Khovanov Homology & The Reidemeister Moves

M. Skilleter

Abstract

We prove the invariance of Khovanov homology under the first and second Reidemeister moves, primarily following [2]. To keep this a succinct account, we assume knowledge of the construction of Khovanov homology in Chapter 1 of [1].

1 Introduction

Khovanov homology is a well-known method for constructing a bi-graded complex of vector spaces from an oriented link D. Because the graded Euler characteristic of this complex $\mathrm{Kh}(D)$ is the unnormalised Jones polynomial, which has long been known to be invariant under the Reidemeister moves, one might hope that $\mathrm{Kh}(D)$ is also invariant; this is what we shall prove in this excerpt. Along with the proof of functoriality of $\mathrm{Kh}(-)$, this shows that Khovanov homology is a "categorification" of the Jones polynomial.

This paper is intended as a successor to a talk which was given based on Chapter 1 of [1]. As such, we will assume not only knowledge of the construction of Khovanov homology, but also of the conventions chosen in this resource. There is no standard convention as to indexing of Khovanov homology, so the reader should take time to familiarise themselves with our definitions.

2 Assumed Background & Reductions

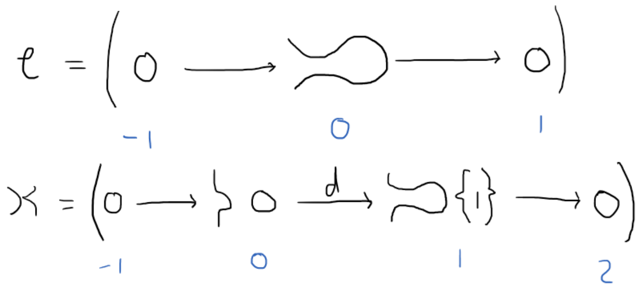
The reader must be comfortable with the notion of cobordisms as morphisms between Khovanov chain complexes. Recall that a *cobordism* between two $(n-1)$-manifolds N_1, N_2 is an n-manifold M in which $\partial M = N_1 \coprod N_2$. We consider cobordisms to be oriented, so that we may speak of a cobordism *from* N_1 *to* N_2. In this paper, we choose the convention that cobordisms move down the page, so N_1 is at the top and N_2 is at the bottom. The composition $C_2 \circ C_1$ of cobordisms is then simply stacking C_1 on top of C_2. At various points in the proof of invariance, we will need to use relations to simplify the cobordisms which make up our morphisms; for further details on this, see Section 3 of [2].

To simplify our calculations, we make the claim that it is enough to prove invariance under the Reidemeister moves locally. That is, we will only consider the diagrams for the Reidemeister moves themselves, and not how they may be embedded into a larger link diagram. The reader should consider why this is sufficient, but the technical details of the proof are not enlightening and so we omit them.

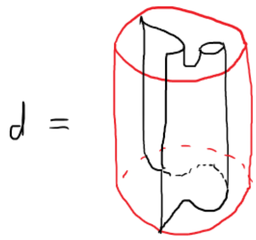
3 Reidemeister (I)

We start by recalling the first Reidemeister move ($\Omega 1$), which is given by

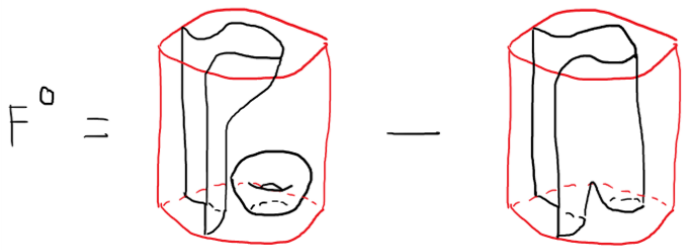
To show that we can perform ($\Omega 1$) and get the same result in Khovanov homology, we must construct a homotopy equivalence between the chain complexes

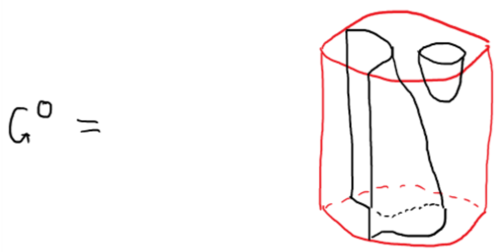


where d is the differential given by the cobordism

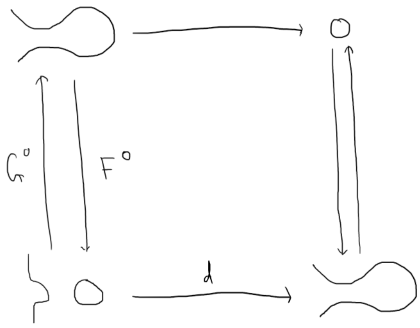


To construct our chain maps $F : \mathcal{C} \to \mathcal{K}$ and $G : \mathcal{K} \to \mathcal{C}$, we set $F^{\neq 0}, G^{\neq 0} = 0$, with the degree 0 maps given by



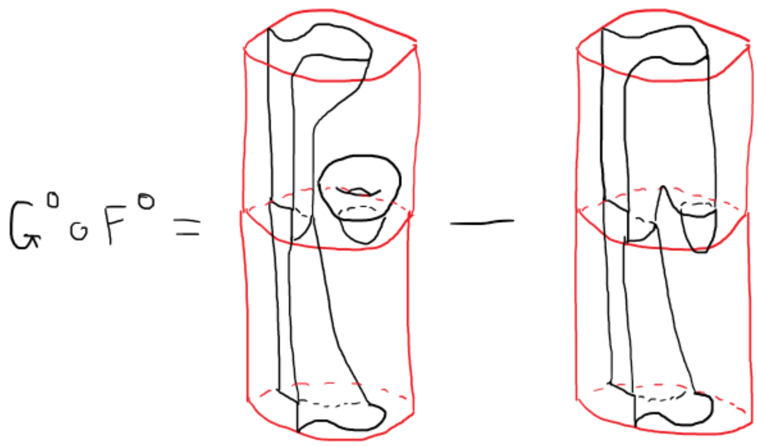


There are two things to check: first, that F and G are actually chain maps; and second, that they are homotopy inverses. The first condition says exactly that we must verify commutativity of the diagram



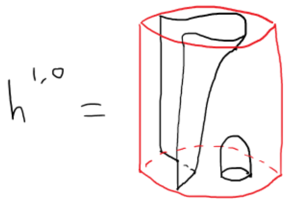
The only non-trivial commutativity relation is $d \circ F^0 = 0$. This follows by simply drawing out the associated composition of cobordisms and observing that both summands are vertical curtains with a single handle, so they cancel.

It remains to show that F and G are inverse (up to chain homotopy). In fact, the stronger statement that $G \circ F = \mathrm{id}$ is true.



When formally adding cobordisms, there is a rule called the T *relation* which says that whenever a diagram contains a closed toroidal component, we can discard the torus at the cost of multiplying our remaining diagram by 2. In this case, we can simplify the first cobordism above by discarding the torus, and the result is 2 times the second diagram. Subtracting them leaves the identity cobordism, as claimed.

Unfortunately, the reverse composition is not the identity, so we must construct a chain homotopy $h : \mathcal{K} \to \mathcal{K}$ from $F \circ G - \mathrm{id}$ to 0. The only degrees where it can possibly be nonzero are $\mathcal{K}^1 \to \mathcal{K}^0$, where we define



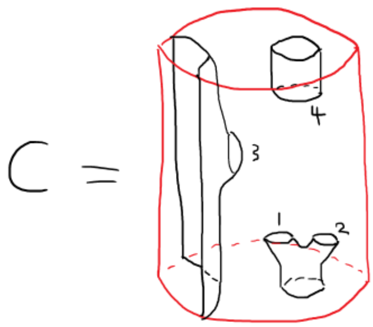
It is immediate that

$$F^1 \circ G^1 - \mathrm{id} + d \circ h^{1,0} = - \mathrm{id} + d \circ h^{1,0} = 0.$$

The other composition to check is that

$$F^0 \circ G^0 - \mathrm{id} + h^{1,0} \circ d = 0$$

and this is less trivial. Let us consider the "shell" cobordism



Denote by $C_{i,j}$ the cobordism which is constructed from C by gluing a tube between disk i and disk j and capping off the other disks. By drawing out the associated cobordism diagrams, one can verify that $C_{1,2}$ and $C_{1,3}$ are the first and second summands respectively in $F^0 \circ G^0$, while $C_{2,4} = \mathrm{id}$ and $C_{3,4} = h^{1,0} \circ d$. We may therefore rewrite our equation as

$$F^0 \circ G^0 - \mathrm{id} + h^{1,0} \circ d = C_{1,2} - C_{1,3} - C_{2,4} + C_{3,4}.$$

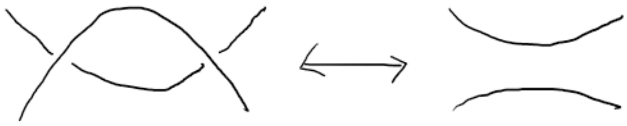
We now invoke another relation from cobordism theory, called the *4 tube relation*. Suppose we have a cobordism C, and its intersection with some ball is the disjoint union of four disks. We enumerate these disks (in an arbitrary order) and let $C_{i,j}$ be as above. Then we have the relation

$$C_{1,2} + C_{3,4} = C_{1,3} + C_{2,4}.$$

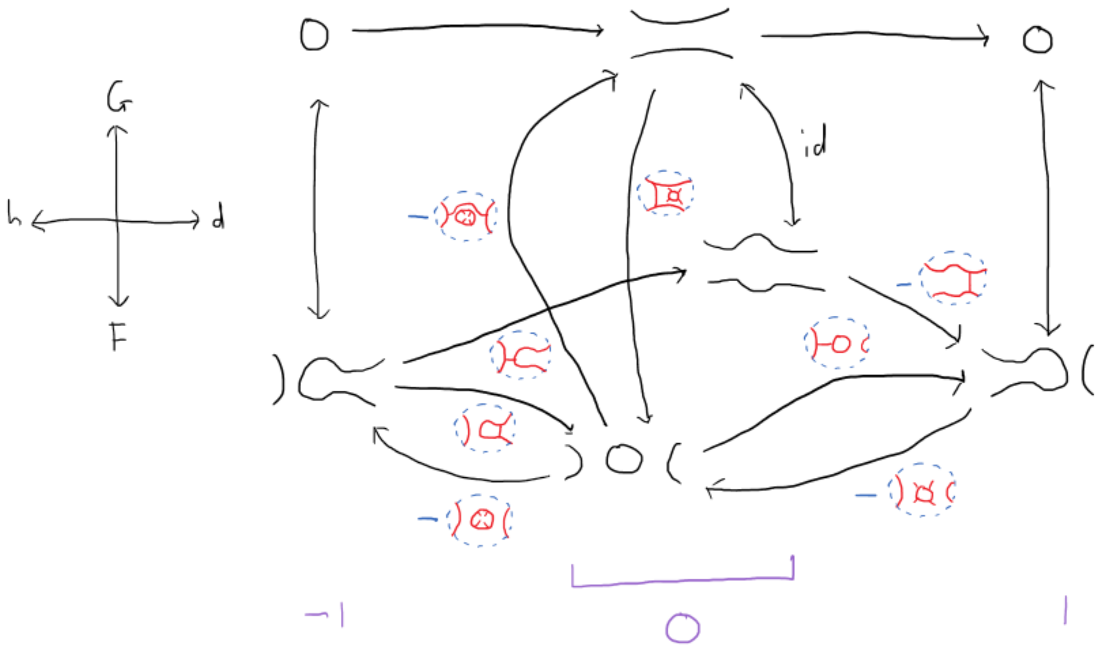
Applying this in our above situation, the right-hand side is 0 and so h is indeed a homotopy equivalence. Thus the complexes $\mathcal{C}$ and $\mathcal{K}$ have isomorphic homology and so performing the first Reidemeister move does not affect Khovanov homology. $\square$

4 Reidemeister (II)

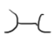
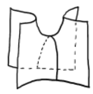
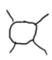
The second Reidemeister move ($\Omega 2$) is given by



The proof of invariance is similar in spirit to the proof of ($\Omega 1$): we will construct chain maps F and G which will prove to be homotopy inverses. The relevant diagram of chain complexes (which we will elaborate further upon momentarily) is given by



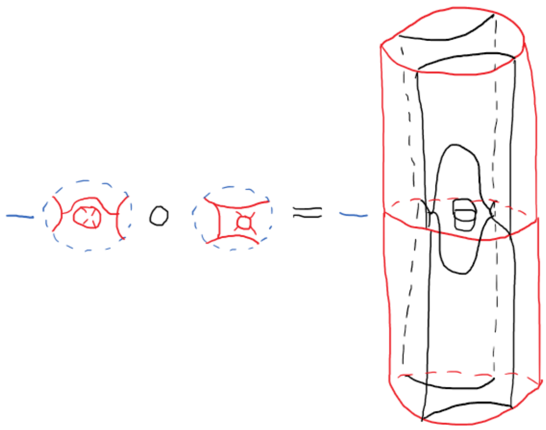
The first and foremost feature of this diagram is the dotted circles, which are a shorthand notation for cobordisms. A key to decipher this notation is

Symbol	Name	Cobordism
	Saddle	
	Cap	
	Cup	

Also note the “compass” in the diagram, which details the maps in a concise way. For example, the rightward moving arrows in the lower complex are the differentials d, which is reflected by the right arrow of the compass.

We claim that this diagram describes a chain homotopy equivalence between the top and bottom complexes. As before, we must verify that F and G are chain maps, and are also homotopy inverses. The former claim is simply that $d \circ F = 0$ and $G \circ d = 0$, which is immediate from drawing out the cobordisms. The latter statement, however, requires more work.

Once again, it is true that $G \circ F = \mathrm{id}$. Observe from the diagram that we already have an identity summand coming from the back vertical arrow, so it suffices to show that the rest of the composition is zero. In this case, the remainder is given by

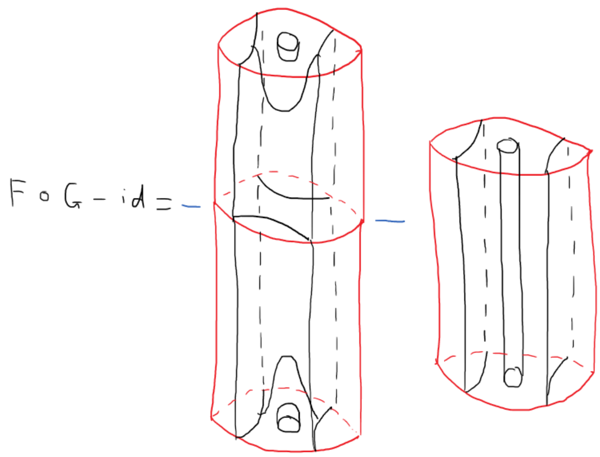


Yet another cobordism relation, called the S *relation*, says that whenever a cobordism contains a closed sphere component, it is set to zero. We can see a sphere component in the above coming from the composition of the cup and the cap, so the remainder is indeed zero. We conclude that $G \circ F = \mathrm{id}$, as claimed.

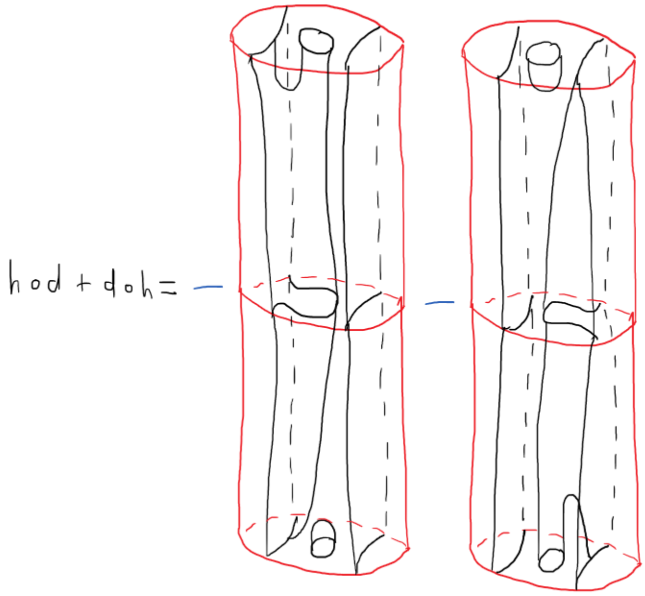
The opposite composition is again more difficult, and requires the use of our chain homotopy h. In this case, we will show that

$$F \circ G - \mathrm{id} = h \circ d + d \circ h.$$

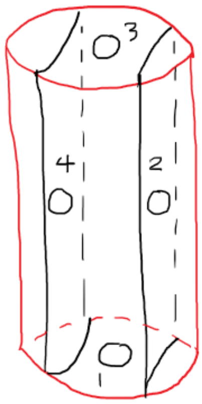
The left-hand side is given by



while the right-hand side is



We will again consider a shell cobordism and apply the 4 tube relation. In this case, our shell is



We now see that

$$F \circ G - \mathrm{id} = -C_{2,4} - C_{1,3} \quad \text{and} \quad h \circ d + d \circ h = -C_{3,4} - C_{1,2}$$

and so by the 4 tube relation, they are indeed equal. We conclude that F and G are homotopy inverses and hence isomorphisms on homology, so Khovanov homology is also invariant under (Ω2). $\square$

5 Remarks on Reidemeister (III)

We have now seen that Khovanov homology is invariant under the first two Reidemeister moves, so all that remains is Reidemeister (III). Because the diagram for ($\Omega 3$) is comparatively complicated, proving invariance using the same techniques as above is quite difficult. Instead, the standard proof is to repeatedly use invariance under ($\Omega 2$) to reduce the chain complexes to a manageable state; those familiar with the proof of invariance of the Kauffman bracket would likely find it very familiar.

Although this is an interesting approach, it is dissimilar enough from the proofs of ($\Omega 1$) and ($\Omega 2$) that we refer the reader elsewhere for the details; a proof using topological quantum field theories can be found in [1], and yet another explicit proof is in Section 4 of [2].

References

[1] P. Turner. *Five Lectures on Khovanov Homology*. Available at `https://arxiv.org/pdf/math/0606464.pdf`.

[2] D. Bar-Natan. *Khovanov's Homology for Tangles and Cobordisms*. Available at `https://www.math.toronto.edu/~drorbn/papers/Cobordism/Cobordism.pdf`.