1 Setup

1.0.1 Tips Test session: Check `__int128` and GNU builtins.

C++ var. limits: int -2^{31}, $2^{31}-1$
ll -2^{63}, $2^{63}-1$
ull 0, $2^{64}-1$
`__int128` -2^{127}, $2^{127}-1$
ld $-1.7e308$, $1.7e308$, 18 digits precision

1.0.2 header.h

```cpp
#pragma once
#include <bits/stdc++.h>
using namespace std;

#define ll long long
#define ull unsigned ll
#define ld long double
#define pl pair<ll, ll>
#define pi pair<int, int>
#define vl vector<ll>
#define vi vector<int>
#define vb vector<bool>
#define vvi vector<vi>
#define vvl vector<vl>
#define vpl vector<pl>
#define vpi vector<pi>
#define vld vector<ld>
#define vvpi vector<vpi>
#define in(el, cont) (cont.find(el) != cont.end()
    )// sets/maps
#define all(x) x.begin(), x.end()

constexpr int INF  = 2000000010;
constexpr ll LLINF = 9000000000000000010LL;

// int main() {
//   ios::sync_with_stdio(false);  // do not use
    cout + printf
//   cin.tie(NULL);
//   cout << fixed << setprecision(12);
//   return 0;
// }
```

1.0.3 Aux. helper C++

```cpp
#include "header.h"

int main() {
    // Read in a line including white space
    string line;
    getline(cin, line);
    // When doing the above read numbers as
        follows:
    int n;
    getline(cin, line);
    stringstream ss(line);
    ss >> n;

    // Count the number of 1s in binary
        represnatation of a number
    ull number;
    __builtin_popcountll(number);
}

// __int128
using lll = __int128;
ostream& operator<<( ostream& o, __int128 n ) {
    auto t = n<0 ? -n : n; char b[128], *d = end(b)
        ;
    do *--d = '0'+t%10, t /= 10; while (t);
    if(n<0) *--d = '-';
    o.rdbuf()->sputn(d,end(b)-d);
    return o;
}
```

1.0.4 Aux. helper python

```python
from functools import lru_cache

# Read until EOF
while True:
    try:
        pattern = input()
    except EOFError:
        break

@lru_cache(maxsize=None)
def smth_memoi(i, j, s):
    # Example in-built cache
    return "sol"

# Fast I
import io, os
def fast_io():
    finput = io.BytesIO(os.read(0,
        os.fstat(0).st_size)).readline
    s = finput().decode()
    return s

# Fast O
import sys
def fast_out():
    n = 5
    sys.stdout.write(str(n)+"\n")
```

2 Python

2.1 Graphs

2.1.1 BFS

```python
from collections import deque
def bfs(g, roots, n):
    q = deque(roots)
    explored = set()
    distances = [0 if v in roots else float('inf'
        ) for v in range(n)]

    while len(q) != 0:
        node = q.popleft()
        if node in explored: continue
        explored.add(node)
        for neigh in g[node]:
            if neigh not in explored:
                q.append(neigh)
                distances[neigh] = distances[node
                    ] + 1
    return distances
```

2.1.2 Dijkstra

```python
from heapq import *
def dijkstra(n, root, g):  # g = {node: (cost,
    neigh)}
    dist = [float("inf")]*n
    dist[root] = 0
    prev = [-1]*n

    pq = [(0, root)]
    heapify(pq)
    visited = set([])

    while len(pq) != 0:
        _, node = heappop(pq)

        if node in visited: continue
        visited.add(node)

        # In case of disconnected graphs
        if node not in g:
            continue

        for cost, neigh in g[node]:
            alt = dist[node] + cost
```

```
23        if alt < dist[neigh]:
24          dist[neigh] = alt
25          prev[neigh] = node
26          heappush(pq, (alt, neigh))
27    return dist
```

2.1.3 Topological Sort topological sorting of a DAG

```
1  from collections import defaultdict
2
3  class Graph:
4      def __init__(self,vertices):
5          self.graph = defaultdict(list) #adjacency
               List
6          self.V = vertices #No. V
7
8      def addEdge(self,u,v):
9          self.graph[u].append(v)
10
11     def topologicalSortUtil(self,v,visited,stack)
               :
12         visited[v] = True
13         # Recur for all the vertices adjacent to
               this vertex
14         for i in self.graph[v]:
15             if visited[i] == False:
16                 self.topologicalSortUtil(i,
                       visited,stack)
17         stack.insert(0,v)
18
19     def topologicalSort(self):
20         visited = [False]*self.V
21         stack =[]
22         for i in range(self.V):
23             if visited[i] == False:
24                 self.topologicalSortUtil(i,
                       visited,stack)
25         return stack
26
27     def isCyclicUtil(self, v, visited, recStack):
28         visited[v] = True
29         recStack[v] = True
30         for neighbour in self.graph[v]:
31             if visited[neighbour] == False:
32                 if self.isCyclicUtil(neighbour,
                       visited, recStack) == True:
33                     return True
34             elif recStack[neighbour] == True:
35                 return True
36         recStack[v] = False
37         return False
38
39     def isCyclic(self):
40         visited = [False] * (self.V + 1)
```

```
41         recStack = [False] * (self.V + 1)
42         for node in range(self.V):
43             if visited[node] == False:
44                 if self.isCyclicUtil(node,
                       visited, recStack) == True:
45                     return True
46         return False
```

2.1.4 Kruskal (UnionFind) Min. span. tree

```
1  class UnionFind:
2      def __init__(self, n):
3          self.parent = [-1]*n
4
5      def find(self, x):
6          if self.parent[x] < 0:
7              return x
8          self.parent[x] = self.find(self.parent[x
               ])
9          return self.parent[x]
10
11     def connect(self, a, b):
12         ra = self.find(a)
13         rb = self.find(b)
14         if ra == rb:
15             return False
16         if self.parent[ra] > self.parent[rb]:
17             self.parent[rb] += self.parent[ra]
18             self.parent[ra] = rb
19         else:
20             self.parent[ra] += self.parent[rb]
21             self.parent[rb] = ra
22         return True
23
24 # Full MST is len(spanning==n-1)
25 def kruskal(n, edges):
26     uf = UnionFind(n)
27     spanning = []
28     edges.sort(key = lambda d: -d[2])
29     while edges and len(spanning) < n-1:
30         u, v, w = edges.pop()
31         if not uf.connect(u, v):
32             continue
33         spanning.append((u, v, w))
34     return spanning
```

2.1.5 Prim Min. span. tree - good for dense graphs

```
1  from heapq import heappush, heappop, heapify
2  def prim(G, n):
3    s = next(iter(G.keys()))
4    V = set([s])
5    M = []
```

```
6    c = 0
7
8    E = [(w,s,v) for v,w in G[s].items()]
9    heapify(E)
10
11   while E and len(M) < n-1:
12     w,u,v = heappop(E)
13     if v in V: continue
14     V.add(v)
15     M.append((u,v))
16     c += w
17     u = v
18     [heappush(E,(w,u,v)) for v,w in G[u].items()
             if v not in V]
19
20   if len(M) == n-1:
21     return M, c
22   else:
23     return None, None
```

2.2 Num. Th. / Comb.

2.2.1 nCk % prime p must be prime and k < p

```
1  def fermat_binom(n, k, p):
2      if k > n:
3          return 0
4      num = 1
5      for i in range(n-k+1, n+1):
6          num *= i % p
7      num %= p
8      denom = 1
9      for i in range(1,k+1):
10         denom *= i % p
11     denom %= p
12     # numerator * denominator^(p-2) (mod p)
13     return (num * pow(denom, p-2, p)) % p
```

2.2.2 Sieve of E. $O(n)$ so actually faster than C++ version, but more memory

```
1  MAX_SIZE = 10**8+1
2  isprime = [True] * MAX_SIZE
3  prime = []
4  SPF = [None] * (MAX_SIZE)
5
6  def manipulated_seive(N): # Up to N (not
       included)
7    isprime[0] = isprime[1] = False
8    for i in range(2, N):
9      if isprime[i] == True:
10       prime.append(i)
```

```
11        SPF[i] = i
12     j = 0
13     while (j < len(prime) and
14     i * prime[j] < N and
15        prime[j] <= SPF[i]):
16        isprime[i * prime[j]] = False
17        SPF[i * prime[j]] = prime[j]
18        j += 1
```

2.2.3   Modular Inverse of a mod b

```
1 def modinv(a, b):
2   if b == 1: return 1
3   b0, x0, x1 = b, 0, 1
4   while a > 1:
5     q, a, b = a//b, b, a%b
6     x0, x1 = x1 - q * x0, x0
7   if x1 < 0: x1 += b0
8   return x1
```

2.2.4   Chinese rem. an x such that $\forall$ y,m: yx = 1 mod m requires all m,m' to be ¿=1 and coprime

```
1 def chinese_remainder(ys, ms):
2   N, x = 1, 0
3   for m in ms: N*=m
4   for y,m in zip(ys,ms):
5     n = N // m
6     x += n * y * modinv(n, m)
7   return x % N
```

2.2.5   Bezout

```
1 def bezout_id(a, b):
2   r,x,s,y,t,z = b,a,0,1,1,0
3   while r:
4       q = x // r
5       x, r = r, x % r
6       y, s = s, y - q * s
7       z, t = t, z - q * t
8     return y % (b // x), z % (-a // x)
```

2.2.6   Gen. chinese rem.

```
1 def general_chinese_remainder(a,b,m,n):
2   g = gcd(m,n)
3
4   if a == b and m == n:
5     return a, m
6   if (a % g) != (b % g):
```

```
7     return None, None
8
9   u,v = bezout_id(m,n)
10   x = (a*v*n + b*u*m) // g
11   return int(x) % lcm(m,n), int(lcm(m,n))
```

2.3   Strings

2.3.1   Longest   common   substr.   (Consecutive) $O(mn)$ time, $O(m)$ space

```
1 from functools import lru_cache
2 @lru_cache
3 def lcs(s1, s2):
4     if len(s1) == 0 or len(s2) == 0:
5         return 0
6     return max(
7       lcs(s1[:-1], s2), lcs(s1, s2[:-1]),
8       (s1[-1] == s2[-1]) + lcs(s1[:-1], s2[:-1])
9     )
```

2.3.2   Longest common subseq. (Non-consecutive)

```
1 def longestCommonSubsequence(text1, text2):
2     n = len(text1)
3     m = len(text2)
4     prev = [0] * (m + 1)
5     cur = [0] * (m + 1)
6     for idx1 in range(1, n + 1):
7         for idx2 in range(1, m + 1):
8             # matching
9             if text1[idx1 - 1] == text2[idx2 -
                 1]:
10                cur[idx2] = 1 + prev[idx2 - 1]
11            else:
12                # not matching
13                cur[idx2] = max(cur[idx2 - 1],
                    prev[idx2])
14        prev = cur.copy()
15    return cur[m]
```

2.3.3   KMP Return all matching pos. of P in T

```
1 class KMP:
2     def partial(self, pattern):
3         """ Calc. partial match table: String ->
            [Int]"""
4         ret = [0]
5         for i in range(1, len(pattern)):
6             j = ret[i - 1]
```

```
7             while j > 0 and pattern[j] != pattern
                 [i]: j = ret[j - 1]
8             ret.append(j + 1 if pattern[j] ==
                 pattern[i] else j)
9         return ret
10
11    def search(self, T, P):
12        """KMPString -> String -> [Int]"""
13        partial, ret, j = self.partial(P), [], 0
14        for i in range(len(T)):
15            while j > 0 and T[i] != P[j]: j =
                 partial[j - 1]
16            if T[i] == P[j]: j += 1
17            if j == len(P):
18                ret.append(i - (j - 1))
19                j = partial[j - 1]
20        return ret
```

2.3.4   Suffix Array

```
1 class Entry:
2     def __init__(self, pos, nr):
3         self.p = pos
4         self.nr = nr
5
6     def __lt__(self, other):
7         return self.nr < other.nr
8
9 class SA:
10    def __init__(self, s):
11        self.P = []
12        self.n = len(s)
13        self.build(s)
14
15    def build(self, s): # n log log n
16        n = self.n
17        L = [Entry(0, 0) for _ in range(n)]
18        self.P = []
19        self.P.append([ord(c) for c in s])
20        step = 1
21        count = 1
22
23        # self.P[step][i] stores the position
24        # of the i-th longest suffix
25        # if suffixes are sorted according to
26        # their first 2^step characters.
27        while count < 2 * n:
28            self.P.append([0] * n)
29            for i in range(n):
30                nr = (self.P[step - 1][i],
31                    self.P[step - 1][i +
                        count]
32                    if i + count < n else -1)
33                L[i].p = i
```

```
34              L[i].nr = nr
35          L.sort()
36          for i in range(n):
37              if i > 0 and L[i].nr == L[i -
                    1].nr:
38                  self.P[step][L[i].p] = \
39                      self.P[step][L[i - 1].p]
40              else:
41                  self.P[step][L[i].p] = i
42          step += 1
43          count *= 2
44
45      self.sa = [0] * n
46      for i in range(n):
47          self.sa[self.P[-1][i]] = i
```

2.3.5 Longest common pref.

with the suffix array built we can do, e.g., longest common prefix of x, y with suffixarray where x,y are suffixes of the string used $O(\log n)$

```
1 def lcp(x, y, P):
2     res = 0
3     if x == y:
4         return n - x
5     for k in range(len(P) - 1, -1, -1):
6         if x >= n or y >= n:
7             break
8         if P[k][x] == P[k][y]:
9             x += 1 << k
10            y += 1 << k
11            res += 1 << k
12    return res
```

2.3.6 Edit distance

```
1 def editDistance(str1, str2):
2   m = len(str1)
3   n = len(str2)
4   curr = [0] * (n + 1)
5   for j in range(n + 1):
6     curr[j] = j
7   previous = 0
8   # dp rows
9   for i in range(1, m + 1):
10    previous = curr[0]
11    curr[0] = i
12
13    # dp cols
14    for j in range(1, n + 1):
15      temp = curr[j]
16      if str1[i - 1] == str2[j - 1]:
17        curr[j] = previous
18      else:
19        curr[j] = 1 + min(previous, curr[j - 1],
                    curr[j])
20    previous = temp
21  return curr[n]
```

2.3.7 Bitstring

Slower than a set for many elements, but hashable

```
1 def add_element(bit_string, index):
2     return bit_string | (1 << index)
3
4 def remove_element(bit_string, index):
5     return bit_string & ~(1 << index)
6
7 def contains_element(bit_string, index):
8     return (bit_string & (1 << index)) != 0
```

2.4 Geometry

2.4.1 Convex Hull

```
1 def vec(a,b):
2     return (b[0]-a[0],b[1]-a[1])
3 def det(a,b):
4     return a[0]*b[1] - b[0]*a[1]
5
6 def convexhull(P):
7     if (len(P) == 1):
8         return [(p[0][0], p[0][1])]
9
10    h = sorted(P)
11    lower = []
12    i = 0
13    while i < len(h):
14        if len(lower) > 1:
15            a = vec(lower[-2], lower[-1])
16            b = vec(lower[-1], h[i])
17            if det(a,b) <= 0 and len(lower) > 1:
18                lower.pop()
19                continue
20        lower.append(h[i])
21        i += 1
22
23    upper = []
24    i = 0
25    while i < len(h):
26        if len(upper) > 1:
27            a = vec(upper[-2], upper[-1])
28            b = vec(upper[-1], h[i])
29            if det(a,b) >= 0:
30                upper.pop()
31                continue
32        upper.append(h[i])
33        i += 1
34
35    reversedupper = list(reversed(upper[1:-1:]))
36    reversedupper.extend(lower)
37    return reversedupper
```

2.4.2 Geometry

```
1
2 def vec(a,b):
3     return (b[0]-a[0],b[1]-a[1])
4
5 def det(a,b):
6     return a[0]*b[1] - b[0]*a[1]
7
8     lower = []
9     i = 0
10    while i < len(h):
11        if len(lower) > 1:
12            a = vec(lower[-2], lower[-1])
13            b = vec(lower[-1], h[i])
14            if det(a,b) <= 0 and len(lower) > 1:
15                lower.pop()
16                continue
17        lower.append(h[i])
18        i += 1
19
20    # find upper hull
21    # det <= 0 -> replace
22    upper = []
23    i = 0
24    while i < len(h):
25        if len(upper) > 1:
26            a = vec(upper[-2], upper[-1])
27            b = vec(upper[-1], h[i])
28            if det(a,b) >= 0:
29                upper.pop()
30                continue
31        upper.append(h[i])
32        i += 1
```

2.5 Other Algorithms

2.5.1 Rotate matrix

```
1 def rotate_matrix(m):
2     return [[m[j][i] for j in range(len(m))] for
            i in range(len(m[0])-1,-1,-1)]
```

2.6 Other Data Structures

2.6.1 Segment Tree

```python
N = 100000 # arr max size
tree = [0] * (2 * N) # tre max size

def build(arr, n):
    for i in range(n):
        tree[n + i] = arr[i]

    for i in range(n - 1, 0, -1):
        tree[i] = tree[i << 1] + tree[i << 1 | 1]

def updateTreeNode(p, value, n):
    tree[p + n] = value
    p = p + n

    i = p   # move upward, update parents
    while i > 1:
        tree[i >> 1] = tree[i] + tree[i ^ 1]
        i >>= 1

def query(l, r, n):   # sum [l, r)
    res = 0
    l += n
    r += n
    while l < r:
        if l & 1:
            res += tree[l]
            l += 1
        if r & 1:
            r -= 1
            res += tree[r]
        l >>= 1
        r >>= 1
    return res
```

2.6.2 Trie

```python
class TrieNode:
    def __init__(self):
        self.children = [None]*26
        self.isEndOfWord = False

class Trie:
    def __init__(self):
        self.root = self.getNode()

    def getNode(self):
        return TrieNode()

    def _charToIndex(self,ch):
        return ord(ch)-ord('a')
```

```python
    def insert(self,key):
        pCrawl = self.root
        length = len(key)
        for level in range(length):
            index = self._charToIndex(key[level])
            if not pCrawl.children[index]:
                pCrawl.children[index] = self.
                    getNode()
            pCrawl = pCrawl.children[index]
        pCrawl.isEndOfWord = True

    def search(self, key):
        pCrawl = self.root
        length = len(key)
        for level in range(length):
            index = self._charToIndex(key[level])
            if not pCrawl.children[index]:
                return False
            pCrawl = pCrawl.children[index]

        return pCrawl.isEndOfWord
```

3 C++

3.1 Graphs

3.1.1 BFS

```cpp
#include "header.h"
#define graph unordered_map<ll, unordered_set<ll
    >>
vi bfs(int n, graph& g, vi& roots) {
    vi parents(n+1, -1); // nodes are 1..n
    unordered_set<int> visited;
    queue<int> q;
    for (auto x: roots) {
        q.emplace(x);
        visited.insert(x);
    }
    while (not q.empty()) {
        int node = q.front();
        q.pop();

        for (auto neigh: g[node]) {
            if (not in(neigh, visited)) {
                parents[neigh] = node;
                q.emplace(neigh);
                visited.insert(neigh);
            }
        }
    }
```

```cpp
    return parents;
}
vi reconstruct_path(vi parents, int start, int
    goal) {
    vi path;
    int curr = goal;
    while (curr != start) {
        path.push_back(curr);
        if (parents[curr] == -1) return vi(); //
            No path, empty vi
        curr = parents[curr];
    }
    path.push_back(start);
    reverse(path.begin(), path.end());
    return path;
}
```

3.1.2 DFS Cycle detection / removal

```cpp
#include "header.h"
void removeCyc(ll node, unordered_map<ll, vector<
    pair<ll, ll>>>& neighs, vector<bool>& visited
    ,
vector<bool>& recStack, vector<ll>& ans) {
    if (!visited[node]) {
        visited[node] = true;
        recStack[node] = true;
        auto it = neighs.find(node);
        if (it != neighs.end()) {
            for (auto util: it->second) {
                ll nnode = util.first;
                if (recStack[nnode]) {
                    ans.push_back(util.second);
                } else if (!visited[nnode]) {
                    removeCyc(nnode, neighs,
                        visited, recStack, ans);
                }
            }
        }
    }
    recStack[node] = false;
}
```

3.1.3 Dijkstra

```cpp
#include "header.h"
vector<int> dijkstra(int n, int root, map<int,
    vector<pair<int, int>>>& g) {
    unordered_set<int> visited;
    vector<int> dist(n, INF);
    priority_queue<pair<int, int>> pq;
    dist[root] = 0;
    pq.push({0, root});
```

```
8      while (!pq.empty()) {
9          int node = pq.top().second;
10         int d = -pq.top().first;
11         pq.pop();
12
13         if (in(node, visited)) continue;
14         visited.insert(node);
15
16         for (auto e : g[node]) {
17             int neigh = e.first;
18             int cost = e.second;
19             if (dist[neigh] > dist[node] + cost)
                   {
20                 dist[neigh] = dist[node] + cost;
21                 pq.push({-dist[neigh], neigh});
22             }
23         }
24     }
25     return dist;
26 }
```

3.1.4   Floyd-Warshall

```
1  #include "header.h"
2  // g[i][j] = infty if not path from i to j
3  // if g[i][i] < 0, i is contained in a negative
      cycle
4  void warshall(vvl& g) {
5      for (int k=0; k<g.size(); ++k) {
6          for (int i=0; i<g.size(); ++i) {
7              for (int j=0; j<g.size(); ++j) {
8                  if (g[i][k] < LLONG_MAX and g[k][
                       j] < LLONG_MAX and g[i][j] >
                       g[i][k] + g[k][j]) {
9                      g[i][j] = g[i][k] + g[k][j];
10 }}}}}
```

3.1.5   Kruskal   Minimum spanning tree of undirected weighted graph. $O(E \log E)$

```
1  #include "header.h"
2  #include "disjoint_set.h"
3  pair<set<pair<ll, ll>>, ll>  kruskal(vector<tuple
      <ll, ll, ll>>& edges, ll n) {
4      set<pair<ll, ll>> ans;
5      ll cost = 0;
6
7      sort(edges.begin(), edges.end());
8      DisjointSet<ll> fs(n);
9
10     ll dist, i, j;
11     for (auto edge: edges) {
12         dist = get<0>(edge);
```

```
13         i = get<1>(edge);
14         j = get<2>(edge);
15
16         if (fs.find_set(i) != fs.find_set(j)) {
17             fs.union_sets(i, j);
18             ans.insert({i, j});
19             cost += dist;
20         }
21     }
22     return pair<set<pair<ll, ll>>, ll> {ans, cost
          };
23 }
```

3.1.6   Hungarian algorithm   Given J jobs and W workers ($J <= W$), computes the minimum cost to assign each prefix of jobs to distinct workers.

```
1  #include "header.h"
2  template <class T> bool ckmin(T &a, const T &b) {
       return b < a ? a = b, 1 : 0; }
3  /**
4   * @tparam T: type large enough to represent
         integers of O(J * max(|C|))
5   * @param C: JxW matrix such that C[j][w] = cost
         to assign j-th
6   * job to w-th worker (possibly negative)
7   * @return a vector (length J), with the j-th
         entry = min. cost
8   * to assign the first (j+1) jobs to distinct
         workers
9   */
10 template <class T> vector<T> hungarian(const
       vector<vector<T>> &C) {
11     const int J = (int)size(C), W = (int)size(C
          [0]);
12     assert(J <= W);
13     // a W-th worker added for convenience
14     vector<int> job(W + 1, -1);
15     vector<T> ys(J), yt(W + 1);   // potentials
16     vector<T> answers;
17     const T inf = numeric_limits<T>::max();
18     for (int j_cur = 0; j_cur < J; ++j_cur) {
19         int w_cur = W;
20         job[w_cur] = j_cur;
21         vector<T> min_to(W + 1, inf);
22         vector<int> prv(W + 1, -1);
23         vector<bool> in_Z(W + 1);
24         while (job[w_cur] != -1) {   // runs at
              most j_cur + 1 times
25             in_Z[w_cur] = true;
26             const int j = job[w_cur];
27             T delta = inf;
28             int w_next;
29             for (int w = 0; w < W; ++w) {
```

```
30                 if (!in_Z[w]) {
31                     if (ckmin(min_to[w], C[j][w]
                           - ys[j] - yt[w]))
32                         prv[w] = w_cur;
33                     if (ckmin(delta, min_to[w]))
34                         w_next = w;
35                 }
36             }
37             for (int w = 0; w <= W; ++w) {
38                 if (in_Z[w]) ys[job[w]] += delta,
                       yt[w] -= delta;
39                 else min_to[w] -= delta;
40             }
41             w_cur = w_next;
42         }
43         for (int w; w_cur != W; w_cur = w) job[
              w_cur] = job[w = prv[w_cur]];
44         answers.push_back(-yt[W]);
45     }
46     return answers;
47 }
```

3.1.7   Suc. shortest path   Calculates max flow, min cost

```
1  #include "header.h"
2  // map<node, map<node, pair<cost, capacity>>>
3  #define graph unordered_map<int, unordered_map<
      int, pair<ld, int>>>
4  graph g;
5  const ld infty = 1e60l;  // Change if necessary
6  ld fill(int n, vld& potential) {  // Finds max
      flow, min cost
7      priority_queue<pair<ld, int>> pq;
8      vector<bool> visited(n+2, false);
9      vi parent(n+2, 0);
10     vld dist(n+2, infty);
11     dist[0] = 0.1;
12     pq.emplace(make_pair(0.1, 0));
13     while (not pq.empty()) {
14         int node = pq.top().second;
15         pq.pop();
16         if (visited[node]) continue;
17         visited[node] = true;
18         for (auto& x : g[node]) {
19             int neigh = x.first;
20             int capacity = x.second.second;
21             ld cost = x.second.first;
22             if (capacity and not visited[neigh]) {
23                 ld d = dist[node] + cost + potential[node
                      ] - potential[neigh];
24                 if (d + 1e-10l < dist[neigh]) {
25                     dist[neigh] = d;
26                     pq.emplace(make_pair(-d, neigh));
```

```
27        parent[neigh] = node;
28  }}}}
29
30  for (int i = 0; i < n+2; i++) {
31    potential[i] = min(infty, potential[i] + dist
        [i]);
32  }
33  if (not parent[n+1]) return infty;
34  ld ans = 0.l;
35  for (int x = n+1; x; x=parent[x]) {
36    ans += g[parent[x]][x].first;
37    g[parent[x]][x].second--;
38    g[x][parent[x]].second++;
39  }
40  return ans;
41 }
```

3.1.8 Bipartite check

```
1 #include "header.h"
2 int main() {
3     int n;
4     vvi adj(n);
5
6     vi side(n, -1);     // will have 0's for one
          side 1's for other side
7     bool is_bipartite = true;   // becomes false
          if not bipartite
8     queue<int> q;
9     for (int st = 0; st < n; ++st) {
10        if (side[st] == -1) {
11            q.push(st);
12            side[st] = 0;
13            while (!q.empty()) {
14                int v = q.front();
15                q.pop();
16                for (int u : adj[v]) {
17                    if (side[u] == -1) {
18                        side[u] = side[v] ^ 1;
19                        q.push(u);
20                    } else {
21                        is_bipartite &= side[u]
                            != side[v];
22                    }
23 }}}}}
```

3.1.9 Find cycle directed

```
1 #include "header.h"
2 int n;
3 const int mxN = 2e5+5;
4 vvi adj(mxN);
5 vector<char> color;
```

```
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v) {
9     color[v] = 1;
10    for (int u : adj[v]) {
11        if (color[u] == 0) {
12            parent[u] = v;
13            if (dfs(u)) return true;
14        } else if (color[u] == 1) {
15            cycle_end = v;
16            cycle_start = u;
17            return true;
18        }
19    }
20    color[v] = 2;
21    return false;
22 }
23 void find_cycle() {
24    color.assign(n, 0);
25    parent.assign(n, -1);
26    cycle_start = -1;
27    for (int v = 0; v < n; v++) {
28        if (color[v] == 0 && dfs(v))break;
29    }
30    if (cycle_start == -1) {
31        cout << "Acyclic" << endl;
32    } else {
33        vector<int> cycle;
34        cycle.push_back(cycle_start);
35        for (int v = cycle_end; v != cycle_start;
              v = parent[v])
36            cycle.push_back(v);
37        cycle.push_back(cycle_start);
38        reverse(cycle.begin(), cycle.end());
39
40        cout << "Cycle␣Found:␣";
41        for (int v : cycle) cout << v << "␣";
42        cout << endl;
43    }
44 }
```

3.1.10 Find cycle undirected

```
1 #include "header.h"
2 int n;
3 const int mxN = 2e5 + 5;
4 vvi adj(mxN);
5 vector<bool> visited;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v, int par) { // passing vertex and
      its parent vertex
9     visited[v] = true;
10    for (int u : adj[v]) {
```

```
11        if(u == par) continue; // skipping edge
              to parent vertex
12        if (visited[u]) {
13            cycle_end = v;
14            cycle_start = u;
15            return true;
16        }
17        parent[u] = v;
18        if (dfs(u, parent[u]))
19            return true;
20    }
21    return false;
22 }
23 void find_cycle() {
24    visited.assign(n, false);
25    parent.assign(n, -1);
26    cycle_start = -1;
27    for (int v = 0; v < n; v++) {
28        if (!visited[v] && dfs(v, parent[v]))
              break;
29    }
30    if (cycle_start == -1) {
31        cout << "Acyclic" << endl;
32    } else {
33        vector<int> cycle;
34        cycle.push_back(cycle_start);
35        for (int v = cycle_end; v != cycle_start;
              v = parent[v])
36            cycle.push_back(v);
37        cycle.push_back(cycle_start);
38        cout << "Cycle␣Found:␣";
39        for (int v : cycle) cout << v << "␣";
40        cout << endl;
41    }
42 }
```

3.1.11 Tarjan's SCC

```
1 #include "header.h"
2
3 struct Tarjan {
4   vvi &edges;
5   int V, counter = 0, C = 0;
6   vi n, l;
7   vector<bool> vs;
8   stack<int> st;
9   Tarjan(vvi &e) : edges(e), V(e.size()), n(V,
        -1), l(V, -1), vs(V, false) {}
10  void visit(int u, vi &com) {
11    l[u] = n[u] = counter++;
12    st.push(u);
13    vs[u] = true;
14    for (auto &&v : edges[u]) {
15      if (n[v] == -1) visit(v, com);
```

```
16        if (vs[v]) l[u] = min(l[u], l[v]);
17      }
18      if (l[u] == n[u]) {
19        while (true) {
20          int v = st.top();
21          st.pop();
22          vs[v] = false;
23          com[v] = C;   //<== ACT HERE
24          if (u == v) break;
25        }
26        C++;
27      }
28    }
29    int find_sccs(vi &com) {   // component indices
         will be stored in 'com'
30      com.assign(V, -1);
31      C = 0;
32      for (int u = 0; u < V; ++u)
33        if (n[u] == -1) visit(u, com);
34      return C;
35    }
36    // scc is a map of the original vertices of the
         graph to the vertices of the SCC graph,
         scc_graph is its adjacency list. SCC
         indices and edges are stored in 'scc' and '
         scc_graph'.
37    void scc_collapse(vi &scc, vvi &scc_graph) {
38      find_sccs(scc);
39      scc_graph.assign(C, vi());
40      set<pi> rec;  // recorded edges
41      for (int u = 0; u < V; ++u) {
42        assert(scc[u] != -1);
43        for (int v : edges[u]) {
44          if (scc[v] == scc[u] ||
45            rec.find({scc[u], scc[v]}) != rec.end()
                ) continue;
46          scc_graph[scc[u]].push_back(scc[v]);
47          rec.insert({scc[u], scc[v]});
48        }
49      }
50    }
51    // The number of edges needed to be added is
         max(sources.size(), sinks.())
52    void findSourcesAndSinks(const vvi &scc_graph,
         vi &sources, vi &sinks) {
53      vi in_degree(C, 0), out_degree(C, 0);
54      for (int u = 0; u < C; u++) {
55        for (auto v : scc_graph[u]) {
56          in_degree[v]++;
57          out_degree[u]++;
58        }
59      }
60      for (int i = 0; i < C; ++i) {
61        if (in_degree[i] == 0) sources.push_back(i)
             ;
```

```
62        if (out_degree[i] == 0) sinks.push_back(i);
63      }
64    }
65  };
```

3.1.12   SCC edges Prints out the missing edges to make the input digraph strongly connected

```
1  #include "header.h"
2  const int N=1e5+10;
3  int n,a[N],cnt[N],vis[N];
4  vector<int> hd,tl;
5  int dfs(int x){
6      vis[x]=1;
7      if(!vis[a[x]])return vis[x]=dfs(a[x]);
8      return vis[x]=x;
9  }
10 int main(){
11     scanf("%d",&n);
12     for(int i=1;i<=n;i++){
13         scanf("%d",&a[i]);
14         cnt[a[i]]++;
15     }
16     int k=0;
17     for(int i=1;i<=n;i++){
18         if(!cnt[i]){
19             k++;
20             hd.push_back(i);
21             tl.push_back(dfs(i));
22         }
23     }
24     int tk=k;
25     for(int i=1;i<=n;i++){
26         if(!vis[i]){
27             k++;
28             hd.push_back(i);
29             tl.push_back(dfs(i));
30         }
31     }
32     if(k==1&&!tk)k=0;
33     printf("%d\n",k);
34     for(int i=0;i<k;i++)printf("%d %d\n",tl[i],hd
          [(i+1)%k]);
35     return 0;
36 }
```

3.1.13   Find Bridges

```
1  #include "header.h"
2  int n; // number of nodes
3  vvi adj; // adjacency list of graph
4  vector<bool> visited;
5  vi tin, low;
```

```
6  int timer;
7  void dfs(int v, int p = -1) {
8      visited[v] = true;
9      tin[v] = low[v] = timer++;
10     for (int to : adj[v]) {
11         if (to == p) continue;
12         if (visited[to]) {
13             low[v] = min(low[v], tin[to]);
14         } else {
15             dfs(to, v);
16             low[v] = min(low[v], low[to]);
17             if (low[to] > tin[v])
18                 IS_BRIDGE(v, to);
19         }
20     }
21 }
22 void find_bridges() {
23     timer = 0;
24     visited.assign(n, false);
25     tin.assign(n, -1);
26     low.assign(n, -1);
27     for (int i = 0; i < n; ++i) {
28         if (!visited[i]) dfs(i);
29     }
30 }
```

3.1.14   Articulation points (i.e. cut off points)

```
1  #include "header.h"
2  int n; // number of nodes
3  vvi adj; // adjacency list of graph
4  vector<bool> visited;
5  vi tin, low;
6  int timer;
7  void dfs(int v, int p = -1) {
8      visited[v] = true;
9      tin[v] = low[v] = timer++;
10     int children=0;
11     for (int to : adj[v]) {
12         if (to == p) continue;
13         if (visited[to]) {
14             low[v] = min(low[v], tin[to]);
15         } else {
16             dfs(to, v);
17             low[v] = min(low[v], low[to]);
18             if (low[to] >= tin[v] && p!=-1)
19                 IS_CUTPOINT(v);
20             ++children;
21         }
22     }
23     if(p == -1 && children > 1)
24         IS_CUTPOINT(v);
25 }
26 void find_cutpoints() {
```

```
26      timer = 0;
27      visited.assign(n, false);
28      tin.assign(n, -1);
29      low.assign(n, -1);
30      for (int i = 0; i < n; ++i) {
31          if (!visited[i]) dfs (i);
32      }
33 }
```

3.1.15 Topological sort

```
1 #include "header.h"
2 int n; // number of vertices
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi ans;
6 void dfs(int v) {
7     visited[v] = true;
8     for (int u : adj[v]) {
9         if (!visited[u]) dfs(u);
10    }
11    ans.push_back(v);
12 }
13 void topological_sort() {
14    visited.assign(n, false);
15    ans.clear();
16    for (int i = 0; i < n; ++i) {
17        if (!visited[i]) dfs(i);
18    }
19    reverse(ans.begin(), ans.end());
20 }
```

3.1.16 Bellmann-Ford Same as Dijkstra but allows neg. edges

```
1 #include "header.h"
2 // Switch vi and vvpi to vl and vvpl if necessary
3 void bellmann_ford_extended(vvpi &e, int source,
      vi &dist, vb &cyc) {
4   dist.assign(e.size(), INF);
5   cyc.assign(e.size(), false); // true when u is
        in a <0 cycle
6   dist[source] = 0;
7   for (int iter = 0; iter < e.size() - 1; ++iter)
       {
8     bool relax = false;
9     for (int u = 0; u < e.size(); ++u)
10    if (dist[u] == INF) continue;
11    else for (auto &e : e[u])
12      if(dist[u]+e.second < dist[e.first])
13        dist[e.first] = dist[u]+e.second, relax
                = true;
14    if(!relax) break;
```

```
15   }
16   bool ch = true;
17   while (ch) { // keep going untill no more
        changes
18    ch = false; // set dist to -INF when in cycle
19    for (int u = 0; u < e.size(); ++u)
20      if (dist[u] == INF) continue;
21      else for (auto &e : e[u])
22        if (dist[e.first] > dist[u] + e.second
            && !cyc[e.first]) {
23          dist[e.first] = -INF;
24          ch = true; //return true for cycles
25          cyc[e.first] = true;
26        }
27   }
28 }
29 }
```

3.1.17 Ford-Fulkerson Basic Max. flow

```
1 #include "header.h"
2 #define V 6  // Num. of vertices in given graph
3 /* Returns true if there is a path from source 's
    ' to sink
4 't' in residual graph. Also fills parent[] to
    store the
5 path */
6 bool bfs(int rGraph[V][V], int s, int t, int
    parent[]) {
7   bool visited[V];
8   memset(visited, 0, sizeof(visited));
9   queue<int> q;
10  q.push(s);
11  visited[s] = true;
12  parent[s] = -1;
13  while (!q.empty()) {
14    int u = q.front();
15    q.pop();
16
17    for (int v = 0; v < V; v++) {
18      if (visited[v] == false && rGraph[u][v] >
          0) {
19        if (v == t) {
20          parent[v] = u;
21          return true;
22        }
23        q.push(v);
24        parent[v] = u;
25        visited[v] = true;
26      }
27    }
28  }
29  return false;
30 }
31 // Returns the maximum flow from s to t
```

```
32 int fordFulkerson(int graph[V][V], int s, int t)
   {
33   int u, v;
34   int rGraph[V]
35      [V];
36   for (u = 0; u < V; u++)
37     for (v = 0; v < V; v++)
38       rGraph[u][v] = graph[u][v];
39
40   int parent[V]; // BFS-filled (to store path)
41   int max_flow = 0; // no flow initially
42   while (bfs(rGraph, s, t, parent)) {
43     int path_flow = INT_MAX;
44     for (v = t; v != s; v = parent[v]) {
45       u = parent[v];
46       path_flow = min(path_flow, rGraph[u][v]);
47     }
48     for (v = t; v != s; v = parent[v]) {
49       u = parent[v];
50       rGraph[u][v] -= path_flow;
51       rGraph[v][u] += path_flow;
52     }
53     max_flow += path_flow;
54   }
55   return max_flow;
56 }
```

3.1.18 Dinic max flow $O(V^2E), O(Ef)$

```
1 #include "header.h"
2 using F = ll; using W = ll; // types for flow and
      weight/cost
3 struct S{
4    const int v;    // neighbour
5    const int r;    // index of the reverse edge
6    F f;            // current flow
7    const F cap;    // capacity
8    const W cost;   // unit cost
9    S(int v, int ri, F c, W cost = 0) :
10      v(v), r(ri), f(0), cap(c), cost(cost) {}
11   inline F res() const { return cap - f; }
12 };
13 struct FlowGraph : vector<vector<S>> {
14   FlowGraph(size_t n) : vector<vector<S>>(n) {}
15   void add_edge(int u, int v, F c, W cost = 0){
16     auto &t = *this;
17     t[u].emplace_back(v, t[v].size(), c, cost
          );
18     t[v].emplace_back(u, t[u].size()-1, c, -
          cost);
19   }
20   void add_arc(int u, int v, F c, W cost = 0){
21     auto &t = *this;
22     t[u].emplace_back(v, t[v].size(), c, cost
          );
```

```cpp
21        t[v].emplace_back(u, t[u].size()-1, 0, -
              cost);
22      }
23    void clear() { for (auto &E : *this) for (
            auto &e : E) e.f = 0LL; }
24 };
25 struct Dinic{
26    FlowGraph &edges; int V,s,t;
27    vi l; vector<vector<S>::iterator> its; //
            levels and iterators
28    Dinic(FlowGraph &edges, int s, int t) :
29        edges(edges), V(edges.size()), s(s), t(t)
            , l(V,-1), its(V) {}
30    ll augment(int u, F c) { // we reuse the same
             iterators
31        if (u == t) return c; ll r = 0LL;
32        for(auto &i = its[u]; i != edges[u].end()
            ; i++){
33            auto &e = *i;
34            if (e.res() && l[u] < l[e.v]) {
35                auto d = augment(e.v, min(c, e.
                    res()));
36                if (d > 0) { e.f += d; edges[e.v
                    ][e.r].f -= d; c -= d;
37                    r += d; if (!c) break; }
38            }    }
39        return r;
40    }
41    ll run() {
42        ll flow = 0, f;
43        while(true) {
44            fill(l.begin(), l.end(),-1); l[s]=0;
45            queue<int> q; q.push(s);
46            while(!q.empty()){
47                auto u = q.front(); q.pop(); its[
                    u] = edges[u].begin();
48                for(auto &&e : edges[u]) if(e.res
                    () && l[e.v]<0)
49                    l[e.v] = l[u]+1, q.push(e.v);
50            }
51            if (l[t] < 0) return flow;
52            while ((f = augment(s, INF)) > 0)
                    flow += f;
53        }    }
54 };
```

3.1.19 Edmonds-Karp (Max) flow algorithm with time $O(VE^2)$. To get edge flow values, compare capacities before and after, and take the positive values only.

```cpp
1 #include "header.h"
2 template<class T> T edmondsKarp(vector<
      unordered_map<int, T>>&
3    graph, int source, int sink) {
4    assert(source != sink);
5    T flow = 0;
6    vi par(sz(graph)), q = par;
7
8    for (;;) {
9        fill(all(par), -1);
10       par[source] = 0;
11       int ptr = 1;
12       q[0] = source;
13
14       rep(i,0,ptr) {
15           int x = q[i];
16           for (auto e : graph[x]) {
17               if (par[e.first] == -1 && e.second > 0) {
18                   par[e.first] = x;
19                   q[ptr++] = e.first;
20                   if (e.first == sink) goto out;
21               }
22           }
23       }
24       return flow;
25 out:
26       T inc = numeric_limits<T>::max();
27       for (int y = sink; y != source; y = par[y])
28           inc = min(inc, graph[par[y]][y]);
29
30       flow += inc;
31       for (int y = sink; y != source; y = par[y]) {
32           int p = par[y];
33           if ((graph[p][y] -= inc) <= 0) graph[p].
                erase(y);
34           graph[y][p] += inc;
35       }
36    }
37 }
```

3.2 Dynamic Programming

3.2.1 Longest Incr. Subseq.

```cpp
1 #include "header.h"
2 template<class T>
3 vector<T> index_path_lis(vector<T>& nums) {
4    int n = nums.size();
5    vector<T> sub;
6     vector<int> subIndex;
7    vector<T> path(n, -1);
8    for (int i = 0; i < n; ++i) {
9        if (sub.empty() || sub[sub.size() - 1] <
              nums[i]) {
10           path[i] = sub.empty() ? -1 : subIndex[sub.
                size() - 1];
11           sub.push_back(nums[i]);
12           subIndex.push_back(i);
```

```cpp
13       } else {
14           int idx = lower_bound(sub.begin(), sub.end(),
                  nums[i]) - sub.begin();
15           path[i] = idx == 0 ? -1 : subIndex[idx - 1];
16           sub[idx] = nums[i];
17           subIndex[idx] = i;
18       }
19    }
20    vector<T> ans;
21    int t = subIndex[subIndex.size() - 1];
22    while (t != -1) {
23        ans.push_back(t);
24        t = path[t];
25    }
26    reverse(ans.begin(), ans.end());
27    return ans;
28 }
29 // Length only
30 template<class T>
31 int length_lis(vector<T> &a) {
32    set<T> st;
33    typename set<T>::iterator it;
34    for (int i = 0; i < a.size(); ++i) {
35        it = st.lower_bound(a[i]);
36        if (it != st.end()) st.erase(it);
37        st.insert(a[i]);
38    }
39    return st.size();
40 }
```

3.2.2 0-1 Knapsack Given a number of coins, calculate all possible distinct sums

```cpp
1 #include "header.h"
2 int main() {
3    int n;
4    vi coins(n); // possible coins to use
5    int sum = 0; // their sum of the coins
6    vi dp(sum + 1, 0); // dp[x] = 1 if sum x can be
          made
7    dp[0] = 1;
8    for (int c = 0; c < n; ++c)
9        for (int x = sum; x >= 0; --x)
10           if (dp[x]) dp[x + coins[c]] = 1;
11 }
```

3.2.3 Coin change Total distinct ways to make sum using n coins of different vals

```cpp
1 #include "header.h"
2 int count(vi& coins, int n, int sum) {
3    vvi dp(n + 1, vi(sum + 1, 0));
```

```
4    dp[0][0] = 1;
5    for (int i = 1; i <= n; i++) {
6        for (int j = 0; j <= sum; j++) {
7            // without using the current coin,
8            dp[i][j] += dp[i - 1][j];
9            // using the current coin
10           if ((j - coins[i - 1]) >= 0)
11               dp[i][j] += dp[i][j - coins[i -
                     1]];
12       }
13   }
14   return dp[n][sum];
15 }
```

3.3 Numerical

3.3.1 Template (for this section)

```
1 #include <bits/stdc++.h>
2 using namespace std;
3 #define rep(i, a, b) for(int i = a; i < (b); ++i)
4 #define all(x) begin(x), end(x)
5 #define sz(x) (int)(x).size()
6 typedef long long ll;
7 typedef pair<int, int> pii;
8 typedef vector<int> vi;
```

3.3.2 Polynomial

```
1 #include "template.cpp"
2 struct Poly {
3   vector<double> a;
4   double operator()(double x) const {
5     double val = 0;
6     for (int i = sz(a); i--;) (val *= x) += a[i];
7     return val;
8   }
9   void diff() {
10    rep(i,1,sz(a)) a[i-1] = i*a[i];
11    a.pop_back();
12  }
13  void divroot(double x0) {
14    double b = a.back(), c; a.back() = 0;
15    for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i
          +1]*x0+b, b=c;
16    a.pop_back();
17  }
18 };
```

3.3.3 Poly Roots
Finds the real roots to a polynomial. $O(n^2 \log(1/\epsilon))$

```
1 // Usage: polyRoots({{2,-3,1}},-1e9,1e9) = solve
      x^2-3x+2 = 0
2 #include "Polynomial.h"
3 #include "template.cpp"
4 vector<double> polyRoots(Poly p, double xmin,
      double xmax) {
5   if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
6   vector<double> ret;
7   Poly der = p;
8   der.diff();
9   auto dr = polyRoots(der, xmin, xmax);
10  dr.push_back(xmin-1);
11  dr.push_back(xmax+1);
12  sort(all(dr));
13  rep(i,0,sz(dr)-1) {
14    double l = dr[i], h = dr[i+1];
15    bool sign = p(l) > 0;
16    if (sign ^ (p(h) > 0)) {
17      rep(it,0,60) { // while (h - l > 1e-8)
18        double m = (l + h) / 2, f = p(m);
19        if ((f <= 0) ^ sign) l = m;
20        else h = m;
21      }
22      ret.push_back((l + h) / 2);
23    }
24  }
25  return ret;
26 }
```

3.3.4 Golden Section Search
Finds the argument minimizing the function f in the interval $[a, b]$ assuming f is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is *eps*. Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version. $O(\log((b-a)/\epsilon))$

```
1 /** Usage:
2   double func(double x) { return 4+x+.3*x*x; }
3   double xmin = gss(-1000,1000,func); */
4 #include "template.cpp"
5 // It is important for r to be precise, otherwise
         we don't necessarily maintain the inequality
         a < x1 < x2 < b.
6 double gss(double a, double b, double (*f)(double
      )) {
7   double r = (sqrt(5)-1)/2, eps = 1e-7;
8   double x1 = b - r*(b-a), x2 = a + r*(b-a);
9   double f1 = f(x1), f2 = f(x2);
10  while (b-a > eps)
11    if (f1 < f2) { //change to > to find maximum
12      b = x2; x2 = x1; f2 = f1;
```

```
13      x1 = b - r*(b-a); f1 = f(x1);
14    } else {
15      a = x1; x1 = x2; f1 = f2;
16      x2 = a + r*(b-a); f2 = f(x2);
17    }
18  return a;
19 }
```

3.3.5 Hill Climbing
Poor man's optimization for unimodal functions.

```
1 #include "template.cpp"
2 typedef array<double, 2> P;
3 template<class F> pair<double, P> hillClimb(P
      start, F f) {
4   pair<double, P> cur(f(start), start);
5   for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
6     rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
7       P p = cur.second;
8       p[0] += dx*jmp;
9       p[1] += dy*jmp;
10      cur = min(cur, make_pair(f(p), p));
11    }
12  }
13  return cur;
14 }
```

3.3.6 Integration
Simple integration of a function over an interval using Simpson's rule. The error should be proportional to h^4, although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

```
1 #include "template.cpp"
2 template<class F>
3 double quad(double a, double b, F f, const int n
      = 1000) {
4   double h = (b - a) / 2 / n, v = f(a) + f(b);
5   rep(i,1,n*2)
6     v += f(a + i*h) * (i&1 ? 4 : 2);
7   return v * h / 3;
8 }
```

3.3.7 Integration Adaptive
Fast integration using an adaptive Simpson's rule.

```
1 /** Usage:
2 double sphereVolume = quad(-1, 1, [](double x) {
3 return quad(-1, 1, [\&](double y) {
4 return quad(-1, 1, [\&](double z) {
```

```cpp
5    return x*x + y*y + z*z < 1; });});}); */
6  #include "template.cpp"
7
8  typedef double d;
9  #define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (
       b-a) / 6
10
11 template <class F>
12 d rec(F& f, d a, d b, d eps, d S) {
13   d c = (a + b) / 2;
14   d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
15   if (abs(T - S) <= 15 * eps || b - a < 1e-10)
16     return T + (T - S) / 15;
17   return rec(f, a, c, eps / 2, S1) + rec(f, c, b,
       eps / 2, S2);
18 }
19 template<class F>
20 d quad(d a, d b, F f, d eps = 1e-8) {
21   return rec(f, a, b, eps, S(a, b));
22 }
```

3.4   Num. Th. / Comb.

3.4.1   Basic stuff

```cpp
1  #include "header.h"
2  ll gcd(ll a, ll b) { while (b) { a %= b; swap(a,
       b); } return a; }
3  ll lcm(ll a, ll b) { return (a / gcd(a, b)) * b;
       }
4  ll mod(ll a, ll b) { return ((a % b) + b) % b; }
5  // Finds x, y s.t. ax + by = d = gcd(a, b).
6  void extended_euclid(ll a, ll b, ll &x, ll &y, ll
       &d) {
7    ll xx = y = 0;
8    ll yy = x = 1;
9    while (b) {
10     ll q = a / b;
11     ll t = b; b = a % b; a = t;
12     t = xx; xx = x - q * xx; x = t;
13     t = yy; yy = y - q * yy; y = t;
14   }
15   d = a;
16 }
17 // solves ab = 1 (mod n), -1 on failure
18 ll mod_inverse(ll a, ll n) {
19   ll x, y, d;
20   extended_euclid(a, n, x, y, d);
21   return (d > 1 ? -1 : mod(x, n));
22 }
23 // All modular inverses of [1..n] mod P in O(n)
       time.
24 vi inverses(ll n, ll P) {
25   vi I(n+1, 1LL);
```

```cpp
26   for (ll i = 2; i <= n; ++i)
27     I[i] = mod(-(P/i) * I[P%i], P);
28   return I;
29 }
30 // (a*b)%m
31 ll mulmod(ll a, ll b, ll m){
32   ll x = 0, y=a%m;
33   while(b>0){
34     if(b&1) x = (x+y)%m;
35     y = (2*y)%m, b /= 2;
36   }
37   return x % m;
38 }
39 // Finds b^e % m in O(lg n) time, ensure that b <
       m to avoid overflow!
40 ll powmod(ll b, ll e, ll m) {
41   ll p = e<2 ? 1 : powmod((b*b)%m,e/2,m);
42   return e&1 ? p*b%m : p;
43 }
44 // Solve ax + by = c, returns false on failure.
45 bool linear_diophantine(ll a, ll b, ll c, ll &x,
       ll &y) {
46   ll d = gcd(a, b);
47   if (c % d) {
48     return false;
49   } else {
50     x = c / d * mod_inverse(a / d, b / d);
51     y = (c - a * x) / b;
52     return true;
53   }
54 }
55
56 // Description: Tonelli-Shanks algorithm for
       modular square roots. Finds $x$ s.t. $x^2 = a
       \pmod p$ ($-x$ gives the other solution). O
       (\log^2 p) worst case, O(\log p) for most $p$
57 ll sqrtmod(ll a, ll p) {
58   a %= p; if (a < 0) a += p;
59   if (a == 0) return 0;
60   assert(powmod(a, (p-1)/2, p) == 1); // else no
       solution
61   if (p % 4 == 3) return powmod(a, (p+1)/4, p);
62   // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if
       p % 8 == 5
63   ll s = p - 1, n = 2;
64   int r = 0, m;
65   while (s % 2 == 0)
66     ++r, s /= 2;
67   /// find a non-square mod p
68   while (powmod(n, (p - 1) / 2, p) != p - 1) ++n;
69   ll x = powmod(a, (s + 1) / 2, p);
70   ll b = powmod(a, s, p), g = powmod(n, s, p);
71   for (;; r = m) {
72     ll t = b;
73     for (m = 0; m < r && t != 1; ++m)
```

```cpp
74     t = t * t % p;
75     if (m == 0) return x;
76     ll gs = powmod(g, 1LL << (r - m - 1), p);
77     g = gs * gs % p;
78     x = x * gs % p;
79     b = b * g % p;
80   }
81 }
```

3.4.2   Mod. exponentiation Or use pow() in python

```cpp
1  #include "header.h"
2  ll mod_pow(ll base, ll exp, ll mod) {
3    if (mod == 1) return 0;
4    if (exp == 0) return 1;
5    if (exp == 1) return base;
6
7    ll res = 1;
8    base %= mod;
9    while (exp) {
10     if (exp % 2 == 1) res = (res * base) % mod;
11     exp >>= 1;
12     base = (base * base) % mod;
13   }
14
15   return res % mod;
16 }
```

3.4.3   GCD Or math.gcd in python, std::gcd in C++

```cpp
1  #include "header.h"
2  ll gcd(ll a, ll b) {
3    if (a == 0) return b;
4    return gcd(b % a, a);
5  }
```

3.4.4   Sieve of Eratosthenes

```cpp
1  #include "header.h"
2  vl primes;
3  void getprimes(ll n) {  // Up to n (not included)
4    vector<bool> p(n, true);
5    p[0] = false;
6    p[1] = false;
7    for(ll i = 0; i < n; i++) {
8      if(p[i]) {
9        primes.push_back(i);
10       for(ll j = i*2; j < n; j+=i) p[j] =
         false;
11 }}}
```

3.4.5 Fibonacci % prime

```cpp
#include "header.h"
const ll MOD = 1000000007;
unordered_map<ll, ll> Fib;
ll fib(ll n) {
    if (n < 2) return 1;
    if (Fib.find(n) != Fib.end()) return Fib[n];
    Fib[n] = (fib((n + 1) / 2) * fib(n / 2) + fib
        ((n - 1) / 2) * fib((n - 2) / 2)) % MOD;
    return Fib[n];
}
```

3.4.6 nCk % prime

```cpp
#include "header.h"
ll binom(ll n, ll k) {
    ll ans = 1;
    for(ll i = 1; i <= min(k,n-k); ++i) ans = ans
        *(n+1-i)/i;
    return ans;
}
ll mod_nCk(ll n, ll k, ll p ){
    ll ans = 1;
    while(n){
        ll np = n%p, kp = k%p;
        if(kp > np) return 0;
        ans *= binom(np,kp);
        n /= p; k /= p;
    }
    return ans;
}
```

3.5 Strings

3.5.1 Z alg. KMP alternative (same complexities)

```cpp
#include "../header.h"
void Z_algorithm(const string &s, vi &Z) {
  Z.assign(s.length(), -1);
  int L = 0, R = 0, n = s.length();
  for (int i = 1; i < n; ++i) {
    if (i > R) {
      L = R = i;
      while (R < n && s[R - L] == s[R]) R++;
      Z[i] = R - L; R--;
    } else if (Z[i - L] >= R - i + 1) {
      L = i;
      while (R < n && s[R - L] == s[R]) R++;
      Z[i] = R - L; R--;
    } else Z[i] = Z[i - L];
  }
}
```

3.5.2 KMP

```cpp
#include "header.h"
void compute_prefix_function(string &w, vi &
    prefix) {
  prefix.assign(w.length(), 0);
  int k = prefix[0] = -1;

  for(int i = 1; i < w.length(); ++i) {
    while(k >= 0 && w[k + 1] != w[i]) k = prefix[
        k];
    if(w[k + 1] == w[i]) k++;
    prefix[i] = k;
  }
}
void knuth_morris_pratt(string &s, string &w) {
  int q = -1;
  vi prefix;
  compute_prefix_function(w, prefix);
  for(int i = 0; i < s.length(); ++i) {
    while(q >= 0 && w[q + 1] != s[i]) q = prefix[
        q];
    if(w[q + 1] == s[i]) q++;
    if(q + 1 == w.length()) {
      // Match at position (i - w.length() + 1)
      q = prefix[q];
    }
  }
}
```

3.5.3 Aho-Corasick Also can be used as Knuth-Morris-Pratt algorithm

```cpp
#include "header.h"

map<char, int> cti;
int cti_size;
template <int ALPHABET_SIZE, int (*mp)(char)>
struct AC_FSM {
  struct Node {
    int child[ALPHABET_SIZE], failure = 0,
        match_par = -1;
    vi match;
    Node() { for (int i = 0; i < ALPHABET_SIZE;
        ++i) child[i] = -1; }
  };
  vector<Node> a;
  vector<string> &words;
  AC_FSM(vector<string> &words) : words(words) {
    a.push_back(Node());
    construct_automaton();
  }
  void construct_automaton() {
    for (int w = 0, n = 0; w < words.size(); ++w,
        n = 0) {
      for (int i = 0; i < words[w].size(); ++i) {
        if (a[n].child[mp(words[w][i])] == -1) {
          a[n].child[mp(words[w][i])] = a.size();
          a.push_back(Node());
        }
        n = a[n].child[mp(words[w][i])];
      }
      a[n].match.push_back(w);
    }
    queue<int> q;
    for (int k = 0; k < ALPHABET_SIZE; ++k) {
      if (a[0].child[k] == -1) a[0].child[k] = 0;
      else if (a[0].child[k] > 0) {
        a[a[0].child[k]].failure = 0;
        q.push(a[0].child[k]);
      }
    }
    while (!q.empty()) {
      int r = q.front(); q.pop();
      for (int k = 0, arck; k < ALPHABET_SIZE; ++
          k) {
        if ((arck = a[r].child[k]) != -1) {
          q.push(arck);
          int v = a[r].failure;
          while (a[v].child[k] == -1) v = a[v].
              failure;
          a[arck].failure = a[v].child[k];
          a[arck].match_par = a[v].child[k];
          while (a[arck].match_par != -1
              && a[a[arck].match_par].match.empty
                  ())
            a[arck].match_par = a[a[arck].
                match_par].match_par;
        }
      }
    }
  }
  void aho_corasick(string &sentence, vvi &
      matches){
    matches.assign(words.size(), vi());
    int state = 0, ss = 0;
    for (int i = 0; i < sentence.length(); ++i,
        ss = state) {
      while (a[ss].child[mp(sentence[i])] == -1)
        ss = a[ss].failure;
      state = a[state].child[mp(sentence[i])]
          = a[ss].child[mp(sentence[i])];
      for (ss = state; ss != -1; ss = a[ss].
          match_par)
        for (int w : a[ss].match)
          matches[w].push_back(i + 1 - words[w].
              length());
    }
  }
};
```

```
67  int char_to_int(char c) {
68    return cti[c];
69  }
70  int main() {
71    ll n;
72    string line;
73    while(getline(cin, line)) {
74      stringstream ss(line);
75      ss >> n;
76
77      vector<string> patterns(n);
78      for (auto& p: patterns) getline(cin, p);
79
80      string text;
81      getline(cin, text);
82
83      cti = {}, cti_size = 0;
84      for (auto c: text) {
85        if (not in(c, cti)) {
86          cti[c] = cti_size++;
87        }
88      }
89      for (auto& p: patterns) {
90        for (auto c: p) {
91          if (not in(c, cti)) {
92            cti[c] = cti_size++;
93          }
94        }
95      }
96
97      vvi matches;
98      AC_FSM <128+1, char_to_int> ac_fms(patterns);
99      ac_fms.aho_corasick(text, matches);
100     for (auto& x: matches) cout << x << endl;
101   }
102
103 }
```

3.5.4 Long. palin. subs Manacher - $O(n)$

```
1  #include "header.h"
2  void manacher(string &s, vi &pal) {
3    int n = s.length(), i = 1, l, r;
4    pal.assign(2 * n + 1, 0);
5    while (i < 2 * n + 1) {
6      if ((i&1) && pal[i] == 0) pal[i] = 1;
7      l = i / 2 - pal[i] / 2; r = (i-1) / 2 + pal[i] / 2;
8
9      while (l - 1 >= 0 && r + 1 < n && s[l - 1] == s[r + 1])
10       --l, ++r, pal[i] += 2;
11
12     for (l = i - 1, r = i + 1; l >= 0 && r < 2 * n + 1; --l, ++r) {
```

```
13       if (l <= i - pal[i]) break;
14       if (l / 2 - pal[l] / 2 > i / 2 - pal[i] / 2)
15         pal[r] = pal[l];
16       else {  if (l >= 0)
17           pal[r] = min(pal[l], i + pal[i] - r);
18         break;
19       }
20     }
21     i = r;
22   } }
```

3.5.5 Bitstring Slower than an unordered set (for many elements), but hashable

```
1  #include "../header.h"
2
3  template<size_t len>
4  struct pair_hash { // To make it hashable (pair<int, bitset<len>>)
5    std::size_t operator()(const std::pair<int, std::bitset<len>>& p) const {
6      std::size_t h1 = std::hash<int>{}(p.first);
7      std::size_t h2 = std::hash<std::bitset<len>>{}(p.second);
8      return h1 ^ (h2 << 1);
9    }
10 };
11 #define MAXN 1000
12 std::bitset<MAXN> bs;
13 // bs.set(idx) <- set idx-th bit (1)
14 // bs.reset(idx) <- reset idx-th bit (0)
15 // bs.flip(idx) <- flip idx-th bit
16 // bs.test(idx) <- idx-th bit == 1
17 // bs.count() <- number of 1s
18 // bs.any() <- any bit == 1
```

3.6 Geometry

3.6.1 essentials.cpp

```
1  #include "../header.h"
2  using C = ld; // could be ll or ld
3  constexpr C EPS = 1e-10; // change to 0 for C=ll
4  struct P {      // may also be used as a 2D vector
5    C x, y;
6    P(C x = 0, C y = 0) : x(x), y(y) {}
7    P operator+ (const P &p) const { return {x + p.x, y + p.y}; }
8    P operator- (const P &p) const { return {x - p.x, y - p.y}; }
```

```
9  P operator* (C c) const { return {x * c, y * c}; }
10 P operator/ (C c) const { return {x / c, y / c}; }
11 C operator* (const P &p) const { return x*p.x + y*p.y; }
12 C operator^ (const P &p) const { return x*p.y - p.x*y; }
13 P perp() const { return P{y, -x}; }
14 C lensq() const { return x*x + y*y; }
15 ld len() const { return sqrt((ld)lensq()); }
16 static ld dist(const P &p1, const P &p2) {
17   return (p1-p2).len(); }
18 bool operator==(const P &r) const {
19   return ((*this)-r).lensq() <= EPS*EPS; }
20 };
21 C det(P p1, P p2) { return p1^p2; }
22 C det(P p1, P p2, P o) { return det(p1-o, p2-o); }
23 C det(const vector<P> &ps) {
24   C sum = 0; P prev = ps.back();
25   for(auto &p : ps) sum += det(p, prev), prev = p;
26   return sum;
27 }
28 // Careful with division by two and C=ll
29 C area(P p1, P p2, P p3) { return abs(det(p1, p2, p3))/C(2); }
30 C area(const vector<P> &poly) { return abs(det(poly))/C(2); }
31 int sign(C c){ return (c > C(0)) - (c < C(0)); }
32 int ccw(P p1, P p2, P o) { return sign(det(p1, p2, o)); }
33
34 // Only well defined for C = ld.
35 P unit(const P &p) { return p / p.len(); }
36 P rotate(P p, ld a) { return P{p.x*cos(a)-p.y*sin(a), p.x*sin(a)+p.y*cos(a)}; }
```

3.6.2 Two segs. itersec.

```
1  #include "header.h"
2  #include "essentials.cpp"
3  bool intersect(P a1, P a2, P b1, P b2) {
4    if (max(a1.x, a2.x) < min(b1.x, b2.x)) return false;
5    if (max(b1.x, b2.x) < min(a1.x, a2.x)) return false;
6    if (max(a1.y, a2.y) < min(b1.y, b2.y)) return false;
7    if (max(b1.y, b2.y) < min(a1.y, a2.y)) return false;
8    bool l1 = ccw(a2, b1, a1) * ccw(a2, b2, a1) <= 0;
```

```
9    bool l2 = ccw(b2, a1, b1) * ccw(b2, a2, b1) <=
         0;
10   return l1 && l2;
11 }
```

3.6.3 Convex Hull

```
1  #include "header.h"
2  #include "essentials.cpp"
3  struct ConvexHull {  // O(n lg n) monotone chain.
4    size_t n;
5    vector<size_t> h, c;  // Indices of the hull
         are in 'h', ccw.
6    const vector<P> &p;
7    ConvexHull(const vector<P> &_p) : n(_p.size()),
         c(n), p(_p) {
8      std::iota(c.begin(), c.end(), 0);
9      std::sort(c.begin(), c.end(), [this](size_t l
         , size_t r) -> bool { return p[l].x != p[
         r].x ? p[l].x < p[r].x : p[l].y < p[r].y;
         });
10     c.erase(std::unique(c.begin(), c.end(), [this
         ](size_t l, size_t r) { return p[l] == p[
         r]; }), c.end());
11     for (size_t s = 1, r = 0; r < 2; ++r, s = h.
         size()) {
12       for (size_t i : c) {
13         while (h.size() > s && ccw(p[h.end()
             [-2]], p[h.end()[-1]], p[i]) <= 0)
14           h.pop_back();
15         h.push_back(i);
16       }
17       reverse(c.begin(), c.end());
18     }
19     if (h.size() > 1) h.pop_back();
20   }
21   size_t size() const { return h.size(); }
22   template <class T, void U(const P &, const P &,
         const P &, T &)>
23   void rotating_calipers(T &ans) {
24     if (size() <= 2)
25       U(p[h[0]], p[h.back()], p[h.back()], ans);
26     else
27       for (size_t i = 0, j = 1, s = size(); i < 2
             * s; ++i) {
28         while (det(p[h[(i + 1) % s]] - p[h[i % s
             ]], p[h[(j + 1) % s]] - p[h[j]]) >=
             0)
29           j = (j + 1) % s;
30         U(p[h[i % s]], p[h[(i + 1) % s]], p[h[j
             ]], ans);
31       }
32   }
33 };
```

```
34 // Example: furthest pair of points. Now set ans
       = 0LL and call
35 // ConvexHull(pts).rotating_calipers<ll, update>(
       ans);
36 void update(const P &p1, const P &p2, const P &o,
        ll &ans) {
37   ans = max(ans, (ll)max((p1 - o).lensq(), (p2 -
       o).lensq()));
38 }
39 int main() {
40   ios::sync_with_stdio(false);  // do not use
         cout + printf
41   cin.tie(NULL);
42
43   int n;
44   cin >> n;
45   while (n) {
46     vector<P> ps;
47       int x, y;
48     for (int i = 0; i < n; i++) {
49         cin >> x >> y;
50         ps.push_back({x, y});
51     }
52
53       ConvexHull ch(ps);
54       cout << ch.h.size() << endl;
55       for(auto& p: ch.h) {
56           cout << ps[p].x << "␣" << ps[p].y <<
                 endl;
57       }
58     cin >> n;
59   }
60
61   return 0;
62 }
```

3.7 Other Algorithms

3.7.1 2-sat

```
1  #include "../header.h"
2  #include "../Graphs/tarjan.cpp"
3  struct TwoSAT {
4    int n;
5    vvi imp; // implication graph
6    Tarjan tj;
7
8    TwoSAT(int _n) : n(_n), imp(2 * _n, vi()), tj(
         imp) { }
9
10   // Only copy the needed functions:
11   void add_implies(int c1, bool v1, int c2, bool
         v2) {
12     int u = 2 * c1 + (v1 ? 1 : 0),
```

```
13         v = 2 * c2 + (v2 ? 1 : 0);
14     imp[u].push_back(v);     //  u => v
15     imp[v^1].push_back(u^1);  // -v => -u
16   }
17   void add_equivalence(int c1, bool v1, int c2,
         bool v2) {
18     add_implies(c1, v1, c2, v2);
19     add_implies(c2, v2, c1, v1);
20   }
21   void add_or(int c1, bool v1, int c2, bool v2) {
22     add_implies(c1, !v1, c2, v2);
23   }
24   void add_and(int c1, bool v1, int c2, bool v2)
         {
25     add_true(c1, v1); add_true(c2, v2);
26   }
27   void add_xor(int c1, bool v1, int c2, bool v2)
         {
28     add_or(c1, v1, c2, v2);
29     add_or(c1, !v1, c2, !v2);
30   }
31   void add_true(int c1, bool v1) {
32     add_implies(c1, !v1, c1, v1);
33   }
34
35   // on true: a contains an assignment.
36   // on false: no assignment exists.
37   bool solve(vb &a) {
38     vi com;
39     tj.find_sccs(com);
40     for (int i = 0; i < n; ++i)
41       if (com[2 * i] == com[2 * i + 1])
42         return false;
43
44     vvi bycom(com.size());
45     for (int i = 0; i < 2 * n; ++i)
46       bycom[com[i]].push_back(i);
47
48     a.assign(n, false);
49     vb vis(n, false);
50     for(auto &&component : bycom){
51       for (int u : component) {
52         if (vis[u / 2]) continue;
53         vis[u / 2] = true;
54         a[u / 2] = (u % 2 == 1);
55       }
56     }
57     return true;
58   }
59 };
```

3.7.2 Matrix Solve

```
1  #include "header.h"
```

```cpp
2  #define REP(i, n) for(auto i = decltype(n)(0); i
       < (n); i++)
3  using T = double;
4  constexpr T EPS = 1e-8;
5  template<int R, int C>
6  using M = array<array<T,C>,R>;  // matrix
7  template<int R, int C>
8  T ReducedRowEchelonForm(M<R,C> &m, int rows) {
      // return the determinant
9    int r = 0; T det = 1;              // MODIFIES
         the input
10   for(int c = 0; c < rows && r < rows; c++) {
11     int p = r;
12     for(int i=r+1; i<rows; i++) if(abs(m[i][c]) >
           abs(m[p][c])) p=i;
13     if(abs(m[p][c]) < EPS){ det = 0; continue; }
14     swap(m[p], m[r]);    det = -det;
15     T s = 1.0 / m[r][c], t; det *= m[r][c];
16     REP(j,C) m[r][j] *= s;         // make leading
           term in row 1
17     REP(i,rows) if (i!=r){ t = m[i][c]; REP(j,C)
           m[i][j] -= t*m[r][j]; }
18     ++r;
19   }
20   return det;
21 }
22 bool error, inconst;  // error => multiple or
      inconsistent
23 template<int R,int C> // Mx = a; M:R*R, v:R*C =>
      x:R*C
24 M<R,C> solve(const M<R,R> &m, const M<R,C> &a,
      int rows){
25   M<R,R+C> q;
26   REP(r,rows){
27     REP(c,rows) q[r][c] = m[r][c];
28     REP(c,C) q[r][R+c] = a[r][c];
29   }
30   ReducedRowEchelonForm<R,R+C>(q,rows);
31   M<R,C> sol; error = false, inconst = false;
32   REP(c,C) for(auto j = rows-1; j >= 0; --j){
33     T t=0; bool allzero=true;
34     for(auto k = j+1; k < rows; ++k)
35       t += q[j][k]*sol[k][c], allzero &= abs(q[j]
           ][k]) < EPS;
36     if(abs(q[j][j]) < EPS)
37       error = true, inconst |= allzero && abs(q[j]
           ][R+c]) > EPS;
38     else sol[j][c] = (q[j][R+c] - t) / q[j][j];
           // usually q[j][j]=1
39   }
40   return sol;
41 }
```

3.7.3   Matrix Exp.

```cpp
1  #include "header.h"
2  #define ITERATE_MATRIX(w) for (int r = 0; r < (w)
      ; ++r) \
3                    for (int c = 0; c < (w); ++c)
4  template <class T, int N>
5  struct M {
6    array<array<T,N>,N> m;
7    M() { ITERATE_MATRIX(N) m[r][c] = 0; }
8    static M id() {
9      M I; for (int i = 0; i < N; ++i) I.m[i][i] =
         1; return I;
10   }
11   M operator*(const M &rhs) const {
12     M out;
13     ITERATE_MATRIX(N) for (int i = 0; i < N; ++i)
14       out.m[r][c] += m[r][i] * rhs.m[i][c];
15     return out;
16   }
17   M raise(ll n) const {
18     if(n == 0) return id();
19     if(n == 1) return *this;
20     auto r = (*this**this).raise(n / 2);
21     return (n%2 ? *this*r : r);
22   }
23 };
```

3.7.4   Finite field For FFT

```cpp
1  #include "header.h"
2  #include "../Number Theory/elementary.cpp"
3  template<ll p,ll w> // prime, primitive root
4  struct Field { using T = Field; ll x; Field(ll x
      =0) : x{x} {}
5    T operator+(T r) const { return {(x+r.x)%p}; }
6    T operator-(T r) const { return {(x-r.x+p)%p};
         }
7    T operator*(T r) const { return {(x*r.x)%p}; }
8    T operator/(T r) const { return (*this)*r.inv()
         ; }
9    T inv() const { return {mod_inverse(x,p)}; }
10   static T root(ll k) { assert( (p-1)%k==0 );
         // (p-1)%k == 0?
11     auto r = powmod(w,(p-1)/abs(k),p);      // k-
         th root of unity
12     return k>=0 ? T{r} : T{r}.inv();
13   }
14   bool zero() const { return x == 0LL; }
15 };
16 using F1 = Field<1004535809,3 >;
17 using F2 = Field<1107296257,10>; // 1<<30 + 1<<25
      + 1
18 using F3 = Field<2281701377,3 >; // 1<<31 + 1<<27
      + 1
```

3.7.5   Complex field For FFR

```cpp
1  #include "header.h"
2  const double m_pi = M_PIf64x;
3  struct Complex { using T = Complex; double u,v;
4    Complex(double u=0, double v=0) : u{u}, v{v} {}
5    T operator+(T r) const { return {u+r.u, v+r.v};
         }
6    T operator-(T r) const { return {u-r.u, v-r.v};
         }
7    T operator*(T r) const { return {u*r.u - v*r.v,
         u*r.v + v*r.u}; }
8    T operator/(T r) const {
9      auto norm = r.u*r.u+r.v*r.v;
10     return {(u*r.u + v*r.v)/norm, (v*r.u - u*r.v)
         /norm};
11   }
12   T operator*(double r) const { return T{u*r, v*r
         }; }
13   T operator/(double r) const { return T{u/r, v/r
         }; }
14   T inv() const { return T{1,0}/ *this; }
15   T conj() const { return T{u, -v}; }
16   static T root(ll k){ return {cos(2*m_pi/k), sin
         (2*m_pi/k)}; }
17   bool zero() const { return max(abs(u), abs(v))
         < 1e-6; }
18 };
```

3.7.6   FFT

```cpp
1  #include "header.h"
2  #include "complex_field.cpp"
3  #include "fin_field.cpp"
4  void brinc(int &x, int k) {
5    int i = k - 1, s = 1 << i;
6    x ^= s;
7    if ((x & s) != s) {
8      --i; s >>= 1;
9      while (i >= 0 && ((x & s) == s))
10       x = x &~ s, --i, s >>= 1;
11     if (i >= 0) x |= s;
12   }
13 }
14 using T = Complex;  // using T=F1,F2,F3
15 vector<T> roots;
16 void root_cache(int N) {
17   if (N == (int)roots.size()) return;
18   roots.assign(N, T{0});
19   for (int i = 0; i < N; ++i)
20     roots[i] = ((i&-i) == i)
21       ? T{cos(2.0*m_pi*i/N), sin(2.0*m_pi*i/N)}
22       : roots[i&-i] * roots[i-(i&-i)];
23 }
24 void fft(vector<T> &A, int p, bool inv = false) {
```

```cpp
25    int N = 1<<p;
26    for(int i = 0, r = 0; i < N; ++i, brinc(r, p))
27      if (i < r) swap(A[i], A[r]);
28 //  Uncomment to precompute roots (for T=Complex)
      . Slower but more precise.
29 //  root_cache(N);
30 //          , sh=p-1      , --sh
31    for (int m = 2; m <= N; m <<= 1) {
32      T w, w_m = T::root(inv ? -m : m);
33      for (int k = 0; k < N; k += m) {
34        w = T{1};
35        for (int j = 0; j < m/2; ++j) {
36 //        T w = (!inv ? roots[j<<sh] : roots[j<<
      sh].conj());
37          T t = w * A[k + j + m/2];
38          A[k + j + m/2] = A[k + j] - t;
39          A[k + j] = A[k + j] + t;
40          w = w * w_m;
41        }
42      }
43    }
44    if(inv){ T inverse = T(N).inv(); for(auto &x :
        A) x = x*inverse; }
45 }
46 // convolution leaves A and B in frequency domain
        state
47 // C may be equal to A or B for in-place
        convolution
48 void convolution(vector<T> &A, vector<T> &B,
      vector<T> &C){
49    int s = A.size() + B.size() - 1;
50    int q = 32 - __builtin_clz(s-1), N=1<<q;  //
        fails if s=1
51    A.resize(N,{}); B.resize(N,{}); C.resize(N,{});
52    fft(A, q, false); fft(B, q, false);
53    for (int i = 0; i < N; ++i) C[i] = A[i] * B[i];
54    fft(C, q, true); C.resize(s);
55 }
56 void square_inplace(vector<T> &A) {
57    int s = 2*A.size()-1, q = 32 - __builtin_clz(s
        -1), N=1<<q;
58    A.resize(N,{}); fft(A, q, false);
59    for(auto &x : A) x = x*x;
60    fft(A, q, true); A.resize(s);
61 }
```

3.7.7   Polyn. inv. div.

```cpp
1 #include "header.h"
2 #include "fft.cpp"
3 vector<T> &rev(vector<T> &A) { reverse(A.begin(),
      A.end()); return A; }
4 void copy_into(const vector<T> &A, vector<T> &B,
      size_t n) {
5    std::copy(A.begin(), A.begin()+min({n, A.size()
        , B.size()}), B.begin());
6 }
7
8 // Multiplicative inverse of A modulo x^n.
      Requires A[0] != 0!!
9 vector<T> inverse(const vector<T> &A, int n) {
10   vector<T> Ai{A[0].inv()};
11   for (int k = 0; (1<<k) < n; ++k) {
12     vector<T> As(4<<k, T(0)), Ais(4<<k, T(0));
13     copy_into(A, As, 2<<k); copy_into(Ai, Ais, Ai
          .size());
14     fft(As, k+2, false); fft(Ais, k+2, false);
15     for (int i = 0; i < (4<<k); ++i) As[i] = As[i
          ]*Ais[i]*Ais[i];
16     fft(As, k+2, true); Ai.resize(2<<k, {});
17     for (int i = 0; i < (2<<k); ++i) Ai[i] = T(2)
          * Ai[i] - As[i];
18   }
19   Ai.resize(n);
20   return Ai;
21 }
22 // Polynomial division. Returns {Q, R} such that
      A = QB+R, deg R < deg B.
23 // Requires that the leading term of B is nonzero
      .
24 pair<vector<T>, vector<T>> divmod(const vector<T>
      &A, const vector<T> &B) {
25   size_t n = A.size()-1, m = B.size()-1;
26   if (n < m) return {vector<T>(1, T(0)), A};
27
28   vector<T> X(A), Y(B), Q, R;
29   convolution(rev(X), Y = inverse(rev(Y), n-m+1),
        Q);
30   Q.resize(n-m+1); rev(Q);
31
32   X.resize(Q.size()), copy_into(Q, X, Q.size());
33   Y.resize(B.size()), copy_into(B, Y, B.size());
34   convolution(X, Y, X);
35
36   R.resize(m), copy_into(A, R, m);
37   for (size_t i = 0; i < m; ++i) R[i] = R[i] - X[
        i];
38   while (R.size() > 1 && R.back().zero()) R.
        pop_back();
39   return {Q, R};
40 }
41 vector<T> mod(const vector<T> &A, const vector<T>
      &B) {
42   return divmod(A, B).second;
43 }
```

3.7.8   Linear recurs.
Given a linear recurrence of the form
$$a_n = \sum_{i=0}^{k-1} c_i a_{n-i-1}$$
this code computes a_n in $O(k \log k \log n)$ time.

```cpp
1 #include "header.h"
2 #include "poly.cpp"
3 // x^k mod f
4 vector<T> xmod(const vector<T> f, ll k) {
5    vector<T> r{T(1)};
6    for (int b = 62; b >= 0; --b) {
7      if (r.size() > 1)
8        square_inplace(r), r = mod(r, f);
9      if ((k>>b)&1) {
10       r.insert(r.begin(), T(0));
11       if (r.size() == f.size()) {
12         T c = r.back() / f.back();
13         for (size_t i = 0; i < f.size(); ++i)
14           r[i] = r[i] - c * f[i];
15         r.pop_back();
16       }
17     }
18   }
19   return r;
20 }
21 // Given A[0,k) and C[0, k), computes the n-th
      term of:
22 // A[n] = \sum_i C[i] * A[n-i-1]
23 T nth_term(const vector<T> &A, const vector<T> &C
      , ll n) {
24   int k = (int)A.size();
25   if (n < k) return A[n];
26
27   vector<T> f(k+1, T{1});
28   for (int i = 0; i < k; ++i)
29     f[i] = T{-1} * C[k-i-1];
30   f = xmod(f, n);
31
32   T r = T{0};
33   for (int i = 0; i < k; ++i)
34     r = r + f[i] * A[i];
35   return r;
36 }
```

3.7.9   Convolution
Precise up to 9e15

```cpp
1 #include "header.h"
2 #include "fft.cpp"
3 void convolution_mod(const vi &A, const vi &B, ll
      MOD, vi &C) {
4    int s = A.size() + B.size() - 1; ll m15 = (1LL
        <<15)-1LL;
```

```cpp
int q = 32 - __builtin_clz(s-1), N=1<<q;  //
    fails if s=1
vector<T> Ac(N), Bc(N), R1(N), R2(N);
for (size_t i = 0; i < A.size(); ++i) Ac[i] = T
    {A[i]&m15, A[i]>>15};
for (size_t i = 0; i < B.size(); ++i) Bc[i] = T
    {B[i]&m15, B[i]>>15};
fft(Ac, q, false); fft(Bc, q, false);
for (int i = 0, j = 0; i < N; ++i, j = (N-1)&(N
    -i)) {
  T as = (Ac[i] + Ac[j].conj()) / 2;
  T al = (Ac[i] - Ac[j].conj()) / T{0, 2};
  T bs = (Bc[i] + Bc[j].conj()) / 2;
  T bl = (Bc[i] - Bc[j].conj()) / T{0, 2};
  R1[i] = as*bs + al*bl*T{0,1}, R2[i] = as*bl +
      al*bs;
}
fft(R1, q, true); fft(R2, q, true);
ll p15 = (1LL<<15)%MOD, p30 = (1LL<<30)%MOD; C.
    resize(s);
for (int i = 0; i < s; ++i) {
  ll l = llround(R1[i].u), m = llround(R2[i].u)
      , h = llround(R1[i].v);
  C[i] = (l + m*p15 + h*p30) % MOD;
}
}
```

3.7.10 Partitions of n Finds all possible partitions of a number

```cpp
#include "header.h"
void printArray(int p[], int n) {
  for (int i = 0; i < n; i++)
    cout << p[i] << " ";
  cout << endl;
}

void printAllUniqueParts(int n) {
  int p[n]; // array to store a partition
  int k = 0;  // idx of last element in a
      partition
  p[k] = n;

  // The loop stops when the current partition
      has all 1s
  while (true) {
    printArray(p, k + 1);
    int rem_val = 0;
    while (k >= 0 && p[k] == 1) {
      rem_val += p[k];
      k--;
    }
    // no more partitions
    if (k < 0) return;
    p[k]--;
    rem_val++;

    // sorted order is violated (fix)
    while (rem_val > p[k]) {
      p[k + 1] = p[k];
      rem_val = rem_val - p[k];
      k++;
    }

    p[k + 1] = rem_val;
    k++;
  }
}
```

3.7.11 Ternary search Find the smallest i in $[a, b]$ that maximizes $f(i)$, assuming that $f(a) < \cdots < f(i) \geq \cdots \geq f(b)$. To reverse which of the sides allows non-strict inequalities, change the $<$ marked with (A) to $<=$, and reverse the loop at (B). To minimize f, change it to $>$, also at (B). $O(\log(b - a))$

```cpp
// Usage: int ind = ternSearch(0,n-1,[\&](int i){
    return a[i];});
#include "../Numerical/template.cpp"
template<class F>
int ternSearch(int a, int b, F f) {
  assert(a <= b);
  while (b - a >= 5) {
    int mid = (a + b) / 2;
    if (f(mid) < f(mid+1)) a = mid; // (A)
    else b = mid+1;
  }
  rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
  return a;
}
```

3.8 Other Data Structures

3.8.1 Disjoint set (i.e. union-find)

```cpp
template <typename T>
class DisjointSet {
    typedef T * iterator;
    T *parent, n, *rank;
  public:
    // O(n), assumes nodes are [0, n)
    DisjointSet(T n) {
      this->parent = new T[n];
      this->n = n;
      this->rank = new T[n];
      for (T i = 0; i < n; i++) {
        parent[i] = i;
        rank[i] = 0;
      }
    }

    // O(log n)
    T find_set(T x) {
      if (x == parent[x]) return x;
      return parent[x] = find_set(parent[x
          ]);
    }

    // O(log n)
    void union_sets(T x, T y) {
      x = this->find_set(x);
      y = this->find_set(y);

      if (x == y) return;
      if (rank[x] < rank[y]) {
        T z = x;
        x = y;
        y = z;
      }
      parent[y] = x;
      if (rank[x] == rank[y]) rank[x]++;
    }
};
```

3.8.2 Fenwick tree (i.e. BIT) eff. update + prefix sum calc. Can be generalized to arbitrary dimensions by duplicating loops.

```cpp
// #include "header.h"
template < class T >
struct FenwickTree { // use 1 based indices !!!
    int n ; vector <T > tree ;
    FenwickTree ( int n ) : n ( n ) { tree .
        assign ( n + 1 , 0) ; }
    T query ( int l , int r ) { return query ( r
        ) - query ( l - 1) ; }
    T query ( int r ) {
      T s = 0;
      for (; r > 0; r -= ( r & ( - r ) ) ) s +=
          tree [ r ];
      return s ;
    }
    void update ( int i , T v ) {
      for (; i <= n ; i += ( i & ( - i ) ) )
          tree [ i ] += v ;
    }
};
```

3.8.3 Trie

```cpp
#include "header.h"
const int ALPHABET_SIZE = 26;
inline int mp(char c) { return c - 'a'; }

struct Node {
  Node* ch[ALPHABET_SIZE];
  bool isleaf = false;
  Node() {
    for(int i = 0; i < ALPHABET_SIZE; ++i) ch[i]
        = nullptr;
  }

  void insert(string &s, int i = 0) {
    if (i == s.length()) isleaf = true;
    else {
      int v = mp(s[i]);
      if (ch[v] == nullptr)
        ch[v] = new Node();
      ch[v]->insert(s, i + 1);
    }
  }

  bool contains(string &s, int i = 0) {
    if (i == s.length()) return isleaf;
    else {
      int v = mp(s[i]);
      if (ch[v] == nullptr) return false;
      else return ch[v]->contains(s, i + 1);
    }
  }

  void cleanup() {
    for (int i = 0; i < ALPHABET_SIZE; ++i)
      if (ch[i] != nullptr) {
        ch[i]->cleanup();
        delete ch[i];
      }
  }
};
```

3.8.4 Treap
A binary tree whose nodes contain two values, a key and a priority, such that the key keeps the BST property

```cpp
#include "header.h"
struct Node {
  ll v;
  int sz, pr;
  Node *l = nullptr, *r = nullptr;
  Node(ll val) : v(val), sz(1) { pr = rand(); }
};
int size(Node *p) { return p ? p->sz : 0; }
```

```cpp
void update(Node* p) {
  if (!p) return;
  p->sz = 1 + size(p->l) + size(p->r);
  // Pull data from children here
}
void propagate(Node *p) {
  if (!p) return;
  // Push data to children here
}
void merge(Node *&t, Node *l, Node *r) {
  propagate(l), propagate(r);
  if (!l)       t = r;
  else if (!r)  t = l;
  else if (l->pr > r->pr)
      merge(l->r, l->r, r), t = l;
  else  merge(r->l, l, r->l), t = r;
  update(t);
}
void spliti(Node *t, Node *&l, Node *&r, int
    index) {
  propagate(t);
  if (!t) { l = r = nullptr; return; }
  int id = size(t->l);
  if (index <= id) // id \in [index, \infty), so
      move it right
    spliti(t->l, l, t->l, index), r = t;
  else
    spliti(t->r, t->r, r, index - id), l = t;
  update(t);
}
void splitv(Node *t, Node *&l, Node *&r, ll val)
    {
  propagate(t);
  if (!t) { l = r = nullptr; return; }
  if (val <= t->v) // t->v \in [val, \infty), so
      move it right
    splitv(t->l, l, t->l, val), r = t;
  else
    splitv(t->r, t->r, r, val), l = t;
  update(t);
}
void clean(Node *p) {
  if (p) { clean(p->l), clean(p->r); delete p; }
}
```

3.8.5 Segment tree

```cpp
#include "../header.h"
// example: SegmentTree<int, min> st(n, INT_MAX);
const int& addOp(const int& a, const int& b) {
    static int result;
    result = a + b;
    return result;
}
```

```cpp
template <class T, const T&(*op)(const T&, const
    T&)>
struct SegmentTree {
  int n; vector<T> tree; T id;
  SegmentTree(int _n, T _id) : n(_n), tree(2 * n,
      _id), id(_id) { }
  void update(int i, T val) {
    for (tree[i+n] = val, i = (i+n)/2; i > 0; i
        /= 2)
      tree[i] = op(tree[2*i], tree[2*i+1]);
  }
  T query(int l, int r) {
    T lhs = T(id), rhs = T(id);
    for (l += n, r += n; l < r; l >>= 1, r >>= 1)
        {
      if (  l&1 ) lhs = op(lhs, tree[l++]);
      if (!(r&1)) rhs = op(tree[r--], rhs);
    }
    return op(l == r ? op(lhs, tree[l]) : lhs,
        rhs);
  }
};
```

3.8.6 Lazy segment tree
Uptimizes range updates

```cpp
#include "../header.h"
using T=int; using U=int; using I=int;    //
    exclusive right bounds
T t_id; U u_id;
T op(T a, T b){ return a+b; }
void join(U &a, U b){ a+=b; }
void apply(T &t, U u, int x){ t+=x*u; }
T convert(const I &i){ return i; }
struct LazySegmentTree {
  struct Node { int l, r, lc, rc; T t; U u;
    Node(int l, int r, T t=t_id):l(l),r(r),lc(-1)
        ,rc(-1),t(t),u(u_id){}
  };
  int N; vector<Node> tree; vector<I> &init;
  LazySegmentTree(vector<I> &init) : N(init.size
      ()), init(init){
    tree.reserve(2*N-1); tree.push_back({0,N});
        build(0, 0, N);
  }
  void build(int i, int l, int r) { auto &n =
      tree[i];
    if (r > l+1) { int m = (l+r)/2;
      n.lc = tree.size();     n.rc = n.lc+1;
      tree.push_back({l,m});    tree.push_back({m
          ,r});
      build(n.lc,l,m);     build(n.rc,m,r);
      n.t = op(tree[n.lc].t, tree[n.rc].t);
    } else n.t = convert(init[l]);
  }
```

```
24  void push(Node &n, U u){ apply(n.t, u, n.r-n.l)
        ; join(n.u,u); }
25  void push(Node &n){push(tree[n.lc],n.u);push(
        tree[n.rc],n.u);n.u=u_id;}
26  T query(int l, int r, int i = 0) { auto &n =
        tree[i];
27      if(r <= n.l || n.r <= l) return t_id;
28      if(l <= n.l && n.r <= r) return n.t;
29      return push(n), op(query(l,r,n.lc),query(l,r,
          n.rc));
30  }
31  void update(int l, int r, U u, int i = 0) {
        auto &n = tree[i];
32      if(r <= n.l || n.r <= l) return;
33      if(l <= n.l && n.r <= r) return push(n,u);
34      push(n); update(l,r,u,n.lc); update(l,r,u,n.
          rc);
35      n.t = op(tree[n.lc].t, tree[n.rc].t);
36  }
37  };
```

3.8.7   Suffix tree

```
1  #include "../header.h"
2  using T = char;
3  using M = map<T,int>; // or array<T,ALPHABET_SIZE
        >
4  using V = string; // could be vector<T> as well
5  using It = V::const_iterator;
6  struct Node{
7    It b, e; M edges; int link; // end is exclusive
8    Node(It b, It e) : b(b), e(e), link(-1) {}
9    int size() const { return e-b; }
10 };
11 struct SuffixTree{
12   const V &s; vector<Node> t;
13   int root,n,len,remainder,llink; It edge;
14   SuffixTree(const V &s) : s(s) { build(); }
15   int add_node(It b, It e){ return t.push_back({b
        ,e}), t.size()-1; }
16   int add_node(It b){ return add_node(b,s.end());
        }
17   void link(int node){ if(llink) t[llink].link =
        node; llink = node; }
18   void build(){
19     len = remainder =  0; edge = s.begin();
20     n = root = add_node(s.begin(), s.begin());
21     for(auto i = s.begin(); i != s.end(); ++i){
22       ++remainder; llink = 0;
23       while(remainder){
24         if(len == 0) edge = i;
25         if(t[n].edges[*edge] == 0){
26           t[n].edges[*edge] = add_node(i); link(n
              );
```

```
27       } else {
28         auto x = t[n].edges[*edge];
29         if(len >= t[x].size()){
30           len -= t[x].size(); edge += t[x].size
                (); n = x;
31           continue;
32         }
33         if(*(t[x].b + len) == *i){
34           ++len; link(n); break;
35         }
36         auto split = add_node(t[x].b, t[x].b+
              len);
37         t[n].edges[*edge] = split;
38         t[x].b += len;
39         t[split].edges[*i] = add_node(i);
40         t[split].edges[*t[x].b] = x;
41         link(split);
42       }
43       --remainder;
44       if(n == root && len > 0)
45         --len, edge = i - remainder + 1;
46       else n = t[n].link > 0? t[n].link: root;
47     }
48   }
49  }
50 };
```

3.8.8   UnionFind

```
1  #include "header.h"
2  struct UnionFind {
3    std::vector<int> par, rank, size;
4    int c;
5    UnionFind(int n) : par(n), rank(n, 0), size(n,
        1), c(n) {
6      for(int i = 0; i < n; ++i) par[i] = i;
7    }
8    int find(int i) { return (par[i] == i ? i : (
        par[i] = find(par[i]))); }
9    bool same(int i, int j) { return find(i) ==
        find(j); }
10   int get_size(int i) { return size[find(i)]; }
11   int count() { return c; }
12   int merge(int i, int j) {
13     if((i = find(i)) == (j = find(j))) return -1;
14     --c;
15     if(rank[i] > rank[j]) swap(i, j);
16     par[i] = j;
17     size[j] += size[i];
18     if(rank[i] == rank[j]) rank[j]++;
19     return j;
20   }
21 };
```

3.8.9   Indexed set Similar to set, but allows accessing elements by index using `find_by_order()` in $O(\log n)$

```
1  #include "../header.h"
2  #include <ext/pb_ds/assoc_container.hpp>
3  using namespace __gnu_pbds;
4  using namespace std;
5
6  typedef tree<int,null_type,less<int>,rb_tree_tag,
        tree_order_statistics_node_update>
        indexed_set;
```

4   Other Mathematics

4.1   Helpful functions

4.1.1   Euler's Totient Fucntion $n = p_1^{k_1-1} \cdot (p_1 - 1) \cdot \ldots \cdot p_r^{k_r-1} \cdot (p_r - 1)$, where $p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ is the prime factorization of n.

```
1  # include "header.h"
2  ll phi(ll n) {  // \Phi(n)
3      ll ans = 1;
4      for (ll i = 2; i*i <= n; i++) {
5          if (n % i == 0) {
6              ans *= i-1;
7              n /= i;
8              while (n % i == 0) {
9                  ans *= i;
10                 n /= i;
11             }
12         }
13     }
14     if (n > 1) ans *= n-1;
15     return ans;
16 }
17 vi phis(int n) {  // All \Phi(i) up to n
18   vi phi(n + 1, 0LL);
19   iota(phi.begin(), phi.end(), 0LL);
20   for (ll i = 2LL; i <= n; ++i)
21     if (phi[i] == i)
22       for (ll j = i; j <= n; j += i)
23         phi[j] -= phi[j] / i;
24   return phi;
25 }
```

4.1.2   Totient (again but .py)

```
1  def totatives(n):
2      if n == 1:
3          return 1
```

```
4      phi = int(n > 1 and n)
5      for p in range(2, int(n ** .5) + 1):
6          if not n % p:
7              phi -= phi // p
8              while not n % p:
9                  n //= p
10     #if n is > 1 it means it is prime
11     if n > 1: phi -= phi // n
12     return phi
```

Formulas $\Phi(n)$ counts all numbers in $1, \ldots, n-1$ coprime to n.

$a^{\varphi(n)} \equiv 1 \mod n$, a and n are coprimes.

$\forall e > \log_2 m : \ n^e \mod m = n^{\Phi(m)+e \mod \Phi(m)} \mod m$.

$\gcd(m,n) = 1 \Rightarrow \Phi(m \cdot n) = \Phi(m) \cdot \Phi(n)$.

4.1.3 Pascal's trinagle $\binom{n}{k}$ is k-th element in the n-th row, indexing both from 0

```
1  #include "header.h"
2  void printPascal(int n) {
3      for (int line = 1; line <= n; line++) {
4          int C = 1; // used to represent C(line, i
               )
5          for (int i = 1; i <= line; i++) {
6              cout << C << "␣";
7              C = C * (line - i) / i;
8          }
9          cout << "\n";
10     }
11 }
```

4.2 Theorems and definitions

Subfactorial (Derangements) Permutations of a set such that none of the elements appear in their original position:

$$!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

$$!(0) = 1, \ !n = n \cdot !(n-1) + (-1)^n$$

$$!n = (n-1)(!(n-1) + !(n-2)) = \left[\frac{n!}{e}\right] \quad (1)$$

$$!n = 1 - e^{-1}, \ n \to \infty \quad (2)$$

Binomials and other partitionings

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \prod_{i=1}^{k} \frac{n-i+1}{i}$$

This last product may be computed incrementally since any product of k' consecutive values is divisibleby $k'!$.

Basic identities: The hockeystick identity:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

or

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Also

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

For $n, m \geq 0$ and p prime: write n, m in base p, i.e. $n = n_k p^k + \cdots + n_1 p + n_0$ and $m = m_k p^k + \ldots m_1 p + m_0$. Then by Lucas theorem we have $\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \mod p$, with the convention that $n_i < m_i \implies \binom{n_i}{m_i} = 0$.

Fibonacci (See also number theory section)

$$\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\sum_{i=1}^{n} F_i = F_{n+2} - 1$, $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$$

Bit stuff $a + b = a \oplus b + 2(a \& b) = a | b + a \& b$.

kth bit is set in x iff $x \mod 2^{k-1} \geq 2^k$, or iff $x \mod 2^{k-1} - x \mod 2^k \neq 0$ (i.e. $= 2^k$) It comes handy when you need to look at the bits of the numbers which are pair sums or subset sums etc.

$n \mod 2^i = n \& (2^i - 1)$.

$\forall k : \ 1 \oplus 2 \oplus \ldots \oplus (4k-1) = 0$

4.3 Geometry Formulas

$$\text{Euler:} \qquad 1 + CC = V - E + F$$

$$\text{Pick:} \qquad \text{Area} = \text{itr pts} + \frac{\text{bdry pts}}{2} - 1$$

Given a non-self-intersecting closed polygon on n vertices, given as (x_i, y_i), its centroid (C_x, C_y) is given as:

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i),$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) = \text{polygon area}$$

Inclusion-Exclusion For appropriate f compute $\sum_{S \subseteq T} (-1)^{|T \setminus S|} f(S)$, or if only the size of S matters, $\sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} f(s)$. In some contexts we might use Stirling numbers, not binomial coefficients!

Some useful applications:

Graph coloring Let $I(S)$ count the number of independent sets contained in $S \subseteq V$ ($I(\emptyset) = 1$, $I(S) = I(S \setminus v) + I(S \setminus N(v))$). Let $c_k = \sum_{S \subseteq V} (-1)^{|V \setminus S|} I(S)$. Then V is k-colorable iff $v > 0$. Thus we can compute the chromatic number of a graph in $O^*(2^n)$ time.

Burnside's lemma Given a group G acting on a set X, the number of elements in X up to symmetry is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

with X^g the elements of X invariant under g. For example, if $f(n)$ counts "configurations" of some sort of length n, and we want to count them up to rotational symmetry using $G = \mathbb{Z}/n\mathbb{Z}$, then

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n} \sum_{k \| n} f(k)\phi(n/k)$$

I.e. for coloring with c colors we have $f(k) = k^c$.

Relatedly, in Pólya's enumeration theorem we imagine X as a set of n beads with G permuting the beads (e.g. a necklace, with G all rotations and reflections of the n-cycle, i.e. the dihedral group D_n). Suppose further that we had Y colors, then the number of G-invariant colorings Y^X/G is counted by

$$\frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)}$$

with $c(g)$ counting the number of cycles of g when viewed as a permutation of X. We can generalize this to a weighted version: if the color i can occur exactly r_i times, then this is counted by the coefficient of $t_1^{r_1} \ldots t_n^{r_n}$ in the polynomial

$$Z(t_1, \ldots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \cdots + t_n^m)^{c_m(g)}$$

where $c_m(g)$ counts the number of length m cycles in g acting as a permutation on X. Note we get the original formula by setting all $t_i = 1$. Here Z is the cycle index. Note: you can cleverly deal with even/odd sizes by setting some t_i to -1.

Lucas Theorem If p is prime, then:

$$\frac{p^a}{k} \equiv 0 (\mod p)$$

Thus for non-negative integers $m = m_k p^k + \ldots + m_1 p + m_0$ and $n = n_k p^k + \ldots + n_1 p + n_0$:

$$\frac{m}{n} = \Pi_{i=0}^k \frac{m_i}{n_i} \mod p$$

Note: The fraction's mean integer division.

4.4 Recurrences

If $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$, and $r_1, \ldots, r_k$ are distinct roots of $x^k - c_1 x^{k-1} - \cdots - c_k$, there are $d_1, \ldots, d_k$ s.t.

$$a_n = d_1 r_1^n + \cdots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1 n + d_2) r^n$.

4.5 Sums
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

4.6 Series
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \ldots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, (-\infty < x < \infty)$$

4.7 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle θ, area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin\theta = F\tan\theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is $180°$, $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

4.8 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \dfrac{a+b+c}{2}$

Area:

$$[ABC] = rp = \frac{1}{2}ab\sin\gamma$$
$$= \frac{abc}{4R} = \sqrt{p(p-a)(p-b)(p-c)} = \frac{1}{2}\left|(B-A, C-A)^T\right|$$

Circumradius: $R = \dfrac{abc}{4A}$, Inradius: $r = \dfrac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two): $s_a = $

$$\sqrt{bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]}$$

Law of tangents: $\dfrac{a+b}{a-b} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$

4.9 Trigonometry
$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

$$\sin v + \sin w = 2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$\cos v + \cos w = 2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$(V+W)\tan(v-w)/2 = (V-W)\tan(v+w)/2$$

where V, W are lengths of sides opposite angles v, w.

$$a\cos x + b\sin x = r\cos(x - \phi)$$
$$a\sin x + b\cos x = r\sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}, \phi = \text{atan2}(b, a)$.

4.10 Combinatorics

Combinations and Permutations
$P(n,r) = \frac{n!}{(n-r)!}$
$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
$C(n,r) = C(n, n-r)$

4.11 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$$

4.12 Labeled unrooted trees

\# on n vertices: n^{n-2}
\# on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
\# with degrees d_i: $(n-2)!/((d_1-1)! \cdots (d_n-1)!)$

4.13 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \ p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

4.14 Numbers

Bernoulli numbers EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able). $B[0,\ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$
Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m)$$

$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

Stirling's numbers **First kind:** $S_1(n,k)$ count permutations on n items with k cycles. $S_1(n,k) = S_1(n-1,k-1) + (n-1)S_1(n-1,k)$ with $S_1(0,0) = 1$. Note:

$$\sum_{k=0}^{n} S_1(n,k)x^k = x(x+1)\ldots(x+n-1)$$

$$\sum_{k=0}^{n} S_1(n,k) = n!$$

$S_1(8,k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$S_1(n,2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$
Second kind: $S_2(n,k)$ count partitions of n distinct elements into exactly k non-empty groups.

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k)$$

$$S_2(n,1) = S_2(n,n) = 1$$

$$S_2(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$$

Catalan Numbers - Number of correct bracket sequence consisting of n opening and n closing brackets.
The number of ways to completely parenthesize $n+1$ factors.
The number of triangulations of a convex polygon with $n+2$ sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).
The number of ways to connect the $2n$ points on a circle to form n disjoint i.e. non-intersecting chords.

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$C_0 = 1, \ C_1 = 1, \ C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Narayana numbers The number of expressions containing n pairs of parentheses, which are correctly matched and which contain k distinct nestings.

$$N(n,k) = \frac{1}{n} \frac{n}{k} \frac{n}{k-1}$$

Eulerian numbers Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j}(k+1-j)^n$$

Bell numbers Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \ C_{n+1} = \frac{2(2n+1)}{n+2}C_n, \ C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

4.15 Probability

Stochastic variables
$P(X = r) = C(n,r) \cdot p^r \cdot (1-p)^{n-r}$

Bayes' Theorem $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|\bar{B})P(\bar{B})}$

$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1)\cdot\ldots\cdot P(A|B_n)P(B_n)}$

Expectation Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x. It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

4.16 Number Theory

Bezout's Theorem

$$a, b \in \mathbb{Z}^+ \implies \exists s, t \in \mathbb{Z} : \gcd(a, b) = sa + tb$$

Bézout's identity For $a \neq$, $b \neq 0$, then $d = gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If (x, y) is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a, b)}, y - \frac{ka}{\gcd(a, b)} \right), \quad k \in \mathbb{Z}$$

Partial Coprime Divisor Property

$$(\gcd(a, b) = 1) \wedge (a \mid bc) \implies (a \mid c)$$

Coprime Modulus Equivalence Property

$$(\gcd(c, m) = 1) \wedge (ac \equiv bc \mod m) \implies (a \equiv b \mod m)$$

Fermat's Little Theorem

$$(\text{prime}(p)) \wedge (p \nmid a) \implies (a^{p-1} \equiv 1 \mod p)$$

$$(\text{prime}(p)) \implies (a^p \equiv a \mod p)$$

Pythagorean Triples The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \ b = k \cdot (2mn), \ c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

Primes $p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than $1\,000\,000$.

Primitive roots exist modulo any prime power p^a, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

Estimates $\sum_{d|n} d = O(n \log\log n)$.

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, $200\,000$ for $n < 1e19$.

Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n) g(d)$

$g(n) = \sum_{1 \le m \le n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \le m \le n} \mu(m) g(\lfloor \frac{n}{m} \rfloor)$

4.17 Discrete distributions

Binomial distribution The number of successes in n independent yes/no experiments, each which yields success with probability p is $\text{Bin}(n, p)$, $n = 1, 2, \ldots, 0 \le p \le 1$.

$$p(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mu = np, \ \sigma^2 = np(1 - p)$$

$\text{Bin}(n, p)$ is approximately $\text{Po}(np)$ for small p.

First success distribution The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability p is $\text{Fs}(p)$, $0 \le p \le 1$.

$$p(k) = p(1 - p)^{k-1}, \ k = 1, 2, \ldots$$

$$\mu = \frac{1}{p}, \ \sigma^2 = \frac{1 - p}{p^2}$$

Poisson distribution The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\text{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

$$\mu = \lambda, \ \sigma^2 = \lambda$$

4.18 Continuous distributions

Uniform distribution If the probability density function is constant between a and b and 0 elsewhere it is $U(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a + b}{2}, \ \sigma^2 = \frac{(b - a)^2}{12}$$

Exponential distribution The time between events in a Poisson process is $\text{Exp}(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \ \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$