Third compulsory exercise

Martin Mikkelsen
–201706771–

May 3, 2021

a) The task in the third exercise is to read one of the papers listed below and write a summary of it containing the most important (nuclear) physics points and results mentioned in the paper. The summary must be less than two typed pages.

The Interacting boson model

The interacting boson model (IBM) is a model in which nucleons pair up, essentially acting as a single particle with boson properties, with a spin of 0, 2 or 4. In this assignment, I only consider even-even complex nuclei and focus on the $s-d$ model. I will also skip most of the algebra of boson operators and highlight the main results.
The low-energy excitations in complex nuclei are of collective character (rotation, vibration). In this model, we describe this behaviour by artificially introducing boson. The relative violation of Fermi statistics is small. The model is justified from the shell model and must reduce to a liquid-drop type Hamiltonian for large N. The main idea of the IBM is to take into account only the monopole and quadrupole bosons with their interactions. This means nucleon pairs coupled to $L = 0$ and pairs excited to $L = 2$. These pairs are called s and d bosons respectively. The total number of bosons is then given by

$$N = N_s + N_d. \tag{1}$$

One can relax this condition and some corrections can be added later. The Boson Hilbert space consists of states derived from the boson vacuum $|0\rangle$ by applying the creation operator successively. This leads to the expression for any tranformation in a subspace given by

$$\hat{N} = \sum_i \hat{N}_i = \sum_i b_i^\dagger b_i, \tag{2}$$

where b_i is the creation operator. This transformation can be done by a sequence of generators given by

$$B_{ij} = b_i^\dagger b_j = B_{ji}^\dagger, \tag{3}$$

which replaces the boson i by j keeping the total number fixed. Skipping all the mathematics concerning group theory, I will focus on the Casimir operator (commuting with all generators of the subgroup). For a given subgroup one can construct a Casimir operator quadratic in generators (B_{ij})

$$C_{U(D)} = \sum_{ij} B_{ij}^\dagger B_{ij} = \sum_{ij} B_{ji} B_{ij} = 4\sum B_{ij}^{(+)\dagger} B_{ij}^{(+)} + 4\sum B_{ij}^{(-)\dagger} B_{ij}^{(-)}. \tag{4}$$

The unitary and orthogonal groups of interest (for instance rotation) the Casimir operator must be a scalar. Using this and the number operator equation (2) leads to the spectrum of the eigenvalues of $C_{\mathcal{O}(D)}$ given by

$$C_{\mathcal{O}(D)} = \sigma(\sigma + D - 2), \quad \sigma = 0, 1, 2, ..., \tag{5}$$

where σ is the number of unpaired bosons. Here σ is analogous to the three-dimensional angular momentum J. I will now focus on the $s-d$ model which is based on a unitary spectrum generating operator, $SU(6)$ and an orthogonal angular momentum symmetry, $O(3)$ and some additional terms I will expand upon after introducing the Hamiltonian. To construct a Hamiltonian for the $s-d$ model I assume the number of bosons is conserved. The total Hamiltonian can be expressed as

$$H = H_s^\circ + H_s' + H_d^\circ + H_d' + H_{sd}' \tag{6}$$

$$= \epsilon N_d + \alpha C_{U(5)} + \beta C_{SU(3)} + \gamma C_{\mathcal{O}(6)} + \delta C_{\mathcal{O}(5)} + \eta C_{\mathcal{O}(3)} + \text{const.}, \tag{7}$$

where the Hamiltonians are expressed in terms of the creation and annihilation operators of the s and d bosons given by

$$s^\dagger, \quad d_m^\dagger, \quad m = 0, \pm 1, \pm 2. \tag{8}$$

These operators can be organized into sets of generators for different groups (page 456). For instance, six pairs of annihilation and creation operators, equation (8), form 36 two-boson combinations, equation (3). This leads to three chain algebras for the IBM given by

$$\text{Vibational limit} : SU(6) \supset SU(5) \supset \mathcal{O}(5) \supset \mathcal{O}(3) \tag{9}$$

$$\gamma\text{-unstable limit} : SU(6) \supset \mathcal{O}(6) \supset \mathcal{O}(5) \supset \mathcal{O}(3) \tag{10}$$

$$\text{Rotational limit} : SU(6) \supset SU(3) \supset \mathcal{O}(3) \tag{11}$$

The IBM Hamiltonian, equation (6), provides a new perspective on the general picture of collective dynamics. Without going into depth to each of the three limits I will talk more generally about figure shapes and phase transitions in the IBM. In this framework phase transitions does not refer to the familiar thermodynamic transitions, but are phase transitions in the shape of the nuclear ground state and in the properties of other low-lying levels.

Figure 19.4 shows the colour map of $E(4_1^+)/E(2_1^+)$ for even-even nuclei which can be calculated using the IBM and compared to data. In the $SU(5)$ limit of non-interacting d-bosons we have a pure vibrational spectrum. The $SU(3)$ limit leads to rotational-like spectra. There are many examples of $SU(3)$ symmetries and these are related to the quantum numbers λ, μ which describe quadrupole deformation parameters[1]. Nuclei with quadrupole shapes can also be defined by β and γ; where $\beta = 0$ is a spherical nucleus and $\beta = 0.3$ is a typical prolate and γ reflects the symmetry of the ellipsoidal shape around the symmetry axis.

Figure 19.5 shows the colour map of excitation energy ratios E_4/E_2 but with a similar Hamiltonian (19.119) and $N = 10$ but one can make a similar plot for equation (6). The main point is how each of these symmetries has characteristic signatures and in principle, any collective nucleus can be assigned a location within this triangle and different locations correspond to different shapes and structures.

[1] The book refers to equation (11.57) and (11.58) but it should be (12.57) and (12.58)