VSM167 Finite element method - basics
Re-exam 2022-04-13, 14:00-18:00

Instructor: Jim Brouzoulis (phone 070-261 5494). The instructor will visit the exam around 15:00 and 17:00.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on 4 May 2022 the latest.

Review: It will be possible to review the grading at the Division of Material and Computational Mechanics. Please contact Martin Fagerström to schedule an appointment for reviewing the exam corrections.

Permissible aids: Chalmers type approved pocket calculator and MATLAB with the CALFEM toolbox. On the computer, you can find the CALFEM manual (excluding the examples section) and the CALFEM finite element method functions. **Note**: A formula sheet is also appended to this exam paper.

Exam instructions

Please note that the solutions to some of the problems require or benefit from being solved by use of MATLAB and CALFEM. Be extra careful to read the instructions for these problems.

The CALFEM finite element files are provided under the directory `C:\__Exam__`. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__`. Please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid possible conflicts with the built-in MATLAB-function `extract.m`).

Please also note that we will collect all files saved under the directory `C:\__Exam__\VSM167` and subdirectories. This in order to be able to review these files during the exam corrections. Therefore, it is very important that you save all your files under `C:\__Exam__\VSM167` in the appropriate subdirectories created for each problem. **It is also absolutely necessary that you write the name of your computer on the cover page for the exam! Please also write your anonymous exam code inside each file.**

Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

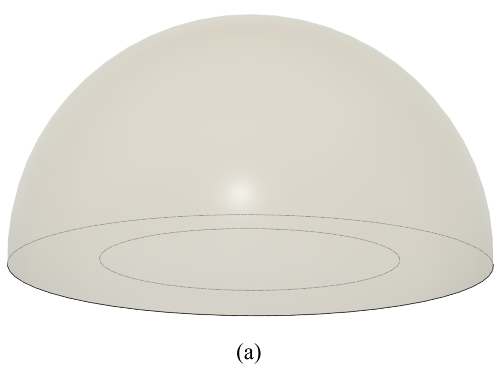
Problem 1

An igloo, as shown in Figure 1a, has the shape of a half-sphere with inner and outer radius r_{in} and r_{out}, respectively. Let T_{in} denote the air temperature on the inside, while T_{out} is the air temperature of the outside. It is assumed that the heat transfer between the air and the wall is convective, both on the inside and the outside. The convection constant is denoted α and is assumed to be same on both inside and outside.

The wall temperature $T(r)$ can then be estimated as the solution to the boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r}\left[k\,r^2\,\dfrac{\mathrm{d}T(r)}{\mathrm{d}r}\right] = 0 \qquad r_{\text{in}} < r < r_{\text{out}} \\ q(r_{\text{in}}) = -\alpha\,(T(r_{\text{in}}) - T_{\text{in}}) \\ q(r_{\text{out}}) = \alpha\,(T(r_{\text{out}}) - T_{\text{out}}) \end{cases} \tag{1}$$

Here, k is the thermal conductivity and $q(r) = -k\,\dfrac{\mathrm{d}u(r)}{\mathrm{d}r}$ is the heat flux according to Fourier's law.



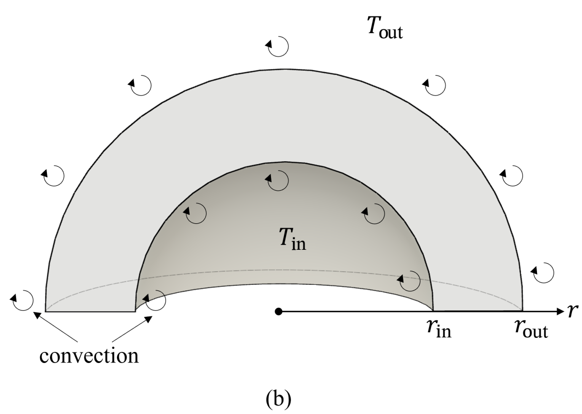


Figure 1: (a) Igloo to be analysed in Problem 1. (b) Section through the igloo.

Tasks on the next page!

Tasks:

(a) **Derive the weak formulation** of the boundary value problem in Eq. (1). (**1.0 p**)

(b) **Derive the global finite element formulation** of the weak form from subtask (a). (**1.0 p**)

(c) Consider a piece-wise linear approximation for the temperature field, $T_h \approx T$. Over an element in the interval $r_i \leq r \leq r_{i+1}$ (between nodes i and $i+1$), the shape functions are given as

$$N_i^e(r) = \frac{r_{i+1} - r}{h^e}, \qquad N_{i+1}^e(r) = \frac{r - r_i}{h^e}$$

where $h^e = r_{i+1} - r_i$. The approximation over an element is then $T_h^e = N_i^e(r)\, a_i^e + N_{i+1}^e(r)\, a_{i+1}^e$. **Show that the element stiffness (or conductivity) matrix becomes**

$$\boldsymbol{K}^e = \frac{k\left(r_{i+1}^3 - r_i^3\right)}{3(h^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

when using this approximation. (**1.0 p**)

(d) **Write a script that solves the FE problem using 2 elements and determine the temperature on the inside wall**, i.e. $T(r_{\text{in}})$. (**3.0 p**)

Start from the provided file `problem1.m` and use the stiffness matrix from subtask (c). Also, do not forget to include the convective contributions.

Use the following numerical values: $r_{\text{in}} = 1.5$ m, $r_{\text{out}} = 2.4$ m, $T_{\text{in}} = 13\,°\text{C}$, $T_{\text{out}} = -15°\text{C}$, $k = 2.25\,\frac{\text{W}}{\text{m}\,°C}$, $\alpha = 5\,\frac{\text{W}}{\text{m}^2\,°C}$ (also given in the provided MATLAB file).

Problem 2

Consider a concrete hot-water pipeline with an internal cylindrical hole for the water, cf. the rectangular cross-section sketch in Figure 2 (right). It is assumed that no heat is transferred along the pipeline whereby a simplification to a 2D heat flow problem can be made.

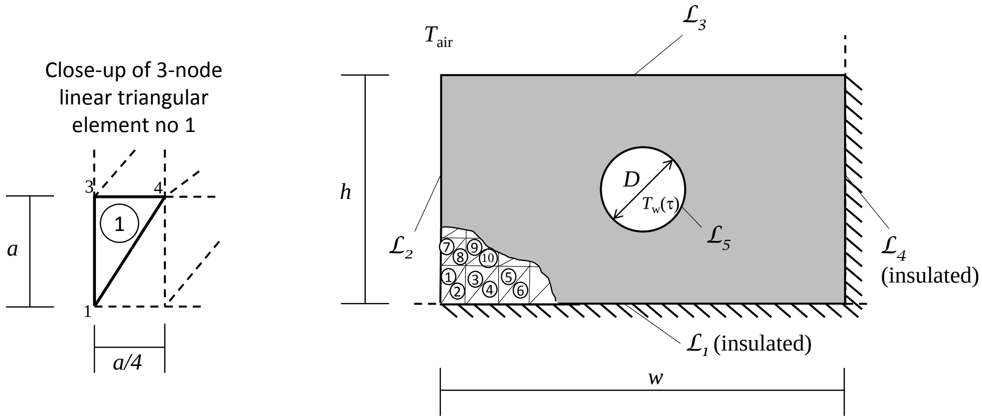


Figure 2: (left): Close-up of element no 1 analysed in task (c). (right): Analysed cross-section of the hot water pipeline where a part of the FE mesh is indicated in the lower left part.

For the whole cross-section, the boundary-initial value problem for $T(x, y, t)$ on strong form becomes

$$
\begin{aligned}
\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{q} + \rho c \dot{T} &= 0 \quad \text{in } \Omega \quad \text{for } 0 < \tau < \tau_{\text{end}}, \\
q_n &= 0 \quad \text{on } \mathcal{L}_1, \mathcal{L}_4 \\
q_n &= \alpha[T - T_{\text{air}}] \quad \text{on } \mathcal{L}_2, \mathcal{L}_3, \\
q_n &= \alpha[T - T_{\text{w}}(\tau)] \quad \text{on } \mathcal{L}_5 \\
T(x, y, 0) &= T_0(x, y).
\end{aligned}
$$

The concrete is assumed to be isotropic (w.r.t heat flow) and obey Fourier's law $\boldsymbol{q} = -k\boldsymbol{\nabla}T$ with the specific heat capacity, c, and thermal conductivity, k, assumed to be constant. For simplicity, the transfer coefficient α is assumed to be the same along all boundaries.

The cross-section is discretised in terms of 3-noded linear triangular elements (as indicated in the lower left corner). In particular, we are in tasks (c) below interested in calculating the element conductivity matrix $\boldsymbol{K}^e$ including its contribution from convection for element no 1 in the lower left corner of the domain, cf. Figure 2 (left), with the nodal coordinates:

$$(x_1, y_1) = (0, 0), \quad (x_3, y_3) = (0, a), \quad (x_4, y_4) = (a/4, a).$$

Tasks on the next page.

(a) Derive the weak form corresponding to the strong form above (for simplicity, heat flow in a "slice" of thickness $t = 1$ (m) can be considered). Be careful to explain how the boundary conditions enter into the weak form. **(2.0p)**

(b) By introducing the approximation

$$T(x, y, \tau) \approx \boldsymbol{N}\boldsymbol{a}(\tau), \ \dot{T}(x, y, \tau) \approx \boldsymbol{N}\dot{\boldsymbol{a}}(\tau)$$

in the weak form, the corresponding semi-discrete FE equation of the problem is obtained by

$$\boldsymbol{C}\dot{\boldsymbol{a}}(\tau) + (\boldsymbol{K} + \boldsymbol{K}_c)\,\boldsymbol{a}(\tau) = \boldsymbol{f}_b(\tau) + \boldsymbol{f}_l,$$

where $\boldsymbol{a}(\tau)$ contains the nodal values of the temperature at the given time τ and where $\dot{\boldsymbol{a}}(\tau)$ denotes the time derivatives of those.

Derive and give the expressions for the matrices $\boldsymbol{C}, \boldsymbol{K}$ and $\boldsymbol{K}_c$ and the vectors $\boldsymbol{f}_b(\tau)$ and $\boldsymbol{f}_l$ for the particular problem considered here. Clearly describe the contents of any additional matrices or vectors you introduce. **(2.0p)**

(c) Calculate, without using any CALFEM functions, the element contributions to $\boldsymbol{K}$ and $\boldsymbol{K}_c$ for element 1, i.e. calculate $\boldsymbol{K}^e_{①}$ and $\boldsymbol{K}^e_{c①}$. Clearly indicate how you locally number the element nodes, i.e. which local node number that is associated with which global node number.

For this task, please consider a unit thickness ($t = 1\,\text{m}$) use the following values for physical, material and geometrical parameters: **(2.0p)**

$\alpha = 10\ \text{W/(m}^2\ °\text{C)}$
$k = 2.25\,\text{W/(m °C)}$
$c = 800\,\text{J/(kg °C)}$
$\rho = 2\,400\ \text{kg/m}^3$
$a = 0.08\,\text{m}$

If you have difficulties solving task (b), you can use that

$$\boldsymbol{K}^e_{①} = \int_{A^e_{①}} \boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B}^e t\,\mathrm{d}A$$

$$\boldsymbol{K}^e_{c①} = \int_{\mathcal{L}^e_{c①}} \alpha\boldsymbol{N}^{eT}\boldsymbol{N}^e t\,\mathrm{d}\mathcal{L}$$

with $\boldsymbol{D} = k\boldsymbol{1}$ ($\boldsymbol{1}$ is a 2 x 2 identity matrix) for which you have to figure out the contents of $\boldsymbol{N}^e$ and $\boldsymbol{B}^e$ on your own.

Problem 3

Consider a hexagonal concrete part of thickness 0.75 m and with a uniform side length $L = 2$ m subjected to an external pressure $p = 0.4$ MPa as shown in Figure 3.

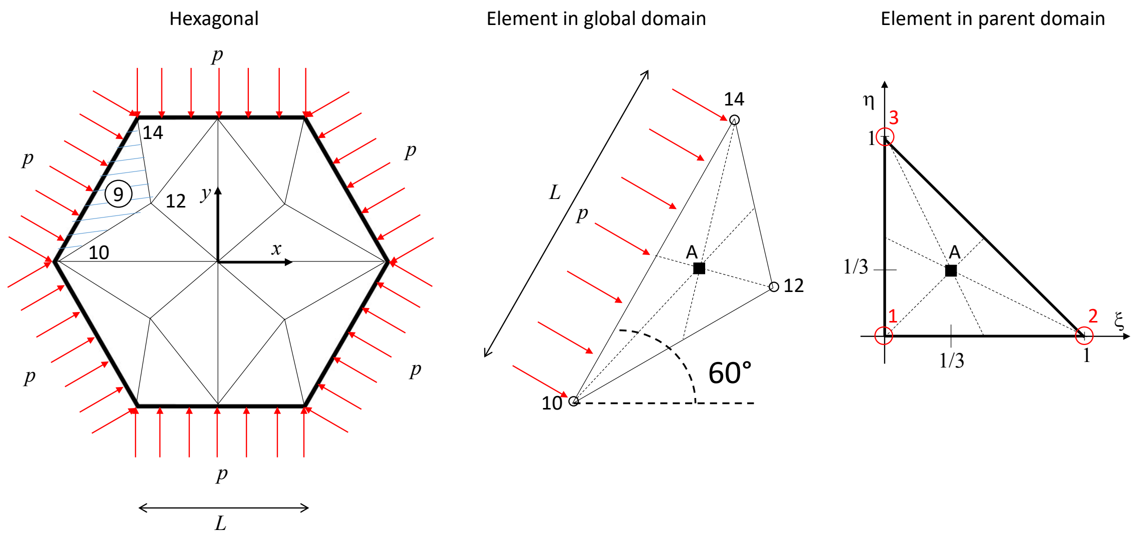


Figure 3: Left: Hexagonal part in Problem 3. Middle: Particular element (no 9) in global domain to be considered in tasks a)-c) where global node numbers are indicated in black. Left: Isoparametric three noded element in the parent domain to be considered in tasks b)-c) where local node numbers are indicated in red.

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\tilde{\nabla}\mathbf{u}\, t\, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\mathbf{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\mathbf{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\mathbf{h}\, t\, \mathrm{d}\mathcal{L}$$

for any domain A with prescribed displacements $\mathbf{u} = \mathbf{g}$ along $\mathcal{L}_g$ and prescribed tractions $\mathbf{t} = \mathbf{h}$ along $\mathcal{L}_h$.

(a) For the resulting FE-problem ($\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$), **calculate the contribution to the global load vector $\boldsymbol{f}$ from the pressure p along the boundary of the dashed element no 9** (also shown in the middle figure) considering the global node numbering in the figure. For full point, both specify the values and the positions of the non-zero contributions in the global load vector. **(2.0p)**

Continued on the next page!

(b) Just as for bilinear quadrilateral elements, an isoparametric formulation can be considered for linear triangular elements, cf. the middle and left part of Figure 3.

For such an isoparametric formulation, **calculate the Jacobian matrix J in the point A for element no 9** (corresponding to the element centre of gravity, $\xi = 1/3$, $\eta = 1/3$). **(2.0p)**

The coordinates for the three nodes are

$$x_{10} = -L, x_{12} = -2L/5, x_{14} = -L/2, y_{10} = 0, y_{12} = \sqrt{3}L/6, y_{14} = \sqrt{3}L/2$$

and the shape functions associated with the local nodes 1-3 can be found in the formula sheet appended.

(c) Given an FE-solution where the nodal displacements for element 9 (dashed in Figure 3) are given as

$$u_{x,10} = 0.02L, u_{x,12} = 0.008L, u_{x,14} = 0.010L,$$
$$u_{y,10} = 0, u_{y,12} = -0.02\sqrt{3}L/6, u_{y,14} = -0.02\sqrt{3}L/2,$$

compute the in-plane strain components (ε_{xx}, ε_{yy} and γ_{xy}) in point A in element 9. (2.0p)

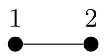
If you have trouble solving task (b), then you may for a reduced number of points describe how you would calculate the strain in point A given the Jacobian matrix on the form:

$$\boldsymbol{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1 Shape functions

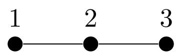
1.1 1D, linear

$$N_1^e = -\frac{1}{L}(x - x_2) \tag{1a}$$
$$N_2^e = \frac{1}{L}(x - x_1) \tag{1b}$$

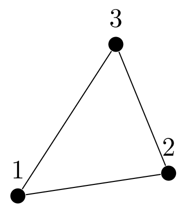


1.2 1D, quadratic

$$N_1^e = \frac{2}{L^2}(x - x_2)(x - x_3) \tag{2a}$$
$$N_2^e = -\frac{4}{L^2}(x - x_1)(x - x_3) \tag{2b}$$
$$N_3^e = \frac{2}{L^2}(x - x_1)(x - x_2) \tag{2c}$$



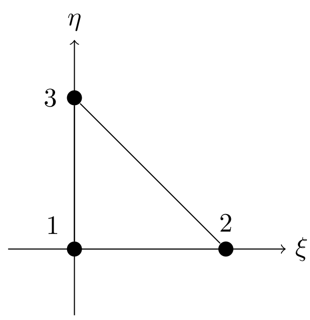
1.3 2D, linear triangle



$$N_1^e = \frac{1}{2A}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y) \tag{3a}$$
$$N_2^e = \frac{1}{2A}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y) \tag{3b}$$
$$N_3^e = \frac{1}{2A}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y) \tag{3c}$$

Parent element:

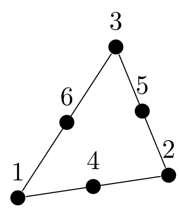
$$\overline{N}_1^e = 1 - \xi - \eta \tag{4a}$$
$$\overline{N}_2^e = \xi \tag{4b}$$
$$\overline{N}_3^e = \eta \tag{4c}$$

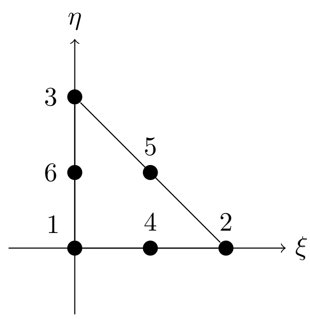


1.4 2D, Quadratic triangle

Parent element:

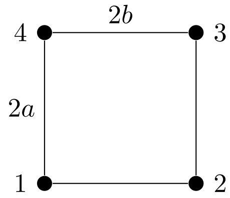
$$\overline{N}_1^e = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \tag{5a}$$
$$\overline{N}_2^e = \xi(2\xi - 1) \tag{5b}$$
$$\overline{N}_3^e = \eta(2\eta - 1) \tag{5c}$$
$$\overline{N}_4^e = 4\xi(1 - \xi - \eta) \tag{5d}$$
$$\overline{N}_5^e = 4\xi\eta \tag{5e}$$
$$\overline{N}_6^e = 4\eta(1 - \xi - \eta) \tag{5f}$$





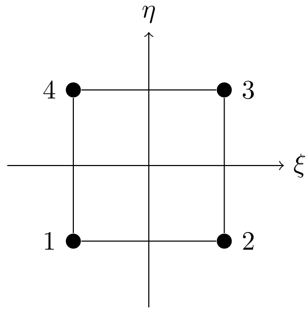
1.5 2D, bilinear

$$N_1^e = \frac{1}{4ab}(x - x_2)(y - y_4) \tag{6a}$$
$$N_2^e = -\frac{1}{4ab}(x - x_1)(y - y_3) \tag{6b}$$
$$N_3^e = \frac{1}{4ab}(x - x_4)(y - y_2) \tag{6c}$$
$$N_4^e = -\frac{1}{4ab}(x - x_3)(y - y_1) \tag{6d}$$



Parent element:

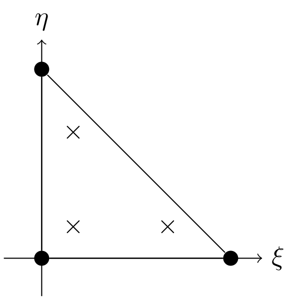
$$\overline{N}_1^e = \frac{1}{4}\left(\xi - 1\right)\left(\eta - 1\right) \tag{7a}$$
$$\overline{N}_2^e = -\frac{1}{4}\left(\xi + 1\right)\left(\eta - 1\right) \tag{7b}$$
$$\overline{N}_3^e = \frac{1}{4}\left(\xi + 1\right)\left(\eta + 1\right) \tag{7c}$$
$$\overline{N}_4^e = -\frac{1}{4}\left(\xi - 1\right)\left(\eta + 1\right) \tag{7d}$$



2 Gauss points

n	ξ_i	W_i
1	0.0000000000000000	2.0000000000000000
2	±0.5773502691896257	1.0000000000000000
3	0.0000000000000000	0.8888888888888889
	±0.7745966692414834	0.5555555555555556
4	±0.3399810435848563	0.6521451548625460
	±0.8611363115940525	0.3478548451374544

Table 1: Position of Gauss points ξ_i and corresponding weight W_i for n Gauss points.



n	(ξ_i, η_i)	W_i
1	$\left(\frac{1}{3}, \frac{1}{3}\right)$	$\frac{1}{2}$
3	$\left(\frac{1}{6}, \frac{1}{6}\right)$	$\frac{1}{6}$
	$\left(\frac{2}{3}, \frac{1}{6}\right)$	$\frac{1}{6}$
	$\left(\frac{1}{6}, \frac{2}{3}\right)$	$\frac{1}{6}$

3 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$.

$$\int_A \phi \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}A + \int_A (\boldsymbol{\nabla}\phi)^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}A = \int_{\mathcal{L}} \boldsymbol{n}^{\mathrm{T}} (\phi \boldsymbol{w}) \, \mathrm{d}\mathcal{L} \tag{8}$$

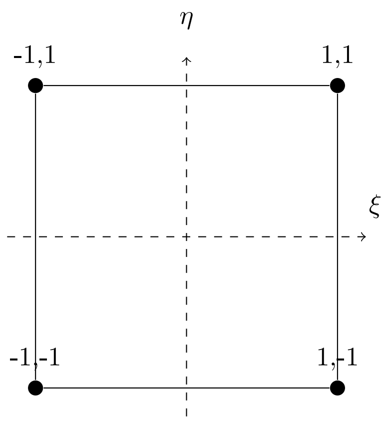
4 Gauss divergence theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$, $\operatorname{div}(\boldsymbol{w}) = \nabla^{\mathrm{T}}\boldsymbol{w}$.

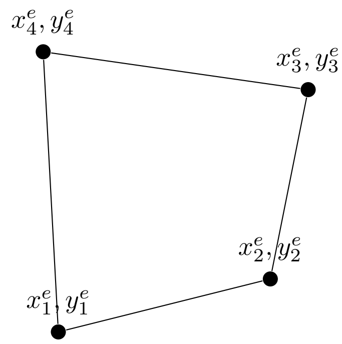
$$\int_A \boldsymbol{\nabla}^{\mathrm{T}}(\phi \boldsymbol{w})\,\mathrm{d}A = \int_{\mathcal{L}} (\phi \boldsymbol{w})^{\mathrm{T}}\,\boldsymbol{n}\,\mathrm{d}\mathcal{L}$$

5 Isoparametric mapping

Parent domain



Global domain



$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

$$x = x(\xi,\eta) = \overline{\mathbf{N}}^e(\xi,\eta)\,\mathbf{x}^e \tag{9}$$

$$y = y(\xi,\eta) = \overline{\mathbf{N}}^e(\xi,\eta)\,\mathbf{y}^e \tag{10}$$

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J}\begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{12}$$

6 Matrix inversion

The inverse of the matrix $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\mathbf{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{13}$$

7 Stresses and strains

Hooke's generalised law: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

2D Strain-displ. relation: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u},\ \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$

Solutions

Problem 1

(a)

Multiply the differential equaition by a test function v(r) and integrate over the domain

$$-\int_{r_{\text{in}}}^{r_{\text{out}}} v(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\left[k\,r^2\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right]\,\mathrm{d}r = 0$$

Rewrite using integration by parts $\Rightarrow$

$$\int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\mathrm{d}v(r)}{\mathrm{d}r}\,\left[k\,r^2\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right]\,\mathrm{d}r - \left[v(r)\,k\,r^2\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right]_{r_{\text{in}}}^{r_{\text{out}}} =$$

$$\begin{aligned}
-\left[v(r)\,k\,r^2\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right]_{r_{\text{in}}}^{r_{\text{out}}} &= \{q = -k\frac{\mathrm{d}T}{\mathrm{d}r}\} = \left[v(r)\,r^2\,q(r)\right]_{r_{\text{in}}}^{r_{\text{out}}} \\
&= v(r_{\text{out}})\,r_{\text{out}}^2\,q(r_{\text{out}}) - v(r_{\text{in}})\,r_{\text{in}}^2\,q(r_{\text{in}}) \\
&= v(r_{\text{out}})\,r_{\text{out}}^2\,\alpha(T(r_{\text{out}}) - T_{\text{out}}) + v(r_{\text{in}})\,r_{\text{in}}^2\,\alpha(T(r_{\text{in}}) - T_{\text{in}})
\end{aligned}$$

Rearranging these terms, the weak formulation can be written as:
Find $T(r)$, such that

$$\begin{aligned}
\int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\mathrm{d}v(r)}{\mathrm{d}r}\,\left[k\,r^2\,\frac{\mathrm{d}T(r)}{\mathrm{d}r}\right]\,\mathrm{d}r + r_{\text{in}}^2\,\alpha\,T(r_{\text{in}})\,v(r_{\text{in}}) + r_{\text{out}}^2\,\alpha\,T(r_{\text{out}})\,v(r_{\text{out}}) \\
= r_{\text{in}}^2\,\alpha\,v(r_{\text{in}})\,T_{\text{in}} + r_{\text{out}}^2\,\alpha\,v(r_{\text{out}})T_{\text{out}}
\end{aligned}$$

(b)

Approximate $T \approx T_h = \boldsymbol{N}\,\boldsymbol{a}$, where $\boldsymbol{N} = [N_1\ N_2\ \ldots N_n]$ is a row vector with shape functions and $\boldsymbol{a} = [a_1\ a_2\ \ldots a_n]^{\mathrm{T}}$ is a column vector with degrees of freedom. We need the derivative of $T(r)$ which is approximated as $\frac{\mathrm{d}T_h}{\mathrm{d}r} = \boldsymbol{B}\,\boldsymbol{a}$ where $\boldsymbol{B} = \frac{\mathrm{d}\boldsymbol{N}}{\mathrm{d}r} = [\frac{\mathrm{d}N_1}{\mathrm{d}r}\quad \frac{\mathrm{d}N_2}{\mathrm{d}r}\ \ldots \frac{\mathrm{d}N_n}{\mathrm{d}r}]$.
Using the Galerkin method, we approximate the test function using the same shape functions as for the primary unknown, and thus we obtain $v = \boldsymbol{N}\,\boldsymbol{c}$ where $\boldsymbol{c} = [c_1\ c_2\ \ldots c_n]^{\mathrm{T}}$ is a column vector with arbitrary constants. Similarly as above, the derivative of the test function is obtained as $\frac{\mathrm{d}v}{\mathrm{d}r} = \boldsymbol{B}\,\boldsymbol{c}$.
Inserting these FE-approximations into the weak form and factoring $\boldsymbol{c}$ (to the left) we get

$$\begin{aligned}
\boldsymbol{c}^{\mathrm{T}} \int_{r_{\text{in}}}^{r_{\text{out}}} \boldsymbol{B}^{\mathrm{T}} k\,r^2\,\boldsymbol{B}\,\mathrm{d}r\,\boldsymbol{a} + \boldsymbol{c}^{\mathrm{T}}\alpha\,r_{\text{in}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\text{in}})\,\boldsymbol{N}(r_{\text{in}}) + \boldsymbol{c}^{\mathrm{T}}\alpha\,r_{\text{out}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\text{out}})\,\boldsymbol{N}(r_{\text{out}}) \\
= \boldsymbol{c}^{\mathrm{T}}\alpha\,r_{\text{in}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\text{in}})\,T_{\text{in}} + \boldsymbol{c}^{\mathrm{T}}\alpha\,r_{\text{out}}^2\,\boldsymbol{N}^{\mathrm{T}}(r_{\text{out}})\,T_{\text{out}}
\end{aligned}$$

Since the constants in $\boldsymbol{c}$ are arbitrary the equation above can only be fulfilled if

$$(\boldsymbol{K} + \boldsymbol{K}_\mathrm{c})\,\boldsymbol{a} = \boldsymbol{f}_\mathrm{c}$$

where

$$\begin{aligned}
\boldsymbol{K} &= \int_{r_\mathrm{in}}^{r_\mathrm{out}} \boldsymbol{B}^\mathrm{T} k\, r^2\, \boldsymbol{B}\, \mathrm{d}r \\
\boldsymbol{K}_\mathrm{c} &= \alpha\, r_\mathrm{in}^2\, \boldsymbol{N}^\mathrm{T}(r_\mathrm{in})\, \boldsymbol{N}(r_\mathrm{in}) + \alpha\, r_\mathrm{out}^2\, \boldsymbol{N}^\mathrm{T}(r_\mathrm{out})\, \boldsymbol{N}(r_\mathrm{out}) \\
f_\mathrm{c} &= \alpha\, r_\mathrm{in}^2\, \boldsymbol{N}^\mathrm{T}(r_1)\, T_\mathrm{in} + \alpha\, r_\mathrm{out}^2\, \boldsymbol{N}^\mathrm{T}(r_\mathrm{out})\, T_\mathrm{out}
\end{aligned}$$

These are the global FE-equations.

(c)

The element stiffness matrix is determined by integrating over one element, defined between r_i and r_{i+1}.

$$\boldsymbol{K}^e = \int_{r_i}^{r_{i+1}} \boldsymbol{B}^\mathrm{T} k\, r^2\, \boldsymbol{B}\, \mathrm{d}r$$

Using linear shape functions (as given in the problem description), the $\boldsymbol{B}$-matrix becomes

$$\boldsymbol{B}^e = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}r} = \begin{bmatrix} \frac{\mathrm{d}N_1^e}{\mathrm{d}r} & \frac{\mathrm{d}N_2^e}{\mathrm{d}r} \end{bmatrix} = \begin{bmatrix} -\frac{1}{h^e} & \frac{1}{h^e} \end{bmatrix}$$

and consequently

$$\boldsymbol{K}^e = \int_{r_i}^{r_{i+1}} r^2\, \mathrm{d}r\, \frac{k}{(h^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{k(r_{i+1}^3 - r_i^3)}{3(h^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(d)

Temperature on the inside wall is 2.30 C. [See separate code for solution]

```matlab
% Short solution to Problem 1, reexam 13-04-2022
% Jim B
clc, clear variables, close all

r_in = 1.5;     % [m]
r_out = 2.4;    % [m]
k = 2.25;       % [W/m C]
alpha_in = 5;   % [W/m^2 C]
alpha_out = 5;  % [W/m*2 C]
T_in = 13;      % [C]
T_out = -15;    % [C]

nel = 2;
nnodes = nel+1;

r = linspace(r_in, r_out, nnodes);
K = zeros(nnodes, nnodes);
f = zeros(nnodes, 1);
he = (r_out-r_in) / nel;
```

Edof (or similar) and r: 0.5p

```matlab
for el=1:nel
    dofs = el:el+1;
    Ke = k*(r(el+1)^3 - r(el)^3)/(3*he^2) * [1 -1
                                             -1  1];
    K(dofs, dofs) = K(dofs, dofs) + Ke;
end
```

Ke and assembly: 0.5p

```matlab
% Convection contribution on the inside
K(1,1) = K(1,1) + alpha_in*r_in^2
f(1) = f(1) + alpha_in*r_in^2*T_in
```

Kc: 0.5p
f: 0.5p

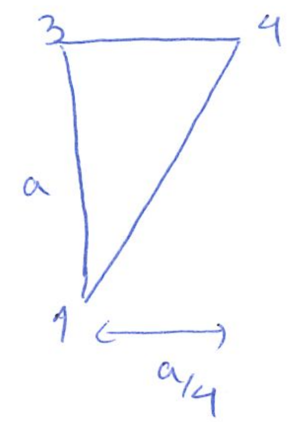
```matlab
% Convection contribution on the outside
K(nnodes,nnodes) = K(nnodes,nnodes) + alpha_out*r_out^2
f(nnodes) = f(nnodes) + alpha_out*r_out^2*T_out

% Solve system of equations
a = K\f

% Plot solution (not part of the task)
figure()
hold on
title('FE-solution')
xlabel('r [m]')
ylabel('T [C]')
plot(r, a, 'bo-')
```

Solve and compute: 0.5p

```matlab
fprintf('Temperature on the inside wall is %4.2f C \n', a(1))
```

Provide answer: 0.5p

2)



a) weak form:

$$\nabla^T q + \rho c \dot{T} = 0 \quad \text{in } \Omega \text{ for } 0 < T < T_{end}$$

$$q = -k \nabla T$$

multiplying by $v(x,y)$ and integrating over A:

$$\int_A v(x,y) \left[\rho c \dot{T} + \mathrm{div}(q) \right] dA = 0$$ 0.5p

$$\left(\int_A v \, \mathrm{div}(q) \, dA = \oint_\Gamma v(x,y) \underbrace{q^T \cdot \hat{n}}_{q_n} \, dL - \int_A (\nabla v)^T \cdot \underbrace{q}_{= -D \nabla T} \, dA \right)$$ 0.5p

$$\Rightarrow \int_A v(x,y) \rho c \frac{\partial T}{\partial t} \, dA + \int_A (\nabla v)^T D (\nabla T) \, dA = -\oint_\Gamma v(x,y) \, q_n \, dL$$

$$= \underbrace{- \underbrace{\int_{\Gamma_1} v(x,y) \, q_n \, dL}_{=0} - \int_{\Gamma_2} v \, q_n \, dL - \int_{\Gamma_3} v \, q_n \, dL - \underbrace{\int_{\Gamma_4} v \, q_n \, dL}_{=0} - \int_{\Gamma_5} v \, q_n \, dL}_{0.5p}$$

P5

weak form:

0.5p

$$\begin{cases} \int_A v \rho c \dot{T} \, dA + \int_A (\nabla v)^T D (\nabla T) \, dA = \\ - \int_{\Gamma_2} v \alpha (T - T_{air}) \, dL - \int_{\Gamma_3} v \alpha (T - T_{air}) \, dL - \int_{\Gamma_5} v \alpha (T - T_w) \, dL \end{cases}$$

b) FE-formulation:

$$\left.\begin{aligned} T(x,y,\tau) &= \underset{\sim}{N} \, \underline{a}(\tau) \\ \dot{T}(x,y,\tau) &= \underset{\sim}{N} \, \dot{a}(\tau) \end{aligned}\right\} \quad C \dot{a}(\tau) + (K + K_c) a(\tau) = f_b(\tau) + f_\ell$$

0.5p

Galerkin's method: $\underbrace{v = v^T = c^T N^T, \quad \nabla v = \ldots Bc \Rightarrow (\nabla v)^T = c^T B^T}$

0.5p

$$\left[\int_A N^T \rho c N \, dA \right] \dot{a}(\tau) + \left[\int_A B^T D B \, dA \right] a(\tau) = - \int_{\Gamma_2} N^T \alpha (N a(\tau) - T_{air})$$

$$- \int_{\Gamma_3} N^T \alpha (N a(\tau) - T_{air}) \, dL - \int_{\Gamma_5} N^T \alpha (N a(\tau) - T_w) \, dL$$

C (0.25p) K (0.25p) K_c (0.25p)

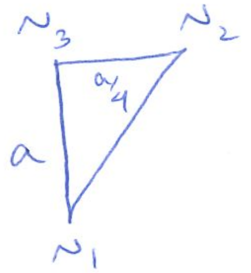
$$\Rightarrow \underbrace{\left[\int_A N^T \rho c N \, dA \right]}_{C} \dot{a}(\tau) + \underbrace{\left[\int_A B^T D B \, dA \right]}_{K} a(\tau) + \underbrace{\left(\int_{\Gamma_2} N^T \alpha N \, dL + \int_{\Gamma_3} N^T \alpha N \, dL \right.}_{}$$

$$\underbrace{\left. + \int_{\Gamma_5} N^T \alpha N \, dL \right)}_{K_c} a(\tau) = \underbrace{\int_{\Gamma_1} \alpha N^T T_{air} \, dL + \int_{\Gamma_2} \alpha N^T T_{air} \, dL + \int_{\Gamma_5} \alpha N^T T_w \, dL}_{f_b(\tau)}$$

(0.25p)

Round to closest 0.5p

1.5

c) 

$$A = \frac{a^2}{4} \times \frac{1}{2}$$

$$\begin{cases} N_1 = \frac{1}{2A}\left(\frac{a^2}{4} - \frac{a}{4}y\right) = 1 - \frac{y}{a} \\ N_2 = \frac{1}{2A}(ax) = \frac{4x}{a} \\ N_3 = \frac{1}{2A}\left(-ax + \frac{ay}{4}\right) = \frac{4}{a}\left(y/4 - x\right) \end{cases}$$

$$N = [N_1 \quad N_2 \quad N_3] \quad , \left(N_{ix} = \frac{\partial N_i}{\partial x} \; , \; N_{iy} = \frac{\partial N_i}{\partial y}\right)$$

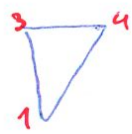
$$B = \begin{bmatrix} N_{1x} & N_{2x} & N_{3x} \\ N_{1y} & N_{2y} & N_{3y} \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{a} & -4/a \\ -1/a & 0 & 1/a \end{bmatrix} = \frac{1}{a}\begin{bmatrix} 0 & 4 & -4 \\ -1 & 0 & 1 \end{bmatrix}$$

(0.25p)

$$K^e_{(1)} = \int_{A_e} B^T \underbrace{D}_{\downarrow \; K I_{2\times 2}} B \, t \, dA = \int_{A_e} \frac{tK}{a^2} \begin{bmatrix} 0 & -1 \\ 4 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 \\ -1 & 0 & 1 \end{bmatrix} dA$$

$$= \int_A \frac{kt}{a^2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 16 & -16 \\ -1 & -16 & 17 \end{bmatrix} dA = \frac{kt}{a^2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 16 & -16 \\ -1 & -16 & 17 \end{bmatrix} \frac{a^2}{8}$$

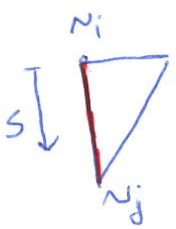
$$= \frac{kt}{8} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 16 & -16 \\ -1 & -16 & 17 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$ (0.5p)



where it is assembled in global matrix as: $\begin{bmatrix} K_{11} & K_{14} & K_{13} \\ K_{41} & K_{44} & K_{43} \\ K_{31} & K_{34} & K_{33} \end{bmatrix}$

(0.25p)

To calculatie K_c ; we have:



$$\begin{cases} N_i = 1 - s/a \\ N_j = s/a \end{cases} \quad 0 < s < a$$

(0.25p)

P7

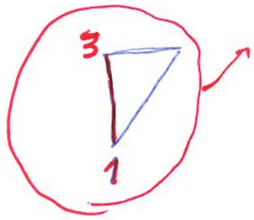
$$k_c^e = \int_0^1 \alpha N^T N \, ds$$

$$= \int_0^1 \alpha \begin{bmatrix} \frac{a-s}{a} \\ s/a \end{bmatrix} \begin{bmatrix} \frac{a-s}{a} & s/a \end{bmatrix} ds$$ (0.5p)

$$= \frac{\alpha}{a^2} \int_0^1 \begin{bmatrix} (a-s)^2 & s(a-s) \\ s(a-s) & s^2 \end{bmatrix} ds = \alpha \frac{a}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} =$$

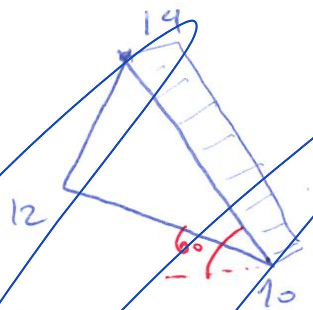
where it sits in global K matrix as: $\begin{bmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{bmatrix}$

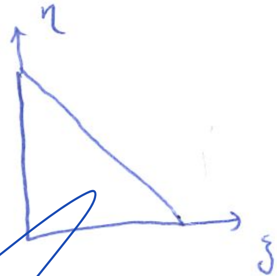
Round to closest to 0.25p



(0.25p)

3





a) we need to calculate $\int_{\Gamma_h} v^T h t \, d\Gamma$

$(0<s<L)$

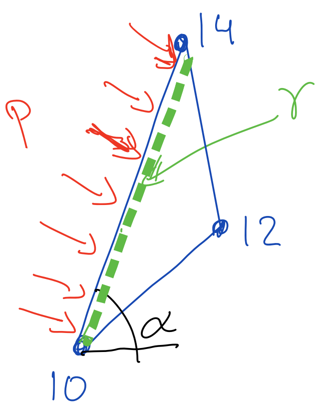
$$\begin{cases} N_i = \frac{L-s}{L} \\ N_j = \frac{s}{L} \end{cases}, \quad N = \begin{bmatrix} N_i & 0 & N_j & 0 \\ 0 & N_i & 0 & N_j \end{bmatrix}$$

$$v = v^T = c^T N^T \Rightarrow f^e = \int_{\Gamma_h} N^T \begin{bmatrix} -p\sin 60 \\ -p\cos 60 \end{bmatrix} t \, ds$$

$$= \int_0^L \begin{bmatrix} \frac{L-s}{L} & 0 \\ 0 & \frac{L-s}{L} \\ s/L & 0 \\ 0 & s/L \end{bmatrix} \begin{bmatrix} -p\sin 60 \\ -p\cos 60 \end{bmatrix} ds$$

P8

3a)



Calculate the contribution to $\mathbb{f}$ from the pressure p along the edge between nodes 10 & 14.

The contribution comes as a boundary contribution

$$\mathbb{f}_b = \int_{\mathcal{L}} \mathbb{N}^T \mathbb{t}\, t\, d\mathcal{L} \quad , t - \text{thickness}$$

For the particular edge γ

$$\mathbb{f}_b^{\gamma} = \int_{\mathcal{L}} \begin{bmatrix} N_{10} & 0 \\ 0 & N_{10} \\ N_{14} & 0 \\ 0 & N_{14} \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} t\, d\mathcal{L} = \left\{ \begin{matrix} t_x = p \sin\alpha \\ t_y = -p\cos\alpha \end{matrix} \right\}$$

0.5p

$$= \int_{\mathcal{L}} \begin{bmatrix} N_{10} & 0 \\ 0 & N_{10} \\ N_{14} & 0 \\ 0 & N_{14} \end{bmatrix} d\mathcal{L} \begin{bmatrix} pt\sin\alpha \\ -pt\cos\alpha \end{bmatrix} = \left\{ \int_{\mathcal{L}} N_{10}\, d\mathcal{L} = \int_{\mathcal{L}} N_{14}\, d\mathcal{L} = L_e/2 \text{ with} \right.$$

0.5p

$$L_e = \sqrt{(x_{14}-x_{10})^2 + (y_{14}-y_{10})^2}$$

0.25p

$$= \frac{Le\,pt}{2}\begin{bmatrix}\sin\alpha\\ -\cos\alpha\\ \sin\alpha\\ -\cos\alpha\end{bmatrix}, \quad \alpha = \arctan\left(\frac{y_{14}-y_{10}}{x_{14}-x_{10}}\right)$$

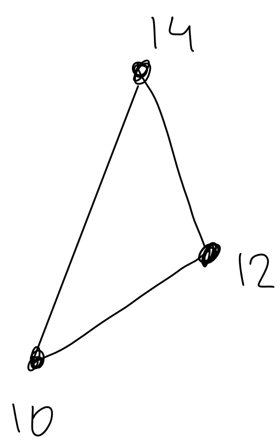
0.25p

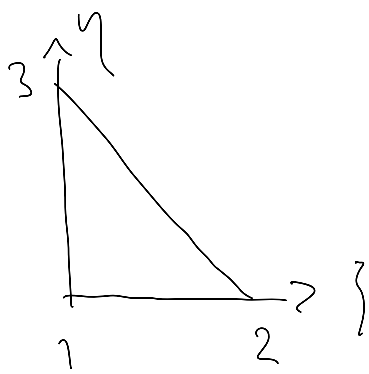
$$= \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \rightarrow \begin{bmatrix} f_{19}\\ f_{20}\\ f_{27}\\ f_{28}\end{bmatrix}$$

0.5p

---

b)





$x = N x^e$

$y = N y^e$

$$\underbrace{x^e = \begin{bmatrix} x_{10}\\ x_{12}\\ x_{14}\end{bmatrix}, \quad y^e = \begin{bmatrix} y_{10}\\ y_{12}\\ y_{14}\end{bmatrix}}_{\text{one example}}$$

0.5p

$$\mathbb{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} , \quad \mathbb{N} = \begin{bmatrix} N_1^e(\xi,\eta) & N_2^e(\xi,\eta) & N_3^e(\xi,\eta) \end{bmatrix}$$

0.5p

$$N_1^e = 1 - \xi - \eta$$

$$N_2^e = \xi$$

$$N_7^e = \eta$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1^e}{\partial \xi} x_1^e + \frac{\partial N_2^e}{\partial \xi} x_2^e + \frac{\partial N_3^e}{\partial \xi} x_3^e$$

$$= -1 \cdot x_{10} + 1 \cdot x_{12} + 0 \cdot x_{14} = 1.2$$

$$\frac{\partial x}{\partial \eta} = \ldots = -x_{10} + x_{14} = 1.0$$

$$\frac{\partial y}{\partial \xi} = \ldots = -y_{10} + y_{12} = 0.5774$$

$$\frac{\partial y}{\partial \eta} = \ldots = -y_{10} + y_{14} = 1.7321$$

1.0p

c

$$\varepsilon = \tilde{\nabla} u = \tilde{\nabla} \mathbb{N}^e a^e = \mathbb{B}^e a^e$$

$$\mathbb{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & 0 & \frac{\partial N_7^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial y} & \frac{\partial N_7^e}{\partial x} \end{bmatrix}$$

0.5p

Components in $\mathbb{B}^e$ can be found from

$$\nabla N^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} \end{bmatrix} = \mathbb{J}^{-T} \begin{bmatrix} \frac{\partial N^e}{\partial \xi} \\ \frac{\partial N^e}{\partial \eta} \end{bmatrix}$$

0.5p

$$= \mathbb{J}^{-T} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} & \frac{\partial N_2^e}{\partial \xi} & \frac{\partial N_3^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} & \frac{\partial N_2^e}{\partial \eta} & \frac{\partial N_3^e}{\partial \eta} \end{bmatrix}$$

$$= \mathbb{J}^{-T} \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{0.5p}$$

$$\Rightarrow \quad \varepsilon = \mathbb{B}^e a^e = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = -\begin{bmatrix} 0.02 \\ 0.02 \\ 0 \end{bmatrix}$$

0.5p

```
%% Task a

L = 2;
x10 = -L;
x12 = -2*L/5
x14 = -L/2;
y10 = 0;
y12 = sqrt(3)*L/6;
y14 = sqrt(3)*L/2;
p = 0.4e6;
t = 0.75;
alpha = atand((y14-y10)/(x14-x10))
Le = sqrt((x10-x14)^2 + (y10-y14)^2)

fe = Le*p*t/2*[cosd(alpha);-sind(alpha);cosd(alpha);-sind(alpha)]


%% Task b
dx_dxi = -x10+x12
dx_deta = -x10+x14
dy_dxi = -y10+y12
dy_deta = -y10+y14

J = [dx_dxi dx_deta;dy_dxi dy_deta]


%% Task c
ae = [0.02 0 0.008 -0.02*sqrt(3)/6 0.01 -0.02*sqrt(3)/2]'*L
gradN = inv(J')*[-1 1 0; -1 0 1]
Be=zeros(3,6);
Be(1,1:2:end) = gradN(1,:);
Be(2,2:2:end) = gradN(2,:);
Be(3,1:2:end) = gradN(2,:);
Be(3,2:2:end) = gradN(1,:)

epsilon = Be*ae
```