Cohomology Theories on Riemann Surfaces

M. Skilleter

Abstract

In lectures, we defined Čech cohomology axiomatically as the family of right-derived functors of the global sections functor. Here we give a more explicit construction, using fine sheaves and Čech covers of our Riemann surface. We then prove the equivalence of the Čech, de Rham and Dolbeault cohomologies on a Riemann surface. We also give another proof of the existence of non-constant meromorphic functions on any compact Riemann surface, different from the one seen in lectures.

1 Introduction

Čech cohomology on manifolds is a powerful tool for studying topological and analytic properties. In this paper, we give an explicit construction of Čech cohomology and prove a practical computational result. Throughout, we assume knowledge of sheaf theory and elementary homological algebra.

We then give a demonstration of the power of Čech cohomology by using it to unify the de Rham and Dolbeault cohomologies we studied in the course. As a consequence, we give yet another proof of the existence of non-constant meromorphic functions on a compact Riemann surface.

2 Čech Cohomology

Fix a Riemann surface X and a sheaf of vector spaces $\mathcal{F}$ on X, which will be our coefficient sheaf. Our goal is to give an explicit definition of the Čech cohomology spaces $\check{H}^n(X, \mathcal{F})$.

Definition 1. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open cover of X, indexed by a totally ordered set I. Associated to $\mathcal{U}$ is an abstract simplicial complex $\mathcal{N}(\mathcal{U})$, called the *nerve* of $\mathcal{U}$, with faces given by finite subsets of $\mathcal{U}$ with nonempty intersection. More explicitly, $\{U_{i_1}, \ldots, U_{i_k}\}$ is a face of $\mathcal{N}(\mathcal{U})$ if $U_{i_1} \cap \ldots \cap U_{i_k} \neq \emptyset$.

Such a locally finite open cover exists on any Riemann surface, and more generally on any topological manifold, because of paracompactness. By studying covers which are so well-behaved locally, we hope to gain global insight into X, and this is made rigorous by Čech cohomology.

Definition 2. Let $\mathcal{U}$ be as in Definition 1. The space of *Čech k-cochains* $\check{C}^k(\mathcal{U}, \mathcal{F})$ is defined by

$$\check{C}^k(\mathcal{U}, \mathcal{F}) := \bigoplus_{i_0 < i_1 < \ldots < i_k} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_k}).$$

Note that we have used the fact that I is totally ordered to even write the above. These will be the terms in our chain complex. We define a differential d from k-cochains to $(k+1)$-cochains by

$$d\alpha(U_{i_0} \cap \ldots \cap U_{i_k} \cap U_{i_{k+1}}) = \sum_{j=0}^{k+1} (-1)^j \alpha(U_{i_0} \cap \ldots \cap \widehat{U_j} \cap \ldots \cap U_{k+1})$$

where the $\widehat{\ }$ indicates omission of the j^{th} open set.

Technically speaking, the above is called the *Čech complex of ordered cochains.* There are other cochain complexes (namely singular and alternating cochains) which ultimately turn out to give the same cohomology, but there are many redundancies on the cochain level and so our definition is often easier to work with in practise.

It is routine to verify that $d^2 = 0$, so this is truly a cochain complex. We now define the *Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$*, denoted $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$, as the cohomology of this complex.

At this point, our definition is dependent on a choice of locally finite cover $\mathcal{U}$. We would like to remove this dependence to get a cohomology theory based solely on the sheaf $\mathcal{F}$. To that end, we consider refinements of our open covers.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be locally finite open covers of X. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every element of $\mathcal{V}$ is a subset of an element of $\mathcal{U}$. A choice of inclusion $\tau : \mathcal{V} \to \mathcal{U}$ is called a *refining map.*

Since any two open covers have a common refinement, we may assume without loss of generality that $\mathcal{V}$ refines $\mathcal{U}$. Given a choice of refining map τ, we have an induced chain map $\tau^* : \check{C}^k(\mathcal{U}, \mathcal{F}) \to \check{C}^k(\mathcal{V}, \mathcal{F})$ given by precomposition with τ.

We have no canonical way to choose a refining map, so it is important to know how much these induced maps depend on τ. The answer is that, on the level of cohomology, they are independent of choice.

Proposition 1. *Let $\tau_1, \tau_2 : \mathcal{V} \to \mathcal{U}$ both be refining maps. Then the induced maps on cohomology are the same. Furthermore, $\tau^* : \check{H}^1(\mathcal{U}, \mathcal{F}) \to \check{H}^1(\mathcal{V}, \mathcal{F})$ is injective.*

Proof. For the first statement, it suffices to construct a chain homotopy between τ_1^* and τ_2^* on the level of chain complexes. That is, we want a collection of homomorphisms h in the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \check{C}^{k-1}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] & \check{C}^k(\mathcal{U},\mathcal{F}) \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "\tau_1^*"'] \arrow[d, shift left, "\tau_2^*"] & \check{C}^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[dl, "h"'] & \ldots \\
\ldots \arrow[r] & \check{C}^{k-1}(\mathcal{V},\mathcal{F}) \arrow[r, "d"] & \check{C}^k(\mathcal{V},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k+1}(\mathcal{V},\mathcal{F}) \arrow[r] & \ldots
\end{tikzcd}$$

such that

$$dh + hd = \tau_1^* - \tau_2^*.$$

It is an easy calculation to verify that the map

$$h(\alpha)(V_{i_0} \cap \ldots \cap V_{i_k}) := \sum_{j=0}^{k} \alpha\left(\tau_1(V_{i_0}) \cap \ldots \cap \tau_1(V_{i_j}) \cap \tau_2(V_{i_j}) \cap \ldots \cap \tau_2(V_{i_k})\right)$$

works.

For the second statement, let $f = (f_{i_0}, f_{i_1}) \in \ker(d^1)$ and assume that $\tau^*[f] = 0$. This means that there exists $(g_j) \in \check{C}^0(\mathcal{V}, \mathcal{F})$ such that

$$f_{\tau(j_0),\tau(j_1)} = g_{j_1} - g_{j_0} \quad \text{on } V_{j_0} \cap V_{j_1}$$

for all tuples (j_0, j_1). Note that we are slightly abusing notation by applying τ not to the open sets in $\mathcal{V}$ but to the indexing set J of $\mathcal{V}$. It should be clear that there is no real difference.

Since $f \in \ker(d^1)$, it follows that

$$f_{\tau(j_0),\tau(j_1)} = f_{i,\tau(j_1)} - f_{i,\tau(j_0)}$$

for every $i \in I$. For all $i \in I, j \in J$, we now define the element

$$h_{i,j} := g_j - f_{i,\tau(j)} \in \mathcal{F}(U_i \cap V_j).$$

For fixed i, the sections $h_{i,j}$ of the cover $\{U_i \cap V_j\}_{j \in J}$ of U_i glue to some $h_i \in \mathcal{F}(U_i)$. Moreover, $f_{i_0,i_1} = h_{i_1} - h_{i_0}$ for all (i_0, i_1) and so f is zero in cohomology, as required. □

An immediate consequence of Proposition 1 is that the projective system of locally finite covers, partially ordered by refinement, gives rise to a canonical direct system of cohomology spaces. We may therefore define the k^{th} Čech cohomology as

$$\check{H}^k(X, \mathcal{F}) := \varinjlim_{\mathcal{U}, \text{ordered by refinement}} \check{H}^k(\mathcal{U}, \mathcal{F})$$

and the result is no longer dependent on the choice of cover.

This definition is not easy to compute in practise. To rectify this, we can choose to impose more conditions on either our sheaf $\mathcal{F}$, or on our open covers $\mathcal{U}$. We now explore the former option, but will use the latter in our proofs of de Rham's and Dolbeault's theorems.

In the former direction, let us restrict ourselves to fine sheaves. Recall that a sheaf is *fine* if its sections can be glued together by partitions of unity. A *fine resolution of* $\mathcal{F}$ is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \ldots \qquad (\dagger)$$

where each $\mathcal{F}^i$ is fine. We take it as given that any sheaf on a Riemann surface has a fine resolution. The remarkable fact is that this can be used to compute the Čech cohomology.

Theorem 1. *Let* $(\dagger)$ *be a fine resolution of* $\mathcal{F}$*, and suppose* $\mathcal{U}$ *is an open cover of* X *such that the sequence of homomorphisms*

$$\mathcal{F}^{j-1}(U_{i_0} \cap \ldots \cap U_{i_k}) \to \mathcal{F}^j(U_{i_0} \cap \ldots \cap U_{i_k}) \to \mathcal{F}^{j+1}(U_{i_0} \cap \ldots \cap U_{i_k})$$

is exact for every face of $\mathcal{N}(\mathcal{U})$*. Then we can recover the Čech cohomology* $\check{H}^k(\mathcal{U}, \mathcal{F})$ *from the chain complex of global sections. Explicitly, there is a canonical isomorphism*

$$\check{H}^k(\mathcal{U}, \mathcal{F}) \cong \frac{\ker\left(\delta : \mathcal{F}^k(X) \to \mathcal{F}^{k+1}(X)\right)}{\operatorname{im}\left(\delta : \mathcal{F}^{k-1}(X) \to \mathcal{F}^k(X)\right)}.$$

Proof. We sketch the argument, but leave many details unchecked, as it is an easy diagram chase. We consider the diagram

$$\begin{tikzcd}
 & \check{C}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \dots \\
\mathcal{F}^0(X) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \dots \\
\mathcal{F}^1(X) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \dots \\
\mathcal{F}^2(X) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Here the horizontal maps between the cochain vector spaces are the differentials, while the vertical maps are induced from the resolution (†) by functoriality of the cochain complexes. The leftmost map in each row simply restricts global sections of $\mathcal{F}$ to the elements of our open cover $\mathcal{U}$.

Our assumption on the open cover $\mathcal{U}$ says that the columns from the second onwards are exact because they are exact on every face. Recalling from lectures that the Čech cohomology of a fine sheaf is trivial for $k > 0$, we see that the rows from the second onwards are also exact. Knowing this, a standard diagram chase allows one to construct a map from $\ker(d^k) \subseteq \check{C}^k(\mathcal{U},\mathcal{F})$ to $\ker(\delta^k) \subseteq \mathcal{F}^k(X)$ by repeatedly moving down the column and pulling back to the left, and this map is well-defined on cohomology. Symmetrically, we get a map $\ker(\delta^k) \to \ker(d^k)$ by repeatedly moving right along the row and pulling upwards. These maps are inverses on cohomology. □

There are many stronger results for computing Čech cohomology, but they often require the techniques of spectral sequences to prove. Fortunately, Theorem 1 will suffice for our purposes.

3 Equivalence of Cohomologies

With the abstract algebra of the previous section out of the way, we are now in a position to prove the equivalence of the three cohomology theories we have seen in the course.

Theorem 2. *(de Rham)*
Let $\underline{\mathbb{R}}$ be the constant $\mathbb{R}$-valued sheaf on a Riemann surface X. Then

$$\check{H}^k(X,\underline{\mathbb{R}}) \cong H^k_{\mathrm{dR}}(X) \quad \textit{for all } k.$$

Proof. Since the sheaves $\Omega^k(X)$ are fine, we have a fine resolution of $\underline{\mathbb{R}}$ by

$$0 \to \underline{\mathbb{R}} \xrightarrow{\text{inclusion}} \Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \to \dots.$$

where d^k is the exterior derivative on k-forms. Note that this sequence is exact locally by Poincaré's Lemma, so it is exact as a sequence of sheaves.

We now recall that Riemann surfaces (and more generally, all topological manifolds) admit a Čech cover by paracompactness. This is an open cover $\mathcal{U}$ of X such that all finite intersections of elements in $\mathcal{U}$ are contractible. Once again invoking Poincare's Lemma, we know that the sequence of homomorphisms

$$\Omega^{j-1}(C) \xrightarrow{d^{j-1}} \Omega^{j}(C) \xrightarrow{d^{j}} \Omega^{j+1}(C)$$

is exact on any contractible set C, so such a Čech cover satisfies the hypothesis of Theorem 1. Thus

$$\check{H}^k(\mathcal{U}, \underline{\mathbb{R}}) \cong \frac{\ker(d^k)}{\operatorname{im}(d^{k-1})} = H^k_{\mathrm{dR}}(X). \qquad (*)$$

We now make a general observation which applies to any topological manifold: covers of the type satisfying Theorem 1 are cofinal among all covers, because a refinement of $\mathcal{U}$ can always be refined again if necessary to satisfy the hypothesis. Since a colimit can be computed using any cofinal system, we can take the colimit over these cohomology groups to compute $\check{H}^k(X, \underline{\mathbb{R}})$. But the right-hand side of $(*)$ is independent of the cover $\mathcal{U}$, so it follows that

$$\check{H}^k(X, \underline{\mathbb{R}}) \cong H^k_{\mathrm{dR}}(X)$$

as required. $\square$

The proof of equivalence with Dolbeault cohomology proceeds along similar lines, although some small adjustments must be made.

Theorem 3. *(Dolbeault)*
Let $\Omega^p_{\mathbb{C}}$ denote the sheaf of complex valued p-forms on X. Then

$$\check{H}^q(X, \Omega^p_{\mathbb{C}}) \cong H^{p,q}_{\mathrm{Dol}}(X) \quad \textit{for all } (p,q).$$

Proof.

We once again consider a fine resolution

$$0 \to \Omega^p_{\mathbb{C}} \xrightarrow{\text{inclusion}} \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \to \dots.$$

Exactness of this sequence follows from the Dolbeault-Grothendieck lemma (see Lemma 1 in the Appendix), a complex analytic analogue of Poincaré's Lemma which says that every $\overline{\partial}$-closed form is locally $\overline{\partial}$-exact. We now proceed exactly as in the proof of Theorem 2. $\square$

Theorems 2 and 3 show that Čech cohomology is a true generalisation of the de Rham and Dolbeault cohomologies. Not only that, but they give us a way to relate these two cohomologies directly, which was a major step in the proof of the existence of meromorphic functions on a compact Riemann surface.

4 The Riemann Existence Theorem

We will now use Čech cohomology to prove the existence of non-constant meromorphic functions on a compact Riemann surface, needing nothing more complicated than the fact that $H^{0,1}_{\mathrm{Dol}}(X)$ is a subspace of $H^1_{\mathrm{dR}}(X)$. Since X is compact, the latter is finite-dimensional and hence so is the former. Notice that we have avoided calculating the Dolbeault cohomology explicitly, so we do not need to invoke the Main Theorem from lectures.

Theorem 4. *(Riemann Existence)*

For any point $p \in X$, there exists a meromorphic function f on X with a pole at p.

Proof. We invoke Theorem 3 and the above to see that $\check{H}^1(X, \Omega^0_{\mathbb{C}}) \cong H^{0,1}_{\mathrm{Dol}}(X)$ is finite-dimensional. To emphasize the fact that $\Omega^0_{\mathbb{C}}$ is the sheaf of holomorphic functions on X, we will henceforth denote it by $\mathcal{O}$.

Choose a chart U_0 around p with coordinate z, and set $U_1 := X \setminus \{p\}$. Then $\mathcal{U} := \{U_0, U_1\}$ is an open cover of X and by Proposition 1, the map $H^1(\mathcal{U}, \mathcal{O}) \to H^1(X, \mathcal{O})$ is injective.

Remark. Technically speaking, Proposition 1 only tells us that the individual maps between the $\check{H}^0(\mathcal{V}, \mathcal{O})$ groups in the direct system are injective. However, it is a general category theoretic result that if all the maps in a direct system are monomorphisms then so are the maps to the colimit, so the above fact follows immediately.

Thus the vector space $H^1(\mathcal{U}, \mathcal{O})$ is finite-dimensional, say of dimension n. We now consider the holomorphic functions

$$\frac{1}{z^j} \in \mathcal{O}(U_0 \cap U_1), \quad j = 1, \ldots, n+1.$$

Note that these are elements of $\check{C}^1(\mathcal{U}, \mathcal{O})$. Furthermore, because our cover $\mathcal{U}$ only has two open sets, $\check{C}^2(\mathcal{U}, \mathcal{O}) = 0$ and so these functions are cocycles.

Looking at their classes in cohomology, our dimension argument says that they become linearly dependent. Therefore there exist complex numbers $c_1, \ldots, c_{k+1}$, not all zero, and a cochain $(f_0, f_1) \in \mathcal{O}(U_0) \oplus \mathcal{O}(U_1) = \check{C}^0(\mathcal{U}, \mathcal{O})$ such that

$$\sum_{j=1}^{n+1} \frac{c_j}{z^j} = d(f_0, f_1) = f_1 - f_0.$$

Using this relation, we may therefore extend f_0 to a meromorphic function on all of X, with a pole only at p of order between 1 and $n+1$. $\square$

One might be tempted to believe that this approach has allowed us to avoid any serious complex analysis. However, recall that in proving Theorem 3, we needed to make use of the Dolbeault-Grothendieck Lemma. This is a non-trivial result, and at the very least requires the Cauchy Integral Formula. Nevertheless, it does not require either the Main Theorem from lectures or the existence of meromorphic functions, so our proof is genuinely different and is not circular.

5 Appendix

In the interest of keeping this paper self-contained, we include here a proof of the Dolbeault-Grothendieck Lemma for open sets in $\mathbb{C}$. We will assume without proof the following generalised form of the Cauchy integral formula for smooth functions:

Proposition 2. *Let $U \subseteq \mathbb{C}$ be open and suppose that $\overline{B_\varepsilon(0)} \subseteq U$. For $f \in C^\infty(U)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} \frac{f(\zeta)}{\zeta - z}\ d\zeta + \frac{1}{2\pi i} \int_{B_\varepsilon(0)} \frac{\partial f}{\partial \overline{\zeta}} \left(\frac{1}{\zeta - z} \right)\ d\zeta \wedge d\overline{\zeta}.$$

Using this, we can now prove:

Lemma 1. *(Dolbeault-Grothendieck)*
Let $B_\varepsilon(0)$ and U be as above, and let $\alpha = f\ d\overline{z} \in \Omega^{0,1}(U)$ be a smooth (0,1)-form. Then

$$g(z) := \frac{1}{2\pi i} \int_{B_\varepsilon(0)} \frac{f(\zeta)}{\zeta - z}\ d\zeta \wedge d\overline{\zeta}$$

satisfies $\alpha = \overline{\partial} g$ on $B_\varepsilon(0)$. In other words, every locally $\overline{\partial}$-closed form is $\overline{\partial}$-exact.

Proof. First we must show that g is smooth so that $g \in \Omega^0(B_\varepsilon(0))$, and then we will prove that $\alpha = \overline{\partial} g$. For the former claim, we choose a point $w \in B_\varepsilon(0)$, an open neighbourhood $w \in V \subseteq B_\varepsilon(0)$ and a smooth function $\rho : B_\varepsilon(0) \to \mathbb{R}$ which is compactly supported and with $\rho\ |_V = 1$. We can then write f as

$$f = f_1 + f_2 := \rho f + (1 - \rho) f$$

and define

$$g_i(z) := \frac{1}{2\pi i} \int_{B_\varepsilon(0)} \frac{f_i(\zeta)}{\zeta - z}\ d\zeta \wedge d\overline{\zeta}.$$

Since $f_2\ |_V = 0$, it is obvious that g_2 is well-defined and smooth. On the other hand, we can explicitly compute

$$\begin{aligned} g_1(z) &= \int_{B_\varepsilon(0)} \frac{f_1(\zeta)}{\zeta - z}\ d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta)}{\zeta - z}\ d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} \frac{f_1(z + re^{i\theta})}{e^{i\theta}}\ d\theta dr \end{aligned}$$

and this is clearly well-defined and smooth. Thus $g = g_1 + g_2$ is too.
It remains to show that $\overline{\partial} g = \alpha$. We first consider

$$\frac{\partial g_2}{\partial \overline{z}} = \frac{1}{2\pi i} \int_{B_\varepsilon(0)} f_2(\zeta) \frac{\partial}{\partial \overline{z}} \left(\frac{1}{\zeta - z} \right)\ d\zeta \wedge d\overline{\zeta}.$$

Because the function $\frac{1}{\zeta - z}$ is holomorphic on $B_\varepsilon(0) \setminus V$, we conclude that $\frac{\partial g_2}{\partial \overline{z}}\ |_{B_\varepsilon(0) \setminus V} = 0$. But recalling that $f_2 \mid V = 0$, it follows that $\frac{\partial g_2}{\partial \overline{z}} = 0$ on all of $B_\varepsilon(0)$.

Using the above expression for g_1 and undoing the change into polar coordinates, we also find that

$$\begin{aligned}\frac{\partial g_1}{\partial \bar{z}} &= \frac{1}{\pi}\int_{\mathbb{C}}\left(\frac{\partial f_1}{\partial \bar{z}}\right)\left(\frac{z+re^{i\theta}}{e^{i\theta}}\right)\, d\theta \wedge dr \\ &= \frac{1}{2\pi i}\int_{B_\varepsilon(0)} \frac{\partial f_1}{\partial \bar{\zeta}}\left(\frac{1}{\zeta - z}\right)\, d\zeta \wedge d\bar{\zeta}\end{aligned}$$

We are now almost done. Applying Proposition 2 to the smooth function f_1, we have

$$f_1(z) = \frac{1}{2\pi i}\int_{\partial B_\varepsilon(0)} \frac{f_1(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{B_\varepsilon(0)} \frac{\partial f_1}{\partial \bar{\zeta}}\left(\frac{1}{\zeta - z}\right)\, d\zeta \wedge d\bar{\zeta}.$$

Because ρ is compactly supported inside $B_\varepsilon(0)$, we see that $f_1 \mid_{\partial B_\varepsilon(0)} = 0$ and so the first integral vanishes. But the second integral is exactly our expression for $\frac{\partial g_1}{\partial \bar{z}}$. It follows that

$$f = f_1 = \frac{\partial g_1}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}}$$

on $B_\varepsilon(0)$, as required. □

References

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