Mandatory Assignment 2

FYS-2000 - Quantum Mechanics

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Task 1

1a)

The stationary solution for the first region is

$$\psi_1(x) = Re^{-ik_1x} + Ie^{ik_1x} \tag{1}$$

For the second region we have

$$\psi_2(x) = Be^{-ik_2x} + Te^{ik_2x} \tag{2}$$

By using the time independent Schrödinger equation(TISE) we can solve for k.
The derivatives gives us

$$\frac{\partial^2\psi}{\partial x^2} = -k_1^2\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = -k_2^2\psi$$

Putting this into the TISE and using the fact that there is no potential in the first region and a potential V_0 in region 2

$$\frac{\hbar k_1^2}{2m} = E$$

$$\frac{\hbar k_2^2}{2m} + V_0 = E$$

Therefore

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

1b)

Now since we know there is no current moving from the left we can remove the term which states some wave coming from the left, so we have now

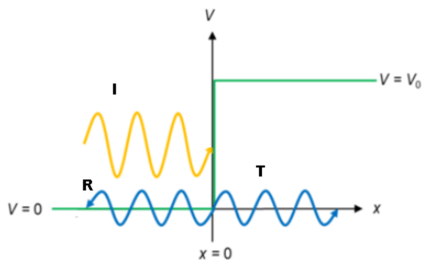
$$\psi_1(x) = Re^{-ik_1x} + Ie^{ik_1x}$$

For the second region we have

$$\psi_2(x) = Te^{ik_2x} \tag{3}$$

which the amplitude is strategically been named I for incident amplitude, R for reflected amplitude and T for transmission amplitude. The exponent of the reflected wave is negative because the wave will move to the left, both the incident wave and transmitted wave has positive exponent because it goes to right.

Figur 1: The quantum situation



1c)

Since we have two stationary state functions which are multi separated

$$\psi = \begin{cases} Re^{-ik_1x} + Ie^{ik_1x} & \text{if } x < 0 \\ Te^{ik_2x} & \text{if } x \geq 0 \end{cases}$$

we have the following condition for it to be continuous.

$$\psi_1(0) = \psi_2(0)$$

and

$$\frac{d}{dx}\psi_1\bigg|_{x=0} = \frac{d}{dx}\psi_2\bigg|_{x=0}$$

If we solve them we find

$$I + R = T \tag{4}$$

$$(R - I)k_1 = k_2T \tag{5}$$

Now lets solve for Transmission and Reflection amplitude in terms of Incident amplitude.

$$T = \frac{k_1}{k_2}(R - I) \tag{6}$$

putting this into eq(4)

$$I + R = \frac{k_1}{k_2}(R - I)$$

$$I + \frac{k_1}{k_2}I = -\frac{k_1}{k_2}R - R$$

$$I(1 + \frac{k_1}{k_2}) = R(-1 - \frac{k_1}{k_2})$$

Solving for R

$$\texttt{ANSWER:} R = I(\frac{k_1 - k_2}{k_1 + k_2})$$

By using this into eq(6) and solving for T we get:

$$\texttt{ANSWER:} T = I(\frac{2k_1}{k_1 + k_2})$$

Task 2

Lets first find the time derivative of the bra and ket of the state given in the Schrödinger equation.

$$|\dot{\psi} >= -\frac{i}{\hbar}\hat{H}|\psi > \tag{7}$$

and for the bra

$$< \dot{\psi}| = \frac{i}{\hbar} < \psi|\hat{H}^\dagger \tag{8}$$

So the time derivative of the inner product gives us

$$\frac{d}{dt} < \psi|\hat{X}|\psi >=< \dot{\psi}|\hat{X}|\psi > + < \psi|\dot{\hat{X}}|\psi > + < \psi|\hat{X}|\dot{\psi} >$$

The $\hat{X}$ operator has no explicit time dependence and therefore its zero, we now have:

$$\frac{d}{dt} < \psi|\hat{X}|\psi >=< \dot{\psi}|\hat{X}|\psi > + < \psi|\hat{X}|\dot{\psi} >$$

by replacing the time derivatives with eq(7) and (8) we get.

$$\frac{i}{\hbar}(< \psi|\hat{H}^\dagger\hat{X}|\psi > - < \psi|\hat{X}\hat{H}|\psi >)$$

Here we can observe that we can clean up by using commutators

$$[\hat{H}, \hat{X}] = \hat{H}\hat{X} - \hat{X}\hat{H}$$

and since the Hamiltonian is hermitian which means that

$$H^{\dagger} = H$$

We can then replace and clean up the equation, we now have

$$\frac{i}{\hbar} < [\hat{H}, \hat{X}] >$$

Now by connecting this with the momentum operator we can find the connection with the Hamiltonian which is

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(x)$$

we can then replace the commutation we had with the Hamiltonian and we have

$$\frac{d}{dt} < x >= \frac{i}{2m\hbar} < [\hat{P^2}, \hat{X}] > + \frac{i}{\hbar} < [V\hat{(x)}, \hat{X}] >$$

The last term does commute because both the potential and $\hat{X}$ are functions of x, therefore it cancels out, we now have:

$$\frac{d}{dt} < x >= \frac{i}{2m\hbar} < [\hat{P^2}, \hat{X}] >$$

which is

$$\frac{d}{dt} < x >= \frac{i}{2m\hbar} < \hat{P}[\hat{P}, \hat{X}] + [\hat{P}, \hat{X}]\hat{P} >$$

and we know that the commutation of

$$[\hat{P}, \hat{X}] = -i\hbar$$

therefore

$$\frac{d}{dt} < x >= \frac{i}{2m\hbar} < -i\hbar\hat{P} - i\hbar\hat{P} >$$

$$\texttt{ANSWER:} \frac{d}{dt} < x >= \frac{1}{m} < \hat{P} >$$

which is what we wanted to find.

Task 3

3a)

We are given the state vectors:

$$|\psi_1 >= 6i|\phi_1 > -3i|\phi_2 > \tag{9}$$

$$|\psi_2 >= -2i|\phi_1 > +4i|\phi_2 > \tag{10}$$

Since the ϕ form a orthonormal basis we can solve this with matrices,

$$|\psi_1 >= \begin{bmatrix} 6i \\ -3i \end{bmatrix}$$

$$|\psi_2 >= \begin{bmatrix} -2i \\ 4i \end{bmatrix}$$

By forming the inner product we get

$$< \psi_1|\psi_2 >= \begin{bmatrix} 6i \\ -3i \end{bmatrix}^{*T} \cdot \begin{bmatrix} -2i \\ 4i \end{bmatrix} = \begin{bmatrix} -6i & 3i \end{bmatrix} \cdot \begin{bmatrix} -2i \\ 4i \end{bmatrix} = -24$$

$$\texttt{ANSWER:} -24$$

3b)

Lets do the same procedure

$$|\psi_1 >= \begin{bmatrix} 6i \\ -3i \end{bmatrix}$$

$$|\psi_2 >= \begin{bmatrix} -2i \\ 4i \end{bmatrix}$$

By forming the inner product we get

$$< \psi_2|\psi_1 >= \begin{bmatrix} -2i \\ 4i \end{bmatrix}^{*T} \cdot \begin{bmatrix} 6i \\ -3i \end{bmatrix} = \begin{bmatrix} 2i & -4i \end{bmatrix} \cdot \begin{bmatrix} 6i \\ -3i \end{bmatrix} = -24$$

Now since the answer is real then the assumption that

$$< \psi_1|\psi_2 >^* = < \psi_2|\psi_1 >$$

is true.

Task 4

Since

$$\hat{A}|\psi >= |\psi' >$$

Taking the dagger on both sides (which is the complex conjugate)

$$(\hat{A}|\psi >)^\dagger = |\psi' >^\dagger$$

gives us

$$< \psi|\hat{A}^\dagger =< \psi'|$$

Task 5

5a)

We have the following relations

$$(\hat{A}^\dagger)^\dagger = (\hat{A}^{*T})^{*T} = \hat{A}$$

5b)

$$(\lambda\hat{A})^\dagger = \lambda^*\hat{A}^\dagger$$

5c)

$$< (\hat{A}+\hat{B})^\dagger\phi|\psi >^* =< \psi|(\hat{A}+\hat{B})\phi >)$$
$$< \phi|\hat{A}\psi > + < \phi|\hat{B}\psi >$$

So,

$$< \phi\hat{A}^\dagger|\psi > + < \hat{B}^\dagger\phi|\psi >$$

We then have,

$$< (\hat{A}^\dagger + \hat{B}^\dagger)\phi|\psi >$$

5d)

$$(\hat{A}\cdot\hat{B})^\dagger = \hat{B}^{*T}\hat{A}^{*T}$$

The reason for the operators to change side is because the dimensions change when we transpose them, therefore for it to be an allowed operation the dimension need to match.

Task 6

6a)

We have

$$|\psi >, |\psi' >= e^{i\theta}|\psi >$$

So the complex conjugate of ψ' is:

$$< \psi'| =< \psi|e^{-i\theta}$$

By forming the inner product we get:

$$< \psi'|\psi' >= e^{-i\theta+i\theta} < \psi|\psi >$$

Which is.

$$< \psi'|\psi' >=< \psi|\psi >$$

Therefore when if one of them are normalized the other is too

$$< \psi'|\psi' >=< \psi|\psi >= 1$$

6b)

Since

$$< \psi'|\hat{A}|\psi' >= e^{i\theta}e^{-i\theta} < \psi|\hat{A}|\psi >$$

where A is arbitrary operator we have

$$< \psi'|\hat{A}|\psi' >= e^{0} < \psi|\hat{A}|\psi >$$

Therefore

$$< \psi'|\hat{A}|\psi' >=< \psi|\hat{A}|\psi >$$

So the measurement is the same for both

6c)

This means that they both have the same quantum state

Task 7

7a)

The creation and annihilation operators are given:

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) \tag{11}$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) \tag{12}$$

Now since commutators are so fine and dandy, lets find some which we want to use. lets find:

$$[a_-, a_+]$$

By replacing with the equation (11) and (12)

$$\frac{1}{2\hbar m\omega}[m\omega\hat{X} + i\hat{P}, m\omega\hat{X} - i\hat{P}]$$

this gives us

$$\frac{1}{2\hbar m\omega}(m\omega[\hat{X},\hat{X}] - m\omega i[\hat{X},\hat{P}] + m\omega i[\hat{P},\hat{X}] + i[\hat{P},\hat{P}])$$

And by knowing the following commutation relations

$$[\hat{X},\hat{P}] = i\hbar$$

$$[\hat{X},\hat{X}] = 0$$

$$[\hat{P},\hat{X}] = -i\hbar$$

$$[\hat{P},\hat{P}] = 0$$

we can solve easily

$$\frac{2m\omega\hbar}{2\hbar m\omega} = 1$$

Therefore

$$[a_-, a_+] = 1 \tag{13}$$

Now lets make a new operator called the number operator

$$\hat{N} = [a_+, a_-]$$

which has its eigenvector/value relation

$$\hat{N}|n >= n|n >$$

And if we solve out the commutation of eq(13)

$$[a_-, a_+] = a_-a_+ - a_+a_- = 1$$

By replacing with N we get

$$a_-a_+ = 1 + \hat{N}$$

which gives us

$$1 + \hat{N}|n >= n + 1|n >$$

Now lets use what we have just found first lets try it on some system states

$$a_+|n >= C_n|n + 1 >$$

Where C_n is some fudge constant (the operator a does not yield a eigenvalue) and since a_+ is a creation operator we now went up in energy level which is showed by $a + 1$. now by adding the complex conjugate we get

$$< n|a_+^\dagger a_+|n >= C_nC_n^* < n|n >$$

What happened? why didnt we go up to level 2? as we can easily see is that

$$a_+^\dagger = a_-$$

Therefore we have

$$< n|a_-a_+|n >= |C_n|^2 < n|n + 1 - 1 >$$

This means that we created and annihilated the energy and went back to state n. Now we observe that we have already given a_-a_+ another name $\hat{N} + 1$, now lets replace.

$$< n|N + 1|n >= n + 1 < n|n >$$

Now since the we must find it somewhere in our region it has to be normalized which means that $< n|n >= 1$ Now we see that

$$|C_n|^2 = n + 1$$

therefore

$$C_n = \sqrt{n + 1}$$

Now lets do the same for a_-

$$a_-|n >= D_n|n - 1 >$$

Which now means we went down in energy. Now by adding the complex conjugate

$$< n|a_-^\dagger a_-|n >= |D_n|^2 < n|n >$$

Since $a_-^\dagger = a_+$ we went down and up to same level As previously solved

$$a_+a_- = \hat{N}$$

Therefore

$$|D_n|^2 = n => D_n = \sqrt{n}$$

So now we know that

$$a_-|n >= \sqrt{n}|n - 1 > \tag{14}$$

$$a_+|n >= \sqrt{n + 1}|n + 1 > \tag{15}$$

To find the expectation value we solve it in terms of the annhilation and creation operators. By multiplying the operators we find

$$a_+ + a_- = \frac{1}{\sqrt{2\hbar m\omega}}(2m\omega\hat{X})$$

Solving for $\hat{X}$

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \tag{16}$$

to find the momentum operator we subtract and switch:

$$a_- - a_+ = \frac{1}{\sqrt{2m\omega\hbar}}(2i\hat{P})$$

which gives us

$$\hat{P} = -i\sqrt{\frac{m\omega\hbar}{2}}(a_- - a_+) \tag{17}$$

Now lets use these to find the expectation values

$$< \psi_n|\hat{X}|\psi_n >= \sqrt{\frac{\hbar}{2m\omega}} < \psi_n|a_+ + a_-|\psi_n >)$$

Separation yields

$$< x >= \sqrt{\frac{\hbar}{2m\omega}}(< \psi_n|a_+|\psi_n + < \psi_n|a_-|\psi_n >)$$

By using what we found earlier we now have their created and annihilated energies

$$< x >= \sqrt{\frac{\hbar}{2m\omega}}(\cancelto{orthogonal}{< \psi_n|\psi_{n+1} >} \sqrt{n+1} + \cancelto{orthogonal}{< \psi_n|\psi_{n-1} >} \sqrt{n})$$

Now since each level is orthogonal to the other, each of these inner products cancel out and we get:

$$\texttt{ANSWER:} < x >= 0$$

Now for $< x^2 >$

$$\hat{X}^2 = \frac{\hbar}{2m\omega}((a_+ + a_-)(a_+ + a_-)) \tag{18}$$

$$(a_+ + a_-)(a_+ + a_-) = a_+a_+ + a_+a_- + a_-a_+ + a_-a_-$$

We then get

$$< x^2 >= \frac{\hbar}{2m\omega}(< \psi_n|a_+a_+|\psi_n > + < \psi_n|a_+a_-|\psi_n > + < \psi_n|a_-a_+|\psi_n > + < \psi_n|a_-a_-|\psi_n >)$$

Here we add inn for N and N+1 as for previously solved

$$< x^2 >= \frac{\hbar}{2m\omega}(< \psi_n|a_+a_+|\psi_n > + < \psi_n|N|\psi_n > + < \psi_n|N+1|\psi_n > + < \psi_n|a_-a_-|\psi_n >)$$

this gives us

$$< x^2 >= \frac{\hbar}{2m\omega}(< \psi_n|\psi_{n+2} > \sqrt{n+1}\sqrt{n+2} + < \psi_n|\psi_{n-1+1} > n$$
$$+ < \psi_n|\psi_{n+1-1} > n+1 + < \psi_n|a_-a_-|\psi_{n-2} > \sqrt{n}\sqrt{n-1})$$

Again all the orthogonal cases cancel out

$$< x^2 >= \frac{\hbar}{2m\omega}(\cancelto{1}{< \psi_n|\psi_n >} n + \cancelto{1}{< \psi_n|\psi_n >} n+1)$$

Again by using the fact that they are normalized we get

$$< x^2 >= \frac{\hbar}{2m\omega}(n + n + 1)$$

$$\texttt{ANSWER:}\ < x^2 >= \frac{\hbar}{m\omega}(n + \frac{1}{2})$$

Now for momentum

$$< p >= -i\sqrt{\frac{m\omega\hbar}{2}} < \psi_n|a_- a_+|\psi_n >$$

Splitting up we have

$$< p >= -i\sqrt{\frac{m\omega\hbar}{2}}(\cancelto{0}{< \psi_n|a_-|\psi_n >} - \cancelto{0}{< \psi_n|a_+}|\psi_n >)$$

Now we observe that each of these will now cancel out because they are orthogonal when we apply the operators,therefore

$$\texttt{ANSWER:}\ < p >= 0$$

For momentum squared

$$p^2 = i^2\frac{m\omega\hbar}{2}((a_+ - a_-)(a_+ - a_-))$$

$$p^2 = -\frac{m\omega\hbar}{2}(a_+a_+ - a_+a_- - a_+a_+ - a_-a_-)$$

Solving the expectation value gives

$$\begin{aligned}< p^2 >= -\frac{m\omega\hbar}{2} < \psi_n|\psi_{n+2} > \sqrt{n+1}\sqrt{n+2} - < \psi_n|\psi_{n-1+1} > n \\ - < \psi_n|\psi_{n+1-1} > n + 1 + < \psi_n|\psi_{n-2} > \sqrt{n}\sqrt{n-1}\end{aligned}$$

Again the ones being orthogonal cancel out and the others are normalized

$$< p^2 >= -\frac{m\omega\hbar}{2}(-n - n - 1)$$

$$< p^2 >= \frac{m\omega\hbar}{2}(n + n + 1)$$

$$\texttt{ANSWER:}\ < p^2 >= m\omega\hbar(n + \frac{1}{2})$$

7b)

For the quantum harmonic oscillator we have the Hamiltonian on the form

$$\hat{H}_{xy} = \hat{H}_x + \hat{H}_y = \frac{\hat{P}^2{}_x}{2m} + \frac{m\omega^2\hat{X}_x^2}{2} + \frac{\hat{P}^2{}_y}{2m} + \frac{m\omega^2\hat{Y}_y^2}{2}$$

Here we see some similarity with the creation and annihilation operators

$$a_{+_x}a_{-_x} = \frac{1}{2m\hbar\omega}(m\omega\hat{X} - i\hat{P})(m\omega\hat{X} + i\hat{P})$$

Gives us

$$a_{+_x}a_{-_x} = \frac{1}{2m\hbar\omega}(m^2\omega^2\hat{X}^2 + im\omega[\hat{X}, \hat{P}] + \hat{P}^2)$$

As we know $[x, p] = i\hbar$, therefore

$$a_{+_x}a_{-_x} = \frac{1}{2\hbar}(m\omega\hat{X} - \hbar + \frac{\hat{P}^2}{m\omega})$$

cleaning up gives

$$a_{+_x}a_{-_x} = \frac{m\omega\hat{X}}{2\hbar} + \frac{\hat{P}^2}{2\hbar m\omega} - \frac{1}{2}$$

So now we see that by adding $\hbar\omega + \frac{1}{2}$ we have the Hamiltonian, therefore

$$\hat{H}_x = \hbar\omega(a_{+_x}a_{-_x} + \frac{1}{2})$$

We also said that $a_+a_- = N$ So we have

$$\hat{H}_x = \hbar\omega(N_x + \frac{1}{2})$$

Now by applying this on a state we have

$$\hat{H}|\psi >= E|\psi >$$

By changing the Hamiltonian with what we found, we get

$$\hbar\omega(N_x + \frac{1}{2})|\psi >= \hbar\omega(n_x + \frac{1}{2})|\psi >$$

where the term in front is of course the eigenvalue and the energy observed

$$E_{n_x} = \hbar\omega(n_x + \frac{1}{2})$$

Now by applying the one for y and add it we get the total energy

$$E_{n_y} = \hbar\omega(n_y + \frac{1}{2})$$

The total energy is then

$$E_n = \hbar\omega(n_y + n_x + 1)$$

`ANSWER:` Here we have M degeneracy, where M is the energy level.

For the eigenfunction we have that,

$$a_+|\psi_n >= \sqrt{n+1}|\psi_{n+1} >$$

and ,

$$a_-|\psi_n >= \sqrt{n}|\psi_{n-1} >$$

therefore we have,

$$|\psi_n >= \frac{a_+}{\sqrt{n}}|\psi_{n-1}$$

then by this fact we can show that,

$$|\psi_n >= \frac{a_+}{\sqrt{n}}|\psi_{n-1} = \frac{a_+^2}{\sqrt{n(n-1)}}|\psi_{n-2} = ...\frac{a^n}{\sqrt{n!}}|\psi_{n-n=0} >$$

where ψ_0 is the ground state.
To find the ground state we can use the fact that if we annihilate the ground state we should get zero. Therefore,

$$a_-|0 >= 0 = \frac{1}{\sqrt{2\hbar m\omega}}m\omega\hat{X} + i\hat{P}$$

We can multiply away the fraction

$$(m\omega\hat{X} + i\hat{P})\psi_0 = 0$$

this is a separable differential equation, there we have

$$\frac{d}{dx}\psi_0 = \frac{-m\omega\hat{X}}{\hbar}\psi_0$$

We see that this is a repeated function, therefore we guess with e^x which gives us,

$$\psi_0 = e^{-m\omega\hat{X}^2/(2\hbar)}$$

This can be shown to be true by taking the derivative. Therefore,

$$\psi_0 = Ae^{-m\omega\hat{X}^2/(2\hbar)}$$

With the normalization constant A. Now lets normalize the function, because we need be able to find it at every level. By combining the dimensions we have

$$\psi_{0,0} = A^2 e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

normalization,

$$< \psi_{0,0}|\psi_{0,0} >= 1$$

which is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^4 e^{-\frac{m\omega}{\hbar}(x^2+y^2)} = 1$$

Here we observe that $x^2 + y^2$ also can be written in terms of polar coordinates, so lets perform the transformation. $dxdy = rdrd\theta$ where r is the Jacobian, $x^2 + y^2 = r^2$ so we now have

$$A^4 \int_0^{2\pi}\int_0^{\infty} e^{-\frac{m\omega}{\hbar}(r^2)} rdrd\theta = 1$$

Now lets get the θ integral out of the way,

$$A^4 2\pi \int_0^{\infty} e^{-\frac{m\omega}{\hbar}(r^2)} rdr = 1$$

Now lets substitute with $w = -\frac{-m\omega}{\hbar}r^2$, then we have
$\frac{dw}{dr} = -\frac{-2m\omega}{\hbar}r$, which gives us $dr = -\frac{dw}{2m\omega r}\hbar$ Now lets use the substitution in our integral,

$$A^4 2\pi \int re^w dr = 1$$

Substitute dr,

$$A^4\pi \int e^w - \frac{dw}{m\omega}\hbar = 1$$

Now if we substitute for w the old term and perform the integral

$$-\frac{\hbar\pi}{m\omega}A^4\left[e^{-\frac{m\omega}{\hbar}r^2}\right]_0^{\infty} = 1$$

Which yield,

$$-\frac{\hbar\pi}{m\omega}A^4[-1] = 1$$

We then finally have,

$$A = \sqrt[4]{\frac{m\omega}{\hbar\pi}}$$

Which gives us the ground state function

$$\psi_{0,0} = \sqrt{\frac{m\omega}{\hbar\pi}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \tag{19}$$

Now by using what we previously found in this task,

$$|\psi_n > \frac{a^n}{\sqrt{n!}} |\psi_0 >$$

But this is for one dimension therefore we have,

$$|\psi_{n_x,n_y} > \frac{a_x^n}{\sqrt{n_x!}} \frac{a_y^n}{\sqrt{n_y!}} |\psi_{0,0} >$$

Then finally we have the eigenfunction,

$$\texttt{ANSWER:} \psi_{n_x,n_y} = \frac{a_x^n a_y^n}{\sqrt{n_x! n_y!}} \psi_{0,0} \tag{20}$$

or

$$\psi_{n_x,n_y} = \frac{a_x^n a_y^n}{\sqrt{n_x! n_y!}} \sqrt{\frac{m\omega}{\hbar\pi}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \tag{21}$$

Task 8

8a)

The parity operator has the following relation:

$$\hat{\Pi}|r >= | - r >$$

Therefore we have

$$\hat{\Pi}|\psi(x) >= |\psi(-x) >$$

8b)

We have that

$$< \phi|\hat{\Pi}|\psi >= \int_{-\infty}^{\infty} \phi(x)^* \hat{\Pi}\psi(x)dx$$

Now we complex conjugate it on this form

$$(< \psi|\hat{\Pi}|\phi >)^* = (\int_{-\infty}^{\infty} \psi(x)^* \hat{\Pi}\phi(x)dx)^*$$

If we absorb the operator into ϕ we get.

$$(< \psi|\hat{\Pi}|\phi >)^* = (\int_{-\infty}^{\infty} \phi(-x)^* \psi(x)dx)$$

Now lets substitute $-x = t$ So we now have $t = -x$, $dx = -dt$ and our new limits become $\infty_x- >$-∞ and $-\infty$ -¿ ∞ so we now have

$$(< \psi|\hat{\Pi}|\phi >)^* = -(\int_{\infty}^{-\infty} \phi(t)^* \psi(-t)dt)$$

We can switch the limits if we put a - in front, and using the fact that $\psi(-t) = \hat{\Pi}\psi(t)$

$$(<\psi|\hat{\Pi}|\phi>)^* = (\int_{-\infty}^{\infty} \phi(t)^*\hat{\Pi}\psi(t)dt) =<\phi|\hat{\Pi}|\psi>$$

Therefore

$$<\phi|\hat{\Pi}|\psi>= (<\psi|\hat{\Pi}|\phi>)^*$$

Therefore the operator is hermitian.

8c)

$$\hat{\Pi}\hat{\Pi}|\psi>= \hat{\Pi}|-\psi>= |\psi>$$

Hence it is its own inverse

$$\hat{\Pi}^2 = 1$$

Now by applying to a eigenfunction we get its eigenvalue

$$\hat{\Pi}|\pi>= \pi|\pi>$$

therefore

$$\hat{\Pi}^2|\pi>= \pi^2|\pi>$$

Now since $\pi^2 = 1$, we have

$$\pi = \pm 1$$

8d)

We know the following

$$\hat{P}_+ = \frac{1}{2}(\hat{I} + \hat{\Pi})$$

$$\hat{P}_- = \frac{1}{2}(\hat{I} - \hat{\Pi})$$

$$|\psi_+>= \hat{P}_+|\psi>$$

$$|\psi_->= \hat{P}_-|\psi>$$

Now lets apply the parity operator on the state vector

$$\hat{\Pi}|\psi_+>= \hat{\Pi}\hat{P}_+|\psi>$$

Which is

$$\hat{\Pi}|\psi_+>= \frac{1}{2}\hat{\Pi}(\hat{I} + \hat{\Pi})|\psi>$$

$$\hat{\Pi}|\psi_+>= \frac{1}{2}(\hat{I}\hat{\Pi} + \hat{\Pi}^2)|\psi>$$

Now since the identity operator spits out whatever it takes in we get $\hat{I}\hat{\Pi} = \hat{\Pi}$ and $\hat{\Pi}^2) = 1 = \hat{I}$ as we found previous, we then have:

$$\hat{\Pi}|\psi_+>= \frac{1}{2}(\hat{\Pi}) + \hat{I}|\psi>$$

Which is then

$$\texttt{ANSWER:}\hat{\Pi}|\psi_+>= |\psi_+>$$

therefore the eigenvalue is just 1 and that is the eigenvalue we found in task 8c)
We do again for $\hat{P}_-$

$$\hat{\Pi}|\psi_->= \hat{\Pi}\hat{P}_-|\psi>$$

Which is

$$\hat{\Pi}|\psi_- >= \frac{1}{2}\hat{\Pi}(\hat{I} - \hat{\Pi})|\psi >$$

$$\hat{\Pi}|\psi_- >= \frac{1}{2}(\hat{I}\hat{\Pi} - \hat{\Pi}^2)|\psi >$$

Now since the identity operator spits out whatever it takes in we get $\hat{I}\hat{\Pi} = \hat{\Pi}$ and $\hat{\Pi}^2) = 1 = \hat{I}$ as we found previous, we then have:

$$\hat{\Pi}|\psi_- >= -\frac{1}{2}(\hat{I} - \hat{\Pi}|\psi >)$$

Which is then

$$\texttt{ANSWER:}\hat{\Pi}|\psi_- >= -|\psi_- >$$

so here the eigenvalue is -1 which is also what we found.

8e)

If we try the parity operator on the eigenfunction

$$< x|\hat{\Pi}|\psi_+ >=< x|\psi_+ >$$

and

$$< -x|\psi_+ >=< \hat{\Pi}x|\psi_+ >$$

and since the parity operator is hermitian

$$< \hat{\Pi}x|\psi_+ >=< x|\hat{\Pi}|\psi_+ >$$

which is

$$< x|\hat{\Pi}|\psi_+ >=< x|\psi_+ >$$

therefore it is an even function.
For the other function we have

$$< x|\hat{\Pi}|\psi_- >= - < x|\psi_- >$$

and

$$< -x|\psi_- >= - < x|\psi_- >$$

Therefore it is an odd function.