Satellite Servicing Equations of Motion

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This project will solve for the equations of motion of a target satellite and a robotic servicing satellite so that the servicing satellite can orient itself with thrusters and determine when the target satellite is reachable.

I. Nomenclature

B_S	=	servicer satellite body fixed frame
B_T	=	target satellite body fixed frame
c	=	cosine
l	=	side length of cube
L	=	length of one robotic arm
m	=	mass of one robotic arm
M	=	mass of servicer satellite cube
$\vec{T}$	=	torque
$\mathcal{RSW}$	=	intertial local satellite frame
$\mathcal{R}$	=	along position vector
s	=	sine
$\mathcal{S}$	=	along direciton of velocity
$\mathcal{W}$	=	normal to orbital plane
ω	=	angular velocity in body fixed coordinates

II. Problem Description

SATELLITE servicing is an emerging field within the space domain and its importance continues to grow as space traffic becomes more crowded. This project will analyze the dynamics of a servicer satellite that will use bang-bang thrusters to position itself so that it can use its robotic arm to grab onto a slowly rotating target satellite. Explicitly, this project will solve for the equations of motion of both the target satellite and the servicing satellite to determine how the servicer satellite can use the thrusters to orient itself and decide when the target satellite is reachable with the robotic arm.

A. Assumptions and Simplifications

The assumptions and simplifications necessary to approach satellite servicing within the scope of this class are:

- There are no perturbations acting on the system of satellite, i.e no drag, earth oblateness, solar radiation pressure
- Mass of the satellites remains constant
- The target satellite is rotating with a non negligible angular velocity with respect to the inertial frame
- Both satellites are perfect cubes with uniform mass distribution
- Robotic arm is made of two identical uniform density slender rods
- All bodies are rigid, i.e the distance between particles in bodies remains constant
- Both satellites are in the same circular orbit and separated by a distance shorter than the outstretched length of the robotic arm
- Thrusters are massless and perform ideally
- Time scale of dynamics is short enough such that the RSW coordinate system can be used as an inertial reference frame
- Ignore gravity gradient effect on extended arm

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B. Drawing

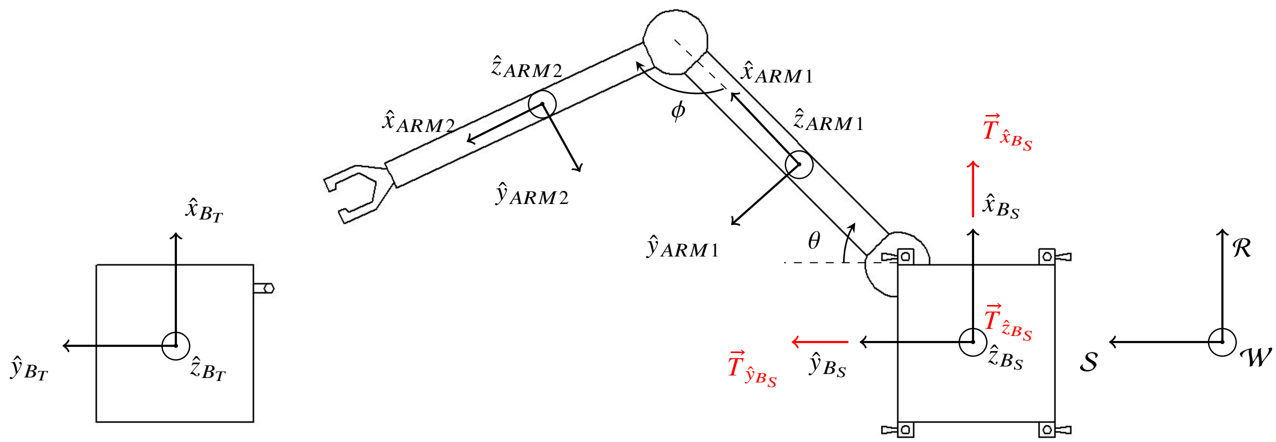

Fig. 1 Free Body Diagram

III. Technical Approach

A. Problem setup

The servicer satellite and the target satellite will be treated as two independent systems where the target system is made up of the target cube and the servicer system contains the servicer cube and the two part robotic arm. The target satellite will have 3 degrees of freedom that will be represented as the euler angles θ, ϕ and γ through a 3-2-1 rotation. The servicer satellite will also use these three euler angles to describe its attitude, but it will have two additional degrees of freedom (θ and ϕ from figure 1, not to be confused with euler angles) because of the robotic arm. However, since the robotic arm is fixed until the servicer satellite determines that the target satellite is in a reachable position, the angles describing the position of the arm can be excluded from the degrees of freedom, thereby leaving the servicer satellite with three degrees of freedom as well. The number of equations generated from the solution methods will be equal to the degrees of freedom, thus this report does not explore any system constraints. The necessary reference frames and coordinate systems are enumerated below.

1) $\mathcal{RSW}$: Interial local reference frame. Necessary to define the euler angles of the servicer satellite and the target satellite
2) B_S: Body fixed servicer frame. Necessary to define inertia tensor and attitude of servicer satellite
3) ($\hat{x}_{B_S}$, $\hat{y}_{B_S}$, $\hat{z}_{B_S}$): Body fixed servicer coordinates. Necessary to fill out servicer inertia tensor and angular velocity in body coordinates
4) $B_T 0$: Body fixed target frame. Necessary to define inertia tensor and attitude of target satellite
5) ($\hat{x}_{B_T}$, $\hat{y}_{B_T}$, $\hat{z}_{B_T}$): Body Fixed target coordinates. Necessary to fill out target inertia tensor and angular velocity in body coordinates
6) $ARM1$: Body fixed frame of ARM1. Necessary to define inertia tensor about center of mass of ARM1
7) ($\hat{x}_{ARM_1}$, $\hat{y}_{ARM_1}$, $\hat{z}_{ARM_1}$): Body fixed coordinate system of ARM1. Necessary to define inertia tensor about center of mass of ARM1
8) $ARM2$: Body fixed frame of ARM2. Necessary to define inertia tensor about center of mass of ARM2
9) ($\hat{x}_{ARM_2}$, $\hat{y}_{ARM_2}$, $\hat{z}_{ARM_2}$): Body fixed coordinate system of ARM2. Necessary to define inertia tensor about center of mass of ARM2

All of the solution methods will require the inertia tensor of the satellite servicer and the target satellite. The latter is straightforward. With the body coordinates as principal axes, the inertia tensor of the target satellite is simply

$$\{I_{target}\} = \begin{bmatrix} \frac{mL^2}{6} & 0 & 0 \\ 0 & \frac{mL^2}{6} & 0 \\ 0 & 0 & \frac{mL^2}{6} \end{bmatrix}$$

The satellite servicer is significantly more complicated because of the two part robotic arm. The procedure for finding the inertia tensor about the B_S frame in B_S coordinates will be as follows.

1) Find inertia tensor of Arm1 and Arm2 about their respective centers of mass in terms of their body fixed coordinate systems
2) Rotate both tensors so that their axes are parallel to the servicer satellite body fixed frame (B_S)
3) Shift the inertia tensors of Arm1 and Arm2 to the center of the servicer cube using parallel axis theorem
4) Sum the three inertia tensors (Arm1, Arm2, and cube) to generate the inertia tensor about the center of the servicer cube in body fixed coordinates

The robotic arm is treated as two slender rods, thus the inertia tensor about their respective centers of mass is the same for both rods.

$$I_{ARM1} = I_{ARM2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix}$$

To use the parallel axis theorem for each of the axes, it is first necessary to rotate the inertia tensors. The z axes are already parallel, as is shown in figure 1. Thus the rotation for both Arm1 and Arm2 will be a rotation about the z axis. From inspection, the rotation for arm1 will be a $R_3(-(90-\theta))$ and the rotation for arm will be a $R_3(-(270-(\theta+\phi)))$. To simplify the expressions of the tensors, the following associations will be made and kept throughout the rest of the report: $C = sin((\theta+\phi)-270)$ $\quad \mathcal{D} = cos((\theta+\phi)-270)$. The inertia rotations for both arms are expressed below

$$\{I'_{ARM1}\} = \begin{bmatrix} c(\theta-90) & s(\theta-90) & 0 \\ -s(\theta-90) & c(\theta-90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix} \begin{bmatrix} c(\theta-90) & -s(\theta-90) & 0 \\ s(\theta-90) & c(\theta-90) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{I'_{ARM1}\} = \begin{bmatrix} \frac{mL^2}{12}s^2(\theta-90) & \frac{mL^2}{12}c(\theta-90)s(\theta-90) & 0 \\ \frac{mL^2}{12}c(\theta-90)s(\theta-90) & \frac{mL^2}{12}c^2(\theta-90) & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix}$$

$$\{I'_{ARM2}\} = \begin{bmatrix} \mathcal{D} & C & 0 \\ -C & \mathcal{D} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix} \begin{bmatrix} \mathcal{D} & -C & 0 \\ C & \mathcal{D} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{I'_{ARM2}\} = \begin{bmatrix} \frac{mL^2}{12}C^2 & \frac{mL^2}{12}C\mathcal{D} & 0 \\ \frac{mL^2}{12}C\mathcal{D} & \mathcal{D}^2 & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix}$$

The inertia tensors with respect to the body fixed frames of the robotic arms are now parallel to B_S frame, so the parallel axis theorem can be applied to each of the Inertia Tensors of Arm 1 and Arm2 to shift them to the body fixed servicer frame B_S. The parallel axis theorem for non principal inertia tensors is $J_{ij} = I_{ij}m(|\mathbf{R}|^2\delta_{ij} + R_iR_j)$ where δ_{ij} is 0 when $i = j$ and 1 when $i \neq j$. The displacement vectors for Arm1 and Arm 2 are

$$R_{ARM1} = \left[\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big), \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big), \sqrt{\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)^2 + \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)^2} \right]$$

$$R_{ARM2} = \left[\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}s(\phi - 90 - \theta)\big), \big(\frac{l}{2} + Ls(\theta) - \frac{L}{2}c(\phi - 90 - \theta)\big), \right.$$
$$\left. \sqrt{\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}s(\phi - 90 - \theta)\big)^2 + \big(\frac{l}{2} + Ls(\theta) - \frac{L}{2}c(\phi - 90 - \theta)\big)^2} \right]$$

The assignments $\mathcal{A} = sin(\phi - 90 - \theta)$ $\mathcal{B} = cos(\phi - 90 - \theta)$ and $\mathcal{R}_3 = R_{ARM2}(3)$ are introduced to simplify the matrices. Perform the summations:

$$I_{ARM1} = \begin{bmatrix} \frac{mL^2}{12}s^2(\theta - 90) + m\Big(\frac{l^2}{4} + \frac{lL}{2}c(\theta) + \frac{L^2}{4}c^2(\theta)\Big) & \frac{mL^2}{12}c(\theta - 90)s(\theta - 90) - m\Big[\frac{l^2}{4} + \frac{lL}{4}\big(s(\theta)c(\theta)\big) + \frac{L^2}{4}\big(c(\theta)s(\theta)\big)\Big] & \frac{-m\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)*}{\sqrt{\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)^2 + \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)^2}} \\ \frac{mL^2}{12}c(\theta - 90)s(\theta - 90) - m\Big[\frac{l^2}{4} + \frac{lL}{4}\big(s(\theta)c(\theta)\big) + \frac{L^2}{4}\big(c(\theta)s(\theta)\big)\Big] & \frac{mL^2}{12}c^2(\theta - 90) + m\Big(\frac{l^2}{4} + \frac{lL}{2}s(\theta) + \frac{L^2}{4}s^2(\theta)\Big) & \frac{-m\big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)*}{\sqrt{\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)^2 + \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)^2}} \\ \frac{-m\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)*}{\sqrt{\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)^2 + \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)^2}} & \frac{-m\big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)*}{\sqrt{\big(\frac{l}{2} + \frac{L}{2}c(\theta)\big)^2 + \big(\frac{l}{2} + \frac{L}{2}s(\theta)\big)^2}} & \frac{mL^2}{12} + m\Big(\frac{l^2}{2} + \frac{lL}{2}\big(c(\theta) + s(\theta)\big) + \frac{L^2}{4}\Big) \end{bmatrix}$$

$$I_{ARM2} = \begin{bmatrix} \frac{mL^2}{12}C^2 + m\Big(\frac{l^2}{4} + lLc(\theta) + L^2c^2(\theta) + \frac{lL}{2}\mathcal{A} + L^2c(\theta)\mathcal{A} + \frac{L^2}{4}\mathcal{A}^2\Big) & \frac{mL^2}{12}C\mathcal{D} - m\Big[\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}\mathcal{A}\big)\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}\mathcal{B}\big)\Big] & -m\big(\tfrac{l}{2} + Lc(\theta) - \tfrac{L}{2}\mathcal{A}\big)\mathcal{R}_3 \\ \frac{mL^2}{12}C\mathcal{D} - m\Big[\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}\mathcal{A}\big)\big(\frac{l}{2} + Lc(\theta) + \frac{L}{2}\mathcal{B}\big)\Big] & \frac{mL^2}{12}\mathcal{D}^2 + m\Big(\frac{l^2}{4} + lLs(\theta) + L^2s^2(\theta) - \frac{lL}{2}\mathcal{B} - L^2s(\theta)\mathcal{B} + \frac{L^2}{4}\mathcal{B}^2\Big) & -m\big(\tfrac{l}{2} + Ls(\theta) - \tfrac{L}{2}\mathcal{B}\big)\mathcal{R}_3 \\ -m\big(\tfrac{l}{2} + Lc(\theta) - \tfrac{L}{2}\mathcal{A}\big)\mathcal{R}_3 & -m\big(\tfrac{l}{2} + Ls(\theta) - \tfrac{L}{2}\mathcal{B}\big)\mathcal{R}_3 & \frac{mL^2}{12} + m\Big(\frac{l^2}{2} + \frac{5}{4}L^2 + \frac{lL}{2}(\mathcal{A} - \mathcal{B}) + lL\big(c(\theta)s(\theta) + L^2(c(\theta)\mathcal{A} - s(\theta)\mathcal{B})\Big) \end{bmatrix}$$

Inertia tensor of satellite servicer cube in in B_S coordinates with respect to B_S frame:

$$\{I_{cube}\} = \begin{bmatrix} \frac{mL^2}{6} & 0 & 0 \\ 0 & \frac{mL^2}{6} & 0 \\ 0 & 0 & \frac{mL^2}{6} \end{bmatrix}$$

Sum everything together for the final Inertia Tensor of the satellite servicer with respect to the B_S frame in B_S coordinates:

$$\{I_{servicer}\} = \{I_{cube}\} + \{I_{ARM1}\} + \{I_{ARM2}\}$$

B. Solutions

1. Conservation

- Angular momentum
  - Satellite Servicer: Not conserved because the satellite thrusters will impart an external torque on the system
  - Target Satellite: Conserved because no external torque on system
- Linear momentum is conserved for both systems because the there are no external forces that act along the direction of motion. This is unfortunately not a useful observation because the problem is concerned purely with the rotational dynamics and not the translational dynamics.
- Energy
  - Satellite Servicer: not conserved because the thrusters will introduce energy into the system when they induce a satellite rotation and then successively remove energy from the system when they halt a satellite rotation. Energy is also not conserved because the setup does not consider the internal processes of the thrusters.
  - Target Satellite: Conserved because there are no dissipative forces acting on the target satellite.

The equations of motion for the satellite servicer cannot be derived using a conservation approach. Both energy and angular momentum conservation can be used to solve for the target satellite equations of motion. Using angular momentum

$$\vec{H} = \{I_{target}\}\,\vec{\omega} \quad \left(\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\right)_S = \vec{0}$$

The derivative is with respect to the inertial frame, so the golden rule is necessary

$$\left(\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\right)_B + \vec{\omega}\times\vec{H} = \vec{0} \quad \vec{\omega}\times\vec{H} = \begin{vmatrix} \hat{x}_{B_T} & \hat{y}_{B_T} & \hat{z}_{B_T} \\ \omega_1 & \omega_2 & \omega_3 \\ I_{11}\omega_1 & I_{22}\omega_2 & I_{33}\omega_3 \end{vmatrix}$$

The cross product results in all 0s because the moments of inertia of a cube are all the same, thus the derivative with respect to the body frame is zero for all components, leaving the equations of motion for the target satellite as

$$\dot{\omega}_1 = 0 \quad \dot{\omega}_2 = 0 \quad \dot{\omega}_3 = 0$$

2. Force Balance

Begin with the definition of external moments as the dot product of the angular momentum vector. Since the time derivative is with respect to the space frame, it is necessary to apply the golden rule.

$$\vec{M} = \dot{\vec{H}}_s \quad \vec{M} = \left(\frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\right)_B + \vec{\omega}\times\vec{H}$$

Recall the equation for angular momentum $\vec{H} = \{I_{servicer}\}\omega$. The inertia tensor $\{I_{servicer}\}$ is not about a principal axes, and rotating it to a principal axes would eliminate the benefit of having the external torques being exactly analogous to the torques applied by the coupled thrusters.

$$\vec{H} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1\hat{x}_{B_s} \\ \omega_2\hat{y}_{B_s} \\ \omega_3\hat{z}_{B_S} \end{bmatrix} \quad \vec{H} = \begin{bmatrix} I_{11}\omega_1\hat{x}_{B_s} + I_{12}\omega_2\hat{y}_{B_s} + I_{13}\omega_3\hat{z}_{B_s} \\ I_{21}\omega_1\hat{x}_{B_s} + I_{22}\omega_2\hat{y}_{B_s} + I_{23}\omega_3\hat{z}_{B_s} \\ I_{31}\omega_1\hat{x}_{B_s} + I_{32}\omega_2\hat{y}_{B_s} + I_{33}\omega_3\hat{z}_{B_s} \end{bmatrix}$$

With an expression for angular momentum, it is now possible to perform the derivative and cross product.

$$\dot{\vec{H}}_B = \left[(I_{11}\dot{\omega}_1 + I_{21}\dot{\omega}_1 + I_{31}\dot{\omega}_1)\hat{x}_{B_s} + (I_{12}\dot{\omega}_2 + I_{22}\dot{\omega}_2 + I_{32}\dot{\omega}_2)\hat{y}_{B_s} + (I_{13}\dot{\omega}_3 + I_{23}\dot{\omega}_3 + I_{33}\dot{\omega}_3)\hat{z}_{B_s}\right]$$

$$\vec{\omega}\times\vec{H} = \begin{vmatrix} \hat{x}_{B_s} & \hat{y}_{B_s} & \hat{z}_{B_S} \\ \omega_1 & \omega_2 & \omega_3 \\ (I_{11}\omega_1 + I_{21}\omega_1 + I_{31}\omega_1) & (I_{12}\omega_2 + I_{22}\omega_2 + I_{32}\omega_2) & (I_{13}\omega_3 + I_{23}\omega_3 + I_{33}\omega_3) \end{vmatrix}$$

The vector $\vec{M} = \begin{bmatrix} \vec{T}_{\hat{x}_{B_s}} & \vec{T}_{\hat{y}_{B_s}} & \vec{T}_{\hat{z}_{B_s}} \end{bmatrix}$ consists of all the external torques and can thus be equated to $\dot{\vec{H}}_B + \vec{\omega} \times \vec{H}$. This results in the Equations of Motion for the servicer satellite as presented below. The external torques are force couples applied by the thrusters.

$$\begin{aligned}
\vec{T}_{\hat{x}_{B_s}} &= \dot{\omega}_1(I_{11} + I_{21} + I_{31}) + \omega_1\Big(\omega_2(I_{13}\omega_3 + I_{23}\omega_3 + I_{33}\omega_3) - \omega_3(I_{12}\omega_2 + I_{22}\omega_2 + I_{32}\omega_2)\Big) \\
\vec{T}_{\hat{y}_{B_s}} &= \dot{\omega}_2(I_{12} + I_{22} + I_{32}) - \omega_2\Big(\omega_1(I_{13}\omega_3 + I_{23}\omega_3 + I_{33}\omega_3) - \omega_3(I_{11}\omega_1 + I_{21}\omega_1 + I_{31}\omega_1)\Big) \\
\vec{T}_{\hat{z}_{B_s}} &= \dot{\omega}_3(I_{12} + I_{22} + I_{33}) + \omega_3\Big(\omega_1(I_{12}\omega_2 + I_{22}\omega_2 + I_{32}\omega_2) - \omega_2(I_{11}\omega_1 + I_{21}\omega_1 + I_{31}\omega_1)\Big)
\end{aligned}$$

The Force Balance method results in the Euler Equations of Motion with the external moments as the force couples applied by the satellite thrusters. This is an incomplete solution because the $\vec{\omega}$ is expressed in body coordinates and is thus non observable. However, following the Lagrangian approach, it will become clear that using the Euler's EOM to propagate the dynamics instead of using the analytical solution (arrived at from Lagrange) will be necessary. The external moments from Euler's EOM are conveniently setup as the thruster force couples and the instantaneous Euler angles and rates can be pulled from this propagation at every time step. Although this is definitely a sub optimal approach, it is significantly simpler than implementing the analytical approach from Lagrange.

The force balance approach is much simpler for the target satellite since the inertia tensor is expressed in principal axes. The process is the same as for the servicer satellite, and since all the diagonal elements are the same for a cube and the motion is torque free, the equations of motion for the target satellite are

$$\dot{\omega}_1 = 0 \quad \dot{\omega}_2 = 0 \quad \dot{\omega}_3 = 0$$

3. Lagrange

Given the assumptions and simplifications mentioned in the beginning of this report, the Lagrangian will only contain rotational kinetic energy. The Lagrangian for the target satellite is straightforward because the body axes are principal axes.

$$\begin{aligned}
\mathcal{L} &= T_{rot} = \frac{1}{2}\vec{\omega}^T\{I_{target}\}\vec{\omega} \\
\mathcal{L} &= \frac{1}{2}(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2)
\end{aligned}$$

But we cannot work with values of $\vec{\omega}$ because they are non observable values and do not correspond to any generalized coordinates. So let us define the euler angles θ, ϕ, and γ as the generalized coordinates and follow a 3-2-1 rotation sequence to generate relations between these coordinates and the values of $\vec{\omega}$. From a 3-2-1 euler rotation, these relations are as follows:

$$\begin{aligned}
\omega_1 &= \dot{\phi} - \dot{\gamma}s(\theta) \\
\omega_2 &= \dot{\theta}c(\phi) + \dot{\gamma}c(\theta)s(\phi) \\
\omega_3 &= \dot{\gamma}c(\theta)c(\phi) - \dot{\theta}s(\phi)
\end{aligned}$$

Performing the substitution results in

$$\mathcal{L} = \frac{1}{2}\left[I_{11}\Big(\dot{\phi} - \dot{\gamma}s(\theta)\Big)^2 + I_{22}\Big(\dot{\theta}c(\phi) + \dot{\gamma}c(\theta)s(\phi)\Big)^2 + I_{33}\big(\dot{\gamma}c(\theta)c(\phi) - \dot{\theta}s(\phi)\big)^2\right]$$

Evaluate Lagrange's equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = Q_j$ for each of the Euler Angles θ, ϕ, and γ with Q_j as 0 for each of the equations because the target satellite is torque free.

Lagrange's equation for θ :

$$0 = \frac{1}{2}\Bigg\{ I_{22}\Big[2\ddot{\theta}c^2(\theta) - 4\dot{\theta}\dot{\phi} + 2\ddot{\gamma}c(\phi)c(\theta)s(\phi) - 2\dot{\gamma}\dot{\phi}s^2(\phi)c(\theta) - 2\dot{\gamma}\dot{\theta}c(\phi)s(\theta)s(\phi) + 2\dot{\gamma}\dot{\phi}c^2(\phi)c(\theta)\Big] + \ldots$$

$$\ldots + I_{33}\Big[- 2\ddot{\gamma}c(\theta)c(\phi)s(\phi) + 2\dot{\gamma}\dot{\theta}s(\theta)c(\phi)s(\phi) + 2\dot{\gamma}\dot{\phi}c(\theta)s^2(\phi) - 2\dot{\gamma}\dot{\phi}c(\theta)c^2(\phi) + 2\ddot{\theta}s^2(\phi) + 4\dot{\theta}\dot{\phi}s(\theta)c(\phi)\Big] + \ldots$$

$$\ldots - \Bigg[I_{11}\Big[- 2\dot{\gamma}c(\theta) + 2\dot{\gamma}^2 s(\theta)c(\theta)\Big] + I_{22}\Big[- 2\dot{\theta}\dot{\gamma}c(\phi)s(\theta)s(\phi) - 2\dot{\gamma}^2 c(\theta)s(\theta)s^2(\phi)\Big] + \ldots$$

$$\ldots + I_{33}\Big[- 2\dot{\gamma}^2 s(\theta)c(\theta)c^2(\phi) + 2\dot{\gamma}\dot{\theta}s(\theta)c(\phi)s(\phi)\Big]\Bigg]\Bigg\}$$

Lagrange's equation for ϕ :

$$0 = \frac{1}{2}\Bigg\{ I_{11}\ddot{\phi} - \Bigg[I_{22}\Big[- 2c(\phi)s(\phi) - 2\dot{\theta}\dot{\gamma}s^2(\phi)c(\theta) + 2\dot{\theta}\dot{\gamma}c^2(\phi)c(\theta) + 2\dot{\gamma}c^2(\theta)c(\phi)s(\phi)\Big] + \ldots$$

$$\ldots + I_{33}\Big[- 2\dot{\gamma}^2 c^2(\theta)c(\phi)s(\phi) + 2\dot{\gamma}\dot{\theta}c(\theta)s^2(\phi) - 2\dot{\gamma}\dot{\theta}c(\theta)c^2(\phi) + 2\dot{\theta}^2 c(\phi)s(\phi)\Big]\Bigg]\Bigg\}$$

Lagrange's equation for γ :

$$0 = \frac{1}{2}\Bigg\{ I_{11}\Big[- 2\dot{\theta}c(\theta) + 2\ddot{\gamma}s^2(\theta) + 4\dot{\gamma}\dot{\theta}c(\theta)c(\theta)\Big] + I_{22}\Big[2\ddot{\theta}c(\phi)c(\theta)s(\phi) + 2\dot{\theta}\dot{\phi}c^2(\phi)c(\theta) - 2\dot{\theta}\dot{\phi}s^2(\phi)c(\theta) + \ldots$$

$$\ldots - 2\dot{\theta}^2 c(\phi)s(\theta)s(\phi) + 2\ddot{\gamma}c^2(\theta)s^2(\phi) - 4\dot{\gamma}\dot{\theta}c(\theta)s(\theta)s^2(\phi) + 4\dot{\gamma}\dot{\phi}c^2(\theta)s(\phi)c(\phi)\Big] + I_{33}\Big[2\ddot{\gamma}c^2(\theta)c^2(\phi) + \ldots$$

$$\ldots - 4\dot{\gamma}\dot{\theta}c(\theta)s(\theta)c^2(\phi) - 4\dot{\gamma}\dot{\phi}c^2(\theta)c(\phi)s(\phi) - 2\ddot{\theta}c(\theta)c(\phi)s(\phi) + 2\dot{\theta}^2 s(\theta)c(\phi)s(\phi) + 2\dot{\theta}\dot{\phi}c(\theta)s^2(\phi) - 2\dot{\theta}\dot{\phi}c(\theta)c^2(\phi)\Big]\Bigg\}$$

The target satellite is simply a cube with $I_{11} = I_{22} = I_{33} = \frac{mL^2}{6}$ and the equations of motion reduce nicely into

$$\ddot{\theta} = -2\dot{\gamma}c(\theta) \quad \ddot{\phi} = 0 \quad \ddot{\gamma} = \frac{c(\theta)}{s^2(\theta)}$$

The Lagrangian for the servicer can be setup as follows:

$$\mathcal{L} = T_{rot} = \frac{1}{2}\vec{\omega}^T\{I_{servicer}\}\vec{\omega}$$
$$\mathcal{L} = \frac{1}{2}\Big[I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2 + \omega_2\omega_1(I_{12} + I_{21}) + \omega_3\omega_1(I_{13} + I_{31}) + \omega_2\omega_3(I_{23} + I_{32})\Big]$$

Performing the substitutions for $\vec{\omega}$ leaves us with

$$\mathcal{L} = \frac{1}{2}\Bigg[I_{11}\Big(\dot{\phi} - \dot{\gamma}s(\theta)\Big)^2 + I_{22}\Big(\dot{\theta}c(\phi) + \dot{\gamma}c(\theta)s(\phi)\Big)^2 + I_{33}\big(\dot{\gamma}c(\theta)c(\phi) - \dot{\theta}s(\phi)\big)^2 + \Big(I_{12} + I_{21}\Big)\Big(\dot{\phi} - \dot{\gamma}s(\theta)\Big)\Big(\dot{\theta}c(\phi) + \dot{\gamma}c(\theta)s(\phi)\Big) +$$

$$\Big(I_{13} + I_{31}\Big)\Big(\dot{\phi} - \dot{\gamma}s(\theta)\Big)\Big(\dot{\gamma}c(\phi)c(\theta) - \dot{\theta}c(\theta)\Big) + \Big(I_{23} + I_{32}\Big)\Big(\dot{\theta}c(\phi) + \dot{\gamma}c(\theta)s(\phi)\Big)\Big(\dot{\gamma}c(\phi)c(\theta) - \dot{\theta}s(\phi)\Big)\Bigg]$$

Evaluate Lagrange's equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = Q_j$ for each of the Euler Angles θ, ϕ, and γ with Q_j as the torque about the $\dot{\theta}, \dot{\phi}$, and $\dot{\gamma}$ axes respectively. This will generate three coupled equations that can be rearranged to analytically solve for the acceleration of each of the Euler angles. Solving the rest of Lagrange is exceptionally difficult because it needs to take the derivatives of an additonal three terms because of the complicated inertia tensor. Additionally, none of the terms will cancel out nicely, further complicating the calculation of the equations of motion using Lagrange. Fortunately, generating the equations of motion for the servicer from Lagrange is not necessary for the goal of the report. To reiterate, the goal of the report is to solve for the equations of motion for the servicer and target satellite *such that the servicer satellite can orient itself and know when the target satellite is reachable*. This is important for a couple of reasons. First, the satellite needs to be able to orient itself with the target satellite. The Euler equations of motion are nice for this requirement because the external torques are simply the force couples from the thrusters, the method with which the servicer satellite will orient itself. The Lagrange approach results in generalized torques that are aligned with the instantaneous $\dot{\theta}$, $\dot{\phi}$, and $\dot{\gamma}$ axes. These are not directly analogous to the force couples applied by the bang-bang thrusters. Therefore, an external moment about one of the instantaneous Euler rate axes requires several force couple burns from the thrusters. Additionally, the implementation of the lagrange equations of motion with the servicer inertia tensor is a grueling task that presents a marginal benefit over using Euler's equation of motion arrived at from force balance. This method also needs to implement the complicated inertia tensor, but it has significantly easier equations to work with. Once the tensor values are imported, the Euler angles and Euler rates can be taken from the propagation of the Euler equations of motion at any times of interest.

IV. Summary

The force balance approach proved the most effective in generating equations of motion for the servicer satellite. Force balance resulted in Euler equations of motion that each correlated exactly to the torques applied by the coupled thrusters about the three body axes. Although the Euler equations of motion are incomplete, the Euler angles and Euler rates can be backed out from the propagation of the Euler equations of motion at any time of interest. A conservation approach was not possible for the servicer satellite because no quantities were conserved. The lagrange approach for the servicer satellite was infeasible given the goal of the project. The products of inertia of the servicer satellite introduced several more terms into the Lagrangian that made evaluating Lagrange's equations for each of the Euler angles excessively difficult. Additionally, Lagrange's method results in external moments that are not aligned with the body axes of the servicer satellites, making the calculation of the required torques significantly more difficult. On the other hand, the Lagrange approach was the most effective method for the target satellite. The inertia tensor of the target satellite was about principal axes and contained identical entries for the diagonal elements, resulting in nice equations of motion from the Lagrange approach. These equations can propagate the Euler angles exactly and compare those to the Euler angles of the servicer satellite. With this information, the servicer can orient itself using the bang bang thrusters and figure out when the target is within reach of its robotic arm.

V. Future Work

Future work on the servicing equations of motion could be vastly simplified by storing the arm within the cube during orientation with the bang bang thrusters and then extending the arm when the target satellite is within reach. This would make the inertia tensor diagonal and the body axes would also become prinicpal axes, allowing for a feasible implementation of the Lagrange approach for the servicer satellite.