Week 5

1. Q — Prove equation (3.10) via explicit calculation.

A — Starting from equation (3.2)

$$S_{12}(\mathbf{n}) = 3(\sigma_1 \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) - (\sigma_1 \cdot \sigma_2) \tag{1}$$

And use the following relation

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B}) + i\sigma \cdot (\mathbf{A} \times \mathbf{B}) \tag{2}$$

This leads to

$$\begin{aligned}(\sigma \cdot \mathbf{n})^2 &= (\mathbf{n} \cdot \mathbf{n}) + i\sigma \cdot \mathbf{r} \times \mathbf{r} \\ &= \mathbf{n}^2\end{aligned}$$

And this leads to

$$\begin{aligned}2(\sigma_1 \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) &= (\sigma_1 \cdot \mathbf{n} + \sigma_2 \cdot \mathbf{n})^2 - (\sigma_1 \cdot \mathbf{n})^2 - (\sigma_2 \cdot \mathbf{n})^2 \\ &= (\sigma_1 \cdot \mathbf{n} + \sigma_2 \cdot \mathbf{n})^2 - 2\mathbf{n}^2\end{aligned}$$

From equation (2.4): $\mathbf{S} = \frac{1}{2}(\sigma_1 + \sigma_2)$. And now

$$(\sigma_1 \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) = 2(\mathbf{S} \cdot \mathbf{n})^2 - \mathbf{n}^2 \tag{3}$$

This means the final expression for equation (3.2) is

$$\begin{aligned}S_{12}(\mathbf{n}) &= 3(\sigma_1 \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) - (\sigma_1 \cdot \sigma_2) \\ &= 3\left[2(\mathbf{S} \cdot \mathbf{n})^2 - \mathbf{n}^2\right] - 2\mathbf{S} + 3 \\ &= 2\left[3(\mathbf{S} \cdot \mathbf{n})^2 - \mathbf{S}\right]\end{aligned}$$

Where I used $\sigma_{\mathbf{1}} \cdot \sigma_2 = 2\mathbf{S} - 3$.