Optical Bloch Equations

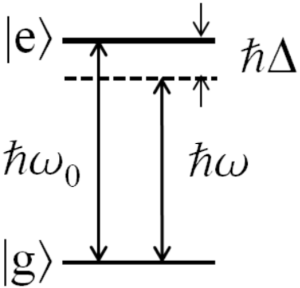
Quantum Engineering II

3rd of June 2021

Light-matter interaction (semi-classical)

Consider a 2 level system with classical light.
Using **dipole approximation** ($r \sim$ Å and $\lambda \sim$ 100-1000 nm)

$$\hat{H} = \hat{H}_0 + \hat{\mathbf{d}} \cdot \mathbf{E}_0 \cos(\omega t)$$



Figur: $\omega_0 = \frac{E_e - E_g}{\hbar}$, $\Delta = \omega_0 - \omega$

Two Level System

Two level eigenstates

$$H_0|g\rangle = E_g|g\rangle, \quad H_0|e\rangle = E_e|e\rangle$$

Coherent superposition of $|g\rangle$ and the excited level $|e\rangle$,

$$|\psi\rangle = c_g|g\rangle + c_e|e\rangle.$$

Inserting into Schrödinger equaton

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = [H_0 + \hat{\mathbf{d}}\cdot\mathbf{E}_0\cos(\omega t)]|\psi(t)\rangle$$

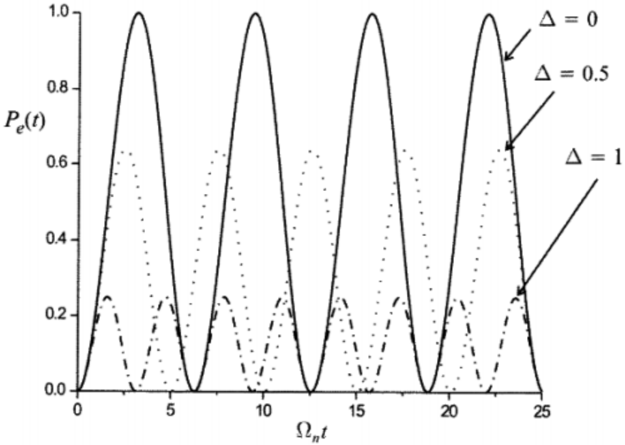
Differential Equation

$$\dot{c}_g = -i\Omega e^{-i\omega_0 t}\cos(\omega t) c_e$$
$$\dot{c}_e = -i\Omega e^{i\omega_0 t}\cos(\omega t) c_g$$

where $\Omega = \langle e|\hat{\mathbf{d}} \cdot E_0|g\rangle/\hbar$ Rotating wave approximation

$$\dot{c}_g = -\frac{i\Omega}{2} e^{-i\Delta t} c_e$$
$$\dot{c}_e = -\frac{i\Omega}{2} e^{i\Delta t} c_g$$

Rabi Oscillations



Denisty Matrix

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} |c_g|^2 & c_g c_e^* \\ c_g^* c_e & |c_e|^2 \end{bmatrix}.$$

Using Heisenbergs Equation

$$i\hbar \frac{d}{dt}\hat{\rho} = [H, \rho]$$

Optical Bloch Equations

$$\frac{d\rho_{11}}{dt} = i\frac{\Omega}{2}(\tilde{\rho}_{12} - \tilde{\rho}_{21})$$
$$\frac{d\rho_{22}}{dt} = i\frac{\Omega}{2}(\tilde{\rho}_{21} - \tilde{\rho}_{12})$$
$$\frac{d\tilde{\rho}_{12}}{dt} = i\Delta\tilde{\rho}_{12} + i\frac{\Omega}{2}(\rho_{11} - \rho_{22})$$
$$\frac{d\tilde{\rho}_{21}}{dt} = -i\Delta\tilde{\rho}_{21} + i\frac{\Omega}{2}(\rho_{22} - \rho_{11})$$

where $\tilde{\rho}_{12} = \rho_{12}e^{i\Delta t}$ and $\tilde{\rho}_{21} = \rho_{21}e^{-i\Delta t}$.

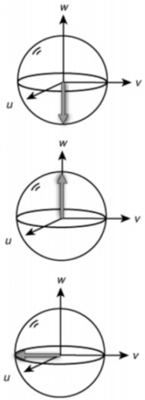
Bloch Vector

3D-vector representation

$$\mathbf{R} = u\hat{\mathbf{e}}_1 + v\hat{\mathbf{e}}_2 + w\hat{\mathbf{e}}_3,$$

with

$$u = \tilde{\rho}_{12} + \tilde{\rho}_{21}$$
$$v = i(\tilde{\rho}_{12} - \tilde{\rho}_{21})$$
$$w = \rho_{22} - \rho_{11},$$

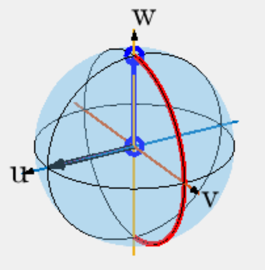


Evolution of the Bloch Vector

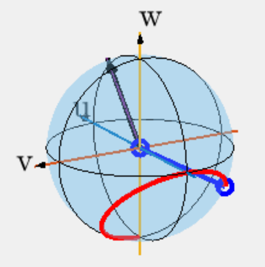
$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \Omega \\ 0 \\ \Delta \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\dot{\mathbf{R}} = \mathbf{W} \times \mathbf{R}$$

Bloch Sphere without Damping



Figur: $t_\pi = \frac{\pi}{\Omega}$, $\Delta = 0$, $\Gamma = 0$



Figur: $\Omega = \Delta$, $\Gamma = 0$

$$\rho_{22} = \frac{\Omega^2}{\Omega_R^2} \sin^2\left(\frac{\Omega_R t}{2}\right), \text{ where } \Omega_R = \sqrt{\Omega^2 + \Delta^2}^2$$

Evolution of the Bloch vector (with damping)

Adding Spontanoues Emission

$$\frac{d}{dt}\rho_{22} = -\Gamma\rho_{22} \tag{1}$$

Leads to decoherence

Optical Bloch equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \Omega \\ 0 \\ \Delta \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \Gamma\begin{pmatrix} u/2 \\ v/2 \\ w+1 \end{pmatrix} \tag{2}$$

The Bloch vector rotates and shrinks.

Saturation

In steady state ($t \to \infty$), $\frac{d}{dt}\rho_{ij} = 0$

$$\frac{d}{dt}\tilde{\rho}_{12} = -(\Gamma + i\Delta)\tilde{\rho}_{12} + i\frac{\Omega}{2}(\rho_{11} - \rho_{22}) = 0 \tag{3}$$

if we define

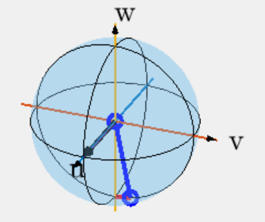
$$w = \rho_{22} - \rho_{11} = -\frac{1}{1+s} \tag{4}$$

we get the saturation parameter s

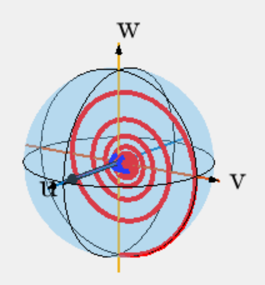
$$s = \frac{\Omega^2/2}{\Gamma^2/4 + \Delta^2} = \frac{s_0}{1 + 2\frac{\Delta^2}{\Gamma^2}} \quad \text{with} \quad s_0 = \frac{2\Omega^2}{\Gamma^2} \tag{5}$$

low saturation ($s \ll 1$): $w \simeq -1$, high saturation ($s \gg 1$): $w \simeq 0$

Bloch Sphere with Damping



Figur: $\Omega \ll \Gamma \neq 0$, (s$\ll$ 1)



Figur: $\Omega \gg \Gamma \neq 0$, (s$\gg$ 1)

Quantum computing

Hadamard transformation

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{6}$$