Week 6

1. Q — Problem 10 – Derivation of the potential energy contribution to the symmetry energy (alternative to the one in the book page 145).

A — The potential energy contribution to the symmetry energy is given by

$$U_{\rm sym}^{\rm pot} = \frac{1}{8}\frac{U_1}{A}(N-Z)^2 \tag{1}$$

Remembering the total Hamiltonian given by

$$H_s' = U_W(r) + U_M(r)\mathcal{P}^r + U_B(r)\mathcal{P}^\sigma + U_H(r)\mathcal{P}^r\mathcal{P}\sigma \tag{2}$$

We now add an isospin term, $U_i\mathcal{P}^\tau$, where $\mathcal{P}^\tau = \frac{1}{2}[1 + (\tau_1 \cdot \tau_2)]$. Remembering the relations for isospin

$$T_3 = \frac{1}{2}(N-Z) = \frac{A}{2} - Z \tag{3}$$

$$\frac{1}{2}|N-Z| \leq T \leq \frac{1}{2}(N+Z) \tag{4}$$

$$\tag{5}$$

This leads to

$$\begin{aligned}
U_i\mathcal{P}^\tau &= U_i(A)\frac{1}{2}\left(T^2 + T - \frac{1}{2}\right)\\
&= U_i(A)\frac{1}{2}\left(\frac{1}{4}(N+Z)^2 + \frac{1}{2}(N+Z) - \frac{1}{2}\right)\\
&= U_i(A)\frac{1}{4}\left(\frac{1}{2}(N+Z)^2 + (N+Z) - 1\right)\\
&= U_i(A)\left(\frac{1}{8}(N+Z)^2 + \frac{N}{4} + \frac{Z}{4} - \frac{1}{4}\right)\\
&= \frac{U_1}{A}\left(\frac{1}{8}(N+Z)^2 + \frac{N}{4} + \frac{Z}{4} - \frac{1}{4}\right)\\
&= \frac{1}{8}\frac{U_1}{A}(N+Z)^2
\end{aligned}$$

Alternatively you could evaluate the following sum

$$U_i\sum_{i\neq j}\tau_{\mathbf{i}} \cdot \tau_{\mathbf{j}}. \tag{6}$$

This is a sum over all pairs, so you would have to include a factor 1/2 to avoid overcounting. This leads to

$$
\begin{aligned}
\hat{H} &= \frac{1}{2}U_i \sum \langle TT_3|t_m t_n|TT_3\rangle \\
&\propto \frac{1}{2}U_i \Big(\langle TT_3|T^2|TT_3\rangle - \langle TT_3|t_m^2|TT_3\rangle \Big) \\
&= \frac{U_i}{8A}(N-Z)^2 + \frac{U_i}{4A}(N-Z) - \frac{3U_i}{8A} \\
&\approx \frac{U_i}{8A}(N-Z)^2
\end{aligned}
$$

where the second term is the Wigner term, which can be including in the semi-empirical mass formula.

2. Q — Problem 13 – A-dependence of $u_{\ell s}$

A — Starting from the Woods-Saxon potential

$$
\begin{aligned}
U &= U(R) + f(r)(\ell \cdot \mathbf{s}) \\
&= \frac{U_0}{1 + e^{\frac{r-R}{a}}} + f(r)(\ell \cdot \mathbf{s})
\end{aligned}
$$

Considering the limits

$$
\begin{aligned}
\lim_{a \to 0^+} &= \begin{cases} U_0, & r < R \\ \frac{U_0}{2}, & r = R \\ 0, & r > R \end{cases} \\
&= U_0\Big(\theta(r-R) - 1\Big),
\end{aligned}
$$

where θ is the Heaviside step function. This leads to

$$
\frac{\mathrm{d}U(r)}{\mathrm{d}r} = U_0 \delta(r-R), \tag{7}
$$

since the derivative of the Heaviside step function is the delta function. Now consider the expectation value of the distribution function in the Woods-Saxon potential.

$$
\begin{aligned}
f(r) &= \frac{-\lambda\hbar}{2M^2c^2}\frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r} \\
&= \frac{kU_0}{r}\delta(r-R)(\ell \cdot \mathbf{s})).
\end{aligned}
$$

The expectation value is then

$$
\Big\langle f(r) \Big\rangle = \int \mathrm{d}r\, f(r)^* f(r) \propto \left(\frac{kU_0}{R}\right)^2 \propto A^{-2/3}, \tag{8}
$$

Where I have used the translational property of the delta function[1] and $R = r_0 A^{1/3}$.

3. Q — Problem 12 – The $0s_{1/2}$ level in figure 8.5 increases rather than decreases in energy when the mass number grows beyond 150. Can you see why ?

[1] $\int_{-\infty}^{\infty} f(t)\delta(t-T)\,\mathrm{d}t = f(T)$

A — In the Woods-Saxon potential the scaling factor, U_0 changes sign depending on protons or neutrons. For stable nuclei (A>150) the excess of neutrons pushes the potential up. The same effect would happen for other energies much higher mass numbers are needed.

4. Q — Problem 18 – Consider a two level system where the Hamiltonian on the diagonal has the unperturbed energies E_1 and E_2 and the two off-diagonal terms V are equal. Find the eigenvalues of the system, i.e. the eigenenergies as a function of V and $\Delta E = E_2 - E_1$, and the mixed wavefunctions.

A — The Hamiltonian is given by

$$\hat{H} = \begin{bmatrix} E_1 & V \\ V & E_2 \end{bmatrix} \tag{9}$$

In words, the states ϕ_1, ϕ_2 interact through the off-diagonal elements. The mixing of these states result in ϕ_+ and ϕ_- as illustrated on figure 1.

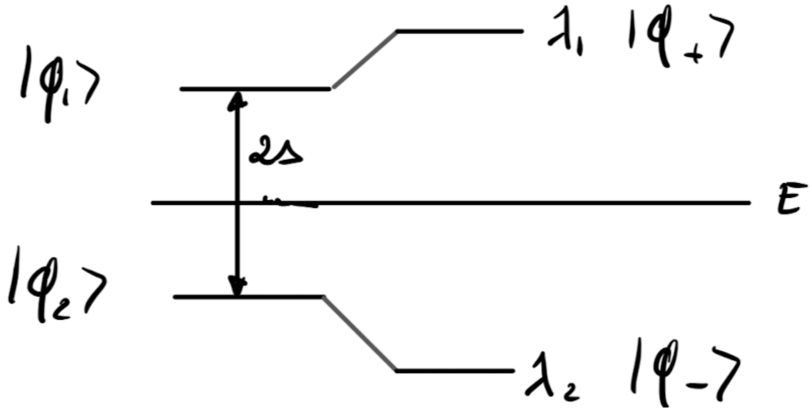

Figure 1: States

The figure also illustrates the two variables Δ and E given by

$$\Delta = \frac{E_1 - E_2}{2}, \quad E = \frac{E_1 + E_2}{2} \tag{10}$$

Now find the eigenvalues

$$\left|\hat{H} - \lambda \mathbb{1}\right| = \begin{vmatrix} E_1 - \lambda & V \\ V & E_2 - \lambda \end{vmatrix} = (E_1 - \lambda)(E_2 - \lambda) - V^2 = 0 \tag{11}$$

This leads to

$$\begin{aligned} \lambda_{1,2} &= \frac{E_1 + E_2}{2} \pm \sqrt{\frac{E_1 - E_2}{2} + V^2} \\ &= E \pm \kappa, \quad k = \sqrt{\Delta^2 + V^2} \end{aligned}$$

Now introduce a mixing angle as the ratio between the magnitudes of V and Δ

$$\frac{V}{\Delta} = \tan(2\theta) \tag{12}$$

In terms of these new variables the Hamiltonian becomes

$$\hat{H}' = \begin{bmatrix} E & \tan(2\theta) \\ \tan(2\theta) & E \end{bmatrix} \tag{13}$$

With the new eigenvalues

$$\lambda'_{1,2} = E \pm \Delta \sec(2\theta) \tag{14}$$

Now find a transformation, S defined by

$$\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \hat{S} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{15}$$

After some calculations one can show

$$\hat{S} = \begin{bmatrix} \cos(\theta) & 1 & \sin(\theta) & 1 \\ -\sin(\theta) & 1 & \cos(\theta) & 1 \end{bmatrix} \tag{16}$$

Considering two limits; one for weak coupling and one for strong coupling. The energies of the states is shown in figure 2.

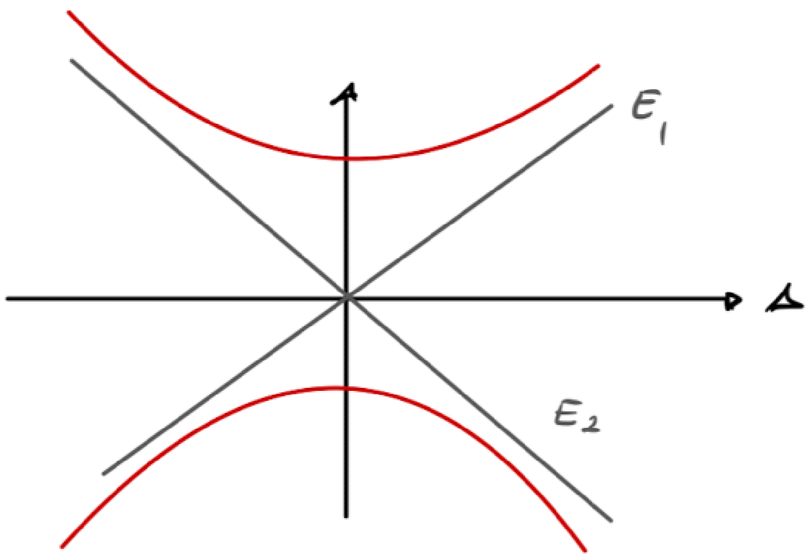

Figure 2: Avoided crossing

The energies (eigenvalues) of the Hamiltonian can never have the same value and hence the name avoided crossing. The same thing is seen on figure 12.4 in the book. When $V = 0$ the system is a normal two-level system increasing V makes the red lines approach the grey lines.