Selected Solutions to Cunningham's "Set Theory: A First Course"

Martín Caputo

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1 Introduction

1.5 The Zermelo–Fraenkel Axioms

Exercise 1

Since $\{u\}$ and $\{v, w\}$ are sets, there exists a set A such that for every set x

$$x \in A \iff x = \{u\} \vee x = \{v, w\}.$$

By extension, this set is unique and we call it $A := \{\{u\}, \{v, w\}\}$. Using the union axiom, given that A is a set, then there exists a set B such that for every set x,

$$
\begin{aligned}
x \in B &\iff \exists z(x \in z \wedge z \in A) \\
&\iff \exists z(x \in z \wedge z \in \{\{u\}, \{v, w\}\}) \\
&\iff \exists z\big(x \in z \wedge (z = \{u\} \vee z = \{v, w\})\big) \\
&\iff \exists z\big((x \in z \wedge z = \{u\}) \vee (x \in z \wedge z = \{v, w\})\big) \\
&\iff \exists z\big(x \in \{u\} \vee x \in \{v, w\}\big) \\
&\iff x \in \{u\} \vee x \in \{v, w\} \\
&\iff x = u \vee (x = v \vee x = w) \\
&\iff x = u \vee x = v \vee x = w
\end{aligned}
$$

Where we have simplified the quantifier which does not appear in the expression. Thus B is the unique set whose elements are exactly u, v, w and we call it $B := \{u, v, w\}$.

Exercise 2

Since A is a set, then by pairing there exists the unique set $\{A, A\}$, we shall prove this set is equal to what we call $\{A\}$, the set whose only element is A. Given $x \in \{A\}$ we conclude $x = A$, therefore $x \in \{A, A\}$. Given $x \in \{A, A\}$ we have that $x = A \vee x = A$, therefore $x = A$, which implies $x \in \{A\}$. Thus $\{A\} = \{A, A\}$. Since the sets are equal and one of them exists, they both exist.

Exercise 3

Forst we prove $A \cap \{A\} = \emptyset$. For the sake of contradiction, suppose that the set A is such that $A \cap \{A\} \neq \emptyset$. Then $B := A \cap \{A\}$ is non-empty. Using the axiom of regularity, this means that there exists a set x such that $x \in B$ and $x \cap B = \emptyset$. But the first condition means $x \in A \cap \{A\}$, which means that $x \in A$ and $x \in \{A\}$, which means that $x \in A$ and $x = A$. But then, the second is $x \cap B = A \cap A \cap \{A\} = A \cap \{A\} \neq \emptyset$, a contradiction. Therefore, $A \cap \{A\} = \emptyset$.

Finally, this means that $A \notin A$, since $A \in A$ would imply that there exists $x \in A$ such that, $x = A$ which is equivalent to $x \in \{A\}$. Thus there would exist $x \in A \wedge x \in \{A\} \implies x \in A \cap \{A\} \implies A \cap \{A\} \neq \emptyset$, which contradicts the previous result.

Exercise 4

Using the regularity axiom, since $\{A, B\}$ is non-empty, then by the regularity axiom

$$\exists x(x \in \{A, B\} \wedge x \cap \{A, B\} = \emptyset).$$

Therefore either $x = A \wedge A \cap \{A, B\} = \emptyset$ or $x = B \wedge B \cap \{A, B\} = \emptyset$ or both. Since $A \in B$, then it cannot be that $x = B$, since this would mean $B \cap \{A, B\} = \emptyset$, but of course, there exists $A \in B$ and also $A \in \{A, B\}$, so the intersection $B \cap \{A, B\} = \emptyset$ cannot be empty. Therefore $x = A \wedge A \cap \{A, B\} = \emptyset$, which means that it is false that

$$\begin{aligned}
\exists z(z \in A \wedge z \in \{A, B\}) &\iff \exists z(z \in A \wedge (z = A \vee z = B)) \\
&\iff \exists z((z \in A \wedge z = A) \vee (z \in A \wedge z = B)) \\
&\iff \exists z(A \in A \vee B \in A) \\
&\iff \exists z(B \in A) \\
&\iff B \in A.
\end{aligned}$$

Where we simplified $A \in A$ since we know by Exercise 3 that $A \notin A$. Therefore, it is false that $B \in A$, thus it is true that $B \notin A$.

Exercise 5

From previous exercises we know that $\{A, B, C\}$ is a set. Since it is non-empty, there exists an element x of this set such that $x \cap \{A, B, C\} = \emptyset$. It cannot happen that $x = B$ or $x = C$, since $A \in B$ and $B \in C$, so that the intersections are $B \cap \{A, B, C\}$ and $C \cap \{A, B, C\}$ are both nonempty. Therefore $x = A$ and $A \cap \{A, B, C\} = \emptyset$. Thus, it is false that there exists a $z \in A \wedge z \in \{A, B, C\}$, that is

$$\begin{aligned}
z \in A \wedge (z = A \vee z = B \vee z = C) &\iff (z \in A \wedge z = A) \vee (z \in A \wedge z = B) \vee (z \in A \wedge z = C) \\
&\iff (A \in A) \vee (B \in A) \vee (C \in A) \\
&\iff C \in A
\end{aligned}$$

Where, $B \in A$ is true by hypothesis and $A \in A$ false by Exercise 3. Therefore, it is false that $C \in A$, which means that it is true that $C \notin A$.

Exercise 6

By the powerset axiom the set $\mathcal{P}(A)$ exists. Let $\psi(x) := x \in B$, then there exists a (unique by extension) set S, where for all sets x,

$$\begin{aligned}
x \in S &\iff x \in \mathcal{P}(A) \wedge \psi(x) \\
&\iff x \in \mathcal{P}(A) \wedge x \in B \\
&\iff x \in \mathcal{P}(A) \cap B.
\end{aligned}$$

Exercise 7

By the subset axiom, there exists a set S such that for all sets x, $x \in S$, if and only if, $x \in A \wedge \psi(x)$. Define $\psi(x) := \neg(x \in B)$ thus $A \backslash B$ exists.

Exercise 8

First we show $\{\emptyset\} \neq \emptyset$. Suppose it is true that $x \in \{\emptyset\}$. Since $x \in \emptyset$ is always false, then it is false that $x \in \{\emptyset\} \implies x \in \emptyset$, therefore, $\{\emptyset\} \neq \emptyset$.

Now we show $\{\emptyset, \{\emptyset\}\} \neq \{\emptyset\}$. Let $x = \{\emptyset\}$, then clearly $x \in \{\emptyset, \{\emptyset\}\}$, but $x \notin \{\emptyset\}$, because if it were true that $x \in \{\emptyset\}$ then $x = \emptyset \neq \{\emptyset\}$ (by the previous part) which contradicts our choice of $x = \{\emptyset\}$. Therefore $\{\emptyset, \{\emptyset\}\} \neq \{\emptyset\}$.

Now we show that $\{\emptyset, \{\emptyset\}\} \neq \emptyset$. Let $x = \emptyset$, then $x \in \{\emptyset, \{\emptyset\}\}$. But $x \notin \emptyset$, therefore $\{\emptyset, \{\emptyset\}\} \neq \emptyset$.

Exercise 9

For any x, consider $x \in A$. Since A has no elements, this statement is false, therefore the implication $x \in A \implies x \in \emptyset$ is true. Similarly, $x \in \emptyset \implies x \in A$ is true. Therefore $x \in \emptyset \iff x \in A$, thus, $A = \emptyset$.

Exercise 10

We know that for any set x, there exists the unique set $\{x\}$. In particular for every $x \in A$, there exists the unique set y such that $y = \{x\}$, thus the formula $\forall z(z \in y \iff z = x)$ is uniquely satisfied for all $x \in A$. Therefore we can conclude that there exists a set S where for any y, $y \in S \iff (\exists x \in A)(y = \{x\})$, which means that there is a set S which contains exactly those elements y for which $y = \{x\}$ for some $x \in A$, which is precisely how $\{\{x\} : x \in A\}$ is defined. Therefore $\{\{x\} : x \in A\}$ is a set.