Solutions to exam in:
MHA021 Finite Element Method
12-01-2023

Problem 1

Strong form

$$\begin{cases} -\frac{1}{r}\frac{d}{dr}\left[kr\frac{dT}{dr}\right] = 0 & r_{in} \le r \le r_{out} \\ T(r_{in}) = T_{in} \\ q(r_{out}) = \alpha\left(T(r_{out}) - T_{out}\right) \end{cases}$$

a) Weak form:

- multiply both sides with r $\Rightarrow -\frac{d}{dr}\left[kr\frac{dT}{dr}\right] = 0$
- multiply with an arbitrary test function $v(r)$ & integrate over the domain $\Rightarrow$

$$-\int_{r_{in}}^{r_{out}} v(r)\frac{d}{dr}\left[kr\frac{dT}{dr}\right]dr = \int_{r_{in}}^{r_{out}} \frac{dv}{dr}\left[kr\frac{dT}{dr}\right]dr - \left[v(r)\,r\underbrace{k\frac{dT}{dr}}_{-q(r)}\right]_{r_{in}}^{r_{out}} \qquad (*)$$

- Expand the boundary term

$$\left[v(r)\,r\,q(r)\right]_{r_{in}}^{r_{out}} = v(r_{out})\,r_{out}\underbrace{q(r_{out})}_{\alpha\left(T(r_{out}) - T_{out}\right)} - v(r_{in})\,r_{in}\,q(r_{in}) \Rightarrow \text{insert into} (*)$$

$$\Rightarrow \begin{cases} \int_{r_{in}}^{r_{out}} \frac{dv}{dr}kr\frac{dT}{dr}dr + v(r_{out})\,r_{out}\,\alpha\,T(r_{out}) = v(r_{out})\,r_{out}\,\alpha\,T_{out} - v(r_{in})\,r_{in}\,q(r_{in}) \\ T(r_{in}) = T_{in} \end{cases}$$

which is the weak form

b) Introduce an FE-approximation for $T(r) \approx T_h = \mathbb{N}\mathbf{a}$ and $v(r) = \mathbb{N}\mathbf{c}$, where $\mathbb{N} = [N_1, \dots N_n]$ is a vector of shape functions, $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ contains the dofs, and $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is a vector of arbitrary coefficients.

$\Rightarrow \quad \frac{dT}{dr} = \frac{d\mathbb{N}}{dr} a = \mathbb{B}a \quad$ and $\quad \frac{dv}{dr} = \mathbb{B}c$

$\Rightarrow$ insert into the weak form $\Rightarrow$

$$c^T \left[\underbrace{\int_{r_{in}}^{r_{out}} \mathbb{B}^T k r \mathbb{B} dr}_{\mathbb{K}} + \underbrace{\mathbb{N}^T(r_{out}) r_{out} \alpha \mathbb{N}(r_{out})}_{\mathbb{K}_c} \right] a =$$

$$c^T \left[\underbrace{\mathbb{N}^T(r_{out}) r_{out} \alpha T_{out} - \mathbb{N}^T(r_{in}) r_{in} q(r_{in})}_{f_b} \right]$$

Since this equation should hold for arbitrary c $\Rightarrow (\mathbb{K} + \mathbb{K}_c) a = f_b$

with $\mathbb{K} = \int_{r_{in}}^{r_{out}} \mathbb{B}^T k r \mathbb{B} dr$

The element stiffness matrix is obtained by splitting up the integration over each el.

$$\mathbb{K} = \int_{r_{in}}^{r_{out}} \cdot \, dr = \int_{\mathcal{L}^1} \cdot \, dr + \ldots + \underbrace{\int_{\mathcal{L}^e} \cdot \, dr}_{\mathbb{K}^e} + \ldots + \int_{\mathcal{L}^N} \cdot \, dr$$

where $\mathbb{K}^e = \int_{\mathcal{L}^e} \mathbb{B}^T k r \mathbb{B} dr = \int_{r_i}^{r_{i+1}} \mathbb{B}^T k r \mathbb{B} dr = \left\{ \begin{matrix} \text{linear shape} \\ \text{functions} \end{matrix} \right\}$

$$\Rightarrow \left.\begin{matrix} \frac{dN_1^e}{dr} = -\frac{1}{L^e} \\ \frac{dN_2^e}{dr} = \frac{1}{L^e} \end{matrix}\right\} \Rightarrow \mathbb{B} = \frac{1}{L^e}\begin{bmatrix} -1 & 1 \end{bmatrix} \Rightarrow \mathbb{K}^e = \frac{k}{(L^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underbrace{\int_{r_i}^{r_{i+1}} r \, dr}$$

$$\frac{r_{i+1}^2 - r_i^2}{2} = \frac{(r_{i+1} + r_i) L^e}{2}$$

$$= \frac{k(r_{i+1} + r_i)}{2L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} //$$

c) See Matlab code for implementation.
When the solution vector has been determined the heat flux is evaluated using the convection bnd.c.

$$q(r_{out}) = \alpha (T(r_{out}) - T_{out}) = \alpha (a(\text{end}) - T_{out}) \approx 78 \, W/m^2 \; //$$

↑ vector in Matlab

For MHA021, ignore all terms in red (they corresponds to transient heat flow which is only for VSM167)

P2

a) Strong form equation:

$$\rho c \dot{T} + \nabla^T \boldsymbol{q} = 0 \quad \text{in } \Omega \quad \text{for } 0 \le \tau \le \tau_{end}$$

Multiply with weight function v & integrate.

$$\int_A v \rho c \dot{T} t \, dA + \underbrace{\int_A v \nabla^T \boldsymbol{q} \, t \, dA}_{} = 0$$

$$\int_A v \nabla^T \boldsymbol{q} \, t \, dA = \int_A \nabla^T (v t \boldsymbol{q}) \, dA - \int_A (\nabla^T v) \boldsymbol{q} \, t \, dA$$

$$\Rightarrow \int_A v \rho c \dot{T} t \, dA - \int_A (\nabla^T v) \boldsymbol{q} \, t \, dA = - \underbrace{\int_A \nabla^T (v t \boldsymbol{q}) \, dA}_{\oint_{\mathcal{L}} v t q_n \, d\mathcal{L}}$$

$$\boldsymbol{q} = -\mathbb{D} \nabla T$$

$\Rightarrow$ Find T such that:

$$\int_A v \rho c \dot{T} t \, dA + \int_A (\nabla^T v) \mathbb{D} t \, \nabla T \, dA = - \oint_{\mathcal{L}_1 \cup \mathcal{L}_3} v t \cdot 0 \, d\mathcal{L} - \int_{\mathcal{L}_2} v t \alpha (T - T_{left}) \, d\mathcal{L}$$

$$- \int_{\mathcal{L}_4} v t \alpha (T - T_{right}) \, d\mathcal{L}$$

$$- \int_{\mathcal{L}_5} v t \alpha (T - T_w) \, d\mathcal{L}$$

$$T(x, y, 0) = T_0$$

b) Introduce approximation for T as

$$T_h = \sum_i N_i a_i = \mathbb{N} \mathbb{a} \quad \text{with} \quad \mathbb{N} = [N_1 \; N_2 \; .. \; N_n]$$

$$\mathbb{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\dot{T}_h = \mathbb{N} \dot{\mathbb{a}}$$

$$\nabla T_h = \nabla \mathbb{N} \mathbb{a} = \nabla \mathbb{N} \, \mathbb{a} = \mathbb{B} \mathbb{a}$$

$$\Rightarrow \mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

Galerkins method $\Rightarrow$

$$v = \mathbb{N} \mathbb{c}$$

$$\nabla v = \mathbb{B} \mathbb{c} \quad , \quad \nabla^T v = (\nabla v)^T = \mathbb{c}^T \mathbb{B}^T$$

Insert all above in weak form:

$$\mathbb{c}^T \int_A \rho c t \, \mathbb{N}^T \mathbb{N} \, dA \, \dot{\mathbb{a}} + \mathbb{c}^T \int_A \mathbb{B}^T \mathbb{D} t \mathbb{B} \, dA \, \mathbb{a} = \mathbb{c}^T \Big[-\int_{\mathcal{L}_2} t \alpha \mathbb{N}^T (\mathbb{N} \mathbb{a} - T_{left}) \, d\mathcal{L}$$

$$- \int_{\mathcal{L}_4} t \alpha \mathbb{N}^T (\mathbb{N} \mathbb{a} - T_{right}) \, d\mathcal{L}$$

$$- \int_{\mathcal{L}_5} t \alpha \mathbb{N}^T (\mathbb{N} \mathbb{a} - T_w) \, d\mathcal{L} \Big]$$

$\Rightarrow$

$$\mathbb{c}^T \int_A \rho c t \, \mathbb{N}^T \mathbb{N} \, dA \, \dot{\mathbb{a}} + \mathbb{c}^T \left[\int_A \mathbb{B}^T \mathbb{D} t \mathbb{B} \, dA + \int_{\mathcal{L}_2 \cup \mathcal{L}_4 \cup \mathcal{L}_5} t \alpha \mathbb{N}^T \mathbb{N} \, d\mathcal{L} \right] \mathbb{a} = \mathbb{c}^T \Big[\int_{\mathcal{L}_2} t \alpha T_{left} \mathbb{N}^T d\mathcal{L}$$

$$+ \int_{\mathcal{L}_4} t \alpha T_{right} \mathbb{N}^T d\mathcal{L}$$

$$+ \int_{\mathcal{L}_5} t \alpha T_w \mathbb{N}^T d\mathcal{L} \Big]$$

$\Longleftrightarrow$

$$\mathbb{c}^T \mathbb{C}\dot{\mathbb{a}} + \mathbb{c}^T \mathbb{K}\mathbb{a} = \mathbb{c}^T \mathbb{f}$$

$$\Rightarrow \mathbb{c}^T \left[\mathbb{C}\dot{\mathbb{a}} + \mathbb{K}\mathbb{a} - \mathbb{f} \right] = 0$$

$$\Rightarrow \mathbb{C}_1 \dot{\mathbb{a}} + \mathbb{K}\mathbb{a} = \mathbb{f} \quad \text{with} \quad \mathbb{C}_1 = \int_A \rho c t \, \mathbb{N}^T \mathbb{N} \, dA$$

$$\mathbb{K} = \int_A \mathbb{B}^T \mathbb{D} t \mathbb{B} \, dA + \int_{\mathcal{L}_2 \cup \mathcal{L}_4 \cup \mathcal{L}_5} t \alpha \mathbb{N}^T \mathbb{N} \, d\mathcal{L}$$

$$\mathbb{f} = \int_{\mathcal{L}_2} t \alpha T_{left} \mathbb{N}^T \, d\mathcal{L} + \int_{\mathcal{L}_4} t \alpha T_{right} \mathbb{N}^T \, d\mathcal{L}$$

$$+ \int_{\mathcal{L}_5} t \alpha T_w \mathbb{N}^T \, d\mathcal{L}$$

c) $\mathbb{K}_c^e = \int_{\mathcal{L}_{26}} t\alpha\, \mathbb{N}^{e^T} \mathbb{N}^e \, d\mathcal{L}$

edge between nodes 2 & 6

$\mathbb{N}^e = [N_2^e \;\; N_6^e]$

$$\Rightarrow \mathbb{K}_c^e = \int_{\mathcal{L}_{26}} t\alpha \begin{bmatrix} N_2^e N_2^e & N_2^e N_6^e \\ N_6^e N_2^e & N_6^e N_6^e \end{bmatrix} d\mathcal{L}$$

Note $\int_{\mathcal{L}_{26}} N_2^e N_2^e d\mathcal{L} = \int_{\mathcal{L}_{26}} N_6^e N_6^e \, d\mathcal{L} = \ldots = \frac{L_{26}^e}{3}$

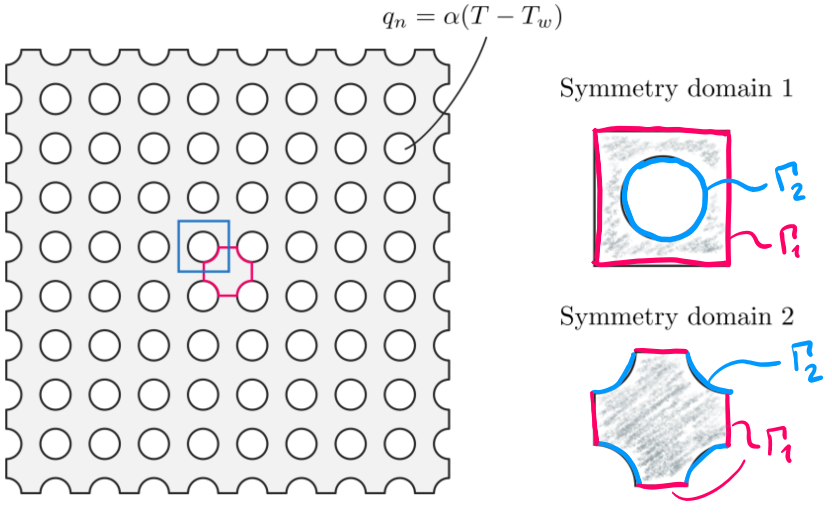
$$\int_{\mathcal{L}_{26}} N_2^e N_6^e \, d\mathcal{L} = \ldots = \frac{L_{26}^e}{6}$$

$$\mathbb{K}_c^e = \frac{t\alpha L_{26}^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad L_{26}^e = \sqrt{(x_6 - x_2)^2 + (y_6 - y_2)^2}$$

$$\mathbb{K}_c^e = \{\text{insert values}\} = \begin{bmatrix} 81.92 & 40.96 \\ 40.96 & 81.92 \end{bmatrix} \text{W/°C}$$

Problem 2

d)



Boundary conditions:

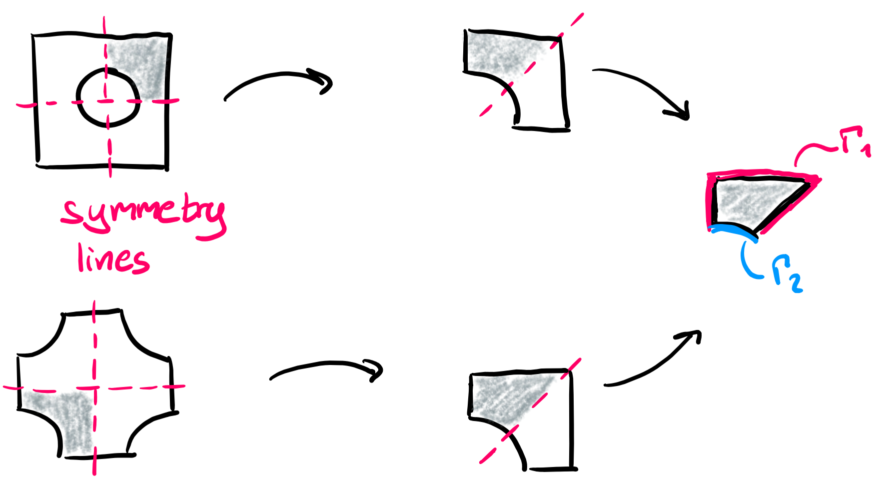
$q_n = 0$ on Γ_1

$q_n = \alpha(T - T_w)$ on Γ_2

$q_n = 0$ on Γ_1

$q_n = \alpha(T - T_w)$ on Γ_2

Smallest domain



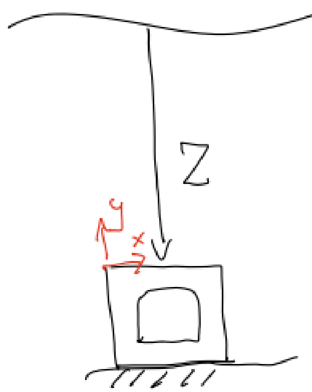
Again, we have

$q_n = 0$ on Γ_1

$q_n = \alpha(T - T_w)$ on Γ_2

P3

a)

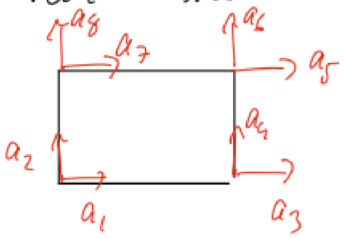


For node i

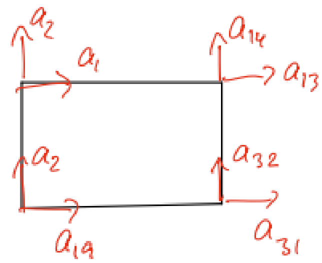
$u_{x,i} = a_{2i-1}$

$u_{y,i} = a_{2i}$

Local numbering



Global numbering



$$a = \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ \vdots \\ u_{n,x} \\ u_{n,y} \end{bmatrix}$$

b) Approximate u

$$u \approx u_h = \sum_i N_i a_i = N a \quad , \quad N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix}$$

$$\varepsilon = \tilde{\nabla} N a = B a \quad , \quad B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Galerkin

$v = N c$

$(\tilde{\nabla} v)^T = c^T B^T$

Insert in weak form:

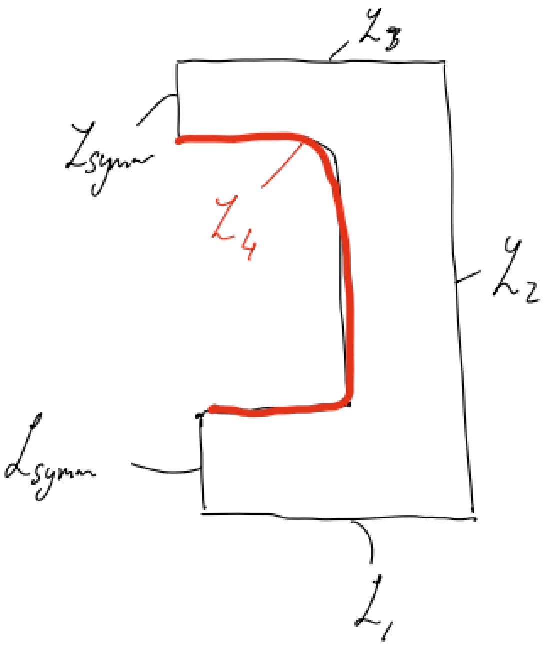
$$c^T \underbrace{\int_A \mathbb{B}^T \mathbb{D} t \mathbb{B}\, dA}_{\mathbb{K}}\, a = c^T \underbrace{\int_A \mathbb{N}^T \mathbb{b}\, t\, dA}_{f_l} + c^T \underbrace{\int_{\mathcal{L}_g} \mathbb{N}^T \mathbb{t}\, t\, d\mathcal{L}}_{f_b^g} + c^T \underbrace{\int_{\mathcal{L}_h} \mathbb{N}^T \mathbb{h}\, t\, d\mathcal{L}}_{f_b^h}$$

$\Rightarrow$

$$c^T \left(\mathbb{K} a - f_l - f_b^g - f_b^h \right) = 0$$

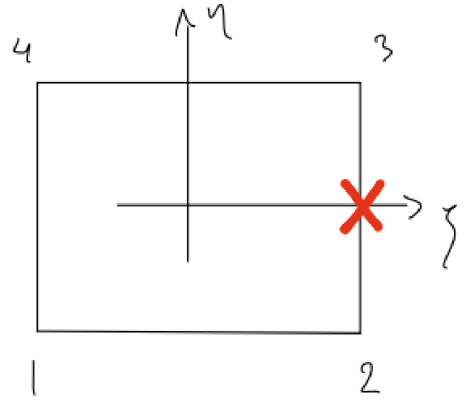
$$\Rightarrow \mathbb{K} a = f_l + f_b^g + f_b^h$$

For the half domain

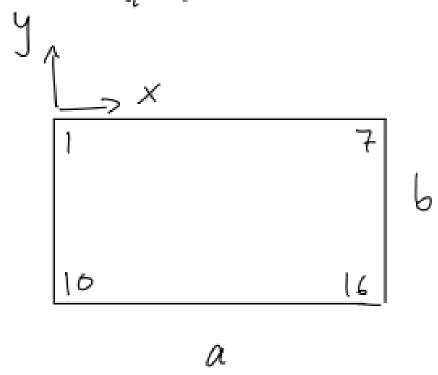


$$\boxed{\begin{array}{l} \mathbb{K} a = f_l + f_b^g + f_b^h \\ u = 0 \text{ along } \mathcal{L}_1 \\ u_x = 0 \text{ along } \mathcal{L}_{symm} \\ t_y = 0 \quad \text{— " —} \\ \text{with:} \\ f_b^h = \displaystyle\int_{\mathcal{L}_2} \mathbb{N}^T \begin{bmatrix} -p \\ 0 \end{bmatrix} t\, d\mathcal{L} + \displaystyle\int_{\mathcal{L}_3} \mathbb{N}^T \begin{bmatrix} 0 \\ -p \end{bmatrix} t\, d\mathcal{L} \end{array}}$$

5) Parent domain



Global



$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$x = N^e x^e$$

$$y = N^e y^e$$

$$x^e = \begin{bmatrix} 0 \\ a \\ a \\ 0 \end{bmatrix}$$

$$y^e = \begin{bmatrix} -b \\ -b \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N^e}{\partial \xi} x^e$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N^e}{\partial \eta} x^e$$

$$\frac{\partial N^e}{\partial \xi} = \begin{bmatrix} \frac{1}{4}(\eta-1) & -\frac{1}{4}(\eta-1) & \frac{1}{4}(\eta+1) & -\frac{1}{4}(\eta+1) \end{bmatrix}$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial N^e}{\partial \xi} y^e$$

$$\frac{\partial N^e}{\partial \xi}(\xi = 1, \eta = 0) = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial N^e}{\partial \eta} y^e$$

$$\frac{\partial N^e}{\partial \eta} = \begin{bmatrix} \frac{1}{4}(\xi-1) & -\frac{1}{4}(\xi+1) & \frac{1}{4}(\xi+1) & -\frac{1}{4}(\xi-1) \end{bmatrix}$$

$$\frac{\partial N^e}{\partial \eta}(\xi = 1, \eta = 0) = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\Rightarrow J = \{\text{MATLAB}\} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.5 \end{bmatrix}$$

d)

$\sigma = \mathbb{D}\,\epsilon$

$\epsilon = \mathbb{B}^e_{el}\,a^e$

Note! $a^e = \begin{bmatrix} u_{10,x} \\ u_{10,y} \\ u_{16,x} \\ u_{16,y} \\ u_{7,x} \\ u_{7,y} \\ u_{1,x} \\ u_{1,y} \end{bmatrix}$

$$\mathbb{B}^e_{th} = \left(\mathbb{J}^T\right)^{-1} \begin{bmatrix} \frac{\partial N^e}{\partial \xi} \\ \frac{\partial N^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} & \frac{\partial N_4^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_4^e}{\partial y} \end{bmatrix}$$

$$\mathbb{B}^e_{el} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \cdots & \frac{\partial N_4^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & & 0 & \frac{\partial N_4^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & & \frac{\partial N_4^e}{\partial x} & \frac{\partial N_4^e}{\partial y} \end{bmatrix}$$

$\mathbb{D} = \text{hooke}(2, E, \nu)$

$\mathbb{D} = \mathbb{D}([1\ 2\ 4], [1\ 2\ 4])$

$$\sigma = \mathbb{D}\,\mathbb{B}^e a^e = \ldots = \begin{pmatrix} -32.5 \\ -42.5 \\ -0.4 \end{pmatrix} \text{MPa}$$