## Problem 1

We will now study the phenomenon of *spectral leakage*. We have seen that the *windowed periodogram* can be used to reduce the leakage, however, this lowers the *resolution*. The windowed periodogram is given by

$$\hat{S}_{XX}^{(per)}(f) = \frac{\Delta t}{NU} \left| \sum_{n=0}^{N-1} w_n x_n e^{-i2\pi f n \Delta t} \right|^2 \tag{1}$$

where $w_n$, $n = 0, 1, \ldots, N-1$, is called a data window ("taper"), and $U = \frac{1}{N} \sum_{n=0}^{N-1} w_n^2$.

**a)** A popular data window is the Hann (Hanning) window given by $w_n = \frac{1}{2}\left[1 - \cos(\frac{2\pi n}{N-1})\right]$ for $n = 0, 1, \ldots, N-1$. Plot the Hann window.

**b)** Estimate analytically $W(f) = \text{DTFT}\{w_n\}$ of the Hann window, and plot $|W(f)|^2$ (corresponding to $Q(f)$, which we have seen during lectures) on a dB–scale.

Generate $N = 32$ samples of the process $X_t = A\cos(2\pi f_0 t + \Theta) + W_t$, where $\Theta \sim U[0, 2\pi]$, $W_t \sim \mathcal{N}(0, \sigma_w^2)$ is independent of $\Theta$ with $\sigma_w = 0.5$, $A = 2$, $f_0 = 0.1 s^{-1}$ and $\Delta t = 1$s.

**c)** Plot the periodogram and the Hann window periodogram, and comment on the differences.

**d)** The Hamming window is also well–known. It is given by $w_n = 0.54 - 0.46\cos(\frac{2\pi n}{N-1})$. Plot this window.

**e)** Estimate analytically $W(f) = \text{DTFT}\{w_n\}$ of the Hamming window, and plot $|W(f)|^2$ on a dB–scale. Compare with the result in b).

**f)** Generate data from the process $X_t = \cos(2\pi f_0 t + \Theta) + 0.001\cos(2\pi f_1 t + \Theta)$, for $n = 0, 1, \ldots, N-1$, and $N = 128$. Here, $\Theta$ is defined as above, and $\Delta t = 1$s. Plot the periodogram and the Hamming window periodogram, and comment on possible differences.

During the lectures, we have seen that the periodogram (and the windowed periodogram) have variance–problems. This problem occurs because we have ignored the expectation operator when attempting to approximate the definition of the power spectral density. We shall see that an *averaging* of the power spectral density estimates can be useful to limit this problem.

Generate $N = 256$ samples of the process $X_t = A\cos(2\pi f_0 t + \Theta) + W_t$, where $\Theta \sim U[0, 2\pi]$, $W_t \sim \mathcal{N}(0, \sigma_w^2)$ is independent of $\Theta$ with $\sigma_w^2 = 0.5$, $A = 1/2$, $f_0 = 0.1 s^{-1}$ and $\Delta t = 1$s.

**g)** Segment the dataset in non–overlapping blocks of length $M = 32$, use a data window on each segment, and compute the average of the resulting estimates. Plot the result of this operation, and compare with the results above.

**h)** Repeat g), but use $M = 64$. Comment on the result.