Lecture Notes

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Course overview

February: The connections to the underlying particle physics. Isospin, the mass formula, Fermi gas model and simple shell model.

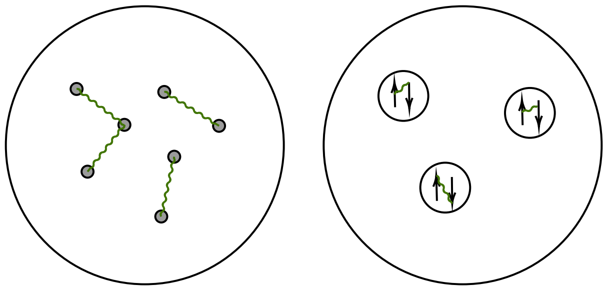
March: Nuclear structure models. Collective vibrations. Then elaborations on the shell-model, including an update on how nuclear shells may change. Also deformations and the theory of pairing.

April: Nuclear decays. Gamma decay, fission, alpha decay and beta decay. Also collective modes, in particular at rotations.

May: Nuclear astrophysics and nuclear reactions

1 Lecture 1: Free quarks and the strong nuclear force

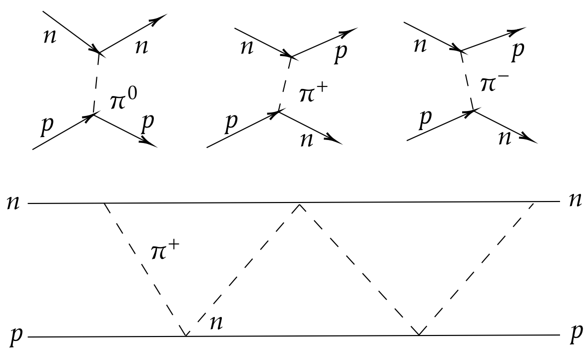
Nuclei consist of protons and neutrons, but why is this configuration energetically favored over free quarks. The nuclear forces that keep the nucleus together are induced through the exchange by mediating quanta – mesons. Considering two quarks with spin. The spin configurations can be aligned or anti-aligned where the energy is highest for the aligned state. This leads to an energetically favored state when the spins are anti-aligned. This can explain why the nuclei consists of protons and neutrons and not of free quarks since this would add more degrees of freedom. When the quarks are confined in protons and neutrons the gluons can only interact within the proton or neuton instead of all the free quarks in the nuclei.
At higher energies the bound system of protons and neutrons break down since the energy is now minimal for a free soup of quarks. This is the quark-gluon plasma.



Nucleons take part in all known interactions. The main feature of the nuclear world are sharped by the strong force – particles participating in strong interactions are called hadrons. Nucleons are the lightest fermions among the hadrons. The bosons among the hadrons are called mesons. The range of the strong force can be estimated by the uncertainty relation and the mass of π^0

$$\Delta R \sim c\Delta t \sim \frac{\hbar}{mc} = 1.46\,\mathrm{fm} \tag{1}$$

Inside the nucleus one cannot distinguish between protons and neutrons since they interchange through the pion. This means one can only distintguish a proton from a neutron when the particle decays. This means one consideres nucleons because it is an interacting system. This motivates the isospin formalism since this shows the net number of protons or neutrons.



2 Lecture 2: Isospin

Since one cannot distinguish the proton and the neutron inside the nucleous we can consider the two particles as two different states of the same strongly interacting object. This leads to a two-level system: the nucleon.

$$\left|p\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

This leads to three different objects to look at, nn, np, pp. Also, the wavefunction consists of three parts: the radial term, the angular part and the spin part. The symmetry with respect to the exchange of spin varialbes is tested by the Bartlett operator. The triplet states are symmetric and the singlet states is anti-symmetric under spin exchange.

$$\mathcal{P}^{\sigma} = (1-)^{S+1} \tag{3}$$

The space inversion chances the sign of the relative coordinate

$$\mathcal{P}^{r} = (-1)^{l} \tag{4}$$

Both spin and orbital is given by the Heisenberg exchange operator

$$\mathcal{P} = (-1)^{l+S+1} \tag{5}$$

An example of symmetric and anti-symmetric

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S}_2, \quad \mathbf{S_1} = \frac{1}{2}, \quad \mathbf{S}_2 = \frac{1}{2} \tag{6}$$

To get the expectation value of the spin product: square both sides. This leads to

$$\langle \mathbf{S_1} \cdot \mathbf{S}_2 \rangle = \begin{cases} -\frac{3}{4}, & \text{singlet, anti-symmetric} \\ \frac{1}{4}, & \text{triplet, symmetric} \end{cases} \tag{7}$$

In gerneral the relative angular momentum, l, and the spin $\mathbf{S}$ is not conserved, but the rotational invariance makes the total angular momentum conserved. Another argument is that this operator commutes with the Hamiltonian.

$$\mathbf{J} = \mathbf{l} + \mathbf{S} \tag{8}$$

So the conserved quantum number is J^{π}. For a two-nucleon state:

$$^{2S+1}l_J \tag{9}$$

To introduce isospin one can consider Noether's theorem. This also means that an approximate symmetry yields some perfered quantites, i.e selection rules. The strong interaction cannot distinguish the proton from the neutron but the Coulomb interaction can. This leads to an approximate symmetry which leads to the isospin formalism.

Again considering the nucleon from equation (2) the isospin acts on the basis with a certain charge given by

$$Q = \frac{1}{2} - t_3 \tag{10}$$

Here t_i is analogous to the Pauli matrices for regular spin given. This also means there are three different operator so isospin is a vector with three components.

$$t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

This also leads to lowering and raising operators in the same way one cna lowe and raise components of angular momentum $J_\pm = J_x \pm iJ_y$. For isospin the relation is given by

$$\tau_\pm = \tau_1 \pm i\tau_2 \tag{12}$$

The strong force cannot distinguish the proton from the neuton so isospin is invariant. This means the totalt isospin is a conserved quantum number

$$T^2 = \sum_a \mathbf{t_a} \tag{13}$$

The eigenvalue of the isospin "length" is $\mathbf{T}^2 = T(T+1)$. One can expand the wavefunction by A factors for each nucleus and this is still an invariant

$$[\hat{\mathbf{T}}, \hat{H}_{\text{strong}}] \tag{14}$$

One can also consider the minimum and maximum of the isospin projection, T_3. This quantity is related to electrical charge

$$Q = \sum_a \left(\frac{1}{2} - t_{3a}\right) = \frac{A}{2} - T_3 \tag{15}$$

All states in a given nucleus (vertical scale) have the same projection

$$T_3 = \frac{1}{2}(N - Z) = \frac{A}{2} - Z \tag{16}$$

And belong to the horizontal scale. This is illustrated in figure 2.1 in the book. To conclude, the total wavefunction for the nucleus must also include Isospin

$$\Psi = RY_{lm}\chi\Omega. \tag{17}$$

3 Lecture 3: Selection rules and liquid drop model

Selection rules

Consider an arbitrary one-body operator, $\hat{O}$ which is a sum of single particle operators

$$Q = \sum_1^A q_a \tag{18}$$

This can be split into different terms if the operator can distinguish neutrons and protons

$$\begin{aligned}
Q &= \sum_n q_n + \sum_p q_p \\
&= \sum_a \left(q_n \frac{1+\tau_{3a}}{2} + q_p \frac{1-\tau_{3a}}{2} \right) \\
&= \sum_a \frac{q_n + q_p}{2} + \sum_a \frac{q_n - q_p}{2} \tau_{3a}
\end{aligned}$$

The first equality is on a neutron, proton level but using the isospin formalism we can move to a nucleon level since the strong force cannot distinguish protons from neutrons. This leads to the second equality where denominator is 1 if a neutron and 0 if a proton. The other way around for the next term. This leads to the third equality where the terms are called the isoscalar ($\tau = 0$) and the isovector $\tau = 1$ respectively. The physical interpretation of this is that for the isoscalar the neutrons and protons oscilllate in phase and out of phase for the isovector. This also leads to $\hat{\tau}_\pm$ being interpreted as a β-decay operator, look at equation (12).

Liquid drop model

The binding energy is traditionally given by

$$B(A,Z) = (ZM_p + NM_n)c^2 - E_{\text{tot}}(A,Z) \tag{19}$$

But in practice it is usefull to include the mass of the electrons

$$B(A,Z) = (ZM_p + NM_n)c^2 - E_{\text{tot}}(A,Z). \tag{20}$$

The dynamical features requires a reasonable chouse of dynamic variables. We consider a continous medium where excitations should resemble propagating waves. The lowest limit for the wavelength us of order r_0 (mean interparticle distance) which is close to the range of the nuclear force.

$$\frac{1}{k} > r_0 \simeq \frac{R}{A^{1/3}}. \tag{21}$$

This is rewriting by multiplaying $\hbar$

$$\frac{\hbar k R}{\hbar} < A^{1/3}, \tag{22}$$

and since $\hbar k$ is momentum, $\hbar k R$ is angular momentum. This motivates the use of spherical harmonics, $\alpha\lambda\mu Y_{\lambda\mu}(\hat{n})$. This means in a collective wave of deformation the radius R can be representated by a superposition of the spherical harmonics

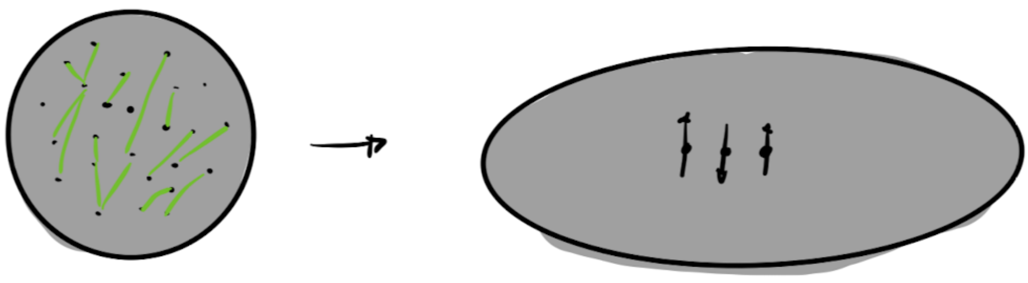
$$R(\mathbf{n}) = R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\mathbf{n}) \right]. \tag{23}$$

4 Lecture 4: Fermi gas model

Introduce mean-field approximation and split the Hamiltonian into a Hamiltonian for each particle and a Hamiltonian for the interaction.

$$\hat{H} = \hat{H}_{\text{each particle}} + \hat{H}_{\text{interaction}} \longrightarrow \sum_{ij} v_{ij} + \sum_{ijk} v_{ijk} = \sum \left(\hat{H}_{\text{each particle}} + \hat{H}_{\text{interaction}} \right) - \sum \hat{H}_{\text{residual}} \tag{24}$$

Also introduce another quark model structure



In this section we considered a cubic box of size $L = V^{1/3}$. Each single-particle orbit in the Fermi gas model is characterized by the momentum $\mathbf{p}$ and spin-isospin quantum numbers – also normalized by the volume, V

$$\psi_{\mathbf{p}\sigma\tau} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r})} \chi_\sigma \Omega_\tau \tag{25}$$

So we assumed plane wave solution and isotropic in momentum-space.

$$\epsilon(p) = \frac{p^2}{2m^*} \tag{26}$$

This leads to the single-particle level density

$$\nu(\varepsilon) = \int \frac{\mathrm{d}^3p\mathrm{d}^3r}{(2\pi\hbar)^3}\delta(\epsilon - \epsilon(p)) = \frac{Vg}{2\pi^2\hbar^3}p^2\frac{\mathrm{d}p}{\mathrm{d}\epsilon}, \tag{27}$$

where g is the degeneracy factor and $(2\pi\hbar)^3$ is the volume the particle occupies. Now move from first quantization to second quantization and express the wavefunction in terms of the occupation number.

$$Z = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma-1/2}, \quad N = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma 1/2} \tag{28}$$

Introduce the Fermi energy diagram and express A in terms of p_F. This leads to the final expression

$$p_F = \hbar k_F = \hbar\left(6\pi^2\frac{n}{g}\right)^{1/3}, \tag{29}$$

where the fraction is the number density per degree of freedom. This means that all you need is the number density to get the Fermi momentum. Also that the Fermi gas is highy degenerate. From the Fermi energy you can get the seperation energy which leads to the potential, $\mathcal{U}_0 = 40 - 50$ MeV depending on m^*.

5 Lecture 5: Spherical mean field

We now examine the clustering of single-particle level called shell structure. This is illustrated in figure 5.1. The shell structure also implies dynamical properties such as excitations, capture etc. In nuclear physics there are some numbers for protons and neutrons that lead to especially stable nuclear systems – analogous to the noble gases in atomic physics. Those magic numbers for nuclei are placed along the valley of stabily and are given by

$$2, \quad 8, \quad 20, \quad 50. \quad 82, \quad 126. \tag{30}$$

Since the shape of the mean field is nearly the same for protons and neutrons the magic numbers turn out to be the same as well. The most stable nucleai are of course the double magic numbered nuclei. Keep in mind the magic numbers are not invariant since the potential changes when you add more protons or neutrons.

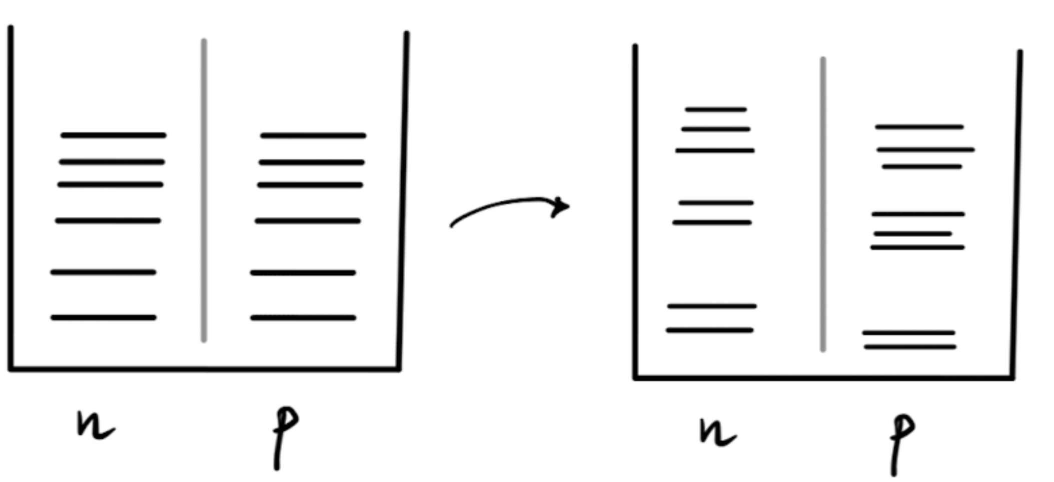


Figure 5.1: Energy gaps

Nuclei with one nucleon on top of a filled major shell have diminished nucleon seperation energy given by

$$S_n(A,Z) = B(A,Z) - B(A-1,Z)$$
$$S_p(A,Z) = B(A,Z) - B(A-1,Z-1)$$

Considering the shell structure the simplest approach to solve the model of mean field is the isopropic harmonic oscillator. This is easily done analytically but the drawbacks are in its unrealistic features that include excessive symmtry. This can be seen from the Hamiltonian

$$\hat{H}_{\text{HO}} = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega_0^2\mathbf{r}^2, \quad \mathbf{r}^2 = (x+y+z)^2 \tag{31}$$

This means symmetric excitations in $N = n_x, n_y, n_z$. The energi levels ar

$$E_n = \hbar\omega_0\left(N + \frac{3}{2}\right) \tag{32}$$

And the parity of the three-dimensional states is given by

$$\Pi = (-1)^N \tag{33}$$

It is important that all states in a major shell have the same parity. Therefore, the linear combinations of the major shell orbits with proper rotational symmetry should have only even (odd) l for even even (odd) N.
A more realistic potential is the Woods-Saxon potential which only consideres the nearest interaction and spin-orbit coupling. This means that the potential is an improvement to the harmonic oscillator potential since it includes surface effects. Another arguement for the surface effects is that angular momentum is with respect to a point. This is seen on figure 8.4. Also consider how degeneracy leads to shell structures from an energy perspective. The distance from $s \to p$ decreases as the nucleus becomes larger because the wavelength becomes larger and the kinetic energy decreases. The energy as a function of major shell number and angular momentum is given by

$$\begin{aligned} E(n,l) &= E(n_0,l_0) + (n-n')\frac{\mathrm{d}E}{\mathrm{d}n} + (l-l_0)\frac{\mathrm{d}E}{\mathrm{d}l} \\ &\simeq b\frac{\mathrm{d}E}{\mathrm{d}n} + a\frac{\mathrm{d}E}{\mathrm{d}l}, \end{aligned}$$

where the second term is the energy to add for a radial excitation and the third term is the energy you must add for an angular excitation. Also, a, b are small. This leads to degeneracy which leads to shell stuctures.

6 Lecture 6: Two-body dynamics

Low-energy nuclear forces

In the low-energy domain we can almost always limit ourselves to considerations of nucleon degrees of freedom. This means we only need a Hamiltonian expressed in terms of the nucleon variables: the coordinates $\mathbf{r}$, momenta $\mathbf{p}$ and spins $\mathbf{s}$. To construct a spin dependent Hamiltonian we need a combinations of spins and does not change sign under spatial inversion or time reversal. Another argument is that $\mathbf{r} \times \mathbf{p}$ does not change under parity. This means we are left with

$$\sigma_{\mathbf{1}}\sigma_2(\sigma_{\mathbf{1}} \cdot \mathbf{n_1})(\sigma_2 \cdot \mathbf{n_2}), \tag{34}$$

where the first product is scalar and does not depend on the orientation of $\mathbf{n}$ and can therefore be ignored when we average over all angles. The rest are tensor forces. The angular average is

$$\overline{n_k n_l} = \frac{1}{4\pi} \int \mathrm{d}o \; n_k n_l = \frac{1}{3}\delta_{kl}. \tag{35}$$

This means the average part of the operator is $1/3(\sigma_{\mathbf{1}} \cdot \sigma_2)$ and belongs to spin-spin forces. We can now define a pure tensor operator

$$S_{12}(\mathbf{n}) = 3(\sigma_{\mathbf{1}} \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) - (\sigma_{\mathbf{1}} \cdot \sigma_2) = 2[3(\mathbf{S} \cdot \mathbf{n})^2 - \mathbf{S}^2] \tag{36}$$

And this operator has angular average of 0. This pure tensor operator is responsible for any noncentral forces. The general momentum-independant interaction of two spin-1/2 particles may only contain three types of forces each with their own radial depenence: central, spin and tensor. The Hamiltonian for the momentum-independent interaction is given by

$$H_s(\mathbf{r}, \sigma_{\mathbf{1}}\sigma_2) = U_c(r) + U_\sigma(r)(\sigma_{\mathbf{1}} \cdot \sigma_2) + U_t(r)S_{12} \tag{37}$$

And this Hamiltonian is symmetric under spin exchange

$$[\mathcal{P}^\sigma, H_s] = 0 \tag{38}$$

We also need to add an isospin term to this Hamiltonian. This leads to a new Hamiltonian which can be expressed in terms of four forces: a central Wigner, Majorana ($\propto \mathcal{P}^r$), Bartlett ($\propto \mathcal{P}^\sigma$)m and Heisenberg ($\propto \mathcal{P}^r\mathcal{P}^\sigma$)

$$H_s' = U_W(r) + U_M(r)\mathcal{P}^r + U_B(r)\mathcal{P}^\sigma + U_H(r)\mathcal{P}^r\mathcal{P}^\sigma \tag{39}$$

Meson exchange

Mesons act as mediators of strong forces. The free meson field $\phi^\alpha(\mathbf{r}, t)$, where α is the isospin (charge) characteristic $t_3 = -Q$ of the meson that satisfies the Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)\phi^\alpha = 0, \quad \mu = \frac{mc}{\hbar} \tag{40}$$

We now consider the Coulomb potential mediated by the photon with mass equal to zero and solve the Poisson equation. We do this since we are doing a similar approach for the Klein-Gorden equation. The Poisson equation is

$$\nabla^2\phi = -4\pi\delta(\mathbf{r}) \tag{41}$$

With the solution

$$\phi(\mathbf{r}) = \frac{e}{|\mathbf{r}-\mathbf{r_1}|} \tag{42}$$

For the Klein-Gorden equation we obtain

$$\left(\nabla^2 - \mu^2\right)\phi^\alpha(\mathbf{r}) = -\frac{g}{\mu}\tau^{alpha}(\sigma\cdot\nabla)\delta(\mathbf{r}-\mathbf{r_1}) \tag{43}$$

With solution

$$\phi^\alpha = \frac{g}{4\pi\mu}\tau^\alpha(\sigma\cdot\nabla)\frac{\mathrm{e}^{\mu|\mathbf{r}-\mathbf{r_1}|}}{|\mathbf{r}-\mathbf{r_1}|} \tag{44}$$

The final result is the pion-exchange potential that is symmetric with respect to nucleons

$$U_\pi = \frac{g^2}{4\pi\mu^2}(\tau_\mathbf{1}\cdot\tau_2)(\sigma_\mathbf{1}\cdot\nabla)(\sigma_2\cdot\nabla)\frac{\mathrm{e}^{-\mu r}}{r} \tag{45}$$

Tensor forces and d-wave

Now also taking the tensor terms into account. The two tensor terms arises from the Majorana operator

$$H_s'' = [\mathcal{U}_{tW}(r) + \mathcal{U}_{tM}(r)\mathcal{P}^r]S_{12} \tag{46}$$

This yields the noncentral spin-dependent potential

$$\mathcal{U}(r) = \mathcal{U}_0 + \mathcal{U}_t(r)S_{12}, \quad \mathcal{U}_t(r) = \mathcal{U}_{tW}(r) + \mathcal{U}_{tM}(r) \tag{47}$$

The complete deuteron wavefunction now contains two radial parts. Factoring out the wave factor, $1/r$ yields

$$\Psi_M = \frac{1}{\sqrt{4\pi}}\frac{1}{r}\left(u_0(r) + \frac{1}{\sqrt{8}}u_2(r)S_{12}\right)\chi_{1M} \tag{48}$$

This equation was rewritten using the convient coupling represented by

$$\Theta_M = \frac{1}{\sqrt{32\pi}}S_{12}\chi_{1M} \tag{49}$$

Plugging this into the Schrödinger equation

$$u_0'' - [\kappa^2 + \mathcal{U}_0(r)]u_0 - \sqrt{8}\tilde{\mathcal{U}}_t(r)u_2 = 0 \tag{50}$$

$$u_2'' - [\kappa^2 + \frac{6}{r^2} + \tilde{\mathcal{U}}_0(r) - 2\tilde{\mathcal{U}}_t(r)]u_2 - \sqrt{8}\tilde{\mathcal{U}}_t u_0 = 0, \tag{51}$$

where the tilded potentials include the factor $2m/\hbar^2$. Now these are almost the same but what is the coupling terms? $S_{12}\frac{u_0}{r}Y_{02}\chi_{1M}$ is the s-wave but transforms into a d-wave contribution. This can be seen if one compares equation (48) to the last term in equation (50). Do the same consideration for the other terms.

$$r_{12}\frac{u_2}{r}\frac{1}{\sqrt{8}}S_{12}Y_{00}Y_{1M} = \frac{u_2}{r}\frac{8}{\sqrt{8}}Y_{00}\chi_{1M} - \frac{u_0}{r}\frac{2}{\sqrt{8}}S_{12}Y_{00}\chi_{1M} \tag{52}$$

Plug in potential and you get figure 3.5 in the book.

7 Lecture 7: Two-body scattering

Scattering theory

Consider elastic scattering in the center-of-mass frame, where $\mathbf{r}$ is the relative distance between the particles. We neglect interaction forces and consider free motion. This means the asympotic form of the wave function of the relative motion can be written as a combination of the incident plane wacve and the outgoing spherical wave

$$\psi(r) \simeq e^{i(\mathbf{k}\cdot\mathbf{r})} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikr}}{r}, \tag{53}$$

where $k = k' = \sqrt{2mE/\hbar^2}$ is the wave vector, m is the reduced mass and $f(\mathbf{k}',\mathbf{k})$ is the scattering amplitude of dimension length. This is related to the differential cross section of scattering given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}o} = \left|f(\mathbf{k}',\mathbf{k}\right|^2. \tag{54}$$

Also, θ is the angle between $\mathbf{k}'$ and $\mathbf{k}$ For low k one can use partial wave expansion which means to new expressions for $e^{i\mathbf{k}\cdot\mathbf{r}}$ and $f(\theta)$

$$e^{i(\mathbf{k}\cdot\mathbf{r})} = e^{ikr\cos(\theta)} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(kr)P_{\ell}(\cos(\theta)) \tag{55}$$

$$f(\theta) = \sum_{\ell}(2\ell+1)P_{\ell}(\cos(\theta))f_{\ell}, \quad f_{\ell} \in \mathbb{C} \tag{56}$$

Plugging this in

$$\psi(\mathbf{r}) \simeq \frac{i}{2kr}\sum_{\ell}(2\ell+1)P_{\ell}(\cos(\theta))[(-1)^{\ell}e^{-ikr} - (1+2ikf_{\ell})e^{ikr}] \tag{57}$$

Now, the outcomming wave is distorted – its amplitude is not equal to 1.