

1 Setup	1	3.1.3 Dijkstra	6	3.5 Strings	12
1.1 header.h	1	3.1.4 Floyd-Warshall	6	3.5.1 Z alg.	12
1.2 Bash for c++ compile with header.h	2	3.1.5 Kruskal	6	3.5.2 KMP	12
1.3 Bash for run tests c++	2	3.1.6 Hungarian algorithm	6	3.5.3 Aho-Corasick	12
1.4 Bash for run tests python	2	3.1.7 Suc. shortest path	7	3.5.4 Long. palin. subs	13
1.4.1 Aux. helper C++	2	3.1.8 Bipartite check	7	3.6 Geometry	13
1.4.2 Aux. helper python	2	3.1.9 Find cycle directed	7	3.6.1 essentials.cpp	13
2 Python	2	3.1.10 Find cycle directed	7	3.6.2 Two segs. itersec.	14
2.1 Graphs	2	3.1.11 Tarjan's SCC	8	3.6.3 Convex Hull	14
2.1.1 BFS	2	3.1.12 SCC edges	8	3.7 Other Algorithms	14
2.1.2 Dijkstra	2	3.1.13 Find Bridges	8	3.7.1 2-sat	14
2.1.3 Topological Sort	2	3.1.14 Artic. points	9	3.7.2 Matrix Solve	15
2.1.4 Kruskal	3	3.1.15 Topological sort	9	3.7.3 Matrix Exp.	15
2.2 Num. Th. / Comb.	3	3.1.16 Bellmann-Ford	9	3.7.4 Finite field	15
2.2.1 nCk % prime	3	3.1.17 Ford-Fulkerson	9	3.7.5 Complex field	15
2.2.2 Sieve of E.	3	3.1.18 Dinic max flow	10	3.7.6 FFT	15
2.3 Strings	3	3.2 Dynamic Programming	10	3.7.7 Polyn. inv. div.	16
2.3.1 LCS	3	3.2.1 Longest Incr. Subseq.	10	3.7.8 Linear recurs.	16
2.3.2 KMP	4	3.2.2 0-1 Knapsack	10	3.7.9 Convolution	17
2.3.3 Edit distance	4	3.2.3 Coin change	11	3.7.10 Partitions of n	17
2.4 Other Algorithms	4	3.3 Trees	11	3.8 Other Data Structures	17
2.4.1 Rotate matrix	4	3.3.1 Tree diameter	11	3.8.1 Disjoint set	17
2.5 Geometry	4	3.3.2 Tree Node Count	11	3.8.2 Fenwick tree	17
2.5.1 Convex Hull	4	3.4 Num. Th. / Comb.	11	3.8.3 Fenwick2d tree	18
2.5.2 Geometry	4	3.4.1 Basic stuff	11	3.8.4 Trie	18
2.6 Other Data Structures	5	3.4.2 Mod. exponentiation	11	3.8.5 Treap	18
2.6.1 Segment Tree	5	3.4.3 GCD	11	4 Other Mathematics	18
2.6.2 Trie	5	3.4.4 Sieve of Eratosthenes	12	4.1 Helpful functions	18
3 C++	5	3.4.5 Fibonacci % prime	12	4.1.1 Euler's Totient Fuction	18
3.1 Graphs	5	3.4.6 nCk % prime	12	4.1.2 Pascal's trinagle	19
3.1.1 BFS	5	3.4.7 Chin. rem. th.	12	4.2 Theorems and definitions	20
3.1.2 DFS	5			4.3 Geometry Formulas	20

1 Setup

1.1 header.h

```

1 #pragma once // Delete this when copying this file
2 #include <bits/stdc++.h>
3 using namespace std;
4
5 #define ll long long
6 #define ull unsigned ll
7 #define ld long double
8 #define pl pair<ll, ll>

```

```

9 #define pi pair<int, int> // use pl where possible/
   necessary
10 #define vl vector<ll>
11 #define vi vector<int> // change to vl where
   possible/necessary
12 #define vb vector<bool>
13 #define vvi vector<vi>
14 #define vvl vector<vl>
15 #define vpl vector<pl>
16 #define vpi vector<pi>
17 #define vld vector<ld>
18 #define vvpi vector<vpi>
19 #define in_fast(el, cont) (cont.find(el) != cont.end
   ())
20 #define in(el, cont) (find(cont.begin(), cont.end(),

```

```

   el) != cont.end())
21
22 constexpr int INF = 2000000010;
23 constexpr ll LLINF = 900000000000000010LL;
24
25 template <typename T, template <typename ELEM,
   typename ALLOC = std::allocator<ELEM> > class
   Container>
26 std::ostream& operator<<(std::ostream& o, const
   Container<T>& container) {
27     typename Container<T>::const_iterator beg =
   container.begin();
28     if (beg != container.end()) {
29         o << *beg++;
30         while (beg != container.end()) {

```

```

31     o << " " << *beg++;
32 }
33 }
34 return o;
35 }
36
37 // int main() {
38 //   ios::sync_with_stdio(false); // do not use cout
39 //   + printf
40 //   cin.tie(NULL);
41 //   cout << fixed << setprecision(12);
42 //   return 0;
43 // }

```

1.2 Bash for c++ compile with header.h

```

1 #!/bin/bash
2 if [ $# -ne 1 ];then echo "Usage: $0 <input_file>";
3   exit 1;fi
4 f="$1";d=code/o=a.out
5 [ -f $d/$f ] || { echo "Input file not found: $f";
6   exit 1; }
7 g++ -I$d $d/$f -o $o && echo "Compilation successful
8   . Executable '$o' created." || echo "Compilation
9   failed."

```

1.3 Bash for run tests c++

```

1 g++ $1/$1.cpp -o $1/$1.out
2 for file in $1/*.in; do diff <($1/$1.out < "$file")
3   "${file%.in}.ans"; done

```

1.4 Bash for run tests python

```

1 for file in $1/*.in; do diff <(python3 $1/$1.py < "
2   $file") "${file%.in}.ans"; done

```

1.4.1 Aux. helper C++

```

1 #include "header.h"
2
3 int main() {
4   // Read in a line including white space

```

```

5   string line;
6   getline(cin, line);
7   // When doing the above read numbers as follows:
8   int n;
9   getline(cin, line);
10  stringstream ss(line);
11  ss >> n;
12
13  // Count the number of 1s in binary
14  // represnatation of a number
15  ull number;
16  __builtin_popcountll(number);

```

1.4.2 Aux. helper python

```

1 from functools import lru_cache
2
3 # Read until EOF
4 while True:
5     try:
6         pattern = input()
7     except EOFError:
8         break
9
10 @lru_cache(maxsize=None)
11 def smth_memoi(i, j, s):
12     # Example in-built cache
13     return "sol"

```

2 Python

2.1 Graphs

2.1.1 BFS

```

1 from collections import deque
2 def bfs(g, roots, n):
3     q = deque(roots)
4     explored = set(roots)
5     distances = [float("inf")]*n
6     distances[0][0] = 0
7
8     while len(q) != 0:
9         node = q.popleft()
10        if node in explored: continue
11        explored.add(node)
12        for neigh in g[node]:
13            if neigh not in explored:
14                q.append(neigh)
15                distances[neigh] = distances[node] +
16
17

```

```

16     return distances

```

2.1.2 Dijkstra

```

1 from heapq import *
2 def dijkstra(n, root, g): # g = {node: (cost, neigh)}
3     dist = [float("inf")]*n
4     dist[root] = 0
5     prev = [-1]*n
6
7     pq = [(0, root)]
8     heapify(pq)
9     visited = set([])
10
11     while len(pq) != 0:
12         _, node = heappop(pq)
13
14         if node in visited: continue
15         visited.add(node)
16
17         # In case of disconnected graphs
18         if node not in g:
19             continue
20
21         for cost, neigh in g[node]:
22             alt = dist[node] + cost
23             if alt < dist[neigh]:
24                 dist[neigh] = alt
25                 prev[neigh] = node
26                 heappush(pq, (alt, neigh))
27     return dist

```

2.1.3 Topological Sort

```

1 #Python program to print topological sorting of a
2 # DAG
3 from collections import defaultdict
4
5 #Class to represent a graph
6 class Graph:
7     def __init__(self,vertices):
8         self.graph = defaultdict(list) #dictionary
9         #containing adjacency List
10        self.V = vertices #No. of vertices
11
12    # function to add an edge to graph
13    def addEdge(self,u,v):
14        self.graph[u].append(v)
15
16    # A recursive function used by topologicalSort
17    def topologicalSortUtil(self,v,visited,stack):

```

```

17 # Mark the current node as visited.
18 visited[v] = True
19
20 # Recur for all the vertices adjacent to
    this vertex
21 for i in self.graph[v]:
22     if visited[i] == False:
23         self.topologicalSortUtil(i,visited,
            stack)
24
25 # Push current vertex to stack which stores
    result
26 stack.insert(0,v)
27
28 # The function to do Topological Sort. It uses
    recursive
29 # topologicalSortUtil()
30 def topologicalSort(self):
31     # Mark all the vertices as not visited
32     visited = [False]*self.V
33     stack =[]
34
35     # Call the recursive helper function to
        store Topological
36     # Sort starting from all vertices one by one
37     for i in range(self.V):
38         if visited[i] == False:
39             self.topologicalSortUtil(i,visited,
                stack)
40
41     # Print contents of stack
42     return stack
43
44 def isCyclicUtil(self, v, visited, recStack):
45
46     # Mark current node as visited and
47     # adds to recursion stack
48     visited[v] = True
49     recStack[v] = True
50
51     # Recur for all neighbours
52     # if any neighbour is visited and in
53     # recStack then graph is cyclic
54     for neighbour in self.graph[v]:
55         if visited[neighbour] == False:
56             if self.isCyclicUtil(neighbour,
                visited, recStack) == True:
57                 return True
58             elif recStack[neighbour] == True:
59                 return True
60
61     # The node needs to be popped from
62     # recursion stack before function ends
63     recStack[v] = False
64     return False

```

```

65 # Returns true if graph is cyclic else false
66 def isCyclic(self):
67     visited = [False] * (self.V + 1)
68     recStack = [False] * (self.V + 1)
69     for node in range(self.V):
70         if visited[node] == False:
71             if self.isCyclicUtil(node, visited,
                recStack) == True:
72                 return True
73
74     return False

```

2.1.4 Kruskal

```

1 class UnionFind:
2     def __init__(self, n):
3         self.parent = [-1]*n
4
5     def find(self, x):
6         if self.parent[x] < 0:
7             return x
8         self.parent[x] = self.find(self.parent[x])
9         return self.parent[x]
10
11     def connect(self, a, b):
12         ra = self.find(a)
13         rb = self.find(b)
14         if ra == rb:
15             return False
16         if self.parent[ra] > self.parent[rb]:
17             self.parent[rb] += self.parent[ra]
18             self.parent[ra] = rb
19         else:
20             self.parent[ra] += self.parent[rb]
21             self.parent[rb] = ra
22         return True
23
24 # Full MST is len(spanning)==n-1
25 def kruskal(n, edges):
26     uf = UnionFind(n)
27     spanning = []
28     edges.sort(key = lambda d: -d[2])
29     while edges and len(spanning) < n-1:
30         u, v, w = edges.pop()
31         if not uf.connect(u, v):
32             continue
33         spanning.append((u, v, w))
34     return spanning
35
36 # Example
37 edges = [(1, 2, 10), (2, 3, 20)]

```

2.2 Num. Th. / Comb.

2.2.1 nCk % prime

```

1 # Note: p must be prime and k < p
2 def fermat_binom(n, k, p):
3     if k > n:
4         return 0
5     # calculate numerator
6     num = 1
7     for i in range(n-k+1, n+1):
8         num *= i % p
9     num %= p
10    # calculate denominator
11    denom = 1
12    for i in range(1,k+1):
13        denom *= i % p
14    denom %= p
15    # numerator * denominator^(p-2) (mod p)
16    return (num * pow(denom, p-2, p)) % p

```

2.2.2 Sieve of E. $O(n)$ so actually faster than C++ version, but more memory

```

1 MAX_SIZE = 10**8+1
2 isprime = [True] * MAX_SIZE
3 prime = []
4 SPF = [None] * (MAX_SIZE)
5
6 def manipulated_seive(N): # Up to N (not included)
7     isprime[0] = isprime[1] = False
8     for i in range(2, N):
9         if isprime[i] == True:
10             prime.append(i)
11             SPF[i] = i
12             j = 0
13             while (j < len(prime) and
14                 i * prime[j] < N and
15                 prime[j] <= SPF[i]):
16                 isprime[i * prime[j]] = False
17                 SPF[i * prime[j]] = prime[j]
18                 j += 1

```

2.3 Strings

2.3.1 LCS

```

1 def longestCommonSubsequence(text1, text2): # O(m*n
    ) time, O(m) space
2     n = len(text1)
3     m = len(text2)
4

```

```

5 # Initializing two lists of size m
6 prev = [0] * (m + 1)
7 cur = [0] * (m + 1)
8
9 for idx1 in range(1, n + 1):
10     for idx2 in range(1, m + 1):
11         # If characters are matching
12         if text1[idx1 - 1] == text2[idx2 - 1]:
13             cur[idx2] = 1 + prev[idx2 - 1]
14         else:
15             # If characters are not matching
16             cur[idx2] = max(cur[idx2 - 1], prev[
17                             idx2])
18
19     prev = cur.copy()
20
21 return cur[m]

```

2.3.2 KMP

```

1 class KMP:
2     def partial(self, pattern):
3         """ Calculate partial match table: String ->
4             [Int] """
5         ret = [0]
6         for i in range(1, len(pattern)):
7             j = ret[i - 1]
8             while j > 0 and pattern[j] != pattern[i
9                 ]: j = ret[j - 1]
10            ret.append(j + 1 if pattern[j] ==
11                pattern[i] else j)
12        return ret
13
14    def search(self, T, P):
15        """KMP search main algorithm: String ->
16            String -> [Int]
17        Return all the matching position of pattern
18        string P in T"""
19        partial, ret, j = self.partial(P), [], 0
20        for i in range(len(T)):
21            while j > 0 and T[i] != P[j]: j =
22                partial[j - 1]
23            if T[i] == P[j]: j += 1
24            if j == len(P):
25                ret.append(i - (j - 1))
26                j = partial[j - 1]
27        return ret

```

2.3.3 Edit distance

```

1 def editDistance(str1, str2):
2     # Get the lengths of the input strings
3     m = len(str1)

```

```

4     n = len(str2)
5
6     # Initialize a list to store the current row
7     curr = [0] * (n + 1)
8
9     # Initialize the first row with values from 0 to n
10    for j in range(n + 1):
11        curr[j] = j
12
13    # Initialize a variable to store the previous
14    # value
15    previous = 0
16
17    # Loop through the rows of the dynamic programming
18    # matrix
19    for i in range(1, m + 1):
20        # Store the current value at the beginning of
21        # the row
22        previous = curr[0]
23        curr[0] = i
24
25        # Loop through the columns of the dynamic
26        # programming matrix
27        for j in range(1, n + 1):
28            # Store the current value in a temporary
29            # variable
30            temp = curr[j]
31
32            # Check if the characters at the current
33            # positions in str1 and str2 are the same
34            if str1[i - 1] == str2[j - 1]:
35                curr[j] = previous
36            else:
37                # Update the current cell with the minimum
38                # of the three adjacent cells
39                curr[j] = 1 + min(previous, curr[j - 1],
40                                curr[j])
41
42            # Update the previous variable with the
43            # temporary value
44            previous = temp
45
46    # The value in the last cell represents the
47    # minimum number of operations
48    return curr[n]

```

2.4 Other Algorithms

2.4.1 Rotate matrix

```

1 def rotate_matrix(m):
2     return [[m[j][i] for j in range(len(m))] for i
3             in range(len(m[0])-1,-1,-1)]

```

2.5 Geometry

2.5.1 Convex Hull

```

1 def vec(a,b):
2     return (b[0]-a[0],b[1]-a[1])
3 def det(a,b):
4     return a[0]*b[1] - b[0]*a[1]
5
6 def convexhull(P):
7     if (len(P) == 1):
8         return [(p[0][0], p[0][1])]
9
10    h = sorted(P)
11    lower = []
12    i = 0
13    while i < len(h):
14        if len(lower) > 1:
15            a = vec(lower[-2], lower[-1])
16            b = vec(lower[-1], h[i])
17            if det(a,b) <= 0 and len(lower) > 1:
18                lower.pop()
19                continue
20            lower.append(h[i])
21            i += 1
22
23    upper = []
24    i = 0
25    while i < len(h):
26        if len(upper) > 1:
27            a = vec(upper[-2], upper[-1])
28            b = vec(upper[-1], h[i])
29            if det(a,b) >= 0:
30                upper.pop()
31                continue
32            upper.append(h[i])
33            i += 1
34
35    reversedupper = list(reversed(upper[1:-1]))
36    reversedupper.extend(lower)
37    return reversedupper

```

2.5.2 Geometry

```

1
2 def vec(a,b):
3     return (b[0]-a[0],b[1]-a[1])
4
5 def det(a,b):
6     return a[0]*b[1] - b[0]*a[1]
7
8     lower = []
9     i = 0
10    while i < len(h):

```

```

11     if len(lower) > 1:
12         a = vec(lower[-2], lower[-1])
13         b = vec(lower[-1], h[i])
14         if det(a,b) <= 0 and len(lower) > 1:
15             lower.pop()
16             continue
17     lower.append(h[i])
18     i += 1
19
20 # find upper hull
21 # det <= 0 -> replace
22 upper = []
23 i = 0
24 while i < len(h):
25     if len(upper) > 1:
26         a = vec(upper[-2], upper[-1])
27         b = vec(upper[-1], h[i])
28         if det(a,b) >= 0:
29             upper.pop()
30             continue
31     upper.append(h[i])
32     i += 1

```

2.6 Other Data Structures

2.6.1 Segment Tree

```

1 N = 100000 # limit for array size
2 tree = [0] * (2 * N) # Max size of tree
3
4 def build(arr, n): # function to build the tree
5     # insert leaf nodes in tree
6     for i in range(n):
7         tree[n + i] = arr[i]
8
9     # build the tree by calculating parents
10    for i in range(n - 1, 0, -1):
11        tree[i] = tree[i << 1] + tree[i << 1 | 1]
12
13 def updateTreeNode(p, value, n): # function to
14     # update a tree node
15     # set value at position p
16     tree[p + n] = value
17     p = p + n
18
19     i = p # move upward and update parents
20     while i > 1:
21         tree[i >> 1] = tree[i] + tree[i ^ 1]
22         i >>= 1
23
24 def query(l, r, n): # function to get sum on
25     # interval [l, r)
26     res = 0
27     # loop to find the sum in the range

```

```

26     l += n
27     r += n
28     while l < r:
29         if l & 1:
30             res += tree[l]
31             l += 1
32         if r & 1:
33             r -= 1
34             res += tree[r]
35         l >>= 1
36         r >>= 1
37     return res

```

2.6.2 Trie

```

1 class TrieNode:
2     def __init__(self):
3         self.children = [None]*26
4         self.isEndOfWord = False
5
6 class Trie:
7     def __init__(self):
8         self.root = self.getNode()
9
10    def getNode(self):
11        return TrieNode()
12
13    def _charToIndex(self, ch):
14        return ord(ch)-ord('a')
15
16    def insert(self, key):
17        pCrawl = self.root
18        length = len(key)
19        for level in range(length):
20            index = self._charToIndex(key[level])
21            if not pCrawl.children[index]:
22                pCrawl.children[index] = self.
23                getNode()
24            pCrawl = pCrawl.children[index]
25            pCrawl.isEndOfWord = True
26
27    def search(self, key):
28        pCrawl = self.root
29        length = len(key)
30        for level in range(length):
31            index = self._charToIndex(key[level])
32            if not pCrawl.children[index]:
33                return False
34            pCrawl = pCrawl.children[index]
35
36        return pCrawl.isEndOfWord

```

3 C++

3.1 Graphs

3.1.1 BFS

```

1 #include "header.h"
2 #define graph unordered_map<ll, unordered_set<ll>>
3 vi bfs(int n, graph& g, vi& roots) {
4     vi parents(n+1, -1); // nodes are 1..n
5     unordered_set<int> visited;
6     queue<int> q;
7     for (auto x: roots) {
8         q.emplace(x);
9         visited.insert(x);
10    }
11    while (not q.empty()) {
12        int node = q.front();
13        q.pop();
14
15        for (auto neigh: g[node]) {
16            if (not in(neigh, visited)) {
17                parents[neigh] = node;
18                q.emplace(neigh);
19                visited.insert(neigh);
20            }
21        }
22    }
23    return parents;
24 }
25 vi reconstruct_path(vi parents, int start, int goal)
26 {
27     vi path;
28     int curr = goal;
29     while (curr != start) {
30         path.push_back(curr);
31         if (parents[curr] == -1) return vi(); // No
32         path, empty vi
33         curr = parents[curr];
34     }
35     path.push_back(start);
36     reverse(path.begin(), path.end());
37     return path;
38 }

```

3.1.2 DFS Cycle detection / removal

```

1 #include "header.h"
2 void removeCyc(ll node, unordered_map<ll, vector<
3     pair<ll, ll>>& neighs, vector<bool>& visited,
4     vector<bool>& recStack, vector<ll>& ans) {
5     if (!visited[node]) {
6         visited[node] = true;

```

```

6   recStack[node] = true;
7   auto it = neighs.find(node);
8   if (it != neighs.end()) {
9       for (auto util: it->second) {
10          ll nnode = util.first;
11          if (recStack[nnode]) {
12              ans.push_back(util.second);
13          } else if (!visited[nnode]) {
14              removeCyc(nnode, neighs, visited
15                      , recStack, ans);
16          }
17      }
18  }
19  recStack[node] = false;
20 }

```

3.1.3 Dijkstra

```

1 #include "header.h"
2 vector<int> dijkstra(int n, int root, map<int,
3   vector<pair<int, int>>& g) {
4   unordered_set<int> visited;
5   vector<int> dist(n, INF);
6   priority_queue<pair<int, int>> pq;
7   dist[root] = 0;
8   pq.push({0, root});
9   while (!pq.empty()) {
10      int node = pq.top().second;
11      int d = -pq.top().first;
12      pq.pop();
13
14      if (in(node, visited)) continue;
15      visited.insert(node);
16
17      for (auto e : g[node]) {
18          int neigh = e.first;
19          int cost = e.second;
20          if (dist[neigh] > dist[node] + cost) {
21              dist[neigh] = dist[node] + cost;
22              pq.push({-dist[neigh], neigh});
23          }
24      }
25      return dist;
26 }

```

3.1.4 Floyd-Warshall

```

1 #include "header.h"
2 // g[i][j] = infy if not path from i to j
3 // if g[i][i] < 0, i is contained in a negative
   cycle

```

```

4 void warshall(vvl g) {
5     for (int i=0; i<g.size(); ++i) {
6         for (int j=0; j<g.size(); ++j) {
7             for (int k=0; k<g.size(); ++k) {
8                 if (g[i][k] < LLINF and g[k][j] <
9                     LLINF and g[i][j] > g[i][k] + g[
10                        k][j]) {
11                     g[i][j] = g[i][k] + g[k][j];
12                 }
13             }
14         }
15     }
16 }

```

3.1.5 Kruskal Minimum spanning tree of undirected weighted graph

```

1 #include "header.h"
2 #include "disjoint_set.h"
3 // O(E log E)
4 pair<set<pair<ll, ll>>, ll> kruskal(vector<tuple<ll
5   , ll, ll>>& edges, ll n) {
6     set<pair<ll, ll>> ans;
7     ll cost = 0;
8
9     sort(edges.begin(), edges.end());
10    DisjointSet<ll> fs(n);
11
12    ll dist, i, j;
13    for (auto edge: edges) {
14        dist = get<0>(edge);
15        i = get<1>(edge);
16        j = get<2>(edge);
17
18        if (fs.find_set(i) != fs.find_set(j)) {
19            fs.union_sets(i, j);
20            ans.insert({i, j});
21            cost += dist;
22        }
23    }
24    return pair<set<pair<ll, ll>>, ll> {ans, cost};
25 }

```

3.1.6 Hungarian algorithm

```

1 #include "header.h"
2
3 template <class T> bool ckmin(T &a, const T &b) {
4     return b < a ? a = b, 1 : 0; }
5
6 /**
7  * Given J jobs and W workers (J <= W), computes the
8  * minimum cost to assign each
9  * prefix of jobs to distinct workers.
10  * @tparam T a type large enough to represent
11  * integers on the order of J *
12  * max(|C|)

```

```

9  * @param C a matrix of dimensions JxW such that C[j
10  * ][w] = cost to assign j-th
11  * job to w-th worker (possibly negative)
12  *
13  * @return a vector of length J, with the j-th entry
14  * equaling the minimum cost
15  * to assign the first (j+1) jobs to distinct
16  * workers
17  */
18 template <class T> vector<T> hungarian(const vector<
19   vector<T>> &C) {
20     const int J = (int)size(C), W = (int)size(C[0]);
21     assert(J <= W);
22     // job[w] = job assigned to w-th worker, or -1
23     // if no job assigned
24     // note: a W-th worker was added for convenience
25     vector<int> job(W + 1, -1);
26     vector<T> ys(J), yt(W + 1); // potentials
27     // -yt[W] will equal the sum of all deltas
28     vector<T> answers;
29     const T inf = numeric_limits<T>::max();
30     for (int j_cur = 0; j_cur < J; ++j_cur) { //
31         assign j_cur-th job
32         int w_cur = W;
33         job[w_cur] = j_cur;
34         // min reduced cost over edges from Z to
35         // worker w
36         vector<T> min_to(W + 1, inf);
37         vector<int> prv(W + 1, -1); // previous
38         // worker on alternating path
39         vector<bool> in_Z(W + 1); // whether
40         // worker is in Z
41         while (job[w_cur] != -1) { // runs at most
42             j_cur + 1 times
43             in_Z[w_cur] = true;
44             const int j = job[w_cur];
45             T delta = inf;
46             int w_next;
47             for (int w = 0; w < W; ++w) {
48                 if (!in_Z[w]) {
49                     if (ckmin(min_to[w], C[j][w] -
50                             ys[j] - yt[w]))
51                         prv[w] = w_cur;
52                     if (ckmin(delta, min_to[w]))
53                         w_next = w;
54                 }
55             }
56             // delta will always be non-negative,
57             // except possibly during the first time
58             // this loop runs
59             // if any entries of C[j_cur] are
60             // negative
61             for (int w = 0; w <= W; ++w) {
62                 if (in_Z[w]) ys[job[w]] += delta, yt
63                     [w] -= delta;
64             }
65         }
66     }
67 }

```

```

49         else min_to[w] -= delta;
50     }
51     w_cur = w_next;
52 }
53 // update assignments along alternating path
54 for (int w; w_cur != W; w_cur = w) job[w_cur
    ] = job[w = prv[w_cur]];
55 answers.push_back(-yt[W]);
56 }
57 return answers;
58 }

```

3.1.7 Suc. shortest path Calculates max flow, min cost

```

1 #include "header.h"
2 // map<node, map<node, pair<cost, capacity>>>
3 #define graph unordered_map<int, unordered_map<int,
    pair<ld, int>>>
4 graph g;
5 const ld infy = 1e601; // Change if necessary
6 ld fill(int n, vld& potential) { // Finds max flow,
    min cost
7     priority_queue<pair<ld, int>> pq;
8     vector<bool> visited(n+2, false);
9     vi parent(n+2, 0);
10    vld dist(n+2, infy);
11    dist[0] = 0.1;
12    pq.emplace(make_pair(0.1, 0));
13    while (not pq.empty()) {
14        int node = pq.top().second;
15        pq.pop();
16        if (visited[node]) continue;
17        visited[node] = true;
18        for (auto& x : g[node]) {
19            int neigh = x.first;
20            int capacity = x.second.second;
21            ld cost = x.second.first;
22            if (capacity and not visited[neigh]) {
23                ld d = dist[node] + cost + potential[node] -
                    potential[neigh];
24                if (d + 1e-101 < dist[neigh]) {
25                    dist[neigh] = d;
26                    pq.emplace(make_pair(-d, neigh));
27                    parent[neigh] = node;
28                }
29            }
30            for (int i = 0; i < n+2; i++) {
31                potential[i] = min(infy, potential[i] + dist[i
                    ]);
32            }
33            if (not parent[n+1]) return infy;
34            ld ans = 0.1;
35            for (int x = n+1; x; x=parent[x]) {

```

```

36        ans += g[parent[x]][x].first;
37        g[parent[x]][x].second--;
38        g[x][parent[x]].second++;
39    }
40    return ans;
41 }

```

3.1.8 Bipartite check

```

1 #include "header.h"
2 int main() {
3     int n;
4     vvi adj(n);
5
6     vi side(n, -1); // will have 0's for one side
7                     // 1's for other side
8     bool is_bipartite = true; // becomes false if
9                             // not bipartite
10    queue<int> q;
11    for (int st = 0; st < n; ++st) {
12        if (side[st] == -1) {
13            q.push(st);
14            side[st] = 0;
15            while (!q.empty()) {
16                int v = q.front();
17                q.pop();
18                for (int u : adj[v]) {
19                    if (side[u] == -1) {
20                        side[u] = side[v] ^ 1;
21                        q.push(u);
22                    } else {
23                        is_bipartite &= side[u] !=
                            side[v];
24                    }
25                }
26            }
27        }
28    }
29 }

```

3.1.9 Find cycle directed

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5+5;
4 vvi adj(mxN);
5 vector<char> color;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v) {
9     color[v] = 1;
10    for (int u : adj[v]) {
11        if (color[u] == 0) {
12            parent[u] = v;
13            if (dfs(u)) return true;
14        } else if (color[u] == 1) {
15            cycle_end = v;

```

```

16        cycle_start = u;
17        return true;
18    }
19 }
20 color[v] = 2;
21 return false;
22 }
23 void find_cycle() {
24     color.assign(n, 0);
25     parent.assign(n, -1);
26     cycle_start = -1;
27     for (int v = 0; v < n; v++) {
28         if (color[v] == 0 && dfs(v)) break;
29     }
30     if (cycle_start == -1) {
31         cout << "Acyclic" << endl;
32     } else {
33         vector<int> cycle;
34         cycle.push_back(cycle_start);
35         for (int v = cycle_end; v != cycle_start; v
            = parent[v])
36             cycle.push_back(v);
37         cycle.push_back(cycle_start);
38         reverse(cycle.begin(), cycle.end());
39
40         cout << "Cycle Found: ";
41         for (int v : cycle) cout << v << " ";
42         cout << endl;
43     }
44 }

```

3.1.10 Find cycle directed

```

1 #include "header.h"
2 int n;
3 const int mxN = 2e5 + 5;
4 vvi adj(mxN);
5 vector<bool> visited;
6 vi parent;
7 int cycle_start, cycle_end;
8 bool dfs(int v, int par) { // passing vertex and its
    parent vertex
9     visited[v] = true;
10    for (int u : adj[v]) {
11        if (u == par) continue; // skipping edge to
            parent vertex
12        if (visited[u]) {
13            cycle_end = v;
14            cycle_start = u;
15            return true;
16        }
17        parent[u] = v;
18        if (dfs(u, parent[u]))
19            return true;

```



```

20 }
21 return false;
22 }
23 void find_cycle() {
24     visited.assign(n, false);
25     parent.assign(n, -1);
26     cycle_start = -1;
27     for (int v = 0; v < n; v++) {
28         if (!visited[v] && dfs(v, parent[v])) break;
29     }
30     if (cycle_start == -1) {
31         cout << "Acyclic" << endl;
32     } else {
33         vector<int> cycle;
34         cycle.push_back(cycle_start);
35         for (int v = cycle_end; v != cycle_start; v
            = parent[v])
36             cycle.push_back(v);
37         cycle.push_back(cycle_start);
38         cout << "Cycle Found: ";
39         for (int v : cycle) cout << v << " ";
40         cout << endl;
41     }
42 }

```

3.1.11 Tarjan's SCC

```

1 #include "header.h"
2
3 struct Tarjan {
4     vvi &edges;
5     int V, counter = 0, C = 0;
6     vi n, l;
7     vector<bool> vs;
8     stack<int> st;
9     Tarjan(vvi &e) : edges(e), V(e.size()), n(V, -1),
        l(V, -1), vs(V, false) {}
10 void visit(int u, vi &com) {
11     l[u] = n[u] = counter++;
12     st.push(u);
13     vs[u] = true;
14     for (auto &&v : edges[u]) {
15         if (n[v] == -1) visit(v, com);
16         if (vs[v]) l[u] = min(l[u], l[v]);
17     }
18     if (l[u] == n[u]) {
19         while (true) {
20             int v = st.top();
21             st.pop();
22             vs[v] = false;
23             com[v] = C; //<== ACT HERE
24             if (u == v) break;
25         }
26         C++;

```

```

27 }
28 }
29 int find_sccs(vi &com) { // component indices
    will be stored in 'com'
30     com.assign(V, -1);
31     C = 0;
32     for (int u = 0; u < V; ++u)
33         if (n[u] == -1) visit(u, com);
34     return C;
35 }
36 // scc is a map of the original vertices of the
    graph to the vertices
37 // of the SCC graph, scc_graph is its adjacency
    list.
38 // SCC indices and edges are stored in 'scc' and '
    scc_graph'.
39 void scc_collapse(vi &scc, vvi &scc_graph) {
40     find_sccs(scc);
41     scc_graph.assign(C, vi());
42     set<pi> rec; // recorded edges
43     for (int u = 0; u < V; ++u) {
44         assert(scc[u] != -1);
45         for (int v : edges[u]) {
46             if (scc[v] == scc[u] ||
                rec.find({scc[u], scc[v]}) != rec.end())
47                 continue;
48             scc_graph[scc[u]].push_back(scc[v]);
49             rec.insert({scc[u], scc[v]});
50         }
51     }
52 }
53 // Function to find sources and sinks in the SCC
    graph
54 // The number of edges needed to be added is max(
    sources.size(), sinks.size())
55 void findSourcesAndSinks(const vvi &scc_graph, vi
    &sources, vi &sinks) {
56     vi in_degree(C, 0), out_degree(C, 0);
57     for (int u = 0; u < C; ++u) {
58         for (auto v : scc_graph[u]) {
59             in_degree[v]++;
60             out_degree[u]++;
61         }
62     }
63     for (int i = 0; i < C; ++i) {
64         if (in_degree[i] == 0) sources.push_back(i);
65         if (out_degree[i] == 0) sinks.push_back(i);
66     }
67 }
68 };

```

3.1.12 SCC edges Prints out the missing edges to make the input digraph strongly connected

```

1 #include "header.h"
2 const int N=1e5+10;
3 int n,a[N],cnt[N],vis[N];
4 vector<int> hd,tl;
5 int dfs(int x){
6     vis[x]=1;
7     if(!vis[a[x]])return vis[x]=dfs(a[x]);
8     return vis[x]=x;
9 }
10 int main(){
11     scanf("%d",&n);
12     for(int i=1;i<=n;i++){
13         scanf("%d",&a[i]);
14         cnt[a[i]]++;
15     }
16     int k=0;
17     for(int i=1;i<=n;i++){
18         if(!cnt[i]){
19             k++;
20             hd.push_back(i);
21             tl.push_back(dfs(i));
22         }
23     }
24     int tk=k;
25     for(int i=1;i<=n;i++){
26         if(!vis[i]){
27             k++;
28             hd.push_back(i);
29             tl.push_back(dfs(i));
30         }
31     }
32     if(k==1&&!tk)k=0;
33     printf("%d\n",k);
34     for(int i=0;i<k;i++)printf("%d %d\n",tl[i],hd[(i
        +1)%k]);
35     return 0;
36 }

```

3.1.13 Find Bridges

```

1 #include "header.h"
2 int n; // number of nodes
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi tin, low;
6 int timer;
7 void dfs(int v, int p = -1) {
8     visited[v] = true;
9     tin[v] = low[v] = timer++;
10    for (int to : adj[v]) {
11        if (to == p) continue;
12        if (visited[to]) {
13            low[v] = min(low[v], tin[to]);
14        } else {

```



```

15     dfs(to, v);
16     low[v] = min(low[v], low[to]);
17     if (low[to] > tin[v])
18         IS_BRIDGE(v, to);
19 }
20 }
21 }
22 void find_bridges() {
23     timer = 0;
24     visited.assign(n, false);
25     tin.assign(n, -1);
26     low.assign(n, -1);
27     for (int i = 0; i < n; ++i) {
28         if (!visited[i]) dfs(i);
29     }
30 }

```

3.1.14 Artic. points (i.e. cut off points)

```

1 #include "header.h"
2 int n; // number of nodes
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi tin, low;
6 int timer;
7 void dfs(int v, int p = -1) {
8     visited[v] = true;
9     tin[v] = low[v] = timer++;
10    int children=0;
11    for (int to : adj[v]) {
12        if (to == p) continue;
13        if (visited[to]) {
14            low[v] = min(low[v], tin[to]);
15        } else {
16            dfs(to, v);
17            low[v] = min(low[v], low[to]);
18            if (low[to] >= tin[v] && p!=-1)
19                IS_CUTPOINT(v);
20            ++children;
21        }
22    }
23    if(p == -1 && children > 1)
24        IS_CUTPOINT(v);
25 }
26 void find_cutpoints() {
27     timer = 0;
28     visited.assign(n, false);
29     tin.assign(n, -1);
30     low.assign(n, -1);
31     for (int i = 0; i < n; ++i) {
32         if (!visited[i]) dfs (i);
33     }

```

3.1.15 Topological sort

```

1 #include "header.h"
2 int n; // number of vertices
3 vvi adj; // adjacency list of graph
4 vector<bool> visited;
5 vi ans;
6 void dfs(int v) {
7     visited[v] = true;
8     for (int u : adj[v]) {
9         if (!visited[u]) dfs(u);
10    }
11    ans.push_back(v);
12 }
13 void topological_sort() {
14     visited.assign(n, false);
15     ans.clear();
16     for (int i = 0; i < n; ++i) {
17         if (!visited[i]) dfs(i);
18     }
19     reverse(ans.begin(), ans.end());
20 }

```

3.1.16 Bellmann-Ford Same as Dijkstra but allows neg. edges

```

1 #include "header.h"
2 // Switch vi and vvpj to vl and vvpj if necessary
3 void bellmann_ford_extended(vvpj &e, int source, vi
    &dist, vb &cyc) {
4     dist.assign(e.size(), INF);
5     cyc.assign(e.size(), false); // true when u is in
6     a <0 cycle
7     dist[source] = 0;
8     for (int iter = 0; iter < e.size() - 1; ++iter){
9         bool relax = false;
10        for (int u = 0; u < e.size(); ++u)
11            if (dist[u] == INF) continue;
12            else for (auto &e : e[u])
13                if(dist[u]+e.second < dist[e.first])
14                    dist[e.first] = dist[u]+e.second, relax =
15                        true;
16            if(!relax) break;
17        }
18    }
19    bool ch = true;
20    while (ch) { // keep going untill no more
21        changes
22        ch = false; // set dist to -INF when in
23        cycle
24        for (int u = 0; u < e.size(); ++u)
25            if (dist[u] == INF) continue;
26            else for (auto &e : e[u])
27                if (dist[e.first] > dist[u] + e.second
28                    && !cyc[e.first]) {

```

```

24     dist[e.first] = -INF;
25     ch = true; //return true for cycle
26         detection only
27     cyc[e.first] = true;
28 }
29 }

```

3.1.17 Ford-Fulkerson Basic Max. flow

```

1 #include "header.h"
2 #define V 6 // Num. of vertices in given graph
3
4 /* Returns true if there is a path from source 's'
5    to sink
6    't' in residual graph. Also fills parent[] to store
7    the
8    path */
9 bool bfs(int rGraph[V][V], int s, int t, int parent
10    []) {
11    bool visited[V];
12    memset(visited, 0, sizeof(visited));
13    queue<int> q;
14    q.push(s);
15    visited[s] = true;
16    parent[s] = -1;
17
18    // Standard BFS Loop
19    while (!q.empty()) {
20        int u = q.front();
21        q.pop();
22
23        for (int v = 0; v < V; v++) {
24            if (visited[v] == false && rGraph[u][v] > 0) {
25                if (v == t) {
26                    parent[v] = u;
27                    return true;
28                }
29                q.push(v);
30                parent[v] = u;
31                visited[v] = true;
32            }
33        }
34    }
35    return false;
36 }
37
38 // Returns the maximum flow from s to t in the given
39 graph
40 int fordFulkerson(int graph[V][V], int s, int t) {
41     int u, v;
42     int rGraph[V][V];
43     for (u = 0; u < V; u++)

```

```

41 for (v = 0; v < V; v++)
42     rGraph[u][v] = graph[u][v];
43
44 int parent[V]; // This array is filled by BFS and
45     to
46     // store path
47 int max_flow = 0; // There is no flow initially
48 while (bfs(rGraph, s, t, parent)) {
49     int path_flow = INT_MAX;
50     for (v = t; v != s; v = parent[v]) {
51         u = parent[v];
52         path_flow = min(path_flow, rGraph[u][v]);
53     }
54     for (v = t; v != s; v = parent[v]) {
55         u = parent[v];
56         rGraph[u][v] -= path_flow;
57         rGraph[v][u] += path_flow;
58     }
59     max_flow += path_flow;
60 }
61 return max_flow;
62 }

```

3.1.18 Dinic max flow $O(V^2E)$, $O(Ef)$

```

1 using F = ll; using W = ll; // types for flow and
2     weight/cost
3 struct S{
4     const int v; // neighbour
5     const int r; // index of the reverse edge
6     F f; // current flow
7     const F cap; // capacity
8     const W cost; // unit cost
9     S(int v, int ri, F c, W cost = 0) :
10         v(v), r(ri), f(0), cap(c), cost(cost) {}
11     inline F res() const { return cap - f; }
12 };
13 struct FlowGraph : vector<vector<S>> {
14     FlowGraph(size_t n) : vector<vector<S>>(n) {}
15     void add_edge(int u, int v, F c, W cost = 0){
16         auto &t = *this;
17         t[u].emplace_back(v, t[v].size(), c, cost);
18         t[v].emplace_back(u, t[u].size()-1, c, -cost);
19     }
20     void add_arc(int u, int v, F c, W cost = 0){
21         auto &t = *this;
22         t[u].emplace_back(v, t[v].size(), c, cost);
23         t[v].emplace_back(u, t[u].size()-1, 0, -cost);
24     }
25     void clear() { for (auto &E : *this) for (auto &
26         e : E) e.f = 0LL; }

```

```

24 };
25 struct Dinic{
26     FlowGraph &edges; int V,s,t;
27     vi l; vector<vector<S>::iterator> its; // levels
28     and iterators
29     Dinic(FlowGraph &edges, int s, int t) :
30         edges(edges), V(edges.size()), s(s), t(t), l
31         (V,-1), its(V) {}
32     ll augment(int u, F c) { // we reuse the same
33         iterators
34         if (u == t) return c; ll r = 0LL;
35         for(auto &i = its[u]; i != edges[u].end(); i
36             ++){
37             auto &e = *i;
38             if (e.res() && l[u] < l[e.v]) {
39                 auto d = augment(e.v, min(c, e.res())
40                     );
41                 if (d > 0) { e.f += d; edges[e.v][e.
42                     r].f -= d; c -= d;
43                     r += d; if (!c) break; }
44             }
45         }
46         return r;
47     }
48     ll run() {
49         ll flow = 0, f;
50         while(true) {
51             fill(l.begin(), l.end(),-1); l[s]=0; //
52             recalculate the layers
53             queue<int> q; q.push(s);
54             while(!q.empty()){
55                 auto u = q.front(); q.pop(); its[u]
56                 = edges[u].begin();
57                 for(auto &&e : edges[u]) if(e.res()
58                     && l[e.v]<0)
59                     l[e.v] = l[u]+1, q.push(e.v);
60             }
61             if (l[t] < 0) return flow;
62             while ((f = augment(s, INF)) > 0) flow
63                 += f;
64         }
65     }
66 };

```

3.2 Dynamic Programming

3.2.1 Longest Incr. Subseq.

```

1 #include "header.h"
2 template<class T>
3 vector<T> index_path_lis(vector<T>& nums) {
4     int n = nums.size();
5     vector<T> sub;
6     vector<int> subIndex;
7     vector<T> path(n, -1);
8     for (int i = 0; i < n; ++i) {

```

```

9         if (sub.empty() || sub[sub.size() - 1] < nums[
10             i]) {
11             path[i] = sub.empty() ? -1 : subIndex[sub.size()
12                 - 1];
13             sub.push_back(nums[i]);
14             subIndex.push_back(i);
15         } else {
16             int idx = lower_bound(sub.begin(), sub.end(),
17                 nums[i]) - sub.begin();
18             path[i] = idx == 0 ? -1 : subIndex[idx - 1];
19             sub[idx] = nums[i];
20             subIndex[idx] = i;
21         }
22     }
23     vector<T> ans;
24     int t = subIndex[subIndex.size() - 1];
25     while (t != -1) {
26         ans.push_back(t);
27         t = path[t];
28     }
29     reverse(ans.begin(), ans.end());
30     return ans;
31 }
32 // Length only
33 template<class T>
34 int length_lis(vector<T> &a) {
35     set<T> st;
36     typename set<T>::iterator it;
37     for (int i = 0; i < a.size(); ++i) {
38         it = st.lower_bound(a[i]);
39         if (it != st.end()) st.erase(it);
40         st.insert(a[i]);
41     }
42     return st.size();
43 }

```

3.2.2 0-1 Knapsack

```

1 #include "header.h"
2 // given a number of coins, calculate all possible
3     distinct sums
4 int main() {
5     int n;
6     vi coins(n); // all possible coins to use
7     int sum = 0; // sum of the coins
8     vi dp(sum + 1, 0); // dp[x] = 1 if sum x
9         can be made
10     dp[0] = 1; // sum 0 can be made
11     for (int c = 0; c < n; ++c) // first
12         iteration: sums with first
13         for (int x = sum; x >= 0; --x) // coin,
14             next first 2 coins etc
15         if (dp[x]) dp[x + coins[c]] = 1; // if sum x
16             valid, x+c valid

```

12 }

3.2.3 Coin change Number of coins required to achieve a given value

```
1 #include "header.h"
2 // Returns total distinct ways to make sum using n
  coins of
3 // different denominations
4 int count(vector<int> coins, int n, int sum) {
5     // 2d dp array where n is the number of coin
6     // denominations and sum is the target sum
7     vector<vector<int>> dp(n + 1, vector<int>(sum +
8         1, 0));
9     dp[0][0] = 1;
10    for (int i = 1; i <= n; i++) {
11        for (int j = 0; j <= sum; j++) {
12            // without using the current coin,
13            dp[i][j] += dp[i - 1][j];
14
15            // using the current coin
16            if ((j - coins[i - 1]) >= 0)
17                dp[i][j] += dp[i][j - coins[i - 1]];
18        }
19    }
20    return dp[n][sum];
21 }
```

3.3 Trees

3.3.1 Tree diameter

```
1 #include "header.h"
2 const int mxN = 2e5 + 5;
3 int n, d[mxN]; // distance array
4 vi adj[mxN]; // tree adjacency list
5 void dfs(int s, int e) {
6     d[s] = 1 + d[e]; // recursively calculate the
7     distance from the starting node to each node
8     for (auto u : adj[s]) { // for each adjacent node
9         if (u != e) dfs(u, s); // don't move backwards
10        in the tree
11    }
12 }
13 int main() {
14     // read input, create adj list
15     dfs(0, -1); // first dfs call to
16     find farthest node from arbitrary node
17     dfs(distance(d, max_element(d, d + n)), -1); //
18     second dfs call to find farthest node from
19     that one
```

```
15 cout << *max_element(d, d + n) - 1 << '\n'; //
    distance from second node to farthest is the
    diameter
16 }
```

3.3.2 Tree Node Count

```
1 #include "header.h"
2 // calculate amount of nodes in each node's subtree
3 const int mxN = 2e5 + 5;
4 int n, cnt[mxN];
5 vi adj[mxN];
6 void dfs(int s = 0, int e = -1) {
7     cnt[s] = 1; // count leaves as one
8     for (int u : adj[s]) {
9         dfs(u, s);
10        cnt[s] += cnt[u]; // add up nodes of the
11        subtrees
12 }
```

3.4 Num. Th. / Comb.

3.4.1 Basic stuff

```
1 #include "header.h"
2 ll gcd(ll a, ll b) { while (b) { a %= b; swap(a, b); }
3   return a; }
4 ll lcm(ll a, ll b) { return (a / gcd(a, b)) * b; }
5 ll mod(ll a, ll b) { return ((a % b) + b) % b; }
6 // Finds x, y s.t. ax + by = d = gcd(a, b).
7 void extended_euclid(ll a, ll b, ll &x, ll &y, ll &d
8   ) {
9     ll xx = y = 0;
10    ll yy = x = 1;
11    while (b) {
12        ll q = a / b;
13        ll t = b; b = a % b; a = t;
14        t = xx; xx = x - q * xx; x = t;
15        t = yy; yy = y - q * yy; y = t;
16    }
17    d = a;
18 }
19 // solves ab = 1 (mod n), -1 on failure
20 ll mod_inverse(ll a, ll n) {
21     ll x, y, d;
22     extended_euclid(a, n, x, y, d);
23     return (d > 1 ? -1 : mod(x, n));
24 }
25 // All modular inverses of [1..n] mod P in O(n) time
26 vi inverses(ll n, ll P) {
27     vi I(n+1, 1LL);
```

```
26 for (ll i = 2; i <= n; ++i)
27     I[i] = mod(-(P/i) * I[P%i], P);
28 return I;
29 }
30 // (a*b)%m
31 ll mulmod(ll a, ll b, ll m){
32     ll x = 0, y=a%m;
33     while(b>0){
34         if(b&1) x = (x+y)%m;
35         y = (2*y)%m, b /= 2;
36     }
37     return x % m;
38 }
39 // Finds b^e % m in O(lg n) time, ensure that b < m
40 // to avoid overflow!
41 ll powmod(ll b, ll e, ll m) {
42     ll p = e<2 ? 1 : powmod((b*b)%m,e/2,m);
43     return e&1 ? p*b%m : p;
44 }
45 // Solve ax + by = c, returns false on failure.
46 bool linear_diophantine(ll a, ll b, ll c, ll &x, ll
47   &y) {
48     ll d = gcd(a, b);
49     if (c % d) {
50         return false;
51     } else {
52         x = c / d * mod_inverse(a / d, b / d);
53         y = (c - a * x) / b;
54         return true;
55     }
56 }
```

3.4.2 Mod. exponentiation Or use pow() in python

```
1 #include "header.h"
2 ll mod_pow(ll base, ll exp, ll mod) {
3     if (mod == 1) return 0;
4     if (exp == 0) return 1;
5     if (exp == 1) return base;
6
7     ll res = 1;
8     base %= mod;
9     while (exp) {
10        if (exp % 2 == 1) res = (res * base) % mod;
11        exp >>= 1;
12        base = (base * base) % mod;
13    }
14    return res % mod;
15 }
16 }
```

3.4.3 GCD Or math.gcd in python, std::gcd in C++

```

1 #include "header.h"
2 ll gcd(ll a, ll b) {
3     if (a == 0) return b;
4     return gcd(b % a, a);
5 }

```

3.4.4 Sieve of Eratosthenes

```

1 #include "header.h"
2 vl primes;
3 void getprimes(ll n) { // Up to n (not included)
4     vector<bool> p(n, true);
5     p[0] = false;
6     p[1] = false;
7     for(ll i = 0; i < n; i++) {
8         if(p[i]) {
9             primes.push_back(i);
10            for(ll j = i*2; j < n; j+=i) p[j] =
11                false;
12        }
13    }
14 }

```

3.4.5 Fibonacci % prime

```

1 #include "header.h"
2 const ll MOD = 1000000007;
3 unordered_map<ll, ll> Fib;
4 ll fib(ll n) {
5     if (n < 2) return 1;
6     if (Fib.find(n) != Fib.end()) return Fib[n];
7     Fib[n] = (fib((n + 1) / 2) * fib(n / 2) + fib((n
8         - 1) / 2) * fib((n - 2) / 2)) % MOD;
9     return Fib[n];
10 }

```

3.4.6 nCk % prime

```

1 #include "header.h"
2 ll binom(ll n, ll k) {
3     ll ans = 1;
4     for(ll i = 1; i <= min(k, n-k); ++i) ans = ans*(n
5         +1-i)/i;
6     return ans;
7 }
8 ll mod_nCk(ll n, ll k, ll p){
9     ll ans = 1;
10    while(n){
11        ll np = n%p, kp = k%p;
12        if(kp > np) return 0;
13        ans *= binom(np, kp);
14        n /= p; k /= p;
15    }
16 }

```

```

15     return ans;
16 }

```

3.4.7 Chin. rem. th.

```

1 #include "header.h"
2 #include "elementary.cpp"
3 // Solves x = a1 mod m1, x = a2 mod m2, x is unique
4 // modulo lcm(m1, m2).
5 // Returns {0, -1} on failure, {x, lcm(m1, m2)}
6 // otherwise.
7 pair<ll, ll> crt(ll a1, ll m1, ll a2, ll m2) {
8     ll s, t, d;
9     extended_euclid(m1, m2, s, t, d);
10    if (a1 % d != a2 % d) return {0, -1};
11    return {mod(s*a2 % m2 * m1 + t*a1 % m1 * m2, m1 * m2
12        ) / d, m1 / d * m2};
13 }
14 // Solves x = ai mod mi. x is unique modulo lcm mi.
15 // Returns {0, -1} on failure, {x, lcm mi} otherwise
16 .
17 pair<ll, ll> crt(vector<ll> &a, vector<ll> &m) {
18     pair<ll, ll> res = {a[0], m[0]};
19     for (ull i = 1; i < a.size(); ++i) {
20         res = crt(res.first, res.second, mod(a[i], m[i])
21             , m[i]);
22         if (res.second == -1) break;
23     }
24     return res;
25 }

```

3.5 Strings

3.5.1 Z alg. KMP alternative

```

1 #include "../header.h"
2 void Z_algorithm(const string &s, vi &Z) {
3     Z.assign(s.length(), -1);
4     int L = 0, R = 0, n = s.length();
5     for (int i = 1; i < n; ++i) {
6         if (i > R) {
7             L = R = i;
8             while (R < n && s[R - L] == s[R]) R++;
9             Z[i] = R - L; R--;
10        } else if (Z[i - L] >= R - i + 1) {
11            L = i;
12            while (R < n && s[R - L] == s[R]) R++;
13            Z[i] = R - L; R--;
14        } else Z[i] = Z[i - L];
15    }
16 }

```

3.5.2 KMP

```

1 #include "header.h"
2 void compute_prefix_function(string &w, vi &prefix)
3 {
4     prefix.assign(w.length(), 0);
5     int k = prefix[0] = -1;
6
7     for(int i = 1; i < w.length(); ++i) {
8         while(k >= 0 && w[k + 1] != w[i]) k = prefix[k];
9         if(w[k + 1] == w[i]) k++;
10        prefix[i] = k;
11    }
12 }
13 void knuth_morris_pratt(string &s, string &w) {
14     int q = -1;
15     vi prefix;
16     compute_prefix_function(w, prefix);
17     for(int i = 0; i < s.length(); ++i) {
18         while(q >= 0 && w[q + 1] != s[i]) q = prefix[q];
19         if(w[q + 1] == s[i]) q++;
20         if(q + 1 == w.length()) {
21             // Match at position (i - w.length() + 1)
22             q = prefix[q];
23         }
24     }
25 }

```

3.5.3 Aho-Corasick Also can be used as Knuth-Morris-Pratt algorithm

```

1 #include "header.h"
2
3 map<char, int> cti;
4 int cti_size;
5 template <int ALPHABET_SIZE, int (*mp)(char)>
6 struct AC_FSM {
7     struct Node {
8         int child[ALPHABET_SIZE], failure = 0, match_par
9             = -1;
10        vi match;
11        Node() { for (int i = 0; i < ALPHABET_SIZE; ++i)
12            child[i] = -1; }
13    };
14    vector<Node> a;
15    vector<string> &words;
16    AC_FSM(vector<string> &words) : words(words) {
17        a.push_back(Node());
18        construct_automaton();
19    }
20    void construct_automaton() {
21        for (int w = 0, n = 0; w < words.size(); ++w, n
22            = 0) {
23            for (int i = 0; i < words[w].size(); ++i) {

```

```

21     if (a[n].child[mp(words[w][i])] == -1) {
22         a[n].child[mp(words[w][i])] = a.size();
23         a.push_back(Node());
24     }
25     n = a[n].child[mp(words[w][i])];
26 }
27 a[n].match.push_back(w);
28 }
29 queue<int> q;
30 for (int k = 0; k < ALPHABET_SIZE; ++k) {
31     if (a[0].child[k] == -1) a[0].child[k] = 0;
32     else if (a[0].child[k] > 0) {
33         a[a[0].child[k]].failure = 0;
34         q.push(a[0].child[k]);
35     }
36 }
37 while (!q.empty()) {
38     int r = q.front(); q.pop();
39     for (int k = 0, arck; k < ALPHABET_SIZE; ++k)
40     {
41         if ((arck = a[r].child[k]) != -1) {
42             q.push(arck);
43             int v = a[r].failure;
44             while (a[v].child[k] == -1) v = a[v].
45                 failure;
46             a[arck].failure = a[v].child[k];
47             a[arck].match_par = a[v].child[k];
48             while (a[arck].match_par != -1
49                 && a[a[arck].match_par].match.empty())
50                 a[arck].match_par = a[a[arck].match_par
51                     ].match_par;
52         }
53     }
54 }
55 void aho_corasick(string &sentence, vvi &matches){
56     matches.assign(words.size(), vi());
57     int state = 0, ss = 0;
58     for (int i = 0; i < sentence.length(); ++i, ss =
59         state) {
60         while (a[ss].child[mp(sentence[i])] == -1)
61             ss = a[ss].failure;
62         state = a[state].child[mp(sentence[i])]
63             = a[ss].child[mp(sentence[i])];
64         for (ss = state; ss != -1; ss = a[ss].
65             match_par)
66             for (int w : a[ss].match)
67                 matches[w].push_back(i + 1 - words[w].
68                     length());
69     }
70 }
71 int char_to_int(char c) {
72     return cti[c];
73 }

```

```

70 int main() {
71     ll n;
72     string line;
73     while(getline(cin, line)) {
74         stringstream ss(line);
75         ss >> n;
76
77         vector<string> patterns(n);
78         for (auto& p: patterns) getline(cin, p);
79
80         string text;
81         getline(cin, text);
82
83         cti = {}, cti_size = 0;
84         for (auto c: text) {
85             if (not in(c, cti)) {
86                 cti[c] = cti_size++;
87             }
88         }
89         for (auto& p: patterns) {
90             for (auto c: p) {
91                 if (not in(c, cti)) {
92                     cti[c] = cti_size++;
93                 }
94             }
95         }
96
97         vvi matches;
98         AC_FSM <128+1, char_to_int> ac_fms(patterns);
99         ac_fms.aho_corasick(text, matches);
100         for (auto& x: matches) cout << x << endl;
101     }
102 }
103 }

```

3.5.4 Long. palin. subs Manacher - $O(n)$

```

1 #include "header.h"
2 void manacher(string &s, vi &pal) {
3     int n = s.length(), i = 1, l, r;
4     pal.assign(2 * n + 1, 0);
5     while (i < 2 * n + 1) {
6         if ((i&1) && pal[i] == 0) pal[i] = 1;
7         l = i / 2 - pal[i] / 2; r = (i-1) / 2 + pal[i] /
8             2;
9         while (l - 1 >= 0 && r + 1 < n && s[l - 1] == s[
10             r + 1])
11             --l, ++r, pal[i] += 2;
12         for (l = i - 1, r = i + 1; l >= 0 && r < 2 * n +
13             1; --l, ++r) {
14             if (l <= i - pal[i]) break;
15             if (l / 2 - pal[l] / 2 > i / 2 - pal[i] / 2)

```

```

15         pal[r] = pal[l];
16         else { if (l >= 0)
17             pal[r] = min(pal[l], i + pal[i] - r);
18             break;
19         }
20     }
21     i = r;
22 } }

```

3.6 Geometry

3.6.1 essentials.cpp

```

1 #include "../header.h"
2 using C = ld; // could be long long or long double
3 constexpr C EPS = 1e-10; // change to 0 for C=ll
4 struct P { // may also be used as a 2D vector
5     C x, y;
6     P(C x = 0, C y = 0) : x(x), y(y) {}
7     P operator+ (const P &p) const { return {x + p.x,
8         y + p.y}; }
9     P operator- (const P &p) const { return {x - p.x,
10         y - p.y}; }
11     P operator* (C c) const { return {x * c, y * c}; }
12     P operator/ (C c) const { return {x / c, y / c}; }
13     C operator* (const P &p) const { return x*p.x + y*
14         p.y; }
15     C operator^ (const P &p) const { return x*p.y - p.
16         x*y; }
17     P perp() const { return P{y, -x}; }
18     C lensq() const { return x*x + y*y; }
19     ld len() const { return sqrt((ld)lensq()); }
20     static ld dist(const P &p1, const P &p2) {
21         return (p1-p2).len(); }
22     bool operator==(const P &r) const {
23         return ((*this)-r).lensq() <= EPS*EPS; }
24 };
25 C det(P p1, P p2) { return p1^p2; }
26 C det(P p1, P p2, P o) { return det(p1-o, p2-o); }
27 C det(const vector<P> &ps) {
28     C sum = 0; P prev = ps.back();
29     for(auto &p : ps) sum += det(p, prev), prev = p;
30     return sum;
31 }
32 // Careful with division by two and C=ll
33 C area(P p1, P p2, P p3) { return abs(det(p1, p2, p3
34     ))/C(2); }
35 C area(const vector<P> &poly) { return abs(det(poly)
36     )/C(2); }
37 int sign(C c){ return (c > C(0)) - (c < C(0)); }
38 int ccw(P p1, P p2, P o) { return sign(det(p1, p2, o
39     )); }
40 // Only well defined for C = ld.

```

3.6.2 Two segs. itersec.

```

1 #include "header.h"
2 #include "essentials.cpp"
3 bool intersect(P a1, P a2, P b1, P b2) {
4     if (max(a1.x, a2.x) < min(b1.x, b2.x)) return
        false;
5     if (max(b1.x, b2.x) < min(a1.x, a2.x)) return
        false;
6     if (max(a1.y, a2.y) < min(b1.y, b2.y)) return
        false;
7     if (max(b1.y, b2.y) < min(a1.y, a2.y)) return
        false;
8     bool l1 = ccw(a2, b1, a1) * ccw(a2, b2, a1) <= 0;
9     bool l2 = ccw(b2, a1, b1) * ccw(b2, a2, b1) <= 0;
10    return l1 && l2;
11 }

```

3.6.3 Convex Hull

```

1 #include "header.h"
2 #include "essentials.cpp"
3 struct ConvexHull { // O(n lg n) monotone chain.
4     size_t n;
5     vector<size_t> h, c; // Indices of the hull are
        in `h`, ccw.
6     const vector<P> &p;
7     ConvexHull(const vector<P> &p) : n(p.size()), c(
        n), p(p) {
8         std::iota(c.begin(), c.end(), 0);
9         std::sort(c.begin(), c.end(), [this](size_t l,
            size_t r) -> bool { return p[l].x != p[r].x
                ? p[l].x < p[r].x : p[l].y < p[r].y; });
10        c.erase(std::unique(c.begin(), c.end(), [this](
            size_t l, size_t r) { return p[l] == p[r];
                }), c.end());
11        for (size_t s = 1, r = 0; r < 2; ++r, s = h.size()
            ()) {
12            for (size_t i : c) {
13                while (h.size() > s && ccw(p[h.end()[-2]], p
                    [h.end()[-1]], p[i]) <= 0)
14                    h.pop_back();
15                h.push_back(i);
16            }
17            reverse(c.begin(), c.end());
18        }
19        if (h.size() > 1) h.pop_back();
20    }
21    size_t size() const { return h.size(); }

```

```

22 template <class T, void U(const P &, const P &,
        const P &, T &)>
23 void rotating_calipers(T &ans) {
24     if (size() <= 2)
25         U(p[h[0]], p[h.back()], p[h.back()], ans);
26     else
27         for (size_t i = 0, j = 1, s = size(); i < 2 *
            s; ++i) {
28             while (det(p[h[(i + 1) % s]] - p[h[i % s]],
                p[h[(j + 1) % s]] - p[h[j]]) >= 0)
29                 j = (j + 1) % s;
30             U(p[h[i % s]], p[h[(i + 1) % s]], p[h[j]],
                ans);
31         }
32     }
33 };
34 // Example: furthest pair of points. Now set ans = 0
        LL and call
35 // ConvexHull(pts).rotating_calipers<ll, update>(ans
        );
36 void update(const P &p1, const P &p2, const P &o, ll
        &ans) {
37     ans = max(ans, (ll)max((p1 - o).lensq(), (p2 - o).
        lensq()));
38 }
39 int main() {
40     ios::sync_with_stdio(false); // do not use cout +
        printf
41     cin.tie(NULL);
42
43     int n;
44     cin >> n;
45     while (n) {
46         vector<P> ps;
47         int x, y;
48         for (int i = 0; i < n; i++) {
49             cin >> x >> y;
50             ps.push_back({x, y});
51         }
52
53         ConvexHull ch(ps);
54         cout << ch.h.size() << endl;
55         for(auto& p: ch.h) {
56             cout << ps[p].x << " " << ps[p].y <<
                endl;
57         }
58         cin >> n;
59     }
60
61     return 0;
62 }

```

3.7 Other Algorithms

3.7.1 2-sat

```

1 #include "../header.h"
2 #include "../Graphs/tarjan.cpp"
3 struct TwoSAT {
4     int n;
5     vvi imp; // implication graph
6     Tarjan tj;
7
8     TwoSAT(int _n) : n(_n), imp(2 * _n, vi()), tj(imp)
        { }
9
10    // Only copy the needed functions:
11    void add_implies(int c1, bool v1, int c2, bool v2)
        {
12        int u = 2 * c1 + (v1 ? 1 : 0),
13            v = 2 * c2 + (v2 ? 1 : 0);
14        imp[u].push_back(v); // u => v
15        imp[v^1].push_back(u^1); // -v => -u
16    }
17    void add_equivalence(int c1, bool v1, int c2, bool
        v2) {
18        add_implies(c1, v1, c2, v2);
19        add_implies(c2, v2, c1, v1);
20    }
21    void add_or(int c1, bool v1, int c2, bool v2) {
22        add_implies(c1, !v1, c2, v2);
23    }
24    void add_and(int c1, bool v1, int c2, bool v2) {
25        add_true(c1, v1); add_true(c2, v2);
26    }
27    void add_xor(int c1, bool v1, int c2, bool v2) {
28        add_or(c1, v1, c2, v2);
29        add_or(c1, !v1, c2, !v2);
30    }
31    void add_true(int c1, bool v1) {
32        add_implies(c1, !v1, c1, v1);
33    }
34
35    // on true: a contains an assignment.
36    // on false: no assignment exists.
37    bool solve(vb &a) {
38        vi com;
39        tj.find_sccs(com);
40        for (int i = 0; i < n; ++i)
41            if (com[2 * i] == com[2 * i + 1])
42                return false;
43
44        vvi bycom(com.size());
45        for (int i = 0; i < 2 * n; ++i)
46            bycom[com[i]].push_back(i);
47
48        a.assign(n, false);
49        vb vis(n, false);

```



```

50 for(auto &&component : bycom){
51     for (int u : component) {
52         if (vis[u / 2]) continue;
53         vis[u / 2] = true;
54         a[u / 2] = (u % 2 == 1);
55     }
56 }
57 return true;
58 }
59 };

```

3.7.2 Matrix Solve

```

1 #include "header.h"
2 #define REP(i, n) for(auto i = decltype(n)(0); i < (
    n); i++)
3 using T = double;
4 constexpr T EPS = 1e-8;
5 template<int R, int C>
6 using M = array<array<T,C>,R>; // matrix
7 template<int R, int C>
8 T ReducedRowEchelonForm(M<R,C> &m, int rows) { //
    return the determinant
9     int r = 0; T det = 1; // MODIFIES the
        input
10     for(int c = 0; c < rows && r < rows; c++) {
11         int p = r;
12         for(int i=r+1; i<rows; i++) if(abs(m[i][c]) >
            abs(m[p][c])) p=i;
13         if(abs(m[p][c]) < EPS){ det = 0; continue; }
14         swap(m[p], m[r]); det = -det;
15         T s = 1.0 / m[r][c]; t; det *= m[r][c];
16         REP(j,C) m[r][j] *= s; // make leading
            term in row 1
17         REP(i,rows) if (i!=r){ t = m[i][c]; REP(j,C) m[i]
            ][j] -= t*m[r][j]; }
18         ++r;
19     }
20     return det;
21 }
22 bool error, inconst; // error => multiple or
    inconsistent
23 template<int R,int C> // Mx = a; M:R*R, v:R*C => x:R
    *C
24 M<R,C> solve(const M<R,R> &m, const M<R,C> &a, int
    rows){
25     M<R,R+C> q;
26     REP(r,rows){
27         REP(c,rows) q[r][c] = m[r][c];
28         REP(c,C) q[r][R+c] = a[r][c];
29     }
30     ReducedRowEchelonForm<R,R+C>(q,rows);
31     M<R,C> sol; error = false, inconst = false;
32     REP(c,C) for(auto j = rows-1; j >= 0; --j){

```

```

33     T t=0; bool allzero=true;
34     for(auto k = j+1; k < rows; ++k)
35         t += q[j][k]*sol[k][c], allzero &= abs(q[j][k]
            ]) < EPS;
36     if(abs(q[j][j]) < EPS)
37         error = true, inconst |= allzero && abs(q[j][R
            +c]) > EPS;
38     else sol[j][c] = (q[j][R+c] - t) / q[j][j]; //
        usually q[j][j]=1
39 }
40 return sol;
41 }

```

3.7.3 Matrix Exp.

```

1 #include "header.h"
2 #define ITERATE_MATRIX(w) for (int r = 0; r < (w);
    ++r) \
3         for (int c = 0; c < (w); ++c)
4 template <class T, int N>
5 struct M {
6     array<array<T,N>,N> m;
7     M() { ITERATE_MATRIX(N) m[r][c] = 0; }
8     static M id() {
9         M I; for (int i = 0; i < N; ++i) I.m[i][i] = 1;
            return I;
10    }
11    M operator*(const M &rhs) const {
12        M out;
13        ITERATE_MATRIX(N) for (int i = 0; i < N; ++i)
14            out.m[r][c] += m[r][i] * rhs.m[i][c];
15        return out;
16    }
17    M raise(ll n) const {
18        if(n == 0) return id();
19        if(n == 1) return *this;
20        auto r = (*this**this).raise(n / 2);
21        return (n%2 ? *this*r : r);
22    }
23 };

```

3.7.4 Finite field For FFT

```

1 #include "header.h"
2 #include "../Number Theory/elementary.cpp"
3 template<ll p,ll w> // prime, primitive root
4 struct Field { using T = Field; ll x; Field(ll x=0)
    : x{x} {}
5     T operator+(T r) const { return {(x+r.x)%p}; }
6     T operator-(T r) const { return {(x-r.x+p)%p}; }
7     T operator*(T r) const { return {(x*r.x)%p}; }
8     T operator/(T r) const { return (*this)*r.inv(); }
9     T inv() const { return {mod_inverse(x,p)}; }

```

```

10 static T root(ll k) { assert( (p-1)%k==0 ); // (
    p-1)%k == 0?
11     auto r = powmod(w,(p-1)/abs(k),p); // k-th
        root of unity
12     return k>0 ? T{r} : T{r}.inv();
13 }
14 bool zero() const { return x == 0LL; }
15 };
16 using F1 = Field<1004535809,3 >;
17 using F2 = Field<1107296257,10>; // 1<<30 + 1<<25 +
    1
18 using F3 = Field<2281701377,3 >; // 1<<31 + 1<<27 +
    1

```

3.7.5 Complex field For FFR

```

1 #include "header.h"
2 const double m_pi = M_PIf64x;
3 struct Complex { using T = Complex; double u,v;
4     Complex(double u=0, double v=0) : u{u}, v{v} {}
5     T operator+(T r) const { return {u+r.u, v+r.v}; }
6     T operator-(T r) const { return {u-r.u, v-r.v}; }
7     T operator*(T r) const { return {u*r.u - v*r.v, u*
        r.v + v*r.u}; }
8     T operator/(T r) const {
9         auto norm = r.u*r.u+r.v*r.v;
10        return {(u*r.u + v*r.v)/norm, (v*r.u - u*r.v)/
            norm};
11    }
12    T operator*(double r) const { return T{u*r, v*r}; }
13    T operator/(double r) const { return T{u/r, v/r}; }
14    T inv() const { return T{1,0}/ *this; }
15    T conj() const { return T{u, -v}; }
16    static T root(ll k){ return {cos(2*m_pi/k), sin(2*
        m_pi/k)}; }
17    bool zero() const { return max(abs(u), abs(v)) < 1
        e-6; }
18 };

```

3.7.6 FFT

```

1 #include "header.h"
2 #include "complex_field.cpp"
3 #include "fin_field.cpp"
4 void brinc(int &x, int k) {
5     int i = k - 1, s = 1 << i;
6     x ^= s;
7     if ((x & s) != s) {
8         --i; s >>= 1;
9         while (i >= 0 && ((x & s) == s))
10             x = x &~ s, --i, s >>= 1;

```



```

11     if (i >= 0) x |= s;
12 }
13 }
14 using T = Complex; // using T=F1,F2,F3
15 vector<T> roots;
16 void root_cache(int N) {
17     if (N == (int)roots.size()) return;
18     roots.assign(N, T{0});
19     for (int i = 0; i < N; ++i)
20         roots[i] = ((i&-i) == i)
21             ? T{cos(2.0*m_pi*i/N), sin(2.0*m_pi*i/N)}
22             : roots[i&-i] * roots[i-(i&-i)];
23 }
24 void fft(vector<T> &A, int p, bool inv = false) {
25     int N = 1<<p;
26     for(int i = 0, r = 0; i < N; ++i, brinc(r, p))
27         if (i < r) swap(A[i], A[r]);
28     // Uncomment to precompute roots (for T=Complex).
29     // Slower but more precise.
30     // root_cache(N);
31     // , sh=p-1 , --sh
32     for (int m = 2; m <= N; m <= 1) {
33         T w, w_m = T::root(inv ? -m : m);
34         for (int k = 0; k < N; k += m) {
35             w = T{1};
36             for (int j = 0; j < m/2; ++j) {
37                 T w = (!inv ? roots[j<<sh] : roots[j<<sh].
38                     conj());
39                 A[k + j + m/2] = A[k + j] - t;
40                 A[k + j] = A[k + j] + t;
41                 w = w * w_m;
42             }
43         }
44         if(inv){ T inverse = T(N).inv(); for(auto &x : A)
45             x = x*inverse; }
46 // convolution leaves A and B in frequency domain
47 // state
48 // C may be equal to A or B for in-place convolution
49 void convolution(vector<T> &A, vector<T> &B, vector<
50     T> &C){
51     int s = A.size() + B.size() - 1;
52     int q = 32 - __builtin_clz(s-1), N=1<<q; // fails
53     if s=1
54     A.resize(N,{}); B.resize(N,{}); C.resize(N,{});
55     fft(A, q, false); fft(B, q, false);
56     for (int i = 0; i < N; ++i) C[i] = A[i] * B[i];
57     fft(C, q, true); C.resize(s);
58 }
59 void square_inplace(vector<T> &A) {
60     int s = 2*A.size()-1, q = 32 - __builtin_clz(s-1),
61         N=1<<q;
62     A.resize(N,{}); fft(A, q, false);

```

```

59     for(auto &x : A) x = x*x;
60     fft(A, q, true); A.resize(s);
61 }

```

3.7.7 Polyn. inv. div.

```

1 #include "header.h"
2 #include "fft.cpp"
3 vector<T> &rev(vector<T> &A) { reverse(A.begin(), A.
4     end()); return A; }
5 void copy_into(const vector<T> &A, vector<T> &B,
6     size_t n) {
7     std::copy(A.begin(), A.begin()+min({n, A.size(), B
8         .size()}), B.begin());
9 }
10 // Multiplicative inverse of A modulo x^n. Requires
11 // A[0] != 0!!
12 vector<T> inverse(const vector<T> &A, int n) {
13     vector<T> Ai{A[0].inv()};
14     for (int k = 0; (1<<k) < n; ++k) {
15         vector<T> As(4<<k, T(0)), Ais(4<<k, T(0));
16         copy_into(A, As, 2<<k); copy_into(Ai, Ais, Ai.
17             size());
18         fft(As, k+2, false); fft(Ais, k+2, false);
19         for (int i = 0; i < (4<<k); ++i) As[i] = As[i]*
20             Ais[i]*Ais[i];
21         fft(As, k+2, true); Ai.resize(2<<k, {});
22         for (int i = 0; i < (2<<k); ++i) Ai[i] = T(2) *
23             Ai[i] - As[i];
24     }
25     Ai.resize(n);
26     return Ai;
27 }
28 // Polynomial division. Returns {Q, R} such that A =
29 // QB+R, deg R < deg B.
30 // Requires that the leading term of B is nonzero.
31 pair<vector<T>, vector<T>> divmod(const vector<T> &A
32     , const vector<T> &B) {
33     size_t n = A.size()-1, m = B.size()-1;
34     if (n < m) return {vector<T>(1, T(0)), A};
35     vector<T> X(A), Y(B), Q, R;
36     convolution(rev(X), Y = inverse(rev(Y), n-m+1), Q)
37     ;
38     Q.resize(n-m+1); rev(Q);
39     X.resize(Q.size()), copy_into(Q, X, Q.size());
40     Y.resize(B.size()), copy_into(B, Y, B.size());
41     convolution(X, Y, X);
42     R.resize(m), copy_into(A, R, m);
43     for (size_t i = 0; i < m; ++i) R[i] = R[i] - X[i];
44     while (R.size() > 1 && R.back().zero()) R.pop_back
45     ();

```

```

39     return {Q, R};
40 }
41 vector<T> mod(const vector<T> &A, const vector<T> &B
42     ) {
43     return divmod(A, B).second;
44 }

```

3.7.8 Linear recurs. Given a linear recurrence of the form

$$a_n = \sum_{i=0}^{k-1} c_i a_{n-i-1}$$

this code computes a_n in $O(k \log k \log n)$ time.

```

1 #include "header.h"
2 #include "poly.cpp"
3 // x^k mod f
4 vector<T> xmod(const vector<T> f, ll k) {
5     vector<T> r{T(1)};
6     for (int b = 62; b >= 0; --b) {
7         if (r.size() > 1)
8             square_inplace(r), r = mod(r, f);
9         if ((k>>b)&1) {
10             r.insert(r.begin(), T(0));
11             if (r.size() == f.size()) {
12                 T c = r.back() / f.back();
13                 for (size_t i = 0; i < f.size(); ++i)
14                     r[i] = r[i] - c * f[i];
15                 r.pop_back();
16             }
17         }
18     }
19     return r;
20 }
21 // Given A[0,k) and C[0, k), computes the n-th term
22 // of:
23 // A[n] = \sum_i C[i] * A[n-i-1]
24 T nth_term(const vector<T> &A, const vector<T> &C,
25     ll n) {
26     int k = (int)A.size();
27     if (n < k) return A[n];
28     vector<T> f(k+1, T{1});
29     for (int i = 0; i < k; ++i)
30         f[i] = T{-1} * C[k-i-1];
31     f = xmod(f, n);
32     T r = T{0};
33     for (int i = 0; i < k; ++i)
34         r = r + f[i] * A[i];
35     return r;
36 }

```

3.7.9 Convolution Precise up to 9e15

```

1 #include "header.h"
2 #include "fft.cpp"
3 void convolution_mod(const vi &A, const vi &B, ll
    MOD, vi &C) {
4     int s = A.size() + B.size() - 1; ll m15 = (1LL
        <<15)-1LL;
5     int q = 32 - __builtin_clz(s-1), N=1<<q; // fails
        if s=1
6     vector<T> Ac(N), Bc(N), R1(N), R2(N);
7     for (size_t i = 0; i < A.size(); ++i) Ac[i] = T{A[
        i]&m15, A[i]>>15};
8     for (size_t i = 0; i < B.size(); ++i) Bc[i] = T{B[
        i]&m15, B[i]>>15};
9     fft(Ac, q, false); fft(Bc, q, false);
10    for (int i = 0, j = 0; i < N; ++i, j = (N-1)&(N-i)
        ) {
11        T as = (Ac[i] + Ac[j].conj()) / 2;
12        T al = (Ac[i] - Ac[j].conj()) / T{0, 2};
13        T bs = (Bc[i] + Bc[j].conj()) / 2;
14        T bl = (Bc[i] - Bc[j].conj()) / T{0, 2};
15        R1[i] = as*bs + al*bl*T{0,1}, R2[i] = as*bl + al
            *bs;
16    }
17    fft(R1, q, true); fft(R2, q, true);
18    ll p15 = (1LL<<15)%MOD, p30 = (1LL<<30)%MOD; C.
        resize(s);
19    for (int i = 0; i < s; ++i) {
20        ll l = llround(R1[i].u), m = llround(R2[i].u), h
            = llround(R1[i].v);
21        C[i] = (l + m*p15 + h*p30) % MOD;
22    }
23 }

```

3.7.10 Partitions of n Finds all possible partitions of a number

```

1 #include "header.h"
2 void printArray(int p[], int n) {
3     for (int i = 0; i < n; i++)
4         cout << p[i] << " ";
5     cout << endl;
6 }
7
8 void printAllUniqueParts(int n) {
9     int p[n]; // An array to store a partition
10    int k = 0; // Index of last element in a
        partition
11    p[k] = n; // Initialize first partition as number
        itself
12
13    // This loop first prints current partition then
        generates next

```

```

14 // partition. The loop stops when the current
    partition has all 1s
15 while (true) {
16     printArray(p, k + 1);
17
18     // Find the rightmost non-one value in p[]. Also
        , update the
19     // rem_val so that we know how much value can be
        accommodated
20     int rem_val = 0;
21     while (k >= 0 && p[k] == 1) {
22         rem_val += p[k];
23         k--;
24     }
25
26     // if k < 0, all the values are 1 so there are
        no more partitions
27     if (k < 0) return;
28
29     // Decrease the p[k] found above and adjust the
        rem_val
30     p[k]--;
31     rem_val++;
32
33     // If rem_val is more, then the sorted order is
        violated. Divide
34     // rem_val in different values of size p[k] and
        copy these values at
35     // different positions after p[k]
36     while (rem_val > p[k]) {
37         p[k + 1] = p[k];
38         rem_val = rem_val - p[k];
39         k++;
40     }
41
42     // Copy rem_val to next position and increment
        position
43     p[k + 1] = rem_val;
44     k++;
45 }
46 }

```

3.8 Other Data Structures**3.8.1 Disjoint set (i.e. union-find)**

```

1 template <typename T>
2 class DisjointSet {
3     typedef T * iterator;
4     T *parent, n, *rank;
5 public:
6     // O(n), assumes nodes are [0, n)
7     DisjointSet(T n) {
8         this->parent = new T[n];

```

```

9         this->n = n;
10        this->rank = new T[n];
11
12        for (T i = 0; i < n; i++) {
13            parent[i] = i;
14            rank[i] = 0;
15        }
16    }
17
18    // O(log n)
19    T find_set(T x) {
20        if (x == parent[x]) return x;
21        return parent[x] = find_set(parent[x]);
22    }
23
24    // O(log n)
25    void union_sets(T x, T y) {
26        x = this->find_set(x);
27        y = this->find_set(y);
28
29        if (x == y) return;
30
31        if (rank[x] < rank[y]) {
32            T z = x;
33            x = y;
34            y = z;
35        }
36
37        parent[y] = x;
38        if (rank[x] == rank[y]) rank[x]++;
39    }
40 };

```

3.8.2 Fenwick tree (i.e. BIT) eff. update + prefix sum calc.

```

1 #include "header.h"
2 #define maxn 200010
3 int t,n,m,tree[maxn],p[maxn];
4
5 void update(int k, int z) {
6     while (k <= maxn) {
7         tree[k] += z;
8         k += k & (-k);
9     }
10 }
11
12 int sum(int k) {
13     int ans = 0;
14     while(k) {
15         ans += tree[k];
16         k -= k & (-k);
17     }
18     return ans;

```

3.8.3 Fenwick2d tree

```

1 #include "header.h"
2 template <class T>
3 struct FenwickTree2D {
4     vector< vector<T> > tree;
5     int n;
6     FenwickTree2D(int n) : n(n) { tree.assign(n + 1,
7         vector<T>(n + 1, 0)); }
8     T query(int x1, int y1, int x2, int y2) {
9         return query(x2,y2)+query(x1-1,y1-1)-query(x2,y1
10             -1)-query(x1-1,y2);
11     }
12     T query(int x, int y) {
13         T s = 0;
14         for (int i = x; i > 0; i -= (i & (-i)))
15             for (int j = y; j > 0; j -= (j & (-j)))
16                 s += tree[i][j];
17         return s;
18     }
19     void update(int x, int y, T v) {
20         for (int i = x; i <= n; i += (i & (-i)))
21             for (int j = y; j <= n; j += (j & (-j)))
22                 tree[i][j] += v;
23     }
24 };

```

3.8.4 Trie

```

1 #include "header.h"
2 const int ALPHABET_SIZE = 26;
3 inline int mp(char c) { return c - 'a'; }
4
5 struct Node {
6     Node* ch[ALPHABET_SIZE];
7     bool isleaf = false;
8     Node() {
9         for(int i = 0; i < ALPHABET_SIZE; ++i) ch[i] =
10             nullptr;
11     }
12     void insert(string &s, int i = 0) {
13         if (i == s.length()) isleaf = true;
14         else {
15             int v = mp(s[i]);
16             if (ch[v] == nullptr)
17                 ch[v] = new Node();
18             ch[v]->insert(s, i + 1);
19         }
20     }
21 };

```

```

22 bool contains(string &s, int i = 0) {
23     if (i == s.length()) return isleaf;
24     else {
25         int v = mp(s[i]);
26         if (ch[v] == nullptr) return false;
27         else return ch[v]->contains(s, i + 1);
28     }
29 }
30
31 void cleanup() {
32     for (int i = 0; i < ALPHABET_SIZE; ++i)
33         if (ch[i] != nullptr) {
34             ch[i]->cleanup();
35             delete ch[i];
36         }
37 }
38 };

```

3.8.5 Treap A binary tree whose nodes contain two values, a key and a priority, such that the key keeps the BST property

```

1 #include "header.h"
2 struct Node {
3     ll v;
4     int sz, pr;
5     Node *l = nullptr, *r = nullptr;
6     Node(ll val) : v(val), sz(1) { pr = rand(); }
7 };
8 int size(Node *p) { return p ? p->sz : 0; }
9 void update(Node* p) {
10     if (!p) return;
11     p->sz = 1 + size(p->l) + size(p->r);
12     // Pull data from children here
13 }
14 void propagate(Node *p) {
15     if (!p) return;
16     // Push data to children here
17 }
18 void merge(Node *&t, Node *l, Node *r) {
19     propagate(l), propagate(r);
20     if (!l) t = r;
21     else if (!r) t = l;
22     else if (l->pr > r->pr)
23         merge(l->r, l->r, r), t = l;
24     else merge(r->l, l, r->l), t = r;
25     update(t);
26 }
27 void spliti(Node *t, Node *&l, Node *&r, int index) {
28     propagate(t);
29     if (!t) { l = r = nullptr; return; }
30     int id = size(t->l);

```

```

31     if (index <= id) // id \in [index, \infty), so
32         move it right
33         spliti(t->l, l, t->l, index), r = t;
34     else
35         spliti(t->r, t->r, r, index - id), l = t;
36     update(t);
37 }
38 void splitv(Node *t, Node *&l, Node *&r, ll val) {
39     propagate(t);
40     if (!t) { l = r = nullptr; return; }
41     if (val <= t->v) // t->v \in [val, \infty), so
42         move it right
43         splitv(t->l, l, t->l, val), r = t;
44     else
45         splitv(t->r, t->r, r, val), l = t;
46     update(t);
47 }
48 void clean(Node *p) {
49     if (p) { clean(p->l), clean(p->r); delete p; }
50 }

```

4 Other Mathematics

4.1 Helpful functions

4.1.1 Euler's Totient Fucntion $n = p_1^{k_1-1} \cdot (p_1 - 1) \cdot \dots \cdot p_r^{k_r-1} \cdot (p_r - 1)$, where $p_1^{k_1} \cdot \dots \cdot p_r^{k_r}$ is the prime factorization of n .

```

1 # include "header.h"
2 ll phi(ll n) { // \Phi(n)
3     ll ans = 1;
4     for (ll i = 2; i*i <= n; i++) {
5         if (n % i == 0) {
6             ans *= i-1;
7             n /= i;
8             while (n % i == 0) {
9                 ans *= i;
10                n /= i;
11            }
12        }
13    }
14    if (n > 1) ans *= n-1;
15    return ans;
16 }
17 vi phis(int n) { // All \Phi(i) up to n
18     vi phi(n + 1, 0LL);
19     iota(phi.begin(), phi.end(), 0LL);
20     for (ll i = 2LL; i <= n; ++i)
21         if (phi[i] == i)
22             for (ll j = i; j <= n; j += i)
23                 phi[j] -= phi[j] / i;

```

```

24     return phi;
25 }

```

Formulas $\Phi(n)$ counts all numbers in $1, \dots, n-1$ coprime to n .

$a^{\varphi(n)} \equiv 1 \pmod n$, a and n are coprimes.

$\forall e > \log_2 m : n^e \pmod m = n^{\Phi(m)+e \pmod{\Phi(m)}} \pmod m$.

$\gcd(m, n) = 1 \Rightarrow \Phi(m \cdot n) = \Phi(m) \cdot \Phi(n)$.

4.1.2 Pascal's trinagle $\binom{n}{k}$ is k -th element in the n -th row, indexing both from 0

```

1  #include "header.h"
2  void printPascal(int n) {
3      for (int line = 1; line <= n; line++) {
4          int C = 1; // used to represent C(line, i)
5          for (int i = 1; i <= line; i++) {
6
7              // The first value in a line is always 1
8              cout << C << " ";
9              C = C * (line - i) / i;
10         }
11         cout << "\n";
12     }
13 }

```

4.2 Theorems and definitions

Fermat's little theorem

$$a^p \equiv a \pmod{p}$$

Subfactorial

$$!n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}$$

$$!(0) = 1, !n = n!(n-1) + (-1)^n$$

Binomials and other partitionings

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \prod_{i=1}^k \frac{n-i+1}{i}$$

This last product may be computed incrementally since any product of k' consecutive values is divisible by $k'!$.

Basic identities: The hockeystick identity:

$$\sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}$$

or

$$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}$$

Also

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{i=0}^n \binom{n}{i} = 2^n$$

For $n, m \geq 0$ and p prime: write n, m in base p , i.e. $n = n_k p^k + \dots + n_1 p + n_0$ and $m = m_k p^k + \dots + m_1 p + m_0$. Then by Lucas theorem we have $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$, with the convention that $n_i < m_i \implies \binom{n_i}{m_i} = 0$.

Fibonacci (See also number theory section)

$$\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\sum_{i=1}^n F_i = F_{n+2} - 1, \sum_{i=1}^n F_i^2 = F_n F_{n+1}$$

$$\gcd(F_m, F_n) = F_{\gcd(m, n)}$$

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$$

Bit stuff $a + b = a \oplus b + 2(a \& b) = a|b + a \& b$.

k th bit is set in x iff $x \bmod 2^{k-1} \geq 2^k$, or iff $x \bmod 2^{k-1} - x \bmod 2^k \neq 0$ (i.e. $= 2^k$) It comes handy when you need to look at the bits of the numbers which are pair sums or subset sums etc.

$n \bmod 2^i = n \& (2^i - 1)$.

$$\forall k: 1 \oplus 2 \oplus \dots \oplus (4k-1) = 0$$

Stirling's numbers First kind: $S_1(n, k)$ count permutations on n items with k cycles. $S_1(n, k) = S_1(n-1, k-1) + (n-1)S_1(n-1, k)$ with $S_1(0, 0) = 1$. Note:

$$\sum_{k=0}^n S_1(n, k) x^k = x(x+1) \dots (x+n-1)$$

$$\sum_{k=0}^n S_1(n, k) = n!$$

Second kind: $S_2(n, k)$ count partitions of n distinct elements into exactly k non-empty groups.

$$S_2(n, k) = S_2(n-1, k-1) + k S_2(n-1, k)$$

$$S_2(n, 1) = S_2(n, n) = 1$$

$$S_2(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

4.3 Geometry Formulas

$$[ABC] = rs = \frac{1}{2} ab \sin \gamma$$

$$= \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} |(B-A, C-A)^T|$$

$$s = \frac{a+b+c}{2}$$

$$2R = \frac{a}{\sin \alpha}$$

cosine rule:

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Euler:

$$1 + CC = V - E + F$$

Pick:

$$\text{Area} = \text{itr pts} + \frac{\text{bdry pts}}{2} - 1$$

$$p \cdot q = |p||q| \cos(\theta) \quad |p \times q| = |p||q| \sin(\theta)$$

Given a non-self-intersecting closed polygon on n vertices, given as (x_i, y_i) , its centroid (C_x, C_y) is given as:

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \quad C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) = \text{polygon area}$$

Inclusion-Exclusion For appropriate f compute $\sum_{S \subseteq T} (-1)^{|T \setminus S|} f(S)$, or if only the size of S matters, $\sum_{s=0}^n (-1)^{n-s} \binom{n}{s} f(s)$. In some contexts we might use Stirling numbers, not binomial coefficients!

Some useful applications:

Graph coloring Let $I(S)$ count the number of independent sets contained in $S \subseteq V$ ($I(\emptyset) = 1$, $I(S) = I(S \setminus v) + I(S \setminus N(v))$). Let $c_k = \sum_{S \subseteq V} (-1)^{|V \setminus S|} I(S)$. Then V is k -colorable iff $v > 0$. Thus we can compute the chromatic number of a graph in $O^*(2^n)$ time.

Burnside's lemma Given a group G acting on a set X , the number of elements in X up to symmetry is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

with X^g the elements of X invariant under g . For example, if $f(n)$ counts “configurations” of some sort of length n , and we want to count them up to rotational symmetry using $G = \mathbb{Z}/n\mathbb{Z}$, then

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k \parallel n} f(k) \phi(n/k)$$

I.e. for coloring with c colors we have $f(k) = k^c$.

Relatedly, in Pólya's enumeration theorem we imagine X as a set of n beads with G permuting the beads (e.g. a necklace, with G all rotations and reflections of the n -cycle, i.e. the dihedral group D_n). Suppose further that we had Y colors, then the number of G -invariant colorings Y^X/G is counted by

$$\frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)}$$

with $c(g)$ counting the number of cycles of g when viewed as a permutation of X . We can generalize this to a weighted version: if the color i can occur exactly r_i times, then this is counted by the coefficient of $t_1^{r_1} \dots t_n^{r_n}$ in the polynomial

$$Z(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \dots + t_n^m)^{c_m(g)}$$

where $c_m(g)$ counts the number of length m cycles in g acting as a permutation on X . Note we get the original formula by setting all $t_i = 1$. Here Z is the cycle index. Note: you can cleverly deal with even/odd sizes by setting some t_i to -1 .

Lucas Theorem If p is prime, then:

$$\frac{p^a}{k} \equiv 0 \pmod{p}$$

Thus for non-negative integers $m = m_k p^k + \dots + m_1 p + m_0$ and $n = n_k p^k + \dots + n_1 p + n_0$:

$$\frac{m}{n} = \prod_{i=0}^k \frac{m_i}{n_i} \pmod{p}$$

Note: The fraction's mean integer division.

Catalan Numbers - Number of correct bracket sequence consisting of n opening and n closing brackets.

The number of ways to completely parenthesize $n+1$ factors.

The number of triangulations of a convex polygon with $n+2$ sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).

The number of ways to connect the $2n$ points on a circle to form n disjoint i.e. non-intersecting chords.

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$C_0 = 1, C_1 = 1, C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Narayana numbers The number of expressions containing n pairs of parentheses, which are correctly matched and which contain k distinct nestings.

$$N(n, k) = \frac{1}{n} \frac{n}{k} \frac{n}{k-1}$$