Lecture 9 - Networks for Sequential Data RNNs & LSTMs

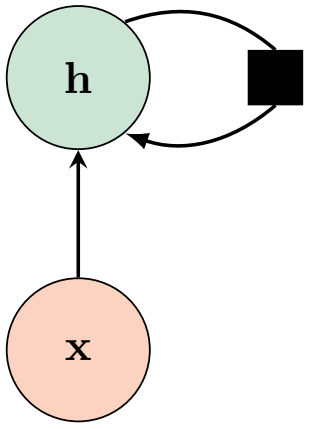
DD2424

April 9, 2025

Recurrent Neural Networks (RNNs)

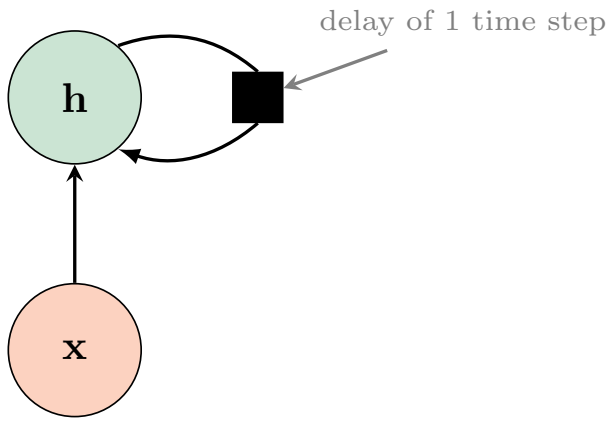
- RNNs are a family of networks for processing sequential data.
- A RNN applies the same function recursively when traversing network's graph structure.
- RNN encodes a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\tau$ into fixed length hidden vector $\mathbf{h}_\tau$.
- The size of $\mathbf{h}_\tau$ is independent of τ.
- Amazingly flexible and powerful high-level architecture.

RNN with no outputs



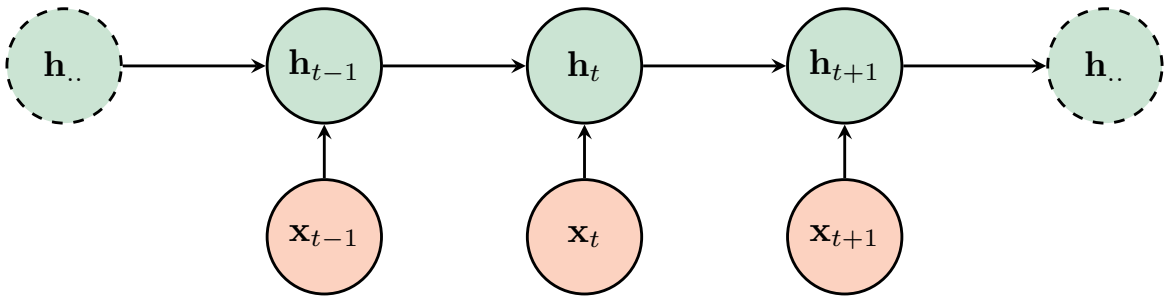
- Graph displays processing of information for each time step.
- Information from input $\mathbf{x}$ is incorporated into state $\mathbf{h}$.

RNN with no outputs



- Graph displays processing of information for each time step.
- Information from input $\mathbf{x}$ is incorporated into state $\mathbf{h}$.
- State $\mathbf{h}$ is passed forward in time.

RNN with no outputs



Unrolled visualization of the RNN

RNN: How hidden states generated

- Most **recurrent neural networks** have a function f

$$\mathbf{h}_t = f(\mathbf{h}_{t-1}, \mathbf{x}_t; \boldsymbol{\theta})$$

that defines their hidden state over time where

  - $\mathbf{h}_t$ the hidden state at time t (a vector)
  - $\mathbf{x}_t$ the input vector at time t
  - $\boldsymbol{\theta}$ the parameters of f.

- θ remains constant as t changes.

Apply the **same function** with the **same parameter** values at **each iteration**.

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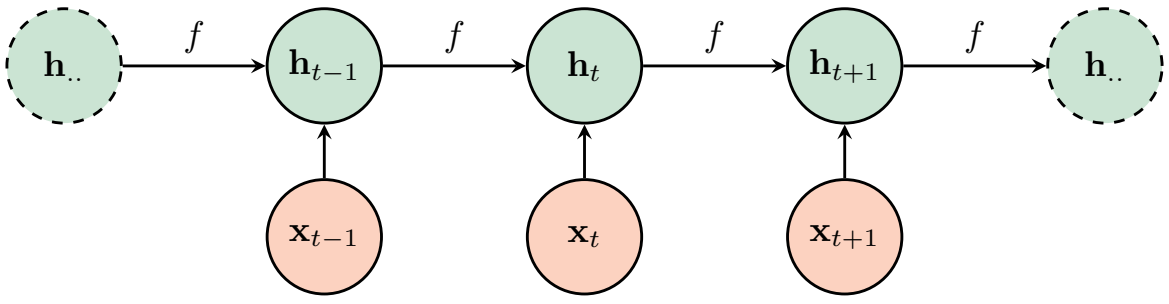
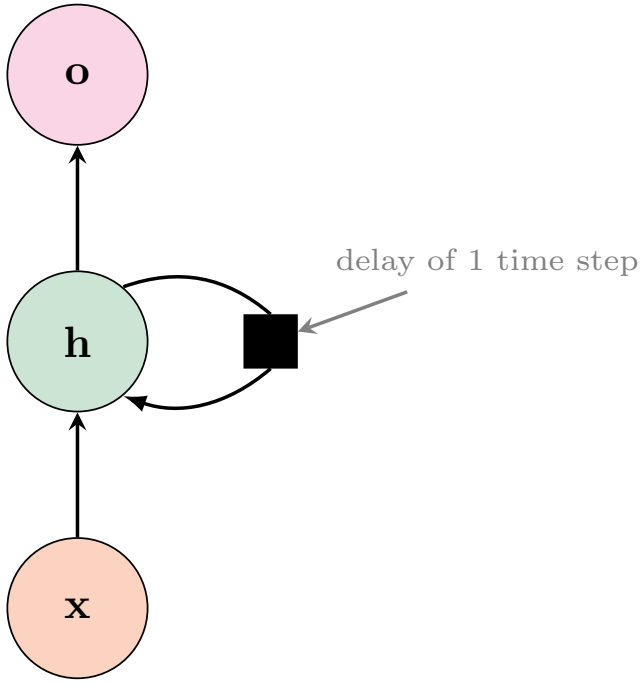
RNN with no outputs



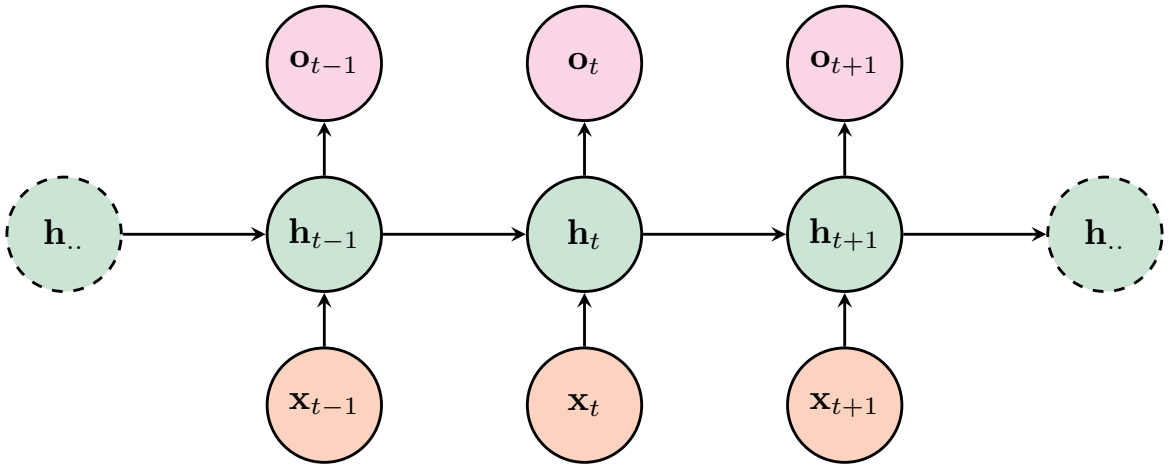
Unrolled visualization of the RNN

RNN with outputs



- Usually also predict an output vector at each time step

RNN with outputs



Unrolled visualization of the RNN

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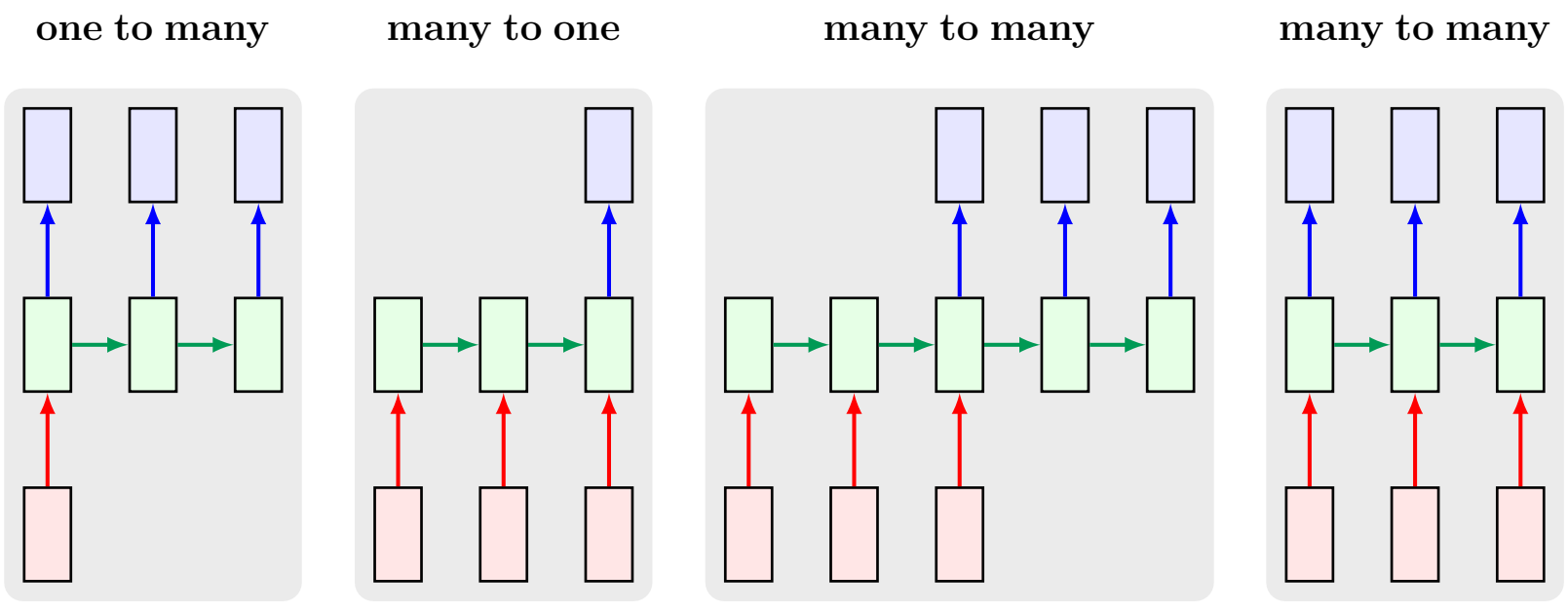
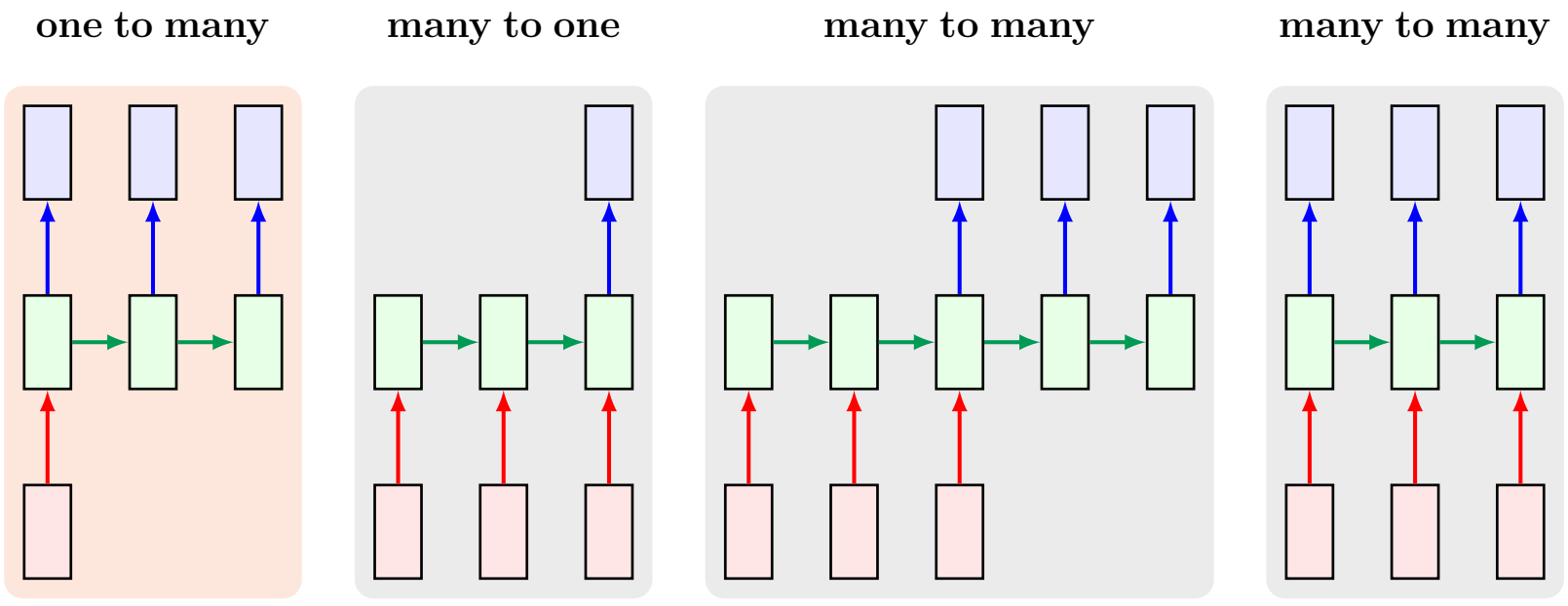
Use cases of RNNs

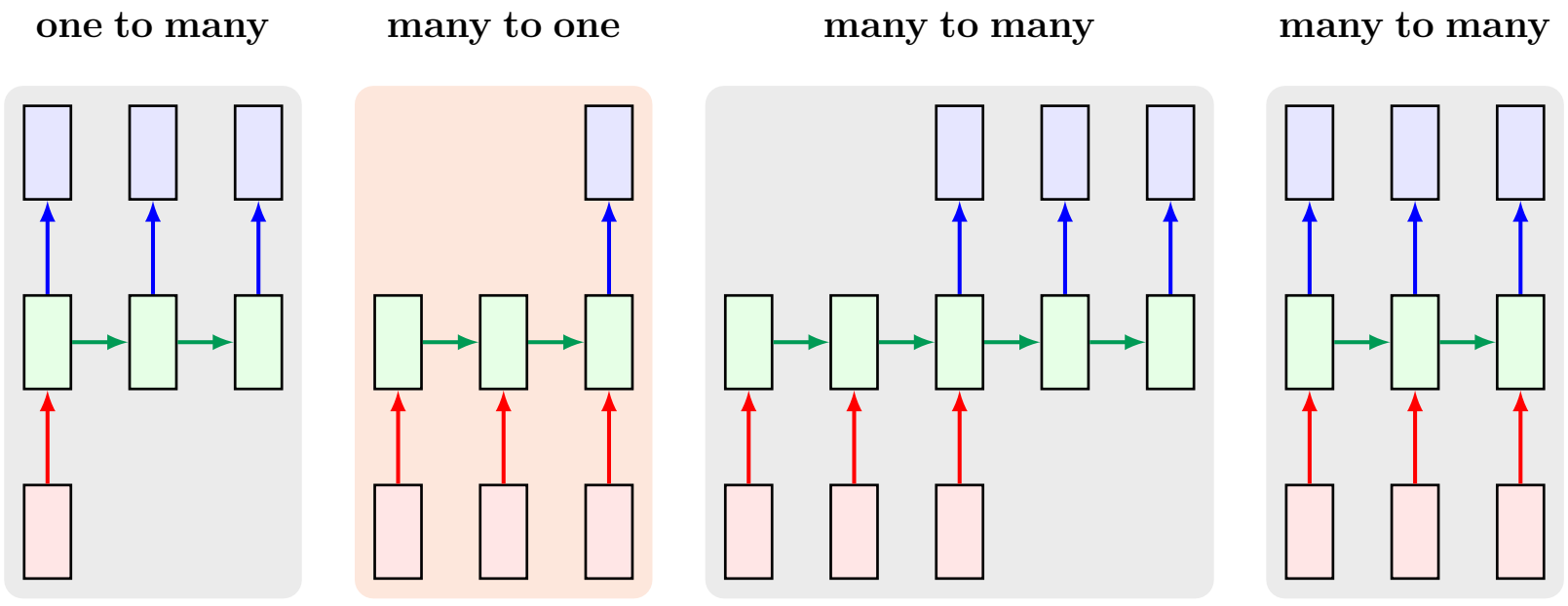


Use cases of RNNs



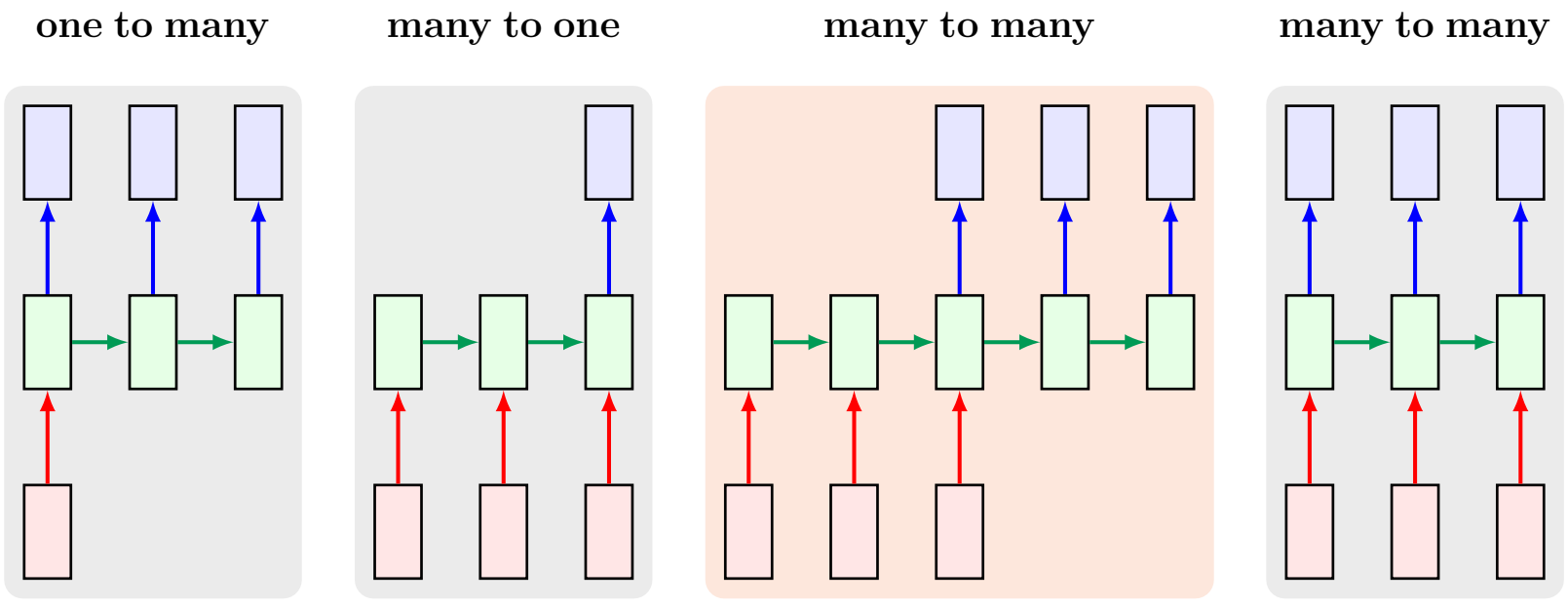
Example - **Image captioning**: image $\rightarrow$ sequence of words

Use cases of RNNs



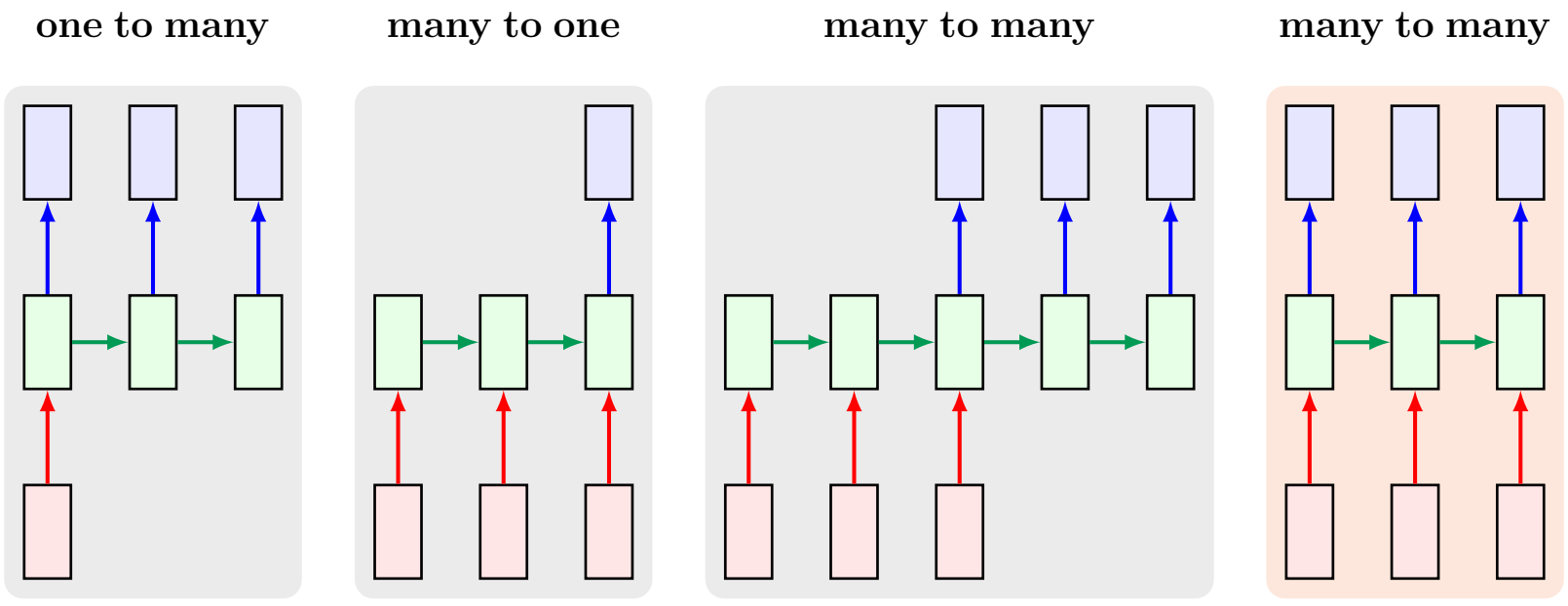
Example - **Sentiment Classification**: seq. of words $\rightarrow$ sentiment

Use cases of RNNs



Example - **Machine Translation**: seq. of words $\rightarrow$ seq. of words

Use cases of RNNs



Example - **Video classification on frame level**

Back to Vanilla RNN

- The state consists of a single **hidden** vector $\mathbf{h}_t$:
- Initial hidden state $\mathbf{h}_0$ is assumed given.
- For $t = 1, \ldots, \tau$ the RNN equations are

$$\begin{aligned}
\mathbf{a}_t &= W\mathbf{h}_{t-1} + U\mathbf{x}_t + \mathbf{b} \\
\mathbf{h}_t &= \tanh(\mathbf{a}_t) \\
\mathbf{o}_t &= V\mathbf{h}_t + \mathbf{c} \\
\mathbf{p}_t &= \text{softmax}(\mathbf{o}_t)
\end{aligned}$$

Network's input

- $\mathbf{h}_0$ initial hidden state has size $m \times 1$
- $\mathbf{x}_t$ input vector at time t has size $d \times 1$

Vanilla RNN equations

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\end{aligned}$$

Network's output and hidden vectors

- $\mathbf{a}_t$ hidden state at time t of size $m \times 1$ before non-linearity
- $\mathbf{h}_t$ hidden state at time t of size $m \times 1$
- $\mathbf{o}_t$ output vector (of unnormalized log probabilities for each class) at time t of size $C \times 1$
- $\mathbf{p}_t$ output probability vector at time t of size $C \times 1$

Vanilla RNN equations

- The state consists of a single **hidden** vector **h**:
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Parameters of the network

- W weight matrix of size $m \times m$ applied to $\mathbf{h}_{t-1}$ (hidden-to-hidden connection)
- U weight matrix of size $m \times d$ applied to $\mathbf{x}_t$ (input-to-hidden connection)
- $\mathbf{b}$ bias vector of size $m \times 1$ in equation for $\mathbf{a}_t$
- V weight matrix of size $C \times m$ applied to $\mathbf{a}_t$ (hidden-to-output connection)
- $\mathbf{c}$ bias vector of size $C \times 1$ in equation for $\mathbf{o}_t$

Simple RNN example: Character-level language model

Vocabulary: $\{h, e, l, o\}$, **Example training sequence**: "*hello*"

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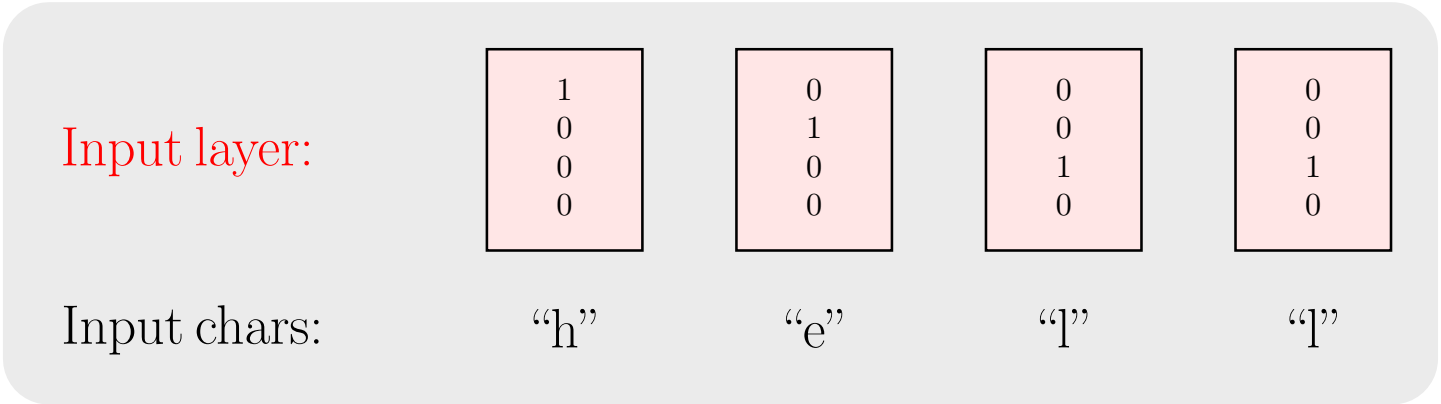

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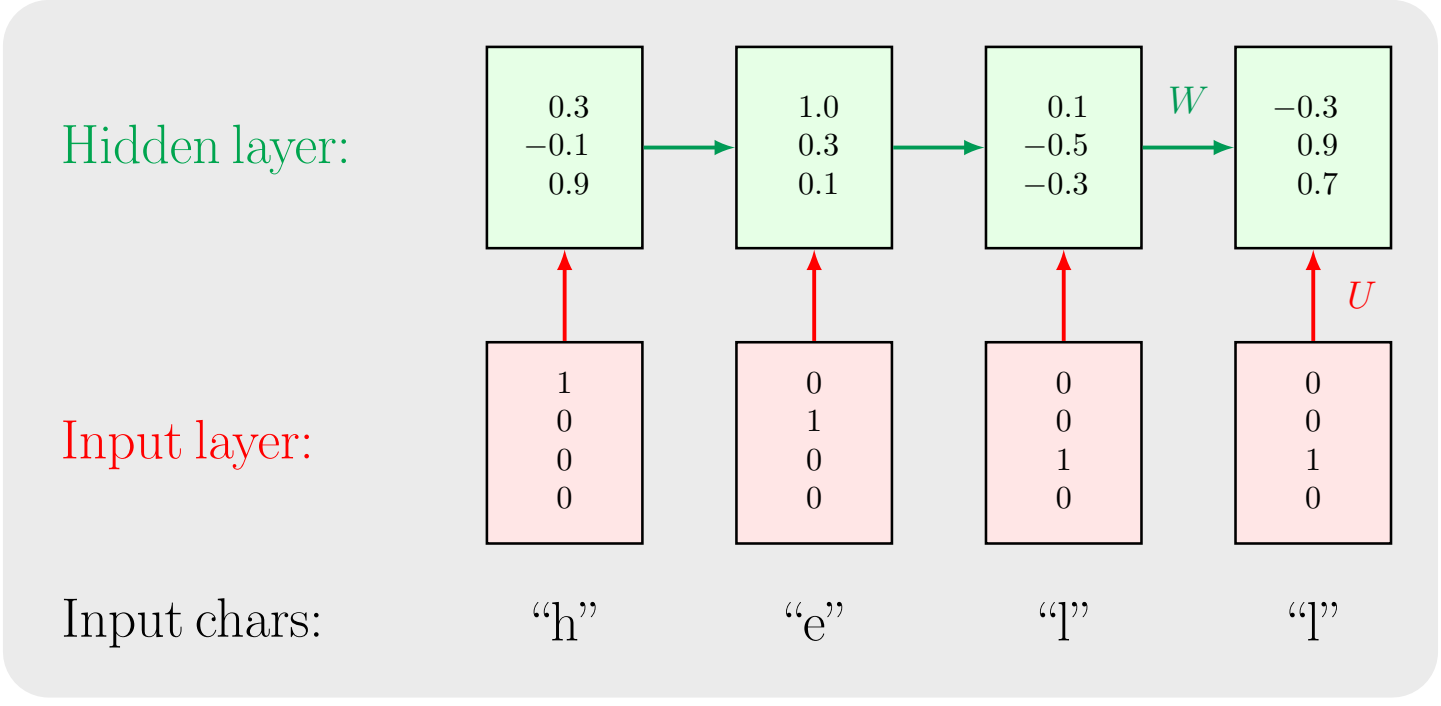
Vocabulary: $\{h, e, l, o\}$, **Example training sequence**: "*hello*"

$$\mathbf{h}_t = \tanh\left(W\mathbf{h}_{t-1} + U\mathbf{x}_t + \mathbf{b}\right)$$



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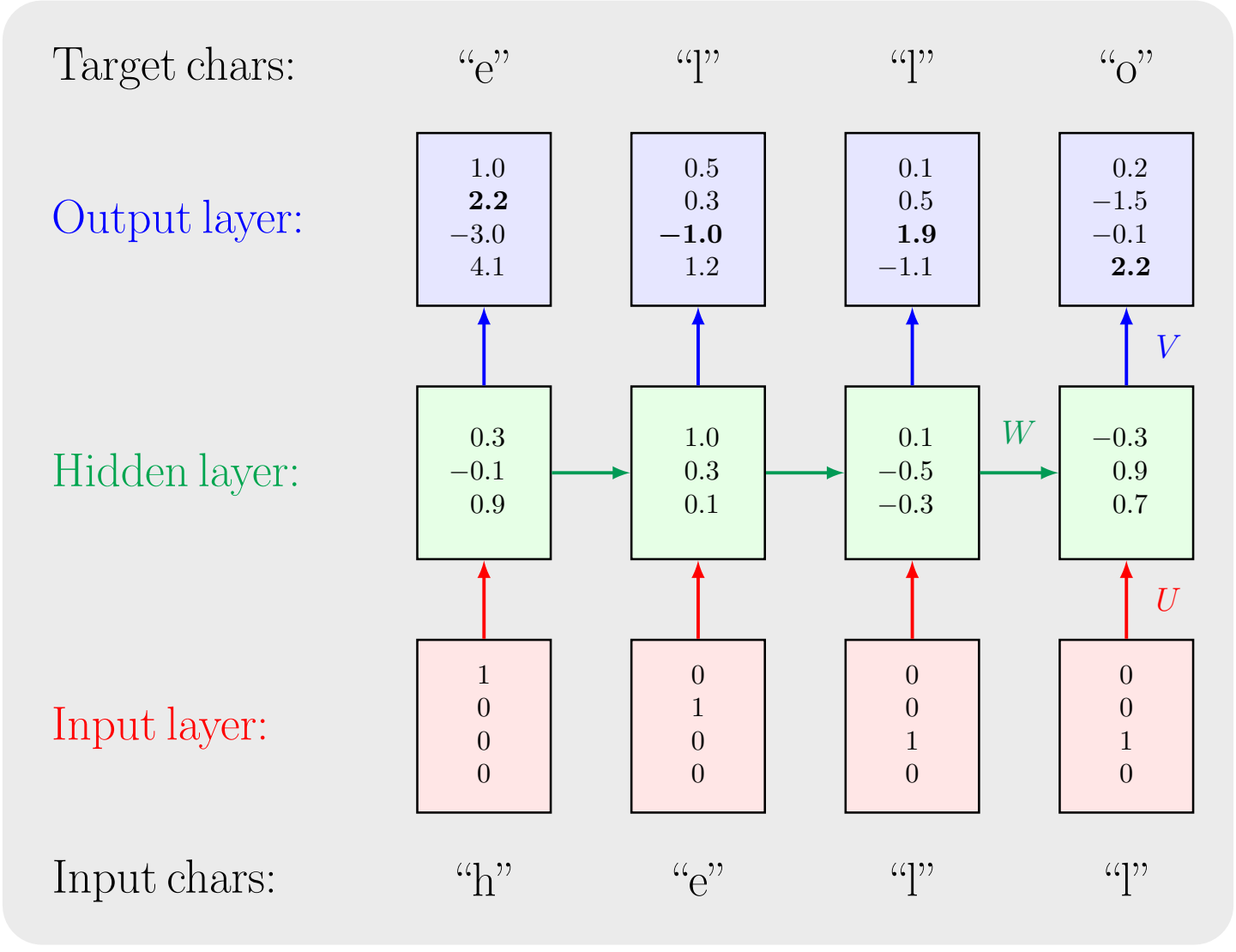

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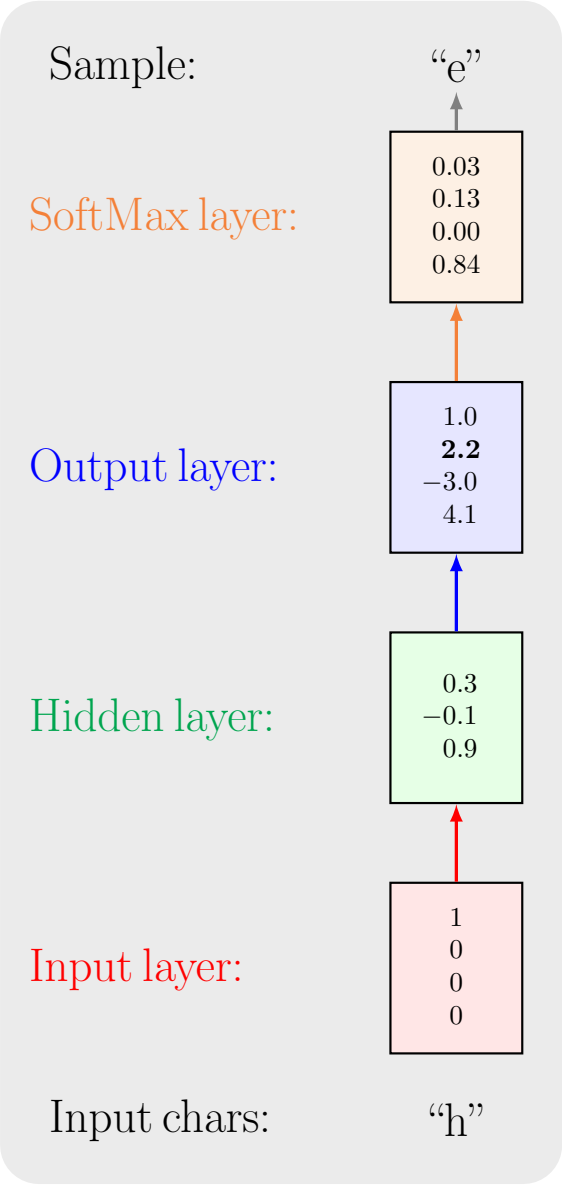

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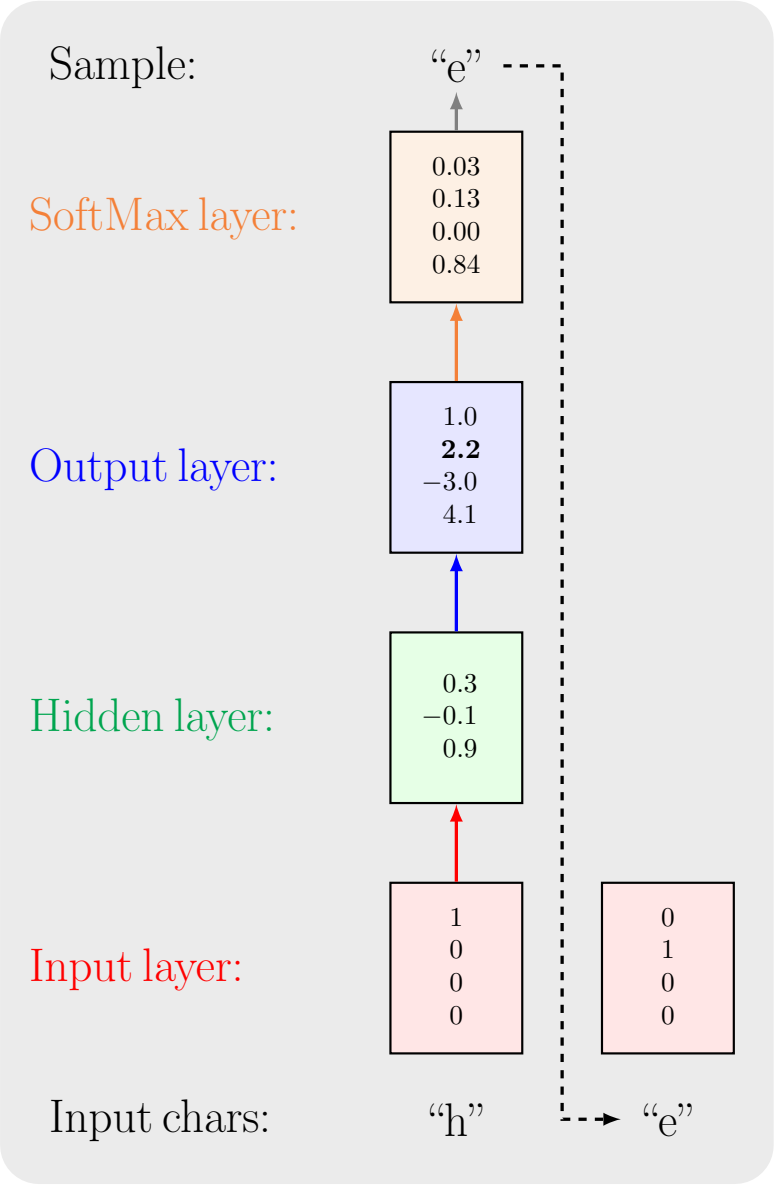

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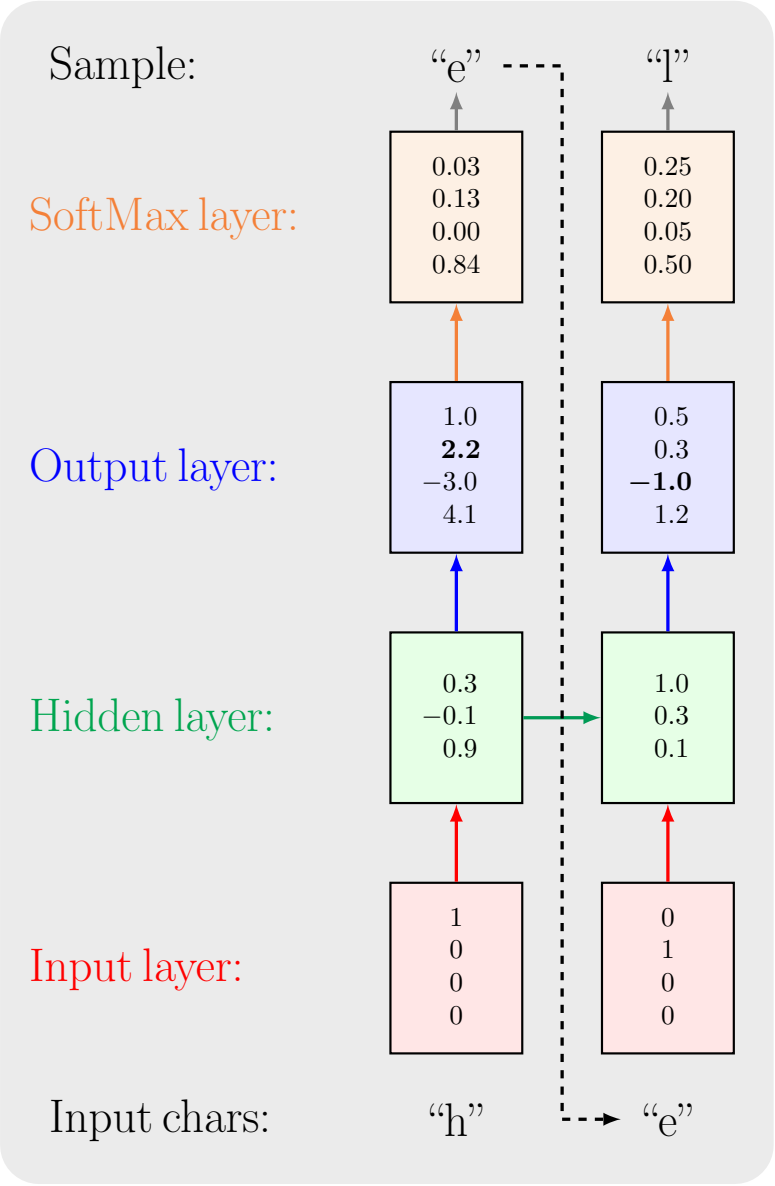

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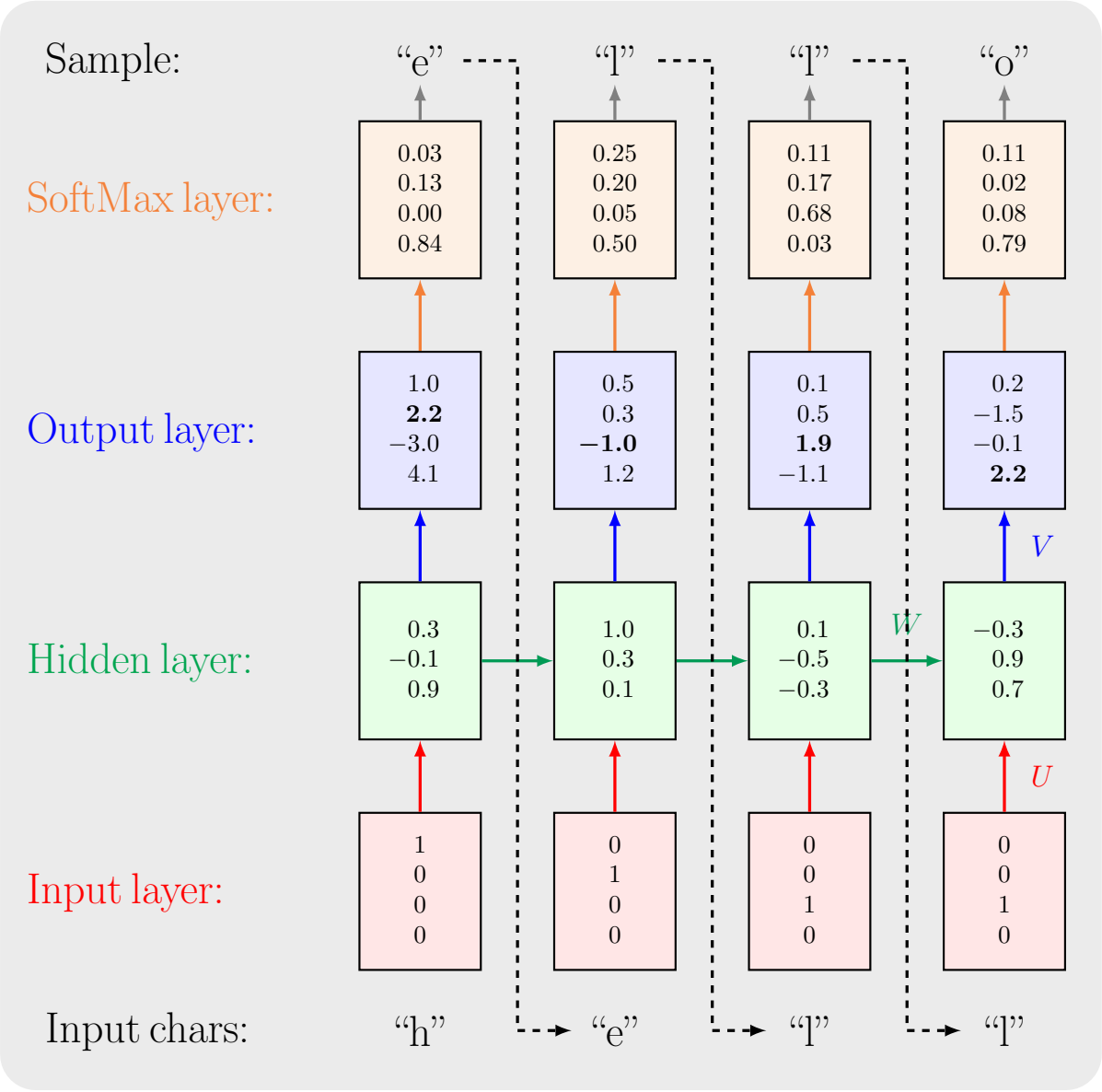
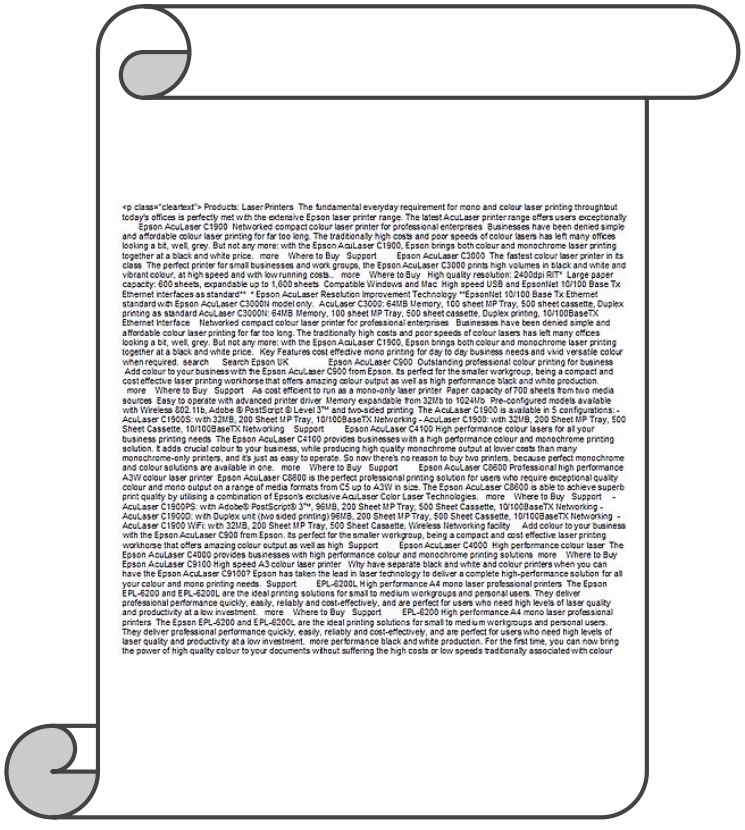


Extend this simple approach to full alphabet and punctuation characters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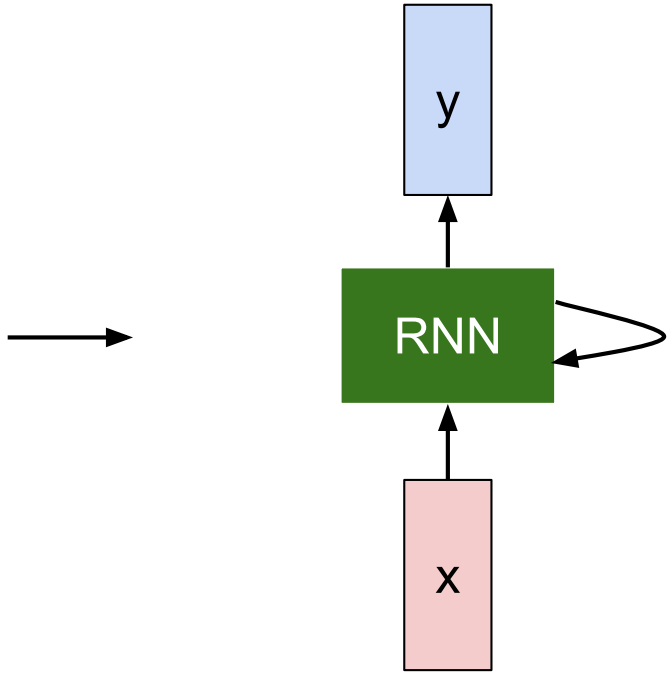

y
RNN
x

Sonnet 116 - Let me not ...

by William Shakespeare

Let me not to the marriage of true minds
Admit impediments. Love is not love
Which alters when it alteration finds,
Or bends with the remover to remove:
O no! it is an ever-fixed mark
That looks on tempests and is never shaken;
It is the star to every wandering bark,
Whose worth's unknown, although his height be taken.
Love's not Time's fool, though rosy lips and cheeks
Within his bending sickle's compass come:
Love alters not with his brief hours and weeks,
But bears it out even to the edge of doom.
If this be error and upon me proved,
I never writ, nor no man ever loved.

at first:

```
tyntd-iafhatawiaoihrdemot  lytdws  e ,tfti, astai f ogoh eoase rrranbyne 'nhthnee e
plia tklrgd t o idoe ns,smtt   h ne etie h,hregtrs nigtike,aoaenns lng
```

train more

```
"Tmont thithey" fomesscerliund
Keushey. Thom here
sheulke, anmerenith ol sivh I lalterthend Bleipile shuwy fil on aseterlome
coaniogennc Phe lism thond hon at. MeiDimorotion in ther thize."
```

train more

```
Aftair fall unsuch that the hall for Prince Velzonski's that me of
her hearly, and behs to so arwage fiving were to it beloge, pavu say falling misfort
how, and Gogition is so overelical and ofter.
```

train more

```
"Why do what that day," replied Natasha, and wishing to himself the fact the
princess, Princess Mary was easier, fed in had oftened him.
Pierre aking his soul came to the packs and drove up his father-in-law women.
```

```
PANDARUS:
Alas, I think he shall be come approached and the day
When little srain would be attain'd into being never fed,
And who is but a chain and subjects of his death,
I should not sleep.

Second Senator:
They are away this miseries, produced upon my soul,
Breaking and strongly should be buried, when I perish
The earth and thoughts of many states.

DUKE VINCENTIO:
Well, your wit is in the care of side and that.

Second Lord:
They would be ruled after this chamber, and
my fair nues begun out of the fact, to be conveyed,
Whose noble souls I'll have the heart of the wars.

Clown:
Come, sir, I will make did behold your worship.

VIOLA:
I'll drink it.
```

```
VIOLA:
Why, Salisbury must find his flesh and thought
That which I am not aps, not a man and in fire,
To show the reining of the raven and the wars
To grace my hand reproach within, and not a fair are hand,
That Caesar and my goodly father's world;
When I was heaven of presence and our fleets,
We spare with hours, but cut thy council I am great,
Murdered and by thy master's ready there
My power to give thee but so much as hell:
Some service in the noble bondman here,
Would show him to her wine.

KING LEAR:
O, if you were a feeble sight, the courtesy of your law,
Your sight and several breath, will wear the gods
With his heads, and my hands are wonder'd at the deeds,
So drop upon your lordship's head, and your opinion
Shall be against your honour.
```

How do we train a vanilla RNN?

Supervised learning via a loss function & mini-batch gradient descent.

Loss defined for one training sequence.

- Have a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\tau$ of input vectors.
- For each $\mathbf{x}_t$ in sequence have a target output y_t.
- Define loss l_t between the y_t and $\mathbf{p}_t$ for each t.
- Sum the loss over all time-steps

$$L(\mathbf{x}_1, \ldots, \mathbf{x}_\tau, y_1, \ldots, y_\tau, W, U, V, \mathbf{b}, \mathbf{c}) = \sum_{t=1}^{\tau} l_t$$

Loss function for a vanilla RNN

Common to use the cross-entropy loss:

$$l_t = -\mathbf{y}_t^T \log(\mathbf{p}_t)$$

thus

$$L(\mathbf{x}_{1:\tau}, y_{1:\tau}, W, U, V, \mathbf{b}, \mathbf{c}) = -\sum_{t=1}^{\tau} \mathbf{y}_t^T \log(\mathbf{p}_t)$$

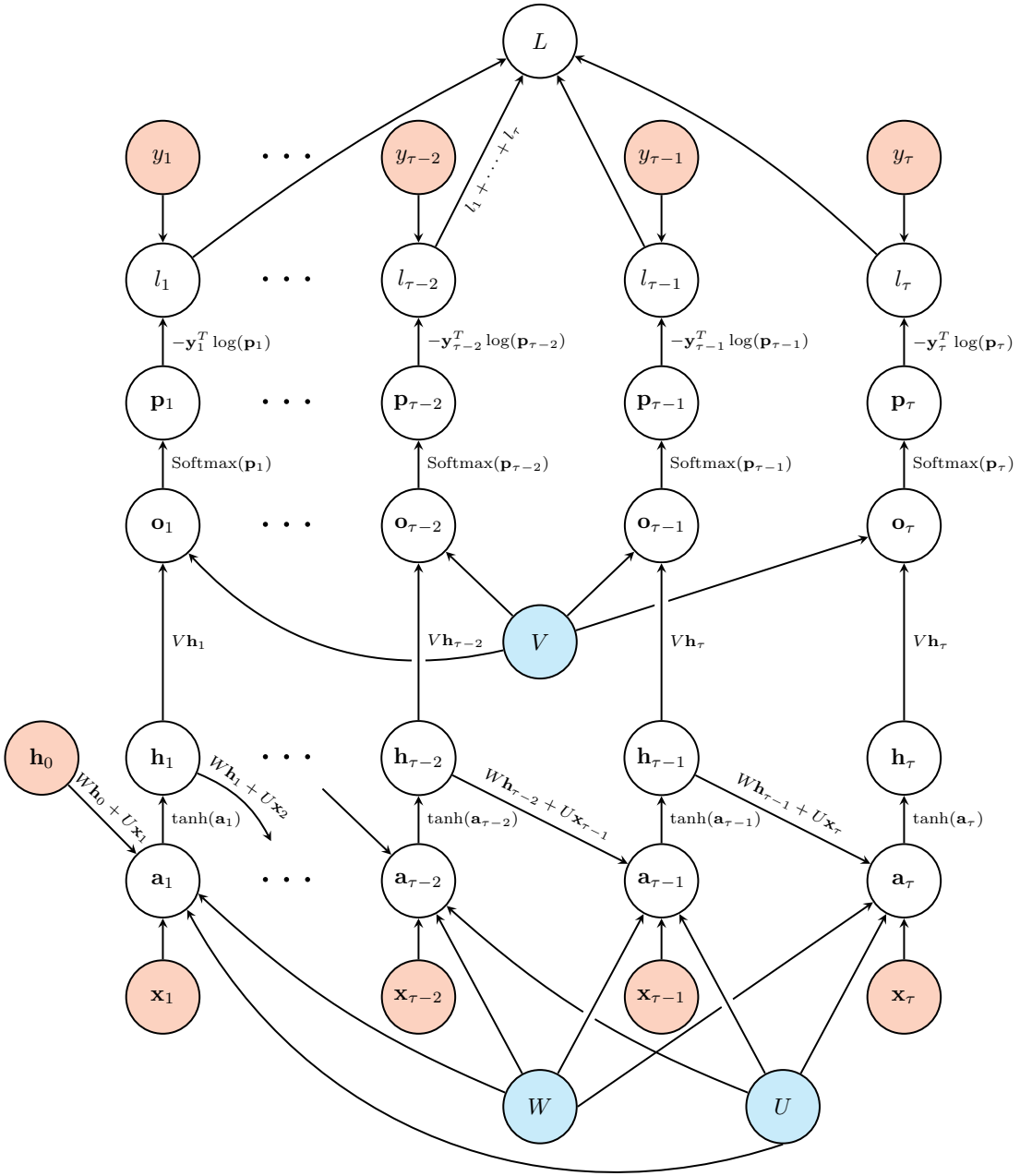
where $\mathbf{x}_{1:\tau} = \{\mathbf{x}_1, \ldots, \mathbf{x}_\tau\}$ and $y_{1:\tau} = \{y_1, \ldots, y_\tau\}$.

- To implement mini-batch gradient descent need to compute

$$\frac{\partial L(\mathbf{x}_{1:\tau}, y_{1:\tau}, W, U, V, \mathbf{b}, \mathbf{c})}{\partial W}, \frac{\partial L(\mathbf{x}_{1:\tau}, y_{1:\tau}, W, U, V, \mathbf{b}, \mathbf{c})}{\partial U}, \ldots$$

- You've guessed it, use back-prop...

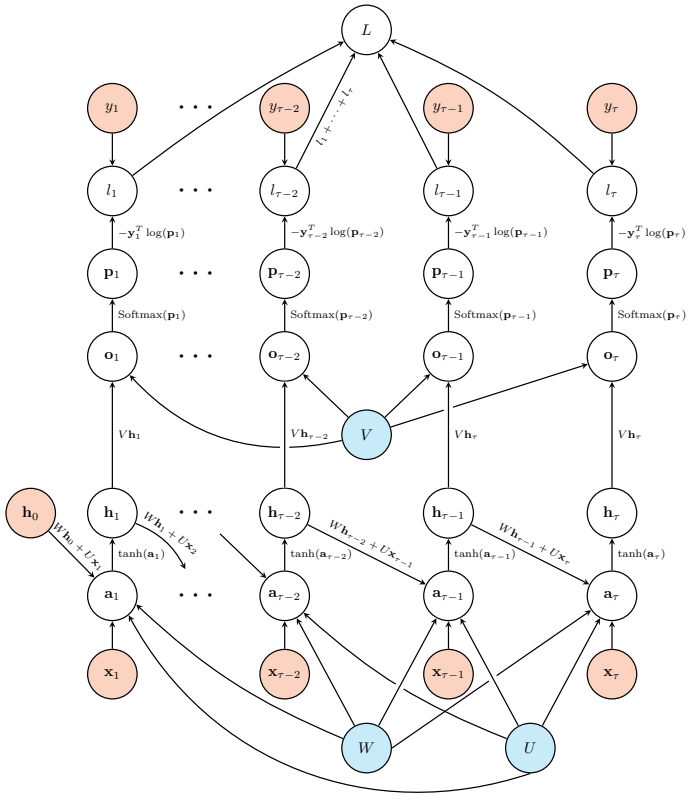
Computational Graph for vanilla RNN loss



- Loss for one labelled training sequence $\mathbf{x}_1, \ldots \mathbf{x}_\tau$
- Bias vectors have been omitted for clarity.

Back-prop for a vanilla RNN

Gradient of loss for the cross-entropy & softmax layers



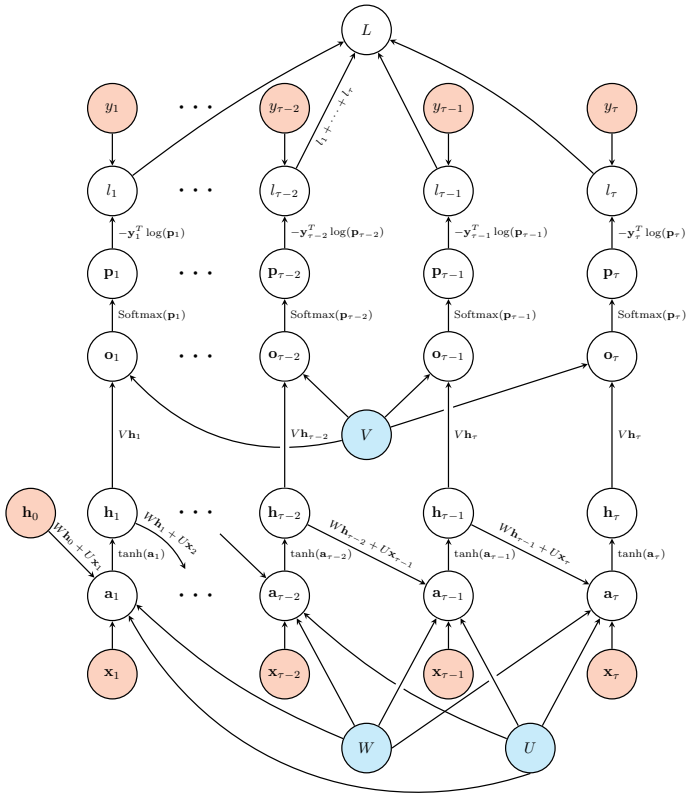
Know from prior dealings with cross-entropy loss:

for $t = 1, \ldots \tau$

$$\frac{\partial L}{\partial l_t} = 1$$

$$\frac{\partial L}{\partial \mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{p}_t} \frac{\partial \mathbf{p}_t}{\partial \mathbf{o}_t} = -(\mathbf{y}_t - \mathbf{p}_t)^T$$

Gradient of loss w.r.t. V



Children of node V are $\mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_\tau$. Thus

$$\frac{\partial L}{\partial \text{vec}(V)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{o}_t} \frac{\partial \mathbf{o}_t}{\partial \text{vec}(V)}$$

Know

$$\mathbf{o}_t = V\mathbf{h}_t \implies \mathbf{o}_t = \left(I_C \otimes \mathbf{h}_t^T\right) \text{vec}(V)$$

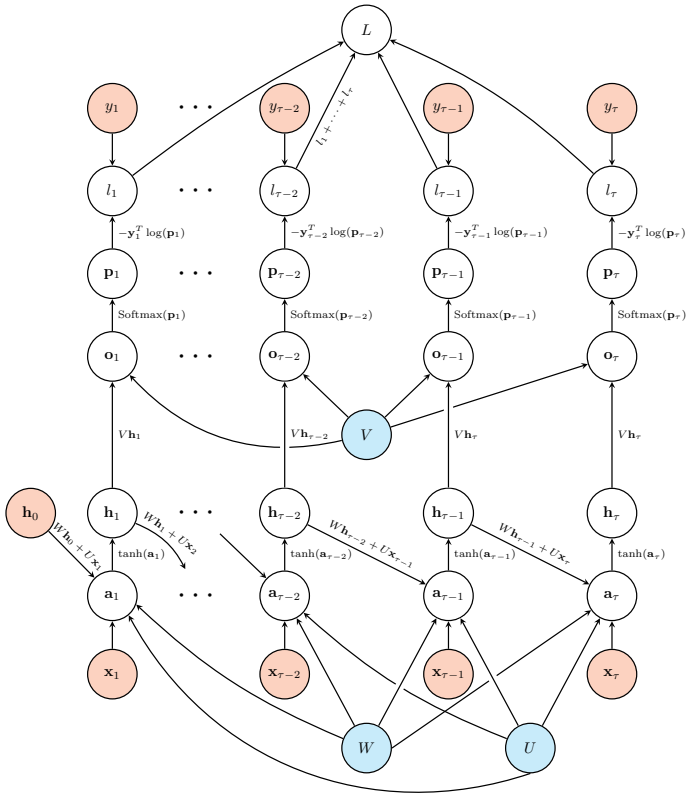
$$\implies \frac{\partial \mathbf{o}_t}{\partial \text{vec}(V)} = I_C \otimes \mathbf{h}_t^T$$

From previous reshapings know:

$$\frac{\partial L}{\partial V} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{h}_t^T$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{o}_t}$.

Gradient of loss w.r.t. V



Children of node V are $\mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_\tau$. Thus

$$\frac{\partial L}{\partial \text{vec}(V)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{o}_t} \frac{\partial \mathbf{o}_t}{\partial \text{vec}(V)}$$

Know

$$\mathbf{o}_t = V\mathbf{h}_t \implies \mathbf{o}_t = \left(I_C \otimes \mathbf{h}_t^T\right) \text{vec}(V)$$

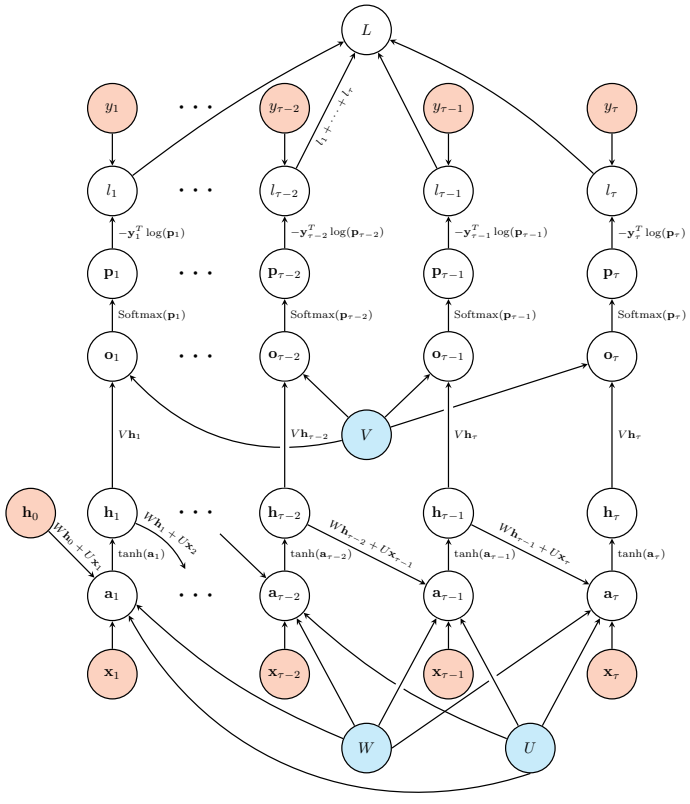
$$\implies \frac{\partial \mathbf{o}_t}{\partial \text{vec}(V)} = I_C \otimes \mathbf{h}_t^T$$

From previous reshapings know:

$$\frac{\partial L}{\partial V} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{h}_t^T \leftarrow \text{gradient needed for training network}$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{o}_t}$.

Gradient of loss w.r.t. $\mathbf{h}_\tau$



$\mathbf{h}_\tau$ (last hidden state) has one child $\mathbf{o}_\tau$ thus

$$\frac{\partial L}{\partial \mathbf{h}_\tau} = \frac{\partial L}{\partial \mathbf{o}_\tau}\frac{\partial \mathbf{o}_\tau}{\partial \mathbf{h}_\tau}$$

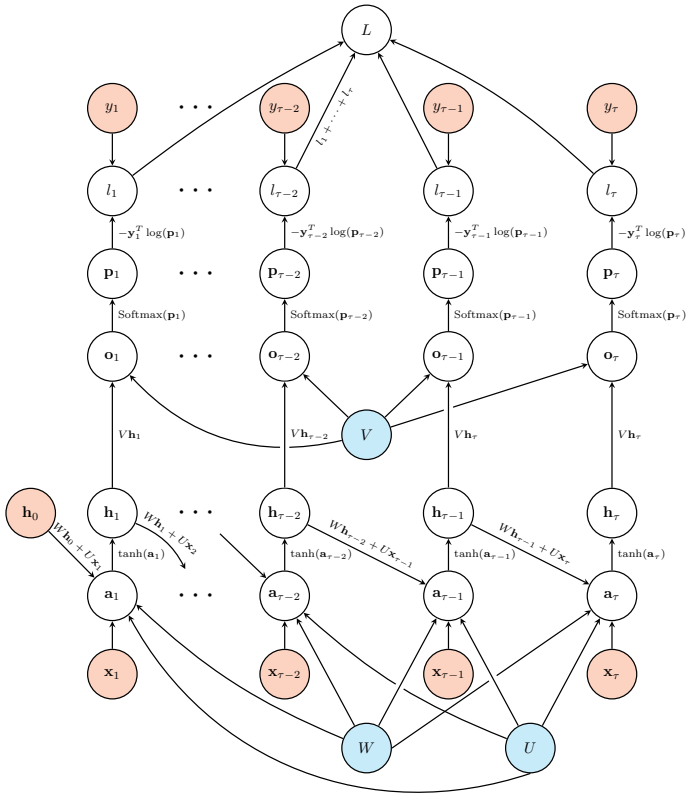
Know

$$\mathbf{o}_\tau = V\mathbf{h}_\tau \implies \frac{\partial \mathbf{o}_\tau}{\partial \mathbf{h}_\tau} = V$$

Thus

$$\frac{\partial L}{\partial \mathbf{h}_\tau} = \frac{\partial L}{\partial \mathbf{o}_\tau}V$$

Gradient of loss w.r.t. $\mathbf{h}_t$



If $1 \leq t \leq \tau - 1$ then $\mathbf{h}_t$ has children $\mathbf{o}_t$ and $\mathbf{a}_{t+1}$

$$\frac{\partial L}{\partial \mathbf{h}_t} = \frac{\partial L}{\partial \mathbf{o}_t}\frac{\partial \mathbf{o}_t}{\partial \mathbf{h}_t} + \frac{\partial L}{\partial \mathbf{a}_{t+1}}\frac{\partial \mathbf{a}_{t+1}}{\partial \mathbf{h}_t}$$

Know

$$\mathbf{o}_t = V\mathbf{h}_t \implies \frac{\partial \mathbf{o}_t}{\partial \mathbf{h}_t} = V$$

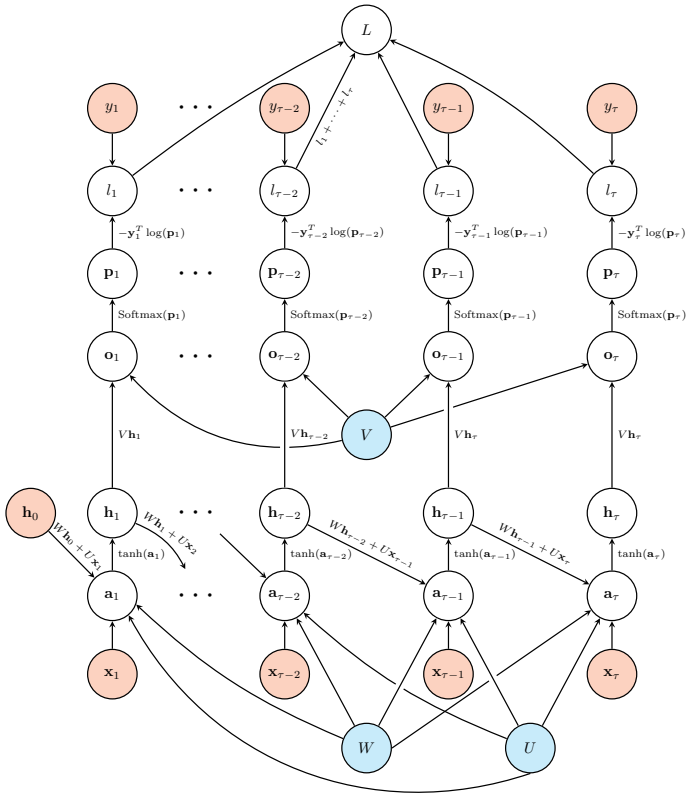
and

$$\mathbf{a}_{t+1} = W\mathbf{h}_t + U\mathbf{x}_{t+1} \implies \frac{\partial \mathbf{a}_{t+1}}{\partial \mathbf{h}_t} = W$$

Thus

$$\frac{\partial L}{\partial \mathbf{h}_t} = \frac{\partial L}{\partial \mathbf{o}_t}V + \frac{\partial L}{\partial \mathbf{a}_{t+1}}W$$

Gradient of loss w.r.t. $\mathbf{h}_t$



If $1 \leq t \leq \tau - 1$ then $\mathbf{h}_t$ has children $\mathbf{o}_t$ and $\mathbf{a}_{t+1}$

$$\frac{\partial L}{\partial \mathbf{h}_t} = \frac{\partial L}{\partial \mathbf{o}_t}\frac{\partial \mathbf{o}_t}{\partial \mathbf{h}_t} + \frac{\partial L}{\partial \mathbf{a}_{t+1}}\frac{\partial \mathbf{a}_{t+1}}{\partial \mathbf{h}_t}$$

Know

$$\mathbf{o}_t = V\mathbf{h}_t \implies \frac{\partial \mathbf{o}_t}{\partial \mathbf{h}_t} = V$$

and

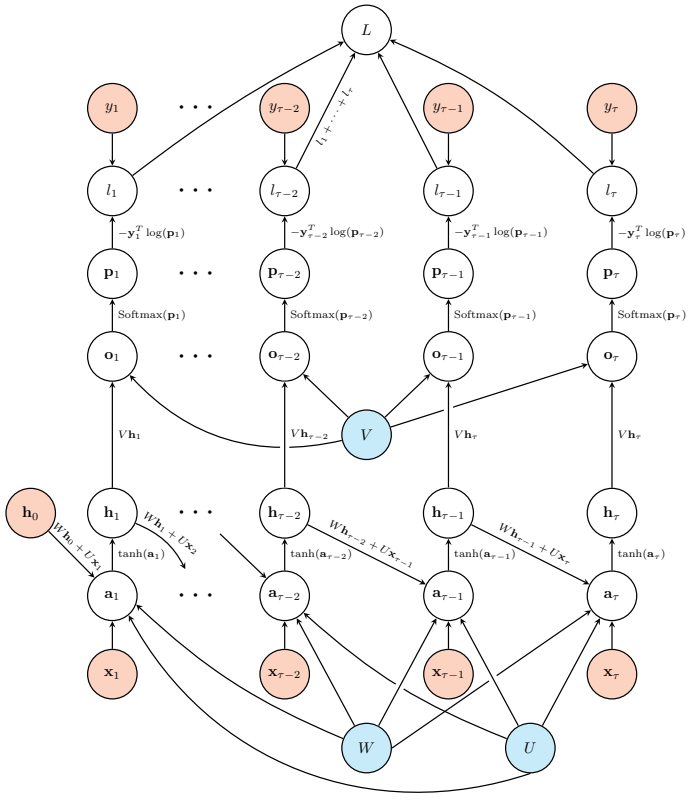
$$\mathbf{a}_{t+1} = W\mathbf{h}_t + U\mathbf{x}_{t+1} \implies \frac{\partial \mathbf{a}_{t+1}}{\partial \mathbf{h}_t} = W$$

Thus

$$\frac{\partial L}{\partial \mathbf{h}_t} = \frac{\partial L}{\partial \mathbf{o}_t}V + \frac{\partial L}{\partial \mathbf{a}_{t+1}}W$$

Have two different time steps in expression $\implies$ must iterate backwards in time to compute all $\frac{\partial L}{\partial \mathbf{h}_t}$

Gradient of loss w.r.t. $\mathbf{a}_t$



The gradient w.r.t. $\mathbf{a}_t$

$$\frac{\partial L}{\partial \mathbf{a}_t} = \frac{\partial L}{\partial \mathbf{h}_t}\frac{\partial \mathbf{h}_t}{\partial \mathbf{a}_t}$$

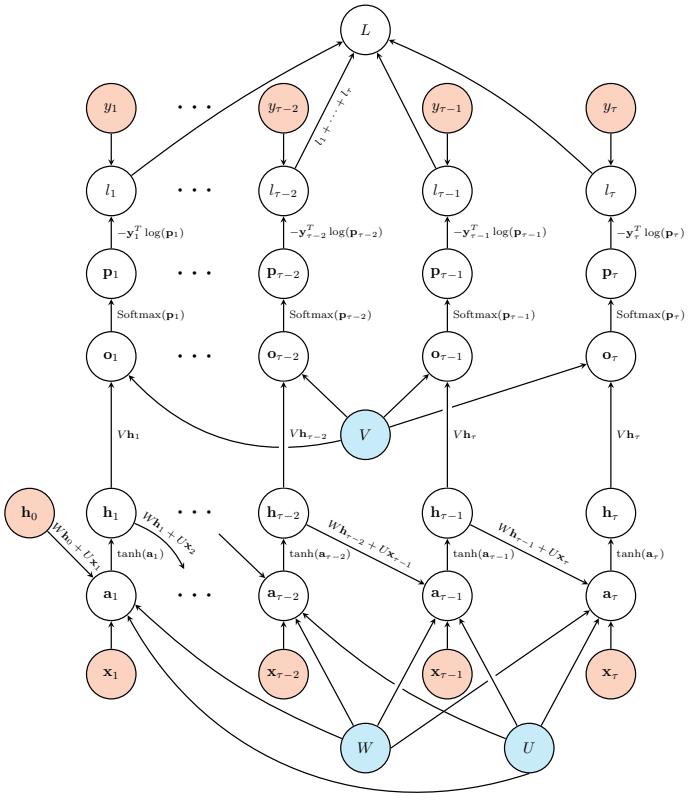
Know

$$\mathbf{h}_t = \tanh(\mathbf{a}_t) \implies \frac{\partial \mathbf{h}_t}{\partial \mathbf{a}_t} = \text{diag}\left(\tanh'(\mathbf{a}_t)\right)$$
$$= \text{diag}\left(1 - \tanh^2(\mathbf{a}_t)\right)$$

Thus

$$\frac{\partial L}{\partial \mathbf{a}_t} = \frac{\partial L}{\partial \mathbf{h}_t}\text{diag}\left(1 - \tanh^2(\mathbf{a}_t)\right)$$

Recursively compute gradients for all $\mathbf{a}_t$ and $\mathbf{h}_t$



- Assume $\frac{\partial L}{\partial \mathbf{o}_t}$ calculated for $1 \leq t \leq \tau$.
- Calculate

$$\frac{\partial L}{\partial \mathbf{h}_\tau} = \frac{\partial L}{\partial \mathbf{o}_\tau} V \quad \& \quad \frac{\partial L}{\partial \mathbf{a}_\tau} = \frac{\partial L}{\partial \mathbf{h}_\tau} \text{diag}\left(1 - \tanh^2(\mathbf{a}_\tau)\right)$$

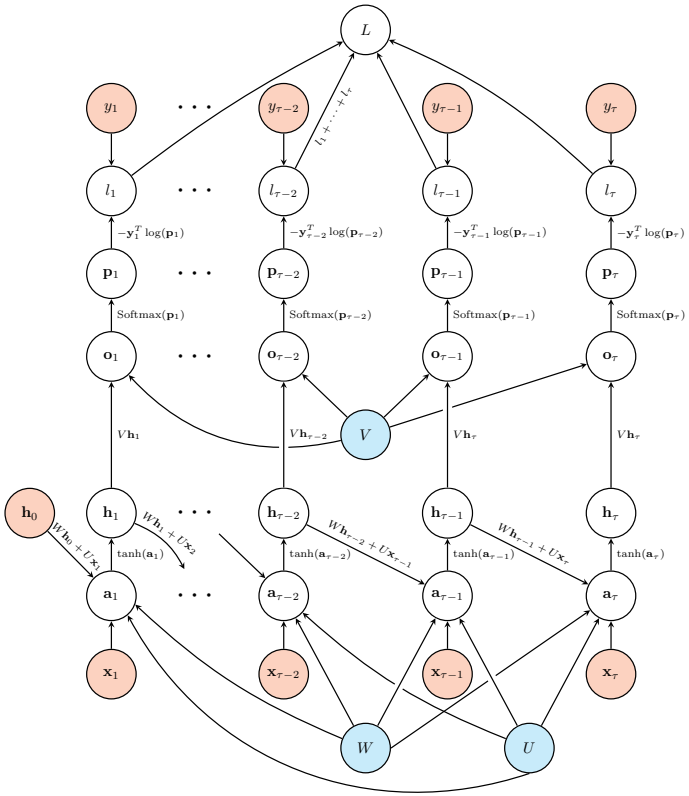
- for $t = \tau - 1, \tau - 2, \ldots, 1$
  - Compute

$$\frac{\partial L}{\partial \mathbf{h}_t} = \frac{\partial L}{\partial \mathbf{o}_t} V + \frac{\partial L}{\partial \mathbf{a}_{t+1}} W$$

  - Compute

$$\frac{\partial L}{\partial \mathbf{a}_t} = \frac{\partial L}{\partial \mathbf{h}_t} \text{diag}\left(1 - \tanh^2(\mathbf{a}_t)\right)$$

Gradient of loss w.r.t. W



The gradient of the loss w.r.t. node W.

Children of W are $\mathbf{a}_1, \dots \mathbf{a}_\tau$ thus

$$\frac{\partial L}{\partial \text{vec}(W)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{a}_t} \frac{\partial \mathbf{a}_t}{\partial \text{vec}(W)}$$

Know

$$\mathbf{a}_t = W\mathbf{h}_{t-1} + U\mathbf{x}_t \implies \mathbf{a}_t = (I_m \otimes \mathbf{h}_{t-1}^T)\text{vec}(W) + U\mathbf{x}_t$$

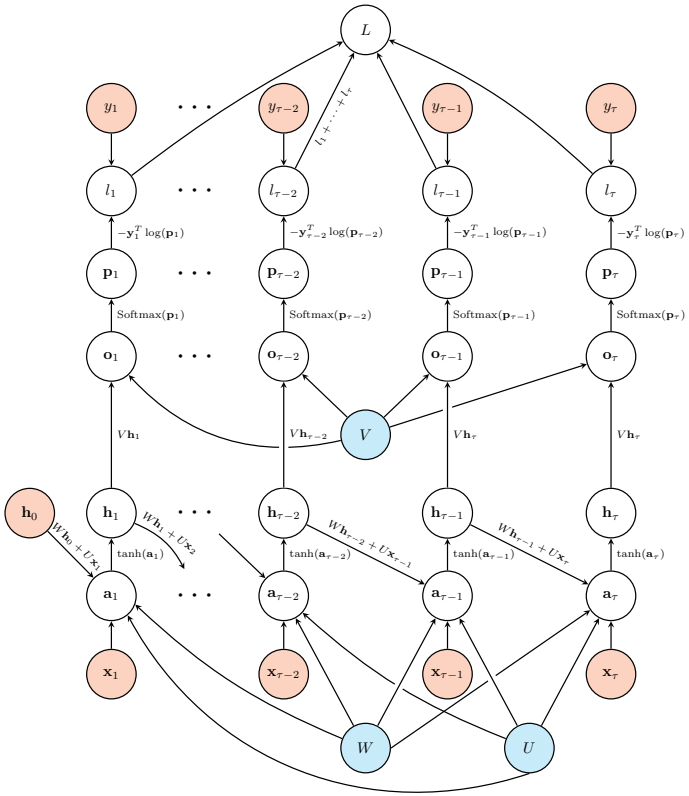
$$\implies \frac{\partial \mathbf{a}_t}{\partial \text{vec}(W)} = I_m \otimes \mathbf{h}_{t-1}^T$$

From previous reshapings know:

$$\frac{\partial L}{\partial W} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{h}_{t-1}^T$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{a}_t}$.

Gradient of loss w.r.t. W



The gradient of the loss w.r.t. node W.

Children of W are $\mathbf{a}_1, \ldots \mathbf{a}_\tau$ thus

$$\frac{\partial L}{\partial \text{vec}(W)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{a}_t} \frac{\partial \mathbf{a}_t}{\partial \text{vec}(W)}$$

Know

$$\mathbf{a}_t = W\mathbf{h}_{t-1} + U\mathbf{x}_t \implies \mathbf{a}_t = (I_m \otimes \mathbf{h}_{t-1}^T)\text{vec}(W) + U\mathbf{x}_t$$

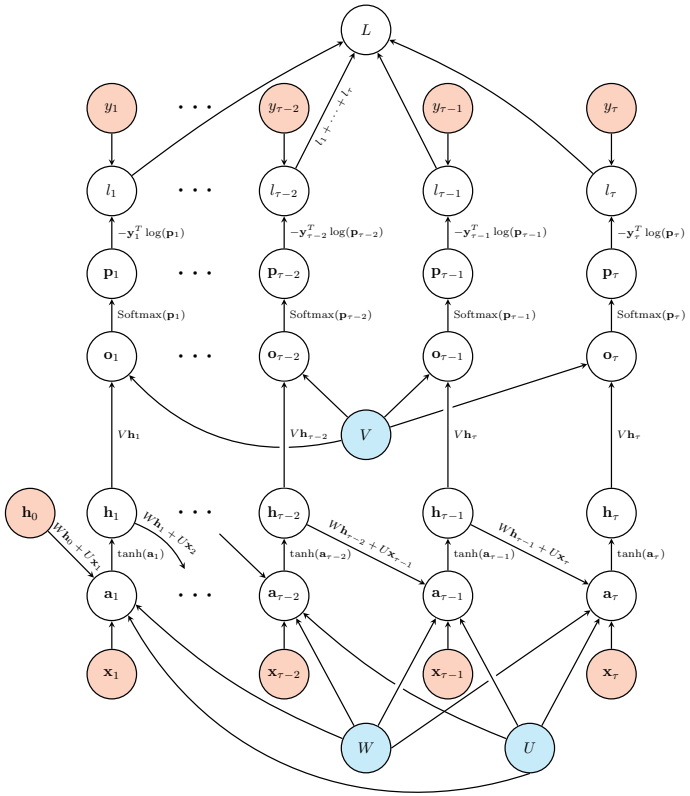
$$\implies \frac{\partial \mathbf{a}_t}{\partial \text{vec}(W)} = I_m \otimes \mathbf{h}_{t-1}^T$$

From previous reshapings know:

$$\frac{\partial L}{\partial W} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{h}_{t-1}^T \leftarrow \text{gradient needed for training network}$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{a}_t}$.

Gradient of loss w.r.t. U



The gradient of the loss w.r.t. node U.

Children of V are $\mathbf{a}_1, \dots \mathbf{a}_\tau$ thus

$$\frac{\partial L}{\partial \text{vec}(U)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{a}_t} \frac{\partial \mathbf{a}_t}{\partial \text{vec}(U)}$$

Know

$$\mathbf{a}_t = W\mathbf{h}_{t-1} + U\mathbf{x}_t \implies \mathbf{a}_t = W\mathbf{h}_{t-1} + (I_m \otimes \mathbf{x}_t^T)\text{vec}(U)$$

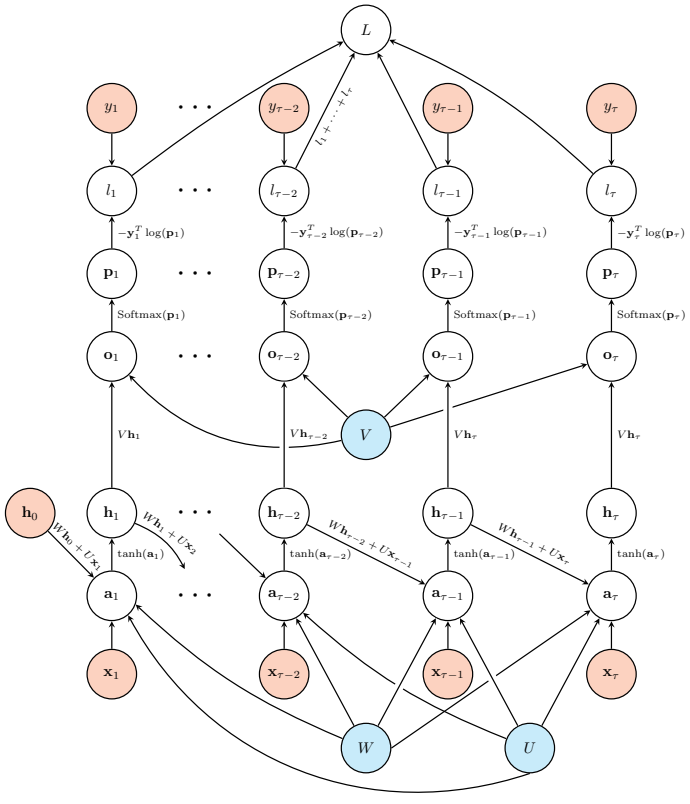
$$\implies \frac{\partial \mathbf{a}_t}{\partial \text{vec}(U)} = I_m \otimes \mathbf{x}_t^T$$

From previous reshapings know:

$$\frac{\partial L}{\partial U} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{x}_t^T$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{a}_t}$.

Gradient of loss w.r.t. U



The gradient of the loss w.r.t. node U.

Children of V are $\mathbf{a}_1, \ldots \mathbf{a}_\tau$ thus

$$\frac{\partial L}{\partial \text{vec}(U)} = \sum_{t=1}^{\tau} \frac{\partial L}{\partial \mathbf{a}_t} \frac{\partial \mathbf{a}_t}{\partial \text{vec}(U)}$$

Know

$$\mathbf{a}_t = W\mathbf{h}_{t-1} + U\mathbf{x}_t \implies \mathbf{a}_t = W\mathbf{h}_{t-1} + (I_m \otimes \mathbf{x}_t^T)\text{vec}(U)$$

$$\implies \frac{\partial \mathbf{a}_t}{\partial \text{vec}(U)} = I_m \otimes \mathbf{x}_t^T$$

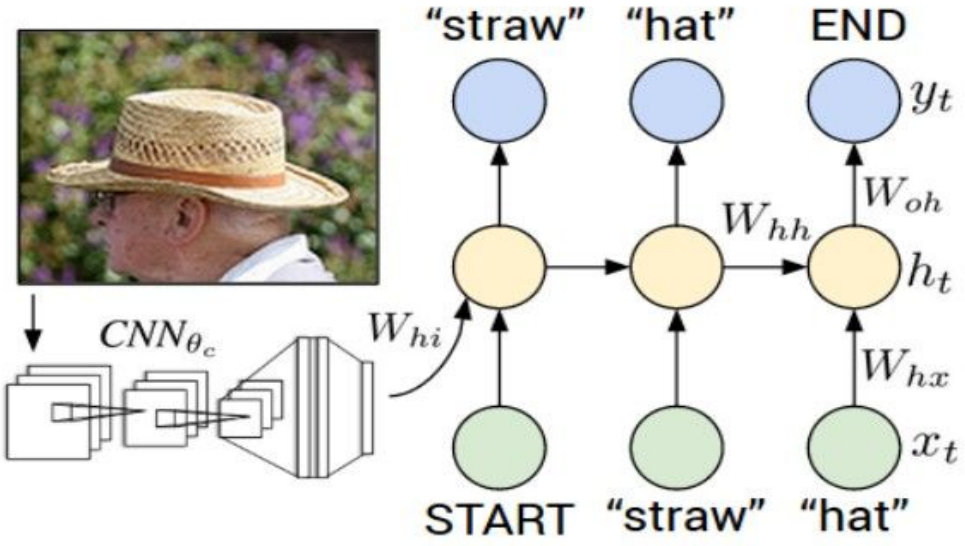
From previous reshapings know:

$$\frac{\partial L}{\partial U} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{x}_t^T \leftarrow \text{gradient needed for training network}$$

where $\mathbf{g}_t = \frac{\partial L}{\partial \mathbf{a}_t}$.

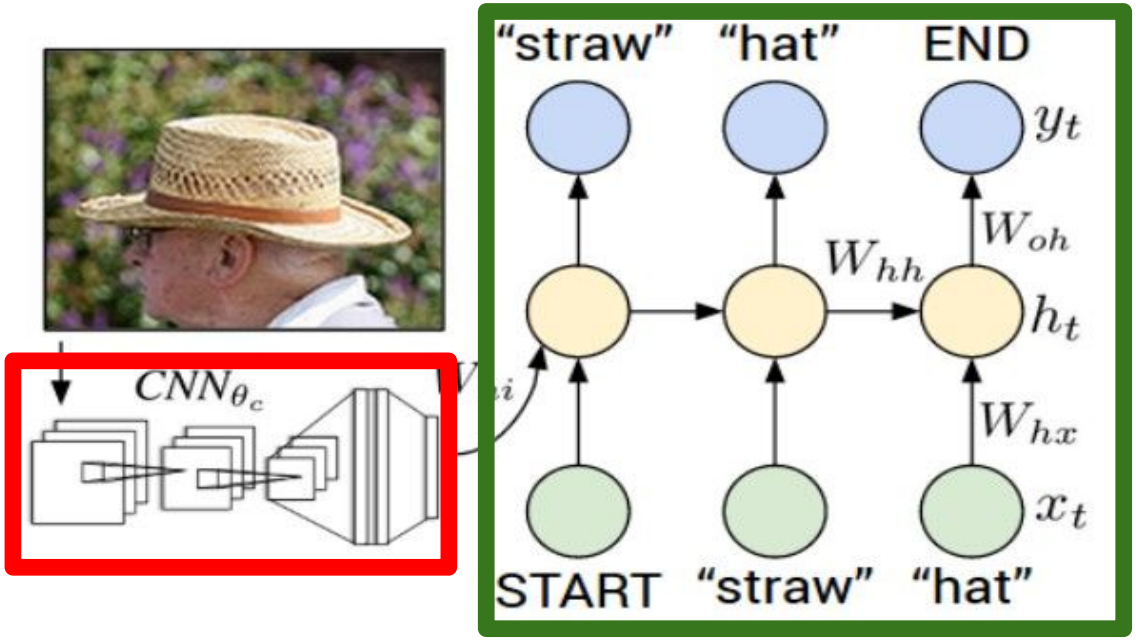
RNNs in Image Translation Application

Image Captioning



Explain Images with Multimodal Recurrent Neural Networks, Mao et al.
Deep Visual-Semantic Alignments for Generating Image Descriptions, Karpathy and Fei-Fei
Show and Tell: A Neural Image Caption Generator, Vinyals et al.
Long-term Recurrent Convolutional Networks for Visual Recognition and Description, Donahue et al.
Learning a Recurrent Visual Representation for Image Caption Generation, Chen and Zitnick

Recurrent Neural Network



Convolutional Neural Network

test image

image

conv-64
conv-64
maxpool

conv-128
conv-128
maxpool

conv-256
conv-256
maxpool

conv-512
conv-512
maxpool

conv-512
conv-512
maxpool

FC-4096
FC-4096
FC-1000
softmax

test image


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
FC-1000
softmax

test image

image

conv-64

conv-64

maxpool

conv-128

conv-128

maxpool

conv-256

conv-256

maxpool

conv-512

conv-512

maxpool

conv-512

conv-512

maxpool

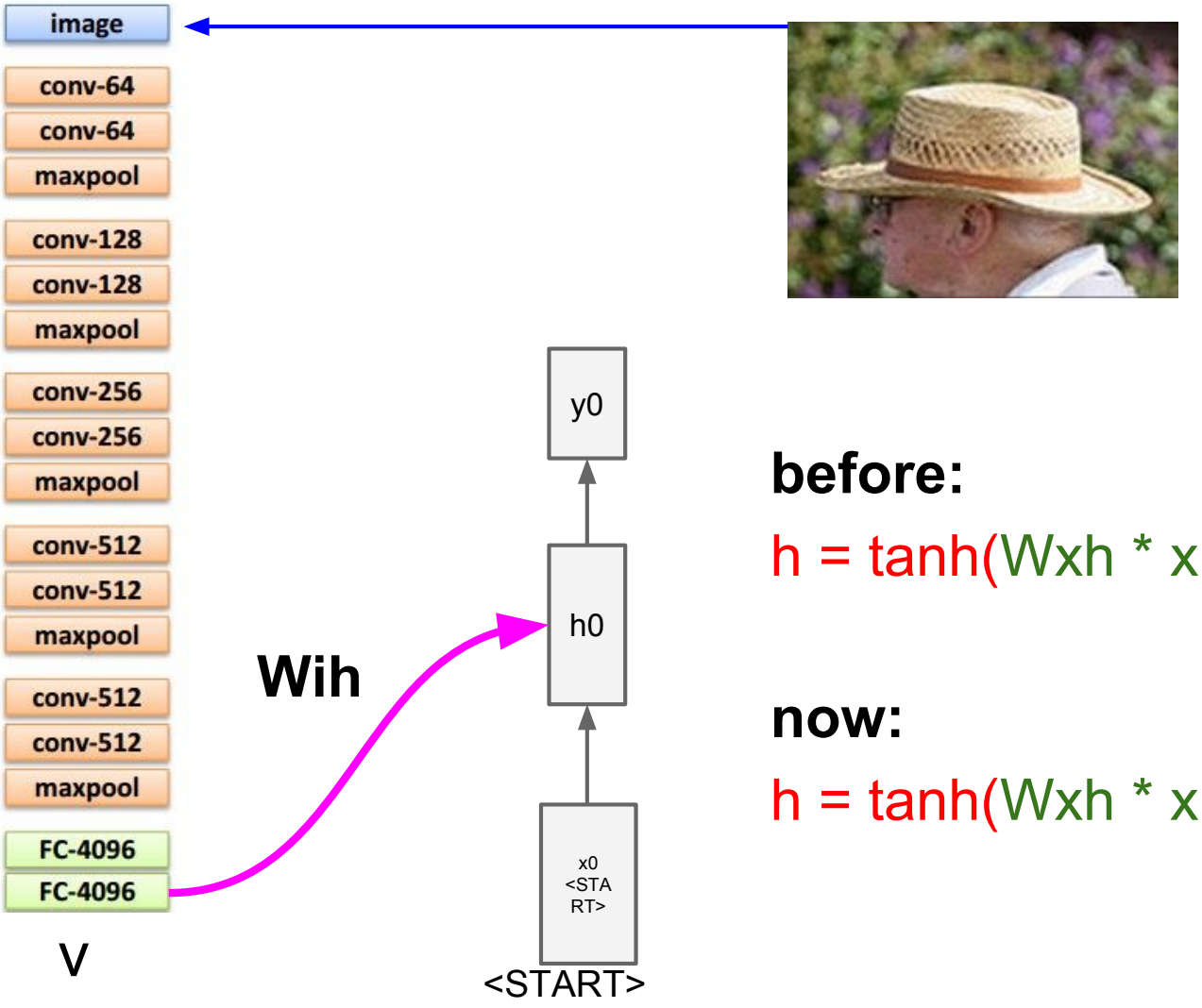
FC-4096

FC-4096

test image

x0
<STA
RT>

<START>



before:

h = tanh(Wxh * x + Whh * h)

now:

h = tanh(Wxh * x + Whh * h **+ Wih * v**)

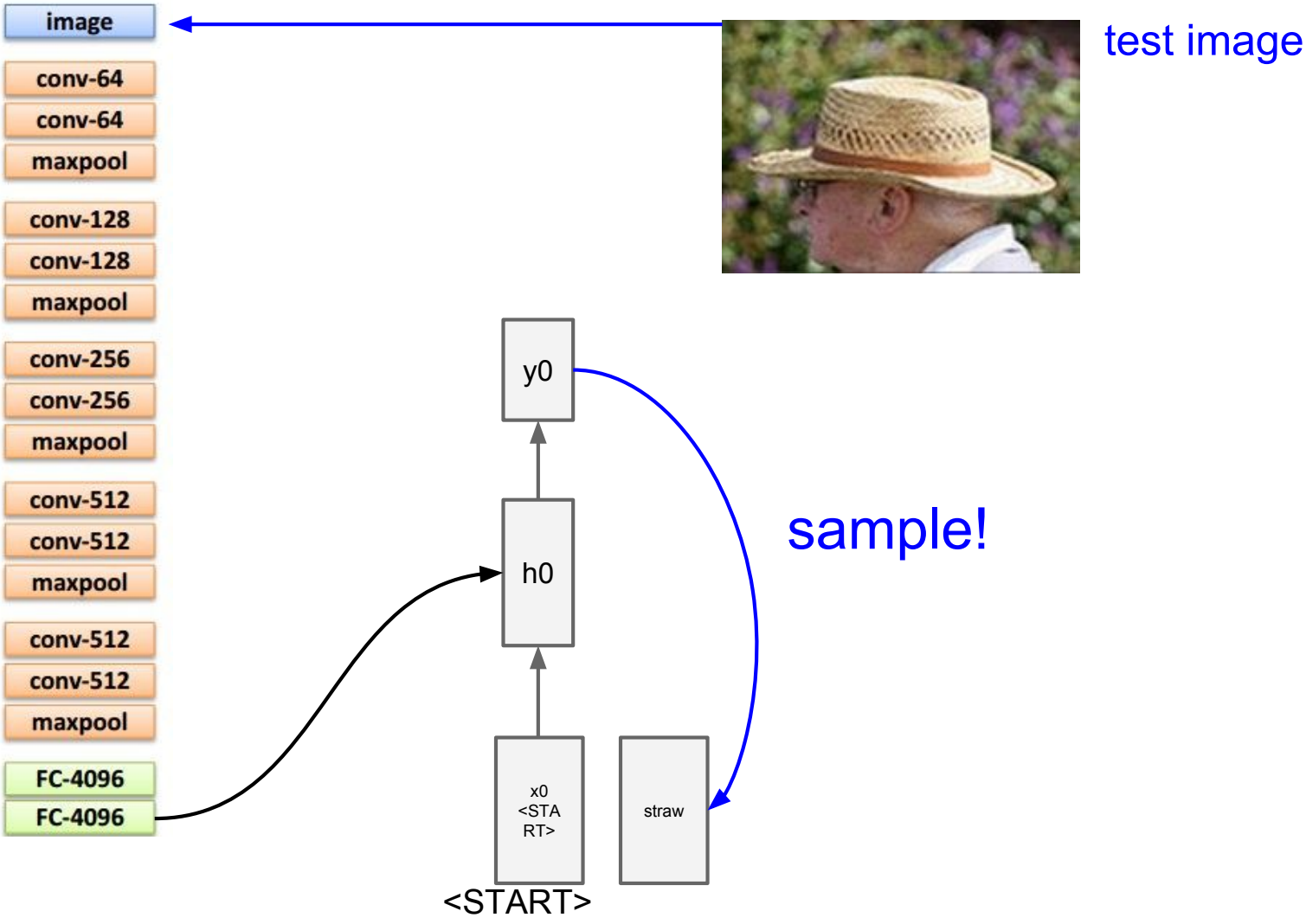


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
test image
y0
h0
x0
<STA
RT>
straw
<START>
sample!

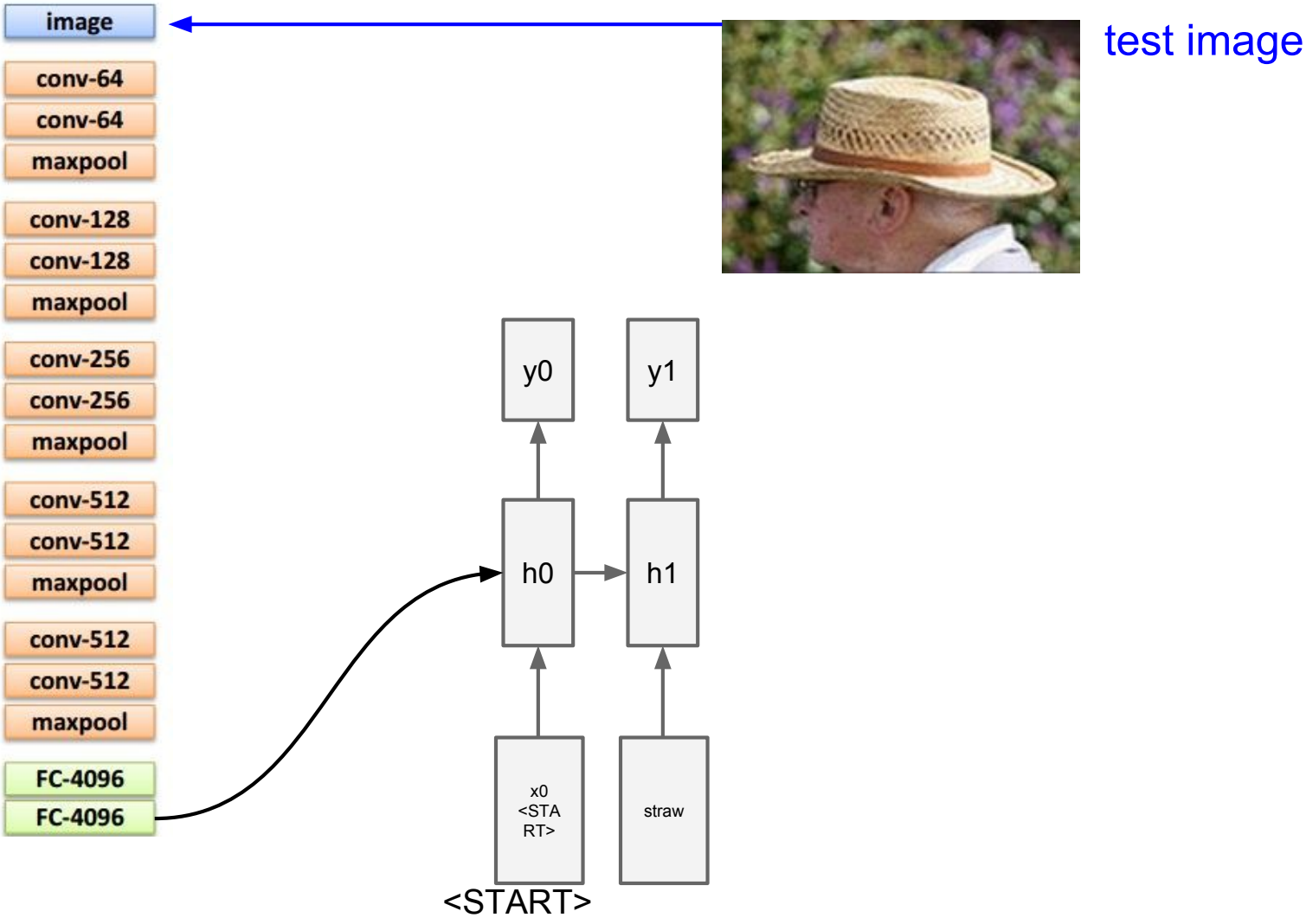


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
test image
y0
y1
h0
h1
x0
<STA
RT>
straw
<START>

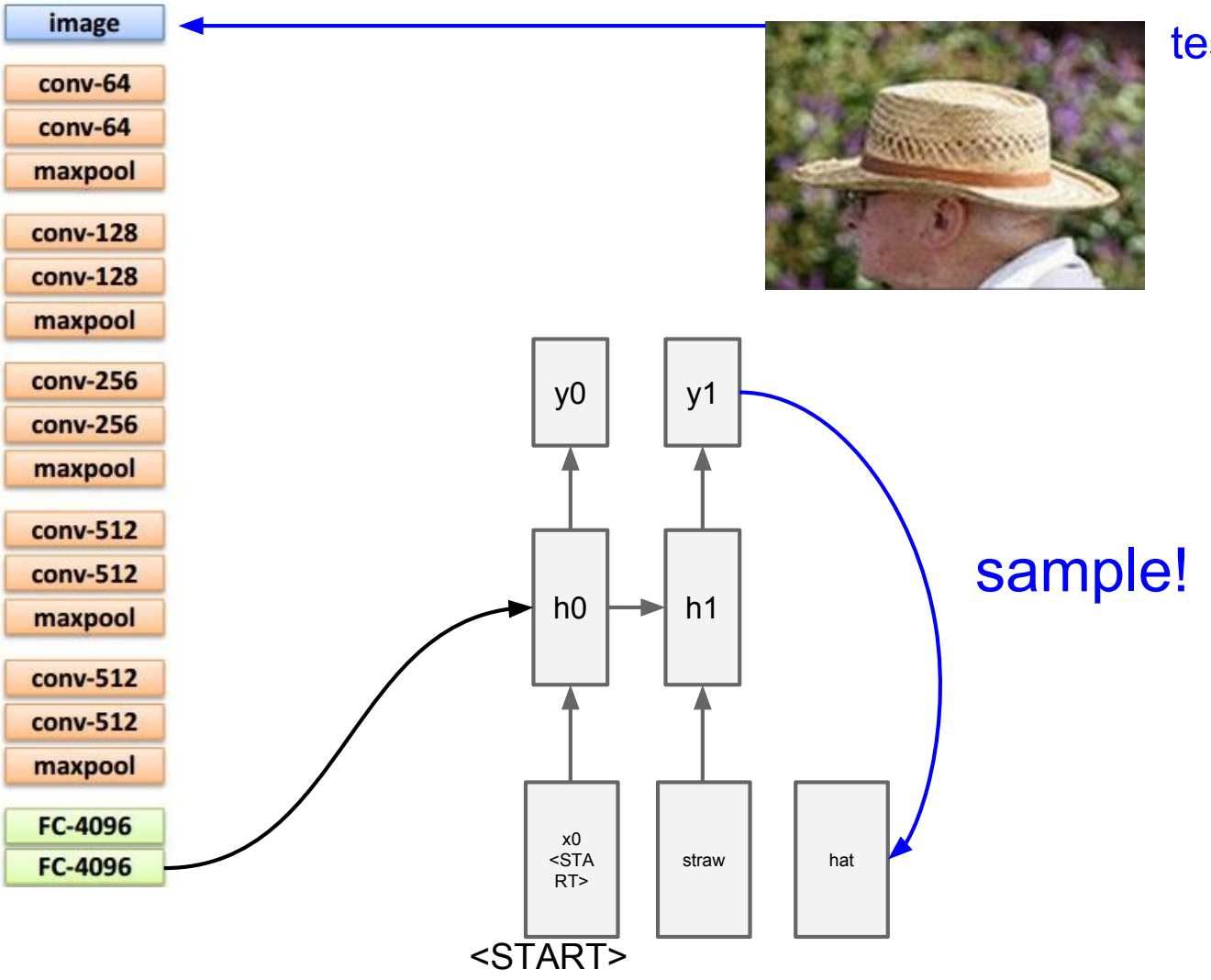


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
test image
y0
y1
h0
h1
x0 <START>
straw
hat
<START>
sample!

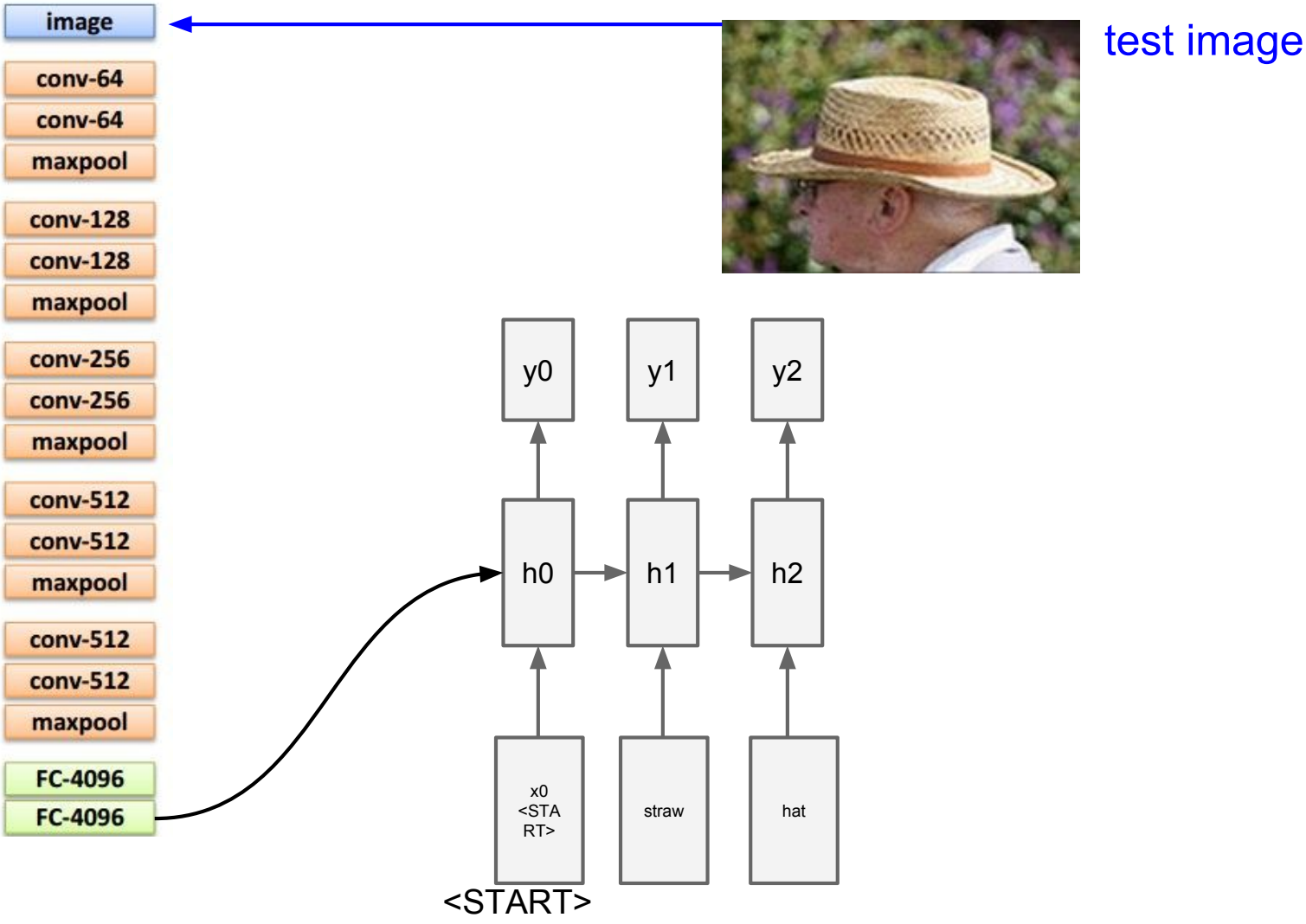


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
test image
y0
y1
y2
h0
h1
h2
x0
<STA
RT>
straw
hat
<START>

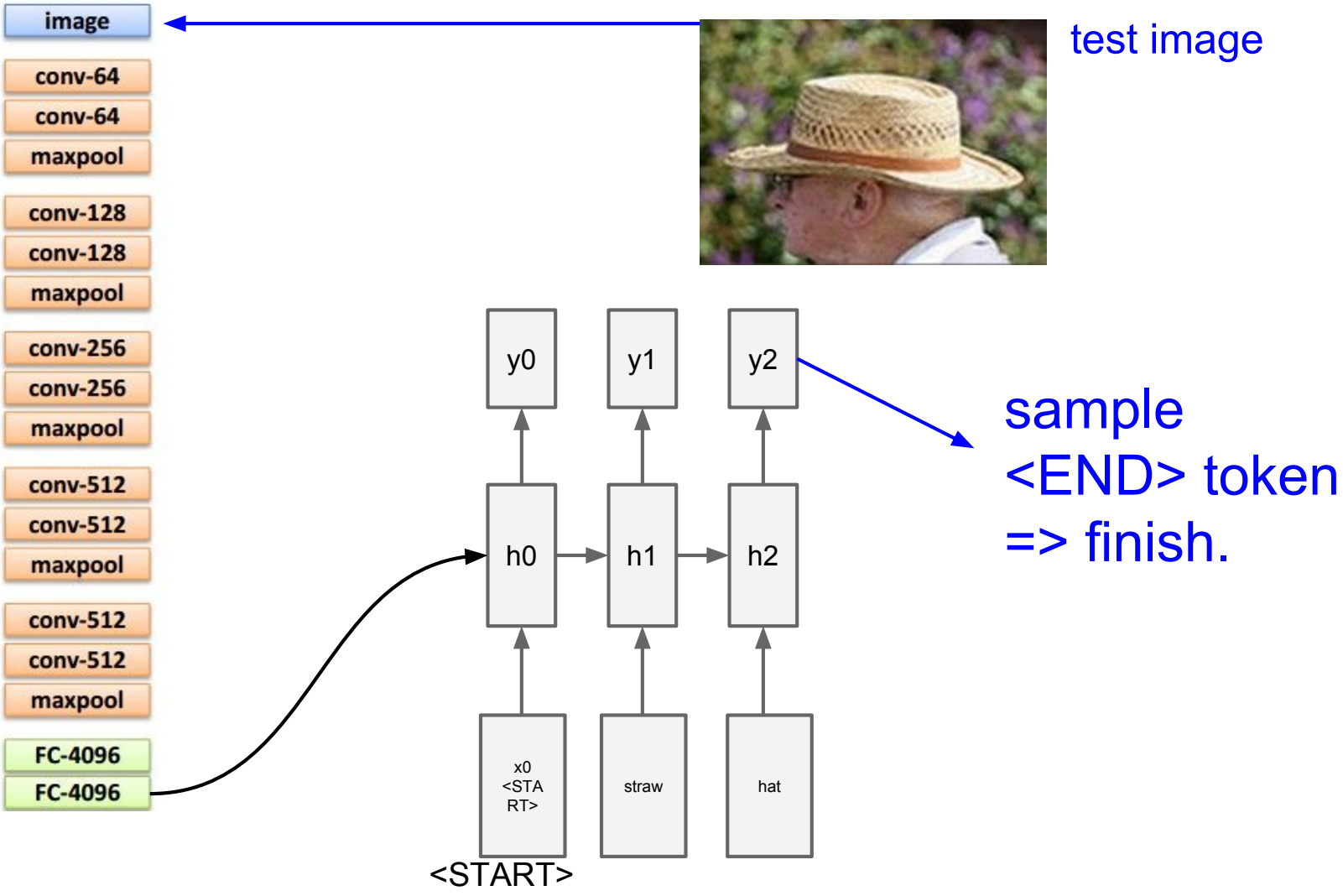


image
conv-64
conv-64
maxpool
conv-128
conv-128
maxpool
conv-256
conv-256
maxpool
conv-512
conv-512
maxpool
conv-512
conv-512
maxpool
FC-4096
FC-4096
test image
y0
y1
y2
h0
h1
h2
x0
<STA
RT>
straw
hat
<START>
sample
<END> token
=> finish.

"man in black shirt is playing guitar."

"construction worker in orange safety vest is working on road."

"two young girls are playing with lego toy."

"boy is doing backflip on wakeboard."

"man in black shirt is playing guitar."

"construction worker in orange safety vest is working on road."

"two young girls are playing with lego toy."

"boy is doing backflip on wakeboard."

"a young boy is holding a baseball bat."

"a cat is sitting on a couch with a remote control."

"a woman holding a teddy bear in front of a mirror."

"a horse is standing in the middle of a road."

Evaluation of text translation results

- Tricky to do automatically!
- Ideally want humans to evaluate
  - What do you ask?
  - Can't use human evaluation for validating models - too slow and expensive.
- Use standard machine translation metrics instead
  - BLEU
  - ROUGE CIDER
  - Meteor

Small interlude with explanation of BLEU score for measuring quality of predicted text.

Slides taken from Philipp Koehn's slides for chapter 8 of the book *Statistical Machine Translation*

Ten Translations of a Chinese Sentence

这个 机场 的 安全 工作 由 以色列 方面 负责 .

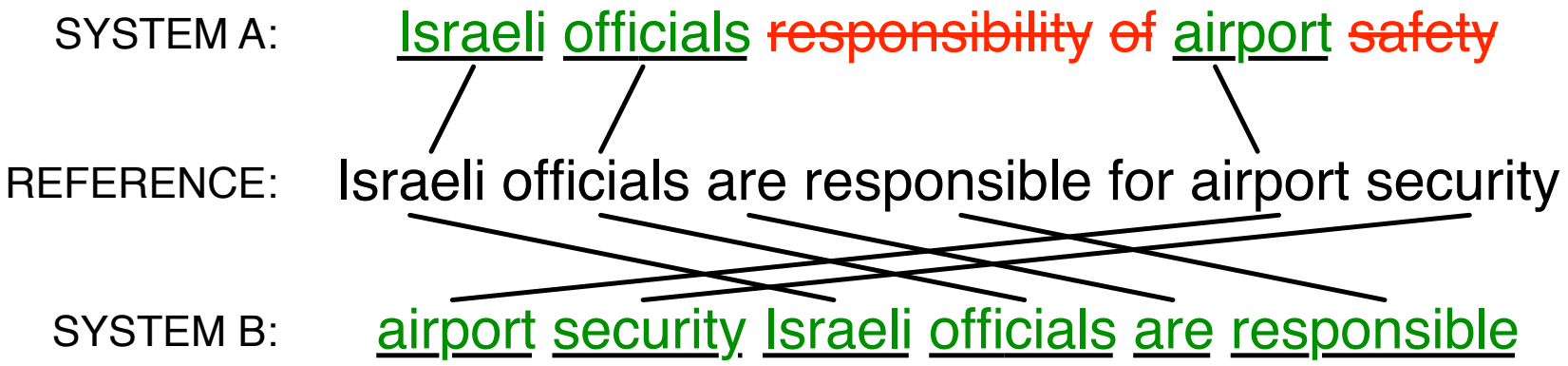
Israeli officials are responsible for airport security.
Israel is in charge of the security at this airport.
The security work for this airport is the responsibility of the Israel government.
Israeli side was in charge of the security of this airport.
Israel is responsible for the airport's security.
Israel is responsible for safety work at this airport.
Israel presides over the security of the airport.
Israel took charge of the airport security.
The safety of this airport is taken charge of by Israel.
This airport's security is the responsibility of the Israeli security officials.

(a typical example from the 2001 NIST evaluation set)

Precision and Recall of Words

- Precision $\frac{correct}{output\text{-}length} = \frac{3}{6} = 50\%$
- Recall $\frac{correct}{reference\text{-}length} = \frac{3}{7} = 43\%$
- F-measure $\frac{precision \times recall}{(precision + recall)/2} = \frac{.5 \times .43}{(.5 + .43)/2} = 46\%$

Precision and Recall



Metric	System A	System B
precision	50%	100%
recall	43%	86%
f-measure	46%	92%

flaw: no penalty for reordering

BLEU

- N-gram overlap between machine translation output and reference translation
- Compute precision for n-grams of size 1 to 4
- Add brevity penalty (for too short translations)

$$\text{BLEU} = \min\left(1, \frac{\textit{output-length}}{\textit{reference-length}}\right) \left(\prod_{i=1}^{4} \textit{precision}_i\right)^{\frac{1}{4}}$$

- Typically computed over the entire corpus, not single sentences

Example

SYSTEM A: Israeli officials (2-GRAM MATCH) responsibility of airport (1-GRAM MATCH) safety

REFERENCE: Israeli officials are responsible for airport security

SYSTEM B: airport security (2-GRAM MATCH) Israeli officials are responsible (4-GRAM MATCH)

Metric	System A	System B
precision (1gram)	3/6	6/6
precision (2gram)	1/5	4/5
precision (3gram)	0/4	2/4
precision (4gram)	0/3	1/3
brevity penalty	6/7	6/7
BLEU	0%	52%

Multiple Reference Translations

- To account for variability, use multiple reference translations
  - n-grams may match in any of the references
  - closest reference length used (usually)
- Example

SYSTEM: Israeli officials (2-GRAM MATCH) responsibility of (2-GRAM MATCH) airport (1-GRAM) safety

REFERENCES:
Israeli officials are responsible for airport security
Israel is in charge of the security at this airport
The security work for this airport is the responsibility of the Israel government
Israeli side was in charge of the security of this airport

Image Sentence Datasets

a man riding a bike on a dirt path through a forest.
bicyclist raises his fist as he rides on desert dirt trail.
this dirt bike rider is smiling and raising his fist in triumph.
a man riding a bicycle while pumping his fist in the air.
a mountain biker pumps his fist in celebration.

Microsoft COCO

[Tsung-Yi Lin et al. 2014]

mscoco.org

currently:

~120K images

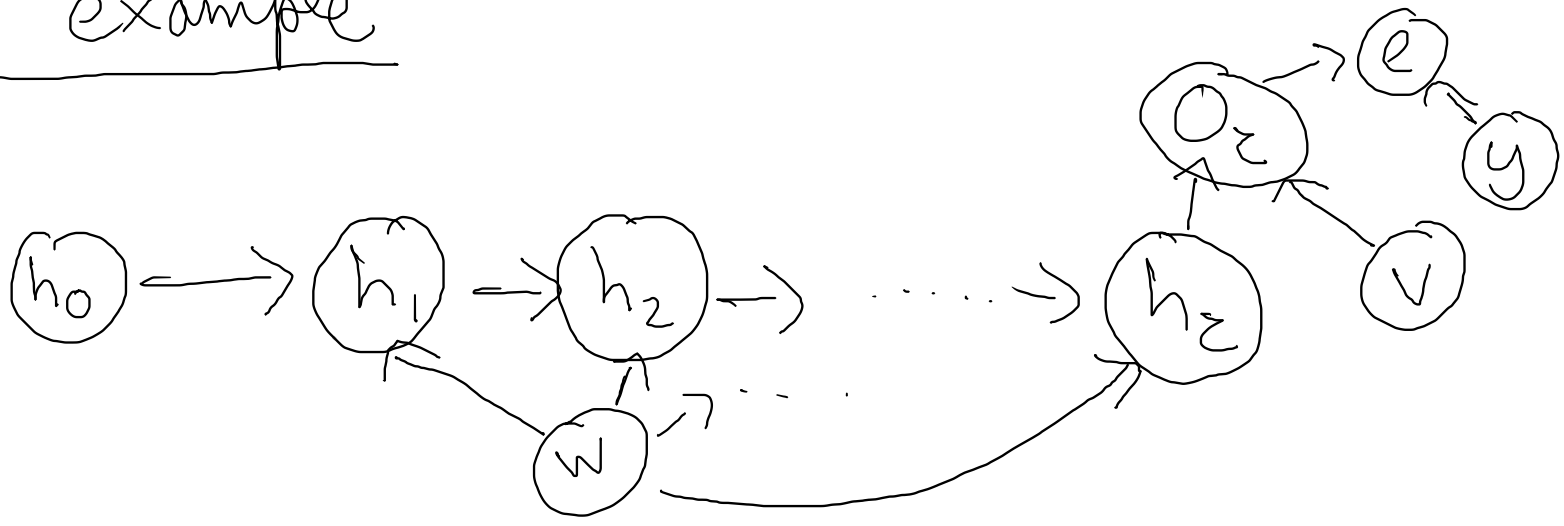
~5 sentences each

More on translations, sequence-to-sequence models, deep learning for NLP in later lectures.

Problem of exploding and vanishing gradients in an RNN

Problem of exploding & vanishing gradients in RNNs

1D example



Eqns: for $t = 1, \ldots, T$

$$h_t = \tanh(w h_{t-1})$$

end

$$o_T = v h_T, \quad e = \frac{1}{2}(o_T - y)^2$$

Eqns: $h_t = \tanh(w h_{t-1})$ for $t = 1, \ldots, T$

$$\frac{\partial \ell}{\partial w} = \sum_{t=1}^{T} \frac{\partial \ell}{\partial h_t} \frac{\partial h_t}{\partial w}$$

as $h_1, \ldots, h_T$ are the children of w

What is $\frac{\partial h_t}{\partial w}$?

$$\frac{\partial h_t}{\partial w} = h_{t-1} \tanh'(w h_{t-1})$$

what about $\frac{\partial \ell}{\partial h_t}$ for $t = T, T-1, \ldots, 1$?

Eqns: $h_t = \tanh(w h_{t-1})$ for $t = 1, \ldots, z$; $o_z = v h_z$

$$\frac{\partial \ell}{\partial h_t} = \frac{\partial \ell}{\partial h_{t+1}} \frac{\partial h_{t+1}}{\partial h_t} \quad \text{for} \quad t = 1, \ldots, z-1$$

$$\frac{\partial \ell}{\partial h_z} = \frac{\partial \ell}{\partial o_z} \frac{\partial o_z}{\partial h_z} = \frac{\partial}{\partial o_z} \left(\tfrac{1}{2} (o_z - y)^2 \right) v$$

$$= (o_z - y) v$$

$$\frac{\partial \ell}{\partial h_{z-1}} = \frac{\partial \ell}{\partial h_z} \frac{\partial h_z}{\partial h_{z-1}} = \frac{\partial \ell}{\partial h_z} w \tanh'(w h_{z-1})$$

$$\frac{\partial \ell}{\partial h_{z-2}} = \frac{\partial \ell}{\partial h_{z-1}} \frac{\partial h_{z-1}}{\partial h_{z-2}} = \frac{\partial \ell}{\partial h_{z-1}} w \tanh'(w h_{z-2})$$

$$\frac{\partial \ell}{\partial h_{\tau-2}} = (o_\tau - y) v \, w^2 \tanh'(w h_{\tau-1}) \tanh'(w h_{\tau-2})$$

$$\vdots$$

$$\frac{\partial \ell}{\partial h_t} = (o_\tau - y) v \, w^{\tau-t} \prod_{i=t}^{\tau-2} \tanh'(w h_i)$$

$$\Rightarrow \frac{\partial \ell}{\partial w} = \sum_{t=1}^{\tau} \frac{\partial \ell}{\partial h_t} \frac{\partial h_t}{\partial w} = (o_\tau - y) v \sum_{t=1}^{\tau} w^{\tau-t} \prod_{i=t}^{\tau-2} \tanh'(w h_i) \frac{\partial h_t}{\partial w}$$

How does $\left|\frac{\partial \ell}{\partial w}\right|$ vary with $|w|$?

$$\frac{\partial l}{\partial w} = (o_\tau - y) \, v \sum_{t=1}^{\tau} w^{\tau-t} \prod \ldots$$

(A) If $|w| > 1$ then $|w^{\tau-t}| \to \infty$ as $\tau - t \to \infty$

$\Rightarrow \left| \frac{\partial l}{\partial w} \right|$ could become large if τ is large

$\Rightarrow$ the gradient explodes.

(B) If $|w| < 1$ then $|w^{\tau-t}| \to 0$ as $\tau - t \to \infty$

$\Rightarrow \frac{\partial l}{\partial w} \approx (o_\tau - y) \, v \sum_{t=\tau-j}^{\tau} w^{\tau-t} \prod \ldots$ where j is small

$\Rightarrow$ gradient only depends on last few hidden states.

Focus on gradient of loss w.r.t. W

- Take a closer look at

$$\frac{\partial L}{\partial W} = \sum_{t=1}^{\tau} \mathbf{g}_t^T \mathbf{h}_{t-1}^T \quad \text{where } \mathbf{g}_t = \frac{\partial L}{\partial \mathbf{a}_t}$$

- $\implies$ $\frac{\partial L}{\partial W}$ depends on $\mathbf{h}_{t-1}$ and $\frac{\partial L}{\partial \mathbf{a}_t}$ for $t = 1, \ldots, \tau$.

- Let's take a closer look at $\frac{\partial L}{\partial \mathbf{a}_t}$....

Focus on $\frac{\partial L}{\partial \mathbf{a}_t}$

- Denote

$$\mathbf{g}_{\mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{o}_t} \quad \text{and} \quad D(\mathbf{a}_t) = \text{diag}(1 - \tanh^2(\mathbf{a}_t))$$

- Remember

$$\frac{\partial L}{\partial \mathbf{h}_\tau} = \mathbf{g}_{\mathbf{o}_\tau} V \implies \frac{\partial L}{\partial \mathbf{a}_\tau} = \mathbf{g}_{\mathbf{o}_\tau} V D(\mathbf{a}_\tau)$$

and for $t = \tau - 1, \tau - 2, \ldots, 1$:

$$\frac{\partial L}{\partial \mathbf{h}_t} = \mathbf{g}_{\mathbf{o}_t} V + \frac{\partial L}{\partial \mathbf{a}_{t+1}} W \quad \text{and} \quad \frac{\partial L}{\partial \mathbf{a}_t} = \frac{\partial L}{\partial \mathbf{h}_t} D(\mathbf{a}_t)$$

- Then you can show by recursive substitution that

$$\frac{\partial L}{\partial \mathbf{h}_t} = \sum_{j=t}^{\tau} \mathbf{g}_{\mathbf{o}_j} V \left(\prod_{k=1}^{j-t} D(\mathbf{a}_{t+k}) \right) W^{j-t}$$

and

$$\frac{\partial L}{\partial \mathbf{a}_t} = \sum_{j=t}^{\tau} \mathbf{g}_{\mathbf{o}_j} V \left(\prod_{k=0}^{j-t} D(\mathbf{a}_{t+k}) \right) W^{j-t}$$

Focus on $\frac{\partial L}{\partial \mathbf{a}_t}$

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and

$$\frac{\partial L}{\partial \mathbf{a}_t} = \sum_{j=t}^{\tau} \mathbf{g}_{\mathbf{o}_j} V \underbrace{\left(\prod_{k=0}^{j-t} D(\mathbf{a}_{t+k}) \right)}_{\text{likely has small values on diagonal}} W^{j-t}$$

Why? Each matrix $D(\mathbf{a}_{t+k})$ has $\tanh'(\mathbf{a}_{t+k})$ on its diagonal and $0 \leq \tanh'(a) \leq 1$. Thus $(\tanh'(a))^{j-t+1}$ is highly likely to have a small value even for not too large $j - t + 1$.

$\implies$ $\frac{\partial L}{\partial \mathbf{a}_t}$ only depends on first few entries in the sum.

Focus on $\frac{\partial L}{\partial \mathbf{a}_t}$

- Denote

$$\mathbf{g}_{\mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{o}_t} \quad \text{and} \quad D(\mathbf{a}_t) = \text{diag}(1 - \tanh^2(\mathbf{a}_t))$$

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and

$$\frac{\partial L}{\partial \mathbf{a}_t} = \sum_{j=t}^{\tau} \mathbf{g}_{\mathbf{o}_j} V \left(\prod_{k=0}^{j-t} D(\mathbf{a}_{t+k}) \right) \underbrace{W^{j-t}}_{\text{potentially has very large or small values}}$$

Why? .....

Focus on W^n

- Remember W has size $m \times m$.
- Assume W is diagonalizable.
- Let its eigen-decomposition be

$$W = P\Lambda P^{-1}$$

  where P is invertible and Λ is a diagonal matrix containing the eigenvalues of W.

- Then

$$W^n = P\Lambda^n P^{-1}$$

- Let $\lambda_1, \ldots, \lambda_m$ be the e-values of W. Thus
  - If $|\lambda_i| > 1 \implies \lambda_i^n$ will explode as n increases.
  - If $|\lambda_i| < 1 \implies \lambda_i^n \to 0$ as n increases.

Focus on $\frac{\partial L}{\partial \mathbf{a}_t}$

- Denote

$$\mathbf{g}_{\mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{o}_t} \quad \text{and} \quad D(\mathbf{a}_t) = \text{diag}(1 - \tanh^2(\mathbf{a}_t))$$

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and

$$\frac{\partial L}{\partial \mathbf{a}_t} = \sum_{j=t}^{\tau} \mathbf{g}_{\mathbf{o}_j} V \left(\prod_{k=0}^{j-t} D(\mathbf{a}_{t+k}) \right) \underbrace{W^{j-t}}_{\text{potentially has very large or small values}}$$

Thus for sufficiently large $j - t$ either entries in W^{j-t} can explode or vanish.

Summary on $\frac{\partial L}{\partial \mathbf{a}_t}$

- Denote

$$\mathbf{g}_{\mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{o}_t} \quad \text{and} \quad D(\mathbf{a}_t) = \text{diag}(1 - \tanh^2(\mathbf{a}_t))$$

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- If W^{j-t} explodes for $j - t > N \implies \frac{\partial L}{\partial \mathbf{a}_t}$ explodes $\implies \frac{\partial L}{\partial W}$ explodes.

- If W^{j-t} vanishes for $j - t > N$
  $\implies \frac{\partial L}{\partial \mathbf{a}_t}$ only has contributions from nearby $\mathbf{g}_{\mathbf{o}_{t'}}$ where $t \leq t' \leq t + N$
  $\implies \frac{\partial L}{\partial W}$ is based on aggregation of gradients from subsets of temporally nearby states.

Summary on $\frac{\partial L}{\partial \mathbf{a}_t}$

- Denote

$$\mathbf{g}_{\mathbf{o}_t} = \frac{\partial L}{\partial \mathbf{o}_t} \quad \text{and} \quad D(\mathbf{a}_t) = \text{diag}(1 - \tanh^2(\mathbf{a}_t))$$

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  $\implies \frac{\partial L}{\partial \mathbf{a}_t}$ only has contributions from nearby $\mathbf{g}_{\mathbf{o}_{t'}}$ where $t \leq t' \leq t + N$
  $\implies \frac{\partial L}{\partial W}$ is based on aggregation of gradients from subsets of temporally nearby states.
  $\implies$ Cannot learn long-range dependencies between states.

Solution to Exploding & Vanishing Gradients

Easy solution to exploding gradients

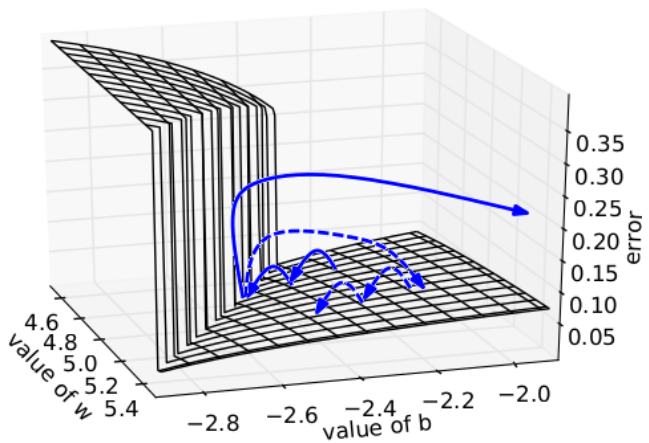
- **Gradient clipping**

  Let $G = \frac{\partial L}{\partial W}$ then

$$G = \begin{cases} \frac{\theta}{\|G\|} G & \text{if } \|G\| \geq \theta \\ G & \text{otherwise} \end{cases}$$

  where θ is some sensible threshold.

- A simple heuristic first introduced by Thomas Mikolov.



Dashed arrow shows what happens when the gradient is rescaled to a fixed size when its norm is above a threshold.

Easy partial solutions to vanishing gradients

- **Solution 1**: Initialize W as the identity matrix as opposed a random initialization.
- **Solution 2**: Use ReLU instead of $\tanh$ as the non-linear activation function.

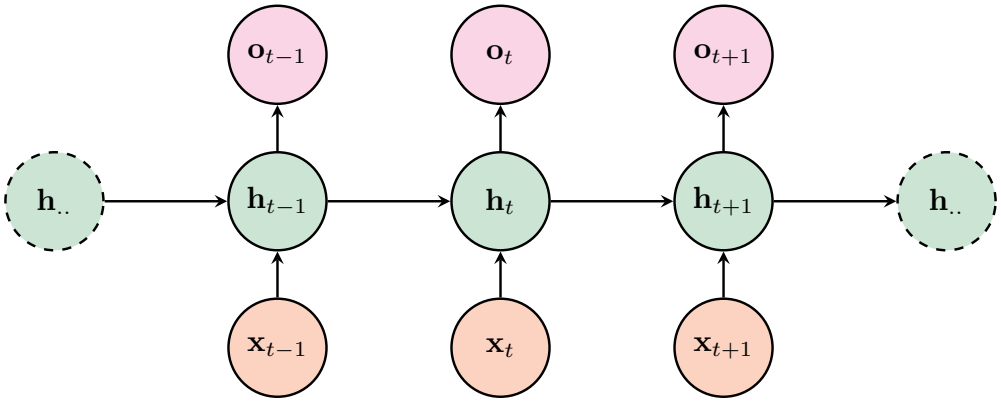
Easy partial solutions to vanishing gradients

- **Solution 1**: Initialize W as the identity matrix as opposed a random initialization.
- **Solution 2**: Use ReLU instead of $\tanh$ as the non-linear activation function.

Still hard for an RNN to capture long-term dependencies.

Long-Short-Term-Memories (LSTMs) - capturing long-range dependencies

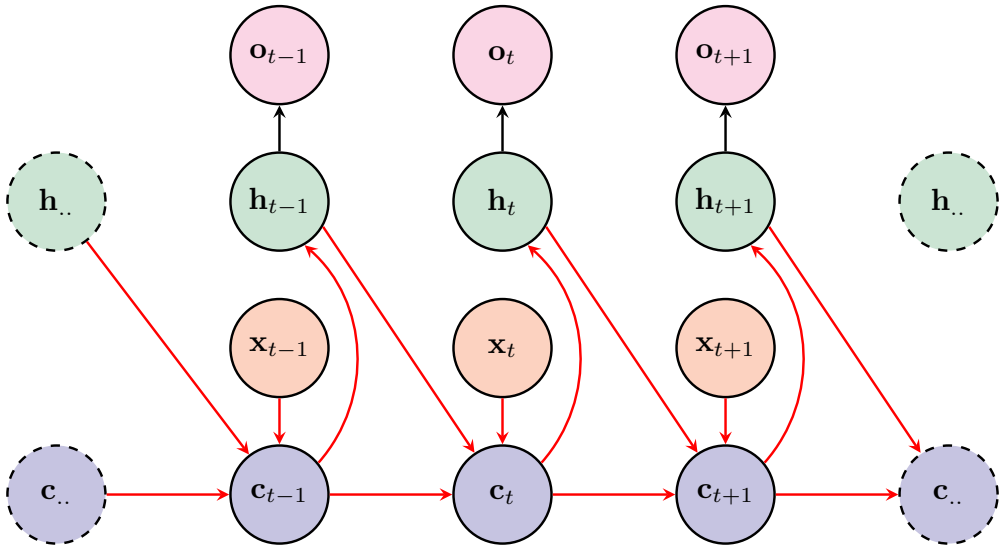
LSTMs Core Idea: Introduce a memory cell



High-level graphic of an RNN

- LSTMs similar to RNN but they introduce a memory cell state $\mathbf{c}_t$.

LSTMs Core Idea: Introduce a memory cell



High-level graphic of a LSTM

- LSTMs similar to RNN but they introduce a memory cell state $\mathbf{c}_t$.
- LSTMs have the ability to read, write & reset information to $\mathbf{c}_t$ regulated by structures called gates based on context.
- Update of $\mathbf{c}_t$ designed so gradient flows through these nodes backward in time easily.
- $\mathbf{c}_t$ then controls what information from $\mathbf{h}_{t-1}$ and $\mathbf{x}_t$ and $\mathbf{c}_{t-1}$ should be used to generate $\mathbf{h}_t$.

LSTMs formal details

- LSTMs (Hochreiter & Schmidhuer, 1997) better at capturing long term dependencies.
- Introduces gates to calculate $\mathbf{h}_t, \mathbf{c}_t$ from $\mathbf{c}_{t-1}, \mathbf{h}_{t-1}$ and $\mathbf{x}_t$.
- Formal description of a LSTM unit:

$$\mathbf{i}_t = \sigma(W_i \mathbf{h}_{t-1} + U_i \mathbf{x}_t) \quad \text{Input gate}$$
$$\mathbf{f}_t = \sigma(W_f \mathbf{h}_{t-1} + U_f \mathbf{x}_t) \quad \text{Forget gate}$$
$$\mathbf{o}_t = \sigma(W_o \mathbf{h}_{t-1} + U_o \mathbf{x}_t) \quad \text{Output/Exposure gate}$$
$$\tilde{\mathbf{c}}_t = \tanh(W_c \mathbf{h}_{t-1} + U_c \mathbf{x}_t) \quad \text{New memory cell}$$
$$\mathbf{c}_t = \mathbf{f}_t \odot \mathbf{c}_{t-1} + \mathbf{i}_t \odot \tilde{\mathbf{c}}_t \quad \text{Final memory cell}$$
$$\mathbf{h}_t = \mathbf{o}_t \odot \tanh(\mathbf{c}_t)$$

where

- $\sigma(\cdot)$ is the sigmoid function and
- $\odot$ denotes element by element multiplication.

LSTM basic unit to help with visualization

- Let

$$W_{\text{all}} = \begin{pmatrix} W_f \\ W_i \\ W_o \\ W_c \end{pmatrix} \quad U_{\text{all}} = \begin{pmatrix} U_f \\ U_i \\ U_o \\ U_c \end{pmatrix}$$

- Begin to compute all the gates and the new memory cell

$$\mathbf{a}_t = W_{\text{all}}\mathbf{h}_{t-1} + U_{\text{all}}\mathbf{x}_t$$

- Extract the individual gate vectors and the new memory cell

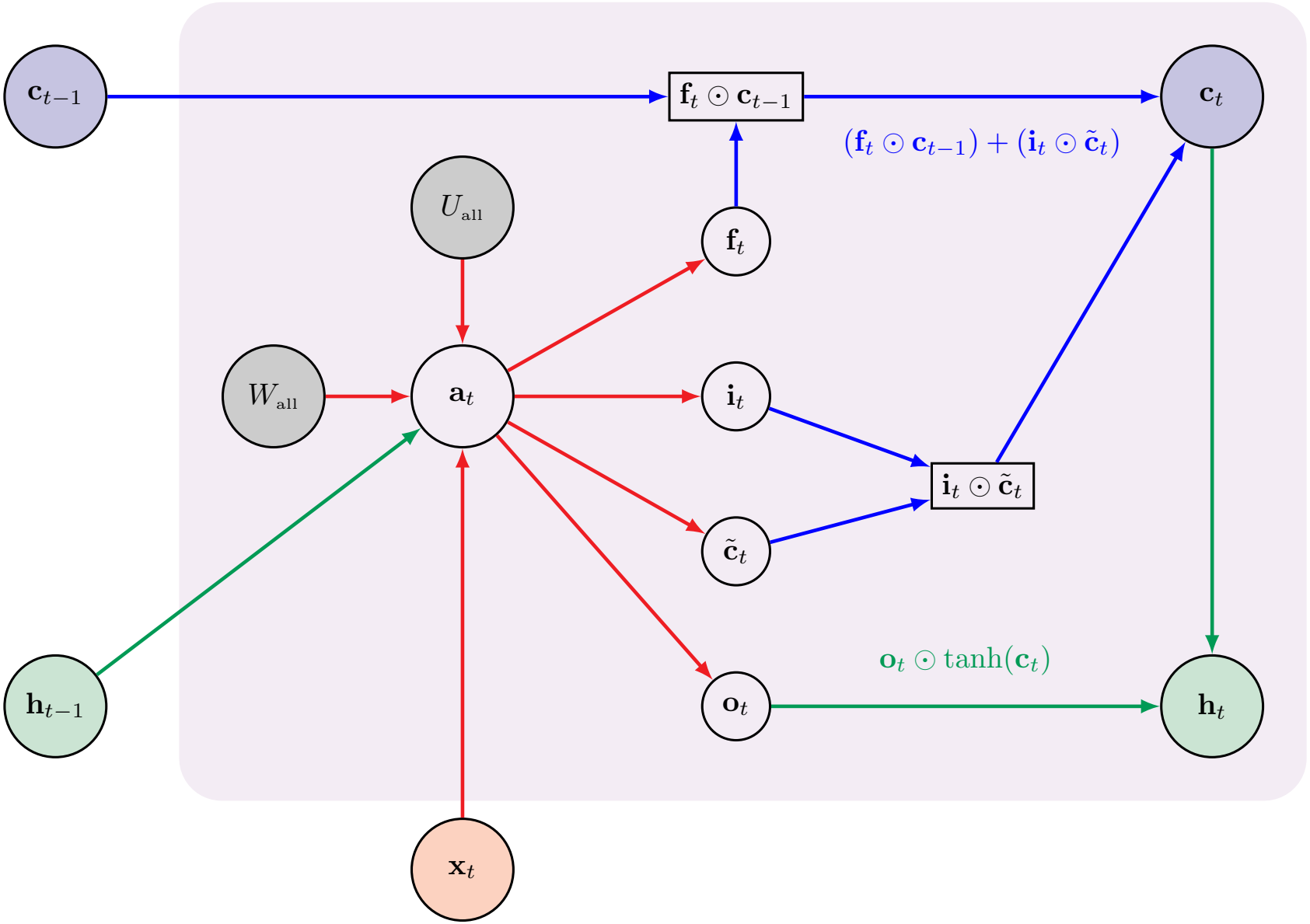
$$\begin{aligned} \mathbf{f}_t &= \sigma(E_1\mathbf{a}_t) \\ \mathbf{i}_t &= \sigma(E_2\mathbf{a}_t) \\ \mathbf{o}_t &= \sigma(E_3\mathbf{a}_t) \\ \tilde{\mathbf{c}}_t &= \tanh(E_4\mathbf{a}_t) \end{aligned}$$

where

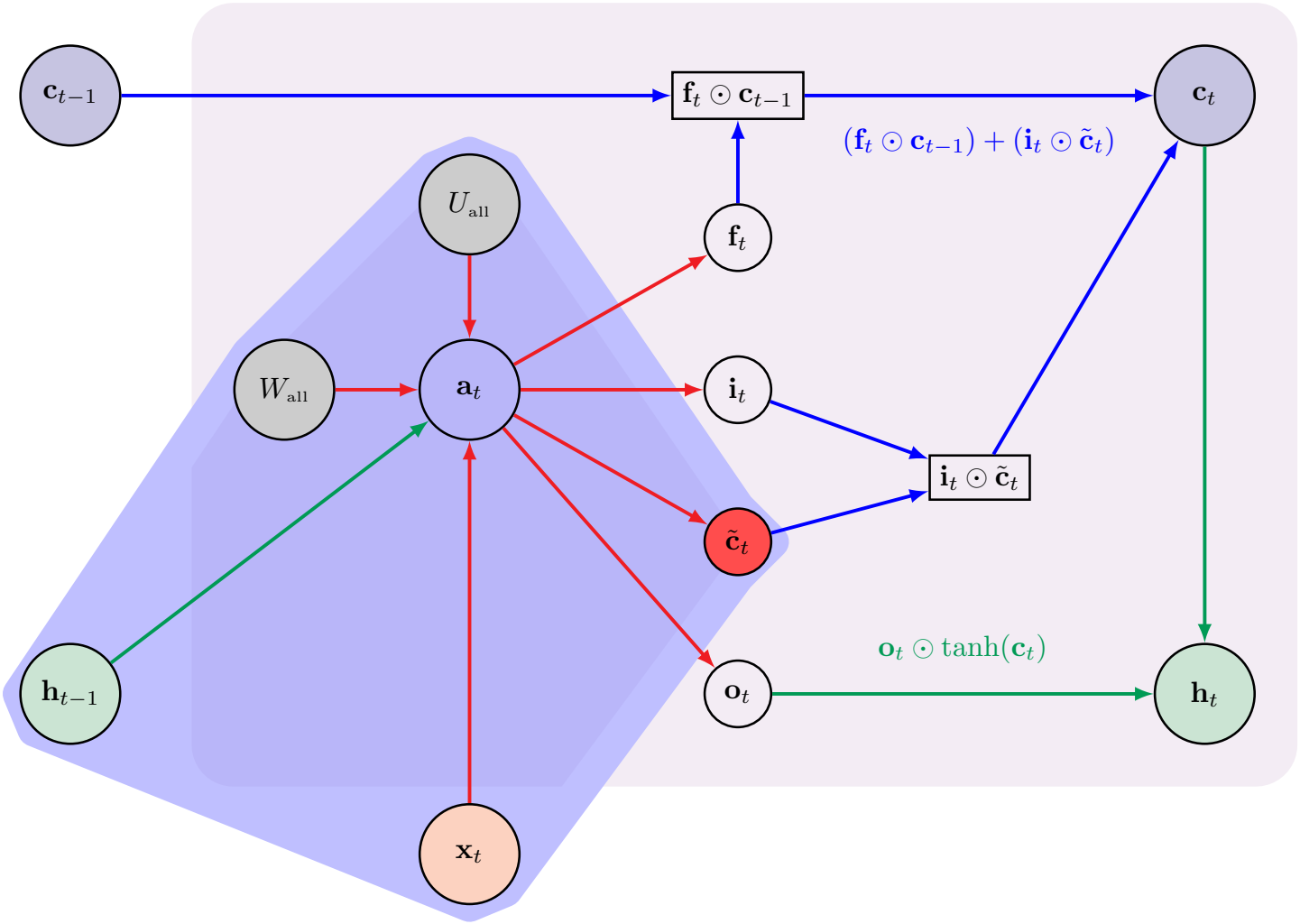
$$\begin{aligned} E_1 &= \begin{pmatrix} I_m & 0_{m\times m} & 0_{m\times m} & 0_{m\times m} \end{pmatrix}, \\ E_2 &= \begin{pmatrix} 0_{m\times m} & I_m & 0_{m\times m} & 0_{m\times m} \end{pmatrix}, \end{aligned}$$

etc

LSTMs basic unit

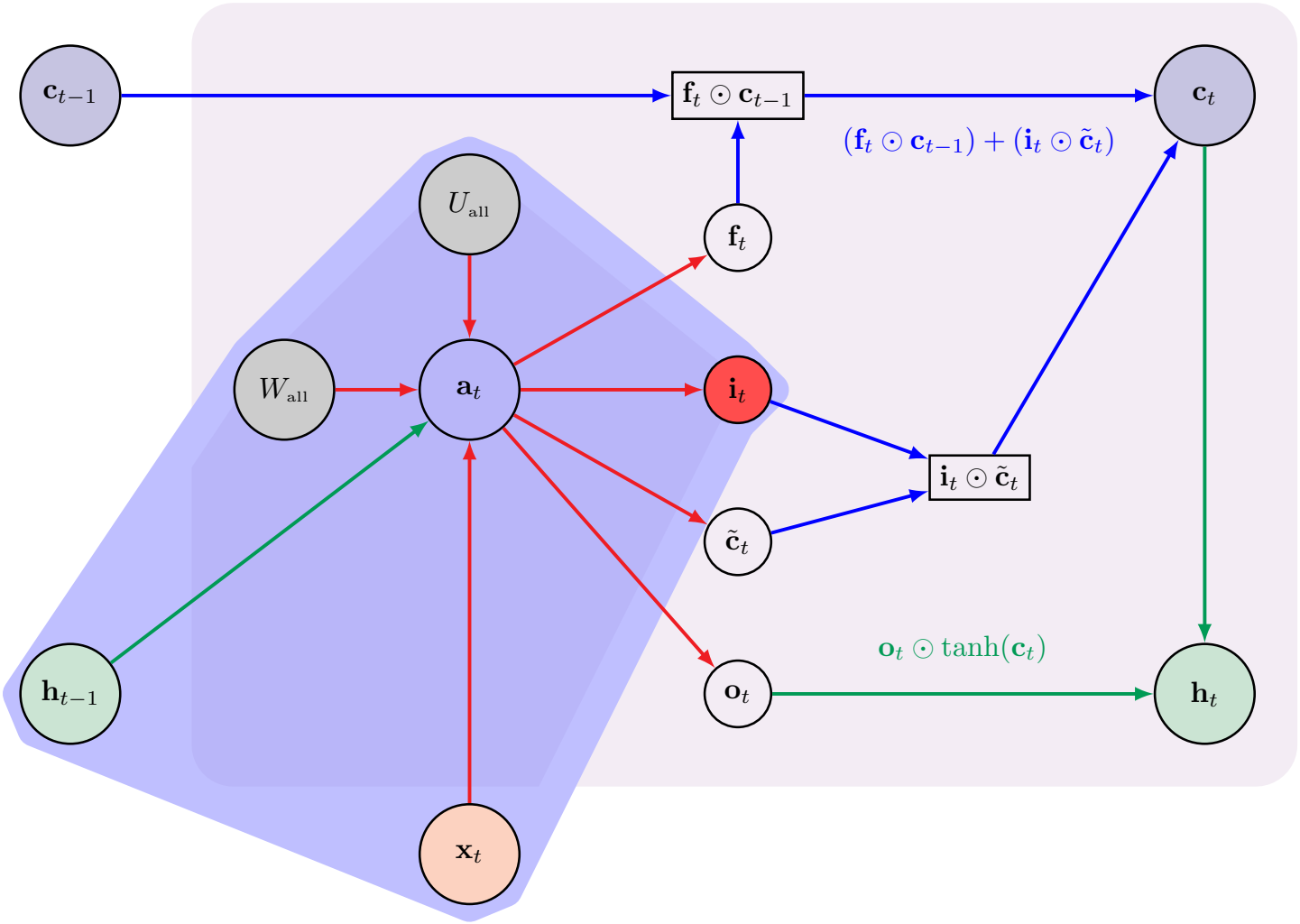


LSTMs basic unit



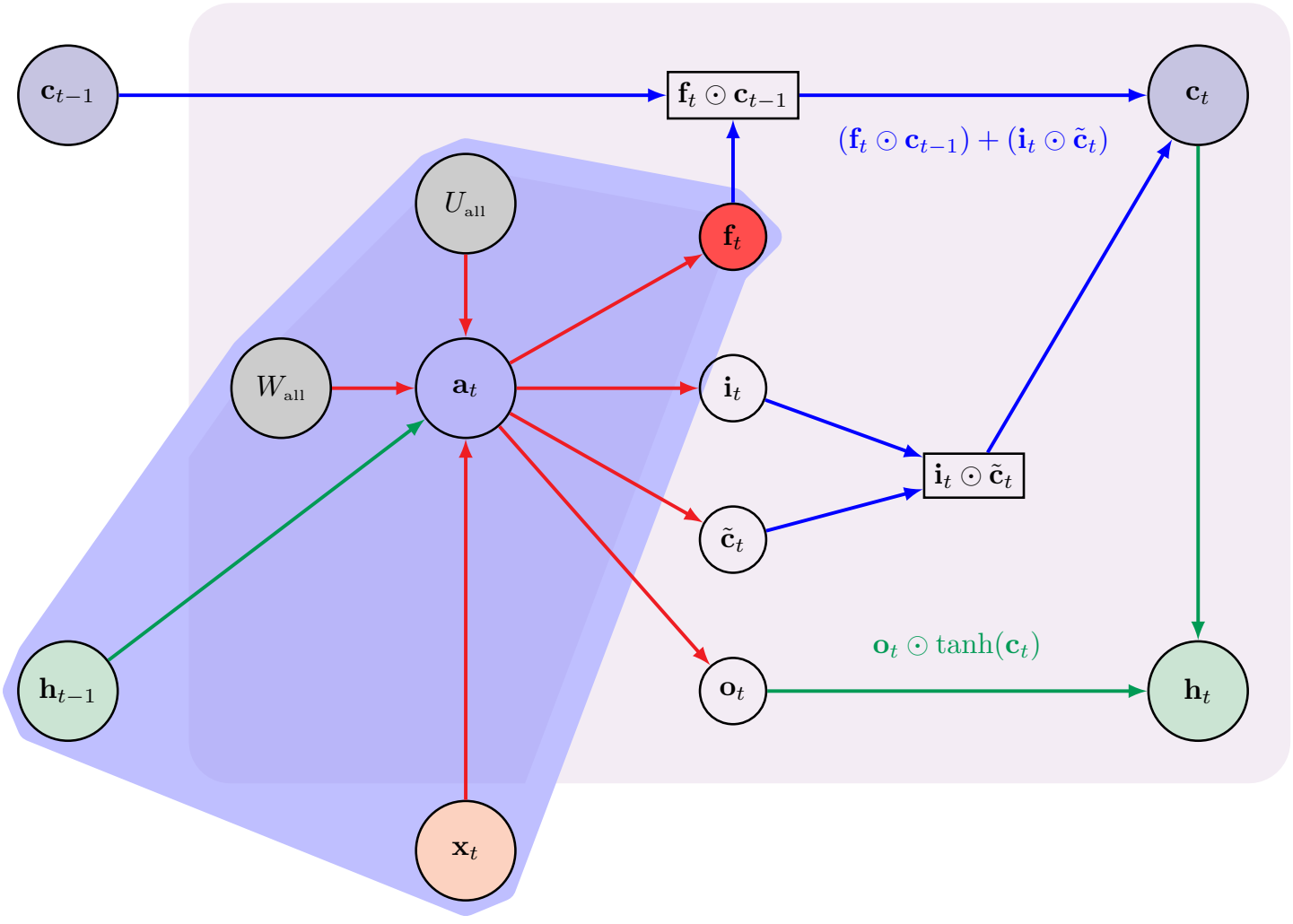
New temporary memory: Use $\mathbf{x}_t$ and $\mathbf{h}_{t-1}$ to generate new memory that includes aspects of $\mathbf{x}_t$.

LSTMs basic unit



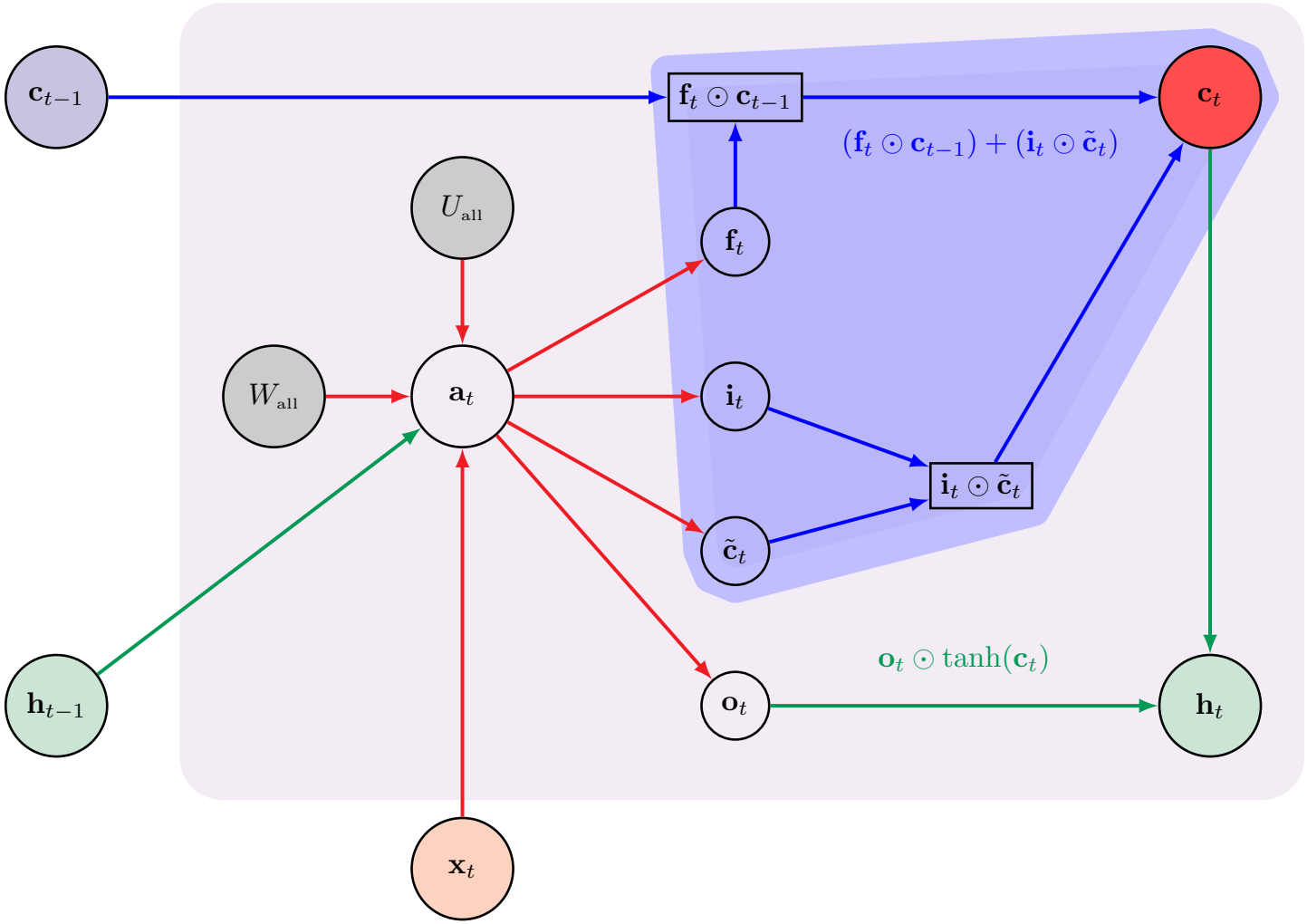
Input gate: Use $\mathbf{x}_t$ and $\mathbf{h}_{t-1}$ to determine whether the temporary memory $\tilde{\mathbf{c}}_t$ is worth preserving.

LSTMs basic unit



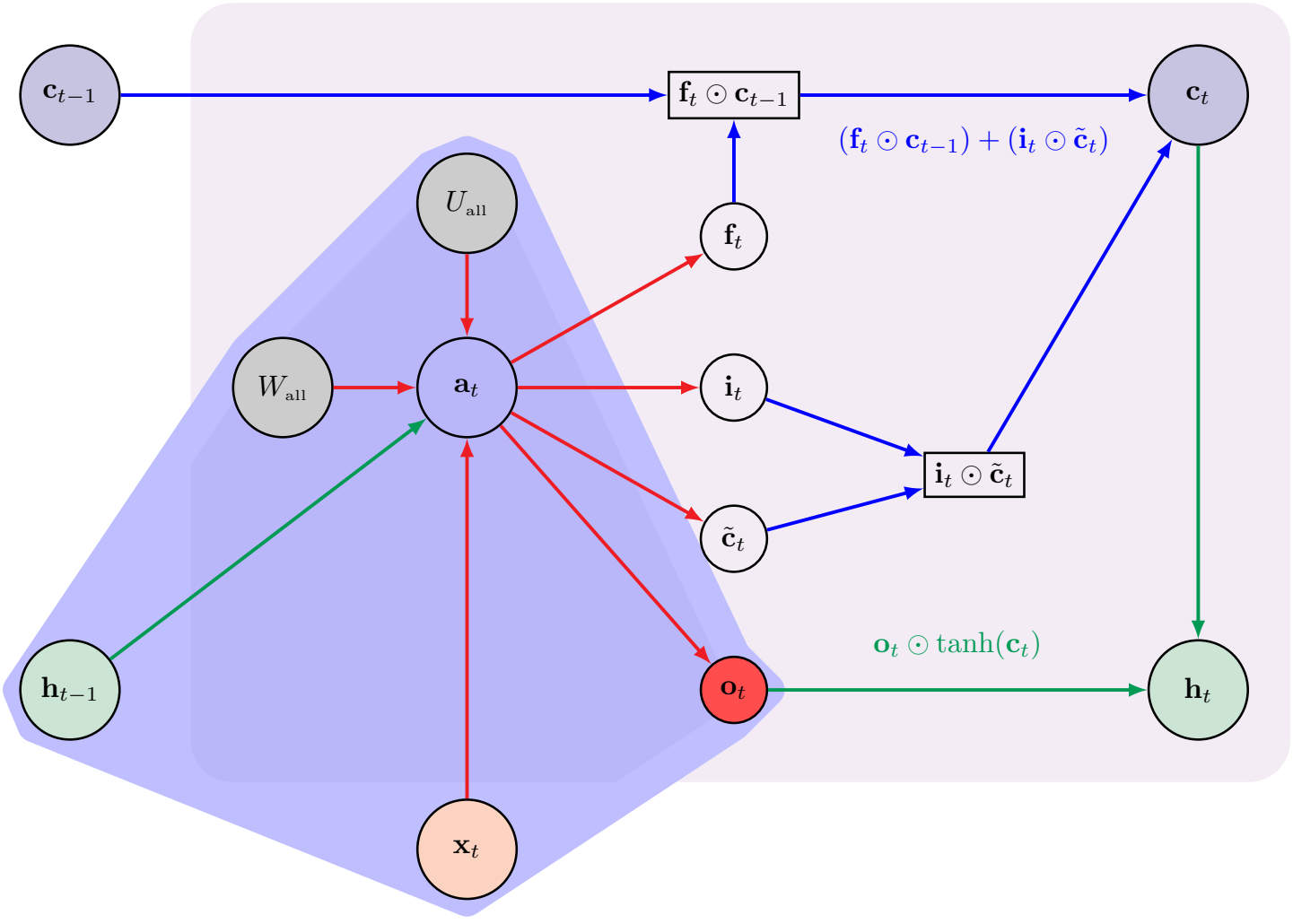
Forget gate: Assess which parts of past memory cell $\mathbf{c}_{t-1}$ should be included in $\mathbf{c}_t$.

LSTMs basic unit



Updated memory state: Use the forget and input gates to combine the new temporary memory and the current memory cell state to get $\mathbf{c}_t$.

LSTMs basic unit

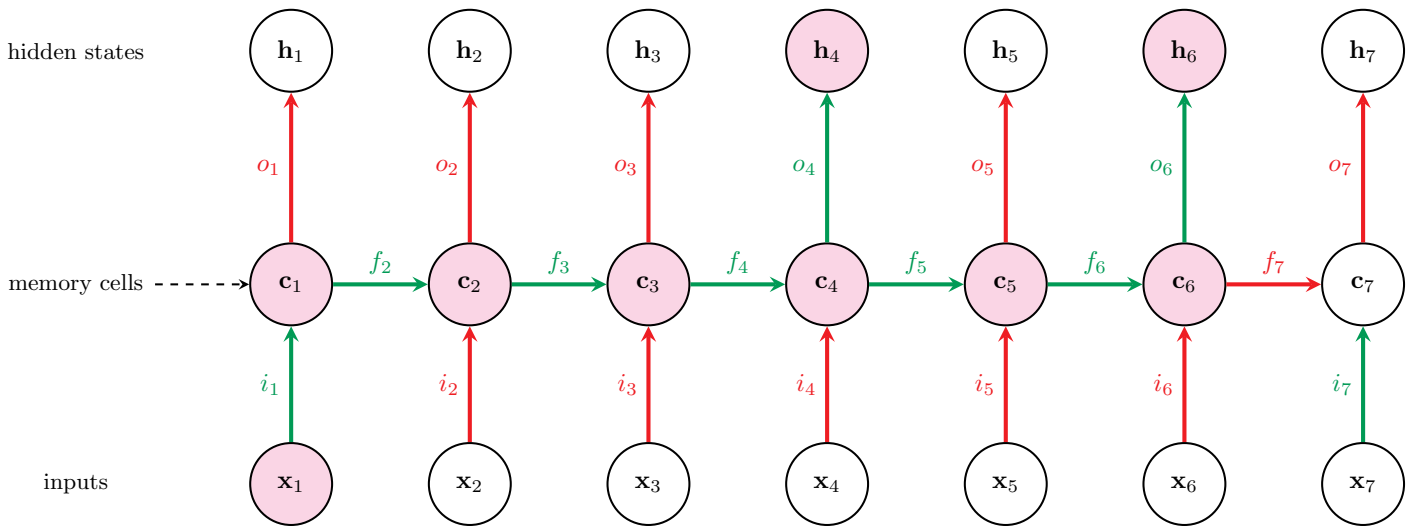


Output gate: Decides which part of $\mathbf{c}_t$ should be exposed to $\mathbf{h}_t$.

LSTM example: flow of information in LSTM

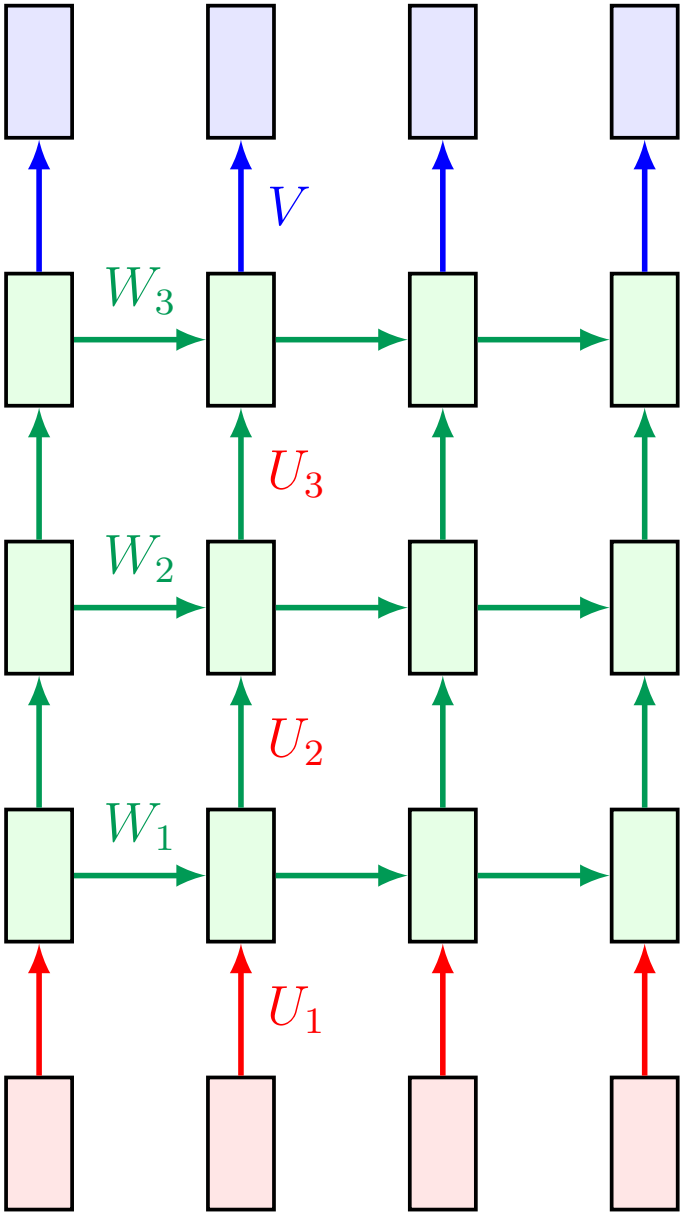
Simple example of sensitivity of memory and hidden states to input

- Colour of node indicates its sensitivity to input $\mathbf{x}_1$.
- In example gates are either entirely open or entirely closed.
- The memory cell remembers $\mathbf{x}_1$ if the forget gate is open and the input gate from subsequent inputs is closed.
- Sensitivity of hidden states can be switched on and off by the output gate without affecting the memory cell.



Can go deep with LSTMs & RNNs

Multi-layer RNN



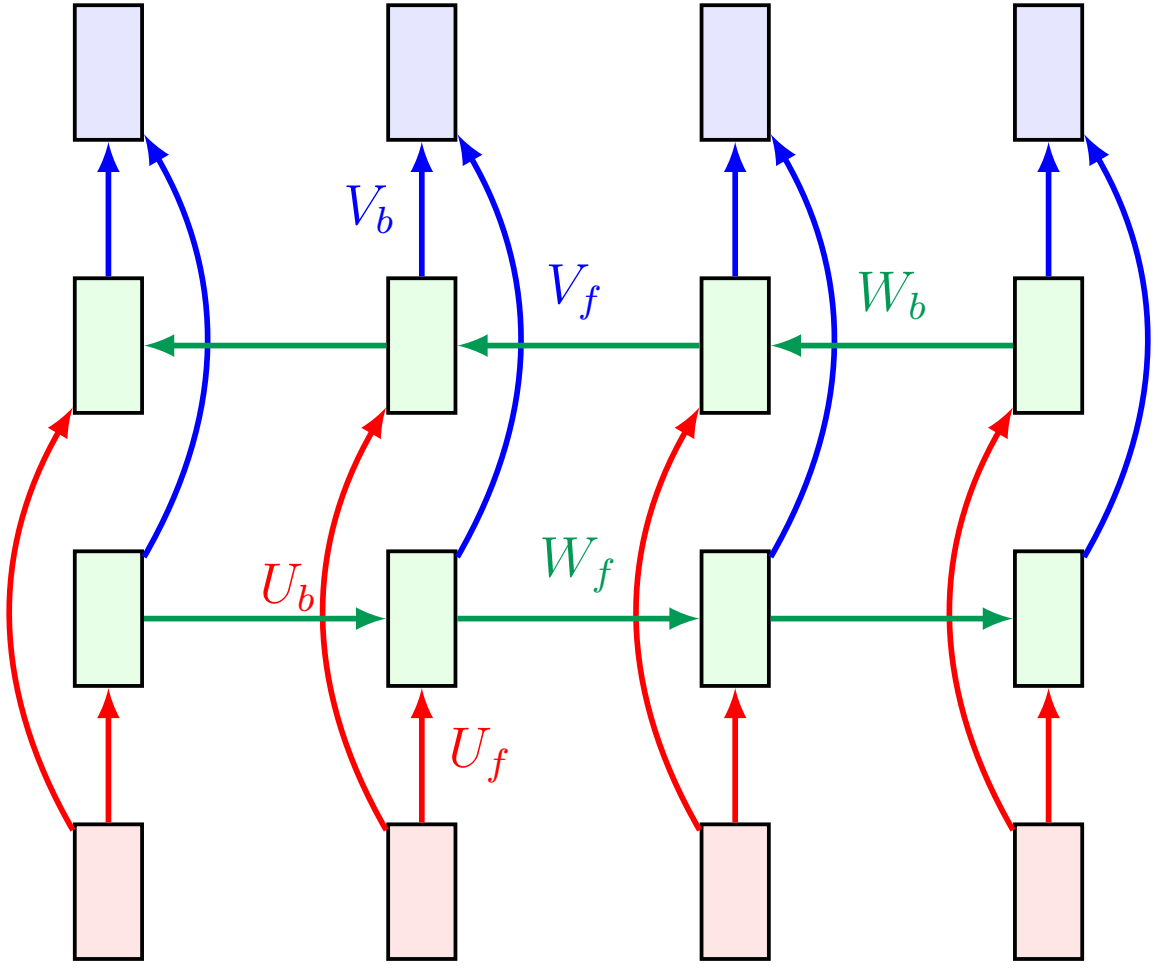
for $l = 1, 2, 3$

$$\mathbf{h}_t^{(l)} = \tanh\left(W_l \mathbf{h}_{t-1}^{(l)} + U_l \mathbf{h}_t^{(l-1)} + \mathbf{b}_l\right)$$

where $h_t^{(0)} = \mathbf{x}_t$

Can go forward & backward in time with LSTMs & RNNs

Bidirectional RNN



- Have separate RNNs in each direction and then jointly make prediction.

Summary

- RNNs allow a lot of flexibility in architecture design
- Backward flow of gradients in RNN can explode or vanish.
- Vanilla RNNs are simple but find it hard to learn long-term dependencies.
- Exploding is controlled with gradient clipping. Vanishing is controlled with additive interactions (LSTM)
- Common to use LSTMs: their additive interactions improve gradient flow