Spectral density and filtering.

↳ Analysis in the frequency domain ⇒ Fourier Transform!

↳ driven by periodic components
Data
- Speech
- Climate (e.g. El Niño)

↳ Linear filters, Analysis (LTI-systems)

↳ Estimation (compendium)

Spectral Density

Consider deterministic signal x_t

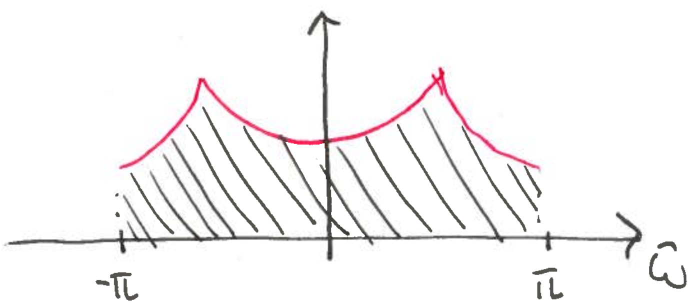
If: $|X(\hat{\omega})| \equiv \left|\sum_{t=-\infty}^{\infty} x_t e^{-j\hat{\omega}t}\right| \leq \sum_{t=-\infty}^{\infty} |x_t| < \infty$

then $\exists$ DTFT $X(\hat{\omega})$ of x_t.

Signal energy

PARSEVAL

$$\mathcal{E} \equiv \sum_{t=-\infty}^{\infty} |x_t|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\hat{\omega})|^2 d\hat{\omega}$$

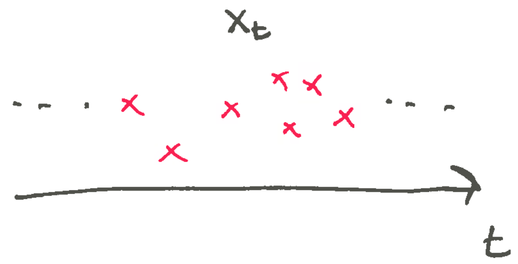


Note: $|X(\hat{\omega})|^2$ is called the energy spectrum of x_t

$\Rightarrow$ distribution of energy as a function of frequency

$$\left[\hat{\omega} = 2\pi \underset{\uparrow \text{ frequency}}{f} \cdot \underset{}{T_s} \leftarrow \text{Sampling period}\right]$$

Realization of r.s. x_t



I. Extends ∞ in time

II. $\sum_{t=-\infty}^{\infty} |x_t| \to \infty$!

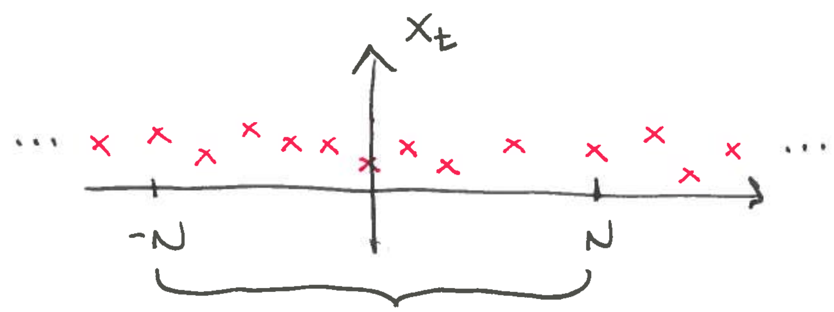
Hence: $\mathcal{E} \to \infty$, energy $\nexists$ and DTFT $\nexists$ for r.s.?

What about: Power $= \frac{\text{energy}}{\text{time}}$?

↳ Power spectral density

Truncated r.s.

Def: $x_{N,t} = \begin{cases} x_t\,; & t = -N, \dots, 0, \dots, N \\ 0\,; & \text{otherwise} \end{cases}$

Realization 

$x_{N,t}$: $2N+1$ points

Can define DTFT of $\tilde{x}_{N,t}$:

$$\underbrace{\tilde{X}_N(\hat{\omega})}_{\text{Random}} = \sum_{t=-\infty}^{\infty} \underbrace{\tilde{x}_{N,t}}_{\text{Random}} e^{-j\hat{\omega}t} = \sum_{t=-N}^{N} \tilde{x}_t e^{-j\hat{\omega}t}.$$

$|\hat{\omega}| \leq \pi$.

Expected energy

in $\tilde{x}_t$ on $|t| \leq N$ is then

$$\mathcal{E}(N) = E\left\{ \sum_{t=-\infty}^{\infty} |\tilde{x}_{N,t}|^2 \right\}$$

PARSEVAL

$$= E\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{X}_N(\hat{\omega})|^2 d\hat{\omega} \right\}$$

Expected power

in $\tilde{x}_t$ on $|t| \leq N$

$$P(N) = \frac{\mathcal{E}(N)}{2N+1} = E\left\{ \frac{1}{2N+1} \sum_{t=-\infty}^{\infty} |\tilde{x}_{N,t}|^2 \right\}$$

$$= E\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\tilde{X}_N(\hat{\omega})|^2}{2N+1} d\hat{\omega} \right\}$$

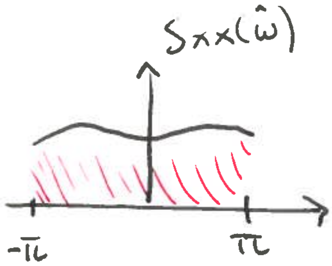
Def. The expected power in x_t is

$$P_{xx} = \lim_{N\to\infty} P(N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{E\{|X_N(\hat{\omega})|^2\}}{2N+1}\, d\hat{\omega}$$

$$\overset{!}{\equiv} \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\hat{\omega})\, d\hat{\omega},$$

where $\boxed{S_{xx}(\hat{\omega}) \equiv \lim_{N\to\infty} \frac{E\{|X_N(\hat{\omega})|^2\}}{2N+1}}$

is the power spectral density.



Note: Assume stationary sequences!

PSD vs. $\gamma_x(h)$

Define $Y_t = X_t - \mu_x$.

Look at $E\{|Y_N(\hat{\omega})|^2\} = E\{Y_N(\hat{\omega})\, Y_N^*(\hat{\omega})\}$

$$\overset{FT}{=} E\left\{\sum_{k=-N}^{N}\sum_{l=-N}^{N} Y_k Y_l\, e^{-j\hat{\omega}k} e^{j\hat{\omega}l}\right\}$$

$$= \sum_{k=-N}^{N}\sum_{l=-N}^{N} E\{Y_k Y_l\}\, e^{-j\hat{\omega}(k-l)}$$

$$= \sum_{k=-N}^{N}\sum_{l=-N}^{N} E\{(X_k-\mu_x)(X_l-\mu_x)\}\, e^{-j\hat{\omega}(k-l)}$$ WOW!

$$\overset{stat.}{=} \sum_{k=-N}^{N}\sum_{l=-N}^{N} \gamma_x(k-l)\, e^{-j\hat{\omega}(k-l)}$$

$\ell \backslash k$	$-N$	$-N+1$	$-N+2$		N
$-N$	$\boxed{-N-(-N)=0 \quad \gamma_x(0)}$	$-N+1-(-N)=1$ $\gamma_x(1)$	$-N+2-(-N)=2$ $\gamma_x(2)$	$\cdots$	$N-(-N)=2N$ $\gamma_x(2N)$
$-N+1$	$-N-(-N+1)=-1$ $\gamma_x(-1)$	$\boxed{-N+1-(-N+1)=0 \quad \gamma_x(0)}$	$-N+2-(-N+1)=1$ $\gamma_x(1)$	$\cdots$	$N-(-N+1)=2N-1$ $\gamma_x(2N-1)$
$-N+2$	$\gamma_x(-2)$	$\gamma_x(-1)$	$\boxed{\gamma_x(0)}$		
$\vdots$					
N	$\gamma_x(-2N)$	$\gamma_x(-2N+1)$	$\gamma_x(-2N+2)$	$\cdots$	$\boxed{\gamma_x(0)}$

$$\overset{h=k-\ell}{=} \sum_{h=-2N}^{2N} (2N+1-|h|)\, \gamma_x(h)\, e^{-j\hat{\omega} h}$$

Hence:

$$S_{yy}(\hat{\omega}) = \lim_{N\to\infty} \frac{E\{|Y_N(\hat{\omega})|^2\}}{2N+1}$$

$$= \lim_{N\to\infty} \sum_{h=-2N}^{2N} \left(1 - \frac{|h|}{2N+1}\right) \gamma_x(h)\, e^{-j\hat{\omega} h}$$

What's this?

$$\overset{\sum_{h=-\infty}^{\infty}|h||\gamma_x(h)|<\infty}{=} \sum_{h=-\infty}^{\infty} \gamma_x(h)\, e^{-j\hat{\omega} h} \quad ; \qquad |\hat{\omega}| \le \pi$$

$$\equiv f_x(\hat{\omega})$$ (notation from book)

Conclusion:

Spectral density $f_x(\hat{\omega}) \xleftrightarrow{\text{DTFT}} \gamma_x(h)$ $\left(= S_{yy}(\hat{\omega}),\ y_t = x_t - \mu_x \right)$

$$\boxed{\begin{aligned} f_x(\hat{\omega}) &= \sum_{h=-\infty}^{\infty} \gamma_x(h)\, e^{-j\hat{\omega}h}; \qquad |\hat{\omega}| \le \pi \\ \gamma_x(h) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_x(\hat{\omega})\, e^{j\hat{\omega}h}\, d\hat{\omega} \end{aligned}}$$

$f_x(\hat{\omega})$: power spectral density of centered t.s. $y_t = x_t - \mu_x$

Differences from the book:

$\hat{\omega} = 2\pi\omega$ ← frequency

$$\Rightarrow f_x(\omega) = \sum_{h=-\infty}^{\infty} \gamma_x(h)\, e^{-j2\pi\omega h}, \qquad |\omega| \le \frac{1}{2}$$

$$\gamma_x(h) = \int_{-1/2}^{1/2} f_x(\omega)\, e^{j2\pi\omega h}\, d\omega \qquad \text{(substitution).}$$

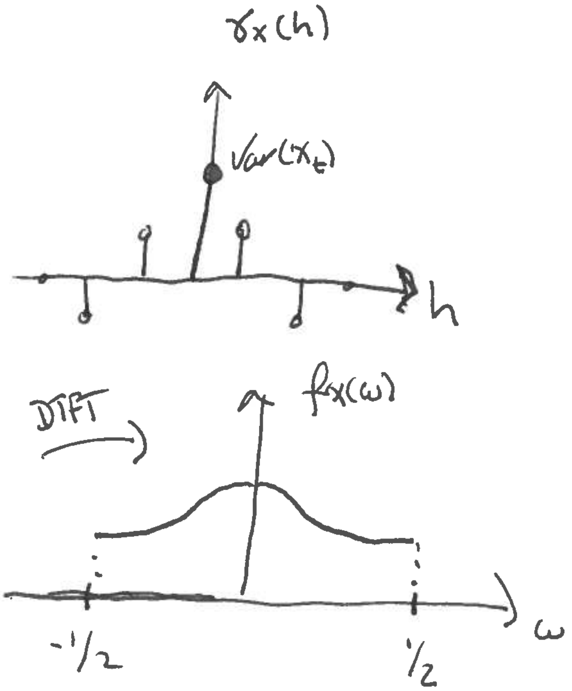
<u>Note</u>: $$\gamma_x(0) = \int_{-1/2}^{1/2} f_x(\omega)\,d\omega = Var(X_t) \quad (=P_{xx})$$

Properties of $f_x(\omega)$

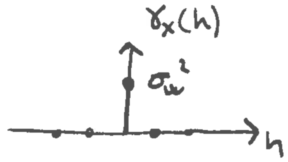
1) $f_x(\omega) \in \mathbb{R}$

2) $f_x(-\omega) = f_x(\omega)$

3) $f_x(\omega) \geq 0$

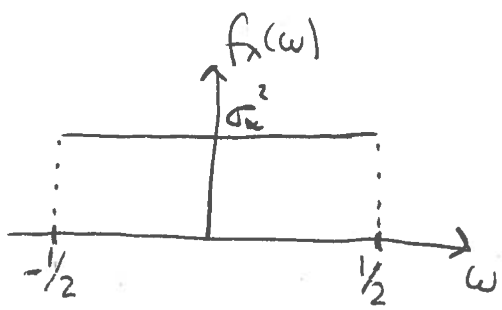


<u>Ex</u> White noise: $\gamma_x(h) = \sigma_w^2\,\delta(h)$



$$f_x(\omega) = \sum_{h=-\infty}^{\infty} \sigma_w^2\,\delta(h)e^{-j2\pi\omega h} = \underline{\sigma_w^2}.$$

↳ All frequencies equally important



<u>Note</u>: $\gamma_x(0) = \sigma_w^2 = \int_{-1/2}^{1/2} \sigma_w^2\,d\omega$

Ex A periodic stationary process

$$x_t = u_1 \cos(2\pi\omega_0 t) + u_2 \sin(2\pi\omega_0 t)$$

u_1, u_2: Uncorrelated

$E(u_1) = E(u_2) = 0$

$Var(u_1) = Var(u_2) = \sigma^2$.

Result from the book: $\gamma_x(h) = \sigma^2 \cos(2\pi\omega_0 h)$

Find $f_x(\omega)$.

What do we expect?

Solution: Know your FT-pairs...

$$\gamma_x(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-j2\pi\omega_0 h} + \frac{\sigma^2}{2} e^{j2\pi\omega_0 h} \qquad \text{(Euler)}$$

Look at

$$IFT\{A\,\delta(\omega-\omega_0)\} = A \cdot IFT\{\delta(\omega-\omega_0)\}$$

$$= A\int_{-1/2}^{1/2} \delta(\omega-\omega_0)\, e^{j2\pi\omega h}\, d\omega$$

$$= A\int_{-1/2}^{1/2} \delta(\omega-\omega_0)\, e^{j2\pi\omega_0 h}\, d\omega$$

$$= A e^{j2\pi\omega_0 h} \underbrace{\int_{-1/2}^{1/2} \delta(\omega-\omega_0)\, d\omega}_{=1}$$

$$= \underline{A e^{j2\pi\omega_0 h}}$$

Try:
$$IFT\left\{\frac{\sigma^2}{2}\delta(\omega+\omega_0) + \frac{\sigma^2}{2}\delta(\omega-\omega_0)\right\} = \frac{\sigma^2}{2} IFT\{\delta(\omega+\omega_0)\} + \frac{\sigma^2}{2} IFT\{\delta(\omega-\omega_0)\}$$

$$= \frac{\sigma^2}{2} e^{-j2\pi\omega_0 h} + \frac{\sigma^2}{2} e^{j2\pi\omega_0 h}$$

$$= \sigma^2 \cos(2\pi\omega_0 h)$$

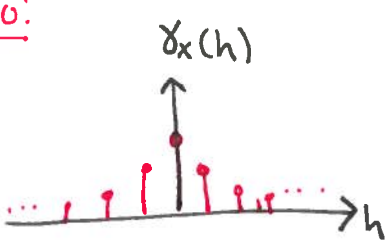
$$= \underline{\gamma_x(h)} \Rightarrow f_x(\omega) = \frac{\sigma^2}{2}\delta(\omega+\omega_0) + \frac{\sigma^2}{2}\delta(\omega-\omega_0)$$

Ex AR(1) process

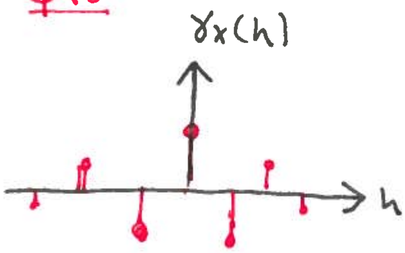
$$x_t = \varphi x_{t-1} + w_t, \quad w_t \sim wn(0, \sigma_w^2), \quad E\{x_t\} = 0, \quad |\varphi| < 1.$$

$$\Rightarrow \gamma_x(h) = \frac{\sigma_w^2}{1-\varphi^2}\varphi^{|h|} = a\varphi^{|h|}, \quad a = \frac{\sigma_w^2}{1-\varphi^2}$$

$\varphi > 0$:



$\varphi < 0$:



Solution:

$$f_x(\omega) = \sum_{h=-\infty}^{\infty} \gamma_x(h) e^{-j2\pi\omega h}$$

$$= a \sum_{h=-\infty}^{\infty} \varphi^{|h|} e^{-j2\pi\omega h}$$

$$= a \sum_{h=-\infty}^{-1} \varphi^{-h} e^{-j2\pi\omega h} + a \sum_{h=0}^{\infty} \varphi^{h} e^{-j2\pi\omega h}$$

$$= a \sum_{h=1}^{\infty} \varphi^{h} e^{j2\pi\omega h} + a \sum_{h=0}^{\infty} \varphi^{h} e^{-j2\pi\omega h}$$

$$= a\left[\sum_{h=0}^{\infty} \varphi^{h} e^{j2\pi\omega h} - 1\right] + a \sum_{h=0}^{\infty} \varphi^{h} e^{-j2\pi\omega h}$$

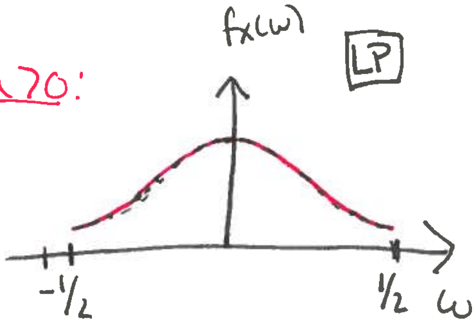
$$= a \sum_{h=0}^{\infty} \left[\varphi e^{j2\pi\omega}\right]^{h} + a \sum_{h=0}^{\infty} \left[\varphi e^{-j2\pi\omega}\right]^{h} - a$$

$$\overset{\text{G.S.}}{=} a\left[\frac{1}{1-\varphi e^{j2\pi\omega}} + \frac{1}{1-\varphi e^{-j2\pi\omega}} - 1\right]$$

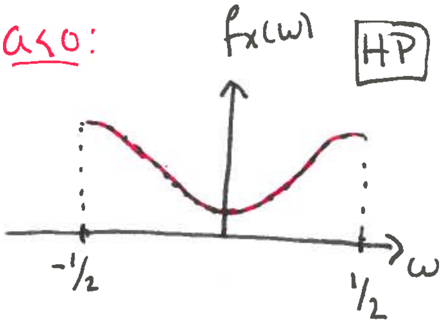
$$= a\,\frac{1-\varphi^2}{1-2\varphi\cos(2\pi\omega)+\varphi^2}$$

$$= \frac{\sigma_w^2}{1-2\varphi\cos(2\pi\omega)+\varphi^2}$$

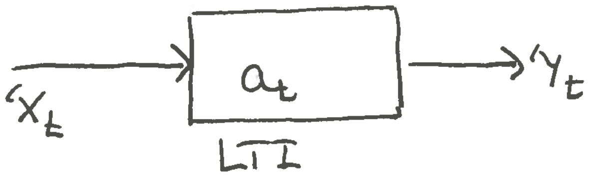
$\underline{a>0}$:



$\underline{a<0}$:



Linear filtering (LTI systems)



↳ A linear filter (LTI system) is completely characterized by its impulse response a_t.

↳ Output given by convolution

$$y_t = a_t * x_t = \sum_{k=-\infty}^{\infty} a_k x_{t-k}$$

Def. transfer function (freq. resp.)

$$A(\omega) = \mathrm{DTFT}\{a_t\} = \sum_{t=-\infty}^{\infty} a_t e^{-j2\pi\omega t}.$$

$$\left[\begin{array}{l} Y(\omega) = A(\omega)\,X(\omega) \\ \Rightarrow A(\omega) = \dfrac{Y(\omega)}{X(\omega)} \end{array}\right]$$

Mean function

$$\mu_y(t) = E\{y_t\} = \sum_{k=-\infty}^{\infty} a_k \underbrace{E\{x_{t-k}\}}_{\substack{\text{Stationary:} \\ \mu_x}}$$

$$= \mu_x \sum_{k=-\infty}^{\infty} a_k = \mu_x A(0) = \mu_y$$

No time dependency!

Autocovariance function

$$\gamma_Y(t+h,t) = E\{(Y_{t+h}-\mu_Y)(Y_t-\mu_Y)\}$$

$$= E\left\{\left(\sum_{r=-\infty}^{\infty} a_r X_{t+h-r} - \sum_{r=-\infty}^{\infty} a_r \mu_X\right)\left(\sum_{s=-\infty}^{\infty} a_s X_{t-s} - \sum_{s=-\infty}^{\infty} a_s \mu_X\right)\right\}$$

$$= E\left\{\sum_r a_r (X_{t+h-r}-\mu_X) \sum_s a_s (X_{t-s}-\mu_X)\right\}$$

$$= \sum_r \sum_s a_r a_s E\{(X_{t+h-r}-\mu_X)(X_{t-s}-\mu_X)\}$$

$$= \sum_r \sum_s a_r a_s \gamma_X(h-r+s)$$

$$= \gamma_Y(h)$$

$$\left(= a_h * a_{-h} * \gamma_X(h)\right)$$

Only depends on h!

$\Rightarrow$ Stationary

Spectral density

$$f_Y(\omega) = \mathrm{DTFT}\{\gamma_Y(h)\}$$

$$= \mathrm{DTFT}\left\{\sum_r \sum_s a_r a_s \gamma_X(h-r+s)\right\}$$

$$= \sum_r \sum_s a_r a_s \mathrm{DTFT}\{\gamma_X(h-(r-s))\}$$

<u>Note</u>: $DTFT\{\gamma(h-h_0)\} = e^{-j2\pi\omega h_0}\, DTFT\{\gamma(h)\}$

(exercise problem)

$$\Rightarrow f_Y(\omega) = \sum_r \sum_s a_r a_s e^{-j2\pi\omega(r-s)} f_X(\omega)$$

$$= \sum_r \sum_s a_r a_s e^{-j2\pi\omega r} e^{j2\pi\omega s} f_X(\omega)$$

$$= \sum_r a_r e^{-j2\pi\omega r} \sum_s a_s e^{j2\pi\omega s} f_X(\omega)$$

$$= \underbrace{\left(\sum_r a_r e^{-j2\pi\omega r}\right)}_{=A(\omega)} \underbrace{\left(\sum_s a_s e^{-j2\pi\omega s}\right)^*}_{=A^*(\omega)} f_X(\omega)$$

$$= A(\omega) A^*(\omega) f_X(\omega)$$

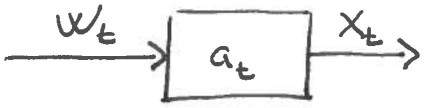
$$\boxed{= |A(\omega)|^2 f_X(\omega)}$$

<u>Filter</u>

Ex MA(1)

$$x_t = w_t + 0.5\,w_{t-1}, \qquad w_t \sim wn(0, \sigma_w^2).$$

Linear filter!



$$a_t = \begin{cases} 1; & t = 0 \\ 0.5; & t = 1 \\ 0; & \text{otherwise} \end{cases}$$

Have:

frequency response

$$A(\omega) = \text{DTFT}\{a_t\} = \sum_{t=-\infty}^{\infty} a_t e^{-j2\pi\omega t} = 1 \cdot e^{-j2\pi\omega\cdot 0} + 0.5\,e^{-j2\pi\omega\cdot 1}$$

$$= \underline{1 + 0.5\,e^{-j2\pi\omega}}.$$

$$\mu_x = A(0)\cdot\mu_w = 1.5\cdot\underbrace{\mu_w}_{=0} = \underline{0}.$$

$$f_x(\omega) = |A(\omega)|^2 \underbrace{f_w(\omega)}_{=\sigma_w^2}$$

$$= (1 + 0.5\,e^{-j2\pi\omega})(1 + 0.5\,e^{-j2\pi\omega})^* \sigma_w^2$$

$$= \sigma_w^2\Big(1 + \underbrace{0.5\,e^{j2\pi\omega} + 0.5\,e^{-j2\pi\omega}}_{=\cos(2\pi\omega)} + 0.25\underbrace{e^{j2\pi\omega}e^{-j2\pi\omega}}_{=1}\Big)$$

$$= \underline{\underline{\sigma_w^2\big(1.25 + \cos(2\pi\omega)\big)}}$$

The spectral density of ARMA (prop. 4.4)

If x_t is ARMA(p, q), its spectral density is given by

$$f_x(\omega) = \sigma_w^2 \frac{|1 + \sum_{k=1}^{q} \theta_k e^{-j2\pi\omega k}|^2}{|1 - \sum_{\ell=1}^{p} \varphi_\ell e^{-j2\pi\omega \ell}|^2} .$$

(proof as an exercise problem)

Ex AR(1) process

Use prop. 4.4!

$$f_x(\omega) = \sigma_w^2 \frac{1}{|1 - \varphi \cdot e^{-j2\pi\omega}|^2}$$

Have:

$$|1 - \varphi e^{-j2\pi\omega\ell}|^2 = (1 - \varphi e^{-j2\pi\omega})(1 - \varphi e^{-j2\pi\omega})^*$$

$$= 1 - \varphi e^{+j2\pi\omega} - \varphi e^{-j2\pi\omega} + \varphi^2 \underbrace{e^{-j2\pi\omega} e^{j2\pi\omega}}_{=1}$$

$$= 1 - \varphi[e^{j2\pi\omega} + e^{-j2\pi\omega}] + \varphi^2$$

$$= \underline{1 - 2\varphi \cos(2\pi\omega) + \varphi^2} .$$

$$\Rightarrow f_x(\omega) = \frac{\sigma_w^2}{\underline{\underline{1 - 2\varphi\cos(2\pi\omega) + \varphi^2}}}$$

(same as before ☺)