Problem 1

Consider the following boundary value problem:
Find $u(x)$ in $0 \le x \le L$ such that

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left[k\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}\right] + v_x\dfrac{\mathrm{d}u(x)}{\mathrm{d}x} = f_0 \\ u(0) = u_0 \\ u(L) = 0 \end{cases}$$

where $k > 0$, $v_x \neq 0$, $u_0 \neq 0$ and f_0 are given constants.

Tasks:

(a) Staring from the boundary value problem above, **derive the weak form and the global FE-form.** Note that integration by parts should not be applied to the last term $v_x\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}$. Make sure to be specific on the structure and contents of any matrices you introduce. **(3p)**

(b) Assume the problem is discretized using linear shape functions and consider one element with the nodes $x = x_i$ and $x = x_{i+1}$, where $0 < x_i < x_{i+1} < L$. From the global FE-form, **derive the local FE-form and show that the element stiffness matrix and load vector become**

$$\mathbf{K}^e = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{v_x}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{f}^e = \frac{f_0\, h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $h = x_{i+1} - x_i$. **(3p)**

a) Multiply d.e. with an arbitrary test function $v(x)$ and integrate over the domain $\Rightarrow$

$$-\int_0^L v(x)\left(k u'(x)\right)' dx + \int_0^L v(x) v_x u'(x)\, dx = \int_0^L v(x) f_0\, dx$$

Use I.B.P. on the first term $\Rightarrow$

$$-\int_0^L v\,(ku')' dx = \int_0^L v' k u'\, dx - \left[v k u'\right]_0^L =$$

$$\int_0^L v' k u' dx - v(L) k u'(L) + v(0) k u'(L)$$

⇒ Weak form:

Find $u(x)$ such that

$$\begin{cases} \int_0^L v' k u' dx + \int_0^L v \cdot v_x u' dx = v(L) k u'(L) - v(0) k u'(0) + \int_0^L v f_0 dx \\ u(0) = u_0 \\ u(L) = 0 \end{cases}$$

FE-form

Introduce approximations: $u(x) \approx u_h(x) = \sum_{i=1}^{n} N_i(x) a_i = [N_1, \dots, N_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbb{N}\mathbb{a}$

$$\Rightarrow u_h'(x) = (\mathbb{N}\mathbb{a})' = \mathbb{N}'\mathbb{a} + \mathbb{N}\underbrace{\mathbb{a}'}_{=0} = \mathbb{N}'\mathbb{a} = \left[\frac{dN_1}{dx}, \dots, \frac{dN_n}{dx}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbb{B}\mathbb{a}$$

Use the same shape functions $\mathbb{N}$ to describe the test function

$v(x) = \sum_{i=1}^{n} N_i(x) c_i = [N_1, \dots, N_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbb{N}\mathbb{c}$, where c_i are arbitrary constants

$$\Rightarrow v'(x) = (\mathbb{N}\mathbb{c})' = \mathbb{N}'\mathbb{c} + \mathbb{N}\underbrace{\mathbb{c}'}_{=0} = \mathbb{N}'\mathbb{c} = \mathbb{B}\mathbb{c}$$

Insert into W.F. ⇒

$$\mathbb{c}^T \int_0^L \mathbb{B}^T k \mathbb{B} dx\, \mathbb{a} + \mathbb{c}^T \int_0^L \mathbb{N}^T v_x \mathbb{B} dx\, \mathbb{a} = \mathbb{c}^T \mathbb{N}^T(L) k u'(L) - \mathbb{c}^T \mathbb{N}^T(0) k u'(0) + \mathbb{c}^T \int_0^L \mathbb{N}^T f_0 dx$$

$$\Rightarrow \mathbb{c}^T \underbrace{\left(\text{FE-form} \right)}_{\text{must be equal to } 0 \text{ since } \mathbb{c} \text{ is arbitrary}} = 0$$

must be equal to 0 since $\mathbb{c}$ is arbitrary ⇒

$$\underbrace{\left[\int_0^L \mathbb{B}^T k \mathbb{B} dx + \int_0^L \mathbb{N}^T v_x \mathbb{B} dx \right]}_{\mathbb{K}} \mathbb{a} = \underbrace{\mathbb{N}^T(L) k u'(L) - \mathbb{N}^T(0) k u'(0) + \int_0^L \mathbb{N}^T f_0 dx}_{\mathbb{f}}$$

or $\mathbb{K}\mathbb{a} = \mathbb{f}$

b) Use linear approximation $\Rightarrow$ {formula sheet} $\Rightarrow$

$$\begin{cases} N_1^e(x) = -\frac{1}{h}(x - x_{i+1}) \\ N^e(x) = \frac{1}{h}(x - x_i) \end{cases} \quad \text{with} \quad h = x_{i+1} - x_i$$

$$\Rightarrow u^e(x) = \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} \begin{bmatrix} a_1^e \\ a_2^e \end{bmatrix} = \mathbb{N}^e \mathbb{a}^e \Rightarrow \mathbb{B}^e = \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} = \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

Localize the global FE-form over one element $\Rightarrow$

$$\mathbb{K}^e = \int_{x_i}^{x_{i+1}} (\mathbb{B}^e)^T k \mathbb{B}^e dx + \int_{x_i}^{x_{i+1}} (\mathbb{N}^e)^T v_x \mathbb{B}^e dx$$

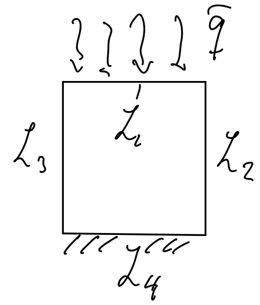
$$= \int_{x_i}^{x_{i+1}} k \begin{bmatrix} -1/h \\ 1/h \end{bmatrix} \begin{bmatrix} -1/h & 1/h \end{bmatrix} dx + \int_{x_i}^{x_{i+1}} \begin{bmatrix} -1/h(x - x_{i+1}) \\ 1/h(x - x_i) \end{bmatrix} \begin{bmatrix} -1/h & 1/h \end{bmatrix} dx \cdot v_x = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{h} \underbrace{\int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{(x - x_{i+1})}{h} \\ \frac{(x - x_i)}{h} \end{bmatrix} dx}_{(*)} \begin{bmatrix} -1 & 1 \end{bmatrix} v_x$$

$$= \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{h} \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot v_x = \frac{k}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{v_x}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} //$$

$$\mathbb{f}^e = \int_{x_i}^{x_{i+1}} (\mathbb{N}^e)^T f_0 dx = \underbrace{\int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{(x - x_{i+1})}{h} \\ \frac{(x - x_i)}{h} \end{bmatrix} dx}_{\text{same as } (*)} \cdot f_0 = \begin{bmatrix} h/2 \\ h/2 \end{bmatrix} f_0 = \frac{f_0 h}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} //$$

Problem 2

a) Heat balance of 2D body



$$\int_A Q t \, dA = \oint_{\mathcal{L}} q_n t \, d\mathcal{L}$$

$$\left\{ \oint_{\mathcal{L}} q_n t \, d\mathcal{L} = \text{div theorem} = \int_A \operatorname{div}(t\boldsymbol{q}) \, dA \right\}$$

$$\Rightarrow \int_A Q t \, dA = \int_A \operatorname{div}(t\boldsymbol{q}) \, dA \Rightarrow$$

$$\int_A -\operatorname{div}(t\boldsymbol{q}) + Qt \;\; dA = 0$$

should hold for arbitrary domain $\Rightarrow$

$$-\operatorname{div}(t\boldsymbol{q}) + Qt = 0$$ 0,5p

Add BC:s

0.5p
$$\begin{cases} q_n = -\bar{q} & \text{along } \mathcal{L}_1 \\ q_n = \alpha(T - T_{air}) & \text{along } \mathcal{L}_2 \cup \mathcal{L}_3 \\ q_n = 0 & \text{along } \mathcal{L}_4 \end{cases}$$

strong form

b)

General weak form:

$$\int_A (\nabla v)^T \mathbb{D} \nabla T \, t \, dA = \int_A v Q t \, dA - \int_{\mathcal{L}} v q_n t \, d\mathcal{L}$$

For this problem:

$$Q = 0$$

$q_n = -\bar{q}$ along $\mathcal{L}_1$

$q_n = \alpha (T - T_{air})$ along $\mathcal{L}_2 \cup \mathcal{L}_3$

$q_n = 0$ along $\mathcal{L}_4$

$\Rightarrow$

$$\int_A (\nabla v)^T \mathbb{D} \nabla T \, t \, dA = \int_{\mathcal{L}_1} v \bar{q} t \, d\mathcal{L} - \int_{\mathcal{L}_2 \cup \mathcal{L}_3} v \alpha (T - T_{air}) t \, d\mathcal{L}$$

c)

$T \approx \mathbb{N} \mathbf{a}$ with $\mathbb{N} = [N_1 \; N_2 \; .. \; N_n]$

$\Rightarrow \nabla T = (\nabla \mathbb{N}) \mathbf{a} = \mathbb{B} \mathbf{a}$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ (arbitrary)}$$

Galerkin's method

$v = \mathbb{N} \mathbf{c}$

$\nabla v = \mathbb{B} \mathbf{c}$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & -- & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & -- & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

Insert in weak form:

$$c^T \int_A B^T D B \, t \, dA \, a = c^T \left[\int_{\mathcal{L}_1} N^T \bar{q} \, t \, d\mathcal{L} - \int_{\mathcal{L}_2 \cup \mathcal{L}_7} \alpha N^T (N a - T_{\infty}) t \, d\mathcal{L} \right]$$

$$\iff$$

$$c^T \left[\Big(\underbrace{\int_A B^T D B \, t \, dA}_{K} + \underbrace{\int_{\mathcal{L}_2 \cup \mathcal{L}_7} \alpha N^T N \, t \, d\mathcal{L}}_{K_c} \Big) a - \overbrace{\Big(- \int_{\mathcal{L}_2 \cup \mathcal{L}_7} \alpha N^T T_{\infty} \, t \, d\mathcal{L} - \int_{\mathcal{L}_1} N^T \bar{q} \, t \, d\mathcal{L} \Big)}^{3 \quad f_b} \right] = 0$$

c^T - arbitrary $\Rightarrow$

$$(K + K_c) a - f_b = 0 \Rightarrow$$

$$(K + K_c) a = f_b \qquad \text{FE-form}$$

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d)
$$K^e = \int_{-1}^{1} \int_{-1}^{1} B^{e^T} D B^e \det(J) \, d\xi \, d\eta$$

B^e ~ linear
D ~ quadratic
$\det(J)$ - constant (for this shape)

$\Rightarrow$ integrand of order ~ 4 $\Rightarrow$ 3 intgr points in each direction
$(2 \cdot 3 - 1 = 5)$

Integration points (9 in total) from formula sheet.

$\xi_1 = -0.7746 \quad \eta_1 = -0.7746 \quad \text{weight} = (0.5556)^2$

$\xi_2 = -0.7746 \quad \eta_2 = 0 \quad \text{weight} = 0.5556 \times 0.8889$

$\xi_3 = -0.7746 \quad \eta_3 = 0.7746 \quad \text{weight} = (0.5556)^2$

$\xi_4 = 0 \quad \eta_4 = -0.7746 \quad \text{weight} = 0.5556 \times 0.8889$

$\xi_5 = 0 \quad \eta_5 = 0 \quad \text{weight} = 0.8889^2$

$\xi_6 = 0 \quad \eta_6 = 0.7746 \quad \text{weight} = 0.5556 \times 0.8889$

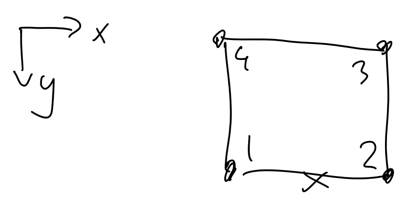
$\xi_7 = 0.7746 \quad \eta_7 = -0.7746 \quad \text{weight} = 0.5556^2$

$\xi_8 = 0.7746 \quad \eta_8 = 0 \quad \text{weight} = 0.5556 \times 0.8889$

$\xi_9 = 0.7746 \quad \eta_9 = 0.7746 \quad \text{weight} = 0.5556^2$

c) Heat flow

$$q = -k \nabla T = -k B^e a^e$$



$y = 5\text{mm}$

$$N_1^e = -\frac{1}{ab}(x - x_2)(y - y_4) \Rightarrow \frac{\partial N_1^e}{\partial x} = -\frac{1}{ab}(y - y_4)$$

$$\frac{\partial N_1^e}{\partial y} = -\frac{1}{ab}(x - x_2)$$

$$N_2^e = \frac{1}{ab}(x-x_1)(y-y_3) \Rightarrow \frac{\partial N_2^e}{\partial x} = \frac{1}{ab}(y-y_3)$$

$$\frac{\partial N_2^e}{\partial y} = \frac{1}{ab}(x-x_1)$$

$$N_3^e = -\frac{1}{ab}(x-x_4)(y-y_2) \Rightarrow \frac{\partial N_3^e}{\partial x} = -\frac{1}{ab}(y-y_2)$$

$$\frac{\partial N_3^e}{\partial y} = -\frac{1}{ab}(x-x_4)$$

$$N_4^e = \frac{1}{ab}(x-x_3)(y-y_1) \Rightarrow \frac{\partial N_4^e}{\partial x} = \frac{1}{ab}(y-y_1)$$

$$\frac{\partial N_4^e}{\partial y} = \frac{1}{ab}(x-x_3)$$

$$B^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & - & \frac{\partial N_4^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & & \frac{\partial N_4^e}{\partial y} \end{bmatrix}$$

MATLAB: $\Rightarrow q = \begin{bmatrix} -6.44 \\ 38.7 \end{bmatrix} \times 10^4 \ W/m^2$ (with $\bar{y} = 15mm$)

```
a = 5e-3;
b = 6e-3;
y = a;
x = b/2;
ybar = 3*a; %unfortunately not given in the problem description
k0 = 200;

x7 = 0; y7 = 0;
x8 = b; y8 = 0;
x22 = 0; y22 = a;
x23 = b; y23 = a;

x1 = x22; y1 = y22;
x2 = x23; y2 = y23;
x3 = x8; y3 = y8;
x4 = x7; y4 = y7;

Be = 1/(a*b)*[-(y-y4), (y-y3), -(y-y2), (y-y1);
              -(x-x2), (x-x1), -(x-x4), (x-x3)];
ae = [294;296;306;304];

k = k0*(1-0.3*(y/ybar)^2);

q = -k*Be*ae
```

Problem 3

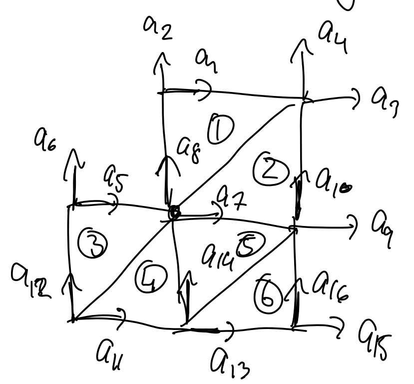
a) Numbering scheme for degrees of freedom.

For node k

$$u_{x,k} = a_{2k-1}$$

$$u_{y,k} = a_{2k}$$

$\Rightarrow$



b) K^e and f^e can be assembled into the global K and f using a topology matrix Edof:

$$
\begin{array}{c}
\quad \text{elno} \quad \text{dof1} \quad \text{dof2} \quad \text{dof3} \quad \text{dof4} \quad \text{dof5} \quad \text{dof6} \\
\text{Edof} = \begin{bmatrix}
1 & 1 & 2 & 7 & 8 & 3 & 4 \\
2 & 3 & 4 & 7 & 8 & 9 & 10 \\
3 & 5 & 6 & 11 & 12 & 7 & 8 \\
4 & 11 & 12 & 13 & 14 & 7 & 8 \\
5 & 13 & 14 & 9 & 10 & 7 & 8 \\
6 & 13 & 14 & 15 & 16 & 9 & 10
\end{bmatrix}
\end{array}
$$

For element i this can be done in MATLAB:

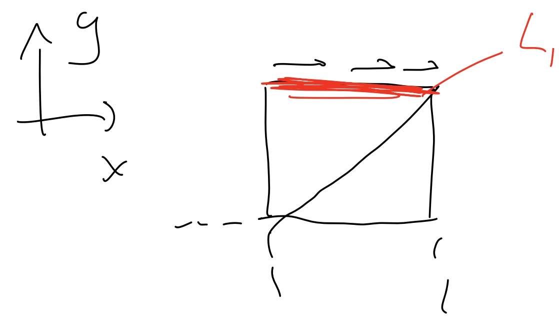
$$\mathbb{K}(Edof(i,2{:}7),Edof(i,2{:}7)) = \mathbb{K}(Edof(i,2{:}7),Edof(i,2{:}7)) + \mathbb{K}^e$$

element stiffness for element i

$$\mathbb{f}(Edof(i,2{:}7)) = \mathbb{f}(Edof(i,2{:}7)) + \mathbb{f}_l^e$$

element load vector for element i

c)



$$\mathbb{f}_b^e = \int_{L_1} \mathbb{N}^T \mathbb{t}\, t\, dL \qquad \mathbb{t} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

$t = 4\,\text{mm}$

$$= \int_{L_1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \tau \\ 0 \end{bmatrix} t\, dL = \frac{a\tau t}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{array}{l} \xrightarrow{\text{added to}} f_1 \\ \longrightarrow f_2 \\ \longrightarrow f_3 \\ \longrightarrow f_4 \end{array}$$

$$= \frac{5.4 \cdot 100}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \rightarrow f_1 \\ \rightarrow f_2 \\ \rightarrow f_3 \\ \rightarrow f_4 \end{matrix}$$

d) see next page

For node 2:

$u_x = 1.73$ mm

$u_y = -5.39$ μm

```
a = 5e-5;
t = 4e-3;
ptype = 2;
E = 200e9;
nu = 0.3;

Coord = [
     a, 2*a;
   2*a, 2*a;
     0,   a;
     a,   a;
   2*a,   a;
     0,   0;
     a,   0;
   2*a,   0];

Dof = [2*[1:8]'-1 2*[1:8]'];

nen = 3;

Edof = [1  1  2  7  8  3  4;
        2  3  4  7  8  9 10;
        3  5  6 11 12  7  8;
        4 11 12 13 14  7  8;
        5 13 14  9 10  7  8;
        6 13 14 15 16  9 10];

bc = [11 0;
      12 0;
      13 0;
      14 0;
      15 0;
      16 0];

[Ex,Ey] = coordxtr(Edof,Coord,Dof,nen);

D = hooke(ptype,E,nu);
K = zeros(max(max(Edof)));
f = zeros(max(max(Edof)),1);

for i=1:size(Edof,1)
   [Ke,fe] = plante(Ex(i,:),Ey(i,:),[ptype t],D,[0;0]);
   [K,f] = assem(Edof(i,:),K,Ke,f,fe);
end

f(1) = f(1) + 1e3;
f(3) = f(3) + 1e3;

[a,Q]=solveq(K,f,bc);

disp('For node 2:')
```

```
ux = a(3)
uy = a(4)
```