The deuteron and other s-wave nuclei

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This problem deals with the deuteron and other s-wave nuclei. To keep calculations simple we shall only consider square well potentials.

a) Solve the Schrödinger equation, plot the wavefunction and find a relation between the binding energy, U_0 and R.

Considering the central potential between the proton and the neutron given by

$$U(r) = \begin{cases} -U_0, & r \leq R \\ 0 & r > R \end{cases}$$

The radial equation is given by

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} + \left[U(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u(r) = Eu(r). \tag{1}$$

This is identical to the one-dimensional Schrodinger equation with an effective potential, where the centrifugal term pushes the particle outwards. To solve this analytically I rewrite the equation and consider the boundary conditions.

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} + \frac{M}{\hbar^2}\left[E - U(r)\right]u(r) = 0, \tag{2}$$

where I plugged in the expression for the reduced mass, $m = M/2$. For the deuteron I use $E = -E_B = -2.225$ MeV [p. 51]. This leads to the following expressions

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} + \frac{M}{\hbar^2}(U_0 - E_B)\,u(r) = 0, \quad r \leq R, \tag{3}$$

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} - \frac{M}{\hbar^2}E_B u(r) = 0, \quad r > R. \tag{4}$$

I introduce two variables given by

$$k = \sqrt{\frac{M}{\hbar^2}(U_0 - E_B)}, \quad \kappa = \sqrt{\frac{ME_B}{\hbar^2}}. \tag{5}$$

Rewriting equation (3) in terms of (5) and solving the differential equation yields

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} = -ku(r) \Rightarrow u(r) = A\sin(kr) + B\cos(kr). \tag{6}$$

Since $R(r) = u(r)/r$ and $\cos(kr)/r$ blows up as $r \to 0 \Rightarrow B = 0$ and the solution is

$$u(r) = A\sin(kr), \quad r \leq R \tag{7}$$

Now, considering equation (4)

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} = \kappa^2 u(r) \Rightarrow u(r) = Ce^{\kappa r} + De^{-\kappa r} \tag{8}$$

Here $Ce^{\kappa r}$ blows up as $r \to \infty$. The wavefunction must be continuous and this means the solutions (6) and (8) must match at $r = R$. The same applies for the derivative. This leads to two equations for $r = R$.

$$A\sin(kR) = De^{-\kappa R} \tag{9}$$

$$Ak\cos(kR) = -D\kappa e^{-\kappa R} \tag{10}$$

Dividing equation (10) by equation (9) leads to

$$-\cot(kR) = \frac{\kappa}{k} \tag{11}$$

This equation is solved by requiring $kR = \pi/2$. Plugging in an appropiate value for $R = 1.7$ fm yields

$$\begin{aligned} U_0 &= \frac{\hbar^2\pi^2}{2mR^2} - E \\ &= \frac{\hbar^2\pi^2}{2mR^2} + E_B \\ &= 37.2\,\mathrm{MeV} \end{aligned}$$

This means the depth of the potential is 37.2 MeV. The wavefunction is plotted in figure 1

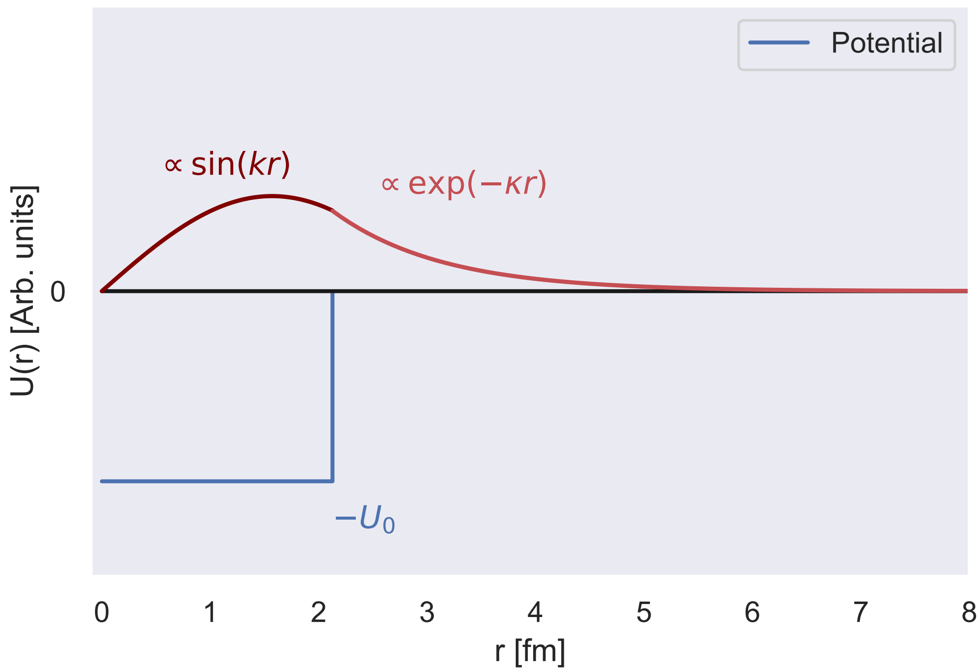

Figure 1: The wavefunction for the deuteron and the square well potential.

b) Extend the calculation to neutron s-wave particles in an arbitrary nucleus of mass number A.

I rewrite the solution to equation (11) using a potential depth of $U_0 = 40$ MeV and radius $R = A^{1/3} r_0$ which leads to the following expression

$$E = \frac{\pi^2 \hbar^2}{2m_n (A^{1/3} r_0)^2} - U_0, \tag{12}$$

where m_n is the mass of the neutron. This is illutrated on figure 2. The figure also includes the energy eigenvalues for a neutron in an infinite square well (dashed lines). I used the script `squareb.py` to calculates energy eigenvalues for a neutron in a square well.

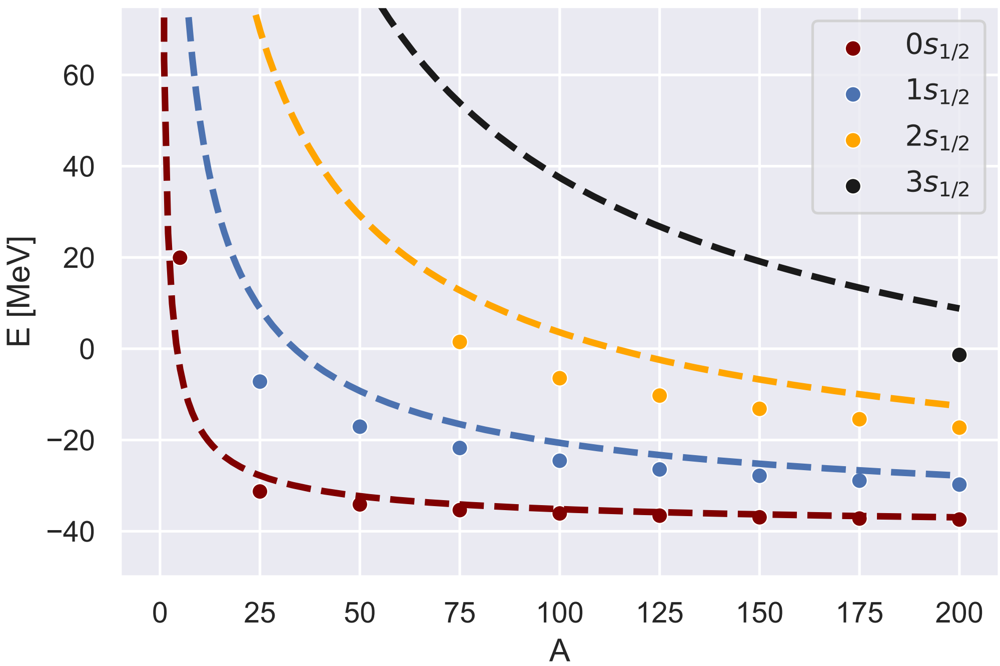

Figure 2: Energies of the neutron single-particle level given by the finite square well potential (blue) as a function of the mass number.

c) Compare to the infinite square well

In figure 2 I compared the energy levels of the infinite square well (dashed lines) and the finite square well (dots) as a function of the mass number. I included the four lowest energy levels to compare to figure 8.5 in the book (Woods-Saxon potential).
The figures are similar but especially the lowest energy state is different – this curve increases for $A > 150$ in figure 8.5. This behaviour is not present in figure 2. The energies of the neutron single-particle levels are generally lower for the finite square well compared to the infinite square well. This energy gap diminishes for increasing A – as the potential length increases the probability of quantum tunnelling decreases and the finite square well approaches the infinite square well.
Similar reasoning can be applied to the depth of the potential. If the potential depth, U_0, is much greater than the energy levels, the finite square well is very similar to the infinite square well.