2. Let $G = \mathbb{Z}$ and impose the following topology: $U \subseteq G$ is open if either $0 \notin U$ or $G - U$ is finite. Show that G is *not* a topological group with respect to this topology.

Proof. Suppose for sake of contradiction that G is a topological group, and let $\tau : G \to G$ denote the map $a \mapsto a + 1$, which is a homeomorphism of G.

Consider the set $U = \{-1\} \subseteq G$, which is open because $0 \notin U$. But $\tau(U) = \{0\}$ is not open because $0 \in \tau(U)$, and $G - \tau(U) = \mathbb{Z} - \{0\}$ is not finite. Since τ is a homeomorphism and hence an open map, this is a contradiction. $\square$

4. Give an example of a topological group with a closed subgroup that is *not* open.

Proof. Consider $G = \mathrm{GL}_n(\mathbb{C})$. Then $H = \mathrm{SL}_n(\mathbb{C})$ is a closed subgroup since it is the kernel of $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$. However, it cannot be open because G is connected, so cannot have a proper clopen subset.

$\square$

6. Let $G = \mathrm{GL}_n(\mathbb{R})$. Show that G^0 is the set of $n \times n$ matrices with positive determinant.

Proof. Let G^+ and G^- denote the $n \times n$ invertible matrices with positive and negative determinant respectively. Observe that G^+ is homeomorphic to G^- via the map which is multiplication by

$$\begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Hence it suffices to show that G^+ is connected, so that G^+ and G^- are the connected components of G. Clearly $e \in G^+$, so we then have $G^0 = G^+$. $\square$