1 Part A

Cosider the two series

$$x_t = w_t$$
$$y_t = w_t - \theta w_{t-1} + u_t,$$

where w_t and u_t are independent white noise series with variances σ_w^2 and σ_u^2, respectively, and θ is an unspecified constant.

1.1 Part a

Express the ACF, $\rho_y(h)$, for $h = 0, \pm 1, \pm 2, ...$ of the series y_t as a function of σ_w^2, σ_u^2, and θ.

x_t is just white noise, so $\rho_x(0) = 1$ and $\rho_x(h) = 0$ for $h \neq 0$. For y_t, we know that the mean is zero so

$$\begin{aligned}
\gamma_y(h) &= E[y_{t+h}y_t] \\
&= E[w_{t+h} - \theta w_{t+h-1} + u_{t+h}][w_t - \theta w_{t-1} + u_t] \\
&= E[w_{t+h}w_t] - \theta E[w_{t+h-1}w_t] - \theta E[w_{t+h}w_{t-1}] + \theta^2 E[w_{t+h-1}w_{t-1}] + E[u_{t+h}u_t] \\
&= \begin{cases} \sigma_w^2(1+\theta^2) + \sigma_u^2 & \text{if } h = 0 \\ -\theta\sigma_w^2 & \text{if } |h| = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Therefore the ACF is

$$\rho_y(h) = \begin{cases} 1 & \text{if } h = 0 \\ -\frac{\theta\sigma_w^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2} & \text{if } |h| = 1 \\ 0 & \text{otherwise} \end{cases}$$

1.2 Part b

Determine the CCF, $\rho_{xy}(h)$ relating x_t and y_t.

$$\begin{aligned}
\gamma_{xy}(h) &= E[x_{t+h}y_t] \\
&= E[w_{t+h}(w_t - \theta w_{t-1} + u_t)] \\
&= \begin{cases} \sigma_w^2 & \text{if } h = 0 \\ -\theta\sigma_w^2 & \text{if } h = -1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Then the CCF is

$$\rho_{xy}(h) = \begin{cases} \frac{\sigma_w}{\sqrt{\sigma_w^2(1+\theta^2)+\sigma_u^2}} & \text{if } h = 0 \\ -\frac{\theta\sigma_w}{\sqrt{\sigma_w^2(1+\theta^2)+\sigma_u^2}} & \text{if } h = -1 \\ 0 & \text{otherwise} \end{cases}$$

2 Part B

For a time series

$$x_t = \mu + \sum_{i=-\infty}^{\infty} \psi_i w_{t-i}$$

2.1 Part a

Find the autocovariance function.

$$\begin{aligned}
\gamma_x(h) &= E[(x_{t+h} - \mu)(x_t - \mu)] \\
&= E[x_{t+h}x_t] \\
&= E\left[\left(\sum_{i=-\infty}^{\infty} \psi_i w_{t+h-i}\right)\left(\sum_{i=-\infty}^{\infty} \psi_i w_{t-i}\right)\right] \\
&= \sigma_w^2 \sum_{i=-\infty}^{\infty} \psi_{i-h}\psi_i
\end{aligned}$$

2.2 Part b

Use the result to find $\gamma_x(h)$ and $\rho_x(h)$ of a moving average process

$$x_t = w_{t-1} + 2w_t + w_{t+1}$$

$$x_t = \sum_{i=-1}^{1} \psi_i w_i \quad \psi = \begin{cases} 2 & \text{if } i = 0 \\ 1 & \text{if } |i| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_x(h) = \sigma_w^2 \sum_{i=-1}^{1} \psi_{i-h}\psi_i$$

$$\gamma_x(0) = \sigma_w^2(1 + 4 + 1)$$

$$\gamma_x(1) = \sigma_w^2(2 + 2 + 0)$$

$$\gamma_x(2) = \sigma_w^2(1 + 0 + 0)$$

$$\gamma_x(3) = \sigma_w^2(0 + 0 + 0)$$

$$\gamma_x(h) = \begin{cases} 6\sigma_w^2 & \text{if } h = 0 \\ 4\sigma_w^2 & \text{if } |h| = 1 \\ \sigma_w^2 & \text{if } |h| = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_x(h) = \begin{cases} 1 & \text{if } h = 0 \\ \frac{2}{3} & \text{if } |h| = 1 \\ \frac{1}{3} & \text{if } |h| = 2 \\ 0 & \text{otherwise} \end{cases}$$