Heavy tails in financial markets and real economies

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Introduction

Heavy-tailed distributions have been a subject of close attention in the economic literature given their different implications for econometric studies based on most popular parametric approaches. Indeed, such distributions undermine the reliability and the interpretation of econometric tests that rely on normal distribution (Rama Cont, 2000). For example, confidence thresholds or intervals could be misleading and disastrous in a Value-at-Risk model that would not properly take into account the over-appearance of extreme losses.

The economic literature suggests that this phenomenon can be modelled with Power Law which seems to exhibit a remarkable ability to appear in different sets of phenomena across time and space (Gabaix, 2003 and 2006). However, Franke (2015) argues that one should perform tests across time within a given sample to be aware of "spurious fatness" appearing in data aggregated from samples with different macroeconomic conditions. While heavy-tails are interpreted by Gabaix (2006) as the results of large trade funds movements on the market, it has been suggested by Warusawitharana (2018) that volatility clustering could also generate power law distributions, a hypothesis that can be assessed with models designed to capture time-varying volatility. Incidentally, Alfrano & Lux (2010) define the theoretical foundation of power law to be extreme value theory which provides a statistical justification for the emergence of power laws.

In this paper, we analyse different samples of growth rates and stock returns and try to assess the assumption of a Power Law fit on the tails of their distributions. For this purpose, we use python power law function based on Maximum Likelihood estimation in order to compute both alphas and xmin that are the most plausible candidates of the underlying process given the empirical distribution of our samples. Then, with a method inspired from Clauset et al (2009), we try to perform Kolmogorov-Smirnoff robustness tests to compare the empirical distribution on the tails of each sample with random power law distributions based on the parameters obtained from the power law fit function of the same sample. In a second time, we follow a filtering method based on time-varying volatility modeling (Engle, 1982) and reproduce the same methodology as above on the standardized residuals to assess the effect on the fatness of the distribution's tails. Finally, we briefly present an application of extreme value theory that derives generalized extreme values on our stock returns based on the method of block maxima by Gilli & Kellezi, 2006.

We find that both stock return and growth rates exhibit fat tails and that the Power Law fitting appears as a plausible candidate for each of them according to our methodology. However, we cannot advocate for a convergence on the alpha of the power law as they seem to be different across time and space. The filtering suggests a phenomenon of volatility clustering as the standardized residuals on the CAC 40 conditional heteroscedasticity model seem to be freed of clusters.

Data collection

The power law effect is popular for its ability to represent different kinds of phenomena. For this reason, we import data from different indices, stock market and growth rates, and from different countries. We use quarterly US since 1962 and French GDP since 1975 as in Fagiolo (2008) which were both imported from FRED database. In the stock market, we focus on CAC 40 since its beginning with data available on the Euronext website. However, as suggested by Franke (2015), we divide our US GDP growth rates sample across time to try to investigate the possibility of different data distribution processes in different macroeconomic periods as in the Great Inflation and the Great Moderation and to be aware of "spurious fatness" . For this purpose, we divide our US sample in two different subsample, one from 1962 to 1982 and the other one from 1982 to 2019.

Econometric tests

Unit roots

The study and comparison of time series require us to pay attention to the possible non-stationarity of some variables. If some variables follow a trend over time, we might do spurious regression when performing a regression of one variable on the other one and miss the effect of the variations that are not due to past results of the observed variables. Also, for what concerns our study we need to check the stationary of the variables defined as returns in order to assess their distribution in a meaningful way. As explained by Rama Cont, we want to study the properties of variables that are stable over time.

With basic assumption on the existence of a trend on US GDP growth we perform an augmented Dickey Fuller test to check for the existence of a unit root. With a p-value of 0.81 we fail to reject the hypothesis H_0 according to which there is a unit root and we obviously accept the hypothesis that US GDP is a non-stationary variable. However, we reject the null hypothesis when checking for unit root in growth rates over time which allows us to perform modelling and make assumptions on the returns distribution given the stationarity hypothesis. Following the same tests, both CAC 40 returns and French GDP growth rate also appear stationary.

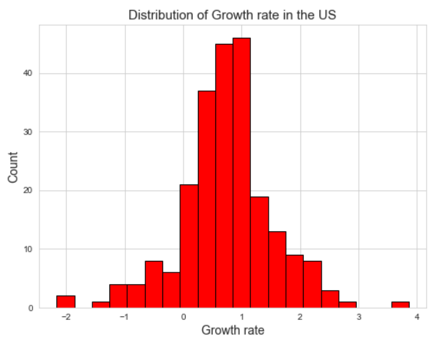
Independent and identically distributed variables

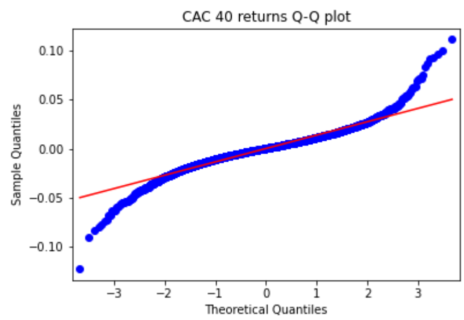
Independent and identically distributed (i.i.d) are a set of variables that follow the same law of probability and whose realization are independent from each other. If one wants to suppose a linear relation between two or more variables, independence of the errors is an important assumption to check with a view to detecting potential seasonality. Independence is also important in stationary time series to understand their process and interpret their distribution. Here, the independence Ljung-Box tests and autocorrelation function plot suggest that CAC 40 returns are uncorrelated from lag one which is consistent with the widely-shared hypothesis of stocks returns following a random walk. However, growth rates exhibit visually significant autocorrelation until lag two for French GDP and lag four for US GDP which suggests our data are not perfectly independent although they are in the long run.

Normal distribution

One way to describe a dataset is to assess whether its distribution is similar to a Gaussian distribution which will help us interpret statistical results obtained on the dataset. To do this, one can calculate skewness and kurtosis which are standardized centred moments of third and fourth order of our series. Skewness captures the symmetry of the distribution while kurtosis is a measure of how fat the tails are.

We visualize the data as a first step. For this purpose, we plot a histogram of US growth rates as an example and a QQ-plot in order to compare the match between the empirical growth rates distribution and a theoretical normal distribution. Visually, the data does not appear to be normally distributed and exhibits outliers. We then perform a Shapiro-Wilk normality test which confirms that the sample does not look Gaussian with a near 0 p-value to wrongly reject the null hypothesis that the sample follows a normal distribution law. Similar econometric tests confirm that both French growth rates and CAC 40 return are not normal.





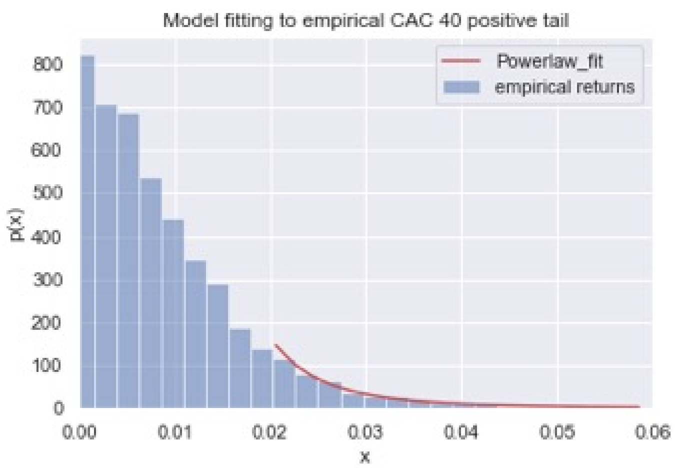
Modelling

French and US GDP growth rates and CAC 40 stock returns

The literature review suggests growth rates and stock returns can be modelled with power law distribution on their tails. We use python power law function to try to find the best possible power law fit on our different samples meaning that we are looking for both the xmin from which the power law is best and the alpha that best captures the heavy tails phenomenon with a higher alpha meaning a steeper probability distribution function and a heavier tail. The analysis of the alpha suggests that stock markets are heavier tailed than growth rates, which could be explained by the volatility clustering phenomenon.

We follow the methodology suggested in our readings and fit a power law on both tails of each of our samples from which it clearly appears that most of time our samples exhibit alphas that are consistent with the literature.

Then we try to apply a Kolmogorov Smirnoff goodness of fit test following Fagiolo (2008). We draw 2500 distributions of a theoretical power law supplied with the parameters from our maximum likelihood fit and compare them with our empirical returns on the tails. With the null hypothesis being that the samples that are compared come from the same distribution, we set that if the p-value of the test is above 0.10, then the Power Law is a plausible fit as in Clauset and al (2009).



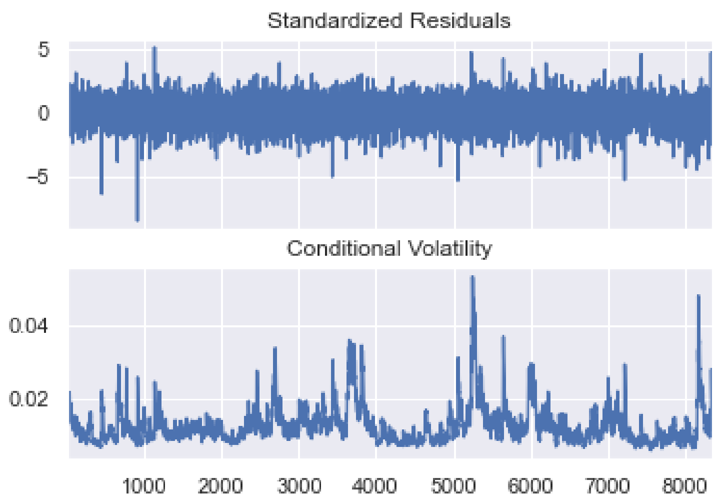
Modelling with GARCH on CAC 40 returns

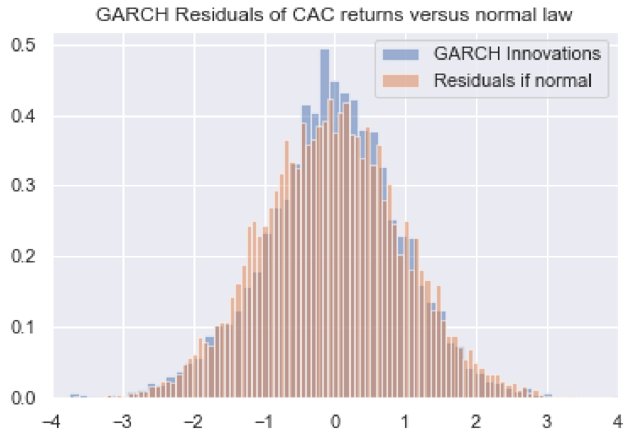
The identification of power law distributions across economic and social sciences has led economists to explore the reason for their emergence. An explanation from Gabaix is that excess volatility is due to movements from large institutional investors. Another way to understand heavy tails comes with the identification of volatility clustering in times of stress which could generate power law (Warusawitharana, 2018). In this section we will try to assess the hypothesis that power law in stock market returns might be partially due to an increase in volatility for extreme returns by trying to model the time-varying volatility component of the underlying process of stock returns.

We apply Engle (1982) autoregressive conditional heteroskedasticity (ARCH) model in its generalized form (GARCH) and assess if it fits our volatility process. This model predicts returns based on past information and a time-varying volatility scaling component applied to a series of innovations which are unobservable. For this purpose, we use maximum likelihood methodology with python to try to find the best parameters for a GARCH (1,1) given empirical returns and find that all three coefficients are significant which reinforce our hypothesis of time-varying volatility.

We then plot the innovations series computed as standardized residuals which seem to be freed of volatility clusters and seem to be uncorrelated according to the autocorrelation function plot. These observations are confirmed by a Ljung-Box autocorrelation test for which we fail to reject the hypothesis H0 of independence of innovations. Then, we apply Jarque-Bera normality test on the innovations and reject the hypothesis of normality with the innovations exhibiting both excess kurtosis (4.84) and a slight negative skewness (-0.27). A Kolmogorov-Smirnoff test also suggests that the data are not drawn from a normal distribution.

We try to fit a power law on the residuals and find that both alphas on positive and negative tails are above 5. Even though the fit successfully passes the Kolmogorov-Smirnoff test with the methodology described before we found found alphas that are above literature standards for power law phenomenon (Rama Cont, 2000). These results suggest that the power law fit might not be the best candidate and that the innovations are less subject to less extreme abnormal returns than the original process.

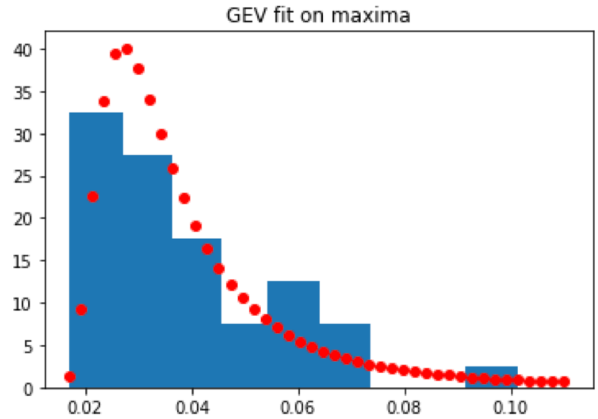




Extreme Value Theory application - Generalized Extreme Values

Extreme Value Theory (EVT) assesses the probability of extreme events in financial data and can be relied on for computing tail risk measures. Introduced by Fisher and Tippett (1928), it includes three families: Gumbel, Frechet, and Weibull, that Jenkinson (1955) combined into the Generalized Extreme Values distribution (GEV) with a cdf. Based on the EVT that derives the GEV distribution, this latter is the limit distribution of normalized maxima of iid random variables.

To apply it, we use the Method of Block Maxima specified by Gilli & Kellezi (2006) to fit a sample of extremes to the GEV distribution by splitting our sample in n parts, collecting the maximum value in each block and fitting the GEV distribution to the set of maxima. We obtain our three parameters (shape = -0.415, location = 0.030 and scale = 0.009). Since our shape parameter $<$ 0, our GEV is equivalent to type III EVT and our CAC 40 daily returns may be fitted by a Gumbel distribution (Gilli & Kellezi, 2006). Our location parameter indicates that our distribution is 0.03 units shifted to the right on the horizontal axis from the normal distribution. From the fitted distribution, we can estimate how often the extreme quantiles occur on a certain return level. We then obtain our CDF of the GEV distribution on the graph below:



Results summary

Table of alpha fits

	FranceGDP	USGDP	USGDP GI	USGDP GM	CAC40	FilteredCAC40
Postal, xmin	**3.76**, 0.75	**3.36**, 0.9	**9.3**, 2.2	**4.36**, 0.92	**4**, 0.021	**6.2**, 2
Negtail	**2**, 1.339	**3.9**, 0.91	**4.33**, 0.94	**3.78**, 0.91	**3.95**, 0.023	**5.4**, 2
Fitted data volume, positive/negative (if available)	48	88	9	51	431/335	140/109

Goodness of fit test, empirical distributions against 2500 random PL generated from the parameters of the fit on the empirical distribution:

Positive/ Negative (if available)	FranceGDP	USGDP	USGDP GI	USGDP GM	CAC40	FilteredCAC 40
Failure to reject H_0 rate	0.948	0.8828	0.998	0.97	0.98/0.81	0.97/0.97
P-value rate	0.54945	0.40878	0.78847	0.76622	0.6721/0.37040	0.59308/0.639233
Fitted data volume	48	88	9	51	431/335	149/260

Conclusion and Discussion

While we have been able to show that distributions exhibit fat tails across time and space, the quality and relevance of our model depends a lot on the number of values in the tails that are fitted by the model. In some cases, we happen to have very few of such values which undermines the interest and the meaning of our fit as in the Great Inflation US subsample. This particular result could also advocate for the fact that power law may just not appear on this sample and confirm the intuition developed by Franke (2015), As explained in Clauset et al., the methodology employed in this paper to check for robustness could generate biases because all the draws are compared with the original sample from which they were generated. In fine, determining the right shape of the tail of a distribution is an important issue from a practical point of view and from an academic one. Studying the behaviour of power laws may “lay the foundations for an entirely novel type of theory” according to Gabaix (2009), and while economists continuously innovate, power laws will always be a powerful class of tool to make sense of ever-changing phenomena in our world.

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