Planning, Learning and Decision Making

Lecture 5. Expected utility

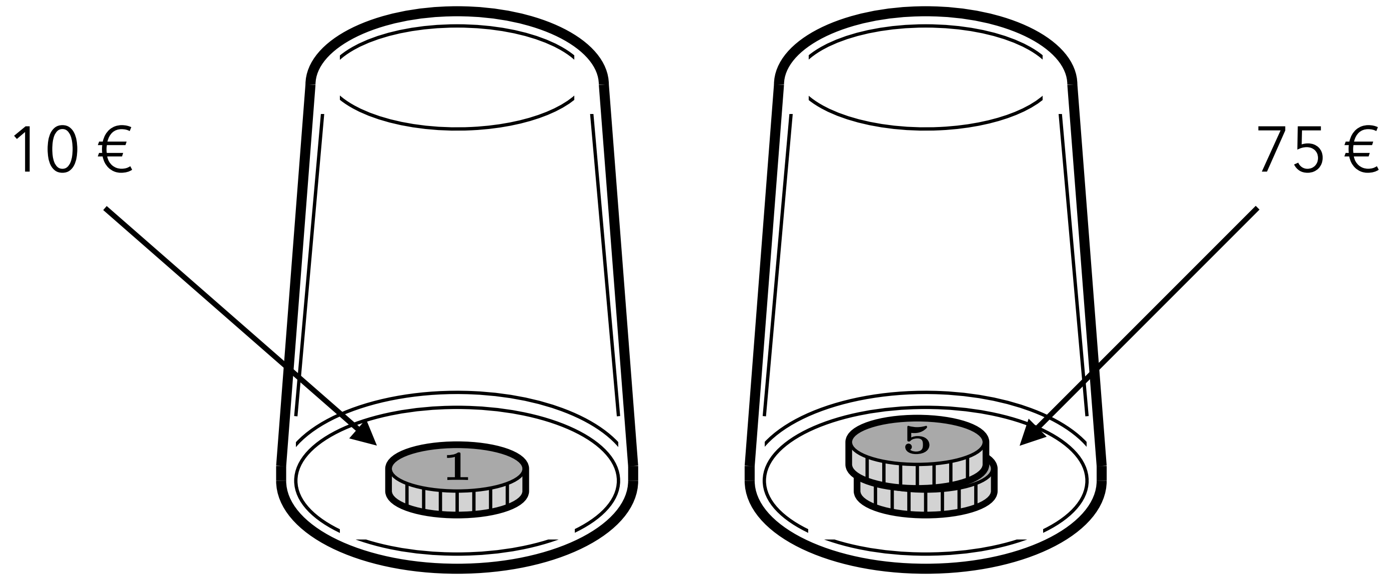
Making decisions

Decision I

- You are a participant in a contest
  - You are currently winning **50€**
  - You are at the last stage of the game

Decision I

- You are offered a choice:
  - Keep your prize
  - Choose one of the cups



Which one will you choose?

Decision II

- You are a participant in a contest
  - You are currently winning a **50€-valued meal** in a steak house
  - You are at the last stage of the game

Decision I

- You are offered a choice:
  - Keep your price
  - Choose one of the cups

Which one will you choose?

Decisions and preferences

- The prizes in both situations have the same monetary value
- In the first situation, the more valuable prize will be selected
- In the second situation, different choices are possible…

Decisions and preferences

- If you are allergic to cheese, you may prefer to keep your current prize
- If you love pizza, you may prefer to go for the pizza house meal

Is this "irrational" in some way?

Decisions and preferences

- Decisions often involve factors other than value:
  - Many decisions are based on personal **preferences**

Can we program “preferences” into a computer?

Can we treat decision making algorithmically?

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Preferences

What is a strict preference?

- A strict preference is a **binary relation** between **outcomes**

Binary relation

A **binary relation R** on a set of outcomes $\mathscr{X}$ is a subset of $\mathscr{X} \times \mathscr{X}$. A pair (x, y) in such subset is said to verify the relation, and we write it

$$x \xrightarrow{R} y$$

What is a strict preference?

- A strict preference is a **binary relation** between **outcomes**
  - When outcome x is **preferred** to y, we write $x > y$
  - If an individual prefers outcome x to y, that means that it would be willing to pay some "fair amount" to have x instead of y

What is a strict preference?

- Is any binary relation a valid preference?
  - **No!**

Example 1.

Suppose that, for some decision maker, $x \succ y$ and $y \succ x$.

We could charge the agent some amount a to swap x for y. And then swap some amount b to swap y for x. And then charge a to swap x for y again, and so on…

Not reasonable

What is a strict preference?

- Is any binary relation a valid preference?
  - **No!**

Preferences must not be **symmetric**

Example 1.

Suppose that, for some decision maker, $x > y$ and $y > x$.

We could charge the agent some amount *a* to swap *x* for *y*. And then swap some amount *b* to swap *y* for *x*. And then charge *a* to swap *x* for *y* again, and so on…

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What is a strict preference?

Anti-symmetric relation

A binary relation R on a set of outcomes $\mathcal{X}$ is **anti-symmetric** if, whenever $x \overset{R}{\to} y$ it holds that $y \overset{R}{\not\to} x$.

What is a strict preference?

- Is any binary relation a valid preference?
  - **No!**

Example 2.

Suppose that, for some decision maker, $x > y$, $y > z$ and $z > x$.

We could charge the agent some amount a to swap z for x. And then pay some amount b to swap y for z. And then charge c to swap x for y, and so on…

Not reasonable

What is a strict preference?

- Is any binary relation a valid preference?
  - **No!**

Preferences must be **negative transitive**

Example 2.

Suppose that, for some decision maker, $x > y$, $y > z$ *and* $z > x$.

We could charge the agent some amount *a* to swap *z* for *x*. And then pay some amount *b* to swap *y* for *z*. And then charge *c* to swap *x* for *y*, and so on…

What is a strict preference?

Negative transitive relation

A binary relation R on a set of outcomes $\mathscr{X}$ is **negative transitive** if, whenever $y \overset{R}{\not\to} z$ and $x \overset{R}{\not\to} y$ it holds that $x \overset{R}{\not\to} z$.

Equivalently, if $x \overset{R}{\to} y$, then either $z \overset{R}{\to} y$ or $x \overset{R}{\to} z$.

What is a strict preference?

- A **strict preference** is a **binary relation** between **outcomes** such that
  - It is anti-symmetric
  - It is negative transitive

Related relations

- If "$\succ$" is a strict preference on some set of outcomes $\mathcal{X}$…
  - … we write "$x \succ y$" if outcome ***x* is preferred to *y*** ("x is better than y")
  - … we write "$x \prec y$" if outcome ***y* is preferred to *x*** (or "x is worse than y")
  - … if $x \nsucc y$ and $x \nprec y$ (x is neither better nor worse than y), we say that **the two outcomes are indifferent**, and write $x \sim y$

Related relations

- If "$\succ$" is a strict preference on some set of outcomes $\mathcal{X}$…
  - … if "$x \succ y$" or "$x \sim y$" we can write "$x \succsim y$" (***x* is not worse than *y***).
  - … if "$x \prec y$" or "$x \sim y$", we can write "$x \precsim y$" (***x* is not better than *y***).

Properties of preferences

- Given any two outcomes, exactly one of the following holds:
  - $x \succ y$
  - $x \sim y$
  - $x \prec y$
- **Why?**

4 possible cases:

- $x \succ y$ and $y \succ x$ → *impossible*
- $x \succ y$ and $y \not\succ x$ → $x \succ y$
- $x \not\succ y$ and $y \succ x$ → $x \prec y$
- $x \not\succ y$ and $y \not\succ x$ → $x \sim y$

Properties of preferences

- Given any two outcomes, exactly one of the following holds:
  - $x \succ y$
  - $x \sim y$
  - $x \prec y$

- Equivalently, given any two outcomes, either $x \succcurlyeq y$ or $y \succcurlyeq x$

The relation $\succcurlyeq$ is complete

Properties of preferences

- Given any outcomes x, y and z, if $x > y$ and $y > z$, then $x > z$
- **Why?**

- if $x > y$, by negative transitivity, either $x > z$ or $z > y$ (or both)
- if $y > z$, by negative transitivity, either $y > x$ or $x > z$ (or both)
- Then it must be that $x > z$

Properties of preferences

- Given any outcomes x, y and z, if $x > y$ and $y > z$, then $x > z$

The relation $>$ is transitive

Properties of preferences

- Given any outcomes x, y and z, if $x \succcurlyeq y$ and $y \succcurlyeq z$, then $x \succcurlyeq z$

The relation $\succcurlyeq$ is transitive

Properties of preferences

- Indifference is
  - Reflexive (i.e., $x \sim x$)
  - Symmetric (i.e., if $x \sim y$, then $y \sim x$)
  - Transitive (i.e., if $x \sim y$ and $y \sim z$, then $x \sim z$)
- **Why?**

- it must hold that $x \not\succ x$
- since $x \not\succ y$ and $y \not\succ x$
- let $x \sim y$ and $y \sim z$, suppose $x \succ z$; then, either $x \succ y$ or $y \succ z$, which is a contradiction

Properties of preferences

- Indifference is
  - Reflexive (i.e., $x \sim x$)
  - Symmetric (i.e., if $x \sim y$, then $y \sim x$)
  - Transitive (i.e., if $x \sim y$ and $y \sim z$, then $x \sim z$)

The relation $\sim$ is an equivalence relation

Properties of preferences

- If $x \succ y$ and $x \sim z$, then $z \succ y$
- **Why?**

- if $x \succ y$, then either $x \succ z$ or $z \succ y$
- then, $z \succ y$

Properties of preferences

Summarizing:

- $\succ$ is **anti-symmetric**, **transitive** and **negative transitive**
- $\succcurlyeq$ is **complete** and **transitive**
- $\sim$ is **reflexive**, **symmetric** and **transitive** (i.e., is an equivalence relation)

Rational preference

- A relation is called a **rational preference** if it is **complete** and **transitive**
  - **The relation $\succcurlyeq$ is complete and transitive and hence rational**

Utility

Computational considerations

- Computationally, preferences are cumbersome to maintain
- Preferences express an **ordering** between outcomes

Order preserving function

Existence

- Does an order preserving function exist?
  - **Yes!**
  - … if the preference is rational

- **Why?**
  - When the preference is rational, we can sort all outcomes consistently
  - Given $N + 1$ outcomes, sort them, assign a value of 0 to the first, 1 to the second, …, N to the last one

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Existence

Theorem

Let $\mathscr{X}$ be a set of possible outcomes, and $\succcurlyeq$ a rational preference on $\mathscr{X}$. Then, there is a function $u : \mathscr{X} \to \mathbb{R}$ such that $u(x) \geq u(y)$ if and only if $x \succcurlyeq y$, for all $x, y \in \mathscr{X}$.

***u* is called a *utility* function**

Making decisions

- We can use utility functions in computational decision-making
  - Given a set of alternatives/actions $\mathscr{A}$
  - $X(a)$ is the outcome associated with action $a \in \mathscr{A}$
  - The **value** of action a is

$$Q(a) \stackrel{\text{def}}{=} u(X(a))$$

Utility of associated outcome

Making decisions

- We can use utility functions in computational decision-making
  - Given a set of alternatives/actions $\mathscr{A}$
  - $X(a)$ is the outcome associated with action $a \in \mathscr{A}$
  - The **value** of action *a* is

$$Q(a) = \sum_{x \in \mathcal{X}} u(x)\mathbb{I}[x = X(a)]$$

1 if condition is true
0 otherwise

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Making decisions

- Select actions with maximum value:

$$\underset{a \in \mathcal{A}}{\operatorname{argmax}}\, Q(a)$$

Uncertainty

Handling uncertainty

- What if action outcomes are uncertain?
  - $P(x \mid a)$ denotes the probability of outcome x when action a is selected
  - The **value** of an action was:

$$Q(a) = \sum_{x \in \mathcal{X}} u(x)\mathbb{I}[x = X(a)]$$

Handling uncertainty

- What if action outcomes are uncertain?
  - $P(x \mid a)$ denotes the probability of outcome x when action a is selected
  - The **expected value** of an action is now:

$$
\begin{aligned}
Q(a) &= \sum_{x \in \mathcal{X}} u(x) P(x \mid a) \\
&= \mathbb{E}\left[u(\mathrm{x}) \mid a\right]
\end{aligned}
$$

Formulating our problem

- One such decision problem is described as a tuple

$$(\mathscr{X}, \mathscr{A}, P, u)$$

  - $\mathscr{X}$ is the set of possible outcomes
  - $\mathscr{A}$ is the set of available actions
  - For each outcome x and action a, $P(x \mid a)$ is the probability of outcome x when action a is selected
  - For each outcome x, $u(x)$ is the utility of x

Examples

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The weather example

The weather example

- You must decide whether to take an umbrella before leaving home
  - Carrying an umbrella is inconvenient
  - You don't want to get soaked because of rain

The weather example

- The weather forecast is:
  - Rain with probability 0.3
  - Sun with probability 0.7

The weather example

- At any moment,
  - You will be outside with a probability 0.5
  - You will be indoors with a probability 0.5

Outcomes

- What are the possible outcomes?

  (A) Don't carry umbrella; get home dry

  (B) Carry umbrella; get home dry

  (C) Don't carry umbrella; get home soaked

  (D) Carry umbrella; get home soaked

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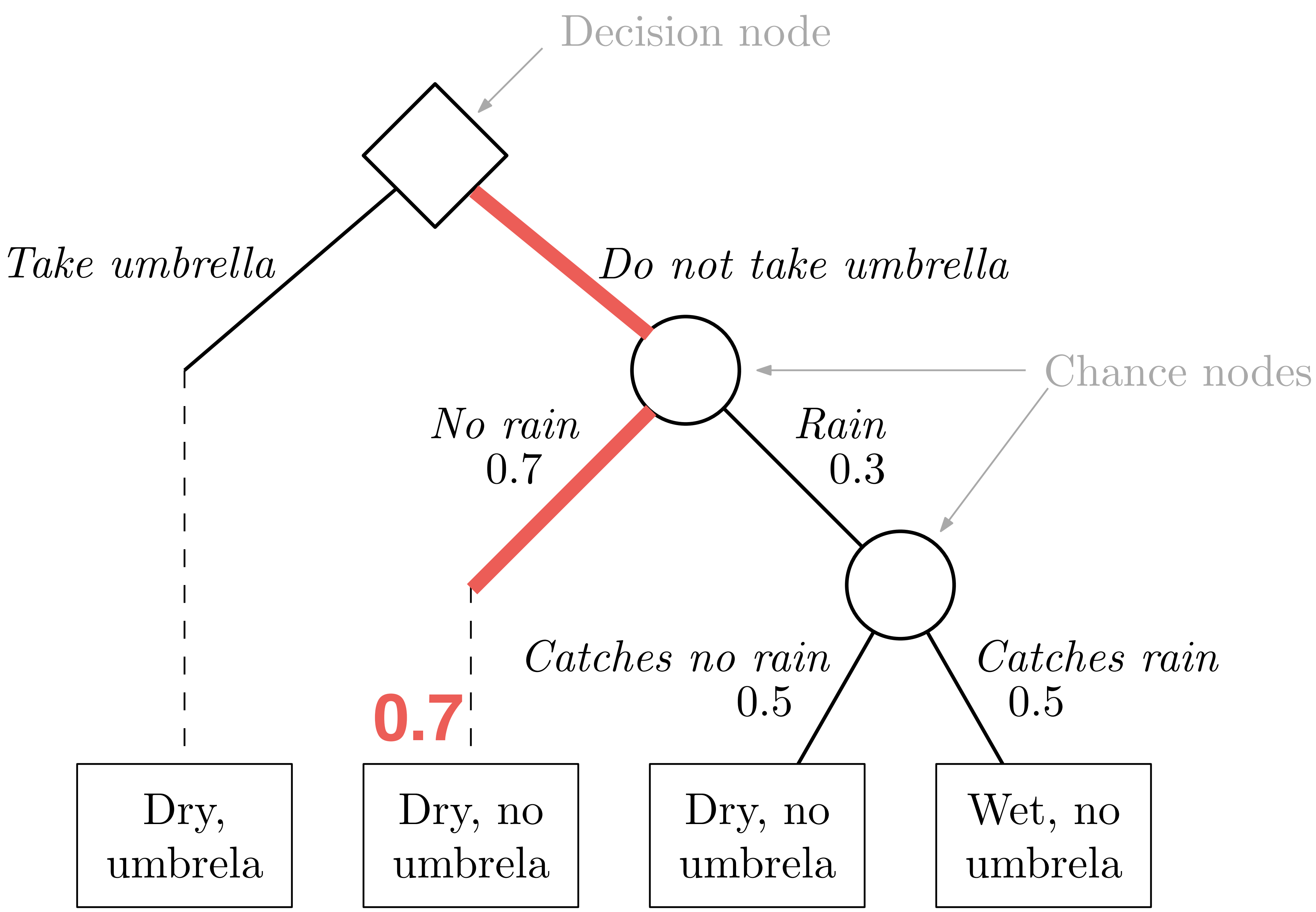
Actions

- What are the possible actions?

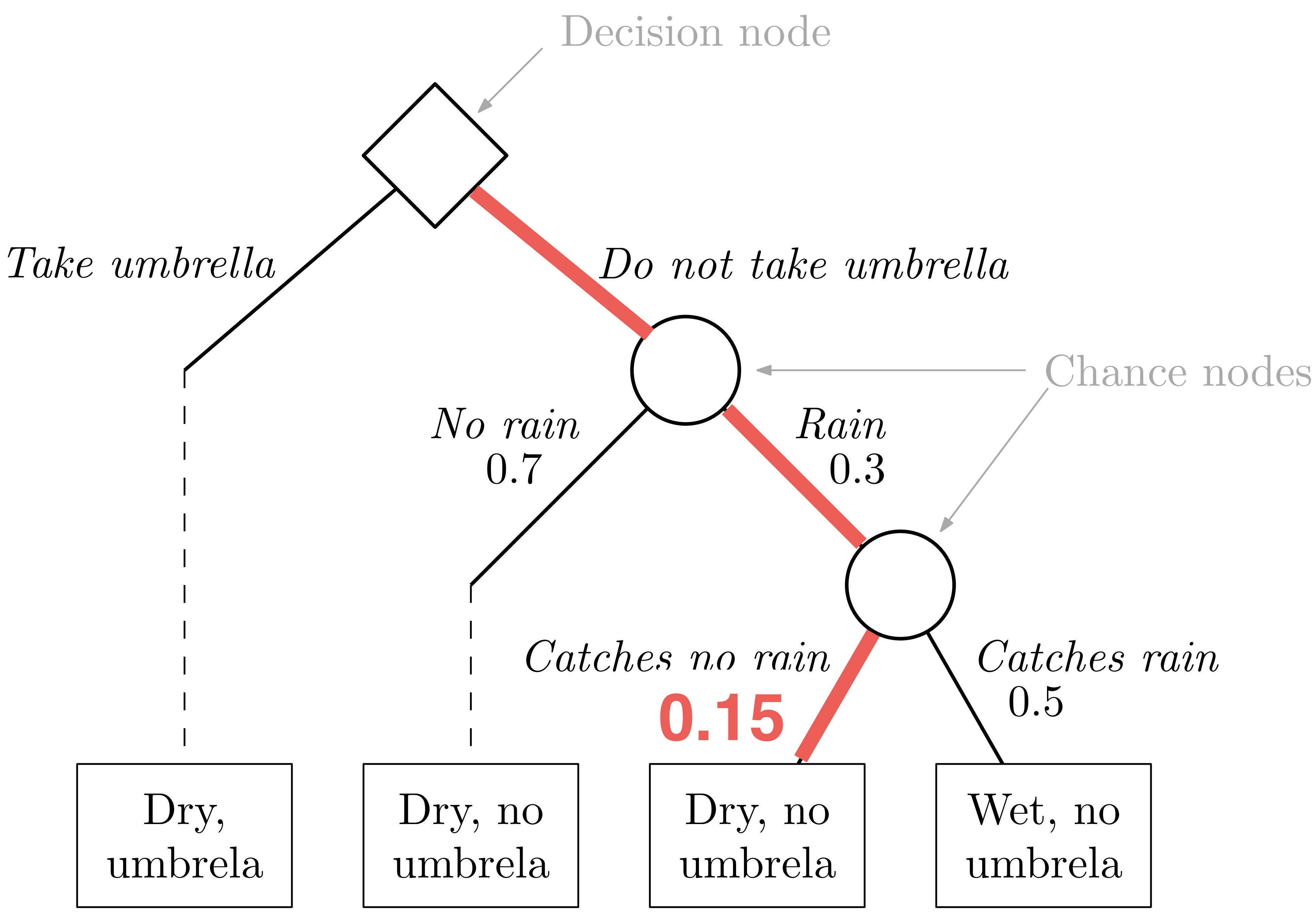
  (A) Take the umbrella

  (B) Don't take the umbrella

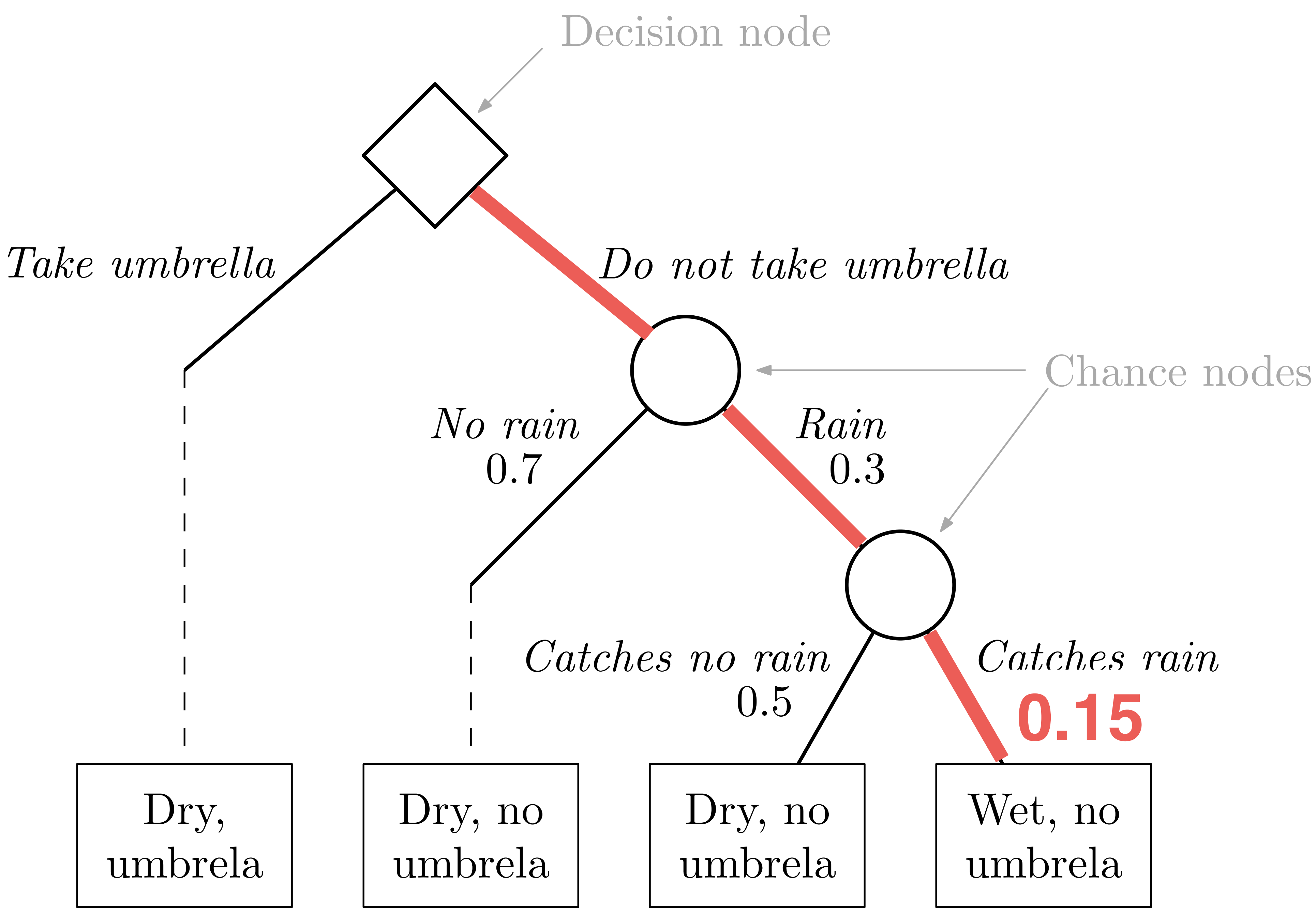
Decision tree



Decision tree



Decision tree



Outcome probabilities

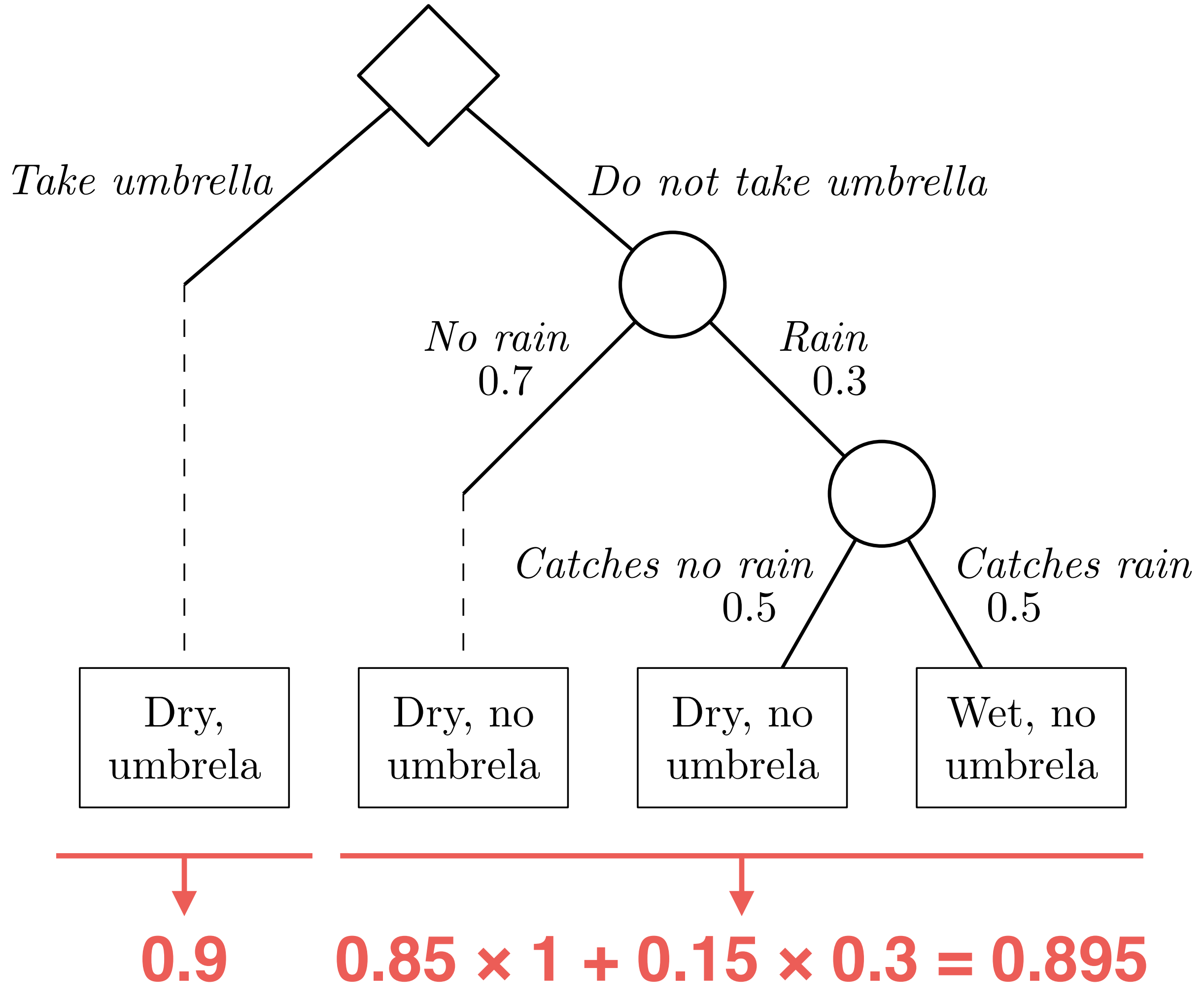
- If a = "Take the umbrella",
  - P(Carry umbrella, get home dry | a) = 1
  - $P(x \mid a) = 0$, otherwise

- If a = "Don't take the umbrella",
  - P(Don't carry umbrella, get home dry | a) = 0.85
  - P(Don't carry umbrella, get home soaked | a) = 0.15
  - $P(x \mid a) = 0$, otherwise

Expected value

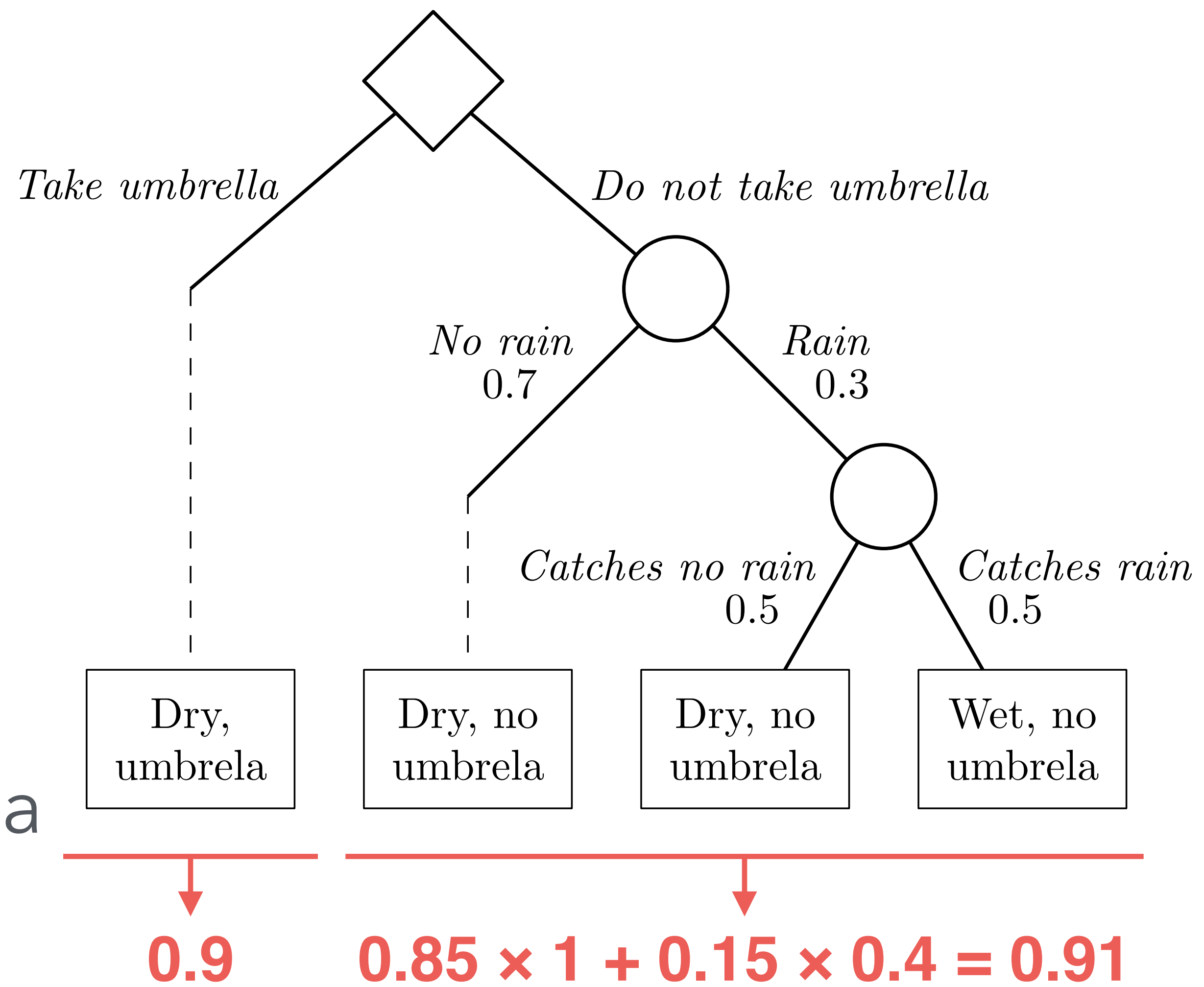
- Multiply leave probabilities by corresponding value
  - u(No umbrella, dry) = 1
  - u(Umbrella, dry) = 0.9
  - u(No umbrella, wet) = 0.3
  - u(Umbrella, wet) = 0

- Best action: Take umbrella



Expected value

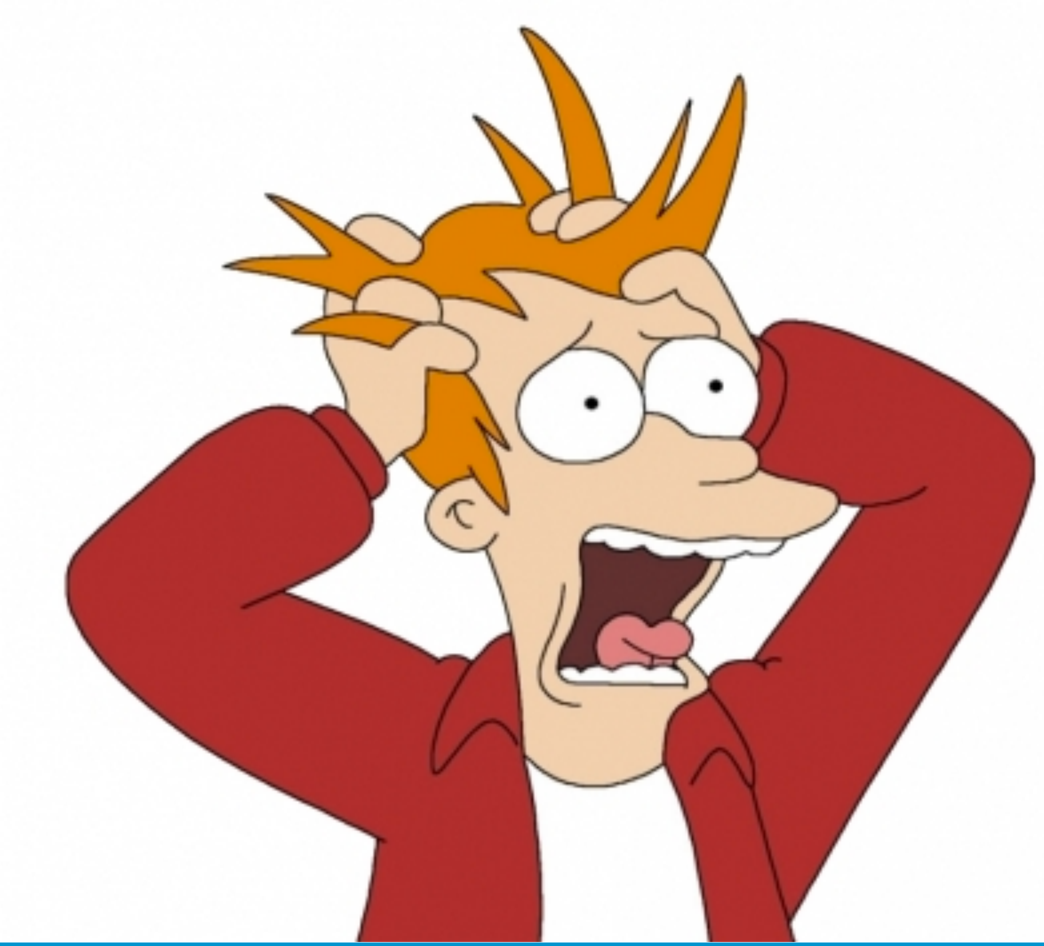
- Multiply leave probabilities by corresponding value
  - u(No umbrella, dry) = 1
  - u(Umbrella, dry) = 0.9
  - u(No umbrella, wet) = 0.4
  - u(Umbrella, wet) = 0
- Best action: Don’t take umbrella



The student example

The student example

- A freshman finished the final project for a course
- On her way to submit the project report, she realizes that half the pages are missing!



Alternatives

- She has two alternatives:

  (A) Return home and print the remaining pages

  (B) Print the remaining pages at the University

The student example

- If she returns home…
    - There is a 0.6 probability that she'll arrive late at the University, due to traffic

The student example

- If she prints in the University…
  - There is a 0.3 probability that she can't find a printer in time (she'll submit an incomplete report)
  - There is a 0.5 chance that the printer is busy (she'll submit the report late)

The student example

- If she submits the report late, she'll lose 2 points
- If she submits an incomplete report, she'll lose 3 points

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Outcomes

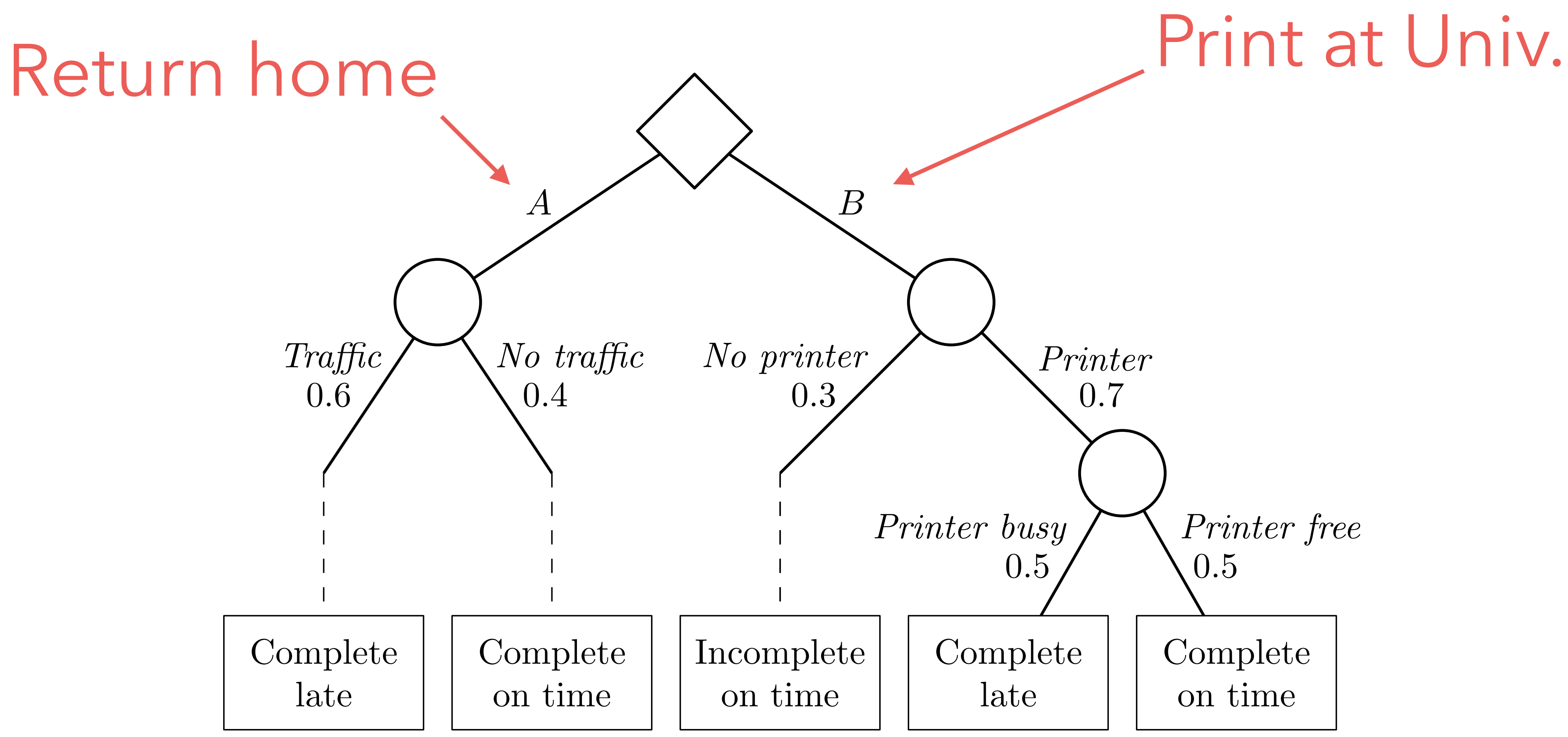
- What are the possible outcomes?

  (A) Report is complete and on time (CT, utility of 0)

  (B) Report is complete but late (CL, utility of -2)

  (C) Report is incomplete (IT, utility of -3)

Decision tree



P(CT | H) = 0.4

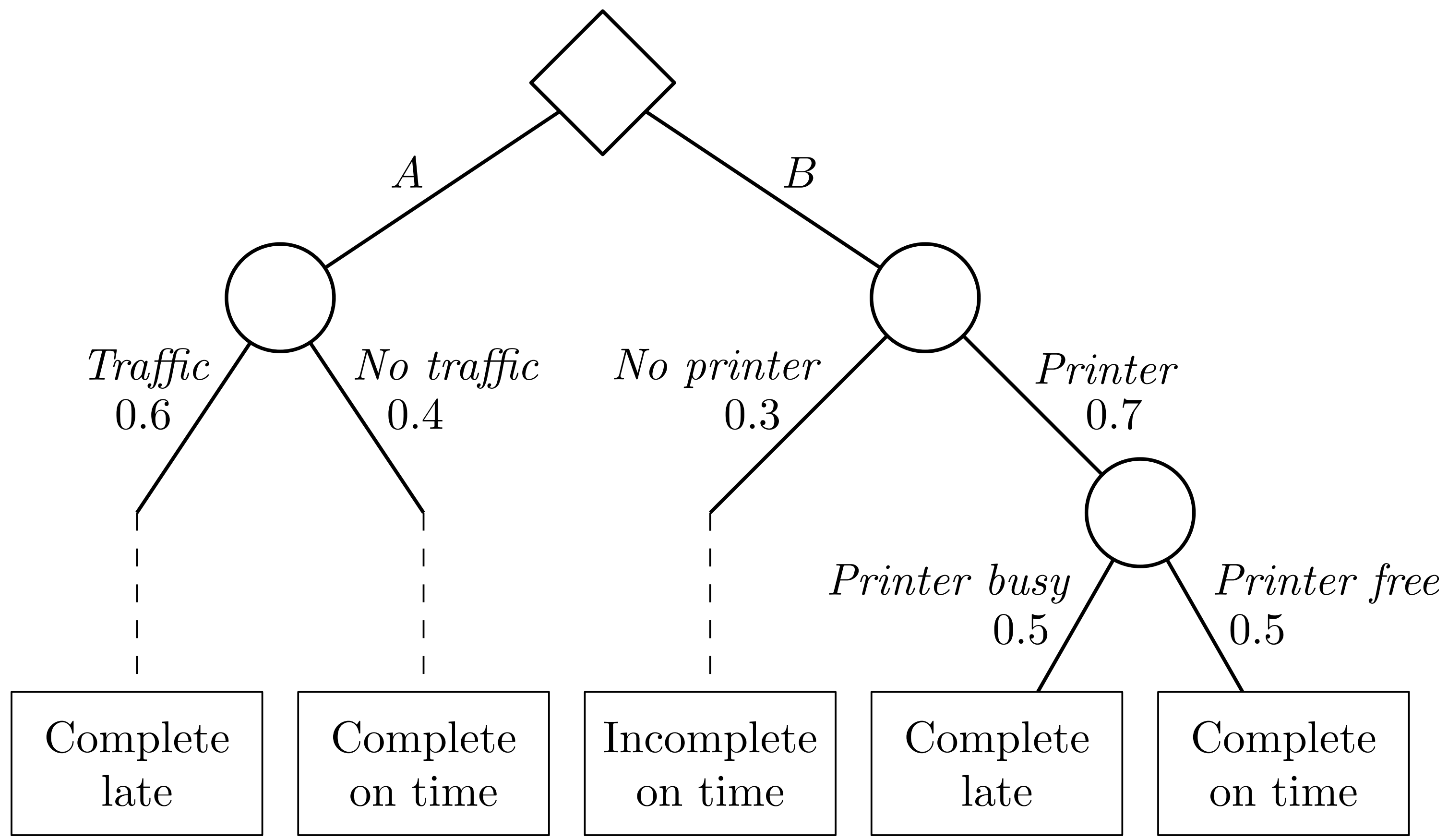
P(CL | H) = 0.6

P(CT | U) = 0.7 × 0.5 = 0.35

P(CL | U) = 0.7 × 0.5 = 0.35

P(IT | U) = 0.3

Expected value



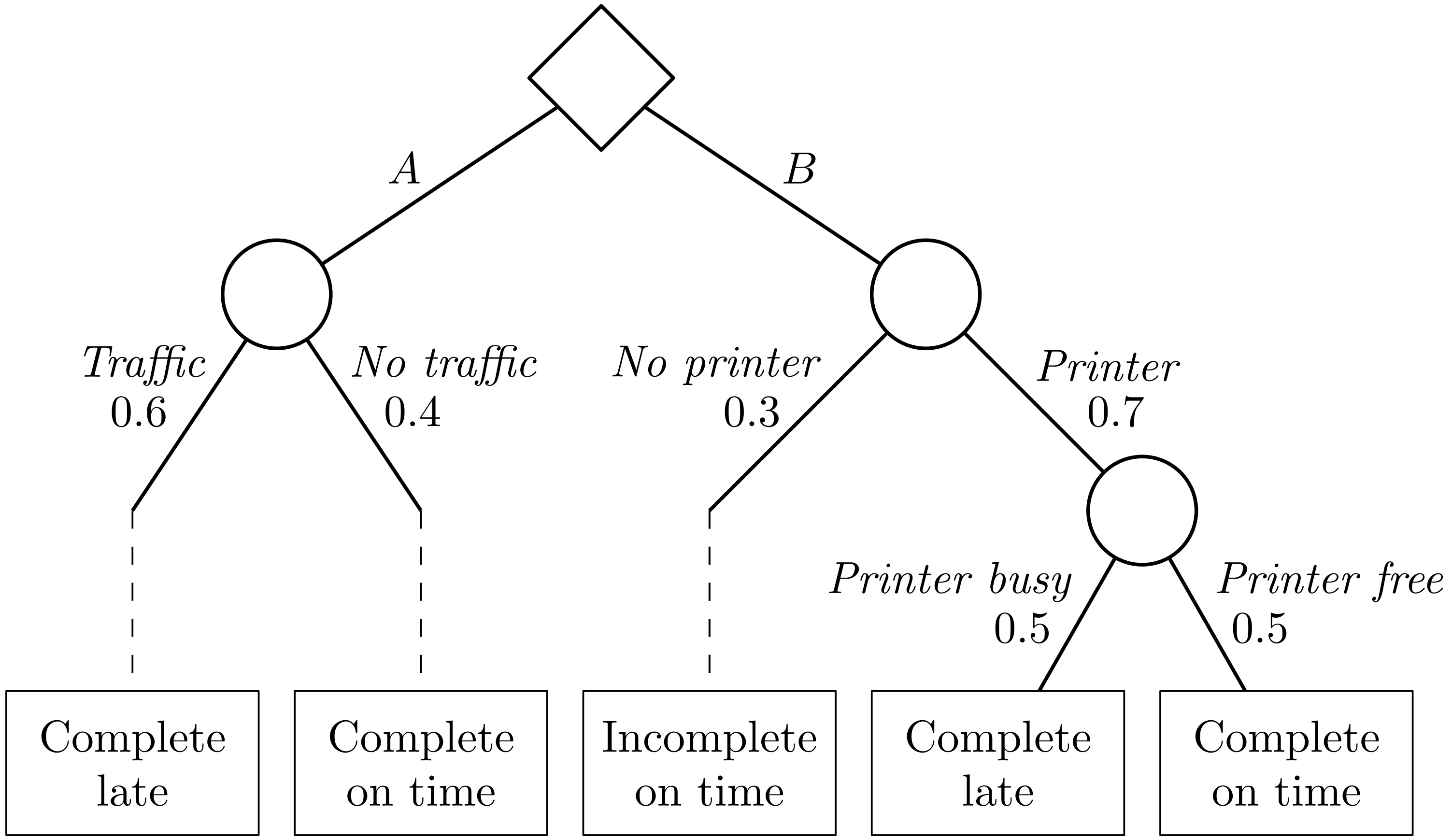
$Q(H) = 0.4 \times 0 + 0.6 \times (-2) = -1.2$

$P(CT \mid U) = 0.7 \times 0.5 = 0.35$

$P(CL \mid U) = 0.7 \times 0.5 = 0.35$

$P(IT \mid U) = 0.3$

Expected value



Q(H) = 0.4 × 0 + 0.6 × (-2) = -1.2 Q(U) = 0.35 × 0 + 0.35 × (-2) + 0.3 × (-3) = -1.6

She should return home!

St. Petersburg paradox

St. Petersburg paradox

- A casino in St. Petersburg offers the following game
  - The initial prize is 2 rubles
  - A fair coin is tossed
  - If it comes out “tails”, the game ends, and you get the prize
  - If it comes “heads”, the prize doubles and the game continues

How much should a player pay to enter for the game to be fair?

Expected value

- Expected value:

  P(T) = 1/2 $\longrightarrow$ Game ends with prize 2

  P(HT) = 1/4 $\longrightarrow$ Game ends with prize 4

  P(HHT) = 1/8 $\longrightarrow$ Game ends with prize 8

  $\ldots$

  P(HH…HT) = $1/2^n$ $\longrightarrow$ Game ends with prize 2^n

$$\text{Expected value} = \frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \ldots + \frac{1}{2^n} \times 2^n + \ldots = \sum_{n=1}^{\infty} 1 = \infty$$

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Would you take this bet?