The Jaynes-Cummings Model

Quantum Engineering II

4th of June 2021

Light-matter interaction

Consider a 2 level system in a single mode cavity

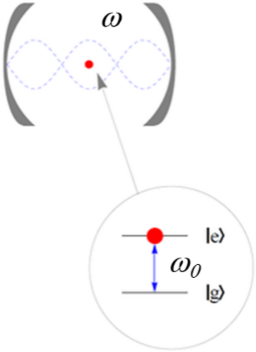
$$\hat{H} = \hat{H}_0 + \hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$$

Two level eigenstates

$$H_0|g\rangle = E_g|g\rangle, \quad H_0|e\rangle = E_e|e\rangle$$

Quantized electric field

$$\hat{\mathbf{E}} = \mathbf{e}\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)\sin(kz)$$



Jaynes-Cummings Hamiltonian Derivation

Fully quantum treatment of light-atom interaction,

$$\hat{H}_{(I)} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}} = g\hat{d}(\hat{a} + \hat{a}^\dagger)$$

where

$$\hat{\mathbf{E}} = \mathbf{e}\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)\sin(kz)$$

$$g = -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\sin(kz)$$

$$\hat{d} = \hat{\mathbf{d}} \cdot \mathbf{e}.$$

Jaynes-Cummings Hamiltonian Derivation (continued)

Introducing atomic transition operators

$$\hat{\sigma}_+ = |e\rangle\langle g|$$
$$\hat{\sigma}_- = |g\rangle\langle e|$$
$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|.$$

Then

$$\hat{d} = d(\hat{\sigma}_+ + \hat{\sigma}_-),$$

where $d = \langle e|\hat{d}|g\rangle = \langle g|\hat{d}|e\rangle$ is the matrix element, which we assume is real.

Jaynes-Cummings Hamiltonian Derivation (continued)

Interaction, atom, and cavity Hamiltonians

$$\hat{H}_{(I)} = \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger), \quad \lambda = dg/\hbar$$
$$\hat{H}_{(A)} = \frac{1}{2}\hbar\omega_0\sigma_z$$
$$\hat{H}_{(C)} = \hbar\omega\hat{a}^\dagger\hat{a}$$

Putting it all together yields

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger).$$

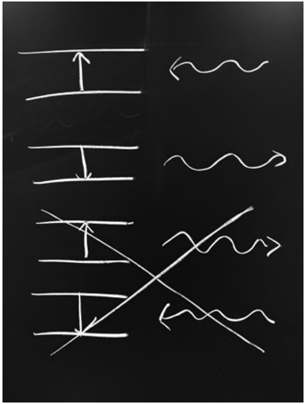
Jaynes-Cummings Hamiltonian Derivation (continued)

Time evolution of operators

$$\hat{\sigma}_+\hat{a} \sim e^{i(\omega_0-\omega)t}$$
$$\hat{\sigma}_-\hat{a}^\dagger \sim e^{-i(\omega_0-\omega)t}$$
$$\hat{\sigma}_+\hat{a}^\dagger \sim e^{i(\omega_0+\omega)t}$$
$$\hat{\sigma}_-\hat{a} \sim e^{-i(\omega_0+\omega)t}$$

Last terms rotate very quickly and does not conserve energy

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)$$



Rabi Oscillations

Consider $\Delta = 0$ and $|i\rangle = |e\rangle|n\rangle$ only couples to $|f\rangle = |g\rangle|n+1\rangle$. State vector

$$|\psi(t)\rangle = c_i(t)|i\rangle + c_f(t)|f\rangle$$

with energy

$$E_i = \frac{1}{2}\hbar\omega + n\hbar\omega = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega = E_f$$

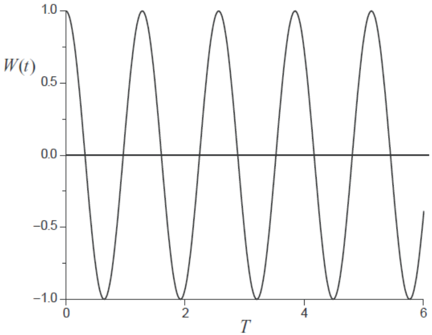
Solving the Schrödinger equation

$$|\psi(t)\rangle = \cos(\lambda t\sqrt{n+1})|i\rangle - i\sin(\lambda t\sqrt{n+1})|f\rangle$$

Rabi Oscillations

$$
\begin{aligned}
W(t) &= \langle \psi(t) | \hat{\sigma}_z | \psi(t) \rangle \\
&= \cos(2\lambda t \sqrt{n+1})
\end{aligned}
$$

For $n = 0$ no light there are still oscillations.



For a general pure state solution of light

$$|\psi\rangle_{\text{light}} = \sum_{n}^{\infty} C_n |n\rangle$$

we get the atomic inversion to

$$W(t) = \sum_{n=0}^{\infty} |C_n|^2 \cos(2\lambda t\sqrt{n+1})$$

For coherent state the atomic inversion becomes

$$W(t) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^2}{n!} \cos(2\lambda t\sqrt{n+1})$$

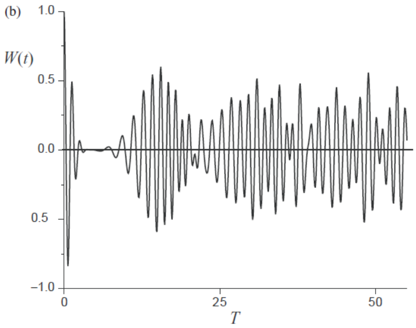
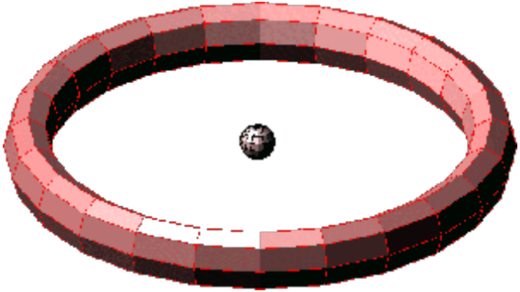
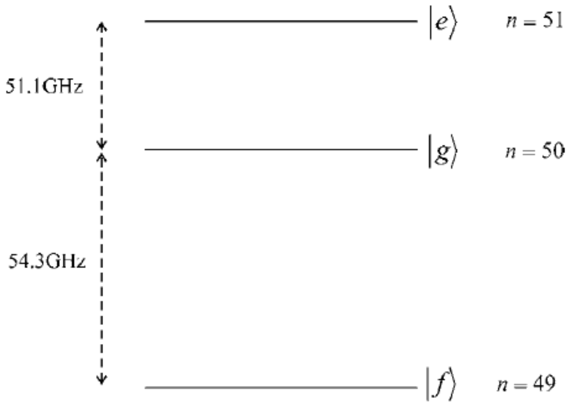


Figure: Coherent state, $\bar{n} = 5$

Rydbergs Atoms as a Two Level System



A Direct Test of Field Quantization in a Cavity



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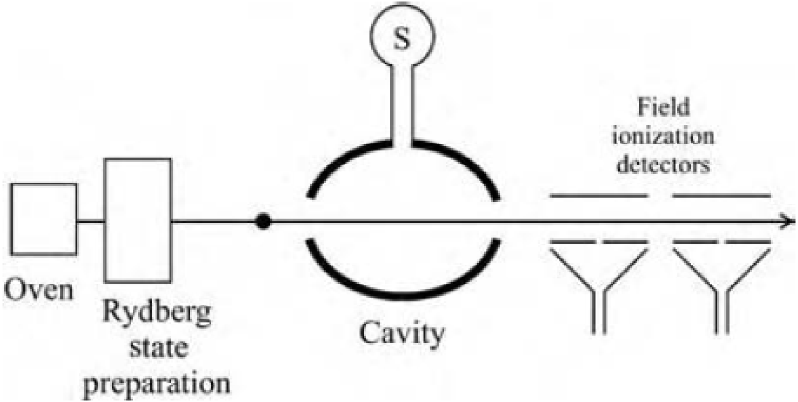

Results

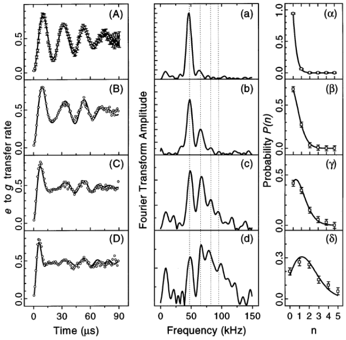

Figure: Frequency: ν, $\sqrt{2}\nu$, $\sqrt{3}\nu$