K-THEORY OF C*-ALGEBRAS

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Abstract. We follow [1] to develop the elementary theory of C^*-algebras, objects which often arise in the study of linear operators on a Hilbert space. We then turn to operator K-theory, an analogue of topological K-theory which functorially assigns groups K_0 and K_1 to a C^*-algebra. We culminate with the Serre-Swan theorem, which shows that topological K-theory, as defined for compact Hausdorff spaces in [2], is a special case of operator K-theory of C^*-algebras.

Introduction

Topological K-theory allows for elegant proofs of many difficult results, so it is natural to ask in what way it can be generalized. In this paper, we present one such generalization: operator K-theory, which is a K-theory for a class of objects called C^*-algebras. C^*-algebras have a rich analytic theory and often arise in the study of Hilbert spaces. No familiarity with these objects will be assumed, though some examples will make reference to Hilbert spaces.

We present two formulations of operator K-theory: the analytic treatment, better suited to spectral analysis, is defined in terms of projections in a matrix algebra. More algebraically, operator K-theory can also be defined using projective modules. We will demonstrate the equivalence of these formulations.

We also state several important properties of operator K-theory, drawing comparisons with topological K-theory. Unfortunately, the proofs of these are rather involved, so we omit them to keep this paper as a concise summary. The reader can find a fully comprehensive development of (analytic) operator K-theory in [1].

Finally, we prove the Serre-Swan theorem, which makes precise the statement that "finitely-generated projective modules over a ring are like vector bundles on a topological space." This shows that operator K-theory is a true generalization of topological K-theory.

For an introduction to the theory of vector bundles and topological K-theory, we direct the reader to [2]. We will make free use of the results therein.

C^*-Algebras

The first two sections of this paper will be devoted to the basic ideas necessary to construct operator K-theory. With this in mind, let us define our main object of study:

Definition. A C^*-algebra $\mathcal{C}$ is a Banach algebra over $\mathbb{C}$ equipped with a map $* : \mathcal{C} \to \mathcal{C}$ (called the star map) satisfying the following axioms for any $a, b \in \mathcal{C}$ and $\alpha \in \mathbb{C}$:

(i) $(a^*)^* = a$
(ii) $(a+b)^* = a^* + b^*$
(iii) $(\alpha a)^* = \overline{\alpha} a^*$
(iv) $(ab)^* = b^* a^*$
(v) $\|a^* a\| = \|a\|^2$.

It is worth remarking that we do not require the multiplication in $\mathcal{C}$ to be commutative. If it is, we say that $\mathcal{C}$ is *abelian*.

If $\mathcal{C}$ has a multiplicative unit $\mathbf{1}$, it is called *unital*. In this case, it follows from (iv) that $\mathbf{1}^* = \mathbf{1}$.

Example 1. The following are important examples of C^*-algebras:

(a) For each $n \in \mathbb{Z}_{>0}$, the collection $M_n(\mathbb{C})$ of $n \times n$ complex matrices becomes a C^*-algebra when equipped with the conjugate transpose as $*$. More generally, for any C^*-algebra $\mathcal{C}$, $M_n(\mathcal{C})$ can be given the structure of a $\mathcal{C}^*$-algebra by taking the transpose and applying the involution entrywise. This C^*-algebra is generally not abelian, but it will be unital if $\mathcal{C}$ is.

(b) Fix a compact topological space X and denote by $C(X)$ the collection of continuous functions $X \to \mathbb{C}$. Then $C(X)$ becomes a C^*-algebra under pointwise addition, multiplication and conjugation. This example plays an important role in the Serre-Swan theorem.

For X not necessarily compact, $C(X)$ is not the correct object to consider. We replace it by $C_0(X)$, the space of maps that *vanish at infinity*, meaning that for every $\varepsilon > 0$, there exists some compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ for $x \notin K$. Clearly if X is compact then these two spaces coincide.

(c) A motivating example in the development of operator K-theory is the *Calkin algebra*. Fix an infinite-dimensional Hilbert space $\mathcal{H}$ and let $B(\mathcal{H})$ denote the set of bounded linear endomorphisms. When equipped with pointwise addition and function composition, $B(\mathcal{H})$ is a non-commutative algebra. We also give $B(\mathcal{H})$ the structure of a C^*-algebra by taking T^* to be the adjoint of T.

Inside $B(\mathcal{H})$ is the set $K(\mathcal{H})$ of compact linear endomorphisms, which is a two-sided ideal. Thus we can form the quotient $B(\mathcal{H})/K(\mathcal{H})$, and this inherits a C^*-algebra structure from $B(\mathcal{H})$. This quotient is the Calkin algebra.

We also define a *$*$-homomorphism* $\varphi : \mathcal{C} \to \mathcal{C}'$ to be a linear multiplicative map commuting with the involutions on $\mathcal{C}$ and $\mathcal{C}'$. Note that we do not require continuity of φ because it can be shown (by spectral analysis) that maps satisfying these axioms are continuous; for this reason, $*$-homomorphisms are often said to be *automatically continuous*. With this definition, it is easy to verify that C^*-algebras with $*$-homomorphisms form a category.

Projective Modules and Projections

Projective modules and projections also play an important role in any construction of operator K-theory. Recall that an R-module over a (not necessarily commutative or unital) ring R is called *projective* if it has the following property: for every morphism $g : P \to N$ and surjection $f : M \twoheadrightarrow N$, there exists some morphism $h : P \to M$ such that $f \circ h = g$. Such a map h is said to be a lift of g over f. Note that we do not require h to be unique, so this is not a universal property.

The information of a projective module is summarised by the diagram

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"'] & N \end{tikzcd}$$

This is the categorical definition of a projective object. However, in the category of R-modules, we also have the following characterization of projective modules:

Proposition 1. *A finitely-generated R-module P is projective if and only if it is a direct summand of some free R-module of finite rank. That is, if there exists some free R-module F and another R-module Q such that $F \cong P \oplus Q$.*

Proof. Suppose first that P is finitely-generated and projective. We will show that P is a direct summand of some free module. With this in mind, let $\pi : F \twoheadrightarrow P$ be a surjection from a free module of finite rank onto P. We then have the following diagram:

$$\begin{tikzcd} & F \arrow[d, two heads, "\pi"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "\mathrm{id}"'] & P \end{tikzcd}$$

Since P is projective, there exists some $h : P \to F$ which makes the diagram commute. We now consider the following short exact sequence:

$$\begin{tikzcd} \ker(\pi) \arrow[r, hook] & F \arrow[r, two heads, "\pi"] & P \arrow[l, dashed, bend left, "h"] \end{tikzcd}$$

The map h provides a section of π, so by the splitting lemma, $F \cong \ker(\pi) \oplus P$. Thus P is a summand of a free module of finite rank.

Conversely, suppose $F \cong P \oplus Q$ and let $i : P \hookrightarrow F$ and $p : F \twoheadrightarrow P$ be the inclusion and projection respectively. Suppose that we have a morphism $g : P \to N$ and a surjection $f : M \twoheadrightarrow N$. Because F is free (hence clearly projective), we have the following commutative diagram:

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ F \arrow[ur, dashed, "h"] \arrow[r, "g \circ p"'] & N \end{tikzcd}$$

This means that

$$\begin{aligned} f \circ h \circ i &= g \circ p \circ i \\ &= g \end{aligned}$$

Thus $h \circ i$ is a lift of g, so P is projective. It is finite rank because of the surjection $F \twoheadrightarrow P$. $\square$

The other concept we will need is that of a projection. An endomorphism $p \in M_n(\mathcal{C})$ is called a *projection* if $p^2 = p$. The proof of Proposition 1 shows that there is a relationship between projective modules and projections: given a projective module P, the map π gives a corresponding projection in $\mathrm{End}(F) \cong M_n(\mathcal{C})$. Conversely, if $p \in M_n(\mathcal{C})$ is a projection then $\mathrm{im}(p)$ is a projective module.

Unfortunately, this is not a one-to-one correspondence because we can always embed a projective module into multiple free modules. As a trivial example, take $P = \mathcal{C}$ embedding into the free modules $F_1 = \mathcal{C}$ and $F_2 = \mathcal{C} \oplus \mathcal{C}$. Then the two projections $p_1 = (1)$ and $p_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ both give P under the above construction.

To address this, we introduce the *Murray von Neumann* equivalence relation for projections. We say that two projections $p \in M_n(\mathcal{C})$ and $q \in M_m(\mathcal{C})$ are Murray von Neumann equivalent, written $p \sim_0 q$, if there exists some $v \in M_{m,n}(\mathcal{C})$ such that $p = v^*v$ and $q = vv^*$. One can verify that this defines an equivalence relation on the set of all finite-dimensional projections.

The importance of this equivalence relation stems from the fact that $\mathrm{im}(p)$ and $\mathrm{im}(q)$ are isomorphic as projective modules if and only if $p \sim_0 q$. The proof of this is elementary linear algebra, so we omit it; it can be found as Lemma A.4.4 in [5].

It is here that the relationship to spectral analysis also starts to become apparent; a linear operator $T : V \to V$ over a finite-dimensional complex inner product space is said to be *normal* if $TT^* = T^*T$. The finite-dimensional spectral theorem says that these are precisely the operators which can be diagonalised with respect to an orthonormal eigenbasis. One can think of Murray von Neumann equivalence as forcing projections to be normal.

There are several other equivalence relations that are often defined on projections, not because they are used in the construction of operator K-theory, but because it is easier to show that two objects are equivalent under these relations. For example, if $p, q \in M_1(\mathcal{C})$ are 1-dimensional projections, then $p \sim_0 q$ if and only if there exists a path between p and q. Since $M_1(M_n(\mathcal{C})) \cong M_n(\mathcal{C})$, this extends to any two projections of the same dimension. This is a useful tool for showing that two projections are equivalent.

Example 2. Let $\mathcal{H}$ be a separable and infinite-dimensional Hilbert space. Using the path equivalence defined above, one can show that two projections p, q over the C^*-algebra $K(\mathcal{H})$ are Murray von Neumann equivalent if and only if they are of the same dimension.

Operator K-theory

We now turn to K-theory fo C^*-algebras. For simplicity, we will only consider the case of unital C^*-algebras, since there are some subtleties surrounding K-theory of non-unital C^*-algebras. These essentially arise because there are not enough projections over a non-unital C^*-algebra, so the unital construction is always trivial.

As in the construction of topological K-theory, K_0 is defined by taking the Grothendieck completion of a certain abelian semigroup. To this end, define

$$\mathcal{P}_n(\mathcal{C}) := \mathcal{P}(M_n(\mathcal{C})) \qquad \text{and} \qquad \mathcal{P}_\infty(\mathcal{C}) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{C}).$$

One should intuitively think of $\mathcal{P}_n(\mathcal{C})$ as playing the role of $\mathrm{Vect}^n(X)$, the set of (isomorphism classes of) n-dimensional vector bundles. Then $\mathcal{P}_\infty(\mathcal{C})$ is analagous to $\mathrm{Vect}(X)$, the set of finite-dimensional bundles, no longer grouped by dimension. The Serre-Swan theorem makes precise this connection.

The set $\mathcal{P}_\infty(\mathcal{C})$ is "too large" to give a useful invariant, so we quotient by Murray von Neumman equivalence. Define

$$\mathcal{D}(\mathcal{C}) := \mathcal{P}_\infty(\mathcal{C}) / \sim_0 .$$

This procedure of quotienting to get a smaller object is analagous to taking isomorphism classes of vector bundles on a space, as opposed to the vector bundles themselves.

We can give $\mathcal{D}(\mathcal{C})$ the structure of an abelian semigroup under the direct sum operation $\oplus$: for $p, q \in \mathcal{D}(\mathcal{C})$, we set

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

This gives a well-defined semigroup operation on $\mathcal{D}(\mathcal{C})$. Furthermore, it is clear that if p, q are projections then $\mathrm{im}(p \oplus q) \cong \mathrm{im}(p) \oplus \mathrm{im}(q)$. Because of this correspondence, we can define K_0 in the following two equivalent ways:

$$K_0(\mathcal{C}) := \mathcal{G}(\mathcal{D}(\mathcal{C}), \oplus) \cong \mathcal{G}(\mathrm{ProjMod}, \oplus)$$

where ProjMod denotes the set of isomorphism classes of finitely-generated projective modules over $\mathcal{C}$.

Example 3. Example 2 says exactly that $\mathcal{D}(K(\mathcal{H})) \cong \mathbb{Z}_{>0}$, so $K_0(K(\mathcal{H})) \cong \mathbb{Z}$. One can also show that $\mathcal{D}(B(\mathcal{H})) \cong \mathbb{Z}_{>0} \cup \{\infty\}$ and hence $K_0(B(\mathcal{H})) = 0$.

To give K_0 the structure of a functor, we need to define it on morphisms. We will use the definition of K_0 via projective modules: for $\varphi : \mathcal{C} \to \mathcal{C}'$ a $*$-homomorphism and P a finitely-generated projective module over $\mathcal{C}$, we define

$$\begin{aligned} K_0(\varphi)(P) &= P \otimes_\varphi \mathcal{C}' \\ &= P \otimes_{\mathbb{C}} \mathcal{C}' / (ma) \otimes b \sim m \otimes (\varphi(a) b) \end{aligned}$$

Then $K_0(\varphi)(P)$ is a finitely-generated projective module over $\mathcal{C}'$. This turns K_0 into a functor from the category of C^*-algebras to the category of abelian groups. Note the subscript 0 because K_0 is a covariant functor, in contrast with the case of topological K-theory.

Analytically, this corresponds to taking a $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{C}'$ and inducing it to $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{C}')$ by entrywise application of φ. One can show that this takes projections to projections, and in fact induces a morphism $\mathcal{D}(\mathcal{C}) \to \mathcal{D}(\mathcal{C}')$. Hence by functoriality of the Grothendieck construction, we get a morphism $K_0(\mathcal{C}) \to K_0(\mathcal{C}')$. However, this construction takes significantly more work than in the algebraic case.

To define the higher K-theory groups, we would like an analogue of the suspension functor from topological K-theory. Because $\mathcal{C}$ is a topological space under the topology induced by the norm, it is reasonable to talk about the free loop space $\Omega\mathcal{C}$. Recall that this is the space of maps $S^1 \to \mathcal{C}$. One can verify that $\mathcal{C}$ induces a C^*-algebra structure on $\Omega\mathcal{C}$ under pointwise application of the involution. Thus we can define

$$K_n(\mathcal{C}) := K_0(\Omega^n\mathcal{C}).$$

Functoriality of K_n follows from functoriality of K_0 and Ω.

As in the topological K-theory setting, we have a Bott Periodicity theorem. This says that operator K-theory is 2-periodic i.e. $K_{n+2}(\mathcal{C}) \cong K_n(\mathcal{C})$ for all n. Thus only the groups K_0 and K_1 need to be considered.

We also have many of the other tools from topological K-theory available to us. One important example is the six-term exact sequence: if

$$0 \to \mathcal{C} \xrightarrow{\varphi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \to 0$$

is a short exact sequence of C^*-algebras, then there is an induced exact sequence in K-theory, given by

$$\begin{tikzcd}
K_0(\mathcal{C}) \arrow[r, "K_0(\varphi)"] & K_0(\mathcal{D}) \arrow[r, "K_0(\psi)"] & K_0(\mathcal{E}) \arrow[d, "\partial_0"] \\
K_1(\mathcal{E}) \arrow[u, "\partial_1"] & K_1(\mathcal{D}) \arrow[l, "K_1(\psi)"] & K_1(\mathcal{C}) \arrow[l, "K_1(\varphi)"]
\end{tikzcd}$$

The ∂_i are called the *index maps*. We will not give their construction because it relies on an alternate definition of $K_1(\mathcal{C})$ in terms of *unitary matrices*. These are elements $a \in M_n(\mathcal{C})$ such that $a^*a = aa^* = I_n$. Although this construction of $K_1(\mathcal{C})$ is quite similar to the analytic definition of $K_0(\mathcal{C})$, we will omit it because there is no obvious analogue for projective modules.

Remark 2. As a consequence of the Serre-Swan theorem, we will construct an explicit isomorphism between operator K-theory and topological K-theory (Corollary 13). Tracing the isomorphism in the operator Bott Periodicity theorem through to $K^n(X)$ gives exactly the standard isomorphism in topological Bott Periodicity. Similarly, all of the maps in the six-term exact sequence are taken to their topological variants under the Serre-Swan isomorphism.

The Serre-Swan Theorem

We now arrive at our main result, the Serre-Swan theorem. This relates the vector bundles on a compact Hausdorff space X to the projective modules over the ring $C(X)$, showing the relationship between the topological and operator K-theories.

Let $\pi : E \to X$ be a complex vector bundle on X and recall that a *section* of the vector bundle π is a map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. We denote the set of all such sections by $\Gamma(E)$, which can be given the structure of a $C(X)$-module in the following way: the statement that s is a section exactly says that $s(x) \in \pi^{-1}(x)$ for all x. Since $\pi^{-1}(x)$ is a $\mathbb{C}$-vector space, there is a notion of addition and scalar multiplication. Thus it makes sense to define $(s_1 + s_2)(x) = s_1(x) + s_2(x)$ and $(a \cdot s)(x) = a(x)s(x)$ for $a \in C(X)$.

For a morphism of vector bundles $f : E_1 \to E_2$, we also define $\Gamma(f) : \Gamma(E_1) \to \Gamma(E_2)$ by $\Gamma(f)(s) = f \circ s$. In this way, Γ is a covariant functor from the category of complex vector bundles on X to the category of finitely-generated modules over $C(X)$. The universal property of products shows that Γ is additive.

The Serre-Swan theorem can then be stated as:

Theorem 3. *Let X be a compact Hausdorff topological space. The functor Γ defines an equivalence of categories between the category of finite-dimensional complex vector bundles on X and the category of finitely-generated projective modules on $C(X)$.*

Remark 4. We could equally well take real vector bundles on X and continuous functions $X \to \mathbb{R}$. The proof is identical, so for simplicity we only consider the case $k = \mathbb{C}$.

Proof.

The proof we give here is due to Swan (see [3]), but rephrased in modern category theoretic language. To show that Γ is an equivalence of categories, we will show that it is faithful, full and essentially surjective. In the course of this proof, we will make heavy use of the fact that a compact Hausdorff space is normal, as well as some properties of normal topological spaces. Proofs of these facts can be found in [4].

Before we prove that Γ is an equivalence of categories, we first must show that $\Gamma(E)$ is a finitely-generated projective module, so that Γ actually defines a functor between the categories we claimed. Proposition 2.1 in [2] says that there is some complementary vector bundle E' such that $E \oplus E' \cong \varepsilon^m$, where $\varepsilon^m = X \times \mathbb{C}^m$ is the trivial m-dimensional vector bundle. Since Γ is additive, $\Gamma(E) \oplus \Gamma(E') \cong \Gamma(\varepsilon^m)$. Observe that $\Gamma(\varepsilon^m) \cong C(X)^m$ is a free module over $C(X)$, so $\Gamma(E)$ is a direct summand of a free module and hence projective by Proposition 1.

We now show that Γ is faithful. The underlying idea will be to show that locally defined sections can be extended to global sections in a way that respects local behaviour, so that we can compare morphisms pointwise.

Lemma 5. *Let $U \subseteq X$ be an open neighbourhood of x and let $s : U \to E$ be a local section of π. Then there exists a global section $r : X \to E$ such that r and s agree on some neighbourhood of x.*

Proof. Let $V, W \ni x$ be such that $\overline{V} \subseteq U$ and $\overline{W} \subseteq V$; these exist by Proposition 2.2 in [4]. Then $\overline{W} \cap (X \backslash V) = \emptyset$ so by Urysohn's Lemma, there exists some $\omega : X \to \mathbb{R}$ such that $\omega \mid_{\overline{W}} = 1$ and $\omega \mid_{X \backslash V} = 0$. Define r by

$$r(y) = \begin{cases} \omega(y)s(y) & \text{if } y \in U \\ 0_y & \text{if } y \notin U \end{cases}$$

where 0_y denotes the zero element in the fiber $\pi^{-1}(y)$. The map r is continuous by the assumption that $\omega \mid_{X \backslash V} = 0$ and $\overline{V} \subseteq U$, so r continuously becomes 0 before moving outside U.

It is also clear that r is a global section of π, and $r \mid_W = s \mid_W$ by the assumption that $\omega \mid_{\overline{W}} = 1$, so r is the required global extension of s.

$\square$

Corollary 6. Γ *is faithful.*

Proof. Suppose that $f, g : E_1 \to E_2$ are two morphisms of vector bundles such that $\Gamma(f) = \Gamma(g)$. To show that $f = g$, let $e \in E_1$ and $x = \pi_1(e)$. Using local trivializations of π_1, we can find some $U \ni x$ and a local section $s : U \to E_1$ of π_1 such that $s(x) = e$. By Lemma 5, there exists a global section $r : X \to E_1$ such that $r(x) = s(x) = e$. Then

$$\begin{aligned} f(e) &= f(r(x)) \\ &= (f \circ r)(x) \\ &= \Gamma(f)(r)(x) \\ &= \Gamma(g)(r)(x) \\ &= g(e) \end{aligned}$$

Since $e \in E_1$ was arbitrary, $f = g$. $\square$

We now turn to the problem of showing that Γ is full. The idea will again be to work locally, where this time locally means on fibers. Using local trivializations of π, we see that for every $x \in X$, there is some $U \ni x$ and local sections $s_1, ..., s_n : U \to E$ such that $s_1(y), ..., s_n(y)$ form a $\mathbb{C}$-basis for $\pi^{-1}(y)$ for every $y \in U$. We will call $s_1, ..., s_n$ a *local base at* x. For any other local section $s : U \to E$, we can write $s(y) = a_1(y)s_1(y) + ... + a_n(y)s_n(y)$ where $y \in U$ and $a_i(y) \in \mathbb{C}$. In this sense, a local base at x also forms a base for local sections.

Using Lemma 5, we have the following:

Corollary 7. *For every* $x \in X$*, there exist global sections* $s_1, ..., s_n \in \Gamma(E)$ *which form a local base at* x*.*

Proof.

Take local sections $s_1, ..., s_n : U \to E$ which form a local base at x. By Lemma 5, these extend to global sections $r_1, ..., r_n : X \to E$ which agree with the s_i when restricted to some neighbourhood $V \ni x$. Then $r_1(y), ..., r_n(y)$ form a $\mathbb{C}$-basis for $\pi^{-1}(y)$ for every $y \in V$ and hence are also a local base at x.

$\square$

For $x \in X$, let $\varepsilon_x : C(X) \to \mathbb{C}$ be the evaluation at x map, with $\delta_x : \Gamma(E) \to \mathbb{C}$ defined similarly. We will use these to get a map on fibers, given a morphism of projective modules. First, though, let us establish a relationship between δ_x and ε_x.

Lemma 8. *For every $x \in X$, $\ker(\varepsilon_x)\Gamma(E) = \ker(\delta_x)$. In words, the sections which vanish at x are exactly those which have coefficients in $C(X)$ which vanish at x.*

Proof.
It is immediate that $\ker(\varepsilon_x)\Gamma(E) \subseteq \ker(\delta_x)$ because all coefficients vanish at x. Conversely, suppose that $s \in \ker(\delta_x)$, so $s(x) = 0$. By Corollary 7, there exist global sections $s_1, ..., s_n \in \Gamma(E)$ which form a local base at x. Write $s(y) = b_1(y)s_1(y) + ... + b_n(y)s_n(y)$ for y in a neighbourhood of x and $b_i(y) \in \mathbb{C}$. A section of the trivial bundle is just a function $X \to \mathbb{C}$, so the b_i are local sections of the trivial bundle ε^1. By Lemma 5, there exist global sections $a_1, ..., a_n \in \Gamma(\varepsilon^1)$ which agree with the b_i when restricted to some neighbourhood $U \ni x$. As sections of ε^1, we can consider the a_i as maps $X \to \mathbb{C}$ and hence $a_i \in C(X)$.

Define $s' = s - \sum_i a_i s_i \in \Gamma(E)$. Since X is normal, we can find some $V \ni x$ such that $\overline{V} \subseteq U$ and some $a \in C(X)$ such that $a(x) = 0$ and $a \mid_{X\setminus V}= 1$. By checking pointwise, one can verify that

$$s = as' + \sum_i a_i s_i.$$

Because the $s_i(x)$ form a $\mathbb{C}$-basis for $\pi^{-1}(x)$, we can only have $s(x) = 0$ if $a_i(x) = 0$ for all i. Since we also have $a(x) = 0$ by construction, we have written s as a $C(X)$-linear combination of global sections, with coefficients in $\ker(\varepsilon_x)$. Thus $s \in \ker(\varepsilon_x)\Gamma(E)$. $\square$

Corollary 9. Γ *is full.*

Proof.
Let $F : \Gamma(E_1) \to \Gamma(E_2)$ be a $C(X)$-module morphism. By composing with the quotient map, we get $F : \Gamma(E_1) \to \Gamma(E_2)/\ker(\delta_x)$. We claim that $\ker(\delta_x) \subseteq \ker(F)$, so F descends to a map $F_x : \Gamma(E_1)/\ker(\delta_x) \to \Gamma(E_2)/\ker(\delta_x)$.

Given some $s \in \ker(\delta_x)$, Lemma 8 says that we can write $s = a_1s_1 + ... + a_ns_n$ for $a_i \in \ker(\varepsilon_x)$ and $s_i \in \Gamma(E_1)$. Then

$$F(s) = F(a_1s_1 + ... + a_ns_n) = a_1F(s_1) + ... + a_nF(s_n).$$

Thus $F(s) \in \ker(\varepsilon_x)\Gamma(E_2) = \ker(\delta_x)$ and so $F(s) = 0$ in the quotient, as claimed.

Now observe that the map

$$\begin{aligned} \Gamma(E_i) &\to \pi_i^{-1}(x) \\ s &\mapsto s(x) \end{aligned}$$

is a surjective linear map and has kernel exactly $\ker(\delta_x)$. Thus
$\Gamma(E_i)/\ker(\delta_x) \cong \pi_i^{-1}(x)$ and so we have a linear map $F_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x)$.

We now define $f : E_1 \to E_2$ on fibers by $f \mid_{\pi_1^{-1}(x)}= F_x$. This commutes with the bundle maps because it takes fibers over π_1 to fibers over π_2, and

$$\Gamma(f)(s)(x) = F_x(s(x)) = F(s)(x)$$

so $\Gamma(f) = F$. Thus it only remains to show that f is continuous, so that f is a morphism of vector bundles.

Let $x \in X$ and let $s_1, ..., s_n \in \Gamma(E_1)$ be a local base at x. Then for $e \in E_1$ such that $\pi_1(e)$ is near x, we can write

$$e = \sum_i a_i(e)s_i(\pi_1(e))$$

where the a_i are continuous $\mathbb{C}$-valued functions. Because f is linear on fibers, this gives

$$f(e) = \sum_i a_i(e)(f \circ s_i)(\pi_1(e)) = \sum_i a_i(e)F(s_i)(\pi_1(e)).$$

Because F is a morphism $\Gamma(E_1) \to \Gamma(E_2)$, $F(s_i)$ is a continuous section of π_2 and so we have written f as a linear combination of continuous functions. Thus f is continuous in a neighbourhood of every point and hence is globally continuous.

□

Finally, we turn to the problem of showing that Γ is essentially surjective. The main difficulty will lie in the fact that the category of vector bundles on X is not abelian, since the kernel and image of a morphism need not be subbundles. However, we will show that for f a morphism such that $f^2 = f$ (intuitively, a projection), in fact the image is a subbundle.

Lemma 10. *Let $f : E_1 \to E_2$ be a morphism of vector bundles and suppose that the dimensions of the fibers of* $\mathrm{im}(f)$ *are locally constant. Then* $\mathrm{im}(f)$ *is a subbundle of* E_2.

Proof.
Let $x \in X$. We will construct an open neighbourhood $U \ni x$ over which $\mathrm{im}(f)$ is locally trivial.

Let $s_1, ..., s_n \in \Gamma(E_1)$ be a local base for E_1 at x, with $t_1, ..., t_m \in \Gamma(E_2)$ a local base for E_2 at x. Let k be the dimension of $\mathrm{im}(f)_x$. By renumbering the s_i if necessary, we may assume that $(f \circ s_1)(x), ..., (f \circ s_k)(x)$ span $\mathrm{im}(f)_x$. Hence the same k vectors are linearly independent in $\mathrm{im}(f)_x$.

We now consider these vectors as living in the fiber $\pi_2^{-1}(x) \supseteq \mathrm{im}(f)_x$, where they are still linearly independent. By renumbering again if necessary, we may assume that $(f \circ s_1)(x), ..., (f \circ s_k)(x), t_{k+1}(x), ..., t_m(x)$ are linearly independent. For sake of notation, define

$$r_i = \begin{cases} f \circ s_i & \text{if } 1 \leq i \leq k \\ t_i & \text{if } k < i \leq m \end{cases}$$

so $r_1(x), ..., r_m(x)$ are linearly independent in $\pi_2^{-1}(x)$. We claim that $r_1(y), ..., r_m(y)$ are linearly independent in $\pi_2^{-1}(y)$ for y in a neighbourhood of x.

Using our local base $t_1, ..., t_m$ at x, we can write

$$r_i(y) = \sum_j a_{i,j}(y)t_j(y)$$

for y in a neighbourhood of x and $a_{i,j}(y) \in \mathbb{C}$. The statement that $r_1(x), ..., r_m(x)$ is linearly independent then means that the matrix $(a_{i,j}(x))$ is invertible. Since the dimensions of the fibers are locally constant, the function $y \mapsto (a_{i,j}(y))$ is a continuous function into some fixed matrix algebra. By continuity of the determinant and this function, there is some neighbourhood $U \ni x$ on which the same matrix is invertible, and hence the elements $r_1(y), ..., r_m(y)$ are linearly independent for $y \in U$. By restricting focus to $r_1(y), ..., r_k(y)$, this gives a local trivialization of $\mathrm{im}(f)$ over x for each $x \in X$, and the corresponding U form an open cover of X. Thus $\mathrm{im}(f)$ is a vector bundle.

□

Remark 11. If $\dim_{\mathbb{C}}(\mathrm{im}(f)_x) = m$ then the proof of Lemma 10 shows, without any other hypotheses on the dimensions of the fibers, that $\dim_{\mathbb{C}}(\mathrm{im}(f)_y) \geq m$ for y in a neighbourhood of x.

Corollary 12. *Suppose $f : E \to E$ is an endomorphism of vector bundles such that $f^2 = f$. Then $\mathrm{im}(f)$ is a subbundle of E.*

Proof. By Lemma 10, it suffices to show that $x \mapsto \dim_{\mathbb{C}}(\mathrm{im}(f)_x)$ is a locally constant function. Since $f^2 = f$, $\ker(f)_x = \mathrm{im}(1-f)_x$ and hence

$$\pi^{-1}(x) = \mathrm{im}(f)_x \oplus \mathrm{im}(1-f)_x.$$

Let $h = \dim_{\mathbb{C}}(\mathrm{im}(f)_x), k = \dim_{\mathbb{C}}(\mathrm{im}(1-f)_x)$. By Remark 11 applied to f and $1-f$ respectively, we have that $\dim_{\mathbb{C}}(\mathrm{im}(f)_y) \geq h$ and $\dim_{\mathbb{C}}(\mathrm{im}(f)_y) \geq k$ for all y in a neighbourhood of x. But we also know that $\dim_{\mathbb{C}}(\mathrm{im}(f)_y) + \dim_{\mathbb{C}}(\mathrm{im}(f)_y) = \dim_{\mathbb{C}}(\pi^{-1}(y)) = h + k$, which is locally constant. Thus $\dim_{\mathbb{C}}(\mathrm{im}(f)_x)$ is locally constant and so $\mathrm{im}(f)$ is a subbundle. □

With this corollary, we can finally show that Γ is essentially surjective. Let P be a finitely-generated projective module. By Proposition 1, we can embed P into a free module F; write $F \cong P \oplus Q$. Let $p \in \mathrm{End}(F)$ be projection onto P, so $p^2 = p$.

Suppose F has rank m. Then $F \cong \Gamma(\varepsilon^m)$, so we can consider p as a morphism $\Gamma(\varepsilon^m) \to \Gamma(\varepsilon^m)$. Because Γ is full, there exists some $f : \varepsilon^m \to \varepsilon^m$ such that $\Gamma(f) = p$. Since Γ is faithful, $f^2 = f$ and hence by Corollary 12, $\mathrm{im}(f)$ is a subbundle. Thus we conclude that

$$P \cong \mathrm{im}(p) \cong \mathrm{im}(\Gamma(f)) \cong \Gamma(\mathrm{im}(f))$$

and so P is isomorphic to an object in the image of Γ, as required.

□

Some consider the Serre-Swan theorem to have more philosophical than practical applications, using it to justify the definition of algebraic vector bundles and operator K-theory. For us, however, there is one immediate benefit:

Corollary 13. *Let X be a compact Hausdorff space. Then $K^0(X) \cong K_0(C(X))$.*

Proof. We use the algebraic definition of $K_0(C(X))$, in terms of projective modules. Because Γ is additive, it induces a group homomorphism $K^0(X) \to K_0(C(X))$. This homomorphism is surjective because Γ is essentially surjective, and is injective because Γ is injective on isomorphism classes. □

Thus operator K-theory is a true generalization of topological K-theory. This is a powerful statement, because it turns often difficult topological problems into algebraic ones. The trade-off is that the topological work is moved into the proof of the Serre-Swan theorem. Having completed this proof, though, we now have access to all results from operator K-theory.

This concludes our summary of operator K-theory. Although we have (necessarily) omitted many details, the comparisons with topological K-theory should hopefully provide some motivation for why operator K-theory is defined the way it is. This paper should also provide a framework for the interested reader to pursue further investigation on their own.

References

[1] R. Matthes, W. Szymański. (2017) *Lecture Notes on the K-theory of Operator Algebras*. Available at `https://www.ime.usp.br/~toscano/disc/k-teoria/KtheoryMatthesSzymanski.pdf`.

[2] A. Hatcher. (2017) *Vector Bundles and K-Theory*. Available at `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`.

[3] R. Swan. (1962) *Vector Bundles and Projective Modules*. Available at `https://www.ams.org/journals/tran/1962-105-02/S0002-9947-1962-0143225-6/S0002-9947-1962-0143225-6.pdf`.

[4] University of Toronto. (2018) *Urysohn's Lemma*. Available at `http://www.math.toronto.edu/ivan/mat327/docs/notes/13-urysohn.pdf`.

[5] N. Higson, J. Roe. (2000) *Analytic K-Homology*. Published by OUP Oxford.

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