CTfP Chapter 13 Challenges

1. *Show that an isomorphism between monoids that preserves multiplication must automatically preserve unit.*

   Let X and Y be monoids, and h an isomorphism between them. Write $\mathbf{1}_X$ and $\mathbf{1}_Y$ as their respective unit elements. Then the following diagram commutes:

$$
\begin{tikzcd}
Y \arrow[r, "h^{-1}"] \arrow[rr, bend left, "\mathbf{id}_Y"] \arrow[d, "\mathbf{1}_Y"'] & X \arrow[r, "h"] \arrow[d, "\mathbf{1}_X"] & Y \arrow[d, "h\mathbf{1}_X"] \\
Y \arrow[r, "h^{-1}"] \arrow[rr, bend right, "\mathbf{id}_Y"'] & X \arrow[r, "h"] & Y
\end{tikzcd}
$$

   The left and right cells commute due to the preservation of multiplication. The top and bottom cells commute due to isomorphism. From the diagram, we can see that $h\mathbf{1}_X \circ \mathbf{id}_Y = \mathbf{id}_Y \circ \mathbf{1}_Y$, and therefore $h\mathbf{1}_X = \mathbf{1}_Y$.

2. *Consider a monoid homomorphism from lists of integers with concatenation to integers with multiplication*

   What is the image of the empty list `[]`? Write $\oplus$ for the concatenation of lists, and h for the homomorphism from lists of integers with concatenation to integers with multiplication. As h must preserve the unit, $h\,\texttt{[]} = 1$.

   Assume that all singleton lists are mapped to the integers they contain, that is `[3]` is mapped to 3, etc. What's the image of `[1, 2, 3, 4]`?

$$
\begin{aligned}
\texttt{[1, 2, 3, 4]} &= \texttt{[1]} \oplus \texttt{[2]} \oplus \texttt{[3]} \oplus \texttt{[4]} \\
h\,\texttt{[1, 2, 3, 4]} &= h(\texttt{[1]} \oplus \texttt{[2]} \oplus \texttt{[3]} \oplus \texttt{[4]}) \\
&= h\,\texttt{[1]} \times h\,\texttt{[2]} \times h\,\texttt{[3]} \times h\,\texttt{[4]} \\
&= 1 \times 2 \times 3 \times 4 \\
&= 12
\end{aligned}
$$

   How many different lists map to the integer 12? Infinitely many. If `[1]` is not a unit element under concatenation of lists, but h maps it to the unit element under multiplication of integers, then for every integer, we can get an infinite amount of unique lists that maps to it, just my repeated concatenation with `[1]`. That is, h`[1, 12]` $= h$`[1, 1, 12]` $=$ h`[1, 1, 1, 12]` $\cdots = 12$.

Is there any other homomorphism between the two monoids? I think we could map each list `[n]` to $n+1$. Then `[]` would map to 1, `[1]` would map to 2, and so on. Of course, this would lose the intuition about lists of multiple integers; `[1, 2]` would map, paradoxically, to 6.

3. *What is the free monoid generated by a one-element set? Can you see what it's isomorphic to?* Let X be the monoid $(\{\bullet\}, +, \mathbf{u})$. Assuming strict associativity, we find $X(\bullet, \bullet) = \{\mathbf{u}, \bullet, \bullet + \bullet, \bullet + \bullet + \bullet, \dots\}$. If we substitute n for $\bullet$, $succ()$ for $- + \bullet$, and $zero$ for $\mathbf{u}$, it seems like we could obtain the Peano axioms. In any case, there is a clear one-to-one correspondence between $X(\bullet, \bullet)$ and $\mathbb{N}$