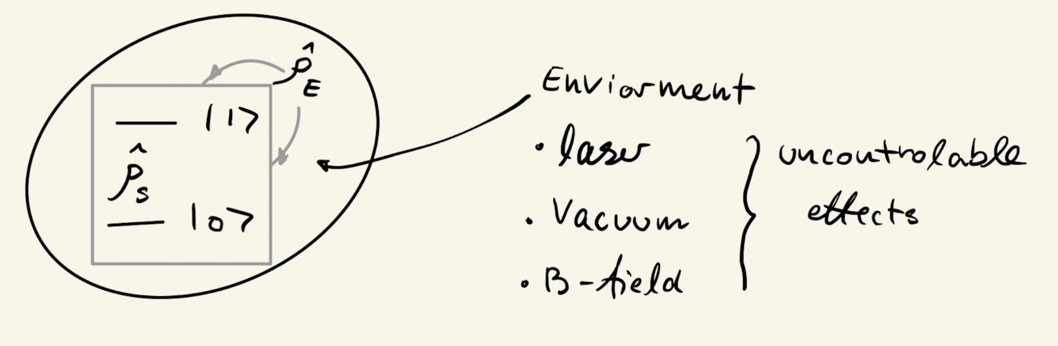
Errors in Quantum Computations and how to Correct Them

Quantum Engineering II

4th of June 2021

Typical sources of errors in quantum computations



Typical sources of errors in quantum computations

Classically: Clone Bits Majority Vote
Quantum Mechanically: No-cloning theorem

$$U(|\phi\rangle \otimes |\psi\rangle) = |\phi\rangle \otimes |\phi\rangle, \quad \forall\, |\phi\rangle$$

Because

$$U|0\rangle|0\rangle \to |0\rangle|0\rangle, \quad U|1\rangle|0\rangle \to |1\rangle|1\rangle$$

Superposition

$$U(|0\rangle + |1\rangle)|0\rangle \to |0\rangle|0\rangle + |1\rangle|1\rangle \neq (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

Evolution

Errors are described by unitary transformation on our total system

$$\mathcal{E}(\hat{\rho}) = \hat{U}\hat{\rho}\hat{U}^\dagger$$

Suppose we have a system $\hat{\rho}_S$ and and environment $\hat{\rho}_E$

$$\rho_S \to \mathcal{E}(\hat{\rho}_S) = \mathrm{Tr}_E[\hat{U}(\hat{\rho}_S \otimes \hat{\rho}_E)\hat{U}^\dagger]$$

Bit Flip Error

Bit flip $|1\rangle \rightarrow |0\rangle$ or $|0\rangle \rightarrow |1\rangle$ with probabilty p.

$$\mathcal{E}(\hat{\rho}_S) = (1-p)\hat{\rho}_S + p\hat{\sigma}_x\hat{\rho}_S\hat{\sigma}_x^\dagger$$

Changes the bloch sphere $\rho = \frac{1}{2}(\mathbb{I} + \mathbf{r} \cdot \boldsymbol{\sigma})$

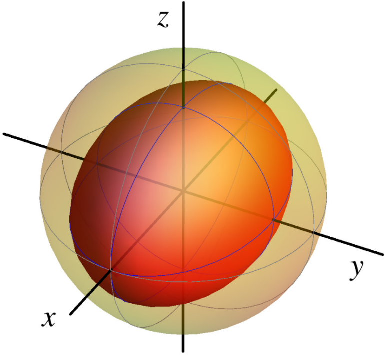


Figure: Contraction in y and z not x

Phase Flip Error

Bit flip $|0\rangle \rightarrow -|0\rangle$ or $|1\rangle \rightarrow -|1\rangle$ with probabilty p.

$$\mathcal{E}(\hat{\rho}_S) = (1-p)\hat{\rho}_S + p\hat{\sigma}_z\hat{\rho}_S\hat{\sigma}_z^\dagger$$

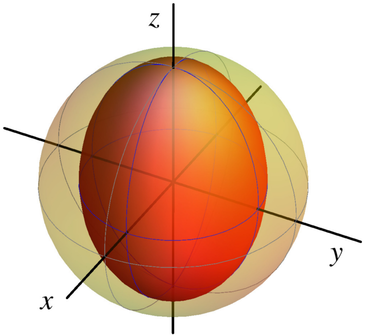


Figure: Contraction in x and y not z

Other Errors

Combined bit and phase flip $|0\rangle \rightarrow -|1\rangle$

$$\mathcal{E}(\hat{\rho}_S) = (1-p)\hat{\rho}_S + p\hat{\sigma}_y\hat{\rho}_S\hat{\sigma}_y^\dagger$$

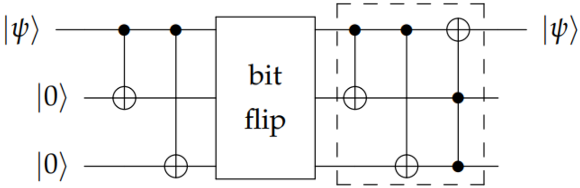
Depolarization Damping (maximally mixed states) $\rho \rightarrow \frac{\mathbb{I}}{2}$.

$$\mathcal{E}(\hat{\rho}_S) = (1-p)\hat{\rho}_S + p\frac{\mathbb{I}}{2}$$

Amplitude Damping (Thermal Equilibrium loss of energy)

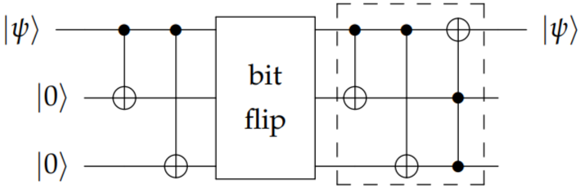
Bit Flip Error Correction

$$|\psi\rangle|0\rangle|0\rangle = \alpha|000\rangle + \beta|100\rangle$$



Bit Flip Error Correction

$$
\begin{aligned}
|\psi\rangle|0\rangle|0\rangle &= \alpha|000\rangle + \beta|100\rangle \\
&\rightarrow \alpha|000\rangle + \beta|110\rangle
\end{aligned}
$$



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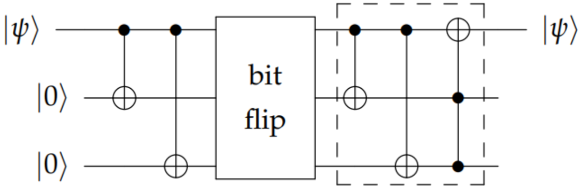

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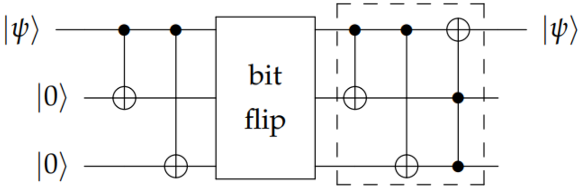

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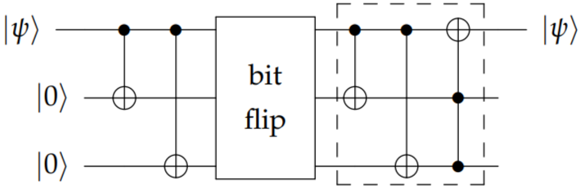
&\rightarrow \alpha|100\rangle + \beta|011\rangle \\
&\rightarrow \alpha|110\rangle + \beta|011\rangle
\end{aligned}
$$



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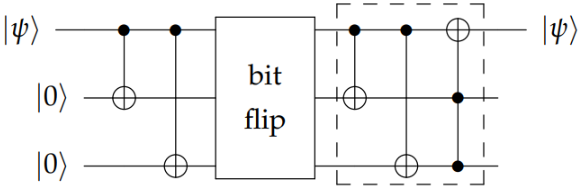
&\to \alpha|100\rangle + \beta|011\rangle \\
&\to \alpha|110\rangle + \beta|011\rangle \\
&\to \alpha|111\rangle + \beta|011\rangle
\end{aligned}$$



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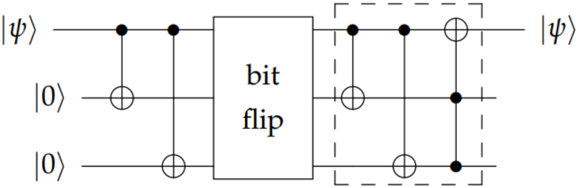

Bit Flip Error Correction

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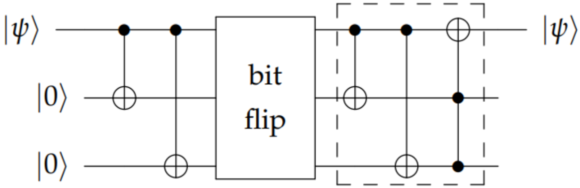
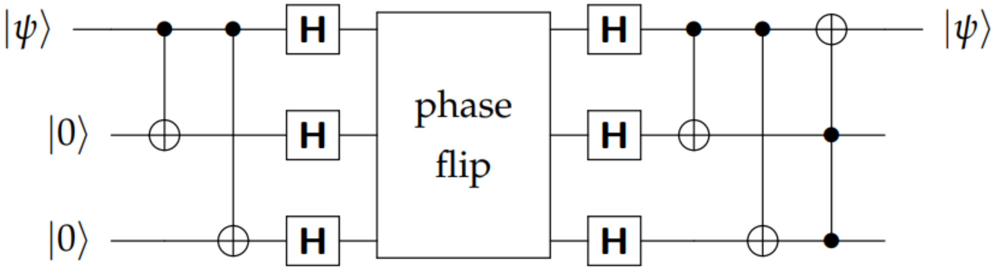
&\rightarrow \alpha|000\rangle + \beta|111\rangle \\
&\rightarrow \alpha|100\rangle + \beta|011\rangle \\
&\rightarrow \alpha|110\rangle + \beta|011\rangle \\
&\rightarrow \alpha|111\rangle + \beta|011\rangle \\
&\rightarrow \alpha|011\rangle + \beta|111\rangle \\
&= |\psi\rangle|1\rangle|1\rangle
\end{aligned}
$$



Table

Error Location	Final State
No Error	$\|\psi\rangle\|0\rangle\|0\rangle$
Bit 1	$\|\psi\rangle\|1\rangle\|1\rangle$
Bit 2	$\|\psi\rangle\|1\rangle\|0\rangle$
Bit 3	$\|\psi\rangle\|0\rangle\|1\rangle$

Phase Flip Error Correction



Pros and Cons

Pro

We can correct bit flip errors

Cons

1. We can only correct bit flip errors
2. More qubits are needed
3. We cannot control when the error occurs

Shor Code