

<b>1</b>	<b>Setup</b>	<b>2</b>							
1.0.1	Tips . . . . .	2	3.1.7	Suc. shortest path . . . . .	7	3.7	Other Algorithms . . . . .	16	
1.0.2	header.h . . . . .	2	3.1.8	Bipartite check . . . . .	8	3.7.1	2-sat . . . . .	16	
1.0.3	Aux. helper C++ . . . . .	2	3.1.9	Find cycle directed . . . . .	8	3.7.2	Matrix Solve . . . . .	16	
1.0.4	Aux. helper python . . . . .	2	3.1.10	Find cycle undirected . . . . .	8	3.7.3	Matrix Exp. . . . .	17	
<b>2</b>	<b>Python</b>	<b>2</b>	3.1.11	Tarjan's SCC . . . . .	8	3.7.4	Finite field . . . . .	17	
2.1	Graphs . . . . .	2	3.1.12	SCC edges . . . . .	9	3.7.5	Complex field . . . . .	17	
2.1.1	BFS . . . . .	2	3.1.13	Find Bridges . . . . .	9	3.7.6	FFT . . . . .	17	
2.1.2	Dijkstra . . . . .	2	3.1.14	Articulation points . . . . .	9	3.7.7	Polyn. inv. div. . . . .	18	
2.1.3	Topological Sort . . . . .	3	3.1.15	Topological sort . . . . .	10	3.7.8	Linear recurs. . . . .	18	
2.1.4	Kruskal (UnionFind) . . . . .	3	3.1.16	Bellmann-Ford . . . . .	10	3.7.9	Convolution . . . . .	18	
2.1.5	Prim . . . . .	3	3.1.17	Ford-Fulkerson . . . . .	10	3.7.10	Partitions of $n$ . . . . .	19	
2.2	Num. Th. / Comb. . . . .	3	3.1.18	Dinic max flow . . . . .	10	3.7.11	Ternary search . . . . .	19	
2.2.1	nCk % prime . . . . .	3	3.1.19	Edmonds-Karp . . . . .	11	<b>3.8</b>	<b>Other Data Structures . . . . .</b>	<b>19</b>	
2.2.2	Sieve of E. . . . .	3	<b>3.2</b>	<b>Dynamic Programming . . . . .</b>	<b>11</b>	3.8.1	Disjoint set . . . . .	19	
2.2.3	Modular Inverse . . . . .	4	3.2.1	Longest Incr. Subseq. . . . .	11	3.8.2	Fenwick tree . . . . .	19	
2.2.4	Chinese rem. . . . .	4	3.2.2	0-1 Knapsack . . . . .	11	3.8.3	Trie . . . . .	20	
2.2.5	Bezout . . . . .	4	3.2.3	Coin change . . . . .	11	3.8.4	Treap . . . . .	20	
2.2.6	Gen. chinese rem. . . . .	4	<b>3.3</b>	<b>Numerical . . . . .</b>	<b>12</b>	3.8.5	Segment tree . . . . .	20	
2.3	Strings . . . . .	4	3.3.1	Template (for this section) . . . . .	12	3.8.6	Lazy segment tree . . . . .	20	
2.3.1	Longest common substr. . . . .	4	3.3.2	Polynomial . . . . .	12	3.8.7	Suffix tree . . . . .	21	
2.3.2	Longest common subseq. . . . .	4	3.3.3	Poly Roots . . . . .	12	3.8.8	UnionFind . . . . .	21	
2.3.3	KMP . . . . .	4	3.3.4	Golden Section Search . . . . .	12	3.8.9	Indexed set . . . . .	21	
2.3.4	Suffix Array . . . . .	4	3.3.5	Hill Climbing . . . . .	12	<b>4</b>	<b>Other Mathematics</b>	<b>21</b>	
2.3.5	Longest common pref. . . . .	5	3.3.6	Integration . . . . .	12	4.1	Helpful functions . . . . .	21	
2.3.6	Edit distance . . . . .	5	3.3.7	Integration Adaptive . . . . .	12	4.1.1	Euler's Totient Fucntion . . . . .	21	
2.3.7	Bitstring . . . . .	5	<b>3.4</b>	<b>Num. Th. / Comb. . . . .</b>	<b>13</b>	4.1.2	Totient (again but .py) . . . . .	21	
2.4	Geometry . . . . .	5	3.4.1	Basic stuff . . . . .	13	4.1.3	Pascal's trinagle . . . . .	22	
2.4.1	Convex Hull . . . . .	5	3.4.2	Mod. exponentiation . . . . .	13	4.2	Theorems and definitions . . . . .	22	
2.4.2	Geometry . . . . .	5	3.4.3	GCD . . . . .	13	4.3	Geometry Formulas . . . . .	22	
2.5	Other Algorithms . . . . .	5	3.4.4	Sieve of Eratosthenes . . . . .	13	4.4	Recurrences . . . . .	23	
2.5.1	Rotate matrix . . . . .	5	3.4.5	Fibonacci % prime . . . . .	14	4.5	Sums . . . . .	23	
2.6	Other Data Structures . . . . .	6	3.4.6	nCk % prime . . . . .	14	4.6	Series . . . . .	23	
2.6.1	Segment Tree . . . . .	6	<b>3.5</b>	<b>Strings . . . . .</b>	<b>14</b>	4.7	Quadrilaterals . . . . .	23	
2.6.2	Trie . . . . .	6	3.5.1	Z alg. . . . .	14	4.8	Triangles . . . . .	23	
<b>3</b>	<b>C++</b>	<b>6</b>	3.5.2	KMP . . . . .	14	4.9	Trigonometry . . . . .	23	
3.1	Graphs . . . . .	6	3.5.3	Aho-Corasick . . . . .	14	4.10	Combinatorics . . . . .	23	
3.1.1	BFS . . . . .	6	3.5.4	Long. palin. subs . . . . .	15	4.11	Cycles . . . . .	24	
3.1.2	DFS . . . . .	6	3.5.5	Bitstring . . . . .	15	4.12	Labeled unrooted trees . . . . .	24	
3.1.3	Dijkstra . . . . .	6	<b>3.6</b>	<b>Geometry . . . . .</b>	<b>15</b>	4.13	Partition function . . . . .	24	
3.1.4	Floyd-Warshall . . . . .	7	3.6.1	essentials.cpp . . . . .	15	4.14	Numbers . . . . .	24	
3.1.5	Kruskal . . . . .	7	3.6.2	Two segs. itersec. . . . .	15	4.15	Probability . . . . .	24	
3.1.6	Hungarian algorithm . . . . .	7	3.6.3	Convex Hull . . . . .	16	4.16	Number Theory . . . . .	25	
						4.17	Discrete distributions . . . . .	25	
						4.18	Continuous distributions . . . . .	25	

## 1 Setup

**1.0.1 Tips Test session:** Check `__int128` and GNU builtins.

**C++ var. limits:** `int`  $-2^{31}$ ,  $2^{31} - 1$

`ll`  $-2^{63}$ ,  $2^{63} - 1$

`ull`  $0$ ,  $2^{64} - 1$

`__int128`  $-2^{127}$ ,  $2^{127} - 1$

`ldd`  $-1.7e308$ ,  $1.7e308$ , 18 digits precision

### 1.0.2 header.h

```
1 #pragma once
2 #include <bits/stdc++.h>
3 using namespace std;
4
5 #define ll long long
6 #define ull unsigned ll
7 #define ld long double
8 #define pl pair<ll, ll>
9 #define pi pair<int, int>
10 #define vl vector<ll>
11 #define vi vector<int>
12 #define vb vector<bool>
13 #define vvi vector<vi>
14 #define vvl vector<vl>
15 #define vpl vector<pl>
16 #define vpi vector<pi>
17 #define vld vector<ld>
18 #define vvp vector<vp>
19 #define in(el, cont) (cont.find(el) != cont.end()
20 // sets/maps
21 #define all(x) x.begin(), x.end()
22
23 constexpr int INF = 2000000010;
24 constexpr ll LLINF = 90000000000000000010LL;
25
26 // int main() {
27 //   ios::sync_with_stdio(false); // do not use
28 //   cout + printf
29 //   cin.tie(NULL);
30 //   cout << fixed << setprecision(12);
31 //   return 0;
32 // }
```

### 1.0.3 Aux. helper C++

```
1 #include "header.h"
2
3 int main() {
4     // Read in a line including white space
```

```
5     string line;
6     getline(cin, line);
7     // When doing the above read numbers as
8     // follows:
9     int n;
10    getline(cin, line);
11    stringstream ss(line);
12    ss >> n;
13
14    // Count the number of 1s in binary
15    // representation of a number
16    ull number;
17    __builtin_popcountll(number);
18 }
19
20 // __int128
21 using lll = __int128;
22 ostream& operator<<(ostream& o, __int128 n) {
23     auto t = n<0 ? -n : n; char b[128], *d = end(b)
24     ;
25     do *--d = '0'+t%10, t /= 10; while (t);
26     if(n<0) *--d = '-';
27     o.rdbuf()->sputn(d, end(b)-d);
28     return o;
29 }
```

### 1.0.4 Aux. helper python

```
1 from functools import lru_cache
2
3 # Read until EOF
4 while True:
5     try:
6         pattern = input()
7     except EOFError:
8         break
9
10 @lru_cache(maxsize=None)
11 def smth_memoi(i, j, s):
12     # Example in-built cache
13     return "sol"
14
15 # Fast I
16 import io, os
17 def fast_io():
18     finput = io.BytesIO(os.read(0,
19 os.fstat(0).st_size)).readline
20     s = finput().decode()
21     return s
22
23 # Fast O
24 import sys
25 def fast_out():
26     n = 5
```

```
27 sys.stdout.write(str(n)+"\n")
```

## 2 Python

### 2.1 Graphs

#### 2.1.1 BFS

```
1 from collections import deque
2 def bfs(g, roots, n):
3     q = deque(roots)
4     explored = set()
5     distances = [0 if v in roots else float('inf')
6                 ] for v in range(n)
7
8     while len(q) != 0:
9         node = q.popleft()
10        if node in explored: continue
11        explored.add(node)
12        for neigh in g[node]:
13            if neigh not in explored:
14                q.append(neigh)
15                distances[neigh] = distances[node] + 1
16
17    return distances
```

#### 2.1.2 Dijkstra

```
1 from heapq import *
2 def dijkstra(n, root, g): # g = {node: (cost,
3                             neigh)}
4     dist = [float("inf")]*n
5     dist[root] = 0
6     prev = [-1]*n
7
8     pq = [(0, root)]
9     heapify(pq)
10    visited = set([])
11
12    while len(pq) != 0:
13        _, node = heappop(pq)
14
15        if node in visited: continue
16        visited.add(node)
17
18        # In case of disconnected graphs
19        if node not in g:
20            continue
21
22        for cost, neigh in g[node]:
23            alt = dist[node] + cost
```

```

23     if alt < dist[neigh]:
24         dist[neigh] = alt
25         prev[neigh] = node
26         heappush(pq, (alt, neigh))
27     return dist

```

### 2.1.3 Topological Sort topological sorting of a DAG

```

1 from collections import defaultdict
2
3 class Graph:
4     def __init__(self, vertices):
5         self.graph = defaultdict(list) #adjacency
6             List
7         self.V = vertices #No. V
8
9     def addEdge(self, u, v):
10         self.graph[u].append(v)
11
12     def topologicalSortUtil(self, v, visited, stack):
13         :
14         visited[v] = True
15         # Recur for all the vertices adjacent to
16         # this vertex
17         for i in self.graph[v]:
18             if visited[i] == False:
19                 self.topologicalSortUtil(i,
20                     visited, stack)
21         stack.insert(0, v)
22
23     def topologicalSort(self):
24         visited = [False]*self.V
25         stack = []
26         for i in range(self.V):
27             if visited[i] == False:
28                 self.topologicalSortUtil(i,
29                     visited, stack)
30         return stack
31
32     def isCyclicUtil(self, v, visited, recStack):
33         visited[v] = True
34         recStack[v] = True
35         for neighbour in self.graph[v]:
36             if visited[neighbour] == False:
37                 if self.isCyclicUtil(neighbour,
38                     visited, recStack) == True:
39                     return True
40             elif recStack[neighbour] == True:
41                 return True
42         recStack[v] = False
43         return False
44
45     def isCyclic(self):
46         visited = [False] * (self.V + 1)

```

```

41     recStack = [False] * (self.V + 1)
42     for node in range(self.V):
43         if visited[node] == False:
44             if self.isCyclicUtil(node,
45                 visited, recStack) == True:
46                 return True
47     return False

```

### 2.1.4 Kruskal (UnionFind) Min. span. tree

```

1 class UnionFind:
2     def __init__(self, n):
3         self.parent = [-1]*n
4
5     def find(self, x):
6         if self.parent[x] < 0:
7             return x
8         self.parent[x] = self.find(self.parent[x])
9         return self.parent[x]
10
11     def connect(self, a, b):
12         ra = self.find(a)
13         rb = self.find(b)
14         if ra == rb:
15             return False
16         if self.parent[ra] > self.parent[rb]:
17             self.parent[rb] += self.parent[ra]
18             self.parent[ra] = rb
19         else:
20             self.parent[ra] += self.parent[rb]
21             self.parent[rb] = ra
22         return True
23
24 # Full MST is len(spanning)==n-1
25 def kruskal(n, edges):
26     uf = UnionFind(n)
27     spanning = []
28     edges.sort(key = lambda d: -d[2])
29     while edges and len(spanning) < n-1:
30         u, v, w = edges.pop()
31         if not uf.connect(u, v):
32             continue
33         spanning.append((u, v, w))
34     return spanning

```

### 2.1.5 Prim Min. span. tree - good for dense graphs

```

1 from heapq import heappush, heappop, heapify
2 def prim(G, n):
3     s = next(iter(G.keys()))
4     V = set([s])
5     M = []

```

```

6     c = 0
7
8     E = [(w,s,v) for v,w in G[s].items()]
9     heapify(E)
10
11     while E and len(M) < n-1:
12         w,u,v = heappop(E)
13         if v in V: continue
14         V.add(v)
15         M.append((u,v))
16         c += w
17         u = v
18         [heappush(E, (w,u,v)) for v,w in G[u].items()
19             if v not in V]
20
21     if len(M) == n-1:
22         return M, c
23     else:
24         return None, None

```

## 2.2 Num. Th. / Comb.

### 2.2.1 nCk % prime p must be prime and k < p

```

1 def fermat_binom(n, k, p):
2     if k > n:
3         return 0
4     num = 1
5     for i in range(n-k+1, n+1):
6         num *= i % p
7     num %= p
8     denom = 1
9     for i in range(1, k+1):
10         denom *= i % p
11     denom %= p
12     # numerator * denominator^(p-2) (mod p)
13     return (num * pow(denom, p-2, p)) % p

```

### 2.2.2 Sieve of E. $O(n)$ so actually faster than C++ version, but more memory

```

1 MAX_SIZE = 10**8+1
2 isprime = [True] * MAX_SIZE
3 prime = []
4 SPF = [None] * (MAX_SIZE)
5
6 def manipulated_seive(N): # Up to N (not
7     included)
8     isprime[0] = isprime[1] = False
9     for i in range(2, N):
10         if isprime[i] == True:
11             prime.append(i)

```

```

11     SPF[i] = i
12     j = 0
13     while (j < len(prime) and
14            i * prime[j] < N and
15            prime[j] <= SPF[i]):
16         isprime[i * prime[j]] = False
17         SPF[i * prime[j]] = prime[j]
18         j += 1

```

### 2.2.3 Modular Inverse of a mod b

```

1 def modinv(a, b):
2     if b == 1: return 1
3     b0, x0, x1 = b, 0, 1
4     while a > 1:
5         q, a, b = a//b, b, a%b
6         x0, x1 = x1 - q * x0, x0
7     if x1 < 0: x1 += b0
8     return x1

```

**2.2.4 Chinese rem.** an  $x$  such that  $\forall y, m: yx = 1 \bmod m$  requires all  $m, m'$  to be  $i=1$  and coprime

```

1 def chinese_remainder(ys, ms):
2     N, x = 1, 0
3     for m in ms: N *= m
4     for y, m in zip(ys, ms):
5         n = N // m
6         x += n * y * modinv(n, m)
7     return x % N

```

### 2.2.5 Bezout

```

1 def bezout_id(a, b):
2     r, x, s, y, t, z = b, a, 0, 1, 1, 0
3     while r:
4         q = x // r
5         x, r = r, x % r
6         y, s = s, y - q * s
7         z, t = t, z - q * t
8     return y % (b // x), z % (-a // x)

```

### 2.2.6 Gen. chinese rem.

```

1 def general_chinese_remainder(a, b, m, n):
2     g = gcd(m, n)
3
4     if a == b and m == n:
5         return a, m
6     if (a % g) != (b % g):

```

```

7         return None, None
8
9     u, v = bezout_id(m, n)
10    x = (a*v*n + b*u*m) // g
11    return int(x) % lcm(m, n), int(lcm(m, n))

```

## 2.3 Strings

### 2.3.1 Longest common substr. (Consecutive)

$O(mn)$  time,  $O(m)$  space

```

1 from functools import lru_cache
2 @lru_cache
3 def lcs(s1, s2):
4     if len(s1) == 0 or len(s2) == 0:
5         return 0
6     return max(
7         lcs(s1[:-1], s2), lcs(s1, s2[:-1]),
8         (s1[-1] == s2[-1]) + lcs(s1[:-1], s2[:-1])
9     )

```

### 2.3.2 Longest common subseq. (Non-consecutive)

```

1 def longestCommonSubsequence(text1, text2):
2     n = len(text1)
3     m = len(text2)
4     prev = [0] * (m + 1)
5     cur = [0] * (m + 1)
6     for idx1 in range(1, n + 1):
7         for idx2 in range(1, m + 1):
8             # matching
9             if text1[idx1 - 1] == text2[idx2 - 1]:
10                cur[idx2] = 1 + prev[idx2 - 1]
11            else:
12                # not matching
13                cur[idx2] = max(cur[idx2 - 1],
14                               prev[idx2])
15            prev = cur.copy()
16    return cur[m]

```

### 2.3.3 KMP Return all matching pos. of P in T

```

1 class KMP:
2     def partial(self, pattern):
3         """ Calc. partial match table: String -> [Int] """
4         ret = [0]
5         for i in range(1, len(pattern)):
6             j = ret[i - 1]

```

```

7             while j > 0 and pattern[j] != pattern[i]: j = ret[j - 1]
8             ret.append(j + 1 if pattern[j] == pattern[i] else j)
9         return ret
10
11     def search(self, T, P):
12         """KMPString -> String -> [Int]"""
13         partial, ret, j = self.partial(P), [], 0
14         for i in range(len(T)):
15             while j > 0 and T[i] != P[j]: j = partial[j - 1]
16             if T[i] == P[j]: j += 1
17             if j == len(P):
18                 ret.append(i - (j - 1))
19                 j = partial[j - 1]
20         return ret

```

### 2.3.4 Suffix Array

```

1 class Entry:
2     def __init__(self, pos, nr):
3         self.p = pos
4         self.nr = nr
5
6     def __lt__(self, other):
7         return self.nr < other.nr
8
9 class SA:
10     def __init__(self, s):
11         self.P = []
12         self.n = len(s)
13         self.build(s)
14
15     def build(self, s): # n log log n
16         n = self.n
17         L = [Entry(0, 0) for _ in range(n)]
18         self.P = []
19         self.P.append([ord(c) for c in s])
20         step = 1
21         count = 1
22
23         # self.P[step][i] stores the position
24         # of the i-th longest suffix
25         # if suffixes are sorted according to
26         # their first 2^step characters.
27         while count < 2 * n:
28             self.P.append([0] * n)
29             for i in range(n):
30                 nr = (self.P[step - 1][i],
31                     self.P[step - 1][i + count])
32                 if i + count < n else -1)
33             L[i].p = i

```

```

34         L[i].nr = nr
35     L.sort()
36     for i in range(n):
37         if i > 0 and L[i].nr == L[i -
38             1].nr:
39             self.P[step][L[i].p] = \
40                 self.P[step][L[i - 1].p]
41         else:
42             self.P[step][L[i].p] = i
43     step += 1
44     count *= 2
45
46     self.sa = [0] * n
47     for i in range(n):
48         self.sa[self.P[-1][i]] = i

```

**2.3.5 Longest common pref.** with the suffix array built we can do, e.g., longest common prefix of  $x$ ,  $y$  with suffixarray where  $x, y$  are suffixes of the string used  $O(\log n)$

```

1 def lcp(x, y, P):
2     res = 0
3     if x == y:
4         return n - x
5     for k in range(len(P) - 1, -1, -1):
6         if x >= n or y >= n:
7             break
8         if P[k][x] == P[k][y]:
9             x += 1 << k
10            y += 1 << k
11            res += 1 << k
12    return res

```

### 2.3.6 Edit distance

```

1 def editDistance(str1, str2):
2     m = len(str1)
3     n = len(str2)
4     curr = [0] * (n + 1)
5     for j in range(n + 1):
6         curr[j] = j
7     previous = 0
8     # dp rows
9     for i in range(1, m + 1):
10        previous = curr[0]
11        curr[0] = i
12
13    # dp cols
14    for j in range(1, n + 1):
15        temp = curr[j]
16        if str1[i - 1] == str2[j - 1]:

```

```

17        curr[j] = previous
18    else:
19        curr[j] = 1 + min(previous, curr[j - 1],
20            curr[j])
21    previous = temp
22    return curr[n]

```

**2.3.7 Bitstring** Slower than a set for many elements, but hashable

```

1 def add_element(bit_string, index):
2     return bit_string | (1 << index)
3
4 def remove_element(bit_string, index):
5     return bit_string & ~(1 << index)
6
7 def contains_element(bit_string, index):
8     return (bit_string & (1 << index)) != 0

```

## 2.4 Geometry

### 2.4.1 Convex Hull

```

1 def vec(a,b):
2     return (b[0]-a[0],b[1]-a[1])
3 def det(a,b):
4     return a[0]*b[1] - b[0]*a[1]
5
6 def convexhull(P):
7     if (len(P) == 1):
8         return [(p[0][0], p[0][1])]
9
10    h = sorted(P)
11    lower = []
12    i = 0
13    while i < len(h):
14        if len(lower) > 1:
15            a = vec(lower[-2], lower[-1])
16            b = vec(lower[-1], h[i])
17            if det(a,b) <= 0 and len(lower) > 1:
18                lower.pop()
19                continue
20            lower.append(h[i])
21            i += 1
22
23    upper = []
24    i = 0
25    while i < len(h):
26        if len(upper) > 1:
27            a = vec(upper[-2], upper[-1])
28            b = vec(upper[-1], h[i])
29            if det(a,b) >= 0:

```

```

30            upper.pop()
31            continue
32            upper.append(h[i])
33            i += 1
34
35    reversedupper = list(reversed(upper[1:-1]))
36    reversedupper.extend(lower)
37    return reversedupper

```

### 2.4.2 Geometry

```

1
2 def vec(a,b):
3     return (b[0]-a[0],b[1]-a[1])
4
5 def det(a,b):
6     return a[0]*b[1] - b[0]*a[1]
7
8     lower = []
9     i = 0
10    while i < len(h):
11        if len(lower) > 1:
12            a = vec(lower[-2], lower[-1])
13            b = vec(lower[-1], h[i])
14            if det(a,b) <= 0 and len(lower) > 1:
15                lower.pop()
16                continue
17            lower.append(h[i])
18            i += 1
19
20    # find upper hull
21    # det <= 0 -> replace
22    upper = []
23    i = 0
24    while i < len(h):
25        if len(upper) > 1:
26            a = vec(upper[-2], upper[-1])
27            b = vec(upper[-1], h[i])
28            if det(a,b) >= 0:
29                upper.pop()
30                continue
31            upper.append(h[i])
32            i += 1

```

## 2.5 Other Algorithms

### 2.5.1 Rotate matrix

```

1 def rotate_matrix(m):
2     return [[m[j][i] for j in range(len(m))] for
3         i in range(len(m[0])-1,-1,-1)]

```

## 2.6 Other Data Structures

### 2.6.1 Segment Tree

```

1 N = 100000 # arr max size
2 tree = [0] * (2 * N) # tre max size
3
4 def build(arr, n):
5     for i in range(n):
6         tree[n + i] = arr[i]
7
8     for i in range(n - 1, 0, -1):
9         tree[i] = tree[i << 1] + tree[i << 1 | 1]
10
11 def updateTreeNode(p, value, n):
12     tree[p + n] = value
13     p = p + n
14
15     i = p # move upward, update parents
16     while i > 1:
17         tree[i >> 1] = tree[i] + tree[i ^ 1]
18         i >>= 1
19
20 def query(l, r, n): # sum [l, r]
21     res = 0
22     l += n
23     r += n
24     while l < r:
25         if l & 1:
26             res += tree[l]
27             l += 1
28         if r & 1:
29             r -= 1
30             res += tree[r]
31         l >>= 1
32         r >>= 1
33     return res

```

### 2.6.2 Trie

```

1 class TrieNode:
2     def __init__(self):
3         self.children = [None]*26
4         self.isEndOfWord = False
5
6 class Trie:
7     def __init__(self):
8         self.root = self.getNode()
9
10    def getNode(self):
11        return TrieNode()
12
13    def _charToIndex(self, ch):
14        return ord(ch)-ord('a')

```

```

15
16
17    def insert(self, key):
18        pCrawl = self.root
19        length = len(key)
20        for level in range(length):
21            index = self._charToIndex(key[level])
22            if not pCrawl.children[index]:
23                pCrawl.children[index] = self.
24                    getNode()
25                pCrawl = pCrawl.children[index]
26                pCrawl.isEndOfWord = True
27
28    def search(self, key):
29        pCrawl = self.root
30        length = len(key)
31        for level in range(length):
32            index = self._charToIndex(key[level])
33            if not pCrawl.children[index]:
34                return False
35            pCrawl = pCrawl.children[index]
36
37        return pCrawl.isEndOfWord

```

## 3 C++

### 3.1 Graphs

#### 3.1.1 BFS

```

1 #include "header.h"
2 #define graph unordered_map<ll, unordered_set<ll
3 >>
4 vi bfs(int n, graph& g, vi& roots) {
5     vi parents(n+1, -1); // nodes are 1..n
6     unordered_set<int> visited;
7     queue<int> q;
8     for (auto x: roots) {
9         q.emplace(x);
10        visited.insert(x);
11    }
12    while (not q.empty()) {
13        int node = q.front();
14        q.pop();
15
16        for (auto neigh: g[node]) {
17            if (not in(neigh, visited)) {
18                parents[neigh] = node;
19                q.emplace(neigh);
20                visited.insert(neigh);
21            }
22        }
23    }
24 }

```

```

23     return parents;
24 }
25 vi reconstruct_path(vi parents, int start, int
26 goal) {
27     vi path;
28     int curr = goal;
29     while (curr != start) {
30         path.push_back(curr);
31         if (parents[curr] == -1) return vi(); //
32         No path, empty vi
33         curr = parents[curr];
34     }
35     path.push_back(start);
36     reverse(path.begin(), path.end());
37     return path;
38 }

```

#### 3.1.2 DFS Cycle detection / removal

```

1 #include "header.h"
2 void removeCyc(ll node, unordered_map<ll, vector<
3 pair<ll, ll>>>& neighs, vector<bool>& visited
4 ,
5 vector<bool>& recStack, vector<ll>& ans) {
6     if (!visited[node]) {
7         visited[node] = true;
8         recStack[node] = true;
9         auto it = neighs.find(node);
10        if (it != neighs.end()) {
11            for (auto util: it->second) {
12                ll nnode = util.first;
13                if (recStack[nnode]) {
14                    ans.push_back(util.second);
15                } else if (!visited[nnode]) {
16                    removeCyc(nnode, neighs,
17                        visited, recStack, ans);
18                }
19            }
20        }
21        recStack[node] = false;
22    }
23 }

```

#### 3.1.3 Dijkstra

```

1 #include "header.h"
2 vector<int> dijkstra(int n, int root, map<int,
3 vector<pair<int, int>>>& g) {
4     unordered_set<int> visited;
5     vector<int> dist(n, INF);
6     priority_queue<pair<int, int>> pq;
7     dist[root] = 0;
8     pq.push({0, root});
9 }

```

```

13         i = get<1>(edge);
14         j = get<2>(edge);
15
16         if (fs.find_set(i) != fs.find_set(j)) {
17             fs.union_sets(i, j);
18             ans.insert({i, j});
19             cost += dist;
20         }
21     }
22     return pair<set<pair<ll, ll>>, ll> {ans, cost};
23 }

```

**3.1.6 Hungarian algorithm** Given  $J$  jobs and  $W$  workers ( $J \leq W$ ), computes the minimum cost to assign each prefix of jobs to distinct workers.

**3.1.7 Suc. shortest path** Calculates max flow, min cost

```

1 #include "header.h"
2 // map<node, map<node, pair<cost, capacity>>>
3 #define graph unordered_map<int, unordered_map<
    int, pair<ld, int>>>
4 graph g;
5 const ld infity = 1e60l; // Change if necessary
6 ld fill(int n, vld& potential) { // Finds max
    flow, min cost
7     priority_queue<pair<ld, int>> pq;
8     vector<bool> visited(n+2, false);
9     vi parent(n+2, 0);
10    vld dist(n+2, infity);
11    dist[0] = 0.1;
12    pq.emplace(make_pair(0.1, 0));
13    while (not pq.empty()) {
14        int node = pq.top().second;
15        pq.pop();
16        if (visited[node]) continue;
17        visited[node] = true;
18        for (auto& x : g[node]) {
19            int neigh = x.first;
20            int capacity = x.second.second;
21            ld cost = x.second.first;
22            if (capacity and not visited[neigh]) {
23                ld d = dist[node] + cost + potential[node]
                    - potential[neigh];
24                if (d + 1e-10l < dist[neigh]) {
25                    dist[neigh] = d;
26                    pq.emplace(make_pair(-d, neigh));

```

```

1 #include "header.h"
2 // g[i][j] = inf if not path from i to j
3 // if g[i][i] < 0, i is contained in a negative
  cycle
4 void warshall(vvl g) {
5     for (int i=0; i<g.size(); ++i) {
6         for (int j=0; j<g.size(); ++j) {
7             for (int k=0; k<g.size(); ++k) {
8                 if (g[i][k] < LLINF and g[k][j] <
                    LLINF and g[i][j] > g[i][k]
                    + g[k][j]) {
9                     g[i][j] = g[i][k] + g[k][j];
10                }
            }
        }
    }
}

```

```

1 #include "header.h"
2 #include "disjoint_set.h"
3 pair<set<pair<ll, ll>>, ll> kruskal(vector<tuple
   <ll, ll, ll>>& edges, ll n) {
4     set<pair<ll, ll>> ans;
5     ll cost = 0;
6
7     sort(edges.begin(), edges.end());
8     DisjointSet<ll> fs(n);
9
10    ll dist, i, j;
11    for (auto edge: edges) {
12        dist = get<0>(edge);

```



```

27     parent[neigh] = node;
28 }}}}
29
30 for (int i = 0; i < n+2; i++) {
31     potential[i] = min(infty, potential[i] + dist
32         [i]);
33 }
34 if (not parent[n+1]) return infty;
35 ld ans = 0.1;
36 for (int x = n+1; x; x=parent[x]) {
37     ans += g[parent[x]][x].first;
38     g[parent[x]][x].second--;
39     g[x][parent[x]].second++;
40 }
41 return ans;
42 }

```

### 3.1.8 Bipartite check

```

1 #include "header.h"
2 int main() {
3     int n;
4     vvi adj(n);
5
6     vi side(n, -1);    // will have 0's for one
7                         // side 1's for other side
8     bool is_bipartite = true;    // becomes false
9     if not bipartite
10     queue<int> q;
11     for (int st = 0; st < n; ++st) {
12         if (side[st] == -1) {
13             q.push(st);
14             side[st] = 0;
15             while (!q.empty()) {
16                 int v = q.front();
17                 q.pop();
18                 for (int u : adj[v]) {
19                     if (side[u] == -1) {
20                         side[u] = side[v] ^ 1;
21                         q.push(u);
22                     } else {
23                         is_bipartite &= side[u]
24                             != side[v];
25                     }
26                 }
27             }
28         }
29     }
30 }

```

### 3.1.9 Find cycle directed

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5+5;
4 vvi adj(mxN);
5 vector<char> color;

```

```

6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v) {
9     color[v] = 1;
10    for (int u : adj[v]) {
11        if (color[u] == 0) {
12            parent[u] = v;
13            if (dfs(u)) return true;
14        } else if (color[u] == 1) {
15            cycle_end = v;
16            cycle_start = u;
17            return true;
18        }
19    }
20    color[v] = 2;
21    return false;
22 }
23 void find_cycle() {
24     color.assign(n, 0);
25     parent.assign(n, -1);
26     cycle_start = -1;
27     for (int v = 0; v < n; v++) {
28         if (color[v] == 0 && dfs(v)) break;
29     }
30     if (cycle_start == -1) {
31         cout << "Acyclic" << endl;
32     } else {
33         vector<int> cycle;
34         cycle.push_back(cycle_start);
35         for (int v = cycle_end; v != cycle_start;
36             v = parent[v])
37             cycle.push_back(v);
38         cycle.push_back(cycle_start);
39         reverse(cycle.begin(), cycle.end());
40
41         cout << "Cycle Found: ";
42         for (int v : cycle) cout << v << " ";
43         cout << endl;
44     }
45 }

```

### 3.1.10 Find cycle undirected

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5 + 5;
4 vvi adj(mxN);
5 vector<bool> visited;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v, int par) { // passing vertex and
9     its parent vertex
10    visited[v] = true;
11    for (int u : adj[v]) {

```

```

11        if(u == par) continue; // skipping edge
12        to parent vertex
13        if (visited[u]) {
14            cycle_end = v;
15            cycle_start = u;
16            return true;
17        }
18        parent[u] = v;
19        if (dfs(u, parent[u]))
20            return true;
21    }
22    return false;
23 }
24 void find_cycle() {
25     visited.assign(n, false);
26     parent.assign(n, -1);
27     cycle_start = -1;
28     for (int v = 0; v < n; v++) {
29         if (!visited[v] && dfs(v, parent[v]))
30             break;
31     }
32     if (cycle_start == -1) {
33         cout << "Acyclic" << endl;
34     } else {
35         vector<int> cycle;
36         cycle.push_back(cycle_start);
37         for (int v = cycle_end; v != cycle_start;
38             v = parent[v])
39             cycle.push_back(v);
40         cycle.push_back(cycle_start);
41         cout << "Cycle Found: ";
42         for (int v : cycle) cout << v << " ";
43         cout << endl;
44     }
45 }

```

### 3.1.11 Tarjan's SCC

```

1 #include "header.h"
2
3 struct Tarjan {
4     vvi &edges;
5     int V, counter = 0, C = 0;
6     vi n, l;
7     vector<bool> vs;
8     stack<int> st;
9     Tarjan(vvi &e) : edges(e), V(e.size()), n(V,
10         -1), l(V, -1), vs(V, false) {}
11     void visit(int u, vi &com) {
12         l[u] = n[u] = counter++;
13         st.push(u);
14         vs[u] = true;
15         for (auto &v : edges[u]) {

```



```

16     if (vs[v]) l[u] = min(l[u], l[v]);
17 }
18 if (l[u] == n[u]) {
19     while (true) {
20         int v = st.top();
21         st.pop();
22         vs[v] = false;
23         com[v] = C; //<== ACT HERE
24         if (u == v) break;
25     }
26     C++;
27 }
28 }
29 int find_sccs(vi &com) { // component indices
30     // will be stored in 'com'
31     com.assign(V, -1);
32     C = 0;
33     for (int u = 0; u < V; ++u)
34         if (n[u] == -1) visit(u, com);
35     return C;
36 }
37 // scc is a map of the original vertices of the
38 // graph to the vertices of the SCC graph,
39 // scc_graph is its adjacency list. SCC
40 // indices and edges are stored in 'scc' and '
41 // scc_graph'.
42 void scc_collapse(vi &scc, vvi &scc_graph) {
43     find_sccs(scc);
44     scc_graph.assign(C, vi());
45     set<pi> rec; // recorded edges
46     for (int u = 0; u < V; ++u) {
47         assert(scc[u] != -1);
48         for (int v : edges[u]) {
49             if (scc[v] == scc[u] ||
50                 rec.find({scc[u], scc[v]}) != rec.end())
51                 continue;
52             scc_graph[scc[u]].push_back(scc[v]);
53             rec.insert({scc[u], scc[v]});
54         }
55     }
56 }
57 // The number of edges needed to be added is
58 // max(sources.size(), sinks.size())
59 void findSourcesAndSinks(const vvi &scc_graph,
60     vi &sources, vi &sinks) {
61     vi in_degree(C, 0), out_degree(C, 0);
62     for (int u = 0; u < C; ++u) {
63         for (auto v : scc_graph[u]) {
64             in_degree[v]++;
65             out_degree[u]++;
66         }
67     }
68 }
69 for (int i = 0; i < C; ++i) {
70     if (in_degree[i] == 0) sources.push_back(i);
71 }

```

```

62     if (out_degree[i] == 0) sinks.push_back(i);
63 }
64 }
65 };

```

**3.1.12 SCC edges** Prints out the missing edges to make the input digraph strongly connected

```

1 #include "header.h"
2 const int N=1e5+10;
3 int n,a[N],cnt[N],vis[N];
4 vector<int> hd,tl;
5 int dfs(int x){
6     vis[x]=1;
7     if(!vis[a[x]])return vis[x]=dfs(a[x]);
8     return vis[x]=x;
9 }
10 int main(){
11     scanf("%d",&n);
12     for(int i=1;i<=n;i++){
13         scanf("%d",&a[i]);
14         cnt[a[i]]++;
15     }
16     int k=0;
17     for(int i=1;i<=n;i++){
18         if(!cnt[i]){
19             k++;
20             hd.push_back(i);
21             tl.push_back(dfs(i));
22         }
23     }
24     int tk=k;
25     for(int i=1;i<=n;i++){
26         if(!vis[i]){
27             k++;
28             hd.push_back(i);
29             tl.push_back(dfs(i));
30         }
31     }
32     if(k==1&&!tk)k=0;
33     printf("%d\n",k);
34     for(int i=0;i<k;i++)printf("%d_%d\n",tl[i],hd
35         [(i+1)%k]);
36     return 0;

```

**3.1.13 Find Bridges**

```

1 #include "header.h"
2 int n; // number of nodes
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi tin, low;

```

```

6 int timer;
7 void dfs(int v, int p = -1) {
8     visited[v] = true;
9     tin[v] = low[v] = timer++;
10    for (int to : adj[v]) {
11        if (to == p) continue;
12        if (visited[to]) {
13            low[v] = min(low[v], tin[to]);
14        } else {
15            dfs(to, v);
16            low[v] = min(low[v], low[to]);
17            if (low[to] > tin[v])
18                IS_BRIDGE(v, to);
19        }
20    }
21 }
22 void find_bridges() {
23     timer = 0;
24     visited.assign(n, false);
25     tin.assign(n, -1);
26     low.assign(n, -1);
27     for (int i = 0; i < n; ++i) {
28         if (!visited[i]) dfs(i);
29     }
30 }

```

**3.1.14 Articulation points** (i.e. cut off points)

```

1 #include "header.h"
2 int n; // number of nodes
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi tin, low;
6 int timer;
7 void dfs(int v, int p = -1) {
8     visited[v] = true;
9     tin[v] = low[v] = timer++;
10    int children=0;
11    for (int to : adj[v]) {
12        if (to == p) continue;
13        if (visited[to]) {
14            low[v] = min(low[v], tin[to]);
15        } else {
16            dfs(to, v);
17            low[v] = min(low[v], low[to]);
18            if (low[to] >= tin[v] && p!=-1)
19                IS_CUTPOINT(v);
20            ++children;
21        }
22    }
23    if(p == -1 && children > 1)
24        IS_CUTPOINT(v);
25 void find_cutpoints() {

```

```

26 timer = 0;
27 visited.assign(n, false);
28 tin.assign(n, -1);
29 low.assign(n, -1);
30 for (int i = 0; i < n; ++i) {
31     if (!visited[i]) dfs(i);
32 }
33 }

```

### 3.1.15 Topological sort

```

1 #include "header.h"
2 int n; // number of vertices
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi ans;
6 void dfs(int v) {
7     visited[v] = true;
8     for (int u : adj[v]) {
9         if (!visited[u]) dfs(u);
10    }
11    ans.push_back(v);
12 }
13 void topological_sort() {
14     visited.assign(n, false);
15     ans.clear();
16     for (int i = 0; i < n; ++i) {
17         if (!visited[i]) dfs(i);
18     }
19     reverse(ans.begin(), ans.end());
20 }

```

### 3.1.16 Bellmann-Ford Same as Dijkstra but allows neg. edges

```

1 #include "header.h"
2 // Switch vi and vvp1 to vl and vvpl if necessary
3 void bellmann_ford_extended(vvp1 &e, int source,
4     vi &dist, vb &cyc) {
5     dist.assign(e.size(), INF);
6     cyc.assign(e.size(), false); // true when u is
7     // in a <0 cycle
8     dist[source] = 0;
9     for (int iter = 0; iter < e.size() - 1; ++iter) {
10        {
11            bool relax = false;
12            for (int u = 0; u < e.size(); ++u)
13                if (dist[u] == INF) continue;
14            else for (auto &e : e[u])
15                if (dist[u] + e.second < dist[e.first])
16                    dist[e.first] = dist[u] + e.second, relax
17                    = true;
18            if (!relax) break;
19        }
20    }

```

```

15 }
16 bool ch = true;
17 while (ch) { // keep going untill no more
18     // changes
19     ch = false; // set dist to -INF when in cycle
20     for (int u = 0; u < e.size(); ++u)
21         if (dist[u] == INF) continue;
22         else for (auto &e : e[u])
23             if (dist[e.first] > dist[u] + e.second
24                 && !cyc[e.first]) {
25                 dist[e.first] = -INF;
26                 ch = true; //return true for cycles
27                 cyc[e.first] = true;
28             }
29 }

```

### 3.1.17 Ford-Fulkerson Basic Max. flow

```

1 #include "header.h"
2 #define V 6 // Num. of vertices in given graph
3 /* Returns true if there is a path from source 's'
4    't' to sink
5    't' in residual graph. Also fills parent[] to
6    store the
7    path */
8 bool bfs(int rGraph[V][V], int s, int t, int
9     parent[]) {
10    bool visited[V];
11    memset(visited, 0, sizeof(visited));
12    queue<int> q;
13    q.push(s);
14    visited[s] = true;
15    parent[s] = -1;
16    while (!q.empty()) {
17        int u = q.front();
18        q.pop();
19        for (int v = 0; v < V; v++) {
20            if (visited[v] == false && rGraph[u][v] >
21                0) {
22                if (v == t) {
23                    parent[v] = u;
24                    return true;
25                }
26                q.push(v);
27                parent[v] = u;
28                visited[v] = true;
29            }
30        }
31    }
32    return false;
33 }
34 // Returns the maximum flow from s to t

```

```

32 int fordFulkerson(int graph[V][V], int s, int t)
33 {
34     int u, v;
35     int rGraph[V][V];
36     for (u = 0; u < V; u++)
37         for (v = 0; v < V; v++)
38             rGraph[u][v] = graph[u][v];
39
40     int parent[V]; // BFS-filled (to store path)
41     int max_flow = 0; // no flow initially
42     while (bfs(rGraph, s, t, parent)) {
43         int path_flow = INT_MAX;
44         for (v = t; v != s; v = parent[v]) {
45             u = parent[v];
46             path_flow = min(path_flow, rGraph[u][v]);
47         }
48         for (v = t; v != s; v = parent[v]) {
49             u = parent[v];
50             rGraph[u][v] -= path_flow;
51             rGraph[v][u] += path_flow;
52         }
53         max_flow += path_flow;
54     }
55     return max_flow;
56 }

```

### 3.1.18 Dinic max flow $O(V^2E)$ , $O(Ef)$

```

1 #include "header.h"
2 using F = ll; using W = ll; // types for flow and
3 // weight/cost
4 struct S {
5     const int v; // neighbour
6     const int r; // index of the reverse edge
7     F f; // current flow
8     const F cap; // capacity
9     const W cost; // unit cost
10    S(int v, int ri, F c, W cost = 0) :
11        v(v), r(ri), f(0), cap(c), cost(cost) {}
12    inline F res() const { return cap - f; }
13 };
14 struct FlowGraph : vector<vector<S>> {
15     FlowGraph(size_t n) : vector<vector<S>>(n) {}
16     void add_edge(int u, int v, F c, W cost = 0) {
17         auto &t = *this;
18         t[u].emplace_back(v, t[v].size(), c, cost);
19         t[v].emplace_back(u, t[u].size() - 1, c, -cost);
20     }
21     void add_arc(int u, int v, F c, W cost = 0) {
22         auto &t = *this;
23         t[u].emplace_back(v, t[v].size(), c, cost);
24     }
25 }

```

```

21     t[v].emplace_back(u, t[u].size()-1, 0, -
        cost);
22 }
23 void clear() { for (auto &E : *this) for (
        auto &e : E) e.f = 0LL; }
24 };
25 struct Dinic{
26     FlowGraph &edges; int V,s,t;
27     vi l; vector<vector<S>::iterator> its; //
        levels and iterators
28     Dinic(FlowGraph &edges, int s, int t) :
        edges(edges), V(edges.size()), s(s), t(t)
        , l(V,-1), its(V) {}
29     ll augment(int u, F c) { // we reuse the same
        iterators
30         if (u == t) return c; ll r = 0LL;
31         for(auto &i = its[u]; i != edges[u].end()
            ; i++){
32             auto &e = *i;
33             if (e.res() && l[u] < l[e.v]) {
34                 auto d = augment(e.v, min(c, e.
                    res()));
35                 if (d > 0) { e.f += d; edges[e.v
                    ][e.r].f -= d; c -= d;
36                     r += d; if (!c) break; }
37             }
38         }
39         return r;
40     }
41     ll run() {
42         ll flow = 0, f;
43         while(true) {
44             fill(l.begin(), l.end(),-1); l[s]=0;
45             queue<int> q; q.push(s);
46             while(!q.empty()){
47                 auto u = q.front(); q.pop(); its[
                    u] = edges[u].begin();
48                 for(auto &&e : edges[u]) if(e.res
                    () && l[e.v]<0)
49                     l[e.v] = l[u]+1, q.push(e.v);
50             }
51             if (l[t] < 0) return flow;
52             while ((f = augment(s, INF)) > 0)
                flow += f;
53         }
54 };

```

**3.1.19 Edmonds-Karp** (Max) flow algorithm with time  $O(VE^2)$ . To get edge flow values, compare capacities before and after, and take the positive values only.

```

1 #include "header.h"
2 template<class T> T edmondsKarp(vector<
    unordered_map<int, T>>&
3     graph, int source, int sink) {

```

```

4     assert(source != sink);
5     T flow = 0;
6     vi par(sz(graph)), q = par;
7
8     for (;;) {
9         fill(all(par), -1);
10        par[source] = 0;
11        int ptr = 1;
12        q[0] = source;
13
14        rep(i,0,ptr) {
15            int x = q[i];
16            for (auto e : graph[x]) {
17                if (par[e.first] == -1 && e.second > 0) {
18                    par[e.first] = x;
19                    q[ptr++] = e.first;
20                    if (e.first == sink) goto out;
21                }
22            }
23        }
24        return flow;
25    out:
26        T inc = numeric_limits<T>::max();
27        for (int y = sink; y != source; y = par[y])
28            inc = min(inc, graph[par[y]][y]);
29
30        flow += inc;
31        for (int y = sink; y != source; y = par[y]) {
32            int p = par[y];
33            if ((graph[p][y] -= inc) <= 0) graph[p].
                erase(y);
34            graph[y][p] += inc;
35        }
36    }
37 }

```

## 3.2 Dynamic Programming

### 3.2.1 Longest Incr. Subseq.

```

1 #include "header.h"
2 template<class T>
3 vector<T> index_path_lis(vector<T>& nums) {
4     int n = nums.size();
5     vector<T> sub;
6     vector<int> subIndex;
7     vector<T> path(n, -1);
8     for (int i = 0; i < n; ++i) {
9         if (sub.empty() || sub[sub.size() - 1] <
            nums[i]) {
10            path[i] = sub.empty() ? -1 : subIndex[sub.
                size() - 1];
11            sub.push_back(nums[i]);
12            subIndex.push_back(i);

```

```

13        } else {
14            int idx = lower_bound(sub.begin(), sub.end(),
                nums[i]) - sub.begin();
15            path[i] = idx == 0 ? -1 : subIndex[idx - 1];
16            sub[idx] = nums[i];
17            subIndex[idx] = i;
18        }
19    }
20    vector<T> ans;
21    int t = subIndex[subIndex.size() - 1];
22    while (t != -1) {
23        ans.push_back(t);
24        t = path[t];
25    }
26    reverse(ans.begin(), ans.end());
27    return ans;
28 }
29 // Length only
30 template<class T>
31 int length_lis(vector<T> &a) {
32     set<T> st;
33     typename set<T>::iterator it;
34     for (int i = 0; i < a.size(); ++i) {
35         it = st.lower_bound(a[i]);
36         if (it != st.end()) st.erase(it);
37         st.insert(a[i]);
38     }
39     return st.size();
40 }

```

**3.2.2 0-1 Knapsack** Given a number of coins, calculate all possible distinct sums

```

1 #include "header.h"
2 int main() {
3     int n;
4     vi coins(n); // possible coins to use
5     int sum = 0; // their sum of the coins
6     vi dp(sum + 1, 0); // dp[x] = 1 if sum x can be
        made
7     dp[0] = 1;
8     for (int c = 0; c < n; ++c)
9         for (int x = sum; x >= 0; --x)
10             if (dp[x]) dp[x + coins[c]] = 1;
11 }

```

**3.2.3 Coin change** Total distinct ways to make sum using  $n$  coins of different vals

```

1 #include "header.h"
2 int count(vi& coins, int n, int sum) {
3     vvi dp(n + 1, vi(sum + 1, 0));

```

```

4 dp[0][0] = 1;
5 for (int i = 1; i <= n; i++) {
6     for (int j = 0; j <= sum; j++) {
7         // without using the current coin,
8         dp[i][j] += dp[i - 1][j];
9         // using the current coin
10        if ((j - coins[i - 1]) >= 0)
11            dp[i][j] += dp[i][j - coins[i - 1]];
12    }
13 }
14 return dp[n][sum];
15 }

```

## 3.3 Numerical

### 3.3.1 Template (for this section)

```

1 #include <bits/stdc++.h>
2 using namespace std;
3 #define rep(i, a, b) for(int i = a; i < (b); ++i)
4 #define all(x) begin(x), end(x)
5 #define sz(x) (int)(x).size()
6 typedef long long ll;
7 typedef pair<int, int> pii;
8 typedef vector<int> vi;

```

### 3.3.2 Polynomial

```

1 #include "template.cpp"
2 struct Poly {
3     vector<double> a;
4     double operator()(double x) const {
5         double val = 0;
6         for (int i = sz(a); i--;) (val += x) += a[i];
7         return val;
8     }
9     void diff() {
10        rep(i, 1, sz(a)) a[i-1] = i*a[i];
11        a.pop_back();
12    }
13    void divroot(double x0) {
14        double b = a.back(), c; a.back() = 0;
15        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
16        a.pop_back();
17    }
18 };

```

**3.3.3 Poly Roots** Finds the real roots to a polynomial.  $O(n^2 \log(1/\epsilon))$

```

1 // Usage: polyRoots({{2,-3,1}},-1e9,1e9) = solve
2 // x^2-3x+2 = 0
3 #include "Polynomial.h"
4 #include "template.cpp"
5 vector<double> polyRoots(Poly p, double xmin,
6 double xmax) {
7     if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
8     vector<double> ret;
9     Poly der = p;
10    der.diff();
11    auto dr = polyRoots(der, xmin, xmax);
12    dr.push_back(xmin-1);
13    dr.push_back(xmax+1);
14    sort(all(dr));
15    rep(i, 0, sz(dr)-1) {
16        double l = dr[i], h = dr[i+1];
17        bool sign = p(l) > 0;
18        if (sign ^ (p(h) > 0)) {
19            rep(it, 0, 60) { // while (h - l > 1e-8)
20                double m = (l + h) / 2, f = p(m);
21                if ((f <= 0) ^ sign) l = m;
22                else h = m;
23            }
24            ret.push_back((l + h) / 2);
25        }
26    }
27    return ret;
28 }

```

**3.3.4 Golden Section Search** Finds the argument minimizing the function  $f$  in the interval  $[a, b]$  assuming  $f$  is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is  $\epsilon$ . Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version.  $O(\log((b-a)/\epsilon))$

```

1 /** Usage:
2 double func(double x) { return 4+x+.3*x*x; }
3 double xmin = gss(-1000,1000,func); */
4 #include "template.cpp"
5 // It is important for r to be precise, otherwise
6 // we don't necessarily maintain the inequality
7 // a < x1 < x2 < b.
8 double gss(double a, double b, double (*f)(double)) {
9     double r = (sqrt(5)-1)/2, eps = 1e-7;
10    double x1 = b - r*(b-a), x2 = a + r*(b-a);
11    double f1 = f(x1), f2 = f(x2);
12    while (b-a > eps)
13        if (f1 < f2) { //change to > to find maximum
14            b = x2; x2 = x1; f2 = f1;
15        }
16    }

```

```

13 x1 = b - r*(b-a); f1 = f(x1);
14 } else {
15     a = x1; x1 = x2; f1 = f2;
16     x2 = a + r*(b-a); f2 = f(x2);
17 }
18 return a;
19 }

```

**3.3.5 Hill Climbing** Poor man's optimization for unimodal functions.

```

1 #include "template.cpp"
2 typedef array<double, 2> P;
3 template<class F> pair<double, P> hillClimb(P start, F f) {
4     pair<double, P> cur(f(start), start);
5     for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
6         rep(j, 0, 100) rep(dx, -1, 2) rep(dy, -1, 2) {
7             P p = cur.second;
8             p[0] += dx*jmp;
9             p[1] += dy*jmp;
10            cur = min(cur, make_pair(f(p), p));
11        }
12    }
13    return cur;
14 }

```

**3.3.6 Integration** Simple integration of a function over an interval using Simpson's rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

```

1 #include "template.cpp"
2 template<class F>
3 double quad(double a, double b, F f, const int n = 1000) {
4     double h = (b - a) / 2 / n, v = f(a) + f(b);
5     rep(i, 1, n*2)
6         v += f(a + i*h) * (i&1 ? 4 : 2);
7     return v * h / 3;
8 }

```

**3.3.7 Integration Adaptive** Fast integration using an adaptive Simpson's rule.

```

1 /** Usage:
2 double sphereVolume = quad(-1, 1, [](double x) {
3     return quad(-1, 1, [&](double y) {
4         return quad(-1, 1, [&](double z) {

```

```

5 return x*x + y*y + z*z < 1; }));}); */
6 #include "template.cpp"
7
8 typedef double d;
9 #define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (
    b-a) / 6
10
11 template <class F>
12 d rec(F& f, d a, d b, d eps, d S) {
13     d c = (a + b) / 2;
14     d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
15     if (abs(T - S) <= 15 * eps || b - a < 1e-10)
16         return T + (T - S) / 15;
17     return rec(f, a, c, eps / 2, S1) + rec(f, c, b,
        eps / 2, S2);
18 }
19 template<class F>
20 d quad(d a, d b, F f, d eps = 1e-8) {
21     return rec(f, a, b, eps, S(a, b));
22 }

```

## 3.4 Num. Th. / Comb.

### 3.4.1 Basic stuff

```

1 #include "header.h"
2 ll gcd(ll a, ll b) { while (b) { a %= b; swap(a,
    b); } return a; }
3 ll lcm(ll a, ll b) { return (a / gcd(a, b)) * b;
    }
4 ll mod(ll a, ll b) { return ((a % b) + b) % b; }
5 // Finds x, y s.t. ax + by = d = gcd(a, b).
6 void extended_euclid(ll a, ll b, ll &x, ll &y, ll
    &d) {
7     ll xx = y = 0;
8     ll yy = x = 1;
9     while (b) {
10         ll q = a / b;
11         ll t = b; b = a % b; a = t;
12         t = xx; xx = x - q * xx; x = t;
13         t = yy; yy = y - q * yy; y = t;
14     }
15     d = a;
16 }
17 // solves ab = 1 (mod n), -1 on failure
18 ll mod_inverse(ll a, ll n) {
19     ll x, y, d;
20     extended_euclid(a, n, x, y, d);
21     return (d > 1 ? -1 : mod(x, n));
22 }
23 // All modular inverses of [1..n] mod P in O(n)
    time.
24 vi inverses(ll n, ll P) {
25     vi I(n+1, 1LL);

```

```

26 for (ll i = 2; i <= n; ++i)
27     I[i] = mod(-(P/i) * I[P%i], P);
28 return I;
29 }
30 // (a*b)%m
31 ll mulmod(ll a, ll b, ll m){
32     ll x = 0, y=a%m;
33     while(b>0){
34         if(b&1) x = (x+y)%m;
35         y = (2*y)%m, b /= 2;
36     }
37     return x % m;
38 }
39 // Finds b^e % m in O(lg n) time, ensure that b <
    m to avoid overflow!
40 ll powmod(ll b, ll e, ll m) {
41     ll p = e<2 ? 1 : powmod((b*b)%m,e/2,m);
42     return e&1 ? p*b%m : p;
43 }
44 // Solve ax + by = c, returns false on failure.
45 bool linear_diophantine(ll a, ll b, ll c, ll &x,
    ll &y) {
46     ll d = gcd(a, b);
47     if (c % d) {
48         return false;
49     } else {
50         x = c / d * mod_inverse(a / d, b / d);
51         y = (c - a * x) / b;
52         return true;
53     }
54 }
55
56 // Description: Tonelli-Shanks algorithm for
    modular square roots. Finds $x$ s.t. $x^2 = a
    \pmod p$ ($-x$ gives the other solution). 0
    ($\log^2 p$) worst case, 0($\log p$) for most $p$
57 ll sqrtmod(ll a, ll p) {
58     a %= p; if (a < 0) a += p;
59     if (a == 0) return 0;
60     assert(powmod(a, (p-1)/2, p) == 1); // else no
        solution
61     if (p % 4 == 3) return powmod(a, (p+1)/4, p);
62     // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if
        p % 8 == 5
63     ll s = p - 1, n = 2;
64     int r = 0, m;
65     while (s % 2 == 0)
66         ++r, s /= 2;
67     /// find a non-square mod p
68     while (powmod(n, (p - 1) / 2, p) != p - 1) ++n;
69     ll x = powmod(a, (s + 1) / 2, p);
70     ll b = powmod(a, s, p), g = powmod(n, s, p);
71     for (;;) r = m) {
72         ll t = b;
73         for (m = 0; m < r && t != 1; ++m)

```

```

74         t = t * t % p;
75         if (m == 0) return x;
76         ll gs = powmod(g, 1LL << (r - m - 1), p);
77         g = gs * gs % p;
78         x = x * gs % p;
79         b = b * g % p;
80     }
81 }

```

### 3.4.2 Mod. exponentiation Or use pow() in python

```

1 #include "header.h"
2 ll mod_pow(ll base, ll exp, ll mod) {
3     if (mod == 1) return 0;
4     if (exp == 0) return 1;
5     if (exp == 1) return base;
6
7     ll res = 1;
8     base %= mod;
9     while (exp) {
10         if (exp % 2 == 1) res = (res * base) % mod;
11         exp >>= 1;
12         base = (base * base) % mod;
13     }
14
15     return res % mod;
16 }

```

### 3.4.3 GCD Or math.gcd in python, std::gcd in C++

```

1 #include "header.h"
2 ll gcd(ll a, ll b) {
3     if (a == 0) return b;
4     return gcd(b % a, a);
5 }

```

### 3.4.4 Sieve of Eratosthenes

```

1 #include "header.h"
2 vl primes;
3 void getprimes(ll n) { // Up to n (not included)
4     vector<bool> p(n, true);
5     p[0] = false;
6     p[1] = false;
7     for(ll i = 0; i < n; i++) {
8         if(p[i]) {
9             primes.push_back(i);
10            for(ll j = i*2; j < n; j+=i) p[j] =
                false;
11        }}

```

### 3.4.5 Fibonacci % prime

```

1 #include "header.h"
2 const ll MOD = 1000000007;
3 unordered_map<ll, ll> Fib;
4 ll fib(ll n) {
5     if (n < 2) return 1;
6     if (Fib.find(n) != Fib.end()) return Fib[n];
7     Fib[n] = (fib((n + 1) / 2) * fib(n / 2) + fib
8         ((n - 1) / 2) * fib((n - 2) / 2)) % MOD;
9     return Fib[n];
10 }

```

### 3.4.6 nCk % prime

```

1 #include "header.h"
2 ll binom(ll n, ll k) {
3     ll ans = 1;
4     for(ll i = 1; i <= min(k, n-k); ++i) ans = ans
5         *(n+1-i)/i;
6     return ans;
7 }
8 ll mod_nCk(ll n, ll k, ll p){
9     ll ans = 1;
10    while(n){
11        ll np = n%p, kp = k%p;
12        if(kp > np) return 0;
13        ans *= binom(np, kp);
14        n /= p; k /= p;
15    }
16    return ans;
17 }

```

## 3.5 Strings

### 3.5.1 Z alg. KMP alternative (same complexities)

```

1 #include "../header.h"
2 void Z_algorithm(const string &s, vi &Z) {
3     Z.assign(s.length(), -1);
4     int L = 0, R = 0, n = s.length();
5     for (int i = 1; i < n; ++i) {
6         if (i > R) {
7             L = R = i;
8             while (R < n && s[R - L] == s[R]) R++;
9             Z[i] = R - L; R--;
10        } else if (Z[i - L] >= R - i + 1) {
11            L = i;
12            while (R < n && s[R - L] == s[R]) R++;
13            Z[i] = R - L; R--;
14        } else Z[i] = Z[i - L];
15    }
16 }

```

### 3.5.2 KMP

```

1 #include "header.h"
2 void compute_prefix_function(string &w, vi &
3     prefix) {
4     prefix.assign(w.length(), 0);
5     int k = prefix[0] = -1;
6     for(int i = 1; i < w.length(); ++i) {
7         while(k >= 0 && w[k + 1] != w[i]) k = prefix[
8             k];
9         if(w[k + 1] == w[i]) k++;
10        prefix[i] = k;
11    }
12 }
13 void knuth_morris_pratt(string &s, string &w) {
14     int q = -1;
15     vi prefix;
16     compute_prefix_function(w, prefix);
17     for(int i = 0; i < s.length(); ++i) {
18         while(q >= 0 && w[q + 1] != s[i]) q = prefix[
19             q];
20         if(w[q + 1] == s[i]) q++;
21         if(q + 1 == w.length()) {
22             // Match at position (i - w.length() + 1)
23             q = prefix[q];
24         }
25     }
26 }

```

### 3.5.3 Aho-Corasick Also can be used as Knuth-Morris-Pratt algorithm

```

1 #include "header.h"
2
3 map<char, int> cti;
4 int cti_size;
5 template <int ALPHABET_SIZE, int (*mp)(char)>
6 struct AC_FSM {
7     struct Node {
8         int child[ALPHABET_SIZE], failure = 0,
9             match_par = -1;
10        vi match;
11        Node() { for (int i = 0; i < ALPHABET_SIZE;
12            ++i) child[i] = -1; }
13    };
14    vector<Node> a;
15    vector<string> &words;
16    AC_FSM(vector<string> &words) : words(words) {
17        a.push_back(Node());
18        construct_automaton();
19    }
20    void construct_automaton() {
21        for (int w = 0, n = 0; w < words.size(); ++w,
22            n = 0) {

```

```

20        for (int i = 0; i < words[w].size(); ++i) {
21            if (a[n].child[mp(words[w][i])] == -1) {
22                a[n].child[mp(words[w][i])] = a.size();
23                a.push_back(Node());
24            }
25            n = a[n].child[mp(words[w][i])];
26        }
27        a[n].match.push_back(w);
28    }
29    queue<int> q;
30    for (int k = 0; k < ALPHABET_SIZE; ++k) {
31        if (a[0].child[k] == -1) a[0].child[k] = 0;
32        else if (a[0].child[k] > 0) {
33            a[a[0].child[k]].failure = 0;
34            q.push(a[0].child[k]);
35        }
36    }
37    while (!q.empty()) {
38        int r = q.front(); q.pop();
39        for (int k = 0, arck; k < ALPHABET_SIZE; ++
40            k) {
41            if ((arck = a[r].child[k]) != -1) {
42                q.push(arck);
43                int v = a[r].failure;
44                while (a[v].child[k] == -1) v = a[v].
45                    failure;
46                a[arck].failure = a[v].child[k];
47                a[arck].match_par = a[v].child[k];
48                while (a[arck].match_par != -1
49                    && a[a[arck].match_par].match.empty
50                        ())
51                    a[arck].match_par = a[a[arck].
52                        match_par].match_par;
53            }
54        }
55    }
56 }
57 void aho_corasick(string &sentence, vvi &
58     matches){
59     matches.assign(words.size(), vi());
60     int state = 0, ss = 0;
61     for (int i = 0; i < sentence.length(); ++i,
62         ss = state) {
63         while (a[ss].child[mp(sentence[i])] == -1)
64             ss = a[ss].failure;
65         state = a[state].child[mp(sentence[i])]
66             = a[ss].child[mp(sentence[i])];
67         for (ss = state; ss != -1; ss = a[ss].
68             match_par)
69             for (int w : a[ss].match)
70                 matches[w].push_back(i + 1 - words[w].
71                     length());
72     }
73 }

```



```

67 int char_to_int(char c) {
68     return cti[c];
69 }
70 int main() {
71     ll n;
72     string line;
73     while(getline(cin, line)) {
74         stringstream ss(line);
75         ss >> n;
76
77         vector<string> patterns(n);
78         for (auto& p: patterns) getline(cin, p);
79
80         string text;
81         getline(cin, text);
82
83         cti = {}, cti_size = 0;
84         for (auto c: text) {
85             if (not in(c, cti)) {
86                 cti[c] = cti_size++;
87             }
88         }
89         for (auto& p: patterns) {
90             for (auto c: p) {
91                 if (not in(c, cti)) {
92                     cti[c] = cti_size++;
93                 }
94             }
95         }
96
97         vvi matches;
98         AC_FSM <128+1, char_to_int> ac_fms(patterns);
99         ac_fms.aho_corasick(text, matches);
100         for (auto& x: matches) cout << x << endl;
101     }
102 }
103 }

```

### 3.5.4 Long. palin. subs Manacher - $O(n)$

```

1 #include "header.h"
2 void manacher(string &s, vi &pal) {
3     int n = s.length(), i = 1, l, r;
4     pal.assign(2 * n + 1, 0);
5     while (i < 2 * n + 1) {
6         if ((i&1) && pal[i] == 0) pal[i] = 1;
7         l = i / 2 - pal[i] / 2; r = (i-1) / 2 + pal[i] / 2;
8
9         while (l - 1 >= 0 && r + 1 < n && s[l - 1] == s[r + 1])
10             --l, ++r, pal[i] += 2;
11
12         for (l = i - 1, r = i + 1; l >= 0 && r < 2 * n + 1; --l, ++r) {

```

```

13         if (l <= i - pal[i]) break;
14         if (l / 2 - pal[l] / 2 > i / 2 - pal[i] / 2)
15             pal[r] = pal[l];
16         else { if (l >= 0)
17             pal[r] = min(pal[l], i + pal[i] - r);
18             break;
19         }
20     }
21     i = r;
22 } }

```

### 3.5.5 Bitstring Slower than an unordered set (for many elements), but hashable

```

1 #include "../header.h"
2
3 template<size_t len>
4 struct pair_hash { // To make it hashable (pair<
5     int, bitset<len>>)
6     std::size_t operator()(const std::pair<int,
7         std::bitset<len>>& p) const {
8         std::size_t h1 = std::hash<int>{}(p.first);
9         std::size_t h2 = std::hash<std::bitset<
10             len>>{}(p.second);
11         return h1 ^ (h2 << 1);
12     }
13 };
14 #define MAXN 1000
15 std::bitset<MAXN> bs;
16 // bs.set(idx) <- set idx-th bit (1)
17 // bs.reset(idx) <- reset idx-th bit (0)
18 // bs.flip(idx) <- flip idx-th bit
19 // bs.test(idx) <- idx-th bit == 1
20 // bs.count() <- number of 1s
21 // bs.any() <- any bit == 1

```

## 3.6 Geometry

### 3.6.1 essentials.cpp

```

1 #include "../header.h"
2 using C = ld; // could be ll or ld
3 constexpr C EPS = 1e-10; // change to 0 for C=ll
4 struct P { // may also be used as a 2D vector
5     C x, y;
6     P(C x = 0, C y = 0) : x(x), y(y) {}
7     P operator+ (const P &p) const { return {x + p.x, y + p.y}; }
8     P operator- (const P &p) const { return {x - p.x, y - p.y}; }

```

```

9     P operator* (C c) const { return {x * c, y * c}; }
10    P operator/ (C c) const { return {x / c, y / c}; }
11    C operator* (const P &p) const { return x*p.x + y*p.y; }
12    C operator^ (const P &p) const { return x*p.y - p.x*y; }
13    P perp() const { return P{y, -x}; }
14    C lensq() const { return x*x + y*y; }
15    ld len() const { return sqrt((ld)lensq()); }
16    static ld dist(const P &p1, const P &p2) {
17        return (p1-p2).len(); }
18    bool operator==(const P &r) const {
19        return ((*this)-r).lensq() <= EPS*EPS; }
20 };
21 C det(P p1, P p2) { return p1^p2; }
22 C det(P p1, P p2, P o) { return det(p1-o, p2-o); }
23 C det(const vector<P> &ps) {
24     C sum = 0; P prev = ps.back();
25     for(auto &p : ps) sum += det(p, prev), prev = p;
26     return sum;
27 }
28 // Careful with division by two and C=ll
29 C area(P p1, P p2, P p3) { return abs(det(p1, p2, p3))/C(2); }
30 C area(const vector<P> &poly) { return abs(det(poly))/C(2); }
31 int sign(C c){ return (c > C(0)) - (c < C(0)); }
32 int ccw(P p1, P p2, P o) { return sign(det(p1, p2, o)); }
33
34 // Only well defined for C = ld.
35 P unit(const P &p) { return p / p.len(); }
36 P rotate(P p, ld a) { return P{p.x*cos(a)-p.y*sin(a), p.x*sin(a)+p.y*cos(a)}; }

```

### 3.6.2 Two segs. itersec.

```

1 #include "header.h"
2 #include "essentials.cpp"
3 bool intersect(P a1, P a2, P b1, P b2) {
4     if (max(a1.x, a2.x) < min(b1.x, b2.x)) return false;
5     if (max(b1.x, b2.x) < min(a1.x, a2.x)) return false;
6     if (max(a1.y, a2.y) < min(b1.y, b2.y)) return false;
7     if (max(b1.y, b2.y) < min(a1.y, a2.y)) return false;
8     bool l1 = ccw(a2, b1, a1) * ccw(a2, b2, a1) <= 0;

```



```

9  bool l2 = ccw(b2, a1, b1) * ccw(b2, a2, b1) <=
    0;
10  return l1 && l2;
11 }

```

### 3.6.3 Convex Hull

```

1  #include "header.h"
2  #include "essentials.cpp"
3  struct ConvexHull { // O(n lg n) monotone chain.
4      size_t n;
5      vector<size_t> h, c; // Indices of the hull
        are in 'h', ccw.
6      const vector<P> &p;
7      ConvexHull(const vector<P> &p) : n(_p.size()),
        c(n), p(_p) {
8          std::iota(c.begin(), c.end(), 0);
9          std::sort(c.begin(), c.end(), [this](size_t l
        , size_t r) -> bool { return p[l].x != p[
        r].x ? p[l].x < p[r].x : p[l].y < p[r].y;
        });
10         c.erase(std::unique(c.begin(), c.end(), [this
        ](size_t l, size_t r) { return p[l] == p[
        r]; }), c.end());
11         for (size_t s = 1, r = 0; r < 2; ++r, s = h.
        size()) {
12             for (size_t i : c) {
13                 while (h.size() > s && ccw(p[h.end()
        ][-2]], p[h.end()[-1]], p[i]) <= 0)
14                     h.pop_back();
15                 h.push_back(i);
16             }
17             reverse(c.begin(), c.end());
18         }
19         if (h.size() > 1) h.pop_back();
20     }
21     size_t size() const { return h.size(); }
22     template <class T, void U(const P &, const P &,
        const P &, T &)>
23     void rotating_calipers(T &ans) {
24         if (size() <= 2)
25             U(p[h[0]], p[h.back()], p[h.back()], ans);
26         else
27             for (size_t i = 0, j = 1, s = size(); i < 2
        * s; ++i) {
28                 while (det(p[h[(i + 1) % s]] - p[h[i % s
        ]], p[h[(j + 1) % s]] - p[h[j]]) >=
        0)
29                     j = (j + 1) % s;
30                 U(p[h[i % s]], p[h[(i + 1) % s]], p[h[j
        ]], ans);
31             }
32     }
33 };

```

```

34 // Example: furthest pair of points. Now set ans
    = 0LL and call
35 // ConvexHull(pts).rotating_calipers<ll, update>(
    ans);
36 void update(const P &p1, const P &p2, const P &o,
    ll &ans) {
37     ans = max(ans, (ll)max((p1 - o).lensq(), (p2 -
    o).lensq()));
38 }
39 int main() {
40     ios::sync_with_stdio(false); // do not use
        cout + printf
41     cin.tie(NULL);
42
43     int n;
44     cin >> n;
45     while (n) {
46         vector<P> ps;
47         int x, y;
48         for (int i = 0; i < n; i++) {
49             cin >> x >> y;
50             ps.push_back({x, y});
51         }
52
53         ConvexHull ch(ps);
54         cout << ch.h.size() << endl;
55         for(auto& p: ch.h) {
56             cout << ps[p].x << " " << ps[p].y <<
        endl;
57         }
58         cin >> n;
59     }
60
61     return 0;
62 }

```

## 3.7 Other Algorithms

### 3.7.1 2-sat

```

1  #include "../header.h"
2  #include "../Graphs/tarjan.cpp"
3  struct TwoSAT {
4      int n;
5      vvi imp; // implication graph
6      Tarjan tj;
7
8      TwoSAT(int _n) : n(_n), imp(2 * _n, vi()), tj(
        imp) { }
9
10     // Only copy the needed functions:
11     void add_implies(int c1, bool v1, int c2, bool
        v2) {
12         int u = 2 * c1 + (v1 ? 1 : 0),

```

```

13         v = 2 * c2 + (v2 ? 1 : 0);
14         imp[u].push_back(v); // u => v
15         imp[v^1].push_back(u^1); // -v => -u
16     }
17     void add_equivalence(int c1, bool v1, int c2,
        bool v2) {
18         add_implies(c1, v1, c2, v2);
19         add_implies(c2, v2, c1, v1);
20     }
21     void add_or(int c1, bool v1, int c2, bool v2) {
22         add_implies(c1, !v1, c2, v2);
23     }
24     void add_and(int c1, bool v1, int c2, bool v2)
        {
25         add_true(c1, v1); add_true(c2, v2);
26     }
27     void add_xor(int c1, bool v1, int c2, bool v2)
        {
28         add_or(c1, v1, c2, v2);
29         add_or(c1, !v1, c2, !v2);
30     }
31     void add_true(int c1, bool v1) {
32         add_implies(c1, !v1, c1, v1);
33     }
34
35     // on true: a contains an assignment.
36     // on false: no assignment exists.
37     bool solve(vb &a) {
38         vi com;
39         tj.find_sccs(com);
40         for (int i = 0; i < n; ++i)
41             if (com[2 * i] == com[2 * i + 1])
42                 return false;
43
44         vvi bycom(com.size());
45         for (int i = 0; i < 2 * n; ++i)
46             bycom[com[i]].push_back(i);
47
48         a.assign(n, false);
49         vb vis(n, false);
50         for(auto &&component : bycom){
51             for (int u : component) {
52                 if (vis[u / 2]) continue;
53                 vis[u / 2] = true;
54                 a[u / 2] = (u % 2 == 1);
55             }
56         }
57         return true;
58     }
59 };

```

### 3.7.2 Matrix Solve

```

1  #include "header.h"

```

```

2 #define REP(i, n) for(auto i = decltype(n)(0); i
  < (n); i++)
3 using T = double;
4 constexpr T EPS = 1e-8;
5 template<int R, int C>
6 using M = array<array<T,C>,R>; // matrix
7 template<int R, int C>
8 T ReducedRowEchelonForm(M<R,C> &m, int rows) {
  // return the determinant
9   int r = 0; T det = 1; // MODIFIES
    the input
10  for(int c = 0; c < rows && r < rows; c++) {
11    int p = r;
12    for(int i=r+1; i<rows; i++) if(abs(m[i][c]) >
      abs(m[p][c])) p=i;
13    if(abs(m[p][c]) < EPS){ det = 0; continue; }
14    swap(m[p], m[r]); det = -det;
15    T s = 1.0 / m[r][c], t; det *= m[r][c];
16    REP(j,C) m[r][j] *= s; // make leading
      term in row 1
17    REP(i,rows) if (i!=r){ t = m[i][c]; REP(j,C)
      m[i][j] -= t*m[r][j]; }
18    ++r;
19  }
20  return det;
21 }
22 bool error, inconst; // error => multiple or
  inconsistent
23 template<int R,int C> // Mx = a; M:R*R, v:R*C =>
  x:R*C
24 M<R,C> solve(const M<R,R> &m, const M<R,C> &a,
  int rows){
25   M<R,R+C> q;
26   REP(r,rows){
27     REP(c,rows) q[r][c] = m[r][c];
28     REP(c,C) q[r][R+c] = a[r][c];
29   }
30   ReducedRowEchelonForm<R,R+C>(q,rows);
31   M<R,C> sol; error = false, inconst = false;
32   REP(c,C) for(auto j = rows-1; j >= 0; --j){
33     T t=0; bool allzero=true;
34     for(auto k = j+1; k < rows; ++k)
35       t += q[j][k]*sol[k][c], allzero &= abs(q[j]
        ][k]) < EPS;
36     if(abs(q[j][j]) < EPS)
37       error = true, inconst |= allzero && abs(q[j]
        ][R+c]) > EPS;
38     else sol[j][c] = (q[j][R+c] - t) / q[j][j];
      // usually q[j][j]=1
39   }
40   return sol;
41 }

```

### 3.7.3 Matrix Exp.

```

1 #include "header.h"
2 #define ITERATE_MATRIX(w) for (int r = 0; r < (w)
  ; ++r) \
3       for (int c = 0; c < (w); ++c)
4 template <class T, int N>
5 struct M {
6   array<array<T,N>,N> m;
7   M() { ITERATE_MATRIX(N) m[r][c] = 0; }
8   static M id() {
9     M I; for (int i = 0; i < N; ++i) I.m[i][i] =
      1; return I;
10  }
11  M operator*(const M &rhs) const {
12    M out;
13    ITERATE_MATRIX(N) for (int i = 0; i < N; ++i)
14      out.m[r][c] += m[r][i] * rhs.m[i][c];
15    return out;
16  }
17  M raise(ll n) const {
18    if(n == 0) return id();
19    if(n == 1) return *this;
20    auto r = (*this**this).raise(n / 2);
21    return (n%2 ? *this*r : r);
22  }
23 };

```

### 3.7.4 Finite field For FFT

```

1 #include "header.h"
2 #include "../NumberTheory/elementary.cpp"
3 template<ll p,ll w> // prime, primitive root
4 struct Field { using T = Field; ll x; Field(ll x
  =0) : x{x} {}
5   T operator+(T r) const { return {(x+r.x)%p}; }
6   T operator-(T r) const { return {(x-r.x+p)%p};
  }
7   T operator*(T r) const { return {(x*r.x)%p}; }
8   T operator/(T r) const { return (*this)*r.inv()
  ; }
9   T inv() const { return {mod_inverse(x,p)}; }
10  static T root(ll k) { assert( (p-1)%k==0 );
    // (p-1)%k == 0?
11    auto r = powmod(w,(p-1)/abs(k),p); // k-
      th root of unity
12    return k>0 ? T{r} : T{r}.inv();
13  }
14  bool zero() const { return x == 0LL; }
15 };
16 using F1 = Field<1004535809,3 >;
17 using F2 = Field<1107296257,10>; // 1<<30 + 1<<25
  + 1
18 using F3 = Field<2281701377,3 >; // 1<<31 + 1<<27
  + 1

```

### 3.7.5 Complex field For FFR

```

1 #include "header.h"
2 const double m_pi = M_PI/64x;
3 struct Complex { using T = Complex; double u,v;
4   Complex(double u=0, double v=0) : u{u}, v{v} {}
5   T operator+(T r) const { return {u+r.u, v+r.v};
  }
6   T operator-(T r) const { return {u-r.u, v-r.v};
  }
7   T operator*(T r) const { return {u*r.u - v*r.v,
      u*r.v + v*r.u}; }
8   T operator/(T r) const {
9     auto norm = r.u*r.u+r.v*r.v;
10    return {(u*r.u + v*r.v)/norm, (v*r.u - u*r.v)
      /norm};
11  }
12  T operator*(double r) const { return T{u*r, v*r
  }; }
13  T operator/(double r) const { return T{u/r, v/r
  }; }
14  T inv() const { return T{1,0}/ *this; }
15  T conj() const { return T{u, -v}; }
16  static T root(ll k){ return {cos(2*m_pi/k), sin
      (2*m_pi/k)}; }
17  bool zero() const { return max(abs(u), abs(v))
      < 1e-6; }
18 };

```

### 3.7.6 FFT

```

1 #include "header.h"
2 #include "complex_field.cpp"
3 #include "fin_field.cpp"
4 void brinc(int &x, int k) {
5   int i = k - 1, s = 1 << i;
6   x ^= s;
7   if ((x & s) != s) {
8     --i; s >>= 1;
9     while (i >= 0 && ((x & s) == s))
10      x = x &~ s, --i, s >>= 1;
11     if (i >= 0) x |= s;
12   }
13 }
14 using T = Complex; // using T=F1,F2,F3
15 vector<T> roots;
16 void root_cache(int N) {
17   if (N == (int)roots.size()) return;
18   roots.assign(N, T{0});
19   for (int i = 0; i < N; ++i)
20     roots[i] = ((i&-i) == i)
21       ? T{cos(2.0*m_pi*i/N), sin(2.0*m_pi*i/N)}
22       : roots[i&-i] * roots[i-(i&-i)];
23 }
24 void fft(vector<T> &A, int p, bool inv = false) {

```

```

25 int N = 1<<p;
26 for(int i = 0, r = 0; i < N; ++i, brinc(r, p))
27     if (i < r) swap(A[i], A[r]);
28 // Uncomment to precompute roots (for T=Complex)
29 // . Slower but more precise.
30 // root_cache(N);
31 // , sh=p-1 , --sh
32 for (int m = 2; m <= N; m <= 1) {
33     T w, w_m = T::root(inv ? -m : m);
34     for (int k = 0; k < N; k += m) {
35         w = T{1};
36         for (int j = 0; j < m/2; ++j) {
37             T w = (!inv ? roots[j<<sh] : roots[j<<
38                 sh].conj());
39             T t = w * A[k + j + m/2];
40             A[k + j + m/2] = A[k + j] - t;
41             A[k + j] = A[k + j] + t;
42             w = w * w_m;
43         }
44     }
45 }
46 // convolution leaves A and B in frequency domain
47 // state
48 // C may be equal to A or B for in-place
49 // convolution
50 void convolution(vector<T> &A, vector<T> &B,
51     vector<T> &C){
52     int s = A.size() + B.size() - 1;
53     int q = 32 - __builtin_clz(s-1), N=1<<q; //
54     fails if s=1
55     A.resize(N,{}); B.resize(N,{}); C.resize(N,{});
56     fft(A, q, false); fft(B, q, false);
57     for (int i = 0; i < N; ++i) C[i] = A[i] * B[i];
58     fft(C, q, true); C.resize(s);
59 }
60 void square_inplace(vector<T> &A) {
61     int s = 2*A.size()-1, q = 32 - __builtin_clz(s
62         -1), N=1<<q;
63     A.resize(N,{}); fft(A, q, false);
64     for(auto &x : A) x = x*x;
65     fft(A, q, true); A.resize(s);
66 }

```

### 3.7.7 Polyn. inv. div.

```

1 #include "header.h"
2 #include "fft.cpp"
3 vector<T> &rev(vector<T> &A) { reverse(A.begin(),
4     A.end()); return A; }
5 void copy_into(const vector<T> &A, vector<T> &B,
6     size_t n) {

```

```

5 std::copy(A.begin(), A.begin()+min({n, A.size()
6     }, B.size()}), B.begin());
7 }
8 // Multiplicative inverse of A modulo x^n.
9 // Requires A[0] != 0!!
10 vector<T> inverse(const vector<T> &A, int n) {
11     vector<T> Ai{A[0].inv()};
12     for (int k = 0; (1<<k) < n; ++k) {
13         vector<T> As(4<<k, T(0)), Ais(4<<k, T(0));
14         copy_into(A, As, 2<<k); copy_into(Ai, Ais, Ai
15             .size());
16         fft(As, k+2, false); fft(Ais, k+2, false);
17         for (int i = 0; i < (4<<k); ++i) As[i] = As[i]
18             *Ais[i]*Ais[i];
19         fft(As, k+2, true); Ai.resize(2<<k, {});
20         for (int i = 0; i < (2<<k); ++i) Ai[i] = T(2)
21             * Ai[i] - As[i];
22     }
23     Ai.resize(n);
24     return Ai;
25 }
26 // Polynomial division. Returns {Q, R} such that
27 // A = QB+R, deg R < deg B.
28 // Requires that the leading term of B is nonzero
29 // .
30 pair<vector<T>, vector<T>> divmod(const vector<T>
31     &A, const vector<T> &B) {
32     size_t n = A.size()-1, m = B.size()-1;
33     if (n < m) return {vector<T>(1, T(0)), A};
34     vector<T> X(A), Y(B), Q, R;
35     convolution(rev(X), Y = inverse(rev(Y), n-m+1),
36         Q);
37     Q.resize(n-m+1); rev(Q);
38     X.resize(Q.size()), copy_into(Q, X, Q.size());
39     Y.resize(B.size()), copy_into(B, Y, B.size());
40     convolution(X, Y, X);
41     R.resize(m), copy_into(A, R, m);
42     for (size_t i = 0; i < m; ++i) R[i] = R[i] - X[
43         i];
44     while (R.size() > 1 && R.back().zero()) R.
45         pop_back();
46     return {Q, R};
47 }
48 vector<T> mod(const vector<T> &A, const vector<T>
49     &B) {
50     return divmod(A, B).second;
51 }

```

**3.7.8 Linear recurs.** Given a linear recurrence of the form

$$a_n = \sum_{i=0}^{k-1} c_i a_{n-i-1}$$

this code computes  $a_n$  in  $O(k \log k \log n)$  time.

```

1 #include "header.h"
2 #include "poly.cpp"
3 // x^k mod f
4 vector<T> xmod(const vector<T> f, ll k) {
5     vector<T> r{T(1)};
6     for (int b = 62; b >= 0; --b) {
7         if (r.size() > 1)
8             square_inplace(r), r = mod(r, f);
9         if ((k>>b)&1) {
10             r.insert(r.begin(), T(0));
11             if (r.size() == f.size()) {
12                 T c = r.back() / f.back();
13                 for (size_t i = 0; i < f.size(); ++i)
14                     r[i] = r[i] - c * f[i];
15                 r.pop_back();
16             }
17         }
18     }
19     return r;
20 }
21 // Given A[0,k) and C[0, k), computes the n-th
22 // term of:
23 // A[n] = \sum_i C[i] * A[n-i-1]
24 T nth_term(const vector<T> &A, const vector<T> &C
25     , ll n) {
26     int k = (int)A.size();
27     if (n < k) return A[n];
28     vector<T> f(k+1, T{1});
29     for (int i = 0; i < k; ++i)
30         f[i] = T{-1} * C[k-i-1];
31     f = xmod(f, n);
32     T r = T{0};
33     for (int i = 0; i < k; ++i)
34         r = r + f[i] * A[i];
35     return r;
36 }

```

### 3.7.9 Convolution Precise up to 9e15

```

1 #include "header.h"
2 #include "fft.cpp"
3 void convolution_mod(const vi &A, const vi &B, ll
4     MOD, vi &C) {
5     int s = A.size() + B.size() - 1; ll m15 = (1LL
6         <<15)-1LL;

```

```

5  int q = 32 - __builtin_clz(s-1), N=1<<q;  //
    fails if s=1
6  vector<T> Ac(N), Bc(N), R1(N), R2(N);
7  for (size_t i = 0; i < A.size(); ++i) Ac[i] = T
    {A[i]&m15, A[i]>>15};
8  for (size_t i = 0; i < B.size(); ++i) Bc[i] = T
    {B[i]&m15, B[i]>>15};
9  fft(Ac, q, false); fft(Bc, q, false);
10 for (int i = 0, j = 0; i < N; ++i, j = (N-1)&(N
    -i)) {
11     T as = (Ac[i] + Ac[j].conj()) / 2;
12     T al = (Ac[i] - Ac[j].conj()) / T{0, 2};
13     T bs = (Bc[i] + Bc[j].conj()) / 2;
14     T bl = (Bc[i] - Bc[j].conj()) / T{0, 2};
15     R1[i] = as*bs + al*bl*T{0,1}, R2[i] = as*bl +
        al*bs;
16 }
17 fft(R1, q, true); fft(R2, q, true);
18 ll p15 = (1LL<<15)%MOD, p30 = (1LL<<30)%MOD; C.
    resize(s);
19 for (int i = 0; i < s; ++i) {
20     ll l = llround(R1[i].u), m = llround(R2[i].u)
        , h = llround(R1[i].v);
21     C[i] = (1 + m*p15 + h*p30) % MOD;
22 }
23 }

```

### 3.7.10 Partitions of $n$ Finds all possible partitions of a number

```

1 #include "header.h"
2 void printArray(int p[], int n) {
3     for (int i = 0; i < n; i++)
4         cout << p[i] << " ";
5     cout << endl;
6 }
7
8 void printAllUniqueParts(int n) {
9     int p[n]; // array to store a partition
10    int k = 0; // idx of last element in a
        partition
11    p[k] = n;
12
13    // The loop stops when the current partition
        has all 1s
14    while (true) {
15        printArray(p, k + 1);
16        int rem_val = 0;
17        while (k >= 0 && p[k] == 1) {
18            rem_val += p[k];
19            k--;
20        }
21        // no more partitions
22        if (k < 0) return;

```

```

23    p[k]--;
24    rem_val++;
25
26    // sorted order is violated (fix)
27    while (rem_val > p[k]) {
28        p[k + 1] = p[k];
29        rem_val = rem_val - p[k];
30        k++;
31    }
32
33    p[k + 1] = rem_val;
34    k++;
35 }
36 }
37 }

```

**3.7.11 Ternary search** Find the smallest  $i$  in  $[a, b]$  that maximizes  $f(i)$ , assuming that  $f(a) < \dots < f(i) \geq \dots \geq f(b)$ . To reverse which of the sides allows non-strict inequalities, change the  $<$  marked with (A) to  $\leq$ , and reverse the loop at (B). To minimize  $f$ , change it to  $>$ , also at (B).  $O(\log(b-a))$

```

1 // Usage: int ind = ternSearch(0,n-1,[\&](int i){
    return a[i];});
2 #include "../Numerical/template.cpp"
3 template<class F>
4 int ternSearch(int a, int b, F f) {
5     assert(a <= b);
6     while (b - a >= 5) {
7         int mid = (a + b) / 2;
8         if (f(mid) < f(mid+1)) a = mid; // (A)
9         else b = mid+1;
10    }
11    rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
12    return a;
13 }

```

## 3.8 Other Data Structures

### 3.8.1 Disjoint set (i.e. union-find)

```

1 template <typename T>
2 class DisjointSet {
3     typedef T * iterator;
4     T *parent, n, *rank;
5     public:
6         // O(n), assumes nodes are [0, n)
7         DisjointSet(T n) {
8             this->parent = new T[n];
9             this->n = n;

```

```

10         this->rank = new T[n];
11         for (T i = 0; i < n; i++) {
12             parent[i] = i;
13             rank[i] = 0;
14         }
15     }
16
17     // O(log n)
18     T find_set(T x) {
19         if (x == parent[x]) return x;
20         return parent[x] = find_set(parent[x]
            );
21     }
22
23     // O(log n)
24     void union_sets(T x, T y) {
25         x = this->find_set(x);
26         y = this->find_set(y);
27
28         if (x == y) return;
29         if (rank[x] < rank[y]) {
30             T z = x;
31             x = y;
32             y = z;
33         }
34         parent[y] = x;
35         if (rank[x] == rank[y]) rank[x]++;
36     }
37 };

```

**3.8.2 Fenwick tree** (i.e. BIT) eff. update + prefix sum calc. Can be generalized to arbitrary dimensions by duplicating loops.

```

1 // #include "header.h"
2 template < class T >
3 struct FenwickTree { // use 1 based indices !!!
4     int n; vector<T> tree;
5     FenwickTree ( int n ) : n ( n ) { tree .
        assign ( n + 1 , 0 ) ; }
6     T query ( int l , int r ) { return query ( r
            ) - query ( l - 1 ) ; }
7     T query ( int r ) {
8         T s = 0;
9         for ( ; r > 0; r -= ( r & ( - r ) ) ) s +=
            tree [ r ] ;
10        return s ;
11    }
12    void update ( int i , T v ) {
13        for ( ; i <= n ; i += ( i & ( - i ) ) )
            tree [ i ] += v ;
14    }
15 };

```

### 3.8.3 Trie

```

1 #include "header.h"
2 const int ALPHABET_SIZE = 26;
3 inline int mp(char c) { return c - 'a'; }
4
5 struct Node {
6     Node* ch[ALPHABET_SIZE];
7     bool isleaf = false;
8     Node() {
9         for(int i = 0; i < ALPHABET_SIZE; ++i) ch[i]
10            = nullptr;
11     }
12     void insert(string &s, int i = 0) {
13         if (i == s.length()) isleaf = true;
14         else {
15             int v = mp(s[i]);
16             if (ch[v] == nullptr)
17                 ch[v] = new Node();
18             ch[v]->insert(s, i + 1);
19         }
20     }
21
22     bool contains(string &s, int i = 0) {
23         if (i == s.length()) return isleaf;
24         else {
25             int v = mp(s[i]);
26             if (ch[v] == nullptr) return false;
27             else return ch[v]->contains(s, i + 1);
28         }
29     }
30
31     void cleanup() {
32         for (int i = 0; i < ALPHABET_SIZE; ++i)
33             if (ch[i] != nullptr) {
34                 ch[i]->cleanup();
35                 delete ch[i];
36             }
37     }
38 };

```

**3.8.4 Treap** A binary tree whose nodes contain two values, a key and a priority, such that the key keeps the BST property

```

1 #include "header.h"
2 struct Node {
3     ll v;
4     int sz, pr;
5     Node *l = nullptr, *r = nullptr;
6     Node(ll val) : v(val), sz(1) { pr = rand(); }
7 };
8 int size(Node *p) { return p ? p->sz : 0; }

```

```

9 void update(Node* p) {
10     if (!p) return;
11     p->sz = 1 + size(p->l) + size(p->r);
12     // Pull data from children here
13 }
14 void propagate(Node *p) {
15     if (!p) return;
16     // Push data to children here
17 }
18 void merge(Node *&t, Node *l, Node *r) {
19     propagate(l), propagate(r);
20     if (!l) t = r;
21     else if (!r) t = l;
22     else if (l->pr > r->pr)
23         merge(l->r, l->r, r), t = l;
24     else merge(r->l, l, r->l), t = r;
25     update(t);
26 }
27 void spliti(Node *t, Node *&l, Node *&r, int
28     index) {
29     propagate(t);
30     if (!t) { l = r = nullptr; return; }
31     int id = size(t->l);
32     if (index <= id) // id \in [index, \infty), so
33         move it right
34         spliti(t->l, l, t->l, index), r = t;
35     else
36         spliti(t->r, t->r, r, index - id), l = t;
37     update(t);
38 }
39 void splitv(Node *t, Node *&l, Node *&r, ll val)
40 {
41     propagate(t);
42     if (!t) { l = r = nullptr; return; }
43     if (val <= t->v) // t->v \in [val, \infty), so
44         move it right
45         splitv(t->l, l, t->l, val), r = t;
46     else
47         splitv(t->r, t->r, r, val), l = t;
48     update(t);
49 }
50 void clean(Node *p) {
51     if (p) { clean(p->l), clean(p->r); delete p; }
52 }

```

### 3.8.5 Segment tree

```

1 #include "../header.h"
2 // example: SegmentTree<int, min> st(n, INT_MAX);
3 const int& addOp(const int& a, const int& b) {
4     static int result;
5     result = a + b;
6     return result;
7 }

```

```

8 template <class T, const T&(*op)(const T&, const
9     T&)>
10 struct SegmentTree {
11     int n; vector<T> tree; T id;
12     SegmentTree(int _n, T _id) : n(_n), tree(2 * n,
13         _id), id(_id) {}
14     void update(int i, T val) {
15         for (tree[i+n] = val, i = (i+n)/2; i > 0; i
16             /= 2)
17             tree[i] = op(tree[2*i], tree[2*i+1]);
18     }
19     T query(int l, int r) {
20         T lhs = T(id), rhs = T(id);
21         for (l += n, r += n; l < r; l >= 1, r >= 1)
22             if (l&1) lhs = op(lhs, tree[l++]);
23             if (!(r&1)) rhs = op(tree[r--], rhs);
24         return op(l == r ? op(lhs, tree[l]) : lhs,
25             rhs);
26     }
27 };

```

### 3.8.6 Lazy segment tree Optimizes range updates

```

1 #include "../header.h"
2 using T=int; using U=int; using I=int; //
3 exclusive right bounds
4 T t_id; U u_id;
5 T op(T a, T b){ return a+b; }
6 void join(U &a, U b){ a+=b; }
7 void apply(T &t, U u, int x){ t+=x*u; }
8 T convert(const I &i){ return i; }
9 struct LazySegmentTree {
10     struct Node { int l, r, lc, rc; T t; U u;
11         Node(int l, int r, T t=t_id):l(l),r(r),lc(-1)
12             ,rc(-1),t(t),u(u_id){}
13     };
14     int N; vector<Node> tree; vector<I> &init;
15     LazySegmentTree(vector<I> &init) : N(init.size()
16         ), init(init){
17         tree.reserve(2*N-1); tree.push_back({0,N});
18         build(0, 0, N);
19     }
20     void build(int l, int r) { auto &n =
21         tree[l];
22         if (r > l+1) { int m = (l+r)/2;
23             n.lc = tree.size(); n.rc = n.lc+1;
24             tree.push_back({l,m}); tree.push_back({m
25                 ,r});
26             build(n.lc,l,m); build(n.rc,m,r);
27             n.t = op(tree[n.lc].t, tree[n.rc].t);
28         } else n.t = convert(init[l]);
29     }

```

```

24 void push(Node &n, U u){ apply(n.t, u, n.r-n.l)
    ; join(n.u,u); }
25 void push(Node &n){push(tree[n.lc],n.u);push(
    tree[n.rc],n.u);n.u=u_id;}
26 T query(int l, int r, int i = 0) { auto &n =
    tree[i];
27 if(r <= n.l || n.r <= l) return t_id;
28 if(l <= n.l && n.r <= r) return n.t;
29 return push(n), op(query(l,r,n.lc),query(l,r,
    n.rc));
30 }
31 void update(int l, int r, U u, int i = 0) {
    auto &n = tree[i];
32 if(r <= n.l || n.r <= l) return;
33 if(l <= n.l && n.r <= r) return push(n,u);
34 push(n); update(l,r,u,n.lc); update(l,r,u,n.
    rc);
35 n.t = op(tree[n.lc].t, tree[n.rc].t);
36 }
37 };

```

### 3.8.7 Suffix tree

```

1 #include "../header.h"
2 using T = char;
3 using M = map<T,int>; // or array<T,ALPHABET_SIZE>
4 using V = string; // could be vector<T> as well
5 using It = V::const_iterator;
6 struct Node{
7     It b, e; M edges; int link; // end is exclusive
8     Node(It b, It e) : b(b), e(e), link(-1) {}
9     int size() const { return e-b; }
10 };
11 struct SuffixTree{
12     const V &s; vector<Node> t;
13     int root,n,len,remainder,llink; It edge;
14     SuffixTree(const V &s) : s(s) { build(); }
15     int add_node(It b, It e){ return t.push_back({b,
        e}), t.size()-1; }
16     int add_node(It b){ return add_node(b,s.end());
        }
17     void link(int node){ if(llink) t[llink].link =
        node; llink = node; }
18     void build(){
19         len = remainder = 0; edge = s.begin();
20         n = root = add_node(s.begin(), s.begin());
21         for(auto i = s.begin(); i != s.end(); ++i){
22             ++remainder; llink = 0;
23             while(remainder){
24                 if(len == 0) edge = i;
25                 if(t[n].edges[*edge] == 0){
26                     t[n].edges[*edge] = add_node(i); link(n)

```

```

27 } else {
28     auto x = t[n].edges[*edge];
29     if(len >= t[x].size()){
30         len -= t[x].size(); edge += t[x].size()
            (); n = x;
31         continue;
32     }
33     if(*(t[x].b + len) == *i){
34         ++len; link(n); break;
35     }
36     auto split = add_node(t[x].b, t[x].b+
        len);
37     t[n].edges[*edge] = split;
38     t[x].b += len;
39     t[split].edges[*i] = add_node(i);
40     t[split].edges[*t[x].b] = x;
41     link(split);
42 }
43 --remainder;
44 if(n == root && len > 0)
45     --len, edge = i - remainder + 1;
46 else n = t[n].link > 0? t[n].link: root;
47 }
48 }
49 }
50 };

```

### 3.8.8 UnionFind

```

1 #include "header.h"
2 struct UnionFind {
3     std::vector<int> par, rank, size;
4     int c;
5     UnionFind(int n) : par(n), rank(n, 0), size(n,
        1), c(n) {
6         for(int i = 0; i < n; ++i) par[i] = i;
7     }
8     int find(int i) { return (par[i] == i ? i : (
        par[i] = find(par[i]))); }
9     bool same(int i, int j) { return find(i) ==
        find(j); }
10    int get_size(int i) { return size[find(i)]; }
11    int count() { return c; }
12    int merge(int i, int j) {
13        if((i = find(i)) == (j = find(j))) return -1;
14        --c;
15        if(rank[i] > rank[j]) swap(i, j);
16        par[i] = j;
17        size[j] += size[i];
18        if(rank[i] == rank[j]) rank[j]++;
19        return j;
20    }
21 };

```

**3.8.9 Indexed set** Similar to set, but allows accessing elements by index using `find_by_order()` in  $O(\log n)$

```

1 #include "../header.h"
2 #include <ext/pb_ds/assoc_container.hpp>
3 using namespace __gnu_pbds;
4 using namespace std;
5
6 typedef tree<int,null_type,less<int>,rb_tree_tag,
    tree_order_statistics_node_update>
    indexed_set;

```

## 4 Other Mathematics

### 4.1 Helpful functions

**4.1.1 Euler's Totient Function**  $n = p_1^{k_1-1} \cdot (p_1 - 1) \cdot \dots \cdot p_r^{k_r-1} \cdot (p_r - 1)$ , where  $p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$  is the prime factorization of  $n$ .

```

1 # include "header.h"
2 ll phi(ll n) { // \Phi(n)
3     ll ans = 1;
4     for (ll i = 2; i*i <= n; i++) {
5         if (n % i == 0) {
6             ans *= i-1;
7             n /= i;
8             while (n % i == 0) {
9                 ans *= i;
10                n /= i;
11            }
12        }
13    }
14    if (n > 1) ans *= n-1;
15    return ans;
16 }
17 vi phis(int n) { // All \Phi(i) up to n
18     vi phi(n+1, 0LL);
19     iota(phi.begin(), phi.end(), 0LL);
20     for (ll i = 2LL; i <= n; ++i)
21         if (phi[i] == i)
22             for (ll j = i; j <= n; j += i)
23                 phi[j] -= phi[j] / i;
24     return phi;
25 }

```

### 4.1.2 Totient (again but .py)

```

1 def totatives(n):
2     if n == 1:
3         return 1

```



```

4  phi = int(n > 1 and n)
5  for p in range(2, int(n **.5) + 1):
6      if not n % p:
7          phi -= phi // p
8          while not n % p:
9              n //= p
10     #if n is > 1 it means it is prime
11     if n > 1: phi -= phi // n
12     return phi

```

**Formulas**  $\Phi(n)$  counts all numbers in  $1, \dots, n-1$  coprime to  $n$ .

$a^{\varphi(n)} \equiv 1 \pmod n$ ,  $a$  and  $n$  are coprimes.

$\forall e > \log_2 m: n^e \pmod m = n^{\Phi(m)+e \pmod \Phi(m)} \pmod m$ .

$\gcd(m, n) = 1 \Rightarrow \Phi(m \cdot n) = \Phi(m) \cdot \Phi(n)$ .

**4.1.3 Pascal's trinagle**  $\binom{n}{k}$  is  $k$ -th element in the  $n$ -th row, indexing both from 0

```

1 #include "header.h"
2 void printPascal(int n) {
3     for (int line = 1; line <= n; line++) {
4         int C = 1; // used to represent C(line, i)
5         for (int i = 1; i <= line; i++) {
6             cout << C << " ";
7             C = C * (line - i) / i;
8         }
9         cout << "\n";
10    }
11 }

```

## 4.2 Theorems and definitions

**Subfactorial (Derangements)** Permutations of a set such that none of the elements appear in their original position:

$$!n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}$$

$$!(0) = 1, !n = n \cdot !(n-1) + (-1)^n$$

$$!n = (n-1)(!(n-1) + !(n-2)) = \left\lfloor \frac{n!}{e} \right\rfloor \quad (1)$$

$$!n = 1 - e^{-1}, \quad n \rightarrow \infty \quad (2)$$

## Binomials and other partitionings

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \prod_{i=1}^k \frac{n-i+1}{i}$$

This last product may be computed incrementally since any product of  $k'$  consecutive values is divisible by  $k'!$ .

Basic identities: The hockeystick identity:

$$\sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}$$

or

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Also

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

For  $n, m \geq 0$  and  $p$  prime: write  $n, m$  in base  $p$ , i.e.  $n = n_k p^k + \dots + n_1 p + n_0$  and  $m = m_k p^k + \dots + m_1 p + m_0$ . Then by Lucas theorem we have  $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod p$ , with the convention that  $n_i < m_i \Rightarrow \binom{n_i}{m_i} = 0$ .

**Fibonacci** (See also number theory section)

$$\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$$

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

$$\sum_{i=1}^n F_i = F_{n+2} - 1, \quad \sum_{i=1}^n F_i^2 = F_n F_{n+1}$$

$$\gcd(F_m, F_n) = F_{\gcd(m, n)}$$

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$$

**Bit stuff**  $a + b = a \oplus b + 2(a \& b) = a|b + a \& b$ .

$k$ th bit is set in  $x$  iff  $x \bmod 2^{k-1} \geq 2^k$ , or iff  $x \bmod 2^{k-1} - x \bmod 2^k \neq 0$  (i.e.  $= 2^k$ ). It comes handy when you need to look at the bits of the numbers which are pair sums or subset sums etc.

$n \bmod 2^i = n \& (2^i - 1)$ .

$\forall k: 1 \oplus 2 \oplus \dots \oplus (4k-1) = 0$

## 4.3 Geometry Formulas

Euler:  $1 + CC = V - E + F$

Pick:  $\text{Area} = \text{itr pts} + \frac{\text{bdry pts}}{2} - 1$

Given a non-self-intersecting closed polygon on  $n$  vertices, given as  $(x_i, y_i)$ , its centroid  $(C_x, C_y)$  is given as:

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i),$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) = \text{polygon area}$$

**Inclusion-Exclusion** For appropriate  $f$  compute  $\sum_{S \subseteq T} (-1)^{|T \setminus S|} f(S)$ , or if only the size of  $S$  matters,  $\sum_{s=0}^n (-1)^{n-s} \binom{n}{s} f(s)$ . In some contexts we might use Stirling numbers, not binomial coefficients!

Some useful applications:

**Graph coloring** Let  $I(S)$  count the number of independent sets contained in  $S \subseteq V$  ( $I(\emptyset) = 1$ ,  $I(S) = I(S \setminus v) + I(S \setminus N(v))$ ). Let  $c_k = \sum_{S \subseteq V} (-1)^{|V \setminus S|} I(S)$ . Then  $V$  is  $k$ -colorable iff  $v > 0$ . Thus we can compute the chromatic number of a graph in  $O^*(2^n)$  time.



**Burnside's lemma** Given a group  $G$  acting on a set  $X$ , the number of elements in  $X$  up to symmetry is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

with  $X^g$  the elements of  $X$  invariant under  $g$ . For example, if  $f(n)$  counts “configurations” of some sort of length  $n$ , and we want to count them up to rotational symmetry using  $G = \mathbb{Z}/n\mathbb{Z}$ , then

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k)$$

I.e. for coloring with  $c$  colors we have  $f(k) = k^c$ .

Relatedly, in Pólya's enumeration theorem we imagine  $X$  as a set of  $n$  beads with  $G$  permuting the beads (e.g. a necklace, with  $G$  all rotations and reflections of the  $n$ -cycle, i.e. the dihedral group  $D_n$ ). Suppose further that we had  $Y$  colors, then the number of  $G$ -invariant colorings  $Y^X/G$  is counted by

$$\frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)}$$

with  $c(g)$  counting the number of cycles of  $g$  when viewed as a permutation of  $X$ . We can generalize this to a weighted version: if the color  $i$  can occur exactly  $r_i$  times, then this is counted by the coefficient of  $t_1^{r_1} \dots t_n^{r_n}$  in the polynomial

$$Z(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \dots + t_n^m)^{c_m(g)}$$

where  $c_m(g)$  counts the number of length  $m$  cycles in  $g$  acting as a permutation on  $X$ . Note we get the original formula by setting all  $t_i = 1$ . Here  $Z$  is the cycle index. Note: you can cleverly deal with even/odd sizes by setting some  $t_i$  to  $-1$ .

**Lucas Theorem** If  $p$  is prime, then:

$$\frac{p^a}{k} \equiv 0 \pmod{p}$$

Thus for non-negative integers  $m = m_k p^k + \dots + m_1 p + m_0$  and  $n = n_k p^k + \dots + n_1 p + n_0$ :

$$\frac{m}{n} = \prod_{i=0}^k \frac{m_i}{n_i} \pmod{p}$$

Note: The fraction's mean integer division.

#### 4.4 Recurrences

If  $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$ , and  $r_1, \dots, r_k$  are distinct roots of  $x^k - c_1 x^{k-1} - \dots - c_k$ , there are  $d_1, \dots, d_k$  s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.  $a_n = (d_1 n + d_2) r^n$ .

#### 4.5 Sums

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

#### 4.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

#### 4.7 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ .

#### 4.8 Triangles

Side lengths:  $a, b, c$

Semiperimeter:  $p = \frac{a+b+c}{2}$

Area:

$$[ABC] = rp = \frac{1}{2} ab \sin \gamma$$

$$= \frac{abc}{4R} = \sqrt{p(p-a)(p-b)(p-c)} = \frac{1}{2} |(B-A, C-A)^T|$$

Circumradius:  $R = \frac{abc}{4A}$ , Inradius:  $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):  $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):  $s_a =$

$$\sqrt{bc \left[ 1 - \left( \frac{a}{b+c} \right)^2 \right]}$$

Law of tangents:  $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

#### 4.9 Trigonometry

$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

$$\sin v + \sin w = 2 \sin \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$\cos v + \cos w = 2 \cos \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$(V+W) \tan(v-w)/2 = (V-W) \tan(v+w)/2$$

where  $V, W$  are lengths of sides opposite angles  $v, w$ .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where  $r = \sqrt{a^2 + b^2}$ ,  $\phi = \text{atan2}(b, a)$ .

#### 4.10 Combinatorics

Combinations and Permutations

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$C(n, r) = C(n, n-r)$$

### 4.11 Cycles

Let  $g_S(n)$  be the number of  $n$ -permutations whose cycle lengths all belong to the set  $S$ . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp \left( \sum_{n \in S} \frac{x^n}{n} \right)$$

### 4.12 Labeled unrooted trees

# on  $n$  vertices:  $n^{n-2}$

# on  $k$  existing trees of size  $n_i$ :  $n_1 n_2 \cdots n_k n^{k-2}$

# with degrees  $d_i$ :  $(n-2)! / ((d_1-1)! \cdots (d_n-1)!)$

### 4.13 Partition function

Number of ways of writing  $n$  as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$n$	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	$\sim 2e5$	$\sim 2e8$

### 4.14 Numbers

**Bernoulli numbers** EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t - 1}$  (FFT-able).  $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$   
Sums of powers:

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^{\infty} f(i) &= \int_m^{\infty} f(x) dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^{\infty} f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

**Stirling's numbers First kind:**  $S_1(n, k)$  count permutations on  $n$  items with  $k$  cycles.  $S_1(n, k) = S_1(n-1, k-1) + (n-1)S_1(n-1, k)$  with  $S_1(0, 0) = 1$ . Note:

$$\sum_{k=0}^n S_1(n, k) x^k = x(x+1) \cdots (x+n-1)$$

$$\sum_{k=0}^n S_1(n, k) = n!$$

$$S_1(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$$

$$S_1(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$$

**Second kind:**  $S_2(n, k)$  count partitions of  $n$  distinct elements into exactly  $k$  non-empty groups.

$$S_2(n, k) = S_2(n-1, k-1) + k S_2(n-1, k)$$

$$S_2(n, 1) = S_2(n, n) = 1$$

$$S_2(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

**Catalan Numbers** - Number of correct bracket sequence consisting of  $n$  opening and  $n$  closing brackets.

The number of ways to completely parenthesize  $n+1$  factors.

The number of triangulations of a convex polygon with  $n+2$  sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).

The number of ways to connect the  $2n$  points on a circle to form  $n$  disjoint i.e. non-intersecting chords.

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$C_0 = 1, \quad C_1 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

**Narayana numbers** The number of expressions containing  $n$  pairs of parentheses, which are correctly matched and which contain  $k$  distinct nestings.

$$N(n, k) = \frac{1}{n} \frac{n}{k} \frac{n}{k-1}$$

**Eulerian numbers** Number of permutations  $\pi \in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k$   $j$ :s s.t.  $\pi(j) > \pi(j+1)$ ,  $k+1$   $j$ :s s.t.  $\pi(j) \geq j$ ,  $k$   $j$ :s s.t.  $\pi(j) > j$ .

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

**Bell numbers** Total number of partitions of  $n$  distinct elements.  $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$   
For  $p$  prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

**Catalan numbers**

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad C_{n+1} = \sum C_i C_{n-i}$$

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

- sub-diagonal monotone paths in an  $n \times n$  grid.
- strings with  $n$  pairs of parenthesis, correctly nested.
- binary trees with  $n+1$  leaves (0 or 2 children).
- ordered trees with  $n+1$  vertices.
- ways a convex polygon with  $n+2$  sides can be cut into triangles by connecting vertices with straight lines.
- permutations of  $[n]$  with no 3-term increasing subseq.

### 4.15 Probability

Stochastic variables

$$P(X=r) = C(n, r) \cdot p^r \cdot (1-p)^{n-r}$$

**Bayes' Theorem**  $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{P(A|B_1)P(B_1) \cdots P(A|B_n)P(B_n)}$$

**Expectation** Let  $X$  be a discrete random variable with probability  $p_X(x)$  of assuming the value  $x$ . It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$  is the standard deviation. If  $X$  is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent  $X$  and  $Y$ ,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

## 4.16 Number Theory

### Bezout's Theorem

$$a, b \in \mathbb{Z}^+ \implies \exists s, t \in \mathbb{Z} : \gcd(a, b) = sa + tb$$

**Bézout's identity** For  $a \neq 0, b \neq 0$ , then  $d = \gcd(a, b)$  is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If  $(x, y)$  is one solution, then all solutions are given by

$$\left( x + \frac{kb}{\gcd(a, b)}, y - \frac{ka}{\gcd(a, b)} \right), \quad k \in \mathbb{Z}$$

### Partial Coprime Divisor Property

$$(\gcd(a, b) = 1) \wedge (a \mid bc) \implies (a \mid c)$$

### Coprime Modulus Equivalence Property

$$(\gcd(c, m) = 1) \wedge (ac \equiv bc \pmod{m}) \implies (a \equiv b \pmod{m})$$

### Fermat's Little Theorem

$$(\text{prime}(p)) \wedge (p \nmid a) \implies (a^{p-1} \equiv 1 \pmod{p})$$

$$(\text{prime}(p)) \implies (a^p \equiv a \pmod{p})$$

**Pythagorean Triples** The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with  $m > n > 0, k > 0, m \perp n$ , and either  $m$  or  $n$  even.

**Primes**  $p = 962592769$  is such that  $2^{21} \mid p - 1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power  $p^a$ , except for  $p = 2, a > 2$ , and there are  $\phi(\phi(p^a))$  many. For  $p = 2, a > 2$ , the group  $\mathbb{Z}_{2^a}^\times$  is instead isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ .

**Estimates**  $\sum_{d \mid n} d = O(n \log \log n)$ .

The number of divisors of  $n$  is at most around 100 for  $n < 5e4$ , 500 for  $n < 1e7$ , 2000 for  $n < 1e10$ , 200 000 for  $n < 1e19$ .

### Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d \mid n} f(d) \Leftrightarrow f(n) = \sum_{d \mid n} \mu(d) g(n/d)$$

Other useful formulas/forms:

$$\begin{aligned} \sum_{d \mid n} \mu(d) &= [n = 1] \text{ (very useful)} \\ g(n) &= \sum_{n \mid d} f(d) \Leftrightarrow f(n) = \sum_{n \mid d} \mu(d/n) g(d) \\ g(n) &= \sum_{1 \leq m \leq n} f\left(\left\lfloor \frac{n}{m} \right\rfloor\right) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m) g\left(\left\lfloor \frac{n}{m} \right\rfloor\right) \end{aligned}$$

## 4.17 Discrete distributions

**Binomial distribution** The number of successes in  $n$  independent yes/no experiments, each which yields success with probability  $p$  is  $\text{Bin}(n, p)$ ,  $n = 1, 2, \dots, 0 \leq p \leq 1$ .

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$  is approximately  $\text{Po}(np)$  for small  $p$ .

**First success distribution** The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability  $p$  is  $\text{Fs}(p)$ ,  $0 \leq p \leq 1$ .

$$p(k) = p(1-p)^{k-1}, \quad k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

**Poisson distribution** The number of events occurring in a fixed period of time  $t$  if these events occur with a known average rate  $\kappa$  and independently of the time since the last event is  $\text{Po}(\lambda)$ ,  $\lambda = t\kappa$ .

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

## 4.18 Continuous distributions

**Uniform distribution** If the probability density function is constant between  $a$  and  $b$  and 0 elsewhere it is  $\text{U}(a, b)$ ,  $a < b$ .

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

**Exponential distribution** The time between events in a Poisson process is  $\text{Exp}(\lambda)$ ,  $\lambda > 0$ .

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

**Normal distribution** Most real random values with mean  $\mu$  and variance  $\sigma^2$  are well described by  $\mathcal{N}(\mu, \sigma^2)$ ,  $\sigma > 0$ .

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If  $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$