Lecture Notes

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Course overview

February: The connections to the underlying particle physics. Isospin, the mass formula, Fermi gas model and simple shell model.

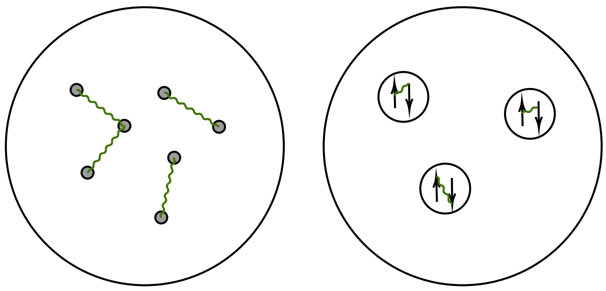
March: Nuclear structure models. Collective vibrations. Then elaborations on the shell-model, including an update on how nuclear shells may change. Also deformations and the theory of pairing.

April: Nuclear decays. Gamma decay, fission, alpha decay and beta decay. Also collective modes, in particular at rotations.

May: Nuclear astrophysics and nuclear reactions

1 Lecture 1: Free quarks and the strong nuclear force

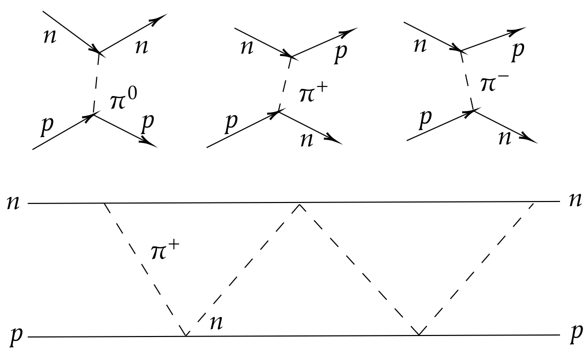
Nuclei consist of protons and neutrons, but why is this configuration energetically favored over free quarks. The nuclear forces that keep the nucleus together are induced through the exchange by mediating quanta – mesons. Considering two quarks with spin. The spin configurations can be aligned or anti-aligned where the energy is highest for the aligned state. This leads to an energetically favored state when the spins are anti-aligned. This can explain why the nuclei consists of protons and neutrons and not of free quarks since this would add more degrees of freedom. When the quarks are confined in protons and neutrons the gluons can only interact within the proton or neuton instead of all the free quarks in the nuclei.
At higher energies the bound system of protons and neutrons break down since the energy is now minimal for a free soup of quarks. This is the quark-gluon plasma.



Nucleons take part in all known interactions. The main feature of the nuclear world are sharped by the strong force – particles participating in strong interactions are called hadrons. Nucleons are the lightest fermions among the hadrons. The bosons among the hadrons are called mesons. The range of the strong force can be estimated by the uncertainty relation and the mass of π^0

$$\Delta R \sim c\Delta t \sim \frac{\hbar}{mc} = 1.46\,\text{fm} \tag{1}$$

Inside the nucleus one cannot distinguish between protons and neutrons since they interchange through the pion. This means one can only distintguish a proton from a neutron when the particle decays. This means one consideres nucleons because it is an interacting system. This motivates the isospin formalism since this shows the net number of protons or neutrons.



2 Lecture 2: Isospin

Since one cannot distinguish the proton and the neutron inside the nucleous we can consider the two particles as two different states of the same strongly interacting object. This leads to a two-level system: the nucleon.

$$\left|p\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

This leads to three different objects to look at, *nn*, *np*, *pp*. Also, the wavefunction consists of three parts: the radial term, the angular part and the spin part. The symmetry with respect to the exchange of spin varialbes is tested by the Bartlett operator. The triplet states are symmetric and the singlet states is anti-symmetric under spin exchange.

$$\mathcal{P}^{\sigma} = (1-)^{S+1} \tag{3}$$

The space inversion chances the sign of the relative coordinate

$$\mathcal{P}^{r} = (-1)^{l} \tag{4}$$

Both spin and orbital is given by the Heisenberg exchange operator

$$\mathcal{P} = (-1)^{l+S+1} \tag{5}$$

An example of symmetric and anti-symmetric

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S}_2, \quad \mathbf{S_1} = \frac{1}{2}, \quad \mathbf{S}_2 = \frac{1}{2} \tag{6}$$

To get the expectation value of the spin product: square both sides. This leads to

$$\langle \mathbf{S_1} \cdot \mathbf{S}_2 \rangle = \begin{cases} -\frac{3}{4}, & \text{singlet, anti-symmetric} \\ \frac{1}{4}, & \text{triplet, symmetric} \end{cases} \tag{7}$$

In gerneral the relative angular momentum, l, and the spin $\mathbf{S}$ is not conserved, but the rotational invariance makes the total angular momentum conserved. Another argument is that this operator commutes with the Hamiltonian.

$$\mathbf{J} = \mathbf{l} + \mathbf{S} \tag{8}$$

So the conserved quantum number is J^{π}. For a two-nucleon state:

$$^{2S+1}l_J \tag{9}$$

To introduce isospin one can consider Noether's theorem. This also means that an approximate symmetry yields some perfered quantites, i.e selection rules. The strong interaction cannot distinguish the proton from the neutron but the Coulomb interaction can. This leads to an approximate symmetry which leads to the isospin formalism.

Again considering the nucleon from equation (2) the isospin acts on the basis with a certain charge given by

$$Q = \frac{1}{2} - t_3 \tag{10}$$

Here t_i is analogous to the Pauli matrices for regular spin given. This also means there are three different operator so isospin is a vector with three components.

$$t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

This also leads to lowering and raising operators in the same way one cna lowe and raise components of angular momentum $J_\pm = J_x \pm iJ_y$. For isospin the relation is given by

$$\tau_\pm = \tau_1 \pm i\tau_2 \tag{12}$$

The strong force cannot distinguish the proton from the neuton so isospin is invariant. This means the totalt isospin is a conserved quantum number

$$T^2 = \sum_a \mathbf{t_a} \tag{13}$$

The eigenvalue of the isospin "length" is $\mathbf{T}^2 = T(T+1)$. One can expand the wavefunction by A factors for each nucleus and this is still an invariant

$$[\hat{\mathbf{T}}, \hat{H}_{\text{strong}}] \tag{14}$$

One can also consider the minimum and maximum of the isospin projection, T_3. This quantity is related to electrical charge

$$Q = \sum_a \left(\frac{1}{2} - t_{3a}\right) = \frac{A}{2} - T_3 \tag{15}$$

All states in a given nucleus (vertical scale) have the same projection

$$T_3 = \frac{1}{2}(N - Z) = \frac{A}{2} - Z \tag{16}$$

And belong to the horizontal scale. This is illustrated in figure 2.1 in the book. To conclude, the total wavefunction for the nucleus must also include Isospin

$$\Psi = RY_{lm}\chi\Omega. \tag{17}$$

3 Lecture 3: Selection rules and liquid drop model

Selection rules

Consider an arbitrary one-body operator, $\hat{O}$ which is a sum of single particle operators

$$Q = \sum_1^A q_a \tag{18}$$

This can be split into different terms if the operator can distinguish neutrons and protons

$$\begin{aligned} Q &= \sum_n q_n + \sum_p q_p \\ &= \sum_a \left(q_n \frac{1+\tau_{3a}}{2} + q_p \frac{1-\tau_{3a}}{2} \right) \\ &= \sum_a \frac{q_n + q_p}{2} + \sum_a \frac{q_n - q_p}{2} \tau_{3a} \end{aligned}$$

The first equality is on a neutron, proton level but using the isospin formalism we can move to a nucleon level since the strong force cannot distinguish protons from neutrons. This leads to the second equality where denominator is 1 if a neutron and 0 if a proton. The other way around for the next term. This leads to the third equality where the terms are called the isoscalar ($\tau = 0$) and the isovector $\tau = 1$ respectively. The physical interpretation of this is that for the isoscalar the neutrons and protons oscilllate in phase and out of phase for the isovector. This also leads to $\hat{\tau}_\pm$ being interpreted as a β-decay operator, look at equation (12).

Liquid drop model

The binding energy is traditionally given by

$$B(A,Z) = (ZM_p + NM_n)c^2 - E_{\text{tot}}(A,Z) \tag{19}$$

But in practice it is usefull to include the mass of the electrons

$$B(A,Z) = (ZM_p + NM_n)c^2 - E_{\text{tot}}(A,Z). \tag{20}$$

The dynamical features requires a reasonable chouse of dynamic variables. We consider a continous medium where excitations should resemble propagating waves. The lowest limit for the wavelength us of order r_0 (mean interparticle distance) which is close to the range of the nuclear force.

$$\frac{1}{k} > r_0 \simeq \frac{R}{A^{1/3}}. \tag{21}$$

This is rewriting by multiplaying $\hbar$

$$\frac{\hbar k R}{\hbar} < A^{1/3}, \tag{22}$$

and since $\hbar k$ is momentum, $\hbar k R$ is angular momentum. This motivates the use of spherical harmonics, $\alpha\lambda\mu Y_{\lambda\mu}(\hat{n})$. This means in a collective wave of deformation the radius R can be representated by a superposition of the spherical harmonics

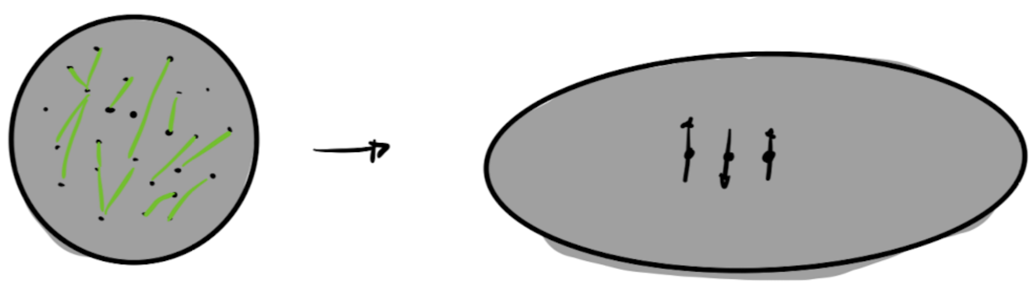
$$R(\mathbf{n}) = R_0\left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\mathbf{n})\right]. \tag{23}$$

4 Lecture 4: Fermi gas model

Introduce mean-field approximation and split the Hamiltonian into a Hamiltonian for each particle and a Hamiltonian for the interaction.

$$\hat{H} = \hat{H}_{\text{each particle}} + \hat{H}_{\text{interaction}} \longrightarrow \sum_{ij} v_{ij} + \sum_{ijk} v_{ijk} = \sum\left(\hat{H}_{\text{each particle}} + \hat{H}_{\text{interaction}}\right) - \sum \hat{H}_{\text{residual}} \tag{24}$$

Also introduce another quark model structure



In this section we considered a cubic box of size $L = V^{1/3}$. Each single-particle orbit in the Fermi gas model is characterized by the momentum $\mathbf{p}$ and spin-isospin quantum numbers – also normalized by the volume, V

$$\psi_{\mathbf{p}\sigma\tau} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r})} \chi_\sigma \Omega_\tau \tag{25}$$

So we assumed plane wave solution and isotropic in momentum-space.

$$\epsilon(p) = \frac{p^2}{2m^*} \tag{26}$$

This leads to the single-particle level density

$$\nu(\varepsilon) = \int \frac{\mathrm{d}^3p\mathrm{d}^3r}{(2\pi\hbar)^3}\delta(\epsilon - \epsilon(p)) = \frac{Vg}{2\pi^2\hbar^3}p^2\frac{\mathrm{d}p}{\mathrm{d}\epsilon}, \tag{27}$$

where g is the degeneracy factor and $(2\pi\hbar)^3$ is the volume the particle occupies. Now move from first quantization to second quantization and express the wavefunction in terms of the occupation number.

$$Z = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma -1/2}, \quad N = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma 1/2} \tag{28}$$

Introduce the Fermi energy diagram and express A in terms of p_F. This leads to the final expression

$$p_F = \hbar k_F = \hbar\left(6\pi^2\frac{n}{g}\right)^{1/3}, \tag{29}$$

where the fraction is the number density per degree of freedom. This means that all you need is the number density to get the Fermi momentum. Also that the Fermi gas is highy degenerate. From the Fermi energy you can get the seperation energy which leads to the potential, $\mathcal{U}_0 = 40-50$ MeV depending on m^*.

5 Lecture 5: Spherical mean field

We now examine the clustering of single-particle level called shell structure. This is illustrated in figure 5.1. The shell structure also implies dynamical properties such as excitations, capture etc. In nuclear physics there are some numbers for protons and neutrons that lead to especially stable nuclear systems – analogous to the noble gases in atomic physics. Those magic numbers for nuclei are placed along the valley of stabily and are given by

$$2, \quad 8, \quad 20, \quad 50. \quad 82, \quad 126. \tag{30}$$

Since the shape of the mean field is nearly the same for protons and neutrons the magic numbers turn out to be the same as well. The most stable nucleai are of course the double magic numbered nuclei. Keep in mind the magic numbers are not invariant since the potential changes when you add more protons or neutrons.

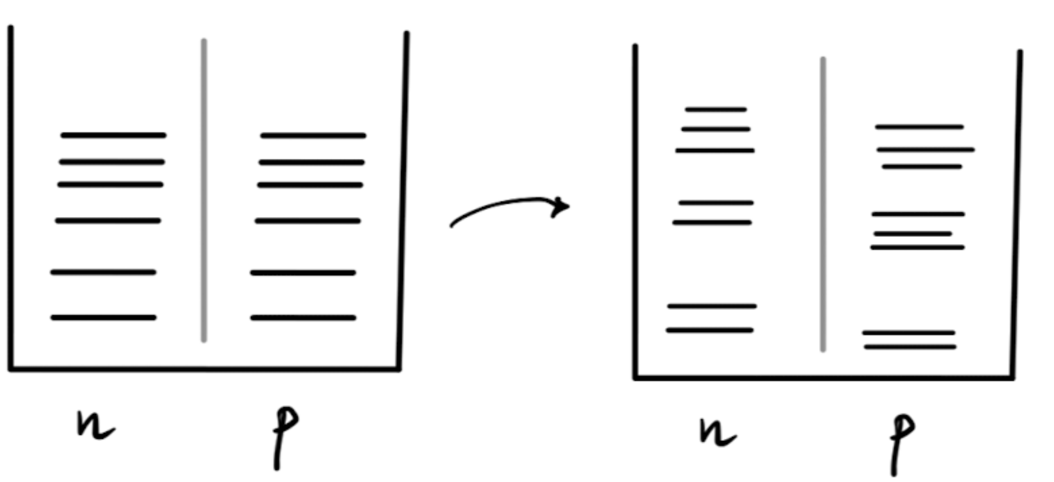


Figure 5.1: Energy gaps

Nuclei with one nucleon on top of a filled major shell have diminished nucleon seperation energy given by

$$S_n(A,Z) = B(A,Z) - B(A-1,Z)$$
$$S_p(A,Z) = B(A,Z) - B(A-1,Z-1)$$

Considering the shell structure the simplest approach to solve the model of mean field is the isopropic harmonic oscillator. This is easily done analytically but the drawbacks are in its unrealistic features that include excessive symmtry. This can be seen from the Hamiltonian

$$\hat{H}_{\mathrm{HO}} = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega_0^2\mathbf{r}^2, \quad \mathbf{r}^2 = (x+y+z)^2 \tag{31}$$

This means symmetric excitations in $N = n_x, n_y, n_z$. The energi levels ar

$$E_n = \hbar\omega_0\left(N + \frac{3}{2}\right) \tag{32}$$

And the parity of the three-dimensional states is given by

$$\Pi = (-1)^N \tag{33}$$

It is important that all states in a major shell have the same parity. Therefore, the linear combinations of the major shell orbits with proper rotational symmetry should have only even (odd) l for even even (odd) N.

A more realistic potential is the Woods-Saxon potential which only consideres the nearest interaction and spin-orbit coupling. This means that the potential is an improvement to the harmonic oscillator potential since it includes surface effects. Another arguement for the surface effects is that angular momentum is with respect to a point. This is seen on figure 8.4. Also consider how degeneracy leads to shell structures from an energy perspective. The distance from $s \to p$ decreases as the nucleus becomes larger because the wavelength becomes larger and the kinetic energy decreases. The energy as a function of major shell number and angular momentum is given by

$$\begin{aligned} E(n,l) &= E(n_0,l_0) + (n-n')\frac{\mathrm{d}E}{\mathrm{d}n} + (l-l_0)\frac{\mathrm{d}E}{\mathrm{d}l} \\ &\simeq b\frac{\mathrm{d}E}{\mathrm{d}n} + a\frac{\mathrm{d}E}{\mathrm{d}l}, \end{aligned}$$

where the second term is the energy to add for a radial excitation and the third term is the energy you must add for an angular excitation. Also, a, b are small. This leads to degeneracy which leads to shell stuctures.

6 Lecture 6: Two-body dynamics

Low-energy nuclear forces

In the low-energy domain we can almost always limit ourselves to considerations of nucleon degrees of freedom. This means we only need a Hamiltonian expressed in terms of the nucleon variables: the coordinates $\mathbf{r}$, momenta $\mathbf{p}$ and spins $\mathbf{s}$. To construct a spin dependent Hamiltonian we need a combinations of spins and does not change sign under spatial inversion or time reversal. Another argument is that $\mathbf{r} \times \mathbf{p}$ does not change under parity. This means we are left with

$$\sigma_{\mathbf{1}}\sigma_2(\sigma_{\mathbf{1}} \cdot \mathbf{n_1})(\sigma_2 \cdot \mathbf{n_2}), \tag{34}$$

where the first product is scalar and does not depend on the orientation of $\mathbf{n}$ and can therefore be ignored when we average over all angles. The rest are tensor forces. The angular average is

$$\overline{n_k n_l} = \frac{1}{4\pi} \int \mathrm{d}o \; n_k n_l = \frac{1}{3}\delta_{kl}. \tag{35}$$

This means the average part of the operator is $1/3(\sigma_{\mathbf{1}} \cdot \sigma_2)$ and belongs to spin-spin forces. We can now define a pure tensor operator

$$S_{12}(\mathbf{n}) = 3(\sigma_{\mathbf{1}} \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) - (\sigma_{\mathbf{1}} \cdot \sigma_2) = 2[3(\mathbf{S} \cdot \mathbf{n})^2 - \mathbf{S}^2] \tag{36}$$

And this operator has angular average of 0. This pure tensor operator is responsible for any noncentral forces. The general momentum-independant interaction of two spin-1/2 particles may only contain three types of forces each with their own radial depenence: central, spin and tensor. The Hamiltonian for the momentum-independent interaction is given by

$$H_s(\mathbf{r}, \sigma_{\mathbf{1}}\sigma_2) = U_c(r) + U_\sigma(r)(\sigma_{\mathbf{1}} \cdot \sigma_2) + U_t(r)S_{12} \tag{37}$$

And this Hamiltonian is symmetric under spin exchange

$$[\mathcal{P}^\sigma, H_s] = 0 \tag{38}$$

We also need to add an isospin term to this Hamiltonian. This leads to a new Hamiltonian which can be expressed in terms of four forces: a central Wigner, Majorana ($\propto \mathcal{P}^r$), Bartlett ($\propto \mathcal{P}^\sigma$)m and Heisenberg ($\propto \mathcal{P}^r\mathcal{P}^\sigma$)

$$H_s' = U_W(r) + U_M(r)\mathcal{P}^r + U_B(r)\mathcal{P}^\sigma + U_H(r)\mathcal{P}^r\mathcal{P}^\sigma \tag{39}$$

Meson exchange

Mesons act as mediators of strong forces. The free meson field $\phi^\alpha(\mathbf{r}, t)$, where α is the isospin (charge) characteristic $t_3 = -Q$ of the meson that satisfies the Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2\right)\phi^\alpha = 0, \quad \mu = \frac{mc}{\hbar} \tag{40}$$

We now consider the Coulomb potential mediated by the photon with mass equal to zero and solve the Poisson equation. We do this since we are doing a similar approach for the Klein-Gorden equation. The Poisson equation is

$$\nabla^2 \phi = -4\pi\delta(\mathbf{r}) \tag{41}$$

With the solution

$$\phi(\mathbf{r}) = \frac{e}{|\mathbf{r} - \mathbf{r_1}|} \tag{42}$$

For the Klein-Gorden equation we obtain

$$\left(\nabla^2 - \mu^2\right)\phi^\alpha(\mathbf{r}) = -\frac{g}{\mu}\tau^{alpha}(\sigma \cdot \nabla)\delta(\mathbf{r} - \mathbf{r_1}) \tag{43}$$

With solution

$$\phi^\alpha = \frac{g}{4\pi\mu}\tau^\alpha(\sigma \cdot \nabla)\frac{\mathrm{e}^{\mu|\mathbf{r}-\mathbf{r_1}|}}{|\mathbf{r} - \mathbf{r_1}|} \tag{44}$$

The final result is the pion-exchange potential that is symmetric with respect to nucleons

$$U_\pi = \frac{g^2}{4\pi\mu^2}(\tau_\mathbf{1} \cdot \tau_2)(\sigma_\mathbf{1} \cdot \nabla)(\sigma_2 \cdot \nabla)\frac{\mathrm{e}^{-\mu r}}{r} \tag{45}$$

Tensor forces and d-wave

Now also taking the tensor terms into account. The two tensor terms arises from the Majorana operator

$$H_s'' = [\mathcal{U}_{tW}(r) + \mathcal{U}_{tM}(r)\mathcal{P}^r]S_{12} \tag{46}$$

This yields the noncentral spin-dependent potential

$$\mathcal{U}(r) = \mathcal{U}_0 + \mathcal{U}_t(r)S_{12}, \quad \mathcal{U}_t(r) = \mathcal{U}_{tW}(r) + \mathcal{U}_{tM}(r) \tag{47}$$

The complete deuteron wavefunction now contains two radial parts. Factoring out the wave factor, $1/r$ yields

$$\Psi_M = \frac{1}{\sqrt{4\pi}}\frac{1}{r}\left(u_0(r) + \frac{1}{\sqrt{8}}u_2(r)S_{12}\right)\chi_{1M} \tag{48}$$

This equation was rewritten using the convient coupling represented by

$$\Theta_M = \frac{1}{\sqrt{32\pi}}S_{12}\chi_{1M} \tag{49}$$

Plugging this into the Schrödinger equation

$$u_0'' - [\kappa^2 + \mathcal{U}_0(r)]u_0 - \sqrt{8}\tilde{\mathcal{U}}_t(r)u_2 = 0 \tag{50}$$

$$u_2'' - [\kappa^2 + \frac{6}{r^2} + \tilde{\mathcal{U}}_0(r) - 2\tilde{\mathcal{U}}_t(r)]u_2 - \sqrt{8}\tilde{\mathcal{U}}_t u_0 = 0, \tag{51}$$

where the tilded potentials include the factor $2m/\hbar^2$. Now these are almost the same but what is the coupling terms? $S_{12}\frac{u_0}{r}Y_{02}\chi_{1M}$ is the s-wave but transforms into a d-wave contribution. This can be seen if one compares equation (48) to the last term in equation (50). Do the same consideration for the other terms.

$$r_{12}\frac{u_2}{r}\frac{1}{\sqrt{8}}S_{12}Y_{00}Y_{1M} = \frac{u_2}{r}\frac{8}{\sqrt{8}}Y_{00}\chi_{1M} - \frac{u_0}{r}\frac{2}{\sqrt{8}}S_{12}Y_{00}\chi_{1M} \tag{52}$$

Plug in potential and you get figure 3.5 in the book.

7 Lecture 7: Two-body scattering

Scattering theory

Consider elastic scattering in the center-of-mass frame, where $\mathbf{r}$ is the relative distance between the particles. We neglect interaction forces and consider free motion. This means the asympotic form of the wave function of the relative motion can be written as a combination of the incident plane wacve and the outgoing spherical wave

$$\psi(r) \simeq e^{i(\mathbf{k}\cdot\mathbf{r})} + f(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{r}, \tag{53}$$

where $k = k' = \sqrt{2mE/\hbar^2}$ is the wave vector, m is the reduced mass and $f(\mathbf{k}', \mathbf{k})$ is the scattering amplitude of dimension length. This is related to the differential cross section of scattering given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}o} = \left|f(\mathbf{k}', \mathbf{k}\right|^2. \tag{54}$$

Also, θ is the angle between $\mathbf{k}'$ and $\mathbf{k}$ For low k one can use partial wave expansion which means to new expressions for $e^{i\mathbf{k}\cdot\mathbf{r}}$ and $f(\theta)$

$$e^{i(\mathbf{k}\cdot\mathbf{r})} = e^{ikr\cos(\theta)} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell} j_{\ell}(kr)P_{\ell}(\cos(\theta)) \tag{55}$$

$$f(\theta) = \sum_{\ell}(2\ell+1)P_{\ell}(\cos(\theta))f_{\ell}, \quad f_{\ell} \in \mathbb{C} \tag{56}$$

Plugging this in

$$\psi(\mathbf{r}) \simeq \frac{i}{2kr}\sum_{\ell}(2\ell+1)P_{\ell}(\cos(\theta))[(-1)^{\ell}e^{-ikr} - (1+2ikf_{\ell})e^{ikr}] \tag{57}$$

Now, the outcomming wave is distorted – its amplitude is not equal to 1. This is the term $\propto e^{ikr}$

$$S_{\ell} = 1 + 2ikf_{\ell} = e^{2i\delta_{\ell}}, \tag{58}$$

where δ_{ℓ} is the phase shift.

Lecture 8: Giant Resonances

Giant resonance is a high-frequency collective excitation of atomic nuclei, as a property of many-body quantum systems. In the macroscopic interpretation of such an excitation in terms of an oscillation, the most prominent giant resonance is a collective oscillation of all protons against all neutrons in a nucleus.

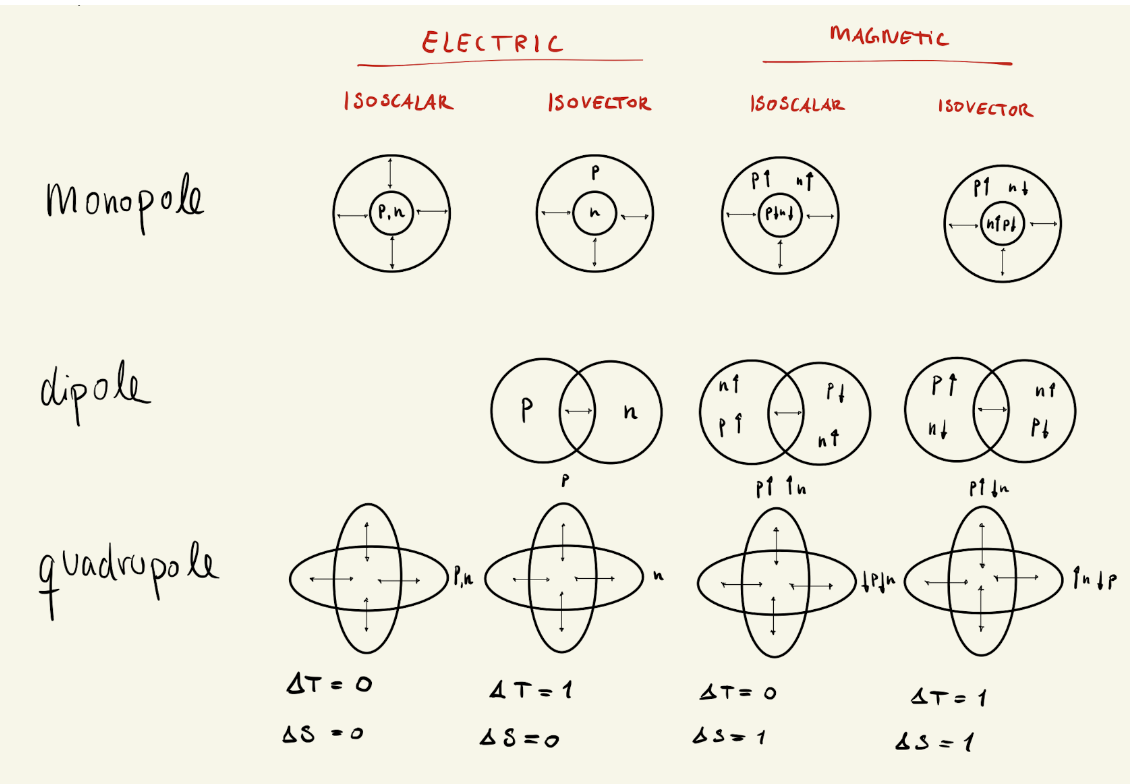


Figure 7.1:

The giant resonances are universal and only weakly correlate with the peculiarities of the individual nuclei. The most striking discrepancy between the liquid drop model and reality is revealed in the low-lying "surface" modes. which are very sensitive to the details of the ground state. This can be seen on figure 6.3 which shows these surface effects (vibrational states).

Lecture 9: Independant Particle Shell Model

Introducing second quantization. The single-particle orbits are given by

$$\lambda = n, \ell, j, m, \tau, \tag{59}$$

where τ is the isospin projection. The simplest many-body state allowed by Fermi-statistics is the Slater determinant given by

$$\Psi = \frac{1}{\sqrt{N}} \begin{vmatrix} \psi_{\lambda_1}(r_1) & \psi_{\lambda_2}(r_1) & \psi_{\lambda_3}(r_1) \\ \psi_{\lambda_1}(r_2) & \psi_{\lambda_2}(r_2) & \psi_{\lambda_3}(r_2) \end{vmatrix}, \tag{60}$$

which of course should be expanded to N-dimensions. For a 2 particles this is exactly the two-body wave function. In general, if you swap to columns you change a sign in the total wave function. The Slater determinant is defined by the set n_λ of occupation numbers $(= 0, 1)$ which fulfils

$$\sum_\lambda n_\lambda = A, \quad E(n_\lambda) = \sum_\lambda \epsilon_\lambda n_\lambda. \tag{61}$$

Section 11.1 introduces the second quantization notation/the occupation number representation. Note the difference between single-particle and many-body wavefunctions (round and angular brackets). The main result is equation (11.13) that gives the expression for a wavefunction of A fermions, and is the second quantization way of writing a Slater determinant.

$$|\Phi\rangle = \prod_{\lambda(n_\lambda=1)} a^\dagger_\lambda |0\rangle \tag{62}$$

It makes use of creation and annihilation operators defined at the bottom of page 204.

$$a_\lambda |0\rangle = 0, \quad \forall \lambda \tag{63}$$

They may seems frightingly deep and obscure if you have never encountered them before - they are not, as you will find out when you have gotten used to them, so think about them as smart ways of changing/probing occupation numbers. The important technical feature is the anticommutation relations in equation (11.8)

$$[a_\lambda, a_{\lambda'}]_+ = 0, \quad [a^\dagger_\lambda, a^\dagger_{\lambda'}]_+ = 0, \quad [a_\lambda, a^\dagger_{\lambda'}]_+ = \delta_{\lambda\lambda'} \tag{64}$$

Check that you understand them by verifying equation (11.23) (and why not also (11.24) that introduces pairwise contractions)! The basis transformations that are discussed around equation (11.5) may seem trivial or unimportant at first.

$$|\lambda) = |\nu) = \sum_\lambda |\lambda)(\lambda|\nu) \tag{65}$$

However, we need them to rewrite the expression for a general one-body operator in section 11.2, i.e. to go from equation (11.30) (make sure you are comfortable with that expression) to equation (11.33). It is not the most important derivation in the course, but at least the theoretically minded should try to follow it. The rest of section 11.2 is examples that are not very essential. Section 11.3 gives the corresponding general expression for a two-body operator in equation (11.39). We skip the derivation of that, but take a moment to ensure that you can see what elements are in (11.39).

$$F = \sum_{1234} (12|f|34) a^\dagger_1 a^\dagger_2 a_4 a_3 \tag{66}$$

Finally, section 11.4 employs the formalism to write down an expression for the two-body interactions in nuclei. The important results are equations (11.46) and (11.50), so check that you understand that they are equal (in (11.49) we only need the negative sign as we deal with fermions).

Lecture 10: Nuclear deformation

Nuclear deformation and collective model

The independent particle shell-model cannot be expected to work for nuclei in general and is seen experimentally to give a good description only for magic and near-magic nuclei in the ground state or in (some) low-lying excited states. (We have seen from the Island of Inversion that it may even fail for some nuclei with magic numbers such as $N = 20$.)

The two most important effects from the residual interaction are the pairing that we shall look at in chapter 13, and the deformation that is the topic of this chapter. We did not look into the detailed dynamical description of nuclear shapes in the second half of chapter 5 and will therefore only look at selected parts of chapter 12. There is a subtle point discussed on page 224 that is conceptually important, but in my experience may take some time to get used to: a deformed nuclear state can have $J = 0$ if the total wavefunction of the system has equal probability for all orientations of the deformed state. (So we think in terms of an "intrinsic frame" where the nucleus is deformed and the "exterior frame" - the one the atomic electrons see - where $J = 0$ and we average over all orientations. If you have difficulties in making sense out of the intrinsic frame, you can think of the deformed state as one that has strong spatial correlations between nucleons leading to the total density being non-spherical.) There are no static properties of a state with $J = 0$ that can tell us that it is deformed, so this information must be deduced from transition matrix elements to other states or from the general pattern of excited states – such as the rotation spectra we shall look at in chapter 16.

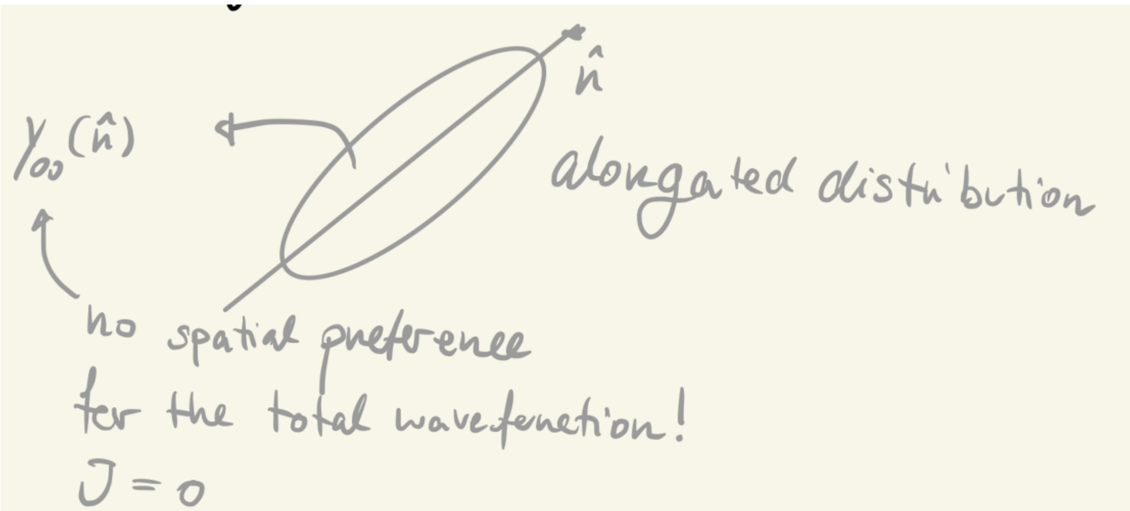


Figure 7.2:

The last paragraph of chapter 12.1 mentions the crucial observation that the system of a single particle outside a spherical core may gain energy if the core is slightly deformed (that would increase the spatial overlap of the single particle with the core which could gain energy as the nucleon-nucleon force is attractive) - when sufficiently many particles (more than one, but a handful or two is often enough) act together this will give permanent deformation. Chapter 12.2 gives a mathematical description of how such a deformation may be expressed. The mean field potential will now no longer be spherical and page 225 derives how one can rewrite. We will not need to use the details of this, but you should note that this (at least for small deformations) is mainly a surface effect - as illustrated in figure 12.1. Equation (12.7) could be the starting point of the classical (Bohr-Mottelson) treatment of deformation, it combines the shape degrees of freedom (from second half of chapter 5, we did not look at them) with a single-particle Hamiltonian and the derived coupling term. The following sections in the chapter explores this in more detail – we shall skip that except for three observations (without proof):

1. The expectation value of the coupling term is given in equation (12.20) for the most important case of a Y_{20} deformation term, note that there is a splitting in m, but that the $+m$ and $-m$ terms are degenerate.

$$\Delta\epsilon_{nljm} = \kappa_{nl}\sqrt{\frac{5}{64\pi}}\frac{3m^2 - j(j+1)}{j(j+1)}\alpha_{20}. \tag{67}$$

2. There are several ways of parametrising deformation, we shall use mainly β or δ, β is defined in equation (12.29)

$$\alpha_0 = \beta\cos\left(\gamma\right), \quad a_2 = a_{-2} = \frac{1}{\sqrt{2}}\beta\sin\left(\gamma\right) \tag{68}$$

(here we only consider Y_{2m} deformation) with equation (12.48) giving a more practical relation between the quadrupole moment Q and β (so β is more "experimental"). δ is almost the same, cf. equation (12.49), but is more convenient in some theoretical derivations.
3. Figure 12.3 shows how widespread deformation is among nuclei.

Single-particle quantum numbers and anisotropic harmonic oscillator

We consider in chapter 12.9 how (single) particles move in a deformed potential. We take from equation (12.20) that a spherical j-level will split up (expected, as the spherical symmetry now is broken and L no longer is a good quantum number). The coupling term, equation (12.59), is proportional to Y20 and is therefore able to mix states with different j but not to change parity - note the selection rules in the text.

$$H_{\text{coupl}} = -\kappa(r)\sqrt{\frac{5}{4\pi}}\beta P_2(\cos(\theta)), \quad \kappa(r) = r\frac{\mathrm{d}U}{\mathrm{d}r} \tag{69}$$

For the case of an axially symmetric (quadrupole) deformation the surviving quantum numbers are parity and m, the angular momentum projection on the symmetry axis. The book outlines how one would still expect simple rules for ground state spin-parity of even-even nuclei and odd-A nuclei, since rotation of the intrinsic frame will give states of higher energy.
Chapter 12.10 is technical and describes how the harmonic oscillator will change when deformed - it gives simply an anisotropic harmonic oscillator - and how the oscillator frequencies and energies will depend on the deformation, here expressed in terms of the delta parameter. The important outcome is figure 12.5 and the fact that there will be "new" magic numbers for certain deformations. For those who wish to follow the technical derivations: show and use that $r^2P_2 = (2z^2 - x^2 - y^2)/2$ to derive (12.62); to show (12.66) start reversely by inserting N, $\overline{\omega}$, δ_{osc} etc in it and show that it reduces to (12.65).
Chapter 12.11 continues the discussion of asymptotic quantum numbers from chapter 12.9, but adds spin-orbit and the fact that real nuclear potentials are flatter than the harmonic oscillator. It is a "discussion in principle" - try to get the gist of it, but we shall not employ these quantum numbers a lot later, so do not overspend time on it. The examples given in the rest of the chapter are more important. Chapter 12.12 outlines the (historically very) important Nilsson model - all the relevant single-particle neutron levels are shown in figure 12.6. You could compare it to figure 8.5 (for a fixed large A) and it is clear that deformation removes a lot of degeneracy of levels. Note how far down in energy some of the levels from the spherical high-j orbits in the upper shells can come at a deformation of 3:2. There are now less clear signatures for "deformed shells", but still a non-uniform level density. Chapter 12.13 discussed this and points to another prominent feature: that of avoided level crossing. Once you realise it is there you will find it is pervasive. We shall touch on avoided level crossing in extra problem 16, but to "tune your eye to it" try to look in figure 12.7 (that gives results starting from a Woods-Saxon potential) and compare the levels emerging from the spherical $g_{9/2}$: the $m = 9/2$ is essentially undisturbed, whereas the lower m values are much neater on the positive delta side than for negative delta where they tend to be bent downwards by similar-m-orbits from $g_{7/2}$ – or by even high-lying levels: all $m = 1/2$ levels in the figure (except for the very lowest one) are clearly perturbed by each other. Figure 12.8 adds octupole deformation and has even more mixing, to the point that wavefunctions at large deformations here are essentially chaotic.
Rather than getting lost in the quantum chaos (a topic the authors like a lot) you should remember that the spherical midpoint and the extreme large-deformation cases both have good (but different)

quantum numbers. The transition region these two limits is where the deformed nuclei sit – and according to figure 12.3 this is a majority of nuclei.

Lecture 11: Pairing

Pairing is a tough subject to get into. In chapter 13.1 we look at one pair of nucleons. Chapter 13.2 gives the technical description of pair wavefunctions, also with more than one pair. We skip chapter 13.3. Chapter 13.4 presents a simplified model of pairing, and the rest of the chapter discusses different aspects of the BCS model of pairing. We shall only look at some of the main features of the BCS model, so will browse through selected parts of chapter 13.5-9 and skip chapter 13.10-11 completely. There are two reasons for this: the first is that it would be a lot of technical material with rather little physics behind; the second is that BCS is actually an approximate theory of nuclear pairing – it naturally leads to some of the main features of pairing (and focusses attention on them), but from e.g. a shell-model point of view, pairing is "just" one part of the residual interactions so a proper shell-model calculation will give the correct answers. It actually does, but seeing simple models - like the one in chapter 13.4 - hopefully will give you a better intuition for pairing.
Chapter 13.1 starts by reminding that pairing is not just the pairwise filling of levels in a Fermi gas picture (and mentions that 0+ for even-even nuclei actually comes out of most models with randomly chosen interactions...). A crucial feature of pairing is the rather large pairing energy associated with it, this is a strong hint that it is a coherent effect. It is of course due to the attractive residual interaction between nucleons. The book makes the observation page 252 that the residual interactions can be expanded in L and gives the explicit formula for a delta-function interaction (it also points out that high L corresponds to small particle distance). More important for the following is the argument on the top of page 253 that two identical nucleons in the same j-shell will couple to $J = 0, 2, ...2j - 1$, i.e. all the odd values of J are excluded. The technical argument is given from the symmetry of Clebsch-Gordan coefficients and hinges partly on the parts of chapter 11 that we skipped (if you have plenty of time, you may look into it, otherwise just accept the result - note the reference to (12.83) on the top of page 253 should be to (11.86)). Let me remind that you have seen this general result already for the two-nucleon system, where nn and pp only existed for $J = S = 0$, not $J = 1$.

$$|JM\rangle = \frac{1}{\sqrt{2}} \sum_{mm'} C^{JM}_{jmjm'} a^\dagger_m a^\dagger_{m'} |0\rangle \tag{70}$$

We will use the result of problem 13.1 - but not do the calculations (perhaps as an exercise) - that using a delta-function interaction the J=0 state will be shifted down in energy with respect to the others, see figure 13.2 that also show a typical experimental example close to a double-magic nucleus. Note that this state appears, cf equation (13.10), by putting the two nucleons in orbits with +m and -m, i.e. orbits that maximise their spatial overlap. It is the "evenly distribution over all m-values" that makes the state coherent and gives the large energy gain. Pairing features are seen in many even-even nuclei, clearly close to magic numbers as illustrated in figure 13.4, but also - as shown in figure 13.3 - in nuclei that also have collective states (vibrational).