Water Supply Network Optimization

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Abstract

1 Introduction

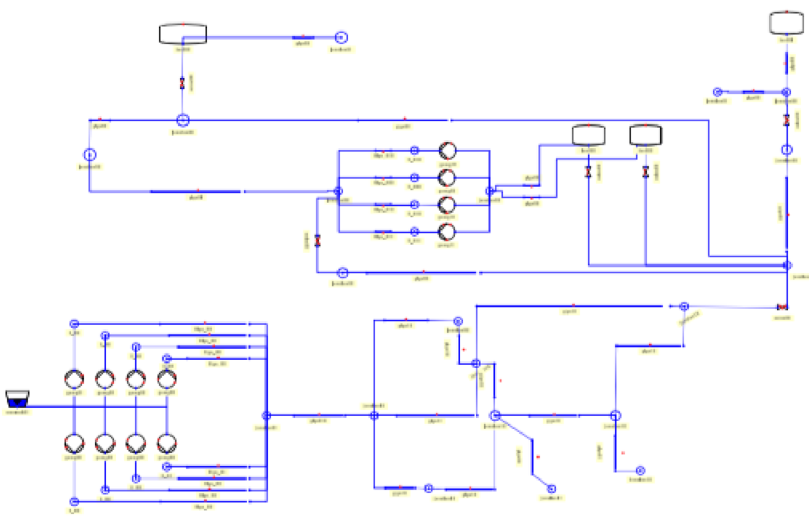

Figure 1: Schematic representation of a water supply network from [Gle+12]

A water supply network is a composition of pipes and pumps that connect water sources to the consumers. Municipal water supply network systems constitute a central part of the public infrastructure and cause substantial costs, both in monetary and energetic terms [BGS09]. Hence, modeling physically sound models of the water supply network is important to avoid unnecessary consumption of the resources, which benefits economically and ecologically. The schematic representation of typical network is shown in Figure 1.

This project is focused on modeling and solving the optimal operative planning of water supply network. This problem is mathematically formulated as a nonconvex *Mixed Integer Nonlinear Program* (MINLP) problem. The detailed formulation of the problem can be found in [Gle+12] and [D+15]. The continuous variables are node pressures and arc flow rates and the discrete variables are device operation modes such as turning on/off a particular pump or valve. The combinatorial nature of the problem leads to a huge search space of variables, which exarcerbated by the slow convergence due to the nonconvexity. Therfore, we are interested in developing convex relaxation of the original problem. One particular model that we explore in this project is the convex *minimum-cost flow* (MCF) model.

2 Formal definition of the problem

General model of a water supply network is based on a directed graph $G = (N, A)$. The set of nodes $N = S \cup J$ consists of *water sources* $s \in S$ and *junctions* $j \in J$. Junctions with nonzero demand $d_i > 0$ correspond to consumers; all other is transitory junction with zero demand $d_i = 0$. The arc set A consists of *pipe segments* $a = (i, j)$ in which water is transported from node i to node j.

2.1 Flow conservation constraint

Each pipe segment $a = (i, j)$ contains a nonnegative flow q_{ij}. For a node i, we denote the set of incoming and outgoing arcs as $\delta^-(i)$ and $\delta^+(i)$ respectively. At each juction j, one has the flow conservation constraints

$$\sum_{ij \in \delta^-(j)} q_{ij} - \sum_{ji \in \delta^+(j)} q_{ji} = d_j \qquad j \in J \tag{1}$$

Usually, the constrait (1) is not imposed on water sources. However, one can model the water source in similar fashion with:

$$\sum_{si \in \delta^+(s)} q_{ji} \geq 0 \qquad s \in S$$

2.2 Pressure-loss constraint

Water flow through the network is induced by different pressure level at the nodes [Gle+12]. Each node i has a potential variable h_i, which represent the water pressure, sometime called hydraulic head, at node i. The pressure loss along the pipe segment $a = (i, j)$ with pressure level h_i and h_j at its ends is described by the *Darcy-Weisbach* equation,

$$h_i - h_j = \lambda_{ij} q_{ij}^2 \tag{2}$$

The loss coefficient λ_{ij} is computed via

$$\lambda_{ij} = \frac{8 \cdot L_{ij} \cdot f_{ij}}{\pi^2 g \cdot D_{ij}^5}$$

which depends on the gravitational acceleration g and constants associated with the pipe characteristics - length L_{ij}, diameter D_{ij}, and the Darcy friction factor f_{ij}.

The Darcy friction factor f_{ij} is approximated using Colebrook-White equation as

$$f_{ij} = \left(2 \log_{10} \frac{k_{ij}}{3.71 \cdot D_{ij}} \right)^2$$

where k_{ij} is the roughness coefficient of the pipe. In context of nonlinear network flow problem, constraint (2) is generally called potential-flow coupling constraints, in which it defines a relationship between the flow value on an arc and the potential levels at the two end nodes [FH14]. It is easy to see that the problem becomes nonconvex when adding (2) to the constraints.

2.3 Minimum pressure constraint

Most junctions have minimum pressure requirement due to the geographic height of the junction. If the junction is located in high altitude areas, then more pressure is required to deliver the water to the junction. We represent the minimum pressure for each node with l_i. Therefore, for each junction j we have the constraint

$$h_j \geq l_j \qquad j \in J \tag{3}$$

2.4 Objective function

The standard objective of the water supply network optimization is to minimized the variable operational costs incurred by purchasing water fed into the network and the energy needed to operate pumps [Gle+12]. The energy consumption of pumps in the objective function and the pressure-loss equation along each pipes are nonconvex, which leads to the MINLP formulation.

However, one convex relaxation of the model that we are interested is to treat the problem as minimum-energy flow problem where the objective is to minimize the energy to transport required flows from the water sources to all consumer nodes. Then, the total energy within a network can be computed by summing up all the energy loss in each pipe segment integrated with respect to the flow.

The total energy loss within a network is given by

$$\sum_{ij \in A} (h_i - h_j) q_{ij} = \sum_{ij \in A} \lambda_{ij} q_{ij}^3 \tag{4}$$

The optimal cost of such objective would provide a valid lower bound for the original MINLP.

2.5 Summary

The complete convex minimum-energy flow problem now reads

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{ij\in E} \lambda_{ij} q_{ij}^3 \\ \text{subject to} \quad & \textstyle\sum_{ij\in\delta^-(j)} q_{ij} - \sum_{ji\in\delta^+(j)} q_{ji} = d_j \\ & h_i - h_j = \lambda_{ij} q_{ij}^2 \\ & h_i \geq l_i \\ & q_{ij} \geq 0 \end{aligned}$$

Let a water supply network consists of m nodes and n edges with the node varaible $h \in \mathbf{R}^m$ represent pressure or potential and $q \in \mathbf{R}^n$ represent flows in the network. The network topology is defined by a node-edge incidence matrix $A \in \mathbf{R}^{m\times n}$ and other properties of the network are encoded in a set of data vectors: $d \in \mathbf{R}^m$ is a vector of demands, $l \in \mathbf{R}^n$ is a vector of minimum pressure requirement, $\lambda \in \mathbf{R}^n$ is a vector of the loss coefficient. Let $\Lambda = \operatorname{diag}(\lambda)$ and q^k denotes a vector of the k^{th} power. Note that, under optimal condition, the objective is monotone increasing with respect to q_{ij}; therefore, constraint (1) is equivalent to $\sum_{ij\in\delta^-(j)} q_{ij} - \sum_{ji\in\delta^+(j)} q_{ji} \geq d_j$.

Then, the convex minimum-energy flow with variables (q, h) can be rewritten in matrix form as

$$\begin{aligned} \text{minimize} \quad & q^T \Lambda q^2 \\ \text{subject to} \quad & Aq \leq d \\ & A^T h = \Lambda q^2 \\ & h \geq l \\ & q \geq 0 \end{aligned} \tag{MEF}$$

Although the objective function of (MEF) is convex, the potential-flow coupling constraint $A^T h = \lambda q^2$ is nonconvex. Hence, (MEF) is nonconvex problem.

3 Convex Relaxation of the Minimum-Energy Flow

3.1 Overview

Our approach is to solve the convex relaxation of (MEF) to get the optimal flow and head pressure, then recover information about integer decision variables from these answer. Our approch divides into 3 parts. In the first part, we find the direction of the optimal flow. Then, in the second part, we use this direction to solve the relaxation of (MEF) to get optimal flow and pressure. Lastly, we recover the solution to the integer decision variables from the flow and pressure we obtained.

3.2 Predirection

We first introduce the *pure flow* problem, which seeks to minimize the overall energy in the network while ignoring all pressure-related constraints.

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{3} q^T \Lambda q^2 \\ \text{subject to} \quad & Aq \leq d \\ & q \geq 0 \end{aligned} \tag{PF}$$

Not only it is convex, the pure flow problem also implicitly produces the head pressure for each node due to its optimality condition.

Theorem 3.1. *Let q^* be the optimal solution for the pure flow problem. For each edge (i, j), we have that*

1. $h_i - h_j = \lambda_{ij} q_{ij}^2 \quad$ *for* $\quad q_{ij} > 0$
2. $h_i - h_j < 0 \quad$ *for* $\quad q_{ij} = 0$

Proof. Let h and ν be the dual variable associated with the constraint $Aq \leq d$ and $q \geq 0$ respectively. Taking the Lagrangian of the pure flow problem, we have

$$\begin{aligned} L(q, h, \nu) &= \frac{1}{3} q^T \Lambda q^2 - h^T (Aq - d) - \nu^T q \\ \nabla_q L(q, h, \nu) &= \Lambda q^2 - A^T h - \nu = 0 \end{aligned}$$

This implies

$$\Lambda q^2 = A^T h + \nu$$

By complementary slackness, we know that $\nu_{ij}^T q_{ij} = 0$. Therefore, for $q_{ij} > 0$, we have $\nu_{ij} = 0$, which yields $\lambda_{ij} q^2 = h_i - h_j$. For $q_{ij} = 0$, we have $\nu_{ij} > 0$, which implies $h_i - h_j < 0$. □

Theorem 3.1 implies that, for $q^* > 0$, the potential-flow coupling constraint is satisfed and the head pressure h can be solved directly from $A^T h = \Lambda (q^*)^2$.

First, we solve for the optimal direction of the flow using pure flow problem on the bi-directional graph. We represent the undirected graph $G(V, E)$ with bi-directional edges to get directed graph $G'(V, E')$ such that $|E'| = 2|E|$. Then, the node-edge incident matrix of G' is $A' = [A \; -A]$ and the friction matrix $\Lambda' = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}$ and let $\hat{q} \in \mathbf{R}^{2|E|}$ to represent the flow in the bi-directional edges. We can solve the pure flow problem on the bi-directional graph as follow

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{3} \hat{q}^T \Lambda' \hat{q}^2 \\ \text{subject to} \quad & A' \hat{q} \leq d \\ & \hat{q} \geq 0 \end{aligned} \tag{5}$$

Then, we select the direction of the flow according to direction of the flow on the bi-directional edges and let A be the new node-edge incident matrix of the directed version of G.

3.3 Convex Relaxation

We want to solve (MEF) to get the optimal flow as well as the head pressure correspond to it; however, (MEF) is non-convex because of the potential-flow coupling constraint, namely $A^T h = \Lambda q^2$. Therefore, we relax the equality constraint to inequality constrain as $A^T h \geq \Lambda q^2$. There is a physical interpretation of this relaxation: the pressure between two junctions need to be large enough to induce the flow in the pipe section between them. Introducing the slack variable s, the inequality constraint $A^T h \geq \Lambda q^2$ can be converted to $A^T h - s = 0$ and $s \geq \Lambda q^2$. Moreover, we penalize s by adding $\mathbf{1}^T s$ to the constraint and also penalzie h to be small by adding $\|h\|_2$. Then, the problem can be reformulated as follow:

$$\begin{array}{ll} \text{minimize} & \alpha(q\Lambda q^2) + \beta \|h\|_2 + \gamma(\mathbf{1}^T s) \\ \text{subject to} & Aq \leq d \\ & q \geq 0 \\ & h \geq l \\ & A^T h - s = 0 \\ & \Lambda q^2 \leq s \end{array} \tag{RMEF}$$

where α, β and γ are the penalty constatnt to adjust the contribution for each terms of the objective. Since the magnitude of each objective terms are largely different we use the penalty constant to normalize the constribution of each terms.

3.4 Solution Recovery

4 Numerical Result

4.1 Overview of Examples

4.1.1 Example 1: Town

We first consider a small water supply network `town` described in [BA91]. This network can be considered as a simplified but reasonably typical municipal water distribution system. The network is shown in Figure 2. The network contains 19 pipe segments, 13 junction nodes, 3 water sources and 4 circuits. The pipe system characteristics and node demands are given by Table 1.

4.1.2 Example 2: Small Network

The second water supply network that we consider is from the small instance, which is a small part of the water supply network from one city. This network is shown in Figure 3. It comprises of 276 edges, 272 nodes, and please refer to Table 2 for the the detailed information about the edges and nodes.

Note that the network is challenging for the MINLP solver due to the problem size and the nonconvex pressure-loss constraints.

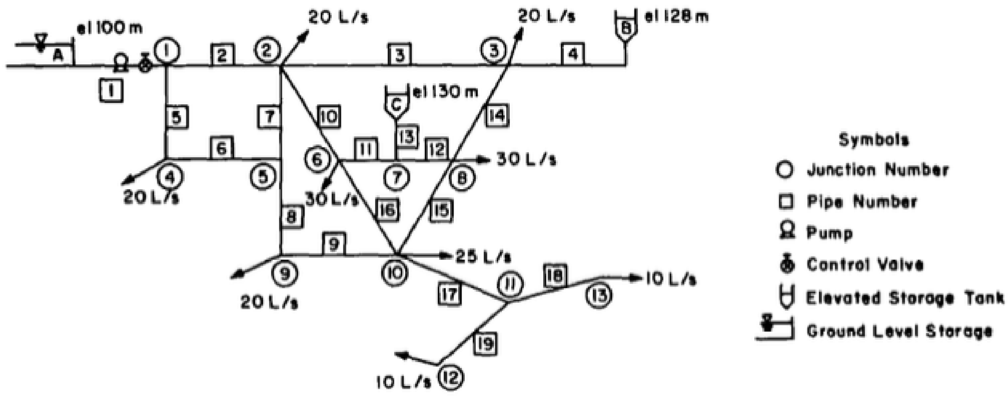

Figure 2: `town` water network

Pipe Number	Head Node	Tail Node	Length (m)	Diameter (mm)	Roughness coefficient	Node	Demand (l/sec)	Head (m)
1	A	1	300.0	300.0	120.0	1	0.0	1005.0
2	1	2	250.0	250.0	120.0	2	20.0	1005.0
3	2	3	450.0	250.0	120.0	3	20.0	1105.0
4	3	B	300.0	200.0	120.0	4	20.0	1105.0
5	1	4	150.0	250.0	120.0	5	0.0	1005.0
6	4	5	250.0	200.0	120.0	6	30.0	905.0
7	5	2	150.0	200.0	120.0	7	0.0	995.0
8	5	9	170.0	200.0	120.0	8	30.0	1005.0
9	9	10	200.0	250.0	120.0	9	20.0	1005.0
10	2	6	160.0	200.0	120.0	10	25.0	1005.0
11	6	7	140.0	200.0	120.0	11	0.0	1005.0
12	8	7	80.0	250.0	120.0	12	10.0	995.0
13	7	C	200.0	250.0	120.0	13	10.0	995.0
14	3	8	200.0	250.0	120.0	A	Source	1100.0
15	10	8	300.0	200.0	120.0	B	Source	1128.0
16	6	10	300.0	200.0	120.0	C	Source	1130.0
17	10	11	300.0	150.0	120.0			
18	11	13	200.0	150.0	120.0			
19	11	12	175.0	150.0	120.0			

Table 1: `town` network characteristics

4.1.3 Example 3: Large Network

The last water supply network that we consider is from the large instance, which is a whole part of the water supply network from one city. This network is shown in Figure 4. As you

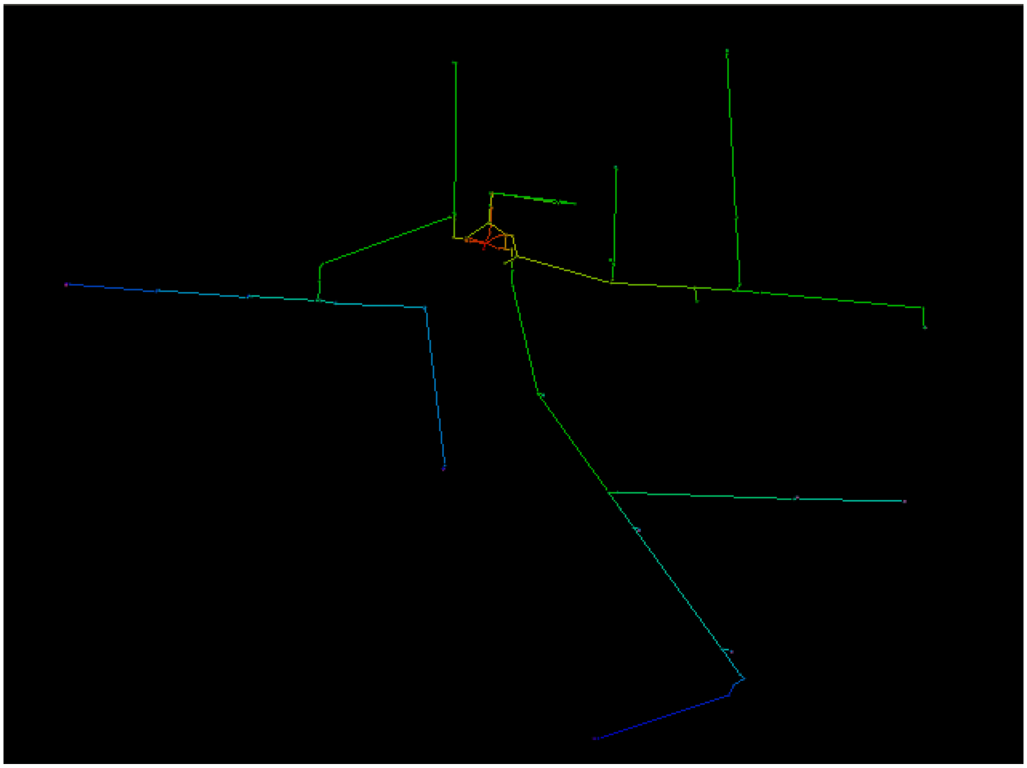

Figure 3: `small` water network

Nodes		Edges	
Reservior	20	Pipes	207
Tank	1	Valves	49
Junction	250	Pumps	20
Consumer	1		

Table 2: `small` network problem size

can see, the pink part in the center is just the small water supply network from Example 2. This large water supply network comprises of 8059 edges, 7876 nodes, and please refer to Table 3 for the the detailed information about the edges and nodes.

Note that the network is challenging for the MINLP solver due to the problem size and the nonconvex pressure-loss constraints.

4.2 Predirection

When we design the optimal water supply network, the directions of each edge would be one of the import decision variables. However, the program would become complicated if we optimize direcion, flow and head pressure at the same time. Thus, we solve for the

Figure 4: `large` water network

Nodes		Edges	
Reservior	20	Pipes	6074
Tank	1	Valves	1961
Junction	7099	Pumps	24
Consumer	846		

Table 3: `large` network problem size

optimal direction first. In this program, we do not consider the pressure-loss constraints. We only care about the flow conservation constraints, and set the energy consumption as the objective. The numerical result of this program is as Table 4.

	Town	Small	Large
Energy	22.734	126.967	261.943

Table 4: Numerical result of predirection

The above result represents the **minimum** energy consumption that we need in total in order to meed the demand of each consumer. These results could serve as a “pseudo” benchmark for our next programs.

In the following, we use the `town` water network example to illustrate the process of predirection. At first, we are given a bidirectional graph, as Figure 5, and we need to solve the pure flow problem to find the direction of each flow.

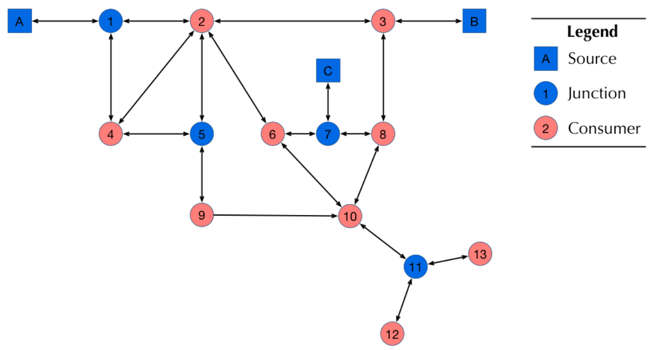

Figure 5: `town` water network(bidirectional)

After sloving the pure flow problem, we can determine the direction of the flow, as Figure 6. This direction is the optimal flow direction without considering the pressure constraints.

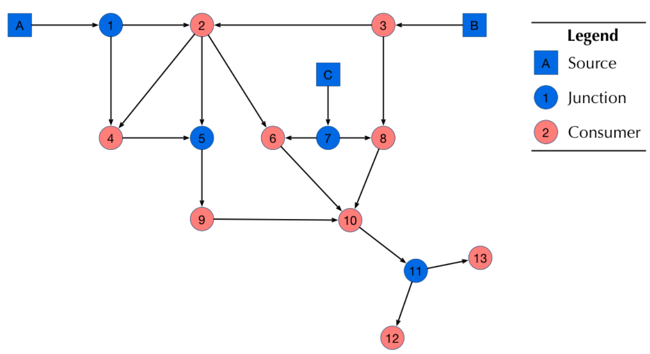

Figure 6: `town` water network(directional)

4.3 Convex Relaxation

In this part, we optimize the flow and head pressure based on the "optimal" directed graph that we find in the predirection step. We do not consider pressure-loss constraints in the previous part, but we need to take them into consideration in this part. Intuitively speaking, the optimal energy consumption increases after we take the pressure-loss constraints into consideration, because we need to generate more power in some edges in order to meet the minimum head pressure constraints. Theoretically speaking, the optimal energy consumption increases, because the feasible region becomes narrower after we consider the pressure-loss constraints.

We define $\left\|A^T h - \Lambda q^2\right\|_2$ as **gap** and define $\frac{1}{3}q^T$ as energy consumption, and this represents the additional energy that we need to generate in order to meet the minimum pressure constraints. Thus, the total cost equals to the sum of gap and energy consumption, i.e. $\left\|A^T h - \Lambda q^2\right\|_2 + \frac{1}{3}q^T$.

Through the numerical experiments, we observe that choosing the right weight for the three terms of the objective in (RMEF), namely $\frac{1}{3}q^T \Lambda q^2$, $\|h\|_2$, and $\mathbf{1}^T s$, affects the gap and energy consumption. Thus, we can modify the (RMEF) a little through putting weight parameters in front of the three terms in the objective, as follow:

$$\begin{array}{ll} \text{minimize} & \alpha \frac{1}{3} q \Lambda q^2 + \beta \|h\|_2 + \gamma \mathbf{1}^T s \\ \text{subject to} & Aq \le d \\ & q \ge 0 \\ & h \ge hc \\ & A^T h - s = 0 \\ & \Lambda q^2 \le s, \end{array} \tag{RMCF$'$}$$

Note that it is important to normalize the contribution of the three terms to the same magnitude before we do the numerical experiments on putting different weight to these three terms. After the normalization, the energy consumption and gap dramatically decrease, as Table 5.

	Town Original	Town Normalized	Small Original	Small Normalized	Large Original	Large Normalized
gap	**0.00**	**0.00**	**2.24**	**0.77**	**3,752**	**1,359**
objective	7,097	68	24,107	582	183,010	904
resource	**24**	**23**	**136**	**130**	**283**	**271**
norm h	5,433	5,438	11,716	12,150	132,930	152,560
sum s	1,640	1,661	12,255	12,033	49,798	37,825

Table 5: Performance before and after normalization

Situation 1: Weight More $\frac{1}{3}q^T \Lambda q^2$. From Figure 7, it shows that as we put more weight to $\frac{1}{3}q^T \Lambda q^2$, the energy consumption definitely decreases, but the pattern of the gap is not significant. In terms of the energy consumption, as the weight becomes larger and larger, the decreasing rate of energy consumption becomes smaller and smaller. Therefore, we need to choose a right weight in order to get a best compromise between energy consumption and gap.

Situation 2: Weight More $\|h\|_2$. From Figure 8, it shows that as we put more weight to $\|h\|_2$, both the energy consumption and gap increase. It means that weighting more on $\|h\|_2$ does not help to decrease either energy consumption or gap. The reason is that $\|h\|_2$ does not have any direct relationship with either energy consumption or gap. Therefore, we should not put more weight to $\|h\|_2$.

Situation 3: Weight More $\mathbf{1}^T s$. From Figure 9, it shows that as we put more weight to $\mathbf{1}^T s$, the energy consumption increases and the gap decreases. It means that it is effective to put more weight to $\mathbf{1}^T s$ in order to optimize the cost. In terms of the gap, as the weight

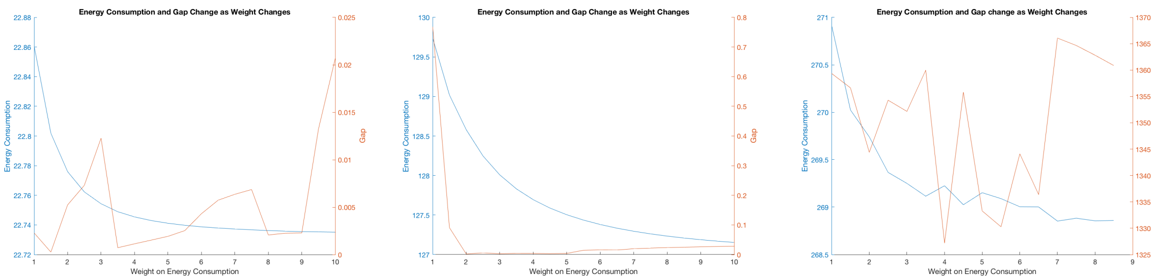

Figure 7: Performance on energy consumption and gap as weight increases ($\frac{1}{3}q^T\Lambda q^2$)

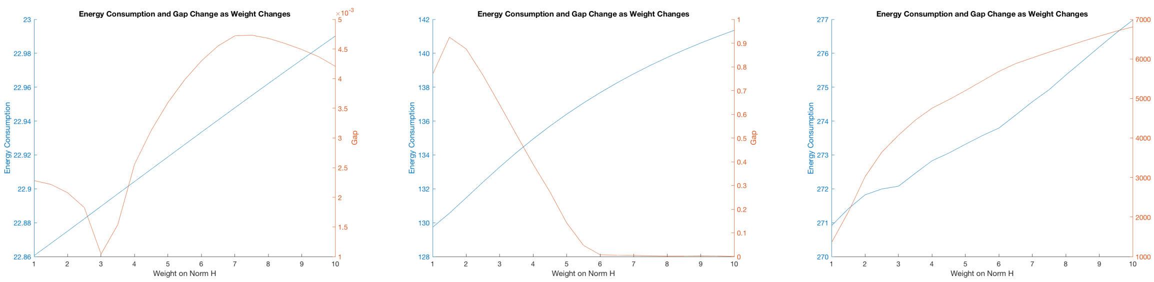

Figure 8: Performance on energy consumption and gap as weight increases ($\|h\|_2$)

becomes larger, the gap decreases first and bounce back latter. Therefore, we need to choose a right weight in order to get a best compromise between energy consumption and gap.

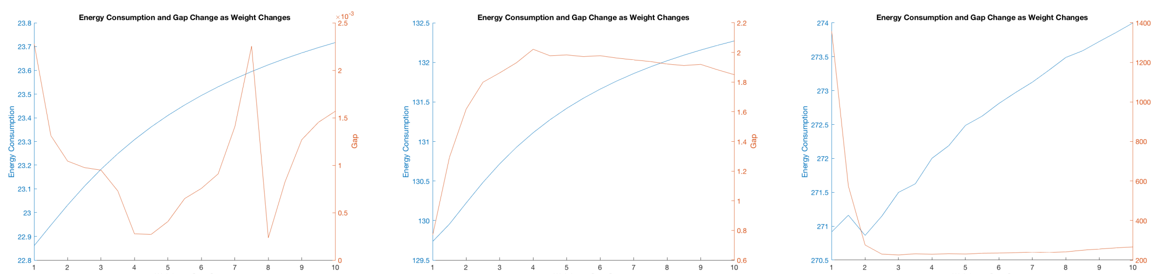

Figure 9: Performance on energy consumption and gap as weight increases($\mathbf{1}^T s$)

In the following, we use the `town` water network example to illustrate the process of convex relaxation. Given the directional graph of the water network, as Figure 6, we solve the convex relaxation model to find the optimal flow of each pipe and the optimal head pressure of each node, as Figure 10.

4.4 Solution Recovery

The important aspect of our approach is to recover the integer decision variables based the solution we obtained from (RMEF). If the edge does not satisfy $A^T h = \Lambda q^2$ and $q = 0$ and

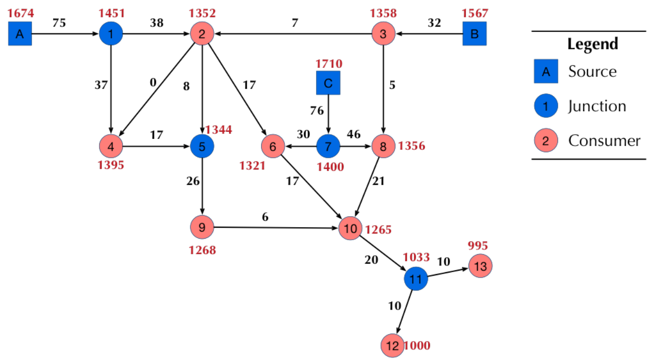

Figure 10: `town`: result of the convex relaxation model

this edge is a valve, the valve should be turned off.

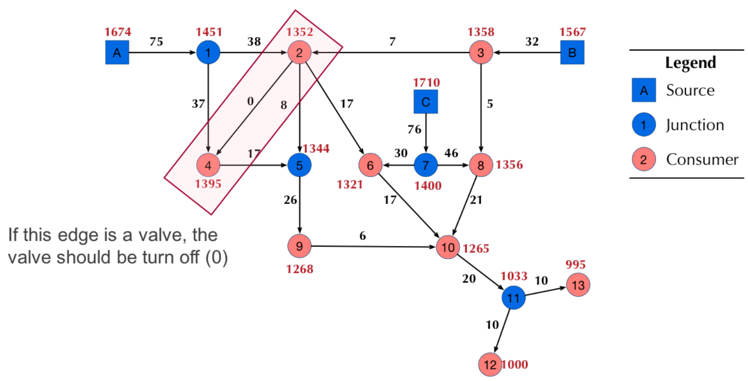

Figure 11: `town`: close the valve

Additionally, we could also detect which pump should be turned on. If the edge does not satisfy $A^T h = \Lambda q^2$ and $q > 0$, then we could conclude that there exist some external force to increase the head pressure. If this edge is a pump, the pump should be turned on to raise the end node pressure.

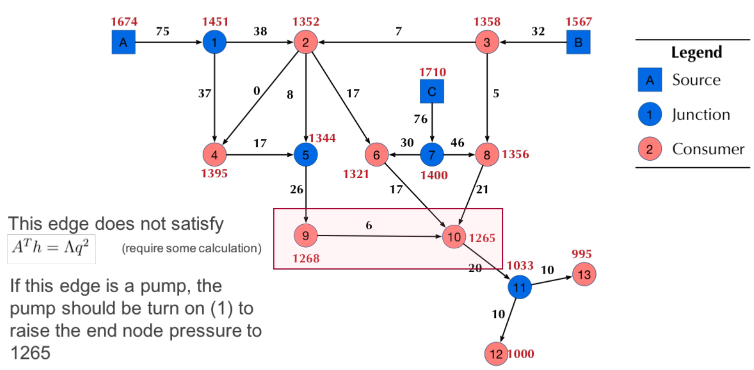

Figure 12: `town`: turn on the pump

5 Discussion

Although (RMEF) is a simple convex relaxation of the MINLP, it does not take into account the devices with multi operation modes such as pumps and valves that correspond to the integer variables in the MINLP. Therefore, the next step is to incorporate the integer variables in the problem.

We want to explore furthur the ADMM method where the potential variable is the dual variable. At each step, for fixed node potentials, we solve the pure flow problem to find an optimal flow and corresponding dual head pressure. Then, we solve the integer program to update the pressure, alternately. Moreover, due to the sparsity of large-scale water supply network, we might be able to efficiently decompose the search space for the discrete variables.

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