Week 13

1. Q — **Problem 29**

A — Since we are only considering allowed beta-decays know the following from figure 24.4

$$\text{allowed}: \begin{cases} \text{F+GT}; & \Delta J = 0 \\ \text{GT}; & \Delta J = 1, \end{cases} \tag{1}$$

and there is no change in the parity. We only know that the ground state in ^{16}O is even-even which means $J^\pi = 0^+$. In order for the α-decay to preserve the parity we require that $J^\pi = 2^+$ for some of the excited states in ^{20}Ne. Since the beta-decay is an allowed decay $\Delta J = 0$. It cannot be $\Delta J = 1$ since this would change the parity. Figure 1 shows the energy diagram.

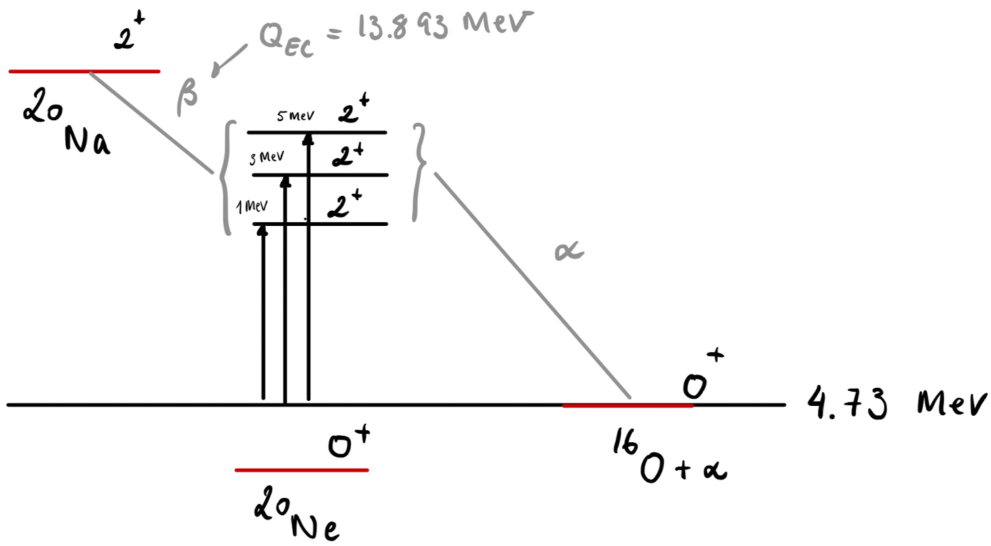

Figure 1: Energy diagram with J^π. Also includes three excited states with energies 1,3,5 MeV above the 4.73 MeV line.

To find the experimental threshold energy in ^{20}Ne for alpha emission I utilize that the Q-value for a decay is the energy (positive or negative) released in the decay. I use the definition of the mass excess which yields

$$\begin{aligned} Q_\alpha &= (M_P - M_D - m_\alpha)c^2 \\ &= -7041.931\,\text{keV} + 4737.001\,\text{keV} - 2.425\,\text{MeV} \\ &= -4.730\,\text{MeV} \end{aligned}$$

This value is also drawn on figure 1.

b) Imagine that three levels in ^{20}Ne are placed so that they all can participate in beta-delayed alpha decay with total centre-of-mass energies of 1, 3 and 5 MeV, respectively, in the α-^{16}O system. Estimate the relevant α penetrability and the beta-decay f-factor for each level.

The α penetrability can be expressed as

$$P_\alpha^{\ell=2} = \exp\big(-\Delta\gamma\big)P_0 \tag{2}$$

$$= \exp\left(-\ell(\ell+1)\sqrt{\frac{2}{\eta k R}}\right)\exp\left(-2\pi\eta - \sqrt{32\eta k R}\right), \quad k = \frac{\sqrt{2\mu E}}{\hbar}, \tag{3}$$

where the value of R is arbitrary. In my calculations I used $R = r_0 A^{1/3}$ and the mass number of ^{16}O. This yields

$$P_\alpha^{\ell=2} = \begin{cases} 4.097\cdot 10^{-4}, & E = 1\,\mathrm{MeV} \\ 1.100\cdot 10^{-2}, & E = 3\,\mathrm{MeV} \\ 0, & E = 5\,\mathrm{MeV} \end{cases} \tag{4}$$

To calculate the beta-decay f-factor you would have to account for the fact that the final state is an excited state and also include the correct relativistic expression.

$$\begin{aligned} E &= T + m_e c^2 \\ &= Q_\beta^* + m_e c^2 \\ &= Q_\beta - E^* \\ &= Q_{EC} - 2m_e c^2 - E^* + m_e c^2 \\ &= Q_{EC} - 2m_e c^2 - (B_E + E_{1,3,5}) + m_e c^2 \\ &= Q_{EC} - m_e c^2 - (B_E + E_{1,3,5}) \end{aligned}$$

2. Q — **Problem 31.** Derive the Breit-Wigner line shape in the following way: start from equation (4.14) and the observation below it that the cross sections peaks for a phase shift δ of $\pi/2$. Do a Taylor expansion around $\pi/2$ of the phase shift as a function of energy E, and denote the first order derivative as $2/\Gamma$. Show then that $\tan(\delta)$ can be written as $\Gamma/2/(E_r - E$ for energies close to the resonance energy E_r. Finally, plug this into (4.14).

A — Equation 4.14 is given by

$$\sigma_{\mathrm{el}} = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2(\delta_\ell) \tag{5}$$

The cross section peaks for a phase shift of δ of $\pi/2$ this means

$$\delta_\ell(E = E_r) = \frac{\pi}{2} \tag{6}$$

It turns out it is easier to the Taylor expansion of $\cot\big(\delta_\ell(E)\big)$ around $\pi/2$ since this converges faster.

$$
\begin{aligned}
\cot\big(\delta_\ell(E)\big) &\simeq \cot(E_r) - (E-E_r)\left.\frac{\partial\cot\big(\delta_\ell(E)\big)}{\partial E}\right|_{E=E_r} + (E-E_r)^2\left.\frac{\partial^2\cot\big(\delta_\ell(E)\big)}{\partial E^2}\right|_{E=E_r} + \ldots \\
&= \cot(E_r) - (E-E_r)\csc^2(E_r) + (E-E_r)^2\cot(E_r)\csc^2(E_r) \\
&= \cot(E_r) - (E-E_r)\frac{2}{\Gamma}(E-E_r)^2\frac{2}{\Gamma}\cot(E_r) \\
&= -(E-E_r)\frac{2}{\Gamma}
\end{aligned}
$$

Using the expansion for $\cot(x)$ for $x = a$

$$\cot(x) \simeq \cot(a) - (x-a)\csc^2(a) + (x-a)^2\cot(a)\csc(a) \tag{7}$$

This leads to

$$\cot\big(\delta_\ell(E)\big) = \frac{E_r - E}{\Gamma/2} \Rightarrow \tan\big(\delta_\ell(E)\big) = \frac{\Gamma/2}{E_r - E} \tag{8}$$

For E close to E_r. Plugging this into equation (5)

$$\sigma_{\text{el}} = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2(\delta_\ell) \tag{9}$$

$$= \sigma_{\text{el}} = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\left(\frac{\Gamma/2}{E_r - E}\right)^2 \tag{10}$$

Dividing by $\cos(\delta_\ell)^2 + \sin(\delta_\ell)^2 = 1$ makes it easier to rewrite the equation above.