboundary group if $n \geq 1$. In addition we define $B^0(G,M) = \{0\}$ - the trivial group.

Consequence:
From $d^{n+1} \circ d^n = 0$ it follows that $B^n(G,M) \subset Z^n(G,M)$ is a subgroup. This fact enables us to consider the following quotient groups.

Definition C.1.7 (Group Cohomology Groups)**:**
Let G a group and M a G-module. Then the n-th group cohomology class is defined by

$$H^n(G,M) := Z^n(G,M)/B^n(G,M) \tag{C.14}$$

or equivalently as the cohomologies of the complex

$$0 \to C^0(G,M) \overset{d^0}{\to} C^1(G,M) \overset{d^1}{\to} C^2(G,M) \overset{d^2}{\to} \dots \tag{C.15}$$

Example C.1.2:
We will now give an example of the above strategy which we will use in the next subsection. Let $G = \Lambda \subset \mathbb{C}$ a lattice. Since $\mathcal{O}^*_{\mathbb{C}}(\mathbb{C})$ is an Abelian group upon multiplication, the following Λ-action makes it a Λ-module

$$\varphi: \Lambda \times \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}) \to \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}) \ , \ (u,e) \mapsto [\widetilde{e}: \mathbb{C} \to \mathbb{C}^* \ , \ z \mapsto f(z+u)] \tag{C.16}$$

Thus we can consider the group cohomologies $H^p(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}))$. One finds

- $C^0(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})) = \{e \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}$
- Let $(e) \in C^0(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}))$. Then its boundary is the function

$$\delta(e): \Lambda \to \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \ , \ u \mapsto [\widetilde{e}_u: \mathbb{C} \to \mathbb{C}^* \ , \ z \mapsto e(z+u)] \tag{C.17}$$

- An element of $C^1(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}))$ can be represented by a collection

$$E = \{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u \in \Lambda} \tag{C.18}$$

- In addition it holds $E \in Z^1(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}))$ precisely if for all $u_1, u_2 \in \Lambda$ and $z \in \mathbb{C}$ it holds

$$e_{u_1+u_2}(z) = e_{u_1}(z) \cdot e_{u_2}(z+u_1) \tag{C.19}$$

Consequently an element of $H^1(\Lambda, \mathcal{O}^*_{\mathbb{C}}(\mathbb{C}))$ is an equivalence class of collections $\{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u \in \Lambda}$ such that

- For all $u_1, u_2 \in \Lambda$ and any $z \in \mathbb{C}$ it holds $e_{u_1+u_2}(z) = e_{u_1}(z) \cdot e_{u_2}(z+u_1)$.
- Two collections $\{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u \in \Lambda}$ and $\{e'_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u \in \Lambda}$ are equivalent precisely if there exists $h \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})$ such that for all $u \in \Lambda$ and $z \in \mathbb{C}$ it holds

$$e'_u(z) = \frac{h(z+u)}{h(z)} \cdot e_u(z) \tag{C.20}$$

C.1.3. Pullback Bundles In Group Cohomology

Construction C.1.1 (Complex Structure On X)**:**
Recall that X is a connected, compact and complex Lie group of complex dimension d. We already found $X \cong V/\Lambda$ for $\Lambda \subset V$ a lattice. In particular we have a projection map

$$\pi : V \to X \tag{C.21}$$

Since X is compact and connected we can find a finite open cover $\mathcal{V} = \{V_i\}_{1 \le i \le n}$ of X in which each V_i is connected. Then note.

- For every $V_i \in \mathcal{V}$ there exists $W_i \subset V$ open and connected such that the following is a disjoint union
$$\pi^{-1}(V_i) = \bigcup_{u \in \Lambda} u + W_i \tag{C.22}$$
- Let $\pi_i = \pi|_{W_i}$. Then $\pi_i : W_i \to V_i$ is a biholomorphism.
- If $V_i \cap V_j \neq \varnothing$, then there exists a unique $u_{ij} \in \Lambda$ such that
$$\pi_j^{-1}(V_i \cap V_j) = \pi_i^{-1}(V_i \cap V_j) + u_{ij} \tag{C.23}$$

By these means we can related the complex structures on X and V.

Remark (Čech 1-Cocycle On X)**:**
Recall that a Čech 1-cocycle $G \in Z^1(\mathcal{V}, \mathcal{O}_X^*)$ is a collection $G = \{g_{ij} \in \mathcal{O}_X^*(V_i \cap V_j)\}_{1 \le i,j \le n}$ such that

- $g_{ij}(z) = \frac{1}{g_{ji}(z)}$ for all $z \in V_i \cap V_j$
- $g_{ij}(z)\, g_{jk}(z) = g_{ik}(z)$ for all $z \in V_i \cap V_j \cap V_k$

We found earlier that certain equivalence classes of such Čech 1-cocycles encode holomorphic line bundles.

Construction C.1.2 (Holomorphic Line Bundles From Group 1-Cycle)**:**
Let $E = \{e_u \in \mathcal{O}_V^*(V)\}_{u \in \Lambda} \in Z^1(\Lambda, \mathcal{O}_V^*(\mathbb{C}^d))$ a group 1-cocycle. We will now construct a Čech 1-cocycle G on $X \cong V/\Lambda$ from E.

1. Whenever $V_i \cap V_j \neq \varnothing$ there is a unique $u_{ij} \in \Lambda$ with the property
$$\pi_j^{-1}(V_i \cap V_j) = \pi_i^{-1}(V_i \cap V_j) + u_{ij} \tag{C.24}$$
Hence we can use this uniquely determined u_{ij} to define a function
$$g_{ij} : V_i \cap V_j \to \mathbb{C}^*\ ,\ x \mapsto \left(e_{u_{ij}} \circ \pi_i^{-1}\right)(x) \tag{C.25}$$
Especially this implies $g_{ij} \in \mathcal{O}_X^*(X)$.
2. Now consider the collection $G = \{g_{ij}\}_{(i,j) \in I^2}$ where $I \subset \{1, \dots, n\}^2$ so that $V_i \cap V_j \neq \varnothing$. It is not too hard to verify from the cocycle conditions on the group 1-cycle E, that the following holds true

- $g_{ij}(x)\,g_{jk}(x) = g_{ik}(x)$ for all $x \in V_i \cap V_j \cap V_k$
- $g_{ij}(x) = \frac{1}{g_{ji}(x)}$ for all $x \in V_i \cap V_j$

Thus G is a Čech 1-cocycle. In fact more is true, as we will learn momentarily.

Definition C.1.8 (Lift Action)**:**
Let $E = \{e_u \in \mathcal{O}_V^*(V)\}_{u\in\Lambda} \in H^1(\Lambda, \mathcal{O}_V^*)$ a representative of the equivalence class. Then this representative induces the following $\widehat{\Lambda}$ lift action

$$\widehat{\Lambda}: \Lambda \times (V \times \mathbb{C}) \mapsto (V \times \mathbb{C}) \ , (u, (z, f)) \mapsto (z + u, e_u(z) \cdot f) \tag{C.26}$$

Remark:
Note that for $h \in \mathcal{O}_V^*(V)$, the equivalent representative

$$E' = \left\{ \frac{h(z+u)}{h(z)} \cdot e_u(z) \in \mathcal{O}_V^*(V) \right\}_{u\in\Lambda} \in H^1(\Lambda, \mathcal{O}_V^*) \tag{C.27}$$

gives a seamingly different lift-action

$$\widehat{\Lambda}: \Lambda \times (V \times \mathbb{C}) \mapsto (V \times \mathbb{C}) \ , (u, (z, f)) \mapsto \left(z + u, \frac{h(z+u)}{h(z)} e_u(z) \cdot f \right) \tag{C.28}$$

Still this action is well-defined in the following sense.

Theorem C.1.2:
There is a natural isomorphism $H^1(\Lambda, \mathcal{O}_{\mathbb{C}}^*) \cong H^1(X, \mathcal{O}_X^*)$ induced by

$$\{e_u \in \mathcal{O}_V^*(V)\}_{u\in\Lambda} \mapsto (V \times \mathbb{C}) / \widehat{\Lambda} \tag{C.29}$$

Proof
A proof can be found in [85]. ■

Consequence:
Group 1-cocycles on V and Čech 1-cocycles on X are thus equivalent descriptions for holomorphic (coordinate) line bundles on X. In the following we will prefer to take the group 1-cocycle perspective. In particular one then terms a group 1-cocycle $\{e_u \in \mathcal{O}_V^*(V)\}$ a multiplier set for the encoded holomorphic line bundle.

C.2. First Chern-Class From Group 1-Cycle

Remark:
A holomorphic line bundle on X is an element of $H^1(X, \mathcal{O}_X^*)$. In addition the sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 0 \tag{C.30}$$

is sheaf exact and thus induces a long exact sequence of cohomology. In particular it induces a connecting homomorphism

$$\delta: H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \tag{C.31}$$

which takes a holomorphic line bundle on X to its first Chern class.

Lemma C.2.1:
Let $X \cong V/\Lambda$ a complex torus. Then there is an isomorphism

$$\Xi_1 \colon H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(\Lambda, \mathbb{Z}) \tag{C.32}$$

Proof
The proof can be found in [85]. ∎

Note:
Let $E = \{e_u \in \mathcal{O}_V^*(V)\} \in H^1(\Lambda, \mathcal{O}_V^*)$ a holomorphic line bundle. Recall that $V \cong \mathbb{C}^d$ and that $\mathbb{C}^d$ is simply connected. From complex calculus it then follows that any $e_u \in \mathcal{O}_V^*(V)$ has a holomorphic logarithm. Thus there exist functions $f_u \in \mathcal{O}_V(V)$ such that

$$e_u(z) = e^{2\pi i f_u(z)} \ , \ \forall z \in V \tag{C.33}$$

The cocycle condition for e_u then implies

$$f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1+u_2}(z) \in \mathbb{Z} \tag{C.34}$$

But the functions f_u are holomorphic, so in particular continous. Since $V \cong \mathbb{C}^d$ is connected, this implies that the above sum is a constant function on all of V. Thus we can consider the collection

$$\{n_{u_1,u_2} \coloneqq f_{u_2}(u_1) + f_{u_1}(0) - f_{u_1+u_2}(0)\}_{(u_1,u_2)\in\Lambda^2} \tag{C.35}$$

where we arbitarily picked $z = 0$ for evaluation, whilst we could also have chosen any other $z \in V$. We claim that this collection gives an element in $H^2(\Lambda, \mathbb{Z})$.

- First we check that this chain is closed, i.e. is a cocycle. This follows since

$$\begin{aligned}
\left(d^2 n\right)(u_1, u_2, u_3) &= u_1 \circ n_{u_2,u_3} - n_{u_1+u_2,u_3} + n_{u_1,u_2+u_3} - n_{u_1,u_2} \\
&= f_{u_3}(z + u_1 + u_2) + f_{u_2}(z + u_1) - f_{u_2+u_3}(z + u_1) \\
&\quad - [f_{u_3}(z + u_1 + u_2) + f_{u_1+u_2}(z) - f_{u_1+u_2+u_3}(z)] \\
&\quad + f_{u_2+u_3}(z + u_1) + f_{u_1}(z) - f_{u_1+u_2+u_3}(z) \\
&\quad - [f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1+u_2}(z)] \\
&= 0
\end{aligned} \tag{C.36}$$

- Next we have to recall that we can alter this collection by a coboundary. To this end we pick a 1-chain $A = \{a_u \in \mathbb{Z}\}_{u\in\Lambda}$ which gives

$$\delta A = \left\{a_{u_2} + a_{u_1} - a_{u_1+u_2}\right\}_{(u_1,u_2)\in\Lambda^2} \tag{C.37}$$

  Thus we can alter the above collection to obtain

$$\left\{n'_{u_1,u_2} = n_{u_1,u_2} + a_{u_2} + a_{u_1} - a_{u_1+u_2}\right\}_{(u_1,u_2)\in\Lambda^2} \tag{C.38}$$

Consequently $\{n_{u_1,u_2}\}_{(u_1,u_2)\in\Lambda^2}$ is a representative of a class in $H^2(\Lambda, \mathbb{Z})$.

Lemma C.2.2:
Let X a complex torus and $E = \{e_u \in \mathcal{O}_V^*(V)\}_{u\in\Lambda} \in H^1(\Lambda, \mathcal{O}_V^*)$ a holomorphic line bundle on X. Then we can find $f_u \in \mathcal{O}_V(V)$ with

$$e_u(z) = e^{2\pi i f_u(z)} \ , \ \forall z \in V \tag{C.39}$$

and via the isomorphism $\Xi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(\Lambda,\mathbb{Z})$ the first Chern class of the line bundle given by E is mapped to the class

$$\{n_{u_1,u_2} \coloneqq f_{u_2}(u_1) + f_{u_1}(0) - f_{u_1+u_2}(0)\}_{(u_1,u_2)\in\Lambda^2} \in H^2(\Lambda,\mathbb{Z}) \tag{C.40}$$

Proof
The proof can be found in [85]. ■

Remark:
In an abuse of notation we can thus write

$$c_1(E) = \{n_{u_1,u_2} \coloneqq f_{u_2}(u_1) + f_{u_1}(0) - f_{u_1+u_2}(0)\}_{(u_1,u_2)\in\Lambda^2} \in H^2(\Lambda,\mathbb{Z}) \tag{C.41}$$

Lemma C.2.3:
Let $X \cong V/\Lambda$ a complex torus. Then there is an isomorphism

$$\Xi_2\colon H^2(\Lambda,\mathbb{Z}) \xrightarrow{\sim} \Lambda^2\mathrm{Hom}(\Lambda,\mathbb{Z}) \tag{C.42}$$

Proof
The proof can be found in [85]. ■

Note:
Let X a complex torus and $E = \{e_u \in \mathcal{O}_V^*(V)\}_{u\in\Lambda} \in H^1(\Lambda, \mathcal{O}_V^*)$ a holomorphic line bundle on X. Then we can find $f_u \in \mathcal{O}_V(V)$ with

$$e_u(z) = e^{2\pi i f_u(z)} \ , \ \forall z \in V \tag{C.43}$$

and the first Chern class of the line bundle given by E can be considered as the alternating 2-form

$$c_1(E)\colon \Lambda\times\Lambda \to \mathbb{Z} \ , \ (u_1,u_2) \mapsto f_{u_2}(z+u_1) + f_{u_1}(z) - f_{u_1}(z+u_2) - f_{u_2}(z) \tag{C.44}$$

where we used $c_1(E)$ in an abuse of notation.
Note that the entire class $H^2(X,\mathbb{Z})$ is mapped via $\Xi_2 \circ \Xi_1$ to the above alternating 2-form and *not* to a class of alternating 2-forms.

C.3. The Appell-Humbert Theorem

Theorem C.3.1 (Appell-Humbert)**:**
Let X a compact, connected and complex Lie group of dimension d. Then we have

$$X \cong V/\Lambda \tag{C.45}$$

with $\Lambda \subset V$ a lattice and $V = T_e(X) \cong \mathbb{C}^d$.

1. Let H a hermitian inner product on V such that

$$E\colon \Lambda \times \Lambda \to \mathbb{Z}\,,\ (x,y) \mapsto \operatorname{im}\left(H\left(u_1,u_2\right)\right) \tag{C.46}$$

   is an integer-valued, skew symmetric form on V.

2. For any such hermitian inner product H there exists a (not necessarily unique) map $\alpha\colon \Lambda \to \{z \in \mathbb{C}\,,\ |z| = 1\} = U(1)$ with the property

$$\alpha\left(u_1+u_2\right) = e^{-i\pi E(u_1,u_2)} \cdot \alpha\left(u_1\right)\alpha\left(u_2\right)\ \forall u_1,u_2 \in \Lambda \tag{C.47}$$

Given the data (H,α) we then have the following important results.

- By setting
$$e_u\colon V \to \mathbb{C}\,,\ z \mapsto \alpha(u) \cdot e^{-\pi H(u,z)-\frac{\pi}{2}H(u,u)} \tag{C.48}$$
  the collection $\{e_u \in \mathcal{O}_V^*(V)\}_{u\in\Lambda}$ gives an element in $H^1\left(\Lambda,\mathcal{O}_V^*\right)$ which defines a holomorphic line bundle $L(H,\alpha)$ on X.
- The first Chern class of $L(H,\alpha)$ is mapped via $\Xi_2 \circ \Xi_1$ to the integer-valued, skew symmetric form E given above.
- Any line bundle on X is isomorphic to a line bundle $L(H,\alpha)$ for a uniquely determined pair (H,α).
- The holomorphic sections of $L(H,\alpha)$ are one-to-one to holomorphic functions $s \in \mathcal{O}_V(V)$ which satisfy for any $z \in V$ and any $u \in \Lambda$

$$s(z+u) = \alpha(u) \cdot e^{-\pi H(z,u)-\frac{\pi}{2}H(u,u)} s(z) \tag{C.49}$$

Proof
The proof can be found in [85]. ■

Remark (Relation With Gauge Theories)**:**
The first Chern class will in the situation of complex tori be easily related to the connection 2-form, which is physically interpreted as the field strength. Then the above theorem tells us that the field strength is not all the data that we need to specify a gauge theory on the torus. Rather we need in addition the function α. This function can be interpretated as a representation of the group $U(1)$ which links the above holomorphic line bundles to $U(1)$ gauge theories. In addition one can set

$$\alpha(u) := e^{i\lambda(u)} \tag{C.50}$$

for some real valued function $\lambda\colon \Lambda \to \mathbb{R}$. Then the sections transform according to

$$s(z+u) = e^{i\lambda(u)-\pi H(z,u)-\frac{\pi}{2}H(u,u)} s(z) \tag{C.51}$$

The quantity $i\lambda(u) - \pi H(z,u) - \frac{\pi}{2}H(u,u)$ is then closely related to the connection 1-form of $L(H,\alpha)$ which in the physics literature is known as the gauge field. More information on this relation can be found in [80].

C.4. Example - Holomorphic Line Bundles On The Complex 2-Torus

C.4.1. Hermitian Forms On The Complex Plane

Remark:
We wish to consider a connected, compact and complex Lie group X of dimension $d = 1$. Then we know

$$X \cong \mathbb{C}/\Lambda \tag{C.52}$$

with $\Lambda \subset \mathbb{C}$ a lattice. Though all complex 1-tori are topologically equivalent, not all of them are biholomorphically equivalent. Rather the lattice Λ can always be written as

$$\Lambda = \mathbb{Z} + \mathbb{Z}\tau \tag{C.53}$$

where $\tau \in F_0 = \left\{z \in \mathbb{H}\ ,\ |z| > 1, |\Re(\tau)| < \frac{1}{2}\right\}$ lies in the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$. In particular note that $\tau_1, \tau_2 \in F_0$ with $\tau_1 \neq \tau_2$ correspond to biholomorphically distinct complex tori, so that τ can be identifed as the complex structure modulus.
In conclusion, there is a one-to-one correspondance between points in F_0 and biholomorphic equivalence classes of complex tori of dimension $d = 1$.

Note (Convention)**:**
We will denote the complex 2-torus with complex structure modulus $\tau \in F_0$ by $\mathbb{C}_{1,\tau}$.

Lemma C.4.1:
Every Hermitian form on $\mathbb{C}$ is given by

$$H_a \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}\ ,\ (u, v) \mapsto a \cdot \overline{u} \cdot v \tag{C.54}$$

for suitable $a \in \mathbb{C}$.

Remark (Associated Skew-Symmetric Form)**:**
To the Hermitian forms H_a ($a \in \mathbb{C}$) we now associate skew-symmetric forms

$$E_a \colon \Lambda \times \Lambda \to \mathbb{C}\ ,\ (u_1, u_2) \mapsto \mathrm{im}\left(H_a\left(u_1, u_2\right)\right) \tag{C.55}$$

However, we have learned in the previous chapter that for the construction of holomorphic line bundles on $\mathbb{C}_{1,\tau}$ we have to restrict to such hermitian forms H_a such that E_a is integer-valued. Thus we require that for any two $u_1, u_2 \in \Lambda$ it holds

$$E_a\left(u_1, u_2\right) \in \mathbb{Z} \tag{C.56}$$

An easy calculation shows that this leads to the requirement

$$a = \frac{m}{\mathrm{im}\left(\tau\right)} \tag{C.57}$$

where $m \in \mathbb{Z}$ is arbitary.

Consequence:
For the construction of holomorphic line bundles on $\mathbb{C}_{1,\tau}$ we consider the Hermitian forms

$$H_m\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}\ ,\ (w,z) \mapsto \frac{mz\overline{w}}{\mathrm{im}\,(\tau)}, \quad m \in \mathbb{Z} \tag{C.58}$$

with associated real-valued, skew-symmetric form

$$E_m\colon \Lambda \times \Lambda \to \mathbb{Z}\ ,\ (a_1 + b_1\tau, a_2 + b_2\tau) \mapsto m\,[a_1 b_2 - a_2 b_1] \tag{C.59}$$

C.4.2. The α-Map And The Group 1-Cycles

Remark:
Now we focus on the map $\alpha\colon \Lambda \to U(1)$ with the property

$$\alpha(u_1 + u_2) = e^{-i\pi E_m(u_1,u_2)} \cdot \alpha(u_1) \cdot \alpha(u_2) \tag{C.60}$$

We will argue that for given E_m these maps are labeled by a point $b \in \mathbb{C}_{1,\tau}$. This implies that all line bundles on $\mathbb{C}_{1,\tau}$ are labeled by a pair (m,b) with $m \in \mathbb{Z}$ and $b \in \mathbb{C}_{1,\tau}$.

Note (Convention)**:**
We can find a map $\lambda\colon \Lambda \to \mathbb{R}$ such that

$$\alpha\colon \Lambda \mapsto \{z \in \mathbb{C}\ ,\ |z| = 1\} = U(1)\ ,\ u \mapsto e^{i\lambda(u)} \tag{C.61}$$

Consequence:
The requirement $\alpha(u_1 + u_2) = e^{-i\pi E_m(u_1,u_2)} \cdot \alpha(u_1) \cdot \alpha(u_2)$ is then equivalent to

$$\lambda(u_1 + u_2) = \lambda(u_1) + \lambda(u_2) - \pi E_m(u_1,u_2) + 2\pi\mathbb{Z} \qquad \forall u_1, u_2 \in \Lambda \tag{C.62}$$

Construction C.4.1 (Group 1-Cycle)**:**
Given the real-valued, skew-symmetry 2-form E_m and the map α, the group 1-cocycle $\{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u \in \Lambda}$ encoding the holomorphic line bundle $L(H_m, \alpha)$ is given by setting

$$e_u(z) = e^{i\lambda(u) - \frac{\pi m \overline{u}}{\mathrm{im}(\tau)} z - \frac{\pi m \overline{u} u}{2\mathrm{im}(\tau)}} \tag{C.63}$$

For ease of notation we define

$$a(u) \coloneqq -\frac{m\pi\overline{u}}{\mathrm{im}\,(\tau)}, \qquad b(u) \coloneqq i\lambda(u) - \frac{\pi m u \overline{u}}{2\mathrm{im}\,(\tau)} \tag{C.64}$$

Then the transformation behaviour of the function α dictates that for $u_1, u_2 \in \Lambda$ it holds

$$\begin{aligned} a(u_1 + u_2) &= a(u_1) + a(u_2) \\ b(u_1 + u_2) &= b(u_1) + b(u_2) + a(u_1) \cdot u_2 + 2\pi i\mathbb{Z} \end{aligned} \tag{C.65}$$

Remark:
Recall that we can alter a group 1-cocycle by a function $\beta \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})$ without changing the encoded element in $H^1(\Lambda, \mathcal{O}^*_{\mathbb{C}})$. We pick

$$\beta\colon \mathbb{C} \to \mathbb{C}^*\ ,\ z \mapsto e^{Az^2+Bz}, \qquad A = \frac{\pi m}{2\mathrm{im}(\tau)}, \quad B = -i \cdot \lambda(1) \tag{C.66}$$

Then one readily verifies that the new group 1-cocycle $\{e'_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u\in\Lambda}$ is given by

$$e'_u(z) = e^{a'(u)z+b'(u)} \tag{C.67}$$

where

$$a'(1) = b'(1) = 0, \qquad a'(\tau) = 2\pi i m, \qquad b'(\tau) = \frac{\pi m \tau^2}{2\mathrm{im}(\tau)} - i\lambda(1)\tau + b(\tau) \tag{C.68}$$

and all other values are determined by Equation C.65. Thus we can represent the original group 1-cocycle in a condensed notation by

$$e'_1(z) = 1, \qquad e'_\tau(z) = e^{2\pi i m z + b'(\tau)} \tag{C.69}$$

Comment (Summary)**:**
We have thus concluded that all line bundles on $\mathbb{C}_{1,\tau}$ are classified by group 1-cocycles $\{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u\Lambda}$ which can always be taken to be of the form

$$e_1(z) = 1, \qquad e_\tau(z) = e^{2\pi i m z + b'(\tau)} \tag{C.70}$$

with all other elements determined by the cocycle condition.
Consequently we can indeed specify all line bundles on $\mathbb{C}_{1,\tau}$ by a pair $(m, b'(\tau))$ with $m \in \mathbb{Z}$ and $b'(\tau) \in \mathbb{C}$. We will find later that $b'(\tau) \in \mathbb{C}_{1,\tau}$.

Notation:
From now on, we agree on the simplified notation

$$b'(\tau) \to b \tag{C.71}$$

Remark (Meromorphic Sections)**:**
Meromorphic sections of the holomorphic line bundle $L(m,b)$ are one-to-one with meromorphic functions $F\colon \mathbb{C} \to \overline{\mathbb{C}}$subject to the condition

$$F(z+1) = F(z), \qquad F(z+\tau) = e^{2\pi i m z + b} F(z)\ \forall z \in \mathbb{C} \tag{C.72}$$

A major task will be to find all holomorphic functions that satisfy these conditions for given (m,b) and also to find (at least) one non-trivial meromorphic section.

Remark:
Holomorphic functions with the above property are known as theta functions. In principle we could now introduce the notion of theta functions and thereby relate the task of finding all holomorphic sections of $L(H,\alpha)$ to the study of theta functions. Details on this can be found in [85], [44] as well as [88]. In this thesis however we prefer to follow a different path and construct all holomorphic and meromorphic sections in a more hands-on fashion.

C.4.3. Elliptic Functions, Divisor Classes And Cohomology Classes

Definition C.4.1 (Elliptic Function)**:**
A function $f \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C})$ is an elliptic function with respect to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ precisely if for all $z \in \mathbb{C}$ it holds

$$f(z+1) = f(z), \qquad f(z+\tau) = f(z) \tag{C.73}$$

Remark:

- The constant function $f\colon \mathbb{C} \to \overline{\mathbb{C}}$, $z \mapsto a$ is an elliptic function.
- Elliptic functions are one-to-one to meromorphic functions on $\mathbb{C}_{1,\tau}$.
- It follows from the third theorem of Liouville, that any holomorphic elliptic function is constant [29].

Proposition:
Let $q \in \mathbb{C}$ with $|q| < 1$. Then consider the product

$$a_q\colon \mathbb{C}^* \to \mathbb{C}, \ t \mapsto (1-t) \cdot \prod_{n=1}^{\infty} (1-q^n t) \cdot \left(1 - q^n t^{-1}\right) \tag{C.74}$$

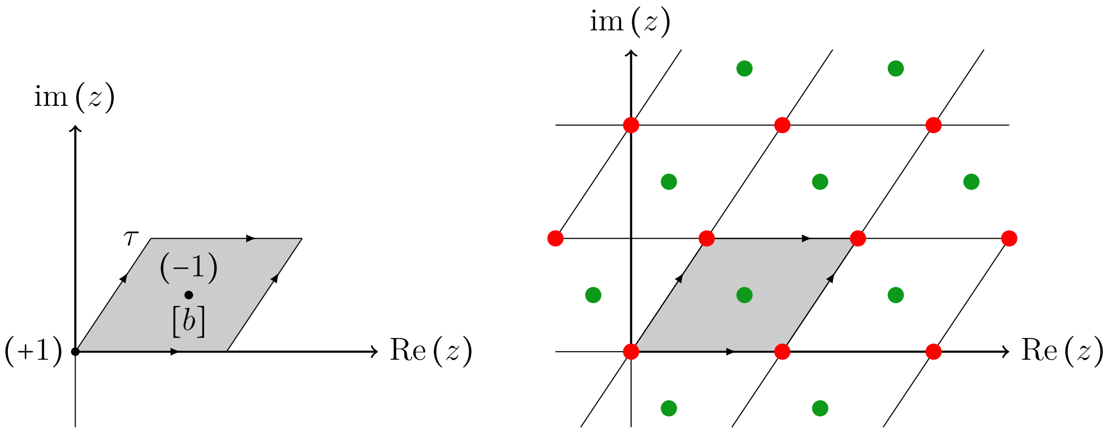
It holds

- The above product converges uniformely on compact subsets of $\mathbb{C}^*$ to a holomorphic function $a_q(t) \in \mathcal{O}(\mathbb{C}^*_{\mathbb{C}^*})$.
- $a_q(t)$ has simple zeros at the points $t = q^n$ for $n \in \mathbb{Z}$ and has no other zeros.
- It holds
$$a_q(qt) = \left(1 - t^{-1}\right) \cdot \frac{a_q(t)}{1-t} = -\frac{1}{t} \cdot a_q(t) \tag{C.75}$$

Proof

- The first two points follows from theorem 15.6 in [89].
- The transformation formula simply follows from computation as follows:

$$\begin{aligned}
a(qt) &= (1-qt) \prod_{n=1}^{\infty} \left(1 - q^{n+1} t\right) \left(1 - q^{n-1} t^{-1}\right) \\
&= (1-qt) \prod_{n=0}^{\infty} \left(1 - q^{n+2} t\right) \left(1 - q^{n} t^{-1}\right) \\
&= \left[(1-qt) \prod_{n=0}^{\infty} \left(1 - q^{n+2} t\right)\right] \cdot \left[\prod_{n=0}^{\infty} \left(1 - q^{n} t^{-1}\right)\right] \\
&= \left[(1-qt)\left(1-q^2 t\right) \prod_{n=1}^{\infty} \left(1 - q^{n+2} t\right)\right] \cdot \left[\left(1 - \frac{1}{t}\right) \cdot \prod_{n=1}^{\infty} \left(1 - q^{n} t^{-1}\right)\right] \\
&= \left[\prod_{n=1}^{\infty} (1 - q^n t)\right] \cdot \left[\left(1 - \frac{1}{t}\right) \cdot \prod_{n=1}^{\infty} \left(1 - q^{n} t^{-1}\right)\right] \\
&= \left(1 - \frac{1}{t}\right) \cdot a(t)
\end{aligned} \tag{C.76}$$



(a) The divisor D as divisor in $\mathbb{C}_{1,\tau}$.

(b) The divisor D as divisor in $\mathbb{C}$.

Figure C.1.: The divisor $D = (+1)[0] + (-1)[b]$ once as divisor in $\mathbb{C}_{1,\tau}$ and once as divisor in $\mathbb{C}$. Note that the gray-shaded $\mathbb{C}_{1,\tau}$ is represented in the standard fashion by a parallelogramm in $\mathbb{C}$ corresponding to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.

This concludes the proof. ∎

Consequence:
Let us set $q(\tau) = e^{2\pi i\tau}$ with $\tau \in F_0$ the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. Then we have $|q(\tau)| < 1$ and can define the function

$$A\colon \mathbb{C} \to \mathbb{C}\,,\ z \mapsto a_{q(\tau)}\left(e^{2\pi iz}\right) \tag{C.77}$$

This function has the following properties

- A is an entire function.
- A has simple zeros at $z \in \Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and no other zeros.
- A satisfies

$$A(z+1) = A(z)\,, \qquad A(z+\tau) = -\exp(-2\pi iz)\cdot A(z) \tag{C.78}$$

Remark (Divisors On $\mathbb{C}_{1,\tau}$)**:**
A divisor D on $\mathbb{C}_{1,\tau}$ is a finite formal sum

$$D = \sum_{i=1}^{n}(a_i)\cdot[p_i] \tag{C.79}$$

where $a_i \in \mathbb{Z}$ and $[p_i] \in \mathbb{C}_{1,\tau}$.

Example C.4.1:
We illustrate the divisor $D = (+1)[0] + (-1)[b]$ with $[b] \in \mathbb{C}_{1,\tau}$ in Figure C.1.

Remark:
We can consider a divisor $D \in \mathrm{Div}\left(\mathbb{C}_{1,\tau}\right)$ also as a divisor on $\mathbb{C}$. To this end $[p_i]$ is to represent an equivalence class of a point in $\mathbb{C}$ with respect to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, namely

$$[p_i] = \{p_i + m + n\tau \ , \ m, n \in \mathbb{Z}\} \tag{C.80}$$

Example C.4.2:
We exemplify this association in Figure C.1.

Remark:
Recall that one can associate to any $f \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C})$ a divisor $\mathrm{div}(f) \in \mathrm{Div}(\mathbb{C})$ which covers the zeros and poles of f counted with their multiplicites a_i.

Claim:
Consider the divisor

$$D = \sum_{i=1}^{n} (a_i)[p_i] \in \mathrm{Div}\left(\mathbb{C}_{1,\tau}\right) \tag{C.81}$$

which satisfies [1]

$$\sum_{i=1}^{n} a_i = 0, \qquad \sum_{i=1}^{n} (n_i)[p_i] = 0 \tag{C.82}$$

Then there exists $f \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C})$ such that $D = \mathrm{div}(f)$.

Proof
First let us separate the set of points in $\mathbb{C}_{1,\tau}$ according to

$$\begin{aligned} Z &:= \{z_i\}_{i\in I} = \{ \text{ points } p_j \text{ in } \mathbb{C}_{1,\tau} \text{ with } a_j > 0 \ \} \\ P &:= \{p_j\}_{j\in J} = \{ \text{ points } p_j \text{ in } \mathbb{C}_{1,\tau} \text{ with } a_j < 0 \ \} \end{aligned} \tag{C.83}$$

Note that the sets Z, P are finite.
We identify the points $z_i, p_j \in \mathbb{C}_{1,\tau}$ with a representative in $\mathbb{C}$. Though the choice of representative is not unique, the following function does not depend on that particular choice

$$f : \mathbb{C} \to \mathbb{C} \ , \ z \mapsto \prod_{i=1}^{|I|} \prod_{j=1}^{|J|} \frac{A(z - z_i)}{A(z - p_j)} \tag{C.84}$$

This is because this function is elliptic with respect to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ which in turn stems from the properties of the function A introduced earlier in this subsection. Finally one easily checks $\mathrm{div}(f) = D$. This proves the statement. ∎

Theorem C.4.1 (Abel Theorem For Complex 2-Torus)**:**
Let $D = \sum_{i=1}^{n} (a_i)[p_i] \in \mathrm{Div}\left(\mathbb{C}_{1,\tau}\right)$. This is the divisor of a meromorphic function on $\mathbb{C}_{1,\tau}$ if and only if

- $\deg(D) = \sum_{i=1}^{n} a_i = 0$
- $\sum_{i=1}^{n} (a_i)[p_i] = 0$ with respect to the additive group action on $\mathbb{C}_{1,\tau}$ respectively.

[1] The second equation is to be read as an equation in the abelian group $\mathbb{C}_{1,\tau}$.

Example C.4.3:
Let us consider some divisors on $\mathbb{C}_{1,\tau}$ and determine whether they are divisors of a meromorphic function on $\mathbb{C}_{1,\tau}$.

- Consider $D_1 = (+1)[p]$ for $[p] \in \mathbb{C}_{1,\tau}$ arbitary. Then however
$$\deg(D_1) = 1 \neq 0 \tag{C.85}$$
So D_1 is not the divisor of an elliptic function.
- Next let $D_2 = (+1)[p] + (-1)[q]$ with $[p] \neq [q] \in \mathbb{C}_{1,\tau}$. Then
$$\deg(D_2) = (+1) + (-1) = 0 \tag{C.86}$$
However with respect to the abelian group structure upon addition on $\mathbb{C}_{1,\tau}$ the divisor D_2 is not trivial since
$$(+1)[p] + (-1)[q] = [p] - [q] \neq [0] \tag{C.87}$$
since we required $[p] \neq [q]$.
- Now consider $D_3 = (+1)[p] + (-1)[q]$ where $[p],[q] \in \mathbb{C}_{1,\tau}$. With the example of D_2 one immediately verifies that D_3 is the divisor of an elliptic function if and only if $[p] = [q]$, i.e. $D_3 \equiv 0$.
- Finally consider
$$D = (-3)[0] + (1)\left[\frac{1}{2}\right] + (1)\left[\frac{\tau}{2}\right] + (1)\left[\frac{1+\tau}{2}\right] \tag{C.88}$$
This non-trivial divisor is easily checked to be the divisor of a meromorphic function.

Remark (Notation)**:**

- The line bundle $L(m,b)$ is encoded by the group 1-cocycle $\{e_u \in \mathcal{O}^*_{\mathbb{C}}(\mathbb{C})\}_{u\in\Lambda}$ where
$$e_1(z) = 1, \qquad e_\tau(z) = e^{2\pi i m z + b} \tag{C.89}$$
and all other elements are given by the cocycle condition.
- For the line bundle $L(m,b)$ we set
$$\begin{aligned} S(m,b) := \{F \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C}) \ , \ F(z+1) = F(z) \ \text{and} \\ F(z+\tau) = e^{2\pi i m z + b} F(z)\} \cup \{0\} \end{aligned} \tag{C.90}$$
- Moreover we set
$$H(m,b) := S(m,b) \cap \mathcal{O}_{\mathbb{C}}(\mathbb{C}) \tag{C.91}$$

Claim:
For every $f \in S(m,b) - \{0\}$ there exists an elliptic function $e: \mathbb{C} \to \overline{\mathbb{C}}$ and a holomorphic section $h \in H(m,b) - \{0\}$ such that
$$f(z) = e(z) \cdot h(z) \ \forall z \in \mathbb{C} \tag{C.92}$$

Proof
Let $f \in S(m,b) - \{0\}$ and $h \in H(m,b) - \{0\}$, then the function

$$e : \mathbb{C} \to \overline{\mathbb{C}}\ ,\ z \mapsto \frac{f(z)}{h(z)} \tag{C.93}$$

is an elliptic function. ■

Remark:
To every $f \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C})$ one can associate a divisor $\operatorname{div}(f) \in \operatorname{Div}(\mathbb{C})$. In particular this applies to all non-trivial meromorphic sections of $L(m,b)$, to which one can even associate a divisor in $\operatorname{Div}(\mathbb{C}_{1,\tau})$ since their poles and zeros are periodic with respect to the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.

Remark (Divisor Class Group)**:**
Let $D, E \in \operatorname{Div}(\mathbb{C}_{1,\tau})$. Then linear equivalence of divisors is defined as

$$D \sim E \iff \exists f \in \mathcal{M}^*(\mathbb{C}_{1,\tau}) : D = E + \operatorname{div}(f) \tag{C.94}$$

The associated equivalence class is denoted $[D]$.

Note:
The Abel-Theorem in its special version to elliptic functions thus gives us an easy means to construct divisor classes and to decide when divisors belong to the same or distinct divisor classes. Note however that the situation is much more involved for connected and compact Riemann surfaces of genus $g \geq 2$. To see this consult [45] or [77] to study the Abel-Theorem in its general form.

Definition C.4.2 (Divisor Of A Line Bundle)**:**
Consider the line bundle $L(m,b)$. Then let $s \in S(m,b) - \{0\}$ a non-trivial meromorphic section. Then we know that the divisor of s is a divisor on $\mathbb{C}_{1,\tau}$ so that we can set

$$\operatorname{div}(s) = D \in \operatorname{Div}(\mathbb{C}_{1,\tau}) \tag{C.95}$$

We now define that $L(m,b)$ has divisor

$$D(m,b) = -D \tag{C.96}$$

Note:

- The minus sign in the above definition appears there on purpose.
- Note that the divisor of $L(m,b)$ depends on the chosen section s. Still this assignment is well-defined in the following sense.

Lemma C.4.2:
The divisor class $[D(m,b)]$ is well-defined, i.e. independent of the choice of the section $s \in S(m,b) - \{0\}$.

Proof
Let us consider $s, t \in S(m, b) - \{0\}$. We will prove that their divisors are linearly equivalent and thus define the same divisor class.
To this end we first note that we can find elliptic functions $e, e' \colon \mathbb{C} \to \overline{\mathbb{C}}$ and holomorphic sections $h, h' \in H(m, b) - \{0\}$ such that for all $z \in \mathbb{C}$ it holds

$$s(z) = e(z) h(z), \qquad t(z) = e'(z) h'(z) \tag{C.97}$$

But then by definition of linear equivalence

$$\operatorname{div}(s) \sim \operatorname{div}(h), \qquad \operatorname{div}(t) \sim \operatorname{div}(h') \tag{C.98}$$

Now note that the quotient $\frac{h}{h'}$ is again an elliptic function, so that $\operatorname{div}(h) \sim \operatorname{div}(h')$.
But since linear equivalence is an equivalence relation, this implies

$$\operatorname{div}(s) \sim \operatorname{div}(t) \tag{C.99}$$

which concludes the proof. ∎

Remark:
Due to the identity theorem of complex calculus the zeros of a meromorphic function are discrete. This has been used to argue that the quotient $\frac{h}{h'}$ is a (special) meromorphic function.
In fact a more general statement holds true - on a connected Riemann surface the set of meromorphic functions does form a field. The prove makes use of the identity theorem on Riemann surfaces. Since $\mathbb{C}$ is a connected Riemann surface, this statement justifies that the inverse of the holomorphic function h' is a meromorphic function on $\mathbb{C}$.

Corollary:
Consider the line bundle $L(m, b)$ with divisor $D(m, b) \in \operatorname{Div}(\mathbb{C}_{1,\tau})$. Then one finds

$$H^0(\mathbb{C}_{1,\tau}, L(m, b)) \cong \{f \in S(m, b)\ ,\ \operatorname{div}(f) \geq -D\} \tag{C.100}$$

Remark:
We agree on the convention, that the function that vanishes identically satisfies this inequality for every divisor $D \in \operatorname{Div}(\mathbb{C}_{1,\tau})$.

Proof
Consider again the line bundle $L(m, b)$ with divisor $D(m, b) \in \operatorname{Div}(\mathbb{C}_{1,\tau})$. By definition, there exists $s \in S(m, b) - \{0\}$ with the property

$$\operatorname{div}(s) = -D \tag{C.101}$$

We can now consider the map

$$\varphi \colon H(m, b) \to H^0(\mathbb{C}_{1,\tau}, L(m, b))\ ,\ f \mapsto f \cdot s \tag{C.102}$$

This map is an isomorphism. Therefore we conclude

$$H(m, b) \cong H^0(\mathbb{C}_{1,\tau}, L(m, b)) \tag{C.103}$$

This suffices for the proof. ∎

Remark (Summary)**:**
We conclude that it suffices to find a single non-trivial meromorphic section of the line bundle $L(m,b)$ to determine its divisor class. As for its first cohomology class we have $H(m,b) \cong H^0(\mathbb{C}_{1,\tau}, L(m,b))$ so that we need to find all of its holomorphic sections.

C.4.4. All Holomorphic Line Bundles With m = 0

A Meromorphic Section For All Degree 0 Bundles

Remark (Task)**:**
Our task is to find a non-trivial meromorphic function $F\colon \mathbb{C} \to \overline{\mathbb{C}}$ such that

$$F(z+1) = F(z), \qquad F(z+\tau) = e^{2\pi i m z + b} \cdot F(z) \overset{m=0}{=} e^b F(z) \tag{C.104}$$

Any such function is a non-trivial meromorphic section of the holomorphic line bundle $L(0,b)$ and allows us to determine the divisor class of this holomorphic line bundle.

Comment:
Recall that we defined back in subsection C.4.3 the function

$$A\colon \mathbb{C} \to \mathbb{C}\ ,\ z \mapsto (1-t)\prod_{n=1}^{\infty}(1-q^n t)\left(1-q^n t^{-1}\right) \tag{C.105}$$

with $t = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. We argued that this function has the following properties.

- A is an entire function
- A has simple zeros at $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and not other zeros
- $A(z+1) = A(z)$ and $A(z+\tau) = -e^{-2\pi i z} \cdot A(z)$

Claim:
Let $b \in \mathbb{C}$ arbitrary but fixed. Then the function

$$F_b\colon \mathbb{C} \to \mathbb{C}\ ,\ z \mapsto \frac{A\left(z - \frac{b}{2\pi i}\right)}{A(z)} \tag{C.106}$$

has the following properties

- $F_b \in \mathcal{M}^*_{\mathbb{C}}(\mathbb{C})$
- F_b has simple poles at $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and simple zeros at $\frac{b}{2\pi i} + \mathbb{Z} + \mathbb{Z}\tau$ but not other poles nor zeros.
- It satisfies
$$F_b(z+1) = F_b(z), \qquad F_b(z+\tau) = e^b \cdot F_b(z) \tag{C.107}$$

Proof
All of these properties follow from the properties of the function $A(z)$. ■

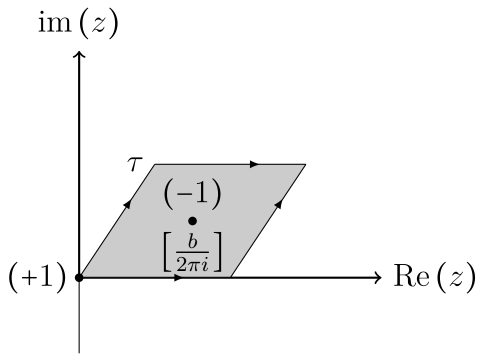


Figure C.2.: The divisor of $L(0,b)$.

Consequence:
The holomorphic line bundle $L(0,b)$ has divisor class represented by the divisor

$$D(0,b) = (+1)[0] + (-1)\left[\frac{b}{2\pi i}\right] \tag{C.108}$$

This divisor is illustrated in Figure C.2.

Remark:
Note that $\deg(D(0,b)) = 0$ which matches m since we are considering the case $m = 0$.

Classification Of Degree 0 Divisors

Claim:
Let $D \in \mathrm{Div}(\mathbb{C}_{1,\tau})$ with $\deg(D) = 0$. Then there exists a unique $[p] \in \mathbb{C}_{1,\tau}$ such that

$$D \sim (+1)[0] + (-1)[p] \tag{C.109}$$

Proof
Let us write

$$D = \sum_{i=1}^{n} (a_i) \cdot [p_i] \tag{C.110}$$

with $a_i \in \mathbb{Z}$ the weights and $[p_i] \in \mathbb{C}_{1,\tau}$. Then we define the point

$$[p] = \left[\sum_{i=1}^{n} -a_i p_i\right] \in \mathbb{C}_{1,\tau} \tag{C.111}$$

and the divisor

$$D' = (+1)\cdot[0] + (-1)[p] \tag{C.112}$$

Then we consider

$$\widetilde{D} = D - D' = \sum_{i\in I} (a_i)\cdot[p_i] + (-1)[0] + (+1)\left[\sum_{i\in I} -a_i p_i\right] \tag{C.113}$$

This is a degree 0 divisor which is trivial in $\mathbb{C}_{1,\tau}$. Thus by the Abel theorem $\widetilde{D}$ is the divisor of a meromorphic function on $\mathbb{C}_{1,\tau}$ and we have $D \sim D'$.
Now let us show that $[p] \in \mathbb{C}_{1,\tau}$ is unique. To this end let us assume that there existed $[p], [q] \in \mathbb{C}_{1,\tau}$ such that

$$D \sim (+1)[0] + (-1)[p], \quad D \sim (+1)[0] + (-1)[q] \tag{C.114}$$

Thus since the equivalence of divisors is an equivalence relation, we must have

$$(+1)[0] + (-1)[p] \sim (+1)[0] + (-1)[q] \tag{C.115}$$

This however implies that

$$\widetilde{\widetilde{D}} := (+1)[0] + (-1)[p] - ((+1)[0] + (-1)[q]) \tag{C.116}$$

must be the divisor of a meromorphic function on $\mathbb{C}_{1,\tau}$. In particular this requires by the Abel theorem

$$\widetilde{\widetilde{D}} = (-1)[p] + (+1)[q] \overset{!}{=} 0 \tag{C.117}$$

which is equivalent to $[p] = [q]$. Thus $[p] \in \mathbb{C}_{1,\tau}$ is uniquely determined which concludes the proof. ∎

Moduli Space Of Degree 0 Bundles

Consequence:
From the preceeding we find the following

- The parameter b can take any value in $\mathbb{C}$ such that $\frac{b}{2\pi i}$ lies in the fundamental parallelogramm of the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Thus we have

$$b \in \{(2\pi i)\lambda + (2\pi i\tau)\mu\ ,\ 0 \leq \lambda, \mu \leq 1\} := \mathcal{M}_0 \tag{C.118}$$

- The following map is an isomorphism.

$$\varphi_0 : \mathcal{M}_0 \to \operatorname{Pic}_0(\mathbb{C}_{1,\tau})\ ,\ b \mapsto L(0,b) \tag{C.119}$$

We have thus classified the degree zero holomorphic line bundles on $\mathbb{C}_{1,\tau}$ as the points of $\mathcal{M}_0 \cong \mathbb{C}_{1,\tau}$.

First Cohomology Class

Remark:
To determine $H^0(\mathbb{C}_{1,\tau}, L(0,b))$ we have to find all holomorphic sections of $L(0,b)$. These are given by holomorphic functions $F : \mathbb{C} \to \mathbb{C}$ such that

$$F(z+1) = F(z), \qquad F(z+\tau) = e^b \cdot F(z) \tag{C.120}$$

where $b \in \mathcal{M}_0$.

Claim:
Let $b \in \mathcal{M}_0$. Then it holds

- $H^0(\mathbb{C}_{1,\tau}, L(0,0)) \cong \mathbb{C}$
- $H^0(\mathbb{C}_{1,\tau}, L(0,b)) = \{0\}$ for $b \neq 0$

Proof
We are looking for all entire functions $F: \mathbb{C} \to \mathbb{C}$ subject to the conditions

$$F(z+1) = F(z), \qquad F(z+\tau) = e^b \cdot F(z) \tag{C.121}$$

If such a function does exist, its Laurent series must be of the form

$$F(z) = \sum_{n \in \mathbb{Z}} a_n t^n \tag{C.122}$$

with $t = e^{2\pi i z}$. In particular this implies

$$F(z+\tau) = \sum_{n \in \mathbb{Z}} a_n q^n t^n \tag{C.123}$$

where $q = e^{2\pi i \tau}$. On the other hand we have by the second requirement

$$F(z+\tau) = \sum_{n \in \mathbb{Z}} a_n e^b t^n \tag{C.124}$$

From the identity theorem of power series we thus conclude that for all $n \in \mathbb{Z}$ it holds

$$a_n = q^n e^b \cdot a_n \tag{C.125}$$

Recall $\mathrm{im}(\tau) > 0$, so that $q \neq 1$. Consequently for $n \neq 0$ it holds $q^n e^b \neq 1$, so that we conclude $a_n = 0$ for $n \neq 0$. As for $n = 0$, we have to satisfy

$$a_0 = e^b \cdot a_0 \tag{C.126}$$

This is trivially satisfied for any $a_0 \in \mathbb{C}$ if $b = 0$. Otherwise it forces $a_0 = 0$. Thus we find

- For $b = 0$ we have $H^0(\mathbb{C}_{1,\tau}, L(0,0)) = \{F: \mathbb{C} \to \mathbb{C}\,,\ z \mapsto a_0 \in \mathbb{C}\} \cong \mathbb{C}$
- For $b \neq 0$ we have $H^0(\mathbb{C}_{1,\tau}, L(0,b)) = \{F: \mathbb{C} \to \mathbb{C}\,,\ z \mapsto 0\} \cong \{0\}$

All so-obtained functions are in fact holomorphic. This concludes the proof. ∎

C.4.5. All Holomorphic Line Bundles With m > 0

A Meromorphic Section For All m > 0 Bundles

Remark:
We are looking for non-trivial meromorphic functions $F: \mathbb{C} \to \overline{\mathbb{C}}$ subject to the conditions

$$F(z+1) = F(z), \qquad F(z+\tau) = e^{2\pi i m z + b} \cdot F(z) \tag{C.127}$$

as these allows us to determine the divisor of the line bundle $L(m,b)$.

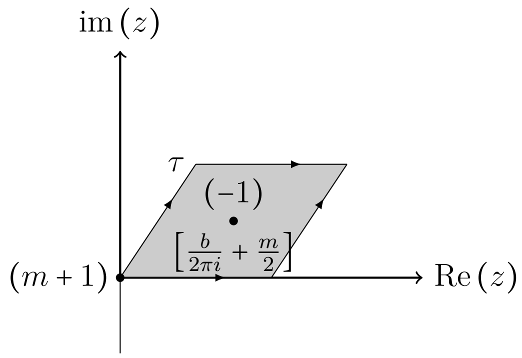


Figure C.3.: Divisor $D(m,b)$ of the holomorphic line bundle $L(m,b)$.

Claim:
The function

$$F_b^{(m)}: \mathbb{C} \to \mathbb{C}\,,\; z \mapsto \frac{A\left(z - \frac{b}{2\pi i} - \frac{m}{2}\right)}{A(z)^{m+1}} \tag{C.128}$$

has the following properties

- $F_b^{(m)} \in \mathcal{M}_{\mathbb{C}}^*(\mathbb{C})$.
- It has simple zeros at $z = \frac{b}{2\pi i} + \frac{m}{2} + \Lambda = \frac{b}{2\pi i} + \frac{m}{2} + \mathbb{Z} + \mathbb{Z}\tau$ and no other zeros.
- It has poles of order $m+1$ at $z \in \Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and no other poles.
- It holds

$$F_b^{(m)}(z+1) = F_b^{(m)}(z)\,, \quad F_b^{(m)}(z+\tau) = e^{2\pi i m z + b} F_b^{(m)}(z) \tag{C.129}$$

Proof
This follows from the properties of the function $A(z)$ as introduced in subsection C.4.3.
■

Consequence:
$L(m,b)$ has divisor class represented by the divisor

$$D(m,b) = (m+1)[0] + (-1)\left[\frac{b}{2\pi i} + \frac{m}{2}\right] \tag{C.130}$$

which is illustrated in Figure C.3.
Note in particular that this divisor has degree m. This proves together with our findings for the line bundles $L(0,b)$ that m is the degree of the holomorphic line bundles $L(m,b)$.

Classification Of Degree m Divisors

Claim:
Let $D \in \mathrm{Div}(\mathbb{C}_{1,\tau})$ with $\deg(D) = m$. Then there exists a unique $[p] \in \mathbb{C}_{1,\tau}$ such that

$$D \sim (m+1)[0] + (-1)[p] \tag{C.131}$$

Proof
Let us write

$$D = \sum_{i \in I} (w_i) \cdot [p_i] \tag{C.132}$$

with $w_i \in \mathbb{Z}$ the weights, $[p_i] \in \mathbb{C}_{1,\tau}$ the points and I an appropriate finite indexing set. Then we define the point

$$[p] = \left[\sum_{i \in I} -w_i p_i\right] \in \mathbb{C}_{1,\tau} \tag{C.133}$$

and the divisor

$$D' = (m+1) \cdot [0] + (-1)[p] \tag{C.134}$$

Then we consider

$$\widetilde{D} = D - D' = \sum_{i \in I} (w_i) \cdot [p_i] + (-m-1)[0] + (+1)\left[\sum_{i \in I} -w_i p_i\right] \tag{C.135}$$

This is a degree 0 divisor which is trivial in $\mathbb{C}_{1,\tau}$. Thus $\widetilde{D}$ is the divisor of a meromorphic function on $\mathbb{C}_{1,\tau}$, so that

$$D \sim D' \tag{C.136}$$

Now let us prove that $[p] \in \mathbb{C}_{1,\tau}$ is unique. To this end let us assume that there existed $[p], [q] \in \mathbb{C}_{1,\tau}$ such that

$$D \sim (m+1)[0] + (-1)[p], \qquad D \sim (m+1)[0] + (-1)[q] \tag{C.137}$$

But since equivalence of divisors is an equivalence relation, this implies that

$$\widetilde{D} = (+1)[p] + (-1)[q] \in \mathrm{Div}(X) \tag{C.138}$$

is the divisor of a meromorphic function on X. In particular this implies that $\widetilde{D}$ is trivial. But then $[p] = [q]$, and the uniqueness property follows. This concludes the proof. ∎

Moduli Space Of Degree m Bundles

Consequence:
From the preceeding we find the following

- The parameter b can take any value in $\mathbb{C}$ such that $\frac{b}{2\pi i}+\frac{m}{2}$ lies in the fundamental parallelogramm of the lattice $\Lambda = \mathbb{Z}+\mathbb{Z}\tau$. Thus we have

$$b \in \{(2\pi i)\,\lambda + (2\pi i\tau)\,\mu - \pi i m\;,\; 0 \leq \lambda, \mu \leq 1\} := \mathcal{M}_m \tag{C.139}$$

- The following map is an isomorphism.

$$\varphi_m\colon \mathcal{M}_m \to \mathrm{Pic}_m\left(\mathbb{C}_{1,\tau}\right)\;,\; b \mapsto L\left(m, b\right) \tag{C.140}$$

We have thus classified all holomorphic line bundles of degree $m > 0$ on $\mathbb{C}_{1,\tau}$ as the points of $\mathcal{M}_m \cong \mathbb{C}_{1,\tau}$.

First Cohomology Class

Remark:
We are now looking for the most general holomorphic functions $F\colon\mathbb{C}\to\mathbb{C}$ such that for $m \in \mathbb{N}_{>0}$ and $b \in \mathcal{M}_m$ it holds

$$F\left(z+1\right) = F\left(z\right), \quad F\left(z+\tau\right) = e^{2\pi i m z + b}\cdot F\left(z\right) \tag{C.141}$$

Claim:
All non-trivial holomorphic functions $F\colon\mathbb{C}\to\mathbb{C}$ which satisfy for $b \in \mathcal{M}_m$ and $m \in \mathbb{N}_{>0}$ the conditions

$$F\left(z+1\right) = F\left(z\right), \quad F\left(z+\tau\right) = e^{2\pi i m z + b}\cdot F\left(z\right) \tag{C.142}$$

can be written as

$$F\left(z\right) = \sum_{l=0}^{m-1} a_l \cdot H_l^{(m)}\left(z\right) \tag{C.143}$$

for $a_0, \ldots, a_{m-1} \in \mathbb{C}$ arbitary and

$$H_l^{(m)}\left(z\right) = \sum_{k\in\mathbb{Z}} e^{2\pi i\tau k\left[\frac{m}{2}(k-1)+l\right]} e^{-b\cdot k} e^{2\pi i(km+l)z} \tag{C.144}$$

Proof
If such a holomorphic does ever exist, it follows from the condition $F\left(z+1\right) = F\left(z\right)$ and F being entire, that its Laurent series is of the form

$$F\left(z\right) = \sum_{n\in\mathbb{Z}} a_n \cdot t^n \tag{C.145}$$

where $t := e^{2\pi i z}$ and $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$. Then we find in particular

$$F\left(z+\tau\right) = \sum_{n\in\mathbb{Z}} a_n q^n t^n \tag{C.146}$$

where $q := e^{2\pi i\tau}$. On the other hand the second condition

$$F\left(z+\tau\right) = e^{2\pi i m z} e^{b}\cdot F\left(z\right) \tag{C.147}$$

is equivalent to

$$F(z+\tau) = \sum_{n\in\mathbb{Z}} a_{n+m} e^b t^n \tag{C.148}$$

Comparing Equation C.146 and Equation C.148 we then find that for any $n \in \mathbb{Z}$ it holds

$$a_{n+m} = q^n e^{-b} a_n \tag{C.149}$$

Thus we can choose the coefficients $a_0, \ldots, a_{m-1} \in \mathbb{C}$ arbitary and describe all other coefficients via

$$a_{km+l} = q^{\frac{km}{2}(k-1)} \cdot q^{kl} e^{-b\cdot k} a_l \tag{C.150}$$

where $k \in \mathbb{Z}$ and $0 \leq l \leq m-1$. Now one easily confirms

$$F(z) = \sum_{l=0}^{m-1} a_l \cdot H_l^{(m)}(z) \tag{C.151}$$

with

$$H_l^{(m)}(z) = \sum_{k\in\mathbb{Z}} e^{2\pi i\tau k\left[\frac{m}{2}(k-1)+l\right]} e^{-b\cdot k} e^{2\pi i(km+l)z} \tag{C.152}$$

Finally we have to check that these functions are in fact holomorphic functions. For this task we refer the interested reader to [85], where a proof ot this is given. This then concludes the proof. ∎

Lemma C.4.3:
The functions $H_l^{(m)}(z)$ are linearly independent over $\mathbb{C}$.

Consequence:
For $m > 0$, the space of holomorphic sections of $L(m,b)$ is an m complex-dimensional vector space with basis $\mathcal{B} = \left\{H_0^{(m)}(z), \ldots, H_{m-1}^{(m)}(z)\right\}$. This implies

$$H^0(\mathbb{C}_{1,\tau}, L(m,b)) \cong \mathbb{C}^m \tag{C.153}$$

C.4.6. Spin Bundles

Remark (Canonical Divisor On Complex 2-Torus)**:**
On $\mathbb{C}_{1,\tau}$ we have a non-trivial meromorophic 1-form given by dz. Its divisor is the trivial divisor. As any divisor linearly equivalent to this divisor is termed a canonical divisor we can state this observation as $D_{\mathcal{K}} \equiv 0$.

Consequence (Spin Divisors)**:**
By definition, a spin divisor $D_s \in \mathrm{Div}(\mathbb{C}_{1,\tau})$ is such that

$$2D_s \sim D_{\mathcal{K}} \equiv 0 \tag{C.154}$$

We know from the above that $\deg(D_{\mathcal{K}}) = 0$. This implies $\deg(D_s) = 0$. By the classification of degree 0 divisors on $\mathbb{C}_{1,\tau}$ we thus conclude

$$D_s \sim (+1)[0] + (-1)[p] \tag{C.155}$$

where $p \in \mathcal{M}_0$. The constraint $2D_s \equiv 0$ has precisely four solutions, namely

- $p = 0$
- $p = \frac{1}{2}$
- $p = \frac{\tau}{2}$
- $p = \frac{1+\tau}{2}$

The four spin bundles on $\mathbb{C}_{1,\tau}$ are thus $L(0,0)$, $L\left(0, \frac{1}{2}\right)$, $L\left(0, \frac{\tau}{2}\right)$, $L\left(0, \frac{1+\tau}{2}\right)$.

D. What Is A Toric Variety?

D.1. Introduction

In this chapter we briefly introduce toric varieties. Our main source of reference will be [52]. Here we omit the proofs of all statements in the spirit of a brief introduction to the material. Therefore we mention that all proofs can be found in [52]. For an alternative careful introduction to toric varieties the interested reader is refered to [90].

D.2. (Toric) Varieties

D.2.1. Varieties

Definition D.2.1 (Affine Variety Of A Finite Number Of Polynomials)**:**
Let $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$. Then

$$V(f_1, \ldots, f_s) := \{x \in \mathbb{C}^n \ , \ f_1(x) = \cdots = f_s(x) = 0\} \tag{D.1}$$

is an affine variety.

Definition D.2.2 (Affine Variety Of An Ideal)**:**
Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ an ideal. Then

$$V(I) := \{x \in \mathbb{C}^n \ , \ f(x) = 0 \ \forall f \in I\} \tag{D.2}$$

is an ideal.

Remark:
The ring $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian by the Hilbert Basis Theorem.

Consequence:
It holds $V(f_1, \ldots, f_s) = V(I)$ with $I = (f_1, \ldots, f_s)$ the ideal generated by the polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$.

Example D.2.1:
$\mathbb{C}^n$ and $\{x \in \mathbb{C}^n \ , \ x_1 \cdot x_2 = 0\}$ are affine varieties.

Definition D.2.3 (Projective Variety)**:**
Let $F_1, \ldots, F_s \in \mathbb{C}[x_1, \ldots, x_n]$ homogeneous polynomials. Then

$$V(F_1, \ldots, F_s) = \{z \in \mathbb{CP}^n \ , \ F_1(z) = \cdots = F_n(z) = 0\} \tag{D.3}$$

is a projective variety.

Note:
When we talk about varieties in the following, we mean either an affine, a projective or abstract variety.

Remark:
In the theory of algebraic groups $(\mathbb{C}^*)^n$ is termed the *n-dimensional complex torus*. This is what will be so 'toric' about toric varieties.

Claim:
$(\mathbb{C}^*)^n$ is an affine variety.

Proof
The map

$$\varphi\colon (\mathbb{C}^*)^n \to \mathbb{C}^{n+1}\ ,\ (t_1, \ldots, t_n) \mapsto \left(t_1, \ldots, t_n, \frac{1}{t_1 \cdot \cdots \cdot t_n}\right) \tag{D.4}$$

gives an isomorphism $(\mathbb{C}^*)^n \cong V(x_1 \cdot \cdots \cdot x_{n+1} - 1)$. ■

Definition D.2.4 (Zariski Open)**:**
Let $W \subset V$ two varieties. Then we term the complement $V - W$ Zariski open.

Note:
This notion extends to a topology - the Zariski topology.

Consequence:
$(\mathbb{C}^*)^n \subset \mathbb{C}^n$ is Zariski open.

Definition D.2.5 (Irreducible)**:**
A variety V is irreducible if V cannot be written as the union of two proper subvarieties V_1, V_2, i.e. $V = V_1 \cup V_2$, such that $V_1 \neq V$ and $V_2 \neq V$.

D.2.2. Toric Varieties

Definition D.2.6 (Toric Variety)**:**
An irreducible variety V is a toric variety precisely if it satisfies in addition the following two requirements.

- $(\mathbb{C}^*)^n$ is a Zariski open subset of V
- the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on V.

Example D.2.2:
$(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are toric varieties.

Claim:
$\mathbb{CP}^n$ is a toric variety.

Proof
$\mathbb{CP}^n$ is an irreducible variety. Let now $x_0, \ldots, x_n$ its homogeneous coordinates and consider the isomorphism

$$\varphi: (\mathbb{C}^*)^n \to \mathbb{CP}^n \ , \ (t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n) \tag{D.5}$$

Thereby we can identify $(\mathbb{C}^*)^n$ with $\mathbb{CP}^n - V(x_0 x_1 \ldots x_n)$ which is Zariski open. Finally we write the torus action as

$$(t_1, \ldots, t_n) \cdot (a_0, a_1, \ldots, a_n) = (a_0, t_1 a_1, \ldots, t_n a_n) \tag{D.6}$$

This then shows that $\mathbb{CP}^n$ is a toric variety. ∎

Comment (General Remarks)**:**
We will see that points $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$ are imporant in the study of toric varieties.

- The Laurent monomial of p is defined by

$$t^p = t_1^{p_1} \cdot \cdots \cdot t_n^{p_n} \tag{D.7}$$

  which gives a function $\chi^p: (\mathbb{C}^*)^n \to \mathbb{C}^*$ which is termed a *character*.
- A 1-parameter subgroup $\lambda^p: \mathbb{C}^* \to (\mathbb{C}^*)^n$ is defined by

$$\lambda^p(t) = (t^{p_1}, \ldots, t^{p_n}) \tag{D.8}$$

In particular toric varieties V are generically just $(\mathbb{C}^*)^n$ with some additional information. Given that the toric variety V is affine, then this addtional information is determined by which Laurent monomials t^p are defined on V.

D.3. Cones And Affine Toric Varieties

D.3.1. Cones

Definition D.3.1 (Rational Polyhedral Cone)**:**
A rational polyhedral cone $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of $\mathbb{Z}^n$ as

$$\sigma = \{\alpha_1 p_1 + \cdots + \alpha p_l \in \mathbb{R}^n \ , \ \alpha_i \geq 0\} \tag{D.9}$$

with $p_1, \ldots, p_l \in \mathbb{Z}^n$.

Definition D.3.2 (Strong Convexity)**:**
Let $\sigma \subset \mathbb{R}^n$ a rational polyhedral cone. Then σ is strongly convex precisely if $\sigma \cap (-\sigma) = \{0\}$

Comment:
We will be mostly interested in strongly convex rational polyhedral cones. These we abbreviate as *scrapc*.

Definition D.3.3 (Dual Cone)**:**
Be $\sigma \subset \mathbb{R}^n$ a scrapc. Then its dual cone $\sigma^\vee \subset \mathbb{R}^n$ is

$$\sigma^v = \{m \in \mathbb{R}^n \ , \ \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\} \tag{D.10}$$

Note that in this expression $\langle m, u \rangle$ is just the usual dot product on $\mathbb{R}^n$.

D.3.2. Affine Toric Varieties

Remark:
Let $\sigma \subset \mathbb{R}^n$ a scrapc. Then we want to associate to it an affine toric variety $U_\sigma \subset \mathbb{C}^l$.

Construction D.3.1:

- By Gordan's lemma, the set $\sigma^\vee \cap \mathbb{Z}^n$ is a finitely generated semigroup over $\mathbb{Z}$, that is there are $m_1, \dots, m_l \in \sigma^\vee \cap \mathbb{Z}^n$ such that every $x \in \sigma^\vee \cap \mathbb{Z}^n$ is of the form

$$x = a_1 m_1 + \cdots + a_l m_l \ , \ a_i \in \mathbb{Z}_{\geq 0} \tag{D.11}$$

- Given the generators $m_1, \dots, m_l \in \sigma^\vee \cap \mathbb{Z}^n$, we consider the map

$$\varphi \colon (\mathbb{C}^*)^n \to \mathbb{C}^l \ , \ (t_1, \dots, t_n) \mapsto (\chi^{m_1}(t_1, \dots, t_n), \dots, \chi^{m_l}(t_1, \dots, t_n)) \tag{D.12}$$

- Now define U_σ as the Zariski closure of the image of $(\mathbb{C}^*)^n$ under the map φ, i.e.

$$U_\sigma = \overline{\varphi((\mathbb{C}^*)^n)} \tag{D.13}$$

Lemma D.3.1:

- The map $\varphi \colon (\mathbb{C}^*)^n \to U_\sigma$ is an inclusion of the algebraic torus, which makes U_σ an affine toric variety.
- The Laurent monomials t^{m_i} extend to functions $U_\sigma \to \mathbb{C}$ given by projection of $U_\sigma \subset \mathbb{C}^l$ onto the i-th coordinate.
- For every $m \in \sigma^\vee \cap \mathbb{Z}^n$, the Laurent monomial t^m extends to a function on U_σ.
- U_σ is the smallest toric variety on which all the t^m are defined.

Definition D.3.4:
The affine toric variety U_σ associated to the scrapc $\sigma \subset \mathbb{R}^n$ is termed the *normal toric variety associated to* σ.

D.3.3. Coordinate Rings

Definition D.3.5 (Coordinate Rings)**:**
Let V an affine variety. Then the ring of polynomial functions over V is the coordinate ring of V.

Example D.3.1:
The affine variety $\mathbb{C}^n$ has coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$.

Lemma D.3.2:
Let $U_\sigma \subset \mathbb{C}^l$ the normal affine variety associated to the scrapc $\sigma \subset \mathbb{R}^n$. Then consider

$$\mathrm{Span}\left\{t^m\ ,\ m \in \sigma^\vee \cap \mathbb{Z}^n\right\} \subset \mathbb{C}\left[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}\right] \tag{D.14}$$

It is not too hard to prove that this forms a ring which is usually denoted by $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. Finally it holds

$$\boxed{\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] \text{ is the coordinate ring of } U_\sigma}$$

Consequence:
The coordinate ring of U_σ is thus made of all polynomial expressions in the Laurent monomials t^{m_i}.

D.3.4. Normality

Definition D.3.6 (Normal Varieties)**:**
A variety V is normal if its local rings are integrally closed in their field of fractions.

Remark:
The toric variety U_σ associated to a scrapc $\sigma \subset \mathbb{R}^n$ is always normal by construction.

D.4. Fans And Toric Varieties

D.4.1. Toric Varieties From Fans

Comment:
We now intend to build more general toric varieties by gluing together affine toric varieties that contain the same algebraic torus $(\mathbb{C}^*)^n$.

Definition D.4.1 (Fan)**:**
A fan Σ in $\mathbb{R}^n$ is a finite collection of scrapcs in $\mathbb{R}^n$ such that

- $\sigma \in \Sigma$ and τ a face of σ, then $\tau \in \Sigma$.
- $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau \in \Sigma$ is a face of σ and τ.

Remark:
Let Σ a fan. Then each $\sigma \in \Sigma$ gives a normal affine toric variety U_σ. Now let τ a face of σ. By definition of a fan, then τ is a scrapc contained in Σ and thus gives another normal affine toric variety U_τ. In particular one can regard U_τ as a Zariski open subset of U_σ.

Definition D.4.2:
Given a fan Σ in $\mathbb{R}^n$. Then the variety X_Σ is obtained from the affine varieties U_σ with $\sigma \in \Sigma$ by gluing together U_σ and U_τ along their common open subset $U_{\sigma\cap\tau}$.

Theorem D.4.1:

- X_Σ is a normal toric variety for a fan Σ in $\mathbb{R}^n$.
- Let V a normal toric variety. Then there exists a fan Σ in $\mathbb{R}^n$ such that $V \cong X_\Sigma$.

Note:
The normal toric variety X_Σ associated to a fan Σ in $\mathbb{R}^n$ is an example of abstract varieties, i.e. it can happen that X_Σ is neither an affine nor a projective variety.

D.4.2. Properties Of Toric Varieties And Properties Of Fans

Note:
There is a close relation between the properties of a fan Σ in $\mathbb{R}^n$ and the structure of the associated normal toric variety X_Σ. It is this relationship among others makes toric varieties easily calculable.

Lemma D.4.1:
There are one-to-one correspondances between the following objects:

- The limits $\lim_{t\to 0}\lambda^u(t)$ for $u \in |\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma$ where $|\Sigma|$ is termed the support of the fan Σ.
- The cones $\sigma \in \Sigma$.
- The orbits of the torus action on X_Σ.

Definition D.4.3 (Smooth And Simplicial Cones)**:**
Let $\sigma \subset \mathbb{R}^n$ a cone. Then we define

- σ is smooth precisely if it is generated by a subsest of a basis of $\mathbb{Z}^n$.
- σ is simplicial precisely if it is generated by a subset of a basis of $\mathbb{R}^n$.

Theorem D.4.2:
Let X_Σ the normal toric variety associated to the fan Σ in $\mathbb{R}^n$. Then we have the following important results.

1. X_Σ is compact precisely if $|\Sigma| = \mathbb{R}^n$
2. X_Σ is smooth precisely if every $\sigma \in \Sigma$ is smooth.
3. X_Σ has at worst finite quotient singularities precisely if every $\sigma \in \Sigma$ is simplicial.

Lemma D.4.2:
Let Σ a fan in $\mathbb{R}^n$ and X_Σ the associated normal toric variety (which need not be smooth nor compact). Assume that there exists $\sigma \in \Sigma$ a scrapc with $\dim_{\mathbb{R}}(\sigma) = n$. Then X_Σ is simply connected.

Consequence:
A normal toric variety X_Σ is compact precisely if the fan Σ in $\mathbb{R}^n$ satisfies the relation

$$|\Sigma| = \mathbb{R}^n \tag{D.15}$$

But since Σ is a *finite* collection of scrapcs, there must then exist $\sigma \in \Sigma$ with $\dim(\sigma) = n$. Consequently all compact toric varieties are simply connected.

D.5. Homogenisation

D.5.1. Homogeneous Coordinates

Remark:
The goal is to describe a normal toric variety X_Σ as a quotient

$$X_\Sigma = (\mathbb{C}^r - Z)/G \tag{D.16}$$

for some variety $Z \subset \mathbb{C}^r$ and some group $G \subset (\mathbb{C}^*)^r$.

Construction D.5.1:
Let Σ a fan in $\mathbb{R}^n$. Then by Gordan's lemma, there are primitive elements of the 1-dimensional cones ρ_i of Σ, i.e. there exist $n_i \in \rho_i \cap \mathbb{Z}^n$ which generate ρ_i over $\mathbb{R}$. We now associate to each primitive element a variable x_i.

Example D.5.1:
Consider the fan Σ in $\mathbb{R}^2$ whose maximal cones are as follows

$$\sigma_1 = \text{Cone}(e_1, e_2), \quad \sigma_2 = \text{Cone}(e_1, -e_1 - e_2), \quad \sigma_3 = \text{Cone}(e_2, -e_1 - e_2) \tag{D.17}$$

This fan is smooth and the primitive elements of the rays are

$$n_1 = e_1, \quad n_2 = e_2, \quad n_3 = -e_1 - e_2 \tag{D.18}$$

By the above prescription we now associate to each primitive element n_i a formal variable x_i.

Definition D.5.1 (The Exceptional Set)**:**
Given a fan Σ in $\mathbb{R}^n$ and having associated variables $x_1, \ldots, x_r$ to the primitive elements $n_1, \ldots, n_r$ of its rays $\rho_1, \ldots, \rho_r$, then we define for $\sigma \in \Sigma$

$$x^{\widehat{\sigma}} = \prod_{n_i \notin \sigma} x_i \tag{D.19}$$

Subsequently we set

$$Z := V\left(x^{\widehat{\sigma}}\ ,\ \sigma \in \Sigma\right) \subset \mathbb{C}^r \tag{D.20}$$

Remark:
It suffices to define Z only using the maximal cones of Σ.

Example D.5.2:
Continuing with the above example, one finds

$$Z = V(x_0, x_1, x_2) = \{(0,0,0)\} \subset \mathbb{C}^3 \tag{D.21}$$

Definition D.5.2 (The Group G)**:**
We now define $G \subset (\mathbb{C}^*)^r$ by

$$G = \left\{(\mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^r\ ,\ \prod_{i=1}^r \mu_i^{\langle m, n_i \rangle} = 1 \text{ for all } m \in \mathbb{Z}^n\right\} \tag{D.22}$$

Lemma D.5.1:
It holds $(\mu_1, \ldots, \mu_n) \in G$ if and only if

$$\prod_{i=1}^{r} \mu_i^{\langle e_1, n_i \rangle} = \prod_{i=1}^{r} \mu_i^{\langle e_2, n_i \rangle} = \cdots = \prod_{i=1}^{r} \mu_i^{\langle e_n, n_i \rangle} = 1 \tag{D.23}$$

Example D.5.3:
Following the preceeding two examples it is readily checked that

$$G = \{(\mu, \mu, \mu) \ , \ \mu \in \mathbb{C}^*\} \cong \mathbb{C}^* \tag{D.24}$$

This also gives the standard action of $\mathbb{C}^*$ on $\mathbb{C}^3$. With Z given in the previous example we conclude that the quotient construction that we have just performed yields

$$\left(\mathbb{C}^3 - 0\right) / \mathbb{C}^* \tag{D.25}$$

Theorem D.5.1:
If X_Σ is the normal toric variety associated to the fan Σ in $\mathbb{R}^n$ such that the primitive elements $n_1, \ldots, n_r$ of its rays span $\mathbb{R}^n$, then it holds.

1. X_Σ is the universal categorical quotient $(\mathbb{C}^r - Z)/G$.
2. X_Σ is a geometric quotient $(\mathbb{C}^r - Z)/G$ if and only if X_Σ is simplicial.

Consequence:
In the remainder of this thesis we want to focus on normal toric varieties X_Σ which are both compact and smooth. Compactness implies that $n_1, \ldots, n_r$ indeed span $\mathbb{R}^n$, whilst smoothness ensures that Σ is smooth and therefore also simplicial. Consequently any smooth and compact normal toric variety can be represented as a geometric quotient $(\mathbb{C}^r - Z)/G$.

Remark:
Consider the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_r]$. Then the action of G induces a natural grading of this ring. To see this consider $f = f(x_1, \ldots, x_r) \in S$ and $(\mu_1, \ldots, \mu_r) \in G$. Then the natural action is given by

$$(\mu_1, \ldots, \mu_r) \cdot f = f(\mu_1 x_1, \ldots, \mu_r x_r) \tag{D.26}$$

This induces a grading on S as follows.

Lemma D.5.2:
It holds $\deg(x_1^{a_1} \cdots x_r^{a_r}) = \deg\left(x_1^{b_1} \cdots x_r^{b_r}\right)$ precisely if there exists $m \in \mathbb{Z}^n$ such that for $1 \leq i \leq r$ it holds

$$a_i = b_i + \langle n_i, m \rangle \tag{D.27}$$

Definition D.5.3:

- The ring $S = \mathbb{C}[x_1, \ldots, x_r]$ together with the above grading is the homogeneous coordinate ring of X_Σ.
- $f \in S$ is homogeneous precisely if all monomials appearing in f have the same degree.

D.5.2. Triangulisations

Remark:
In the physics literature one usually focuses on smooth and compact normal toric varieties. Those, as we have just learned, can always be represented as a geometric quotient $(\mathbb{C}^r - Z)/G$ for some affine variety $Z \subset \mathbb{C}^r$ and a group G. In particular both, the variety Z and the group G need to be known, to specify a smooth and compact normal toric variety uniquely.
Still oftentimes given the group G there is a unique choice for the variety Z such that the above geometric quotient becomes a smooth and compact normal toric variety. This leads to the following definitions.

Definition D.5.4 (Irrelevant Ideal)**:**
For a smooth and compact normal toric variety X_Σ, the exceptional set Z appearing in the geometric quotient is an affine variety in $\mathbb{C}^r$. Therefore we know that there exists and ideal $B_{X_\Sigma} \subset \mathbb{C}[x_1, \ldots, x_r]$ such that $Z = V(B_{X_\Sigma})$. We term this ideal B_{X_Σ} the *irrelevant ideal* of the toric variety X_Σ.

Definition D.5.5 (Stanley-Reisner Ideal)**:**
For a smooth and compact normal toric variety X_Σ, the Stanley-Reisner ideal I_{SR} is the Alexander-dual of the irrelevant ideal B_{X_Σ}.

Consequence (Construction Of The Stanley-Reisner Ideal)**:**
Consider a smooth and compact normal toric variety X_Σ. The primitive elements in the rays ρ_i of Σ be denoted by n_i with $1 \leq i \leq |\Sigma(1)|$. In addition we denote the scrapc generated by $n_{i_1}, \ldots, n_{i_m}$ as $\langle n_{i_1}, \ldots, n_{i_m} \rangle$. Finally we associate to each primitive element n_i a formal variable x_i.
Given this notation, the Stanley-Reisner ideal of X_Σ is given by

$$I_{\mathrm{SR}} = \left\langle \prod_{k=1}^{m} x_{i_j} \; , \; \langle n_{i_1}, \ldots, n_{i_m} \rangle \notin \Sigma \right\rangle \subset \mathbb{C}\left[x_1, \ldots, x_{|\Sigma(1)|}\right] \tag{D.28}$$

Definition D.5.6 (Triangulisation)**:**
Let $G \subset (\mathbb{C}^*)^r$ a group acting on $\mathbb{C}^r$. Then the following set is a triangulisation of the group G

$$\mathcal{T} := \{ I_{\mathrm{SR}} \subset \mathbb{C}[x_1, \ldots, x_r] \; , \; (\mathbb{C}^r - V(I_{\mathrm{SR}}))/G \text{ is a smooth, compact toric variety} \} \tag{D.29}$$

Note:
Triangulisations can be computed with the computer program *Sage* [51]. We give an example of the necessary sourcecode.

Example D.5.4 (Stanley-Reisner Ideal From *Sage* [51])**:**
The Stanley-Reisner ideal for $\mathbb{CP}^4$ can be obtained from *Sage* [51]. To this end we need to know the ray generators of the fan of $\mathbb{CP}^4$. Those can be taken of the following form

$$u_1 = e_1, \quad u_2 = e_2, \quad u_3 = e_3, \quad u_4 = e_4, \quad u_5 = -e_1 - e_2 - e_3 - e_4 \tag{D.30}$$

Given this knowledge we perform the following commands in *Sage* [51]

- ```
  points=matrix([ [ 1, 0, 0, 0,-1],
                  [ 0, 1, 0, 0,-1],
                  [ 0, 0, 1, 0,-1],
                  [ 0, 0, 0, 1,-1] ]).transpose()
  ```
- ```
  p = PointConfiguration(points.transpose()
      .augment(vector([0,0,0,0])).transpose())
  ```
- `p=p.restrict_to_star_triangulations((0,0,0,0))`
- `p=p.restrict_to_fine_triangulations()`
- `p=p.restrict_to_regular_triangulations(True)`
- `tria=p.triangulations_list()`
- `len(tria)`

The last line of code returns '1'. This is the number of triangulisations admitted by the entered ray generators. We will now save all triangulisations in the variables *triangl* and evalute the single triangulisation in this case via '[0]'. If there are more triangulisations one has to replace '[0]' by e.g. '[1]' to take a look at the second triangulisation.

- `triangl=[[i[:-1] for i in j] for j in tria]`
- `fan=Fan(triangl[0],points)`
- `tor=ToricVariety(fan,coordinate_names='x1 x2 x3 x4 x5')`
- `tor.Stanley_Reisner_ideal()`

The last command returns the desired result.

"Ideal (x1*x2*x3*x4*x5) of Multivariate Polynomial Ring in x1, x2, x3, x4, x5 over Rational Field"

Thus we have $I_{\text{SR}} = \langle x_1x_2x_3x_4x_5\rangle \subset \mathbb{C}\left[x_1, x_2, x_3, x_4, x_5\right]$ for $\mathbb{CP}^4$, just as expected.

E. Code Of *Mathematica* Notebook

```
(*Routines From The Koszul-Extension-Notebook*)
(*Routines From The Koszul-Extension-Notebook*)
(*Routines From The Koszul-Extension-Notebook*)

cohomCalgPath = NotebookDirectory[];
(*Please fill in the name of the executable e.g. "cohomcalg.exe" for \
Windows and "./cohomcalg" for Linux*)
(*Default for Windows*)
cohomCalgExecutable = "cohomcalg.exe";

(*Setting directory ... *)
SetDirectory[cohomCalgPath];
(*Generate temp file for more efficent calculations *)
TempFile = FileNameJoin[StringJoin[{"cohomCalgKoszulExtension.in"}]];
stream1 = OpenWrite[TempFile];
Close[stream1];
(*This file contains the monomial data and is deleted once you choose \
a different ambient space*)
MonomFile = StringJoin[TempFile, ".monoms"];

(*Three small routines that convert the Mathematica input to the \
proper input format of the C++ implementation of cohomCalg *)
(*Converts the variety specifications to the proper string needed by \
cohomCalg*)
ConvertMathematicaToCppInput[Variety_] := Module[{i, j, StringVariety},
   StringVariety = "";
   For[i = 1, i <= Length[Variety[[1]]], i++,
    StringVariety =
     StringJoin[StringVariety, "vertex ", ToString[Variety[[1, i]]],
      "|GLSM:(", ListToStringWithoutBracket[Variety[[3, i]]], ");"];
    ];
   StringVariety =
    StringJoin[StringVariety, " srideal [",
     ListToStringWithoutBracket[ReconvertSR[Variety[[2]]]], "];"];
   Return[StringVariety];
   ];
(*Converts a list to as string without the brackets*)
ListToStringWithoutBracket[list_] := Module[{i, StringList},
   StringList = ToString[list[[1]]];
   For[i = 2, i <= Length[list], i++,
    StringList = StringJoin[StringList, ",", ToString[list[[i]]]];
```

```
   ];
  Return[StringList ];
  ];
(*Converts the SR ideal into the correct cohomCalg input string*)
ReconvertSR[SR_] := Module[{i, j, StringSR, ListStringSR },
  ListStringSR = {};
  For[i = 1, i <= Length[SR], i++,
   AppendTo[ListStringSR, ToString[SR[[i, 1]]]];
   For[j = 2, j <= Length[SR[[i]]], j++,
    ListStringSR [[ i ]] = StringJoin[ListStringSR [[ i ]], "*"];
    ListStringSR [[ i ]] =
     StringJoin[ListStringSR [[ i ]], ToString[SR[[i, j ]]]];
    ];
   ];
  Return[ListStringSR];
  ];

(*Generates the requested command for a given variety and line bundle*)


GenerateRequestCommand[AmbientVariety_, LineBundles_] :=
 Module[{i, j, command, StringAmbientVariety, StingLineBundle},
  StringAmbientVariety = ConvertMathematicaToCppInput[AmbientVariety];
  command =
   StringJoin["!", cohomCalgExecutable, " --integrated --in=\" ",
    StringAmbientVariety ];

  For[j = 1, j <= Length[LineBundles], j++,
   StingLineBundle = ToString[LineBundles[[j , 1]]];
   For[i = 2, i <= Length[LineBundles[[j ]]], i++,
    StingLineBundle =
      StringJoin [StingLineBundle, ",", ToString[LineBundles[[j , i ]]]];
    ];
   (*Turn off use of the monomfile as I can then (without having to \
delete it explicitely ) use other ambient spaces*)
   command = command <> " monomialfile off; ";
   command =
    StringJoin [command, " ambientcohom O(", StingLineBundle, "); "];
   ];
  command = StringJoin[command, "\" ", TempFile];
  Return[command];
  ];

(*Useful routines from me*)
(*Useful routines from me*)
(*Useful routines from me*)

(*Gives a vector from weights which allows to rephrase the problem as \
linear equation*)
VarGenerator[Weights_, Variables_] := Module[{var},
 var = Array[
   If [Weights[[#]] == {-1}, {-1 -
```

```
        Variables[[#]]  }, {Variables[[#]]}] &,
    Length[Weights]];
  Return[var];
  ]
(*Print the rationom corresponding to a list *)
Rationom[m_] := Module[{Rat},
  Rat = Array[Symbol[StringJoin[ToString[x], ToString[#]]]^(m[[#]]) &,
     Length[m]];
  Return[Times @@ Rat];
  ]
(*Modify a list to the right signs*)
RightSigns[Solu_] := Module[{res, w},
  (*Weight vector is last element in given list - extract it*)
  w = Solu[[Length[Solu]]];
  (*Modify all other entries in solu according the rules dictated by \
w*)
  res = Array[RightSignVectors[w, Solu[[#]]] &, Length[Solu] - 1];
  Return[ res ];
  ]
(*Modify a given vectors according to the rules dictated by the \
weights*)
RightSignVectors[Weights_, Vec_] := Module[{res},
  res = Array[If[Weights[[#]] == {-1}, -1 - Vec[[#]], Vec[[#]]] &,
    Length[Vec]];
  Return[ res ];
  ]
(*Generate Weights from Polynomial*)
GenerateWeightFromPolynomial[P_, N_] := Module[{weights, vars},
  vars = Table[Symbol["x" <> ToString[i]], {i, N}];
  weights = Array[If[Exponent[P, vars[[#]]] == 0, {1}, {-1}] &, N];
  Return[weights];
  ]
(*Important routine - find all rationoms*)
(*Important routine - find all rationoms*)
FindRationoms[Weights_, BundleCharges_, Relations_, Multi_] :=
 Module[{Vars, MyAssumption, MyVec, sol, i, RationomResult},

  (*Step1 - Create Variables*)
  Vars = Array[Symbol["a" <> ToString[#]] &, Length[Weights]];

  (*Step2 -
  Reformulate the problem such that mathematica can solve it*)
  MyVec = VarGenerator[Weights, Vars];
  MyVec = Transpose[Relations].MyVec;

  (*Step3 - Create Assumption*)
  MyAssumption = True;
  Do[MyAssumption = MyAssumption && Vars[[i]] >= 0, {i, Length[Vars]}];

  (*Step4 - Solve Problem under created assumption*)
  sol = Solve[MyVec == BundleCharges && MyAssumption, Integers];

```

```
(*Step5 –
Extract explicit solutions from rules generated by the solve–
command*)
sol = Vars /. sol;
sol = Join[sol, {Weights}];

(*Step6 – Reformulate solutions as rationoms*)
sol = RightSigns[sol];
sol = Map[Rationom, sol];

(*Step7 –
Determine basis of the space taking into account their multiplicity *)


RationomResult = Array[sol[[#]] &, Length[sol]];
Do[
 RationomResult =
  Join[RationomResult, Array[sol [[#]] &, Length[sol]]],
 {i, Multi – 1}
 ];
(*Return the result *)
Return[RationomResult];
]

(*2 Get Basis Of Vector Bundle Cohomology*)
(*2 Get Basis Of Vector Bundle Cohomology*)
(*2 Get Basis Of Vector Bundle Cohomology*)

(*Get basis of line bundle cohomology by use of cohomCalg*)
GetBasisOfLineBundleCohomology[ambToricSpace_, BundleCharges_] :=
 Module[{cohomology, res, dimA, dim, i, j, P, R, w, help, multi,
   CohomPolynomials},
  (*Get result of cohomcalg computation*)
  cohomology =
   ReadList[GenerateRequestCommand[ambToricSpace, {BundleCharges}]];

  (*Strip of unnessary data*)
  cohomology = cohomology[[1]];

  (*Step 1 – Strip of unnessary data agin*)
  cohomology = cohomology[[2]];
  cohomology = cohomology[[2]];

  (*Step 2 – Initialise cohomology vector*)
  dim = Dimensions[ambToricSpace[[3]]];
  dimA = dim[[1]] – dim [[2]];
  res = Table[{}, {dimA + 1}];

  (*Step 3 –
   Fill in all rationoms contributing to the cohomology vector*)
  For[i = 1, i <= Length[cohomology], i++,
   (* Isolate i–th cohomology contribution*)
```

```
    help = cohomology[[i]];
    (*Check to what cohomology group it does contribute*)
    j = help [[1]];
    (*Get the polynomial delivered by cohomCalg*)
    P = help [[2]];
    (*Generate weight vector from P*)
    w = GenerateWeightFromPolynomial[P, dim[[1]]];
    (*Determine multiplicity of contribution*)
    If[Variables[P] == {}, multi = P,
     multi = Coefficient[P, Times @@ Variables[P]]];
    (*Add the rationoms*)
    R = FindRationoms[w, BundleCharges, ambToricSpace[[3]], multi];
    res [[ j + 1]] = Join[res [[ j + 1]], R];
    ];

  (*Step4 - return the result*)
  Return[res];
  ]
(*Get Basis of vector bundle*)
GetBasisOfVectorBundleCohomology[ambToricSpace_, BundleCharges_] :=
 Module[{bundleNumb, helpcohomologies, fcohom, i, j, k, help,
    helpvector},

  (* Initialise variables*)
  helpcohomologies = {};
  bundleNumb = Length[BundleCharges];
  helpvector = Table[0, {BundleCharges}];

  (*Get Cohomology of the individual bundles*)
  For[i = 1, i <= bundleNumb, i++,
   helpcohomologies =
     Join[helpcohomologies, {GetBasisOfLineBundleCohomology[
        ambToricSpace, BundleCharges[[i]]]}];
   ];

  (*Make vectors*)
  fcohom = Table[{}, {Length[helpcohomologies[[1]]]}];
  For[i = 1, i <= Length[fcohom], i++,
   help = {};
   (*scan over all bundles*)
   For[j = 1, j <= bundleNumb, j++,
    (*scan over the cohomology of each bundle*)

    For[k = 1, k <= Length[helpcohomologies[[j, i]]], k++,
      (*Reset helpvector*)
      helpvector = Table[{0}, {Length[BundleCharges]}];

      (*Make replacement at position j*)
      helpvector [[ j ]] = {helpcohomologies[[j, i, k]]};
      help = Join[help, {helpvector}];
      ];
    ];
```

```
    (* if cohomology trivial *)
    If [Length[help] == 0,
     help = {{Table[{0}, {Length[BundleCharges]}]}, {}};
     ,
     help = {help, {}};
     ];

    (*add the result *)
    fcohom[[i ]] = Join[fcohom[[i ]], help ];
    ];

   (*Return the result *)
   Return[fcohom];
   ]

(*3 Generate Koszul sequence and compute cohomologies therein*)
(*3 Generate Koszul sequence and compute cohomologies therein*)
(*3 Generate Koszul sequence and compute cohomologies therein*)

(*Create ordered tuples − needed for creation of Koszul sequence*)
OrderedTuples[n_, k_] := Module[{List1, List2},
   List1 = Array[# &, k];
   List2 = Table[List1, {i , n }];
   Return[Select[Tuples@List2, Less @@ # &]];
   ]
(*This is a routine used in the creation of the Koszul sequence*)
AddDivisors[ Divisors_ , Labels_, BundleDivisor_] :=
 Module[{i, j , Div, DivLength, ReturnDivisors , indices , index},

   (*Determine length of the divisors *)
   DivLength = Length[Divisors [[1]]];

   (* Initialise ReturnDivisors *)
   ReturnDivisors = Table[{}, {Length[Labels]}];

   For[ i = 1, i <= Length[Labels], i++,

    (* Initialise Div*)
    Div = Table[0, {DivLength}];

    (*Add to Div*)
    indices = Labels[[ i ]];
    For[j = 1, j <= Length[indices], j++,
     Div = Div − Divisors[[ indices [[ j ]]]];
     ];
    Div = Div + BundleDivisor;

    (*Save Result*)
    ReturnDivisors [[ i ]] = Div;
    ];
   Return[ReturnDivisors]
```

```
  ]
(*Generate the Koszul Sequence*)
GenerateKoszulSequence[Divisors_, BundleDivisor_] :=
 Module[{DivisorNumb, Bundles, Matrices, Tuples, RememberTuples, i, j},
  (* Initialise the variable bundles that will be used to save the \
bundles appearing in the Koszul sequence*)
  DivisorNumb = Length[Divisors];
  Bundles = Table[{}, {i, DivisorNumb + 1}];
  Matrices = Table[{}, {i, DivisorNumb}];

  (* Fill in the details of the individual bundles and bridging maps*)


  Bundles[[DivisorNumb + 1]] = {BundleDivisor};
  For[i = 1, i <= DivisorNumb, i++,

   (*Get tuples for bunle i*)
   Tuples = OrderedTuples[DivisorNumb + 1 - i, DivisorNumb];

   (*Create the matrix mapping between bundle i and bundle i-1*)
   If[i > 1,
    Matrices[[i - 1]] = CreateMatrix[RememberTuples, Tuples];
    ];

   (*Create the explicit weights of bundle i*)
   Bundles[[i]] = AddDivisors[Divisors, Tuples, BundleDivisor];

   (*Remember the current tuples for the construction of the next map*)


   RememberTuples = Tuples;
   ];

  (*Add final matrix*)
  Matrices[[
    DivisorNumb]] = {Array[Symbol["s" <> ToString[#]] &,
     Length[Divisors]]};

  Return[{Bundles, Matrices}];
  ]
(*Correct production of the mapping matrices in the Koszul complex*)
CreateMatrix[ITuples_, FTuples_] :=
 Module[{BMatrix, i, j, helpArray, helpVector, helpPosition},

  (*Create BMatrix*)
  BMatrix = {};

  (*Compute image for the i-th tuple in ITuples*)
  For[i = 1, i <= Length[ITuples], i++,

   (* initialise helpVector*)
   helpVector = Table[0, {Length[FTuples]}];
```

```

    (*run over all elements in the i-th tuple in ITuples*)
    For[j = 1, j <= Length[ITuples[[i]]], j++,

     (*Omit the j-th element of the i-th tuple in ITuples*)
     helpArray = Delete[ITuples[[ i ]], j ];

     (*determine position of the image*)
     helpPosition = Position[FTuples, helpArray ];

     (*set the value*)
     helpVector [[ helpPosition [[1, 1]]]] = (-1)^(j - 1)*
       Symbol["s" <> ToString[ITuples[[i, j ]]]];
     ];

    (*add result from the i-th tuple in ITuples to BMatrix*)
    BMatrix = Join[BMatrix, {helpVector}];
    ];

   (*Transpose BMatrix to obtain final result *)
   BMatrix = Transpose[BMatrix];

   (*Return the result *)
   Return[BMatrix];
   ]
(*Calculate all cohomologies that appear in a sequence*)
GetAllCohomologiesInKoszulSequence[ambSpace_, KoszulSequence_] :=
 Module[{i, Cohomology},
  (* initialise variable and add all cohomologies*)
  Cohomology = {};
  For[i = 1, i <= Length[KoszulSequence], i++,
   Cohomology =
     Join[Cohomology, {GetBasisOfVectorBundleCohomology[ambSpace,
        KoszulSequence[[i ]]]}];
   ];
  (*Return the result *)
  Return[Cohomology]
  ]

(*4 Fancy Output*)
(*4 Fancy Output*)
(*4 Fancy Output*)

(*Generate nice output to display charges of a certain direct sum of \
line bundles*)
GenerateBundleOutput[Charges_] :=
 Module[{OutputString, i, j , SingleCharge, helpString },
  OutputString = "";
  For[i = 1, i <= Length[Charges], i++,
   (*Get charges of the i-th bundle in the direct sum*)
   SingleCharge = Charges[[i ]];

```

```
    (*Generate helpString*)
    helpString = "\[ScriptCapitalO]( ";
    For[j = 1, j <= Length[SingleCharge], j++,
     helpString = helpString <> ToString[SingleCharge[[j ]]];
     If [ j < Length[SingleCharge], helpString = helpString <> ","];
     ];
    helpString = helpString <> ")";

    (*Add to OutputString*)
    OutputString = OutputString <> helpString;
    If [ i < Length[Charges],
     OutputString = OutputString <> "\[CirclePlus]" ];
    ];

   (*Return result*)
   Return[OutputString];
   ]
(*Check if a vector is trivial or not*)
CheckTrivial [Vector_] := Module[{i, res},
  (*check for non- trivial entries*)
  res = True;
  For[i = 1, i <= Length[Vector], i++,
   If [Variables[Vector [[ i , 1]]] != {}, res = False];
   If [Vector [[ i , 1]] != 0, res = False];
   ];
  (*return the result*)
  Return[res];
  ]
(*Get the global sections defining the algebraic subvariety*)
GetGlobalSections [ambSpace_, Divisors_] :=
 Module[{helpcohom, helpsec, res, i , j , CCount},

  (*get basis of the zeroth-cohomolgy group*)
  res = {{"Section", " Explicit form", "Charges"}};
  CCount = 0;
  For[i = 1, i <= Length[Divisors], i++,
   helpcohom =
    GetBasisOfLineBundleCohomology[ambSpace, Divisors[[i ]]][[1]];
   helpsec = 0;
   For[j = 1, j <= Length[helpcohom], j++,
    CCount = CCount + 1;
    helpsec =
     helpsec + Symbol["C" <> ToString[CCount]]* helpcohom[[j]];
    ];
   res = Join[
     res , {{Symbol["s" <> ToString[i]], helpsec, Divisors [[ i ]]}}];
   ];

  (*return the result*)
  Return[res];
  ]
MyGenerateRowII[Polynomial_, Numb_] :=
```

```
  ReplacePart[Table[0, {Numb}],
   ParallelMap[
    ToExpression[StringDrop[ToString[#[[1]]], 1]] -> #[[2]] &,
    ParallelMap[Level[#, 1] &, Polynomial, Method -> Automatic],
    Method -> Automatic]];

(*Slow generateRow method replaced by the faster one given above*)
GenerateRow[Polynomial_, Numb_] := Module[{HelpRow, HelpVar, i},
  (* Initialise HelpRow*)
  HelpRow = Table[0, {Numb}];

  (* Initialise Helpvar*)
  HelpVar = Array[Symbol["A" <> ToString[#]] &, Numb];

  (*Get all their coefficients *)
  For[i = 1, i <= Numb, i++,
   HelpRow[[i]] = Coefficient[Polynomial, HelpVar[[i]]];
   ];

  (*Return result*)
  Return[HelpRow];
  ]
(*Draw the first sheet of the Koszul spectral sequence with maps*)
DrawFirstSheetWithMaps[ambSpace_, Divisors_, BundleDivisor_,
  MapSpecifier_] := Monitor[Module[
   {KoszulSequence, Sequence, SequenceMaps, MyCohomologies,
    bundleNumb, Tab1, Tab2, Tab3, Tab5, PCount, i, j, k, l, numbCohom,
     help, helpVector, helpVector2, helpRule, helpRule2, helpRule3,
    helpRule4, helpBasisVec, helpIndex, helpEntry, helpCoefficient ,
     helpCoefficient2 , MyArrow, Sections, helpMatrix, helpMatrix2,
    helpArray, helpArray2, VariableArray, tStart, tEnd, helper},

   (*Step 0 - save system time and start computation*)
   tStart = AbsoluteTime[];
   Print["Computation started"];

   (*Step 1 - Generate the Koszul-Sequence and the rough mappings*)
   KoszulSequence = GenerateKoszulSequence[Divisors, BundleDivisor];
   Sequence = KoszulSequence[[1]];
   SequenceMaps = KoszulSequence[[2]];
   bundleNumb = Length[Sequence];

   (*Step 2 - Get the cohomologies of all direct-sum line bundles*)
   MyCohomologies =
    GetAllCohomologiesInKoszulSequence[ambSpace, Sequence];
   numbCohom = Length[MyCohomologies[[1]]];

   (*Step 3 -
   Get the sections defining the algebraic subvariety and create \
corresponding rule*)
   Tab3 = GetGlobalSections[ambSpace, Divisors];
   helpRule = {};
```

```
For[i = 1, i <= Length[Divisors], i++,
 helpRule = Join[helpRule, {Tab3[[i + 1, 1]] -> Tab3[[i + 1, 2]]}];
 ];

(*Step 4 - Compute explicit mapping matrices*)
helpRule2 =
 Array[Symbol["x" <> ToString[#]] -> 0 &, Length[ambSpace[[3]]]];
Tab5 = Table[Table[0, {numbCohom}], {Length[Sequence] - 1}];
VariableArray =
 Array[Symbol["x" <> ToString[#]] &, Length[ambSpace[[3]]]];
MCount = 0;

(*check if the user wants any matrices coputed*)
If [MapSpecifier [[1, 1]] == "none",

 (*user wants no mappings computed, so*)
 Tab5 =
  Table[Table["not computed", {numbCohom}], {Length[Sequence] - 1}];
 MCount =
  Length[Divisors]*(Length[ambSpace[[3]]] -
     Length[ambSpace[[3, 1]]]) ;


 ,

 (*user wants at least some matrices computed, so do it*)

 (*Compute the mappings between row i and row i+1*)
 For[i = 1, i < Length[Sequence], i++,
   (*and therein the columns j*)
   For[j = 1, j <= numbCohom, j++,

     (*Check if user wants this map computed*)
     If [
      MemberQ[MapSpecifier, {i, j}] ||
       MapSpecifier [[1, 1]] == "all",

      (*user wants this map computed, so do it*)

      (*get principal mapping matrix from Tab4*)
      helpMatrix = SequenceMaps[[i]];

      (*represent the basis space*)
      help = MyCohomologies[[i, j, 1]];
      helpVector = Symbol["A" <> ToString[1]]*help[[1]];
      For[k = 2, k <= Length[help], k++,

        helpVector =
          helpVector + Symbol["A" <> ToString[k]]*help[[k]];
        ];

      (*if the domain space is trivial , the mapping is trivial , so*)
```

```
            If [!  CheckTrivial [helpVector ],

             (*the domain is non− trivial ,
             so proceed and check if the target space is non−
              trivial ... *)


             If [Length[MyCohomologies[[i + 1, j,  1]]]  > 1 || !
                 CheckTrivial [MyCohomologies[[i + 1, j ,  1,   1]]],

               (*the target space is non− trivial ,
               so proceed with the real computation*)
               (*The situation is non− trivial ,
               and the mapping coefficient for each basis element in the \
target space are determind in what follows*)

               (*compute the mapped vector −
               in particular use the expressions for the global sections *)


               helpVector = helpMatrix.helpVector ;
               helpVector = Expand[helpVector /. helpRule];

               (* initialise helpMatrix2*)
               helpMatrix2 = {};

               (*get the mapping coefficient for each basis element of \
the target*)
               For[l = 1, l <= Length[MyCohomologies[[i + 1, j, 1]]], l++,

                (*consider the l−
                th basis vector of the target cohomologies*)
                helpBasisVec = MyCohomologies[[i + 1, j, 1, l ]];

                (* isolate its single entry and position of that entry*)
                helpEntry = Total[helpBasisVec];
                helpEntry = helpEntry [[1]];
                helpIndex = Position[helpBasisVec, helpEntry ][[1,   1]];

                (*compute the coefficient of the l−
                th target space basis vector*)
                (* this is the most time−
                consuming task in the entire notebook*)

                (*Step 1 − Compute the coefficient *)
                If [Variables[helpEntry] != {},

                 (*\[Rule] basis vector non− trivial ,
                 so go through detailed analysis *)

                 (*step 1.1: keep only expression in helpArray2,
```

```
                that have the correct denominator*)

                (* split the long polynomial image expression into its \
additive parts*)
                helpArray2 = Level[helpVector[[helpIndex, 1]], 1];
                (*get variables in the l-th basis vector*)
                helpArray = Variables[helpEntry];
                (*keep only expression of helpArray2,
                that depend on the variables that the l-
                th basis vector does depend on*)


                helpArray2 =
                 Select[helpArray2,
                  SameQ[Denominator[helpEntry], Denominator[#]] &];

                (*step 2.1: keep only expression in helpArray2,
                that depend on the correct variables *)

                helpArray2 =
                 Select[helpArray2,
                  Intersection[Complement[VariableArray, helpArray],
                     Variables[#]] == {} &];

                (*step 3.1:
                compute the coefficient in the reduced expression via \
the 'slow' command coefficient*)
                (*step 3.1: for speed-up, this is parallelised *)
                 helpCoefficient = ParallelMap[
                  If[Variables[Denominator[#]*helpEntry] != {},
                    Coefficient[Numerator[#], Denominator[#]*helpEntry]
                    ,
                    Numerator[#] /. helpRule2
                    ]
                   &, helpArray2, Method -> Automatic];

                (*Step 4.1:
                mathematica might have factorised and thus left \
variables x_i in helpCoefficient2 *)
                (*Step 4.1: those we do not consider real coefficients ,
                thus we get rid of them now*)

                 helpCoefficient =
                 Select[ helpCoefficient ,
                  Intersection[Variables[#], VariableArray ] == {} &];


                ,

                (*otherwise the basis vector is trivial ,
                and the coefficient is simply the constant part*)

                 helpCoefficient =
```

```
                    helpVector [[ helpIndex ,  1]]  /. helpRule2;
                  ];
                (*drop zeros in  helpCoefficient *)
                (* helpCoefficient =Select[ helpCoefficient ,
                Variables [#]\[NotEqual]{}&];*)
                helpCoefficient  = Total[ helpCoefficient ];

                (*Step 2: Use helpCoefficient  to assemble matrix*)

                helpMatrix2 =
                 Join[helpMatrix2, {GenerateRow[helpCoefficient ,
                     Length[MyCohomologies[[i, j,  1]]]]}];

                ];

               ,

               (*otherwise the target space is  trivial ,
               so the mapping is*)

               helpMatrix2 =
                 Table[Table[
                   0, {Length[MyCohomologies[[i, j,  1]]]}],  {Length[
                    MyCohomologies[[i + 1, j ,  1]]]}];
               ];

             ,

             (*otherwise the domain is  trivial , so the mapping is*)

             helpMatrix2 =
               Table[Table[
                 0, {Length[MyCohomologies[[i, j,  1]]]}],  {Length[
                  MyCohomologies[[i + 1, j ,  1]]]}];
             ];

           (*save the user wished matrix*)
           Tab5[[ i ,  j ]]  = MatrixForm[helpMatrix2];

           ,

           (*user does not want this map computed, so*)
           Tab5[[ i ,  j ]]  = "not computed";

           ];

          (* increase the computed−matrix−counter*)
          MCount = MCount + 1;

          ];  (*j−loop finished *)
        ];  (*i−loop finished *)
     ];  (*user check finished *)
```

```
(*Step 5 - Nicely represent the first sheet*)
MyArrow =
 Graphics[{LightBlue, Rectangle[{0, 0}, {1, 1}], Black,
   Arrowheads[0.1], Arrow[{{0.5, 1}, {0.5, 0}}]},
  ImageSize -> Tiny, AspectRatio -> 0.5];
Tab1 = Table[{}, {4*bundleNumb - 2}];
Tab2 = {{"Space", "Basis", "Equivalence Relations",
   "Naive Dimension"}};
PCount = 0;
For[i = 1, i <= bundleNumb, i++,

 (*Get cohomologies of bundle i*)
 help = MyCohomologies[[i]];

 (*Look for non- trivial entries and save them to Tab2*)
 For[k = 1, k <= numbCohom, k++,
  (*Print[help[[k ,1,1]]]; *)
  If [! CheckTrivial [help [[k, 1, 1]]],
    PCount = PCount + 1;
    Tab2 =
     Join[Tab2, {{"P" <> ToString[PCount],
        Map[MatrixForm, help[[k, 1]], {1}],
        Map[MatrixForm, help[[k, 2]], {1}],
        Length[help[[k, 1]]] - Length[help[[k, 2]]]}}];
    help [[k]] = "P" <> ToString[PCount];
    ,
    help [[k]] = {Map[MatrixForm, help[[k, 1]], {1}],
       Map[MatrixForm, help[[k, 2]], {1}]};
    ];
  ];

 (*Add information to Tab1*)
 Tab1[[4*i - 3]] =
  Join[{GenerateBundleOutput[Sequence[[i]]]}, help ];
 If [i < bundleNumb,
  Tab1[[4*i - 2]] = Table[MyArrow, {j, numbCohom + 1}];
  Tab1[[4*i - 1]] =
   Join[{MatrixForm[SequenceMaps[[i]]]}, Tab5[[i ]]];
  (*Tab1[[4*i-1]]=Table["mappings",{numbCohom+1}];*)
  Tab1[[4*i ]] = Table[MyArrow, {j, numbCohom + 1}];
  ];

 ];
(*add cohomology labels*)
Tab1 =
 Join[Tab1, {Array[
    If[# == 1, , Superscript["H", ToString[# - 2]]] &,
    numbCohom + 1]}];

(*Step 6 - Print output*)
```

```
    (*Print global sections*)
    Print[Style[
      Labeled[Grid[Tab3, Frame -> All, Background -> LightBlue,
        Alignment -> Baseline],
       Text@Style[
         "Global sections defining the complete intersection \
subvariety.", "Text"], {{Bottom, Center}}], FontSize -> 16]];

    (*Print legends*)
    Print[Style[
      Labeled[Grid[Tab2, Frame -> All, Background -> LightBlue,
        Alignment -> Baseline],
       Text@Style[
         "Rationom spaces in the first sheet of the Koszul spectral \
sequence.", "Text"], {{Bottom, Center}}], FontSize -> 16]];

    (*Print first sheet*)
    helpRule3 =
     Array[If[Divisible[# + 3, 4], # -> Orange, # -> LightBlue] &,
      Length[Tab1]];
    Print[Style[
      Labeled[Grid[Tab1, Alignment -> Center,
        Frame -> {1 -> True, -1 ->
           True, {{1, Length[Tab1] - 1}, {2, numbCohom + 1}} -> True},
        Background -> {Automatic, helpRule3}],
       Text@Style[
         " First sheet of the Koszul exact sequence and the maps \
therein.", "Text"], {{Bottom, Center}}], FontSize -> 16]];

    (*Step 7 - Signal that the computation is done*)
    tEnd = AbsoluteTime[];
    Return[
     "Computation finished after " <> ToString[tEnd - tStart] <>
      " seconds."];
    ], ProgressIndicator[
    MCount, {0,
     Length[Divisors]*(Length[ambSpace[[3]]] -
        Length[ambSpace[[3, 1]]]) }]]
RandomSection[ambSpace_, Charges_] := Module[{MySec, ReplacementTaple},
  MySec = GetBasisOfLineBundleCohomology[ambSpace, Charges][[1]];
  MySec = Map[RandomReal[]*# &, MySec];
  Return[Total[MySec]];
  ]
(*Compute E2-sheet under simplified assumptions*)
ComputeSheetE2[ambSpace_, Divisors_, BundleDivisor_, GlobalSection_] :=
  Monitor[Module[
    {KoszulSequence, Sequence, SequenceMaps, MyCohomologies,
     bundleNumb, Tab1, Tab1b, Tab1c, Tab3, Tab5, Tab5b, PCount, i, j,
     k, l, numbCohom, help, helpVector, helpVector2, helpRule,
     helpRule2, helpRule3, helpRule4, helpBasisVec, helpIndex,
     helpEntry, helpCoefficient, helpCoefficient2, MyArrow, Sections,
     helpMatrix, helpMatrix2, helpArray, helpArray2, VariableArray,
```

```
    tStart , tEnd, helper},

  (*Step 0 - save system time and start computation*)
  tStart = AbsoluteTime[];
  Print["Computation started"];

  (*Step 1 - Generate the Koszul-Sequence and the rough mappings*)
  KoszulSequence = GenerateKoszulSequence[Divisors, BundleDivisor];
  Sequence = KoszulSequence[[1]];
  SequenceMaps = KoszulSequence[[2]];
  bundleNumb = Length[Sequence];

  (*Step 2 - Get the cohomologies of all direct-sum line bundles*)
  MyCohomologies =
   GetAllCohomologiesInKoszulSequence[ambSpace, Sequence];
  numbCohom = Length[MyCohomologies[[1]]];

  (*Step 3 -
  Get the sections defining the algebraic subvariety and create \
corresponding rule*)
  (*Tab3=GetGlobalSections[ambSpace,Divisors];*)
  Tab3 = GlobalSection;
  helpRule = {};
  For[i = 1, i <= Length[Divisors], i++,
   helpRule = Join[helpRule, {Tab3[[i + 1, 1]] -> Tab3[[i + 1, 2]]}];
   ];

  (*Step 4 - Compute explicit mapping matrices*)
  helpRule2 =
   Array[Symbol["x" <> ToString[#]] -> 0 &, Length[ambSpace[[3]]]];
  Tab5 = Table[Table[0, {numbCohom}], {Length[Sequence] - 1}];
  Tab5b = Table[Table[0, {numbCohom}], {Length[Sequence] - 1}];
  VariableArray =
   Array[Symbol["x" <> ToString[#]] &, Length[ambSpace[[3]]]];
  MCount = 0;

  (*Compute the mappings between row i and row i+1*)
  For[i = 1, i < Length[Sequence], i++,
   (*and therein the columns j*)
   For[j = 1, j <= numbCohom, j++,

     (*get principal mapping matrix from SequenceMaps*)
     helpMatrix = SequenceMaps[[i]];

     (*represent the basis space*)
     help = MyCohomologies[[i, j, 1]];
     helpVector = Symbol["A" <> ToString[1]]*help[[1]];
     For[k = 2, k <= Length[help], k++,
      helpVector = helpVector + Symbol["A" <> ToString[k]]*help[[k]];
      ];

     (*if the domain space is trivial , the mapping is trivial , so*)
```

```
      If [!  CheckTrivial [helpVector],

       (*the domain is non– trivial ,
       so proceed and check if the target space is non– trivial ... *)

        If [
          Length[MyCohomologies[[i + 1, j, 1]]]  > 1 || !
            CheckTrivial [MyCohomologies[[i + 1, j , 1,  1]]],

          (*the target space is non– trivial ,
          so proceed with the real computation*)
          (*The situation is non– trivial ,
          and the mapping coefficient for each basis element in the \
target space are determind in what follows*)

          (*compute the mapped vector –
          in particular use the expressions for the global sections *)
          helpVector = helpMatrix.helpVector;
          helpVector = Expand[helpVector /. helpRule];

          (* initialise  helpMatrix2*)
          helpMatrix2 = {};

          (*get the mapping coefficient for each basis element of the \
target*)
          For[l = 1, l <= Length[MyCohomologies[[i + 1, j, 1]]], l++,

           (*consider the l−th basis vector of the target cohomologies*)


           helpBasisVec = MyCohomologies[[i + 1, j, 1, l ]];

           (* isolate  its single entry and position of that entry*)
           helpEntry = Total[helpBasisVec];
           helpEntry = helpEntry [[1]];
           helpIndex = Position[helpBasisVec, helpEntry ][[1,  1]];

           (*compute the coefficient of the l–
           th target space basis vector*)
           (* this is the most time–
           consuming task in the entire notebook*)

           (*Step 1 – Compute the coefficient *)
           If [Variables[helpEntry] != {},

            (*\[Rule] basis vector non– trivial ,
            so go through detailed analysis *)

            (*step 1.1: keep only expression in helpArray2,
            that have the correct denominator*)

            (* split the long polynomial image expression into its \
```

```
additive parts*)
            helpArray2 = Level[helpVector[[helpIndex, 1]], 1];
            (*get variables in the l-th basis vector*)
            helpArray = Variables[helpEntry];
            (*keep only expression of helpArray2,
            that depend on the variables that the l-
            th basis vector does depend on*)

            (*helper=Array[SameQ[Denominator[helpEntry],Denominator[
            helpArray2[[#]]]]&, Length[helpArray2]];
            helpArray2=Pick[helpArray2,helper];*)

            helpArray2 =
             Select[helpArray2,
              SameQ[Denominator[helpEntry], Denominator[#]] &];

            (*step 2.1: keep only expression in helpArray2,
            that depend on the correct variables*)

            helpArray2 =
             Select[helpArray2,
              Intersection[Complement[VariableArray, helpArray],
                 Variables[#]] == {} &];

            (*step 3.1:
            compute the coefficient in the reduced expression via the \
'slow' command coefficient*)
            (*step 3.1: for speed-up, this is parallelised *)
             helpCoefficient = ParallelMap[
               If[Variables[Denominator[#]*helpEntry] != {},
                 Coefficient[Numerator[#], Denominator[#]*helpEntry]
                 ,
                 Numerator[#] /. helpRule2
                 ]
                &, helpArray2, Method -> Automatic];

            (*Step 4.1:
            mathematica might have factorised and thus left variables \
x_i in helpCoefficient2*)
            (*Step 4.1: those we do not consider real coefficients,
            thus we get rid of them now*)

             helpCoefficient =
             Select[helpCoefficient,
              Intersection[Variables[#], VariableArray] == {} &];


            ,

            (*otherwise the basis vector is trivial,
            and the coefficient is simply the constant part*)
             helpCoefficient = helpVector[[helpIndex, 1]] /. helpRule2;
            ];
```

```
          (*drop zeros in helpCoefficient *)
          (* helpCoefficient =Select[ helpCoefficient ,
          Variables [#]\[NotEqual]{}&];*)
           helpCoefficient = Total[ helpCoefficient ];

          (*Step 2: Use helpCoefficient to assemble matrix*)

          helpMatrix2 =
           Join[helpMatrix2, {GenerateRow[helpCoefficient ,
               Length[MyCohomologies[[i, j, 1]]]]}];

           ];

         (*compute kernel of non- trivial mapping matrix*)
         Tab5[[i , j ]] = Length[NullSpace[helpMatrix2]];
         Tab5b[[i , j ]] = "Ker = " <> ToString[Tab5[[i, j ]]];
         ,

         (*otherwise the target space is trivial ,
         so the mapping is trivial *)
         Tab5[[i , j ]] = Length[MyCohomologies[[i, j, 1]]];
         Tab5b[[i , j ]] = "Ker = " <> ToString[Tab5[[i, j ]]];
         ];

       ,

       (*otherwise the domain is trivial , so the mapping is*)
       Tab5[[i , j ]] = 0;
       Tab5b[[i , j ]] = "Ker = 0";
       ];

      (* increase the computed-matrix-counter*)
      MCount = MCount + 1;

      ]; (*j-loop finished *)
    ]; (*i-loop finished *)

  (*Step 5 - Nicely represent the first sheet*)
  MyArrow =
   Graphics[{LightBlue, Rectangle[{0, 0}, {1, 1}], Black,
     Arrowheads [0.1], Arrow[{{0.5, 1}, {0.5, 0}}]},
    ImageSize -> Tiny, AspectRatio -> 0.5];
  Tab1 = Table[{}, {4*bundleNumb - 2}];
  Tab1b = Table[{}, {4*bundleNumb - 2}];
  PCount = 0;
  For[i = 1, i <= bundleNumb, i++,

   (*Get cohomologies of bundle i *)
   help =
    Map[If[Variables[#[[1, 1]]] != {}, Length [#[[1]]], 0] &,
     MyCohomologies[[i ]]];

```

```
    (*Add information to Tab1*)
    Tab1[[4*i - 3]] =
     Join[{GenerateBundleOutput[Sequence[[i]]]}, help];
    Tab1b[[4*i - 3]] = Tab1[[4*i - 3]];
    If[i < bundleNumb,
     Tab1[[4*i - 2]] = Table[MyArrow, {j, numbCohom + 1}];
     Tab1b[[4*i - 2]] = Table[MyArrow, {j, numbCohom + 1}];
     Tab1[[4*i - 1]] =
      Join[{MatrixForm[SequenceMaps[[i]]]}, Tab5b[[i]]];
     Tab1b[[4*i - 1]] =
      Join[{MatrixForm[SequenceMaps[[i]]]}, Tab5[[i]]];
     Tab1[[4*i]] = Table[MyArrow, {j, numbCohom + 1}];
     Tab1b[[4*i]] = Table[MyArrow, {j, numbCohom + 1}];
     ];

    ];
   (*add cohomology labels*)
   Tab1 =
    Join[Tab1, {Array[
       If[# == 1, , Superscript["H", ToString[# - 2]]] &,
       numbCohom + 1]}];
   Tab1b =
    Join[Tab1b, {Array[
       If[# == 1, , Superscript["H", ToString[# - 2]]] &,
       numbCohom + 1]}];

   (*Step 6 - compute sheet E2*)
   (*use Tab1b as backup data, and compute the entries of Tab1c*)
   Tab1c = Table[Table[, {Length[Tab1b[[1]]]}], {Length[Tab1b]}];
   For[i = 1, i <= bundleNumb, i++,

    (*add legend on the left *)
    Tab1c[[4*i - 3, 1]] = Tab1b[[4*i - 3, 1]];
    If[i != bundleNumb,
     Tab1c[[4*i - 2, 1]] = Tab1b[[4*i - 2, 1]];
     Tab1c[[4*i - 1, 1]] = Tab1b[[4*i - 1, 1]];
     Tab1c[[4*i, 1]] = Tab1b[[4*i, 1]];
     ];

    (*compute dimensions of quotient spaces in sheet E2*)
    For[j = 1, j <= numbCohom, j++,

     (*compute dimensions*)
     If[i == 1,
       (* If we compute the row on top just take the kernel*)
       Tab1c[[4*i - 3, j + 1]] = Tab1b[[4*i - 1, j + 1]];
       ,

       (* If we compute the row at the bottom*)
       If[i == bundleNumb,

         (*otherwise ker - Im*)
```

```
          Tab1c[[4*i - 3, j + 1]] =
           Tab1b[[4*i - 3, j + 1]] + Tab1b[[4*i - 5, j + 1]] -
            Tab1b[[4*i - 7, j + 1]];
          Tab1c[[4*i - 1, j + 1]] = Superscript["H", ToString[j - 1]];
          ,

          (*Otherwise we are somewhere in the middle*)
          Tab1c[[4*i - 3, j + 1]] =
            Tab1b[[4*i - 1, j + 1]] + Tab1b[[4*i - 5, j + 1]] -
             Tab1b[[4*i - 7, j + 1]];
          ];
        ];
      ];
     ];

   (*Print first sheet*)
   helpRule3 =
    Array[If[ Divisible [# + 3, 4], # -> Orange, # -> LightBlue] &,
     Length[Tab1]];
   Print[Style[
     Labeled[Grid[Tab1, Alignment -> Center,
       Frame -> {1 -> True, -1 ->
          True, {{1, Length[Tab1] - 1}, {2, numbCohom + 1}} -> True},
       Background -> {Automatic, helpRule3}],
      Text@Style[
        " First sheet of the Koszul exact sequence and the maps \
therein .", "Text"], {{Bottom, Center}}], FontSize -> 16]];

   (*Print second sheet*)
   Print[Style[
     Labeled[Grid[Tab1c, Alignment -> Center,
       Frame -> {1 -> True, -1 ->
          True, {{1, Length[Tab1c] - 1}, {2, numbCohom + 1}} -> True},
        Background -> {Automatic, helpRule3} (*ItemSize\[Rule]All*)],
      Text@Style[
        "Second sheet of the Koszul exact sequence and the maps \
therein .", "Text"], {{Bottom, Center}}], FontSize -> 16]];

   (*Step 7 - Signal that the computation is done*)
   tEnd = AbsoluteTime[];
   Return[tEnd - tStart];
   ], ProgressIndicator [
   MCount, {0,
    Length[Divisors]*(Length[ambSpace[[3]]] -
       Length[ambSpace[[3, 1]]]) }]]

(*Compute model of our choice*)
(*Compute model of our choice*)
Model[ambSpace_, DB3_, DGUT_, G4_] :=
  Module[{AntiKB3, Da1, Da21, Da32, Da43, DC10, DC5m, DC5H, DL1, DL2,
    DL3, sB3, sGUT, sa1, sa21, sa32, sa43, sC10, sC5m, sC5H,
    OutputTable, InputTable1, InputTable2, InputTable3, t1, t2},
```

```
(*Step 0: Signal that the computation started*)
Print["Started computation."];
t1 = AbsoluteTime[];

(*Step 1: Compute necessary divisor classes *)
AntiKB3 = Total[ambSpace[[3]]] - DB3;
Da1 = AntiKB3;
Da21 = 2*AntiKB3 - 1*DGUT;
Da32 = 3*AntiKB3 - 2*DGUT;
Da43 = 4*AntiKB3 - 3*DGUT;
DC10 = Da1;
DC5m = Da32;
DC5H = 5*AntiKB3 - 3*DGUT;

(*Step 2: Compute some sections*)
sa1 = RandomSection[ambSpace, Da1];
sa21 = RandomSection[ambSpace, Da21];
sa32 = RandomSection[ambSpace, Da32];
sa43 = RandomSection[ambSpace, Da43];

(*Step 3: Compute line bundles L_i*)
DL1 = -G4 + (DC10 + DGUT - AntiKB3)/2;
DL2 = 3*G4 + (DC5m + DGUT - AntiKB3)/2;
DL3 = 2*G4 + (DC5H + DGUT - AntiKB3)/2;

(*Step 4: Compute random sections*)
sB3 = RandomSection[ambSpace, DB3];
sGUT = RandomSection[ambSpace, DGUT];
sC10 = sa1;
sC5m = sa32;
sC5H = Expand[sa32*sa21 - sa43*sa1];

(*Step 5: Organise the above information in a nice output*)
OutputTable = {{"Name", "Charge", "Random Section"}};
OutputTable = Join[OutputTable, {{"Base", DB3, sB3}}];
OutputTable = Join[OutputTable, {{"GUT", DGUT, sGUT}}];
OutputTable = Join[OutputTable, {{"a1", Da1, sa1}}];
OutputTable = Join[OutputTable, {{"a21", Da21, sa21}}];
OutputTable = Join[OutputTable, {{"a32", Da32, sa32}}];
OutputTable = Join[OutputTable, {{"a43", Da43, sa43}}];
OutputTable = Join[OutputTable, {{"C10", DC10, sC10}}];
OutputTable = Join[OutputTable, {{"C5m", DC5m, sC5m}}];
OutputTable = Join[OutputTable, {{"C5H", DC5H, sC5H}}];
OutputTable = Join[OutputTable, {{"L1", DL1, "-"}}];
OutputTable = Join[OutputTable, {{"L2", DL2, "-"}}];
OutputTable = Join[OutputTable, {{"L3", DL3, "-"}}];
Print[Style[
  Labeled[Grid[OutputTable, Frame -> All, Background -> LightBlue,
    Alignment -> Baseline],
   Text@Style[
     "Global sections defining the complete intersection \
```

```
subvariety .", "Text"], {{Bottom, Center}}], FontSize -> 16]];

   (*Step 6: Compute cohomologies till E2-sheet*)

   (*Start computation of second sheets on C_10*)
   InputTable1 = {{"Section", " Explicit form", "Charges"}};
   InputTable1 =
    Join[InputTable1, {{Symbol["s" <> ToString[1]], sB3, DB3}}];
   InputTable1 =
    Join[InputTable1, {{Symbol["s" <> ToString[2]], sGUT, DGUT}}];
   InputTable1 =
    Join[InputTable1, {{Symbol["s" <> ToString[3]], sC10, DC10}}];
   Print["Start computation on C10 curve"];
   Print["Computation finished after " <>
     ToString[
      ComputeSheetE2[ambSpace, {DB3, DGUT, DC10}, DL1, InputTable1]] <>
      " seconds."];

   (*Start computation of second sheets on C_5m*)
   InputTable2 = {{"Section", " Explicit form", "Charges"}};
   InputTable2 =
    Join[InputTable2, {{Symbol["s" <> ToString[1]], sB3, DB3}}];
   InputTable2 =
    Join[InputTable2, {{Symbol["s" <> ToString[2]], sGUT, DGUT}}];
   InputTable2 =
    Join[InputTable2, {{Symbol["s" <> ToString[3]], sC5m, DC5m}}];
   Print["Start computation on C5m curve"];
   Print["Computation finished after " <>
     ToString[
      ComputeSheetE2[ambSpace, {DB3, DGUT, DC5m}, DL2, InputTable2]] <>
      " seconds."];

   (*Start computation of second sheets on C_5H*)
   InputTable3 = {{"Section", " Explicit form", "Charges"}};
   InputTable3 =
    Join[InputTable3, {{Symbol["s" <> ToString[1]], sB3, DB3}}];
   InputTable3 =
    Join[InputTable3, {{Symbol["s" <> ToString[2]], sGUT, DGUT}}];
   InputTable3 =
    Join[InputTable3, {{Symbol["s" <> ToString[3]], sC5H, DC5H}}];
   Print["Start computation on C5H curve"];
   Print["Computation finished after " <>
     ToString[
      ComputeSheetE2[ambSpace, {DB3, DGUT, DC5H}, DL3, InputTable3]] <>
      " seconds."];

   (*Step 7: Signal end of computation*)
   t2 = AbsoluteTime[];
   Return[
    "Finished the computation after " <> ToString[t2 - t1] <>
     " seconds."];
   ];
```

F. List of Tables

G. List of Figures

H. Bibliography

[1] J. Polchinski, *S*tring Theory: Volume 1, An Introduction To The Bosonic String, Cambridge monographs on mathematical physics, Cambridge University Press, 1998, ISBN 9781139457408, URL `http://books.google.de/books?id=jbM3t_usmX0C`.

[2] J. Polchinski, *S*tring Theory: Volume 2, Superstring Theory and Beyond, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2005, ISBN 9780521672283, URL `http://books.google.de/books?id=tJtOMAEACAAJ`.

[3] M. Green, J. Schwarz and E. Witten, *S*uperstring Theory: Volume 1, Introduction, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1988, ISBN 9780521357524, URL `http://books.google.co.in/books?id=ItVsHqjJo4gC`.

[4] M. Green, J. Schwarz and E. Witten, *S*uperstring Theory: Volume 2, Loop Amplitudes, Anomalies and Phenomenology, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1987, ISBN 9780521357531, URL `http://books.google.de/books?id=Z-uz4svcl0QC`.

[5] H. P. Nilles, S. Ramos-Sanchez, M. Ratz and P. K. Vaudrevange, *From strings to the MSSM*, *E*ur.Phys.J. **C**59 (2009) 249–267, [0806.3905].

[6] L. E. Ibanez, F. Marchesano and R. Rabadan, *Getting just the standard model at intersecting branes*, *J*HEP **0**111 (2001) 002, [hep-th/0105155].

[7] F. G. Marchesano Buznego, *Intersecting D-brane models*, *a*rXiv Ph.D. Thesis (Advisor: D.Luis E. Ibanez Santiago), [hep-th/0307252].

[8] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *P*hys.Rept. **4**45 (2007) 1–193, [hep-th/0610327].

[9] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, *Toward realistic intersecting D-brane models*, *A*nn.Rev.Nucl.Part.Sci. **5**5 (2005) 71–139, [hep-th/0502005].

[10] M. Cvetic, G. Shiu and A. M. Uranga, *Three family supersymmetric standard - like models from intersecting brane worlds*, *P*hys.Rev.Lett. **8**7 (2001) 201801, [hep-th/0107143].

[11] M. Cvetic, P. Langacker and G. Shiu, *A Three family standard - like orientifold model: Yukawa couplings and hierarchy*, *N*ucl.Phys. **B**642 (2002) 139–156, [hep-th/0206115].

[12] M. Cvetic, P. Langacker and G. Shiu, *Phenomenology of a three family standard like string model*, Phys.Rev. **D**66 (2002) 066004, [hep-ph/0205252].

[13] D. Lust, *Intersecting brane worlds: A Path to the standard model?*, Class.Quant.Grav. **2**1 (2004) S1399–1424, [hep-th/0401156].

[14] T. Weigand, *Lectures on F-theory compactifications and model building*, Class.Quant.Grav. **2**7 (2010) 214004, [1009.3497].

[15] J. J. Heckman, *Particle Physics Implications of F-theory*, Ann.Rev.Nucl.Part.Sci. **6**0 (2010) 237–265, [1001.0577].

[16] C. Vafa, *Evidence for F theory*, Nucl.Phys. **B**469 (1996) 403–418, [hep-th/9602022].

[17] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 1*, Nucl.Phys. **B**473 (1996) 74–92, [hep-th/9602114].

[18] D. R. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau threefolds. 2.*, Nucl.Phys. **B**476 (1996) 437–469, [hep-th/9603161].

[19] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - I*, JHEP **0**901 (2009) 058, [0802.3391].

[20] C. Beasley, J. J. Heckman and C. Vafa, *GUTs and Exceptional Branes in F-theory - II: Experimental Predictions*, JHEP **0**901 (2009) 059, [0806.0102].

[21] R. Donagi and M. Wijnholt, *Model Building with F-Theory*, Adv.Theor.Math.Phys. **1**5 (2011) 1237–1318, [0802.2969].

[22] R. Donagi and M. Wijnholt, *Breaking GUT Groups in F-Theory*, Adv.Theor.Math.Phys. **1**5 (2011) 1523–1604, [0808.2223].

[23] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New Aspects of Heterotic–F Theory Duality*, Nucl.Phys. **B**806 (2009) 224–299, [0805.1057].

[24] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov et al., *Geometric singularities and enhanced gauge symmetries*, Nucl.Phys. **B**481 (1996) 215–252, [hep-th/9605200].

[25] S. H. Katz and C. Vafa, *Matter from geometry*, Nucl.Phys. **B**497 (1997) 146–154, [hep-th/9606086].

[26] E. Witten, *Nonperturbative superpotentials in string theory*, Nucl.Phys. **B**474 (1996) 343–360, [hep-th/9604030].

[27] P. Deligne, *"Tcourbes elliptiques"*, Deligne, P., 1975, 5375 pp.

[28] J. Milne, *E*lliptic Curves, BookSurge Publishers, 2006, ISBN 1-4196-5257-5, 238+viii pp.

[29] E. Freitag and R. Busam, *C*omplex Analysis, Universitext - Springer-Verlag, Springer, 2009, ISBN 9783540939832, URL `http://books.google.de/books?id=3xBpS-ZKlgsC`.

[30] F. Denef, *Les Houches Lectures on Constructing String Vacua*, arXiv pp. 483–610, [0803.1194].

[31] K. Kodaira, *On compact analytic surfaces*, Annals of Math. 77 p. 563.

[32] T. W. Grimm and T. Weigand, *On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs*, Phys.Rev. **D**82 (2010) 086009, [1006.0226].

[33] J. Marsano, N. Saulina and S. Schafer-Nameki, *Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs*, JHEP **0**908 (2009) 046, [0906.4672].

[34] J. Marsano, N. Saulina and S. Schafer-Nameki, *Compact F-theory GUTs with U(1) (PQ)*, JHEP **1**004 (2010) 095, [0912.0272].

[35] M. J. Dolan, J. Marsano and S. Schafer-Nameki, *Unification and Phenomenology of F-Theory GUTs with $U(1)_PQ$*, JHEP **1**112 (2011) 032, [1109.4958].

[36] E. Palti, *A Note on Hypercharge Flux, Anomalies, and U(1)s in F-theory GUTs*, Phys.Rev. **D**87, no. 8 (2013) 085036, [1209.4421].

[37] T. Weigand, C. Pehle and M. Bies, Massless matter in F-theory via Chow groups and Deligne cohomology, 2014, to be published.

[38] P. S. Aspinwall, *D-branes on Calabi-Yau manifolds*, arxiv pp. 1–152, [hep-th/0403166].

[39] R. Donagi and M. Wijnholt, *MSW Instantons*, JHEP **1**306 (2013) 050, [1005.5391].

[40] R. Donagi and M. Wijnholt, *Gluing Branes, I*, JHEP **1**305 (2013) 068, [1104.2610].

[41] R. Donagi and M. Wijnholt, *Gluing Branes II: Flavour Physics and String Duality*, JHEP **1**305 (2013) 092, [1112.4854].

[42] S. H. Katz and E. Sharpe, *D-branes, open string vertex operators, and Ext groups*, Adv.Theor.Math.Phys. **6** (2003) 979–1030, [hep-th/0208104].

[43] M. F. Atiyah, Riemann surfaces and spin structures, in Annales Scientifiques de L'Ecole Normale Superieure, volume 4, pp. 47–62, Société mathématique de France, 1971.

[44] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley Classics Library, Wiley, 2011, ISBN 9781118030776, URL `http://books.google.de/books?id=Sny48qKdW40C`.

[45] E. Freitag, Riemann Surfaces, Selfpublishing, 2013, URL `http://www.rzuser.uni-heidelberg.de/~t91/skripten/riemfl.pdf`.

[46] S. Krause, C. Mayrhofer and T. Weigand, *Gauge Fluxes in F-theory and Type IIB Orientifolds*, JHEP **1**208 (2012) 119, [1202.3138].

[47] S. Krause, C. Mayrhofer and T. Weigand, *G_4 flux, chiral matter and singularity resolution in F-theory compactifications*, Nucl.Phys. **B**858 (2012) 1–47, [1109.3454].

[48] E. Witten, *Phases of N=2 theories in two-dimensions*, Nucl.Phys. **B**403 (1993) 159–222, [hep-th/9301042].

[49] T. Rahn, *Heterotic Target Space Dualities with Line Bundle Cohomology*, Ph.D. thesis, Ludwig-Maximilians-Universität München, 2012, URL `http://edoc.ub.uni-muenchen.de/14344/2/Rahn_Thorsten.pdf`.

[50] M. Larfors, D. Lust and D. Tsimpis, *Flux compactification on smooth, compact three-dimensional toric varieties*, JHEP **1**007 (2010) 073, [1005.2194].

[51] W. Stein et al., Sage Mathematics Software (Version 5.13), The Sage Development Team, 2014, `http://www.sagemath.org`.

[52] D. Cox, J. Little and H. Schenck, Toric Varieties, Graduate Studies in Mathematics, American Mathematical Society, 2011, ISBN 9780821848197, URL `http://books.google.de/books?id=eXLGwYD4pmAC`.

[53] B. Jurke, T. Rahn, R. Blumenhagen and H. Roschy, "cohomCalg ++ Koszul Extension - Manual v0.31", "May" "2011".

[54] R. Blumenhagen, B. Jurke, T. Rahn and H. Roschy, *Cohomology of Line Bundles: A Computational Algorithm*, J.Math.Phys. **5**1 (2010) 103525, [1003.5217].

[55] R. Blumenhagen, B. Jurke, T. Rahn and H. Roschy, *Cohomology of Line Bundles: Applications*, J.Math.Phys. **5**3 (2012) 012302, [1010.3717].

[56] R. Blumenhagen, B. Jurke and T. Rahn, *Computational Tools for Cohomology of Toric Varieties*, Adv.High Energy Phys. **2**011 (2011) 152749, [1104.1187].

[57] *cohomCalg package*, Download link, 2010, URL `http://wwwth.mppmu.mpg.de/members/blumenha/cohomcalg/`, high-performance line bundle cohomology computation based on [54].

[58] L. B. Anderson, *Heterotic and M-theory Compactifications for String Phenomenology*, arxiv [0808.3621].

[59] A. Grothendieck, Éléments de géométrie algébrique: par A. Grothendieck, rédigés avec la collaboration de J. [Jean] Dieudonné. Étude locale des schémas et des morphismes de schémas. 2e partie, Bd. 4, Presses universitaires, 1965, URL `http://books.google.de/books?id=CjnfQwAACAAJ`.

[60] D. Eisenbud, Commutative Algebra: With a View Toward Algebraic Geometry, Graduate Texts in Mathematics, Springer, 1995, ISBN 9780387942698, URL `http://books.google.de/books?id=Fm_yPgZBucMC`.

[61] M. Cvetic, I. Garcia-Etxebarria and J. Halverson, *Global F-theory Models: Instantons and Gauge Dynamics*, JHEP **1**101 (2011) 073, [1003.5337].

[62] T. Rahn and H. Roschy, *Cohomology of Line Bundles: Proof of the Algorithm*, J.Math.Phys. **5**1 (2010) 103520, [1006.2392].

[63] S.-Y. Jow, *Cohomology of toric line bundles via simplicial Alexander duality*, Journal of Mathematical Physics **5**2, no. 3 (2011) 033506, [1006.0780].

[64] J.-P. Serre, Géométrie algébrique et géométrie analytique, Annales de l'institut Fourier **6** (1956) 1–42, URL `http://eudml.org/doc/73726`.

[65] D. Hilbert, *Über* die Theorie der algebraischen formen, *M*ath. Annalen **3**6 (1890) 473–534.

[66] R. Hartshorne, *A*lgebraic Geometry, Graduate Texts in Mathematics, Springer, 1977, ISBN 9780387902449, URL `http://books.google.de/books?id=3rtX9t-nnvwC`.

[67] J. Distler and B. Greene, *A*spects of (2, 0) string compactifications, *N*uclear Physics B **3**04 (1988) 1–62.

[68] T. Hübsch, *C*alabi-Yau Manifolds: A Bestiary for Physicists, World Scientific Publishing Company Incorporated, 1994, ISBN 9789810219277, URL `http://books.google.de/books?id=Z5zSbFktn1EC`.

[69] H.-C. Wang, *Closed manifolds with homogeneous complex structure.*, *Am. J. Math.* **7**6 (1954) 1–32.

[70] W. Fulton and J. Harris, *R*epresentation Theory: A First Course, Graduate texts in mathematics, Springer-Verlag, 1991, ISBN 9780387974958, URL `http://books.google.de/books?id=6GUH8ARxhp8C`.

[71] E. Freitag, *C*omplex Spaces, Selfpublishing, 2013, URL `http://www.rzuser.uni-heidelberg.de/~t91/skripten/complexspaces.pdf`.

[72] K. Kodaira and D. Spencer, *D*ivisor class groups on algebraic varieties, *P*roceedings of the National Academy of Sciences of the United States of America **3**9, no. 8 (1953) 872.

[73] M. F. Atiyah, *V*ector bundles over an elliptic curve, *P*roc. London Math. Soc **7**, no. 3 (1957) 415–452.

[74] M. Narasimhan, *V*ector bundles on compact Riemann surfaces, *C*omplex analysis and its applications **3** (1976) 335–346.

[75] O. Forster and B. Gilligan, *L*ectures on Riemann Surfaces, Graduate Texts in Mathematics, Springer, 1981, ISBN 9780387906171, URL `http://books.google.de/books?id=iDYBTCVCO_IC`.

[76] R. C. Gunning, *L*ectures on vector bundles over Riemann surfaces, volume 6, Princeton University Press, 1967.

[77] E. Freitag, *F*unktionentheorie 2, *F*unktionentheorie 2: Riemannásche Flächen, Mehrere komplexe Variable, Abelásche Funktionen, Höhere Modulformen, Springer-Lehrbuch. ISBN 978-3-540-87896-4. Springer-Verlag Berlin Heidelberg, 2009 **1**.

[78] S. Cautis, *V*ector Bundles On Riemann Surfaces, 2005, URL `http://www.math.ubc.ca/~cautis/classes/notes-bundles.pdf`, lectures at the Mathematics Department University of Southern California.

[79] A. Bobenko, *C*ompact Riemann Surfaces, URL `http://page.math.tu-berlin.de/~bobenko/Lehre/Skripte/RS.pdf`, lecture notes.

[80] M. Nakahara, *G*eometry, topology and physics, Nakahara, M., 2003, bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).

[81] D. Mumford, *T*heta characteristics of an algebraic curve, *A*nn. Sci. Ecole Norm. Sup **4**, no. 4 (1971) 181–192.

[82] P. Appell, *"Sur les functiones périodiques de deux variables"*, Appell, P., 1891, 157-219 pp., URL http://gallica.bnf.fr/ark:/12148/bpt6k107455z.

[83] G. Humbert, *"Théorie générale des surfaces hyperelliptiques"*, Humbert, G., 1893, 29170, 361475 pp., URL http://gallica.bnf.fr/ark:/12148/bpt6k107457q.

[84] S. Lefschetz, *O*n Certain Numerical Invariants of Algebraic Varieties with Application to Abelian Varieties, American Mathematical Society, 1921, URL http://books.google.de/books?id=luQqQwAACAAJ.

[85] D. Mumford, *A*belian Varieties, Studies in mathematics, Mumbai, 2008, ISBN 9788185931869, URL http://books.google.de/books?id=J81jPwAACAAJ.

[86] B. Hall, *L*ie Groups, Lie Algebras, and Representations: An Elementary Introduction, Graduate Texts in Mathematics, Springer, 2003, ISBN 9780387401225, URL http://books.google.de/books?id=m1VQi8HmEwcC.

[87] K. Brown, *C*ohomology of Groups, Graduate Texts in Mathematics, Springer, 1982, ISBN 9780387906881, URL http://books.google.de/books?id=PMqb2DppvCsC.

[88] R. Bellman, *A* Brief Introduction to Theta Functions, Dover Books on Mathematics, DOVER PUBN Incorporated, 2013, ISBN 9780486492957, URL http://books.google.de/books?id=0TTonAEACAAJ.

[89] W. Rudin, *R*eal and Complex Analysis, McGraw-Hill series in higher mathematics, Tata McGraw-Hill, 1974, ISBN 9780070995574, URL http://books.google.de/books?id=onVoAAAACAAJ.

[90] W. Fulton, *I*ntroduction to Toric Varieties, Annals of mathematics studies, Princeton University Press, 1993, ISBN 9780691000497, URL http://books.google.fr/books?id=CbmaGqCRbhUC.

Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, 01. Februar 2014

Martin Bies