

1 Setup			3.1.7	Suc. shortest path	7	3.7.1	2-sat	16		
	1.0.1	Tips	2	3.1.8	Bipartite check	7	3.7.2	Finite field	17	
	1.0.2	Xmodmap setup	2	3.1.9	Bipartite matching (Hopcroft-Karp)	8	3.7.3	Complex field	17	
	1.0.3	header.h	2	3.1.10	Find cycle directed	8	3.7.4	FFT	17	
	1.0.4	Aux. helper C++	2	3.1.11	Find cycle undirected	8	3.7.5	Polyn. inv. div.	17	
	1.0.5	Aux. helper python	2	3.1.12	Tarjan's SCC	8	3.7.6	Linear recurs.	18	
2 Python			3.1.13	SCC edges	9	3.7.7	Convolution	18		
	2.1	Graphs	2	3.1.14	Topological sort	9	3.7.8	Partitions of n	18	
	2.1.1	BFS	2	3.1.15	Bellmann-Ford	9	3.7.9	Ternary search	18	
	2.1.2	Dijkstra	2	3.1.16	Ford-Fulkerson	10	3.8	Other Data Structures	19	
	2.1.3	Topological Sort	3	3.1.17	Dinic max flow	10		3.8.1	Disjoint set	19
2.1.4	Kruskal (UnionFind)	3	3.1.18	Edmonds-Karp	11	3.8.2		Fenwick tree	19	
2.1.5	Prim	3	3.2	Dynamic Programming	11	3.8.3		Trie	19	
2.2	Num. Th. / Comb.	3		3.2.1	Longest Incr. Subseq.	11		3.8.4	Treap	19
2.2.1	nCk % prime	3		3.2.2	0-1 Knapsack	11	3.8.5	Segment tree	20	
2.2.2	Sieve of E.	3		3.2.3	Coin change	11	3.8.6	Lazy segment tree	20	
2.2.3	Modular Inverse	4		3.2.4	Longest common subseq.	11	3.8.7	Dynamic segment tree	20	
	2.2.4	Chinese rem.	4	3.3	Numerical	12	3.8.8	Suffix tree	20	
	2.2.5	Bezout	4		3.3.1	Template (for this section)	12	3.8.9	UnionFind	21
	2.2.6	Gen. chinese rem.	4		3.3.2	Polynomial	12	3.8.10	Indexed set	21
2.3	Strings	4	3.3.3		Poly Roots	12	3.8.11	Order Statistics Tree	21	
	2.3.1	Longest common substr.	4		3.3.4	Golden Section Search	12	3.8.12	Range minimum queries	21
	2.3.2	Longest common subseq.	4	3.3.5	Hill Climbing	12	4 Other Mathematics		21	
	2.3.3	KMP	4	3.3.6	Integration	12		4.1	Helpful functions	21
	2.3.4	Suffix Array	4	3.3.7	Integration Adaptive	13		4.1.1	Euler's Totient Fuction	21
2.3.5	Longest common pref.	5	3.4	Num. Th. / Comb.	13	4.1.2		Totient (again but .py)	22	
2.3.6	Edit distance	5		3.4.1	Basic stuff	13		4.1.3	Pascal's trinagle	22
2.3.7	Bitstring	5		3.4.2	Mod. exponentiation	13	4.2	Theorems and definitions	22	
2.4	Geometry	5		3.4.3	GCD	13	4.3	Geometry Formulas	22	
	2.4.1	Convex Hull		5	3.4.4	Sieve of Eratosthenes	13	4.4	Recurrences	23
	2.4.2	Geometry	5	3.4.5	Fibonacci % prime	14	4.5	Sums	23	
2.5	Other Algorithms	5	3.4.6	nCk % prime	14	4.6	Series	23		
	2.5.1	Rotate matrix	5	3.5	Strings	14	4.7	Quadrilaterals	23	
2.6	Other Data Structures	6	3.5.1		Z alg.	14	4.8	Triangles	23	
	2.6.1	Trie	6		3.5.2	KMP	14	4.9	Trigonometry	23
3 C++			3.5.3		Aho-Corasick	14	4.10	Combinatorics	24	
	3.1	Graphs	6		3.5.4	Long. palin. subs	15	4.11	Cycles	24
	3.1.1	BFS	6	3.5.5	Bitstring	15	4.12	Labeled unrooted trees	24	
	3.1.2	DFS	6	3.6	Geometry	15	4.13	Partition function	24	
	3.1.3	Dijkstra	6		3.6.1	essentials.cpp	15	4.14	Numbers	24
3.1.4	Floyd-Warshall	6	3.6.2		Two segs. itersec.	15	4.15	Probability	24	
	3.1.5	Kruskal	7	3.6.3	Convex Hull	16	4.16	Number Theory	25	
	3.1.6	Hungarian algorithm	7	3.7	Other Algorithms	16	4.17	Discrete distributions	25	
							4.18	Continuous distributions	25	

1 Setup

1.0.1 Tips Test session: Check `__int128`, GNU builtins, and end of line whitespace requirements.

C++ var. limits: `int` -2^{31} , $2^{31} - 1$

`ll` -2^{63} , $2^{63} - 1$

`ull` 0 , $2^{64} - 1$

`__int128` -2^{127} , $2^{127} - 1$

`ld` $-1.7e308$, $1.7e308$, 18 digits precision

1.0.2 Xmodmap setup remove `Lock = Caps_Lock`

`keysym Escape = Caps_Lock`

`keysym Caps_Lock = Escape`

add `Lock = Caps_Lock`

1.0.3 header.h

```
1 #pragma once
2 #include <bits/stdc++.h>
3 using namespace std;
4
5 #define ll long long
6 #define ull unsigned ll
7 #define ld long double
8 #define pl pair<ll, ll>
9 #define pi pair<int, int>
10 #define vl vector<ll>
11 #define vi vector<int>
12 #define vb vector<bool>
13 #define vvi vector<vi>
14 #define vvl vector<vl>
15 #define vpl vector<pl>
16 #define vpi vector<pi>
17 #define vld vector<ld>
18 #define vvp vector<vp>
19 #define in(e1, cont) (cont.find(e1) != cont.end())
20 // sets/maps
21 #define all(x) x.begin(), x.end()
22
23 constexpr int INF = INT_MAX;
24 constexpr ll LLINF = LONG_LONG_MAX;
25
26 // int main() {
27 //   ios::sync_with_stdio(false); // do not use
28 //   cout << printf
29 //   cin.tie(NULL);
30 //   cout << fixed << setprecision(12);
31 //   return 0;
32 // }
```

1.0.4 Aux. helper C++

```
1 #include "header.h"
2 int main() {
3     // Read in a line including white space
4     string line;
5     getline(cin, line);
6     // When doing the above read numbers as
7     // follows:
8     int n;
9     getline(cin, line);
10    stringstream ss(line);
11    ss >> n;
12
13    // Count the number of 1s in binary
14    // representation of a number
15    ull number;
16    __builtin_popcountll(number);
17 }
18
19 // __int128
20 using lll = __int128;
21 ostream& operator<<(ostream& o, __int128 n) {
22     auto t = n<0 ? -n : n; char b[128], *d = end(b);
23     ;
24     do *--d = '0'+t%10, t /= 10; while (t);
25     if(n<0) *--d = '-';
26     o.rdbuf()->sputn(d, end(b)-d);
27     return o;
28 }
```

1.0.5 Aux. helper python

```
1 from functools import lru_cache
2
3 # Read until EOF
4 while True:
5     try:
6         pattern = input()
7     except EOFError:
8         break
9
10 @lru_cache(maxsize=None)
11 def smth_memoi(i, j, s):
12     # Example in-built cache
13     return "sol"
14
15 # Fast I
16 import io, os
17 def fast_io():
18     finput = io.BytesIO(os.read(0,
19                             os.fstat(0).st_size)).readline
20     s = finput().decode()
21     return s
22
```

```
23 # Fast O
24 import sys
25 def fast_out():
26     n = 5
27     sys.stdout.write(str(n)+"\n")
```

2 Python

2.1 Graphs

2.1.1 BFS

```
1 from collections import deque
2 def bfs(g, roots, n):
3     q = deque(roots)
4     explored = set()
5     distances = [0 if v in roots else float('inf')
6                  for v in range(n)]
7     while len(q) != 0:
8         node = q.popleft()
9         if node in explored: continue
10        explored.add(node)
11        for neigh in g[node]:
12            if neigh not in explored:
13                q.append(neigh)
14                distances[neigh] = float('inf')
15            distances[neigh] = distances[
16                node] + 1
17    return distances
```

2.1.2 Dijkstra

```
1 from heapq import *
2 def dijkstra(n, root, g): # g = {node: (cost,
3                             neigh)}
4     dist = [float("inf")] * n
5     dist[root] = 0
6     prev = [-1] * n
7
8     pq = [(0, root)]
9     heapify(pq)
10    visited = set([])
11
12    while len(pq) != 0:
13        _, node = heappop(pq)
14        if node in visited: continue
15        visited.add(node)
16
17    # In case of disconnected graphs
```

```

18 if node not in g:
19     continue
20
21 for cost, neigh in g[node]:
22     alt = dist[node] + cost
23     if alt < dist[neigh]:
24         dist[neigh] = alt
25         prev[neigh] = node
26         heappush(pq, (alt, neigh))
27 return dist

```

2.1.3 Topological Sort topological sorting of a DAG

```

1 from collections import defaultdict
2 class Graph:
3     def __init__(self, vertices):
4         self.graph = defaultdict(list) #adjacency
5             List
6         self.V = vertices #No. V
7
8     def addEdge(self, u, v):
9         self.graph[u].append(v)
10
11     def topologicalSortUtil(self, v, visited, stack):
12         :
13         visited[v] = True
14         # Recur for all the vertices adjacent to
15         # this vertex
16         for i in self.graph[v]:
17             if visited[i] == False:
18                 self.topologicalSortUtil(i,
19                     visited, stack)
20         stack.insert(0, v)
21
22     def topologicalSort(self):
23         visited = [False]*self.V
24         stack = []
25         for i in range(self.V):
26             if visited[i] == False:
27                 self.topologicalSortUtil(i,
28                     visited, stack)
29         return stack
30
31     def isCyclicUtil(self, v, visited, recStack):
32         visited[v] = True
33         recStack[v] = True
34         for neighbour in self.graph[v]:
35             if visited[neighbour] == False:
36                 if self.isCyclicUtil(neighbour,
37                     visited, recStack) == True:
38                     return True
39             elif recStack[neighbour] == True:
40                 return True
41         recStack[v] = False

```

```

36 return False
37
38 def isCyclic(self):
39     visited = [False] * (self.V + 1)
40     recStack = [False] * (self.V + 1)
41     for node in range(self.V):
42         if visited[node] == False:
43             if self.isCyclicUtil(node,
44                 visited, recStack) == True:
45                 return True
46     return False

```

2.1.4 Kruskal (UnionFind) Min. span. tree

```

1 class UnionFind:
2     def __init__(self, n):
3         self.parent = [-1]*n
4
5     def find(self, x):
6         if self.parent[x] < 0:
7             return x
8         self.parent[x] = self.find(self.parent[x])
9         return self.parent[x]
10
11     def connect(self, a, b):
12         ra = self.find(a)
13         rb = self.find(b)
14         if ra == rb:
15             return False
16         if self.parent[ra] > self.parent[rb]:
17             self.parent[rb] += self.parent[ra]
18             self.parent[ra] = rb
19         else:
20             self.parent[ra] += self.parent[rb]
21             self.parent[rb] = ra
22         return True
23
24 # Full MST is len(spanning)==n-1
25 def kruskal(n, edges):
26     uf = UnionFind(n)
27     spanning = []
28     # Sort edges by asc. weight (check+-)
29     edges.sort(key = lambda d: -d[2])
30     while edges and len(spanning) < n-1:
31         u, v, w = edges.pop()
32         if not uf.connect(u, v):
33             continue
34         spanning.append((u, v, w))
35     return spanning

```

2.1.5 Prim Min. span. tree - good for dense graphs

```

1 from heapq import heappush, heappop, heapify
2 def prim(G, n):
3     s = next(iter(G.keys()))
4     V = set([s])
5     M = []
6     c = 0
7
8     E = [(w,s,v) for v,w in G[s].items()]
9     heapify(E)
10
11     while E and len(M) < n-1:
12         w,u,v = heappop(E)
13         if v in V: continue
14         V.add(v)
15         M.append((u,v))
16         c += w
17         u = v
18         [heappush(E, (w,u,v)) for v,w in G[u].items()
19             if v not in V]
20
21     if len(M) == n-1:
22         return M, c
23     else:
24         return None, None

```

2.2 Num. Th. / Comb.

2.2.1 nCk % prime p must be prime and k < p

```

1 def fermat_binom(n, k, p):
2     if k > n:
3         return 0
4     num = 1
5     for i in range(n-k+1, n+1):
6         num *= i % p
7     num %= p
8     denom = 1
9     for i in range(1, k+1):
10        denom *= i % p
11    denom %= p
12    # numerator * denominator^(p-2) (mod p)
13    return (num * pow(denom, p-2, p)) % p

```

2.2.2 Sieve of E. $O(n)$ so actually faster than C++ version, but more memory

```

1 MAX_SIZE = 10**8+1
2 isprime = [True] * MAX_SIZE
3 prime = []
4 SPF = [None] * (MAX_SIZE)
5 def manipulated_seive(N): # Up to N (not
6     included)

```

```

6 isprime[0] = isprime[1] = False
7 for i in range(2, N):
8     if isprime[i] == True:
9         prime.append(i)
10        SPF[i] = i
11        j = 0
12        while (j < len(prime) and
13              i * prime[j] < N and
14              prime[j] <= SPF[i]):
15            isprime[i * prime[j]] = False
16            SPF[i * prime[j]] = prime[j]
17            j += 1

```

2.2.3 Modular Inverse of a mod b

```

1 def modinv(a, b):
2     if b == 1: return 1
3     b0, x0, x1 = b, 0, 1
4     while a > 1:
5         q, a, b = a//b, b, a%b
6         x0, x1 = x1 - q * x0, x0
7     if x1 < 0: x1 += b0
8     return x1

```

2.2.4 Chinese rem. an x such that $\forall y, m: yx = 1 \pmod m$ requires all m, m' to be ≥ 1 and coprime

```

1 def chinese_remainder(ys, ms):
2     N, x = 1, 0
3     for m in ms: N *= m
4     for y, m in zip(ys, ms):
5         n = N // m
6         x += n * y * modinv(n, m)
7     return x % N

```

2.2.5 Bezout

```

1 def bezout_id(a, b):
2     r, x, s, y, t, z = b, a, 0, 1, 1, 0
3     while r:
4         q = x // r
5         x, r = r, x % r
6         y, s = s, y - q * s
7         z, t = t, z - q * t
8     return y % (b // x), z % (-a // x)

```

2.2.6 Gen. chinese rem.

```

1 def general_chinese_remainder(a, b, m, n):
2     g = gcd(m, n)
3
4     if a == b and m == n:
5         return a, m
6     if (a % g) != (b % g):
7         return None, None
8
9     u, v = bezout_id(m, n)
10    x = (a*v*n + b*u*m) // g
11    return int(x) % lcm(m, n), int(lcm(m, n))

```

2.3 Strings

2.3.1 Longest common substr. (Consecutive)

$O(mn)$ time, $O(m)$ space

```

1 from functools import lru_cache
2 @lru_cache
3 def lcs(s1, s2):
4     if len(s1) == 0 or len(s2) == 0:
5         return 0
6     return max(
7         lcs(s1[:-1], s2), lcs(s1, s2[:-1]),
8         (s1[-1] == s2[-1]) + lcs(s1[:-1], s2[:-1])
9     )

```

2.3.2 Longest common subseq. (Non-consecutive)

```

1 def longestCommonSubsequence(text1, text2):
2     n = len(text1)
3     m = len(text2)
4     prev = [0] * (m + 1)
5     cur = [0] * (m + 1)
6     for idx1 in range(1, n + 1):
7         for idx2 in range(1, m + 1):
8             # matching
9             if text1[idx1 - 1] == text2[idx2 - 1]:
10                cur[idx2] = 1 + prev[idx2 - 1]
11            else:
12                # not matching
13                cur[idx2] = max(cur[idx2 - 1], prev[idx2])
14        prev = cur.copy()
15    return cur[m]

```

2.3.3 KMP Return all matching pos. of P in T

```

1 class KMP:
2     def partial(self, pattern):
3         """ Calc. partial match table: String -> [Int] """
4         ret = [0]
5         for i in range(1, len(pattern)):
6             j = ret[i - 1]
7             while j > 0 and pattern[j] != pattern[i]: j = ret[j - 1]
8             ret.append(j + 1 if pattern[j] == pattern[i] else j)
9         return ret
10
11    def search(self, T, P):
12        """KMPString -> String -> [Int] """
13        partial, ret, j = self.partial(P), [], 0
14        for i in range(len(T)):
15            while j > 0 and T[i] != P[j]: j = partial[j - 1]
16            if T[i] == P[j]: j += 1
17            if j == len(P):
18                ret.append(i - (j - 1))
19                j = partial[j - 1]
20        return ret

```

2.3.4 Suffix Array

```

1 class Entry:
2     def __init__(self, pos, nr):
3         self.p = pos
4         self.nr = nr
5
6     def __lt__(self, other):
7         return self.nr < other.nr
8
9 class SA:
10    def __init__(self, s):
11        self.P = []
12        self.n = len(s)
13        self.build(s)
14
15    def build(self, s): # n log log n
16        n = self.n
17        L = [Entry(0, 0) for _ in range(n)]
18        self.P = []
19        self.P.append([ord(c) for c in s])
20        step = 1
21        count = 1
22
23        # self.P[step][i] stores the position
24        # of the i-th longest suffix
25        # if suffixes are sorted according to
26        # their first 2^step characters.
27        while count < 2 * n:
28            self.P.append([0] * n)

```

```

28     for i in range(n):
29         nr = (self.P[step - 1][i],
30              self.P[step - 1][i +
31                  count]
32              if i + count < n else -1)
33         L[i].p = i
34         L[i].nr = nr
35         L.sort()
36         for i in range(n):
37             if i > 0 and L[i].nr == L[i -
38                 1].nr:
39                 self.P[step][L[i].p] = \
40                     self.P[step][L[i - 1].p]
41             else:
42                 self.P[step][L[i].p] = i
43                 step += 1
44                 count *= 2
45
46     self.sa = [0] * n
47     for i in range(n):
48         self.sa[self.P[-1][i]] = i

```

2.3.5 Longest common pref. with the suffix array built we can do, e.g., longest common prefix of x , y with suffixarray where x, y are suffixes of the string used $O(\log n)$

```

1 def lcp(x, y, P):
2     res = 0
3     if x == y:
4         return n - x
5     for k in range(len(P) - 1, -1, -1):
6         if x >= n or y >= n:
7             break
8         if P[k][x] == P[k][y]:
9             x += 1 << k
10            y += 1 << k
11            res += 1 << k
12    return res

```

2.3.6 Edit distance

```

1 def editDistance(str1, str2):
2     m = len(str1)
3     n = len(str2)
4     curr = [0] * (n + 1)
5     for j in range(n + 1):
6         curr[j] = j
7     previous = 0
8     # dp rows
9     for i in range(1, m + 1):
10        previous = curr[0]

```

```

11        curr[0] = i
12
13        # dp cols
14        for j in range(1, n + 1):
15            temp = curr[j]
16            if str1[i - 1] == str2[j - 1]:
17                curr[j] = previous
18            else:
19                curr[j] = 1 + min(previous, curr[j - 1],
20                                curr[j])
21            previous = temp
22    return curr[n]

```

2.3.7 Bitstring Slower than a set for many elements, but hashable

```

1 def add_element(bit_string, index):
2     return bit_string | (1 << index)
3 def remove_element(bit_string, index):
4     return bit_string & ~(1 << index)
5 def contains_element(bit_string, index):
6     return (bit_string & (1 << index)) != 0

```

2.4 Geometry

2.4.1 Convex Hull

```

1 def vec(a,b):
2     return (b[0]-a[0], b[1]-a[1])
3 def det(a,b):
4     return a[0]*b[1] - b[0]*a[1]
5 def convexhull(P):
6     if (len(P) == 1):
7         return [(p[0][0], p[0][1])]
8
9     h = sorted(P)
10    lower = []
11    i = 0
12    while i < len(h):
13        if len(lower) > 1:
14            a = vec(lower[-2], lower[-1])
15            b = vec(lower[-1], h[i])
16            if det(a,b) <= 0 and len(lower) > 1:
17                lower.pop()
18                continue
19            lower.append(h[i])
20            i += 1
21
22    upper = []
23    i = 0
24    while i < len(h):
25        if len(upper) > 1:

```

```

26            a = vec(upper[-2], upper[-1])
27            b = vec(upper[-1], h[i])
28            if det(a,b) >= 0:
29                upper.pop()
30                continue
31            upper.append(h[i])
32            i += 1
33
34    reversedupper = list(reversed(upper[1:-1]))
35    reversedupper.extend(lower)
36    return reversedupper

```

2.4.2 Geometry

```

1
2 def vec(a,b):
3     return (b[0]-a[0], b[1]-a[1])
4
5 def det(a,b):
6     return a[0]*b[1] - b[0]*a[1]
7
8     lower = []
9     i = 0
10    while i < len(h):
11        if len(lower) > 1:
12            a = vec(lower[-2], lower[-1])
13            b = vec(lower[-1], h[i])
14            if det(a,b) <= 0 and len(lower) > 1:
15                lower.pop()
16                continue
17            lower.append(h[i])
18            i += 1
19
20    # find upper hull
21    # det <= 0 -> replace
22    upper = []
23    i = 0
24    while i < len(h):
25        if len(upper) > 1:
26            a = vec(upper[-2], upper[-1])
27            b = vec(upper[-1], h[i])
28            if det(a,b) >= 0:
29                upper.pop()
30                continue
31            upper.append(h[i])
32            i += 1

```

2.5 Other Algorithms

2.5.1 Rotate matrix

```

1 def rotate_matrix(m):

```

```

2 return [[m[j]][i] for j in range(len(m))] for
   i in range(len(m[0])-1,-1,-1)]

```

2.6 Other Data Structures

2.6.1 Trie

```

1 class TrieNode:
2     def __init__(self):
3         self.children = [None]*26
4         self.isEndOfWord = False
5
6 class Trie:
7     def __init__(self):
8         self.root = self.getNode()
9     def getNode(self):
10        return TrieNode()
11    def _charToIndex(self, ch):
12        return ord(ch)-ord('a')
13    def insert(self, key):
14        pCrawl = self.root
15        length = len(key)
16        for level in range(length):
17            index = self._charToIndex(key[level])
18            if not pCrawl.children[index]:
19                pCrawl.children[index] = self.
20                    getNode()
21            pCrawl = pCrawl.children[index]
22            pCrawl.isEndOfWord = True
23    def search(self, key):
24        pCrawl = self.root
25        length = len(key)
26        for level in range(length):
27            index = self._charToIndex(key[level])
28            if not pCrawl.children[index]:
29                return False
30            pCrawl = pCrawl.children[index]
31        return pCrawl.isEndOfWord

```

3 C++

3.1 Graphs

3.1.1 BFS

```

1 #include "header.h"
2 #define graph unordered_map<ll, unordered_set<ll
   >>
3 vi bfs(int n, graph& g, vi& roots) {
4     vi parents(n+1, -1); // nodes are 1..n
5     unordered_set<int> visited;

```

```

6     queue<int> q;
7     for (auto x: roots) {
8         q.emplace(x);
9         visited.insert(x);
10    }
11    while (not q.empty()) {
12        int node = q.front();
13        q.pop();
14
15        for (auto neigh: g[node]) {
16            if (not in(neigh, visited)) {
17                parents[neigh] = node;
18                q.emplace(neigh);
19                visited.insert(neigh);
20            }
21        }
22    }
23    return parents;
24 }
25 vi reconstruct_path(vi parents, int start, int
   goal) {
26     vi path;
27     int curr = goal;
28     while (curr != start) {
29         path.push_back(curr);
30         if (parents[curr] == -1) return vi(); //
           No path, empty vi
31         curr = parents[curr];
32     }
33     path.push_back(start);
34     reverse(path.begin(), path.end());
35     return path;
36 }

```

3.1.2 DFS Cycle detection / removal

```

1 #include "header.h"
2 void removeCyc(ll node, unordered_map<ll, vector<
   pair<ll, ll>>>& neighs, vector<bool>& visited
   ,
3 vector<bool>& recStack, vector<ll>& ans) {
4     if (!visited[node]) {
5         visited[node] = true;
6         recStack[node] = true;
7         auto it = neighs.find(node);
8         if (it != neighs.end()) {
9             for (auto util: it->second) {
10                ll nnode = util.first;
11                if (recStack[nnode]) {
12                    ans.push_back(util.second);
13                } else if (!visited[nnode]) {
14                    removeCyc(nnode, neighs,
15                        visited, recStack, ans);
16                }
17            }
18        }
19    }
20 }

```

```

16     }
17     }
18 }
19 recStack[node] = false;
20 }

```

3.1.3 Dijkstra

```

1 #include "header.h"
2 vector<int> dijkstra(int n, int root, map<int,
   vector<pair<int, int>>>& g) {
3     unordered_set<int> visited;
4     vector<int> dist(n, INF);
5     priority_queue<pair<int, int>> pq;
6     dist[root] = 0;
7     pq.push({0, root});
8     while (!pq.empty()) {
9         int node = pq.top().second;
10        int d = -pq.top().first;
11        pq.pop();
12
13        if (in(node, visited)) continue;
14        visited.insert(node);
15
16        for (auto e : g[node]) {
17            int neigh = e.first;
18            int cost = e.second;
19            if (dist[neigh] > dist[node] + cost)
20                {
21                    dist[neigh] = dist[node] + cost;
22                    pq.push({-dist[neigh], neigh});
23                }
24        }
25        return dist;
26 }

```

3.1.4 Floyd-Warshall

```

1 #include "header.h"
2 // g[i][j] = infity if not path from i to j
3 // if g[i][i] < 0, i is contained in a negative
   cycle
4 void warshall(vvl& g) {
5     for (int k=0; k<g.size(); ++k) {
6         for (int i=0; i<g.size(); ++i) {
7             for (int j=0; j<g.size(); ++j) {
8                 if (g[i][k] < LLONG_MAX and g[k][
5                 j] < LLONG_MAX and g[i][j] >
5                 g[i][k] + g[k][j]) {
5                     g[i][j] = g[i][k] + g[k][j];
6                 }
7             }
8         }
9     }
10 }

```

3.1.5 Kruskal Minimum spanning tree of undirected weighted graph. $O(E \log E)$

```
1 #include "header.h"
2 #include "disjoint_set.h"
3 pair<set<pair<ll, ll>>, ll> kruskal(vector<tuple
   <ll, ll, ll>>& edges, ll n) {
4     set<pair<ll, ll>> ans;
5     ll cost = 0;
6
7     sort(edges.begin(), edges.end());
8     DisjointSet<ll> fs(n);
9
10    ll dist, i, j;
11    for (auto edge: edges) {
12        dist = get<0>(edge);
13        i = get<1>(edge);
14        j = get<2>(edge);
15
16        if (fs.find_set(i) != fs.find_set(j)) {
17            fs.union_sets(i, j);
18            ans.insert({i, j});
19            cost += dist;
20        }
21    }
22    return pair<set<pair<ll, ll>>, ll> {ans, cost
23 }
```

3.1.6 Hungarian algorithm Given J jobs and W workers ($J \leq W$), computes the minimum cost to assign each prefix of jobs to distinct workers.

```
1 #include "header.h"
2 template <class T> bool ckmin(T &a, const T &b) {
3     return b < a ? a = b, 1 : 0; }
4
5 /**
6  * @tparam T: type large enough to represent
7  * integers of  $O(J * \max(|C|))$ 
8  * @param C: JxW matrix such that  $C[j][w] = \text{cost}$ 
9  * to assign j-th
10 * job to w-th worker (possibly negative)
11 * @return a vector (length J), with the j-th
12 * entry = min. cost
13 * to assign the first (j+1) jobs to distinct
14 * workers
15 */
16 template <class T> vector<T> hungarian(const
   vector<vector<T>> &C) {
17     const int J = (int)size(C), W = (int)size(C
18 [0]);
19     assert(J <= W);
20     // a W-th worker added for convenience
21     vector<int> job(W + 1, -1);
```

```
15     vector<T> ys(J), yt(W + 1); // potentials
16     vector<T> answers;
17     const T inf = numeric_limits<T>::max();
18     for (int j_cur = 0; j_cur < J; ++j_cur) {
19         int w_cur = W;
20         job[w_cur] = j_cur;
21         vector<T> min_to(W + 1, inf);
22         vector<int> prv(W + 1, -1);
23         vector<bool> in_Z(W + 1);
24         while (job[w_cur] != -1) { // runs at
25             most j_cur + 1 times
26             in_Z[w_cur] = true;
27             const int j = job[w_cur];
28             T delta = inf;
29             int w_next;
30             for (int w = 0; w < W; ++w) {
31                 if (!in_Z[w]) {
32                     if (ckmin(min_to[w], C[j][w]
33                             - ys[j] - yt[w]))
34                         prv[w] = w_cur;
35                     if (ckmin(delta, min_to[w]))
36                         w_next = w;
37                 }
38             }
39             for (int w = 0; w <= W; ++w) {
40                 if (in_Z[w]) ys[job[w]] += delta,
41                     yt[w] -= delta;
42                 else min_to[w] -= delta;
43             }
44             w_cur = w_next;
45         }
46     }
47     for (int w; w_cur != W; w_cur = w) job[
48         w_cur] = job[w = prv[w_cur]];
49     answers.push_back(-yt[W]);
50 }
```

3.1.7 Suc. shortest path Calculates max flow, min cost

```
1 #include "header.h"
2 // map<node, map<node, pair<cost, capacity>>>
3 #define graph unordered_map<int, unordered_map<
   int, pair<ld, int>>>
4 graph g;
5 const ld inf = 1e60; // Change if necessary
6 ld fill(int n, vld& potential) { // Finds max
7     flow, min cost
8     priority_queue<pair<ld, int>> pq;
9     vector<bool> visited(n+2, false);
10    vi parent(n+2, 0);
11    vld dist(n+2, inf);
12    dist[0] = 0.1;
```

```
13    pq.emplace(make_pair(0.1, 0));
14    while (not pq.empty()) {
15        int node = pq.top().second;
16        pq.pop();
17        if (visited[node]) continue;
18        visited[node] = true;
19        for (auto& x : g[node]) {
20            int neigh = x.first;
21            int capacity = x.second.second;
22            ld cost = x.second.first;
23            if (capacity and not visited[neigh]) {
24                ld d = dist[node] + cost + potential[node]
25                    - potential[neigh];
26                if (d + 1e-10 < dist[neigh]) {
27                    dist[neigh] = d;
28                    pq.emplace(make_pair(-d, neigh));
29                    parent[neigh] = node;
30                }
31            }
32        }
33        for (int i = 0; i < n+2; i++) {
34            potential[i] = min(inf, potential[i] + dist
35                [i]);
36        }
37        if (not parent[n+1]) return inf;
38        ld ans = 0.1;
39        for (int x = n+1; x; x = parent[x]) {
40            ans += g[parent[x]][x].first;
41            g[parent[x]][x].second--;
42            g[x][parent[x]].second++;
43        }
44        return ans;
45    }
46 }
```

3.1.8 Bipartite check

```
1 #include "header.h"
2 int main() {
3     int n;
4     vvi adj(n);
5
6     vi side(n, -1); // will have 0's for one
7     side 1's for other side
8     bool is_bipartite = true; // becomes false
9     if not bipartite
10    queue<int> q;
11    for (int st = 0; st < n; ++st) {
12        if (side[st] == -1) {
13            q.push(st);
14            side[st] = 0;
15            while (!q.empty()) {
16                int v = q.front();
17                q.pop();
18                for (int u : adj[v]) {
19                    if (side[u] == -1) {
```



```

18         side[u] = side[v] ^ 1;
19         q.push(u);
20     } else {
21         is_bipartite &= side[u]
22             != side[v];
23     }
24 }
25 }
26 }
27 }
28 }
29 }
30 }
31 }
32 }
33 }
34 }
35 }
36 }
37 }
38 }
39 }
40 }

```

3.1.9 Bipartite matching (Hopcroft-Karp) Fast bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and $btoa$ should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. $btoa[i]$ will be the match for vertex i on the right side, or -1 if it's not matched. Time: $O(\sqrt{VE})$

```

1 // Usage: vi btoa(m, -1); hopcroftKarp(g, btoa);
2
3 bool dfs(int a, int L, vector<vi>& g, vi& btoa,
4         vi& A, vi& B) {
5     if (A[a] != L) return 0;
6     A[a] = -1;
7     for (int b : g[a]) if (B[b] == L + 1) {
8         B[b] = 0;
9         if (btoa[b] == -1 || dfs(btoa[b], L + 1, g,
10                                btoa, A, B))
11             return btoa[b] = a, 1;
12     }
13     return 0;
14 }
15
16 int hopcroftKarp(vector<vi>& g, vi& btoa) {
17     int res = 0;
18     vi A(g.size()), B(btoa.size()), cur, next;
19     for (;;) {
20         fill(all(A), 0); fill(all(B), 0);
21         /// Find the starting nodes for BFS (i.e.
22         layer 0).
23         cur.clear();
24         for (int a : btoa) if (a != -1) A[a] = -1;
25         rep(a, 0, sz(g)) if (A[a] == 0) cur.push_back(a);
26
27         /// Find all layers using bfs.
28         for (int lay = 1;; lay++) {
29             bool islast = 0;
30             next.clear();
31             for (int a : cur) for (int b : g[a]) {
32                 if (btoa[b] == -1) {
33                     B[b] = lay; islast = 1;
34                 } else if (btoa[b] != a && !B[b]) {
35                     B[b] = lay;
36                     next.push_back(btoa[b]);
37                 }
38             }
39             if (islast) break;
40             if (next.empty()) return res;
41             for (int a : next) A[a] = lay;
42         }
43     }
44 }

```

3.1.10 Find cycle directed

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5+5;
4 vvi adj(mxN);
5 vector<char> color;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v) {
9     color[v] = 1;
10    for (int u : adj[v]) {
11        if (color[u] == 0) {
12            parent[u] = v;
13            if (dfs(u)) return true;
14        } else if (color[u] == 1) {
15            cycle_end = v;
16            cycle_start = u;
17            return true;
18        }
19    }
20    color[v] = 2;
21    return false;
22 }
23 void find_cycle() {
24     color.assign(n, 0);
25     parent.assign(n, -1);
26     cycle_start = -1;
27     for (int v = 0; v < n; v++) {
28         if (color[v] == 0 && dfs(v)) break;
29     }
30     if (cycle_start == -1) {
31         cout << "Acyclic" << endl;
32     } else {
33         vector<int> cycle;
34         cycle.push_back(cycle_start);
35         for (int v = cycle_end; v != cycle_start;
36              v = parent[v])
37             cycle.push_back(v);
38         cycle.push_back(cycle_start);
39         reverse(cycle.begin(), cycle.end());
40
41         cout << "Cycle Found: ";
42         for (int v : cycle) cout << v << " ";
43         cout << endl;
44     }
45 }

```

```

44 }

```

3.1.11 Find cycle undirected

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5 + 5;
4 vvi adj(mxN);
5 vector<bool> visited;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v, int par) { // passing vertex and
9     its parent vertex
10    visited[v] = true;
11    for (int u : adj[v]) {
12        if (u == par) continue; // skipping edge
13        to parent vertex
14        if (visited[u]) {
15            cycle_end = v;
16            cycle_start = u;
17            return true;
18        }
19        parent[u] = v;
20        if (dfs(u, parent[u]))
21            return true;
22    }
23    return false;
24 }
25 void find_cycle() {
26     visited.assign(n, false);
27     parent.assign(n, -1);
28     cycle_start = -1;
29     for (int v = 0; v < n; v++) {
30         if (!visited[v] && dfs(v, parent[v]))
31             break;
32     }
33     if (cycle_start == -1) {
34         cout << "Acyclic" << endl;
35     } else {
36         vector<int> cycle;
37         cycle.push_back(cycle_start);
38         for (int v = cycle_end; v != cycle_start;
39              v = parent[v])
40             cycle.push_back(v);
41         cycle.push_back(cycle_start);
42         cout << "Cycle Found: ";
43         for (int v : cycle) cout << v << " ";
44         cout << endl;
45     }
46 }

```

3.1.12 Tarjan's SCC


```

1 #include "header.h"
2 struct Tarjan {
3     vvi &edges;
4     int V, counter = 0, C = 0;
5     vi n, l;
6     vector<bool> vs;
7     stack<int> st;
8     Tarjan(vvi &e) : edges(e), V(e.size()), n(V,
9         -1), l(V, -1), vs(V, false) {}
10    void visit(int u, vi &com) {
11        l[u] = n[u] = counter++;
12        st.push(u);
13        vs[u] = true;
14        for (auto &&v : edges[u]) {
15            if (n[v] == -1) visit(v, com);
16            if (vs[v]) l[u] = min(l[u], l[v]);
17        }
18        if (l[u] == n[u]) {
19            while (true) {
20                int v = st.top();
21                st.pop();
22                vs[v] = false;
23                com[v] = C; //<== ACT HERE
24                if (u == v) break;
25            }
26            C++;
27        }
28    int find_sccs(vi &com) { // component indices
29        // will be stored in 'com'
30        com.assign(V, -1);
31        C = 0;
32        for (int u = 0; u < V; ++u)
33            if (n[u] == -1) visit(u, com);
34        return C;
35    }
36    // scc is a map of the original vertices of the
37    // graph to the vertices of the SCC graph,
38    // scc_graph is its adjacency list. SCC
39    // indices and edges are stored in 'scc' and '
40    // scc_graph'.
41    void scc_collapse(vi &scc, vvi &scc_graph) {
42        find_sccs(scc);
43        scc_graph.assign(C, vi());
44        set<pi> rec; // recorded edges
45        for (int u = 0; u < V; ++u) {
46            assert(scc[u] != -1);
47            for (int v : edges[u]) {
48                if (scc[v] == scc[u] ||
49                    rec.find({scc[u], scc[v]}) != rec.end())
50                    continue;
51                scc_graph[scc[u]].push_back(scc[v]);
52                rec.insert({scc[u], scc[v]});
53            }
54        }
55    }

```

```

49    }
50    // The number of edges needed to be added is
51    // max(sources.size(), sinks.())
52    void findSourcesAndSinks(const vvi &scc_graph,
53        vi &sources, vi &sinks) {
54        vi in_degree(C, 0), out_degree(C, 0);
55        for (int u = 0; u < C; u++) {
56            for (auto v : scc_graph[u]) {
57                in_degree[v]++;
58                out_degree[u]++;
59            }
60        }
61        for (int i = 0; i < C; ++i) {
62            if (in_degree[i] == 0) sources.push_back(i);
63            if (out_degree[i] == 0) sinks.push_back(i);
64        }
65    }

```

3.1.13 SCC edges Prints out the missing edges to make the input digraph strongly connected

```

1 #include "header.h"
2 const int N=1e5+10;
3 int n,a[N],cnt[N],vis[N];
4 vector<int> hd,tl;
5 int dfs(int x){
6     vis[x]=1;
7     if(!vis[a[x]])return vis[x]=dfs(a[x]);
8     return vis[x]=x;
9 }
10 int main(){
11     scanf("%d",&n);
12     for(int i=1;i<=n;i++){
13         scanf("%d",&a[i]);
14         cnt[a[i]]++;
15     }
16     int k=0;
17     for(int i=1;i<=n;i++){
18         if(!cnt[i]){
19             k++;
20             hd.push_back(i);
21             tl.push_back(dfs(i));
22         }
23     }
24     int tk=k;
25     for(int i=1;i<=n;i++){
26         if(!vis[i]){
27             k++;
28             hd.push_back(i);
29             tl.push_back(dfs(i));
30         }
31     }

```

```

32     if(k==1&&!tk)k=0;
33     printf("%d\n",k);
34     for(int i=0;i<k;i++)printf("%d_%d\n",tl[i],hd
35         [(i+1)%k]);
36     return 0;

```

3.1.14 Topological sort

```

1 #include "header.h"
2 int n; // number of vertices
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi ans;
6 void dfs(int v) {
7     visited[v] = true;
8     for (int u : adj[v]) {
9         if (!visited[u]) dfs(u);
10    }
11    ans.push_back(v);
12 }
13 void topological_sort() {
14     visited.assign(n, false);
15     ans.clear();
16     for (int i = 0; i < n; ++i) {
17         if (!visited[i]) dfs(i);
18     }
19     reverse(ans.begin(), ans.end());
20 }

```

3.1.15 Bellmann-Ford Same as Dijkstra but allows neg. edges

```

1 #include "header.h"
2 // Switch vi and vvpi to vl and vvpl if necessary
3 void bellmann_ford_extended(vvpi &e, int source,
4     int goal, vi &dist, vb &cyc) {
5     dist.assign(e.size(), INF);
6     cyc.assign(e.size(), false); // true when u
7     // is in a <0 cycle
8     dist[source] = 0;
9     // Perform n-1 relaxations
10    for (int iter = 0; iter < e.size() - 1; ++
11        iter) {
12        bool relax = false;
13        for (int u = 0; u < e.size(); ++u) {
14            if (dist[u] == INF) continue;
15            for (auto &edge : e[u]) {
16                int v = edge.first, w = edge.
17                    second;
18                if (dist[u] + w < dist[v]) {
19                    dist[v] = dist[u] + w;

```

```

17         relax = true;
18     }
19 }
20 }
21 if (!relax) break;
22 }
23 // Step to detect any reachable negative
24 // cycles
25 for (int u = 0; u < e.size(); ++u) {
26     if (dist[u] == INF) continue;
27     for (auto &edge : e[u]) {
28         int v = edge.first, w = edge.second;
29         if (dist[u] + w < dist[v]) {
30             // If we can still relax, mark
31             // the node in the negative
32             // cycle
33             dist[v] = -INF;
34             cyc[v] = true;
35         }
36     }
37 }
38 // Propagate neg. cycle detection to all
39 // reachable nodes (if necessary)
40 bool change = true;
41 while (change) {
42     change = false;
43     for (int u = 0; u < e.size(); ++u) {
44         if (!cyc[u]) continue;
45         for (auto &edge : e[u]) {
46             int v = edge.first;
47             if (!cyc[v]) {
48                 cyc[v] = true;
49                 dist[v] = -INF;
50                 change = true;
51             }
52         }
53     }
54 }
55 }
56 }

```

3.1.16 Ford-Fulkerson Basic Max. flow

```

1 #include "header.h"
2 #define V 6 // Num. of vertices in given graph
3 /* Returns true if there is a path from source 's'
4  * to sink 't' in residual graph. Also fills parent[] to
5  * store the path */
6 bool bfs(int rGraph[V][V], int s, int t, int
7  parent[]) {
8     bool visited[V];
9     memset(visited, 0, sizeof(visited));
10    queue<int> q;

```

```

10    q.push(s);
11    visited[s] = true;
12    parent[s] = -1;
13    while (!q.empty()) {
14        int u = q.front();
15        q.pop();
16
17        for (int v = 0; v < V; v++) {
18            if (visited[v] == false && rGraph[u][v] >
19                0) {
20                if (v == t) {
21                    parent[v] = u;
22                    return true;
23                }
24                q.push(v);
25                parent[v] = u;
26                visited[v] = true;
27            }
28        }
29    }
30    return false;
31 }
32 // Returns the maximum flow from s to t
33 int fordFulkerson(int graph[V][V], int s, int t)
34 {
35     int u, v;
36     int rGraph[V][V];
37     for (u = 0; u < V; u++)
38         for (v = 0; v < V; v++)
39             rGraph[u][v] = graph[u][v];
40
41     int parent[V]; // BFS-filled (to store path)
42     int max_flow = 0; // no flow initially
43     while (bfs(rGraph, s, t, parent)) {
44         int path_flow = INT_MAX;
45         for (v = t; v != s; v = parent[v]) {
46             u = parent[v];
47             path_flow = min(path_flow, rGraph[u][v]);
48         }
49         for (v = t; v != s; v = parent[v]) {
50             u = parent[v];
51             rGraph[u][v] -= path_flow;
52             rGraph[v][u] += path_flow;
53         }
54         max_flow += path_flow;
55     }
56     return max_flow;
57 }

```

3.1.17 Dinic max flow $O(V^2E)$, $O(Ef)$

```

1 #include "header.h"
2 using F = ll; using W = ll; // types for flow and
3 weight/cost

```

```

3 struct S{
4     const int v; // neighbour
5     const int r; // index of the reverse edge
6     F f; // current flow
7     const F cap; // capacity
8     const W cost; // unit cost
9     S(int v, int ri, F c, W cost = 0) :
10         v(v), r(ri), f(0), cap(c), cost(cost) {}
11     inline F res() const { return cap - f; }
12 };
13 struct FlowGraph : vector<vector<S>> {
14     FlowGraph(size_t n) : vector<vector<S>>(n) {}
15     void add_edge(int u, int v, F c, W cost = 0){
16         auto &t = *this;
17         t[u].emplace_back(v, t[v].size(), c, cost);
18         t[v].emplace_back(u, t[u].size()-1, c, -
19             cost);
20     }
21     void add_arc(int u, int v, F c, W cost = 0){
22         auto &t = *this;
23         t[u].emplace_back(v, t[v].size(), c, cost);
24         t[v].emplace_back(u, t[u].size()-1, 0, -
25             cost);
26     }
27     void clear() { for (auto &E : *this) for (
28         auto &e : E) e.f = 0LL; }
29 };
30 struct Dinic{
31     FlowGraph &edges; int V,s,t;
32     vi l; vector<vector<S>>::iterator> its; //
33     levels and iterators
34     Dinic(FlowGraph &edges, int s, int t) :
35         edges(edges), V(edges.size()), s(s), t(t),
36         l(V,-1), its(V) {}
37     ll augment(int u, F c) { // we reuse the same
38         iterators
39         if (u == t) return c; ll r = 0LL;
40         for(auto &i = its[u]; i != edges[u].end();
41             i++){
42             auto &e = *i;
43             if (e.res() && l[u] < l[e.v]) {
44                 auto d = augment(e.v, min(c, e.
45                     res()));
46                 if (d > 0) { e.f += d; edges[e.v]
47                     [e.r].f -= d; c -= d;
48                     r += d; if (!c) break; }
49             }
50         }
51         return r;
52     }
53     ll run() {
54         ll flow = 0, f;
55         while(true) {
56             fill(l.begin(), l.end(), -1); l[s]=0;

```

```

45     queue<int> q; q.push(s);
46     while(!q.empty()){
47         auto u = q.front(); q.pop(); its[
            u] = edges[u].begin();
48         for(auto &&e : edges[u]) if(e.res
            () && l[e.v]<0)
49             l[e.v] = l[u]+1, q.push(e.v);
50     }
51     if (l[t] < 0) return flow;
52     while ((f = augment(s, INF)) > 0)
        flow += f;
53 }
54 };

```

3.1.18 Edmonds-Karp (Max) flow algorithm with time $O(VE^2)$. To get edge flow values, compare capacities before and after, and take the positive values only.

```

1 #include "header.h"
2 template<class T> T edmondsKarp(vector<
    unordered_map<int, T>>&
3     graph, int source, int sink) {
4     assert(source != sink);
5     T flow = 0;
6     vi par(sz(graph)), q = par;
7
8     for (;;) {
9         fill(all(par), -1);
10        par[source] = 0;
11        int ptr = 1;
12        q[0] = source;
13
14        rep(i,0,ptr) {
15            int x = q[i];
16            for (auto e : graph[x]) {
17                if (par[e.first] == -1 && e.second > 0) {
18                    par[e.first] = x;
19                    q[ptr++] = e.first;
20                    if (e.first == sink) goto out;
21                }
22            }
23        }
24        return flow;
25    out:
26        T inc = numeric_limits<T>::max();
27        for (int y = sink; y != source; y = par[y])
28            inc = min(inc, graph[par[y]][y]);
29
30        flow += inc;
31        for (int y = sink; y != source; y = par[y]) {
32            int p = par[y];
33            if ((graph[p][y] -= inc) <= 0) graph[p].
                erase(y);
34            graph[y][p] += inc;

```

```

35     }
36 }
37 }

```

3.2 Dynamic Programming

3.2.1 Longest Incr. Subseq.

```

1 #include "header.h"
2 template<class T>
3 vector<T> index_path_lis(vector<T>& nums) {
4     int n = nums.size();
5     vector<T> sub;
6     vector<int> subIndex;
7     vector<T> path(n, -1);
8     for (int i = 0; i < n; ++i) {
9         if (sub.empty() || sub[sub.size() - 1] <
                nums[i]) {
10            path[i] = sub.empty() ? -1 : subIndex[sub.
                size() - 1];
11            sub.push_back(nums[i]);
12            subIndex.push_back(i);
13        } else {
14            int idx = lower_bound(sub.begin(), sub.end(),
                nums[i]) - sub.begin();
15            path[i] = idx == 0 ? -1 : subIndex[idx - 1];
16            sub[idx] = nums[i];
17            subIndex[idx] = i;
18        }
19    }
20    vector<T> ans;
21    int t = subIndex[subIndex.size() - 1];
22    while (t != -1) {
23        ans.push_back(t);
24        t = path[t];
25    }
26    reverse(ans.begin(), ans.end());
27    return ans;
28 }
29 // Length only
30 template<class T>
31 int length_lis(vector<T> &a) {
32     set<T> st;
33     typename set<T>::iterator it;
34     for (int i = 0; i < a.size(); ++i) {
35         it = st.lower_bound(a[i]);
36         if (it != st.end()) st.erase(it);
37         st.insert(a[i]);
38     }
39     return st.size();
40 }

```

3.2.2 0-1 Knapsack Given a number of coins, calculate all possible distinct sums

```

1 #include "header.h"
2 int main() {
3     int n;
4     vi coins(n); // possible coins to use
5     int sum = 0; // their sum of the coins
6     vi dp(sum + 1, 0); // dp[x] = 1 if sum x can be
            made
7     dp[0] = 1;
8     for (int c = 0; c < n; ++c)
9         for (int x = sum; x >= 0; --x)
10             if (dp[x]) dp[x + coins[c]] = 1;
11 }

```

3.2.3 Coin change Total distinct ways to make sum using n coins of different vals

```

1 #include "header.h"
2 int count(vi& coins, int n, int sum) {
3     vvi dp(n + 1, vi(sum + 1, 0));
4     dp[0][0] = 1;
5     for (int i = 1; i <= n; i++) {
6         for (int j = 0; j <= sum; j++) {
7             // without using the current coin,
8             dp[i][j] += dp[i - 1][j];
9             // using the current coin
10            if ((j - coins[i - 1]) >= 0)
11                dp[i][j] += dp[i][j - coins[i -
                    1]];
12        }
13    }
14    return dp[n][sum];
15 }

```

3.2.4 Longest common subseq. Optimization for each unique element appearing k -times

```

1 #include "../header.h"
2 #include "../DataStructures/fenwick_tree.cpp"
3 int lcs(int k, vector<int>& A, vector<int>& B) {
4     int lenA = A.size();
5     int lenB = B.size();
6
7     // Determine the number of distinct elements
8     // from max element in A and B
9     int n = max(*max_element(A.begin(), A.end()),
        *max_element(B.begin(), B.end())) + 1;
10
11    vector<vector<int>> C(n);
12    for (int j = 0; j < lenB; ++j) {

```

```

12     C[B[j]].push_back(j);
13 }
14
15 int ans = 0;
16 FenwickTree<int> fenwick(lenB + 1);
17 for (int i = 0; i < lenA; ++i) {
18     int a = A[i];
19     for (int j = C[a].size() - 1; j >= 0; --j)
20     {
21         int pos = C[a][j];
22         int x = fenwick.query(pos) + 1;
23         fenwick.update(pos + 1, x); //
24         Convert to 1-based index
25         ans = max(ans, x);
26     }
27 }
28 return ans;
29 }

```

3.3 Numerical

3.3.1 Template (for this section)

```

1 #include <bits/stdc++.h>
2 using namespace std;
3 #define rep(i, a, b) for(int i = a; i < (b); ++i)
4 #define all(x) begin(x), end(x)
5 #define sz(x) (int)(x).size()
6 typedef long long ll;
7 typedef pair<int, int> pii;
8 typedef vector<int> vi;

```

3.3.2 Polynomial

```

1 #include "template.cpp"
2 struct Poly {
3     vector<double> a;
4     double operator()(double x) const {
5         double val = 0;
6         for (int i = sz(a); i--;) (val += x) += a[i];
7         return val;
8     }
9     void diff() {
10         rep(i, 1, sz(a)) a[i-1] = i*a[i];
11         a.pop_back();
12     }
13     void divroot(double x0) {
14         double b = a.back(), c; a.back() = 0;
15         for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i]
16             +1*x0+b, b=c;
17         a.pop_back();
18     }
19 };

```

3.3.3 Poly Roots Finds the real roots to a polynomial. $O(n^2 \log(1/\epsilon))$

```

1 // Usage: polyRoots({{2,-3,1}},-1e9,1e9) = solve
2 // x^2-3x+2 = 0
3 #include "Polynomial.h"
4 #include "template.cpp"
5 vector<double> polyRoots(Poly p, double xmin,
6     double xmax) {
7     if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
8     vector<double> ret;
9     Poly der = p;
10    der.diff();
11    auto dr = polyRoots(der, xmin, xmax);
12    dr.push_back(xmin-1);
13    dr.push_back(xmax+1);
14    sort(all(dr));
15    rep(i,0,sz(dr)-1) {
16        double l = dr[i], h = dr[i+1];
17        bool sign = p(l) > 0;
18        if (sign ^ (p(h) > 0)) {
19            rep(it,0,60) { // while (h - l > 1e-8)
20                double m = (l + h) / 2, f = p(m);
21                if ((f <= 0) ^ sign) l = m;
22                else h = m;
23            }
24            ret.push_back((l + h) / 2);
25        }
26    }
27    return ret;
28 }

```

3.3.4 Golden Section Search Finds the argument minimizing the function f in the interval $[a, b]$ assuming f is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is ϵ . Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version. $O(\log((b-a)/\epsilon))$

```

1 /** Usage:
2  * double func(double x) { return 4+x+.3*x*x; }
3  * double xmin = gss(-1000,1000,func); */
4 #include "template.cpp"
5 // It is important for r to be precise, otherwise
6 // we don't necessarily maintain the inequality
7 // a < x1 < x2 < b.
8 double gss(double a, double b, double (*f)(double)) {
9     double r = (sqrt(5)-1)/2, eps = 1e-7;
10    double x1 = b - r*(b-a), x2 = a + r*(b-a);
11    double f1 = f(x1), f2 = f(x2);
12    while (b-a > eps)

```

```

11    if (f1 < f2) { //change to > to find maximum
12        b = x2; x2 = x1; f2 = f1;
13        x1 = b - r*(b-a); f1 = f(x1);
14    } else {
15        a = x1; x1 = x2; f1 = f2;
16        x2 = a + r*(b-a); f2 = f(x2);
17    }
18    return a;
19 }

```

3.3.5 Hill Climbing Poor man's optimization for unimodal functions.

```

1 #include "template.cpp"
2 typedef array<double, 2> P;
3 template<class F> pair<double, P> hillClimb(P
4     start, F f) {
5     pair<double, P> cur(f(start), start);
6     for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
7         rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
8             P p = cur.second;
9             p[0] += dx*jmp;
10            p[1] += dy*jmp;
11            cur = min(cur, make_pair(f(p), p));
12        }
13    }
14    return cur;
15 }

```

3.3.6 Integration Simple integration of a function over an interval using Simpson's rule. The error should be proportional to h^4 , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

```

1 #include "template.cpp"
2 template<class F>
3 double quad(double a, double b, F f, const int n
4     = 1000) {
5     double h = (b - a) / 2 / n, v = f(a) + f(b);
6     rep(i,1,n*2)
7         v += f(a + i*h) * (i&1 ? 4 : 2);
8     return v * h / 3;
9 }

```

3.3.7 Integration Adaptive Fast integration using an adaptive Simpson's rule.

```

1 /** Usage:
2 double sphereVolume = quad(-1, 1, [](double x) {
3 return quad(-1, 1, [&](double y) {
4 return quad(-1, 1, [&](double z) {
5 return x*x + y*y + z*z < 1; });});}); */
6 #include "template.cpp"
7 typedef double d;
8 #define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (
9 b-a) / 6
9 template <class F>
10 d rec(F& f, d a, d b, d eps, d S) {
11     d c = (a + b) / 2;
12     d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
13     if (abs(T - S) <= 15 * eps || b - a < 1e-10)
14         return T + (T - S) / 15;
15     return rec(f, a, c, eps / 2, S1) + rec(f, c, b,
16         eps / 2, S2);
17 }
18 template<class F>
19 d quad(d a, d b, F f, d eps = 1e-8) {
20     return rec(f, a, b, eps, S(a, b));
21 }

```

3.4 Num. Th. / Comb.

3.4.1 Basic stuff

```

1 #include "header.h"
2 ll gcd(ll a, ll b) { while (b) { a %= b; swap(a,
3     b); } return a; }
4 ll lcm(ll a, ll b) { return (a / gcd(a, b)) * b; }
5 ll mod(ll a, ll b) { return ((a % b) + b) % b; }
6 // Finds x, y s.t. ax + by = d = gcd(a, b).
7 void extended_euclid(ll a, ll b, ll &x, ll &y, ll
8     &d) {
9     ll xx = y = 0;
10    ll yy = x = 1;
11    while (b) {
12        ll q = a / b;
13        ll t = b; b = a % b; a = t;
14        t = xx; xx = x - q * xx; x = t;
15        t = yy; yy = y - q * yy; y = t;
16    }
17    d = a;
18 }
19 // solves ab = 1 (mod n), -1 on failure
20 ll mod_inverse(ll a, ll n) {
21     ll x, y, d;
22     extended_euclid(a, n, x, y, d);
23     return (d > 1 ? -1 : mod(x, n));

```

```

24 }
25 // All modular inverses of [1..n] mod P in O(n)
26 // time.
27 vi inverses(ll n, ll P) {
28     vi I(n+1, 1LL);
29     for (ll i = 2; i <= n; ++i)
30         I[i] = mod(-(P/i) * I[P%i], P);
31     return I;
32 }
33 // (a*b)%m
34 ll mulmod(ll a, ll b, ll m){
35     ll x = 0, y=a%m;
36     while(b>0){
37         if(b&1) x = (x+y)%m;
38         y = (2*y)%m, b /= 2;
39     }
40     return x % m;
41 }
42 // Finds b^e % m in O(lg n) time, ensure that b <
43 // m to avoid overflow!
44 ll powmod(ll b, ll e, ll m) {
45     ll p = e<2 ? 1 : powmod((b*b)%m,e/2,m);
46     return e&1 ? p*b%m : p;
47 }
48 // Solve ax + by = c, returns false on failure.
49 bool linear_diophantine(ll a, ll b, ll c, ll &x,
50     ll &y) {
51     ll d = gcd(a, b);
52     if (c % d) {
53         return false;
54     } else {
55         x = c / d * mod_inverse(a / d, b / d);
56         y = (c - a * x) / b;
57         return true;
58     }
59 }
60 // Description: Tonelli-Shanks algorithm for
61 // modular square roots. Finds $x$ s.t. $x^2 = a
62 // \pmod p$ ($-x$ gives the other solution). 0
63 // (\log^2 p) worst case, O(\log p) for most $p$
64 ll sqrtmod(ll a, ll p) {
65     a %= p; if (a < 0) a += p;
66     if (a == 0) return 0;
67     assert(powmod(a, (p-1)/2, p) == 1); // else no
68     // solution
69     if (p % 4 == 3) return powmod(a, (p+1)/4, p);
70     // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if
71     // p % 8 == 5
72     ll s = p - 1, n = 2;
73     int r = 0, m;
74     while (s % 2 == 0)
75         ++r, s /= 2;
76     /// find a non-square mod p
77     while (powmod(n, (p - 1) / 2, p) != p - 1) ++n;

```

```

69     ll x = powmod(a, (s + 1) / 2, p);
70     ll b = powmod(a, s, p), g = powmod(n, s, p);
71     for (; r = m) {
72         ll t = b;
73         for (m = 0; m < r && t != 1; ++m)
74             t = t * t % p;
75         if (m == 0) return x;
76         ll gs = powmod(g, 1LL << (r - m - 1), p);
77         g = gs * gs % p;
78         x = x * gs % p;
79         b = b * g % p;
80     }
81 }

```

3.4.2 Mod. exponentiation Or use pow() in python

```

1 #include "header.h"
2 ll mod_pow(ll base, ll exp, ll mod) {
3     if (mod == 1) return 0;
4     if (exp == 0) return 1;
5     if (exp == 1) return base;
6
7     ll res = 1;
8     base %= mod;
9     while (exp) {
10        if (exp % 2 == 1) res = (res * base) % mod;
11        exp >>= 1;
12        base = (base * base) % mod;
13    }
14
15    return res % mod;
16 }

```

3.4.3 GCD Or math.gcd in python, std::gcd in C++

```

1 #include "header.h"
2 ll gcd(ll a, ll b) {
3     if (a == 0) return b;
4     return gcd(b % a, a);
5 }

```

3.4.4 Sieve of Eratosthenes

```

1 #include "header.h"
2 vl primes;
3 void getprimes(ll n) { // Up to n (not included)
4     vector<bool> p(n, true);
5     p[0] = false;
6     p[1] = false;
7     for(ll i = 0; i < n; i++) {
8         if(p[i]) {

```

```

9     primes.push_back(i);
10    for(11 j = i*2; j < n; j+=i) p[j] =
        false;
11 }}}

```

3.4.5 Fibonacci % prime Starting 1,1,2,3,...

```

1 #include "header.h"
2 const 11 MOD = 1000000007;
3 unordered_map<11, 11> Fib;
4 11 fib(11 n) {
5     if (n < 2) return 1;
6     if (Fib.find(n) != Fib.end()) return Fib[n];
7     Fib[n] = (fib((n + 1) / 2) * fib(n / 2) + fib
        ((n - 1) / 2) * fib((n - 2) / 2)) % MOD;
8     return Fib[n];
9 }

```

3.4.6 nCk % prime

```

1 #include "header.h"
2 11 binom(11 n, 11 k) {
3     11 ans = 1;
4     for(11 i = 1; i <= min(k,n-k); ++i) ans = ans
        *(n+1-i)/i;
5     return ans;
6 }
7 11 mod_nCk(11 n, 11 k, 11 p ){
8     11 ans = 1;
9     while(n){
10         11 np = n%p, kp = k%p;
11         if(kp > np) return 0;
12         ans *= binom(np,kp);
13         n /= p; k /= p;
14     }
15     return ans;
16 }

```

3.5 Strings

3.5.1 Z alg. KMP alternative (same complexities)

```

1 #include "../header.h"
2 void Z_algorithm(const string &s, vi &Z) {
3     Z.assign(s.length(), -1);
4     int L = 0, R = 0, n = s.length();
5     for (int i = 1; i < n; ++i) {
6         if (i > R) {
7             L = R = i;
8             while (R < n && s[R - L] == s[R]) R++;
9             Z[i] = R - L; R--;

```

```

10        } else if (Z[i - L] >= R - i + 1) {
11            L = i;
12            while (R < n && s[R - L] == s[R]) R++;
13            Z[i] = R - L; R--;
14        } else Z[i] = Z[i - L];
15    }
16 }

```

3.5.2 KMP

```

1 #include "header.h"
2 void compute_prefix_function(string &w, vi &
    prefix) {
3     prefix.assign(w.length(), 0);
4     int k = prefix[0] = -1;
5
6     for(int i = 1; i < w.length(); ++i) {
7         while(k >= 0 && w[k + 1] != w[i]) k = prefix[
            k];
8         if(w[k + 1] == w[i]) k++;
9         prefix[i] = k;
10    }
11 }
12 vi knuth_morris_pratt(string &s, string &w) {
13     int q = -1;
14     vi prefix, positions;
15     compute_prefix_function(w, prefix);
16     for(int i = 0; i < s.length(); ++i) {
17         while(q >= 0 && w[q + 1] != s[i]) q = prefix[
            q];
18         if(w[q + 1] == s[i]) q++;
19         if(q + 1 == w.length()) {
20             // Match at position (i - w.length() + 1)
21             positions.push_back(i - w.length() +
                1);
22             q = prefix[q];
23         }
24     }
25     return positions;
26 }

```

3.5.3 Aho-Corasick Also can be used as Knuth-Morris-Pratt algorithm

```

1 #include "header.h"
2 map<char, int> cti;
3 int cti_size;
4 template <int ALPHABET_SIZE, int (*mp)(char)>
5 struct AC_FSM {
6     struct Node {
7         int child[ALPHABET_SIZE], failure = 0,
            match_par = -1;
8         vi match;

```

```

9         Node() { for (int i = 0; i < ALPHABET_SIZE;
            ++i) child[i] = -1; }
10    };
11    vector<Node> a;
12    vector<string> &words;
13    AC_FSM(vector<string> &words) : words(words) {
14        a.push_back(Node());
15        construct_automaton();
16    }
17    void construct_automaton() {
18        for (int w = 0, n = 0; w < words.size(); ++w,
            n = 0) {
19            for (int i = 0; i < words[w].size(); ++i) {
20                if (a[n].child[mp(words[w][i])] == -1) {
21                    a[n].child[mp(words[w][i])] = a.size();
22                    a.push_back(Node());
23                }
24                n = a[n].child[mp(words[w][i])];
25            }
26            a[n].match.push_back(w);
27        }
28        queue<int> q;
29        for (int k = 0; k < ALPHABET_SIZE; ++k) {
30            if (a[0].child[k] == -1) a[0].child[k] = 0;
31            else if (a[0].child[k] > 0) {
32                a[a[0].child[k]].failure = 0;
33                q.push(a[0].child[k]);
34            }
35        }
36        while (!q.empty()) {
37            int r = q.front(); q.pop();
38            for (int k = 0, arck; k < ALPHABET_SIZE; ++
                k) {
39                if ((arck = a[r].child[k]) != -1) {
40                    q.push(arck);
41                    int v = a[r].failure;
42                    while (a[v].child[k] == -1) v = a[v].
                        failure;
43                    a[arck].failure = a[v].child[k];
44                    a[arck].match_par = a[v].child[k];
45                    while (a[arck].match_par != -1
                        && a[a[arck].match_par].match.empty
                            ())
46                        a[arck].match_par = a[a[arck].
                            match_par].match_par;
47                }
48            }
49        }
50    }
51 }
52 void aho_corasick(string &sentence, vvi &
    matches){
53     matches.assign(words.size(), vi());
54     int state = 0, ss = 0;
55     for (int i = 0; i < sentence.length(); ++i,
        ss = state) {

```



```

56 while (a[ss].child[mp(sentence[i])] == -1)
57     ss = a[ss].failure;
58 state = a[state].child[mp(sentence[i])]
59     = a[ss].child[mp(sentence[i])];
60 for (ss = state; ss != -1; ss = a[ss].
    match_par)
61     for (int w : a[ss].match)
62         matches[w].push_back(i + 1 - words[w].
            length());
63 }
64 }
65 };
66 int char_to_int(char c) {
67     return cti[c];
68 }
69 int main() {
70     ll n;
71     string line;
72     while(getline(cin, line)) {
73         stringstream ss(line);
74         ss >> n;
75
76         vector<string> patterns(n);
77         for (auto& p: patterns) getline(cin, p);
78
79         string text;
80         getline(cin, text);
81
82         cti = {}, cti_size = 0;
83         for (auto c: text) {
84             if (not in(c, cti)) {
85                 cti[c] = cti_size++;
86             }
87         }
88         for (auto& p: patterns) {
89             for (auto c: p) {
90                 if (not in(c, cti)) {
91                     cti[c] = cti_size++;
92                 }
93             }
94         }
95
96         vvi matches;
97         AC_FSM <128+1, char_to_int> ac_fms(patterns);
98         ac_fms.aho_corasick(text, matches);
99         for (auto& x: matches) cout << x << endl;
100     }
101 }
102 }

```

3.5.4 Long. palin. subs Manacher - $O(n)$

```

1 #include "header.h"
2 void manacher(string &s, vi &pal) {

```

```

3 int n = s.length(), i = 1, l, r;
4 pal.assign(2 * n + 1, 0);
5 while (i < 2 * n + 1) {
6     if ((i&1) && pal[i] == 0) pal[i] = 1;
7     l = i / 2 - pal[i] / 2; r = (i-1) / 2 + pal[i]
        / 2;
8
9     while (l - 1 >= 0 && r + 1 < n && s[l - 1] ==
        s[r + 1])
10         --l, ++r, pal[i] += 2;
11
12     for (l = i - 1, r = i + 1; l >= 0 && r < 2 *
        n + 1; --l, ++r) {
13         if (l <= i - pal[i]) break;
14         if (l / 2 - pal[l] / 2 > i / 2 - pal[i] /
            2)
15             pal[r] = pal[l];
16         else { if (l >= 0)
17             pal[r] = min(pal[l], i + pal[i] - r);
18             break;
19         }
20     }
21     i = r;
22 } }

```

3.5.5 Bitstring Slower than an unordered set (for many elements), but hashable

```

1 #include "../header.h"
2 template<size_t len>
3 struct pair_hash { // To make it hashable (pair<
    int, bitset<len>>)
4     std::size_t operator()(const std::pair<int,
        std::bitset<len>>& p) const {
5         std::size_t h1 = std::hash<int>{}(p.first
        );
6         std::size_t h2 = std::hash<std::bitset<
            len>>{}(p.second);
7         return h1 ^ (h2 << 1);
8     }
9 };
10 #define MAXN 1000
11 std::bitset<MAXN> bs;
12 // bs.set(idx) <- set idx-th bit (1)
13 // bs.reset(idx) <- reset idx-th bit (0)
14 // bs.flip(idx) <- flip idx-th bit
15 // bs.test(idx) <- idx-th bit == 1
16 // bs.count() <- number of 1s
17 // bs.any() <- any bit == 1

```

3.6 Geometry

3.6.1 essentials.cpp

```

1 #include "../header.h"
2 using C = ld; // could be ll or ld
3 constexpr C EPS = 1e-10; // change to 0 for C=ll
4 struct P { // may also be used as a 2D vector
5     C x, y;
6     P(C x = 0, C y = 0) : x(x), y(y) {}
7     P operator+ (const P &p) const { return {x + p.
        x, y + p.y}; }
8     P operator- (const P &p) const { return {x - p.
        x, y - p.y}; }
9     P operator* (C c) const { return {x * c, y * c
        }; }
10    P operator/ (C c) const { return {x / c, y / c
        }; }
11    C operator* (const P &p) const { return x*p.x +
        y*p.y; }
12    C operator^ (const P &p) const { return x*p.y -
        p.x*y; }
13    P perp() const { return P{y, -x}; }
14    C lensq() const { return x*x + y*y; }
15    ld len() const { return sqrt((ld)lensq()); }
16    static ld dist(const P &p1, const P &p2) {
17        return (p1-p2).len(); }
18    bool operator==(const P &r) const {
19        return ((*this)-r).lensq() <= EPS*EPS; }
20 };
21 C det(P p1, P p2) { return p1^p2; }
22 C det(P p1, P p2, P o) { return det(p1-o, p2-o);
    }
23 C det(const vector<P> &ps) {
24     C sum = 0; P prev = ps.back();
25     for(auto &p : ps) sum += det(p, prev), prev = p
        ;
26     return sum;
27 }
28 // Careful with division by two and C=ll
29 C area(P p1, P p2, P p3) { return abs(det(p1, p2,
    p3))/C(2); }
30 C area(const vector<P> &poly) { return abs(det(
    poly))/C(2); }
31 int sign(C c){ return (c > C(0)) - (c < C(0)); }
32 int ccw(P p1, P p2, P o) { return sign(det(p1, p2
    , o)); }
33
34 // Only well defined for C = ld.
35 P unit(const P &p) { return p / p.len(); }
36 P rotate(P p, ld a) { return P{p.x*cos(a)-p.y*sin
    (a), p.x*sin(a)+p.y*cos(a)}; }

```

3.6.2 Two segs. itersec.

```

1 #include "header.h"
2 #include "essentials.cpp"
3 bool intersect(P a1, P a2, P b1, P b2) {

```



```

4  if (max(a1.x, a2.x) < min(b1.x, b2.x)) return
    false;
5  if (max(b1.x, b2.x) < min(a1.x, a2.x)) return
    false;
6  if (max(a1.y, a2.y) < min(b1.y, b2.y)) return
    false;
7  if (max(b1.y, b2.y) < min(a1.y, a2.y)) return
    false;
8  bool l1 = ccw(a2, b1, a1) * ccw(a2, b2, a1) <=
    0;
9  bool l2 = ccw(b2, a1, b1) * ccw(b2, a2, b1) <=
    0;
10 return l1 && l2;
11 }

```

3.6.3 Convex Hull

```

1 #include "header.h"
2 #include "essentials.cpp"
3 struct ConvexHull { // O(n lg n) monotone chain.
4     size_t n;
5     vector<size_t> h, c; // Indices of the hull
6                             // are in 'h', ccw.
7     const vector<P> &p;
8     ConvexHull(const vector<P> &p) : n(p.size()),
9                                     c(n), p(p) {
10         std::iota(c.begin(), c.end(), 0);
11         std::sort(c.begin(), c.end(), [this](size_t l
12             , size_t r) -> bool { return p[l].x != p[
13                 r].x ? p[l].x < p[r].x : p[l].y < p[r].y;
14             });
15         c.erase(std::unique(c.begin(), c.end(), [this
16             ](size_t l, size_t r) { return p[l] == p[
17                 r]; }, c.end()));
18         for (size_t s = 1, r = 0; r < 2; ++r, s = h.
19             size()) {
20             for (size_t i : c) {
21                 while (h.size() > s && ccw(p[h.end()
22                     ][-2], p[h.end()[-1]], p[i]) <= 0)
23                     h.pop_back();
24                 h.push_back(i);
25             }
26             reverse(c.begin(), c.end());
27         }
28         if (h.size() > 1) h.pop_back();
29     }
30     size_t size() const { return h.size(); }
31     template <class T, void U(const P &, const P &,
32                             const P &, T &)>
33     void rotating_calipers(T &ans) {
34         if (size() <= 2)
35             U(p[h[0]], p[h.back()], p[h.back()], ans);
36         else
37             for (size_t i = 0, j = 1, s = size(); i < 2
38                 * s; ++i) {

```

```

28         while (det(p[h[(i + 1) % s]] - p[h[i % s
29             ]], p[h[(j + 1) % s]] - p[h[j % s]]) >=
30             0)
31             j = (j + 1) % s;
32         U(p[h[i % s]], p[h[(i + 1) % s]], p[h[j
33             % s]], ans);
34     }
35 }
36 // Example: furthest pair of points. Now set ans
37 // = 0LL and call
38 // ConvexHull(pts).rotating_calipers<ll, update>(
39 // ans);
40 void update(const P &p1, const P &p2, const P &o,
41             ll &ans) {
42     ans = max(ans, (ll)max((p1 - o).lensq(), (p2 -
43         o).lensq()));
44 }
45 int main() {
46     ios::sync_with_stdio(false); // do not use
47     cout << printf
48         << "cin.tie(NULL);
49         \n";
50     int n;
51     cin >> n;
52     while (n) {
53         vector<P> ps;
54         for (int x, y;
55             x < n; ++x) {
56             for (int i = 0; i < n; i++) {
57                 cin >> x >> y;
58                 ps.push_back({x, y});
59             }
60             ConvexHull ch(ps);
61             cout << ch.h.size() << endl;
62             for(auto& p: ch.h) {
63                 cout << ps[p].x << " " << ps[p].y <<
64                     endl;
65             }
66             cin >> n;
67         }
68     }
69     return 0;
70 }

```

3.7 Other Algorithms

3.7.1 2-sat

```

1 #include "../header.h"
2 #include "../Graphs/tarjan.cpp"
3 struct TwoSAT {
4     int n;
5     vvi imp; // implication graph

```

```

6     Tarjan tj;
7     TwoSAT(int _n) : n(_n), imp(2 * _n, vi()), tj(
8         n, imp) {}
9     // Only copy the needed functions:
10    void add_implies(int c1, bool v1, int c2, bool
11        v2) {
12        int u = 2 * c1 + (v1 ? 1 : 0),
13            v = 2 * c2 + (v2 ? 1 : 0);
14        imp[u].push_back(v); // u => v
15        imp[v^1].push_back(u^1); // -v => -u
16    }
17    void add_equivalence(int c1, bool v1, int c2,
18        bool v2) {
19        add_implies(c1, v1, c2, v2);
20        add_implies(c2, v2, c1, v1);
21    }
22    void add_or(int c1, bool v1, int c2, bool v2) {
23        add_implies(c1, !v1, c2, v2);
24    }
25    void add_and(int c1, bool v1, int c2, bool v2) {
26        add_true(c1, v1); add_true(c2, v2);
27    }
28    void add_xor(int c1, bool v1, int c2, bool v2) {
29        add_or(c1, v1, c2, v2);
30        add_or(c1, !v1, c2, !v2);
31    }
32    void add_true(int c1, bool v1) {
33        add_implies(c1, !v1, c1, v1);
34    }
35    // on true: a contains an assignment.
36    // on false: no assignment exists.
37    bool solve(vb &a) {
38        vi com;
39        tj.find_sccs(com);
40        for (int i = 0; i < n; ++i)
41            if (com[2 * i] == com[2 * i + 1])
42                return false;
43        vvi bycom(com.size());
44        for (int i = 0; i < 2 * n; ++i)
45            bycom[com[i]].push_back(i);
46        a.assign(n, false);
47        vb vis(n, false);
48        for(auto &&component : bycom){
49            for (int u : component) {
50                if (vis[u / 2]) continue;
51                vis[u / 2] = true;
52                a[u / 2] = (u % 2 == 1);
53            }
54        }
55    }

```

```

56 }
57 return true;
58 }
59 };

```

3.7.2 Finite field For FFT

```

1 #include "header.h"
2 #include "../Number_Theory/elementary.cpp"
3 template<ll p, ll w> // prime, primitive root
4 struct Field { using T = Field; ll x; Field(ll x
=0) : x{x} {}
5 T operator+(T r) const { return {(x+r.x)%p}; }
6 T operator-(T r) const { return {(x-r.x+p)%p};
}
7 T operator*(T r) const { return {(x*r.x)%p}; }
8 T operator/(T r) const { return (*this)*r.inv()
; }
9 T inv() const { return {mod_inverse(x,p)}; }
10 static T root(ll k) { assert( (p-1)%k==0 );
// (p-1)%k == 0?
11 auto r = powmod(w, (p-1)/abs(k), p); // k-
th root of unity
12 return k>0 ? T{r} : T{r}.inv();
13 }
14 bool zero() const { return x == 0LL; }
15 };
16 using F1 = Field<1004535809, 3>;
17 using F2 = Field<1107296257, 10>; // 1<<30 + 1<<25
+ 1
18 using F3 = Field<2281701377, 3>; // 1<<31 + 1<<27
+ 1

```

3.7.3 Complex field For FFR

```

1 #include "header.h"
2 const double m_pi = M_PI/64x;
3 struct Complex { using T = Complex; double u,v;
4 Complex(double u=0, double v=0) : u{u}, v{v} {}
5 T operator+(T r) const { return {u+r.u, v+r.v};
}
6 T operator-(T r) const { return {u-r.u, v-r.v};
}
7 T operator*(T r) const { return {u*r.u - v*r.v,
u*r.v + v*r.u}; }
8 T operator/(T r) const {
9 auto norm = r.u*r.u+r.v*r.v;
10 return {(u*r.u + v*r.v)/norm, (v*r.u - u*r.v)
/norm};
11 }
12 T operator*(double r) const { return T{u*r, v*r
}; }

```

```

13 T operator/(double r) const { return T{u/r, v/r
}; }
14 T inv() const { return T{1,0}/ *this; }
15 T conj() const { return T{u, -v}; }
16 static T root(ll k){ return {cos(2*m_pi/k), sin
(2*m_pi/k)}; }
17 bool zero() const { return max(abs(u), abs(v))
< 1e-6; }
18 };

```

3.7.4 FFT

```

1 #include "header.h"
2 #include "complex_field.cpp"
3 #include "fin_field.cpp"
4 void brinc(int &x, int k) {
5 int i = k - 1, s = 1 << i;
6 x ^= s;
7 if ((x & s) != s) {
8 --i; s >>= 1;
9 while (i >= 0 && ((x & s) == s))
10 x = x &~ s, --i, s >>= 1;
11 if (i >= 0) x |= s;
12 }
13 }
14 using T = Complex; // using T=F1,F2,F3
15 vector<T> roots;
16 void root_cache(int N) {
17 if (N == (int)roots.size()) return;
18 roots.assign(N, T{0});
19 for (int i = 0; i < N; ++i)
20 roots[i] = ((i&-i) == i)
21 ? T{cos(2.0*m_pi*i/N), sin(2.0*m_pi*i/N)}
22 : roots[i&-i] * roots[i-(i&-i)];
23 }
24 void fft(vector<T> &A, int p, bool inv = false) {
25 int N = 1<<p;
26 for(int i = 0, r = 0; i < N; ++i, brinc(r, p))
27 if (i < r) swap(A[i], A[r]);
28 // Uncomment to precompute roots (for T=Complex)
. Slower but more precise.
29 // root_cache(N);
30 // , sh=p-1 , --sh
31 for (int m = 2; m <= N; m <= 1) {
32 T w, w_m = T::root(inv ? -m : m);
33 for (int k = 0; k < N; k += m) {
34 w = T{1};
35 for (int j = 0; j < m/2; ++j) {
36 // T w = (!inv ? roots[j<<sh] : roots[j<<
sh].conj());
37 T t = w * A[k + j + m/2];
38 A[k + j + m/2] = A[k + j] - t;
39 A[k + j] = A[k + j] + t;
40 w = w * w_m;

```

```

41 }
42 }
43 }
44 if(inv){ T inverse = T(N).inv(); for(auto &x :
A) x = x*inverse; }
45 }
46 // convolution leaves A and B in frequency domain
state
47 // C may be equal to A or B for in-place
convolution
48 void convolution(vector<T> &A, vector<T> &B,
vector<T> &C){
49 int s = A.size() + B.size() - 1;
50 int q = 32 - __builtin_clz(s-1), N=1<<q; //
fails if s=1
51 A.resize(N, {}); B.resize(N, {}); C.resize(N, {});
52 fft(A, q, false); fft(B, q, false);
53 for (int i = 0; i < N; ++i) C[i] = A[i] * B[i];
54 fft(C, q, true); C.resize(s);
55 }
56 void square_inplace(vector<T> &A) {
57 int s = 2*A.size()-1, q = 32 - __builtin_clz(s
-1), N=1<<q;
58 A.resize(N, {}); fft(A, q, false);
59 for(auto &x : A) x = x*x;
60 fft(A, q, true); A.resize(s);
61 }

```

3.7.5 Polyn. inv. div.

```

1 #include "header.h"
2 #include "fft.cpp"
3 vector<T> &rev(vector<T> &A) { reverse(A.begin(),
A.end()); return A; }
4 void copy_into(const vector<T> &A, vector<T> &B,
size_t n) {
5 std::copy(A.begin(), A.begin()+min({n, A.size()
, B.size()}), B.begin());
6 }
7 // Multiplicative inverse of A modulo x^n.
Requires A[0] != 0!!
8 vector<T> inverse(const vector<T> &A, int n) {
9 vector<T> Ai{A[0].inv()};
10 for (int k = 0; (1<<k) < n; ++k) {
11 vector<T> As(4<<k, T{0}), Ais(4<<k, T{0});
12 copy_into(A, As, 2<<k); copy_into(Ai, Ais, Ai
.size());
13 fft(As, k+2, false); fft(Ais, k+2, false);
14 for (int i = 0; i < (4<<k); ++i) As[i] = As[i]
*Ais[i]*Ais[i];
15 fft(As, k+2, true); Ai.resize(2<<k, {});
16 for (int i = 0; i < (2<<k); ++i) Ai[i] = T(2)
* Ai[i] - As[i];
17 }

```

```

18 Ai.resize(n);
19 return Ai;
20 }
21 // Polynomial division. Returns {Q, R} such that
22 // A = QB+R, deg R < deg B.
23 // Requires that the leading term of B is nonzero
24 .
25 pair<vector<T>, vector<T>> divmod(const vector<T>
26 &A, const vector<T> &B) {
27     size_t n = A.size()-1, m = B.size()-1;
28     if (n < m) return {vector<T>(1, T(0)), A};
29     vector<T> X(A), Y(B), Q, R;
30     convolution(rev(X), Y = inverse(rev(Y), n-m+1),
31 Q);
32 Q.resize(n-m+1); rev(Q);
33 X.resize(Q.size()), copy_into(Q, X, Q.size());
34 Y.resize(B.size()), copy_into(B, Y, B.size());
35 convolution(X, Y, X);
36 R.resize(m), copy_into(A, R, m);
37 for (size_t i = 0; i < m; ++i) R[i] = R[i] - X[
38 i];
39 while (R.size() > 1 && R.back().zero()) R.
40 pop_back();
41 return {Q, R};
42 }
43
44 vector<T> mod(const vector<T> &A, const vector<T>
45 &B) {
46     return divmod(A, B).second;
47 }

```

3.7.6 Linear recurs. Given a linear recurrence of the form

$$a_n = \sum_{i=0}^{k-1} c_i a_{n-i-1}$$

this code computes a_n in $O(k \log k \log n)$ time.

```

1 #include "header.h"
2 #include "poly.cpp"
3 // x^k mod f
4 vector<T> xmod(const vector<T> f, ll k) {
5     vector<T> r{T(1)};
6     for (int b = 62; b >= 0; --b) {
7         if (r.size() > 1)
8             square_inplace(r), r = mod(r, f);
9         if ((k>>b)&1) {
10             r.insert(r.begin(), T(0));
11             if (r.size() == f.size()) {
12                 T c = r.back() / f.back();
13                 for (size_t i = 0; i < f.size(); ++i)

```

```

14         r[i] = r[i] - c * f[i];
15         r.pop_back();
16     }
17 }
18 }
19 return r;
20 }
21 // Given A[0,k) and C[0, k), computes the n-th
22 // term of:
23 // A[n] = \sum_i C[i] * A[n-i-1]
24 T nth_term(const vector<T> &A, const vector<T> &C
25 , ll n) {
26     int k = (int)A.size();
27     if (n < k) return A[n];
28     vector<T> f(k+1, T{1});
29     for (int i = 0; i < k; ++i)
30         f[i] = T{-1} * C[k-i-1];
31     f = xmod(f, n);
32     T r = T{0};
33     for (int i = 0; i < k; ++i)
34         r = r + f[i] * A[i];
35     return r;
36 }

```

3.7.7 Convolution Precise up to 9e15

```

1 #include "header.h"
2 #include "fft.cpp"
3 void convolution_mod(const vi &A, const vi &B, ll
4 MOD, vi &C) {
5     int s = A.size() + B.size() - 1; ll m15 = (1LL
6 <<15)-1LL;
7     int q = 32 - __builtin_clz(s-1), N=1<<q; //
8     fails if s=1
9     vector<T> Ac(N), Bc(N), R1(N), R2(N);
10    for (size_t i = 0; i < A.size(); ++i) Ac[i] = T
11    {A[i]&m15, A[i]>>15};
12    for (size_t i = 0; i < B.size(); ++i) Bc[i] = T
13    {B[i]&m15, B[i]>>15};
14    fft(Ac, q, false); fft(Bc, q, false);
15    for (int i = 0, j = 0; i < N; ++i, j = (N-1)&(N
16    -i)) {
17        T as = (Ac[i] + Ac[j].conj()) / 2;
18        T al = (Ac[i] - Ac[j].conj()) / T{0, 2};
19        T bs = (Bc[i] + Bc[j].conj()) / 2;
20        T bl = (Bc[i] - Bc[j].conj()) / T{0, 2};
21        R1[i] = as*bs + al*bl*T{0,1}, R2[i] = as*bl +
22        al*bs;
23    }
24    fft(R1, q, true); fft(R2, q, true);
25    ll p15 = (1LL<<15)%MOD, p30 = (1LL<<30)%MOD; C.
26    resize(s);

```

```

19 for (int i = 0; i < s; ++i) {
20     ll l = llround(R1[i].u), m = llround(R2[i].u)
21     , h = llround(R1[i].v);
22     C[i] = (1 + m*p15 + h*p30) % MOD;
23 }

```

3.7.8 Partitions of n Finds all possible partitions of a number

```

1 #include "header.h"
2 void printArray(int p[], int n) {
3     for (int i = 0; i < n; ++i)
4         cout << p[i] << " ";
5     cout << endl;
6 }
7 void printAllUniqueParts(int n) {
8     int p[n]; // array to store a partition
9     int k = 0; // idx of last element in a
10    partition
11    p[k] = n;
12    // The loop stops when the current partition
13    // has all 1s
14    while (true) {
15        printArray(p, k + 1);
16        int rem_val = 0;
17        while (k >= 0 && p[k] == 1) {
18            rem_val += p[k];
19            k--;
20        }
21        // no more partitions
22        if (k < 0) return;
23        p[k]--;
24        rem_val++;
25        // sorted order is violated (fix)
26        while (rem_val > p[k]) {
27            p[k + 1] = p[k];
28            rem_val = rem_val - p[k];
29            k++;
30        }
31    }
32    p[k + 1] = rem_val;
33    k++;
34 }
35 }
36 }

```

3.7.9 Ternary search Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \dots < f(i) \geq \dots \geq$

$f(b)$. To reverse which of the sides allows non-strict inequalities, change the $<$ marked with (A) to \leq , and reverse the loop at (B). To minimize f , change it to $>$, also at (B). $O(\log(b-a))$

```
1 // Usage: int ind = ternSearch(0,n-1,[\&](int i){
  return a[i];});
2 #include "../Numerical/template.cpp"
3 template<class F>
4 int ternSearch(int a, int b, F f) {
5     assert(a <= b);
6     while (b - a >= 5) {
7         int mid = (a + b) / 2;
8         if (f(mid) < f(mid+1)) a = mid; // (A)
9         else b = mid+1;
10    }
11    rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
12    return a;
13 }
```

3.8 Other Data Structures

3.8.1 Disjoint set (i.e. union-find)

```
1 template <typename T>
2 class DisjointSet {
3     typedef T * iterator;
4     T *parent, n, *rank;
5     public:
6         // O(n), assumes nodes are [0, n)
7         DisjointSet(T n) {
8             this->parent = new T[n];
9             this->n = n;
10            this->rank = new T[n];
11            for (T i = 0; i < n; i++) {
12                parent[i] = i;
13                rank[i] = 0;
14            }
15        }
16
17        // O(log n)
18        T find_set(T x) {
19            if (x == parent[x]) return x;
20            return parent[x] = find_set(parent[x]);
21        }
22
23        // O(log n)
24        void union_sets(T x, T y) {
25            x = this->find_set(x);
26            y = this->find_set(y);
27
28            if (x == y) return;
```

```
29         if (rank[x] < rank[y]) {
30             T z = x;
31             x = y;
32             y = z;
33         }
34         parent[y] = x;
35         if (rank[x] == rank[y]) rank[x]++;
36     }
37 };
```

3.8.2 Fenwick tree (i.e. BIT) eff. update + prefix sum calc. Can be generalized to arbitrary dimensions by duplicating loops.

```
1 // #include "header.h"
2 template < class T >
3 struct FenwickTree { // use 1 based indices !!!
4     int n ; vector<T> tree ;
5     FenwickTree ( int n ) : n ( n ) { tree .
6         assign ( n + 1 , 0 ) ; }
7     T query ( int l , int r ) { return query ( r
8         ) - query ( l - 1 ) ; }
9     T query ( int r ) {
10         T s = 0;
11         for ( ; r > 0; r -= ( r & ( - r ) ) ) s +=
12             tree [ r ];
13         return s ;
14     }
15     void update ( int i , T v ) {
16         for ( ; i <= n ; i += ( i & ( - i ) ) )
17             tree [ i ] += v ;
18     }
19 };
```

3.8.3 Trie

```
1 #include "header.h"
2 const int ALPHABET_SIZE = 26;
3 inline int mp(char c) { return c - 'a'; }
4 struct Node {
5     Node* ch[ALPHABET_SIZE];
6     bool isleaf = false;
7     Node() {
8         for(int i = 0; i < ALPHABET_SIZE; ++i) ch[i]
9             = nullptr;
10    }
11
12    void insert(string &s, int i = 0) {
13        if (i == s.length()) isleaf = true;
14        else {
15            int v = mp(s[i]);
16            if (ch[v] == nullptr)
```

```
16         ch[v] = new Node();
17         ch[v]->insert(s, i + 1);
18     }
19 }
20
21 bool contains(string &s, int i = 0) {
22     if (i == s.length()) return isleaf;
23     else {
24         int v = mp(s[i]);
25         if (ch[v] == nullptr) return false;
26         else return ch[v]->contains(s, i + 1);
27     }
28 }
29
30 void cleanup() {
31     for (int i = 0; i < ALPHABET_SIZE; ++i)
32         if (ch[i] != nullptr) {
33             ch[i]->cleanup();
34             delete ch[i];
35         }
36 }
37 };
```

3.8.4 Treap A binary tree whose nodes contain two values, a key and a priority, such that the key keeps the BST property

```
1 #include "header.h"
2 struct Node {
3     ll v;
4     int sz, pr;
5     Node *l = nullptr, *r = nullptr;
6     Node(ll val) : v(val), sz(1) { pr = rand(); }
7 };
8 int size(Node *p) { return p ? p->sz : 0; }
9 void update(Node* p) {
10     if (!p) return;
11     p->sz = 1 + size(p->l) + size(p->r);
12     // Pull data from children here
13 }
14 void propagate(Node *p) {
15     if (!p) return;
16     // Push data to children here
17 }
18 void merge(Node *&t, Node *l, Node *r) {
19     propagate(l), propagate(r);
20     if (!l) t = r;
21     else if (!r) t = l;
22     else if (l->pr > r->pr)
23         merge(l->r, l->l, r), t = l;
24     else merge(r->l, l, r->r), t = r;
25     update(t);
26 }
```

```

27 void spliti(Node *t, Node *&l, Node *&r, int
    index) {
28     propagate(t);
29     if (!t) { l = r = nullptr; return; }
30     int id = size(t->l);
31     if (index <= id) // id \in [index, \infty), so
        move it right
32     spliti(t->l, l, t->l, index), r = t;
33     else
34     spliti(t->r, t->r, r, index - id), l = t;
35     update(t);
36 }
37 void splitv(Node *t, Node *&l, Node *&r, ll val)
    {
38     propagate(t);
39     if (!t) { l = r = nullptr; return; }
40     if (val <= t->v) // t->v \in [val, \infty), so
        move it right
41     splitv(t->l, l, t->l, val), r = t;
42     else
43     splitv(t->r, t->r, r, val), l = t;
44     update(t);
45 }
46 void clean(Node *p) {
47     if (p) { clean(p->l), clean(p->r); delete p; }
48 }

```

3.8.5 Segment tree

```

1 #include "../header.h"
2 // example: SegmentTree<int, min> st(n, INT_MAX);
3 const int& addOp(const int& a, const int& b) {
4     static int result;
5     result = a + b;
6     return result;
7 }
8 template <class T, const T&(*op)(const T&, const
    T&)>
9 struct SegmentTree {
10     int n; vector<T> tree; T id;
11     SegmentTree(int _n, T _id) : n(_n), tree(2 * n,
        _id), id(_id) { }
12     void update(int i, T val) {
13         for (tree[i+n] = val, i = (i+n)/2; i > 0; i
            /= 2)
14             tree[i] = op(tree[2*i], tree[2*i+1]);
15     }
16     T query(int l, int r) {
17         T lhs = T(id), rhs = T(id);
18         for (l += n, r += n; l < r; l >>= 1, r >>= 1)
19             if (l&1) lhs = op(lhs, tree[l++]);
20             if (!(r&1)) rhs = op(tree[r--], rhs);
21     }

```

```

22     return op(l == r ? op(lhs, tree[l]) : lhs,
        rhs);
23 }
24 };

```

3.8.6 Lazy segment tree Optimizes range updates

```

1 #include "../header.h"
2 using T=int; using U=int; using I=int; //
    exclusive right bounds
3 T t_id; U u_id;
4 T op(T a, T b){ return a+b; }
5 void join(U &a, U b){ a+=b; }
6 void apply(T &t, U u, int x){ t+=x*u; }
7 T convert(const I &i){ return i; }
8 struct LazySegmentTree {
9     struct Node { int l, r, lc, rc; T t; U u;
10         Node(int l, int r, T t=t_id):l(l),r(r),lc(-1),rc(-1),t(t),u(u_id){}
11     };
12     int N; vector<Node> tree; vector<I> &init;
13     LazySegmentTree(vector<I> &init) : N(init.size()
        ), init(init){
14         tree.reserve(2*N-1); tree.push_back({0,N});
15         build(0, 0, N);
16     }
17     void build(int i, int l, int r) { auto &n =
        tree[i];
18         if (r > l+1) { int m = (l+r)/2;
19             n.lc = tree.size(); n.rc = n.lc+1;
20             tree.push_back({l,m}); tree.push_back({m,
                r});
21             build(n.lc,l,m); build(n.rc,m,r);
22             n.t = op(tree[n.lc].t, tree[n.rc].t);
23             } else n.t = convert(init[l]);
24     }
25     void push(Node &n, U u){ apply(n.t, u, n.r-n.l)
        ; join(n.u,u); }
26     void push(Node &n){push(tree[n.lc],n.u);push(
        tree[n.rc],n.u);n.u=u_id;}
27     T query(int l, int r, int i = 0) { auto &n =
        tree[i];
28         if(r <= n.l || n.r <= l) return t_id;
29         if(l <= n.l && n.r <= r) return n.t;
30         return push(n), op(query(l,r,n.lc),query(l,r,
            n.rc));
31     }
32     void update(int l, int r, U u, int i = 0) {
33         auto &n = tree[i];
34         if(r <= n.l || n.r <= l) return;
35         if(l <= n.l && n.r <= r) return push(n,u);
36         push(n); update(l,r,u,n.lc); update(l,r,u,n.
            rc);
37         n.t = op(tree[n.lc].t, tree[n.rc].t);

```

```

36 }
37 };

```

3.8.7 Dynamic segment tree Sparse, i.e., large values, i.e., not stored as an array

```

1 #include "../header.h"
2 using T=ll; using U=ll; // exclusive
    right bounds
3 T t_id; U u_id;
4 T op(T a, T b){ return a+b; }
5 void join(U &a, U b){ a+=b; }
6 void apply(T &t, U u, int x){ t+=x*u; }
7 T part(T t, int r, int p){ return t/r*p; }
8 struct DynamicSegmentTree {
9     struct Node { int l, r, lc, rc; T t; U u;
10         Node(int l, int r):l(l),r(r),lc(-1),rc(-1),t(
            t_id),u(u_id){}
11     };
12     vector<Node> tree;
13     DynamicSegmentTree(int N) { tree.push_back({0,N
        }); }
14     void push(Node &n, U u){ apply(n.t, u, n.r-n.l)
        ; join(n.u,u); }
15     void push(Node &n){push(tree[n.lc],n.u);push(
        tree[n.rc],n.u);n.u=u_id;}
16     T query(int l, int r, int i = 0) { auto &n =
        tree[i];
17         if(r <= n.l || n.r <= l) return t_id;
18         if(l <= n.l && n.r <= r) return n.t;
19         if(n.lc < 0) return part(n.t, n.r-n.l, min(n.
            r,r)-max(n.l,l));
20         return push(n), op(query(l,r,n.lc),query(l,r,
            n.rc));
21     }
22     void update(int l, int r, U u, int i = 0) {
23         auto &n = tree[i];
24         if(r <= n.l || n.r <= l) return;
25         if(l <= n.l && n.r <= r) return push(n,u);
26         if(n.lc < 0) { int m = (n.l + n.r) / 2;
27             n.lc = tree.size(); n.rc = n.lc+1;
28             tree.push_back({tree[i].l, m}); tree.
                push_back({m, tree[i].r});
29         }
30         push(tree[i]); update(l,r,u,tree[i].lc);
31         update(l,r,u,tree[i].rc);
32         tree[i].t = op(tree[tree[i].lc].t, tree[tree[
            i].rc].t);
33     }
34 };

```

3.8.8 Suffix tree

```

1 #include "../header.h"
2 using T = char;
3 using M = map<T,int>; // or array<T,ALPHABET_SIZE>
4 using V = string; // could be vector<T> as well
5 using It = V::const_iterator;
6 struct Node{
7     It b, e; M edges; int link; // end is exclusive
8     Node(It b, It e) : b(b), e(e), link(-1) {}
9     int size() const { return e-b; }
10 };
11 struct SuffixTree{
12     const V &s; vector<Node> t;
13     int root,n,len,remainder,llink; It edge;
14     SuffixTree(const V &s) : s(s) { build(); }
15     int add_node(It b, It e){ return t.push_back({b,e}), t.size()-1; }
16     int add_node(It b){ return add_node(b,s.end()); }
17     void link(int node){ if(llink) t[llink].link = node; llink = node; }
18     void build(){
19         len = remainder = 0; edge = s.begin();
20         n = root = add_node(s.begin(), s.begin());
21         for(auto i = s.begin(); i != s.end(); ++i){
22             ++remainder; llink = 0;
23             while(remainder){
24                 if(len == 0) edge = i;
25                 if(t[n].edges[*edge] == 0){
26                     t[n].edges[*edge] = add_node(i); link(n);
27                 } else {
28                     auto x = t[n].edges[*edge];
29                     if(len >= t[x].size()){
30                         len -= t[x].size(); edge += t[x].size(); n = x;
31                         continue;
32                     }
33                     if(*(t[x].b + len) == *i){
34                         ++len; link(n); break;
35                     }
36                     auto split = add_node(t[x].b, t[x].b + len);
37                     t[n].edges[*edge] = split;
38                     t[x].b += len;
39                     t[split].edges[*i] = add_node(i);
40                     t[split].edges[*t[x].b] = x;
41                     link(split);
42                 }
43             }
44             --remainder;
45             if(n == root && len > 0)
46                 --len, edge = i - remainder + 1;
47             else n = t[n].link > 0? t[n].link: root;
48         }
49     }
50 };

```

3.8.9 UnionFind

```

1 #include "header.h"
2 struct UnionFind {
3     std::vector<int> par, rank, size;
4     int c;
5     UnionFind(int n) : par(n), rank(n, 0), size(n, 1), c(n) {
6         for(int i = 0; i < n; ++i) par[i] = i;
7     }
8     int find(int i) { return (par[i] == i ? i : (par[i] = find(par[i]))); }
9     bool same(int i, int j) { return find(i) == find(j); }
10    int get_size(int i) { return size[find(i)]; }
11    int count() { return c; }
12    int merge(int i, int j) {
13        if((i = find(i)) == (j = find(j))) return -1;
14        --c;
15        if(rank[i] > rank[j]) swap(i, j);
16        par[i] = j;
17        size[j] += size[i];
18        if(rank[i] == rank[j]) rank[j]++;
19        return j;
20    }
21 };

```

3.8.10 Indexed set Similar to set, but allows accessing elements by index using `find_by_order()` in $O(\log n)$

```

1 #include "../header.h"
2 #include <ext/pb_ds/assoc_container.hpp>
3 using namespace __gnu_pbds;
4 using namespace std;
5 typedef tree<int,null_type,less<int>,rb_tree_tag,tree_order_statistics_node_update> indexed_set;

```

3.8.11 Order Statistics Tree A set (not multiset!) with support for finding the n 'th element, and finding the index of an element. To get a map, change `null_type` to `O(log N)`

```

1 #include <bits/extc++.h> // !!!!
2 using namespace __gnu_pbds;
3 using namespace std;
4

```

```

5 template<class T>
6 using Tree = tree<T, null_type, less<T>, rb_tree_tag, tree_order_statistics_node_update>;
7
8 void example() {
9     Tree<int> t, t2; t.insert(8);
10    auto it = t.insert(10).first;
11    assert(it == t.lower_bound(9));
12    assert(t.order_of_key(10) == 1);
13    assert(t.order_of_key(11) == 2);
14    assert(*t.find_by_order(0) == 8);
15    t.join(t2); // assuming T < T2 or T > T2, merge t2 into t
16 }

```

3.8.12 Range minimum queries Answers range minimum queries in constant time after $O(V \log V)$ preproc.

```

1 template<class T>
2 struct RMQ {
3     vector<vector<T>> jmp;
4     RMQ(const vector<T>& V) : jmp(1, V) {
5         for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
6             jmp.emplace_back(sz(V) - pw * 2 + 1);
7             rep(j,0,sz(jmp[k]))
8                 jmp[k][j]=min(jmp[k-1][j],jmp[k-1][j+pw]);
9         }
10    }
11    T query(int a, int b) { // returns min(V[a], ..., V[b-1])
12        assert(a<b); // or return inf if a == b
13        int dep = 31 - __builtin_clz(b-a);
14        return min(jmp[dep][a], jmp[dep][b-(1<<dep)]);
15    }
16 };

```

4 Other Mathematics

4.1 Helpful functions

4.1.1 Euler's Totient Function $n = p_1^{k_1-1} \cdot (p_1 - 1) \cdot \dots \cdot p_r^{k_r-1} \cdot (p_r - 1)$, where $p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ is the prime factorization of n .

```

1 # include "header.h"
2 ll phi(ll n) { // \Phi(n)
3     ll ans = 1;

```



```

4   for (ll i = 2; i*i <= n; i++) {
5       if (n % i == 0) {
6           ans *= i-1;
7           n /= i;
8           while (n % i == 0) {
9               ans *= i;
10              n /= i;
11          }
12      }
13  }
14  if (n > 1) ans *= n-1;
15  return ans;
16 }
17 vi phi(int n) { // All \Phi(i) up to n
18     vi phi(n+1, 0LL);
19     iota(phi.begin(), phi.end(), 0LL);
20     for (ll i = 2LL; i <= n; ++i)
21         if (phi[i] == i)
22             for (ll j = i; j <= n; j += i)
23                 phi[j] -= phi[j] / i;
24     return phi;
25 }

```

4.1.2 Totient (again but .py)

```

1 def totatives(n):
2     if n == 1:
3         return 1
4     phi = int(n > 1 and n)
5     for p in range(2, int(n**.5) + 1):
6         if not n % p:
7             phi -= phi // p
8             while not n % p:
9                 n //= p
10    #if n is > 1 it means it is prime
11    if n > 1: phi -= phi // n
12    return phi

```

Formulas $\Phi(n)$ counts all numbers in $1, \dots, n-1$ coprime to n .

$a^{\varphi(n)} \equiv 1 \pmod n$, a and n are coprimes.

$\forall e > \log_2 m: n^e \pmod m = n^{\Phi(m)+e} \pmod{\Phi(m)} \pmod m$.

$\gcd(m, n) = 1 \Rightarrow \Phi(m \cdot n) = \Phi(m) \cdot \Phi(n)$.

4.1.3 Pascal's trinagle $\binom{n}{k}$ is k -th element in the n -th row, indexing both from 0

```

1 #include "header.h"
2 void printPascal(int n) {
3     for (int line = 1; line <= n; line++) {

```

```

4         int C = 1; // used to represent C(line, i)
5         for (int i = 1; i <= line; i++) {
6             cout << C << " ";
7             C = C * (line - i) / i;
8         }
9         cout << "\n";
10    }
11 }

```

4.2 Theorems and definitions

Subfactorial (Derangements) Permutations of a set such that none of the elements appear in their original position:

$$!n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}$$

$$!(0) = 1, !n = n \cdot !(n-1) + (-1)^n$$

$$!n = (n-1)(!(n-1) + !(n-2)) = \left\lfloor \frac{n!}{e} \right\rfloor \quad (1)$$

$$!n = 1 - e^{-1}, n \rightarrow \infty \quad (2)$$

Binomials and other partitionings

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \prod_{i=1}^k \frac{n-i+1}{i}$$

This last product may be computed incrementally since any product of k' consecutive values is divisible by $k'!$.

Basic identities: The hockeystick identity:

$$\sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}$$

or

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Also

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

For $n, m \geq 0$ and p prime: write n, m in base p , i.e. $n = n_k p^k + \dots + n_1 p + n_0$ and $m = m_k p^k + \dots + m_1 p + m_0$. Then by Lucas theorem we have $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod p$, with the convention that $n_i < m_i \Rightarrow \binom{n_i}{m_i} = 0$.

Fibonacci (See also number theory section)

$$\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\sum_{i=1}^n F_i = F_{n+2} - 1, \sum_{i=1}^n F_i^2 = F_n F_{n+1}$$

$$\gcd(F_m, F_n) = F_{\gcd(m, n)}$$

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$$

Bit stuff $a + b = a \oplus b + 2(a \& b) = a|b + a \& b$.

k th bit is set in x iff $x \bmod 2^{k-1} \geq 2^k$, or iff $x \bmod 2^{k-1} - x \bmod 2^k \neq 0$ (i.e. $= 2^k$) It comes handy when you need to look at the bits of the numbers which are pair sums or subset sums etc.

$n \bmod 2^i = n \& (2^i - 1)$.

$\forall k: 1 \oplus 2 \oplus \dots \oplus (4k-1) = 0$

4.3 Geometry Formulas

Euler: $1 + CC = V - E + F$

Pick: $\text{Area} = \text{itr pts} + \frac{\text{bdry pts}}{2} - 1$

Given a non-self-intersecting closed polygon on n vertices, given as (x_i, y_i) , its centroid (C_x, C_y) is given as:

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i),$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) = \text{polygon area}$$

Inclusion-Exclusion For appropriate f compute $\sum_{S \subseteq T} (-1)^{|T \setminus S|} f(S)$, or if only the size of S matters, $\sum_{s=0}^n (-1)^{n-s} \binom{n}{s} f(s)$. In some contexts we might use Stirling numbers, not binomial coefficients!

Some useful applications:

Graph coloring Let $I(S)$ count the number of independent sets contained in $S \subseteq V$ ($I(\emptyset) = 1$, $I(S) = I(S \setminus v) + I(S \setminus N(v))$). Let $c_k = \sum_{S \subseteq V} (-1)^{|V \setminus S|} I(S)$. Then V is k -colorable iff $v > 0$. Thus we can compute the chromatic number of a graph in $O^*(2^n)$ time.

Burnside's lemma Given a group G acting on a set X , the number of elements in X up to symmetry is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

with X^g the elements of X invariant under g . For example, if $f(n)$ counts “configurations” of some sort of length n , and we want to count them up to rotational symmetry using $G = \mathbb{Z}/n\mathbb{Z}$, then

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k)$$

I.e. for coloring with c colors we have $f(k) = k^c$.

Relatedly, in Pólya's enumeration theorem we imagine X as a set of n beads with G permuting the beads (e.g. a necklace, with G all rotations and reflections of the n -cycle, i.e. the dihedral group D_n). Suppose further that we had Y colors, then the number of G -invariant colorings Y^X/G is counted by

$$\frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)}$$

with $c(g)$ counting the number of cycles of g when viewed as a permutation of X . We can generalize this to a weighted version: if the color i can occur exactly r_i times, then this is counted by the coefficient of $t_1^{r_1} \dots t_n^{r_n}$ in the polynomial

$$Z(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \dots + t_n^m)^{c_m(g)}$$

where $c_m(g)$ counts the number of length m cycles in g acting as a permutation on X . Note we get the original formula by setting all $t_i = 1$. Here Z is the cycle index. Note: you can cleverly deal with even/odd sizes by setting some t_i to -1 .

Lucas Theorem If p is prime, then:

$$\frac{p^a}{k} \equiv 0 \pmod{p}$$

Thus for non-negative integers $m = m_k p^k + \dots + m_1 p + m_0$ and $n = n_k p^k + \dots + n_1 p + n_0$:

$$\frac{m}{n} = \prod_{i=0}^k \frac{m_i}{n_i} \pmod{p}$$

Note: The fraction's mean integer division.

4.4 Recurrences

If $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$, and r_1, \dots, r_k are distinct roots of $x^k - c_1 x^{k-1} - \dots - c_k$, there are d_1, \dots, d_k s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1 n + d_2) r^n$.

4.5 Sums

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

4.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

4.7 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

4.8 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \frac{a+b+c}{2}$

Area:

$$[ABC] = rp = \frac{1}{2} ab \sin \gamma$$

$$= \frac{abc}{4R} = \sqrt{p(p-a)(p-b)(p-c)} = \frac{1}{2} |(B-A, C-A)^T|$$

Circumradius: $R = \frac{abc}{4A}$, Inradius: $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two): $s_a =$

$$\sqrt{bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right]}$$

Law of tangents: $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

4.9 Trigonometry $\frac{\tan v + \tan w}{\tan(v+w)} = \frac{1 - \tan v \tan w}{1 - \tan v \tan w}$

$$\sin v + \sin w = 2 \sin \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$\cos v + \cos w = 2 \cos \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$(V+W) \tan(v-w)/2 = (V-W) \tan(v+w)/2$$

where V, W are lengths of sides opposite angles v, w .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}$, $\phi = \text{atan2}(b, a)$.

4.10 Combinatorics

Combinations and Permutations

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$C(n, r) = C(n, n-r)$$

4.11 Cycles

Let $g_S(n)$ be the number of n -permutations whose cycle lengths all belong to the set S . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp \left(\sum_{n \in S} \frac{x^n}{n} \right)$$

4.12 Labeled unrooted trees

on n vertices: n^{n-2}

on k existing trees of size n_i : $n_1 n_2 \dots n_k n^{k-2}$

with degrees d_i : $(n-2)! / ((d_1-1)! \dots (d_n-1)!)$

4.13 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	$\sim 2e5$	$\sim 2e8$

4.14 Numbers

Bernoulli numbers EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t - 1}$ (FFT-able). $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^{\infty} f(i) &= \int_m^{\infty} f(x) dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^{\infty} f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

Stirling's numbers First kind: $S_1(n, k)$ count permutations on n items with k cycles. $S_1(n, k) = S_1(n-1, k-1) + (n-1)S_1(n-1, k)$ with $S_1(0, 0) = 1$. Note:

$$\sum_{k=0}^n S_1(n, k) x^k = x(x+1) \dots (x+n-1)$$

$$\sum_{k=0}^n S_1(n, k) = n!$$

$$S_1(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$$

$$S_1(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$$

Second kind: $S_2(n, k)$ count partitions of n distinct elements into exactly k non-empty groups.

$$S_2(n, k) = S_2(n-1, k-1) + k S_2(n-1, k)$$

$$S_2(n, 1) = S_2(n, n) = 1$$

$$S_2(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

Catalan Numbers - Number of correct bracket sequence consisting of n opening and n closing brackets.

The number of ways to completely parenthesize $n+1$ factors.

The number of triangulations of a convex polygon with $n+2$ sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).

The number of ways to connect the $2n$ points on a circle to form n disjoint i.e. non-intersecting chords.

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$C_0 = 1, \quad C_1 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Narayana numbers The number of expressions containing n pairs of parentheses, which are correctly matched and which contain k distinct nestings.

$$N(n, k) = \frac{1}{n} \frac{n}{k} \frac{n}{k-1}$$

Eulerian numbers Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j :s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j :s s.t. $\pi(j) \geq j$, k j :s s.t. $\pi(j) > j$.

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

Bell numbers Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$ For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad C_{n+1} = \sum C_i C_{n-i}$$

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

4.15 Probability

Stochastic variables

$$P(X = r) = C(n, r) \cdot p^r \cdot (1-p)^{n-r}$$

Bayes' Theorem $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(B)}$$

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)}$$

Expectation Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x . It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y ,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

4.16 Number Theory

Bezout's Theorem

$$a, b \in \mathbb{Z}^+ \implies \exists s, t \in \mathbb{Z} : \gcd(a, b) = sa + tb$$

Bézout's identity For $a \neq 0$, $b \neq 0$, then $d = \gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If (x, y) is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a, b)}, y - \frac{ka}{\gcd(a, b)} \right), \quad k \in \mathbb{Z}$$

Partial Coprime Divisor Property

$$(\gcd(a, b) = 1) \wedge (a \mid bc) \implies (a \mid c)$$

Coprime Modulus Equivalence Property

$$(\gcd(c, m) = 1) \wedge (ac \equiv bc \pmod{m}) \implies (a \equiv b \pmod{m})$$

Fermat's Little Theorem

$$(\text{prime}(p)) \wedge (p \nmid a) \implies (a^{p-1} \equiv 1 \pmod{p})$$

$$(\text{prime}(p)) \implies (a^p \equiv a \pmod{p})$$

Euler's Theorem

$$a^{\phi(m)-1} \equiv a^{-1} \pmod{m}, \text{ if } \gcd(a, m) = 1$$

$$a^{-1} \equiv a^{m-2} \pmod{m}, \text{ if } m \text{ is prime}$$

Pythagorean Triples The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

Primes $p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a , except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

Estimates $\sum_{d|n} d = O(n \log \log n)$.

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$

$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$$

$$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$$

4.17 Discrete distributions

Binomial distribution The number of successes in n independent yes/no experiments, each which yields success with probability p is $\text{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$ is approximately $\text{Po}(np)$ for small p .

First success distribution The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability p is $\text{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\text{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

4.18 Continuous distributions

Uniform distribution If the probability density function is constant between a and b and 0 elsewhere it is $\text{U}(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution The time between events in a Poisson process is $\text{Exp}(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$