Examination

MHA021 Finite Element Method

Date and time:	26-08-2024, 14.00-18.00
Instructors:	Martin Fagerström and Jim Brouzoulis (phone 1300 and 2253). An instructor will visit the exam around 15:00 and 17:00.
Solutions:	Example solutions will be posted within a few days after the exam on the course homepage.
Grading:	The grades will be reported to the registration office on 13 September the latest.
Review:	For a review of the exam corrections, please make an appointment with your examiner.
Permissible aids:	Chalmers type approved pocket calculator. **Note**: A formula sheet is available as a pdf-file alongside with this exam thesis.

Exam instructions

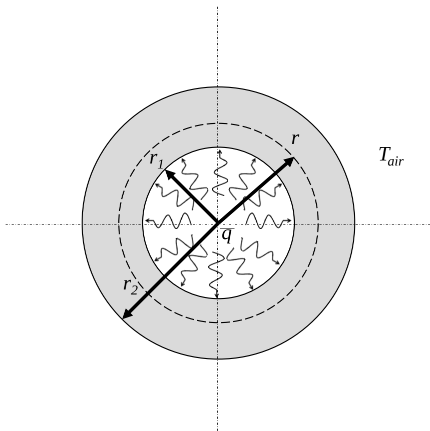
All exam problems require a hand-in on paper. For some of the problems, it may be convenient to also use MATLAB including CALFEM. If you use MATLAB and CALFEM as part of your solutions, you must make sure to also hand in any MATLAB code you have written yourself. You do this by saving your files under `C:\__Exam__\Assignments\` in the appropriate sub-directories created for each problem. When doing so, it is strongly recommended that you write your anonymous exam code in any MATLAB files that you want to hand in. Finally, **it is also absolutely necessary that you write the name of your computer on the cover page for the exam!**

Note that most CALFEM files (but not all) are provided for your convenience. In addition, please note that the CALFEM function `extract.m` also exists in the CALFEM directory as `extract_dofs.m` (to avoid a conflict with a built-in MATLAB function). These CALFEM finite element files can be found under the directory `C:\__Exam__\Assignments`. You can utilize these files by copying appropriate files into the sub-directories for the problem where they are needed. Should you need to refer to the CALFEM manual, you can find this also (excluding the examples section) under `C:\__Exam__\Assignments`.

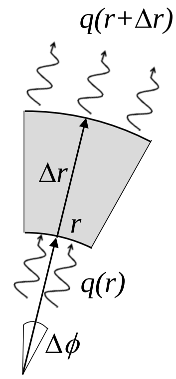
Finally, remember to close MATLAB and log-out from the computer when you are finished with the exam.

Problem 1

Consider a cylinder to be used in a shaft shrink-fit, see Figure 1a. To expand the cylinder such that it can be placed on the outside of the two shafts to be connected, the cylinder is heated by subjecting it to an internal heat flux $\overline{q}$ on the internal surface.



a)



b)

Figure 1: a) Cylinder to be considered in Problem 1 subjected to an internal heat $\overline{q}$. b) Close up of a small piece of the cylinder at radial coordinate r.

The heat is supplied in such a way that no temperature variations occur neither in the circumferential direction, nor in the longitudinal direction. This means that it is enough to consider a 1D heat flow problem (in the radial direction) to calculate the temperature distribution in the cylinder. For this case, the 1D heat flow is given by Fourier's law as $q(r) = -k(r)\frac{dT}{dr}$. Furthermore, the conditions on the outer surface of the cylinder is to be considered as convective, with the heat transfer coefficient α and external temperature T_{air}.

To derive the governing equations, it is helpful to consider a small section of the cylinder (given by a small circumferential angle $\Delta\phi$) at the radial position r, see Figure 1b. As no heat (except for that supplied to the inner surface of the cylinder) is added to the system, a simple heat balance gives at hand that:

$$q(r) \cdot A(r) = q(r + \Delta r) \cdot A(r + \Delta r) = \text{constant}, \text{ or } \frac{d}{dr}\left(q(r)A(r)\right) = 0$$

where $A(r)$ is the surface area at coordinate r equal to:

$$A(r) = \Delta\phi \cdot r \cdot L$$

where L is the length of the cylinder. Since this length is constant over r, and since no variations occur in the circumferential direction, the heat balance equation can be simplified to:

$$\frac{d}{dr}\left(r \cdot q(r)\right) = 0.$$

Tasks:

a) By considering the 1D heat flow balance in the radial direction, derive and state the strong form of the problem, expressed in terms of the temperature T. **(1.0p)**

b) Given the strong form of the problem, derive and state the full weak form of the problem at hand. **(1.0p)**

c) Given the weak form of the problem, derive and state the FE form of the problem at hand. Be careful to explain the contents of any vectors or matrices you introduce. **(1.5p)**

d) Consider specifically a problem with a constant heat conductivity $k(r) = k$, discretised with four 1D elements with linear shape functions. Calculate the element stiffness (or conductivity) matrix for the inner-most element (the element with one node on the inner surface). **(1.5p)**

e) For the same problem, calculate any contribution to the global stiffness (or conductivity) matrix associated with the boundary conditions. **(1.0p)**

Problem 2

Consider the heat exchanger in Figure 2, designed as a block of concrete surrounded by a frame of an insulating material (of width w). The heat exchanger contains two pipes of radius R in which cooling water flows with a temperature of T_{w}. In the centre, there is a channel of width b in which hot water flows that has a temperature of T_{ch} under steady-state conditions. the upper and lower part of the heat exchanger boundaries are insulated and the air temperature on the left and right hand side is T_{out}.

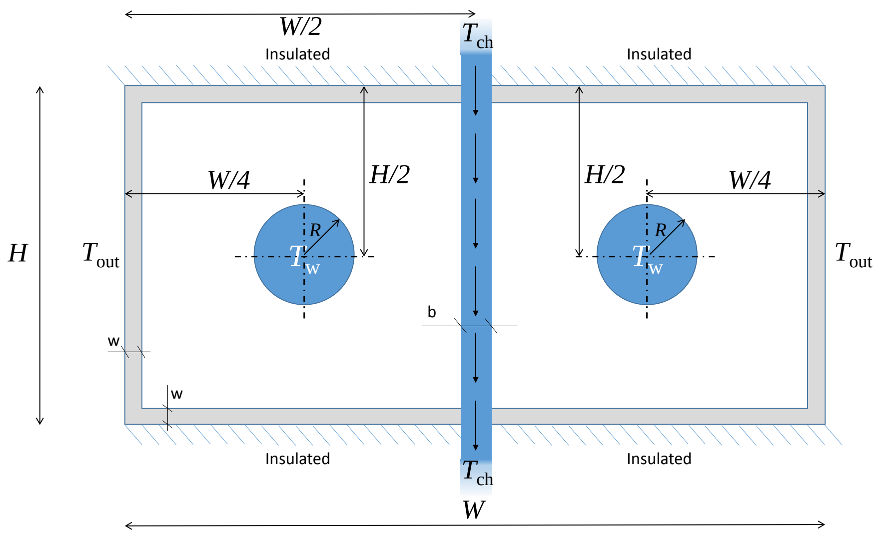


Figure 2: Sketch of the heat exchanger

The concrete and the insulating material are both assumed to be isotropic (w.r.t heat flow) and obey Fourier's law $\mathbf{q} = -k\nabla T$. Furthermore, the concrete and insulating material conductivities are k_{c} and k_{s} respectively, and the heat transfer coefficient between the insulating material and air is α_{air}. Between the concrete and water it is α_{w}.

No heat is assumed to flow out of the plane shown in Figure 2. Thus, the problem can be considered as a 2D heat flow problem and the governing partial differential equation for the heat flow problem thereby becomes

$$\nabla^{\mathrm{T}}\mathbf{q} \;=\; 0 \quad \text{in} \;\; \Omega$$

Tasks:

a) Motivate whether or not symmetry can be utilised for a more efficient solution of the current problem. Then state the full strong form (in terms of T) for the smallest part of the cross-section that can be analysed by FE for this particular problem. Make sure to indicate a sketch of the domian that clearly defines all the boundaries. **(2.0p)**

b) Derive the weak form corresponding to the strong form of the particular problem. Be specific in how the boundary conditions enter in the weak form. **(2.0p)**

c) By introducing the finite element approximation on the form $T(x, y) \approx \mathbf{N}(x, y)\mathbf{a}$ and using Galerkin's method, the discrete form of the problem can be obtained as:

$$[\mathbf{K} + \mathbf{K}_c]\,\mathbf{a} = \mathbf{f}_b$$

Based on this, derive the expressions for $\mathbf{K}, \mathbf{K}_c$ and $\mathbf{f}_b$ for the particular problem at hand. Please note that $\mathbf{f}_b$ may have different contributions so be sure to state which of these will yield non-zero contributions and why. **(2.0p)**

Problem 3

Consider a hexagonal concrete part of thickness 0.5 m and with a uniform side length $L = 1$ m subjected to an external pressure $p = 0.3$ MPa as shown in Figure 3.

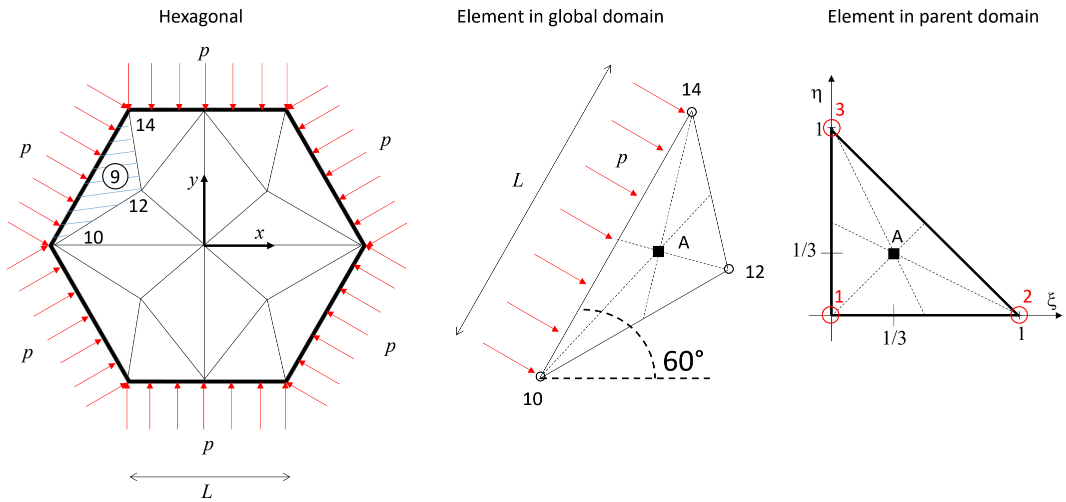


Figure 3: Left: Hexagonal part in Problem 3. Middle: Particular element (no 9) in global domain to be considered in tasks a)-c) where global node numbers are indicated in black. Left: Isoparametric three noded element in the parent domain to be considered in tasks b)-c) where local node numbers are indicated in red.

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\mathbf{v}\right)^{\mathrm{T}} \mathbf{D}\tilde{\nabla}\mathbf{u}\, t\, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\mathbf{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \mathbf{v}^{\mathrm{T}}\mathbf{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \mathbf{v}^{\mathrm{T}}\mathbf{h}\, t\, \mathrm{d}\mathcal{L}$$

for any domain A with prescribed displacements $\mathbf{u} = \mathbf{g}$ along $\mathcal{L}_g$ and prescribed tractions $\mathbf{t} = \mathbf{h}$ along $\mathcal{L}_h$.

Tasks:

a) For the resulting FE-problem ($\mathbf{Ka} = \mathbf{f}$), **calculate the contribution to the global load vector f from the pressure p along the boundary of the dashed element no 9** (also shown in the middle figure) considering the global node numbering in the figure. For full point, both specify the values and the positions of the non-zero contributions in the global load vector. **(2.0p)**

b) Just as for bilinear quadrilateral elements, an isoparametric formulation can be considered for linear triangular elements, cf. the middle and left part of Figure 3.

For such an isoparametric formulation, **calculate the Jacobian matrix J in the point A for element no 9** (corresponding to the element centre of gravity, $\xi = 1/3$, $\eta = 1/3$). **(2.0p)**

The coordinates for the three nodes are

$$x_{10} = -L, x_{12} = -2L/5, x_{14} = -L/2, y_{10} = 0, y_{12} = \sqrt{3}L/6, y_{14} = \sqrt{3}L/2$$

and the shape functions associated with the local nodes 1-3 can be found in the formula sheet appended.

c) Given an FE-solution where the nodal displacements for element 9 (dashed in Figure 3) are given as

$$u_{x,10} = 0.012L, u_{x,12} = 0.005L, u_{x,14} = 0.006L,$$
$$u_{y,10} = 0, u_{y,12} = -0.01\sqrt{3}L/6, u_{y,14} = -0.01\sqrt{3}L/2,$$

calculate the strain components in point A in element 9. (2.0p)

If you have trouble solving task (b), then you may for a reduced number of points describe how you would calculate the strain in point A given the Jacobian matrix on the form:

$$\mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$