The Story of Fourier Inversion

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$$\widehat{f}(\xi) = \int e^{-2\pi i \xi x} f(x)\, dx$$

Fourier analysis is an important computational tool across a range of fields, including in the theory of differential equations and quantum mechanics. There are many variations on the Fourier transform, the most commonly used being the standard, discrete and circular transforms. The beauty of doing Fourier analysis on locally compact abelian groups is that it unifies these approaches, and *transforms* what is a remarkable (but sometimes unmotivated) result into something which is intrinsic to the groups themselves.

The purpose of this note is to provide an approachable introduction to Fourier analysis on locally compact groups. Historically, this is an important area of representation theory which has a fairly high entry-level. We do not give a rigorous treatment, but rather aim to motivate the theory with examples and analogies to the simpler finite group case.

To keep this as a self-contained summary of Fourier Inversion, we have eschewed the proofs of the results in favour of explicit computations; a full treatment can be found in [1] and [2]. By necessity, much of Fourier analysis is reliant on high-level results in functional analysis. Generally, the necessary results are taught at a graduate level, but Fourier analysis has much lower-level applications. To make this theory approachable to those who have only seen representation theory of finite groups, we omit a description of the spectral analysis necessary to prove these results.

For those wishing to explore further, the key point is spectral analysis on Banach algebras. A further generalisation of the Fourier transform is the Gelfand transform, but this approach perhaps loses some of its motivation.

In Chapter 4, we will explore an application of Fourier analysis by studying zeta functions. In particular, we will see how a Fourier analytic perspective gives a natural proof of the functional equation and meromorphic continuation of the Riemann zeta function.

Contents

1 Locally Compact Groups

Before we start developing Fourier analysis on locally compact groups, we should define what a locally compact group *is.* In this section, we give the relevant definitions and discuss important examples and properties of these groups.

1.1 The Definition and Examples

Definition. A locally compact group is a topological group that is locally compact (every point has a compact neighborhood) and Hausdorff.

This is a very general definition, which should start to hint at the strength of the theory we will develop. From here on we will suppress references to Hausdorffness, since all of our topological groups will have this property.

Example 1. The following are important examples of locally compact groups:

(1) Any compact group is locally compact; in particular, finite groups (equipped with the discrete topology) are locally compact.

(2) $\mathbb{R}$ under addition is locally compact. This is the simplest example of a non-compact locally compact group.

(3) $\mathbb{T} = S^1$ under multiplication is locally compact. Although technically included in (1), $\mathbb{T}$ is one of the motivating examples when approaching questions about locally compact groups, and is an important example in its own right.

(4) Lie groups, which are locally Euclidean, are locally compact.

(5) A Hausdorff topological vector space is locally compact if and only if it is finite-dimensional.

It might also be helpful to look at a non-example of a locally compact group: the rational numbers $\mathbb{Q}$ under addition. To see that $\mathbb{Q}$ is not locally compact, note that any open set contains a Cauchy sequence corresponding to some irrational number. Such a sequence cannot have a subsequence converging in $\mathbb{Q}$, hence cannot be contained in any compact set.

In light of Examples (2) and (3) above, it might be surprising that $\mathbb{Q}$ is not locally compact. After all, we often think of the rational numbers as having every desirable property. Our argument says that $\mathbb{Q}$ is not locally compact because it is does not contain all of its limit points, and in some sense, this is the only thing that can go wrong:

Proposition 1. *Let G be a locally compact group and $H \subseteq G$ a subgroup. Then H is itself a locally compact group if and only if H is closed in G.*

Proposition 1 says that as long as we are willing to restrict our attention to *closed* subgroups, we will remain in the class of locally compact groups. This assumption is no great loss, since it is easy to show that if $H \subseteq G$ is a (normal, abelian) subgroup then so is $\overline{H}$.

In standard group theory, there are two main tools for constructing smaller objects out of larger ones, these being the subgroup and the quotient. In our more general case, we again have some version of the quotient operation available:

Proposition 2. *Let G be a locally compact group and $H \subseteq G$ a closed subgroup. Then G/H is a locally compact group with the quotient topology.*

Propositions 1 and 2 strongly suggest that our subobjects should be closed for them to be of any real use. This will be a recurring theme.

There is one final property of locally compact groups that will be useful later, which places strong restrictions on the possible homomorphisms between certain types of locally compact groups. To state it, however, we need one more definition:

Definition. A locally compact group G is called *σ-compact* if G can be written as the union of countably many compact subspaces.

In particular, any compact group is σ-compact. The purpose of this definition is to state the following important result:

Theorem 1. *(Open Mapping Theorem) Let G and H be locally compact groups, with G σ-compact. Let $\phi : G \to H$ be a surjective continuous group homomorphism. Then ϕ is an open map.*

This result is reminiscent of the Open Mapping Theorem in functional analysis and indeed, the proofs are essentially identical; both results follow easily from the Baire Category Theorem.

Theorem 1 will play an important role in later sections when we begin computing dual groups.

1.2 The Haar Measure

Having seen some examples of locally compact groups, one might wonder why they are interesting objects. As we shall see later, locally compact groups are 'nice enough' that we can somewhat mimic the representation theory of finite groups. Most of the properties which make this possible can be traced back to the Haar measure.

Definition. A Haar measure on a locally compact group G is a nonzero Radon measure on the Borel σ-algebra that is left (or right) G-invariant.

The term 'Radon' is a technical condition which we will not discuss in detail, but it essentially means that our measure is finite on compact sets and is well-behaved with respect to approximating open and Borel sets. The important point is the invariance property.

For intuition, one should think of the Lebesgue measure μ on $\mathbb{R}^n$. It is a well-known result that μ is translation invariant, meaning $\mu(S) = \mu(c + S)$ for every measurable $S \subseteq \mathbb{R}^n$ and $c \in \mathbb{R}^n$. This is just the statement that moving a set through space should not change its size.

A Haar measure is a natural generalisation of the Lebesgue measure on $\mathbb{R}^n$. The question then becomes: when does such a Haar measure exist? This is answered neatly by the following theorem:

Theorem 2. *Let G be a locally compact group. Then there exists a Haar measure μ on G. Furthermore, the measure μ is unique up to multiplication by a positive scalar.*

The requirement that μ be finite on compact sets means that if the entire group G is compact then $\mu(G) < \infty$. In this case, we have a canonical choice of Haar measure by normalising so that $\mu(G) = 1$. For non-compact groups, we no longer have such an obvious choice, but we will still speak of 'the' Haar measure on G.

Example 2. Here are a few examples of the Haar measure on a locally compact group:

(1) From our discussion above, it should not be surprising that the Haar measure on $\mathbb{R}^n$ is the Lebesgue measure.

(2) The Lebesgue measure on $\mathbb{R}^2 = \mathbb{C}$ is also rotation-invariant, so its restriction to $\mathbb{T}$ is an invariant measure. By uniqueness, it must be 'the' Haar measure on $\mathbb{T}$. Since $\mathbb{T}$ is compact, we have a canonical choice of Haar measure; normalising so that $\mu(\mathbb{T}) = 1$, the Haar measure is given by

$$\mu(S) = \frac{1}{2\pi}\int_S dx.$$

(3) Consider the multiplicative group $\mathbb{R}^\times$ of real numbers. The Lebesgue measure is no longer invariant under the group operation, but we can modify it slightly. Note that

$$\frac{d(cx)}{cx} = \frac{cdx}{cx} = \frac{dx}{x}$$

so the 1-form $\frac{1}{x}dx$ is invariant. Then the measure

$$\mu(S) = \int_S \frac{1}{|x|_\infty}\, dx$$

is invariant (note that we are taking the absolute value simply so that $\mu(S) \geq 0$), so μ is the Haar measure on $\mathbb{R}^\times$.

(4) If G is a discrete group (for example $G = \mathbb{Z}$) then the counting measure is the Haar measure on G.

(5) We should now address an issue we've been dancing around for a while. When we talk about invariance, we should really specify left or right invariance. In all of the examples we have considered so far, the left and right Haar measures have been the same because the groups have been abelian. However, this is not necessarily true of all locally compact groups.

For example, consider the group G of affine transformations of $\mathbb{R}$ which are of the form

$$x \mapsto ax + b, \qquad a > 0.$$

Then

$$\mu_L(S) = \int_S \frac{1}{a^2}\ da\ db$$

is a left-invariant Haar measure on G. In contrast, a right-invariant Haar measure on G is given by

$$\mu_R(S) = \int_S \frac{1}{a}\ da\ db.$$

The measures μ_L and μ_R are not scalar multiples of one another, so our uniqueness does not extend to comparing left and right Haar measures.

If the left and right Haar measures agree, then G is said to be *unimodular*. There is another definition of unimodularity, in terms of a homomorphism $G \to \mathbb{R}_{>0}$ called a *modular function*, and we will discuss this in Section 2.2. Examples of unimodular groups include abelian groups, compact groups, semisimple Lie groups and connected nilpotent Lie groups, among others.

Let's now return to our non-example of $\mathbb{Q}$ and see if it possible to construct a Haar measure even though $\mathbb{Q}$ is not locally compact; we claim that it is not. Suppose for sake of contradiction that we could find some Haar measure μ on $\mathbb{Q}$. Observe that $\mu(\{x\}) > 0$ for every $x \in \mathbb{Q}$; if not, then $\mu(\{x\}) = 0$ for some $x \in \mathbb{Q}$. By invariance of μ, this means that $\mu(\{x\}) = 0$ for every $x \in \mathbb{Q}$ and so, by countable additivity, μ is uniformly zero since every subset of $\mathbb{Q}$ is countable. But Haar measures are nonzero by assumption, so this is not the case.

We now consider the compact set $E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}\right\}$, which is clearly infinite. By countable additivity, $\mu(E) = \infty$. But a Haar measure must be finite on compact sets. Thus we have reached a contradiction and so there is no Haar measure on $\mathbb{Q}$.

At this point, one might ask if it possible to find non-locally compact groups which have a Haar measure. Is there a converse to Theorem 2? In 1936, Weil provided a partial answer to this question ([6]):

Theorem 3. *Suppose that G is a group (not necessarily topological) with a left invariant measure for which one can define a convolution product. Then there is a topology on G so that the completion G^* of G is locally compact and the given measure is induced by the Haar measure on G^*.*

In this sense, any group with a Haar measure is 'almost' locally compact. Theorem 3 will not be used in later sections, we simply include it as an interesting result.

2 Topological Representation Theory

With the preliminaries of locally compact groups out of the way, we can now turn to representation theory of these groups. There are many new ideas in this section, and we will attempt to relate these back to the finite group case where possible. The final subsection, relating to Schur's Lemma, is only included as motivation for the use of unitary representations, so may be skipped.

2.1 Topological Representations and Irreducibility

Definition. Let G be a locally compact group and let V be a locally convex topological $\mathbb{C}$-vector space. An *abstract representation* of G is a homomorphism $G \to \mathrm{Aut}(V)$. A *topological representation* is an abstract representation which also satisfies the condition that

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned}$$

is continuous.

When V happens to be a Banach space (which will always be the case in the examples we consider), we have the following alternate characterisation of topological representations:

Proposition 3. *Suppose that V is a complex Banach space. Then an abstract representation $G \to \mathrm{Aut}(V)$ is moreover a topological representation if and only if for every $v \in V$, the map*

$$\begin{aligned} G &\to V \\ g &\mapsto g \cdot v \end{aligned}$$

is continuous.

Note that if G is finite, equipped with the discrete topology, then the above condition will always be satisfied because any map out of a discrete topological space is continuous. Since the vector space $\mathbb{C}^n$ is a Banach space, this shows that any finite-dimensional representation of a finite group is also a topological representation.

Example 3. Some examples of representations of locally compact groups:

(1) As discussed above, any finite-dimensional representation of a finite group can be interpreted as a topological representation of a locally compact group.

(2) Consider the group $G = \mathrm{GL}_n(\mathbb{C})$, which is locally compact because it is a Lie group. The obvious action of G on $\mathbb{C}^n$ is continuous in g and hence is a topological representation by Proposition 3.

(3) The real numbers act on the Hilbert space $L^2(\mathbb{R})$ by translation i.e.

$$(a \cdot f)(x) = f(x + a).$$

We shall see that $L^2(\mathbb{R})$ is a *unitary* representation of $\mathbb{R}$, meaning the action of $\mathbb{R}$ preserves the inner product.

We now turn to the problem of irreducibility. Much as with the definition of a representation, there are two types that we will be concerned with.

Definition. Let V be a topological representation of a locally compact group G (note that we have suppressed the map $G \to \mathrm{Aut}(V)$, which we shall often do without reference). We say that V is *algebraically irreducible* if V has no proper G-invariant subspaces. V is *topologically irreducible* if it has no proper **closed** G-invariant subspaces.

Clearly if V is algebraically irreducible then V is topologically irreducible. We will rarely see examples which are topologically irreducible but not algebraically irreducible, so we will often abbreviate this as *irreducible.*

Given that it is difficult to construct examples to show that topological irreducibility is a strictly weaker condition, why do we bother distinguishing between the two? One answer is that in the representation theory of finite groups, the only irreducible representations that arise are necessarily finite-dimensional, so any invariant subspace is also a representation of G. However, this is no longer true if we allow infinite-dimensional representations. For example, an arbitrary subspace of a Banach space need not also be a Banach space, but a *closed* subspace will be. Thus a closed invariant subspace of a Banach representation is itself a Banach representation. This is another example of closed subobjects being better-behaved than arbitrary ones.

Example 4. Let's now look at some more examples of representations, both reducible and irreducible.

(1) If G is finite then the regular representation always decomposes as a direct sum of irreducible representations, each with multiplicity their dimension. In Section 2.2, we will see how to appropriately generalise this idea to the setting of locally compact groups.

(2) The representation given in Example 3.(2) is irreducible, because any subspace can be taken to any other subspace by an appropriate change-of-basis matrix. Thus $\mathbb{C}^n$ has no invariant subspaces.

(3) Consider Example 3.(3) and the subspace $W \leq L^2(\mathbb{R})$ spanned by simple functions (finite sums of indicator functions of measurable sets). The translation of a simple function is itself a simple function, so W is an invariant subspace. This shows that $L^2(\mathbb{R})$ is not algebraically irreducible, but does not show that it is not topologically irreducible because simple functions are dense in $L^2(\mathbb{R})$, so $\overline{W} = L^2(\mathbb{R})$ and hence W is not closed.

    In fact, it turns out that $L^2(\mathbb{R})$ is not topologically irreducible, although it is difficult to write down a closed invariant subspace. This is essentially a consequence of Theorem 7.

The problem of finding all representations can often be reduced to computing just the irreducible representations (irreps).

There is one final fundamental idea in topological representation theory, that of the unitary representation.

Definition. Suppose that H is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and that $\rho : G \to \mathrm{Aut}(H)$ is a representation of G. We say that ρ is *unitary* if $\rho(g)$ is unitary for every $g \in G$ i.e. if

$$\langle \rho(g)(v), \rho(g)(w) \rangle = \langle v, w \rangle, \qquad v, w \in H.$$

A representation is *unitarisable* if there is an inner product on H with respect to which ρ is unitary. Any representation of a finite group is unitarisable because we can apply Weil's averaging trick; pick an inner product $\langle \cdot, \cdot \rangle$, and define a new inner product by

$$(v, w) = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle.$$

This is then a G-invariant non-degenerate sesquilinear form, so G is unitary with respect to $(\cdot, \cdot)$.

In fact, this technique applies more generally. If G is compact, we can take a Haar measure μ on G and define

$$(v, w) = \frac{1}{\mu(G)} \int_G \langle g \cdot v, g \cdot w \rangle \; dg.$$

This does not work for all locally compact groups because $\mu(G)$ need not be finite (in fact, $\mu(G) < \infty$ if and only if G is compact). In light of the above, we have proven:

Proposition 4. *Let G be a compact group and let $\rho : G \to \mathrm{Aut}(H)$ be a complex Hilbert representation of G. Then ρ is unitarisable.*

For non-compact groups, it is much harder to detect if a representation is unitarisable, so one should not make the mistake of thinking that all representations are unitarisable. An important problem in representation theory is the description of the *unitary dual*, which effectively classifies all unitary irreps of real reductive Lie groups.

One advantage of working with unitary representations is that the orthogonal complement of a closed invariant subspace will also be invariant. This means that finite-dimensional unitary representations are completely reducible. A word of caution: it is *not true* that arbitrary representations of a locally compact group need be completely reducible. For instance, consider the two-dimensional representation of $G = \mathbb{R}$ defined by

$$\rho(x)(z_1, z_2) = (z_1 + x z_2, z_2).$$

The subspace $W := \mathrm{Span}_{\mathbb{C}}\{(1, 0)\}$ is the only one-dimensional invariant subspace of $\mathbb{C}^2$, so W cannot have a complementary subspace. Thus $\mathbb{C}^2$ is not completely reducible.

Unitary representations will play an important role in the later section on duality theory for locally compact groups.

2.2 The Group Algebra

In this section, we discuss the group algebra of a locally compact group G. If G is finite, the standard construction of $\mathbb{C}[G]$ is as a free $\mathbb{C}$-vector space with basis indexed by elements of G, subject to the multiplicative law

$$e_g \cdot e_h = e_{gh}.$$

It is a well-known result that there is an equivalence of categories between complex representations of G and left modules of $\mathbb{C}[G]$, with irreducible representations corresponding to simple modules. The advantage of the group algebra is that we have techniques from non-commutative algebra available to us (such as Maschke's Theorem), so we would like an analogue in the case of locally compact groups.

Can we use the same definition? If not, what goes wrong? One answer is that, as asserted in Example 1.(5), a topological vector space is locally compact if and only if it is finite-dimensional. For infinite G, this means we lose access to tools like the Haar measure. Another answer is that the group algebra as defined above does not have anything to do with the topology on G; if we want to study *continuous* representations of G, then our group algebra needs to relate to the topology somehow.

So we would like a new definition of the group algebra which agrees with $\mathbb{C}[G]$ in the case that G is finite. As a first step towards this, let's reformulate our definition of $\mathbb{C}[G]$. Observe that a finite linear combination of vectors of the form e_g is the same as a finitely-supported function $G \to \mathbb{C}$. Hence we can set

$$\mathbb{C}[G] = \{f : G \to \mathbb{C} \mid f \text{ is finitely supported}\}.$$

With this new definition, multiplication corresponds to the convolution

$$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x).$$

Of course, the sum on the right is finite because f, g are finitely-supported, but what if we no longer require this? Then our sum need not be well-defined, but we can use the standard trick of replacing it with an integral i.e.

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)\ dy.$$

To make sense of an integral, we need a measure to integrate with respect to; enter the Haar measure, which naturally restricts our attention to locally compact groups. If f, g are themselves integrable, an application of Fubini's Theorem shows that the convolution is an element of $L^1(G)$, and that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

We will soon see that the convolution gives $L^1(G)$ the structure of a Banach algebra, but in fact more can be said. With this in mind, we now introduce the *modular function*, mentioned briefly in Example 2.(5).

Definition. Let μ be a right Haar measure on a locally compact group G. For each $g \in G$, the measure

$$\mu_g(S) := \mu(g^{-1}S)$$

is right-invariant, hence is also a right Haar measure. Because the Haar measure is unique up to scaling, there is a positive real number $\Delta(g)$ such that

$$\mu_g = \Delta(g)\mu.$$

The function $\Delta : G \to \mathbb{R}_{>0}$ is a continuous homomorphism called the *modular function*, which is independent of the choice of Haar measure μ because the Haar measure is well-defined up to a positive scalar.

In a sense, the modular function quantifies how much the left and right Haar measures differ:

Proposition 5. *Let G be a locally compact group. Then G is unimodular (the left and right Haar measures agree) if and only if $\Delta : G \to \mathbb{R}_{>0}$ is the trivial homomorphism.*

Using the modular function, we can define an involution on $L^1(G)$ by

$$f^*(x) := \Delta(x^{-1})\overline{f(x^{-1})}.$$

Theorem 4. *Let G be a locally compact group. Then $L^1(G)$ is a Banach $*$-algebra under the convolution and involution defined above; we call $L^1(G)$ the group algebra of G.*

Technically speaking, the structure on $L^1(G)$ depends on a choice of Haar measure. However, because the Haar measure is defined up to a positive scalar, any choice will induce the same topology on $L^1(G)$, so we need only keep track of the Haar measure if we care about isometries.

Note that if G is finite then every function is integrable and the integral convolution simplifies to the finite sum convolution from earlier; hence the group algebra $L^1(G)$ agrees with our old definition in the case of G a finite group.

The group algebra encodes several important properties of the group, which we can translate between using the following dictionary:

G	$L^1(G)$
discrete	unital
abelian	commutative
compact	multiplicatively compact

The final condition, multiplicative compactness, means that the maps

$$x \mapsto ux \quad \text{and} \quad x \mapsto xu, \qquad u, x \in L^1(G)$$

are compact linear operators (take weakly-convergent sequences to norm-convergent sequences).

As hoped, there is a correspondence between representations of G and representations of $L^1(G)$, although we will have to restrict attention on either side. Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary representation of G: for $f \in L^1(G)$, we want to define a bounded linear operator $\rho_\pi(f) : H \to H$. For $u \in H$, we define $\rho_\pi(f)(u)$ by specifying its inner product with an arbitrary $v \in H$:

$$\langle \rho_\pi(f)(u), v\rangle = \int_G f(x)\langle \pi(x)(u), v\rangle \, dx.$$

With this definition, we then have the following:

Theorem 5. *Let π be a unitary representation of a locally compact group G. Then ρ_π defined above is a non-degenerate $*$-representation of $L^1(G)$. Moreover, every non-degenerate $*$-representation $\rho : L^1(G) \to \mathrm{Aut}(H)$ arises as ρ_π for a unique unitary representation π of G on H.*

Thus there is a correspondence between unitary representations of G and non-degenerate $*$-representations of $L^1(G)$. In the case that G is finite (so every representation is unitary), this reduces to the standard correspondence between representations of G and left modules over $L^1(G) = \mathbb{C}[G]$.

Example 5. As promised in Example 4.(1), we can now generalise the regular representation of a finite group. Since $L^1(G)$ is a Banach space, Proposition 3 applies to show that the map

$$\begin{gathered} \pi_L : G \to \mathrm{Aut}(L^1(G)) \\ [\pi_L(x)(f)](y) = f(x^{-1}y) \end{gathered}$$

is a topological representation of G. This is called the *left regular representation.* Under the above correspondence, the induced map $\rho_{\pi_L}(f)$ is given by left convolution with f i.e.

$$\rho_{\pi_L}(f)(g) = f * g.$$

In fact, the correspondence given by Theorem 5 is stronger:

Proposition 6. *Let $\pi : G \to \mathrm{Aut}(H)$ be a unitary representation of a locally compact group G. Then*

(a) A linear map $T : H \to H$ is an intertwining operator if and only if

$$T \circ \rho_\pi(f) = \rho_\pi(f) \circ T$$

for every $f \in L^1(G)$.

(b) A closed subspace $M \subseteq H$ is G-invariant if and only if $\rho_\pi(f)(M) \subseteq M$ for every $f \in L^1(G)$.

So in some sense, the correspondence of Theorem 5 also gives information about endomorphisms of the representation.

2.3 Schur's Lemma

We now briefly touch upon one of the key results in the representation theory of locally compact groups: Schur's Lemma. This result plays a less important role in the development of Fourier analysis on locally compact groups, so we include it here more as further motivation for why unitary representations are important.

We first consider the naïve version of Schur's Lemma, which is important but almost immediate from the definition.

Lemma 1. *(Schur's Lemma v. 1) Let G be an arbitrary group and let V and W be $\mathbb{C}$-vector spaces. Suppose that π and ψ are algebraically irreducible representations of G on V and W respectively. If $T \in \mathrm{Hom}_G(V, W)$ then either T is the zero map or T is an isomorphism.*

Given that this result is so useful in the representation theory of finite groups, why do we need another version? One answer is the following: a direct consequence of Schur's Lemma (or sometimes an alternate statement) is that if V is an irreducible representation of G then $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{C}$. If V is finite-dimensional then so is $\mathrm{End}_G(V)$, so this forces $\mathrm{End}_G(V) \cong \mathbb{C}$. Every irreducible representation of a finite group is finite-dimensional, so this says exactly that the only endomorphisms of an irrep of a finite group are the scalar multiples of the identity.

This is no longer true for an arbitrary locally compact group, since irreps need no longer be finite-dimensional. However, a modified version of Schur's Lemma does apply if the representation is *unitary*.

Theorem 6. *(Schur's Lemma v. 2) Now suppose that G is a locally compact group and that H is a complex Hilbert space. Let $\rho : G \to \mathrm{Aut}(H)$ be a topologically irreducible unitary representation of G and let $T \in \mathrm{End}_G(H)$. If T is a normal operator then T is a scalar multiple of the identity.*

This again highlights the importance of unitary representations: they behave in much the same way as finite-dimensional representations.

One important reason for this modified statement of Schur's Lemma is the following theorem, which is an easy corollary:

Theorem 7. *Let G be a locally compact abelian group. Then every irreducible unitary representation of G is 1-dimensional.*

This is of course true in the finite group case, but it is somewhat surprising that the result holds more generally; it is not *a priori* clear that such a unitary irrep should even be finite-dimensional. This result, while innocuous, is one of the biggest motivators for the next section.

3 Duality of Locally Compact Groups

In this section, we introduce the Pontryagin dual of a locally compact group G. We also (finally) develop the theory of Fourier inversion on locally compact groups, and discuss the related result known as Pontryagin Duality.

3.1 The Dual Group

Definition. Let G be a locally compact group and define

$$\hat{G} := \mathrm{Hom}(G, \mathbb{T}),$$

the collection of continuous group homomorphisms $G \to \mathbb{T}$. We call $\hat{G}$ the *dual group* (or *Pontryagin dual*) of G; note that $\hat{G}$ is a group under pointwise multiplication.

We define a topology on $\hat{G}$ as follows: let $K \subseteq G$ be compact and let $V \ni 1$ be an open neighbourhood of the identity in $\mathbb{T}$. Define $W(K, V) \subseteq \hat{G}$ by

$$W(K, V) = \{\chi \in \hat{G} : \chi(K) \subseteq V\}.$$

The sets $W(K, V)$ form a neighbourhood base of the identity in $\hat{G}$ and hence induce a topology on $\hat{G}$, called the *compact-open* topology. Under this topology, $\hat{G}$ is also a topological group.

This is quite an involved definition, so let's spend some time unpacking it. There are several obvious questions to ask when initially encountering $\hat{G}$, the first of which is: why are we considering the characters of G? One reason is that they are closely related to one-dimensional representations of G, which are automatically irreducible. Thus $\hat{G}$ encodes useful information about the representations of G, but it is simple enough to actually compute. In all of the cases we will consider, G will be abelian and so by Theorem 7, $\hat{G}$ actually classifies all of the unitary irreps of G.

Why do we require that our characters land in $\mathbb{T}$ as opposed to, say, $\mathbb{C}^\times$? Firstly, because this is automatic if G is finite: any $g \in G$ has finite order, so $\chi(g)$ must be a root of unity and hence must lie in $\mathbb{T}$. Thus this definition immediately agrees with the ordinary dual of a finite group. Secondly, the compactness of $\mathbb{T}$ plays an important role in several results related to Pontryagin Duality, and we no longer have this tool if our characters are allowed to take values in all of $\mathbb{C}^\times$. Thirdly, thinking of characters 'as' 1-dimensional unitary representations, it is quite reasonable that they land in $U(1) = \mathbb{T}$.

The final question we might ask is: why the compact-open topology? A reader seeing it for the first time might think it is somewhat unmotivated, but in fact it is quite ubiquitous in homotopy theory and functional analysis. These two areas are extremely useful in calculating the dual group (in Example 6.(1), we will see how both of these areas can be used to compute $\hat{\mathbb{R}}$).

As further motivation, note that because $\mathbb{T}$ is a metric space, the compact-open topology is equal to the topology of uniform convergence: a sequence $\{\chi_n\}$ converges to χ in $\hat{G}$ if and only if for every compact subset $K \subseteq G$, $\{\chi_n\}$ converges uniformly to χ on K. This is a useful alternative characterisation of the topology on $\hat{G}$.

Let's now look at some general properties of $\hat{G}$, and we will then check that they hold in a few explicit examples.

Proposition 7. *Let G be a locally compact group. Then the following hold:*

(a) $\hat{G}$ is also locally compact.

(b) If G is compact then $\hat{G}$ is discrete.

(c) If G is discrete then $\hat{G}$ is compact.

Proposition 7.(a) suggests that our definition of $\hat{G}$ is a good one, since taking the dual keeps us in the class of locally compact groups. The rest of the proposition can be interpreted as saying that $\hat{G}$ encodes useful topological properties of G.

Example 6. As promised, we now make some explicit computations of the dual group. For the first example, we give a complete proof of the calculation of the dual group, to show that it is possible to work with the definition and results we have. For the others, however, we will be less rigorous.

(1) Let $G = \mathbb{R}$; we claim that $\hat{\mathbb{R}} \cong \mathbb{R}$ via the map

$$\begin{aligned} \Phi : \mathbb{R} &\to \hat{\mathbb{R}} \\ t &\mapsto (x \mapsto e^{2\pi i t x}) \end{aligned}$$

We first show that Φ is injective, so suppose that $t \neq 0$. Then

$$\Phi(t)\left(\frac{1}{2t}\right) = e^{i\pi} = -1$$

and so $\Phi(t)$ is nontrivial. Thus Φ has trivial kernel.

We now wish to show that Φ is surjective, and we will give two different proofs of this. The first of these (Theorem 4.6.a in [2]) is fundamentally analytic in nature.

Proof 1:

Let $\chi \in \hat{\mathbb{R}}$, so $\chi(0) = 1$ because χ is a homomorphism. Then there exists $a > 0$ such that $A := \int_0^a \chi(t)dt \neq 0$, and

$$A\chi(x) = \int_0^a \chi(x+t)dt = \int_x^{a+x} \chi(t)dt.$$

Thus

$$\chi(x) = \frac{1}{A}\int_x^{a+x} \chi(t)dt$$

and so χ is differentiable, with

$$\chi'(x) = \frac{1}{A}[\chi(a+x) - \chi(x)] = \frac{1}{A}[\chi(a) - 1]\chi(x).$$

Note that the final equality holds because χ is a homomorphism. Setting $t = \frac{\chi(a)-1}{2\pi i A}$, we have

$$\chi'(x) = 2\pi i t \chi(x)$$

and it follows that

$$\chi(x) = e^{2\pi i t x}.$$

Since $|\chi(x)| = 1$ for every $x \in \mathbb{R}$, we must have $t \in \mathbb{R}$ and so $\chi = \Phi(t) \in \mathrm{im}(\Phi)$, as required. $\square$

Proof 2:

We now give an alternate of the surjectivity of Φ, which will have a more algebraic flavour.

Let $\chi \in \hat{\mathbb{R}}$. Since $\mathbb{R}$ is contractible, χ lifts to the universal covering space $\mathbb{R}$ of $\mathbb{T}$. Hence there is some $\varphi : \mathbb{R} \to \mathbb{R}$ such that the diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto e^{2\pi i x}"] \\ \mathbb{R} \arrow[ur, dashed, "\varphi"] \arrow[r, "\chi"'] & \mathbb{T} \end{tikzcd}$$

commutes. Commutativity of this diagram tells us that for arbitrary $a, b \in \mathbb{R}$,

$$\begin{aligned} e^{2\pi i \varphi(a+b)} &= \chi(a+b) \\ &= \chi(a) \cdot \chi(b) \\ &= e^{2\pi i \varphi(a)} \cdot e^{2\pi i \varphi(b)} \\ &= e^{2\pi i (\varphi(a) + \varphi(b))} \end{aligned}$$

Since φ lands in $\mathbb{R}$, the above expressions must be out of phase by some integer multiple of $2\pi i$, so we can write

$$\varphi(a+b) = \varphi(a) + \varphi(b) + k_{a,b}$$

for some $k_{a,b} \in \mathbb{Z}$ which is a priori dependent on a and b. However, because φ is continuous, in fact this integer cannot vary based on a and b i.e. there exists some $k \in \mathbb{Z}$ such that

$$\varphi(a+b) = \varphi(a) + \varphi(b) + k, \qquad a, b \in \mathbb{Z}.$$

Translating so that $k = 0$ (which does not affect commutativity of the above diagram), we then see that φ is additive. It is an easy result that a continuous additive map $\mathbb{R} \to \mathbb{R}$ must be of the form $\varphi(x) = tx$ for some fixed $t \in \mathbb{R}$, so commutativity again tells us that

$$\chi(x) = e^{2\pi i t x}$$

and hence $\chi = \Phi(t) \in \mathrm{im}(\Phi)$, as required. $\square$

We have now shown that Φ is a bijection, but we claimed it was an isomorphism of topological groups. Hence it still remains to check that Φ is continuous with continuous inverse.

For continuity, we use the characterisation of the topology on $\hat{\mathbb{R}}$ as the topology of uniform convergence. Let $t_n \to t$ in $\mathbb{R}$; we will show that $\Phi(t_n) \to \Phi(t)$ in $\hat{\mathbb{R}}$. As discussed, this happens if and only if for every compact $K \subseteq G$, $\Phi(t_n) \to \Phi(t)$ uniformly on K. It is a well-known fact that compact subsets of $\mathbb{R}$ are exactly of the form $[a, b]$ or $[a, b] \setminus \coprod_{i=1}^{\infty} U_i$ where the U_i are open intervals with endpoints in K. A sequence of functions on a set of the second form converges uniformly if and only if it converges uniformly on any closed subinterval, so it suffices to consider the case $K = [a, b]$. With these simplifications, it is then an easy $\varepsilon - N$ argument to show that $e^{2\pi i t_n x} \to e^{2\pi i t x}$ uniformly on $[a, b]$. Thus Φ is continuous.

Finally, we wish to show that that Φ^{-1} is continuous or, equivalently, that Φ is an open map. For this, we invoke Theorem 1; writing $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$ shows that $\mathbb{R}$ is σ-compact, and $\hat{\mathbb{R}}$ is locally compact by Proposition 7.(a). Since we have already shown that Φ is a continuous surjective group homomorphism, Theorem 1 applies and so Φ is indeed open, which concludes the proof.

(2) Consider $G = \mathbb{T}$; we claim that $\hat{\mathbb{T}} \cong \mathbb{Z}$. We use the map

$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\to \mathbb{T} \\ [x] &\mapsto e^{2\pi i x} \end{aligned}$$

which is readily seen to be an isomorphism of topological groups. Under this identification, characters of $\mathbb{T}$ are just characters of $\mathbb{R}$ which are trivial on $\mathbb{Z}$. By (1), these are exactly the maps $x \mapsto e^{2\pi i n x}$ for $n \in \mathbb{Z}$, so characters of $\mathbb{T}$ are parameterized by $\mathbb{Z}$, as claimed.

Note that $\mathbb{T}$ is compact and $\mathbb{Z}$ is discrete, so this is an instance of Proposition 7.(b).

(3) Consider $G = \mathbb{Z}$; we claim that $\hat{\mathbb{Z}} \cong \mathbb{T}$. Since $\mathbb{Z}$ is the free abelian group on one generator, any character is entirely determined by $\chi(1)$, and any value $\chi(1) \in \mathbb{T}$ gives a valid character. Furthermore, $\mathbb{Z}$ has the discrete topology, so any such homomorphism is automatically continuous. Thus characters of $\mathbb{Z}$ are parameterized by elements of $\mathbb{T}$, as claimed.

Again observe that $\mathbb{Z}$ is discrete and $\mathbb{T}$ is compact, so this is an instance of Proposition 7.(c).

(4) Consider $G = \mathbb{Z}/n\mathbb{Z}$; we claim that $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$. The characters of $\mathbb{Z}/n\mathbb{Z}$ are the characters of $\mathbb{Z}$ which are trivial on $n\mathbb{Z}$. By (3), these are exactly the maps

$$\chi_k(x) = e^{\frac{2\pi i k x}{n}}, \qquad k = 0, 1, ..., n-1,$$

which are generated cyclically by χ_1. Thus $\widehat{\mathbb{Z}/n\mathbb{Z}}$ is a cyclic group of order n and hence (non-canonically) isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Examples 6.(1) and 6.(4) show that $\mathbb{R}$ and $\mathbb{Z}/n\mathbb{Z}$ are *self-dual*, meaning $\hat{G} \cong G$. In fact, more can be said: the universal property of coproducts states that characters of $G_1 \oplus G_2$ ($\cong G_1 \times G_2$) are the same as pairs of characters of G_1 and G_2.

Thus taking the dual distributes across finite products. The classification theorem for finite abelian groups says that any finite abelian group can be written as a product of cyclic groups; in conjunction with Example 6.(4), we conclude that any finite abelian group is self-dual.

An observant reader will have noticed that in all of the above cases, the group G is *reflexive*; taking the bidual $\hat{\hat{G}}$ gives back G. This is not a coincidence, and in fact the statement of Pontryagin Duality is that this holds for an arbitrary locally compact abelian group G. For those who have seen a proof that this is true for finite-dimensional vector spaces, this might not be surprising, but it is a major and difficult result in this context.

3.2 The Fourier Inversion Formula

At last we arrive at the Fourier Inversion formula, which plays a major role in the proof of Pontryagin Duality. It is also worth studying for its own sake, and we will see that the theory of Fourier inversion of locally compact groups generalises standard Fourier inversion for functions $\mathbb{R} \to \mathbb{C}$.

Throughout this section, let G denote a locally compact abelian group; as we discussed, G is unimodular and hence has a bi-invariant Haar measure dx.

Definition. Let $f \in L^1(G)$. We define $\hat{f} : \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi}(x)\ dx, \qquad \chi \in \hat{G}.$$

We call $\hat{f}$ the *Fourier transform* of f.

Under the correspondence of Example 6.(1), every $\chi \in \hat{\mathbb{R}}$ can be written as $\chi(x) = e^{2\pi itx}$ for some $t \in \mathbb{R}$. Using this identification to think of $\hat{f}$ as a function $\mathbb{R} \to \mathbb{C}$, we find

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi itx}\ dx$$

which is of course the standard formula for the Fourier transform. Thus this definition agrees in the case of $G = \mathbb{R}$.

Our goal is to recover f from $\hat{f}$, at least under certain hypotheses on f. To this end, we make the following technical definition:

Definition. Let $\varphi : G \to \mathbb{C}$ in $L^\infty(G)$ be a Haar-measurable function. We say that φ is of *positive type* if

$$\int (f^* * f)\varphi \geq 0 \text{ for all } f \in L^1(G).$$

We will only consider functions of positive type which are also continuous, so we define

$$\mathcal{P}(G) = \{\varphi \in C(G) \cap L^\infty(G) : \varphi \text{ is of positive type}\}.$$

This definition can be rather intimidating at first, so let's consider a related class of functions that might make the idea of functions of positive type more transparent.

Definition. We define a subset $\mathcal{E}(G) \subseteq \mathcal{P}(G)$ (called the *elementary functions*) as follows: a function φ is elementary if it is either the zero map or it satisfies

(i) $\varphi \in \mathcal{P}(G)$

(ii) $\varphi(e_G) = 1$

(iii) For any decomposition of φ as $\varphi = \varphi_1 + \varphi_2$ for $\varphi_1, \varphi_2 \in \mathcal{P}(G)$, there exist positive real constants λ_1, λ_2 such that

$$\varphi_1 = \lambda_1 \varphi \quad \text{and} \quad \varphi_2 = \lambda_2 \varphi.$$

Thus the elementary functions are the functions of positive type which cannot be decomposed any further, so they are in some sense the 'building blocks' of the functions of positive type. So far, understanding the elementary functions isn't any easier than understanding arbitrary functions of positive type: after all, we defined $\mathcal{E}(G)$ as a subset of $\mathcal{P}(G)$!

The reason we introduced elementary functions is that they are characterised by the following result:

Theorem 8. *Let G be a locally compact abelian group. Then the nonzero elementary functions on G are exactly the characters of G.*

In Example 6, we saw many instances of characters on a locally comapct abelian group, so these will hopefully give some intuition for what the elementary functions look like.

We can now state the class of functions for which Fourier Inversion will hold. Set

$$V^1(G) = \mathrm{Span}_{\mathbb{C}}(\mathcal{P}(G)) \cap L^1(G).$$

There are many definitions here to unravel, but the important point is that the integrable functions of positive type are exactly the functions for which the Fourier transform is invertible. This is summarised by the following theorem:

Theorem 9. *There exists a Haar measure $d\chi$ on $\hat{G}$ (called the dual measure, sometimes denoted $d\hat{x}$) such that for all $f \in V^1(G)$,*

$$f(x) = \int_{\hat{G}} \hat{f}(\chi)\chi(x)\ d\chi.$$

Furthermore, the Fourier transform $f \mapsto \hat{f}$ is a bijection $V^1(G) \to V^1(\hat{G})$.

It is a somewhat remarkable fact that $V^1(G)$ is dense in $L^1(G)$, so the Fourier transform (being continuous) extends uniquely to a function on all of $L^1(G)$. However, there need not be a dual measure as in Theorem 9 which gives equality for all of $L^1(G)$.

Theorem 9 is the so-called *Fourier Inversion formula.* Knowing that the dual measure exists, the problem then becomes calculating it; since the Haar measure is only defined up to a positive scalar, there is an unknown scalar factor to find. In practise, this is often done by explicitly computing the Fourier transform of a specific function.

Example 7. We will compute the dual measure on $\hat{\mathbb{R}}$, continuing the association $\hat{\mathbb{R}} \cong \mathbb{R}$ from Example 6.(1). We claim that under this isomorphism, the Lebesgue measure is self-dual i.e. $d\chi = dx$.

Consider the function $g(x) = e^{-\pi x^2}$, the so-called *error function.* We will explicitly compute the Fourier transform of g to show that the scalar factor is 1. Why did we choose this particular g? Because it is *its own* Fourier transform:

$$\begin{aligned}
\hat{g}(t) &= \int_{\mathbb{R}} e^{-2\pi i t x - \pi x^2} dx \\
&= \int_{\mathbb{R}} e^{-\pi t^2 - \pi(x+it)^2} dx \\
&= e^{-\pi t^2} \cdot \int_{\mathbb{R}} e^{-\pi x^2} d(x+it) \\
&= e^{-\pi t^2} \cdot \int_{\mathbb{R}} e^{-\pi x^2} dx \\
&= e^{-\pi t^2} \\
&= g(t)
\end{aligned}$$

Noting that g is even, it follows that

$$g(t) = g(-t) = \int_{\mathbb{R}} e^{2\pi i t x - \pi x^2} dx.$$

This shows that dx satisfies the Fourier Inversion formula given in Theorem 9, so $dt = dx$ and the scalar factor is 1, as claimed. $\square$

Note that if we had chosen a different isomorphism $\mathbb{R} \to \hat{\mathbb{R}}$ then it might no longer be true that the Haar measure is self-dual. For example, if we instead use the isomorphism

$$t \mapsto (x \mapsto e^{itx}),$$

the dual measure would be $\frac{1}{2\pi} dt$. When computing the Fourier transform of specific groups, it is important to keep track of isomorphisms.

As seen in Example 7, it can be often be a lot of work to compute the dual measure. Fortunately, the next result will allow us to find the dual measure in a large class of examples:

Proposition 8. *As always, let G be a locally compact abelian group.*

(a) If G is compact and the Haar measure μ is chosen so that $\mu(G) = 1$, then the dual measure on $\hat{G}$ is the counting measure (recall that by Proposition 7.(b), $\hat{G}$ is discrete).

(b) If G is discrete and the Haar measure is chosen to be the counting measure, then the dual measure on $\hat{G}$ is the one such that $\mu(\hat{G}) = 1$.

Example 8. Using Proposition 8, we can now find the dual measures on the remaining groups from Example 6.

(1) The groups $\mathbb{Z}$ and $\mathbb{T}$ are dual via the isomorphisms of Example 6.(2) and 6.(3). We saw in Example 2.(2) that the normalised Haar measure on $\mathbb{T}$ is $\frac{1}{2\pi}d\theta$ and by Proposition 8.(a), the dual measure on $\mathbb{Z}$ is the counting measure. Combining these results, the Fourier transform for $f \in V^1(\mathbb{T})$ is then

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta, \qquad f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}.$$

This is known as the *circular Fourier transform.*

(2) We now compute the Fourier transform on $\mathbb{Z}/n\mathbb{Z}$, which is self-dual by the isomorphism of Example 6.(4). If we pick the Haar measure on $\mathbb{Z}/n\mathbb{Z}$ to be the counting measure, Proposition 8.(b) applies to say that the dual measure on $\widehat{\mathbb{Z}/n\mathbb{Z}}$ is the normalised Haar measure; this is again the counting measure, normalised so that $\mu(\mathbb{Z}/n\mathbb{Z}) = 1$.

In this case, the Fourier transform for $f \in V^1(\mathbb{Z}/n\mathbb{Z})$ is

$$\hat{f}(m) = \sum_{k=0}^{n-1} f(k)e^{-\frac{2\pi imk}{n}}, \qquad f(m) = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}(k)e^{\frac{2\pi imk}{n}}.$$

This is the *discrete Fourier transform.* It is possible to start with the above formula for $\hat{f}$ and, using only elementary methods, recover the formula for f. However, doing so obscures all of the theory we have developed thus far.

We now look at how the Fourier transform can be extended to a function on all of $L^2(G)$. What follows is of fundamental importance in the proof of Pontryagin Duality, but it is also a remarkable result in its own right.

Before we arrive at the main statement, we make the elementary observation that simple functions are dense in $L^p(G)$ for every $1 \leq p \leq \infty$. In particular, this means that $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, since this intersection contains the simple functions.

Theorem 10. *(Plancherel's Theorem) The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$.*

It is quite surprising that so strong a statement can be made. After all, it is not necessarily true that $G \cong \hat{G}$ (consider $G = \mathbb{T}$) and yet $L^2(G) \cong L^2(\hat{G})$, and in a canonical way!

3.3 Pontryagin Duality

We now arrive at the Pontryagin Duality theorem, which has several important corollaries in the theory of Fourier Inversion. For now, let's simply establish the background to state the result.

By definition, the elements of $\hat{G}$ are characters on G. However, we can also regard the elements of G as characters on $\hat{G}$, via the map

$$\begin{aligned} \Phi : G &\to \hat{\hat{G}} \\ \Phi(x)(\chi) &= \chi(x) \end{aligned}$$

This map will likely be familiar, since it is a well-known (and comparatively easy) result that Φ is an isomorphism $V \cong \hat{\hat{V}}$ for V a finite-dimensional vector space. The remarkable result called *Pontryagin Duality* is that the same statement is true in this more general context:

Theorem 11. *(Pontryagin Duality) Let G be a locally compact abelian group. Then the map Φ defined above is an isomorphism of topological groups, whence we conclude that locally compact abelian groups are reflexive.*

Although it is not typically included in the statement of Pontryagin Duality, we now observe that Φ is in fact a *natural* isomorphism, when we regard dualisation as a functor. This means that if $\psi : G \to H$ is a continuous homomorphism between locally compact abelian groups, then the diagram

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\Phi"] & H \arrow[d, "\Phi"] \\ \hat{\hat{G}} \arrow[r, "\hat{\hat{\psi}}"] & \hat{\hat{H}} \end{tikzcd}$$

commutes. Consequently, the double dualisation functor is natural to the identity functor in the category of locally compact groups.

Continuing with this idea, let's reformulate Pontryagin Duality using category theoretic language. Let LCA denote the category of locally compact abelian groups. Then Pontryagin Duality says that the functor $\mathrm{Hom}(-, \mathbb{T})$ is an equivalence of categories $\mathrm{LCA} \to \mathrm{LCA}^{\mathrm{op}}$ which is its own inverse.

One immediate consequence of this is the following: from the above, we conclude that $(\mathrm{Hom}(-, \mathbb{T}), \mathrm{Hom}(-, \mathbb{T}))$ is an adjoint pair since it is a pair of equivalences. This means that $\mathrm{Hom}(-, \mathbb{T})$ has the properties of both a left and right adjoint, hence takes colimits to limits (this is always true of a contravariant Hom functor) and, more remarkably, also takes limits to colimits. This will be extremely useful to us in Chapter 4.

Having seen Pontryagin Duality, the obvious question is whether such a result holds for a larger class of groups. Does Theorem 11 have a converse?

Possibly the neatest answer was proven in 1995 by E. Martin-Peinador ([7]):

Theorem 12. *Suppose that G is a Hausdorff abelian topological group that satisfies Pontryagin Duality, and suppose further that the natural pairing*

$$\begin{aligned} G \times \hat{G} &\to \mathbb{T} \\ (x, \chi) &\mapsto \chi(x) \end{aligned}$$

is continuous. Then G is locally compact.

While it is possible to find examples of non-locally compact abelian groups which satisfy Pontryagin Duality, a corollary of the above is that the natural pairing cannot be continuous in such cases.

We now look at how Pontryagin Duality impacts on Fourier analysis on locally compact groups. One immediate consequence is the following:

Theorem 13. *(Fourier Inversion formula v. 2)*

If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$ then $f(x) = (\hat{\hat{f}})(x^{-1})$ for almost every x; that is,

$$f(x) = \int_{\hat{G}} \chi(x)\hat{f}(\chi)d\chi \text{ for almost every } x.$$

If f is continuous, equality holds for every x.

Note that instead of stating Theorem 13 in terms of $f \in V^1(G)$, we rephrased it as 'if both f and $\hat{f}$ are absolutely integrable'. This is only a slight generalisation, but it means that the formula holds for a larger class of functions than Theorem 9. It is not necessarily true that the Fourier transform of an $L^1(G)$ function is itself integrable, so we cannot omit this hypothesis.

Possibly the most important part of Theorem 13 is the statement in the case of f continuous. This says that equality *of functions* holds, rather than just equality of equvialences classes in $L^1(G)$. This is the situation in which Fourier Inversion is most commonly applied in practise, so this stronger result is quite important for applications.

Yet another consequence is the so-called *Fourier Uniqueness theorem*. While technically a corollary of Theorem 13, they both follow fairly easily from Pontryagin Duality:

Theorem 14. *(Fourier Uniqueness Theorem) If $f, g \in L^1(G)$ and $\hat{f} = \hat{g}$ then $f = g$.*

Thus the Fourier transform is an injective function $L^1(G) \to L^1(\hat{G})$. While this follows immediately for functions in $V^1(G)$ by the invertibility given in Theorem 9, it is not at all obvious that such a statement would hold once we extend the Fourier transform to all of $L^1(G)$.

This brings us to the end of our direct study of Fourier analysis. In the next section, we will see an application of this theory when we begin to investigate Tate's thesis. Even for those not interested in this avenue of investigation, there will be more examples of dual groups and further properties of the Fourier transform.

4 Toward Tate's Thesis

In this section, we will see an application of the Fourier analysis we have developed thus far. We will study zeta functions on number fields, a subject that arose as a generalisation of the original Riemann zeta function. Tate's insight was that the theory of zeta functions could be formulated in terms of the Fourier transform on a locally compact field, and we will see how this approach gives a beautiful description of the Riemann zeta function in terms of all possible metric completions of $\mathbb{Q}$.

In the interests of simplicity, we will not study zeta functions on arbitrary number fields, but rather will specialise to $\mathbb{Q}$.

4.1 p-adic Numbers and Even More Dual Groups

To even begin to approach the formulation of the Riemann zeta function that we want, we will need to understand the dual groups (both additive and multiplicative) of all of the completions of $\mathbb{Q}$. With that in mind, let us now introduce the p-adic integers and p-adic rationals:

Definition. Fix a prime p. We define the p-adic integers $\mathbb{Z}_p$ by

$$\mathbb{Z}_p := \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, ..., p-1\} \right\}$$

which has an obvious ring structure. Equivalently (and we will make use of both formulations), we define

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

where the inverse system is generated by the obvious maps

$$\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}.$$

With the second description, $\mathbb{Z}_p$ is a profinite ring and hence compact and totally disconnected.

Both of these presentations of $\mathbb{Z}_p$ have advantages, as we shall see when we begin to calculate dual groups.

We wish to extend from $\mathbb{Z}_p$ to a field, denoted $\mathbb{Q}_p$, and we will again give two constructions.

Definition. One can show using elementary methods that $\mathbb{Z}_p$ is an integral domain. Thus its fraction field is well-defined, so we set

$$\mathbb{Q}_p := \mathrm{Frac}(\mathbb{Z}_p) = \left\{ \sum_{i=-N}^{\infty} a_i p^i : N \in \mathbb{Z}_{\geq 0}, a_i \in \{0, 1, ..., p-1\} \right\}.$$

When making explicit computations with $\mathbb{Q}_p$, we will essentially only work with this description. However, to understand the uniformity in our approach to zeta functions, it is important to know that $\mathbb{Q}_p$ can also be interpreted as a metric completion of $\mathbb{Q}$.

To this end, let us define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ as follows: given any nonzero $x \in \mathbb{Q}$, x can be written uniquely as

$$x = p^n \frac{a}{b}$$

where a and b are coprime integers not divisible by p. We then define

$$|x|_p := \begin{cases} 0 & \text{if } x = 0 \\ p^{-n} & \text{otherwise} \end{cases}$$

One can show that $|\cdot|_p$ is a metric on $\mathbb{Q}$, so we equivalently define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Our first construction of $\mathbb{Q}_p$ is clearly linked to $\mathbb{Z}_p$, but in the second, the connection might be more subtle. They are related by the following:

Proposition 9. *$\mathbb{Q}_p$ contains $\mathbb{Z}_p$ as a topological ring. Namely, we can realise $\mathbb{Z}_p$ as the closed unit ball*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Furthermore, the group of units of $\mathbb{Z}_p$ is exactly the unit circle

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

As a consequence of Proposition 9, both $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ are closed subsets of $\mathbb{Q}_p$, hence are themselves complete with respect to $|\cdot|_p$. It is important to note that $\mathbb{Q}_p$ is locally compact (translates and scalings of $\mathbb{Z}_p$ form a base for the topology) so all of our theory is available to us.

For those not familiar with $|\cdot|_p$ (or absolute values in general), it has essentially the same properties as the standard Euclidean absolute value $|\cdot|_\infty$. You might wonder, why do we say absolute value instead of metric? The answer is that absolute values satisfy one additional property which arbitrary metrics do not have; they are multiplicative, meaning

$$|xy| = |x| \cdot |y|.$$

Thus absolute values are the metrics which also respect the multiplicative structure.

Let us introduce slightly more terminology, so that we can make sense of the statement "all possible completions of $\mathbb{Q}$". Two absolute values $|\cdot|$ and $|\cdot|_*$ on $\mathbb{Q}$ are said to be *equivalent* if there exists a real number $c > 0$ such that

$$\forall x \in \mathbb{Q} \qquad |x|_* = |x|^c.$$

This notion of equivalence is important because two completions of $\mathbb{Q}$ with respect to different absolute values are isomorphic if and only if the absolute values are equivalent. Let us now check that for distinct primes p and q, the absolute values $|\cdot|_p$ and $|\cdot|_q$ are inequivalent. Indeed, observe that

$$\begin{aligned} \left|\frac{p}{q}\right|_p &= |p|_p \cdot \left|\frac{1}{q}\right|_p \\ &= p^{-1} \cdot 1 && \text{since } p \nmid q \\ &< 1 \end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\frac{p}{q}\right|_q &= |p|_q \cdot \left|\frac{1}{q}\right|_q \\ &= 1 \cdot q && \text{since } q \nmid p \\ &> 1\end{aligned}$$

Thus there is no $c > 0$ which relates these quantities, so the absolute values $|\cdot|_p$ and $|\cdot|_q$ are inequivalent. Similarly, one can show that all $|\cdot|_p$ are inequivalent to the standard Euclidean absolute value $|\cdot|_\infty$.

We have now identified a whole class of distinct absolute values on $\mathbb{Q}$. The natural question is: are there any others?

Theorem 15. *(Ostrowski) Any nontrivial absolute value on $\mathbb{Q}$ is equivalent to either $|\cdot|_\infty$, or to $|\cdot|_p$ for some prime p.*

Thus $\mathbb{R}$ and $\mathbb{Q}_p$ comprise all the nontrivial completions of $\mathbb{Q}$.

Example 9. With the definitions out of the way, we will now compute the (additive and multiplicative) dual groups of $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

(1) We first consider the additive group $\mathbb{Z}_p$. We claim that its dual is the Prüfer group $\mathbb{Z}(p^\infty)$, which consists of all p^nth roots of unity,

$$\mathbb{Z}(p^\infty) = \{z \in \mathbb{C} : z^{(p^n)} = 1 \text{ for some } n \in \mathbb{Z}\}.$$

It is easy to verify that this group is also the colimit of the direct system generated by the maps

$$\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\times p} \mathbb{Z}/p^{n+1}\mathbb{Z}$$

so we can equally well write

$$\mathbb{Z}(p^\infty) = \varinjlim \mathbb{Z}/p^n\mathbb{Z}.$$

We now recall our discussion directly following Pontryagin Duality (Theorem 11), in which we showed that the $\mathrm{Hom}(-, \mathbb{T})$ functor takes limits to colimits. Therefore, we have

$$\begin{aligned}\widehat{\mathbb{Z}_p} &= \mathrm{Hom}\left(\varprojlim \mathbb{Z}/p^n\mathbb{Z}, \mathbb{T}\right) \\ &\cong \varinjlim \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{T}) \\ &= \varinjlim \widehat{\mathbb{Z}/p^n\mathbb{Z}} \\ &\cong \varinjlim \mathbb{Z}/p^n\mathbb{Z} && \text{by Example 6.(4)} \\ &= \mathbb{Z}(p^\infty)\end{aligned}$$

As an interesting side-remark which has no bearing on this computation, note that there is actually a similarity to the real case here (where we say that $\mathbb{Z} \subset \mathbb{R}$ is the analogue of $\mathbb{Z}_p \subset \mathbb{Q}_p$). Yet another description of $\mathbb{Z}(p^\infty)$ is given by

$$\mathbb{Z}(p^\infty) = \mathbb{Q}_p/\mathbb{Z}_p$$

so we have proven that $\mathbb{Z}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$.

Similarly, we showed in Example 6.(2) that $\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$, and then identified the right-hand side with $\mathbb{T}$. Thus $\mathbb{Z}(p^\infty)$ can be thought of as a p-adic analogue of $\mathbb{T}$.

(2) We now consider the dual of $\mathbb{Q}_p$, which we claim is $\mathbb{Q}_p$. We will first construct one character, and then build on this. To that end, define

$$\chi_1\left(\sum_{i=-N}^{\infty} a_i p^i\right) = \exp\left(2\pi i \sum_{i=-N}^{-1} a_i p^i\right).$$

This is often abbreviated to

$$\chi_1(x) = e^{2\pi i x}$$

where it is understood that the non-negative terms of x contribute nothing. Some work needs to be done to make sense of this definition (for example, one needs to define an exponential $\exp : \mathbb{Q}_p \to \mathbb{R}$, which is of course given by a power series) but it can be verified that this gives a continuous group homomorphism $\mathbb{Q}_p \to \mathbb{T}$, so $\chi_1 \in \widehat{\mathbb{Q}_p}$. We extend this definition to all $t \in \mathbb{Q}_p$ by setting

$$\chi_t(x) = \chi_1(tx).$$

This gives rise to an injective map

$$\begin{aligned} \mathbb{Q}_p &\to \widehat{\mathbb{Q}}_p \\ t &\mapsto \chi_t \end{aligned}$$

which we claim is an isomorphism. Note the similarities between this map and the isomorphism from the $\mathbb{R}$ case, given in Example 6.(1). Once again, the difficulty lies in showing that this map is surjective.

In the interests of brevity, we will omit the proof because it is quite lengthy (it spans Lemma 4.10 through Theorem 4.13 in [2]). At its core, the proof is quite similar to the calculation in the $\mathbb{R}$ case, although adaptations need to be made because of the topology on $\mathbb{Q}_p$. This is especially easy to see if we try to adapt Proof 2 from Example 6.(1): $\mathbb{Q}_p$ is totally disconnected, so there is no way to lift a character to the universal covering space of $\mathbb{T}$.

Fundamentally, the proof is based on the idea that a character of $\mathbb{Q}_p$ is fully determined by what it does on various powers of p. An element of $\mathbb{Q}_p$ (considered as a Laurent series) has only finitely many negative terms, so we can control what a character does on the negative powers.

(3) We now look at the multiplicative group $\mathbb{Z}_p^\times$. To give a full calculation of the dual group, a rather involved detour into elementary number theory would be required to understand the structure of $\mathbb{Z}_p^\times$. Again in the interests of brevity, we refer the interested reader to Chapter II of [8] to see a proof of the fact that

$$\mathbb{Z}_p^\times \cong \begin{cases} \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } p \neq 2 \\ \mathbb{Z}_2 \times \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \end{cases}$$

Recalling that the dual group distributes across products, and combining this with example (1) above, we find that

$$\widehat{\mathbb{Z}_p^\times} \cong \begin{cases} \mathbb{Z}(p^\infty) \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } p \neq 2 \\ \mathbb{Z}(2^\infty) \times \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \end{cases}$$

(4) We now look at $\mathbb{Q}_p^\times$. We can reduce this to the case of $\mathbb{Z}_p^\times$ by using the map

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z}_p^\times &\to \mathbb{Q}_p^\times \\ (N, x) &\mapsto p^N x \end{aligned}$$

which is an isomorphism of locally compact groups. The only subtle point is injectivity of this map, which relies on the observation that a power series $\sum_{i=0}^\infty a_i p^i \in \mathbb{Z}_p$ is invertible if and only if $a_0 \neq 0$.

In combination with (3) above, we then find that

$$\widehat{\mathbb{Q}_p^\times} \cong \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}_p^\times} \cong \begin{cases} \mathbb{T} \times \mathbb{Z}(p^\infty) \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } p \neq 2 \\ \mathbb{T} \times \mathbb{Z}(2^\infty) \times \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \end{cases}$$

(5) As a final example, let's return to the real case to calculate $\mathbb{R}^\times$. Much as in example (4), the multiplicative structure is, in some sense, determined by the additive structure.

It is easy to see that we can identify $\mathbb{R}^\times$ with $\mathbb{R}_{>0} \times \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$ keeps track of which component of $\mathbb{R}^\times$ an element lies in (i.e. what the sign of the element is). On the positive real numbers, the exponential and logarithm are an isomorphism of locally compact groups $\mathbb{R} \to \mathbb{R}_{>0}$. Thus we can write

$$\mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$$

as a product of groups which are self-dual by Example 6. We conclude that $\mathbb{R}^\times$ is itself self-dual.

Note that this is in stark contrast to the $\mathbb{Q}_p^\times$ case, and should serve as a warning that not everything true of $\mathbb{R}$ will be true for $\mathbb{Q}_p$. Because $\mathbb{R}$ is contractible, many things are easier of $\mathbb{R}$ than over $\mathbb{Q}_p$.

This brings us to the end of our direct study of $\mathbb{Z}_p$ and $\mathbb{Q}_p$, which will naturally arise in later sections. The important point to take away is that, for intuition, $\mathbb{Q}_p$ should be thought of as the "p prime version of $\mathbb{R}$". Indeed, we will use the notation $\mathbb{Q}_\infty$ to denote $\mathbb{R}$.

4.2 Schwartz Functions

In this section, we will look at another class of functions which is closely related to the Fourier transform: the Schwartz functions. Unfortunately, we will need a different definition for the $\mathbb{R}$ and $\mathbb{Q}_p$ cases, but we will see how these are related. We first consider the $\mathbb{R}$ case:

Definition. A smooth function $f : \mathbb{R} \to \mathbb{C}$ is said to be *rapidly decaying* (or *Schwartz*) if

$$\forall \alpha, \beta \in \mathbb{Z}_{\geq 0} \qquad x^\alpha d^\beta f(x) \to 0 \text{ as } |x| \to \infty.$$

The collection of all such rapidly decaying functions is called the *Schwartz space*, and is denoted $\mathcal{S}(\mathbb{R})$.

We have already seen a Schwartz function in Example 7, given by $e^{-\pi x^2}$. Recall that this was a fixed point under the $\mathbb{R}$-additive Fourier transform with the standard Lebesgue measure. This is the first suggestion that Schwartz functions are related to the Fourier transform, and the purpose of this section is to make precise that connection.

In some sense, $e^{-\pi x^2}$ is "the" example of a Schwartz function on $\mathbb{R}$, as we will see in the next proposition.

Proposition 10. *The following are properties of Schwarz functions:*

(i) $\mathcal{S}(\mathbb{R})$ *is a subalgebra of* $L^1(\mathbb{R})$*, closed with respect to addition, multiplication and convolution. Furthermore, it is invariant under the Fourier transform.*

(ii) $\mathcal{S}(\mathbb{R})$ *is "generated by"* $e^{-\pi x^2}$ *in the sense that*

$$\mathcal{S}(\mathbb{R}) = \overline{\mathbb{C}[x]e^{-\pi x^2}}$$

where the closure is taken in $C^\infty(\mathbb{R}, \mathbb{C})$*.*

(iii) When restricted to $\mathcal{S}(\mathbb{R})$*, the Fourier transform takes differentiation to multiplication by* t*, and vice versa. More precisely,*

$$\widehat{\frac{df}{dx}}(t) = 2\pi i t \hat{f}(t) \quad \textit{and} \quad \widehat{xf}(t) = \frac{1}{2\pi} i \frac{d\hat{f}}{dt}(t).$$

(iv) The Fourier transform is an isomorphism $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$*. The inverse is given by the inverse Fourier transform from Theorem 13.*

We are primarily interested in the Schwartz functions because $\mathcal{S}(\mathbb{R})$ is a space on which the Fourier transform is an automorphism, but it is still broad enough to contain the functions that actually show up in practise. Furthermore, property (iii) above often simplifies calculations of the Fourier transform.

We would like an analogue of the Schwartz functions for $\mathbb{Q}_p$. As a first attempt, consider part (ii) of Proposition 10, in which we saw that the Schwartz functions were generated by $e^{-\pi x^2}$. We have already discussed that $e^{-\pi x^2}$ is a fixed point under the Fourier transform on $\mathbb{R}$, so we can ask: is there a similar fixed point for the Fourier transform on $\mathbb{Q}_p$? For this question to make sense, we need to make a definitive choice of Haar measure on $\mathbb{Q}_p$. If we choose $\mu_{\mathbb{Q}_p}$ to be such that $\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$, then there is indeed an easy fixed point.

Example 10. Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{Q}_p$ and recall that $\mathbb{Q}_p$ is self-dual under the isomorphism of Example 9.(2), so it makes sense to talk about a fixed point of $\mathcal{F}$. Let $\mathbb{1}_{\mathbb{Z}_p}$ be the indicator function for $\mathbb{Z}_p \subset \mathbb{Q}_p$. We claim that

$$\mathcal{F}(\mathbb{1}_{\mathbb{Z}_p}) = \mathbb{1}_{\mathbb{Z}_p}.$$

To prove this, we need the following lemma:

Lemma 2. *Let* $\psi : K \to \mathbb{T}$ *be a continuous group homomorphism on a compact group* K*. Then*

$$\int_K \psi(x)\ dx = \begin{cases} \text{meas}(K) & \textit{if } \psi = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Here, dx *and* $\text{meas}(K)$ *are both defined in terms of the Haar measure.*

This lemma actually holds more generally for any unimodular locally compact group, but this version is sufficient for our purposes.

We now have

$$\begin{aligned}
\mathcal{F}(\mathbb{1}_{\mathbb{Z}_p})(t) &= \int_{\mathbb{Q}_p} \mathbb{1}_{\mathbb{Z}_p}(x) e^{-2\pi i t x}\, dx \\
&= \int_{\mathbb{Z}_p} \chi_t(-x)\, dx \\
&= \begin{cases} 1 & \text{if } \chi_t \mid_{\mathbb{Z}_p} = 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \text{by Lemma 2}
\end{aligned}$$

Thus to show that $\mathcal{F}(\mathbb{1}_{\mathbb{Z}_p}) = \mathbb{1}_{\mathbb{Z}_p}$, it only remains to prove that

$$\chi_t \mid_{\mathbb{Z}_p} = 1 \iff t \in \mathbb{Z}_p.$$

One direction is clear: if t is an element of $\mathbb{Z}_p$ then it has no negative terms in its power series expansion. Thus for any $x \in \mathbb{Z}_p$, tx has no negative terms and so

$$\chi_t(x) = \chi_1(tx) = 1.$$

Conversely, if $t \notin \mathbb{Z}_p$ then there exists some $x \in \mathbb{Z}_p$ such that

$$tx = p^{-1}.$$

But this means that

$$\chi_t(x) = e^{2\pi i p^{-1}} \neq 1$$

and so $\chi_t \mid_{\mathbb{Z}_p} \neq 1$.

Knowing this, we could define the Schwartz functions on $\mathbb{Q}_p$ to be the algebra generated by $\mathbb{1}_{\mathbb{Z}_p}$. For various technical reasons, we cannot use exactly the same definition as in the $\mathbb{R}$ case, but our approach is the same at heart.

Definition. The Schwartz-Bruhat functions on $\mathbb{Q}_p$ are defined to be

$$\mathcal{S}(\mathbb{Q}_p) = \left\{ \sum_{i=1}^{n} c_i \mathbb{1}_{a_i + p^{k_i}\mathbb{Z}_p} : a_i \in \mathbb{Q}_p, c_i \in \mathbb{C}, k_i \in \mathbb{Z} \right\}.$$

One can show that a function $f : \mathbb{Q}_p \to \mathbb{C}$ is Schwartz if and only if it is locally constant with compact support.

Having shown that $\mathbb{1}_{\mathbb{Z}_p}$ is a fixed point under $\mathcal{F}$, an almost immediate corollary is that $\mathcal{S}(\mathbb{Q}_p)$ is invariant under the Fourier transform. Indeed, all of the obvious analogues of Proposition 10 still remain true for $\mathcal{S}(\mathbb{Q}_p)$.

It is worth noting that characters of $\mathbb{Q}_p$ (for $p \leq \infty$) are never Schwartz-Bruhat functions.

4.3 The Adeles and Ideles

We now introduce the object $\mathbb{A}_{\mathbb{Q}}$ over which we will study zeta functions. The definition might seem slightly strange at first, but this object has many advantages. This section will be devoted to cataloguing several important properties of $\mathbb{A}_{\mathbb{Q}}$.

On a surface level, $\mathbb{A}_{\mathbb{Q}}$ is very useful as an organizational tool when working with so many different fields simultaneously (namely, we will need to work with $\mathbb{Q}_p$ for all $p \leq \infty$). Another perspective is that the adeles are to $\mathbb{Q}_p$ as a manifold is to its charts: $\mathbb{A}_{\mathbb{Q}}$ has useful global behaviour that we would like to understand, but we need to understand the local behaviour of its charts and how they interact.

Definition. The *adeles* are the ring given by

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \{(t_p)_{p \text{ prime}} \in \prod_{p \text{ prime}} \mathbb{Q}_p : t_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}$$

equipped with the obvious ring structure. We topologise $\mathbb{A}_{\mathbb{Q}}$ by declaring the open sets to be of the form

$$U_{\mathbb{R}} \times \prod_{p \text{ prime}} U_p, \quad U_p \subset \mathbb{Q}_p \text{ open}, \quad U_p = \mathbb{Z}_p \text{ for almost all } p.$$

With this definition, $\mathbb{A}_{\mathbb{Q}}$ is a topological ring which is locally compact because $\mathbb{Z}_p$ is compact; being locally compact, it has a Haar measure. In the past, we made definitive choices of Haar measures on $\mathbb{Q}_p$ for $p \leq \infty$. These induce the Haar measure μ on $\mathbb{A}_{\mathbb{Q}}$, which is given on open sets by

$$\mu\left(U_{\mathbb{R}} \times \prod_{p \text{ prime}} U_p\right) = \mu_{\mathbb{R}}(U_{\mathbb{R}}) \cdot \prod_{p \text{ prime}} \mu_{\mathbb{Q}_p}(U_p).$$

A moment's thought shows that the right-hand product converges because $U_p = \mathbb{Z}_p$ for almost all p, and we chose the Haar measure on $\mathbb{Q}_p$ to be such that $\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$. Therefore the product is "secretly finite", a recurring theme when working with the adeles.

Let's now look at the dual group of $\mathbb{A}_{\mathbb{Q}}$, which we claim is also $\mathbb{A}_{\mathbb{Q}}$. This should not be surprising, since $\mathbb{A}_{\mathbb{Q}}$ is essentially a product of groups which are self-dual.

Our isomorphism $\mathbb{A}_{\mathbb{Q}} \to \widehat{\mathbb{A}_{\mathbb{Q}}}$ will be defined in terms of the isomorphisms $\mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$, which were not canonical. However, once we fix these isomorphisms, we do have an obvious choice of map.

Fix a tuple $(r, (t_p)) \in \mathbb{A}_{\mathbb{Q}}$. In Example 6.(1), we constructed a character $\chi_r \in \hat{\mathbb{R}}$. Similarly, in Example 9.(1), we constructed a character $\chi_{t_p} \in \widehat{\mathbb{Q}_p}$. We now define

$$\chi_{(r,(t_p))}(r', (t'_p)) = \chi_r(r') \cdot \prod_{p \text{ prime}} \chi_{t_p}(t'_p).$$

Once again, we must ask why this product converges. Recall from Example 10 that for $t \in \mathbb{Z}_p$,

$$\chi_t \mid_{\mathbb{Z}_p} = 1.$$

Because they are tuples in $\mathbb{A}_{\mathbb{Q}}$, almost all of the (t_p) and (t'_p) lie in $\mathbb{Z}_p$, so almost all of the terms on the right-hand side are 1. Knowing that $\chi_{(r,(t_p))}$ is well-defined, it is easy to see that this is a continuous homomorphism, so we have constructed a character of $\mathbb{A}_{\mathbb{Q}}$.

A standard "small subgroup" (cf. Lemma 4.10 in [2]) argument shows that these are all the characters of $\mathbb{A}_{\mathbb{Q}}$, the main point being that, upon restriction to a sufficiently small open set, any character $\chi \in \mathbb{A}_{\mathbb{Q}}$ must behave trivially on almost all factors, and we can analyse the behaviour on the finitely-many nontrivial factors.

So we understand the Haar measure on $\mathbb{A}_{\mathbb{Q}}$, and what the characters of $\mathbb{A}_{\mathbb{Q}}$ look like. An obvious question to ask is: can we relate the Fourier transforms of these characters to the Fourier transforms on the fields $\mathbb{Q}_p$?

Theorem 16. *We call an integrable function $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ simple if*

$$f = \prod_{p \leq \infty} f_p, \quad f_p = \mathbb{1}_{\mathbb{Z}_p} \text{ for almost all } p.$$

For such a simple function f, we have

$$\int_{\mathbb{A}_{\mathbb{Q}}} f \; d\mu = \prod_{p \leq \infty} \int_{\mathbb{Q}_p} f_p \; d\mu_p.$$

In particular,

$$\hat{f} = \prod_{p \leq \infty} \hat{f}_p.$$

We call the functions f_p the *slices of f*, and they will be important when we begin to study zeta functions. With that in mind, we define:

Definition. The Schwartz-Bruhat functions on $\mathbb{A}_{\mathbb{Q}}$ consist of finite linear combinations of simple functions

$$f = \prod_{p \leq \infty} f_p,$$

where f_p is a Schwartz-Bruhat function on $\mathbb{Q}_p$ and $f_p = \mathbb{1}_{\mathbb{Z}_p}$ for almost all p.

Again we see that the Schwartz-Bruhat functions on $\mathbb{A}_{\mathbb{Q}}$ are "built out of" the local Schwartz-Bruhat functions on the $\mathbb{Q}_p$, with the additional requirement that they be simple so that Theorem 16 applies.

Because the characters of $\mathbb{Q}_p$ are not Schwartz-Bruhat functions, neither are the characters of $\mathbb{A}_{\mathbb{Q}}$. This will be a small impediment to us in the future, but it is not insurmountable.

Another important feature of $\mathbb{A}_\mathbb{Q}$ is the canonical diagonal embedding

$$\begin{aligned} \mathbb{Q} &\hookrightarrow \mathbb{A}_\mathbb{Q} \\ q &\mapsto (q, q, ...) \end{aligned}$$

It is not quite immediate that this embedding actually lands in $\mathbb{A}_\mathbb{Q}$. The key observation is that the denominator of a rational number has at most finitely many prime factors, and hence will be an element of $\mathbb{Z}_p$ for any prime p not dividing its denominator.

This embedding will play a prominent role in our study of zeta functions, and we will return to it shortly. For now, we make a slight detour to discuss the related group known as the *ideles*.

Definition. The *ideles* $\mathbb{I}_\mathbb{Q}$ is the group of units of the adeles,

$$\mathbb{I}_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^\times.$$

To give $\mathbb{I}_\mathbb{Q}$ the structure of a topological group, we **do not** use the subspace topology on $\mathbb{A}_\mathbb{Q}$ (inversion is not continuous with this topology). Instead, we notice that we have an embedding

$$\begin{aligned} \mathbb{I}_\mathbb{Q} &\hookrightarrow \mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q} \\ x &\mapsto (x, x^{-1}) \end{aligned}$$

and we give $\mathbb{I}_\mathbb{Q}$ the subspace topology by thinking of $\mathbb{I}_\mathbb{Q}$ as a subspace of $\mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q}$ via this map.

This topology on $\mathbb{I}_\mathbb{Q}$ is strictly finer than the subspace topology coming from $\mathbb{A}_\mathbb{Q}$. An important feature is that the topology on $\mathbb{I}_\mathbb{Q}$, unlike the topology on $\mathbb{A}_\mathbb{Q}$, is induced by an "absolute value" (not an actual absolute value, since $\mathbb{I}_\mathbb{Q}$ is not a ring, but it behaves in the way you expect) given by

$$|(r, (t_p))| = |r|_\infty \cdot \prod_{p \text{ prime}} |t_p|_p.$$

Proposition 9 tells us that the right-hand side converges, since $|t_p|_p \leq 1$ for almost all p. We could have defined $|\cdot|$ on all of $\mathbb{A}_\mathbb{Q}$, but it is not positive-definite and so is not even a metric. However, we could equivalently characterise $\mathbb{I}_\mathbb{Q}$ as the elements of $\mathbb{A}_\mathbb{Q}$ with nonzero absolute value.

Returning to our diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}$, we observe that this restricts to an embedding $\mathbb{Q}^\times \hookrightarrow \mathbb{I}_\mathbb{Q}$. This embedding is particularly well-behaved with respect to the idelic absolute value:

Proposition 11. *Any $\frac{a}{b} \in \mathbb{Q}^\times$, thought of as an element in $\mathbb{I}_\mathbb{Q}$, satisfies*

$$\left|\frac{a}{b}\right| = 1.$$

Proof. We write

$$\frac{a}{b} = \pm \frac{p_1^{m_1} ... p_r{}^{m_r}}{q_1^{n_1} ... q_l^{n_l}}$$

where the p_i and q_j are all primes.

For any prime r not in either of these lists, we have

$$\left|\frac{a}{b}\right|_r = 1$$

so we just need to understand what happens in the p_i, q_j and ∞ terms. By the definition of the p-adic absolute values, we get

$$\left|\frac{a}{b}\right|_{p_i} = p_i^{-m_i}, \qquad \left|\frac{a}{b}\right|_{q_j} = q_j^{n_j}$$

so that

$$\left|\frac{a}{b}\right| = \left|\frac{a}{b}\right|_\infty \cdot \frac{q_1^{n_1} \ldots q_l^{n_l}}{p_1^{m_1} \ldots p_r^{m_r}}.$$

But of course, $\left|\frac{a}{b}\right|_\infty = \frac{p_1^{m_1} \ldots p_r{}^{m_r}}{q_1^{n_1} \ldots q_l^{n_l}}$ so these terms cancel. □

The proof of Proposition 11 is quite illustrative of the idea behind the adeles. If we understand the local calculations (in this case, that means understanding the p-adic norms for $p \leq \infty$) then we can make global calculations on $\mathbb{A}_\mathbb{Q}$.

4.4 Zeta Functions on $\mathbb{A}_\mathbb{Q}$

At last, we arrive at the punchline of Tate's thesis: zeta functions on $\mathbb{A}_\mathbb{Q}$. We will primarily be following the exposition in [9], but Tate's thesis itself, [10], also includes complete proofs of the results herein.

Definition. Denote the Haar measure on $\mathbb{Q}_p^\times$ by $d_p^\times x$. A *local zeta integral* is a function of the form

$$Z_p(f, s) = \int_{\mathbb{Q}_p^\times} f(x)|x|_p^s \, d_p^\times x$$

where f is a Schwartz-Bruhat function on $\mathbb{Q}_p$ (a Schwartz function, if $p = \infty$).

Recall that we found the Haar measure on $\mathbb{R}^\times$ in Example 2.(3), which is given by

$$d_\infty^\times x = \frac{1}{|x|_\infty} \, dx.$$

The Haar measures $d_p^\times x$ can be constructed similarly. An important property of the multiplicative Haar measures is that, because of our choice of additive Haar measure $\mu_{\mathbb{Q}_p}$, we get for free that $\mu_{\mathbb{Q}_p^\times}(\mathbb{Z}_p^\times) = 1$.

The local zeta integral $Z_p(f, s)$ can be considered as a function of s, once we fix a Schwartz-Bruhat function f, but it is somewhat unclear where it is defined. Certainly this integral makes sense for complex s (assuming the integral converges) so we will think of $Z_p(f, s)$ as a complex-valued function, defined on some subset of $\mathbb{C}$. As a first step towards understanding the domain of $Z_p(f, s)$, we have

Proposition 12. *Fix a Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{Q}_p)$.*

(i) The local zeta integral $Z_p(f, s)$ converges absolutely for $\mathrm{Re}(s) > 0$.

(ii) As a function of s, $Z_p(f, s)$ is holomorphic on $\mathrm{Re}(s) > 0$.

Knowing that $Z_p(f, s)$ is holomorphic on the open half-plane $\mathrm{Re}(s) > 0$, we can start to look for a meromorphic continuation. Finding such a continuation in the global setting is one of the main results of Tate's thesis.

Definition. Let $d^\times x$ denote the Haar measure on $\mathbb{I}_\mathbb{Q}$. For a Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$, the *global zeta integral* is

$$Z(f,s) = \int_{\mathbb{I}_\mathbb{Q}} f(x)|x|^s \, d^\times x.$$

Unsurprisingly, the global zeta integral is related to the local zeta integrals by the following:

Proposition 13. *If $f : \mathbb{A}_\mathbb{Q} \to \mathbb{C}$ is a pure Schwartz-Bruhat function, in the sense that*

$$f = \prod_{p \leq \infty} f_p, \quad f_p \in \mathcal{S}(\mathbb{Q}_p), \quad f_p = \mathbb{1}_{\mathbb{Z}_p} \text{ for almost all } p,$$

then

$$Z(f,s) = \prod_{p \leq \infty} Z_p(f_p, s).$$

Proposition 13 shows the importance of our local-to-global perspective in the actual calculation of zeta functions.

Example 11. We have identified specific functions $f_p \in \mathcal{S}(\mathbb{Q}_p)$ which are fixed under their respective Fourier transforms. For $p = \infty$, this is the function $f_\infty(x) = e^{-\pi x^2}$. For p prime, it is $f_p = \mathbb{1}_{\mathbb{Z}_p}$. The function

$$f = \prod_{p \leq \infty} f_p$$

satisfies the hypotheses of Proposition 13, so

$$Z(f,s) = \prod_{p \leq \infty} Z_p(f_p, s).$$

To understand this global zeta integral, we calculate the local zeta integrals. For a prime $p < \infty$, this is

$$\begin{aligned}
Z_p(\mathbb{1}_{\mathbb{Z}_p}, s) &= \int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x)|x|_p^s \, d_p^\times x \\
&= \int_{\mathbb{Z}_p - \{0\}} |x|_p^s \, d_p^\times x \\
&= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} p^{-ks} \, d_p^\times x && \text{because } \mathbb{Z}_p - \{0\} = \coprod_{k=0}^{\infty} \{x : |x|_p = p^{-k}\} \\
&= \sum_{k=0}^{\infty} p^{-ks} && \text{because } \mu_{\mathbb{Q}_p^\times}(\mathbb{Z}_p^\times) = 1 \\
&= \frac{1}{1 - p^{-s}}
\end{aligned}$$

Thus

$$\prod_{p < \infty} f_p = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

For those not familiar with the Riemann zeta function, it is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In his 1737 thesis ([11]), Euler proved the *Euler product formula*,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

which is the product we calculated above. Hence we have shown that

$$Z(f, s) = Z_\infty(f_\infty, s)\zeta(s).$$

It remains to understand the ∞-component. For this factor, we get

$$\begin{aligned} Z_\infty(f_\infty, s) &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|_\infty^s \frac{1}{|x|_\infty} \, dx \\ &= 2 \int_0^\infty e^{-\pi x^2} x^{s-1} \, dx \end{aligned}$$

Making the substitution $u = \pi x^2$ gives

$$\begin{aligned} Z_\infty(f_\infty, s) &= 2 \int_0^\infty e^{-u} \left(\frac{u}{\pi}\right)^{\frac{s}{2} - \frac{1}{2}} \cdot \frac{1}{2} (u\pi)^{-\frac{1}{2}} \, du \\ &= \pi^{-\frac{s}{2}} \int_0^\infty e^{-u} u^{\frac{s}{2} - 1} \, du \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \end{aligned}$$

where $\Gamma(-)$ is the gamma function. Hence

$$Z(f, s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) =: \tilde{\zeta}(s).$$

The right-hand side is called the *extended Riemann zeta function*.

Example 11 shows how the Riemann zeta function can be realised as a specific instance of a global zeta integral. In fact, the Riemann zeta function is distinguished among all global zeta integrals. For example:

Proposition 14. *For any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$, $Z(f, s)$ converges absolutely for* $\mathrm{Re}(s) > 1$.

We saw in Proposition 12 that the local zeta integrals converge for $\mathrm{Re}(s) > 0$, so it might be surprising that we get $\mathrm{Re}(s) > 1$ in the global setting. Without going into too much detail in the proof, the discrepancy arises because we make reductions to bound $Z(f, s)$ by $\zeta(s)$. It can be shown by elementary methods that $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$, so this is the best bound we get.

This is one example of how intimately related global zeta integrals are to $\zeta(s)$. Although they are a vast generalisation, several proofs reduce to elementary properties of $\zeta(s)$. It is important to note that the functional equation and meromorphic continuation are **not** instances of this phenomenon, so the theory we derive here gives an actual proof of these properties for $\tilde{\zeta}(s)$.

We now arrive at the functional equation and meromorphic continuation of $Z(f, s)$. The proof is quite lengthy, spanning Lemma 2.5.5-Theorem 2.5.10 in [9], so we will not include all of it. The crux of the argument, and the well-spring from which the magic of zeta functions flows, is the *Poisson summation formula*:

Theorem 17. *(Poisson Summation Formula) Let $f : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$ be a Schwartz-Bruhat function. Then*

$$\sum_{a \in \mathbb{Q}} f(a) = \sum_{a \in \mathbb{Q}} \hat{f}(a).$$

This result about $\mathbb{A}_{\mathbb{Q}}$ can be stitched together from similar statements about the fields $\mathbb{Q}_p$ for $p \leq \infty$. Indeed, there is a reformulation of Poisson summation that holds for arbitrary locally compact abelian groups, and simplifies for self-dual groups such as $\mathbb{Q}_p$. This reformulation is a fairly easily derived consequence of Pontryagin Duality (see Theorem 4.43 in [2]), but there is an elementary proof for Poisson summation on $\mathbb{R}$ that can be enlightening.

Example 12. The version of Poisson summation for a Schwartz function $f : \mathbb{R} \to \mathbb{C}$ is

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

We define

$$F(x) = \sum_{n \in \mathbb{Z}} f(x + n)$$

which is 1-periodic. Its Fourier coefficients are given by

$$\begin{aligned} \hat{F}_k &= \int_0^1 \sum_{n \in \mathbb{Z}} f(x+n) e^{-2\pi i k x} \, dx \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x} \, dx \end{aligned}$$

Note that we may interchange the infinite sum and integral because f is Schwartz, so all convergence is uniform. Continuing, we get

$$\begin{aligned} \hat{F}_k &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi i k x} \, dx && \text{since } dx \text{ is translation-invariant} \\ &= \int_{\mathbb{R}} f(x) e^{-2\pi i k x} \, dx \\ &= \hat{f}(k) \end{aligned}$$

By definition of the Fourier series for F, this means that

$$F(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}.$$

Substituting $x = 0$ into this formula yields

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

as claimed. □

It is not an immediate consequence, but Poisson summation and a clever locality argument allow us to derive

Theorem 18. *For any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$, the global zeta integral $Z(f,s)$ extends to a meromorphic function on the complex plane, and satisfies the functional equation*

$$Z(f,s) = Z(\hat{f}, 1-s).$$

Furthermore, $Z(f,s)$ is holomorphic everywhere except at $s = 0$ and $s = 1$, where it has simple poles.

Let's talk about the spirit of the proof, if not the technical details. As is always the case with $\mathbb{A}_\mathbb{Q}$, it amounts to gluing together local equations into a global statement. However, the locality we consider here is not the type we have seen in the past.

The key insight is to consider certain slices of $\mathbb{I}_\mathbb{Q}$. Write $\mathbb{I}_t$ to denote the ideles of absolute value exactly t. Recall from Proposition 11 that every element of $\mathbb{Q}^\times$ has absolute value 1, so there is a well-defined multiplicative action of $\mathbb{Q}^\times$ on $\mathbb{I}_t$. Using this action, we can now define

$$\begin{aligned} Z(f,s;t) &= \int_{\mathbb{I}_t} f(x)|x|^s \, d^\times x \\ g(f,s,t) &= Z(f,s;t) + f(0) \int_{\mathbb{I}_t/\mathbb{Q}^\times} |x|^s \, d^\times x \end{aligned}$$

The functions $Z(f,s;t)$ should be thought of as "slices" of $Z(f,s)$, in much the same way that the f_p are slices of a pure Schwartz-Bruhat function f. This idea is made precise by the equation

$$Z(f,s) = \int_0^\infty Z(f,s;t) \, \frac{dt}{t}.$$

We introduce the function $g(f,s,t)$ because it includes a correction term. Intuitively, integrating over all the $Z(f,s;t)$ results in a sum over all of $\mathbb{Q}^\times$ which looks similar to Poisson summation. To properly apply Theorem 17, we need a sum over all of $\mathbb{Q}$, so we introduce the $a = 0$ into the function $g(f,s,t)$.

Poisson summation then allows one to show that

$$g(f,s,t) = g(\hat{f}, 1-s, t^{-1}).$$

These slice-local statements reduce the problem of showing the functional equation for $Z(f,s)$ to checking symmetry in the correction term of $g(f,s,t)$ under the transformations $f \to \hat{f}$ and $s \to 1-s$. It is nothing short of magic that this symmetry holds, and so we will say no more on this piece of witchcraft.

We conclude by deriving the meromorphic continuation and functional equation of the Riemann zeta function, which was the goal of this chapter. In Example 11, we showed that the extended Riemann zeta function can be realised as

$$\tilde{\zeta}(s) = Z(f, s)$$

for a particular function f. We now notice that f is a simple function, so Theorem 16 applies to give

$$\hat{f} = \prod_{p \leq \infty} \hat{f}_p.$$

But we (intentionally) chose the slices f_p of f to be fixed points under their respective Fourier transforms, so

$$\hat{f} = \prod_{p \leq \infty} f_p = f.$$

From Theorem 18, we conclude that

$$\tilde{\zeta}(s) = Z(f, s) = Z(\hat{f}, 1 - s) = Z(f, 1 - s) = \tilde{\zeta}(1 - s)$$

which is the functional equation for the Riemann zeta function. Furthermore, the same Theorem also tells us that $\tilde{\zeta}(s)$ extends to a meromorphic function on the complex plane, holomorphic except for $s = 0$ and $s = 1$.

This is quite a remarkable result. While these properties of $\tilde{\zeta}(s)$ were known prior to Tate's thesis, his approach to their proof was considered revolutionary, and it spawned entire fields dedicated to generalising Theorem 18.

More importantly for us, Tate's thesis illustrates the remarkable power of Fourier analysis, and how wide-spread its applications can be. We have taken a statement about zeta functions, which on the surface has nothing to do with the Fourier transform, and proven an extremely nontrivial result. This is the power of Fourier analysis.

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