DD2380 Artificial Intelligence

Hidden Markov Models (HMMs)

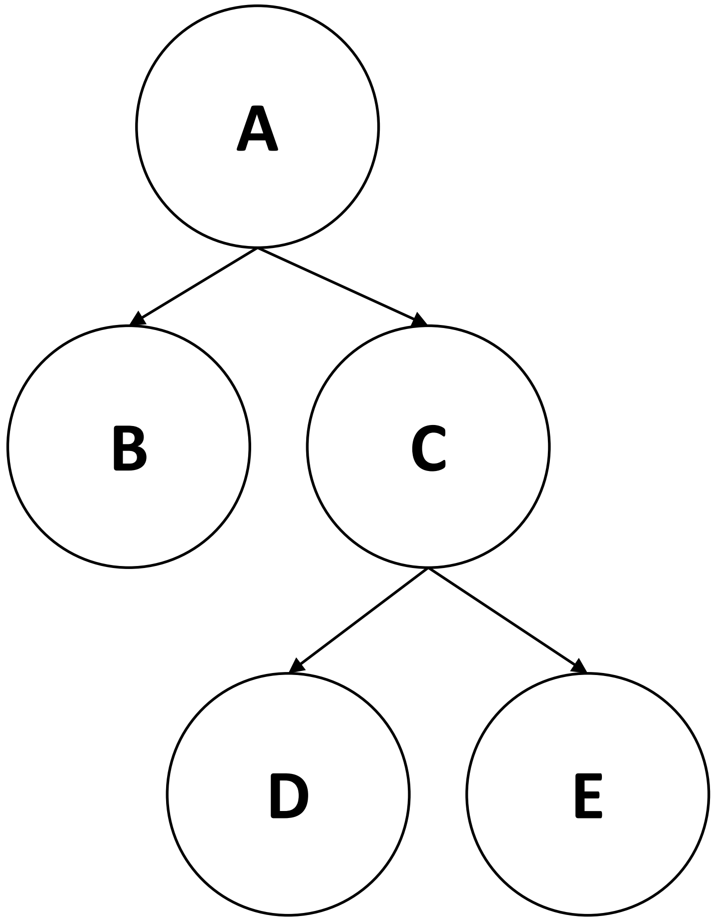
André Pereira (Starts at 15:15)

Reading Instructions

- Chapters 13-15, Russel & Norvig
- Stamp tutorial on HMMs on the course web page

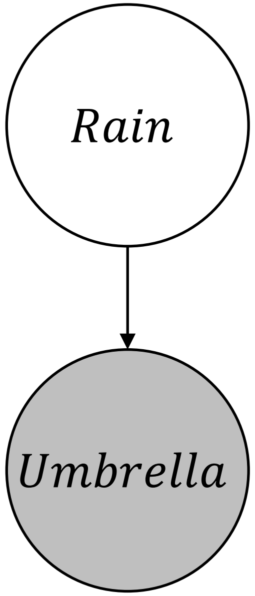
Bayesian Networks

- Aka Probabilistic **Directed Acyclic** Graphical Model
- Represents joint probability distribution (in a compact manner)
  - Helps with analyzing probability information
  - Helps with structuring probability information
- Arrow → "direct influence over"
  - A has direct influence over B
- Each arrow is accompanied with a conditional probability distribution, e.g., $P(B|A)$
- Very hard to gather data to build a model for $P(A, B, C, D, E)$, much easier to look at conditional probabilities such as $P(B|A)$
- Factorization
  - $P(X_1, X_2, \dots, X_n) = \prod_{i=1}^{i=n} P(X_i|Parents(X_i))$

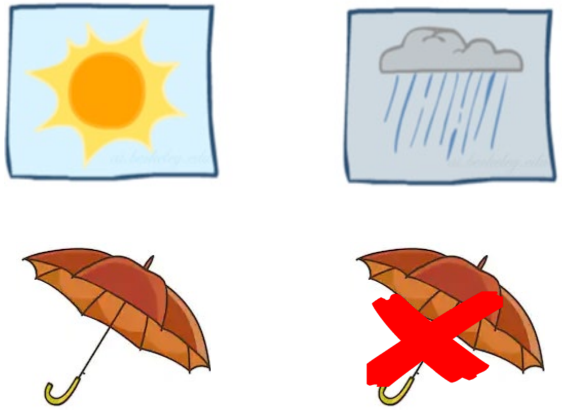


Example: Umbrella world

- A person tries to infer $rain = \{true, false\}$ by observing if a certain person has an $umbrella = \{true, false\}$ that day.
- Draw Bayesian network!
  - What are the variables?
  - Which cause which?



We cannot directly observe if it rains.
This is **hidden** to us.
We observe umbrella and infer if it rains or not
Rain causes the person to use an umbrella.

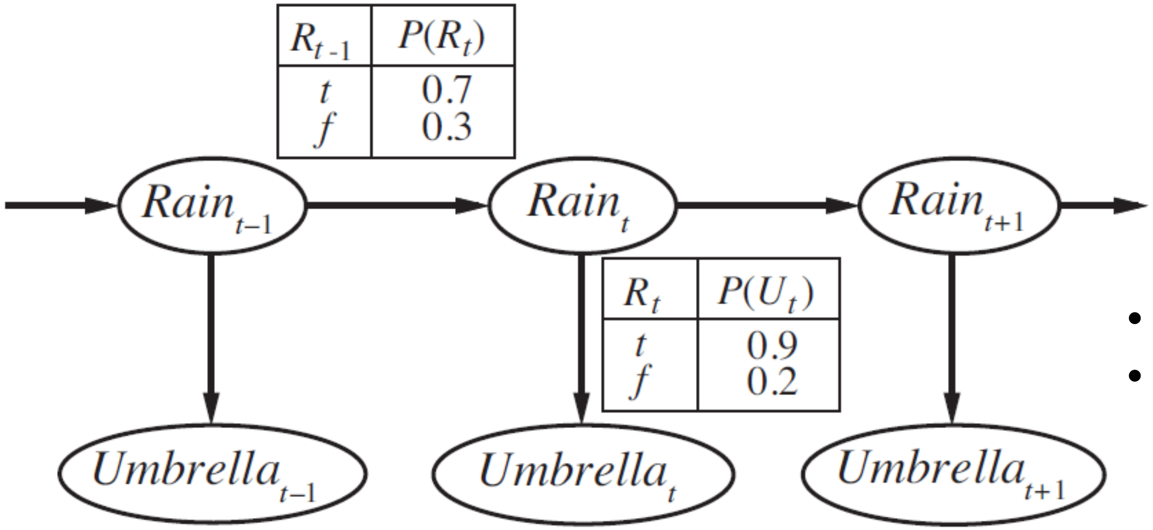


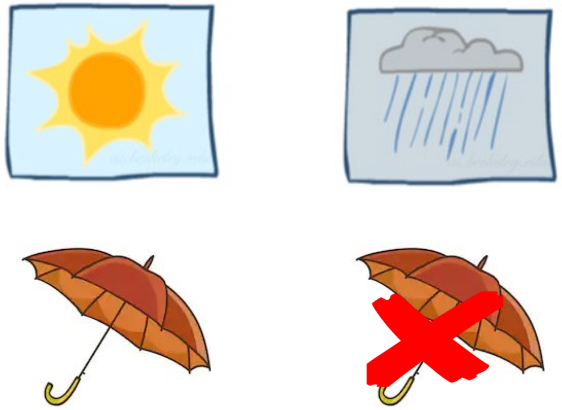
We often cannot directly observe the state!

- Other examples:
  - Cannot observe the weather, only the temperature
    - State = weather, observation = temperature
  - Cannot observe the words spoken, only the sound uttered
    - State = word, observation = sound
  - Cannot observe the position of the car only the laser scanner readings
    - State = position, observation = laser data

Example: Sequential Umbrella world

- A person tries to infer $rain = \{true, false\}$ **every day** by observing if a certain person has an $umbrella = \{true, false\}$ over consecutive days.
- Draw the Bayesian network that corresponds to the **time sequence**
  - What additional assumptions did you make?





- The observation depends directly on the state.
- Additional assumption
  - The next state depends only on the previous state (first order Markov assumption).

Markov Model

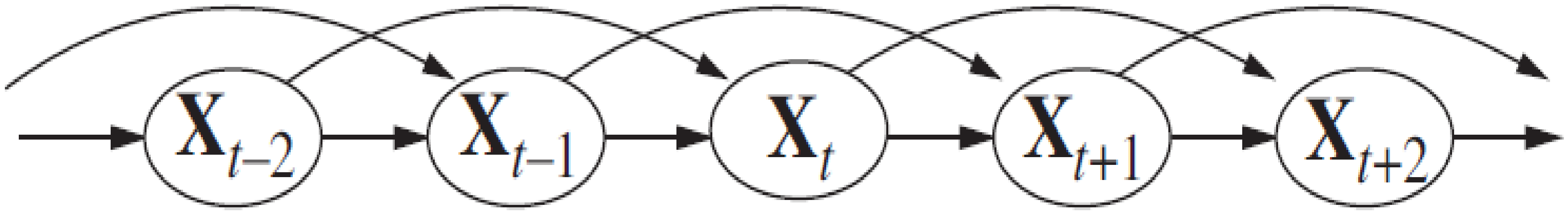
- The (first order) Markov assumption

- The distribution $P(X_t)$ depends only on the distribution $P(X_{t-1})$
  - $p(X_t|X_{t-1}, X_{t-2}, X_{t-3}, \dots) = p(X_t|X_{t-1})$

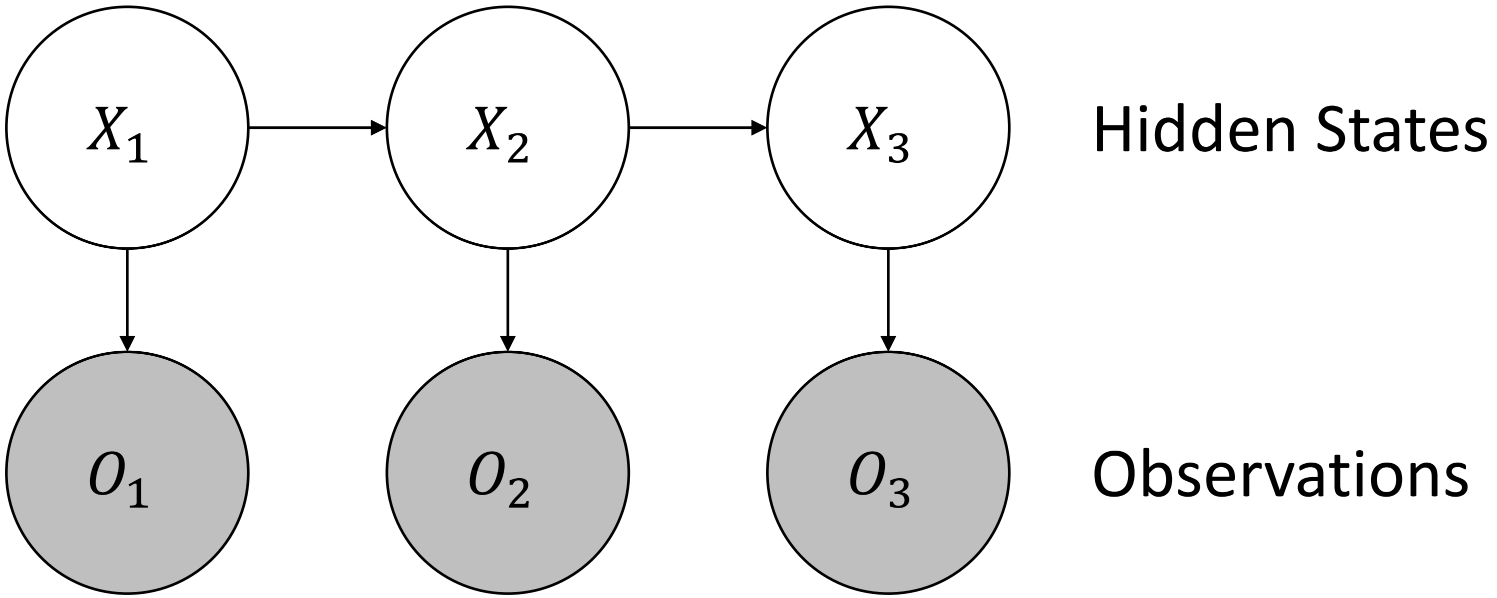
- The present (current state) can be predicted using local knowledge of the past (state at the previous step)

Second-order Markov Model

- State at time k depends on the states at times k-1 and k-2

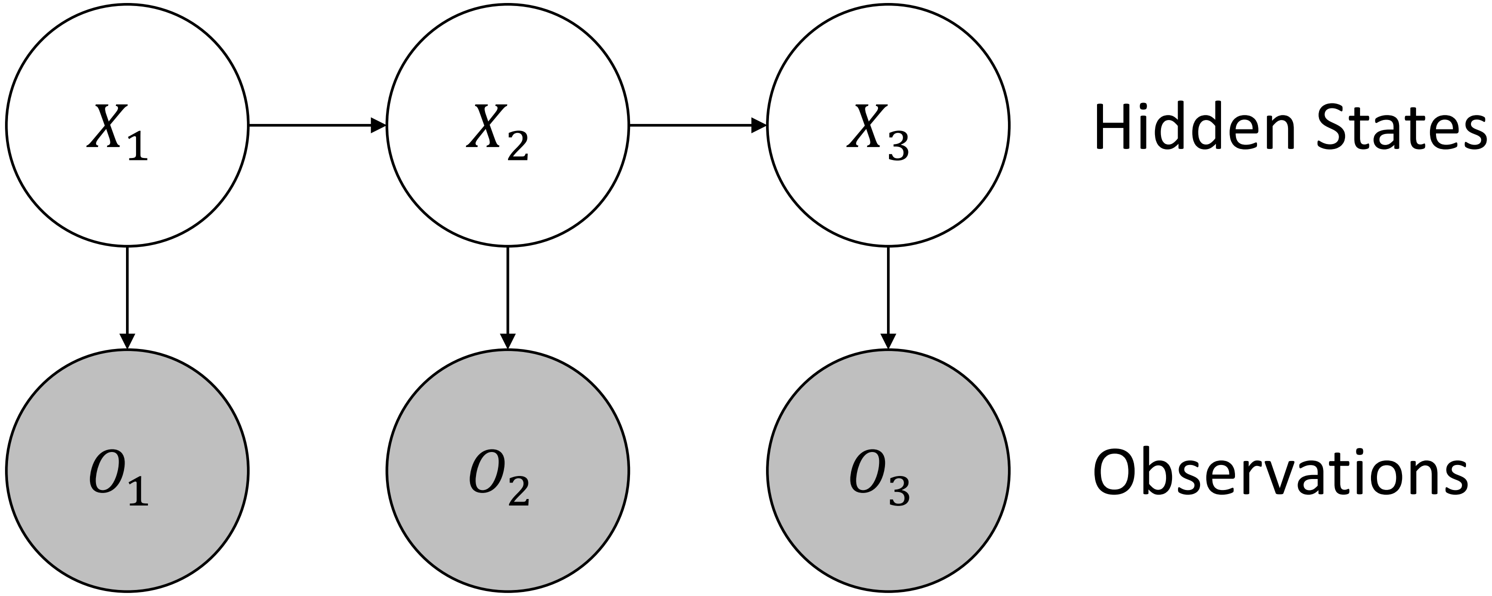


Hidden Markov Models (HMMs)



- Two important (conditional) independence properties:
  - Markov hidden process: future depends on the past via the present
  - Current observation independent of all else given current state

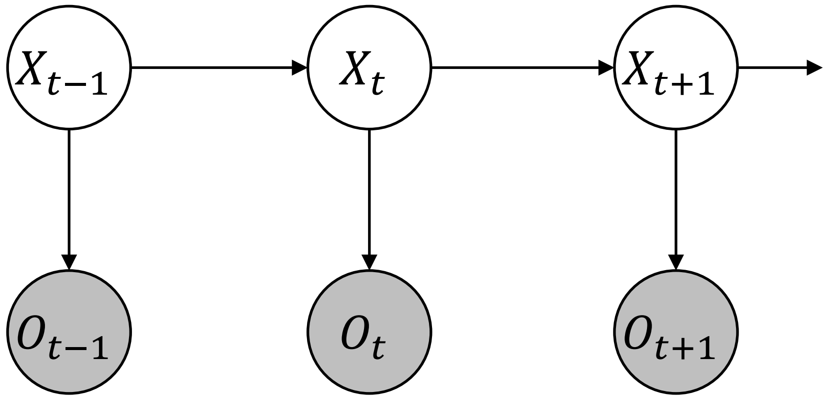
Hidden Markov Models (HMMs)

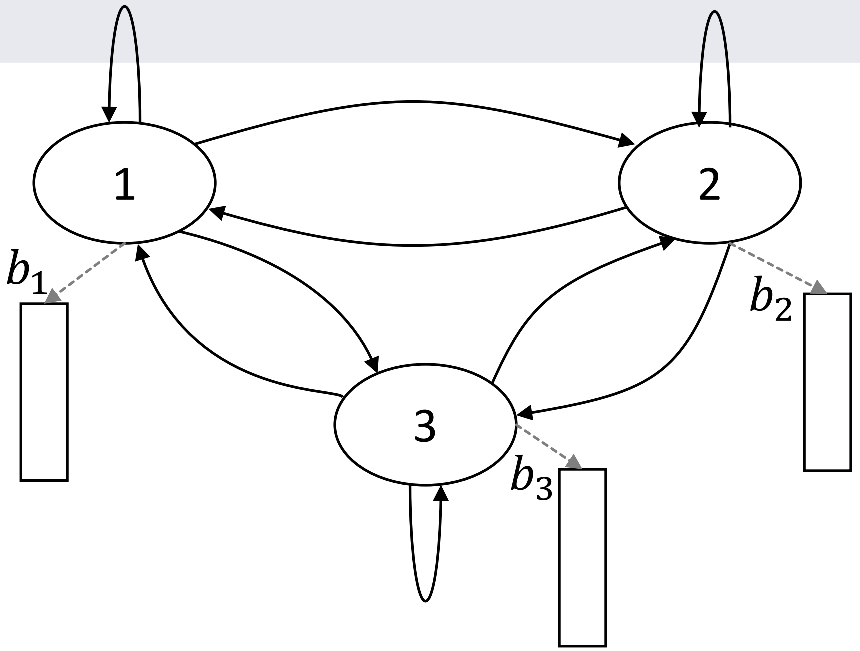


- State transition model: $\mathrm{P}(X_t = j | X_{t-1} = i) = A(i, j) = a_{ij}$
- Observation model: $\mathrm{P}(O_t = j | X_t = i) = b_{ij}$

BN vs State Machine Representation

	Bayesian Network Notation	State Machine Representation
Arrows	Dependency between variables	State transitions and observation probs
Circles	Variables (states at each time step)	The different states



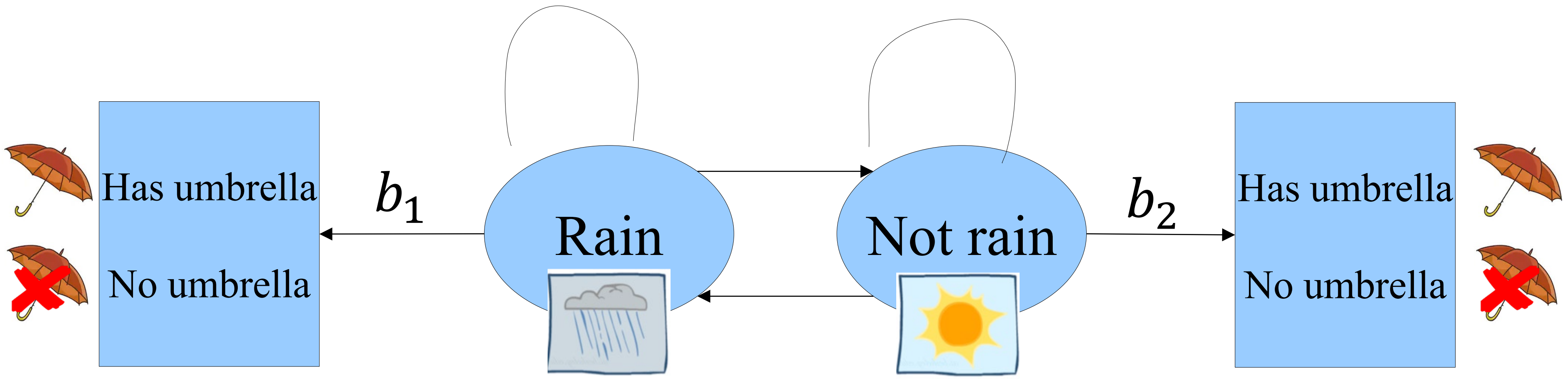


Example: Sequential Umbrella world

- A person tries to infer $rain = \{true, false\}$ **every day** by observing if a certain person has an $umbrella = \{true, false\}$ over consecutive days.
- Formulate using a State Machine Representation
  - How many states? Which?

    Rain = true, Rain = False
  - How many observations? Which?

    Umbrella = true, Umbrella = false



Elements of HMM

1. Number of states $N, x \in \{1, \dots, N\}$

2. Number of events $K, k \in \{1, \dots, K\}$

3. Initial State Probabilities
   $\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \leq i \leq N$

4. State-transition probabilities
   $\mathrm{A} = \{a_{ij}\} = \{\mathrm{P}(x_t = j | x_{t-1} = i)\}$ for $1 \leq i, j \leq N$

5. Discrete Output Probabilities
   $\mathrm{B} = \{b_i(k)\} = \{\mathrm{P}(o_t = k | x_t = i)\}$ for $1 \leq i \leq N$ and $1 \leq k \leq K$

Elements of HMMs, $\boldsymbol{\pi}$

- Initial Distribution
  - Contains the probability of the (hidden) model being in a particular hidden state at time t = 1 (sometimes t=0).

- Often referred to as $\boldsymbol{\pi}$

- Example:

$\pi = [0.5\ 0.2\ 0.3]$, i.e.,
$p(X_1 = 1) = 0.5$
$p(X_1 = 2) = 0.2$
$p(X_1 = 3) = 0.3$

Elements of HMMs, A

- State transition matrix A
  - holding the probability of transitioning from one hidden state to another hidden state.
- a_{21} is represented by $p(X_{t+1} = 1 | X_t = 2)$,
  - i.e., probability to transition from state 2 to state 1

$$
\begin{bmatrix}
 & X_{t+1} = 1 & X_{t+1} = 2 & \dots & X_{t+1} = N \\
X_t = 1 & a_{11} & a_{12} & \dots & a_{1N} \\
X_t = 2 & a_{21} & a_{22} & \dots & a_{2N} \\
\dots & \dots & \dots & \dots & \dots \\
X_t = N & a_{N1} & a_{N2} & \dots & a_{NN}
\end{bmatrix}
$$

Elements of HMMs, B

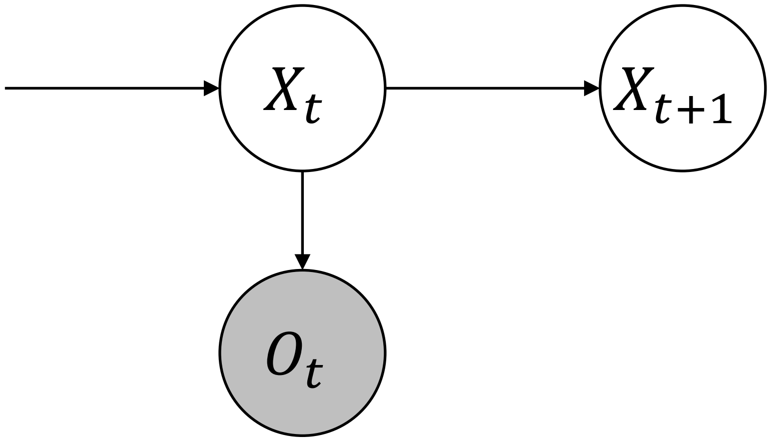
- Output matrix B
  - probability of observing a particular measurement given that the hidden model is in a particular hidden state.
- $b_i(O_t = j)$ is the probability to observe j in state i

$$
\begin{bmatrix}
 & O_t = 1 & O_t = 2 & \dots & O_t = K \\
X_t = 1 & b_1(1) & b_1(2) & \dots & b_1(K) \\
X_t = 2 & b_2(1) & b_2(2) & \dots & b_2(K) \\
\dots & \dots & \dots & \dots & \dots \\
X_t = N & b_N(1) & b_N(2) & \dots & b_N(K)
\end{bmatrix}
$$

The model - λ

- λ sometimes just called M, is the model, i.e.,
- $\lambda = (A, B, \pi)$
  - A: state transition matrix
  - B: observation matrix, output matrix, emission probabilities, emissions
  - π: initial state distribution
- A, B and π are row-stochastic matrices (their rows sum to 1)

Prediction in HMMs



- Assume we have current belief P(X | evidence to date)

  $P(X_t \mid O_{1:t})$

- Then, after one time step passes:

$$p(X_{t+1} \mid O_{1:t}) = \{sum\ rule\} = \sum_{X_t} p\,(X_{t+1}, X_t \mid O_{1:t})$$

$$p(X) = \sum_{Y} P(X,Y)$$

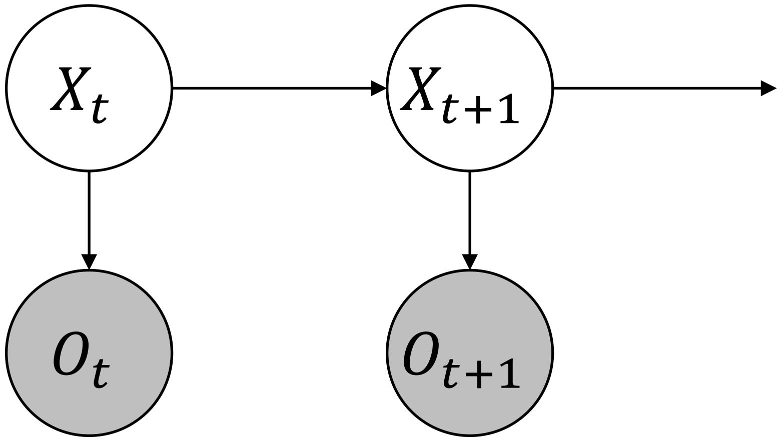
$$= \{product\ rule\} = \sum_{X_t} p\,(X_{t+1} \mid X_t\,, O_{1:t})\ p(X_t \mid O_{1:t})$$

$$P(A,B) = P(A \mid B)P(B)$$
$$P(A,B \mid C) = P(A \mid B,C)P(B|C)$$

$$= \{O_{1:t}\ cond.indep\ of\ X_{t+1}\ given\ X_t\} = \sum_{X_t} p\,(X_{t+1} \mid X_t)\ p(X_t \mid O_{1:t})$$

- Basic idea: beliefs get “pushed” through the transitions

Measurements/observations



- Assume we have current belief P(X | previous evidence)

$$p(X_{t+1} \mid O_{1:t})$$

- Then, after evidence comes in:

$$p(X_{t+1} \mid O_{1:t+1}) = \{\text{"}split\text{"}\ O\} = p(X_{t+1} \mid O_{t+1}, O_{1:t})$$

$$= \{Bayes\ rule, \text{conditioned denominator}\} = \frac{p(O_{t+1} \mid X_{t+1}, O_{1:t})\, p(X_{t+1} \mid O_{1:t})}{\sum_{X_{t+1}} p\,(O_{t+1} \mid X_{t+1}, O_{1:t})\, p(X_{t+1} \mid O_{1:t})}$$

$$P(B|A) = \frac{P(A|B)P(B)}{\sum_B P(A|B)P(B)}$$

$$P(B|A,C) = \frac{P(A|B,C)P(B|C)}{\sum_B P(A|B,C)P(B|C)}$$

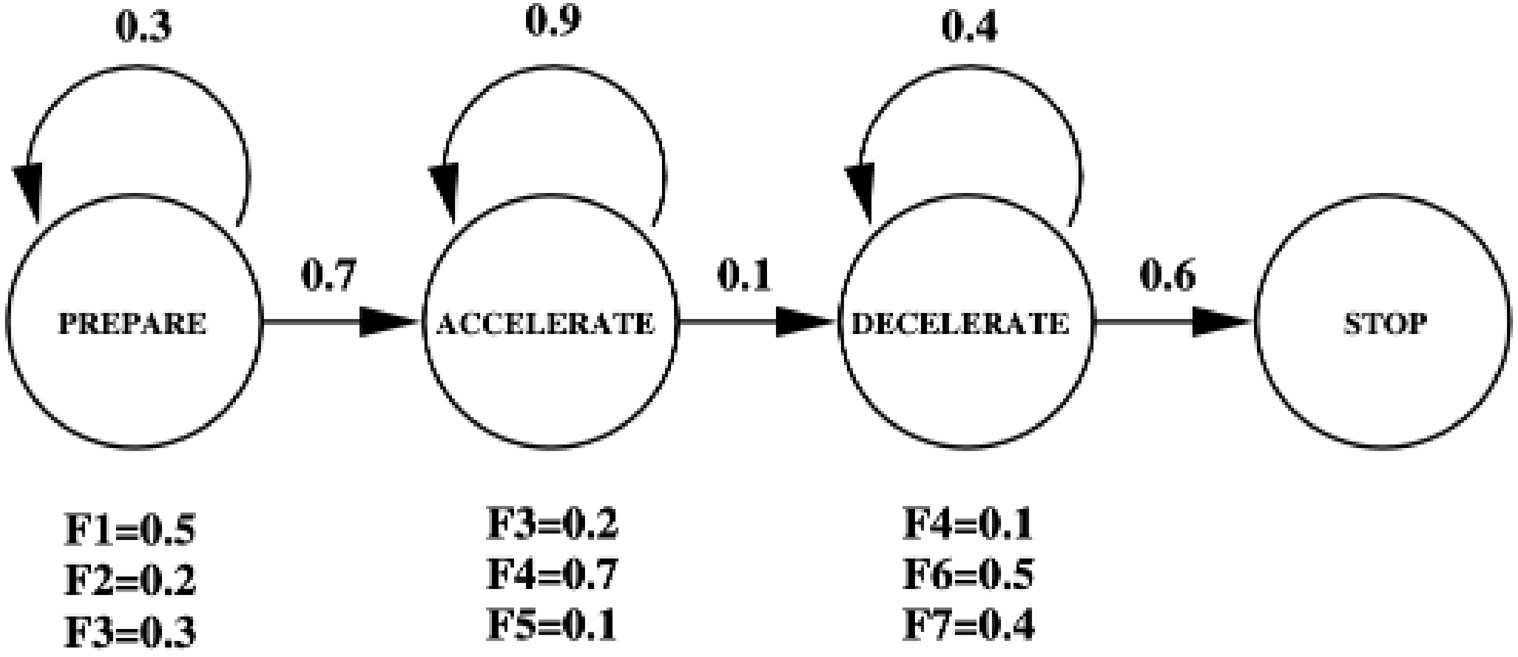
$$= \{\ O_{t+1} \text{ cond. } indep \text{ of } O_{1:t}\ given\ X_{t+1}\} = \frac{p(O_{t+1} \mid X_{t+1})\, p(X_{t+1} \mid O_{1:t})}{\sum_{X_{t+1}} p\,(O_{t+1} \mid X_{t+1}\,)\, p(X_{t+1} \mid O_{1:t})}$$

Example: Usain Bolt

- You have developed an automatic video annotation system for annotating recorded running sequences of Usain Bolt. The system takes images from a video stream of a running sequence as an input, it extracts some visual data and annotates each image as:
  - Usain is preparing for running
  - Usain runs/accelerates
  - Usain decelerates

Example: Usain Bolt

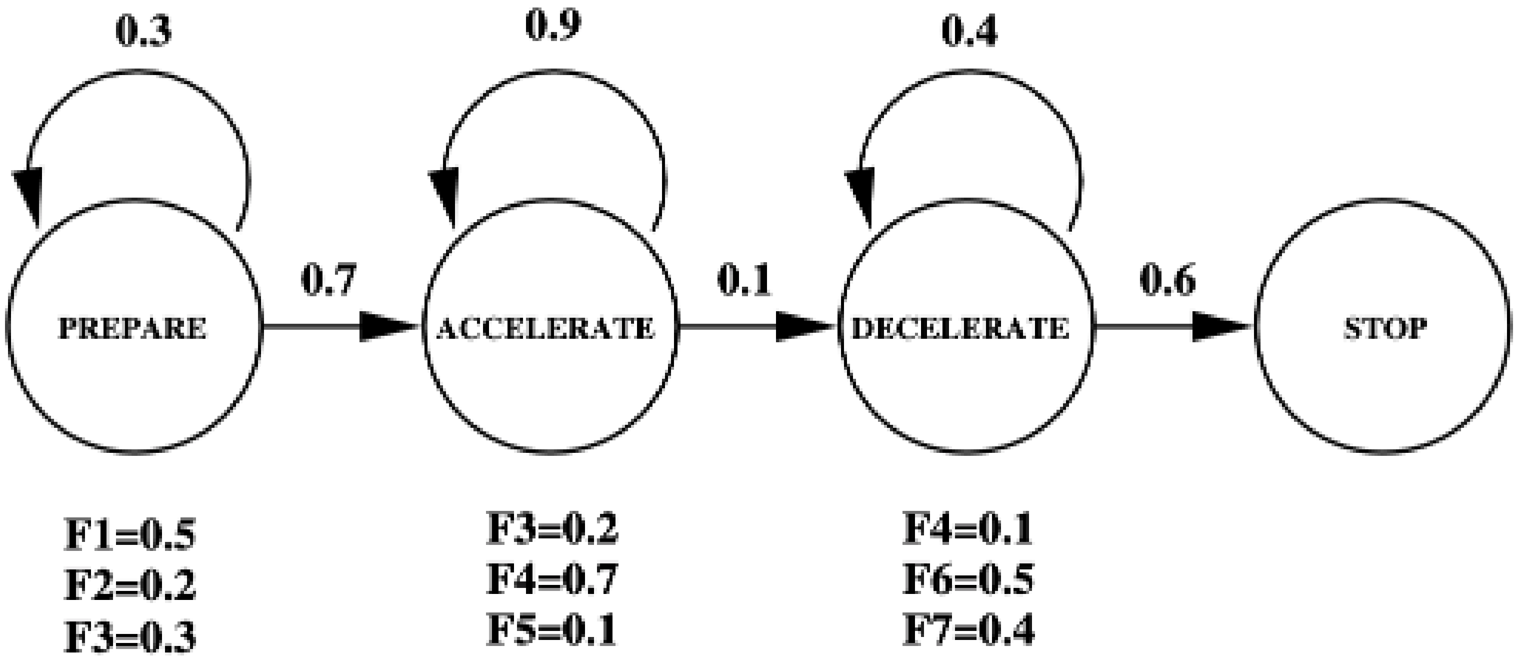
- F1 to 7 represent 7 ways of classifying frames
  - 7 possible observation outcomes that depict Usain's posture and probabilities of observing in each state
  - Observations not mentioned under a state have probability 0 of being observed there.
- STOP is a non-emitting terminating state.
- Video sequences are divided in several video frames classified as F1 to F7.



Example: Usain Bolt

- What does $A, B, and\ \pi$ look like?

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \pi = [1 \quad 0 \quad 0 \quad 0]$$



Example: Usain Bolt

- Given:

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \pi = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

- How to calculate $\mathrm{P}(X_2 = ACCELERATE)$?
  - Start with π and propagate probability one step with A
    - $\pi = X_1 = [1\ 0\ 0\ 0] \ \rightarrow \ X_2 = \pi A = [0.3\ 0.7\ 0\ 0]$
    - $p(X_2 = ACCELERATE) \ = \ 0.7$

Example: Usain Bolt

- Given:

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{array}{c} \begin{array}{ccccccc} & F_2 & & & & & \end{array} \\ \begin{bmatrix} 0.5 & \boxed{0.2} & 0.3 & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0.1 & 0 & 0.5 & 0.4 \\ 0 & \boxed{0} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \pi = [1 \quad 0 \quad 0 \quad 0]$$

- How to calculate $\mathrm{P}(O_2 = F2)$?
  - **We know** $\mathbf{P(X_2) = [0.3\ 0.7\ 0\ 0]}$
  - And we know how probable each measurement is in each state from B. Use sum rule.

$$p(0_2 = F_2) = \sum_{i=1}^{N} p\,(O_2 = F_2 \mid X_2 = i)p(X_2 = i)$$

$$= 0.2 \cdot 0.3 + 0 \cdot 0.7 + 0 \cdot 0 + 0 \cdot 0 = 0.06$$

Example: Usain Bolt

- Given:

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{array}{c} \begin{array}{ccccccc} & F_2 & & & & & \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.5 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \pi = [1 \quad 0 \quad 0 \quad 0]$$

- How to calculate $\mathrm{P}(X_2 = ACCELERATE | O_2 = F_2)$?
  - From previous slides $p(0_2 = F_2) = 0.06$ and $p(X_2 = ACCELERATE) = 0.7$
  - Use Bayes rule

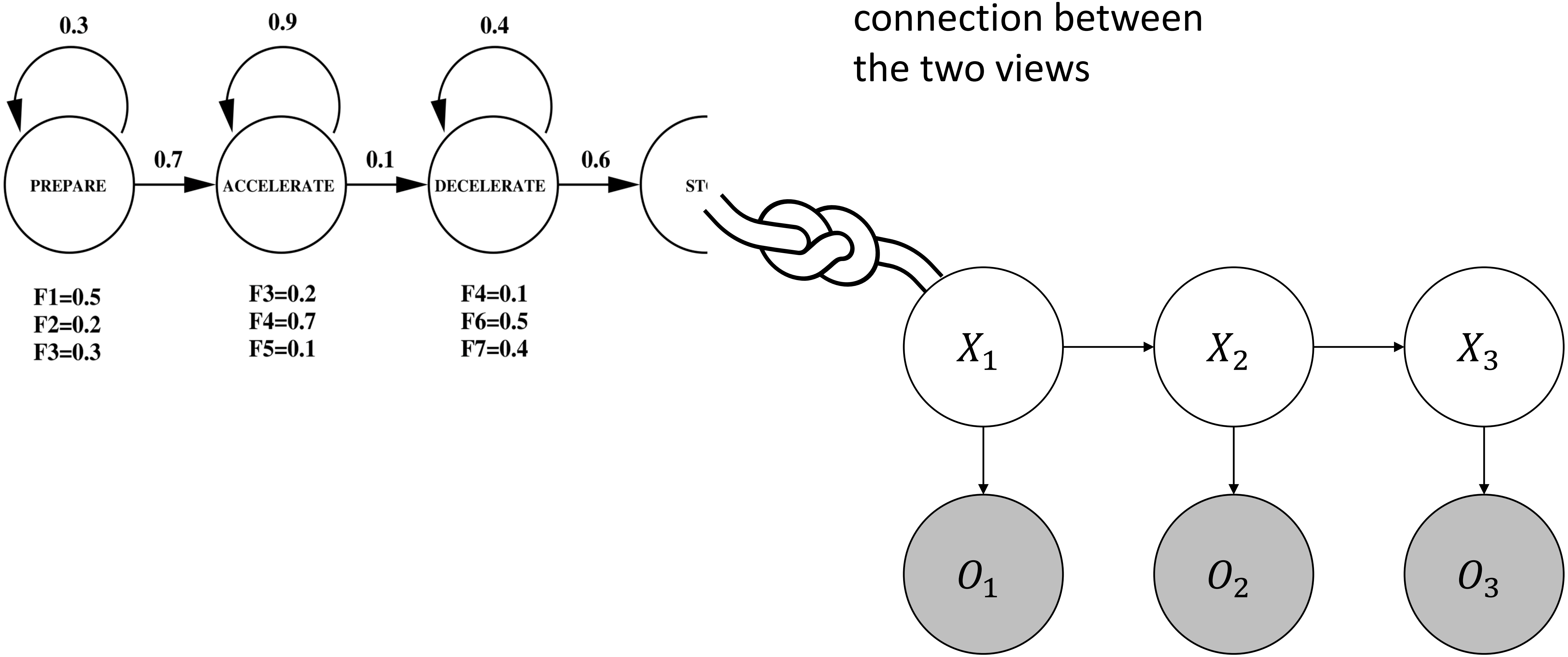
$$p(X_2 = ACCELERATE \mid O_2 = F_2) = \frac{p(O_2 = F_2 \mid X_2 = ACCELERATE)p(X_2 = ACCELERATE)}{p(O_2 = F_2)}$$

$$= \frac{0 \cdot 0.7}{0.06} = 0$$

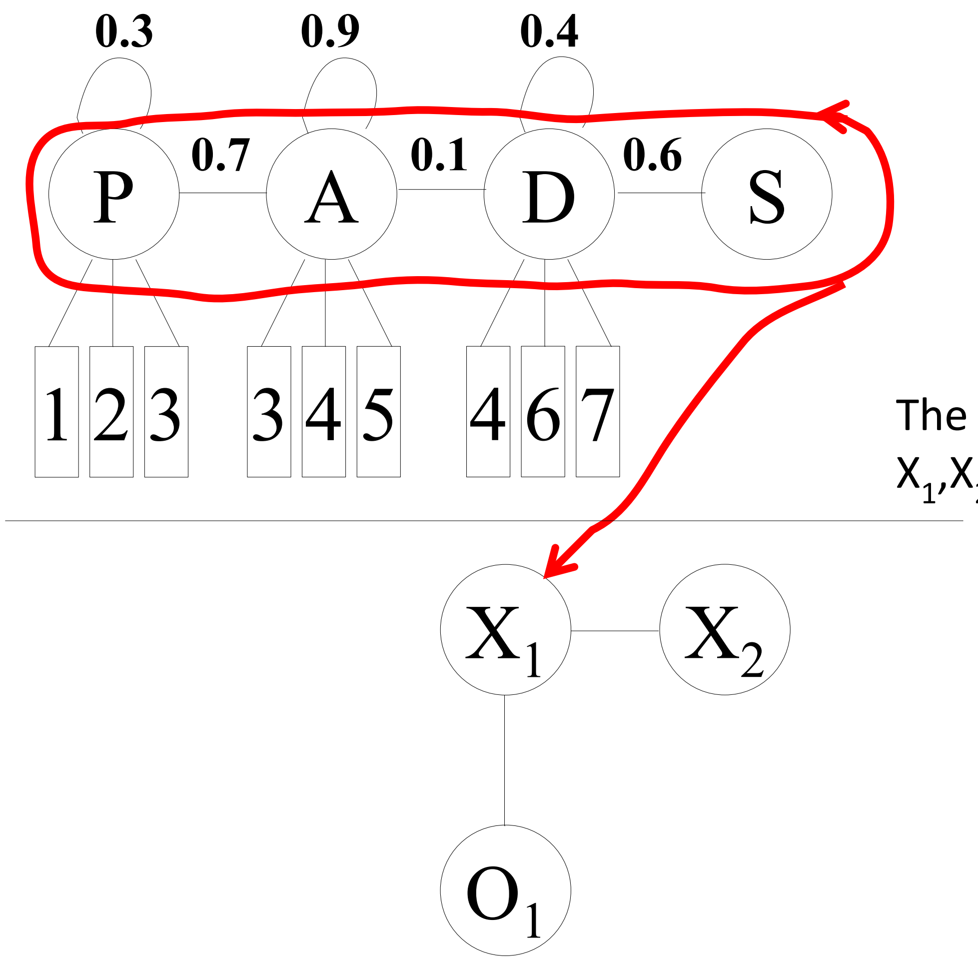
- Could we have seen this immediately?
  - YES. Cannot measure F2 in Accelerate state, i.e., p=0

Example: Usain Bolt

Let's look at the connection between the two views

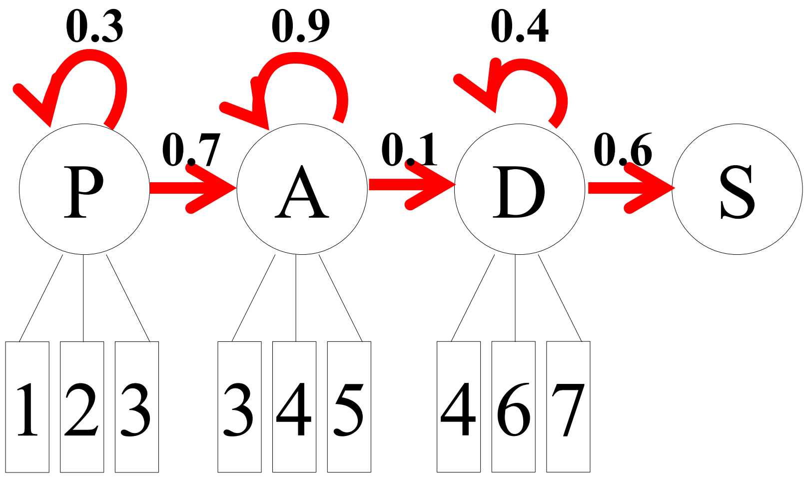


The states

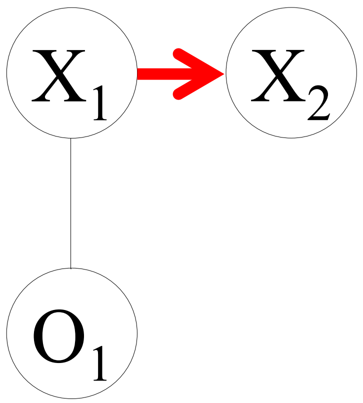


The states that variables X_1,X_2,... can be in (remember discrete)

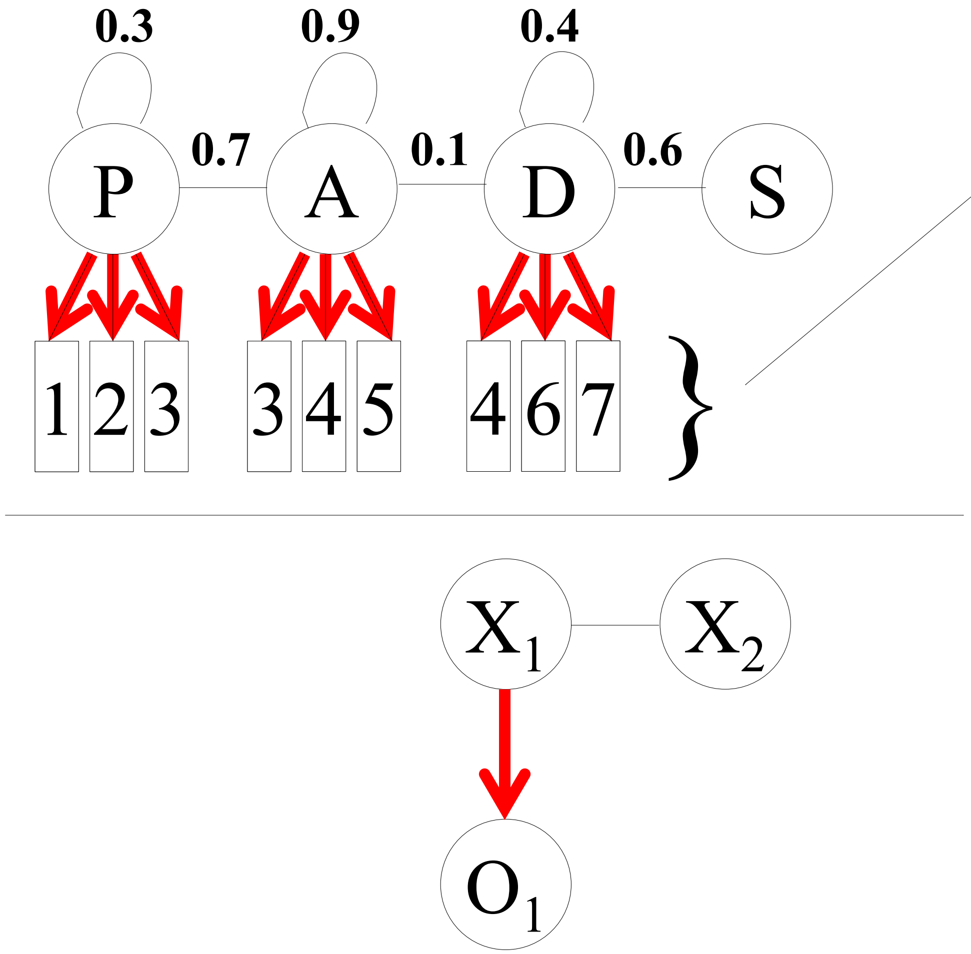
The state transitions



Red lines describe $p(x_{k+1}|x_k)$
i.e., the state transitions.
Captured with the A matrix (top)
and $p(x_{k+1}|x_k)$ (bottom)



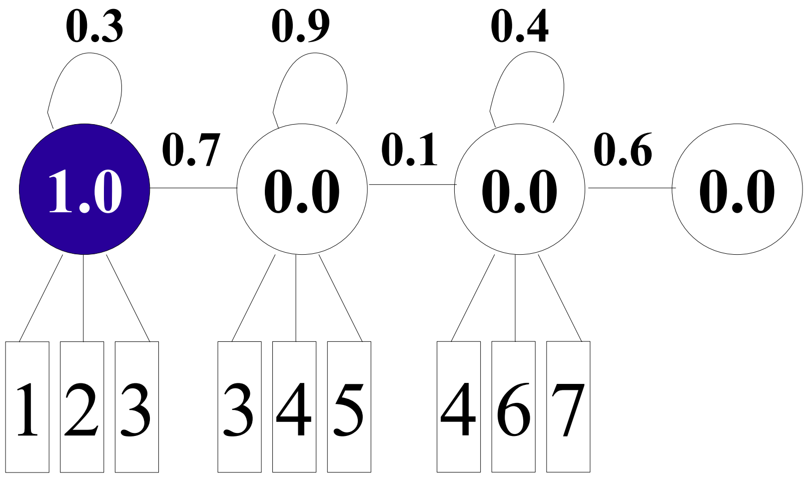
The observation/emission model



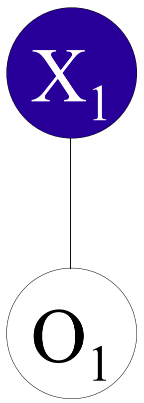
Different possible observations,
(aka “symbols”)

Red lines describe $p(O_k|X_k)$
i.e., observation model.
Captured by matrix B (top)
and $p(O_k|X_k)$ (bottom).

Initial state π=[1 0 0 0]



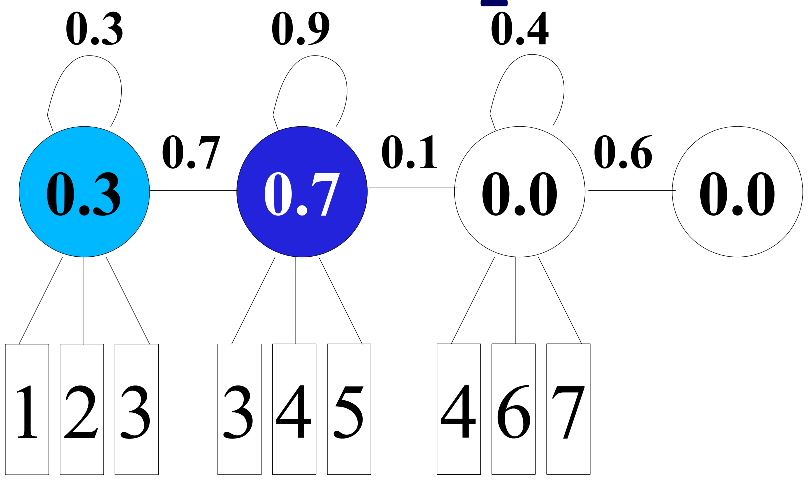
The darker the higher probability



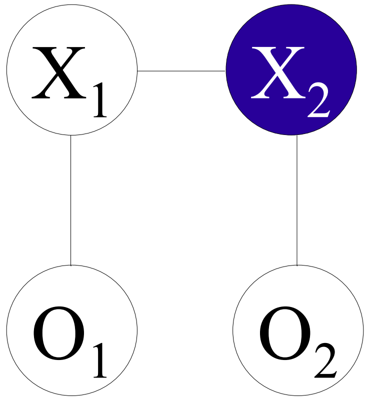
$$\pi = [1 \quad 0 \quad 0 \quad 0]$$

$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prediction for X_2 (= π A)



Must sum to 1 i.e., the system is in some state!!

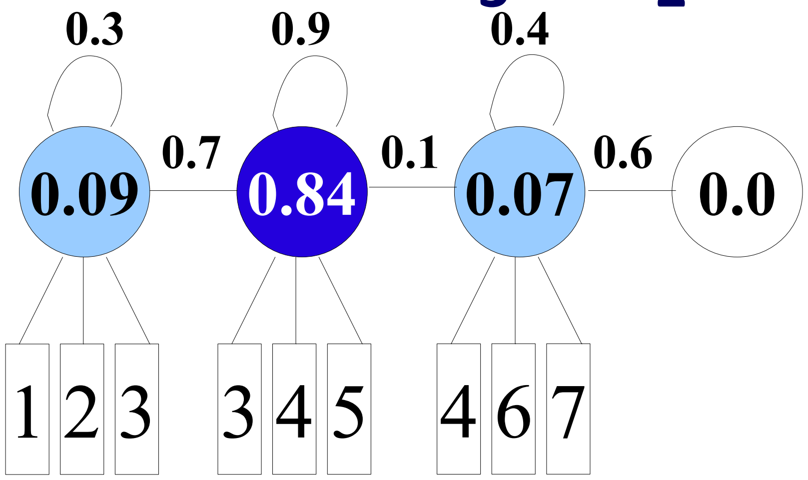


$$\pi = [1 \quad 0 \quad 0 \quad 0]$$

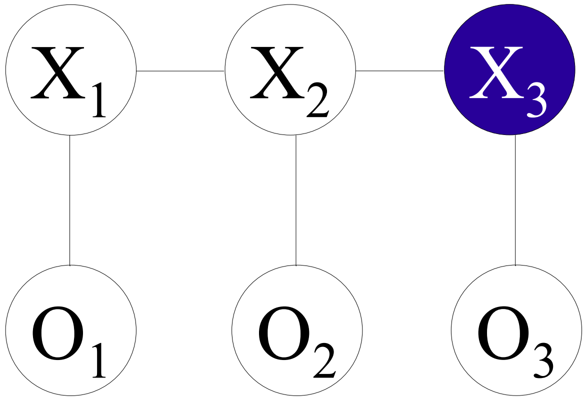
$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\pi}\mathbf{A} = \mathbf{X_2} = [\mathbf{0.3} \quad \mathbf{0.7} \quad \mathbf{0} \quad \mathbf{0}]$$

Prediction for X_3 (=X_2A)



Must sum to 1 i.e., the system is in some state!!

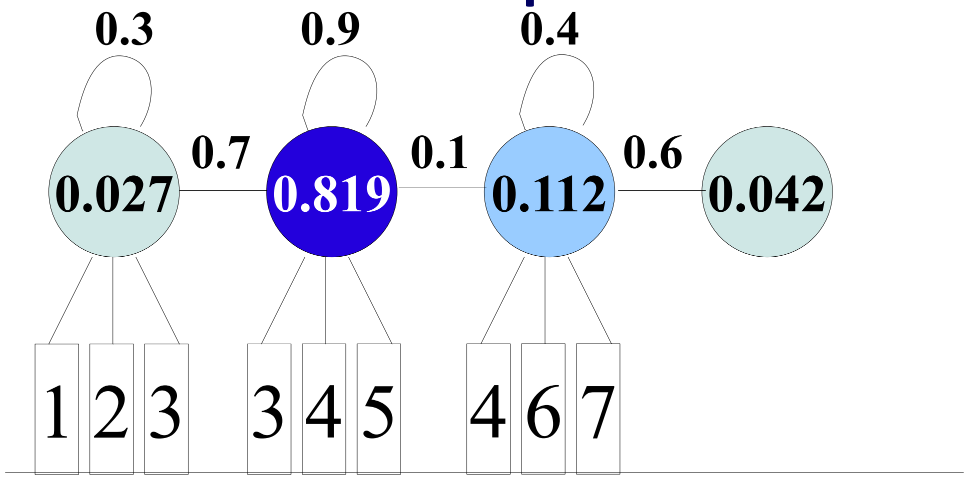


$$X_2 = [0.3 \quad 0.7 \quad 0 \quad 0]$$

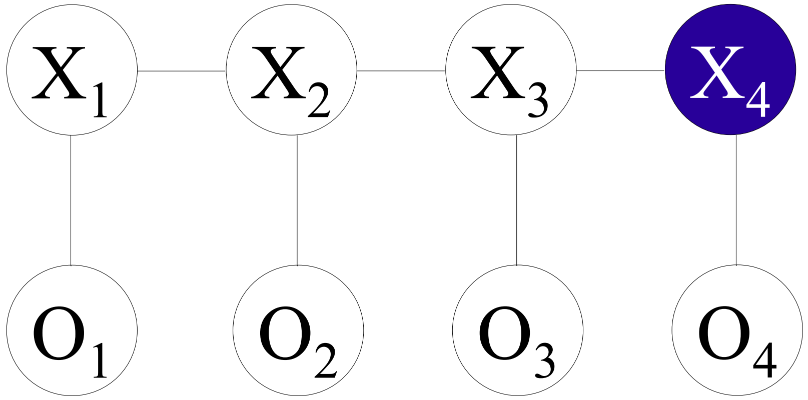
$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X_2A} = \mathbf{X_3} = [\mathbf{0.09} \quad \mathbf{0.84} \quad \mathbf{0.07} \quad \mathbf{0}]$$

Prediction for X_4



Note: We know for sure that we are at t=4 but not in which state we are. This is what we often try to estimate!



$$X_3 = [0.09 \quad 0.84 \quad 0.07 \quad 0]$$

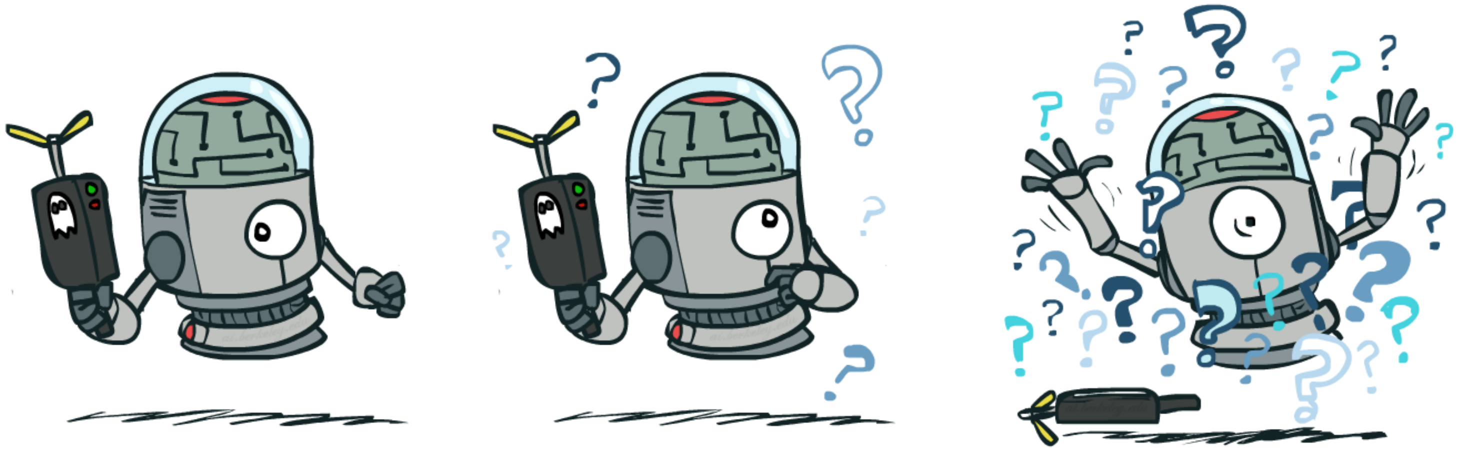
$$A = \begin{bmatrix} 0.3 & 0.7 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{X_3A} = \mathbf{X_4} = [\mathbf{0.027} \quad \mathbf{0.819} \quad \mathbf{0.112} \quad \mathbf{0.042}]$$

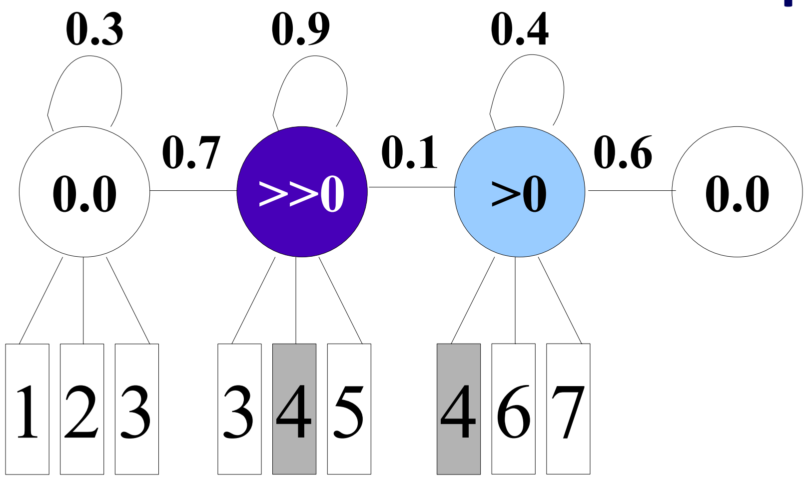
Measurements?

- Without measurements we are doing pure prediction
  - can do that ahead of time!

- Measurements give clues to what state we are in
  - beliefs get reweighted, uncertainty “decreases”



Assume we measure $O_4 = 4$ (i.e. obs 4 at t=4)



- We have $p(X_4)$ from the prediction
- For each state j we can calculate $p(X_4 = j|O_4)$

$$p(X_4 = j \mid O_4) = \{Bayesrule\}$$

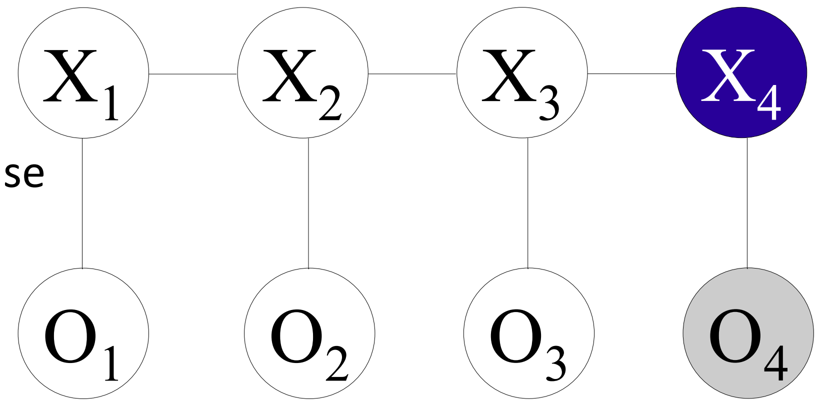
$$= \frac{p(O_4 = 4 \mid X_4 = j)p(X_4 = j)}{p(0_4)}$$

$$= \frac{b_j(4)p(X_4 = j)}{p(0_4)}$$

$$= \eta b_j(4)p(X_4 = j)$$

$$with\ \eta = \frac{1}{\sum_j b_j(4) \times p(X_4 = j)}$$

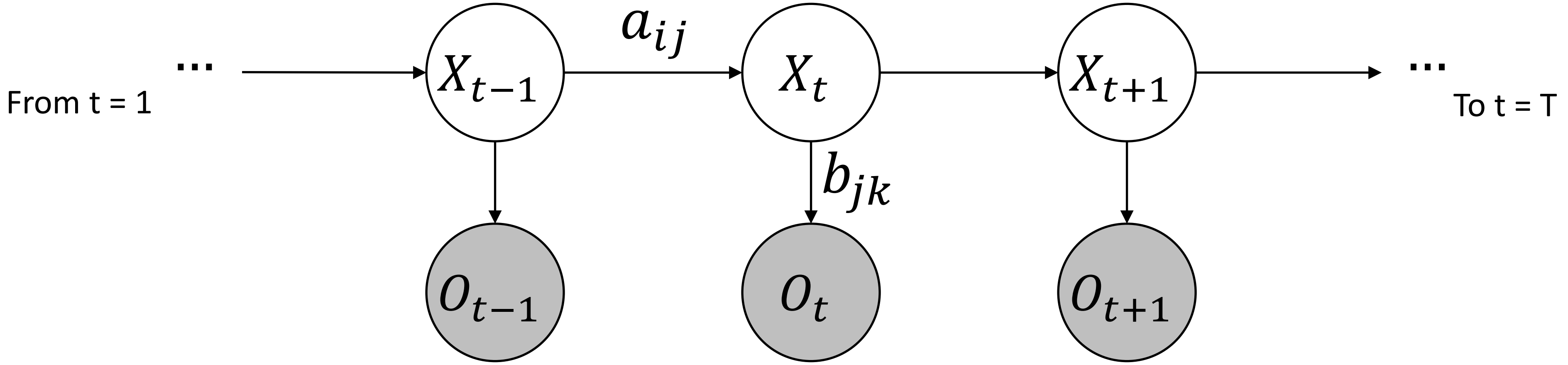
Weight with p(O|X)
→ not in state 1 for sure because we cannot measure 4 there



Observed (grey)

HMM Terminology

Time instants	$t\ in\ \{1,2\ldots T\}$
Hidden States / States / Emitters	X_t
Outputs / Emissions / Observations / Visible States	O_t
All possible states / states set	$X_t\ in\ \{1, 2, \ldots, N\}$
All possible emissions / emissions set	$O_t\ in\ \{1, 2, \ldots, K\}$
Initial state distribution / Initial state probabilities	$p_i\ in\ q\ or\ \pi_i\ in\ \pi$
Transition probabilities / State transition probabilities	$a_{ij}\ in\ row-stochastic\ matrix\ A$
Emission probabilities / Observation probabilities	$b_{jk}\ in\ row-stochastic\ matrix\ B$



Three problems solved with HMMs

1. **Evaluation/**filtering: Compute likelihood $p(O_{1:t}|\lambda)$
   of observation sequence ($O_{1:T}=\{O_1,O_2,\ldots,O_t,\ldots,O_T\}$) given λ
   Forward algorithm
   Backward algorithm

2. **Decoding**: Most likely state sequence $X^*_{1:T}$ given $O_{1:T}$ and λ
   Viterbi algorithm

3. **Learning**: Estimate model parameters
   $\Lambda=\{\lambda\}$ given $O_{1:T}$ such that $p(O_{1:T}|\lambda)$ is maximized
   $\rightarrow$ A and B matrices and π

   Baum-Welch algorithm

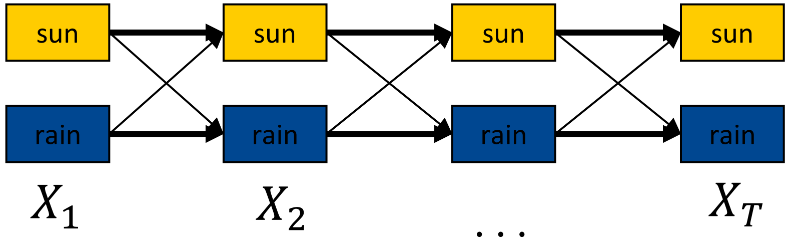
1. Compute likelihood $p(O_{1:t}|\lambda)$ of observed sequence given λ

- Motivating examples
  - Character recognition:
    - Have models for each character
    - Draw a character and **compare it to models of different characters**
    - Pick model that fits best → Recognized char
  - Identify fish species
    - Model swimming patterns of known fish, **compare new fish to the models**

1. Compute likelihood $p(O_{1:t}|\lambda)$ of observed sequence given λ

- **Given:**
  - A, B, π
  - Emission sequence **O** = $\{O_1, O_2 \dots O_T\}$

- **Unknown:**
  - Hidden state sequence **X** = $\{X_1, X_2 \dots X_T\}$ that actually produced **O**.



- **To Find:**
  - Probability that the given sequence **O** occurred **regardless of which X produced the sequence.**

Likelihood of $p(O_{1:T} | \lambda)$

$$p(O_{1:T} \mid \lambda) = \{sum\ rule\} = \sum_{X_{1:T}} p\left(O_{1:T}, X_{1:T}\right)$$

$$= \{product\ rule\} = \sum_{X_{1:T}} p(X_{1:T})p\left(O_{1:T} \mid X_{1:T}\right)$$

$$= \sum_{X_{1:T}} \underbrace{\pi_{X_1} a_{X_1 X_2} a_{X_2 X_3} \ldots a_{X_{T-1} X_T}}_{\substack{p(X_{1:T}) \\ Transition\ Probabilities}} \underbrace{b_{X_1}(O_1) b_{X_2}(O_3) \ldots b_{X_T}(O_T)}_{\substack{p(O_{1:T} | X_{1:T}) \\ Emission\ Probabilities}}$$

- Note that we are summing over all possible permutations of $X_{1:T}$
- Evaluating this requires $O(2TN^T)$ multiplications
- Can be formulated recursively using the forward (and backward) algorithm

Example for Clarity

If you have:

- $N = 2$ states (e.g., A and B),
- $T = 3$ time steps,

then there are $2^3 = 8$ possible sequences, such as:

1. $A \to A \to A$
2. $A \to A \to B$
3. $A \to B \to A$
4. $A \to B \to B$
5. $B \to A \to A$
6. $B \to A \to B$
7. $B \to B \to A$
8. $B \to B \to B$

Forward algorithm (aka α-pass)

- Introduce: $\alpha_t(i) = p(O_{1:t}, X_t = i \mid \lambda) \forall t = 1, \ldots, T$

$\alpha_t(i)$ is the probability of observing a partial sequence of observables $O_1, \ldots O_t$ AND at time t, being in state i

- Initialize as: $\alpha_1(i) = \pi_i b_i(O_1)$

Forward algorithm (aka α-pass)

- Introduce: $\alpha_t(i) = p(O_{1:t}, X_t = i \mid \lambda) \forall t = 1, \ldots, T$

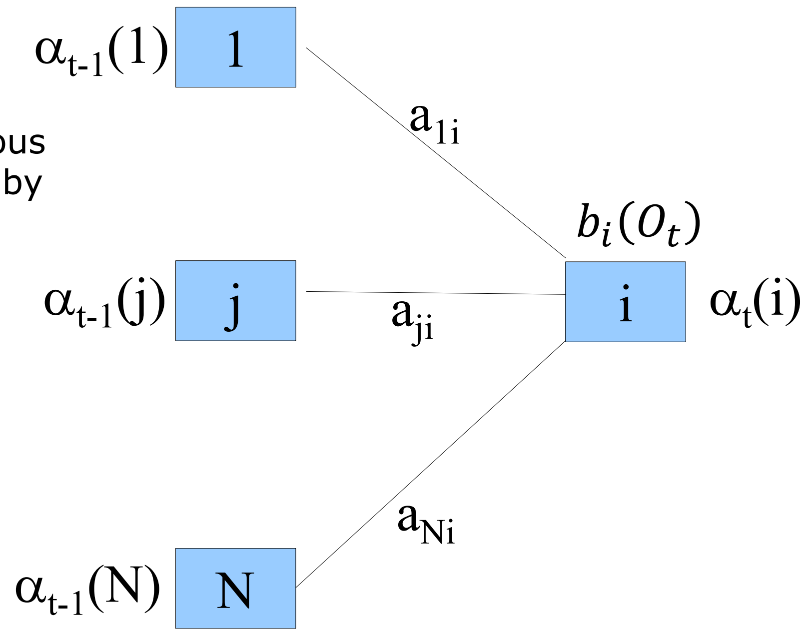
Prediction (sum of the previous partial probabilities multiplied by the transition probabilities)

- Initialize as: $\alpha_1(i) = \pi_i b_i(O_1)$

Weight (observation probability)

- For 2≤t≤T:

$$\alpha_t(i) = \left[\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji}\right] b_i(O_t)$$



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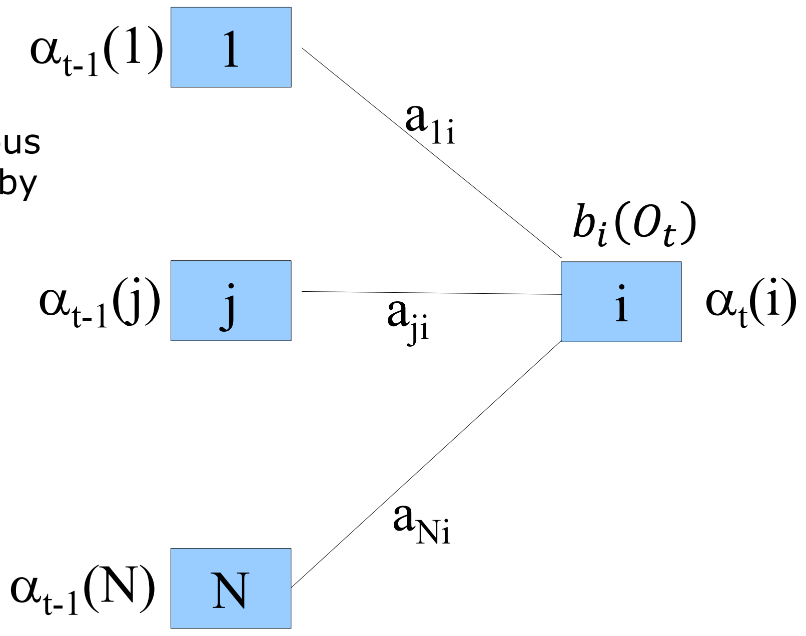
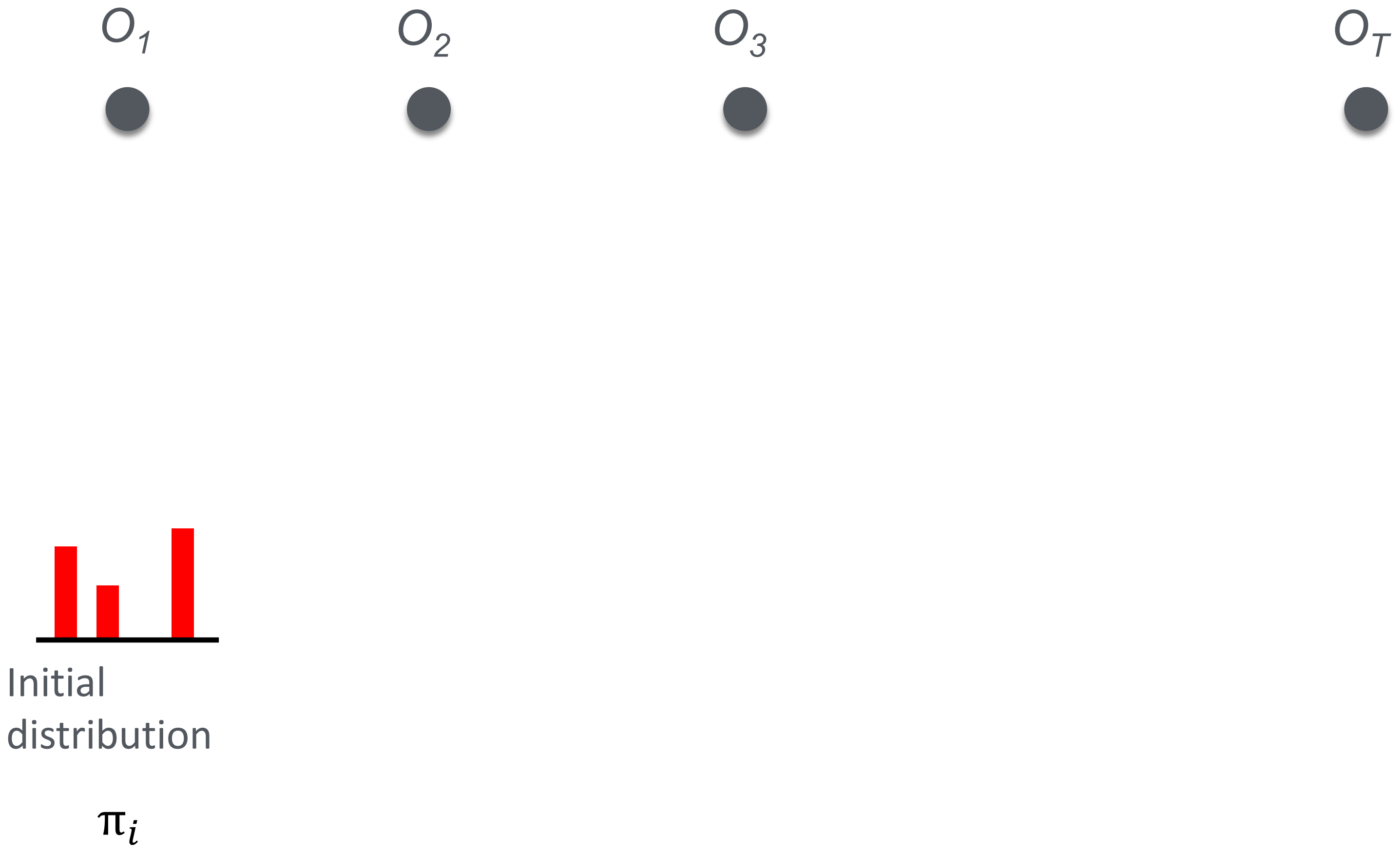


- Which gives us:

$$p(O_{1:T} \mid \lambda) = \sum_{i=1}^{N} p(O_{1:T}, X_t = i \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

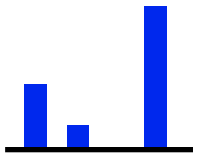
- recursive way to calculate likelihood with only N^2T multiplications (compared to $2TN^T$)

Forward algorithm intuition



Forward algorithm intuition

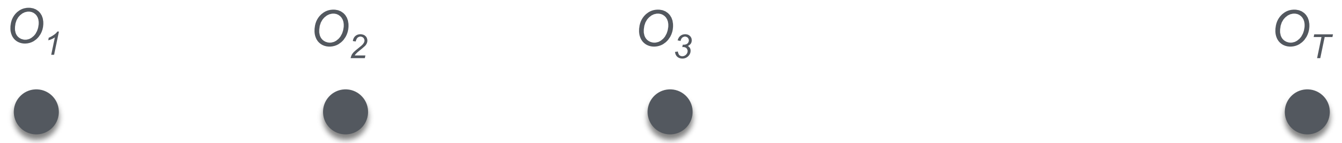
O_1 O_2 O_3 O_T

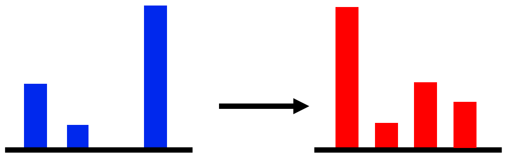


"Improved" distribution

$$\alpha_1(i) = \pi_i b_i(O_1)$$

Forward algorithm intuition

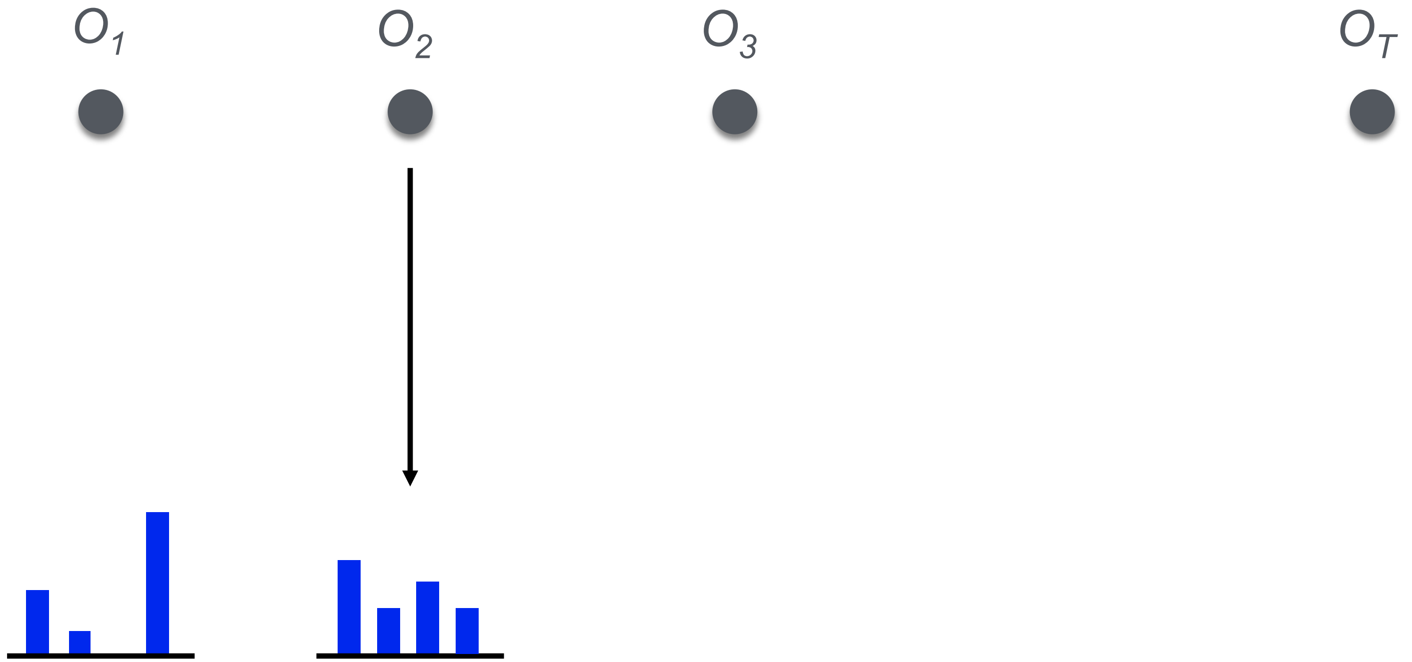




Predicted distribution

$$\left[\sum_{j=1}^{N} \alpha_1(j) a_{ji}\right]$$

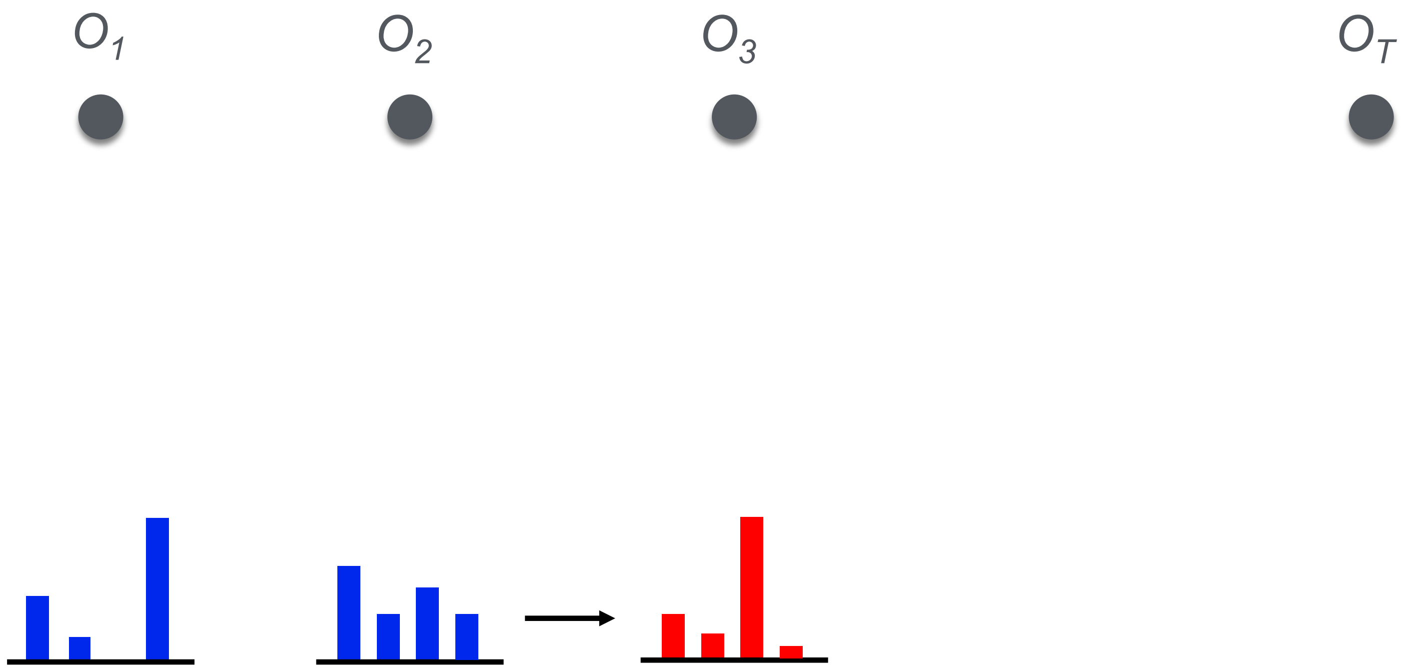
Forward algorithm intuition



"Improved" distribution

$$\left[\sum_{j=1}^{N} \alpha_1(j) a_{ji}\right] b_i(O_2)$$

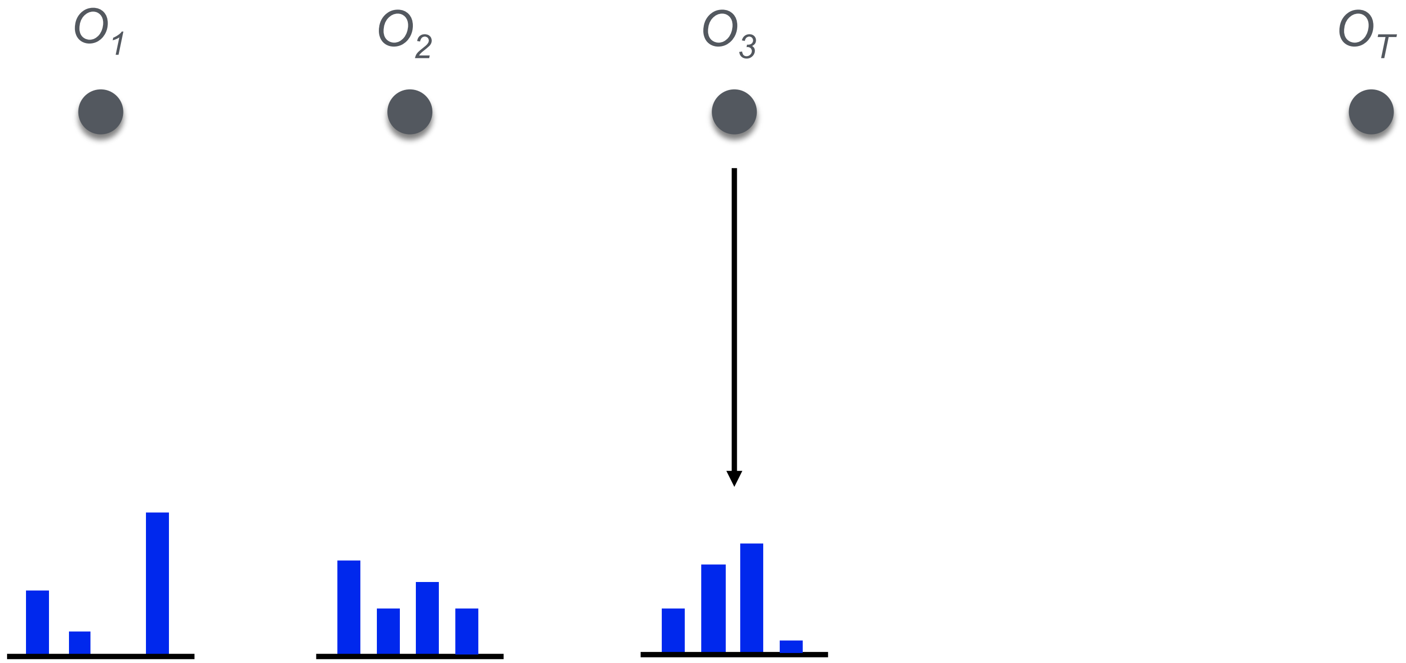
Forward algorithm intuition



Predicted distribution

$$\left[\sum_{j=1}^{N} \alpha_2(j) a_{ji}\right]$$

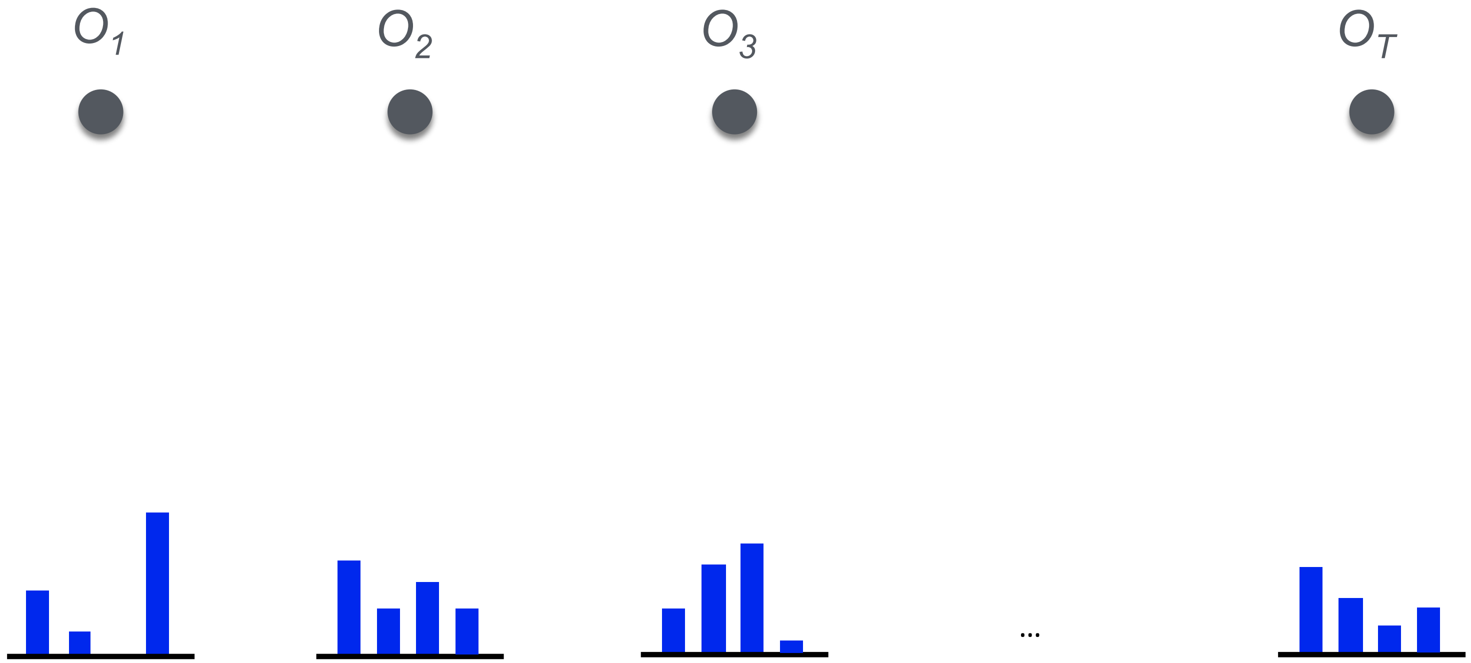
Forward algorithm intuition



"Improved" distribution

$$\left[\sum_{j=1}^{N} \alpha_2 (j) a_{ji}\right] b_i(O_3)$$

Forward algorithm intuition



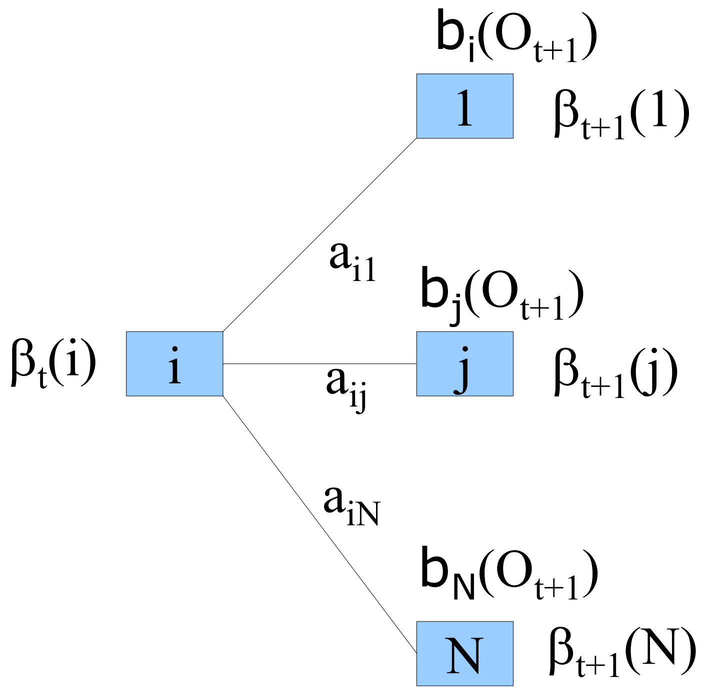
Backward algorithm (aka β-pass)

- βt(i) = Probability that the model is in the hidden state $X_t(i)$ $(i\ in\ [1,2,\ldots,N])$
- **will generate** the remainder of the emission sequence, from Ot+1 to OT, as specified by the emission sequence **O**.

- **Introduce:** $$\beta_t(i) = p(O_{t+1:T} \mid X_t = i, \lambda)$$

- **Initialize:** $$\beta_T(i) = 1, \forall i = 1, \ldots, N$$

- **For t<T:** $$\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(O_{t+1})\beta_{t+1}(j)$$



How the present relates to the future

Three problems solved with HMMs

1. **Evaluation/**filtering: Compute likelihood $p(O_{1:t}|\lambda)$
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   $\rightarrow$ A and B matrices and π

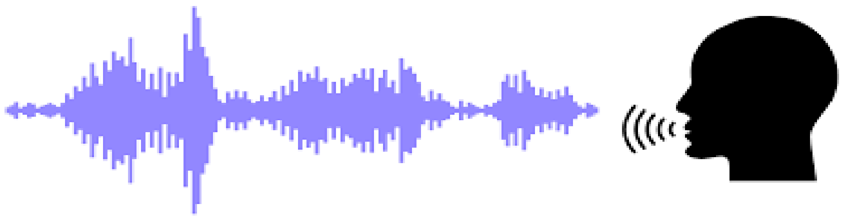
   Baum-Welch algorithm

2. Calculate most likely state sequence

- Motivating examples
  - Parts-Of-Speech (POS) tagging:
    - Given a sentence such as “I love cats and dogs”
    - Find POS tags <pronoun><verb><noun><conjunction><noun>
  - Speech recognition
    - Given a sound recording of spoken words
    - Find which words were spoken

“Recognize speech”

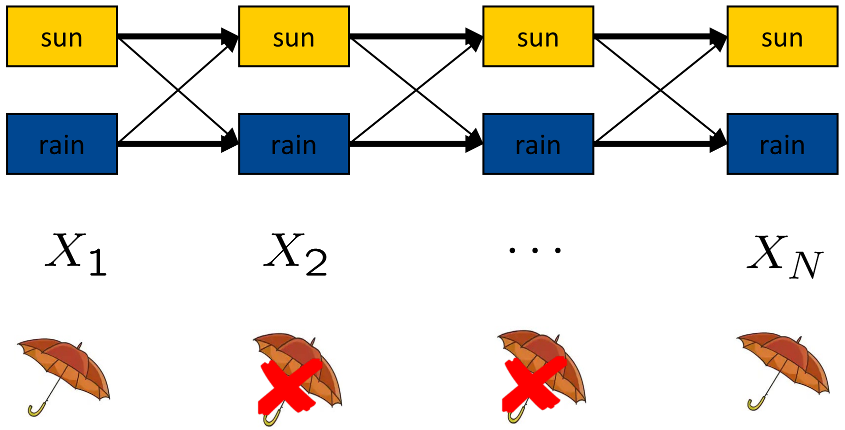
“Wreck a nice beach”



  - HMM will return the **most likely** sequence of words (hidden states)

2. Calc most likely state sequence

- Given:
  - Emission sequence **O** = $\{O_1, O_2 ... O_T\}$
  - A, B, π
- To Find:
  - Hidden state sequence $\boldsymbol{X}^* = \{X_1, X_2 \dots X_T\}$ that most likely produced **O**.
  - Probability of occurrence of $\boldsymbol{X}^*$



2. Calculate most likely state sequence

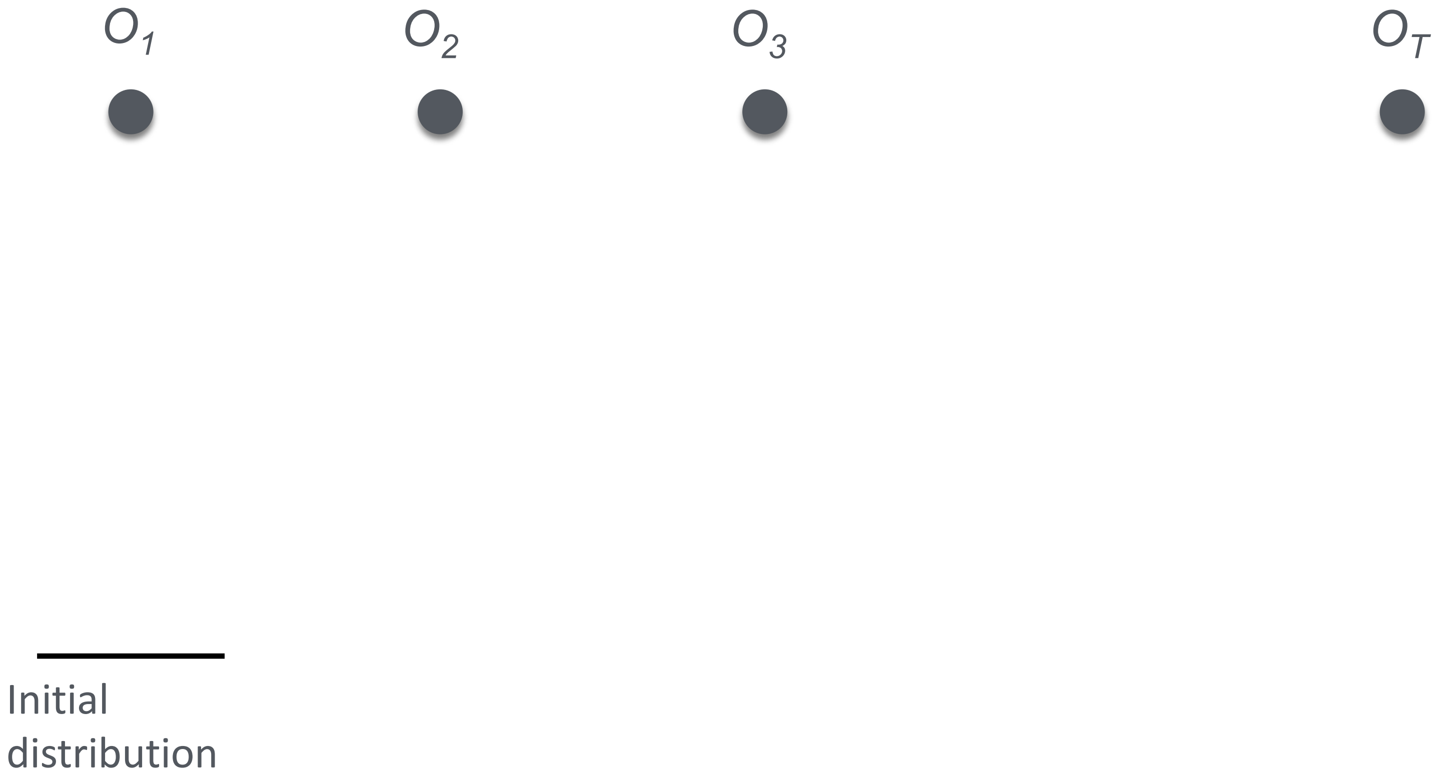
- We can find the most likely sequence by listing all possible sequences and finding the prob of the observed sequence for each of the combinations

$$X_{1:T}^{*} = \underset{X_{1:T}}{argmax}\ p(X_{1:T} \mid O_{1:T}, \lambda)$$

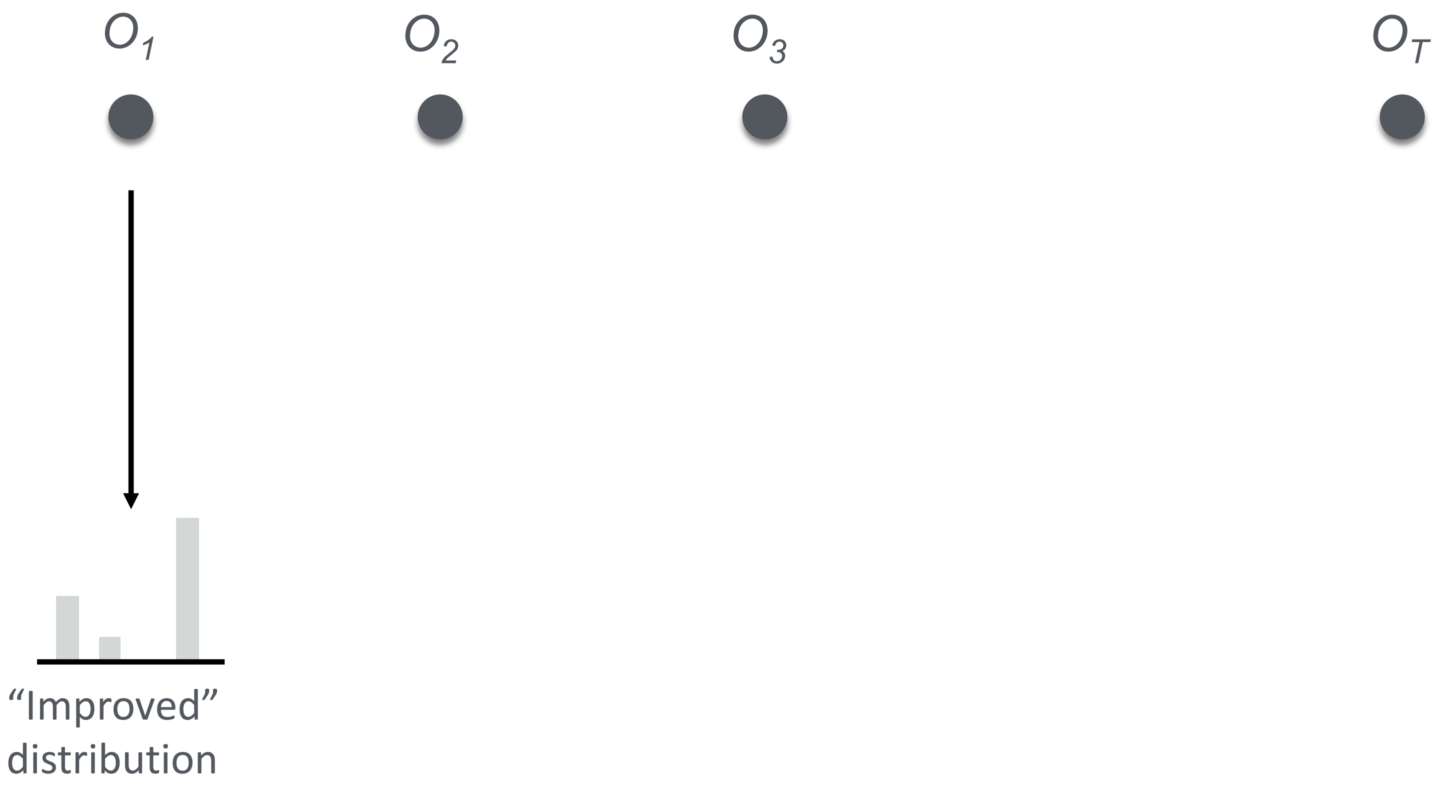
- Cannot solve individually for each time step

- Need to optimize the sequence not just individual states

2. Calc most likely state sequence

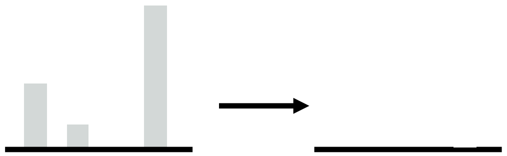


2. Calc most likely state sequence



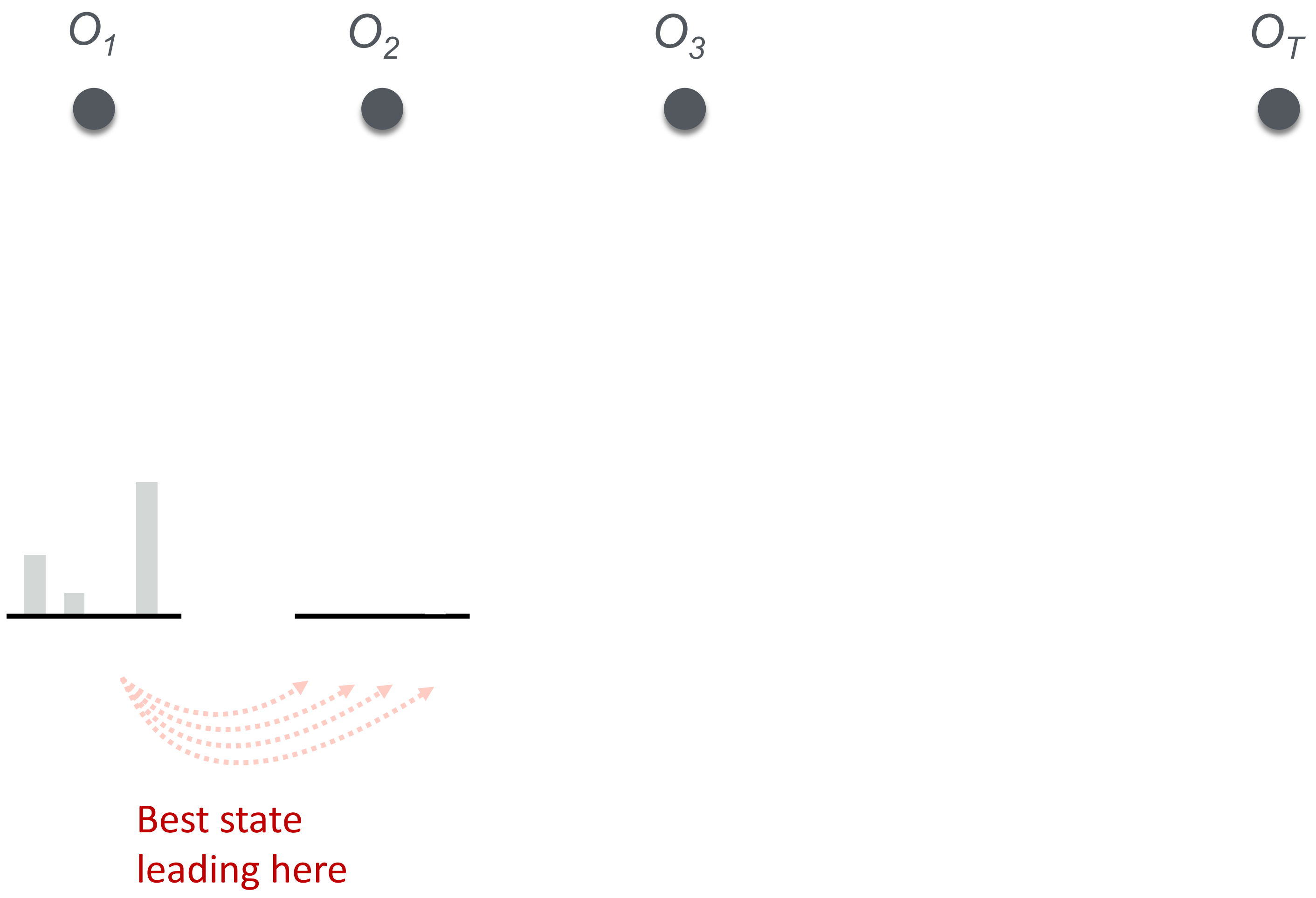
2. Calc most likely state sequence

O_1 O_2 O_3 O_T

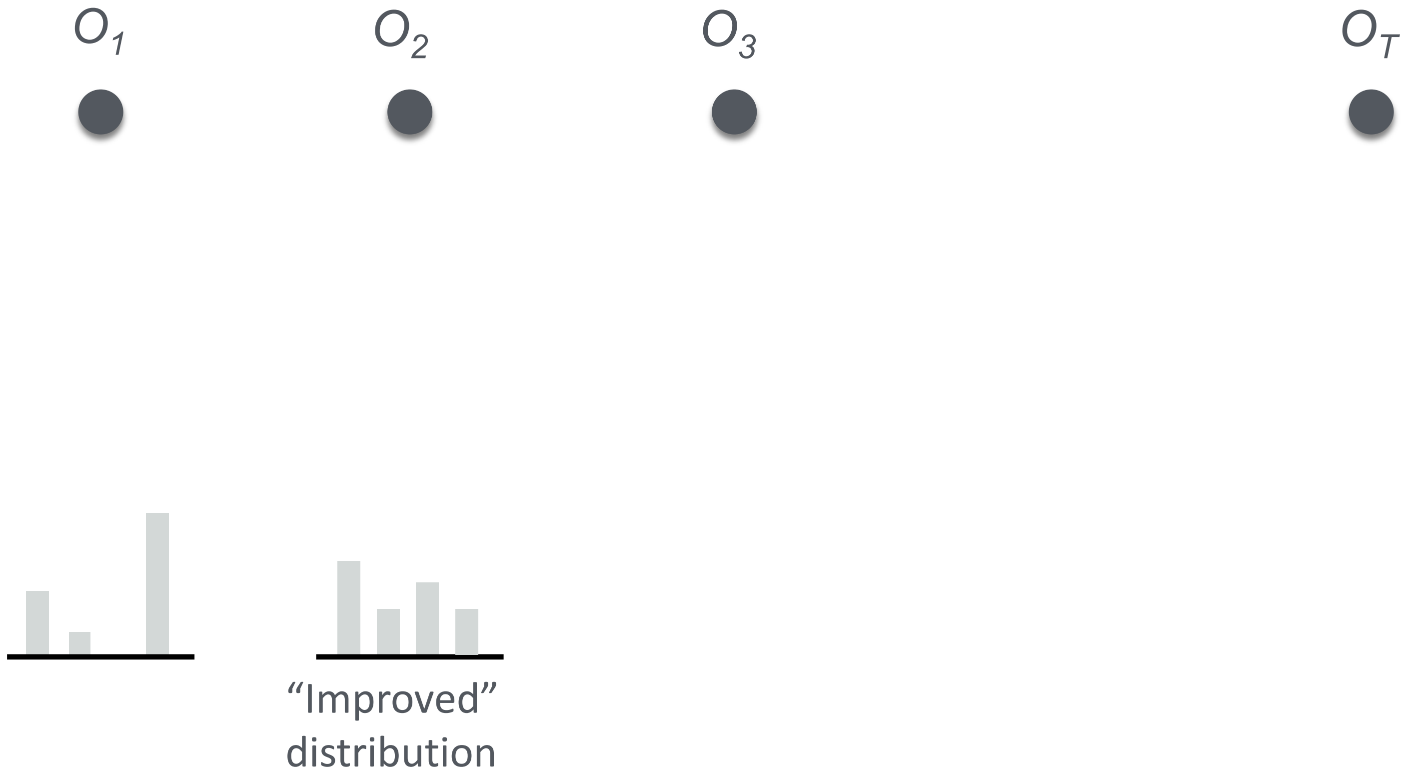


Predicted distribution

2. Calc most likely state sequence



2. Calc most likely state sequence



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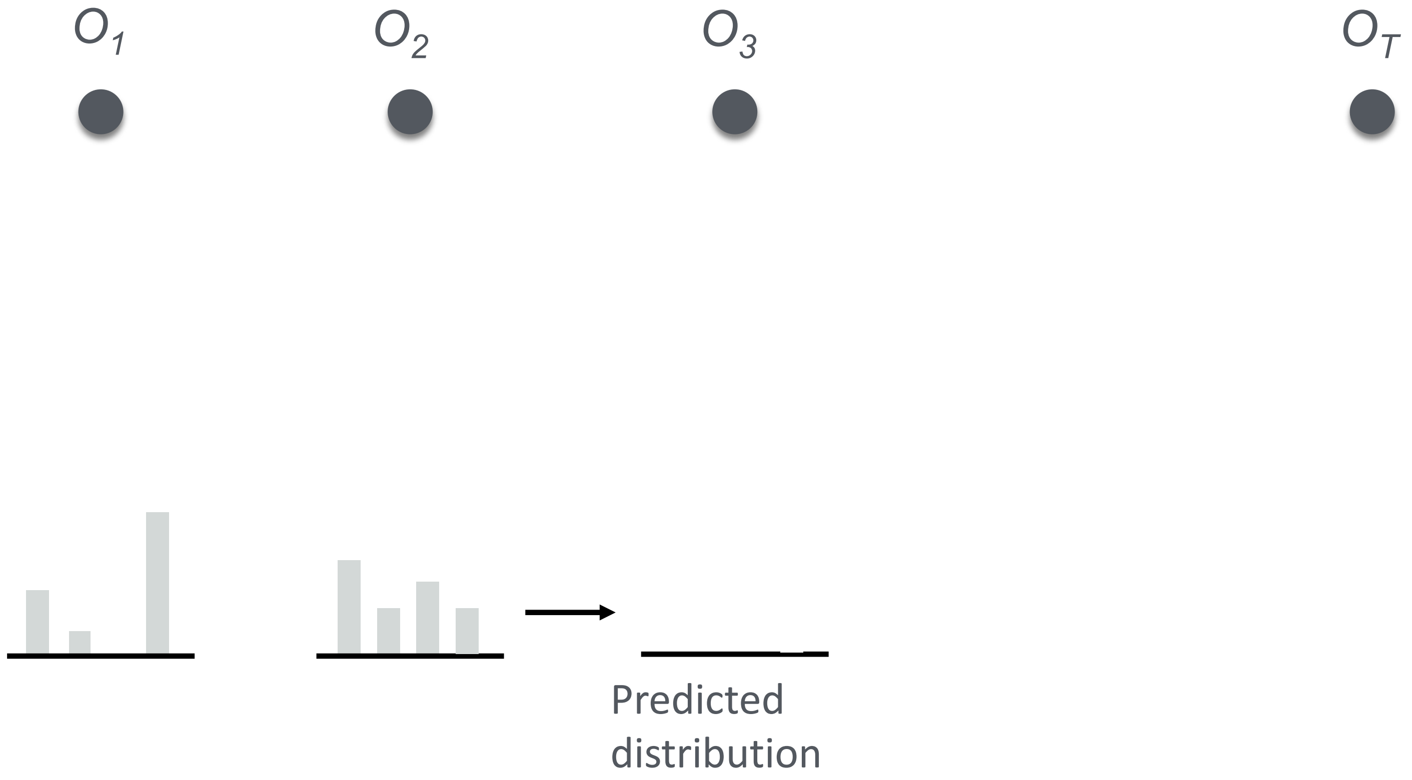

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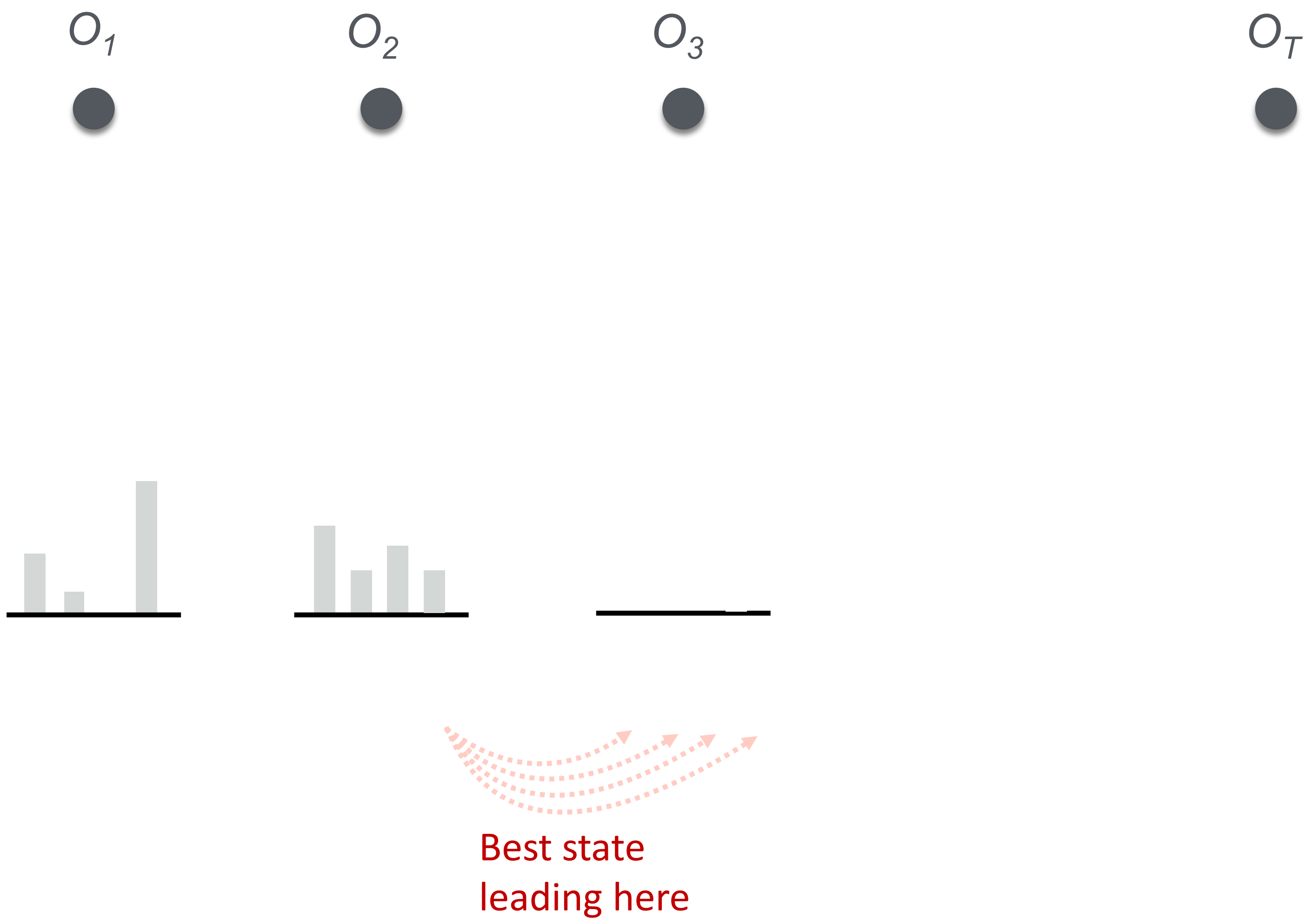

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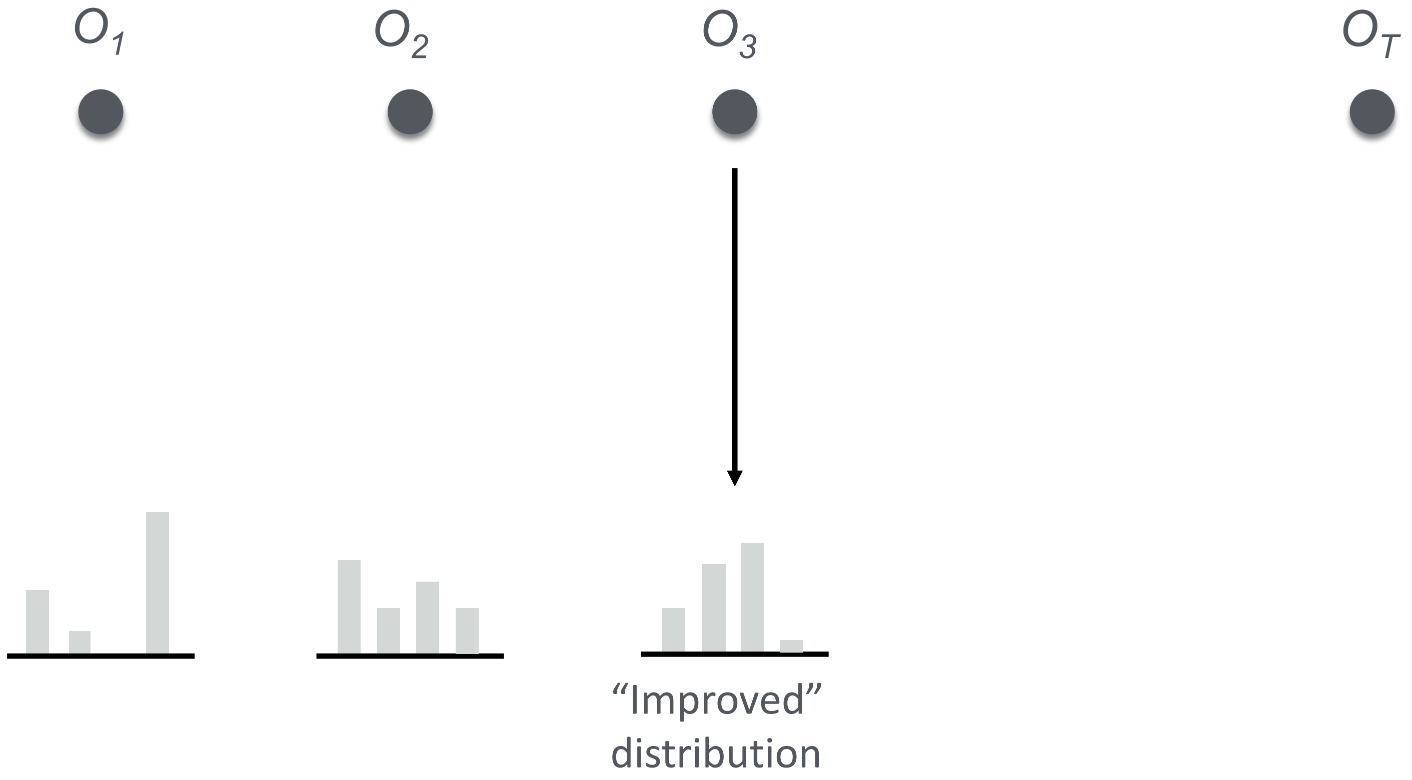

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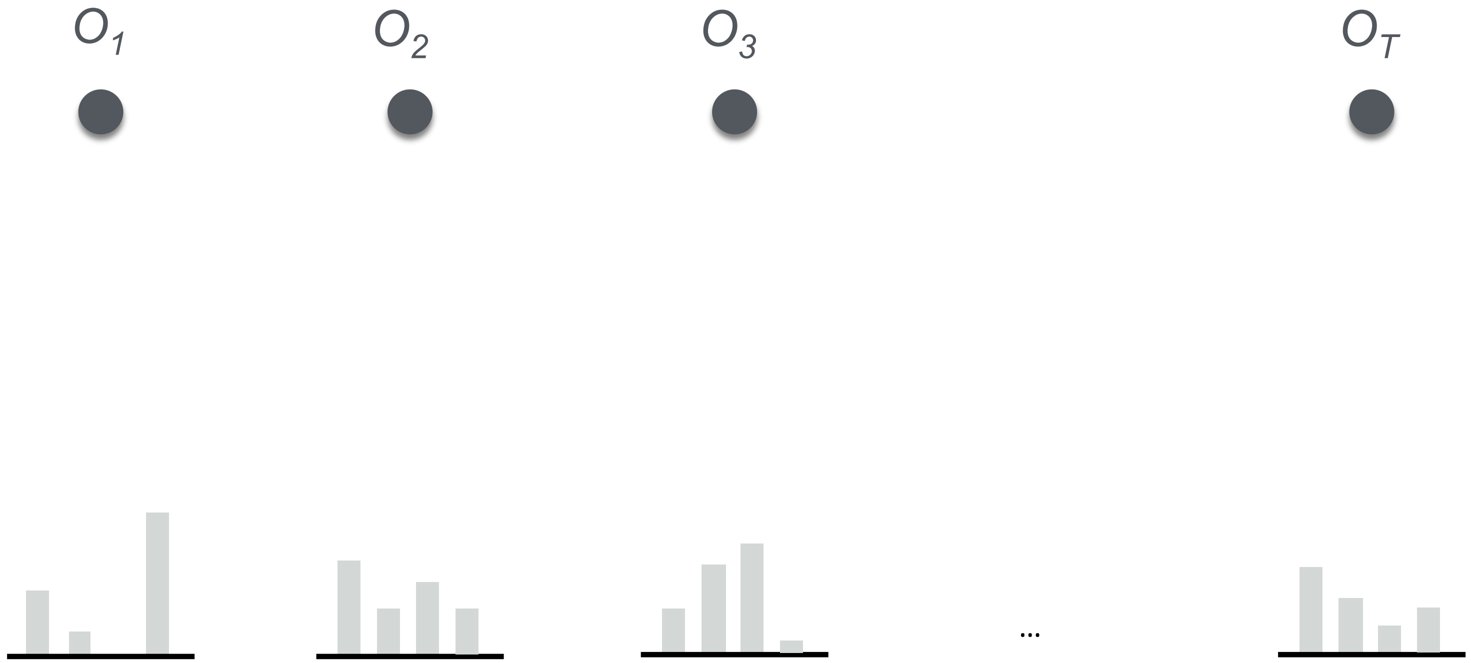

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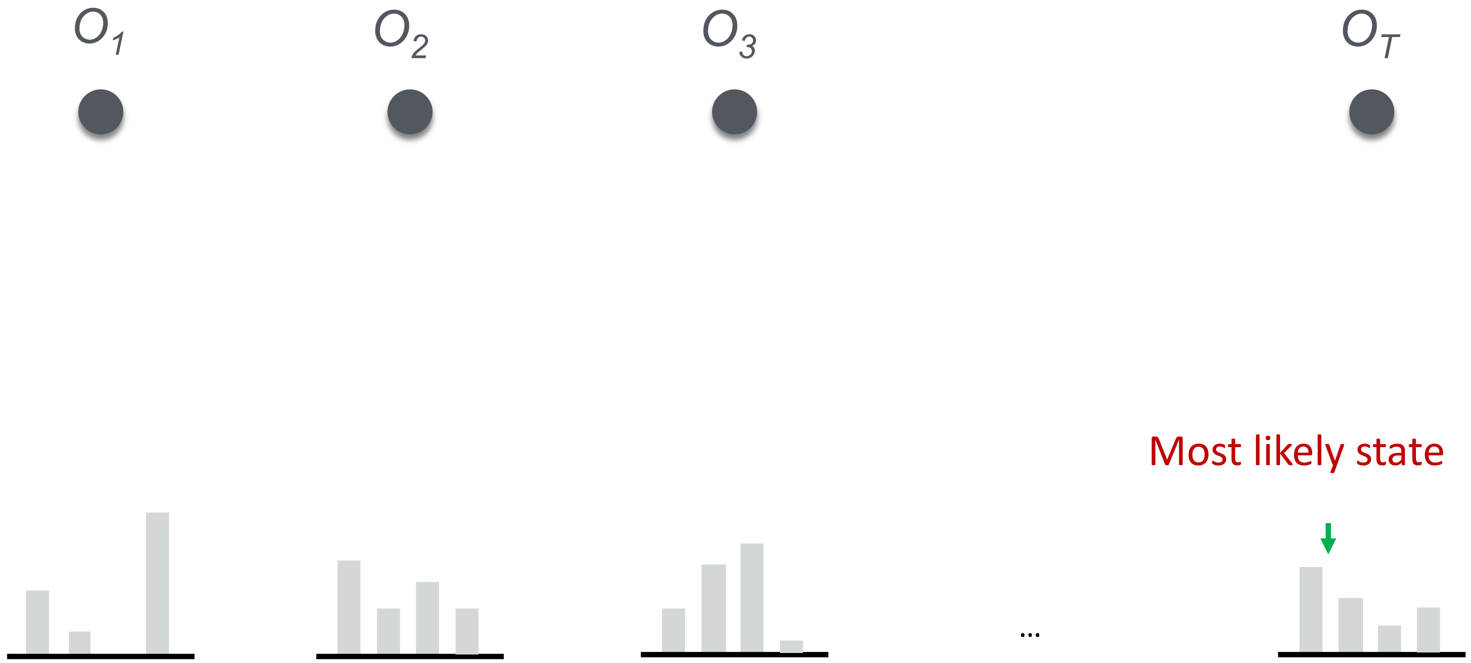

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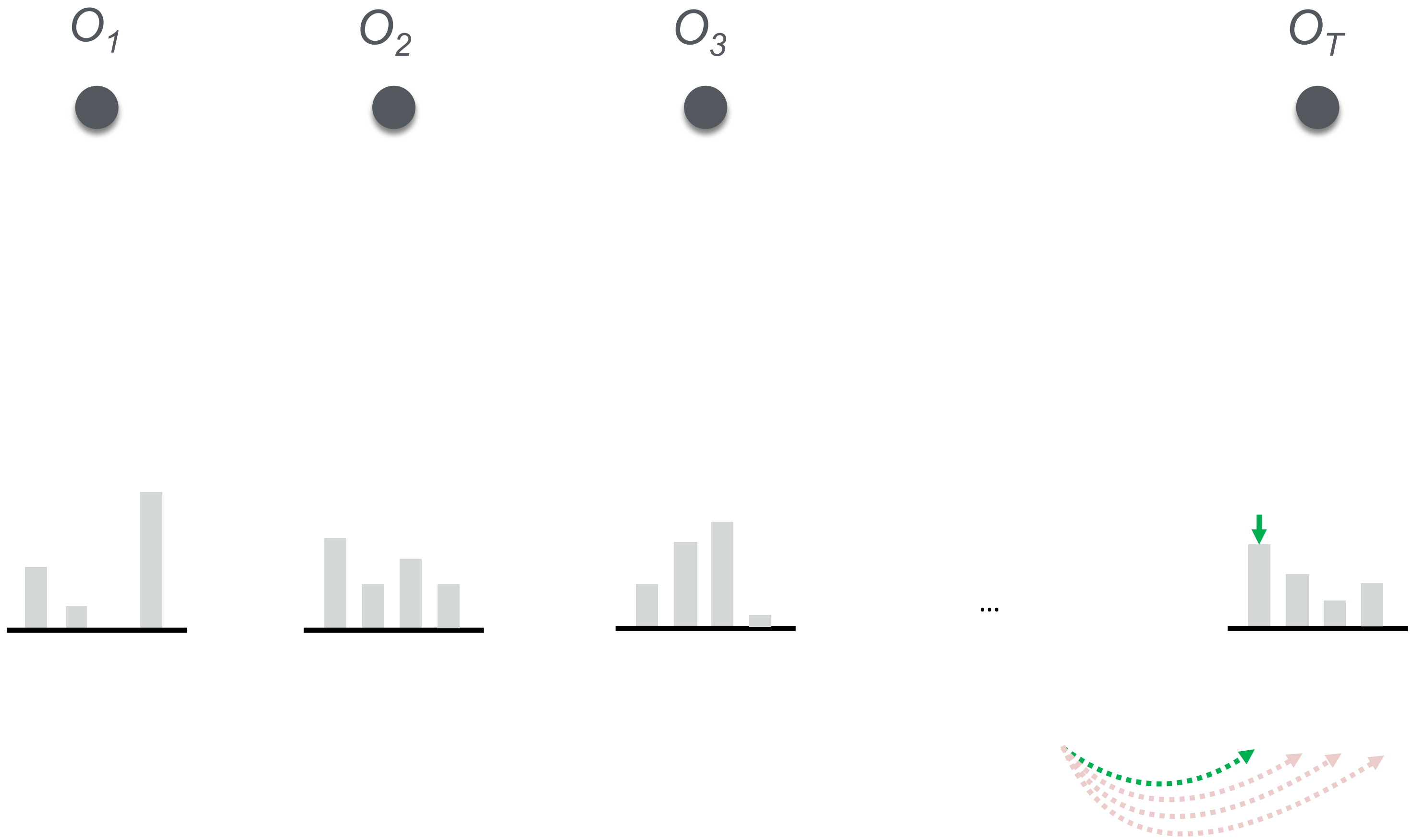

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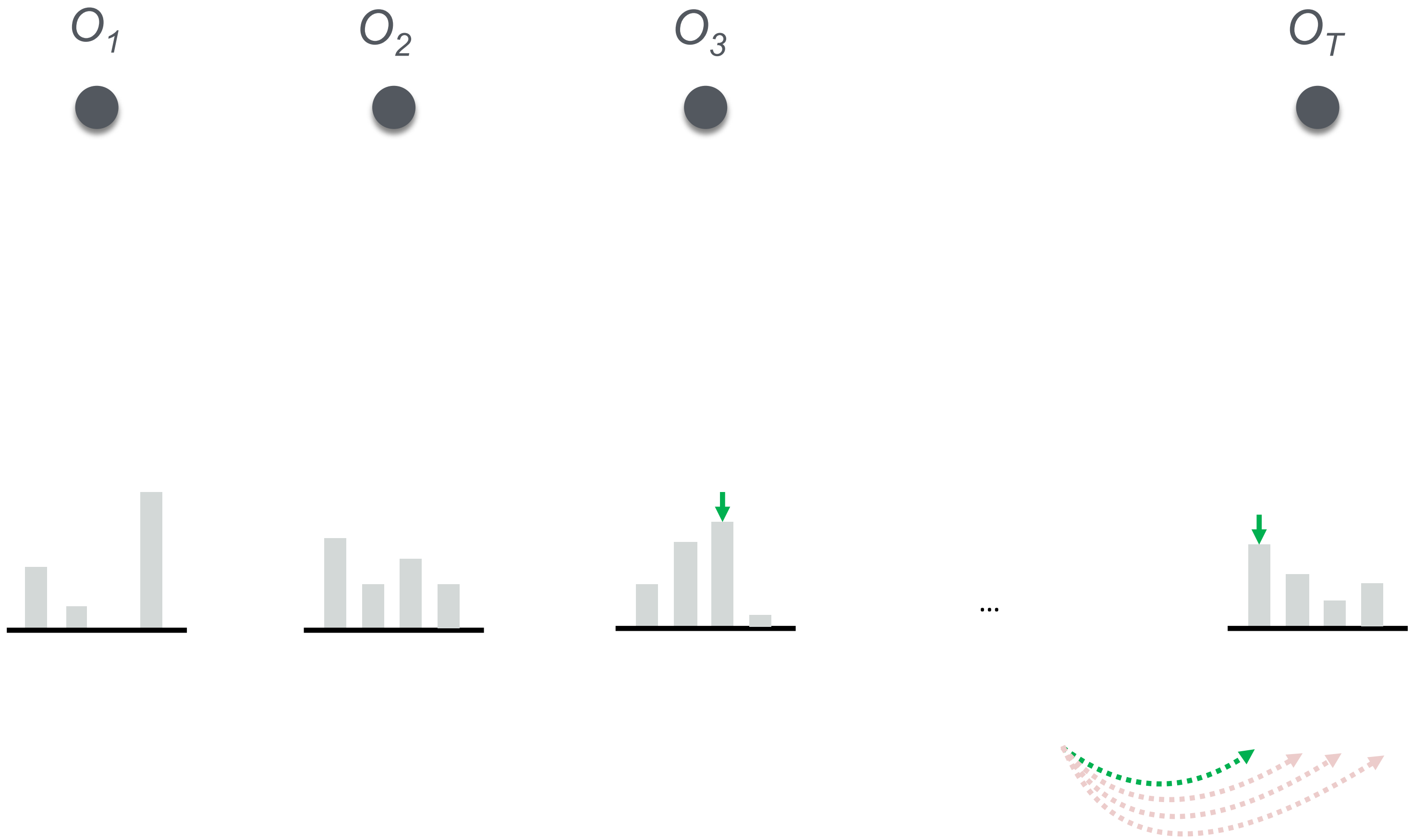

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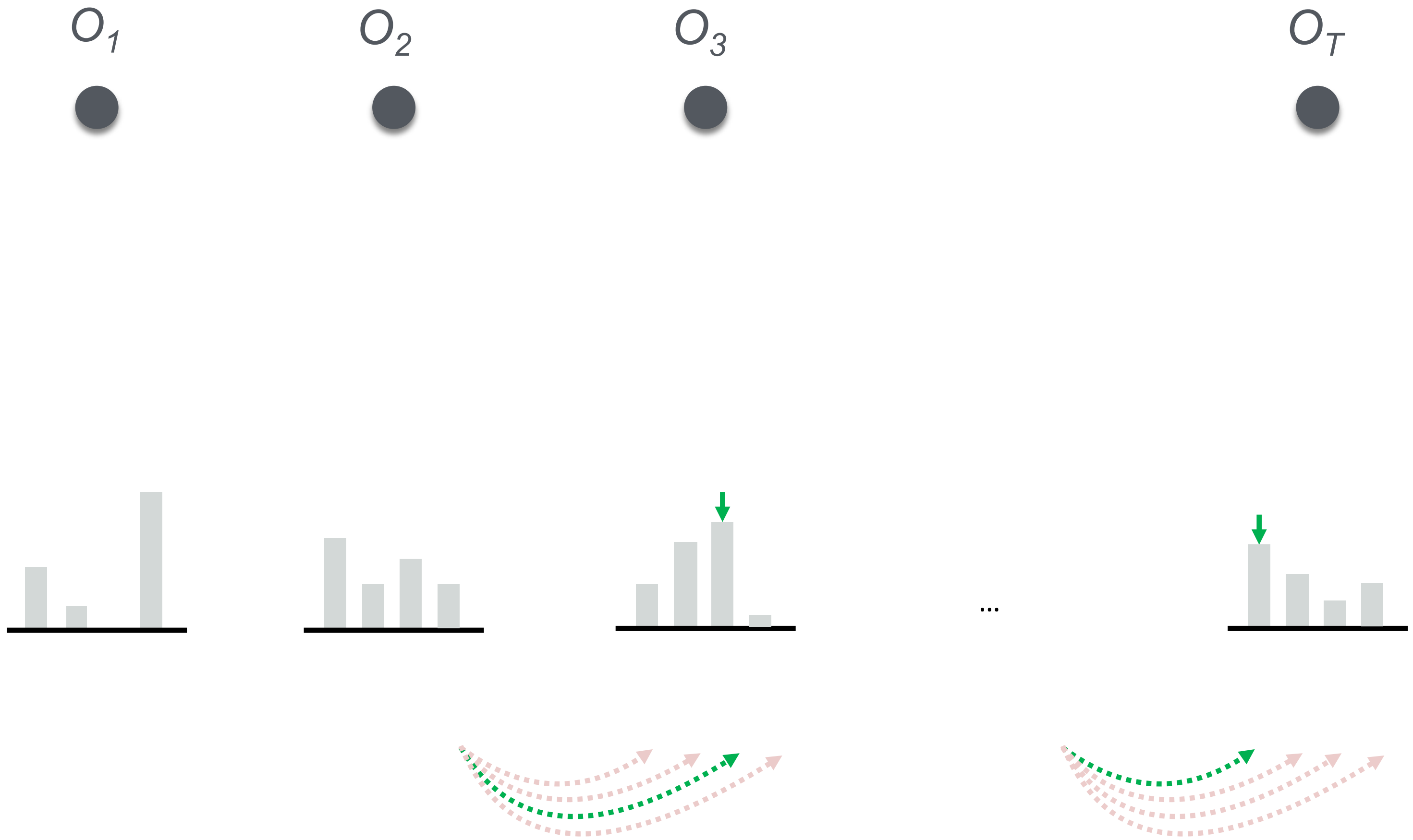

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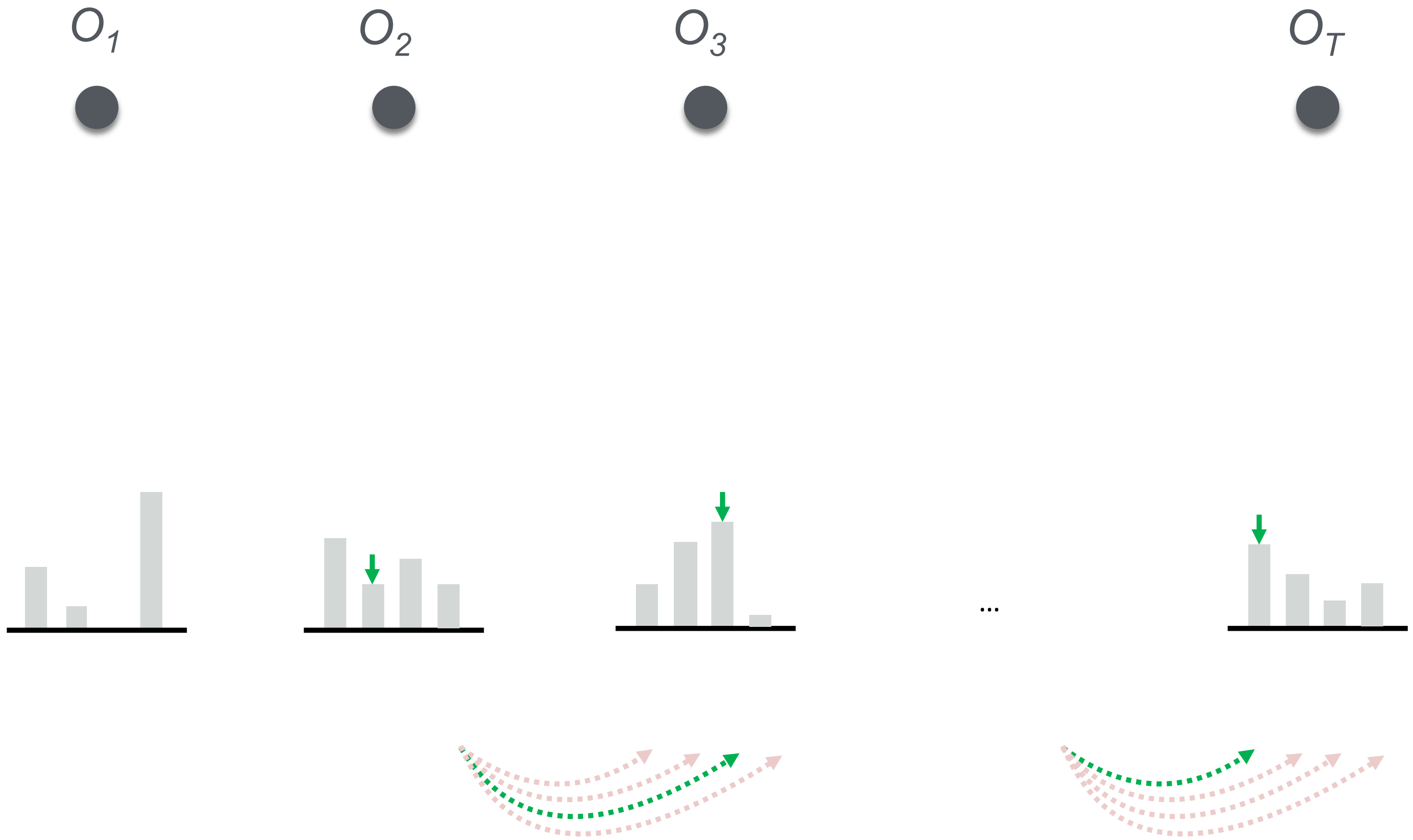

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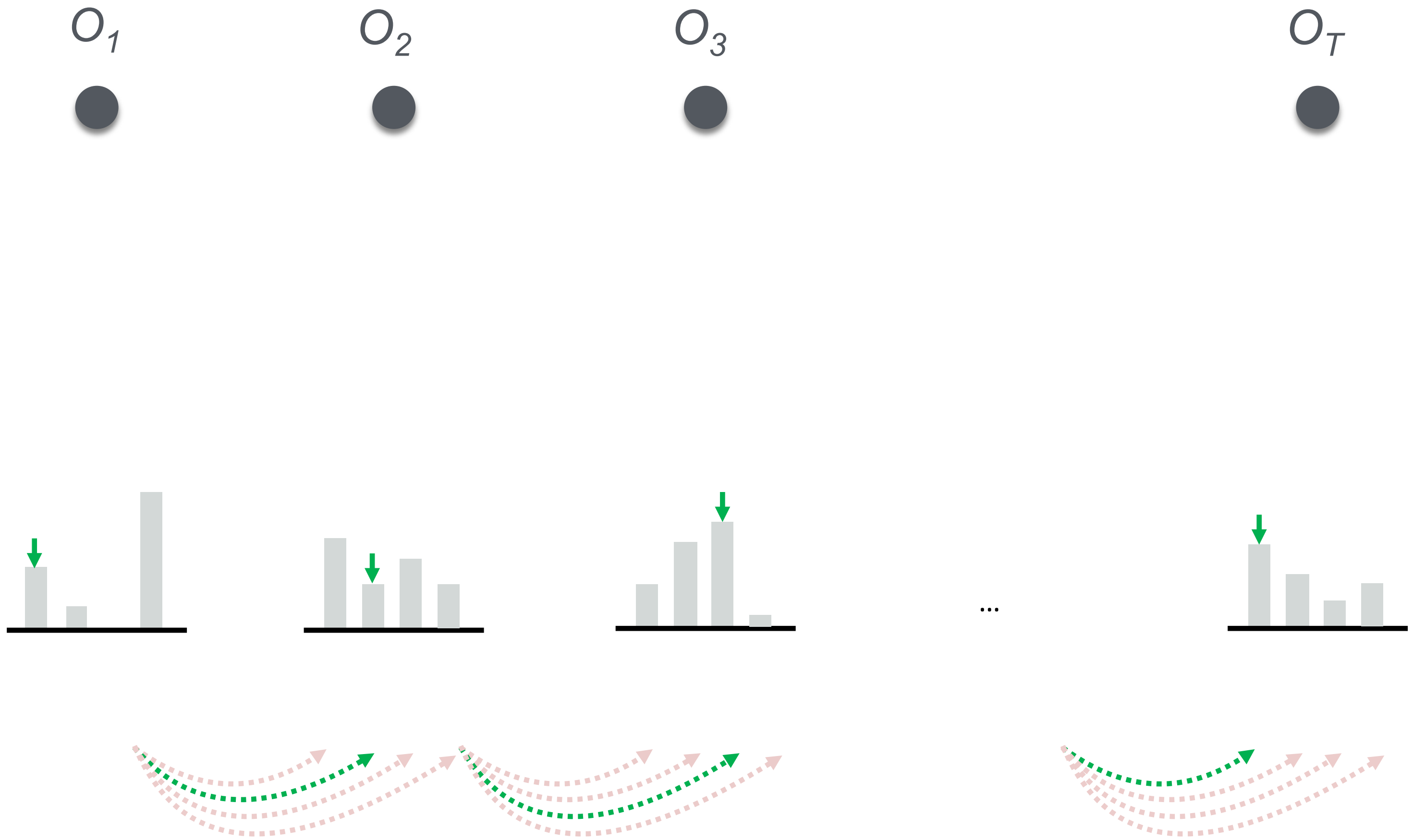

Most likely sequence (Viterbi alg.)

- Solved using dynamic programming (DP) with an algorithm called Viterbi.
  - Initialize: $\delta_1(i) = \pi_i b_i(O_1), i = 1, \ldots, N$
  - For each t>1: $\delta_t(i) = \max_{j \in \{1,\ldots,N\}} [\delta_{t-1}(j) a_{ji}] b_i(O_t)$ Partial prob of one of the most probable paths to state *i* at time *t*
  - Probability of best path: $\max_{j \in \{1,\ldots,N\}} [\delta_{T-1}(j)]$
  - Find path by keeping book of preceding states and trace back from highest-scoring final state: $\Psi = \operatorname{argmax}_{j \in \{1,\ldots,N\}} [\delta_{t-1}(j) a_{ji}]$

Forward vs. Viterbi algorithm

- Both dynamic programming algorithms
  Forward algorithm computes sums of paths,
  Viterbi computes best paths

$$\alpha_t(i) = \left[\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji}\right] b_i(O_t)$$

Forward algorithm (Sum)

$$\delta_t(i) = \max_{j \in \{1,\ldots,N\}} [\delta_{t-1}(j) a_{ji}] b_i(O_t)$$

Viterbi algorithm (Max)

Viterbi cont'd

- Watch out for underflows!!! Multiplying many small numbers (probabilities)

- Better to use
  - Initialize: $\delta_0(i) = \log[\pi_i b_i(O_0)], i = 1, \ldots, N$
  - For t>1: $\widehat{\delta_t}(i) = \max_{j \in \{1,\ldots,N\}} \left[\widehat{\delta_{t-1}}(j) + \log[a_{ji}] + \log[b_i(O_t)] \right]$
  - Probability of best path: $\max_{j \in \{1,\ldots,N\}} \left[\widehat{\delta_{T-1}}(j) \right]$

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   $\rightarrow$ A and B matrices and π
   Baum-Welch algorithm

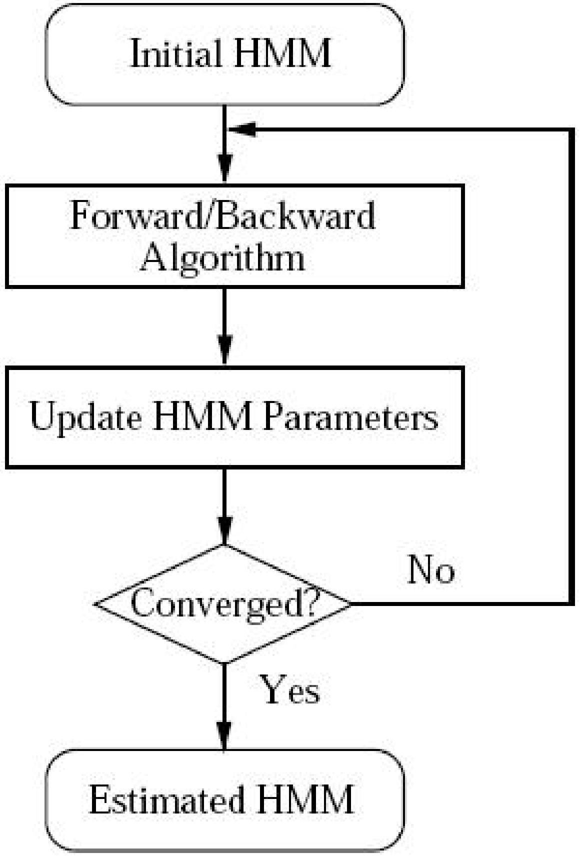
3. Estimate model parameters $\Lambda=\{\lambda\}$ given $O_{1:T}$

- Analogous to the training phase of Machine Learning
- Motivating examples:
  - Learn a model for a letter for recognizing handwritten text
  - Learn a model for the word "Bayesian" from audio data
  - Learn a model for the movement of fish in the Fishing Derby Game

The Baum-Welch Algorithm

- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

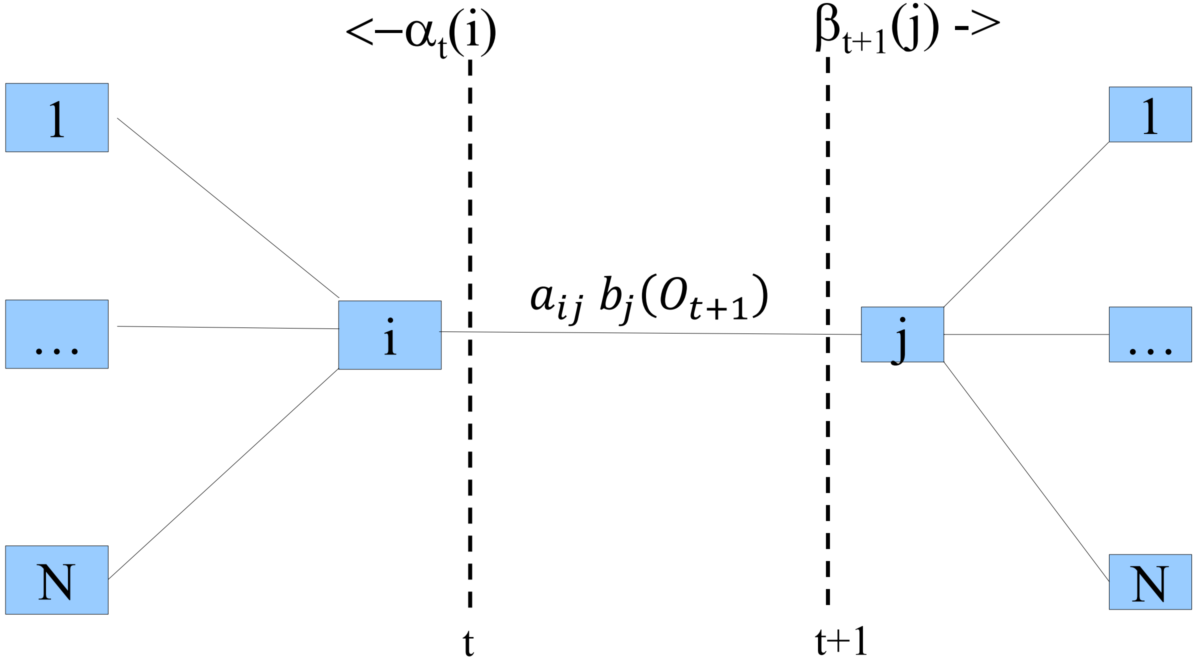
1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$, $\beta_t(k)$, $\gamma_t(i,j)$ and $\gamma_t(i)$
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ converges



GAMMA CALCULATIONS:

1) **Di – Gamma Function**

$$\gamma_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{\sum_{i=1}^{N} \alpha_T(i)} \qquad = p(X_t = i, X_{t+1} = j | O_{1:T}, \lambda)$$



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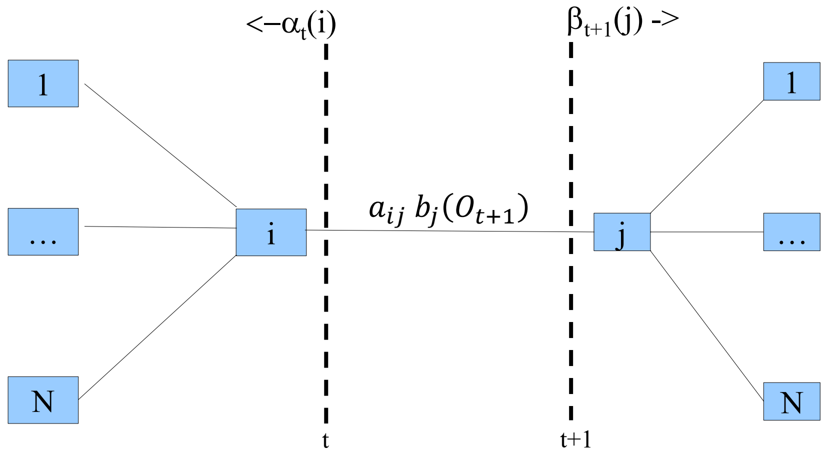

Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i) & at time (t+1) the hidden state is (X_{t+1}=j)?

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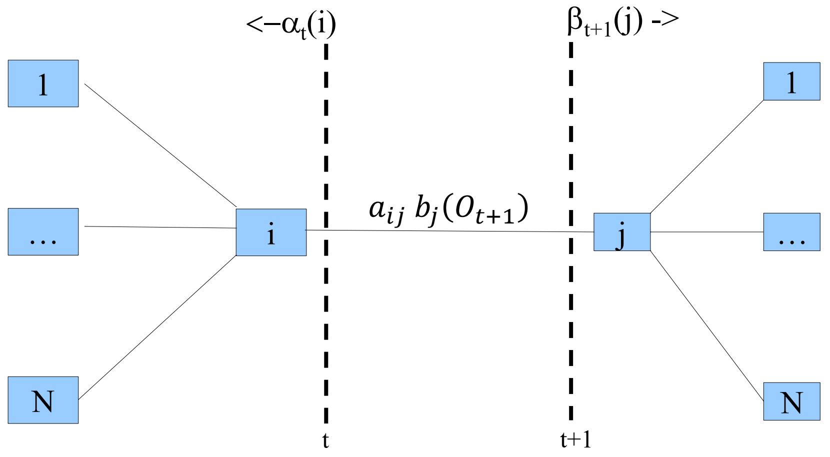
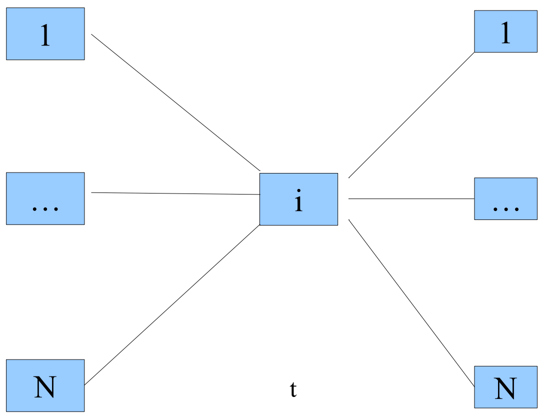


Interpretation: Given the entire observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i) & at time (t+1) the hidden state is (X_{t+1}=j)?

2) Gamma Function (Marginalizing out X_{t+1})

$$\gamma_t(i) = \sum_{j=1}^{N} \gamma_t(i,j) \qquad = p(X_t = i | O_{1:T}, \lambda)$$



Interpretation: Given the observation sequence and current estimate of the HMM, what is the probability that at time (t) the hidden state is (X_t=i)?

Learn the model, i.e. calculating A,B,π

- Given γ and di-gamma, estimate A, B, p using:

A) Transition estimates

$$a_{ij} = \frac{\sum_{t=1}^{T-1} \gamma_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} \quad \forall \quad i,j = 1,\ldots,N$$

expected number of transitions from state i to state j

expected number of times in state i (regardless of what we observe)

B) Emission estimates

$$b_j(k) = \frac{\sum_{\substack{t=1,2,\ldots,T-1 \\ O_t = k}} \gamma_t(i)}{\sum_{t=1}^{T-1} \gamma_t(i)} \quad \forall \quad j = 1,\ldots,N, \quad k = 1,\ldots,K$$

expected number of times in state i and observing k

expected number of times in state i (regardless of what we observe)

q) Initial state probabilities

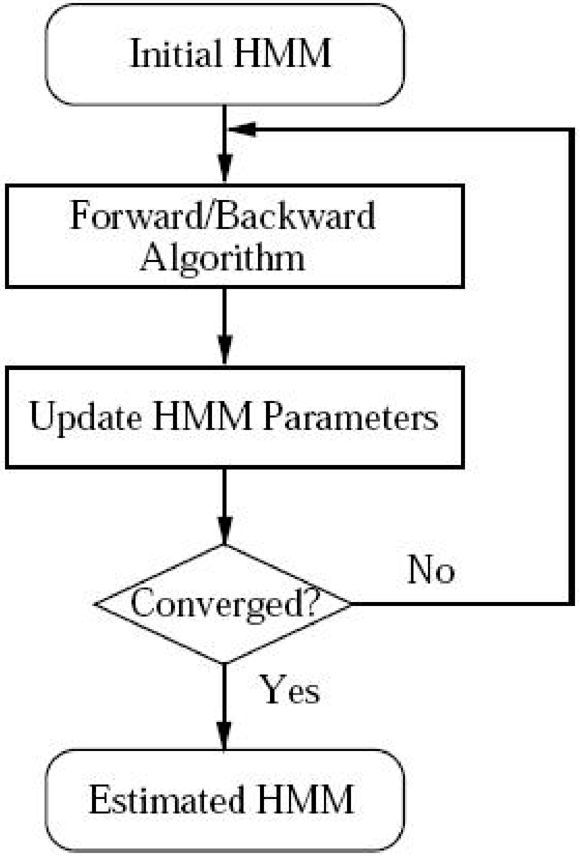
$$\pi_i = \gamma_1(i) \quad \forall \quad i = 1,\ldots,N$$

expected frequency of being in state i at t = 1

The Baum-Welch Algorithm

- Given an observation sequence $O_{1:T}$, the number of states, N, and the number of observation outcomes, M.

1. Initialize $\lambda=(A,B,\pi)$
2. Compute $\alpha_t(i)$, $\beta_t(k)$, $\gamma_t(i,j)$ and $\gamma_t(i)$
3. Re-estimate the model $\lambda=(A,B,\pi)$
4. Repeat from 2 until $p(O|\lambda)$ converges



To keep in mind

- Baum-Welch is an Expectation-Maximization (EM) algorithm used to train HMM parameters. It *uses* the forward-backward algorithm during each iteration.
- The forward-backward algorithm is just a combination of the forward and backward algorithms: one forward pass, one backward pass.
- On their own, the forward and backward algorithms are used for computing the marginal likelihoods of a sequence of states (not learning).

Model initialization and considerations

- Need to make sure that A,B,π are all row stochastic (rows sum to 1)
- Use whatever prior knowledge you have to provide good initial guesses
- If you have no clue, assign the values randomly as

  $a_{ij} \approx 1/N$

  $b_j(k) \approx 1/M$

  $\pi_i \approx 1/N$

  - Make sure that A,B and p are **not** uniform, i.e., that the values are not exactly 1/N and 1/M resp.
    - Otherwise, we are in a local maximum that we cannot get out of, and the method will not converge
    - If B is uniform, a measurement gives no info!
- Stop if too many iterations to avoid deadlocks
- Calculating long products of probabilities
  - very small numbers
  - underflow problems
  - use scaling and log likelihoods

Three problems solved with HMMs

1. **Evaluation/**filtering: Compute likelihood $p(O_{1:t}|\lambda)$
   of observation sequence ($O_{1:T}=\{O_1,O_2,\dots,O_t,\dots,O_T\}$) given λ
   Forward algorithm
   Backward algorithm

2. **Decoding**: Most likely state sequence $X^*_{1:T}$ given $O_{1:T}$ and λ
   Viterbi algorithm

3. **Learning**: Estimate model parameters
   $\Lambda=\{\lambda\}$ given $O_{1:T}$ such that $p(O_{1:T}|\lambda)$ is maximized
   → A and B matrices and π

   Baum-Welch algorithm

Additional (highly recommended) study material

- http://learn-ai.web.app/
  - Page developed by one of the previous course TAs with detailed (and visually appealing) explanations of today's lecture exercises + Forward algorithm
  - Visit the page before the HMM Tutorial sessions to consolidate

- Stamp Tutorial on Canvas containing the algorithmic implementations of these algorithms

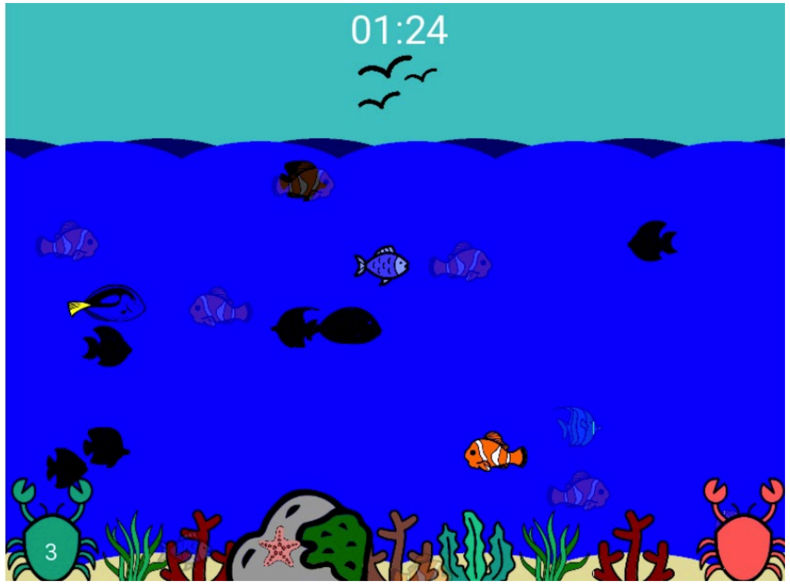
What's NEXT?

- HMM Tutorials
  - Check the schedule on Canvas for these and do not forget to attend next week (Monday)
  - Prepare tutorial exercise sheets (print or .doc)

- New quiz and last Lecture's quiz

- HMM assignment will be up after this lecture at 17:00pm!
  - Arm computers should now be able to run the assignment

- Lab sessions to start working on HMM assignment next week (Tuesday).



Motivation for HMM Assignment

- Solve an actual task, i.e., use the AI methods in a context
- Covers
  - Probabilistic reasoning
  - Machine learning
  - Decision making
  - Implementation
  - Testing and evaluation
- Work in pairs
  - Practice and implement HMMs
  - Using an existing implementation is not allowed

End of Probabilistic Reasoning Part 2/2 - HMMs