# Selected Solutions to Cunningham's "Set Theory: A First Course"

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## Contents

# 1 Introduction

## 1.5 The Zermelo–Fraenkel Axioms

### Exercise 1

Since $\{u\}$ and $\{v,w\}$ are sets, there exists a set $A$ such that for every set $x$

$$x \in A \iff x = \{u\} \vee x = \{v,w\}.$$

By extension, this set is unique and we call it $A := \{\{u\},\{v,w\}\}$. Using the union axiom, given that $A$ is a set, then there exists a set $B$ such that for every set $x$,

$$\begin{aligned}
x \in B &\iff \exists z(x \in z \wedge z \in A)\\
&\iff \exists z(x \in z \wedge z \in \{\{u\},\{v,w\}\})\\
&\iff \exists z\big(x \in z \wedge (z = \{u\} \vee z = \{v,w\})\big)\\
&\iff \exists z\big((x \in z \wedge z = \{u\}) \vee (x \in z \wedge z = \{v,w\})\big)\\
&\iff \exists z\big(x \in \{u\} \vee x \in \{v,w\}\big)\\
&\iff x \in \{u\} \vee x \in \{v,w\}\\
&\iff x = u \vee (x = v \vee x = w)\\
&\iff x = u \vee x = v \vee x = w
\end{aligned}$$

Where we have simplified the quantifier which does not appear in the expression. Thus $B$ is the unique set whose elements are exactly $u, v, w$ and we call it $B := \{u,v,w\}$.

### Exercise 2

Since $A$ is a set, then by pairing there exists the unique set $\{A, A\}$, we shall prove this set is equal to what we call $\{A\}$, the set whose only element is $A$. Given $x \in \{A\}$ we conclude $x = A$, therefore $x \in \{A,A\}$. Given $x \in \{A,A\}$ we have that $x = A \vee x = A$, therefore $x = A$, which implies $x \in \{A\}$. Thus $\{A\} = \{A,A\}$. Since the sets are equal and one of them exists, they both exist.

**Exercise 3**

Forst we prove $A \cap \{A\} = \emptyset$. For the sake of contradiction, suppose that the set $A$ is such that $A \cap \{A\} \neq \emptyset$. Then $B := A \cap \{A\}$ is non-empty. Using the axiom of regularity, this means that there exists a set $x$ such that $x \in B$ and $x \cap B = \emptyset$. But the first condition means $x \in A \cap \{A\}$, which means that $x \in A$ and $x \in \{A\}$, which means that $x \in A$ and $x = A$. But then, the second is $x \cap B = A \cap A \cap \{A\} = A \cap \{A\} \neq \emptyset$, a contradiction. Therefore, $A \cap \{A\} = \emptyset$.

Finally, this means that $A \notin A$, since $A \in A$ would imply that there exists $x \in A$ such that, $x = A$ which is equivalent to $x \in \{A\}$. Thus there would exist $x \in A \wedge x \in \{A\} \implies x \in A \cap \{A\} \implies A \cap \{A\} \neq \emptyset$, which contradicts the previous result.

**Exercise 4**

Using the regularity axiom, since $\{A, B\}$ is non-empty, then by the regularity axiom

$$\exists x(x \in \{A, B\} \wedge x \cap \{A, B\} = \emptyset).$$

Therefore either $x = A \wedge A \cap \{A, B\} = \emptyset$ or $x = B \wedge B \cap \{A, B\} = \emptyset$ or both. Since $A \in B$, then it cannot be that $x = B$, since this would mean $B \cap \{A, B\} = \emptyset$, but of course, there exists $A \in B$ and also $A \in \{A, B\}$, so the intersection $B \cap \{A, B\} = \emptyset$ cannot be empty. Therefore $x = A \wedge A \cap \{A, B\} = \emptyset$, which means that it is false that

$$\begin{aligned}
\exists z(z \in A \wedge z \in \{A, B\}) &\iff \exists z(z \in A \wedge (z = A \vee z = B)) \\
&\iff \exists z((z \in A \wedge z = A) \vee (z \in A \wedge z = B)) \\
&\iff \exists z(A \in A \vee B \in A) \\
&\iff \exists z(B \in A) \\
&\iff B \in A.
\end{aligned}$$

Where we simplified $A \in A$ since we know by Exercise 3 that $A \notin A$. Therefore, it is false that $B \in A$, thus it is true that $B \notin A$.

**Exercise 5**

From previous exercises we know that $\{A, B, C\}$ is a set. Since it is non-empty, there exists an element $x$ of this set such that $x \cap \{A, B, C\} = \emptyset$. It cannot happen that $x = B$ or $x = C$, since $A \in B$ and $B \in C$, so that the intersections are $B \cap \{A, B, C\}$ and $C \cap \{A, B, C\}$ are both nonempty. Therefore $x = A$ and $A \cap \{A, B, C\} = \emptyset$. Thus, it is false that there exists a $z \in A \wedge z \in \{A, B, C\}$, that is

$$\begin{aligned} z \in A \land (z = A \lor z = B \lor z = C) &\iff (z \in A \land z = A) \lor (z \in A \land z = B) \lor (z \in A \land z = C) \\ &\iff (A \in A) \lor (B \in A) \lor (C \in A) \\ &\iff C \in A \end{aligned}$$

Where, $B \in A$ is true by hypothesis and $A \in A$ false by Exercise 3. Therefore, it is false that $C \in A$, which means that it is true that $C \notin A$.

**Exercise 6**

By the powerset axiom the set $\mathcal{P}(A)$ exists. Let $\psi(x) := x \in B$, then there exists a (unique by extension) set $S$, where for all sets $x$,

$$\begin{aligned} x \in S &\iff x \in \mathcal{P}(A) \land \psi(x) \\ &\iff x \in \mathcal{P}(A) \land x \in B \\ &\iff x \in \mathcal{P}(A) \cap B. \end{aligned}$$

**Exercise 7**

By the subset axiom, there exists a set $S$ such that for all sets $x$, $x \in S$, if and only if, $x \in A \land \psi(x)$. Define $\psi(x) := \neg(x \in B)$ thus $A \backslash B$ exists.

**Exercise 8**

First we show $\{\emptyset\} \neq \emptyset$. Suppose it is true that $x \in \{\emptyset\}$. Since $x \in \emptyset$ is always false, then it is false that $x \in \{\emptyset\} \implies x \in \emptyset$, therefore, $\{\emptyset\} \neq \emptyset$.

Now we show $\{\emptyset, \{\emptyset\}\} \neq \{\emptyset\}$. Let $x = \{\emptyset\}$, then clearly $x \in \{\emptyset, \{\emptyset\}\}$, but $x \notin \{\emptyset\}$, because if it were true that $x \in \{\emptyset\}$ then $x = \emptyset \neq \{\emptyset\}$ (by the previous part) which contradicts our choice of $x = \{\emptyset\}$. Therefore $\{\emptyset, \{\emptyset\}\} \neq \{\emptyset\}$.

Now we show that $\{\emptyset, \{\emptyset\}\} \neq \emptyset$. Let $x = \emptyset$, then $x \in \{\emptyset, \{\emptyset\}\}$. But $x \notin \emptyset$, therefore $\{\emptyset, \{\emptyset\}\} \neq \emptyset$.

**Exercise 9**

For any $x$, consider $x \in A$. Since $A$ has no elements, this statement is false, therefore the implication $x \in A \implies x \in \emptyset$ is true. Similarly, $x \in \emptyset \implies x \in A$ is true. Therefore $x \in \emptyset \iff x \in A$, thus, $A = \emptyset$.

**Exercise 10**

We know that for any set $x$, there exists the unique set $\{x\}$. In particular for every $x \in A$, there exists the unique set $y$ such that $y = \{x\}$, thus the formula $\forall z(z \in y \iff z = x)$ is uniquely satisfied for all $x \in A$. Therefore we can conclude that there exists a set $S$ where for any $y$, $y \in S \iff (\exists x \in A)(y = \{x\})$, which means that there is a set $S$ which contains exactly those elements $y$ for which $y = \{x\}$ for some $x \in A$, which is precisely how $\{\{x\} : x \in A\}$ is defined. Therefore $\{\{x\} : x \in A\}$ is a set.

# 2 Basic Set-Building Axioms and Operations

## 2.1 The First Six Axioms

### Exercise 1

If $A = \emptyset$ the claim is vacuously true. Let $x \in A$, then $x \in A \vee x \in B$, therefore $A \subseteq A \cup B$. Notice $A \subseteq B$ is not needed.

### Exercise 2

If $A = \emptyset$ the claim is vacuously true. Let $x \in A$, since $A \subseteq B$, we have that $x \in B$, since $B \subseteq C$, we have that $x \in C$. Therefore $A \subseteq C$.

### Exercise 3

If $B = \emptyset$ the claim is vacuously true. The subset implication $x \in B \implies x \in C$ is equivalent to its contrapositive $x \notin C \implies x \notin B$, which is the same as $x \in A \wedge x \notin C \implies x \in A \wedge x \notin B$, therefore $A \backslash C \subseteq A \backslash B$.

### Exercise 4

If $C = \emptyset$ then, the claim $\emptyset \subseteq A$ and $\emptyset \subseteq B$ and the claim $\emptyset \subseteq A \cap B$ are both always true, therefore equivalent.

Suppose $C \neq \emptyset$.

$\boxed{\implies}$ Suppose $C \subseteq A$ and $C \subseteq B$. Let $x \in C$, since $C \subseteq A$, $x \in A$. Since $C \subseteq B$, $x \in B$ so that $x \in C \implies x \in A \wedge x \in B$. Therefore $C \subseteq A \cap B$.

$\boxed{\impliedby}$ Suppose $C \subseteq A \cap B$. Let $x \in C$, then $x \in A \cap B$, which means $x \in A \wedge x \in B$. But for any propositions, $P \wedge Q \implies P$. Therefore $x \in A \wedge x \in B \implies x \in A$ and also $x \in A \wedge x \in B \implies x \in B$. Therefore $C \subseteq A$ and $C \subseteq B$.

### Exercise 5

For the sake of contradiciton suppose $\forall x(x \in A)$, then $A \in A$, but we know this is false (section 1.5 exercise 3). Therefore $\exists x(x \notin A)$.

**Exercise 6**

Commutativity of $\wedge$.

**Exercise 7**

Commutativity of $\vee$.

**Exercise 8**

Distributivity of $\wedge$ over $\vee$.

**Exercise 9**

Distributivity of $\vee$ over $\wedge$.

**Exercise 10**

Associativity $\vee$.

**Exercise 11**

Associativity $\wedge$.

**Exercise 12**

$$
\begin{aligned}
x \in C \backslash (A \cap B) &\iff x \in C \wedge \neg(x \in A \cap B) \\
&\iff x \in C \wedge \neg(x \in A \wedge x \in B) \\
&\iff x \in C \wedge (x \notin A \vee x \notin B) \\
&\iff (x \in C \wedge x \notin A) \vee (x \in C \wedge x \notin B) \\
&\iff (x \in C \backslash A) \vee (x \in C \backslash B) \\
&\iff x \in (C \backslash A) \cup (C \backslash B)
\end{aligned}
$$

**Exercise 13**

$$
\begin{aligned}
x \in C \backslash (A \cup B) &\iff x \in C \wedge \neg(x \in A \cup B) \\
&\iff x \in C \wedge \neg(x \in A \vee x \in B) \\
&\iff x \in C \wedge x \notin A \wedge x \notin B \\
&\iff x \in C \wedge x \notin A \wedge x \in C \wedge x \notin B \\
&\iff x \in C \backslash A \wedge x \in C \backslash B \\
&\iff x \in (C \backslash A) \cap (C \backslash B)
\end{aligned}
$$

**Exercise 14**

$$\begin{aligned}
x \in (A \backslash B) \cap (C \backslash B) &\iff x \in (A \backslash B) \wedge x \in (C \backslash B) \\
&\iff x \in A \wedge x \notin B \wedge x \in C \wedge x \notin B \\
&\iff x \in A \wedge x \in C \wedge x \notin B \\
&\iff x \in A \cap C \wedge x \notin B \\
&\iff x \in (A \cap C) \backslash B
\end{aligned}$$

**Exercise 15**

Same as exercise 10 and 11.

**Exercise 16**

$$\begin{aligned}
x \in (A \cup B) \backslash (A \cap B) &\iff x \in (A \cup B) \wedge \neg(x \in A \cap B) \\
&\iff (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) \\
&\iff (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) \\
&\iff (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \vee (x \in B \wedge x \notin B) \\
&\iff (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \\
&\iff x \in (A \backslash B) \vee x \in (B \backslash A) \\
&\iff x \in (A \backslash B) \cup (B \backslash A)
\end{aligned}$$

**Exercise 25**

Since $C \in \mathcal{F}$, if $x \in C$, then $x \in C \wedge C \in \mathcal{F}$. Therefore $x \in \bigcup \mathcal{F}$, thus $C \subseteq \bigcup \mathcal{F}$.

**Exercise 26**

Since $C \in \mathcal{F}$, if $x \in \bigcap \mathcal{F}$, then $x \in A, \forall A \in \mathcal{F}$. In particular for $A = C \in \mathcal{F}$, $x \in C$. Therefore $\bigcap \mathcal{F} \subseteq C$.

**Exercise 27**

If $A \subseteq C$ for some $C \in \mathcal{F}$, $x \in A$ then $x \in C$ for some $C \in \mathcal{F}$. Therefore $x \in \bigcup \mathcal{F}$, thus $A \subseteq \bigcup \mathcal{F}$.

**Exercise 28**

Let $x \in A$, then $x \in C$ for all $C \in \mathcal{F}$. Therefore $x \in \bigcap \mathcal{F}$, thus $A \subseteq \bigcap \mathcal{F}$.

**Exercise 29**

Let $x \in \bigcup \mathcal{F}$, then $x \in C$ for some $C \in \mathcal{F}$. But for all $C \in \mathcal{F}$, $C \subseteq A$, therefore $x \in A$. Therefore $\bigcup \mathcal{F} \subseteq A$.

**Exercise 30**

$$\boxed{\subseteq} \quad \begin{aligned} x \in \bigcup \mathcal{P}(A) &\iff \exists C(x \in C \wedge C \in \mathcal{P}(A)) \\ &\iff \exists C(x \in C \wedge C \subseteq A) \\ &\implies \exists C(x \in A) \\ &\iff x \in A \end{aligned}$$

$\boxed{\supseteq}$ Suppose $\bigcup \mathcal{P}(A) \not\supseteq A$, then $\exists x \in A$ such that $x \notin \bigcup \mathcal{P}(A)$. But $x \notin \bigcup \mathcal{P}(A)$, if and only if, for all $C \in \mathcal{P}(A)$, $x \notin C$. But for $C = \{x\}$ we clearly have $x \in C$. And $\{x\} \in \mathcal{P}A$ because $z \in \{x\} \implies z = x \implies z \in A$ (we assumed $x \in A$). Therefore we have a contradiction, since we have found $\{x\} \in \mathcal{P}(A)$ with $x \in \{x\}$. Therefore $A \subseteq \bigcup \mathcal{P}(A)$.

Finally since $\subseteq \wedge \supseteq$, we have that $\bigcup \mathcal{P}(A) = A$.

**Exercise 31**

Suppose $A \not\subseteq \mathcal{P}(\bigcup A)$. Then there exists $x \in A$, such that,

$$\begin{aligned} x \notin \mathcal{P}(\bigcup A) &\iff x \not\subseteq \bigcup A \\ &\iff \exists z(z \in x \wedge z \notin \bigcup A) \\ &\iff \exists z(z \in x \wedge (\forall C \in A, z \notin C)) \end{aligned}$$

Now, we examine what happens when we consider the set $C = x$. Since $x \in A$ (as given in the problem statement), we know that $x \in A$. Therefore, the condition $z \notin C$ (with $C = x$ ) must hold. But since $z \in x$ by assumption, this contradicts the statement that $z \notin x$ (since $z \in x$). Thus, we have reached a contradiction: $z \in x$ and $z \notin x$ at the same time, which is impossible. Therefore $\forall x(x \in A \implies x \in \mathcal{P}(\bigcup A))$, thus $A \subseteq \mathcal{P}(\bigcup A)$.

**Exercise 32**

We prove $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$, which is equivalent to $\mathcal{P}(C) \subseteq \mathcal{P}(\bigcup \mathcal{F})$. But since $C \in \mathcal{F}$, using exercise 25 we have that $C \subseteq \bigcup \mathcal{F}$. So if we can prove $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$, then since we know $C \subseteq \bigcup \mathcal{F}$, we can conclude

$\mathcal{P}(C) \subseteq \mathcal{P}(\bigcup \mathcal{F})$, that is $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$.

We prove $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$. Suppose $A \subseteq B$, let $x \in \mathcal{P}(A)$, then $x \subseteq A$, but $A \subseteq B$, by transitivity $x \subseteq B$. Therefore and $x \in \mathcal{P}(B)$, thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Since $C \subseteq \bigcup \mathcal{F}$, then $\mathcal{P}(C) \subseteq \mathcal{P}(\bigcup \mathcal{F})$, that is $\mathcal{P}(C) \in \mathcal{P}(\mathcal{P}(\bigcup \mathcal{F}))$.

**Exercise 33**

If this was a set then this would be the set of all sets, since for any $x$, there is the singleton set $\{x\}$ such that $x \in \{x\}$. So that the condition is always true.

**Exercise 34**

In formulas

$$\{x : \varphi(x)\} \text{ is a not a set} \iff \forall A \exists x(\psi(x) \wedge x \notin A).$$

$\boxed{\Longleftarrow}$ If for every set $A$, there is an $x$ such that $\varphi(x)$ holds and $x \notin A$, then $\{x : \varphi(x)\}$ cannot be a set, because it cannot be contained in any set $A$. If it were a set, we could choose $A = \{x : \varphi(x)\}$ , and then by assumption, there would be an $x$ such that $\varphi(x)$ holds and $x \notin A$, which is a contradiction. Hence, $\{x : \varphi(x)\}$ must be a proper class.

$\boxed{\Longrightarrow}$ We prove the contrapositive,

$$\exists A \forall x(\neg\psi(x) \vee x \in A) \implies \{x : \varphi(x)\} \text{ is a set.}$$

Which is equivalent to

$$\exists A \forall x(\psi(x) \implies x \in A) \implies \{x : \varphi(x)\} \text{ is a set.}$$

Theorem 2.1.3 States that

$$\forall A\Big(\forall x\big(\psi(x) \implies x \in A\big) \implies \exists! \mathcal{D}\big(x \in \mathcal{D} \iff \psi(x)\big)\Big).$$

Therefore the contrapositive is true, since in particular (from theorem 2.1.3), there exists $A$ such that $\forall x(\psi(x) \implies x \in A)$, which implies $\exists! \mathcal{D}\big(x \in \mathcal{D} \iff \psi(x)\big)$ and clearly $\mathcal{D} = \{x : \varphi(x)\}$.