Problem 1

In this exercise we shall estimate the power spectral density using the periodogram estimator:

$$\hat{S}_{XX}^{(per)}(f) = \frac{\Delta t}{N} \left| \sum_{n=0}^{N-1} x_n e^{-i2\pi f n \Delta t} \right|^2 \tag{1}$$

In practice, we have to evaluate the estimator at a discrete set of frequencies in the range $|f| < f_{NQ}$, where $f_{NQ} = \frac{1}{2\Delta t}$ is the Nyquist frequency. Our choice is to evaluate the estimator at the *Fourier frequencies*. The Fourier frequencies are given as $f_k = k\Delta f, k = 0, 1, \ldots, N-1$, where $\Delta f = \frac{1}{N\Delta t}$. Remember to identify those k corresponding to negative frequencies, and those corresponding to positive frequencies[1].

To start with, we will compute sums directly by means of the sum command.

a) Generate $N = 51$ samples of a white Gaussian process, with mean value equal to zero. Let the variance be $\sigma_w^2 = 1$ and let $\Delta t = 1$.

b) Plot the result from a) on a decibel scale (dB) as a function of frequency (two–sided). Compare with the true power spectral density.

c) Let $N = 101$ and repeat a) and b). Comment on the variance properties of the estimator.

d) Let $N = 401$ and repeat a) and b). Comment on the result.

It is also possible to use the built–in Python/Matlab/R command `fft` in periodogram estimation. `fft` calculates the Fourier transform for positive Fourier frequencies. Therefore you have to use the `fftshift`–command to be able to plot the periodogram estimate in the frequency interval $|f| < f_{NQ}$.

e) Repeat a) and b), but now using `fft`. Check that the results are the same.

f) Repeat c), but now by using `fft`.

g) Repeat d), but now by using `fft`.

The technique called *zero–padding* is used in connection with the `fft`–command to create an *interpolation* among the Fourier frequencies. This yields a smoother periodogram estimate. By using e.g. $\texttt{fft}(x, n = 512)$, the signal x appears in a sense to have 512 samples, while it actually consists of far fewer samples.

h) Consider once more the dataset from a), i.e. $N = 51$. Use the zero–padding $\texttt{fft}(x, n = 512)$, and plot the resulting periodogram estimate. Compare with the results from a) and e). Comment on the result.

[1] See page 32 in the compendium

Generate $N = 40$ samples of the periodic process $X_t = A\cos(2\pi f_0 t + \Theta)$, where $\Theta \sim U[0, 2\pi]$, $A = 1$, $f_0 = 0.6s^{-1}$ and $\Delta t = 0.2$s.

i) Evaluate the periodogram estimate of the process on the Fourier–frequencies. Use a valid frequency axis. Plot the result both on a linear and dB scale. Comment on the result.

j) Repeat e), but this time let $X_t = A\cos(2\pi f_0 t + \Theta) + W_t$, where $W_t \sim \mathcal{N}(0, \sigma_w^2)$ is independent of Θ. Choose different values of σ_w^2 and N, and discuss the result.