# Deligne–Lusztig theory & character sheaves

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# Declaration

The work in this thesis is my own except where otherwise stated.

Martin Skilleter

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Finally, to my partner Isabel: you made this year bearable, and not only bearable but fun. Talking to you is the best part of my day and I am so grateful for having you in my life.

# Abstract

In this thesis, we discuss techniques in geometric representation theory which construct irreducible characters of finite groups of Lie type. Specifically, we will explore Deligne–Lusztig theory and character sheaves, two approaches to this problem.

By defining the étale topology, we give the geometric background necessary to understand the setting for these constructions. We then develop the theory of Deligne–Lusztig characters and, in particular, prove that the construction is exhaustive, in the sense that every irreducible character appears as a constituent of some Deligne–Lusztig character. We conclude by relating character sheaves and the Deligne–Lusztig characters, and discussing how the additional structure of character sheaves assists in decomposing the Deligne–Lusztig characters into their irreducible constituents.

# Contents

# Notation & Terminology

### Notation

| | |
|---|---|
| $p$ | A prime number. |
| $q$ | A power of $p$. |
| $G$ | A connected reductive linear algebraic group. |
| $F$ | A Frobenius endomorphism $G \to G$. |
| $B$ | A Borel subgroup of $G$, see Definition 2.3. |
| $U$ | The unipotent radical of $B$, see Definition 2.6. |
| $T$ | A maximal torus contained in $B$. |
| $\mathbb{1}$ | The trivial irrep. |
| $\theta$ | A character of $T$. |
| $E_{i,j}$ | The $n \times n$ matrix with 1 in the $(i, j)$ entry and all other entries 0. |
| $\overline{x}$ | A geometric point in a $k$-scheme $X$, which is a morphism $\mathrm{Spec}(\overline{k}) \to X$. |
| $\mathbb{C}[H]$ | The group algebra of a finite group $H$. |
| $C(t)$ | The centraliser (in $G$) of a semisimple element $t \in G$. |
| $C^0(t)$ | The connected component of the identity in $C(t)$. |
| ${}^xT$ | Shorthand for the conjugate $xTx^{-1}$. |
| $\mathrm{ad}(x)(\theta)$ | The character of ${}^xT$ defined by $\mathrm{ad}(x)(\theta)(xtx^{-1}) := \theta(t)$. |

| | |
|---|---|
| $N(T,S)$ | The set of $x \in G$ for which ${}^xS = T$. |
| $\mathcal{F}_{\overline{x}}$ | The stalk of the étale sheaf $\mathcal{F}$ at $\overline{x}$. |

**Terminology**

| | |
|---|---|
| Irrep | Shorthand for "irreducible representation". We will also refer to irreducible characters as irreps. |
| Solvable | A group $G$ is solvable if it has a finite resolution $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_k = G$ such that $G_i/G_{i-1}$ is abelian for all $i$. |

# Chapter 1

# Introduction

Constructing representations of finite simple groups is a difficult problem; often some insight is required to find exceptional representations, and such insights rarely generalise even to the finite simple groups in the same family. In this thesis, we study *Deligne–Lusztig theory* and *character sheaves*, two sophisticated methods of constructing representations for a particular class of finite groups called *finite groups of Lie type*. Among the infinite families of finite simple groups, all but the cyclic groups and alternating groups are of Lie type, so these procedures are widely applicable.

Finite groups of Lie type are the fixed points under Frobenius maps of certain groups over $\overline{\mathbb{F}}_p$, called *reductive groups*. The benefit of this perspective is that reductive groups come equipped with geometric structure, since they are also affine varieties in the Zariski topology. Additionally, by varying the Frobenius map, we can treat all the finite groups of fixed points on equal footing. Deligne–Lusztig theory reduces the problem of finding representations of an infinite family to studying one fixed reductive group, which parametrizes the whole family.

While this construction is theoretically exhaustive, there is a serious obstacle in practice: although every irreducible character is contained in some Deligne–Lusztig character, there are no general tools for decomposing the Deligne–Lusztig characters into their irreducible constituents. This problem can be partially overcome by associating to the Deligne–Lusztig characters objects with more geometric structure, called character sheaves. Decomposing character sheaves into their simple subsheaves induces a decomposition of the Deligne–Lusztig characters, and there are more tools for finding a decomposition of the former than the latter.

The purpose of this thesis is to give an approachable introduction to Deligne–Lusztig theory and character sheaves. Especially in later chapters, we eschew proofs in favour of examples and motivation. Lusztig's original program to construct character sheaves spanned 12 papers and nearly 1,000 pages, and those contain fully rigorous proofs of all the results we state. On the other hand, the literature is sorely lacking in any sort of motivation.

In our second chapter, we give a survey of the structure theory of linear algebraic groups as background. Rather than providing proofs, we discuss the specific case of the general linear group $\mathrm{GL}_n$, where the results can all be seen very concretely using only elementary linear algebra. We also study the main precursor to Deligne–Lusztig theory, called parabolic induction, as motivation for the constructions in later chapters.

In our third chapter, we turn to geometry to discuss the étale topology, a Grothendieck topology which is necessary in the construction of both Deligne–Lusztig theory and character sheaves. We will again aim to motivate various definitions by drawing comparisons with standard constructions in topology, with the hope that the reader will develop some intuition regarding the objects in question.

In our fourth chapter, we define the Deligne–Lusztig characters and discuss several important results. As this is high-level algebraic geometry, the proofs are quite technical and so for brevity we do not include them all. Instead, we include a subset of the proofs which are illustrative of the types of techniques which appear in the field, or which elucidate the connection to standard representation theory.

In our final chapter, we introduce the notion of character sheaves, which are a particular class of perverse sheaves. We give two different constructions of character sheaves and discuss how they are related, then outline how to recover the Deligne–Lusztig characters from character sheaves by taking functions on stalks.

As background for this thesis, we assume knowledge of the standard representation theory of finite groups. For an introduction to this area, we refer the reader to Chapters 1 and 4 of [2]. We will also need techniques from both modern and classical algebraic geometry, of which a comprehensive summary can be found in [3].

# Chapter 2

# Linear algebraic groups

We begin with an introduction to linear algebraic groups. Unless otherwise specified, references are to [1], which is an excellent resource in this area.

## 2.1 Structure theory

Throughout this section, let $k$ be an algebraically closed field. We will soon specialise to $k = \overline{\mathbb{F}}_p$, but the first several results are independent of characteristic.

**Definition 2.1.** A linear algebraic group $G$ over $k$ is a group which is also an affine variety over $k$, such that multiplication and inversion are regular maps.

**Example 2.2.** The quintessential example of a linear algebraic group is $\mathrm{GL}_n(k)$. We can view $\mathrm{GL}_n(k)$ as an affine variety by embedding it into affine space $\mathbb{A}_k^{n^2+1}$ as

$$\{(M, y) \in \mathbb{A}_k^{n^2} \times \mathbb{A}_k^1 : \det(M)y = 1\}.$$

Projection onto $\mathbb{A}_k^{n^2}$ then gives an isomorphism to the standard realization of $\mathrm{GL}_n(k)$ inside $\mathbb{A}_k^{n^2}$. Matrix multiplication is clearly a polynomial in the entries of the matrices, and inversion is too by expressing the inverse in terms of the adjugate.

A priori, linear algebraic groups need not take the form of matrix groups. Indeed, another name for such groups is *affine algebraic groups*. However, an algebraic group is affine if and only if it admits an embedding as a closed subgroup of $\mathrm{GL}_n(k)$ for some $n$ (2.4.4)[†]. It is this latter perspective which will be more useful to us, hence our choice of naming convention.

[†]As a reminder, unspecified references are to [1].

The additional geometric information means that linear algebraic groups have many structural properties not found in other groups. Since a common theme in representation theory is building representations from subgroups, it will be important later to know these structural results.

**Definition 2.3.** A Borel subgroup of a linear algebraic group $G$ is a closed connected solvable subgroup which is maximal with respect to these properties.

**Example 2.4.** Once again consider $G = \mathrm{GL}_n(k)$. We claim that the subgroup $B \leq G$ of upper triangular matrices is a Borel. It is clear that $B$ is closed and connected. To show solvability, we produce a subnormal series with abelian quotients.

Observe that we have a surjective homomorphism $\pi : B \twoheadrightarrow (k^\times)^n$ given by projection onto the diagonal. We let $U = \ker(\pi)$ be the subgroup consisting of matrices with all 1s on the diagonal; such matrices are said to be *unipotent* (more on these later). Since $B/U \cong (k^\times)^n$ is abelian, it suffices to show that $U$ is solvable.

We define a subnormal series for $U$ inductively: setting $N_0 = U$, the projection from $N_l$ onto the $(l+1)$-diagonal is a surjective homomorphism $N_l \twoheadrightarrow k^{n-(l+1)}$. We take $N_{l+1}$ to be the kernel of this projection. The series

$$\{1\} \trianglelefteq N_{n-2} \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq U$$

shows that $U$ is solvable.

The proof of maximality of $B$ relies on the fact that every connected solvable subgroup $H$ of $\mathrm{GL}_n(k)$ is conjugate to a subgroup of $B$, which is not a hard result, but the proof is difficult to summarize. We therefore refer the reader to Theorem 4.1 in [7] for details. □

Understanding all of the Borel subgroups will be quite important when constructing representations. Fortunately, relating Borel subgroups is easily accomplished by the following result:

**Theorem 2.5.** *(Borel, 3.4.3)*

*Let $G$ be a linear algebraic group. Then all Borel subgroups of $G$ are conjugate.*

In particular, all Borel subgroups of $\mathrm{GL}_n(k)$ are conjugate to the upper triangular matrices, which are therefore usually called the *standard Borel*. We can add to the above theorem by remarking that Borel subgroups are also self-normalizing (Theorem 0.11.iv in [13]), though this is not normally included in the statement of Borel's theorem since it is comparatively much easier to prove.

As we saw in Example 2.4, the unipotent matrices play an important role in understanding the structure of the Borel $B$. This is not an isolated phenomenon.

**Definition 2.6.** First suppose that $G \subseteq \mathrm{GL}_n(k)$ is a linear algebraic group with a specific embedding as matrices. We say that a matrix $u \in G$ is *unipotent* if all its eigenvalues are 1. This property turns out to be independent of the choice of embedding (see 5.1 in [32]) so we say that an element of an abstract linear algebraic group is unipotent if it is unipotent in any embedding. A subgroup of $G$ is called unipotent if all its elements are unipotent.

There is a distinguished unipotent subgroup, called the *unipotent radical of* $G$ and denoted $R_u(G)$, which is defined to be the maximal connected normal unipotent subgroup of $G$ (the existence and uniqueness of such a subgroup is Proposition 0.16.ii in [13]). It is given explicitly by taking the set of all unipotent elements inside the identity component of the maximal normal solvable subgroup of $G$ (Proposition 0.32.i in [13]).

**Example 2.7.** Let us look at a concrete example of a unipotent group by computing the unipotent radical of $G = \mathrm{GL}_n(k)$. Any connected solvable subgroup $N$ is contained in some Borel subgroup, so by Borel's theorem $N$ is conjugate to a subgroup of the upper triangular matrices. But if $N$ is also normal then it is its own conjugate, so all matrices in $N$ must be upper triangular. Because the lower triangular matrices in $\mathrm{GL}_n(k)$ are also a Borel subgroup, the same argument shows that $N$ must also be lower triangular. Thus any such $N$ consists of diagonal matrices.

By conjugating with matrices of the form $I_n + E_{i,j}$, one easily verifies that any normal subgroup of diagonal matrices must actually consist of scalar matrices. The only unipotent scalar matrix is the identity, so we conclude that $R_u(\mathrm{GL}_n(k)) = \{I_n\}$. □

Groups with trivial unipotent radical are called *reductive*, and they parametrize the finite groups of Lie type. Precisely, let $G$ be a connected reductive group over $\overline{\mathbb{F}}_p$ and let $F : G \to G$ be a Frobenius map, meaning $F$ is the restriction of some standard Frobenius map $F_q$ on $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$, which raises all entries to some $q^{th}$ power. Then the group of fixed points $G^F$ is a finite group of Lie type.

Example 2.7 therefore says that $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is reductive, so $\mathrm{GL}_n(\mathbb{F}_q) = \mathrm{GL}_n(\overline{\mathbb{F}}_p)^{F_q}$ is a finite group of Lie type. Other examples include $\mathrm{SL}_n(\mathbb{F}_q)$, the symplectic group $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, the special orthogonal group $\mathrm{SO}_n(\mathbb{F}_q)$ and more generally, almost every infinite family among the finite simple groups. We will discuss Frobenius maps and finite groups of Lie type further in Chapter 4.

The full strength of the structure theory of linear algebraic groups does not come from unipotent elements alone; there is another closely related concept which we will utilise heavily.

**Definition 2.8.** For a linear algebraic group $G$, pick an embedding $G \subseteq \mathrm{GL}_n(k)$. We say that a matrix $s \in G$ is *semisimple* if it is diagonalisable. This is independent of the choice of embedding (see 5.1 in [32]), so we may speak of an element of an abstract linear algebraic group being semisimple.

The importance of semisimple and unipotent elements stems largely from the following result, known as the *Jordan decomposition.* This often allows us to reduce calculations to the semisimple and unipotent cases, and will play a central role when developing the character formulae in Chapters 4 and 5.

**Theorem 2.9.** *(Jordan decomposition, pg. 206)*

*Let $G$ be a linear algebraic group over $k$. Then every element $g \in G$ can be written uniquely as $g = us = su$ where $u$ and $s$ are unipotent and semisimple elements of $G$ respectively.*

For the rest of the chapter, we specialise to $k = \overline{\mathbb{F}}_p$. Observe that

$$\mathrm{GL}_n(\overline{\mathbb{F}}_p) = \bigcup_{d \geq 1} \mathrm{GL}_n(\mathbb{F}_{p^d})$$

so every element of a linear algebraic group $G$ over $\overline{\mathbb{F}}_p$ lies in some finite subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$, and hence has finite order.

Over $\overline{\mathbb{F}}_p$, semisimplicity and unipotence have equivalent characterisations in terms of order which are often easier to work with. Precisely, a matrix $g \in \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is semisimple if and only if its order is coprime to $p$, and is unipotent if and only if its order is a power of $p$ (3.18 in [13]).

Elements of $G$ having finite order allows us to use techniques from the theory of finite groups. As an example of this, we will soon state a geometric analogue of the Schur-Zassenhaus theorem, which gives a sufficient condition for a finite group to decompose as a semidirect product of a normal subgroup and its quotient. Unlike the case of finite groups, however, we can say something stronger about the non-normal factor in this decomposition.

**Definition 2.10.** A linear algebraic group $T$ over $\overline{\mathbb{F}}_p$ is called a *torus (of rank d)* if $T \cong (\overline{\mathbb{F}}_p^\times)^d$. We say that $T$ is *split over* $\mathbb{F}_q$ if $T^{F_q} \cong (\mathbb{F}_q^\times)^d$, where $F_q$ is the $q^{th}$ power Frobenius.

We have already seen an example of a split torus, namely the group $B/U$ in Example 2.4, which is naturally identified with the diagonal matrices in $B$. Moreover, since every upper triangular matrix can be uniquely expressed as the product of a unipotent matrix and a diagonal matrix, it follows that $B = U \rtimes T$. The next theorem generalises this considerably.

**Theorem 2.11.** *(3.5.6)*
*Let $B$ be a connected solvable linear algebraic group over $\overline{\mathbb{F}}_p$. Let $U = R_u(B)$ be its unipotent radical. Then there exists a torus $T \subseteq B$ for which $B = U \rtimes T$. Moreover, all such complements of $U$ are conjugate in $B$.*

As the notation suggests, we will often apply this in the case where $B$ is a Borel subgroup. Although this result shows the importance of tori in relation to Borel subgroups, they are actually useful in their own right, and will play a central role in the construction of the Deligne-Lusztig characters. In particular, the *maximal* tori (the maximal closed connected subgroups which are tori) inherit many nice properties from the Borel subgroups.

**Theorem 2.12.** *(3.5.7)*
*Let $G$ be a connected linear algebraic group over $\overline{\mathbb{F}}_p$. Then every maximal torus $T$ of $G$ is contained in some Borel subgroup, and all maximal tori are conjugate.*

As the identity component of a linear algebraic group is a connected linear algebraic group, we lose very little by assuming connectedness, and this will be a standing assumption for the rest of the thesis.

The previous theorem suggests that there is a natural group acting on the maximal tori inside $G$.

**Definition 2.13.** For a maximal torus $T$ in a connected linear algebraic group $G$, denote the normalizer of $T$ in $G$ by $N_G(T)$. We define the *Weyl group (of $T$)* as $W(T) = N_G(T)/T$. Since all maximal tori in $G$ are conjugate, this group is well-defined up to isomorphism and so we omit the $T$ from our notation when the choice of torus does not matter.

**Example 2.14.** Let $G = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and consider the torus $D$ of diagonal matrices; $D$ is maximal because, if there were a strictly larger torus, we could find a non-diagonal matrix commuting with all the elements of $D$. But $D$ contains all the diagonal matrices with $n$ distinct eigenvalues, and anything which commutes with all such matrices must itself be diagonal.

The normalizer of $D$ in $G$ is the group $M$ of monomial matrices: that is, matrices with exactly one nonzero entry in each row and column. Monomial matrices can be written as the product of a diagonal matrix and a permutation matrix, so $W := M/D$ can be identified with the permutation matrices, meaning $W \cong S_n$. $\square$

A small word of caution: in the previous example, the Weyl group had a natural embedding in $G$ as the group of permutation matrices. In general, no such embedding exists, so one should not think of $W$ as a subgroup of $G$. If we want to work with elements in $G$, we denote a representative for $w \in W$ by $\dot{w}$.

The Weyl group directly controls the structure of $G$, as the next theorem illustrates.

**Theorem 2.15.** *(Bruhat decomposition, 1.6.3)*

*Let $G$ be a connected linear algebraic group over $\overline{\mathbb{F}}_p$, with $T$ a maximal torus in $G$ and $B$ a Borel subgroup containing $T$. Let $W = W(T)$ be the Weyl group. Then*

$$G = \coprod_{w \in W} B\dot{w}B.$$

Because $B \supseteq T$, $B\dot{w}B$ does not depend on the choice of representative, so the dot is generally omitted. The double coset $BwB$ is called a *Bruhat cell*.

**Example 2.16.** The Bruhat decomposition for $G = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is particularly easy to see, since by Example 2.14 the Weyl group $W \cong S_n$ embeds into $G$ as the permutation matrices. In this case, the Bruhat decomposition is just Gaussian elimination: multiplication on the left and right by upper triangular matrices give column and row operations, except for permuting the rows and columns. We end up with a uniquely determined "generalised reduced row echelon form", which is exactly the permutation matrix required to reduce to the identity. $\square$

## 2.2 Parabolic induction and cuspidal irreps

At this stage, we have developed all of the structure theory we will need throughout the thesis. We will now see an example of applying this to representation theory by studying *parabolic induction*. This was, historically, a first attempt at constructing representations of finite groups of Lie type, but unfortunately it often does not yield all the irreps. Deligne–Lusztig induction was originally conceived to generalize parabolic induction (see Theorem 4.17 in Chapter 4).

**Definition 2.17.** Let $G$ be a connected reductive group with Frobenius $F : G \to G$. Let $T$ be an $F$-stable maximal torus (meaning $F(T) = T$) and let $B \supseteq T$ be an $F$-stable Borel. If $U = R_u(B)$ is the unipotent radical then taking fixed points under $F$ gives rise to finite groups $G^F$ and $B^F = U^F \rtimes T^F$.

For a complex linear character $\theta$ of $T^F$, we may extend $\theta$ to a representation $\tilde{\theta}$ of $B^F$ by letting $U^F$ act trivially. Explicitly, $\tilde{\theta}$ is given by

$$\tilde{\theta}(ut) := \theta(t), \qquad u \in U^F, t \in T^F.$$

Inducing from $B^F$ to $G^F$ then gives a representation of $G^F$.

This construction will often yield reducible representations, but these can be decomposed into their irreducible components. An irreducible representation (irrep) of $G^F$ is said to be *parabolically induced* if it arises as a component in a representation constructed using the above procedure.

Let us discuss the rationale behind parabolic induction. In standard representation theory, inducing a representation from a subgroup $H$ and then decomposing it is an excellent way to generate irreps of $G$. In general, the decomposition of the induced character is by far the hardest part of this procedure. From the formula

$$\dim(\operatorname{Ind}_H^G V) = [G : H] \dim(V),$$

we see that inducing from a smaller subgroup will generally result in a larger representation, which means it is likely to have more constituents and thus be harder to decompose. By going through the intermediate Borel $B$ instead of inducing directly from $T$, the resulting parabolically induced representation will be smaller and should theoretically be easier to decompose.

It is not immediate from the definition, but parabolic induction is (up to isomorphism) independent of $B$. Although this was known before the advent of Deligne–Lusztig theory, we will prove it in Chapter 4 by realising parabolic induction as a special case of Deligne–Lusztig induction, then showing that the more general construction does not depend on $B$ (Corollary 4.28). In this sense, parabolic induction is a functor from representations of $T^F$ to representations of $G^F$ which is intrinsic to the maximal torus.

Unfortunately, it is often the case that not all irreps of $G^F$ are parabolically induced; those irreps which do not appear are called *cuspidal.* The hardest part of finding the irreps of finite groups of Lie type is generally finding the cuspidal representations.

For readers interested in a concrete example of parabolic induction, we have included a calculation of the character table of $\mathrm{GL}_2(\mathbb{F}_q)$ in Appendix A. This example also shows the ad hoc methods that are often required to find the cuspidal irreps. Particularly if the reader finds the results of Deligne–Lusztig theory in Chapter 4 too abstract, we encourage them to draw parallels to this calculation.

# Chapter 3

# Everything étale

We now turn away from pure representation theory to discuss geometry. In this chapter, we introduce the Grothendieck topology[†] known as the étale topology. We then apply this to define algebro-geometric analogues of several topological invariants such as the fundamental group. Throughout this chapter, we take the viewpoint of modern algebraic geometry, discussing schemes rather than varieties.

## 3.1 Étale morphisms and the étale topology

When trying to apply topological methods to problems in algebraic geometry, there is often a significant roadblock: the Zariski topology is simply too coarse. To rectify this, Grothendieck made the insightful observation that the notion of a topology can be categorified; we can equivalently view a topology on a space $X$ as a category $\mathrm{Op}(X)$ where the objects are open sets and the morphisms are inclusions. Together with certain axioms, this information fully encodes a topology on $X$.

From this perspective, there is no need for the objects in our category to actually be open sets in $X$. Motivated by topology, Grothendieck constructed an analogue of covering spaces called *étale maps*. As we will soon see, we can naturally view étale maps as a generalization of open sets (which cover themselves) in the Zariski topology, so this gives a finer 'topology' on a scheme $X$.

We will now define étale morphisms and see some explicit examples; throughout this section, all references are to [9] unless specified otherwise. There are two components to the definition of an étale map: these are the properties of being *unramified* (Definition 3.1) and *flat* (Definition 3.2).

[†]To avoid misconceptions, a Grothendieck topology is not an actual topology.

**Definition 3.1.** Let $X$ and $Y$ be schemes, and let $f : X \to Y$ be a scheme morphism. We say that $f$ is *unramified* if

(a) $f$ is locally of finite presentation.

(b) For each $x \in X$ and $y = f(x)$, the induced map on stalks $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ maps $\mathfrak{m}_y$ surjectively onto $\mathfrak{m}_x$, where $\mathfrak{m}_y$ and $\mathfrak{m}_x$ are the maximal ideals of the local rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ respectively.

(c) With $x$ and $y$ as in (b), the induced map on residue fields $\kappa(y) \to \kappa(x)$ realises $\kappa(x)$ as a finite separable extension of $\kappa(y)$.

As a reminder for (a) above, a ring homomorphism $\phi : R \to S$ is *of finite presentation* if $S$ is isomorphic (with the $R$-algebra structure induced by $\phi$) to $R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ for some variables $x_i$ and polynomials $g_j$. A scheme morphism is *locally of finite presentation* if for all affine opens $U \subseteq X$ and $V \supseteq f(U)$, the ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

**Definition 3.2.** Recall that a ring homomorphism $\phi : R \to S$ gives $S$ the structure of an $R$-module. We say that $\phi$ is *flat* if the functor

$$- \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$$

is exact. A morphism of schemes $f : X \to Y$ is *flat* if for every affine open $U \subseteq X$ and $V \supseteq f(U)$, the ring homomorphism

$$f_{U,V} : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

is flat.

**Remark 3.3.** The notion of flatness of a scheme morphism might not seem intuitive, but it is a very easy property to check in practice. For example, flatness[‡] is local on the source and target, meaning it is enough to check flatness on any compatible affine open covers of $X$ and $Y$ (Proposition 17.4.1). In describing flat morphisms in [4], Mumford said "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

We call a scheme morphism *étale* if it is both flat and unramified. There are many equivalent definitions of an étale map; we have chosen this one because it is the most computable, and we will often need to work with explicit étale maps. It is worth remarking that étale maps are also exactly the smooth and unramified maps (Corollary 17.6.2), which might offer more intuition.

[‡]Being unramified is also local on the source and target.

**Example 3.4.** We will now see two examples of scheme morphisms, the first of which is étale and the second of which is not.

(i) Let $j : \mathbb{R}[s] \hookrightarrow \mathbb{C}[t]$ be the obvious inclusion, and let $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ be the induced map. We claim that $f$ is étale.

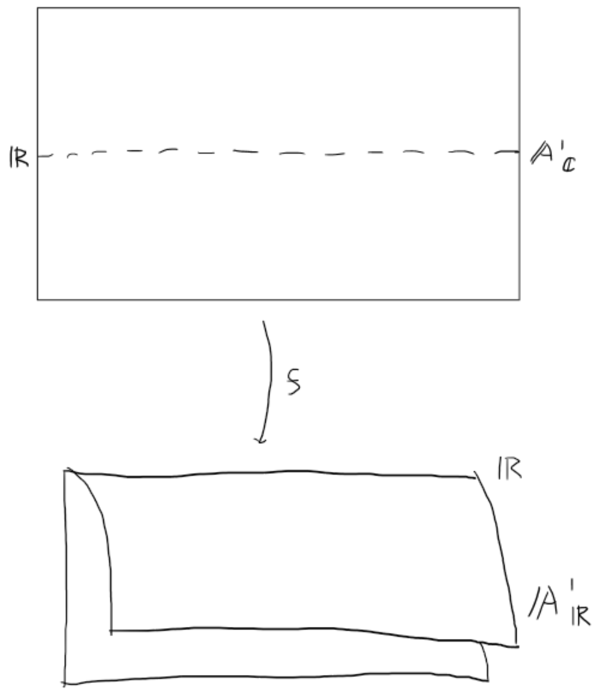


Figure 3.1: Diagram of the 'folding map' $f$

Since we can check flatness on any affine open cover, it is enough to check that the global map $i$ is flat, i.e. we must now show that

$$- \otimes_{\mathbb{R}[s]} \mathbb{C}[t]$$

is an exact functor. We observe that, as $\mathbb{R}[s]$-modules, we have

$$\mathbb{C}[t] = \mathbb{R}[t] + i\mathbb{R}[t] \cong \mathbb{R}[s] \oplus \mathbb{R}[s]$$

so $\mathbb{C}[t]$ is free. It is well-known that tensoring with a free module is exact, so the map $f$ is indeed flat. More generally, tensoring with a projective module is exact, and this is usually how the flatness condition is checked.

We next check that $f$ is unramified. Writing

$$\mathbb{C}[t] \cong \mathbb{R}[s,x]/(x^2+1),$$

we see that $j$ is of finite presentation and so $f$ is locally of finite presentation, proving (a) in Definition 3.1.

Let us now study $f$ on stalks. Recall that nonzero prime ideals in $\mathbb{C}[t]$ are of the form

$$(t-\lambda), \qquad \lambda \in \mathbb{C}.$$

The induced map on stalks is then

$$\begin{aligned} f^{\#} : \mathbb{R}[s]_{j^{-1}(t-\lambda)} &\to \mathbb{C}[t]_{(t-\lambda)} \\ s &\mapsto t \end{aligned}$$

As Figure 3.1 indicates, we should expect different behaviour depending on whether or not $\lambda$ is purely real. Indeed, if $\lambda \in \mathbb{R}$ then

$$j^{-1}(t-\lambda) = (s-\lambda).$$

The maximal ideal $(s-\lambda)$ of $\mathbb{R}[s]_{(s-\lambda)}$ is clearly mapped surjectively to the maximal ideal $(t-\lambda)$ of $\mathbb{C}[t]_{(t-\lambda)}$. On the level of residue fields, we find that

$$\mathbb{R} \cong \mathbb{R}[s]_{(s-\lambda)}/(s-\lambda) \xrightarrow{f^{\#}} \mathbb{C}[t]_{(t-\lambda)}/(t-\lambda) \cong \mathbb{C}$$

does indeed realise $\kappa(t-\lambda)$ as a finite separable extension of $\kappa(s-\lambda)$, so we have proven (b) and (c) of Definition 3.1 in the case $\lambda \in \mathbb{R}$.

Let us now consider the case of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then

$$j^{-1}(t-\lambda) = \big((s-\lambda)(s-\overline{\lambda})\big)$$

and it is perhaps no longer clear that $f^{\#}$ maps the maximal ideal of $\mathbb{R}[s]_{\left((s-\lambda)(s-\overline{\lambda})\right)}$ to the maximal ideal of $\mathbb{C}[t]_{(t-\lambda)}$. However, we note that $\overline{\lambda} \neq \lambda$ by assumption, so $t-\overline{\lambda} \notin (t-\lambda)$ and hence $t-\overline{\lambda}$ is a unit in the localisation. It follows that

$$(t-\lambda) = \big((t-\lambda)(t-\overline{\lambda})\big) = f^{\#}\left((s-\lambda)(s-\overline{\lambda})\right)$$

and (b) is indeed satisfied. On the level of residue fields, we have

$$\mathbb{C} \cong \mathbb{R}[s]_{\left((s-\lambda)(s-\overline{\lambda})\right)}/\left((s-\lambda)(s-\overline{\lambda})\right) \xrightarrow{f^{\#}} \mathbb{C}[t]_{(t-\lambda)}/(t-\lambda) \cong \mathbb{C}$$

which is again a finite separable field extension. Thus (c) holds and $f$ is unramified, as claimed. $\square$

(ii) We now consider the ring homomorphism

$$\begin{aligned} \phi : \mathbb{C}[s] &\to \mathbb{C}[t] \\ s &\mapsto t^2 \end{aligned}$$

and the induced map

$$g : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}.$$

We claim that $g$ is **not** étale. Since $g$ is a 2-sheeted covering map away from 0, we would expect the obstruction to $g$ being étale to arise at the prime ideal $(t)$. Indeed, we will now show that $g$ is ramified at $(t)$.

On the level of stalks, we have the map

$$\begin{aligned} g^{\#} : \mathbb{C}[s]_{(s)} &\to \mathbb{C}[t]_{(t)} \\ s &\mapsto t^2 \end{aligned}$$

Since $t \notin \mathrm{im}(g^{\#})$, $g^{\#}$ does not map the maximal ideal of $\mathbb{C}[s]_{(s)}$ surjectively to the maximal ideal of $\mathbb{C}[t]_{(t)}$ and so $g$ is ramified at $(t)$, as we expected. □

We now have all of the requisite ingredients to construct our new 'topology'.

**Definition 3.5.** Let $X$ be a scheme. An *étale open set* on $X$ is a pair $(U, u)$, where $U$ is a scheme and $u : U \to X$ is an étale morphism. The *étale topology* on $X$ is the category $\mathrm{Et}(X)$ of étale open sets on $X$. A morphism $(U, u) \to (V, v)$ of étale open sets is a scheme morphism $f : U \to V$ such that the diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "u"'] & & V \arrow[dl, "v"] \\ & X & \end{tikzcd}$$

commutes.

We can make precise the statement that the étale topology is finer than the Zariski topology by observing that open immersions are étale, so we have an embedding $\mathrm{Op}(X) \hookrightarrow \mathrm{Et}(X)$.

In general, it is hard to compute the étale topology on a scheme because we would need to find all étale open covers of every open subset of $X$. To rectify this, we often restrict ourselves to the full subcategory $\mathrm{FEt}(X) \subseteq \mathrm{Et}(X)$ of *finite* étale morphisms (recall that $f$ is finite if $f_{U,V} : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely-generated $\mathcal{O}_Y(V)$-module for every affine open $U \subseteq X, V \supseteq f(U)$).

Because étale morphisms are open and finite morphisms are closed, any finite étale map onto a connected scheme $X$ is surjective, so we now need only find all finite étale covers of the whole of $X$. This is a much more tractable problem.

**Example 3.6.** Let $k$ be a field and let $X = \mathrm{Spec}(k)$. One can easily check that if $K/k$ is a finite separable extension then $\mathrm{Spec}(K) \to X$ is a finite étale cover. Conversely, because $X$ is affine, any finite étale open set $(U, u)$ on $X$ is fully determined by the global ring homomorphism

$$k \to \mathcal{O}_U(U).$$

Because this global ring homomorphism is of finite presentation, $\mathcal{O}_U(U)$ must be an integral extension of $k$. By finiteness of $u$, this is also a finite extension, hence a field. Because $u$ is unramified, this extension must be separable. Thus

$$\mathrm{FEt}(X) \simeq \{\text{finite separable extensions of } k\}^{\mathrm{opp}}. \qquad \square$$

## 3.2 Topological motivation

Having constructed $\mathrm{FEt}(X)$, we ultimately aim to study étale sheaves and, in particular, étale local systems on a scheme. It is a well-known result that local systems on a topological space are equivalent to representations of the topological fundamental group $\pi_1^{\mathrm{top}}$, via the *monodromy action* (see Theorem 7 in [15]). We will eventually generalize this to the étale setting, and this equivalence is how we shall construct étale local systems in practice. With that in mind, our next goal is to define the étale fundamental group.

At this point, we take a step back to review the topological fundamental group and rephrase its definition in a way that will naturally generalise to give the étale fundamental group. This is not strictly necessary, so readers only interested in the definitions may skip this section. Complete proofs of the following facts can be found in Chapter 1 of [14].

Recall that for a sufficiently nice topological space $X$[§], $X$ has a universal cover $(\tilde{X}, p)$. Let us fix a basepoint $x_0 \in X$ and a lifted basepoint $\tilde{x}_0 \in \tilde{X}$. If $\gamma : [0,1] \to X$ is a loop at $x_0$ then $\gamma$ lifts to a path $\tilde{\gamma}$ in $\tilde{X}$ which starts and ends at some lifts of $x_0$ (perhaps distinct lifts). For a deck transformation $\rho \in \mathrm{Aut}_X(\tilde{X})$, $\rho \circ \tilde{\gamma}$ is also a path in $\tilde{X}$ which starts and ends at some lifts of $x_0$. Projecting down via $p$, we see that $p \circ (\rho \circ \tilde{\gamma})$ is a loop at $x_0$ and hence defines a class in $\pi_1(X, x_0)$. One can verify that this construction is independent of the choice of lift $\tilde{\gamma}$ and homotopy, so we have a well-defined action

$$\mathrm{Aut}_X(\tilde{X}) \circlearrowright \pi_1(X, x_0).$$

[§]For example, [14] takes $X$ to be path-connected, locally path-connected and semilocally simply-connected.

**Theorem 3.7.** *For $X$ a path-connected, locally path-connected and semilocally simply-connected space, we have*

$$\pi_1(X, x_0) \cong \mathrm{Aut}_X(\tilde{X}).$$

If we can find some meaningful analogue of the universal cover, we could perhaps *define* the étale fundamental group as the group of deck transformations. Unfortunately, there are many instances of schemes which fail to have universal covers.

**Example 3.8.** Consider $X = \mathrm{Spec}\,(\mathbb{C}[x,y]/(xy-1)) = \mathbb{A}^1_{\mathbb{C}} - \{0\}$. On the analytic side, the universal cover is given by

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} - \{0\} \\ x &\mapsto e^x \end{aligned}$$

Because $e^x$ is not an algebraic function, this does not define a cover of $X$ as a scheme and in fact, $X$ does not have a universal cover.

We therefore wish to find a reasonable stand-in for $\tilde{X}$. To that end, we observe that $(\tilde{X}, \tilde{x}_0)$ is characterized by the following universal property:

**Proposition 3.9.** *For any covering space $Y \to X$ and lift $y_0$ of $x_0$, there is a unique covering space map $\tilde{X} \to Y$ which makes the diagram*

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[d] \arrow[dd, bend left, "p"] \\ (Y, y_0) \arrow[d] \\ (X, x_0) \end{tikzcd}$$

*commute.*

This says exactly that for any covering space $Y$ of $X$, morphisms $\tilde{X} \to Y$ are fully determined by lifts of $x_0$ in $Y$. Phrased slightly differently, we have a functor

$$\begin{aligned} \mathrm{Fib} : \{\text{finite covering spaces of } (X, x_0)\} &\to \mathbf{Sets} \\ (Y, \pi) &\mapsto \pi^{-1}(x_0) \end{aligned}$$

and the universal property says that

$$\mathrm{Fib}(-) \simeq \mathrm{Hom}(\tilde{X}, -).$$

In categorical language, $\tilde{X}$ is a *representing object* for Fib. This is how we shall find our 'universal cover' of a scheme.

## 3.3 The étale fundamental group

Motivated by the previous section, we may now define the étale fundamental group of a scheme. It will be a standing assumption throughout the section that our scheme $X$ is connected, but we will highlight where this assumption is necessary.

**Definition 3.10.** A *geometric point* for a $k$-scheme $X$ is a morphism $\mathrm{Spec}(\overline{k}) \to X$; such a geometric point is often denoted $\overline{x}$ and identified with its image in $X$.

Geometric points play the role of basepoints for a scheme. For any such $\overline{x}$, we have a functor

$$\begin{aligned} \mathrm{Fib}_{\overline{x}} : \mathrm{FEt}(X) &\to \mathrm{Sets} \\ (Y, \pi) &\mapsto \mathrm{Hom}(\overline{x}, Y) \end{aligned}$$

which is commonly known as the fiber functor.

For any two geometric points $\overline{x}, \overline{y}$ in a connected scheme $X$, the functors $\mathrm{Fib}_{\overline{x}}$ and $\mathrm{Fib}_{\overline{y}}$ are naturally isomorphic, so we generally omit the subscript. Fib should be thought of as 'taking the fiber over $\overline{x}$', so in keeping with our rephrasing from the last section, we would like to find a representing object for Fib. While such an object does not exist in general, it **is** true that Fib is pro-representable[¶].

To find a pro-system representing Fib, we look to the Galois finite étale covers; these are the covers such that $[X_i : X] = |\,\mathrm{Aut}_X(X_i)|$, where the left-hand side is the degree of the field extension at any geometric point, and the right-hand side is the order of the deck transformation group. Because any finite separable extension can be embedded into a finite Galois extension, Galois covers are cofinal among all finite étale covers, which simply means that every finite étale cover is mapped onto by some Galois cover sitting above it.

The Galois covers are naturally a pro-system as follows: we say that $X_i \leq X_j$ if there is a morphism $X_j \to X_i$. For each $X_i$, pick a lifted basepoint $x_i \in \mathrm{Fib}(X_i)$. Then if $X_i \leq X_j$, we pick the morphism that sends $x_j$ to $x_i$ for use in the pro-system.

**Theorem 3.11.** *Let $\tilde{X} = (X_i)_{i \in I}$ be the pro-system of all Galois finite étale coverings of $X$, defined above. Then $\tilde{X}$ represents* Fib, *in the sense that*

$$\mathrm{Fib}(-) \simeq \varinjlim_{X_i \in \tilde{X}} \mathrm{Hom}(X_i, -).$$

[¶]We will define this by example in Theorem 3.11.

*Proof.* We have a natural transformation

$$\varinjlim \operatorname{Hom}(X_i, Y) \to \operatorname{Fib}(Y)$$

induced by the family of maps that send a morphism $\pi : X_i \to Y$ to $\pi(x_i)$.

Conversely, given any $y \in \operatorname{Fib}(Y)$, we may restrict to the component of $y$ and assume that $Y$ is connected. Some Galois cover sits over $Y$, so we have a surjection $X_i \to Y$. Because $X_i$ is Galois, the group $\operatorname{Aut}_X(X_i)$ acts transitively on $F(X_i)$, so we can precompose with an automorphism to get a morphism $X_i \to Y$ that sends $x_i$ to $y$. This morphism is necessarily unique because $\operatorname{Aut}_X(X_i)$ also acts freely on $F(X_i)$.

We now map $y \in \operatorname{Fib}(Y)$ to the class of the unique morphism described above. Note that we needed to make a choice of Galois cover above $Y$, but all such morphisms become equivalent in $\varinjlim \operatorname{Hom}(X_i, Y)$.

It is routine to verify that these are inverses. □

Our proof shows that we do not need to take **all** Galois coverings: any cofinal subset of $\tilde{X}$ will do. In practise, this is often how one finds a pro-representing system.

Now that we have our pro-system, we would like to define the étale fundamental group to be the group of deck transformations. To give meaning to this, we observe that if $f_{i,j} : X_j \to X_i$ is a morphism in our pro-system, we get an induced map $h_{i,j} : \operatorname{Aut}_X(X_j) \to \operatorname{Aut}_X(X_i)$ as follows:

Given an automorphism $\eta \in \operatorname{Aut}_X(X_j)$, we consider the basepoint

$$\operatorname{Fib}(f_{i,j} \circ \eta)(x_j) = f_{i,j}(\eta(x_j)).$$

Because $X_i$ is Galois, there is a unique $\psi \in \operatorname{Aut}_X(X_i)$ such that $\psi(x_i) = f_{i,j}(\eta(x_j))$. We now define $h_{i,j}(\eta) = \psi$.

One can verify that $h_{i,j}$ is a group homomorphism. Furthermore, it satisfies the following naturality condition: for every $\eta \in \operatorname{Aut}_X(X_j)$, the square

$$\begin{tikzcd}
X_j \arrow[r, "f_{i,j}"] \arrow[d, "\eta"'] & X_i \arrow[d, "h_{i,j}(\eta)"] \\
X_j \arrow[r, "f_{i,j}"] & X_i
\end{tikzcd}$$

commutes.

To check this, we will use the fact that étale lifts of the same map are uniquely determined by their value at a single point (analogously to covering maps), so it suffices to evaluate both sides of the square at $x_j$. We then have

$$(h_{i,j}(\eta) \circ f_{i,j})(x_j) = \psi(f_{i,j}(x_j)) = \psi(x_i) = (f_{i,j} \circ \eta)(x_j)$$

as claimed.

We therefore have a projective system with the groups $\mathrm{Aut}_X(X_i)$ as objects and $h_{i,j}$ as morphisms.

**Definition 3.12.** For a connected $k$-scheme $X$, the *étale fundamental group* of $X$ at $\overline{x}$ is

$$\pi_1(X, \overline{x}) := \varprojlim \mathrm{Aut}_X(X_i)$$

endowed with the profinite topology.

**Example 3.13.** Let us now see some examples of the étale fundamental group.

(i) Let $X = \mathrm{Spec}(k)$. We saw in Example 3.6 that $\mathrm{FEt}(X)$ is equivalent to the category of finite separable extensions of $k$. In this case, we have an obvious pro-representing system for $F$ given by $\tilde{X} = \mathrm{Spec}(k^{\mathrm{sep}})$, where $k^{\mathrm{sep}}$ is the separable closure of $k$. It follows that

$$\pi_1(X, \overline{x}) := \mathrm{Aut}_X(\mathrm{Spec}(k^{\mathrm{sep}})) \cong \mathrm{Gal}(k^{\mathrm{sep}}/k)$$

is the absolute Galois group of $k$. □

(ii) Consider $X = \mathbb{A}^1_{\mathbb{C}} - \{0\}$. Every finite étale covering of $X$ is of the form

$$\begin{aligned} X_n = X &\to X \\ t &\mapsto t^n \end{aligned}$$

for some $n \in \mathbb{N}$. These are all Galois, so we simply take $\tilde{X}$ to be every finite étale covering of $X$. In this case, $\mathrm{Aut}_X(X_n) = \mu_n(\mathbb{C})$ is the group of $n^{th}$ roots of unity, which acts by $x \mapsto \zeta x$. It follows that

$$\pi_1(\mathbb{A}^1_{\mathbb{C}} - \{0\}, \overline{x}) = \varprojlim \mu_n(\mathbb{C}) \cong \hat{\mathbb{Z}}$$

is the profinite completion of $\mathbb{Z}$. □

(iii) Let $X = \mathbb{P}^1_{\mathbb{C}}$. We claim that $\pi_1(X, \overline{x}) = 0$, which we will show by proving the stronger statement that the only finite étale covers of $X$ are by finite disjoint unions of copies of $X$.

Let $\pi : Y \to X$ be a finite étale map, with $Y$ connected. Because $X$ is a non-singular curve, $Y$ must be too (it is essentially a local argument, and $\pi$ is a local isomorphism). We now invoke the Riemann-Hurwitz formula, which says that

$$2g - 2 = -2\deg(\pi) + \sum_{y \in Y}(k_y - 1)$$

where $g$ is the genus of $Y$ and $k_y$ is the ramification index of $\pi$ at $y$. Since $\pi$ is étale, it is unramified and hence $k_y = 1$ everywhere, so the sum vanishes.

We are now looking for non-negative integer solutions to the equation

$$2g - 2 = -2\deg(\pi),$$

so the only possibility is $g = 0, \deg(\pi) = 1$. Therefore $Y$ is a smooth curve of genus 0 and $\pi$ is a degree 1 map, so it is an isomorphism. □

Compare Examples (ii) and (iii) above with the topological fundamental groups of $S^1 \simeq \mathbb{C} - \{0\}$ and $\mathbb{P}^1_{\mathbb{C}}$ respectively: we get $\hat{\mathbb{Z}}$ and 0 in comparison to $\mathbb{Z}$ and 0. Based on this, one might expect that there is a relationship between the topological fundamental group and the étale one. Indeed, if $Y \to X$ is a finite étale cover then we can recover a finite covering space via *analytification*, which informally is the process of turning an algebraic space into an analytic one. More precisely, if $X$ is a scheme over $\mathbb{C}$ then the set $X(\mathbb{C})$ of maximal ideals embeds naturally into complex space, so it can be equipped with the complex analytic topology. Applying this same process to $Y$ and restricting the étale map then gives a topological covering. The remarkable result is that **every** finite covering space of $X(\mathbb{C})$ arises in this way.

**Theorem 3.14.** *(Grothendieck-Riemann existence)*
*Let $X$ be a non-singular $\mathbb{C}$-variety. Then analytification induces an equivalence of categories*

$$\mathrm{FEt}(X) \simeq \{\textit{finite covering spaces of } X(\mathbb{C})\}.$$

This theorem is far outside the scope of this thesis to prove; we include it to demonstrate that much of the topological intuition of the standard fundamental group transfers into the étale setting. In a similar vein, almost all properties of the topological fundamental group (such as the lifting criterion) generalise immediately to the étale setting, regardless of characteristic. We refer the reader to [11] for precise statements and proofs.

## 3.4 Étale sheaves and representations of $\pi_1(X)$

We will now define a suitable analogue of sheaves for the étale topology. For conciseness, we will assume a familiarity with sheaves and the standard constructions, all of which generalise immediately to the étale context.

**Definition 3.15.** Let $R$ be a commutative unital ring. An *étale presheaf of $R$-modules* on a scheme $X$ is a contravariant functor

$$\mathcal{F} : \mathrm{Et}(X) \to R\text{-Mod}.$$

An étale presheaf is an *étale sheaf* if it satisfies the obvious gluing condition (with intersection of open sets replaced by fibered product of étale open sets). A morphism of étale sheaves is a natural transformation between the corresponding functors. The category of étale sheaves on $X$ is denoted $\mathfrak{Sh}_{\mathrm{Et}}(X)$.

**Example 3.16.** The following are some important examples of étale sheaves on a scheme $X$:

(i) Restricting étale sheaves to the subcategory $\mathrm{Op}(X) \subseteq \mathrm{Et}(X)$, we recover the ordinary notion of a sheaf on $X$.

(ii) Let $A$ be an abelian group, endowed with the discrete topology. For each étale open set $U$ on $X$, let $\underline{A}(U)$ be the collection of all continuous maps $U \to A$, i.e. the functions $U \to A$ which are constant on the connected components of $U$. Then $\underline{A}$ is an étale sheaf on $X$, called the *constant étale sheaf attached to $A$*.

In the étale setting, we still have notions of (proper) pushforward, pullback, stalks etc. All of these can be shown to exist and satisfy the appropriate universal properties; the standard constructions in sheaf theory generalise immediately. For example, to define the tensor product $\mathcal{F} \otimes \mathcal{G}$ of two étale sheaves, we take the sheafification of the étale presheaf $(\mathcal{F} \otimes \mathcal{G})^-$ given on étale open sets by $(\mathcal{F} \otimes \mathcal{G})^-(U) := \mathcal{F}(U) \otimes \mathcal{G}(U)$, which is entirely analogous to the tensor product of standard sheaves. The other constructions are similar, so for brevity we do not include them.

For our purposes, constant sheaves are slightly too restrictive, so we will consider a more general class of sheaves. These form the smallest abelian subcategory of $\mathfrak{Sh}_{\mathrm{Et}}(X)$ which contains the constant sheaves.

**Definition 3.17.** A *local system* on $X$ is an étale sheaf that is locally constant, i.e. an étale sheaf $\mathcal{F}$ such that there exists an étale open cover $\{U_i\}_{i\in I}$ with $\mathcal{F}\mid_{U_i}$ a constant sheaf for every $i \in I$.

Recall that we constructed the étale fundamental group with the intention that it would classify finite étale covers of our scheme. Passing to stalks allows us to transfer this classification onto étale sheaves.

**Theorem 3.18.** *(Theorem 3.5 in [16])*

*Let $X$ be a connected scheme, with $\overline{x}$ a geometric point. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is an equivalence of categories between the local systems on $X$ and the finite-dimensional complex $\pi_1(X, \overline{x})$-representations.*

We will not prove this, but let us at least describe how $\pi_1(X, \overline{x})$ acts on the stalk. As an inverse limit, elements of $\pi_1(X, \overline{x})$ can be represented by families of automorphisms of Galois étale open covers satisfying certain compatibility conditions. By picking a Galois étale open cover $U$ containing $\overline{x}$, we can then act on $\overline{x}$ through the automorphism of $U$, and functoriality induces an action on the stalk. The compatibility condition ensures this is independent of the choice of $U$.

Beyond the choice of a geometric point, which only induces an inner automorphism of the étale fundamental group, this equivalence is entirely natural. In particular, it respects all of the natural structure on either side, such as dimension.

## 3.5 Kummer local systems on tori

We conclude the chapter by looking at a particular class of local systems, which will be used in Chapter 5 to construct character sheaves. We include this calculation here because it directly uses the results of the previous section, but it could just as well have been included in Chapter 5. Readers are encouraged to review this section before reading Chapter 5.

**Definition 3.19.** Let $X$ be an $\overline{\mathbb{F}}_p$-scheme. A local system $\mathcal{L}$ on $X$ is said to be *Kummer* if it is rank one and $\mathcal{L}^{\otimes n} \cong \underline{\mathbb{C}}$ for some $n$ which is coprime to $p$.

**Remark 3.20.** Because any two algebraically closed fields of the same uncountable cardinality and characteristic are isomorphic (see Remark 9.17 in [12]), we could choose an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ and this would induce an isomorphism $\underline{\overline{\mathbb{Q}}_l} \cong \underline{\mathbb{C}}$. It is common in the literature to replace $\underline{\mathbb{C}}$ by $\underline{\overline{\mathbb{Q}}_l}$ in the definition of a Kummer local system, for some prime $l$ different to $p$.

Kummer local systems are, in some sense which can be made precise, the semisimple local systems on $X$; the superficial analogy is that semisimple elements over $\overline{\mathbb{F}}_p$ are those which have order coprime to $p$, but in fact this runs much deeper. This is not something we will discuss further, but it is a key step in developing the *Jordan decomposition* of character sheaves, so it is important to mention.

The rest of this chapter will be devoted to finding the Kummer local systems in the particular case that $X$ is an algebraic torus, as this is very important for understanding character sheaves.

**Example 3.21.** Throughout the example, let $T = \overline{\mathbb{F}}_p^\times$ be an algebraic torus of rank 1. The case of arbitrary rank follows from the fact that the Kummer local systems on $T_1 \times T_2$ are all of the form $\mathcal{L}_1 \boxtimes \mathcal{L}_2 := \mathrm{pr}_1^* \mathcal{L}_1 \otimes \mathrm{pr}_2^* \mathcal{L}_2$ for $\mathcal{L}_i$ a Kummer local system on $T_i$ (5.4.1 in [17]).

Observe that the set of (isomorphism classes of) Kummer local systems on $T$ forms a group under $\otimes$, denoted $\mathcal{K}T$. If $X = \mathrm{Hom}(T, \overline{\mathbb{F}}_p^\times)$ is the algebraic character group of $T$, we claim that

$$\mathcal{K}T \cong \left[\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} X\right] / \left[1 \otimes_{\mathbb{Z}} X\right].$$

Here $\mathbb{Z}_{(p)}$ denotes the additive group of the localisation of $\mathbb{Z}$ away from the prime ideal $(p)$, so we can think of it as rational numbers where the denominator is coprime to $p$. The right-hand side, which we shall abbreviate as $\hat{X}$, is an abelian torsion group without $p$-torsion. It is often called the *$p'$-torsion part of* $X$. Our goal is to construct a map $\hat{X} \to \mathcal{K}T$, which we will then show is an isomorphism.

We can simplify the definition of $\hat{X}$ somewhat: all algebraic characters $\overline{\mathbb{F}}_p^\times \to \overline{\mathbb{F}}_p^\times$ are of the form $\chi_n(t) = t^n$ (Lemma B.1 in the appendix), so $X \cong \mathbb{Z}$ . It follows that $\hat{X} \cong \mathbb{Z}_{(p)}/\mathbb{Z}$. Despite this presentation being simpler, we will continue to use our original definition of $\hat{X}$ so that we can keep track of the character.

We will now construct a family of Kummer local systems. Let $m > 0$ be an integer coprime to $p$, and consider the $m^{th}$ power isogeny

$$\begin{aligned} m : T &\to T \\ t &\mapsto t^m \end{aligned}$$

This is a Galois finite étale covering of $T$, and the group of deck transformations ${}_mT$ can be identified with the elements of $T$ with order dividing $m$. We pick an isomorphism $\psi$ of the group of roots of unity in $\overline{\mathbb{F}}_p$ with the group of roots of unity coprime to $p$ in $\mathbb{C}$. For $x \in X$, we now define a (complex) character $\chi_{x,m}$ of ${}_mT$ by

$$\chi_{x,m} = \psi \circ x \mid_{{}_mT} .$$

Because $\pi_1(T, e)$ is an inverse limit, it naturally comes equipped with a map

$$\pi_1(T, e) \to {}_mT.$$

Pulling $\chi_{x,m}$ back under this map now yields a one-dimensional representation $\tilde{\chi}_{x,m}$ of $\pi_1(T, e)$, which by the previous section corresponds to a rank one local system $\mathcal{L}_{x,m}$ on $T$.

**Proposition 3.22.** *The local systems $\mathcal{L}_{x,m}$ satisfy the following properties:*

*(i)* $\mathcal{L}_{nx,nm} = \mathcal{L}_{x,m}$*;*

*(ii)* $\mathcal{L}_{x,m} = \mathcal{L}_{y,n}$ *if and only if* $my - nx \in mnX$*;*

*(iii)* $\mathcal{L}_{x,m}$ *is trivial if and only if* $x \in mX$*.*

*Proof.* It suffices to check these properties on the level of the characters $\tilde{\chi}_{x,m}$ of $\pi_1(T, e)$, since they are preserved by the equivalence of categories.

(i) We consider the diagram

$$\begin{tikzcd} & \pi_1(T,e) \arrow[dl] \arrow[dr] & \\ {}_{nm}T \arrow[rr, "(-)^n"] \arrow[dr, "\chi_{nx,nm}"'] & & {}_mT \arrow[dl, "\chi_{x,m}"] \\ & \mathbb{C}^\times & \end{tikzcd}$$

The lower triangle commutes by construction, and the upper triangle commutes because $\pi_1(T, e)$ is the limit over a projective system containing these maps. We conclude that the outer square commutes, and these are exactly the characters $\tilde{\chi}_{nx,nm}$ and $\tilde{\chi}_{x,m}$ of $\pi_1(T, e)$.

(ii) Applying part (i) shows that $\tilde{\chi}_{x,m} = \tilde{\chi}_{nx,nm}$ and $\tilde{\chi}_{y,n} = \tilde{\chi}_{my,nm}$. Making a change of variables, it therefore suffices to show that $\tilde{\chi}_{\dot{x},d} = \tilde{\chi}_{\dot{y},d}$ if and only if $\dot{y} - \dot{x} \in dX$.

Suppose first that $\dot{y} - \dot{x} \in dX$, meaning $\dot{y} - \dot{x} = \dot{z}^d$ for some $\dot{z} \in X$. For elements $t \in {}_dT$, we now find

$$(\dot{y} - \dot{x})(t) = \dot{z}^d(t) = \dot{z}(t^d) = \dot{z}(1) = 1.$$

Thus $\chi_{\dot{x},d}$ and $\chi_{\dot{y},d}$ are equal even before pulling back to $\pi_1(T, e)$.

Conversely, suppose that $\tilde{\chi}_{\dot{x},d} = \tilde{\chi}_{\dot{y},d}$. As in part (i), we consider the diagram

$$\begin{tikzcd}
 & \pi_1(T,e) \arrow[dl] \arrow[dr] & \\
{}_dT \arrow[dr, "\chi_{\dot{x},d}"'] & & {}_dT \arrow[dl, "\chi_{\dot{y},d}"] \\
 & \mathbb{C}^\times &
\end{tikzcd}$$

Note that this diagram commutes by assumption. We now invoke a general result about inverse systems, the proof of which can be found in Appendix B. The precise statement is:

**Lemma B.2.** Let $(S_i, \pi_{ji})$ be an inverse system of finite sets with limit $S$, and suppose each $\pi_{ji}$ is surjective. Then every projection $S \to S_i$ is surjective.

As $m$ varies, the isogeny maps we defined earlier give a cofinal system of Galois étale coverings of $T$, so in particular we can use them to construct $\pi_1(T, e)$. Since the sets ${}_mT$ are finite and the transition maps ${}_{nm}T \to {}_mT$ are surjective, the lemma applies to show that the map $\pi_1(T,e) \to {}_dT$ in our above diagram is surjective. We may therefore cancel this map to conclude that $\chi_{\dot{x},d} = \chi_{\dot{y},d}$. Since $\dot{y} - \dot{x}$ vanishes on ${}_dT$, it is the $d^{th}$ power of some character $\dot{z} \in X$, as required.

(iii) This follows from (ii) by observing that $\mathcal{L}_{0,1}$ is trivial. □

We may now deduce that $\mathcal{L}_{x,m}$ depends only on the class $\zeta \in \hat{X}$ of $m^{-1} \otimes x$, so we write $\mathcal{L}_\zeta = \mathcal{L}_{x,m}$. Once again checking on the level of characters, we also see that $\mathcal{L}_{\zeta+\eta} = \mathcal{L}_\zeta \otimes \mathcal{L}_\eta$. Since $\zeta \in \hat{X}$ is $p'$-torsion, we conclude that $\mathcal{L}_\zeta$ is too and hence $\mathcal{L}_\zeta$ is indeed Kummer. We therefore have a homomorphism

$$\hat{X} \to \mathcal{K}T$$

which is injective by (iii) in the above proposition.

It remains to show that this map is also surjective, so let $\mathcal{L} \in \mathcal{K}T$ be a Kummer local system. Then $\mathcal{L}$ corresponds to some one-dimensional representation $\rho : \pi_1(T,e) \to \mathbb{C}^\times$. Because $\mathcal{L}$ is Kummer, the image of $\rho$ must lie in the group of $m^{th}$ roots of unity, for some $m$ coprime to $p$.

Much as in the topological fundamental group setting, subgroups of the étale fundamental group correspond to Galois étale covers of $T$ (section 2.2.2 in [25]), so $\ker(\rho)$ defines a Galois étale covering $\pi : Y \to T$. Because the image of $\rho$ is a subgroup of a cyclic group, hence also cyclic, the corresponding extension of function fields $\overline{\mathbb{F}}_p(Y)/\overline{\mathbb{F}}_p(T)$ is cyclic of degree dividing $m$.

By general Kummer theory (see, e.g., Proposition 3.2 in [18]), a cyclic extension $K/F$ of degree dividing $m$ is of the form $K = F(x^{\frac{1}{m}})$ if and only if $K$ contains some $m^{th}$ root of unity. Since $\overline{\mathbb{F}}_p$ is algebraically closed and $m$ is coprime to $p$, this applies to our extension and so we can write $\overline{\mathbb{F}}_p(Y) = \overline{\mathbb{F}}_p(T)(x^{\frac{1}{m}})$ for some $x \in \overline{\mathbb{F}}_p(T)$.

Since $\pi$ is unramified, $x$ can have neither poles nor zeroes in $T$. Because $x$ has no poles, it is an element of the ring of regular functions $\overline{\mathbb{F}}_p[T] = \overline{\mathbb{F}}_p[z, z^{-1}]$ and so it is a sum of monomials. If $x$ has more than one monomial summand then we could factor out some power of $z$ and be left with a polynomial which has a nonzero root, contradicting the fact that $x$ has no zeroes in $T$. Thus $x = z^j$ is in fact a pure monomial.

We now claim that we can lift $m$ over $\pi$, i.e. that we can find a map $M : T \to Y$ such that the diagram

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ T \arrow[ur, dashed, "M"] \arrow[r, "m"'] & T \end{tikzcd} \tag{1}$$

commutes. To find such a lift, we recall that a field homomorphism $\overline{\mathbb{F}}_p(Y) \to \overline{\mathbb{F}}_p(T)$ induces a (rational) map $T \to Y$. We therefore consider the dualised diagram of function fields:

$$\begin{tikzcd} & \overline{\mathbb{F}}_p(Y) = \overline{\mathbb{F}}_p(T)(x^{\frac{1}{m}}) \arrow[dl, dashed] \\ \overline{\mathbb{F}}_p(T) & \overline{\mathbb{F}}_p(T) \arrow[l, "m^*"] \arrow[u, hook] \end{tikzcd} \tag{2}$$

We now work in explicit coordinates. Write $\overline{\mathbb{F}}_p(T) = \overline{\mathbb{F}}_p(z)$ and $x = z^j$, so $\overline{\mathbb{F}}_p(Y) = \overline{\mathbb{F}}_p(z)(z^{\frac{j}{m}})$. The only relation that $x^{\frac{1}{m}}$ and $z$ satisfy is $(x^{\frac{1}{m}})^m = z^j$. Making the change of variables $y = x^{\frac{1}{m}}$, we therefore find that

$$\overline{\mathbb{F}}_p(Y) = \overline{\mathbb{F}}_p(z)[y]/(y^m - z^j).$$

We consider the map $\overline{\mathbb{F}}_p(z)[y] \to \overline{\mathbb{F}}_p(z)$ which sends $y$ to $z^j$ and restricts to $m^*$ (precomposition with $m$) on $\overline{\mathbb{F}}_p(z)$. Because $(z^j)^m = (z^m)^j$, our relation is in the kernel and so this induces a field homomorphism $\overline{\mathbb{F}}_p(Y) \to \overline{\mathbb{F}}_p(T)$ which makes (2) commute. In turn, we now have a rational map $T \dashrightarrow Y$ making (1) commute.

But recall that this rational map is induced by evaluating our field homomorphism on the coordinate function $z$, which gives $z^m$. Since $z^m$ has no denominator in $T$, this actually defines a map on all of $T$, so in fact our lift $M$ is regular.

We next claim that $\mathcal{L}$ is a constituent of $m_*\underline{\mathbb{C}}$: indeed, utilising both the above factorisation of $m$ and the pullback-pushforward adjunction, we have

$$\mathrm{Hom}(\mathcal{L}, m_*\underline{\mathbb{C}}) \cong \mathrm{Hom}(\pi^*\mathcal{L}, M_*\mathcal{L}).$$

We now recall the definition of $\pi$: it is the Galois étale covering corresponding to the kernel of $\rho$, which is itself the representation corresponding to $\mathcal{L}$. Unravelling the equivalences here, this means that $\pi^*\mathcal{L}$ is the trivial local system $\underline{\mathbb{C}}$ on $Y$.

We now find that

$$\mathrm{Hom}(\mathcal{L}, m_*\underline{\mathbb{C}}) \cong \mathrm{Hom}(M^*\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \mathrm{Hom}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \neq 0.$$

Thus $\mathcal{L}$ is indeed a constituent of $m_*\underline{\mathbb{C}}$.

To conclude, we now decompose $m_*\underline{\mathbb{C}}$. We claim that

$$m_*\underline{\mathbb{C}} = \bigoplus_{\substack{\zeta \in \hat{X} \\ m\zeta = 0}} \mathcal{L}_\zeta \qquad (\dagger).$$

Assuming (†), it follows from the fact that $\mathcal{L}$ is constituent of $m_*\underline{\mathbb{C}}$ that $\mathcal{L} = \mathcal{L}_\zeta$ for some $\zeta$ as above, and we are done.

To prove the claim, first suppose that $m\zeta = 0$. From the pullback-pushforward adjunction, we have

$$\mathrm{Hom}(\mathcal{L}_\zeta, m_*\underline{\mathbb{C}}) \cong \mathrm{Hom}(m^*\mathcal{L}_\zeta, \underline{\mathbb{C}}).$$

We now utilise the fact that $m^*\mathcal{L}_\zeta = \mathcal{L}_{m\zeta}$. This is immediate if one knows the functors giving the equivalence between local systems on $T$ and representations of $\pi_1(T, e)$, but we chose not to prove theorem 3.18, so we refer the reader to lemma 2.3.1 in [17] for the proof. Assuming this, we find that

$$\mathrm{Hom}(\mathcal{L}_\zeta, m_*\underline{\mathbb{C}}) \cong \mathrm{Hom}(\mathcal{L}_0, \underline{\mathbb{C}}) \cong \mathrm{Hom}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \neq 0.$$

Thus $\mathcal{L}_\zeta$ is indeed a constituent of $m_*\underline{\mathbb{C}}$. We now observe that $m_*\underline{\mathbb{C}}$ has rank $m$, and there are exactly $m$ such $\zeta$ in $\hat{X} \cong \mathbb{Z}_{(p)}/\mathbb{Z}$, represented by the elements $\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m-1}{m}$. By comparing ranks, these must therefore be all the constituents of $m_*\underline{\mathbb{C}}$, proving (†) and concluding the proof. □

This was quite an involved calculation, and most of the techniques we used will not be seen elsewhere in the thesis. We include it because, even though it is a very important result for computing character sheaves, it is almost entirely absent from the literature. Note that we do not claim this proof is original: it is based on Section 2 in [17], but the details in this source are so lacking that it could at best be called a sketch.

Despite this, [17] is the main reference for this result, for which reason we chose to include a complete proof as our contribution to the literature. The only details we have really omitted are the construction of the covering $\pi$ corresponding to $\rho$, and the consequent fact that $\pi^*\mathcal{L}$ is trivial. The former is a standard fact for which we have provided a reference in the body of the proof, and the latter is immediate if one simply writes down all of the equivalences in question.

# Chapter 4

# Deligne–Lusztig theory

In this chapter, we introduce the Deligne–Lusztig virtual characters for finite groups of Lie type, and give a survey of useful results. Because of the complexity of the theory, we could devote the entire thesis to proving all the results we list. Instead, we will prove a subset of the results which are illustrative of the techniques utilized in the field, or which show how these results relate to the standard representation theory of finite groups.

## 4.1 Motivation for the Deligne–Lusztig characters

The next several sections (4.2-4.4) will be devoted to various technical issues that must be dealt with before defining the Deligne–Lusztig characters. In the hopes that understanding the overarching goal will make these sections more cohesive, we will now give some brief motivation for the construction of the Deligne–Lusztig characters in section 4.5. As before, readers interested only in the definitions are invited to skip this section.

In the standard representation theory of finite groups, there is a particular class of representations, known as *permutation representations*, which are especially easy to work with. Such representations are constructed by taking an action of $G$ on a finite set $X$ and upgrading it to a representation on the free vector space $\mathbb{C}(X)$, which has a basis indexed by $X$. For example, the regular representation is the permutation representation induced by the left multiplication action of $G$ on itself.

Permutation representations are desirable because the character $\chi$ of such a representation is easily computable: $\chi(g)$ counts the number of fixed points of $G$ in $X$. When trying to construct new representations, permutation representations are obvious candidates.

To apply this idea in our geometric setting, we take our connected reductive group $G$ and act upon an algebraic variety $X$. We might then hope that counting fixed points under this action would give us a character. However, we need a more sophisticated notion of 'counting fixed points'. Here we borrow from topology once again by introducing the *Lefschetz number*, which for a continuous map $f : X \to X$ is defined by

$$\mathcal{L}(f) := \sum_i (-1)^i \operatorname{Tr}(f^*, H^i(X, \mathbb{Q})).$$

Here $X$ is some nice-enough topological space, such as a compact triangulable space, and $H^i(X, \mathbb{Q})$ is the $i^{th}$ simplicial cohomology.

Readers familiar with the Lefschetz-Hopf theorem will know that the Lefschetz number counts the number of fixed points of $f$ (weighted by ramification). To define the Deligne–Lusztig characters, we will construct a module whose trace is analogous to the Lefschetz number, so that the resulting character counts fixed points of this virtual permutation representation.

It is worth remarking now that the Deligne–Lusztig characters are not actual characters. Instead, they are $\mathbb{Z}$-linear combinations of the irreducible characters, often called *virtual characters*.

## 4.2 Frobenius maps and Lang's theorem

Throughout the remainder of the chapter, let $G$ be a connected reductive group over $\overline{\mathbb{F}}_p$, with $F : G \to G$ a Frobenius map. The main tool for studying $G^F$ through $G$ is the *Lang map*, defined by

$$\begin{aligned} L : G &\to G \\ g &\mapsto g^{-1}F(g) \end{aligned}$$

The following theorem, originally due to Lang but also often attributed to Steinberg, gives the most important property of $L$.

**Theorem 4.1.** *(Lang's theorem)*

*For $G$ connected, the Lang map $L$ is surjective.*

*Proof.* The classical proof is to show that $L$ is dominant (meaning its image is dense) and finite, hence closed. Recall that finiteness is local on the source and target, and $G$ is affine, so $\{G\}$ is a compatible affine open cover of the domain and codomain. It therefore suffices to show that $\overline{\mathbb{F}}_p[G]$ is finitely-generated over itself with the module structure induced by $L$. Steinberg did this by exhibiting an explicit finite generating set. The construction of such a generating set is not particularly enlightening, so we refer the reader to Theorem 4.1.12 in [1] for full details.

The proof that $L$ is dominant, however, illustrates the connection between the Lang map and $G^F$. Recall that, for every $y \in G$, the fiber $L^{-1}(y)$ satisfies

$$\dim L^{-1}(y) = \dim G - \dim \overline{\operatorname{im} L}.$$

However, the fibers of $L$ are exactly the right cosets of $G^F$, hence are finite and of dimension 0. It follows that

$$\dim \overline{\operatorname{im} L} = \dim G.$$

For linear algebraic groups, connectedness is equivalent to irreducibility (Proposition 1.3.13.c in [1]), so this is only possible if the closed subvariety $\overline{\operatorname{im} L}$ is all of $G$. Thus $L$ is dominant, as claimed. □

As an example of applying Lang's theorem to the descent problem, we will now study how certain actions of $G$ descend to actions of $G^F$. As Deligne–Lusztig theory is derived from the actions of $G$ on various algebraic varieties, this result will play a crucial role.

**Proposition 4.2.** *Assume that a connected linear algebraic group $G$ over $\overline{\mathbb{F}}_p$ acts transitively on a set $X$, and that there is a map $F' : X \to X$ satisfying*

*(i) $F'(g \cdot x) = F(g) \cdot F'(x)$ for all $g \in G, x \in X$;*

*(ii) The stabilizer of any point in $X$ is a closed subgroup of $G$.*

*Then*

*(a) The set $X^{F'}$ is non-empty.*

*(b) Additionally, if $\operatorname{Stab}_G(x_0)$ is connected for any $x_0 \in X^{F'}$, then the finite group $G^F$ acts transitively on $X^{F'}$.*

*Proof.* (a) Take any $x \in X$. Since $G$ acts transitively on $X$, we have $F'(x) = g^{-1} \cdot x$ for some $g \in G$. By Lang's theorem, we can write $g = h^{-1}F(h)$ for some $h \in G$. Then

$$F'(h \cdot x) = F(h) \cdot F'(x) = hg \cdot F'(x) = h \cdot x$$

and so $h \cdot x \in X^{F'}$.

(b) Let $H := \mathrm{Stab}_G(x_0)$, which by hypothesis is a closed connected subgroup of $G$. For any $h \in H$, observe that

$$F(h) \cdot x_0 = F(h) \cdot F'(x_0) = F'(h \cdot x_0) = F'(x_0) = x_0$$

so $F(h) \in H$. We may therefore regard $F$ as a Frobenius $H \to H$. Now let $x \in X^{F'}$, and write $x = g \cdot x_0$ for some $g \in G$. Then

$$g \cdot x_0 = x = F'(x) = F'(g \cdot x_0) = F(g) \cdot F'(x_0) = F(g) \cdot x_0$$

and hence $g^{-1}F(g) \in H$. Since $H$ is connected, we can apply Lang's theorem to $H$ and conclude that $g^{-1}F(g) = h^{-1}F(h)$ for some $h \in H$. But then $gh^{-1} \in G^F$ and

$$x = g \cdot x_0 = gh^{-1} \cdot x_0$$

meaning $x$ and $x_0$ lie in the same $G^F$-orbit, as required. □

**Corollary 4.3.** *Let $G$ be a connected linear algebraic group over $\overline{\mathbb{F}}_p$. Then there exists an $F$-stable Borel subgroup of $G$. Moreover, all $F$-stable Borel subgroups are $G^F$-conjugate.*

*Proof.* Let $\mathfrak{B}$ be the set of all Borel subgroups of $G$, which $G$ acts upon by conjugation. Because all Borel subgroups are conjugate, this action is transitve, and because all Borels are self-normalising, the stabilizer of any $B \in \mathfrak{B}$ is $B$ itself, which is closed.

We claim that the map

$$\begin{aligned} F' : \mathfrak{B} &\to \mathfrak{B} \\ B &\mapsto F(B) \end{aligned}$$

satisfies the hypotheses of Proposition 4.2. Let us first check that $F'$ is well-defined, i.e. that $F(B)$ is also a Borel subgroup of $G$.

Because the fibers of $F$ are finite, hence 0-dimensional, $\dim F(B) = \dim B$. Furthermore, homomorphisms of algebraic groups preserve the properties of being closed, connected and solvable, so $F(B)$ is also a Borel subgroup of $G$ and we actually have a map $F'$.

Since the map $F'$ clearly satisfies condition (i), we conclude that $\mathfrak{B}^{F'}$ is nonempty; in particular, there exists some $B_0 \in \mathfrak{B}^{F'}$. Because $N_G(B_0) = B_0$ is connected, part (b) of the proposition yields

$$\mathfrak{B}^{F'} = \{gB_0g^{-1} \mid g \in G^F\}. \qquad \square$$

## 4.3 $l$-adic cohomology

We will now describe the construction of $l$-adic cohomology, as well as the properties we will need (we will take these as axioms, as the proofs are far outside the scope of this thesis). This is an algebro-geometric analogue of singular cohomology which has many properties that we would expect from the topological setting. However, they are often better behaved in one respect: they tend to be acted upon by Galois groups.

This can be explained (at least intuitively) using the theory of the previous chapter. Galois groups can generally be realised as the étale fundamental group of certain varieties, and just as in the topological setting, there is a natural action of the fundamental group on the cohomology.

$l$-adic cohomology is relevant for us because we will soon construct actions of linear algebraic groups on various algebraic varieties. However, in representation theory, we want actions of these groups on vector spaces. These geometric actions induce actions on the cohomology vector spaces, and these will be our representations.

Before proceeding to $l$-adic cohomology, let us briefly touch upon the more general notion of étale cohomology.

**Definition 4.4.** Let $X$ be a scheme. Because the global sections functor

$$\begin{aligned} \mathfrak{Sh}_{\mathrm{Et}}(X) &\to \mathrm{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}(X) \end{aligned}$$

is left-exact and the category of étale sheaves of abelian groups has enough injectives (Lemma 5.1 in [16]), we can form the right-derived functors, denoted $H^i(X, -)$. These are the étale cohomology functors.

Étale cohomology satisfies many of the properties we would expect from a cohomology theory, such as the existence of a long exact sequence (which is immediate from the definition of the $H^i$ as right-derived functors). However, it lacks many of the more geometric properties we would hope for, such as Poincaré duality. We need a more refined cohomology theory to properly capture the geometry of schemes over $\overline{\mathbb{F}}_p$.

**Definition 4.5.** Suppose that $X$ is a scheme over $\overline{\mathbb{F}}_p$, and let $l$ be any prime different to $p$. For each power $l^k$, we consider the constant étale sheaf $\underline{\mathbb{Z}/l^k}$. The reduction map $\mathbb{Z}/l^{k+1} \to \mathbb{Z}/l^k$ induces a sheaf morphism $\underline{\mathbb{Z}/l^{k+1}} \to \underline{\mathbb{Z}/l^k}$, and functoriality of étale cohomology then gives us an inverse system $\{H^i(X, \underline{\mathbb{Z}/l^k})\}_{k\in\mathbb{N}}$. We set

$$H^i(X, \mathbb{Z}_l) := \varprojlim H^i(X, \underline{\mathbb{Z}/l^k}).$$

To remove torsion, we now define

$$H^i(X, \overline{\mathbb{Q}}_l) := H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l.$$

**Remark 4.6.** There is a notorious trap here. It is **not** true that the modules $H^i(X, \mathbb{Z}_l)$ are the standard étale cohomology with the constant sheaf $\underline{\mathbb{Z}_l}$. We have omitted the underlines on the left to emphasize this, but the reader should keep in mind that $H^i(X, \mathbb{Z}_l)$ and $H^i(X, \underline{\mathbb{Z}_l})$ are not isomorphic in general.

The vector spaces $H^i(X, \overline{\mathbb{Q}}_l)$ are almost what we need, but are generally too large for our purposes (it is possible for $H^i(X, \overline{\mathbb{Q}}_l)$ to be infinite-dimensional, whereas we only want finite-dimensional representations). To rectify this, we introduce $l$-adic cohomology *with compact support.*

**Definition 4.7.** Now let $X$ be a variety and let $j : X \to Y$ be an open immersion of $X$ into a proper variety $Y$. For an étale sheaf of abelian groups $\mathcal{F}$ on $X$, the étale cohomology with compact support is

$$H^i_c(X, \mathcal{F}) := H^i(Y, j_!\mathcal{F}).$$

The $l$-adic cohomology with compact support $H^i_c(X, \overline{\mathbb{Q}}_l)$ is defined by replacing $H^i(X, \mathbb{Z}_l)$ with $H^i_c(X, \mathbb{Z}_l)$ in Definition 4.5.

It is a consequence of *Nagata's compactification theorem* (see Theorem 4.5.6 in [28]) that such a $j$ always exists, and that $H^i_c(X, \mathcal{F})$ does not depend on $j$ or $Y$.

Henceforth, whenever we say $l$-adic cohomology, we mean $l$-adic cohomology with compact support, as this is the only variant that interests us. This will be reflected in the notation $H_c^i(X, \overline{\mathbb{Q}}_l)$.

Rather than attempt to work with these complicated definitions, we will now give an exhaustive list of the properties of $l$-adic cohomology which we will need for the definition of the Deligne–Lusztig characters. Many of them pertain to the following invariant:

**Definition 4.8.** Let $g$ be an automorphism of $X$ of finite order. This induces an action on the $l$-adic cohomology $H_c^i(X, \overline{\mathbb{Q}}_l)$ (it is nontrivial that we get an induced action, but we simply assert it). We define the *Lefschetz number* by

$$\mathcal{L}(g, X) := \sum_i (-1)^i \operatorname{Tr}(g, H_c^i(X, \overline{\mathbb{Q}}_l)).$$

The next theorem will enumerate the properties of $l$-adic cohomology that we need. Their proofs are by no means simple; we refer the reader to [8], [10] and [11].

**Theorem 4.9.** *$l$-adic cohomology and the Lefschetz number satisfy the following properties:*

1. *The Lefschetz number $\mathcal{L}(g, X)$ is an integer independent of the choice of prime $l \neq p$;*

2. *Let $f : X \to Y$ be a regular map of algebraic varieties over $\overline{\mathbb{F}}_p$ and suppose there is some fixed $n \in \mathbb{N}$ such that for every $y \in Y$, the fiber $f^{-1}(y)$ is isomorphic to affine space $\mathbb{A}^n_{\overline{\mathbb{F}}_p}$. Let $g, h$ be automorphisms of $X, Y$ respectively such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

*commutes. Then $\mathcal{L}(g, X) = \mathcal{L}(h, Y)$;*

3. *Suppose $X$ decomposes as*
$$X = \coprod_{j=1}^{m} X_j$$
*where each $X_j$ is a locally closed subscheme of $X$ which is stable under $g$. Then*
$$\mathcal{L}(g, X) = \sum_{j=1}^{m} \mathcal{L}(g, X_j).$$

4. *Suppose $X$ decomposes as*
$$X = \coprod_{j=1}^{m} X_j$$
*where each $X_j$ is a closed subscheme (not just locally closed, as in the previous property). Let $G$ be a finite group permuting the $X_j$ transitively, and define $H := \mathrm{Stab}_G(X_1)$. Then the virtual character $g \mapsto \mathcal{L}(g, X)$ of $G$ is induced by the virtual character $h \mapsto \mathcal{L}(h, X_1)$ of $H$.*

5. *Let $X$ be an affine variety acted upon by a finite automorphism group $G$. Then*
$$H^i_c(X/G, \overline{\mathbb{Q}}_l) \cong H^i_c(X, \overline{\mathbb{Q}}_l)^G.$$

6. *(Künneth formula)*

   *For $X_1, X_2$ algebraic varieties, we have*
$$H^k_c(X_1 \times X_2, \overline{\mathbb{Q}}_l) \cong \bigoplus_{i+j=k} H^i_c(X_1, \overline{\mathbb{Q}}_l) \otimes H^j_c(X_2, \overline{\mathbb{Q}}_l).$$
   *Moreover, if $g_1, g_2$ are automorphisms of $X_1, X_2$ respectively then*
$$\mathcal{L}(g_1 \times g_2, X_1 \times X_2) = \mathcal{L}(g_1, X_1)\mathcal{L}(g_2, X_2).$$

7. *Suppose $G$ is a connected algebraic group acting on an algebraic variety $X$. Then each element of $G$ acts trivially on $H^i_c(X, \overline{\mathbb{Q}}_l)$.*

8. *If $X$ is finite and $g$ is an automorphism of $X$ then*
$$\mathcal{L}(g, X) = |X^g|.$$

9. *Let $g$ be an automorphism of $X$ with Jordan decomposition $g = su = us$, where $s$ is semisimple and $u$ is unipotent. Then*
$$\mathcal{L}(g, X) = \mathcal{L}(u, X^s).$$
   *Recalling that $s$ has order coprime to $p$ and $u$ has order a power of $p$, $s$ and $u$ are often respectively called the $p'$- and $p$-parts of $g$.*

## 4.4 Induction and restriction via bimodules

We are almost at the point where we can define the Deligne–Lusztig characters. The only remaining ingredient is an appropriate abstraction of the notions of induction and restriction of representations.

**Notation 4.10.** Throughout the remainder of the thesis, we will abuse notation by calling a $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule a $(G, H)$-bimodule. We will also often consider such bimodules as $G \times H^{\mathrm{opp}}$-modules.

**Definition 4.11.** Let $G, H$ be finite groups, and fix a $(G, H)$-bimodule $M$. We have a functor from the category of left $\mathbb{C}[H]$-modules to the category of left $\mathbb{C}[G]$-modules given by

$$R_H^G : E \mapsto M \otimes_{\mathbb{C}[H]} E$$

where $G$ acts on $M \otimes_{\mathbb{C}[H]} E$ through its action on $M$. If we set $M^* = \mathrm{Hom}(M, \mathbb{C})$ then the tensor-hom adjunction shows that tensoring with $M^*$ gives a right-adjoint functor ${}^*R_H^G$.

**Remark 4.12.** This is not the best possible notation, as the definition above depends on the $(G, H)$-bimodule $M$, but $R_H^G$ does not reflect this. When constructing Deligne–Lusztig induction, we will have one explicit bimodule $M$ and this will not be a problem, but it is worth keeping in mind.

**Example 4.13.** To demonstrate the usefulness of this construction, let us look at two cases where we have already seen it in practise.

(i) Suppose that $H \subseteq G$ is a subgroup and $M = \mathbb{C}[G]$ is the group algebra, on which $G$ acts by left translation and $H$ acts by right translation. Then $R_H^G = \mathrm{Ind}_H^G$ is the standard induction of representations. The right adjoint ${}^*R_H^G = \mathrm{Res}_H^G$ is restriction.

(ii) We can also recover parabolic induction. Given a finite group of Lie type $G$ and a Borel $B$, recall that $B = U \rtimes T$ for $U$ the unipotent radical of $B$ and $T \subseteq B$ a maximal torus. It follows that $T \cong B/U$, and so we can consider the $(G, T)$-bimodule $\mathbb{C}[G] \otimes_{\mathbb{C}[B]} \mathbb{C}[T]$. The functor $R_T^G$ corresponds to extending a representation of $T$ to one of $B$ by letting $U$ act trivially (in this language, quotienting out by $U$) and then inducing to $G$.

As we will mostly study Deligne–Lusztig characters, rather than the representations, we would like a formula for the trace of automorphisms acting on such modules.

**Proposition 4.14.** *Let $M$ be a $(G,H)$-bimodule and $E$ an $H$-module. For $g \in G$, we have*

$$\mathrm{Tr}(g, R_H^G(E)) = \frac{1}{|H|}\sum_{h\in H}\mathrm{Tr}((g,h^{-1}),M)\,\mathrm{Tr}(h,E).$$

*Proof.* The idea is to replace $M \otimes_{\mathbb{C}[H]} E$ by an isomorphic $G$-module upon which the trace is easier to compute. Consider the element $\frac{1}{|H|}\sum_{h\in H} h^{-1}\otimes h$, which is an idempotent of the group algebra $\mathbb{C}[H \times H^{\mathrm{opp}}] = \mathbb{C}[H] \otimes \mathbb{C}[H^{\mathrm{opp}}]$. Consequently, its image in the representation of $H \times H^{\mathrm{opp}}$ on $M \otimes_{\mathbb{C}} E$ is a projection (note that here we are tensoring over $\mathbb{C}$, not yet over $\mathbb{C}[H]$). We claim that its kernel is generated by elements of the form $mh \otimes x - m \otimes hx$.

Indeed, if $\sum_i \sum_{h\in H} m_i h^{-1} \otimes hx_i = 0$ then

$$\sum_i m_i \otimes x_i = \frac{1}{|H|}\sum_{h\in H}\sum_i \left(m_i \otimes x_i - m_i h^{-1} \otimes hx_i\right)$$

which expresses $\sum_i m_i \otimes x_i$ in the desired form.

But $M \otimes_{\mathbb{C}[H]} E$ is the quotient of $M \otimes_{\mathbb{C}} E$ by elements of this form, so we can identify $M \otimes_{\mathbb{C}[H]} E$ with the image of our projection. Thus

$$\mathrm{Tr}(g, R_H^G(E)) = \mathrm{Tr}\left(\frac{1}{|H|}\sum_{h\in H}(g,h^{-1})\otimes h, M \otimes_{\mathbb{C}} E\right)$$

and the proposition now follows from additivity and multiplicativity of the trace. □

## 4.5 Deligne–Lusztig induction and restriction

We are at last ready to define the Deligne–Lusztig characters. Motivated by Section 4.1, we wish to use the Lefschetz number to construct a virtual permutation representation, so we must find an appropriate variety upon which $G^F$ acts. We fix an $F$-**stable** maximal torus $T$ of $G$ and pick a Borel subgroup $B \supseteq T$. Decomposing $B$ as $B = U \rtimes T$, let us consider the algebraic variety $\tilde{X} := L^{-1}(U)$, where $L : G \to G$ is the Lang map.

We claim that $G^F$ acts on $\tilde{X}$ by left multiplication, and $T^F$ acts by right-multiplication. Indeed, observe that

$$L(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) = L(x) \in U$$

where the middle equality follows from the fact that $g \in G^F$, so $F(g) = g$. Similarly,

$$L(xt) = t^{-1}L(x)F(t) = t^{-1}L(x)t \in t^{-1}Ut = U.$$

We therefore have an action of $G^F \times (T^F)^{\text{opp}}$ on $\tilde{X}$. As this is a finite group, every element has finite order and so we get an induced action of $G^F \times (T^F)^{\text{opp}}$ on the $l$-adic cohomology $H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$.

**Definition 4.15.** As described above, $H^*_c(\tilde{X}, \overline{\mathbb{Q}}_l) := \sum_i (-1)^i H^i_c(\tilde{X}, \overline{\mathbb{Q}}_l)$ is a virtual $(G^F, T^F)$-bimodule. The *Deligne–Lusztig induction* functor $R^G_T$ is the corresponding induction functor, as constructed in Definition 4.11. The *Deligne–Lusztig restriction* functor is the right adjoint ${}^*R^G_T$. These functors can be applied to characters using the equivalence between $G$-representations and $\mathbb{C}[G]$-modules.

**Remark 4.16.** There are several important observations we can immediately make about the Deligne–Lusztig functors:

- From their definition as adjoints, it follows immediately that Deligne–Lusztig induction & restriction satisfy Frobenius reciprocity. Explicitly, for any representations $\chi$ of $G^F$ and $\theta$ of $T^F$, we have

$$\langle R^G_T(\theta), \chi\rangle_{G^F} = \langle \theta, {}^* R^G_T(\chi)\rangle_{T^F}.$$

- A priori, this construction depends on a choice of Borel $B \supseteq T$, since we needed to use the complementary unipotent radical $U$ in the definition of the variety $\tilde{X}$. We shall later see that this construction is independent of choice (Corollary 4.28), so from the start we omit the Borel from our notation.

- While Deligne–Lusztig induction actually gives virtual representations of $G^F$, understanding the actions would involve computing the action of $G^F \times (T^F)^{\text{opp}}$ on the $l$-adic cohomology, which is outside the scope of this thesis. Instead, we will content ourselves with working on the level of characters. Applying Proposition 4.14 immediately gives the following preliminary formula for the Deligne–Lusztig characters:

$$(R^G_T\theta)(g) = \frac{1}{|T^F|}\sum_{t\in T^F} \theta(t^{-1})\mathcal{L}((g,t), \tilde{X})$$

  This currently involves the Lefschetz number, which is difficult to compute, but we will later obtain a formula which does not involve the $l$-adic cohomology at all.

- We want to emphasize again that in all of the above, the torus $T$ is $F$-stable. However, we do **not** assume that the Borel $B$ is $F$-stable; indeed, the fact that some maximal tori are not contained in any $F$-stable Borel is what will allow us to find the cuspidal representations. If $T$ is contained in an $F$-stable Borel then we say that $T$ is *maximally split*.

At this point, it is not clear that the Deligne–Lusztig characters generate useful representations. We will now show that we can recover the notion of parabolic induction when $T$ is particularly simple, meaning it is maximally split, so this construction gives at least as many irreps as we had before.

**Theorem 4.17.** *Let $B \supseteq T$ be a maximally split $F$-stable Borel-torus pair. Let $\theta$ be an irreducible character of $T$, and denote by $\tilde{\theta}$ the extension of $\theta$ to $B$ by letting $U$ act trivially. Then $R_T^G(\theta) = \operatorname{Ind}_{B^F}^{T^F}(\tilde{\theta})$ is a parabolically induced character.*

*Proof.* As in the proof of Corollary 4.3, let $\mathfrak{B}^F$ be the set of $F$-stable Borel subgroups of $G$. We define a map $\Psi : \tilde{X} \to \mathfrak{B}^F$ by $\Psi(g) = gBg^{-1}$. To see that $gBg^{-1}$ is actually an $F$-stable Borel, we observe that $L(g) = g^{-1}F(g) \in U \subset B$ and so

$$F(gBg^{-1}) = F(g)BF(g)^{-1} = g(g^{-1}F(g))B(g^{-1}F(g))^{-1}g^{-1} = gBg^{-1}.$$

Note that $\Psi$ is surjective because all $F$-stable Borel are $G^F$-conjugate, and $G^F = L^{-1}(1) \subseteq L^{-1}(U) = \tilde{X}$.

Let $\tilde{X}_1, \ldots, \tilde{X}_n$ be the fibers of $\Psi$, with $\tilde{X}_1$ the fiber over $B$. Then

$$\tilde{X} = \coprod_{j=1}^{n} \tilde{X}_j.$$

Observe that $G^F$ acts on $\tilde{X}$ by left multiplication and on $\mathfrak{B}^F$ by conjugation, and $\Psi$ commutes with both actions. Because $G^F$ acts transitively on $\mathfrak{B}^F$, commutativity implies that $G^F$ also permutes the fibers of $\Psi$ transitively. Moreover, each fiber $\tilde{X}_j$ is closed. Thus the hypotheses of property 4 of $l$-adic cohomology are satisfied, so the virtual character $\mathcal{L}((g,t), \tilde{X})$ of $G^F \times T^F$ is induced from the virtual character $\mathcal{L}((b,t), \tilde{X}_1)$ of the stabilizer of $\tilde{X}_1$. Because Borel subgroups are self-normalizing, the stabilizer of $\tilde{X}_1 = \Psi^{-1}(B)$ is exactly $B^F \times T^F$.

Now recall the formula

$$(R_T^G\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1})\mathcal{L}((g,t), \tilde{X}).$$

By the above, this is induced from the virtual character

$$b \mapsto \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathcal{L}((b,t), \tilde{X}_1)$$

of $B^F$, so we want to show that this is exactly $\tilde{\theta}$.

Let us start by explicitly describing the fiber $\tilde{X}_1$, which we claim is $T^F U$. Any element of $B$ can be written uniquely as $b = tu$, and then $L(b) = u^{-1}t^{-1}F(t)F(u)$.

We then see that

$$b \in \tilde{X} \iff L(b) \in U \iff t^{-1}F(t) \in U \iff t \in T^F$$

where the final statement follows because $T$ and $U$ intersect trivially, so the only way that $t^{-1}F(t)$ can lie in $U$ is if it is the identity. By observing that $\tilde{X}_1 = \tilde{X} \cap B$, we have proven the claim.

We now consider the natural map $T^F U \to T^F U/U$, which is a regular morphism of varieties. The fibers of this map are all isomorphic to the affine space $\overline{\mathbb{F}}_p^{\dim(U)}$, and clearly the map commutes with the action of $(b,t)$ on either side. We may therefore use property 2 of $l$-adic cohomology to find

$$\mathcal{L}((b,t), \tilde{X}_1) = \mathcal{L}((b,t), T^F U) = \mathcal{L}((b,t), T^F U/U).$$

The final variety is isomorphic to $B^F/U^F$ in an obvious way, so this is all also equal to $\mathcal{L}((b,t), B^F/U^F)$. Since $B^F/U^F$ is finite, the Lefschetz number actually just counts fixed points. Explicitly, property 8 of $l$-adic cohomology yields

$$\mathcal{L}((b,t), B^F/U^F) = \left|(B^F/U^F)^{(b,t)}\right|$$

so it only remains to count these fixed points.

Elements of $B^F/U^F$ are of the form $sU^F$ for $s \in T^F$. Now

$$(b,t) \text{ fixes } sU^F \iff bsU^F t = sU^F \iff bU^F = t^{-1}U^F.$$

Note that in the above, we have used multiple times that $T^F$ normalizes $U^F$, so $sU^F = U^F s$ and $t^{-1}U^F = U^F t^{-1}$. We now see that $\left|(B^F/U^F)^{(b,t)}\right| = 0$ unless $b \in t^{-1}U^F$, in which case $\left|(B^F/U^F)^{(b,t)}\right| = |T^F|$. Combined with the fact that every $b \in B^F$ lies in a unique left coset of $U^F$, we conclude that $R_T^G(\theta)$ is induced from the character of $B^F$ given by

$$b \mapsto \theta(t^{-1})$$

where $t$ is the unique element of $T^F$ satisfying $b \in t^{-1}U^F$. But this exactly the extended character $\tilde{\theta}$, as required. □

This is the archetype for proofs of properties of the Deligne–Lusztig characters: one starts with a natural action on some variety (in this case, the natural action of $G^F$ on the finite variety $\mathfrak{B}^F$ by conjugation), then uses properties of $l$-adic cohomology to reduce the problem to a combinatorial one.

## 4.6 The character formula

The rest of the chapter will be devoted to exploring the properties of Deligne–Lusztig induction & restriction to see how they allow us to construct irreps of finite groups of Lie type. For the most part, we will omit proofs, with one key exception: in this section, we will prove the character formula. This is an explicit formula for the Deligne–Lusztig characters which makes no reference to $l$-adic cohomology and which is eminently computable.

The formula will be in terms of the Jordan decomposition, since unipotent and semisimple elements are comparatively much easier to understand. For the rest of the section, we let $g \in G^F$ have Jordan decomposition $g = su = us$. Also let $t \in T^F$. As a first step towards the character formula, let us rewrite $R_T^G\theta(g)$ in terms of the Jordan decomposition.

**Lemma 4.18.** *Define $\tilde{X}^{(s,t)} := \{x \in \tilde{X} \mid sxt = x\}$. Then*

$$R_T^G\theta(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}(u, \tilde{X}^{(s,t)}).$$

*Proof.* Recall that we have the formula

$$R_T^G\theta(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}((g,t), \tilde{X}).$$

The maps $v \mapsto vt$ and $v \mapsto sv$ both have order coprime to $p$, and they commute, so their composition $v \mapsto svt$ also has order coprime to $p$. The map $v \mapsto uv$ has order a power of $p$. Since these two maps commute and their composition is $v \mapsto suvt = gvt$, it follows that $(s,t)$ and $(u,1)$ are respectively the $p'$- and $p$-parts of $(g,t)$. Property 9 now lets us conclude immediately. □

We have therefore reduced the study of $R_T^G(\theta)$ to understanding the varieties $\tilde{X}^{(s,t)}$. As a step in this direction, the next proposition will construct a map onto $\tilde{X}^{(s,t)}$ and we will use this to pull the calculation of the Lefschetz number back onto even simpler varieties. The proof of this proposition is quite technical, so we have deferred it to the appendix for interested readers.

**Proposition B.3.** Let $t \in T^F$ and suppose $s \in G^F$ is a semisimple element which is $G^F$-conjugate to $t^{-1}$. Define $(G^F)^{(s,t)} := \{k \in G^F \mid skt = k\}$ and $\tilde{Y}_t := \tilde{X} \cap C^0(t)^\dagger$. Then:

(i) The map $(k, z) \mapsto kz$ is a surjective morphism $(G^F)^{(s,t)} \times \tilde{Y}_t \to \tilde{X}^{(s,t)}$;

(ii) $C^0(t)^F$ acts on $(G^F)^{(s,t)} \times \tilde{Y}_t$ by $m \cdot (k, z) = (km^{-1}, mz)$;

(iii) The orbits of $C^0(t)^F$ are exactly the fibers of the map in (i).

We almost have all of the necessary results. The final step is to decompose $\tilde{X}^{(s,t)}$ in a way that allows us to apply the previous proposition.

**Lemma 4.19.** *Let $s, t$ be as above. Then $\tilde{X}^{(s,t)}$ is the disjoint union of $[C(t)^F : C^0(t)^F]$ closed subvarieties, each isomorphic to $\tilde{Y}_t$.*

*Proof.* For $k \in (G^F)^{(s,t)}$, we have $(G^F)^{(s,t)} = kC(t)^F$. Let $z_1, \dots, z_r$ be a set of coset representatives for $C^0(t)^F$ in $C(t)^F$. Then

$$(G^F)^{(s,t)} = \coprod_{j=1}^{r} kz_jC^0(t)^F.$$

Decomposing $\tilde{X}^{(s,t)}$ into the fibers of the map from the previous proposition, which are exactly the orbits of the $C^0(t)^F$ action, it follows that

$$\tilde{X}^{(s,t)} = \coprod_{j=1}^{r} kz_j\tilde{Y}_t.$$

Each $kz_j\tilde{Y}_t$ is a closed subset of $G$, hence also of $\tilde{X}^{(s,t)}$. Clearly $kz_j\tilde{Y}_t \cong \tilde{Y}_t$. □

**Definition 4.20.** For a unipotent $u \in G^F$, we define the *Green function* $Q_T^G$ by $Q_T^G(u) = R_T^G\mathbf{1}(u)$. We will later see that $Q_T^G(u) = R_T^G\theta(u)$ for any character $\theta$ of $T^F$.

**Theorem 4.21.** *(Character formula)*

*Let $g \in G^F$ have Jordan decomposition $g = su = us$. Then*

$$R_T^G\theta(g) = \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx)Q_{xTx^{-1}}^{C^0(s)}(u).$$

† Here $C(t)$ is the centraliser of $t$ in $G$, and $C^0(t)$ is the connected component of the identity in $C(t)$.

**Remark 4.22.** Before proceeding with the proof, let us check that the expression $Q^{C^0(s)}_{xTx^{-1}}(u)$ makes sense. Since $x^{-1}sx \in T^F$, we have $s \in xT^Fx^{-1} = (xTx^{-1})^F$. Thus $xTx^{-1}$ is an $F$-stable maximal torus in $C^0(s)$. Because $u$ commutes with $s$, we have $u \in C(s) \cap U \subseteq C^0(s)$, and the expression is indeed well-defined.

*Proof.* We continue to make use of the varieties $kz_j\tilde{Y}_t$ seen in the previous lemma. To simplify notation, set $k_j := kz_j$, so that $\tilde{X}^{(s,t)} = \coprod_{j=1}^r k_j\tilde{Y}_t$.

By the preceding lemmas, we already have the expressions

$$\begin{aligned} R_T^G\theta(g) &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}(u,\tilde{X}^{(s,t)}) \\ &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}(u,\coprod_{j=1}^r k_i\tilde{Y}_t) \end{aligned}$$

We claim that each of the subvarieties $k_i\tilde{Y}_t$ is invariant under $u$, so that we can split the Lefschetz number into a sum. Indeed, $k_i \in (G^F)^{(s,t)}$ so $sk_it = k_i$.

Hence

$$t^{-1}k_i^{-1}uk_it = k_i^{-1}sus^{-1}k_i = k_i^{-1}uk_i$$

so $k_i^{-1}uk_i \in C(t)$. Since $U$ is normal, $k_i^{-1}uk_i$ is unipotent and so $k_i^{-1}uk_i \in C^0(t)^F$. But recalling that $\tilde{Y}_t = \tilde{X} \cap C^0(t)$, it follows that $k^{-1}uk_i\tilde{Y}_t = \tilde{Y}_t$. Rearranging yields $uk_i\tilde{Y}_t = k_i\tilde{Y}_t$, as claimed.

Property 3 of $l$-adic cohomology now yields

$$R_T^G\theta(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\sum_{j=1}^r\mathcal{L}(u,k_j\tilde{Y}_t).$$

Observe that $u$ acts on $k_i\tilde{Y}_t$ in the same way that $k_i^{-1}uk_i$ acts on $\tilde{Y}_t$. Thus

$$\begin{aligned} R_T^G\theta(g) &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\sum_{j=1}^r\mathcal{L}(k_i^{-1}uk_i,\tilde{Y}_t) \\ &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\frac{1}{|C^0(t)^F|}\sum_{k\in(G^F)^{(s,t)}}\mathcal{L}(k^{-1}uk,\tilde{Y}_t) \\ &= \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\frac{1}{|C^0(t)^F|}\sum_{\substack{k\in G^F \\ k^{-1}sk=t^{-1}}}\mathcal{L}(k^{-1}uk,\tilde{Y}_t) \end{aligned}$$

Let us rewrite this expression by replacing every occurrence of $t^{-1}$ by $k^{-1}sk$. Also observe that

$$C^0(t)^F = C^0(t^{-1})^F = C^0(k^{-1}sk)^F = (k^{-1}C^0(s)k)^F = k^{-1}C^0(s)^Fk.$$

Since conjugate subgroups have the same order, we may therefore also replace $|C^0(t)^F|$ by $|C^0(s)|^F$, which eliminates all occurences of $t$. Making these substitutions now yields

$$R_T^G\theta(g) = \frac{1}{|T^F|}\frac{1}{|C^0(s)^F|}\sum_{\substack{k\in G^F\\ k^{-1}sk\in T^F}}\theta(k^{-1}sk)\mathcal{L}(k^{-1}uk, \tilde{Y}_{k^{-1}sk})$$
$$= \frac{1}{|T^F|}\frac{1}{|C^0(s)^F|}\sum_{\substack{k\in G^F\\ k^{-1}sk\in T^F}}\theta(k^{-1}sk)\mathcal{L}(k^{-1}uk, \tilde{Y}_s)$$

This expression allows us to study the Deligne–Lusztig characters on unipotent elements. Setting $s = 1$ and $\theta = \mathbf{1}$ in the above, we find that

$$Q_T^G(u) = R_T^G\mathbf{1}(u) = \frac{1}{|T^F|}\mathcal{L}(u, \tilde{X}).$$

On the other hand,

$$\tilde{Y}_s = \tilde{X}\cap C^0(s) = L^{-1}(U)\cap C^0(s) = L^{-1}(U\cap C^0(s))\cap C^0(s).$$

Because $U\cap C^0(s)$ is a maximal unipotent subgroup of $C^0(s)$ (Corollary 3.5.5 in [29]), $\tilde{Y}_s$ is the analogue for $C^0(s)$ of $\tilde{X}$ for $G$. In particular, applying the above formula for the Green functions gives

$$Q_{kTk^{-1}}^{C^0(s)}(u) = \frac{1}{|T^F|}\mathcal{L}(u, \tilde{Y}_s).$$

Substituting this into our final expression for $R_T^G\theta(g)$ yields the desired formula. □

**Corollary 4.23.** *If $u \in G^F$ is unipotent then $R_T^G\theta(u)$ does not depend on $\theta$.*

*Proof.* Since $s = 1$, the character formula gives

$$R_T^G\theta(u) = \frac{1}{|G^F|}\sum_{x\in G^F} Q_{xTx^{-1}}^G(u)$$

which does not depend on $\theta$. □

## 4.7 Other results

In the final section of this chapter, we give a survey and discussion of other key results about Deligne–Lusztig induction & restriction. We begin with the Mackey decomposition, which is an analogue of the Mackey decomposition for standard induction & restriction of representations.

For an element $x \in G$, observe that ${}^xT := xTx^{-1}$ is also a maximal torus in $G$, so we also have a Deligne–Lusztig induction functor $R^G_{{}^xT}$. Guided by the Mackey decomposition for standard induction and restriction, we might hope that $R^G_T$ and $R^G_{{}^xT}$ are related by conjugation. More precisely, we can consider the diagram

$$\begin{tikzcd} & G^F \arrow[dr, "{}^*R^G_{{}^xT}"] & \\ T^F \arrow[ur, "R^G_T"] \arrow[rr, "\mathrm{ad}(x)"'] & & ({}^xT)^F \end{tikzcd}$$

where $\mathrm{ad}(x)$ is the functor from $T$ characters to ${}^xT$ characters defined by

$$\mathrm{ad}(x)(\theta)(xtx^{-1}) := \theta(t).$$

Note that this diagram does **not** commute, but it comes as close to commuting as we could hope for, in the sense that summing these diagrams over appropriate choices of $x$ will commute. For two maximal tori $T, S$, we denote by $N(T, S)$ the set of $x \in G$ for which ${}^xS = T$.

**Theorem 4.24.** *(Mackey decomposition)*

*Let $T, S$ be two $F$-stable maximal tori. Then*

$${}^*R^G_T \circ R^G_S = \sum_{T^FxS^F \in T^F\backslash N(T,S)^F/S^F} \mathrm{ad}(x).$$

**Remark 4.25.** There are two important remarks to make about our statement of the Mackey decomposition.

(i) We will not prove the Mackey decomposition as it is quite involved. Readers interested in the proof may find this as Theorem 11.13 in [13], though it is stated there in far more generality than we give here. To recover our formulation, note that both Levi subgroups are maximal tori, so the only way $T \cap^x S$ can contain a maximal torus is if $T = {}^xS$. In this case, the Deligne–Lusztig induction functors $R^T_{T\cap^xS}$ and ${}^*R^{{}^xS}_{T\cap^xS}$ are both trivial, so we have omitted them from our statement.

(ii) It is worth noting that $N(T,S)^F$ can be empty: indeed, if $T$ and $S$ are conjugate by an element $x \in G^F$ then $xS^Fx^{-1} = (xSx^{-1})^F = T^F$. However, Appendix A contains examples of two maximal tori in $\mathrm{GL}_2(\mathbb{F}_q)$ which have different orders, so in particular cannot be conjugate. If $N(T,S)^F = \emptyset$ then both sides of the Mackey decomposition are trivial.

The Mackey decomposition is not particularly useful for computing values of the Deligne–Lusztig characters, since it is essentially subsumed by the character formula. However, since Deligne–Lusztig characters are so controlled by their values on maximal tori, it provides an important tool for understanding their structure. The following theorem is one of the most important instances of this.

**Theorem 4.26.** *(Inner product formula)*

*Let $T, S$ be two maximal tori, with respective characters $\theta, \psi$. Then*

$$\langle R_T^G\theta, R_S^G\psi\rangle = \frac{1}{|T^F|}|\{x \in N(T,S)^F : \mathrm{ad}(x^{-1})(\theta) = \psi\}|.$$

*Proof.* Using Frobenius reciprocity and the Mackey decomposition, we find

$$\begin{aligned}\langle R_T^G\theta, R_S^G\psi\rangle &= \langle \theta, {}^*R_T^G R_S^G\psi\rangle \\ &= \sum_{T^FxS^F \in T^F\backslash N(T,S)^F/S^F} \langle \theta, \mathrm{ad}(x)(\psi)\rangle \\ &= \sum_{T^FxS^F \in T^F\backslash N(T,S)^F/S^F} \langle \mathrm{ad}(x^{-1})(\theta), \psi\rangle\end{aligned}$$

Let us rewrite this without the double cosets. We recall the general formula that if $H, K$ are subgroups of a finite group, then the order of the double coset $HxK$ is

$$|HxK| = \frac{|H|\cdot|K|}{|x^{-1}Hx \cap K|}.$$

We apply this with $H = T^F, K = S^F$. Recall that for $x \in N(T,S)^F$, $x^{-1}T^Fx = S^F$ so the intersection in the denominator will just be $S^F$. It follows that the order of each double coset in the above sum is

$$|T^FxS^F| = |T^F|$$

and so

$$\langle R_T^G\theta, R_S^G\psi\rangle = \frac{1}{|T^F|}\sum_{x\in N(T,S)^F} \langle \mathrm{ad}(x^{-1})(\theta), \psi\rangle.$$

The formula now follows immediately by noting that $\mathrm{ad}(x^{-1})(\theta)$ and $\psi$ are standard linear characters of $S^F$, hence orthonormal, so the summands are 1 if $\mathrm{ad}(x^{-1})(\theta) = \psi$ and 0 otherwise. □

If $T = S$ then $N(T,T) = N_G(T)$ is the normaliser of $T$ in $G$. Notice that because $T^F$ is abelian, $\mathrm{ad}(t^{-1})$ is trivial and so we get an induced action on the characters of $T^F$ by $N_G(T)^F/T^F$. A simple application of Lang's theorem shows that, because $T$ is a closed connected $F$-stable subgroup of $N_G(T)$, we have $N_G(T)^F/T^F = (N_G(T)/T)^F = W^F$, where $W$ is the Weyl group (see Corollary 3.13 in [13]). In this case, we can remove the factor of $\frac{1}{|T^F|}$ to write

$$\langle R_T^G\theta, R_T^G\psi\rangle = |\{\omega \in W^F : \mathrm{ad}(\omega^{-1})(\theta) = \psi\}|.$$

At this point, we make a small historical detour to discuss the significance of *regular characters.* We say that a character $\theta$ of $T^F$ is *regular* if the only element of $W^F$ which fixes it is the identity.

Using a generalization of the techniques in Appendix A, it has long been known that there is a map from the regular characters of $\mathrm{GL}_n(\mathbb{F}_q)$ to its irreps. It was a conjecture of Macdonald, that such a map should exist for all finite groups of Lie type, that motivated Deligne & Lusztig to develop their theory. The inner product formula finally settled the problem to the affirmative.

**Corollary 4.27.** *Let $\theta \in \mathrm{Irr}(T^F)$ be a regular character. Then either $R_T^G\theta$ or $-R_T^G\theta$ is an irrep of $G^F$.*

*Proof.* Since $R_T^G\theta$ is a virtual character of $G^F$, we can write it as a $\mathbb{Z}$-linear combination of irreps, say

$$R_T^G\theta = \sum_i m_i\chi_i.$$

But the irreps form an orthonormal basis for the virtual characters on $G$, so

$$\langle R_T^G\theta, R_T^G\theta\rangle = \sum_i m_i^2.$$

By the inner product formula, if $\theta$ is regular then $\langle R_T^G\theta, R_T^G\theta\rangle = 1$. This is only possible if all but one of the coefficients is zero, and that remaining coefficient is $\pm 1$. □

As yet another consequence of the inner product formula, we can finally establish that the Deligne–Lusztig induction functors are independent of the choice of Borel $B \supseteq T$, justifying our choice of notation.

**Corollary 4.28.** *Let $B, B' \supseteq T$ be two Borel subgroups containing the maximal torus $T$. Denote by $R_{T\subseteq B}^G$ and $R_{T\subseteq B'}^G$ the Deligne–Lusztig induction functors obtained using $B$ and $B'$ respectively. Then $R_{T\subseteq B}^G(\theta) = R_{T\subseteq B'}^G(\theta)$ for any $\theta \in \mathrm{Irr}(T^F)$.*

*Proof.* For a character $\theta$ of $T^F$, we note that the set $\{\omega \in W^F : \mathrm{ad}(w^{-1})(\theta) = \theta\}$ does not involve the Borel containing $T$ at all. It follows from the inner product formula that

$$\langle R^G_{T\subseteq B}(\theta), R^G_{T\subseteq B}(\theta)\rangle = \langle R^G_{T\subseteq B}(\theta), R^G_{T\subseteq B'}(\theta)\rangle = \langle R^G_{T\subseteq B'}(\theta), R^G_{T\subseteq B'}(\theta)\rangle.$$

Thus

$$\langle R^G_{T\subseteq B}(\theta) - R^G_{T\subseteq B'}(\theta), R^G_{T\subseteq B}(\theta) - R^G_{T\subseteq B'}(\theta)\rangle = 0$$

and so because $\langle \cdot, \cdot \rangle$ is an inner product on virtual characters, we conclude that

$$R^G_{T\subseteq B}(\theta) = R^G_{T\subseteq B'}(\theta).$$

□

The final results we will discuss are perhaps the most important, in that they show that the construction of Deligne & Lusztig is exhaustive. We will express the regular representation as a $\mathbb{Z}$-linear combination of Deligne–Lusztig characters, and derive several useful consequences.

Before proceeding to this decomposition, there is a minor technical issue to deal with. As we saw in Corollary 4.27, the Deligne–Lusztig characters often fail to be true characters by some factor of a sign. Unsurprisingly, this sign enters into the picture for geometric reasons.

**Definition 4.29.** Let $T$ be an $F$-stable torus. It is an easy fact (Proposition 8.2 in [13]) that $T$ contains a maximal split subtorus $S$. We define $\mathbb{F}_q$-rank$(T)$ as the dimension of any such subtorus. We also define $\mathbb{F}_q$-rank$(G)$ as the $\mathbb{F}_q$-rank of any maximally split torus in $G$.

Finally, we define the *$\mathbb{F}_q$-sign of $T$* as $\epsilon_T := (-1)^{\mathbb{F}_q\text{-rank}(T)}$, with $\epsilon_G$ defined similarly.

Note that the existence of a maximally split torus in $G$ is Example 4.3.3 in [1], which also shows that all such tori are $G^F$-conjugate, so that their $\mathbb{F}_q$-ranks are the same. The proof is almost identical to Corollary 4.3.

We introduce this sign factor because non-split tori are geometrically 'twisted' inside $G$, and this twisting needs to be accounted for when computing Deligne–Lusztig characters.

With this technical detour out of the way, we can now state our final main result in this chapter. We denote the set of all $F$-stable maximal tori by $\mathcal{T}$.

**Theorem 4.30.** *Let* $\mathrm{reg}_G$ *denote the character of the regular representation of* $G^F$*. Then*

$$\mathrm{reg}_G = \frac{1}{|G^F|_p}\sum_{T\in\mathcal{T}} \epsilon_G\epsilon_T R^G_T(\mathrm{reg}_T) = \frac{1}{|G^F|_p}\sum_{\substack{T\in\mathcal{T}\\ \theta\in\mathrm{Irr}(T^F)}} \epsilon_G\epsilon_T R^G_T(\theta).$$

Moving between the last two expressions is easy, because the regular representation of $T^F$ decomposes as the direct sum of the irreps of $T^F$ (weighted by their dimensions) and all irreps of $T^F$ are 1-dimensional because $T^F$ is abelian.

This theorem is remarkably nontrivial; indeed, it is difficult to show that these two characters even have the same dimension (we define the dimension of a virtual character $\psi$ as $\dim \psi := \psi(1)$). Although we will again not provide a complete proof, this result is so central to the theory that we must at least provide a sketch.

The following is based on the proof of Proposition 7.5 in [31]. The statement given there is again more general than our own, which can be recovered by observing that the regular representation is the characteristic function of $1 \in G^F$.

For ease of notation, let $\rho_G$ denote the rightmost expression in Theorem 4.30. There are three main steps to the proof, which can be broken down as follows:

(i) Recall that $\mathrm{reg}_G$ vanishes away from the identity, so we certainly must check that $\rho_G$ does too. Rewriting $R_T^G(\theta)$ using the character formula, we first consider $\rho_G$ on the complement of the unipotent set (where $s \neq 1$). We find that

$$\begin{aligned}\rho_G(g) &= \frac{1}{|C^0(s)^F||G^F|_p} \sum_{\substack{T\in\mathcal{T}\\ \theta\in\mathrm{Irr}(T^F)}} \epsilon_G\epsilon_T \sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \theta(x^{-1}sx) Q^{C^0(s)}_{xTx^{-1}}(u)\\ &= \frac{1}{|C^0(s)^F||G^F|_p} \sum_{x\in G^F} \sum_{T\subseteq x^{-1}C^0(s)x} \epsilon_G\epsilon_T Q^{C^0(s)}_{xTx^{-1}}(u) \sum_{\theta\in\mathrm{Irr}(T^F)} \theta(x^{-1}sx)\end{aligned}$$

Observe that $\sum_{\theta\in\mathrm{Irr}(T^F)} \theta(x^{-1}sx)$ is exactly the inner product of two columns in the character table of $T^F$, namely the trivial conjugacy class and the conjugacy class of $x^{-1}sx$, which are distinct because $s \neq 1$. By column orthogonality, this sum is 0, whence $\rho_G(g) = 0$.

(ii) The behaviour on the unipotent set is more subtle. The main point is that $R_T^G\theta(u)$ does not depend on $\theta$ (Corollary 4.23). We can therefore rewrite $\rho_G(u)$ so that every summand is $\epsilon_G\epsilon_T R_T^G\mathbb{1}(u)$, and it is then possible to check by hand that $\rho_G(u) = 0$ unless $u = 1$. Checking this requires understanding the $\mathbb{F}_q$-signs $\epsilon_T$ to more of a degree than we wish to develop, but nothing conceptually difficult is required here. We will see an example of the sorts of arguments required when we prove the dimension formula (Corollary 4.32).

(iii) Finally, we know that $\rho_G$ is supported only at the identity, so it is *some* scalar multiple of $\mathrm{reg}_G$, and we must determine this scalar. Equivalently, we want to check that $\mathrm{reg}_G$ and $\rho_G$ have the same dimension. The trick is to compare both $\mathrm{reg}_G$ and $\rho_G$ to a third representation St, known as the *Steinberg representation.*

The important properties of St are that it is supported on the semisimple elements, and $\dim(\mathrm{St}) = |G^F|_{p'}$. For each $T \in \mathcal{T}$, one can show that $\sum_{\theta \in \mathrm{Irr}(T^F} R_T^G \theta(s) = 0$ for semisimple elements $s \neq 1$: one essentially realises this sum as a Lefschetz number and then uses property 9 of $l$-adic cohomology to count fixed points under the left multiplication action by $s$ on a certain variety, of which there are none for $s \neq 1$. It follows that when taking the inner product of $\sum_{\theta \in \mathrm{Irr}(T^F} R_T^G$ with St the only non-vanishing summand is $g = 1$. This allows one to deduce that

$$\sum_{\theta} \epsilon_G \epsilon_T R_T^G \theta(1) = \frac{|G^F|}{|G^F|_p}.$$

On the other hand, a non-trivial result of Steinberg (Theorem 14.14 in [30]) says that there are $|G^F|_p^2$ $F$-stable maximal tori in $G$, so

$$\dim(\rho_G) = \rho_G(1) = \frac{1}{|G^F|_p} \sum_{\substack{T \in \mathcal{T} \\ \theta \in \mathrm{Irr}(T^F)}} \epsilon_G \epsilon_T R_T^G \theta(1) = |G^F| = \dim(\mathrm{reg}_G)$$

and we are done. □

Obviously many details have been omitted in the sketch above, but the substance of the argument (especially of the final step) is that there are ‘enough’ maximal tori in $G$ to allow for a complete description of the regular representation. Although parabolic induction only sees the maximally split tori, all maximal tori contribute equally to Deligne–Lusztig induction.

As an immediate practical consequence, we have:

**Corollary 4.31.** *Every irrep of $G^F$ can be found as a component of some Deligne–Lusztig character $R_T^G(\theta)$. In other words, the construction of Deligne & Lusztig is exhaustive.*

The other result which the decomposition of the regular representation allows us to derive is the *dimension formula.* As the name suggests, this is an explicit formula for the dimensions of the virtual characters $R_T^G \theta$.

To see how this relates to the regular representation, we recall that $\mathrm{reg}_G$ is the permutation representation induced by the left multiplication action of $G^F$ on itself. No element of $G^F$ fixes anything under this action except for the identity, which fixes all $|G^F|$ elements. It follows that

$$\langle \psi, \mathrm{reg}_G \rangle = \frac{1}{|G^F|} \sum_{g \in G^F} \psi(g) \overline{\mathrm{reg}_G(g)} = \psi(1) = \dim(\psi)$$

so we can recover the dimension of a virtual character by comparing it with the regular representation.

**Corollary 4.32.** *(Dimension formula)*

*For $T$ an $F$-stable maximal torus and $\theta \in \mathrm{Irr}(T^F)$, we have*

$$\dim(R_T^G \theta) = \epsilon_G \epsilon_T [G^F : T^F]_{p'}.$$

*Proof.* Combining Frobenius reciprocity, the Mackey decomposition and the first expression for the regular representation yields

$$\begin{aligned}
\dim(R_T^G \theta) &= \langle R_T^G \theta, \mathrm{reg}_G \rangle \\
&= \frac{1}{|G^F|_p} \sum_{S \in \mathcal{T}} \epsilon_G \epsilon_S \langle R_T^G \theta, R_S^G(\mathrm{reg}_S) \rangle \\
&= \frac{1}{|G^F|_p} \sum_{S \in \mathcal{T}} \epsilon_G \epsilon_S \langle {}^*R_S^G R_T^G \theta, \mathrm{reg}_S \rangle \\
&= \frac{1}{|G^F|_p} \sum_{S \in \mathcal{T}} \epsilon_G \epsilon_S \sum_{S^F x T^F \in S^F \backslash N(S,T)^F / T^F} \langle \mathrm{ad}(x)(\theta), \mathrm{reg}_S \rangle
\end{aligned}$$

Note that $\langle \mathrm{ad}(x)(\theta), \mathrm{reg}_S \rangle$ is the dimension of $\mathrm{ad}(x)(\theta)$, which is 1, so every summand of the inner sum is 1. Recall from the proof of the inner product theorem that the order of the double coset $S^F x T^F$ is $|T^F|$. We may therefore rewrite this as

$$\dim(R_T^G \theta) = \frac{1}{|G^F|_p} \sum_{S \in \mathcal{T}} \epsilon_G \epsilon_S \frac{1}{|T^F|} \sum_{x \in N(S,T)^F} 1 = \epsilon_G \frac{1}{|G^F|_p |T^F|} \sum_{S \in \mathcal{T}} \epsilon_S |N(S,T)^F|.$$

We discard the summands such that $N(S,T)^F = \emptyset$, since they contribute nothing. If $S$ and $T$ are $G^F$-conjugate, say by $x$, then conjugating a maximal split subtorus in $S$ by $x$ gives a maximal split subtorus in $T$. It follows that $S$ and $T$ have the same $\mathbb{F}_q$-signs, so we may further simplify this as

$$\dim(R_T^G \theta) = \epsilon_G \epsilon_T \frac{1}{|G^F|_p |T^F|} \sum_{\substack{S \in \mathcal{T} \\ N(S,T)^F \neq \emptyset}} |N(S,T)^F|.$$

We now make two observations. Firstly, the sets $N(S,T)^F$ are clearly disjoint. Secondly, if $x \in G^F$ then ${}^xT$ is an $F$-stable maximal torus and $x \in N({}^xT, T)^F$. It follows that we have a partition

$$G^F = \coprod_{\substack{S \in \mathcal{T} \\ N(S,T)^F \neq \emptyset}} N(S,T)^F.$$

In particular, the orders of both sides are the same, so we now find that

$$\dim(R_T^G \theta) = \epsilon_G \epsilon_T \frac{|G^F|}{|G^F|_p |T^F|} = \epsilon_G \epsilon_T \frac{|G^F|_{p'}}{|T^F|}.$$

Because $T^F$ is a torus, its order must be coprime to $p$: if $p \mid |T^F|$ then, by Cauchy's theorem, there exists some $t \in T^F$ of order $p$, which must be unipotent. Since $T \cong (\overline{\mathbb{F}}_p^\times)^d$, $t$ is also semisimple, so $t = 1$. But the order of the identity is not $p$, so no such $t$ exists and we conclude that $p \nmid |T^F|$. It follows that

$$\dim(R_T^G \theta) = \epsilon_G \epsilon_T \left( \frac{|G^F|}{|T^F|} \right)_{p'} = \epsilon_G \epsilon_T [G^F : T^F]_{p'}. \qquad \square$$

This concludes the chapter on Deligne–Lusztig theory. Although the results we have included are not exhaustive, they are the most crucial ones for computing characters. We could at this point theoretically find the Deligne–Lusztig characters of any finite group of Lie type: for any given $g \in G^F$, finding the Jordan decomposition is simply a matter of computing the order of $g$, and then the character formula allows us to write down exact character values. The difficulty now lies in decomposing the Deligne–Lusztig characters: notice that although we know that all the irreps are constituents of some $R_T^G \theta$, we have no results that let us actually pick them out.

For finite groups of Lie type with a small Weyl group, the inner product formula places strong constraints on how the $R_T^G \theta$ can decompose. In this case, it is usually possible to solve a small system of $\mathbb{Z}$-linear equations to find the irreps. Unfortunately, as $|W^F| \to \infty$, this becomes infeasible and we need more sophisticated techniques to decompose the $R_T^G \theta$. In the final chapter we will explore how character sheaves can be used to assist with this problem.

# Chapter 5

# Character sheaves

In this final chapter, we will define character sheaves and show how to recover the Deligne–Lusztig characters from their construction. Character sheaves are not, as the reader might expect, actual sheaves, but are rather certain complexes of étale sheaves in the derived category. It is worth noting that we will be using the term "sheaf" rather loosely in this chapter, but usually it will refer to objects of the above kind.

This is by far the most technical chapter of the thesis, and by necessity we assume familiarity with derived categories, triangulated categories and $t$-structures. Ideally, the reader would also have knowledge of perverse $t$-structures, but we include a fast treatment of this less-common topic.

## 5.1 Constructible sheaves and the middle perverse $t$-structure

As in Chapter 3, let $\mathfrak{Sh}_{\mathrm{Et}}(X)$ denote the category of étale sheaves on a scheme $X$. This is an abelian category, so we can form its derived category $\mathcal{D}(\mathfrak{Sh}_{\mathrm{Et}}(X))$ in the usual way; we denote $\mathcal{D}(X) := \mathcal{D}(\mathfrak{Sh}_{\mathrm{Et}}(X))$. This category is too large in practice, so we will impose two further conditions.

**Definition 5.1.** An étale sheaf $\mathcal{F}$ is *constructible* if, for any irreducible closed subscheme $Z \subseteq X$, there exists a non-empty open subset $U \subseteq Z$ such that $\iota_U^*(\mathcal{F})$ is locally constant with finite stalks.

Informally, an étale sheaf is constructible if it is 'generically' locally constant with finite stalks. We say that a complex $K$ of étale sheaves is constructible if all its cohomology sheaves $\mathcal{H}^i(K)$ are constructible.

This property passes to the derived category: we set $\mathcal{D}_c(X) \subseteq \mathcal{D}(X)$ to be the full subcategory consisting of the constructible complexes.

Inside $\mathcal{D}_c(X)$, we can also consider those complexes which are *bounded*, meaning they are only supported in finitely many grades. Taking the collection of all bounded constructible complexes gives the category $\mathcal{D}_c^b(X)$, and this is the category we will work with.

$\mathcal{D}_c^b(X)$ has a remarkably interesting $t$-structure, called the *perverse $t$-structure.* For a point $x \in X$, write $d(x)$ for the dimension of $\overline{\{x\}}$ and let $\overline{x}$ be a geometric point with image $x$ in $X$. Denoting the pullback of the cohomology sheaf $\mathcal{H}^i K$ to $\overline{x}$ by $(\mathcal{H}^i K)_{\overline{x}}$, we now define

$$\begin{aligned}
{}^p\mathcal{D}_c^{b,\leq 0}(X) &:= \{K \in \mathcal{D}_c^b(X) : (\mathcal{H}^i K)_{\overline{x}} = 0 \text{ for all } x \in X \text{ with } i > -d(x)\} \\
{}^p\mathcal{D}_c^{b,\geq 0}(X) &:= \{K \in \mathcal{D}_c^b(X) : (\mathcal{H}^i K)_{\overline{x}} = 0 \text{ for all } x \in X \text{ with } i < -d(x)\}
\end{aligned}$$

It is highly non-trivial that these define a $t$-structure on $\mathcal{D}_c^b(X)$, but this is not something we will prove. We refer interested readers to Theorem 3.9 in [20].

Since this is a $t$-structure, its heart is a full abelian subcategory of $\mathcal{D}_c^b(X)$. We now define the category of *middle perverse sheaves* as the heart:

$$\mathcal{M}(X) := {}^p\mathcal{D}_c^{b,\leq 0}(X) \cap {}^p\mathcal{D}_c^{b,\geq 0}(X).$$

It is worth noting that $G$-equivariant analogues of everything in this section exist, and the same results hold. This is an important point in the construction of character sheaves, since we would like our character sheaves to be equivariant with respect to conjugation, but we will try to avoid directly dealing with this issue since it is just an added layer of complexity.

## 5.2 Constructing character sheaves

We will now give two different constructions of character sheaves. Each has its advantages and disadvantages: the first is (comparatively) intuitive but is difficult to compute, while the second is very unmotivated but is easier to work with.

The reference for this section is [21], where the first construction is Section 3 and the second construction is Section 4. The author's style is such that many results are not labelled but are rather discussed in the body of the text. For this reason, we cannot give explicit numerical references. A short summary can be found in [19].

The idea of both constructions is, unsurprisingly, to build a sheaf on $G$ from a sheaf on a maximal torus $T$. The sheaves on $T$ will be Kummer local systems, introduced in Section 3.5, so we encourage the reader to take the time to review the definition before continuing.

We consider the subset of $T$ defined by

$$T_{\text{reg}} = \{t \in T : C^0(t)^\dagger = T\}$$

which we extend to $G$ by

$$G_{\text{rss}} := \bigcup_{h \in G} hT_{\text{reg}}h^{-1}.$$

The elements of $G_{\text{rss}}$ are called *regular semisimple*. Because all maximal tori in $G$ are conjugate, $G_{\text{rss}}$ does not depend on $T$. However, it is worth noting that $T_{\text{reg}}$ **does** depend on $G$: because $T$ is abelian, the centraliser of $t$ is always at least $T$, and it can only be larger if there are elements in $G \setminus T$ commuting with $t$.

Based on the Mackey decomposition for Deligne–Lusztig characters, our character sheaf should behave on tori like a sum of conjugates of the original sheaf on $T$, so we might simply try to extend our sheaf to all tori by conjugation. A potential problem with this approach is elements which lie in multiple maximal tori, since there could theoretically be several incompatible ways to extend the sheaf to these points. This is why we consider regular semisimple elements: a semisimple element is regular if and only if it lies in a unique maximal torus. Moreover $T_{\text{reg}}$ is dense in $T$ and $G_{\text{rss}}$ is dense in $G$, which will be useful for technical reasons. For proofs of these facts, we refer the reader to Theorem 12.3 in [26], as well as the unnumbered Proposition directly preceding it.

The scheme $G_{\text{rss}}$ admits a finite Galois étale cover with Galois group $W = W(T)$, which we will denote by $\tilde{G}_{\text{rss}}$. Explicitly, $\tilde{G}_{\text{rss}}$ is given by

$$\tilde{G}_{\text{rss}} = \{(g, hT) \in G_{\text{rss}} \times G/T : h^{-1}gh \in T_{\text{reg}}\}$$

and the covering map $\pi_{\text{rss}}$ is just projection onto the first coordinate. It is worth noting that we cannot simply replace $G_{\text{rss}}$ by $G$ in the definition of $\tilde{G}_{\text{rss}}$ to get a cover of $G$: not all elements of $G$ live in a maximal torus, and in particular, the unipotent elements would not be covered. The second construction we will give addresses this by replacing the torus $T$ with a Borel subgroup $B$ containing it. This allows us to cover the unipotent elements by projecting off the unipotent radical $U \subseteq B$.

$^\dagger$Recall that $C^0(t)$ is the identity component of the centraliser of $t$ in $G$.

Notice that we also have a map

$$\begin{aligned} \rho_{\mathrm{rss}} : \tilde{G}_{\mathrm{rss}} &\to T_{\mathrm{reg}} \\ (g, hT) &\mapsto h^{-1}gh \end{aligned}$$

With these schemes in mind, there is now a natural diagram to consider:

$$\begin{tikzcd} & \tilde{G}_{\mathrm{rss}} \arrow[dl, two heads, "\pi_{\mathrm{rss}}"'] \arrow[dr, two heads, "\rho_{\mathrm{rss}}"] & \\ G_{\mathrm{rss}} \arrow[d, hook, "j"'] & & T_{\mathrm{reg}} \arrow[d, hook] \\ G & & T \end{tikzcd}$$

We are at last ready to construct the character sheaves on $G$. For a Kummer local system $\mathcal{L} \in \mathcal{K}T$‡, consider the restriction $\mathcal{L}_{\mathrm{reg}}$ of $\mathcal{L}$ to $T_{\mathrm{reg}}$. The $\overline{\mathbb{Q}}_l$-étale sheaf

$$(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$$

is $G$-equivariant with respect to the conjugation action of $G$ on $G_{\mathrm{rss}}$.

Furthermore, it is semisimple as an object of $\mathcal{M}(G_{\mathrm{rss}})[-\dim G]$: the sheaf $\mathcal{L}_{\mathrm{reg}}$ is rank 1 on $T_{\mathrm{reg}}$, hence simple, and pullback functors preserve semisimplicity, so $\rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ is semisimple on $\tilde{G}_{\mathrm{rss}}$. Finally, the *decomposition theorem* for perverse sheaves (Section 1.3.2 in [17]) says that, for proper maps such as $\pi_{\mathrm{rss}}$, the push-forward of a semisimple perverse sheaf 'of geometric origin' remains semisimple. Being of geometric origin is a somewhat complicated technical condition that we do not wish to discuss, but all the perverse sheaves we will see are of geometric origin.

Extending this sheaf to $G$ via $j$ gives an étale sheaf on $G$. This sheaf is not quite perverse because it is concentrated in the wrong degree, so shifting gives our first definition of a character sheaf as

$$K_T^{\mathcal{L}} := j_{!*}(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}[\dim G].$$

One might ask why we go by way of $\tilde{G}_{\mathrm{rss}}$: since $T_{\mathrm{reg}} \subseteq G_{\mathrm{rss}}$, we could simply extend $\mathcal{L}_{\mathrm{reg}}$ to $G_{\mathrm{rss}}$ by pushing forward. Perhaps the simplest answer is that the sheaf $(T_{\mathrm{reg}} \hookrightarrow G_{\mathrm{rss}})_* \mathcal{L}_{\mathrm{reg}}$ is not $G$-equivariant for the conjugation action, and virtual characters of $G$ must be constant on conjugacy classes.

‡We now embed $\mathcal{K}T$ into $\mathcal{D}_c^b(T)$ as complexes supported in degree 0.

The functor $j_{!*}$ is known as the *intermediate extension*. Its construction is quite technical, but thankfully it has an easy equivalent characterisation that we will take as our definition. Interested readers can find the original construction of the intermediate extension in Example 3.10 of [20].

**Definition 5.2.** Let $j : U \hookrightarrow X$ be the inclusion of a subscheme $U$ into $X$. For a perverse sheaf $\mathcal{F}$ on $U$, $j_{!*}\mathcal{F}$ is the unique perverse sheaf on $X$ satisfying the following three conditions:

- $\mathrm{Supp}(j_{!*}\mathcal{F}) \subseteq \overline{U}$,
- $(j_{!*}\mathcal{F}) \mid_U \cong \mathcal{F}$, and
- $j_{!*}\mathcal{F}$ has no subobjects or quotient objects on $\overline{U} \setminus U$.

Note that because $G_{\mathrm{rss}}$ is dense in $G$, the first condition is always satisfied in our situation.

To understand $K_T^{\mathcal{L}}$, let us temporarily ignore $j_{!*}$ and study the sheaf on $G_{\mathrm{rss}}$, and in particular its endomorphisms. Consider the subset of the Weyl group

$$W_{\mathcal{L}} := \{w \in W : (w^{-1})^* \mathcal{L} \cong \mathcal{L}\}.$$

It is shown in Proposition 3.5 of [21] that every endomorphism of $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}[\dim G]$ is automatically $G$-equivariant, and that $\mathrm{End}((\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}[\dim G])$ is isomorphic to the group algebra $\mathbb{C}[W_{\mathcal{L}}]$. The argument is essentially that $G$-equivariant endomorphisms of $\mathcal{L}$ induce (distinct) endomorphisms of $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ by applying the functors in the definition of character sheaves. Since $\mathbb{C}[W_{\mathcal{L}}]$ is the algebra of $G$-equivariant endomorphisms of $\mathcal{L}$, this gives us a map in one direction. Conversely, an endomorphism of $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ 'restricts' to an endomorphism of $\mathcal{L}_{\mathrm{reg}}$, and the density of $T_{\mathrm{reg}}$ in $T$ means that this extends uniquely to a $G$-equivariant endomorphism of $\mathcal{L}$ on $T$. Thus our endomorphism came from $\mathbb{C}[W_{\mathcal{L}}]$.

In analogy to regular characters, we call a Kummer local system *regular* if $W_{\mathcal{L}} = \{1\}$. If $\mathcal{L}$ is regular then, by the above, $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ has no nontrivial endomorphisms. Because it is also semisimple, Schur's lemma now implies $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ is irreducible when $\mathcal{L}$ is regular.

The final step is to understand $j_{!*}$. Although difficult to compute, this functor has the desirable property that if $A$ is simple then so is $j_{!*}A$ (Theorem 3.16 in [20]). In particular, this means that $K_T^{\mathcal{L}}$ is irreducible whenever $\mathcal{L}$ is regular, and moreover $K_T^{\mathcal{L}}$ is always semisimple: the image of a simple decomposition of $(\pi_{\mathrm{rss}})_* \rho_{\mathrm{rss}}^* \mathcal{L}_{\mathrm{reg}}$ is a simple decomposition of $K_T^{\mathcal{L}}$.

In light of this discussion, we have now proven:

**Theorem 5.3.** *For $\mathcal{L}$ a Kummer local system on $T$, $K_T^{\mathcal{L}}$ is a $G$-equivariant semisimple perverse sheaf on $G$. If $\mathcal{L}$ is regular then $K_T^{\mathcal{L}}$ is irreducible.*

We will soon proceed to give the second construction of character sheaves. Before we do, however, let us see an example of these definitions in the simplest possible case.

**Example 5.4.** Let $G = T$ be a torus. Since there are no elements outside $T$, $C^0(t)$ can never be larger than $T$ and so every element of $T$ is regular, i.e. $T_{\mathrm{reg}} = T$. We also see that $G/T$ is trivial and $G_{\mathrm{rss}} = T$, so the maps $\rho_{\mathrm{rss}}, \pi_{\mathrm{rss}}$ and $j$ are all the identity. It follows that

$$K_T^{\mathcal{L}} = \mathrm{id}_{!*} \, \mathrm{id}_* \, \mathrm{id}^* \, \mathcal{L}[\dim T] = \mathcal{L}[\dim T]. \qquad \square$$

As we mentioned earlier, the main problem with this construction is that understanding the value of the character sheaves on unipotent elements is quite difficult, since all elements of $G_{\mathrm{rss}}$ are semisimple. Our second construction will bypass $j$ by making use of a Borel subgroup $B \supseteq T$, essentially replacing instances of $T$ by $B$. Firstly, consider the map

$$\{(g, hT) \in G_{\mathrm{rss}} \times G/T : h^{-1}gh \in T_{\mathrm{reg}}\} \to \{(g, hB) \in G_{\mathrm{rss}} \times G/B : h^{-1}gh \in B\}$$
$$(g, hT) \mapsto (g, hB)$$

and recall that the left-hand side is by definition $\tilde{G}_{\mathrm{rss}}$. Both the left and right-hand sides are Galois étale covers of $G_{\mathrm{rss}}$, and both are $|W|$-sheeted over $G_{\mathrm{rss}}$. Since the map above commutes with the étale covers, it must be a bijection on each fiber and so it is an isomorphism. The right-hand side embeds into

$$\tilde{G} = \{(g, hB) \in G \times G/B : h^{-1}gh \in B\}$$

and we again have a covering map $\pi : \tilde{G} \to G$ via projection. Restricting $\pi$ to the isomorphic copy of $\tilde{G}_{\mathrm{rss}}$ inside $\tilde{G}$ recovers $\pi_{\mathrm{rss}}$.

Similarly, if $\mathrm{pr}_T : B \to T$ denotes the projection, then we can extend $\rho_{\mathrm{rss}}$ to

$$\rho : \tilde{G} \to T$$
$$(g, hB) \mapsto \mathrm{pr}_T(h^{-1}gh)$$

**Theorem 5.5.** *Having chosen a Borel subgroup $B \supseteq T$, there is a canonical isomorphism*

$$K_T^{\mathcal{L}} \cong R\pi_* \rho^* \mathcal{L}[\dim G].$$

*Proof.* (sketch)

Recall that $K_T^{\mathcal{L}}$ is the unique perverse sheaf on $G$ which restricts to $(\pi_{\mathrm{rss}})_*\rho_{\mathrm{rss}}^*\mathcal{L}_{\mathrm{reg}}[\dim G]$ on $G_{\mathrm{rss}}$ and has no sub-or-quotient objects on $G \setminus G_{\mathrm{rss}}$. One can verify that

$$(R\pi_*\rho^*\mathcal{L}) \mid_{G_{\mathrm{rss}}} \cong (\pi_{\mathrm{rss}})_*\rho_{\mathrm{rss}}^*\mathcal{L}_{\mathrm{reg}}$$

by exploiting standard adjunction relations. To check the third condition, one essentially looks at the stalks on $G \backslash G_{\mathrm{rss}}$ and argues that they are not high-enough rank to support either subobjects or quotients. For more details see Section 4.5 of [21]. □

**Corollary 5.6.** *The restriction of $K_T^{\mathcal{L}}$ to the set of unipotent elements in $G$ is independent of $\mathcal{L} \in \mathcal{K}T$.*

*Proof.* Because $G_{\mathrm{unip}} \cap T = \{1\}$, $\mathrm{pr}_T(u) = 1$ for any unipotent $u$. It follows that $\rho^*\mathcal{L}$, when considered as an étale sheaf on the subset of $\tilde{G}$ sitting over $G_{\mathrm{unip}}$, is just the pullback of the constant rank 1 $\overline{\mathbb{Q}}_l$-sheaf on the point $\{1\} \subseteq T$, which is independent of $\mathcal{L}$. □

**Example 5.7.** Let us at least verify Theorem 5.5 in the simplest possible case by again computing the character sheaves on a torus $G = T$. The only Borel subgroup is $B = T$, so $G/B$ is trivial. The maps $\pi$ and $\rho$ are again both the identity, so

$$K_T^{\mathcal{L}} = R\,\mathrm{id}_*\,\mathrm{id}^*\,\mathcal{L}[\dim T] = \mathcal{L}[\dim T].$$ □

The ethos of character sheaves is that the first construction is useful for proving general properties (cf. Theorem 5.3) while the second is better for computing specific values (cf. Corollary 5.6).

## 5.3 From character sheaves to characters

Having defined character sheaves, the next step is to derive characters of $G^F$. Given a Frobenius map $F : G \to G$ and an $F$-stable maximal torus $T$, taking pullbacks induces a homomorphism $F^* : \mathcal{K}T \to \mathcal{K}T$. If $\mathcal{L} \in \mathcal{K}T$ is $F$-stable, meaning $F^*\mathcal{L} \cong \mathcal{L}$, a choice of isomorphism $\tau : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ is called a *Weil structure* on $\mathcal{L}$.

We now fix a Kummer local system $\mathcal{L}$ and Weil structure $\tau$. Observe that if $t \in T^F$ then

$$(F^*\mathcal{L})_t = \mathcal{L}_{F(t)} = \mathcal{L}_t$$

so the Weil structure $\tau$ acts on the stalk at $t$.

We therefore have a function

$$\begin{aligned}\chi_{\mathcal{L},\tau} : T^F &\to \overline{\mathbb{Q}}_l^\times \\ t &\mapsto \operatorname{Tr}(\tau_t : \mathcal{L}_t \to \mathcal{L}_t)\end{aligned}$$

which is commonly known as the *Frobenius trace* of $(\mathcal{L}, \tau)$.

**Theorem 5.8.** *The construction above gives a complete description of the characters of $T^F$, in the sense that:*

*(i) The group of $F$-stable Kummer local systems $(\mathcal{K}T)^F$ is canonically isomorphic to the character group* $\operatorname{Hom}(T^F, \overline{\mathbb{Q}}_l^\times)$*.*

*(ii) Normalise $\tau$ so that $\tau_e = \mathrm{id}$. Then $\chi_{\mathcal{L},\tau}$ is a character of $T^F$, and moreover all characters arise in this way.*

*Proof.* (sketch)

Part (i) requires more theory than we wish to develop, so we refer the reader to Proposition 2.3.1 in [17]. However, part (ii) follows quite easily from part (i) and the fact that Example 3.21 gave a complete description of the Kummer local systems on $T$. By Example 5.7, the character sheaves on $T$ are just shifts of the Kummer local systems, so we can compute all of the Frobenius traces. One then computes the characters of $T^F$ using elementary means (it is a finite abelian group, so this is not difficult) and compares the two lists. □

It is important to note that the proof of (ii) is non-constructive: given a character $\theta$ of $T^F$, we know that there is some pair $(\mathcal{L}, \tau)$ such that $\theta = \chi_{\mathcal{L},\tau}$, but it can be difficult in practice to find such a pair.

Having chosen a Weil structure $\tau$, we get an induced isomorphism $F^* K_T^{\mathcal{L}} \xrightarrow{\sim} K_T^{\mathcal{L}}$; note that either construction induces the same isomorphism since the isomorphism in Theorem 5.5 is canonical. As in the torus case, $\tau$ induces a map on the stalk at every $g \in G^F$. We may therefore define

$$\begin{aligned}\chi_{K_T^{\mathcal{L}},\tau} : G^F &\to \overline{\mathbb{Q}}_l \\ g &\mapsto \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}\left({}^p\mathcal{H}^i(\tau_g)\right)\end{aligned}$$

This is the *characteristic function* of $(\mathcal{L}, \tau)$, and this procedure is how we shall recover the Deligne–Lusztig characters from character sheaves. It is worth remarking that taking characteristic functions commutes with all the standard operations on character sheaves.

The remainder of this section will be devoted to deriving an explicit formula for the value of the characteristic function $\chi_{K_T^{\mathcal{L}},\tau}$. Observe that by Corollary 5.6, the restriction of $\chi_{K_T^{\mathcal{L}},\tau}$ to the unipotent elements of $G$ is independent of $\mathcal{L}$. In analogy to the Green functions, we therefore define

$$\begin{aligned} \tilde{Q}_T^G : G_{\text{unip}} &\to \overline{\mathbb{Q}}_l \\ u &\mapsto \chi_{\underline{\overline{\mathbb{Q}}_l},\text{id}}(u) \end{aligned}$$

where $\underline{\overline{\mathbb{Q}}_l}$ is the constant Kummer local system, upon which $F$ acts trivially.

The key fact is that, with this modified notion of a Green function, the characteristic functions satisfy their own character formula.

**Theorem 5.9.** *Let $g \in G^F$ have Jordan decomposition $g = su = us$. Then*

$$\chi_{K_T^{\mathcal{L}},\tau}(g) = \frac{1}{|C^0(s)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \tilde{Q}_{xTx^{-1}}^{C^0(s)}(u)\chi_{\mathcal{L},\tau}(x^{-1}sx).$$

The proof of this result is far too intricate for this thesis, but we can certainly sketch the key steps. Our ultimate goal is to find an open set $\mathcal{U}$ around $g \in G^F$ on which $K_T^{\mathcal{L}}$ is particularly well-behaved and use this open set to compute the stalk at $g$. We start with the semisimple component $s \in G$ and consider the smooth immersion

$$\begin{aligned} i_s : C^0(s) &\hookrightarrow G \\ x &\mapsto xs = sx \end{aligned}$$

This map translates taking the stalk at $s$ to taking the stalk at the identity $e$, since $i_s(e) = s$.

Given $g \in G$ such that $g^{-1}sg \in T$, observe that ${}^gT := gTg^{-1}$ is a maximal torus in $C^0(s)$ and ${}^g\mathcal{L} := \text{ad}(g^{-1})^*\mathcal{L}$ is a Kummer local system on ${}^gT$. We may therefore replace $(G, T, \mathcal{L})$ by $(C^0(s), {}^gT, {}^g\mathcal{L})$ in our first construction of character sheaves, and this gives a character sheaf on $C^0(s)$.

The resulting character sheaf depends only on the class $o$ of $g$ in the double coset

$$O := C^0(s)\backslash\{g \in G : g^{-1}sg \in T\}/T.$$

We therefore denote the character sheaf by ${}^oK_T^{\mathcal{L}}$. Similarly, the stalk of ${}^g\mathcal{L}$ at $s \in {}^gT$ only depends on $o$, so we denote it by ${}^o\mathcal{L}_s$.

We at last have the language to describe the local structure of $K_T^{\mathcal{L}}$ about $s$, and it only remains to find our open set.

**Lemma 5.10.** *There exists an open neighbourhood $\mathcal{U} \subseteq C^0(s)$ of the identity satisfying the following properties:*

*(a) For every $x \in C^0(s)$, $x\mathcal{U}x^{-1} = \mathcal{U}$.*

*(b) Give $x \in C^0(s)$ with Jordan decomposition $x = x_s x_u$, we have $x \in \mathcal{U}$ if and only if $x_s \in \mathcal{U}$.*

*(c) Given $x \in \mathcal{U}$ and $g \in G$ such that $g^{-1}sxg \in B$ (resp. $T$), then both $g^{-1}sg \in B$ (resp. $T$) and $g^{-1}x_s g \in B$ (resp. $T$).*

The construction of such an open set can be found as Lemma 8.6 in [22]. For $\mathcal{U}$ as in the lemma, we now have:

**Proposition 5.11.** *The restriction of*

$$i_s^* K_T^{\mathcal{L}}[\dim C^0(s) - \dim G]$$

*to $\mathcal{U}$ is canonically isomorphic to*

$$\bigoplus_{o \in O} \left({}^o K_T^{\mathcal{L}} \mid_{\mathcal{U}}\right) \otimes^o \mathcal{L}_s.$$

*Proof.* (sketch)

The idea is to reduce to studying both sides on the open set $\mathcal{U} \cap i_s^{-1}(G_{\mathrm{rss}})$. If $k : \mathcal{U} \cap i_s^{-1}(G_{\mathrm{rss}}) \hookrightarrow \mathcal{U}$ is the inclusion, naturality properties of the intermediate extension give

$$k_{!*}\left(\bigoplus_{o \in O} \left({}^o K_T^{\mathcal{L}} \mid_{\mathcal{U} \cap i_s^{-1}(G_{\mathrm{rss}})}\right) \otimes^o \mathcal{L}_s\right) = \bigoplus_{o \in O} \left({}^o K_T^{\mathcal{L}} \mid_{\mathcal{U}}\right) \otimes^o \mathcal{L}_s.$$

Knowing that the right-hand side is an intermediate extension, it now suffices to show that $i_s^* K_T^{\mathcal{L}}[\dim C^0(s) - \dim G]$ satisfies the three conditions of Definition 5.2. Since $\mathcal{U} \cap i_s^{-1}(G_{\mathrm{rss}})$ is dense in $\mathcal{U}$, the first condition is immediate.

Checking $i_s^* K_T^{\mathcal{L}}[\dim C^0(s) - \dim G]$ has the correct restriction to $\mathcal{U} \cap i_s^{-1}(G_{\mathrm{rss}})$ is most of the proof, and where the technical conditions on $\mathcal{U}$ are necessary. The point is that, because of the Bruhat decomposition, there is a partition of $\tilde{G}$ into 'translates' of $B$ (namely the Bruhat cells $BwB$) and $K_T^{\mathcal{L}}$ is just a twisted translation of $\mathcal{L}$ on these sets, similar to the Mackey decomposition for Deligne–Lusztig characters.

Finally, checking that $i_s^* K_T^{\mathcal{L}}[\dim C^0(s) - \dim G]$ has no sub-or-quotient objects is similar to the proof of Theorem 5.5, where one essentially argues that the stalks could not support them. □

**Corollary 5.12.** *For $u \in C^0(s)$ unipotent,*

$$(K_T^{\mathcal{L}})_{su}[\dim C^0(s) - \dim G] \cong \bigoplus_{o \in O} \left( {}^oK_T^{\overline{\mathbb{Q}}_l} \right)_u \otimes^o \mathcal{L}_s.$$

*Proof.* First observe that the Jordan decomposition of $u$ is $u = e \cdot u$. Since $e \in \mathcal{U}$, condition (b) in Lemma 5.10 implies $u \in \mathcal{U}$, so we may use $\mathcal{U}$ to compute the stalk at $u$. Invoking the proposition now yields

$$(K_T^{\mathcal{L}})_u[\dim C^0(s) - \dim G] \cong \bigoplus_{o \in O} \left( {}^oK_T^{\mathcal{L}} \right)_u \otimes^o \mathcal{L}_s$$

and multiplying by $s$ now gives the desired result, except with ${}^oK_T^{\overline{\mathbb{Q}}_l}$ replaced by ${}^oK_T^{\mathcal{L}}$. But by Corollary 5.6, the restriction of ${}^oK_T^{\mathcal{L}}$ to the unipotent element $u$ does not depend on $\mathcal{L}$, so we are done. □

*Proof.* (of Theorem 5.9)
Note that part of the Jordan decomposition is that the elements $u$ and $s$ commute, so $u \in C^0(s)$. We may therefore use the previous corollary to compute the stalk at $g = su$, from which the formula follows immediately. □

Complete details for the proof of the character formula can be found in Section 8 of [22].

Knowing that both the characteristic functions and the Deligne–Lusztig characters satisfy their own character formulas, it is quite believable that we could recover the Deligne–Lusztig characters from character sheaves. It turns out that (up to a constant scalar) this is the case.

**Theorem 5.13.** *Let $\theta = \chi_{\mathcal{L},\tau}$ be an algebraic character of $T$. Then*

$$\chi_{K_T^{\mathcal{L}},\tau} = (-1)^{\dim T} R_T^G \theta.$$

Note that because all maximal tori in $G$ are conjugate, the sign is actually independent of $T$.

This result, originally stated as Theorem 1.14 and proven in Section 8.12 of [23], spans some 30 pages. The main step is to show that there is *some* scalar $\lambda$ such that

$$\chi_{K_T^{\mathcal{L}},\tau} = \lambda R_T^G \theta. \qquad (\dagger)$$

We then evaluate either side at a particular point $g$ to determine $\lambda$. Since we have a dimension formula for the Deligne–Lusztig character $R_T^G\theta$, and there is an analogous formula for the characteristic function $\chi_{K_T^{\mathcal{L}},\tau}$, evaluating at $g = 1$ allows one to deduce $\lambda = (-1)^{\dim T}$. Note that this strategy is quite similar to the proof of the decomposition of the regular representation that we sketched in Chapter 4.

To prove (†), the idea is to pick a Borel $B \supseteq T$, which then induces a Bruhat decomposition

$$G = \coprod_{w \in W} BwB.$$

On each Bruhat cell $BwB$, one shows that there is a scalar $\lambda_w$, a priori dependent on $w$, such that (†) holds. We then wish to show that all the $\lambda_w$ are equal, using a standard 'local-to-global' argument. The difficulty lies in the fact that the Bruhat cells are disjoint; however, the dense set $G_{\text{rss}}$ intersects each Bruhat cell, and small perturbations do not affect the sheaf $(\pi_{\text{rss}})_*\rho_{\text{rss}}^*\mathcal{L}_{\text{reg}}$ on $G_{\text{rss}}$.

We can therefore pick a discrete 'path' of elements in $G_{\text{rss}}$ that traverses the Bruhat cells. The biggest technical problem with this approach is that the elements in this path must be chosen to lie in $G^F$, since this is where $\chi_{K_T^{\mathcal{L}},\tau}$ and $R_T^G\theta$ are defined. However, using the Lang map and Lang's theorem, one can modify some path in $G$ to lie inside $G^F$.

Readers familiar with the topological monodromy action, mentioned briefly in Section 3.2, may recognise this technique of patching together information on stalks along a path. The ability to use large open sets to analyse stalks (somewhat inverse to the way one normally studies sheaves) is arguably one of the biggest advantages that comes from using character sheaves.

## 5.4 Decomposing character sheaves

In this final section, we will discuss how character sheaves assist in decomposing the Deligne–Lusztig characters into their irreducible constituents. This is not a topic that we (the author) investigated too deeply, so this section should be viewed only as a guide to topics for further exploration.

Informally, the problem of decomposing character sheaves is easier than that of decomposing Deligne–Lusztig characters because, where before we only had information on stalks, we now have information about how the character sheaves decompose on *every étale open set.*

In practice, and especially for simple groups, it is usually possible to piece together this local information to determine how the character sheaf decomposes into irreducible constituents. If more sophisticated techniques are required, there is a classification theorem for irreducible characters sheaves.

The main difficulty lies in the fact that the character sheaves, as semisimple objects in $\mathcal{M}(G)$, can be decomposed as a direct sum of simple *perverse* sheaves. However, these simple perverse summands need not be character sheaves themselves, in which case we do not get a decomposition of the characteristic functions. Instead, one would actually want a decomposition into a direct sum of character sheaves such that the summands are indecomposable as character sheaves. The problem of finding such a decomposition systematically is (to the best of our knowledge) still not completely resolved.

For a more comprehensive survey on the topic of decomposing character sheaves (still omitting proofs), we refer the reader to Chapters 11 and 12 of [17].

# Appendix A

# The character table of $\mathrm{GL}_2(\mathbb{F}_q)$

In this appendix, we will compute the character table of $\mathrm{GL}_2(\mathbb{F}_q)$ using elementary methods. We include this calculation for three reasons: firstly, to give an example of parabolic induction, which is possibly the most fundamental technique in the representation theory of finite groups of Lie type. Secondly, to emphasize that, before the invention of Deligne-Lusztig theory, there was really no uniform approach to finding the cuspidal irreps; the construction of the cuspidal irreps that we give here does not even generalise to $\mathrm{SL}_2(\mathbb{F}_q)$, which is widely considered to be the next simplest finite group of Lie type. Thirdly, to see concretely results which appear in a far more abstract setting in Chapter 4, in the hopes that an explicit example will help keep the reader grounded.

Throughout this appendix, we have omitted the Frobenius $F_q$ from our notation since our approach does not make use of the reductive group $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$. It is worth noting, however, that we are somewhat abusing terminology by doing so. For example, the finite subgroup $B \leq \mathrm{GL}_2(\mathbb{F}_q)$ is technically **not** a Borel subgroup according to the definition we gave in Chapter 2, since it is not defined over an algebraically closed field, but it should be clear from context what we mean.

**Example A.1.** Let $G = \mathrm{GL}_2(\mathbb{F}_q)$, with $B \leq G$ the standard Borel, $U \leq B$ its unipotent radical (upper triangular matrices with 1s on the diagonal) and $D \leq B$ the diagonal matrices. We begin by looking at some structural properties which are specific to $\mathrm{GL}_2(\mathbb{F}_q)$, such as its order and its conjugacy classes.

For the former, we recall that a matrix is invertible if and only if its columns are linearly independent. We have $q^2-1$ choices for the first column (everything except the zero vector) and then $q^2-q$ choices for the second column (anything which is not a scalar multiple of the first column). It follows that

$$|G| = (q^2-1)(q^2-q).$$

The calculation of the conjugacy classes of $G$ is somewhat tedious, so we refer the reader to [5] for details. One essentially uses Jordan canonical form to pass to a degree 2 extension of $\mathbb{F}_q$ where the characteristic polynomial has both its roots, then breaks into cases based on the multiplicity of the eigenvalues and whether they exist in the smaller field $\mathbb{F}_q$.

The result is:

| Name | Representative Element | Number of Classes | Size of Class |
|---|---|---|---|
| Central, $c_1(\alpha)$ | $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in \mathbb{F}_q^\times$ | $q-1$ | $1$ |
| Non-semisimple, $c_2(\alpha)$ | $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in \mathbb{F}_q^\times$ | $q-1$ | $(q-1)(q+1)$ |
| Split regular semisimple, $c_3(\alpha,\beta)$ | $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\alpha \neq \beta \in \mathbb{F}_q^\times$ | $\frac{1}{2}(q-1)(q-2)$ | $q(q+1)$ |
| Anisotropic regular semisimple, $c_4(\delta)$ | $\begin{pmatrix} 0 & -N(\delta) \\ 1 & \mathrm{Tr}(\delta) \end{pmatrix}$, $\delta \in \mathbb{F}_{q^2} - \mathbb{F}_q$ | $\frac{1}{2}q(q-1)$ | $q(q-1)$ |

We now wish to study the parabolically induced representations. Technically we are only going to compute the parabolically induced representations coming from $B = U \rtimes D$ and there may be others coming from different Borel subgroups, but it is a fact that parabolically inducing from any other Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$ gives the same series of representations (this is specific to $\mathrm{GL}_2(\mathbb{F}_q)$, not a general phenomenon).

Because $D \cong (\mathbb{F}_q^\times)^2$, all its linear characters are of the form $\chi_1 \otimes \chi_2$ for $\chi_1, \chi_2$ linear characters of $\mathbb{F}_q^\times$. We denote by $I(\chi_1,\chi_2)$ the parabolically induced representation of $G$ resulting from the input of two such characters. To determine the parabolically induced characters, we need to decompose the $I(\chi_1,\chi_2)$ into their irreducible components.

**Theorem A.2.** *Fix characters* $\chi_1, \chi_2, \mu_1, \mu_2$ *of* $\mathbb{F}_q^\times$. *Then* $\langle I(\chi_1, \chi_2), I(\mu_1, \mu_2)\rangle = e_1 + e_\omega$, *where*

$$e_1 = \begin{cases} 1 & \textit{if } (\chi_1, \chi_2) = (\mu_1, \mu_2) \\ 0 & \textit{otherwise} \end{cases}$$

$$e_\omega = \begin{cases} 1 & \textit{if } (\chi_1, \chi_2) = (\mu_2, \mu_1) \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Recall that

$$\langle I(\chi_1, \chi_2), I(\mu_1, \mu_2)\rangle = \dim_{\mathbb{C}} \operatorname{Hom}_G(I(\chi_1, \chi_2), I(\mu_1, \mu_2)).$$

By Mackey's description of intertwiners (Section 1.4 in [6]), the space of $G$-equivariant maps $I(\chi_1, \chi_2) \to I(\mu_1, \mu_2)$ is isomorphic to

$$D = \{\Delta : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{C} \mid \Delta(b_2 g b_1) = \mu(b_2)\Delta(g)\chi(b_1), \quad \forall b_i \in B, g \in G\}.$$

We now find a basis for $D$. By the Bruhat decomposition, any such $\Delta$ is fully determined by its values at 1 and $\omega$. Taking $g = 1$ in the above, we see that for any $b \in B$,

$$\mu(b)\Delta(1) = \Delta(b) = \Delta(1)\chi(b).$$

Therefore if $\mu \neq \chi$ then $\Delta(1) = 0$. On the other hand, if $\mu = \chi$, let $\Delta_1$ be the function

$$\Delta_1(g) = \begin{cases} \chi(g) & \text{if } g \in B \\ 0 & \text{otherwise} \end{cases}$$

and if $e_1 = 0$, set $\Delta_1 = 0$. This is our first basis vector.

We now consider $g = \omega$, which gives

$$\mu(b)\Delta(\omega) = \Delta(b\omega) = \Delta(\omega(\omega^{-1}b\omega)) = \Delta(\omega)\chi(\omega^{-1}b\omega).$$

**Remark A.3.** For $b = \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix} \in B$, $\omega^{-1}b\omega = \begin{pmatrix} a_2 & c \\ 0 & a_1 \end{pmatrix}$.

From the remark, we now see that

$$\mu_1(a_1)\mu_2(a_2)\Delta(\omega) = \Delta(\omega)\chi_1(a_2)\chi_2(a_1).$$

Therefore if $\mu_1 \neq \chi_2$ or $\mu_2 \neq \chi_1$ then $\Delta(\omega) = 0$. However, if $e_\omega = 1$, let $\Delta_\omega$ be the function such that $\Delta_\omega(b_2\omega b_1) = \chi(b_1)\mu(b_2)$ and $\Delta_\omega\,|_B = 0$. If $e_\omega = 0$, set $\Delta_\omega = 0$.

The above analysis shows that the vectors $\Delta_1$ and $\Delta_\omega$ form a basis for $D$, so $\dim D = e_1 + e_\omega$. □

**Corollary A.4.** *Let $\chi_1, \chi_2, \mu_1, \mu_2$ be characters of $\mathbb{F}_q^\times$. Then $I(\chi_1, \chi_2)$ is an irrep of $\mathrm{GL}_2(\mathbb{F}_q)$ of degree $q+1$ unless $\chi_1 = \chi_2$, in which case it is a direct sum of two irreps of degrees 1 and $q$. We also have*

$$I(\chi_1, \chi_2) \cong I(\mu_1, \mu_2) \iff \{\chi_1, \chi_2\} = \{\mu_1, \mu_2\}.$$

*Proof.* First observe that because $[G : B] = q+1$, the degree of $I(\chi_1, \chi_2)$ is always $q+1$. We now apply Theorem A.2 with $\mu_1 = \chi_1$ and $\mu_2 = \chi_2$ to find

$$\dim_{\mathbb{C}} \mathrm{End}_G(I(\chi_1, \chi_2)) = \begin{cases} 1 & \text{if } \chi_1 \neq \chi_2 \\ 2 & \text{if } \chi_1 = \chi_2 \end{cases}$$

We conclude that $I(\chi_1, \chi_2)$ is indeed a $(q+1)$-dimensional irrep when $\chi_1 \neq \chi_2$.

If $\chi_1 = \chi_2$ then $I(\chi_1, \chi_2)$ must be a direct sum of two irreps because $2 = 1^2+1^2$ is the only way to write 2 as a sum of two squares. One can verify that the subspace spanned by the function $\chi_1 \circ \det = \chi_2 \circ \det$ is a one-dimensional invariant subspace of $I(\chi_1, \chi_2)$, hence forms a one-dimensional irrep. The other component is therefore $q$-dimensional.

The last part now follows by again invoking Theorem A.2 and counting dimensions (recall that by Schur's lemma, a nonzero homomorphism between irreps is necessarily an isomorphism). □

Recall that the number of irreps of $G$ is the same as the number of conjugacy classes. We have now constructed

- $q-1$ one-dimensional irreps,
- $q-1$ irreps of dimension $q$, &
- $\frac{1}{2}(q-1)(q-2)$ irreps of dimension $q+1$.

Comparing with the number of conjugacy classes from our earlier table, we still have $\frac{1}{2}q(q-1)$ irreps left to find. Also recall that

$$\sum_{\rho \in \mathrm{Irr}(G)} \dim(\rho)^2 = |G| = (q^2-1)(q^2-q).$$

The sum of the squares of the irreps we have constructed thus far is

$$(q-1) + (q-1)q^2 + \frac{1}{2}(q-1)(q-2)(q+1)^2$$

and the difference between these two is

$$\frac{1}{2}q(q-1)(q-1)^2.$$

We might therefore hope that each of our remaining $\frac{1}{2}q(q-1)$ irreps has dimension $q-1$ and indeed, this will turn out to be the case.

To find the cuspidal representations, we now need to use something specific to $\mathrm{GL}_2(\mathbb{F}_q)$, which in this case means constructing a special torus which is not contained in any Borel subgroup (we will explore the significance of such tori further in Chapter 4). Since we are looking at $2\times 2$ matrices, we get an obvious candidate by looking at the degree 2 extension of $\mathbb{F}_q$.

We view $\mathbb{F}_{q^2}$ as a 2-dimensional $\mathbb{F}_q$-vector space. Multiplication by elements of $\mathbb{F}_{q^2}^\times$ then gives an embedding of $\mathbb{F}_{q^2}^\times$ into $\mathrm{GL}_2(\mathbb{F}_q)$. Although this embedding exists regardless of characteristic, we can see it explicitly when $p \neq 2$ by picking a non-square $\delta \in \mathbb{F}_q^\times$ and writing $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{\delta}]$. $\mathbb{F}_{q^2}^\times$ then embeds into $\mathrm{GL}_2(\mathbb{F}_q)$ as the subgroup

$$T = \left\{ \begin{pmatrix} a & \delta b \\ b & a \end{pmatrix} : (a,b) \in \mathbb{F}_q \times \mathbb{F}_q - \{(0,0)\} \right\}.$$

Let $\theta \in \mathrm{Irr}(T)$ be a one-dimensional character.

**Definition A.5.** We say that $\theta$ is *regular* if $\overline{\theta} \neq \theta$, where $x \mapsto \overline{x}$ is the Galois involution for $\mathbb{F}_{q^2}/\mathbb{F}_q$, i.e. $\overline{x} = x^q$.

The notion of regularity of characters was introduced more generally in Chapter 4. Note that the definition given there agrees with this one because the nontrivial element of the Weyl group $W \cong S_2$ is realised by the Frobenius.

Let us count the number of regular characters of $T$. Note that $\theta$ is **not** regular if and only if $\theta^{q-1} = \mathbb{1}$. Since the kernel of the Frobenius has $q-1$ elements, we must therefore have

$$(q^2-1)-(q-1) = q^2 - q$$

regular characters.

Regular characters occur in pairs $\{\theta, \overline{\theta}\}$, so there are a total of $\frac{1}{2}q(q-1)$ Galois orbits. Each of these orbits will give rise to a cuspidal irrep.

Now let $ZU := Z(G) \rtimes U$ and fix a nontrivial character $\psi$ of $U$. We define a character $\theta_\psi$ of $ZU$ by

$$\theta_\psi(au) = \theta(a)\psi(u), \quad a \in \mathbb{F}_q^\times, u \in U.$$

**Theorem A.6.** *Let $\theta, \psi, T$ and $ZU$ be as above. Then*

*(a)* $\mathrm{Ind}_{ZU}^G(\theta_\psi) = \mathrm{Ind}_T^G(\theta) \oplus \pi_\theta$ *for an irreducible $(q-1)$-dimensional cuspidal irrep $\pi_\theta$ of $G$;*

(b) *For two regular characters $\theta, \theta'$ of $T$, $\pi_\theta \cong \pi_{\theta'}$ iff $\theta' \in \{\theta, \overline{\theta}\}$; and*

(c) *The irreps $\{\pi_\theta\}$ exhaust the irreducible cuspidal representations of $G$.*

Implicit in the statement of this theorem is that the construction of the $\pi_\theta$ does not actually depend on the choice of $\psi$. Indeed, a different choice of $\psi$ will permute the $\pi_\theta$, but they will all still appear exactly once.

The bulk of the proof falls under the following lemma:

**Lemma A.7.** *Let $\chi_\theta$ be the character of the rep $\pi_\theta$. Then*

$$\begin{aligned}
\chi_\theta(c_1(\alpha)) &= (q-1)\theta(\alpha) \\
\chi_\theta(c_2(\alpha)) &= -\theta(\alpha) \\
\chi_\theta(c_3(\alpha,\beta)) &= 0 \\
\chi_\theta(c_4(\delta)) &= -\big(\theta(\delta) + \overline{\theta}(\delta)\big)
\end{aligned}$$

*Proof.* (sketch)

One first verifies that $\langle \mathrm{Ind}_{ZU}^G(\theta_\psi), \mathrm{Ind}_T^G(\theta)\rangle = 1$ using Frobenius reciprocity and the Mackey formula. We can now recover $\chi_\theta$ as $\mathrm{Ind}_{ZU}^G(\theta_\psi) - \mathrm{Ind}_T^G(\theta)$, both of which we can compute. The lemma follows by explicitly evaluating the characters on each of the conjugacy classes, which is a standard but tedious calculation. $\square$

*Proof.* (of Theorem A.6)

(a) Using the values of $\chi_\theta$ given by Lemma A.7, we can compute $\langle \chi_\theta, \chi_\theta\rangle = 1$, so $\chi_\theta$ is irreducible. It must be $(q-1)$-dimensional by dimension counting.

(b) First recall that $\mathbb{F}_q$ is exactly the set of elements in $\mathbb{F}_{q^2}$ that are fixed under the Frobenius, so $\chi_\theta = \chi_{\overline{\theta}}$ by checking each of the formulae in Lemma A.7. Conversely, we can recover $\theta\,|_{\mathbb{F}_q}$ from the classes $c_2(\alpha)$, and we can recover $\theta\,|_{\mathbb{F}_q^2 - \mathbb{F}_q}$ (up to Galois conjugation) from $c_4(\delta)$.

(c) From (a) and (b), we know that we have constructed $\frac{1}{2}q(q-1)$ distinct irreps. Since none of the parabolically induced irreps were $(q-1)$-dimensional, these must be new. Our earlier counting argument now implies that we have exhausted the cuspidal representations. $\square$

# Appendix B

# Miscellaneous proofs

In this appendix, we give proofs of various results which are hard to find in the standard literature. They are included here, rather than where they are used, because they are generally quite technical and the proofs themselves do not contribute anything to the places where the results are used.

**Lemma B.1.** *Let $k = \overline{\mathbb{F}}_p$ and let $T = k^\times$. Then every algebraic character $T \to T$ is of the form $x \mapsto x^i$ for some $i \in \mathbb{Z}$.*

*Proof.* Let $\chi : T \to T$ be such a character. We can think of $\chi$ as a regular function on $T$ which does not have 0 in its image, so $\chi \in k[T] = k[x, x^{-1}]$. We now write $\chi$ as

$$\chi(x) = \sum_{i=-n}^{m} c_i x^i$$

where $n, m$ are non-negative integers and $c_i \in k$.

Because $\chi$ is a homomorphism, we have

$$\begin{aligned} \chi(xy) &= \chi(x)\chi(y) \\ \sum_{i=-n}^{m} c_i x^i y^i &= \left(\sum_{i=-n}^{m} c_i x^i\right)\left(\sum_{j=-n}^{m} c_j y^j\right) \end{aligned}$$

Expanding the right-hand side and equating coefficients in the monomials $x^i y^j$, we find that $c_i^2 = c_i$ and $c_i c_j = 0$ for $i \neq j$. Since $\chi$ is nonzero, at least one of the $c_i$ is nonzero. From the first condition we find that $c_i = 1$ and from the second condition, we get $c_j = 0$ for $j \neq i$. Thus

$$\chi(x) = x^i$$

as claimed. □

**Lemma B.2.** *Let $(S_i, \pi_{ji})$ be an inverse system of finite sets with limit $S$, and suppose each $\pi_{ji}$ is surjective. Then every projection $S \to S_i$ is surjective.*

*Proof.* First note that because the $S_i$ are finite, equipping them with the discrete topology makes them into compact Hausdorff topological spaces.

Since we are in the category of sets, we have an explicit realisation of $S$ as

$$S = \left\{ (s_i)_i \in \prod_i S_i : s_i = \pi_{ji}(s_j) \text{ for all } j < i \right\}.$$

The projection map $\pi_i : S \to S_i$ is then just projection onto the $i^{th}$ coordinate. Given some $x_i \in S_i$, we now want to show that there is some $x \in S$ such that $\pi_i(x) = x_i$. For every $j > i$, we consider the set

$$T_j := \left\{ (s_l)_l \in \prod_l S_l : s_i = x_i \text{ and } s_l = \pi_{lj}(s_j) \text{ for all } l < j \right\}.$$

Now each $T_j$ is closed in the product topology on $\prod_l S_l$ because the $S_j$ are Hausdorff, and

$$\bigcap_{j>i} T_j \subset S$$

by construction. Furthermore, the $i^{th}$ coordinate of any element of this intersection is $x_i$, so it suffices to show that $\bigcap_{j>i} T_j$ is nonempty.

Observe that each $T_j$ is closed in $\prod_l S_l$ because the $S_j$ are Hausdorff, and this product is itself compact by Tychonoff's theorem. We recall the equivalent characterisation of compactness in terms of the *finite intersection property*, which says that $S$ is compact if and only if for every collection $\mathcal{C}$ of closed sets such that all finite intersections are nonempty, the intersection over all of $\mathcal{C}$ is nonempty. Taking

$$\mathcal{C} = \{T_j\}_{j>i}$$

it now suffices to show that any finite intersection in $\mathcal{C}$ is nonempty. Given $i_1, \ldots, i_n$ all greater than $i$, we take some $k > i_1, \ldots, i_n$. Then

$$T_k \subset T_{i_1} \cap \ldots \cap T_{i_n}.$$

But because all the transition maps $\pi_{lk}$ are surjective, $T_k$ is nonempty and hence so is $T_{i_1} \cap \ldots \cap T_{i_n}$, as required. □

**Proposition B.3.** *Let $G$ be a connected reductive group, with $T \leq G$ a maximal torus and $F : G \to G$ a Frobenius. Let $t \in T^F$ and suppose $s \in G^F$ is a semisimple element which is $G^F$-conjugate to $t^{-1}$. Define $(G^F)^{(s,t)} := \{k \in G^F \mid skt = k\}$ and $\tilde{Y}_t := \tilde{X} \cap C^0(t)$. Then:*

*(i) The map $(k, z) \mapsto kz$ is a surjective morphism $(G^F)^{(s,t)} \times \tilde{Y}_t \to \tilde{X}^{(s,t)}$;*

*(ii) $C^0(t)^F$ acts on $(G^F)^{(s,t)} \times \tilde{Y}_t$ by $m \cdot (k, z) = (km^{-1}, mz)$;*

*(iii) The orbits of $C^0(t)^F$ are exactly the fibers of the map in (i).*

*Proof.* (i) Let us first show that $kz \in \tilde{X}^{(s,t)}$ so that the map is well-defined. We have

$$L(kz) = \underbrace{z^{-1}k^{-1}F(k)F(z) = z^{-1}F(z)}_{\text{since } k \in G^F} \in U$$

and so $kz \in \tilde{X}$. Moreover,

$$\underbrace{skzt = sktz}_{\text{since } z \in C^0(t)} = kz$$

and hence $kz \in \tilde{X}^{(s,t)}$, as claimed.

For surjectivity, let $g \in \tilde{X}^{(s,t)}$. Then $sgt = g$ and so $gt^{-1}g^{-1} = s$. Applying $F$ to this expression then yields

$$F(g)t^{-1}F(g)^{-1} = s = gt^{-1}g^{-1}.$$

Rearranging, we see that $L(g) = g^{-1}F(g)$ commutes with $t$. But $L(g) \in U$ is unipotent and unipotent elements always lie in the connected component of the identity, so $L(g) \in C^0(t)$.

Since $C^0(t)$ is connected, Lang's theorem means that we can pick some $z \in C^0(t)$ such that $z^{-1}F(z) = g^{-1}F(g)$. Such a $z$ lies in $C^0(t) \cap \tilde{X} = \tilde{Y}_t$. Setting $k = gz^{-1}$, we have

$$\begin{aligned} skt &= sgz^{-1}t \\ &= sgtz^{-1} && \text{since } z^{-1} \in C(t) \\ &= gz^{-1} && \text{since } g \in \tilde{X}^{(s,t)} \\ &= k \end{aligned}$$

and hence $k \in (G^F)^{(s,t)}$. This shows that $g = kz$ is in the image, as required.

(ii) Given $m \in C^0(t)^F$ and $k \in (G^F)^{(s,t)}$, we have $km^{-1} \in G^F$ and

$$skm^{-1}t = sktm^{-1} = km^{-1},$$

so $km^{-1} \in (G^F)^{(s,t)}$. Also, for $z \in \tilde{Y}_t$, we have $mz \in C^0(t)$ and

$$L(mz) = z^{-1}m^{-1}F(m)F(z) = z^{-1}F(z) \in U.$$

Hence $mz \in C^0(t) \cap \tilde{X} = \tilde{Y}_t$, so this action is well-defined.

(iii) If $(k, z)$ and $(km, m^{-1}z)$ are in the same $C^0(t)^F$-orbit then clearly they are in the same fiber.

Conversely, suppose that $(k_1, z_1)$ and $(k_2, z_2)$ are in the same fiber, meaning $k_1z_1 = k_2z_2$. Then

$$k_2^{-1}k_1 = z_2z_1^{-1} \in G^F \cap C^0(t) = C^0(t)^F.$$

Setting $m = k_2^{-1}k_1 = z_2z_1^{-1}$, we then have $(k_2, z_2) = m \cdot (k_1, z_1)$. □

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