Threshold pion photo-production off nucleons using the nuclear model with explicit pions

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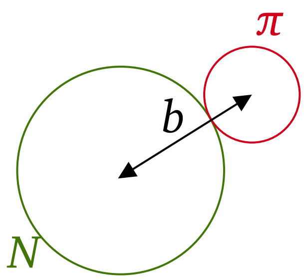

Master's thesis
Supervisor: Dmitri Fedorov

Department of Physics and Astronomy
Aarhus University
Denmark

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Summary

This is a bachelor's project about relativistic kinematics. This means a theoretical description of particle's motion without considering the forces that cause them to move, in the relativistic limit. More specifically this project illuminates three-particle decays with kinematic invariant quantities. The theory introduces an approach to examine an anomaly that occurred in an experiment when a Hungarian research team measured transitions in ^{8}Be. It combines considerations of phase space for three-particle decay with resonance amplitudes and angular distributions. This leads to experimental predictions separated into three parts depending on the spin state of the resonant particle, the X17 particle. This project concludes that the formalism is consistent with the relativistic scattering angles in the laboratory system and how experimental results can be illustrated using kinematic invariants. Three angles are proposed to investigate the X17 particle experimentally.

Resumé

Dette bachelorprojekt omhandler relativistisk kinematisk. Det vil altså sige en teoretisk beskrivelse af partiklers bevægelse uden ydre påvirkning i den relativistiske grænse. Mere specifikt belyser dette projekt 3-partikel henfald ved hjælp af kinematiske invariante størrelser. Teorien i projektet introducerer den tilgang, der er anvendt til at undersøge en anormalitet i et forsøg, som opstod da et ungarnsk forskerhold målte overgange i ^{8}Be. Teorien kombinerer overvejelser omkring faserum for tre-partikel henfald med resonansamplituder og vinkelfordelinger. Dette fører til eksperimentelle forudsigelser, der er inddelt i tre dele afhængig af spintilstanden for resonanspartiklen, X17. Projektet konkluderer, at formalismen er konsistent med den relativistiske spredningsvinkel i laboratoriets hvileramme og viser hvordan eksperimentelle resultater kan repræsenteres ved kinematiske invariante størrelser. Der foreslås tre vinkler, som kan måles eksperimentelt for at undersøge X17 partiklen.

Colophon

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The project is supervised by Dmitri Fedorov

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Code for document and figures can be found on `https://github.com/MartinMikkelsen`

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Chapter 1

Introduction

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1.1 Units

We will work in "God-given" units where

$$\hbar = c = 1. \tag{1.1}$$

Discussions of spin and isospin make use of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.2}$$

which also defines the Pauli vector given by

$$\boldsymbol{\sigma} = \sigma_1 \hat{x}_1 + \sigma_2 \hat{x}_2 + \sigma_3 \hat{x}_3, \tag{1.3}$$

and analogously for the isospin vector. The two level system of the nucleon has the following representation

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.4}$$

Chapter 2

Pion Photoproduction

We consider a nuclear model where the nucleus is held together by emitting and absorbing mesons, and the mesons are treated explicitly [Fedorov 2020]. In this model, the nucleus is a superposition of different states with a different number of mesons and transitions between these states keep the nucleus together.

2.1 Description of the model

In this case we consider pions and the specific case of the deuteron. Here we can apply the one meson approximation and the total system consists of two subsystems given by

$$\psi_p = p \uparrow \frac{1}{\sqrt{V}}, \quad \psi_{N\pi} = (\boldsymbol{\tau} \cdot \boldsymbol{\pi})(\boldsymbol{\sigma} \cdot \mathbf{r})\phi(r) p \uparrow \frac{1}{\sqrt{V}}, \tag{2.1}$$

where V is the volume, $\boldsymbol{\tau}$ is combination of the matrices $\tau_{1,2,3}$ into a matrix vector–completely analogous to the vector $\boldsymbol{\sigma}$ of spin Pauli matrices. Also, we have chosen the proton as isospin up state in the nucleon

$$|p\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.2}$$

From (2.1)

$$\Psi = \begin{bmatrix} \psi_p \\ \psi_{N\pi} \end{bmatrix}, \tag{2.3}$$

To construct a Hamiltonian we define two operators corresponding to creation and annihilation of the meson. We define the operator as

$$W \equiv (\boldsymbol{\tau} \cdot \boldsymbol{\pi})(\boldsymbol{\sigma} \cdot \mathbf{r}) f(r), \tag{2.4}$$

where $f(r)$ is some form factor given by

$$f(r) = \frac{S}{b} e^{-r^2/b^2}, \tag{2.5}$$

where $S \simeq 10\,\text{MeV}$ and $b \simeq 1\,\text{fm}$. This is illustrated in figure 2.1. Note that (2.5) must have units of energy per length such that (2.4) has units of energy.

From (2.4) we can construct the Hamiltonian of the system

$$H \doteq \begin{bmatrix} K_{\mathbf{R}} & W^\dagger \\ W & K_{\mathbf{R}} + K_{\mathbf{r}} + m_\pi c^2 \end{bmatrix}, \tag{2.6}$$

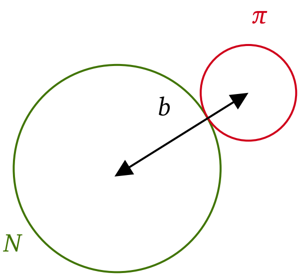

Figure 2.1: Schematic figure of the system to describe the form factor, (2.5). The pion is assumed to sit on the surface.

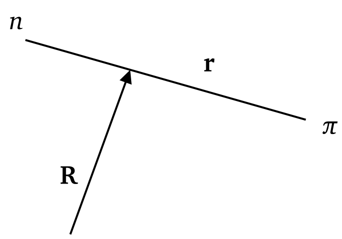

Figure 2.2: Absolute and relative distance.

where K_i represents the kinetic term of particle i. Also, R represents the absolute distance and r is the relative distance as illustrated in figure 2.2. Plugging this into the Schrödinger equation yields

$$\begin{bmatrix} K_{\mathbf{p}} & W^\dagger \\ W & K_{\mathbf{R}} + K_{\mathbf{r}} + m_\pi c^2 \end{bmatrix} \begin{bmatrix} \psi_p \\ \psi_{N\pi} \end{bmatrix} = E \begin{bmatrix} \psi_p \\ \psi_{N\pi} \end{bmatrix}. \tag{2.7}$$

Expanding (2.7) yields two equations

$$K_{\mathbf{p}}\psi_p + W^\dagger \psi_{N\pi} = E\psi_p \tag{2.8}$$

$$W\psi_p + (K_{\mathbf{R}} + K_{\mathbf{r}} + m_\pi)\psi_{N\pi} = E\psi_{N\pi}. \tag{2.9}$$

The first term in (2.8) vanishes and inserting (2.4) yields

$$\int_V \mathrm{d}^3 r\, (\boldsymbol{\tau}\cdot\boldsymbol{\pi})^\dagger (\boldsymbol{\sigma}\cdot\mathbf{r})^\dagger f(r)\phi(r)(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r}) p\frac{1}{\sqrt{V}} = Ep\frac{1}{\sqrt{V}} \tag{2.10}$$

This can be further simplified using relations for the matrix vectors[1]

$$12\pi \int_0^\infty \mathrm{d}r\, f(r)\phi(r) r^4 = E. \tag{2.11}$$

1. $(\boldsymbol{\tau}\cdot\boldsymbol{\pi})^\dagger(\boldsymbol{\tau}\cdot\boldsymbol{\pi}) = 3$ and $(\boldsymbol{\sigma}\cdot\mathbf{r})^\dagger(\boldsymbol{\sigma}\cdot\mathbf{r}) = r^2$

Similarly for (2.9) where the term $K_{\mathbf{R}}\psi_{N\pi}$ vanishes,

$$(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})f(r)p\frac{1}{\sqrt{V}} - \frac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})p\frac{1}{\sqrt{V}}\phi(r) = (E - m_\pi c^2)(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})\phi(r)p\frac{1}{\sqrt{V}}, \tag{2.12}$$

where μ is the reduced mass of the system. This equation can be further simplified by using a vector operator relation[2]

2. $\nabla^2(\mathbf{r}\phi(r)) = \mathbf{r}\left(\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right)$

$$f(r) - \frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right) = (E - m_\pi c^2)\phi(r). \tag{2.13}$$

This means equation (2.11) and (2.13) are the two equations that must be solved numerically.

$$\left.\begin{aligned} &12\pi \textstyle\int_0^\infty \mathrm{d}r\, f(r)\phi(r) r^4 = E \\ &f(r) - \tfrac{\hbar^2}{2\mu}\left(\tfrac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} + \tfrac{4}{r}\tfrac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right) + m_\pi c^2\phi(r) = E\phi(r) \end{aligned}\right\} \tag{2.14}$$

2.1.1 Numerical considerations

To solve the system of equations (2.14) one can consider two different numerical approaches.

For a given E one can solve the second order corresponding to $\phi[E]$. Conversely, for a given $\phi(r)$ one can calculate the integral to find $E[\phi]$. This leads to the fixed-point equation given by

$$E[\phi[\mathcal{E}]] = \mathcal{E}, \tag{2.15}$$

which is a single variable non-linear equation. (2.15) can be solved using a root-finding algorithm.

The second approach consists of reformulating the system (2.14) as a boundary value problem with the following conditions

$$I'(r) = 12\pi f(r)\phi(r) r^4, \quad I(0) = 0,\ I(\infty) = E. \tag{2.16}$$

The equation starts from a singular point and extends to an infinite point. Considerations about how to scale the starting point and stopping point as a function of b are needed. These two points must be proportional to

b with some arbitrary proportionality constant. In the regime of nuclear physics we expect the wavefunction to extend up to a length within the order of magnitude of ten Fermi.

We require the solution to stay finite which means approximations are needed at both limits. At $r \longrightarrow 0$ the differential equation is approximately an Euler-Cauchy equation with basis solutions 1 and r^{-1}. For finite solutions the latter is ignored which means $\phi'(a) = 0$ is the requirement for $a \approx 0$.[3] For $r \longrightarrow \infty$ the dominating term is the differential equation are

3. You would end up with the same conclusion if you consider $\phi = a + r^n$ and plug this into $\phi'' + 4\phi'/r = 0$, which yields $n = 0, -1$.

$$-\phi''(r) + 2\mu(m_\pi c^2 - E)\phi(r) = 0. \tag{2.17}$$

Since we expect a negative value for E the basis solutions are $\exp\left\{\pm\sqrt{2\mu(m_\pi c^2 + |E|)}\right\}r$. In the case of a positive sign the solution diverges. For the basis solution with negative exponents we have

$$\phi'(r) + \sqrt{2\mu(m_\pi c^2 + |E|)}\phi(r) = 0. \tag{2.18}$$

These two conditions are suitable boundary conditions for the left and right boundaries, respectively. The solutions can be seen in figure 2.3.

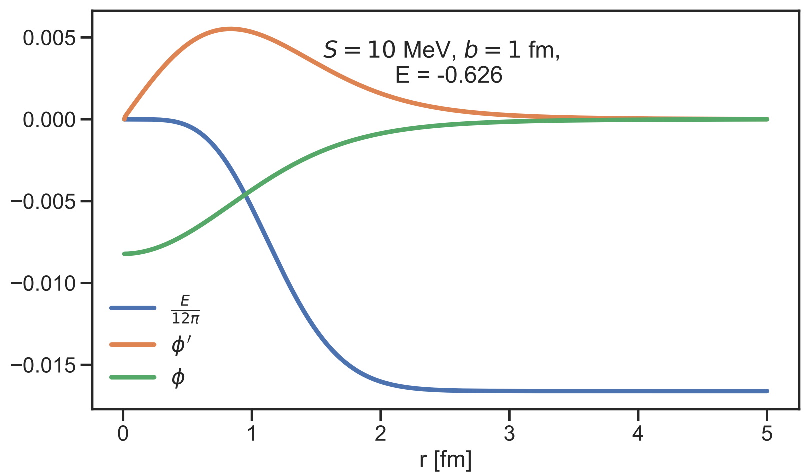

Figure 2.3: Boundary value problem solutions. The plot is generated using a tolerance of 10^{-6}. The RMS values of the relative residuals over each mesh intervals is shown in the Appendix. For clarity, I have scaled the energy.

The algorithm converges and a solution to (2.7) is found. Also, note that since we expect the energy to be less than zero it makes sense for the wavefunction to be negative since all other terms in the integral are positive. This also makes sense since we are adding more degrees of freedom to the system, which means the energy in turn must decrease. In figure 2.3 I have scaled the energy for plotting aesthetics. The actual energy is given by

$$E = -0.626\,\text{MeV}. \tag{2.19}$$

This energy will act as a zero-point energy in the rest of the calculations. Note that this energy is found without any considerations about the velocity of the pion below the potential barrier. Strictly speaking, some considerations about the kinematic limit of the pion is needed to more robustly confirm the energy, which is essential to the rest of the calculations.

2.1.2 Relativistic Expansion

It is customary to assume a nonrelativistic limit when considering regular quantum mechanics. To account for relativistic effect we can replace the kinetic term, $K_{\mathbf{r}}$ in (2.7)

$$K_{\mathbf{r}} \longrightarrow K_{\mathbf{r},\text{rel}} = \sqrt{p^2c^2 + \mu^2c^4} = \mu c^2\left(\sqrt{1+\frac{p^2}{\mu^2c^2}} - 1\right), \tag{2.20}$$

where μ is the reduces mass of the nucleon-pion system. This leads to a new systems of equations and these solutions can be compared found in the nonrelativistic limit to deduce with relativistic regime dominates the physical setup. More specifically we can compare the energy ratios denoted by E_R. Starting from (2.9)

$$f(r)(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})\psi_p + \mu c^2\left(\sqrt{1+\frac{p^2}{\mu^2c^2}} - 1\right)\psi_{N\pi} = (E - m_\pi c^2)\psi_{N\pi}, \tag{2.21}$$

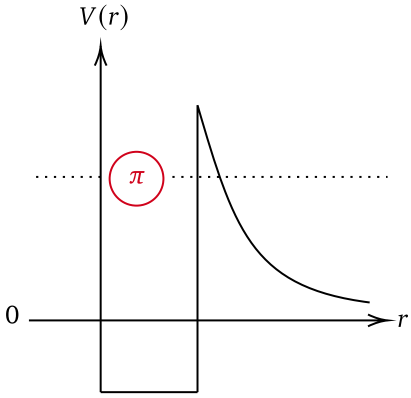

Figure 2.4: Illustration of the pion under the barrier. We are considering the kinetic expansion of the pion in this regime. The behavior of the potential barrier is a sketch.

This equation turns out to be divergent and we must therefor resort to an approximation. The kinetic energy is expanded

$$K_{\mathbf{r},\text{rel}} = \mu c^2\sqrt{1+\frac{p^2}{\mu^2}} - \mu c^2 \approx \frac{p^2}{2\mu} - \frac{p^4}{8\mu^3c^2} \tag{2.22}$$

This means we get an extra term in (2.12) yielding

$$f(r)\mathbf{r} - \left(\frac{p^2}{2\mu} - \frac{p^4}{8\mu^3c^3}\right)\mathbf{r}\phi(r) = (E - m_\pi c^2)\mathbf{r}\phi(r) \tag{2.23}$$

Using the vector operators yields the following expression[4]

$$f(r) - \frac{\hbar^2}{2\mu}\left(\phi^{(2)}(r) + \frac{4}{r}\phi^{(1)}(r)\right) + \frac{\hbar^3}{8\mu^3c^3}\left(\phi^{(4)}(r) + \frac{6}{r}\phi^{(3)}(r)\right) = (E - m_\pi c^2)\phi(r), \tag{2.24}$$

where the exponent, (n), is the order the differentiation. This leads to a system of equations given by

$$\left.\begin{array}{l} 12\pi\int_0^\infty \mathrm{d}r\, f(r)\phi(r)r^4 = E \\ f(r) - \frac{\hbar^2}{2\mu}\left(\phi^{(2)}(r) + \frac{4}{r}\phi^{(1)}(r)\right) + \frac{\hbar^4}{8\mu^3c^3}\left(\phi^{(4)}(r) + \frac{6}{r}\phi^{(3)}(r)\right) = (E - m_\pi c^2)\phi(r) \end{array}\right\} \tag{2.25}$$

This system is a fourth order differential equation coupled to an integrodifferential equation and is solved using the boundary value problem technique. The boundary conditions can be found using the same considerations as in the previous section. For $r \longrightarrow \infty$ the dominating terms are

$$\phi^{(4)}(r) = 8\mu^3(E - m_\pi c^2)\phi^{(1)}(r) + 4\mu\phi^{(2)}(r) \tag{2.26}$$

The solutions are shown in figure 2.5

4.
$$\begin{aligned} \nabla^4(\mathbf{r}\phi(r)) &= \nabla^2(r\phi'' + 4\phi') \\ &= 2\phi'''' + 2(\nabla\phi''\cdot\nabla)r + 4\phi''' \\ &= r\phi'''' + 6\phi''' \end{aligned}$$

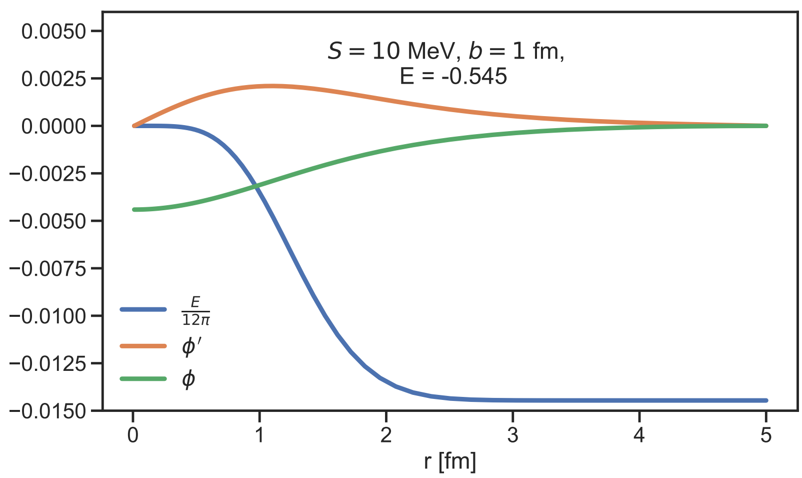

Figure 2.5: Boundary value problem solutions for the relativistic expansion. The plot is generated using a tolerance of 10^{-3}. The energy convergence is scaled.

Let me show a plot for yall

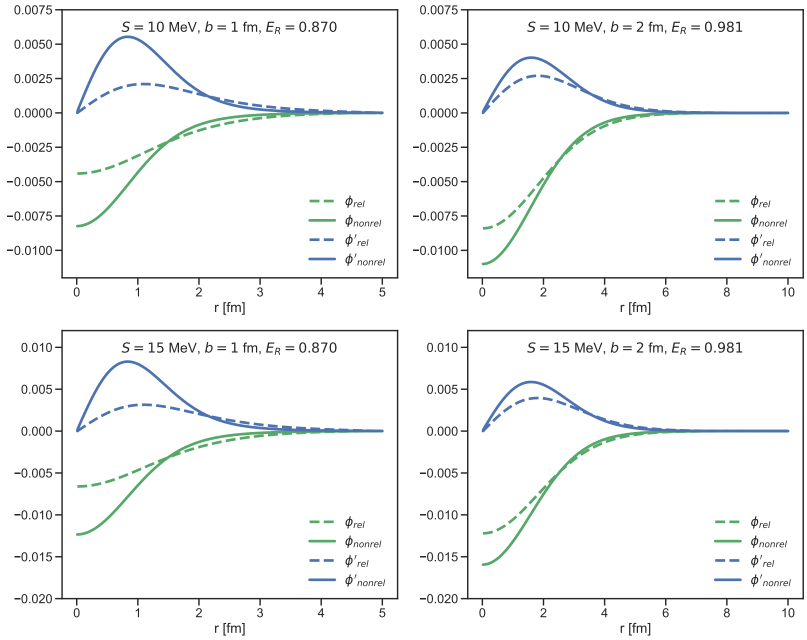

Figure 2.6: Plots for different values of the parameters S and b to illustrate the difference between the nonrelativistic equations and the relativistic equations.

From figure 2.6 one can also deduce the effect of changing the parameters S, b corresponding to changing the physical coupling strength and the form factor. Increasing the parameter b will flatten the wavefunction which will decrease the energy. This corresponds to increasing the distance between the pion and the nucleus. Increasing the coupling strength S will increase the amplitude of the wave function.

2.2 Pion photoproduction

We now consider the case of pion photoproduction. We consider an initial (bound) state given by

$$|\Phi_i\rangle = \begin{bmatrix} \phi_p \\ \phi_{N\pi} \end{bmatrix}, \tag{2.27}$$

where ϕ represents a bound state. The final state consists of the same superposition but in an unbound system represented by ψ, i.e.

$$|\Psi_f\rangle = \begin{bmatrix} \psi_p \\ \psi_{N\pi} \end{bmatrix}. \tag{2.28}$$

2.2.1 Normalization of the initial state

We must also impose some normalization. Starting from (2.27)

$$\Phi = \mathcal{N} \begin{bmatrix} p\uparrow \\ (\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})p\uparrow\phi(r) \end{bmatrix}, \tag{2.29}$$

where $\uparrow$ represents the spin state, $\phi(r)$ is the wavefunction from figure 2.3 and $\mathcal{N}$ is the normalization constant. This leads to

$$\begin{aligned} \langle\Phi|\Phi\rangle &= |\mathcal{N}|^2 \left(\langle\phi_p|\phi_p\rangle + \langle\phi_{N\pi}|\phi_{N\pi}\rangle \right) & (2.30) \\ &= |\mathcal{N}|^2 \left(V + 3V \int \mathrm{d}^3r\, r^2 \phi(r)^2\right) & (2.31) \\ &\stackrel{!}{=} 1. & (2.32) \end{aligned}$$

This leads to the following normalization constant

$$\mathcal{N} = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{1+\epsilon}}, \qquad (2.33)$$

where V is the volume and ϵ is the integral in (2.31). To figure out what particle can be knocked out by a photon via photodisintegration we consider the pion channel in (2.28). This expression is the properly normalized initial state.

2.2.2 Normalization of the final state

The final state consists of the unbound system represented by ψ. Expanding the terms using the isospin operators yields[5]

$$\phi_{N\pi} = c(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})p\uparrow\phi(r) = c(p\pi^0 + \sqrt{2}n\pi^+)(\boldsymbol{\sigma}\cdot\mathbf{r})\uparrow\phi(r). \qquad (2.34)$$

From this equation the first result should be highlighted. We consider two photodisintegration processes

$$p\gamma \longrightarrow p\pi^0 \qquad (2.35)$$

$$p\gamma \longrightarrow n\pi^+, \qquad (2.36)$$

where due to orthogonality of the states in isospin space the process involves the neutron is proportional to $\sqrt{2}$. Since this term will exist throughout the calculations we know the ratio between these two processes must be related by a factor of 2. There are also some corrections related to the mass difference between the proton and the neutron. This will be evident when comparing the matrix elements in section 2.2.3. By expanding the matrices in spin space and using the spherical tensor operator we see

$$(\boldsymbol{\sigma}\cdot\mathbf{r}) = \sqrt{\frac{4\pi}{3}}r\begin{bmatrix} Y_1^0 & \sqrt{2}Y_1^{-1} \\ \sqrt{2}Y_1^1 & Y_1^0 \end{bmatrix}, \qquad (2.37)$$

where, similar to in isospin space, the off-diagonals include a factor $\sqrt{2}$. This expansion will be useful when evaluating matrix elements with the different spin states represented by $(\uparrow\downarrow)$. Considering the final state which consists of a plane wave. It is useful to expand this in terms of spherical harmonics. Using the spherical harmonic decomposition of the plane wave yields

$$\frac{1}{\sqrt{V}}e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{1}{\sqrt{V}}\sum_{\ell,m} 4\pi i^\ell Y_\ell^{*m}(\mathbf{q})Y_\ell^m(\mathbf{r})j_\ell(qr) \qquad (2.38)$$

$$= \frac{1}{\sqrt{V}}\sum_{\ell} 4\pi i^\ell j_\ell(qr)\left(\frac{2\ell+1}{4\pi}\right)P_\ell(\cos\theta), \qquad (2.39)$$

where θ is the angle between $\mathbf{q}$ and $\mathbf{r}$ also illustrated on 2.7 and P_ℓ is the Legendre polynomial of degree ℓ. For now $\mathbf{q}$ is some variable related to the energy of the photon and will act as momentum. This is illustrated on figure 2.8 and is expanded upon in the next section. Also, the spherical harmonic addition theorem has been used.[6] Since we are considering the pion channnel just above the threshold we do the following expansion

$$\frac{1}{\sqrt{V}}e^{i\mathbf{q}\cdot\mathbf{r}} \overset{\ell=0}{=} \frac{1}{\sqrt{V}}j_0(qr), \qquad (2.40)$$

which is assumed to be the dominant contribution

5. $\tau_0\pi^0 + \sqrt{2}\tau_+\pi^- + \sqrt{2}\tau_-\pi^+$

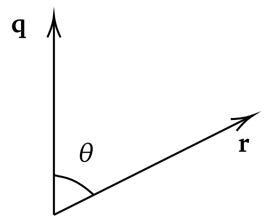

Figure 2.7: Illustration of the angle between the two vectors $\mathbf{q}$ and $\mathbf{r}$ in equation (2.40)

6. $\sum_m Y_\ell^m(\mathbf{p}')Y_\ell^{*m}(\mathbf{r}) = \left(\frac{2\ell+1}{4\pi}\right)P_\ell(\cos\theta)$

2.2.3 Photoproduction: dipole approximation

We now move on to the actual matrix element we want to calculate

$$\left\langle \Psi_f \middle| \mathbf{d} \middle| \Phi_i \right\rangle, \tag{2.41}$$

where $\mathbf{d}$ is the dipole operator, $|\Phi_i\rangle$ and $\left|\Psi_f\right\rangle$ are given by (2.27) and (2.28) respectively. As before, we only consider the pion channel. Since we are considering transitions just above the threshold we only consider the contributions from the electric dipole term. This terms is assumed to dominate and higher order terms can be calculated later if necessary. Considering the pion channel for the following process $p\gamma \longrightarrow n\pi^+$

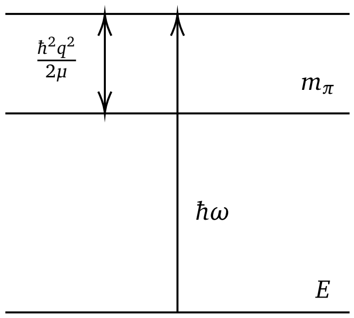

Figure 2.8: Illustration of the energy levels. Here the energy E refers to the energy shown in 2.3 and will act as a zero-point enery.

$$\mathcal{M} = -i\omega_k \sqrt{\frac{2\pi\hbar}{V\omega_{\mathbf{k}}}}\mathbf{e}_{\mathbf{k},\lambda} \left\langle \frac{1}{\sqrt{V}} e^{i\mathbf{q}\cdot\mathbf{r}} n\pi^+(\uparrow\downarrow)|\mathbf{d}|(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathbf{r})p\uparrow\phi(r)\mathcal{N}\right\rangle \tag{2.42}$$

where the two arrows represents the two spin states of the neutron and the proton. Also, q is illustrated in figure 2.8 where the energy, E, is the same as in (2.14). This is the zero point energy in the system we are considering and can be written as

$$\frac{\hbar^2 q^2}{2\mu_{N\pi}} = \hbar\omega - m_\pi c^2, \tag{2.43}$$

where $\mu_{N\pi}$ is the reduced mass of the nucleon-pion system. Compare this equation to (A.22). The front factor arises from the quantization of the electromagnetic field. Using the results from section 2.2.2 and section 2.2.1. The different spin states of the neutron in the final state yields to contributions to the total matrix element given by

$$\mathcal{M}^{\uparrow} = \frac{-i\mathcal{N}\sqrt{2}\omega_k \mathbf{e}_{\mathbf{k},\lambda}}{V}\sqrt{\frac{2\pi\hbar}{V\omega_{\mathbf{k}}}} \langle j_0(qr)|d_0 r_0|\phi(r)\rangle \tag{2.44}$$

$$= \frac{-i\mathcal{N}\sqrt{2}\omega_k \mathbf{e}_{\mathbf{k},\lambda}}{V}\sqrt{\frac{2\pi\hbar}{V\omega_{\mathbf{k}}}}\sqrt{\frac{4\pi}{3}} \langle j_0(qr)|d_0 r Y_1^0|\phi(r)\rangle \tag{2.45}$$

$$\mathcal{M}^{\downarrow} = \frac{-i\mathcal{N}2\omega_k \mathbf{e}_{\mathbf{k},\lambda}}{V}\sqrt{\frac{2\pi\hbar}{V\omega_{\mathbf{k}}}} \left\langle j_0(qr)\middle|dr_+\middle|\phi(r)\right\rangle \tag{2.46}$$

$$= \frac{-i\mathcal{N}2\omega_k \mathbf{e}_{\mathbf{k},\lambda}}{V}\sqrt{\frac{2\pi\hbar}{V\omega_{\mathbf{k}}}}\sqrt{\frac{4\pi}{3}} \langle j_0(qr)|d_0 r Y_1^1|\phi(r)\rangle, \tag{2.47}$$

$$\tag{2.48}$$

where the spin down state picks up a factor $\sqrt{2}$ from equation 2.37. Now we calculate the remaining matrix elements,

$$\left\langle j_0(qr)\middle|d_0 r_0\middle|\phi(r)\right\rangle = \frac{\mu}{m_\pi} e \left\langle j_0(qr)\middle|r_0 r_0\middle|\phi(r)\right\rangle \tag{2.49}$$

$$= \frac{\mu}{m_\pi} e \frac{4\pi}{3} \left\langle j_0\middle|r^2\middle|\phi(r)\right\rangle \tag{2.50}$$

$$= \frac{\mu e}{m_\pi} \int_0^\pi \int_0^{2\pi} \int_0^\infty \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta\, j_0(qr) r^4 \cos^2\theta \sin\theta \phi(r) \tag{2.51}$$

$$= \frac{4\pi\mu e}{3m_\pi} \underbrace{\int_0^\infty \mathrm{d}r\, j_0(qr) r^4 \phi(r)}_{Q}, \tag{2.52}$$

where the dipole operator has been inserted and the angular integrals calculated. Similarly for the next matrix element,

$$\left\langle j_0(qr)\middle|d_- r_+\middle|\phi(r)\right\rangle = \frac{\mu}{m_\pi} e \left\langle j_0(qr)\middle|r_- r_+\middle|\phi(r)\right\rangle \tag{2.53}$$

$$= \frac{4\pi\mu e}{3m_\pi}\left\langle j_0(qr)\middle|r^2 Y_1^{-1} Y_1^1\middle|\phi(r)\right\rangle \tag{2.54}$$

$$= \frac{4\pi\mu e}{3m_\pi} Q. \tag{2.55}$$

This leads to the two final expressions for the two matrix elements

$$\left|\mathcal{M}^{\uparrow}\right| = \left(\frac{4\pi\mu e}{3m_\pi}\right)^2 \frac{2\mathcal{N}^2\omega_k(2\pi\hbar)}{V^2}(\mathbf{e}_{\mathbf{k},\lambda})^0(\mathbf{e}^*_{\mathbf{k},\lambda})^0 Q^2 \tag{2.56}$$

Similarly for the next matrix element

$$\left|\mathcal{M}^{\downarrow}\right| = \left(\frac{4\pi\mu e}{3m_\pi}\right)^2 \frac{4\mathcal{N}\omega_{\mathbf{k}}(2\pi\hbar)}{V^2}(\mathbf{e}_{\mathbf{k},\lambda})^+(\mathbf{e}^*_{\mathbf{k},\lambda})^+ Q^2. \tag{2.57}$$

Calculating the total matrix element using a polarization theorem[7]

$$|\mathcal{M}|^2 = \left|\mathcal{M}^{\uparrow}\right|^2 + \left|\mathcal{M}^{\downarrow}\right|^2 \tag{2.58}$$

$$= \frac{2\pi\hbar\omega_k\mathcal{N}^2 e^2}{V^2}\left(\frac{4\pi\mu}{3m_\pi}\right)^2 Q^2, \tag{2.59}$$

which is the final expression for the matrix element. According to Fermi's golden rule we can calculate the transition probability

$$\mathrm{d}\omega = \frac{2\pi}{\hbar}|\mathcal{M}|^2\,\mathrm{d}\rho, \tag{2.60}$$

where the density of states is given by

$$\mathrm{d}\rho = \frac{Vqm}{\hbar^2(2\pi)^3}\mathrm{d}\Omega. \tag{2.61}$$

This is related to the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{16\pi}{9}\mathcal{N}^2\alpha\left(\frac{\mu}{m_\pi}\right)^2\frac{\omega_k q\mu}{\hbar^2}Q^2 \tag{2.62}$$

This should be compared to experimental data shown in figure 2.9

7. $(\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\mathbf{e}_{\mathbf{k},\lambda}) = \delta_{\lambda,\lambda'}$ and $\mathbf{e}_{\mathbf{k},\mp} = \pm\frac{1}{\sqrt{2}}(\mathbf{e}_{\mathbf{k},1} \pm i\mathbf{e}_{\mathbf{k},2})$. This leads to $(\mathbf{e}^{0*}_{\mathbf{k},\lambda}\cdot\mathbf{e}^0_{\mathbf{k},\lambda'})+(\mathbf{e}^{0+}_{\mathbf{k},\lambda}\cdot\mathbf{e}^+_{\mathbf{k},\lambda'}) = \delta_{\lambda,\lambda'}+\frac{1}{2}\delta_{\lambda,\lambda'}$

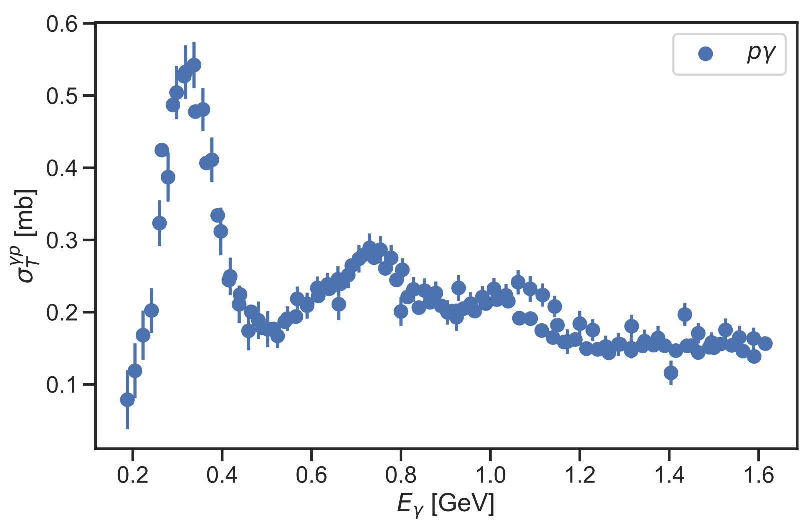

Figure 2.9: Experimental total cross section for γp. Data from

Plotting (2.62) yields

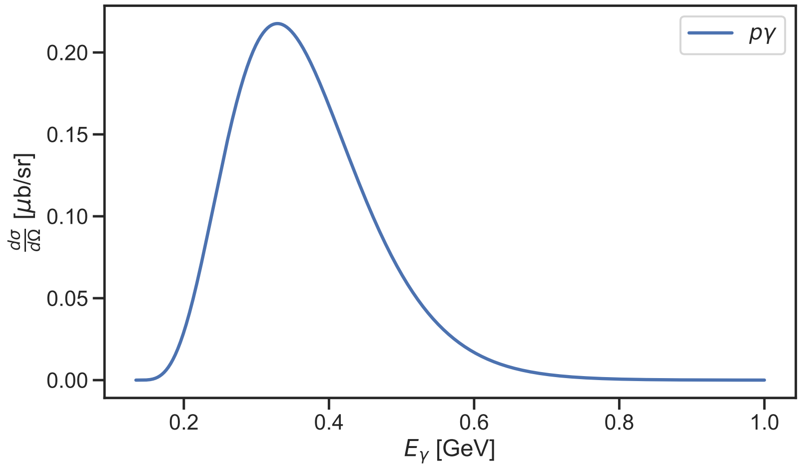

Figure 2.10: The differential cross section

Comparing the two

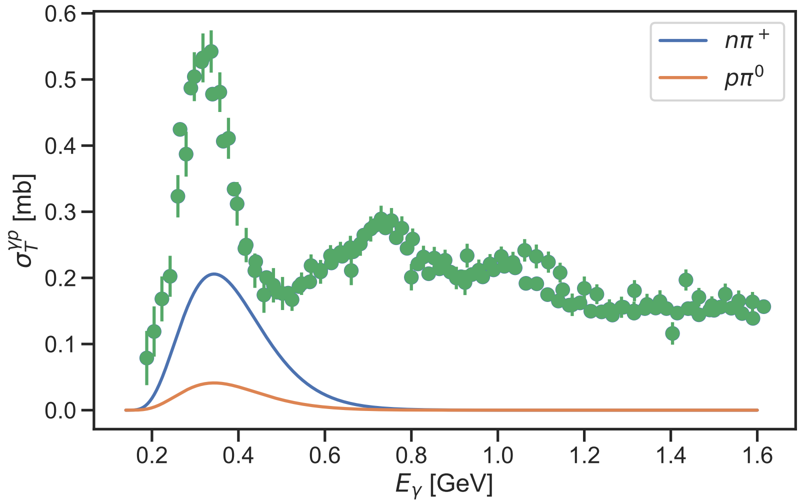

Figure 2.11: Comparing

2.3 Pion Photoproduction exact

In section 2.2 we looked at how to use the model descriped in section 2.1 to get an expression for the cross section which was compared to experimental data. More specifically we used the dipole approximation which introduces a trade-off between the difficulty of the calculations and the regime in which our solution is valid. We expect the dipole approximation to hold true for low energies just above the threshold. To both validate and generalize this result we now do a different approach and calculate the cross section exact and also consider recoil effects. Strictly speaking, recoil effects should also be considering in section 2.2 since the mass ratio between the nucleon and the pion cannot be assumed to yield a stationary nucleon after the pion photoproduction process. To calculate the exact matrix elements we consider a non-relativistic system of particles interacting with the electromagnetic field. The interacting part of the Hamiltonian is given by

$$H = \frac{1}{2m_\pi}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2, \tag{2.63}$$

where $\mathbf{p}$ is the momentum operator and $\mathbf{A}(\mathbf{r})$ is the quantized vector potential at the point $\mathbf{r}$. Note that we have already replaced the usual mass by the mass of the pion, m_π since we are considering the interaction of a pion with charge e with the electromagnetic field.

The electromagnetic interaction is relatively weak compared to the strong force. For our problem this means we can expand the interaction taking only the lowest non-vanishing order of perturbation into account. Since we later want to consider transition probabilites we only keep the first non-linear term of (2.63) which yields

$$V^{(1)} = -\frac{e}{2m_\pi c}\Big(\mathbf{p}\cdot\mathbf{A}(\mathbf{r}_\pi, t) + \mathbf{A}(\mathbf{r}_\pi, t)\cdot\mathbf{p}\Big), \tag{2.64}$$

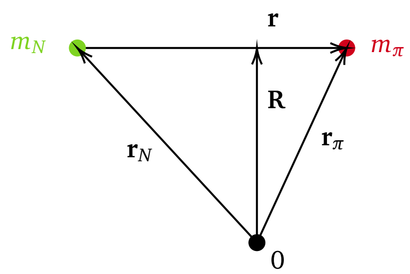

Figure 2.12: Jacobi coordinates illustrating $\mathbf{r}_\pi$ used in the vector potential in equation 2.66. Here we use $\mathbf{r} = \mathbf{r}_\pi - \mathbf{r}_N$ and see that $\mathbf{r}_\pi = \mathbf{R} + \mathbf{r}\frac{m_N}{M}$, where $M = m_N + m_\pi$.

which also means the interacting part is linear in the creation(annihilation) operators corresponding to single-photon emission(absorption). Our choice of gauge is purely conventional and we choose the radiation gauge which imposes a condition on the vector potential given by

$$\nabla\cdot\mathbf{A} = 0, \tag{2.65}$$

and this is a convenient choice of gauge since the commutator in (2.64) is $\nabla\cdot\mathbf{A}$ and we can write

$$V^{(1)} = -\frac{e}{m_\pi c}\mathbf{A}(\mathbf{r}_\pi, t)\cdot\mathbf{p}_\pi. \tag{2.66}$$

Note that (2.66) consists of the pion momentum operator, $\mathbf{p}_\pi$ and the electromagnetic vector potential at a distance r_π. This distance was also mentioned in section 2.1 and can be expressed in term of the Jacobi coordinates illustrated on figure 2.12 where the relative coordinate is given by $\mathbf{r} = \mathbf{r}_\pi - \mathbf{r}_N$ which leads to the following transformation of equation (2.66)

$$V^{(1)} = -\frac{e}{m_\pi c}\mathbf{A}(\mathbf{r}_\pi, t)\cdot\mathbf{p}_\pi = \frac{-e}{m_\pi}\Big(\mathbf{p} + \frac{m_N}{M}\mathbf{P}\Big)\mathbf{A}\Big(\mathbf{R} + \frac{m_N}{M}\mathbf{r}, t\Big), \tag{2.67}$$

where $M = m_N + m_\pi$ is the total mass of the system. We now move on from the general theory of the pion interacting with the electromagnetic field and consider the specific case of pion photoproduction. To recap we consider the following processes

$$p\gamma \longrightarrow p\pi^0 \tag{2.68}$$
$$p\gamma \longrightarrow n\pi^+ \tag{2.69}$$
$$n\gamma \longrightarrow n\pi^0 \tag{2.70}$$
$$n\gamma \longrightarrow p\pi^-, \tag{2.71}$$

where the inital states consist of a dressed proton or neutron and a plane wave photon which in the language of 2nd quantization can be written as $a^\dagger_{\mathbf{k},\lambda}\,|0\rangle$ corresponding to creating a photon with wave vector $\mathbf{k}$ and polarization λ from the vacuum $|0\rangle$. The final states consists of a nucleon and a pion and no photon, i.e electromagnetic vacuum. This leads to the following expression for the matrix element for the transitions in (2.68), (2.69), (2.70) and (2.71)

$$\frac{-e}{m_\pi c}\,\langle 0|\mathbf{A}\big(\mathbf{R} + \frac{m_N}{M}\mathbf{r}, t\big)a^\dagger_{\mathbf{k},\lambda}|0\rangle = \frac{-e}{m_\pi}\sqrt{\frac{2\pi\hbar}{\omega_{\mathbf{k}}V}}\mathbf{e}_{\mathbf{k},\lambda}\mathrm{e}^{i\mathbf{k}(\mathbf{R}+\frac{m_N}{M}\mathbf{r})-i\omega_k t}. \tag{2.72}$$

Completely analogous to section 2.2.3 and specifically equation (2.60) we now want to consider the probability of transition per unit time of going from the initial state $|i\rangle$ to $\langle f|$ according to Fermi's golden rule

$$\mathrm{d}\omega = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \,\mathrm{d}\rho, \tag{2.73}$$

where $\mathrm{d}\rho$ is the density of states also given by (2.61). The difference is that we do not employ the dipole approximation to evaluate the matrix element $\mathcal{M}$ and we also consider recoil. In total this leads to the following matrix element

$$\mathcal{M} = \tag{2.74}$$

Appendix A

Nuclear photoeffect and the deuteron

In this appendix, I wish to go through how to get expressions for the differential cross section and the total cross section from the wave function. The wave function can be obtained analytically or numerically – in this appendix, I will sketch an analytical approach to s-wave calculations. Considering the central potential between the proton and the neutron given by

$$U(r) = \begin{cases} -U_0, & r \leq R \\ 0 & r > R \end{cases}$$

The radial equation is given by

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} + \left[U(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right] u(r) = Eu(r). \tag{A.1}$$

This is identical to the one-dimensional Schrodinger equation with an effective potential, where the centrifugal term pushes the particle outwards. To solve this analytically I rewrite the equation and consider the boundary conditions.

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} + \frac{M}{\hbar^2}\left[E - U(r)\right] u(r) = 0, \tag{A.2}$$

where I plugged in the expression for the reduced mass, $m = M/2$. For the deuteron I use $E = -E_B = -2.225$ MeV [Zelevinsky and Volya 2017, p. 51]. This leads to the following expressions

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} + \frac{M}{\hbar^2}(U_0 - E_B)\, u(r) = 0, \quad r \leq R, \tag{A.3}$$

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} - \frac{M}{\hbar^2} E_B u(r) = 0, \quad r > R. \tag{A.4}$$

I introduce two variables given by

$$k = \sqrt{\frac{M}{\hbar^2}(U_0 - E_B)}, \quad \kappa = \sqrt{\frac{ME_B}{\hbar^2}}. \tag{A.5}$$

Rewriting equation (A.3) in terms of (A.5) and solving the differential equation yields

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} = -ku(r) \Rightarrow u(r) = A\sin(kr) + B\cos(kr). \tag{A.6}$$

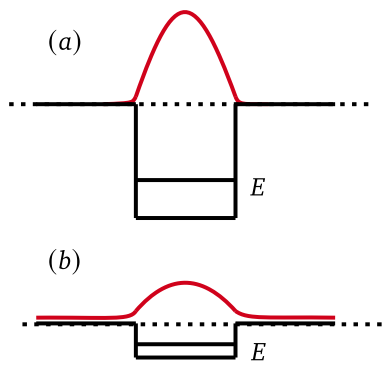

Figure A.1: Behavior of the ground state bound wave function for two potentials. (a) is an illustration of the deeper potential well case and (b) is for a shallower potential well.

Since $R(r) = u(r)/r$ and $\cos(kr)/r$ blows up as $r \longrightarrow 0 \Rightarrow B = 0$ and the solution is

$$u(r) = A\sin(kr), \quad r \le R \tag{A.7}$$

Now, considering equation (A.4)

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} = \kappa^2 u(r) \Rightarrow u(r) = Ce^{\kappa r} + De^{-\kappa r} \tag{A.8}$$

Here $Ce^{\kappa r}$ blows up as $r \longrightarrow \infty$. The wavefunction must be continuous and this means the solutions (A.6) and (A.8) must match at $r = R$. The same applies for the derivative. This leads to two equations for $r = R$.

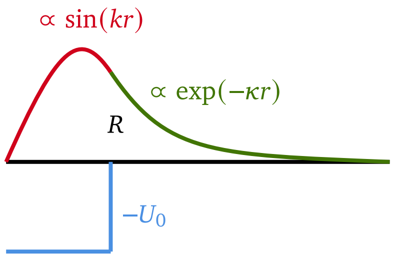

Figure A.2: The s-wave wave function for the deuteron.

$$A\sin(kR) = De^{-\kappa R} \tag{A.9}$$

$$Ak\cos(kR) = -D\kappa e^{-\kappa R} \tag{A.10}$$

Dividing equation (A.10) by equation (A.9) leads to

$$-\cot(kR) = \frac{\kappa}{k} \tag{A.11}$$

This equation is solved by requiring $kR = \pi/2$. Plugging in an appropiate value for $R = 1.7$ fm yields

$$\begin{aligned} U_0 &= \frac{\hbar^2\pi^2}{2mR^2} - E \\ &= \frac{\hbar^2\pi^2}{2mR^2} + E_B \\ &= 37.2\,\mathrm{MeV} \end{aligned}$$

This means the depth of the potential is 37.2 MeV.

	0^+	0^-	1^+	1^-
Singlet	$^1s_0$			$^1p_1$
Triplet		3^p_0	$^3s_1, ^3d_1$	$^3p_1$

Table A.1: Two nucleon states J^Π. The deuteron consists of a wave function superposition of $^3s_1 + ^3d_1$.

Note that this is all for s-wave. Some considerations about the tensor force are also needed. This means we have to consider the Schrödinger equation with noncentral spin-dependent potential given by

$$\mathcal{U}(r) = \mathcal{U}_0(r) + \mathcal{U}_t(r)S_{12}, \tag{A.12}$$

where

$$\mathcal{U}_t(r) = \mathcal{U}_{tW}(r) + \mathcal{U}_{tM}(r), \tag{A.13}$$

for the space-even states we are considering with the deuteron, see table A. Considering the d-wave I introduce the anglular momentum coupling $[Y_2(\mathbf{n})\chi_1]_{1M}$ of spin 1 and $\ell = 1$ to the total deuteron spin $J = 1$ and projection $J_z = M$. The same coupling but properly normalized can be written as

$$\Theta_M = \frac{1}{\sqrt{32\pi}} S_{12}\chi_{1M}. \tag{A.14}$$

To get the complete wave function of the deuteron the expression must contain two radial parts and a spherical wave factor $1/r$

$$\Psi_M = \frac{1}{\sqrt{4\pi}}\frac{1}{r}\left(u_0(r) + \frac{1}{\sqrt{8}}u_2(r)S_{12}\right)\chi_{1M}, \tag{A.15}$$

where the two radial parts are $u_0(r)$ and $u_2(r)$ for s-wave and d-wave respectivly. These must also be normalized as

$$\int_0^\infty \mathrm{d}r\,|u_0|^2 + \int_0^\infty \mathrm{d}r\,|u_2|^2 = 1, \tag{A.16}$$

and the two terms can be interpreted as a weight for the respective wave. Now using the expression for the deuteron wave function equation

(A.15) I want to get an expression for the differential cross section for the nuclear photoeffect. When an absorbed photon frequency is greater than the lowest threshold of nuclear decay the nucleus becomes excited to the continuum states. These states can decay in a number of ways but here are consider the case where the decay happens through particle emission. In other words this is the absorbtion of a photon that results in a particle decay into the continuum. From conservation of energy we have

$$E_i = \hbar\omega = E_f + \epsilon, \tag{A.17}$$

where the nucleus A goes from the inital state with energy E_i to the final state with $A-1$ and energy E_f and the particle in the continuum has energy $\epsilon = \mathbf{p}^2/2m$.

In contrast to the excitation of the discrete states that shows resonance behavior the continuum of energy states makes for a more smooth dependence. This means we can use what we know form the discrete excitation but introduce a level density ρ_f instead of the usual delta function in the expression for the differential cross section. This yields

$$\mathrm{d}\sigma_{fi} = \frac{4\pi^2\hbar}{E_\gamma c}\left|\sum_a \frac{\mathbf{e_a}}{m_a}\left\langle f\middle|(\mathbf{p}_a\cdot\mathbf{e}_{\mathbf{k}\lambda})\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}_a)}\middle|i\right\rangle\right|^2 \rho_f, \tag{A.18}$$

where the level density is given by

$$\rho_f = \frac{Vmp}{(2\pi\hbar)^3}\mathrm{d}o. \tag{A.19}$$

Here the particle is emitted with momentum $\mathbf{p}$ into the solid angle element $\mathrm{d}o$ and E_γ is the energy of the photon. In the case of the deuteron equation (A.18) and be split into different multipolarities and the most simple is the electric dipole transition ($E1$) from the deuteron bound state into the state of continuum motion of the proton and the neutron with reduced mass $m/2$.

In the long wave length limit the plane wave expression reduces to unity which means equation (A.18) for the dipole transition can be written as

$$\mathrm{d}\sigma_{E1} = \frac{\alpha mp\omega}{\hbar^2}\left|\sum_a(\mathbf{e}\cdot\mathbf{r_a})_{fi}\right|^2\frac{\mathrm{d}o}{4\pi}, \tag{A.20}$$

where the solid angle could be the direction along the motion of the proton[1].

When assuming an unpolarized deuteron we can take the average over the spin states $1/3\sum_m$ and count all final polarizations $\sum_{m'}$[2]. The final state is still spin triplet since the dipole operator does not act on the spin variable. This yields

$$\overline{\mathrm{d}\sigma_{E1}} = \frac{1}{4}\frac{\alpha mp\omega}{\hbar^2}\frac{1}{3}\sum_{mm'}\left|(\mathbf{e}\cdot\mathbf{r})_{fi}\right|^2\frac{\mathrm{d}o}{4\pi}. \tag{A.21}$$

The task in now to find an expression for the dot product in the sum. The final spin state after the $E1$ transition remains a triplet with $S = J = 1$, the orbital and parity, however, is not the same. The final state corresponds to the p-wave where the low-energy nuclear forces are weak and the wavelength of the relative motion is much larger than the range of those forces. From conservation of energy we have have

$$\frac{\hbar^2k^2}{2m} = \hbar\omega - \epsilon, \tag{A.22}$$

1. Also, $V = 1$ and $\alpha = e^2/\hbar c$

2. Note m and m'

where ϵ is the binding energy of the deuteron also in equation (A.2). For any direction of the relative momentum vector $\hbar\mathbf{k}$ the p-wave component must be normalized through some Legendre polynomial $P_1(\cos(\theta))$ and the spherical Bessel function $j_{\ell=1}(kr)$.[3]

3. $P_1(\cos(\theta)) = \cos(\theta)$ and $j_1(\rho) = \frac{\sin(\rho)-\rho\cos(\rho)}{\rho^2}$.

Here θ is the angle between the relative coordinate $\mathbf{r}$ and the wave vector $\mathbf{k}$ this yields

$$\psi_f(r,\theta) = 3i\cos(\theta) j_1(kr)\chi_{mm'} \tag{A.23}$$

In the s-wave transition we get the following expression when integration over the angles of the unit vector $\mathbf{n} = \mathbf{r}/r$

$$\frac{1}{2}\left\langle f; m' \middle| (\mathbf{e}\cdot\mathbf{r}) \middle| \ell = 0; m\right\rangle = -i\frac{\sqrt{\pi}}{k}(\mathbf{e}\cdot\mathbf{k}) I_0 \delta_{mm'}. \tag{A.24}$$

The radial integrals for the s-wave and d-wave are given by I_0 and I_2 respectively.[4] For the d-wave we have to reintroduce the tensor operator[5] S_12. Just like in the s-wave case we have to integrate over the unit vector – this time, however, it contains four components

4. $I_\ell = \int_0^\infty \mathrm{d}r\, r^2 j_1(kr) u_\ell(r)$, $\ell = 0, 2$

5.

$$\begin{aligned} S_{12}(\mathbf{n}) &= 3(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2)\cdot\mathbf{n} - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \\ &= 2[3(\mathbf{S}\cdot\mathbf{n})^2 - \mathbf{S}^2] \end{aligned}$$

$$\int \mathrm{d}o\, n_i n_j n_k n_l = \frac{4\pi}{15}(\delta ij\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{A.25}$$

and the d-wave contribution is given by

$$\frac{1}{2}\left\langle f; m' \middle| (\mathbf{e}\cdot\mathbf{r}) \middle| \ell = 2; m\right\rangle = -i\frac{\sqrt{\pi}}{k} C_{m'm} I_2, \tag{A.26}$$

where the spin matrix element $C_{mm'}$ contains

$$C = \frac{2\sqrt{2}}{5}\left[\frac{3}{4}\left[(\mathbf{k}\cdot\mathbf{S})(\mathbf{e}\cdot\mathbf{S}) + (\mathbf{e}\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{S})\right] - (\mathbf{e}\cdot\mathbf{k})\right]. \tag{A.27}$$

Equation (A.27) can be rewritten using a trace identity[6] Skipping the calculation and moving back to equation (A.21) we have

6.

$$\sum_{mm'} |O_{m'm}|^2 = \sum_m (O^\dagger O)_{mm} = \mathrm{Tr}\{O^\dagger O\}$$

$$\frac{1}{3}\sum_{mm'} \left|(\mathbf{e}\cdot\mathbf{r})_{fi}\right|^2 = 4\pi\left(I_0^2\cos^2(\alpha) + \frac{1}{25}I_2^2(3+\cos^2(\alpha))\right), \tag{A.28}$$

where α is the angle between $\mathbf{e}$ and the momentum of the final nucleon $\mathbf{k}$. The final steps involve averaging over the transverse polarizations of the initial photon which also relates the angle α to the experimentally observed angle between the directions of the photon and final nucleus. This means we get the following expression

$$\overline{\cos^2(\alpha)} = \frac{1}{2}\sin^2(\theta) \tag{A.29}$$

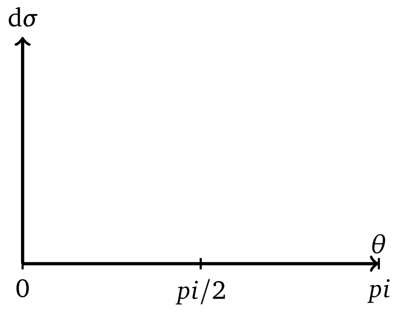

Figure A.3: Behavior of the differential cross section from equation (A.30).

Plugging this into equation (A.20) yields

$$\mathrm{d}\sigma_{E1} = \frac{\pi}{2}\frac{\alpha m p \omega}{\hbar^2}\left[I_0^2\sin^2(\theta) + \frac{1}{25}(6+\sin^2(\theta))I_2^2\right]\frac{\mathrm{d}o}{4\pi}, \tag{A.30}$$

and we arrive at the final expression when integrating over the angle of emitted photons

$$\sigma_{E1} = \frac{\pi}{3}\frac{\alpha m p \omega}{\hbar^2}\left(I_0^2 + \frac{2}{5}I_2^2\right). \tag{A.31}$$

If is also possible to estimate the cross section in equation (A.31) using the initial wave function of the approximation of weak binding. Here the wave function is replaced by its exponential tail outside the range of nuclear forces. Furthermore, the contribution I_2 is neglected. This means the wave function is given by

$$\psi_i = \sqrt{\frac{\kappa}{2\pi}}\frac{e^{-\kappa r}}{r}, \tag{A.32}$$

where κ is defined in equation (A.5). Calculating the integral I_0 yields

$$\sigma = \frac{8\pi}{3}\frac{\alpha\hbar^2}{M}\frac{\sqrt{\epsilon}(\hbar\omega - \epsilon)^{3/2}}{(\hbar\omega)^3}, \tag{A.33}$$

which is rewritten in terms of the photon energy, $\xi = \hbar\omega/\epsilon$ and in terms of numerical estimates

$$\sigma(\xi) \simeq 1.2\frac{(\xi - 1)^{3/2}}{\xi^3} \times 10^{-26}\,\text{cm}^2. \tag{A.34}$$

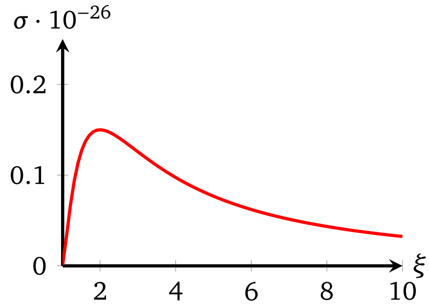

Figure A.4: Behavior of the cross section as a function of photon energy, ξ. Maximum occurs at $0.15 \cdot 10^{-26}$ cm^2 which is equivalent to 1.5 mb.

Appendix B

Special functions and properties

In this appendix I wish to cover the basics of some of the special functions that arises when discussing properties of some operators in quantum mechanics.

$$Y_\ell^m(\theta,\phi) = \frac{(-1)^{\ell l+m}}{(2\ell)!!}\left[\frac{(2\ell+1)(\ell-m)}{4\pi(\ell+m)!}\right](\sin\theta)^m \frac{\mathrm{d}^{\ell+m}}{(d\cos\theta)^{\ell+m}}\left[(\sin\theta)^{2\ell}\right]\exp\{im\phi\}, \tag{B.1}$$

which satisfy

$$Y_\ell^{*m} = (-1)^m Y_l^{-m}. \tag{B.2}$$

The spherical harmonics are connected to the Legendre polynomials

$$P_\ell(\cos\theta) = \left[\frac{4\pi}{2\ell+1}\right]^{1/2} Y_l^0(\theta). \tag{B.3}$$

Another important feature of the spherical harmonics is that they form a complete set of functions over the unit sphere. Furthermore, they form an orthonormal set

$$\int \mathrm{d}\Omega\, Y_\ell^{*m} Y_{l'}^{m'} = \delta_{mm'}\delta_{ll'}. \tag{B.4}$$

Also, there exists an addition theorem for spherical harmonics

$$\sum_{m=-\ell} Y_\ell^{*m}(\theta,\phi) Y_\ell^m(\theta'.\phi') = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} Y_\ell^0(\alpha) \tag{B.5}$$

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