MHA021 Finite element method
Exam 2023-01-12, 08:30-12:30

Examiner: Jim Brouzoulis,

Instructor: Martin Fagerström, Phone 772 1300. Instructor will visit the exam around 09:30 and 11:30.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on Feb. 2nd at the latest.

Review: It will be possible to review the grading at the Division of Dynamics (3rd floor, Mech eng. building) on Wednesday Feb 1st 12.20-13.00.

Aids: No aids. **Note**: A formula sheet is available on the computers next to this thesis.

Exam instructions

Please note that the solutions to some of the problems require or benefit from being solved by use of MATLAB and CALFEM. Be extra careful to read the instructions for these problems.

The CALFEM finite element files are provided under the directory `C:\__Exam__\assignments`. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__\assignments`. Please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid possible conflicts with the built-in MATLAB-function `extract.m`).

Please also note that we will collect all files saved under the directory `C:\__Exam__\assignments\MHA021` and subdirectories. This in order to be able to review these files during the exam corrections. Therefore, it is very important that you save all your files under `C:\__Exam__\assignments\MHA021` in the appropriate subdirectories created for each problem. **It is also absolutely necessary that you write the name of your computer on the cover page for the exam! Please also write your anonymous exam code inside each MATLAB file.**

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

Problem 1

Consider heat flow through a thermos where a large part can be considered cylindrical with inner radius $r_{\text{in}} = 40$ mm and outer radius $r_{\text{out}} = 48$ mm. In this cylindrical section, the heat flow can be considered as 1D in a polar coordinate system. If the temperature on the inner wall is T_{i} and the heat transfer at the outside is considered convective with transfer coefficient $\alpha = 6\text{W/ m}^2K$, the strong form of the 1D heat flow can be written as

$$\begin{cases} -\dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left[k\, r\, \dfrac{\mathrm{d}T(r)}{\mathrm{d}r}\right] = 0 \qquad r_{\text{i}} \leq r \leq r_{\text{o}} \\ T(r_{\text{in}}) = T_{\text{in}} \\ q(r_{\text{out}}) = \alpha\left(T(r_{\text{out}}) - T_{\text{out}}\right) \end{cases}$$

where the inside temperature is $T_{\text{in}} = 50$ °C and outside temperature $T_{\text{out}} = 20$ °C, and $q(r) = -k\,\dfrac{\mathrm{d}T(r)}{\mathrm{d}r}$ is the heat flux according to Fourier's law. The wall of the thermos consists of three layers, the inner and outer layers are made of aluminum with conductivity $k_{\text{al}} = 240$ W/m K, and the middle (insulating) layer is made from a newly designed material with conductivity $k_{\text{new}} = 0.02$ W/m K. The thickness of the aluminum layers is 2 mm and the insulating layer is 4 mm thick.

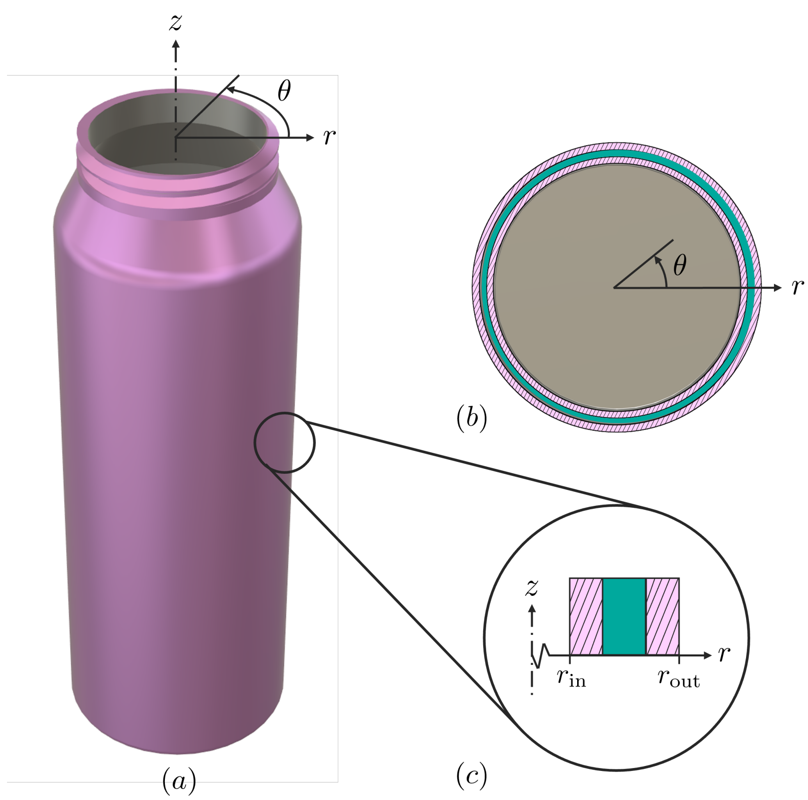


Figure 1: Illustration of the problem. (a) Thermos with a cylindrical coordinate system, (b) top view of the thermos, and (c) a section through the thermos wall showing the two aluminum layers and the insulating (green) layer in between.

Tasks:

(a) From the strong form above, **Derive the weak form and the global FE formulation (using the Galerkin method): (2.0p)**
In your derivation, make sure to clearly show the structure of all introduced matrices etc.

(b) **Derive the local FE formulation** and show that using linear approximation the element stiffness matrix over an element with coordinates r_i and r_{i+1} becomes

$$\boldsymbol{K}^e = \frac{k^e\,(r_{i+1} + r_i)}{2\,L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where $L^e = r_{i+1} - r_i$ is the element length. **(2.0p)**

(c) **Write a FE script that implements and solves the FE form derived in the previous sub-tasks.** then compute the heat flux on the outside wall of the thermos. Use three elements, one for each material layer. In this task you are free to use any CALFEM-routines you like. **(2.0p)**

Problem 2

Consider a concrete hot-water pipeline with an internal centrally placed cylindrical pipe for the water, cf. the rectangular cross-section with a circular hole sketched in Figure 2 (right). It is assumed that no heat is transferred along the pipeline, whereby a simplification to a 2D heat flow problem can be made.

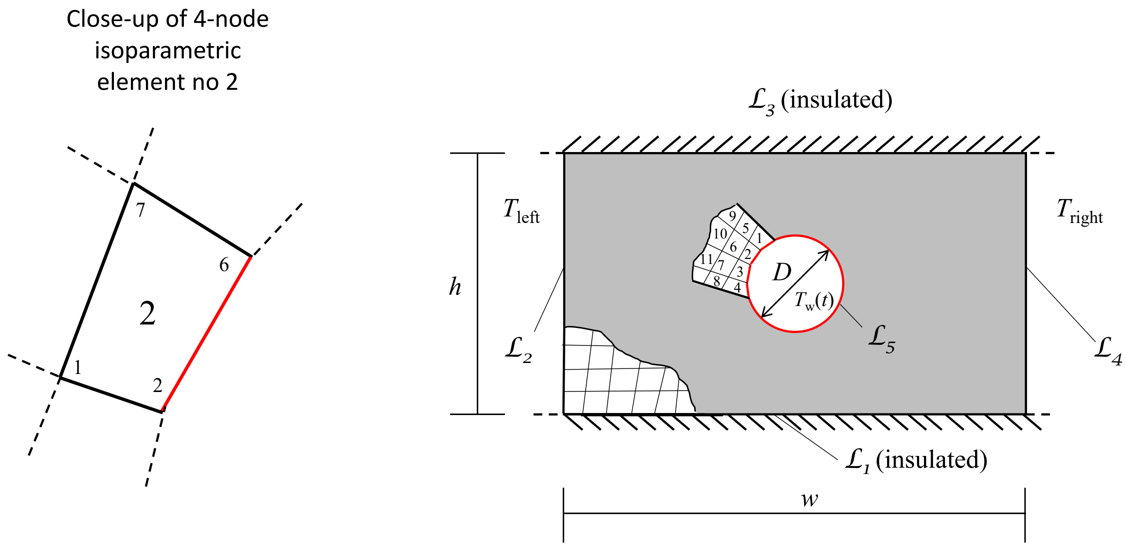


Figure 2: (left): Close-up of element no 2 analysed in subtask (e). (right): Analysed cross-section of the hot water pipeline, indicating the FE mesh (in the lower left corner and close to the pipe).

For the whole cross-section, the boundary-value-problem for $T(x, y)$ becomes

$$\begin{aligned}
\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{q} &= 0 \quad \text{in } \Omega, \\
q_n &= 0 \quad \text{on } \mathcal{L}_1, \mathcal{L}_3 \\
q_n &= \alpha[T - T_{\text{left}}] \quad \text{on } \mathcal{L}_2, \\
q_n &= \alpha[T - T_{\text{right}}] \quad \text{on } \mathcal{L}_4, \\
q_n &= \alpha[T - T_{\text{w}}(t)] \quad \text{on } \mathcal{L}_5
\end{aligned}$$

Here, T_{left} is different from T_{right}.

The concrete is assumed to be isotropic (w.r.t heat flow) and obey Fourier's law $\boldsymbol{q} = -k\boldsymbol{\nabla}T$ with the specific heat capacity, c, and thermal conductivity, k, is assumed to be constant. The heat transfer coefficient α is given to $\alpha_{air} = 5$ W/m$^{2\circ}$C on the outer boundaries and $\alpha_w = 1,000$ W/m$^{2\circ}$C along the inner boundary to the water.

The simulation domain is to be discretised in terms of 4-node isoparametric bilinear elements (as indicated in the figure in lower left corner and close to the water pipe). In particular, we are in subtask (d) below interested in the convective element matrix contributions of element 2, cf. Figure 2 (left), with the nodal coordinates (all dimensions in cm):

$$(x_1, y_1) = (-22, 8), \quad (x_2, y_2) = (-19, 7), \quad (x_6, y_6) = (-17, 13), \quad (x_7, y_7) = (-20, 14).$$

Tasks:

(a) Derive the weak form of the problem. Make sure to insert all natural (Neumann) and mixed (Robin) boundary conditions as well as to clearly indicate boundaries where the heat flux is unknown. For simplicity, the heat flow in a "slice" of thickness $t = 1$ (m) can be considered. **(1.0p)**

(b) Derive the global finite element form of the boundary value problem on the form:

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}.$$

Make sure to clearly specify the contents of any matrices of vectors that you introduce. Also clearly define the expressions for $\boldsymbol{K}$ and $\boldsymbol{f}$ in the expression above. **(1.5p)**

(c) Compute the values of all components in the convective contribution ($\boldsymbol{K}_c^e$) to $\boldsymbol{K}$ from element 2 above. Then explain how these are assembled into $\boldsymbol{K}$. **(2.0p)**

(d) Consider another 2D heat flow problem, this time modeling hot-water flow through a number of small pipes as shown to the left in Figure 3. Also in this case, heat convection is used to model the heat transfer in each pipe as $q_n = \alpha\,(T - T_w)$ with T_w being the water temperature. Since the geometry is large and the geometry shows a repeating pattern the problem can be simplified by assuming symmetry. Regarding symmetry, there are two natural choices of symmetry regions as shown to the right of Figure 3. **For each symmetry domain, introduce boundary labels (Γ_1, Γ_2, etc.) and define all boundary conditions needed to solve the problem over each simplified domain.** The problem can be simplified further by reducing the domain of each symmetry region using more symmetry. **Reduce each symmetry domain to the smallest domain possible and state the corresponding boundary conditions for each of these two simplified domains. (1.5p)**

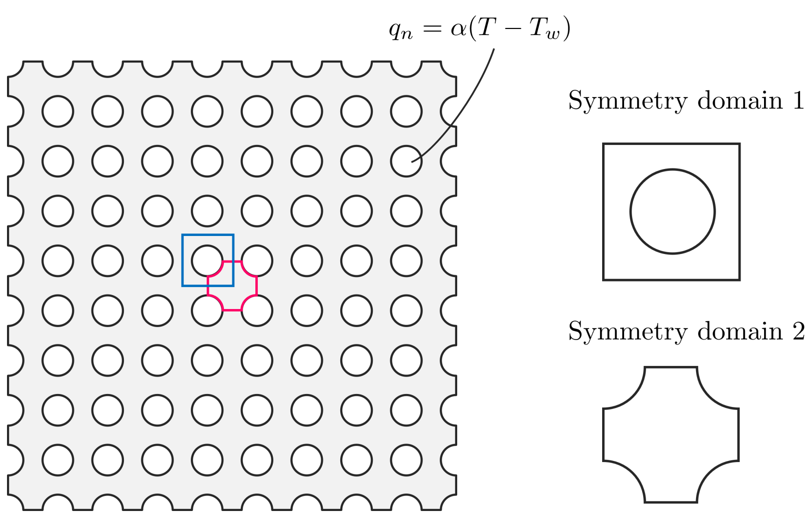
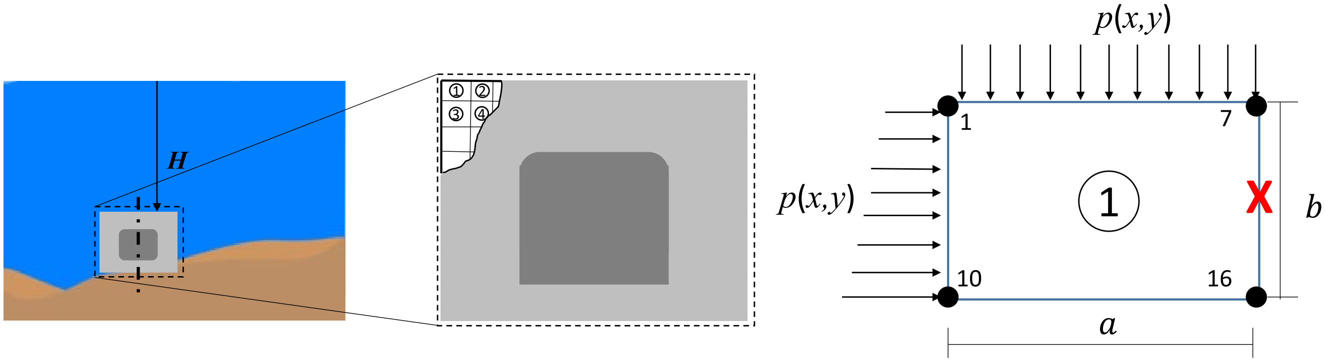


Figure 3: (left) Geometry for problem 2-subtask (d). (right) Symmetry regions.

Problem 3

Consider a concrete tunnel at the bottom of a river such that the distance from the water level to the top of the tunnel is $H = 20$ [m]. The concrete tunnel is along its outer surfaces subjected to the water pressure of magnitude $p = \rho g h$ where $\rho = 1,000\,\mathrm{kg/m^3}$ is the density of water, $g = 9.81\,\mathrm{m/s^2}$ is the standard gravity and h is the distance below the water level. Furthermore, the volume load intensity (due to gravity) for the tunnel is $\boldsymbol{b} = [0, -\rho g]^T$ [N/m^3] and the material is considered to behave linearly elastic wth material parameters $E = 30\,\mathrm{GPa}$ and $\nu = 0.2$.

The tunnel is to be analysed under a 2D plane strain simplification (if you want, you may set t = 1 m for simplicity). For this purpose, the domain is discretised by quadrilateral bilinear elements, as indicated in the upper left corner of Figure 4b and in detail in Figure 4c for the first element. In the 2D problem at hand, each node is associated with two degrees-of-freedom representing the displacement in the horizontal and vertical direction, respectively.



(a) (b) (c)

Figure 4: a) The under-water tunnel to be considered in Problem 3. b) Calculation domain incl. sketch of discretisation with bilinear quadrilateral elements. c) Close-up of element 1 with geometry information and global node numbering.

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\tilde{\nabla}\mathbf{u}\, t\, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\mathbf{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\mathbf{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\mathbf{h}\, t\, \mathrm{d}\mathcal{L}$$

where A denotes the cross section of the tunnel (light gray in Figure 3b), $\mathcal{L}_g$ denotes the parts of the boundary with prescribed displacements $\mathbf{u} = \mathbf{g}$ and $\mathcal{L}_h$ denotes the parts of the boundary with prescribed tractions $\mathbf{t} = \mathbf{h}$. Moreover, the constitutive matrix relating stresses and strains is given by

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}.$$

Tasks on the next page.

(a) Define a coordinate system for the current problem and propose a numbering scheme for the degrees-of-freedom, i.e. how the numbering of the local and global degrees-of-freedom relate to the local and global numbering of the nodes (shown only for element 1 in Figure 3c). **For your specific numbering scheme of your choice, illustrate for element 1 both the local and the global degrees-of-freedom in terms of number and orientation. (1.0p)**

(b) Derive the global FE-form of the problem using Galerkin's method. Specify the contents (in general terms) of any matrices or vectors you introduce. No explicit expressions for shape functions or their derivatives are necessary in this part. **(1.0p)**

(c) Consider an isoparametric element formulation, and **compute the Jacobian matrix $\boldsymbol{J}$ of the isoparametric mapping evaluated at the midpoint of the element edge between nodes 7 and 16** (marked with a red cross). Here, please respect the coordinate axes as introduced in subtask (a)

The dimensions of element 1 are given by $a = 1.5\,\mathrm{m}$ and $b = 1.0\,\mathrm{m}$. **(2.0p)**

(d) For the same point and element as in subtask (c), **compute the resulting in-plane stress components** given a solution that yields the local degrees of freedom as:

$$\boldsymbol{a}^e = \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,7} \\ u_{y,7} \\ u_{x,10} \\ u_{y,10} \\ u_{x,16} \\ u_{y,16} \end{bmatrix} = \begin{bmatrix} 5.1 \\ -11.0 \\ 4.0 \\ -11.1 \\ 5.0 \\ -10.0 \\ 4.0 \\ -10.0 \end{bmatrix} \mathrm{mm},$$

where $u_{x,i}$ and $u_{y,i}$ are the displacements in the $x-$ and $y-$directions, respectively. **(2.0p)**