$$\int_A (\tilde{\nabla}\mathbf{v})^T \mathbf{D}\, \tilde{\nabla}\mathbf{u}\, t\, dA = \int_A \mathbf{v}^T \mathbf{b}\, t\, dA + \int_{\mathcal{L}_g} \mathbf{v}^T \mathbf{t}\, t\, d\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^T \mathbf{h}\, t\, d\mathcal{L}$$

FE-approx:

0.5p

$$\mathbf{u} = \mathbf{N}\mathbf{a}, \quad \mathbf{N} = \begin{bmatrix} N_1 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & \cdots & 0 & N_n \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ \vdots \\ u_{xn} \\ u_{yn} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{bmatrix}$$

$$\tilde{\nabla}\mathbf{u} = \mathbf{B}\mathbf{a}$$

$$\mathbf{B} = \tilde{\nabla}\mathbf{N} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Galerkin

0.5p

$$\mathbf{v} = \mathbf{N}\mathbf{c}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2n} \end{bmatrix} \text{ are arbitrary coefficients}$$

$$\mathbf{v}^T = \mathbf{c}^T \mathbf{N}^T$$

$$\tilde{\nabla}\mathbf{v} = \mathbf{B}\mathbf{c}$$

$$(\tilde{\nabla}\mathbf{v})^T = \mathbf{c}^T \mathbf{B}^T$$

Insert FE-approx in weak form:

$$\mathbf{c}^T \underbrace{\int_A \mathbf{B}^T \mathbf{D}\, t\, \mathbf{B}\, dA}_{\mathbf{K}}\, \mathbf{a} = \mathbf{c}^T \left[\underbrace{\int_A \mathbf{N}^T \mathbf{b}\, t\, dA}_{\mathbf{f}_l} + \underbrace{\int_{\mathcal{L}_g} \mathbf{N}^T \mathbf{t}\, t\, d\mathcal{L}}_{\mathbf{f}_b^g} + \underbrace{\int_{\mathcal{L}_h} \mathbf{N}^T \mathbf{h}\, t\, d\mathcal{L}}_{\mathbf{f}_b^h} \right]$$

0.5p

$\Leftrightarrow$

$$\mathbf{c}^T \left(\mathbf{K}\mathbf{a} - \mathbf{f}_l - \mathbf{f}_b^g - \mathbf{f}_b^h \right) = 0$$

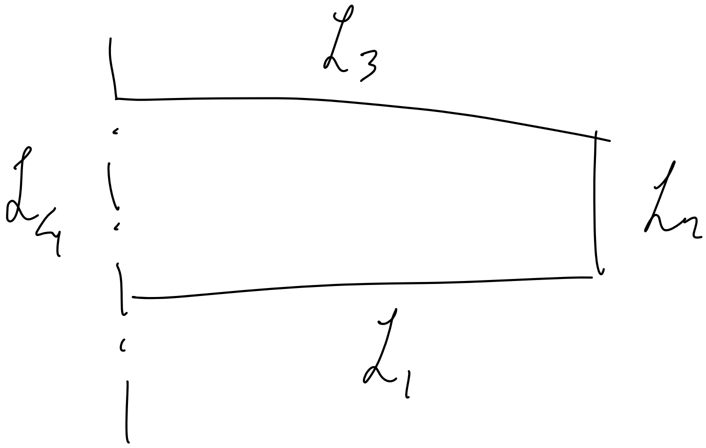
$\mathbf{c}$ - arbitrary $\Rightarrow$

$$\mathbf{K}\mathbf{a} - \mathbf{f}_l - \mathbf{f}_b^g - \mathbf{f}_b^h = 0$$

$\Leftrightarrow$ $\underline{\underline{\mathbf{K}\mathbf{a} = \mathbf{f}}}$, with $\mathbf{f} = \mathbf{f}_l + \mathbf{f}_b$

Problem specific boundary conditions

0.5p



On $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$:

$\mathbf{t} = \mathbf{h} = \mathbf{0}$

On $\mathcal{L}_4$:

$u_x = 0$

$t_y = 0$

Note! For this to be solvable, also the

vertical displacement need to be constrained in at least one point / node

Redistribute points from d_y if solved here

b) Element local vector:

$$\mathbb{f}_l^e = \int_{A^e} \mathbb{N}^{e^T} \mathbb{b} \, t \, dA$$

$$\mathbb{N}^e = \begin{bmatrix} N_1^e & 0 & \cdots & N_4^e & 0 \\ 0 & N_1^e & & 0 & N_4^e \end{bmatrix}$$

$$\mathbb{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} \rho \omega^2 x \\ -\rho g \end{bmatrix}$$

$$\int_{A^e} \mathbb{N}^{e^T} \mathbb{b} \, t \, dA = \int_{-1}^{1} \int_{-1}^{1} \mathbb{N}^{e^T} \mathbb{b} \, t \det(\mathbb{J}) \, d\xi \, d\eta$$

0.5p

* $\mathbb{b} = \begin{bmatrix} \rho \omega x \\ -\rho g \end{bmatrix}$ with $x = \mathbb{N}^e \cancel{x}^e$ where $\mathbb{N}^e = [N_1^e \; N_2^e \; N_3^e \; N_4^e]$, $x^e = \begin{bmatrix} x_9 \\ x_5 \\ x_{18} \\ x_{17} \end{bmatrix}$ (~ linear in ξ)

* $\mathbb{N}^{eT}$ from formula sheet

1.0p from $\partial\xi$

* $J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$

$\frac{\partial x}{\partial \xi} = \frac{\partial \mathbb{N}^e}{\partial \xi} \cdot \cancel{x}^e = \ldots = \frac{1}{2} \cdot 1 \cdot 10^{-2}$ (const)

$\frac{\partial x}{\partial \eta} = \frac{\partial \mathbb{N}^e}{\partial \eta} \cancel{x}^e = \ldots = 0$ (constant)

$\frac{\partial y}{\partial \xi} = \frac{\partial \mathbb{N}^e}{\partial \xi} y^e = \ldots$ linear in η

$\frac{\partial y}{\partial \eta} = \frac{\partial \mathbb{N}^e}{\partial \eta} y^e$ -- linear in ξ

* $\det(J) = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \cancel{\frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi}} \rightarrow 0$

(linear in η)

Here, it is suggested to use numerical integration. Shape functions have linear term in ξ & η + bilinear term $(\xi\eta)$, the load vector varies linearly and the determinant is linear in ξ . Altogether max order 3 in ξ-direction & less in η-dir

0.5P

2x2 integration scheme is sufficient.

$$\approx \sum_{i=1}^{2} \sum_{j=1}^{2} N^{eT}(\xi_i,\eta_j)\, b(\xi_i,\eta_j)\, t\, \det(J(\xi_i,\eta_j))\, H_i H_j$$

0.5P

Implemented in MATLAB:

$$f_l^e = \begin{bmatrix} 5.548 \\ -165.8 \\ 6.568 \\ -178.54 \\ 6.568 \\ -178.54 \\ 5.548 \\ -165.8 \end{bmatrix} \cdot 10^{-2}\, t$$

not given in the assignment

c)

$$f_l^e = \begin{bmatrix} f_{l1}^e \\ f_{l2}^e \\ f_{l3}^e \\ f_{l4}^e \\ f_{l5}^e \\ f_{l6}^e \\ f_{l7}^e \\ f_{l8}^e \end{bmatrix} \begin{matrix} \to f_{l7} \\ \to f_{l8} \\ \to f_{l9} \\ \to f_{l10} \\ \to f_{l35} \\ \to f_{l36} \\ \to f_{l33} \\ \to f_{l34} \end{matrix}$$

Numbering scheme
node i
$u_{x,i} \leftrightarrow a_{2i-1}$
$u_{y,i} \leftrightarrow a_{2i}$

All correct 0.5p

d)

$$\int_{A^e} dA = \int_{-1}^{1}\int_{-1}^{1} \det(J)\, d\xi\, d\eta$$

0.5p

$\det(J)$ linear $\Rightarrow$ 1 integr. point enough $\Rightarrow$

0.5p

$$\int_{-1}^{1}\int_{-1}^{1} \det(J)\, d\xi\, d\eta = \sum_{i=1}^{1}\sum_{j=1}^{1} \det(J(\xi_i, \eta_j))\, H_i \cdot H_j$$

$$= 9.0 \cdot 10^{-5}\ m^2$$

See MATLAB implementation

Computation of J & det(J) gives 1.0p

Unclear if this is included in the solution here or in b), Consider both.