Formula sheet FEM – MHA021

December 30, 2025

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1 Basic math formulas

1.1 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to the boundary, Γ.

$$\int_{\Omega} \phi \ \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}\Omega + \int_{\Omega} [\boldsymbol{\nabla}\phi]^{\mathrm{T}} \, \boldsymbol{w} \, \mathrm{d}\Omega = \int_{\Gamma} \boldsymbol{n}^{\mathrm{T}} [\phi \boldsymbol{w}] \, \mathrm{d}\Gamma$$

As a special case (with a constant $\phi = 1$), we obtain the *divergence theorem*,

$$\int_{\Omega} \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}\Omega = \int_{\Gamma} \boldsymbol{n}^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}\Gamma$$

1.2 Matrix inversion

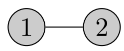
The inverse of the matrix $\boldsymbol{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\boldsymbol{M}^{-1} = \frac{1}{\det(\boldsymbol{M})} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\boldsymbol{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{1}$$

2 Shape functions in global coordinates

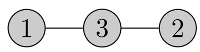
2.1 1D, linear

$$\begin{aligned} N_1^e &= -\frac{1}{L}(x - x_2) \\ N_2^e &= \frac{1}{L}(x - x_1) \end{aligned}$$



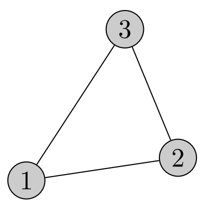
2.2 1D, quadratic

$$\begin{aligned} N_1^e &= \frac{2}{L^2}(x - x_2^e)(x - x_3^e) \\ N_2^e &= \frac{2}{L^2}(x - x_1^e)(x - x_3^e) \\ N_3^e &= -\frac{4}{L^2}(x - x_1^e)(x - x_2^e) \end{aligned}$$



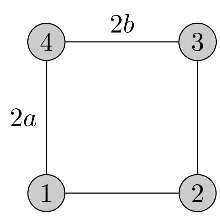
2.3 2D, linear triangle

$$\begin{aligned} N_1^e &= \frac{1}{2A}(x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y) \\ N_2^e &= \frac{1}{2A}(x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y) \\ N_3^e &= \frac{1}{2A}(x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y) \\ A &= \frac{1}{2}\Big[[x_2^e y_3^e - x_3^e y_2^e] - [x_1^e y_3^e - x_3^e y_1^e] + [x_1^e y_2^e - x_2^e y_1^e]\Big] \end{aligned}$$



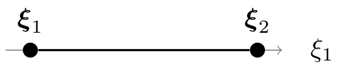
2.4 2D, bilinear quadrilateral

$$\begin{aligned} N_1^e &= \frac{1}{4ab}(x - x_2^e)(y - y_4^e) \\ N_2^e &= -\frac{1}{4ab}(x - x_1^e)(y - y_3^e) \\ N_3^e &= \frac{1}{4ab}(x - x_4^e)(y - y_2^e) \\ N_4^e &= -\frac{1}{4ab}(x - x_3^e)(y - y_1^e) \end{aligned}$$



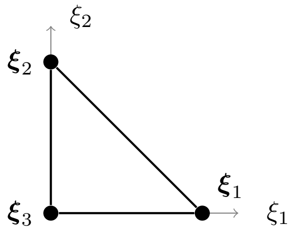
3 Reference shapes

3.1 Line



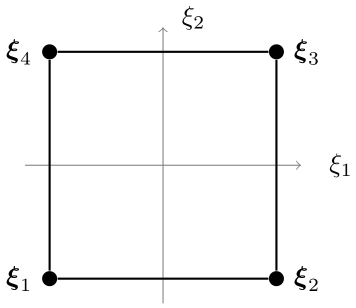
$$\boldsymbol{\xi}_1 = [\ -1\ ]$$
$$\boldsymbol{\xi}_2 = [\ +1\ ]$$

3.2 Triangle



$$\boldsymbol{\xi}_1 = [\ 1,\ 0\ ]^{\mathrm{T}}$$
$$\boldsymbol{\xi}_2 = [\ 0,\ 1\ ]^{\mathrm{T}}$$
$$\boldsymbol{\xi}_3 = [\ 0,\ 0\ ]^{\mathrm{T}}$$

3.3 Quadrilateral



$$\boldsymbol{\xi}_1 = [\ -1,\ -1\ ]^{\mathrm{T}}$$
$$\boldsymbol{\xi}_2 = [\ +1,\ -1\ ]^{\mathrm{T}}$$
$$\boldsymbol{\xi}_3 = [\ +1,\ +1\ ]^{\mathrm{T}}$$
$$\boldsymbol{\xi}_4 = [\ -1,\ +1\ ]^{\mathrm{T}}$$

4 Quadrature points

Reference line
Quadrature points, ξ, and weights, w, for n points on the reference line. Also see the function `gauss_integration_rule`

n	ξ	w
1	0.0000000000000000	2.0000000000000000
2	±0.5773502691896257	1.0000000000000000
3	0.0000000000000000	0.8888888888888889
	±0.7745966692414834	0.5555555555555556
4	±0.3399810435848563	0.6521451548625460
	±0.8611363115940525	0.3478548451374544

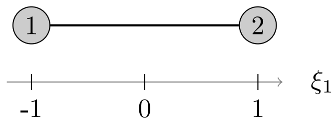
Reference triangle
Quadrature points, ξ, and weights, w, for n points in the reference triangle.

n	$\boldsymbol{\xi}^{\mathrm{T}}$	w
1	[1/3, 1/3]	1/2
3	[1/6, 1/6]	1/6
	[2/3, 1/6]	1/6
	[1/6, 2/3]	1/6

5 Shape functions on reference shapes

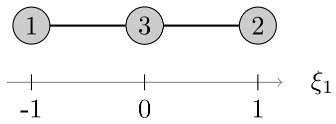
5.1 Line, linear

$$\hat{N}_1^e(\boldsymbol{\xi}) = \frac{1-\xi_1}{2}$$
$$\hat{N}_2^e(\boldsymbol{\xi}) = \frac{1+\xi_1}{2}$$



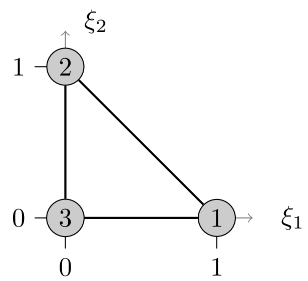
5.2 Line, quadratic

$$\hat{N}_1^e(\boldsymbol{\xi}) = \frac{\xi_1[\xi_1-1]}{2}$$
$$\hat{N}_2^e(\boldsymbol{\xi}) = \frac{\xi_1[\xi_1+1]}{2}$$
$$\hat{N}_3^e(\boldsymbol{\xi}) = 1-\xi_1^2$$



5.3 Triangle, linear

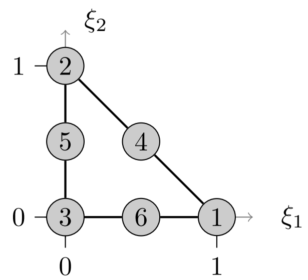
$$\hat{N}_1^e(\boldsymbol{\xi}) = \xi_1$$
$$\hat{N}_2^e(\boldsymbol{\xi}) = \xi_2$$
$$\hat{N}_3^e(\boldsymbol{\xi}) = 1-\xi_1-\xi_2$$



5.4 Triangle, quadratic

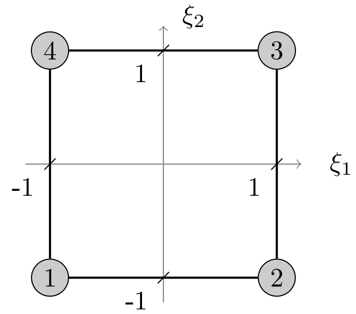
With $u = 1-\xi_1-\xi_2$,

$$\hat{N}_1^e(\boldsymbol{\xi}) = \xi_1[2\xi_1-1]$$
$$\hat{N}_2^e(\boldsymbol{\xi}) = \xi_2[2\xi_2-1]$$
$$\hat{N}_3^e(\boldsymbol{\xi}) = u[2u-1]$$
$$\hat{N}_4^e(\boldsymbol{\xi}) = 4\xi_1\xi_2$$
$$\hat{N}_5^e(\boldsymbol{\xi}) = 4\xi_2 u$$
$$\hat{N}_6^e(\boldsymbol{\xi}) = 4\xi_1 u$$



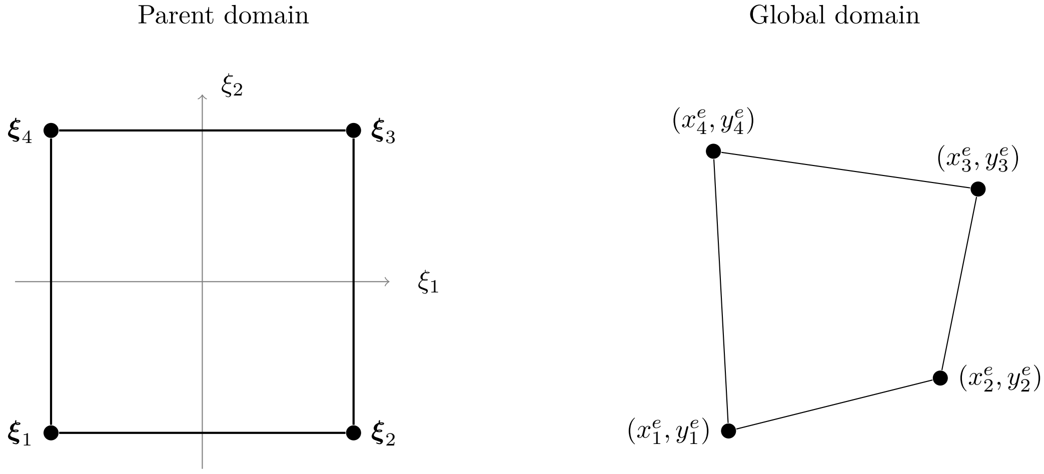
5.5 Quadrilateral, bilinear

$$\hat{N}_1^e(\boldsymbol{\xi}) = \frac{[1-\xi_1][1-\xi_2]}{4}$$
$$\hat{N}_2^e(\boldsymbol{\xi}) = \frac{[1+\xi_1][1-\xi_2]}{4}$$
$$\hat{N}_3^e(\boldsymbol{\xi}) = \frac{[1+\xi_1][1+\xi_2]}{4}$$
$$\hat{N}_4^e(\boldsymbol{\xi}) = \frac{[1-\xi_1][1+\xi_2]}{4}$$



6 Isoparametric mapping

Example for a four node element but the procedure is the same for all elements in 2D:



Nodal element coordinates

$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \ \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

Geometry approximation

$$x = x(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\,\mathbf{x}^e \tag{6}$$
$$y = y(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\,\mathbf{y}^e \tag{7}$$

where $\overline{\mathbf{N}}^e$ are the element shape functions.
Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{8}$$

Derivatives w.r.t. global coordinates

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{9}$$

7 Stresses and strains

General, linear elasticity in Voigt notation, $\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$. For the 2D case, we have,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{bmatrix} \approx \boldsymbol{B}\,\boldsymbol{a}$$

8 B-matrices

The $\boldsymbol{B}$-matrix for a 2D thermal problem is

$$\boldsymbol{B} = \boldsymbol{\nabla} \boldsymbol{N} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x_1} & \frac{\partial N_2^e}{\partial x_1} & \cdots \\ \frac{\partial N_1^e}{\partial x_2} & \frac{\partial N_2^e}{\partial x_2} & \cdots \end{bmatrix}$$

The $\boldsymbol{B}$-matrix for a 2D mechanical problem is

$$\boldsymbol{B} = \tilde{\boldsymbol{\nabla}} \boldsymbol{N} = \begin{bmatrix} \frac{\partial N_1^e}{\partial x_1} & 0 & \frac{\partial N_2^e}{\partial x_1} & 0 & \cdots \\ 0 & \frac{\partial N_1^e}{\partial x_2} & 0 & \frac{\partial N_2^e}{\partial x_2} & \cdots \\ \frac{\partial N_1^e}{\partial x_2} & \frac{\partial N_1^e}{\partial x_1} & \frac{\partial N_2^e}{\partial x_2} & \frac{\partial N_2^e}{\partial x_1} & \cdots \end{bmatrix}$$

9 Dynamics

9.1 Free vibration analysis

Circular natural frequencies ω and (reduced) vibration modes are solved from the generalized eigenvalue problem

$$\mathbf{K}_{\text{red}}\, \boldsymbol{\phi}_{\text{red}} = \omega^2\, \mathbf{M}_{\text{red}}\, \boldsymbol{\phi}_{\text{red}}$$

where $\boldsymbol{K}_{\text{red}}$ and $\boldsymbol{M}_{\text{red}}$ is the reduced stiffness and mass matrix constructed by extracting free DOFs. Frequencies in Hertz are determined as $f = \frac{\omega}{2\pi}$.

```
from mha021 import *
free_dofs = ...
K_red = extract_block(K, free_dofs)
M_red = extract_block(M, free_dofs)
omega2, phi = eigh(K_red, M_red)
```

9.2 Consistent mass matrices

Also see `mha021.py` for additional element functions.

Discrete mass in a node For a point mass m associated to a DOF, it is directly assembled to the corresponding position in the main diagonal of the global mass matrix $\boldsymbol{M}$.

Bar element in local coordinate system Cross-sectional area A, length L and density ρ.

$$\mathbf{M}^e = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Euler-Bernoulli beam element in local coordinate system Cross-sectional area A, length L and density ρ.

$$\mathbf{M}^e = \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

3-node triangle Element area A, thickness t and density ρ.

$$\mathbf{M}^e = \frac{\rho\, t\, A}{12} \begin{bmatrix} 2\mathbf{I} & \mathbf{I} & \mathbf{I} \\ \mathbf{I} & 2\mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} & 2\mathbf{I} \end{bmatrix}$$

4-node bilinear quad Element area A, thickness t and density ρ.

$$\mathbf{M}^e = \frac{\rho\, t\, A}{36} \begin{bmatrix} 4\mathbf{I} & 2\mathbf{I} & 1\mathbf{I} & 2\mathbf{I} \\ 2\mathbf{I} & 4\mathbf{I} & 2\mathbf{I} & 1\mathbf{I} \\ 1\mathbf{I} & 2\mathbf{I} & 4\mathbf{I} & 2\mathbf{I} \\ 2\mathbf{I} & 1\mathbf{I} & 2\mathbf{I} & 4\mathbf{I} \end{bmatrix}$$