Second compulsory exercise

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This problem is concerned with the Island of inversion around $N = 20$ that was mentioned briefly at the end of section II in the Otsuka et al review. Unless stated explicitly, all references to equations etc below are to our textbook: V. Zelivinsky and A. Volya Physics of Atomic Nuclei.

a) The first indication that something unusual was happening in the Island of inversion came from mass measurements. As an example, we know today that the two-neutron separation energies for $^{31-33}$Mg are: 8.66 MeV, 8.09 MeV and 8.06 MeV. Explain why this indicates that N = 20 is not a magic number any more.

The first indication could be found by plotting the two-neutron separations for $^{31-33}$Mg and comparing to figure 8.3 in the book. The two-neutron separation energies have an advantage compared to the single-neutron separation energy since these are not subject to pairing effects. Figure 8.3 in the book shows that S_{2n} falls for increasing A but drops a lot near the magic numbers. Figure 1 shows the observed two-neutron separation energies. The figure does show a decreasing S_{2n} for increasing A but does not show much lower energy which would be expected from a magic number. This indicates that $N = 20$ is not a magic number.

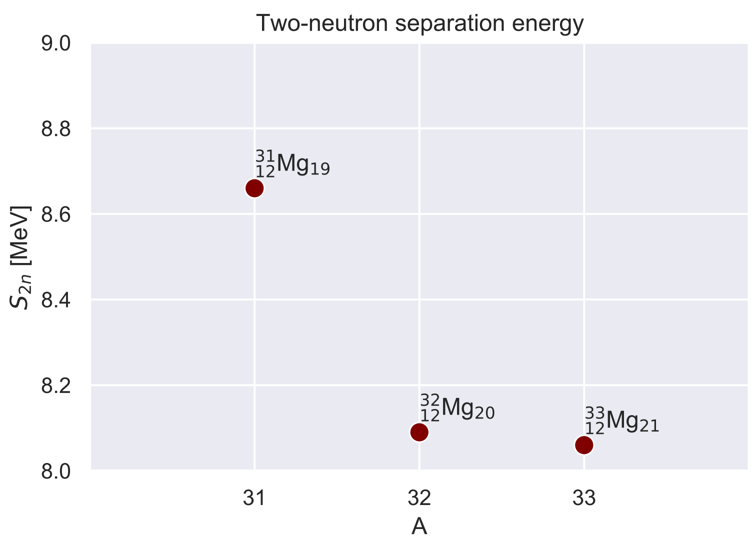

Figure 1: Two-neutron separation energies for $^{31-33}$Mg

b) Explain why this is inconsistent with the standard shell-model picture where N = 20 is a closed shell.

From experiments the spin of ^{31}Mg was measured to be 1/2+ with a magnetic moment of 0.8836 nuclear magnetons; in another the spin of ^{33}Mg was measured to be 3/2 and the magnetic moment 0.497 nuclear magnetons.
From the shell model we know the closed shells do not contribute to the spin – this means we can ignore the protons in these nuclei since $Z = 12$ is a filled shell. This means we only have to consider an extra neutron or a missing neutron for ^{31}Mg and ^{33}Mg respectively. From figure 8.5 this means

$$J^\pi = \begin{cases} 7/2^- & ^{33}\mathrm{Mg} \\ 3/2^+, & ^{31}\mathrm{Mg}, \end{cases} \tag{1}$$

since the extra neutron is in the f-orbital and the missing neutron is in the d-orbital. From the j quantum number the magnetic moment can be calculated using the equations on page 173.

$$\mu_n(j) = \begin{cases} \mu_n, & j = \ell + \frac{1}{2} \\ -\frac{j}{j+1}\mu_n, & j = \ell - \frac{1}{2}, \end{cases} \tag{2}$$

where $\mu_n = \frac{-3.82}{2}$. Inserting numbers yields

$$\mu_n(j) = \begin{cases} -1.91, & ^{33}\mathrm{Mg}, \quad j = \ell + 1/2 \\ 1.15, & ^{31}\mathrm{Mg}, \quad j = \ell - 1/2. \end{cases} \tag{3}$$

$$\tag{4}$$

The inconsistency is clear when comparing the values in equation (1) and equation (3) to the values from experiments.

c1) We look in more detail at ^{32}Mg and only consider the neutron configurations. Normally $N = 20$ would be a closed shell, so we shall use an energy scale where the zero point is the energy of the (unperturbed) 0^+ state that has an $N = 20$ closed shell. Moving two neutrons from the sd-shell up to the fp-shell will give other states: first consider two neutrons in the $f_{7/2}$ orbit and give arguments why among the states they can form, a 0^+ state is lowest in energy. Hint: section 13.1 in the book is relevant for the question.

Considering the neutrons in the $f_{7/2}$ orbit–both neutrons will have an angular momentum of 7/2 and a negative parity according to $l = 3$. This means the system consisting of two neutrons will have positive parity. Due to pairing, there will be some extra binding energy which can be explained by the character of the residual interaction between the particles in the common mean-field. The short-range attraction prefers the nucleons with strongly overlapped wave functions. This is achieved by a nucleon pair in time-conjugate orbitals, which supports the idea of a 0^+ ground state. The short-range potential decreases for larger J since the spin can only couple to either 0 or 1. This means the $J = 0$ will have the lowest energy since the overlap between the neutron wave function is largest. A similar situation is shown in figure 13.2 which shows the energy spectrum of the lowest states in ^{210}Po.

c2) Now focus on the energy of this second 0+ state. In a simple model the (unperturbed) energy of the $(f_{7/2})^2$ state is $E = 10$ MeV E_0, where E_0 is the energy due to the short-range residual interaction between the two $f_{7/2}$ neutrons. (Note that we do not attempt to calculate E_0 and instead use E' as variable.) There will be a small residual interaction V between the closed shell 0^+ state and the $(f_{7/2})2\ 0^+$ state. Assume the matrix element of V between the two states has the value 0.3 MeV, and plot the energy of the two 0^+ states as well as the probability that the lowest 0^+ state is in the closed shell configuration as a function of E' (let E' vary from +2 MeV to 2 MeV).

The Hamiltonian is given by

$$\hat{H} = \begin{bmatrix} E_1 & V \\ V & E_2 \end{bmatrix}. \tag{5}$$

In words, the states ϕ_1, ϕ_2 interact through the off-diagonal elements. The mixing of these states result in ϕ_+ and ϕ_- as illustrated on figure 2.

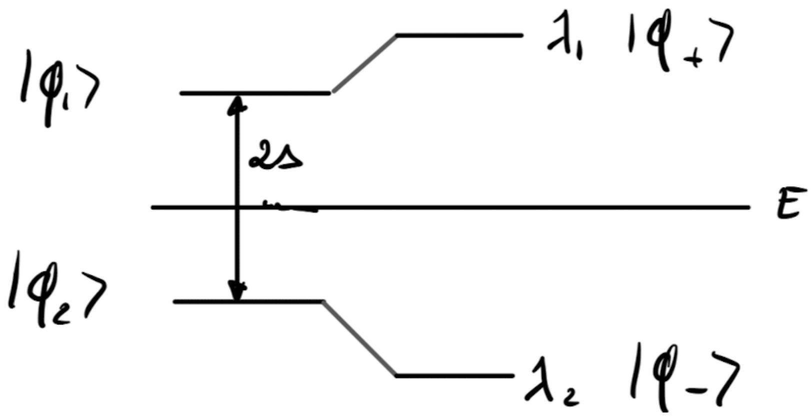

Figure 2: States

The figure also illustrates the two variables Δ and E given by

$$\Delta = \frac{E_1 - E_2}{2}, \quad E = \frac{E_1 + E_2}{2}. \tag{6}$$

Now find the eigenvalues

$$\left|\hat{H} - \lambda\mathbb{1}\right| = \begin{vmatrix} E_1 - \lambda & V \\ V & E_2 - \lambda \end{vmatrix} = (E_1 - \lambda)(E_2 - \lambda) - V^2 = 0. \tag{7}$$

This leads to

$$\lambda_{1,2} = \frac{E_1 + E_2}{2} \pm \sqrt{\frac{E_1 - E_2}{2} + V^2} \tag{8}$$

$$= E \pm \kappa, \quad k = \sqrt{\Delta^2 + V^2}. \tag{9}$$

The two eigenwavefunctions should of course be normalized and we therefore get the following equations that gives the probabilities of being in ϕ_1 and ϕ_2, respectively[1]

$$c_{1,\pm}^2 = \frac{1}{1 + (E_\pm - E_1)^2/V^2}, \quad c_{2,\pm}^2 = \frac{1}{1 + (E_\pm - E_2)^2/V^2}. \tag{10}$$

[1] I have changed the variable $\lambda_{1,2}$ to $E_\pm$ since the two-level mixing note uses this notation.

Figure 3 shows the energy of the two 0^+ states and it plotted using equation (8) with values $V = 0.3$ MeV and E' from -2 to 2 MeV. Also, $E_1 = 0$ is the energy of the closed-shell state and E_2 is the energy of the $(f_{7/2})^2 0^+$ state.

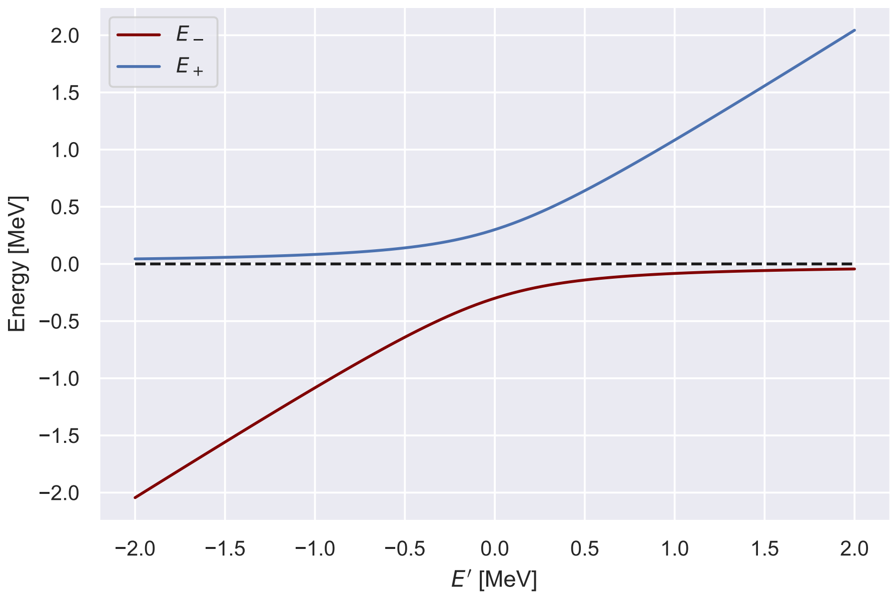

Figure 3: Energies

The probability of the lowest closed-shell state is $c_{1,-}$ from equation (10) is shown on figure 4. The figure also shows $c_{2,-}^2$ and the sum of the two which is equal to 1.

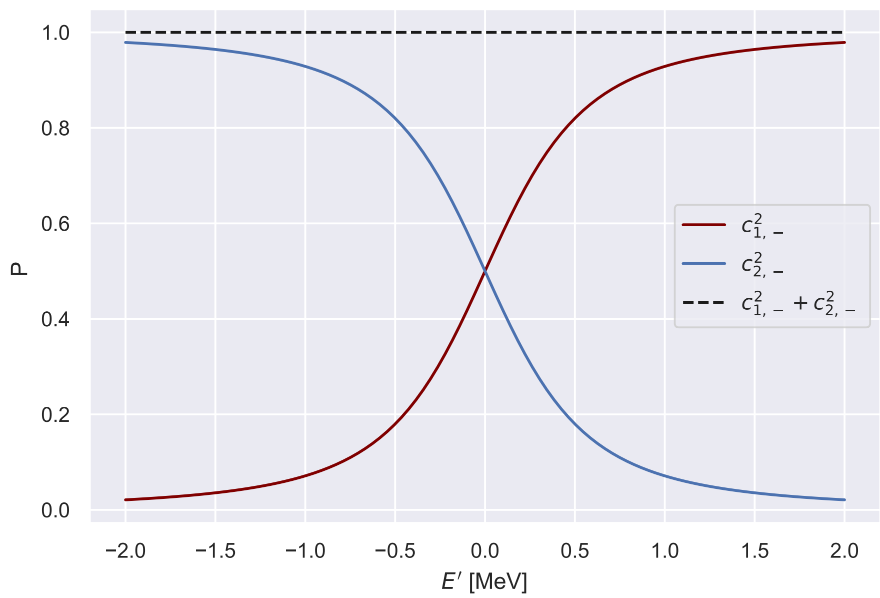

Figure 4: Probability

c3) An experiment at ISOLDE/CERN in 2010 found an excited 0+ state at an energy of 1.058 MeV in ^{32}Mg. The experiment claimed that the ^{32}Mg ground state mainly had the intruder configuration, i.e. in our model the $(f_{7/2})^2$ configuration. What value for E' would the experiment correspond to if $V = 0.3$ MeV? Can you deduce any limit on V based on the energy measured in the experiment?

Since the experiment found an excited 0+ state at 1.058 MeV the following must be true

$$E_+ - E_- = 1.058\,\mathrm{MeV}, \tag{11}$$

which can be solved for E' if I choose $V = 0.3$ MeV. This yields

$$\begin{aligned} E_+ - E_- &= \frac{1}{2}\left(E_1 + E_2 + \sqrt{(E_1+E_2)^2 - 4E_1E_2 + 4V^2}\right) - \frac{1}{2}\left(E_1 + E_2 - \sqrt{(E_1+E_2)^2 - 4E_1E_2 + 4V^2}\right) \\ &= \frac{1}{2}\left(E_2 + \sqrt{E_2^2 - 4V^2} - E^2 + \sqrt{E_2^2 + 4V^2}\right), \end{aligned}$$

with the following solution

$$E' \simeq \pm 0.871. \tag{12}$$

This can be used to deduce a limit on V by considering equation (8) but rewritten to adopt the notation in the note

$$\begin{aligned} E_\pm &= \frac{1}{2}\left(E_1 + E_2 \pm \sqrt{(E_1+E_2)^2 - 4E_1E_2 + 4V^2}\right) && (13) \\ &= \frac{1}{2}\left(E_1 + E_2 \pm \sqrt{\Delta^2 + 4V^2}\right), \quad \Delta = E_2 - E_1 && (14) \\ &= \frac{1}{2}(E_1 + E_2 \pm s). && (15) \end{aligned}$$

The two energies E_1 and E_2 are placed $s/2$ above and below the average energy $(E_1 + E_2)/2$, so s is the difference between them. This means the maximum V is given by $s/2 = 0.529$ MeV.