# VSM167 Finite element method - basics
# Exam 2022-01-12, 8:30-12:30

Instructor: Martin Fagerström (phone 070-224 8731). The instructor will visit the exam around 9:30 and 11:30.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on 1 February 2022 the latest.

Review: It will be possible to review the grading at the Division of Material and Computational Mechanics (floor 3 in M-building) on 2 February 12:00-13:00 and 7 February 12:00-13:00.

Permissible aids: Chalmers type approved pocket calculator and MATLAB with the CALFEM toolbox. On the computer, you can find the CALFEM manual (excluding the examples section) and the CALFEM finite element method functions. **Note**: A formula sheet is also appended to this exam paper.

# Exam instructions

Please note that the solutions to Problems 1 and 3 should (Problem 1) and could (Problem 3) be solved by use of MATLAB and CALFEM. Be extra careful to read the instructions for these problems.

The CALFEM finite element files are provided under the directory `C:\__Exam__`. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__`.

Please also note that we will collect all files saved under the directory `C:\__Exam__` and subdirectories. This in order to be able to review these files during the exam corrections. Therefore, it is very important that you save all your files under `C:\__Exam__` in the appropriate subdirectories created for each problem. **It is also absolutely necessary that you write the name of your computer on the cover page for the exam!**

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

# Problem 1

Consider the plane truss in Figure 1(a), subjected to three point forces. All the members have the same cross-sectional area $A$ and Young's modulus $E$. A finite element model of the structure has been created and consists of 7 bar elements, with nodal numbering and element orientation according to Figure 1(b).

**Tasks:**

**(a) Determine the horisontal displacement at node 5.** Start from the provided file `problem1.m` (see the subdirectory for Problem 1 under `C:\__Exam__`), containing the coordinates for the elements (the `Ex` and `Ey` matrices), and write a script that establishes the system of equations $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$ for the structure, and solves this. It is important that you use the provided node and element numbering.

**Please write your answer to the problem on the hand-in paper**. **(4.0p)**

**(b)** Extend the script from subtask (a) and **determine the minimum required cross-sectional area** $A$, such that the maximum normal stress (magnitude) in any member is below the yield limit $\sigma_{\mathrm{y}}$.

**Please write your answer to the problem on the hand-in paper** **(2.0p)**

Use the following numerical values: $\phi = 70°$, $L = 2$ m, $P = 17$ kN, $E = 210$ GPa, $A = 10^{-4}$ m$^2$ and $\sigma_{\mathrm{y}} = 250$ MPa (also given in the provided MATLAB file).

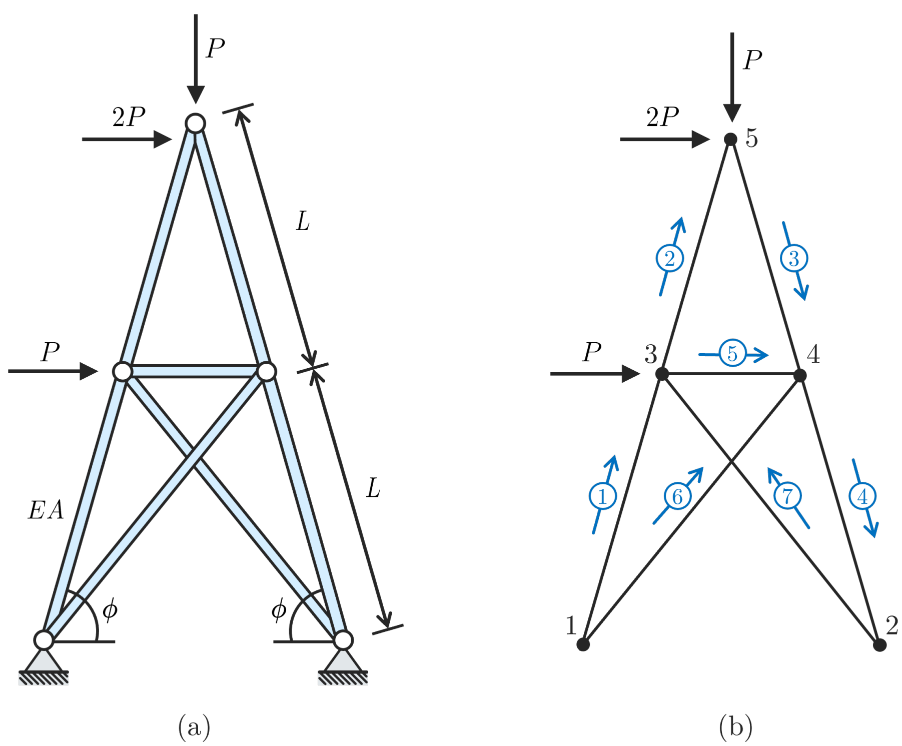



Figure 1: Plane struss structure to be analysed in Problem 1.

# Problem 2

Consider the roof as depicted in Figure 2, in which the temperature distribution is to be computed. Sun radiation leads to a heat supply to the roof along its upper surface. As a result, the amount of inflowing heat per unit surface is $\overline{q}$ [W/m$^2$] (the heat convection along this surface is negligible in comparison to $\overline{q}$). Furthermore, the left and right boundaries can be considered as insulated, and the surrounding air temperature is $T_{\text{out}}$ (to be considered for the convection on the lower surface).

The out-of-plane width of the roof is such that it is sufficient to consider a planar 2D cross section. As a consequence, the roof cross section has been discretised into linear triangular heat flow elements.

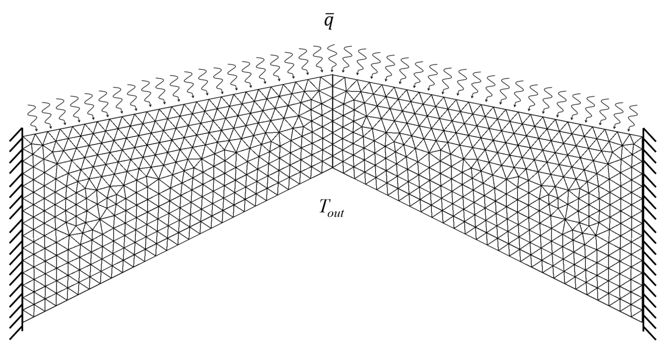



Figure 2: Illustration of the roof construction to be considered in Problem 2

**Tasks:**

**(a)** For a general 2D heat flow problem, the governing partial differential equation is given by:

$$-\text{div}(\boldsymbol{q}t) + Qt = 0.$$

where $\boldsymbol{q}$ is the heat flux vector, $t$ is the thickness and $Q$ is the external heat supply. **Please derive this governing equation by considering heat balance of an arbitrary domain**. **(1.0p)**

**(b)** For the particular problem (Figure 2), **utilise symmetry and make a sketch of the smallest possible simulation domain where you indicate the different boundary parts. Then state the strong form of the current 2D heat flow problem**, which can be solved to determine the temperature distribution inside the roof.

Make sure to clearly define any additional notations you introduce. **(1.0p)**

**(c) From the strong form, derive and state the corresponding weak form of the 2D heat flow problem**. **(2.0p)**

**(d)** Introduce suitable FE approximations for $v$ and $T$ and then **derive and state the discrete FE formulation** of the boundary value problem. **(2.0p)**

## Problem 3

Consider a concrete retaining wall – a landfill supporting structure – ($E = 40$ GPa, $\nu = 0.2$) as depicted in Figure 3. The wall is 3 metres high, and has a varying thickness as shown in the figure (base width is 1 m and top width is 0.53 m).

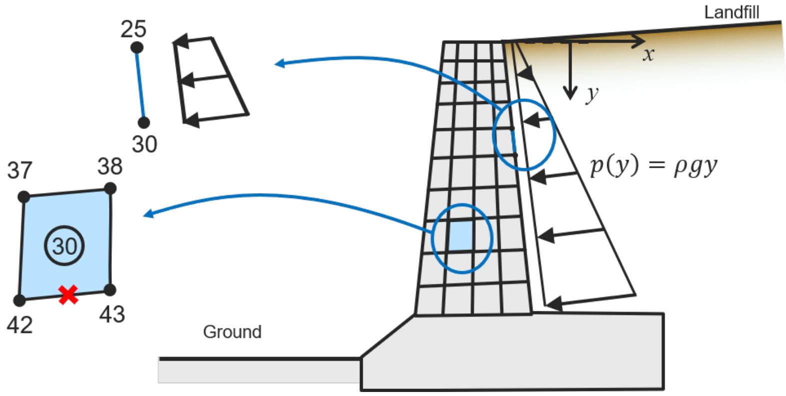



Figure 3: 2D section of a landfill wall to be analysed in Problem 3.

The supported soil gives rise to a pressure load acting (perpendicular) on the right surface of the wall $p(y) = \rho g y$, where $\rho = 2\,000$ kg/m$^3$ is the soil density, $g = 9.81$ m/s$^2$ is the gravitational acceleration constant and $y$ is the depth measured from the top. To clarify, this means that the traction along the right surface is $\boldsymbol{t} = -p\boldsymbol{n}$, where $\boldsymbol{n}$ is an outward pointing surface normal.

In this problem, we will consider the loading on the wall. More particularly, we will study the resulting load contribution along an inclined element edge (between nodes 25 and 30), and the stress in a specific point of element 30 (with nodes 37, 38, 42 and 43).

Since the wall is straight and sufficiently long (10 m long out of the sketched plane) it is sufficient to consider a 2D section under the assumption of plane strain. As a consequence, the wall has been discretised with quadrilateral isoparametric plane strain elements (bilinear approximation).

The general weak form for this type of problem is given by:

$$\int_A \left(\tilde{\nabla}\boldsymbol{v}\right)^T t\boldsymbol{D}\,\tilde{\nabla}\boldsymbol{u}\,\mathrm{d}A = \int_A \boldsymbol{v}^T\boldsymbol{b}\,t\,\mathrm{d}A + \int_{\mathcal{L}_g} \boldsymbol{v}^T\boldsymbol{t}\,t\,\mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \boldsymbol{v}^T\boldsymbol{h}\,t\,\mathrm{d}\mathcal{L},$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ along } \mathcal{L}_g,$$

where $\boldsymbol{v}$ is the weight function, $t$ is the thickness, $\boldsymbol{D}$ is the material (constitutive) 2D stiffness matrix, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{b}$ is the body load (due to gravity), $\boldsymbol{t}$ is the unknown surface traction along $\mathcal{L}_g$ and $\boldsymbol{h}$ is the known (prescribed) surface traction along $\mathcal{L}_h$.

**Continued on the next page**

With standard Voigt notation for the in-plane stress and strain, the 2D plane strain elasticity matrix is given by:

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Nodal coordinates:

Node 25: $x = 8.7$ cm, $y = 100.0$ cm
Node 30: $x = 14.0$ cm, $y = 130.0$ cm

Node 37: $x = -47.5$ cm, $y = 180.0$ cm
Node 38: $x = -26.5$ cm, $y = 175.0$ cm
Node 42: $x = -54.0$ cm, $y = 220.0$ cm
Node 43: $x = -26.5$ cm, $y = 215.0$ cm

***When solving the tasks below, you may benefit from using MATLAB. However, make sure that the entire solution can be followed through what you hand in on paper!***

**Tasks:**

**(a)** For element 30, **calculate the Jacobian matrix associated with the isoparametric mapping in the midpoint of the element edge between nodes 42 and 43** (marked with a red cross). **(1.5p)**

**(b)** Assume that the FE problem above has been solved. For element 30, the nodal displacements have been determined as:

Node 37: $u_x = -0.8$ mm, $u_y = 4.1$ mm
Node 38: $u_x = -1.0$ mm, $u_y = 4.8$ mm
Node 42: $u_x = -0.3$ mm, $u_y = 0.0$ mm
Node 43: $u_x = -0.6$ mm, $u_y = 0.8$ mm

For this deformed state, **calculate the in-plane stress components** $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ **for the same edge midpoint** (marked with the red cross). **(1.5p)**

**(c) Calculate the boundary load vector contributions from the applied pressure along the inclined edge between nodes 25 and 30. Also explain to which positions in the global load vector these are to be added**. **(3.0p)**

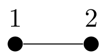
*If you find this problem too challenging, you may solve the problem with the simplification that the pressure distribution is approximated as constant over the element edge between nodes 25 and 30. This can be motivated by the fact that this approach will converge to the correct load as the mesh is refined.*

*If you choose this option, then please choose an appropriate pressure value with a good motivation. This will however limit the maximum points to 2.0p for this problem (c).*

# 1 Shape functions

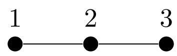
## 1.1 1D, linear

$$N_1^e = -\frac{1}{L}(x - x_2) \tag{1a}$$
$$N_2^e = \frac{1}{L}(x - x_1) \tag{1b}$$

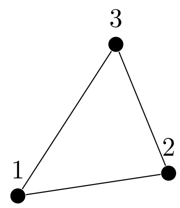


## 1.2 1D, quadratic

$$N_1^e = \frac{2}{L^2}(x - x_2)(x - x_3) \tag{2a}$$
$$N_2^e = -\frac{4}{L^2}(x - x_1)(x - x_3) \tag{2b}$$
$$N_3^e = \frac{2}{L^2}(x - x_1)(x - x_2) \tag{2c}$$



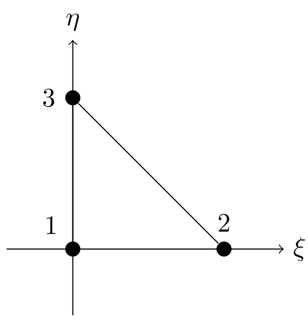
## 1.3 2D, linear triangle



$$N_1^e = \frac{1}{2A}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y) \tag{3a}$$
$$N_2^e = \frac{1}{2A}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y) \tag{3b}$$
$$N_3^e = \frac{1}{2A}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y) \tag{3c}$$

Parent element:

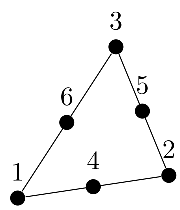
$$\overline{N}_1^e = 1 - \xi - \eta \tag{4a}$$
$$\overline{N}_2^e = \xi \tag{4b}$$
$$\overline{N}_3^e = \eta \tag{4c}$$

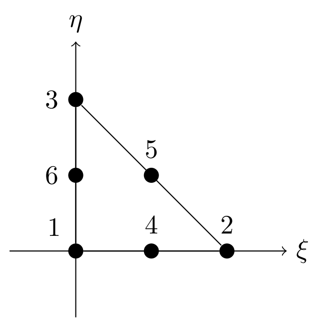


## 1.4 2D, Quadratic triangle

Parent element:

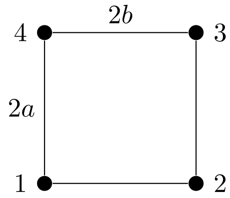
$$\overline{N}_1^e = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \tag{5a}$$
$$\overline{N}_2^e = \xi(2\xi - 1) \tag{5b}$$
$$\overline{N}_3^e = \eta(2\eta - 1) \tag{5c}$$
$$\overline{N}_4^e = 4\xi(1 - \xi - \eta) \tag{5d}$$
$$\overline{N}_5^e = 4\xi\eta \tag{5e}$$
$$\overline{N}_6^e = 4\eta(1 - \xi - \eta) \tag{5f}$$





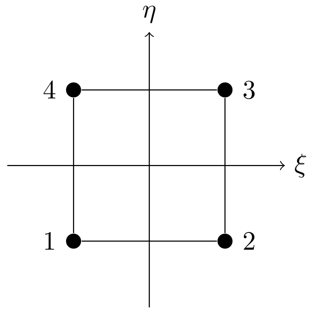
### 1.5 2D, bilinear

$$N_1^e = \frac{1}{4ab}(x - x_2)(y - y_4) \tag{6a}$$
$$N_2^e = -\frac{1}{4ab}(x - x_1)(y - y_3) \tag{6b}$$
$$N_3^e = \frac{1}{4ab}(x - x_4)(y - y_2) \tag{6c}$$
$$N_4^e = -\frac{1}{4ab}(x - x_3)(y - y_1) \tag{6d}$$




Parent element:

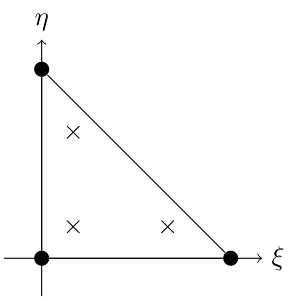
$$\overline{N}_1^e = \frac{1}{4}\,(\xi - 1)\,(\eta - 1) \tag{7a}$$
$$\overline{N}_2^e = -\frac{1}{4}\,(\xi + 1)\,(\eta - 1) \tag{7b}$$
$$\overline{N}_3^e = \frac{1}{4}\,(\xi + 1)\,(\eta + 1) \tag{7c}$$
$$\overline{N}_4^e = -\frac{1}{4}\,(\xi - 1)\,(\eta + 1) \tag{7d}$$




## 2 Gauss points

| $n$ | $\xi_i$ | $W_i$ |
|---|---|---|
| 1 | 0.0000000000000000 | 2.0000000000000000 |
| 2 | ±0.5773502691896257 | 1.0000000000000000 |
| 3 | 0.0000000000000000 | 0.8888888888888889 |
| | ±0.7745966692414834 | 0.5555555555555556 |
| 4 | ±0.3399810435848563 | 0.6521451548625460 |
| | ±0.8611363115940525 | 0.3478548451374544 |

Table 1: Position of Gauss points $\xi_i$ and corresponding weight $W_i$ for $n$ Gauss points.




| $n$ | $(\xi_i, \eta_i)$ | $W_i$ |
|---|---|---|
| 1 | $\left(\frac{1}{3}, \frac{1}{3}\right)$ | $\frac{1}{2}$ |
| 3 | $\left(\frac{1}{6}, \frac{1}{6}\right)$ | $\frac{1}{6}$ |
| | $\left(\frac{2}{3}, \frac{1}{3}\right)$ | $\frac{1}{6}$ |
| | $\left(\frac{1}{6}, \frac{2}{3}\right)$ | $\frac{1}{6}$ |

## 3 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, $\phi$ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$.

$$\int_A \phi \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w} \,\mathrm{d}A + \int_A (\boldsymbol{\nabla}\phi)^{\mathrm{T}} \boldsymbol{w} \,\mathrm{d}A = \int_{\mathcal{L}} \boldsymbol{n}^{\mathrm{T}} (\phi \boldsymbol{w}) \,\mathrm{d}\mathcal{L} \tag{8}$$

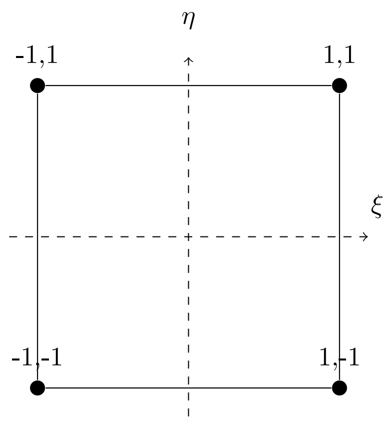
# 4 Gauss divergence theorem

$\boldsymbol{w}$ = vector field, $\phi$ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$, $\mathrm{div}(\boldsymbol{w}) = \nabla^{\mathrm{T}}\boldsymbol{w}$.

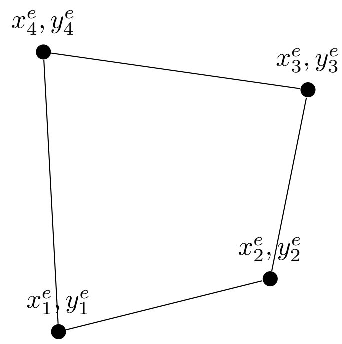
$$\int_A \boldsymbol{\nabla}^{\mathrm{T}}(\phi \boldsymbol{w})\, \mathrm{d}A = \int_{\mathcal{L}} (\phi \boldsymbol{w})^{\mathrm{T}}\, \boldsymbol{n}\, \mathrm{d}\mathcal{L}$$

# 5 Isoparametric mapping

Parent domain




Global domain




$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

$$x = x(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{x}^e \tag{9}$$

$$y = y(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{y}^e \tag{10}$$

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{12}$$

# 6 Matrix inversion

The inverse of the matrix $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\mathbf{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{13}$$

# 7 Stresses and strains

Hooke's generalised law: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

2D Strain-displ. relation: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u},\ \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$

# Problem 1

Consider the plane truss in Figure 1(a), subjected to three point forces. All the members have the same cross-sectional area $A$ and Young's modulus $E$. A finite element model of the structure has been created and consists of 7 bar elements, with nodal numbering and element orientation according to Figure 1(b).

**Tasks:**

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**Please write your answer to the problem on the hand-in paper**. (**4.0p**)

**(b)** Extend the script from subtask (a) and **determine the minimum required cross-sectional area** $A$, such that the maximum normal stress (magnitude) in any member is below the yield limit $\sigma_{\mathrm{y}}$.

**Please write your answer to the problem on the hand-in paper** (**2.0p**)

Use the following numerical values: $\phi = 70°$, $L = 2$ m, $P = 17$ kN, $E = 210$ GPa, $A = 10^{-4}$ m$^2$ and $\sigma_{\mathrm{y}} = 250$ MPa (also given in the provided MATLAB file).

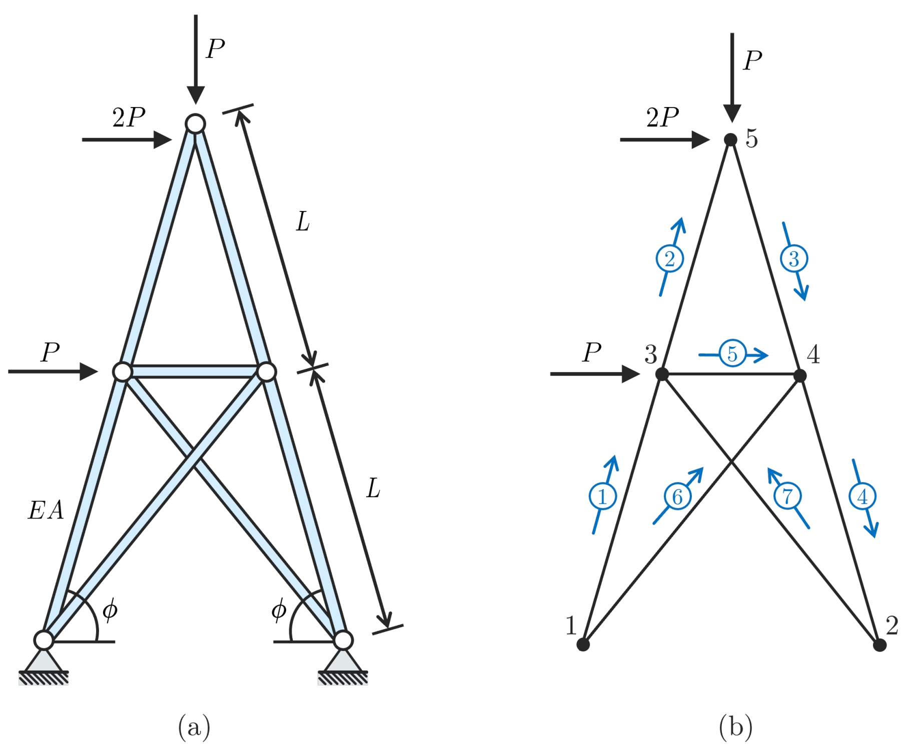



Figure 1: Plane struss structure to be analysed in Problem 1.

```
clc, clear variables, close all

phi = 70; % [degrees]
L = 2; % [m]
c = cosd(phi); s = sind(phi);
sigy = 250e6;

% Element cooridnates
Ex = [0       L*c
      L*c   2*L*c
      2*L*c 3*L*c
      3*L*c 4*L*c
      L*c   3*L*c
      0     3*L*c
      4*L*c   L*c];

Ey = [  0      L*s
       L*s 2*L*s
     2*L*s   L*s
       L*s   0
       L*s   L*s
       0      L*s
       0      L*s ];

eldraw2(Ex, Ey) % Draw the truss structure

E = 210e9;     % Young's modulus [Pa]
A = 1.0e-4;    % cross-sectional area [m^2]
P = 17e3;      % force magnitude [N]
sig_y = 250e6; % material yield limit [Pa]


%% Write you implementation below
% Anonumous code:


% A = 3.0e-4;      % cross-sectional area [m^2]
Edof = [1  1  2  5  6
        2  5  6  9 10
        3  9 10  7  8
        4  7  8  3  4
        5  5  6  7  8
        6  1  2  7  8
        7  3  4  5  6]

ep = [E A];
nel = 7;
ndofs = 10;
K = zeros(ndofs, ndofs);
f = zeros(ndofs, 1);
```

```
for el = 1:nel
    Ke = bar2e(Ex(el,:), Ey(el,:), ep);
    K = assem(Edof(el, :), K, Ke);
end

f(9) = 2*P;
f(10) = -1*P;
f(5) = P;

bc = [1 0
      2 0
      3 0
      4 0];
[a, r] = solveq(K, f, bc);

hor_disp = a(9)
Ed = extract(Edof, a);

hold on
eldisp2(Ex, Ey, Ed);

% Post-processing
N = zeros(nel,1);
for el = 1:nel
    N(el) = bar2s(Ex(el,:), Ey(el,:), ep, Ed(el, :));
end
sigma = N/A
max_sig = max(abs(sigma))
Anew = A*max_sig/sigy
```

a) If answer correct & code ~OK => 4p immediately

Otherwise:

Edef : 1p

Ke : 0.5p

assem : 0.5p

BCs : 1p

Solve with BC: 0.5p

extract answer : 0.5p

answer: a9 = 0.0294 m

b) Answer correct & code OK => 2p

Otherwise:

loop & comput. of all bar forces: 1p

Stress calc : 0.5p

Correct area comp : 0.5p

Answer: A = 2.48e-4

P2

a) Steady-state heat balance

$$\int_{\mathcal{L}} q_n t \, d\mathcal{L} = \int_A Q t \, dA$$

outflow = inflow

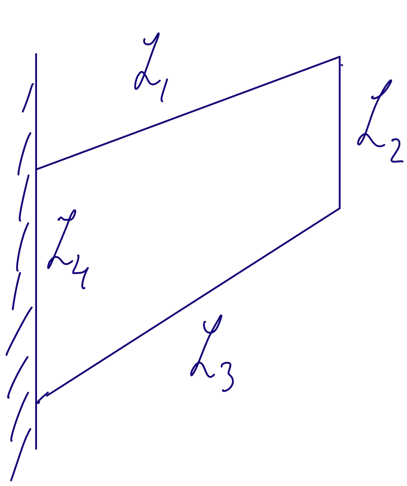
0.25

$$\underbrace{\int_{\mathcal{L}} t q_n \, d\mathcal{L} = \int_A \operatorname{div}(t\mathbf{q}) A}_{0.5p} = \int_A Q t \, dA$$

Must hold for arbitrary $A$ $\Rightarrow$ 0.25p

$$\underline{\underline{-\operatorname{div}(t\mathbf{q}) + Qt = 0}}$$

b)




$Q = 0$, $\mathbf{q} = -D\nabla T$

$\Rightarrow$

$\operatorname{div}(Dt\nabla T) = 0$ 0.2p

$q_n = 0$ along $\mathcal{L}_4$ 0.2p

$q_n = \alpha(T - T_{out})$ along $\mathcal{L}_3$ 0.2p

$q_n = 0$ along $\mathcal{L}_2$ 0.2p

$q_n = -\bar{q}$ along $\mathcal{L}_1$ 0.2p c)

Multiply with arbitrary weight function & integrate over the domain

$$\int_A v \,\mathrm{div}(t q) \, dA = 0 \qquad 0.25p$$

0.75p

$$\int_A v \,\mathrm{div}(t q)\, dA = \int_A \mathrm{div}(v t q)\, dA - \int_A (\nabla^T v)\, t\, q \, dA = 0$$

(not necessary)

$\Rightarrow$ insert $q = -D\nabla T$ & rewrite first term

0.5p

$$\int_{\mathcal{L}} v t q^T n \, d\mathcal{L} + \int_A (\nabla^T v) D t \nabla T \, dA = 0$$

$$\Leftrightarrow \boxed{\begin{aligned}\int_A (\nabla^T v) D t \nabla T \, dA &= -\int_{\mathcal{L}} v t q_n \, d\mathcal{L} \\ &= \int_{\mathcal{L}_1} v t \bar{q} \, d\mathcal{L} - \int_{\mathcal{L}_3} v t \alpha (T - T_{out}) \, d\mathcal{L}\end{aligned}}$$

0.5p for split

−0.5p if Dirichlet BC

d)

Approximate $T \approx T_h = \sum_i N_i a_i = N a$

$\nabla T \approx (\nabla N) a = B a$

0.5p

$$\mathbb{N} = [N_1 \;\; N_2 \;\; \cdots \;\; N_n]$$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

0.25p

$$v \approx v_h = \mathbb{N}c \;, \; c\text{-arbitrary}$$

$$\nabla^T v = (\nabla v)^T = c^T \mathbb{B}^T$$

0.5p

Insert in weak form: 0.25p

$$\underbrace{c^T \int_A \mathbb{B}^T \mathbb{D} t \mathbb{B} \, dA}_{\mathbb{K}} \, a = \underbrace{c^T \int_{\mathcal{L}_1} \mathbb{N}^T t \bar{q} \, d\mathcal{L}}_{f_b^1} - \underbrace{c^T \int_{\mathcal{L}_3} \mathbb{N}^T t \alpha (\mathbb{N}a - T_{out}) \, d\mathcal{L}}$$

$$= -c^T \underbrace{\int_{\mathcal{L}_3} t\alpha \mathbb{N}^T \mathbb{N} \, d\mathcal{L}}_{\mathbb{K}_c} \, a + c^T \underbrace{\int_{\mathcal{L}_3} \mathbb{N}^T t \alpha T_{out} \, d\mathcal{L}}_{f_b^2}$$

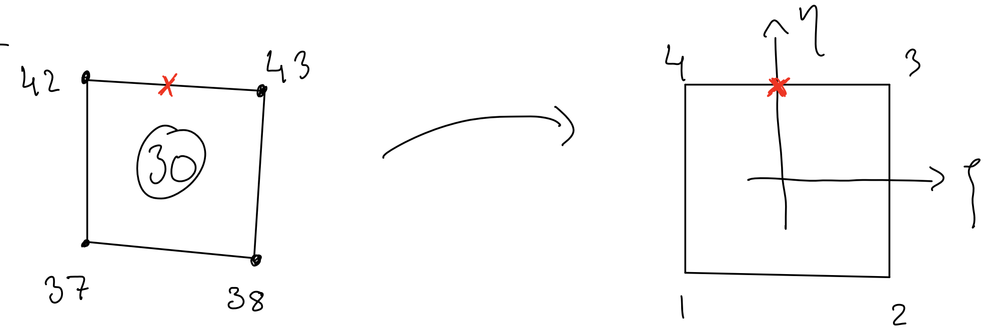
$$\therefore \quad c^T \mathbb{K} a = c^T f_b^1 - c^T \mathbb{K}_c a + c^T f_b^2$$

$$\Longleftrightarrow \quad c^T \left( \mathbb{K}a - f_b^1 - f_b^2 + \mathbb{K}_c a \right) = 0$$

$$c^T\text{- arbitr.} \Rightarrow (\mathbb{K} + \mathbb{K}_c) a = f_b^1 + f_b^2 \quad //$$

0.5p

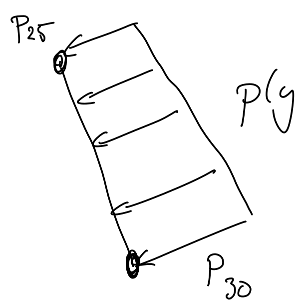
# Problem 3




$$x^e = \begin{bmatrix} -47.5 \\ -26.5 \\ -26.5 \\ -54.0 \end{bmatrix} \cdot 10^{-2}\,m$$

$$y^e = \begin{bmatrix} 180.0 \\ 175.0 \\ 215.0 \\ 220.0 \end{bmatrix} \cdot 10^{-2}\,m$$

Note that other global to local relations can be used. Will influence $J$ but not $\sigma$




$p(y) = \rho g y$

$x_{25} = 8.7 \cdot 10^{-2}\,m$

$y_{25} = 100 \cdot 10^{-2}\,m$

$x_{30} = 14.0 \cdot 10^{-2}\,m$

$y_{30} = 130 \cdot 10^{-2}\,m$

Given: $E = 40\,GPa$

$\nu = 0.2$

a) Compute $J$ at × (in our case for $\xi=0, \eta=1$)

b) Compute $\sigma$ at ×

c) Compute edge load on element edge between nodes 25 & 30

a)
$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \quad x = N^e x^e, \; N^e = [N_1^e \; N_2^e \; N_3^e \; N_4^e], \quad y = N^e y^e$$

A
$$\left\{ \begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{\partial N^e}{\partial \xi} \cdot x^e \\ \frac{\partial x}{\partial \eta} &= \frac{\partial N^e}{\partial \eta} \cdot x^e \\ \frac{\partial y}{\partial \xi} &= \frac{\partial N^e}{\partial \xi} y^e \\ \frac{\partial y}{\partial \eta} &= \frac{\partial N^e}{\partial \eta} y^e \end{aligned} \right.$$

$$\underbrace{\begin{aligned} \frac{\partial N_1^e}{\partial \xi} &= \frac{1}{4}(\eta-1) & \frac{\partial N_1^e}{\partial \eta} &= \frac{1}{4}(\xi-1) \\ \frac{\partial N_2^e}{\partial \xi} &= -\frac{1}{4}(\eta-1) & \frac{\partial N_2^e}{\partial \eta} &= -\frac{1}{4}(\xi+1) \\ \frac{\partial N_3^e}{\partial \xi} &= \frac{1}{4}(\eta+1) & \frac{\partial N_3^e}{\partial \eta} &= \frac{1}{4}(\xi+1) \\ \frac{\partial N_4^e}{\partial \xi} &= -\frac{1}{4}(\eta+1) & \frac{\partial N_4^e}{\partial \eta} &= -\frac{1}{4}(\xi-1) \end{aligned}}_{0.5p}$$

At $\underline{\eta=1, \xi=0}$ this becomes:

(or other depending on student choice of numbering) 0.5p

B $\left\{\begin{aligned} \frac{\partial x}{\partial \xi} &= 0.1375 \\ \frac{\partial x}{\partial \eta} &= -0.0167 \\ \frac{\partial y}{\partial \xi} &= -0.025 \\ \frac{\partial y}{\partial \eta} &= 0.2 \end{aligned}\right\} \Rightarrow \mathbb{J} = \begin{pmatrix} 0.1375 & -0.0167 \\ -0.025 & 0.2 \end{pmatrix}$

A + B = 0.5p

b)

$$\mathbb{\sigma} = \mathbb{D}\varepsilon = \mathbb{D}\tilde{\nabla}\mathbb{u} \qquad \tilde{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}$$

$$\mathbb{u} = \mathbb{N}^e \mathbb{a}^e$$

$$\Rightarrow \varepsilon = \dots = \mathbb{B}^e \mathbb{a}^e \quad \text{with} \quad \mathbb{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \dots & \frac{\partial N_4^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & \text{---} & 0 & \frac{\partial N_4^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \text{--} & \frac{\partial N_4^e}{\partial y} & \frac{\partial N_4^e}{\partial x} \end{bmatrix}$$

$$\mathbb{a}^e = \begin{bmatrix} -0.8 \\ 4.1 \\ -1.0 \\ 4.8 \\ -0.6 \\ 0.8 \\ -0.7 \\ 0 \end{bmatrix} \cdot 10^{-3}\,\mathrm{m}$$

Derivatives in $\mathbb{B}^e$ can be found via

$$\begin{bmatrix} \frac{dN^e}{dx} \\ \frac{dN^e}{dy} \end{bmatrix} = (\mathbb{J}^T)^{-1} \begin{bmatrix} \frac{\partial N^e}{\partial \xi} \\ \frac{\partial N^e}{\partial \eta} \end{bmatrix}$$

(Implemented in MATLAB)

0.5p

0.5p

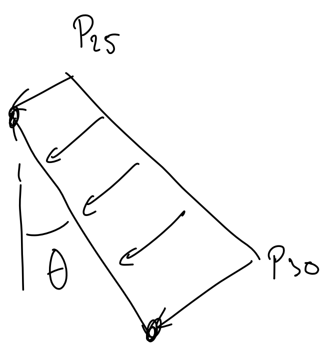
From $\begin{bmatrix} \frac{\partial N^e}{\partial x} \\ \frac{\partial N^e}{\partial y} \end{bmatrix}$, pick the right contributions & place in $\mathbb{B}^e$ (implemented in MATLAB)

Finally, $\sigma$ is obtained as

0.5p

$$\sigma = \mathbb{D}\,\mathbb{B}\,a^e = \ldots = \begin{pmatrix} -151.51 \\ -456.08 \\ 35.6 \end{pmatrix} \text{ MPa}$$

c)




$P_{25} = \rho g y_{25}$

$P_{30} = \rho g y_{30}$

nodal coords

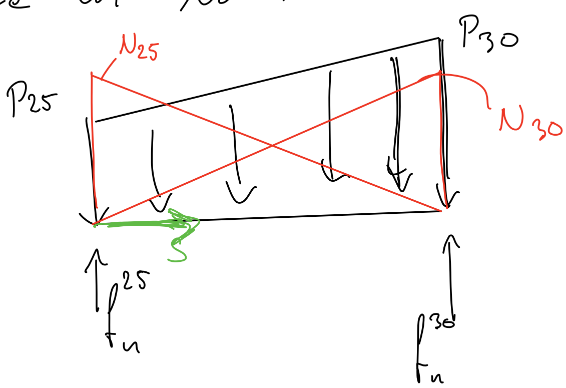
As the pressure is linear in $y$, it will also be linear considering a coordinate along the edge.

One way to solve this question is then to compute the nodal load contributions in the direction normal to the surface (tangential will be $=0$) & thereafter compute the x- & y-components using the angle

$$\theta = \operatorname{atan}\left(\frac{x_{30} - x_{25}}{y_{30} - y_{25}}\right)$$

0.25p

If we compute the nodal contribution in the normal direction, positive outwards, we will have:




0.5p or similar

$$\left\{\begin{aligned} f_n^{25} &= \int_{L^e} (-p)\, N_{25}\, t\, ds \\ f_n^{30} &= \int_{L^e} (-p)\, N_{30}\, t\, ds \end{aligned}\right.$$

$L^e$ = segment length

$$= \sqrt{(x_{30}-x_{25})^2 + (y_{30}-y_{25})^2}$$

0.25p

Without loss of generality, we can for our contribution define the shape function & pressure variation in terms of $s$ as indicated in the figure.

Then:

$$N_{25} = 1 - \frac{s}{L^e}$$

$$N_{30} = \frac{s}{L^e}$$

0.5p

$$p(s) = p_{25} + \frac{\overbrace{(p_{30} - p_{25})}^{\Delta p}}{L^e} \cdot s$$

0.5p

$$f_n^{25} = -\int_0^{L^e} \left(p_{25} + \frac{\Delta p}{L^e} \cdot s\right)\left(1 - \frac{s}{L^e}\right) t \, ds$$

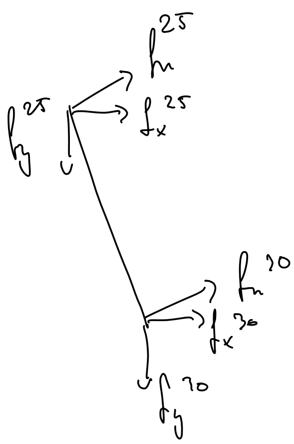
$$= -t\left[ p_{25} s + \left(\frac{\Delta p}{L^e} - \frac{p_{25}}{L^e}\right)\frac{s^2}{2} - \frac{\Delta p \, s^3}{3 L^{e\,2}} \right]_0^{L^e}$$

$$= -t\left( p_{25} L^e + \frac{\Delta p L^e}{2} - \frac{p_{25} L^e}{2} - \frac{\Delta p L^e}{3} \right)$$

$$= -t\left( \frac{p_{25}}{2} \cdot L^e + \frac{\Delta p \cdot L^e}{6} \right)$$

$$f_n^{30} = -\int_0^{L^e} \left(p_{25} + \frac{\Delta p}{L^e} s\right) \frac{s}{L^e} t \, ds = \ldots = -t\left( \frac{p_{25} \cdot L^e}{2} + \frac{\Delta p \cdot L^e}{3} \right)$$

0.5p




$f_x^{25} = f_n^{25} \cdot \cos\theta = -32.4 \text{ kN} \rightarrow$ dof 49

$f_y^{25} = -f_n^{25} \sin\theta = 5.72 \text{ kN} \rightarrow$ dof 50

$f_x^{30} = f_n^{30} \cos\theta = -35.3 \text{ kN} \rightarrow$ dof 59

$f_y^{30} = -f_n^{30} \sin\theta = 6.24 \text{ kN} \rightarrow$ dof 60

0.5p

Alternative solution:

Start from specifying $t = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$ in terms of

$p$ :

$$\mathbb{t}(y=y_{25}) = \begin{bmatrix} -p_{25}\cos\theta \\ p_{25}\sin\theta \end{bmatrix} = \begin{bmatrix} t_x^{25} \\ t_y^{25} \end{bmatrix}$$

$$\mathbb{t}(y=y_{30}) = \begin{bmatrix} -p_{30}\cos\theta \\ p_{25}\sin\theta \end{bmatrix} = \begin{bmatrix} t_x^{30} \\ t_y^{30} \end{bmatrix}$$

Both $t_x$ & $t_y$ will vary linearly between node 25 & node 30

$$\mathbb{f}_b^e = \int_{L^e} \mathbb{N}^{eT}\mathbb{t}\, t\, dL = \dots = \int_0^{L^e} \begin{bmatrix} N_{25} & 0 \\ 0 & N_{25} \\ N_{30} & 0 \\ 0 & N_{30} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} t\, dL$$

$$= \int_{L^e} \begin{bmatrix} N_{25} t_x \\ N_{25} t_y \\ N_{30} t_x \\ N_{30} t_y \end{bmatrix} t\, dL \qquad \Delta t_x = t_x^{30} - t_x^{25}$$

$$\Delta t_y = t_y^{30} - t_y^{25}$$

Example:

$$\int_{L^e} N_{25} t_x\, t\, dL = \int_0^S \left(1 - \frac{s}{L^e}\right)\left(t_x^{25} + \frac{\Delta t_x}{L^e} s\right) t\, ds$$

$$= \dots = -t\left(\frac{p_{25}}{2}L^e + \frac{\Delta p \cdot L^e}{6}\right)\cos\theta$$