Homework 2

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Advanced Methods for Numerical Fluid Dynamics
MVKN70

Task 1

We want to do an eigenvalue decomposition of the non-conservative unsteady Euler equations in 1-D

$$\frac{\partial}{\partial t}\mathbf{V} + [B]\frac{\partial}{\partial x}\mathbf{V} = 0, \mathbf{V} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, [B] = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c^2 & u \end{bmatrix}. \tag{1}$$

We do this decomposition by solving the problem $|[B] - \lambda I|$ for λ and obtaining the eigenvalues $\lambda_i, i \in \{1, 2, 3\}$ sorted in ascending order. We then compute the eigenvector $\mathbf{y}_i$ for each eigenvalue from the system $([B] - \lambda_i I)\mathbf{y}_i = 0$.

From this we obtain the following eigenvalue matrix

$$[\Lambda] = \begin{bmatrix} u-c & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+c \end{bmatrix}$$

and the corresponding eigenvector matrix

$$[Y] = \begin{bmatrix} \rho & 1 & \rho \\ -c & 0 & c \\ \rho c^2 & 0 & \rho c^2 \end{bmatrix}.$$

We then want to obtain the inverse of $[Y]$ denoted $[Y]^{-1}$. We obtain $[Y]^-1$ by solving the system $[Y][Y]^{-1} = I$ and obtain

$$[Y]^{-1} = \begin{bmatrix} 0 & -\frac{1}{2c} & \frac{1}{2\rho c^2} \\ 1 & 0 & -\frac{1}{c^2} \\ 0 & \frac{1}{2c} & \frac{1}{2\rho c^2} \end{bmatrix}.$$

That the eigenvalues, eigenvectors and the inverse are correct was the checked manually by hand.

We now utilize this decomposition to expand the our original equations (1), resulting in

$$\left([Y]^{-1}\frac{\partial}{\partial t}\mathbf{V}\right) + [\Lambda]\left([Y]^{-1}\frac{\partial}{\partial x}\mathbf{V}\right) = 0 \tag{2}$$

and we arrive at the following equations after a full expansion, I am a bit uncertain what is meant by "full expansion", so I settle on the form after matrix multiplication using the variables used in the calculations.

$$-\frac{1}{2c}\frac{\partial u}{\partial t}+\frac{1}{2\rho c^2}\frac{\partial p}{\partial t}+-\frac{u-c}{2c}\frac{\partial u}{\partial x}+\frac{u-c}{2\rho c^2}\frac{\partial p}{\partial x}=0 \tag{3}$$

$$\frac{\partial \rho}{\partial t}-\frac{1}{c^2}\frac{\partial p}{\partial t}+\frac{\partial \rho}{\partial x}-\frac{1}{c^2}\frac{\partial p}{\partial x}=0 \tag{4}$$

$$\frac{1}{2c}\frac{\partial u}{\partial t}+\frac{1}{2\rho c^2}\frac{\partial p}{\partial t}+-\frac{u+c}{2c}\frac{\partial u}{\partial x}+\frac{u+c}{2\rho c^2}\frac{\partial p}{\partial x}=0. \tag{5}$$

Bonus

As a extra part we also derive the Non-conservative form of the Euler equations from the conservative form

$$\frac{\partial}{\partial t}\mathbf{U}+\frac{\partial}{\partial x}\mathbf{F}=0, \mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho(C_vT+\frac{1}{2}u^2)\end{bmatrix}, \mathbf{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho u(C_pT+\frac{1}{2}u^2)\end{bmatrix}. \tag{6}$$

First off, by using the product rule and expanding the first equation we get

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \tag{7}$$

which we see correspond exactly to the first equation of the non-conservative form. As for the second equation after expansion we get

$$u\frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial t}+u^2\frac{\partial \rho}{\partial x}+2\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0. \tag{8}$$

By then using equation (7) to eliminate the $u\frac{\partial \rho}{\partial t}$ and dividing by ρ we get the following equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0. \tag{9}$$

There we have the first two equations of the non-conservative form, as for the third, after derivation we get

$$(C_vT+\frac{1}{2}u^2)\frac{\partial \rho}{\partial t}+\rho u\frac{\partial u}{\partial t}+u(C_pT+\frac{1}{2}u^2)\frac{\partial \rho}{\partial x}+\rho(C_pT+\frac{3}{2}u^2)\frac{\partial u}{\partial x}=0. \tag{10}$$

after multiplying equation (7) by $(-(C_pT+\frac{1}{2}u^2)$ and adding to this equation we get

$$-RT\frac{\partial \rho}{\partial t}+\rho u\frac{\partial u}{\partial t}+\rho u^2\frac{\partial u}{\partial x}=0. \tag{11}$$

where $R=C_p-C_v$. By then multiplying equation (9) by $-\rho u$ and using that $p=RT\rho$ our final result is

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}=0. \tag{12}$$

This however does not correspond with the third non-conservative equation. We are missing the dependence on $\frac{\partial u}{\partial x}$. By instead using the entropy equation I think you can get the correct solution.

Task 2

For this task two initial conditions where set up as to show scenario a and b. The movemement speed of the wave that was measured was done graphically in the graph by observing the distance travelled and dividing by the time that had passed, in case a) 0.617 ms and in b) 1.381 ms. The values are rounded to whole numbers. The initial conditions and the different speeds for a) and b) are shown in Figure 1 and 2 below.

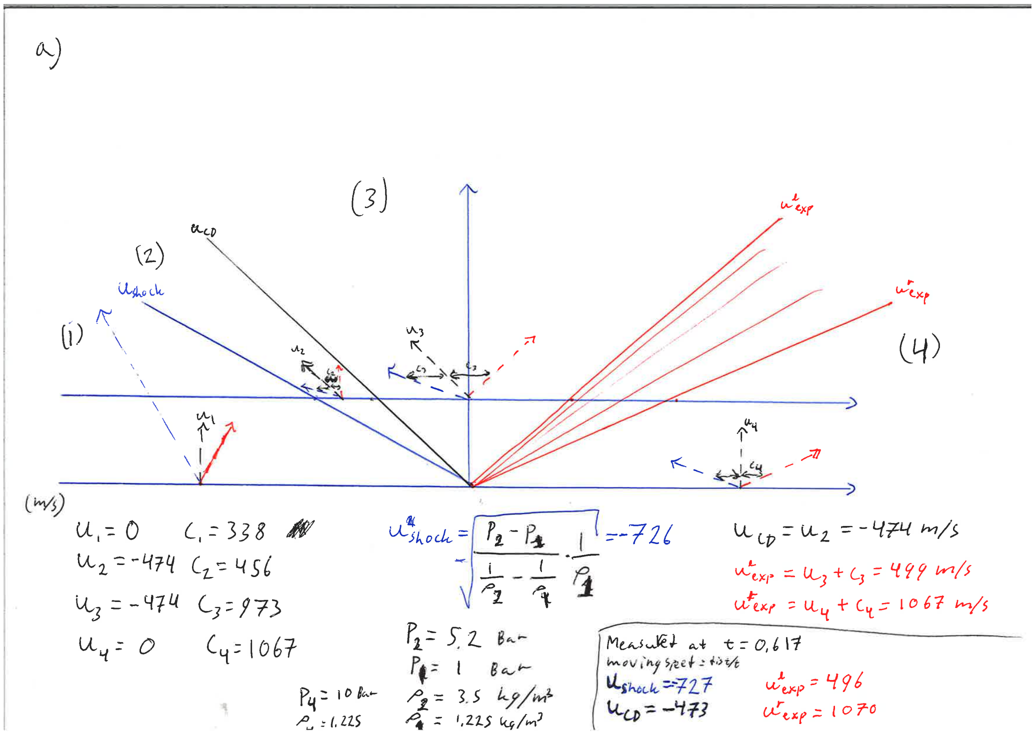

Figure 1: Scenario a) with a shock on the left side, a contact discontinuity in the middle left and a expansion wave going to the right.

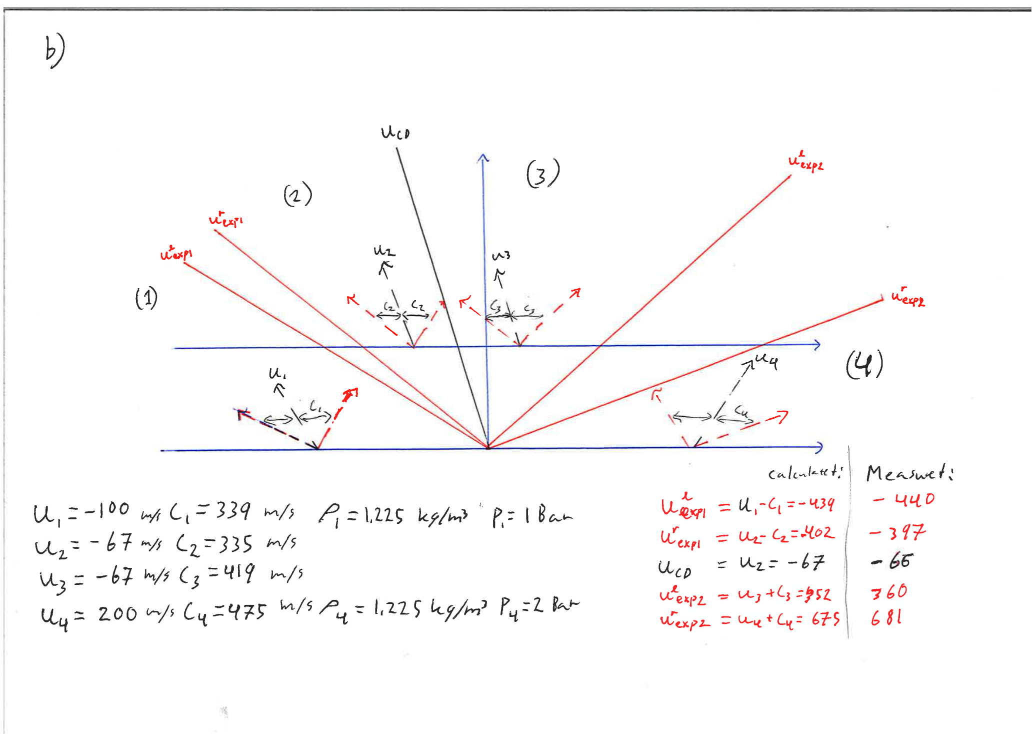

Figure 2: Scenario b) with two expansion waves travelling to the left and right with a contact discontinuity travelling very slowly to the left in the middle.