Planning, Learning and Decision Making

Lecture 2. Markov chains

Sequential models

Weather in Lisbon

Weather in Lisbon

- We want to predict the weather in Lisbon
- Suppose that the weather is either **Sunny** or **Rainy**

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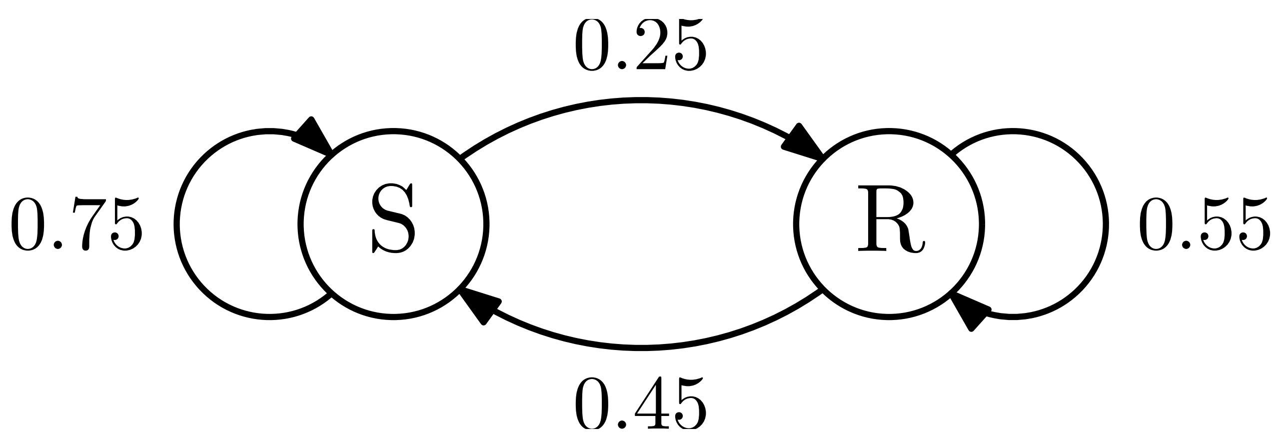
Weather in Lisbon

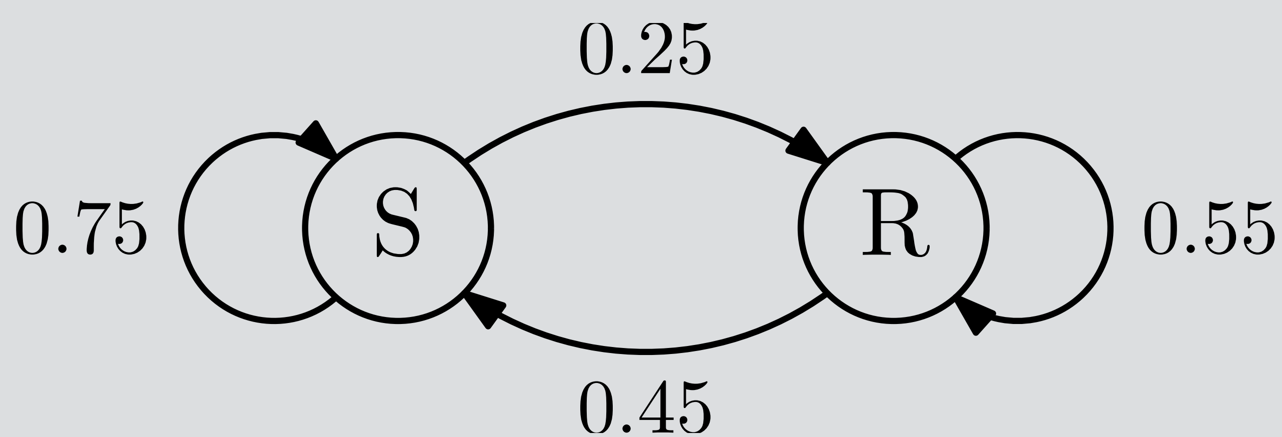
- Sunny day → Sunny day (prob. 75%)
- Rainy day → Rainy day (prob. 55%)
- Let’s draw this:



Transition diagram

Predicting the weather…

- Today is **Sunny**
- What is the weather **tomorrow**?



- **Rainy**, with 25% probability
- **Sunny**, with 75% probability

Predicting the weather…

- Today is **Sunny**
- What is the weather the day after tomorrow?

- If the weather tomorrow is **Sunny**, then
  - **Rainy**, with 25% probability
  - **Sunny**, with 75% probability

75% probability

- If the weather tomorrow is **Rainy**, then
  - **Rainy**, with 55% probability
  - **Sunny**, with 45% probability

25% probability

Predicting the weather…

- Is this mathematically correct?

- x_0: weather today
- x_1: weather tomorrow
- We have:
  - $$\mathbb{P}\left[x_1 = S \mid x_0 = S\right] = 75\%$$
    $$\mathbb{P}\left[x_1 = R \mid x_0 = S\right] = 25\%$$
  - $$\mathbb{P}\left[x_1 = S \mid x_0 = R\right] = 45\%$$
    $$\mathbb{P}\left[x_1 = R \mid x_0 = R\right] = 55\%$$

Predicting the weather...

- Is this mathematically correct?

- x_0: weather today
- x_1: weather tomorrow
- x_2: weather day after tomorrow
    - $$\begin{aligned}&\mathbb{P}\left[x_2 = S \mid x_0 = S\right]\\ &= \mathbb{P}\left[x_2 = S \mid x_1 = S\right]\mathbb{P}\left[x_1 = S \mid x_0 = S\right]\\ &\quad + \mathbb{P}\left[x_2 = S \mid x_1 = R\right]\mathbb{P}\left[x_1 = R \mid x_0 = S\right]\\ &= 0.75 \times 0.75 + 0.45 \times 0.25\\ &= 67.5\%\end{aligned}$$

Total probability law!

Assumptions

- The weather in one day is enough to predict the weather in the next day

- This prediction does not depend on the particular day

Is this always true?

- No.

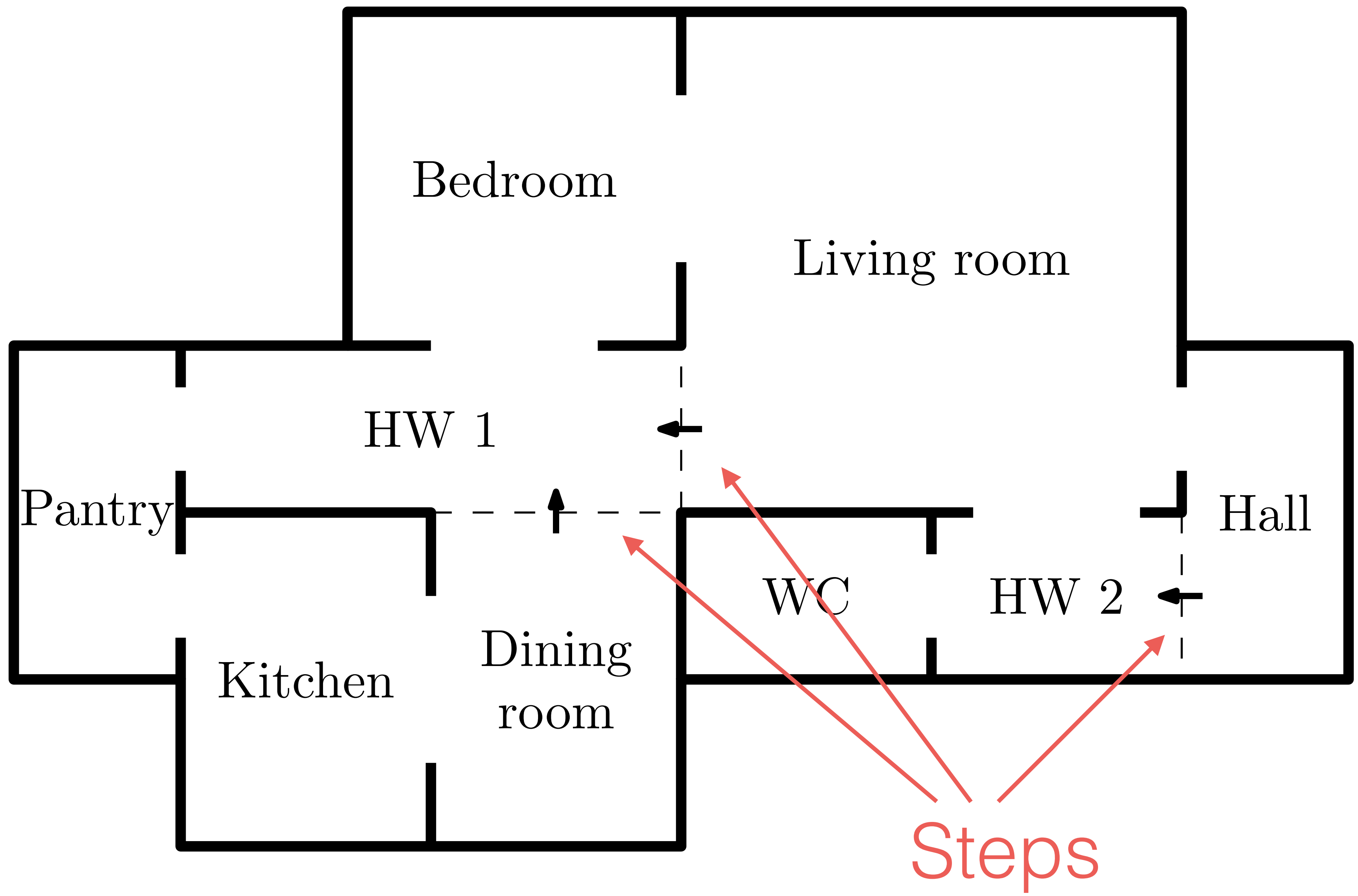
- … however, many times it's almost true 🙂

"Models have limitations;
Stupidity does not."

The household robot

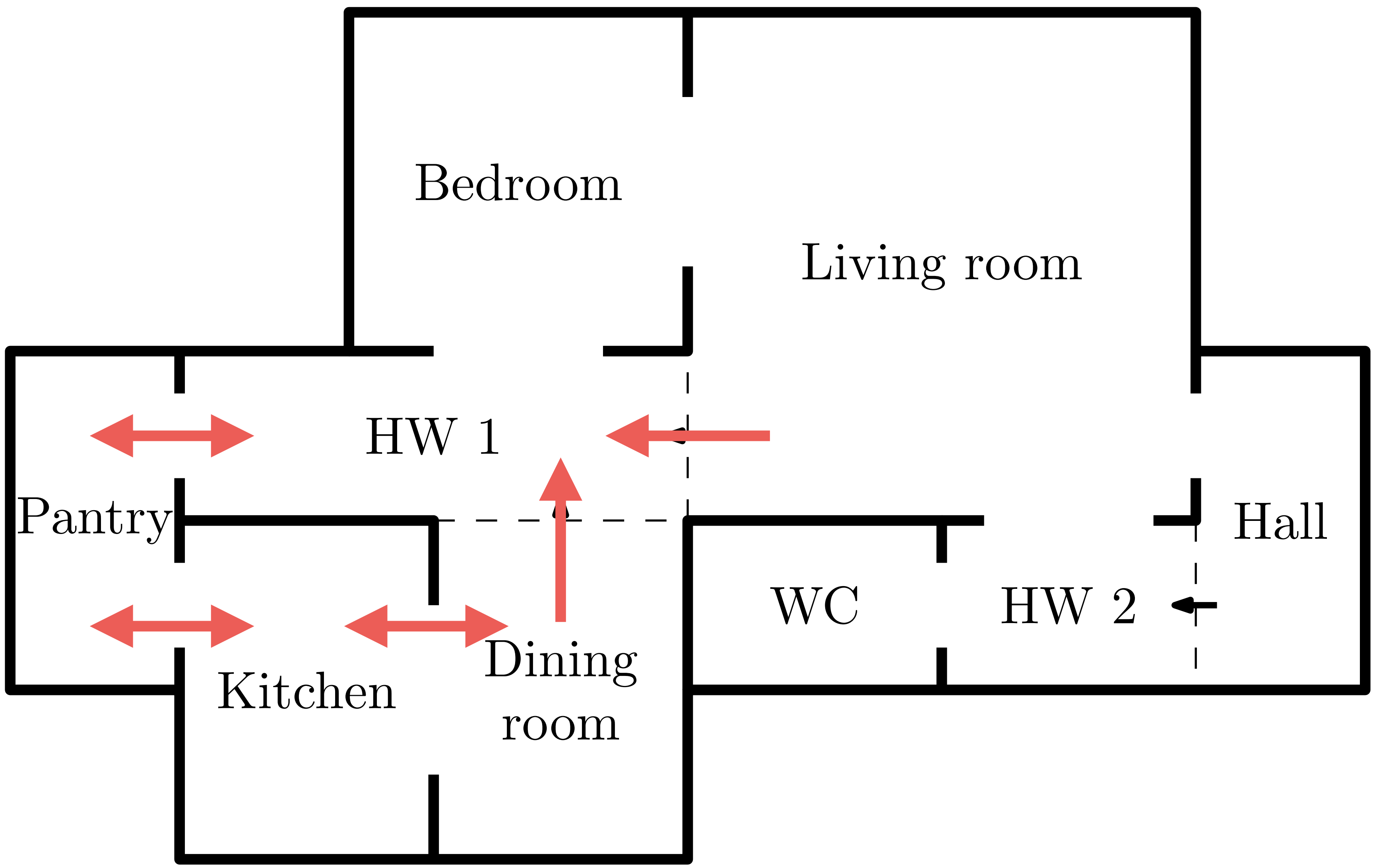
Household robot

- Consider the household



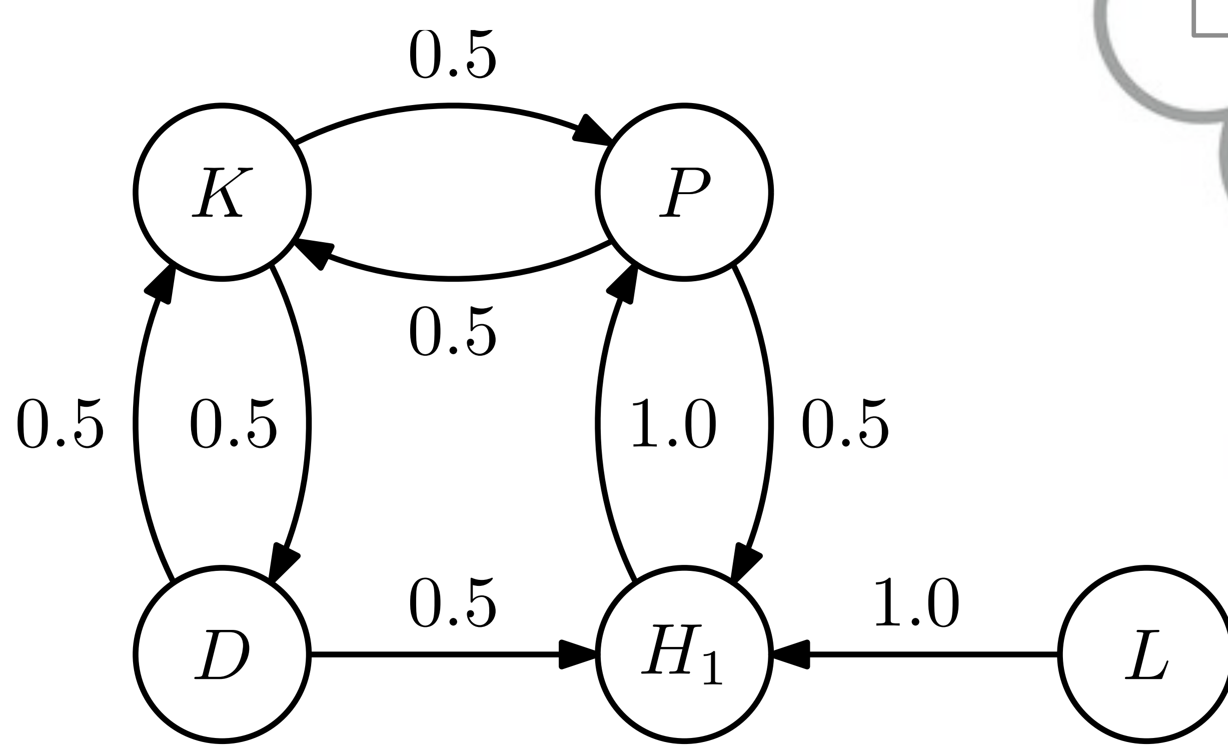
Household robot

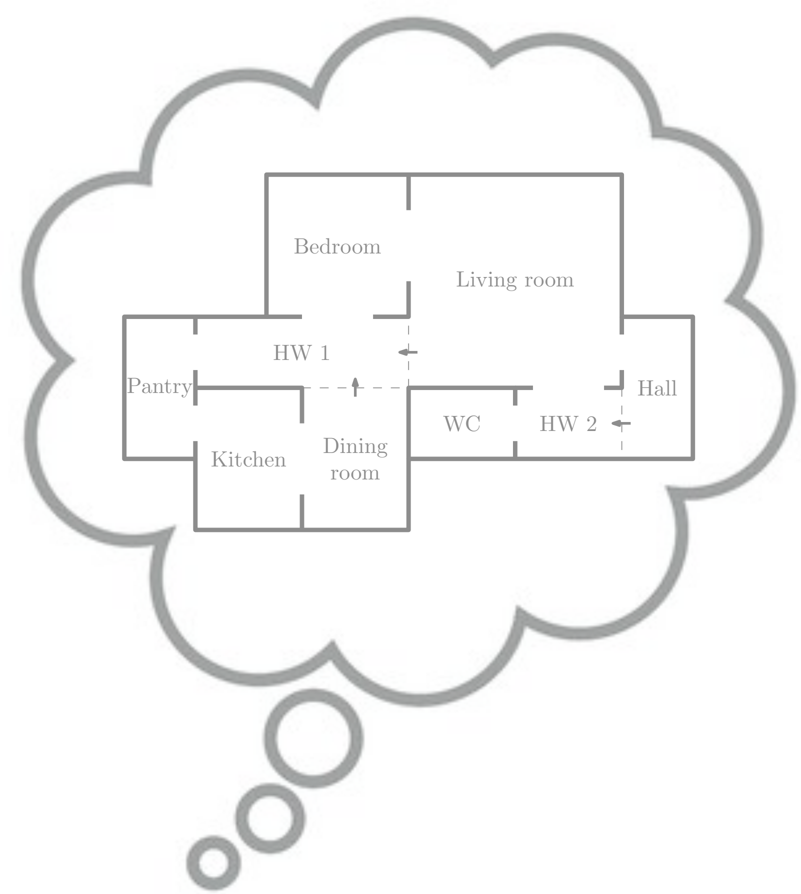
- Robot circulates between **Kitchen**, **Pantry**, **Dining room**, **Hallway 1** and **Living room**



Household robot

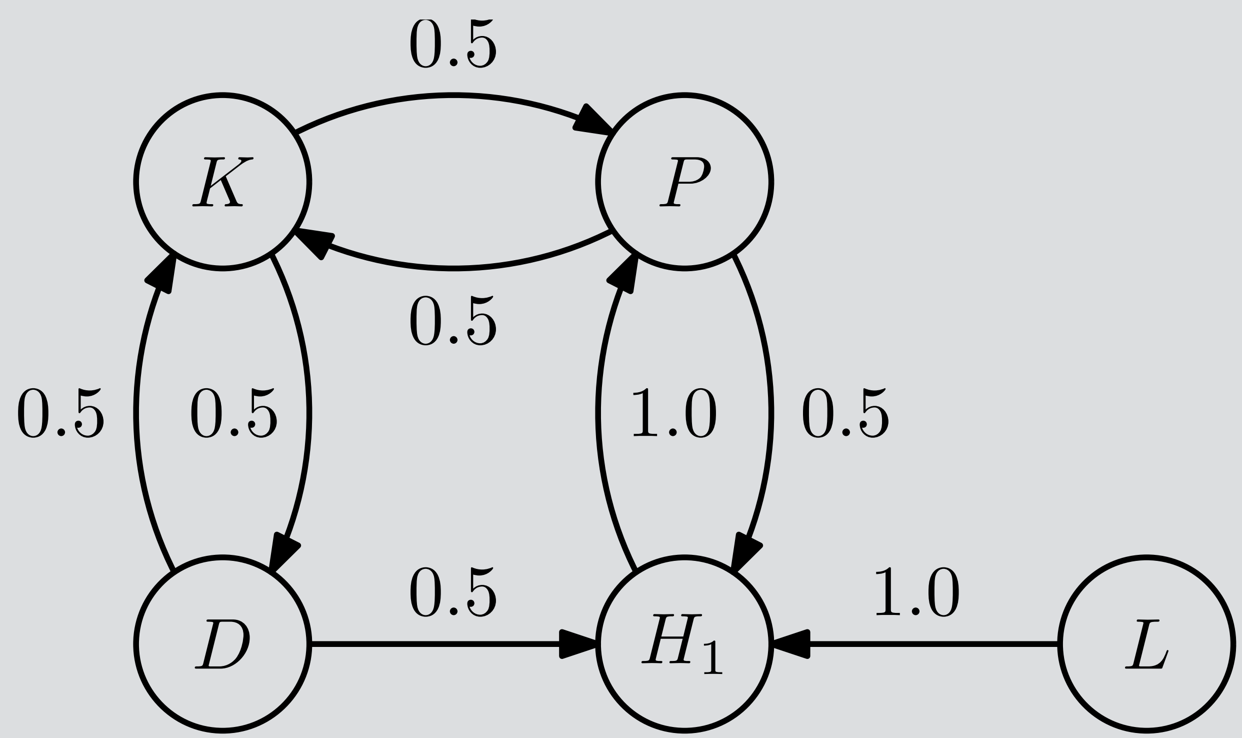
- Robot selects randomly the next room to visit
- Robot can't move back in steps
- Transition diagram:





Predicting the position…

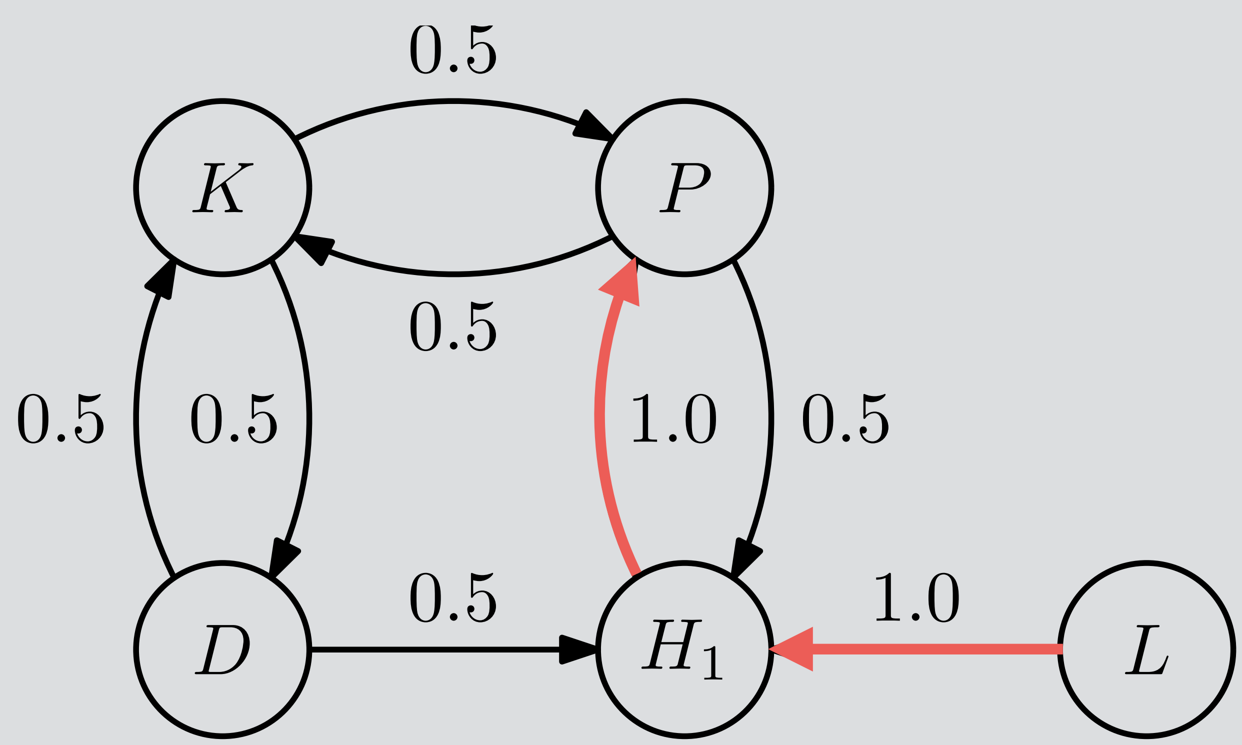
- Robot is in the **Living room**
- Where will it be the next time step?



- **Hallway 1**, with 100% probability

Predicting the position…

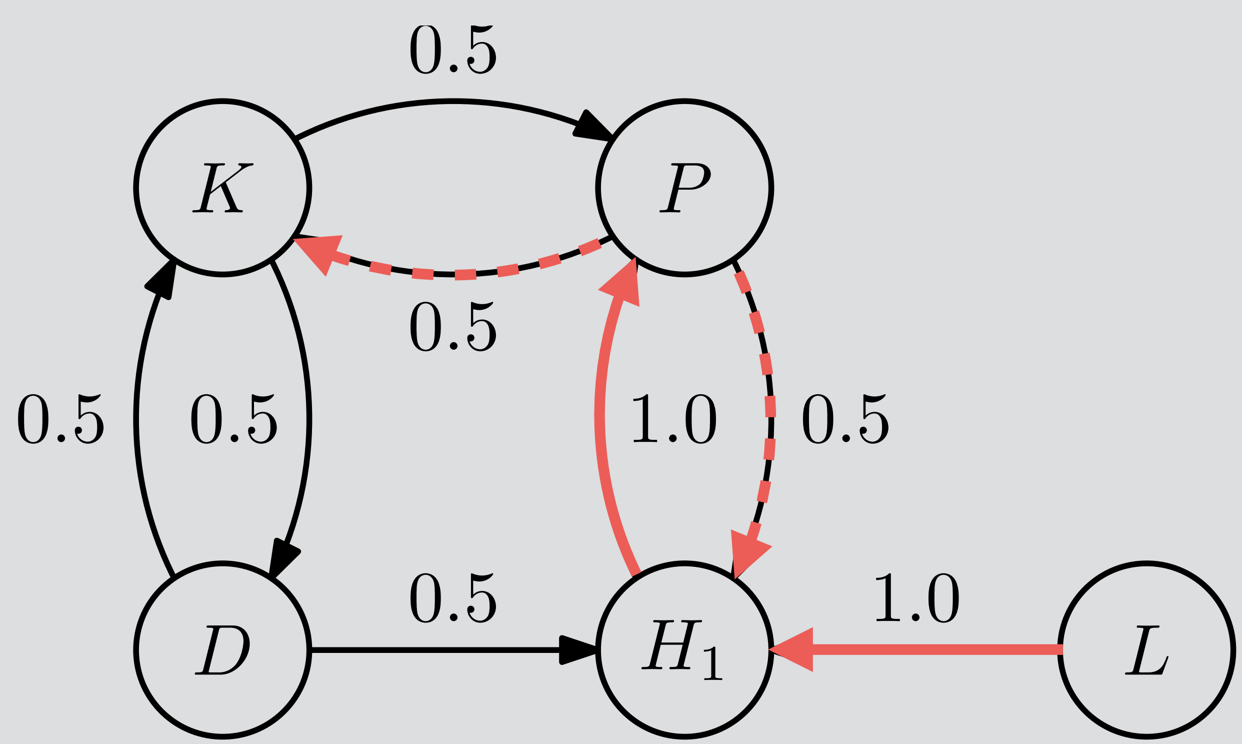
- Robot is in the **Living room**
- Where will it be in **two** time steps?



- **Pantry**, with 100% probability

Predicting the position…

- Robot is in the **Living room**
- Where will it be in **three** time steps?



- **Kitchen**, with 50% prob.; **Hallway 1**, with 50% prob.

Predicting the position…

- Robot is in the **Living room**
- Where will it be 5 time steps from now?

Not practical to list all possibilities…

… but you can have fun at home 😜

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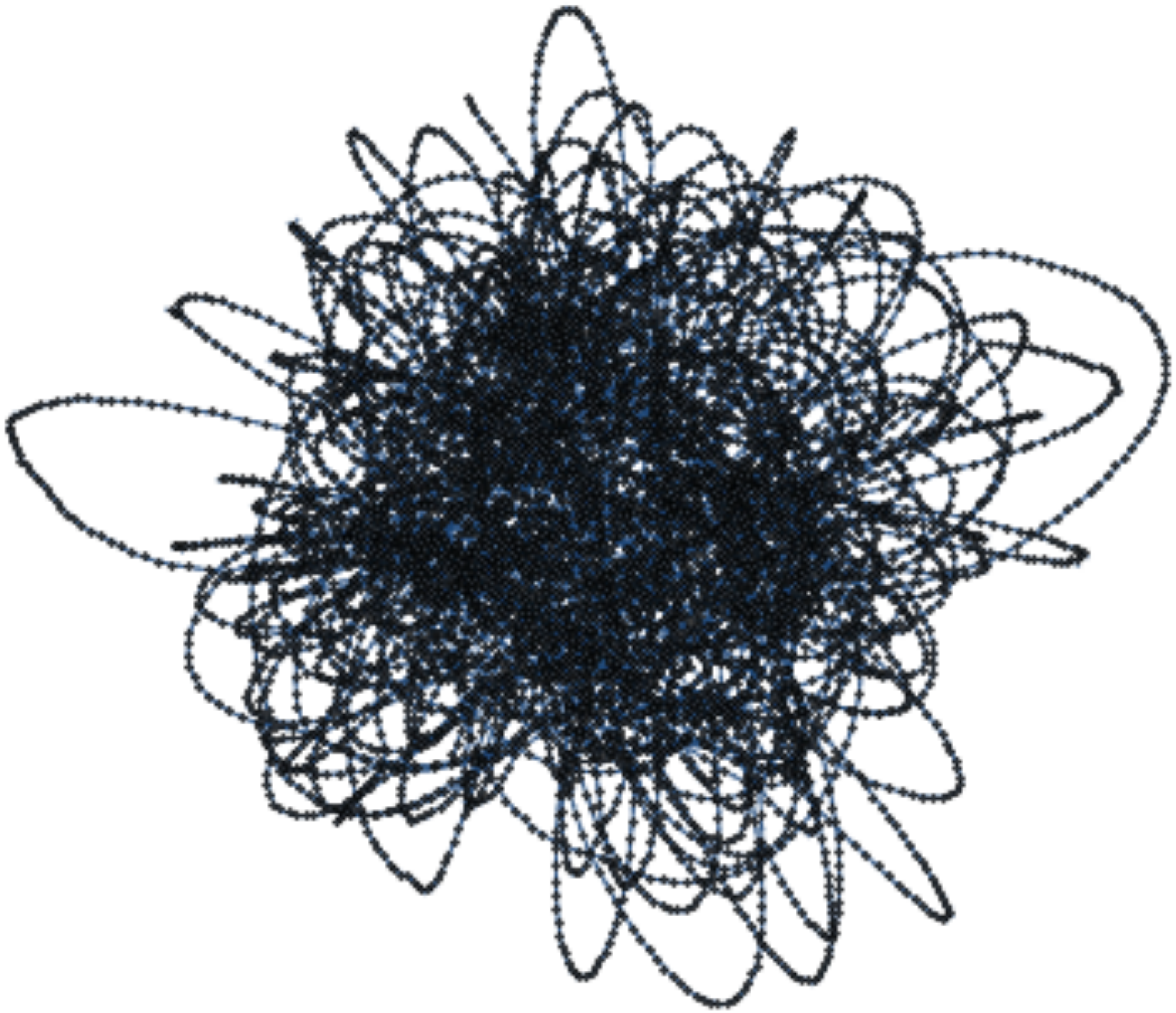
Assumptions

- The position of the robot in an instant is enough to predict its position in the next instant

- This prediction does not depend on the particular instant



Markov chains

Markov chain

- Model for **sequential process**
- Process evolves in **discrete time steps** (hence the chain)
- **State of the process** at time step t: x_t

Quantity of interest (weather, robot position, etc.)

Markov chain

Key Property: Markov property

The state at instant t is enough to predict the state at instant $t + 1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

Markov chain

- Other assumptions (for this most of this course):
  - There is only a **finite number** of possible states
  - $\mathscr{X}$ is the set of possible states (**state space**)

Finite chain

Markov chain

- Other assumptions (for most of this course):
  - The probabilities $\boxed{\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]}$ do not depend on *t*

Transition probability
from *x* to *y*

Homogeneous chain

Transition probability matrix

- Computationally, it is easier to store the transition probabilities in a **matrix P**

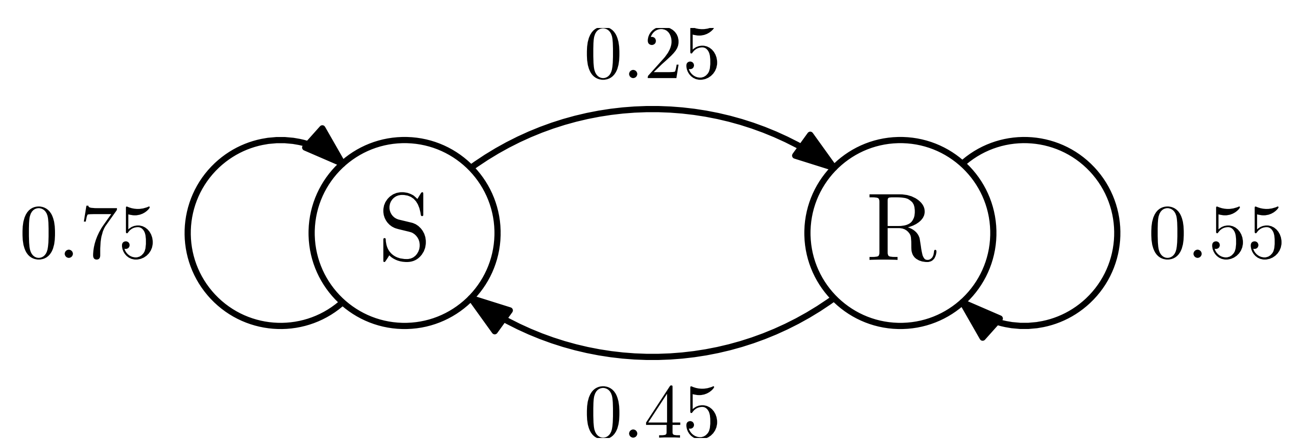
$$[\mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$$

Number in row *x* column *y* is the probability of "moving" from *x* to *y*

Transition probability matrix

- **Example:** the weather in Lisbon

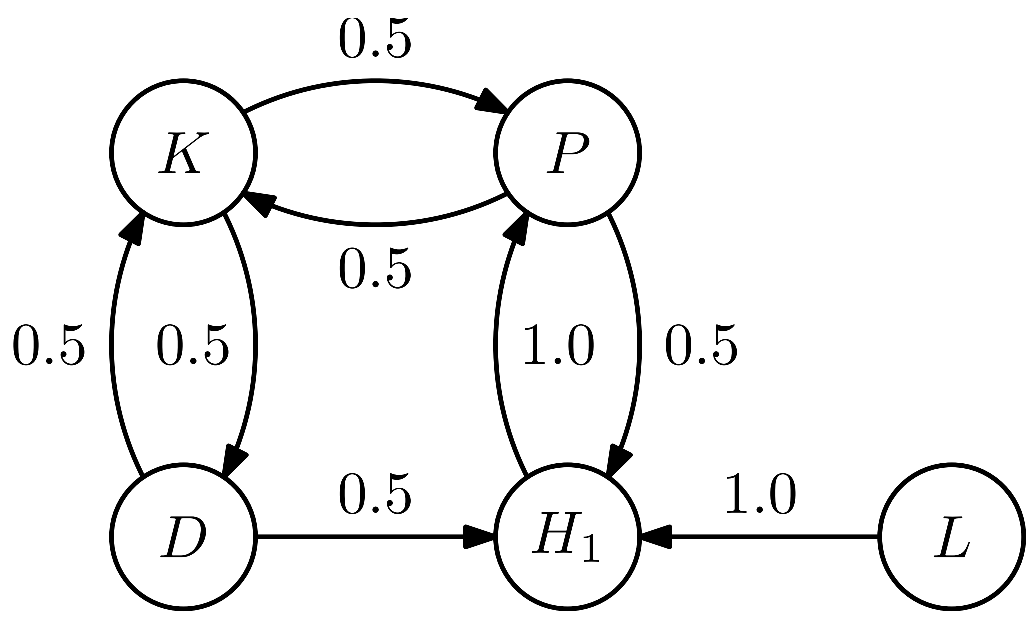
$$
\begin{array}{cc}
 & \begin{array}{cc} \mathrm{S} & \mathrm{R} \end{array} \\
\begin{array}{c} \mathrm{S} \\ \mathrm{R} \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

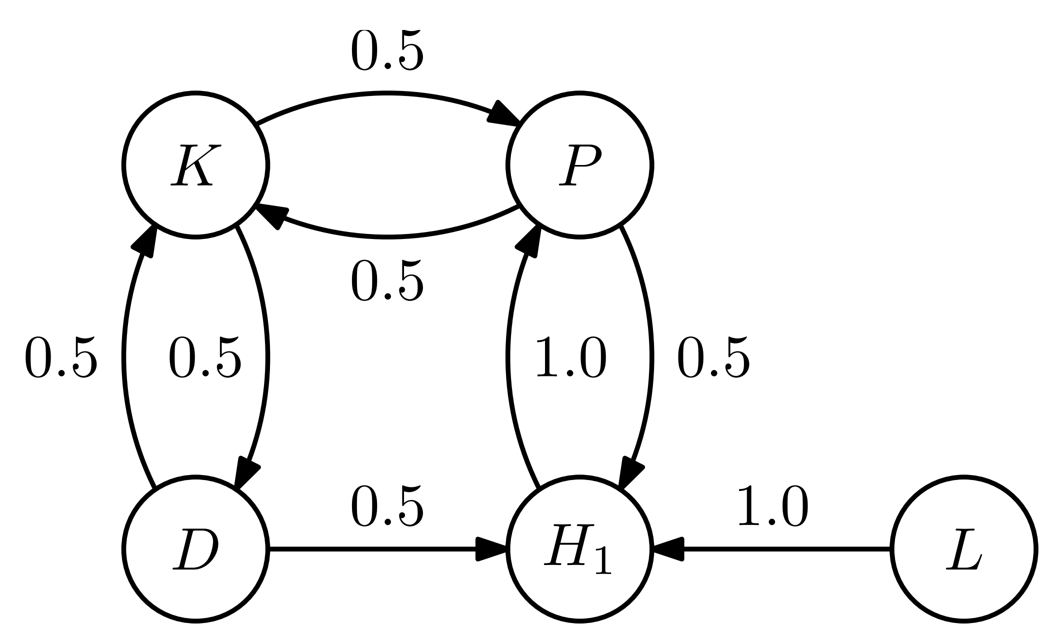
$$\begin{array}{cc} & \begin{array}{ccccc} L & K & P & D & H \end{array} \\ \begin{array}{c} L \\ K \\ P \\ D \\ H \end{array} & \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$



Transition probability matrix

- **Example:** the household robot

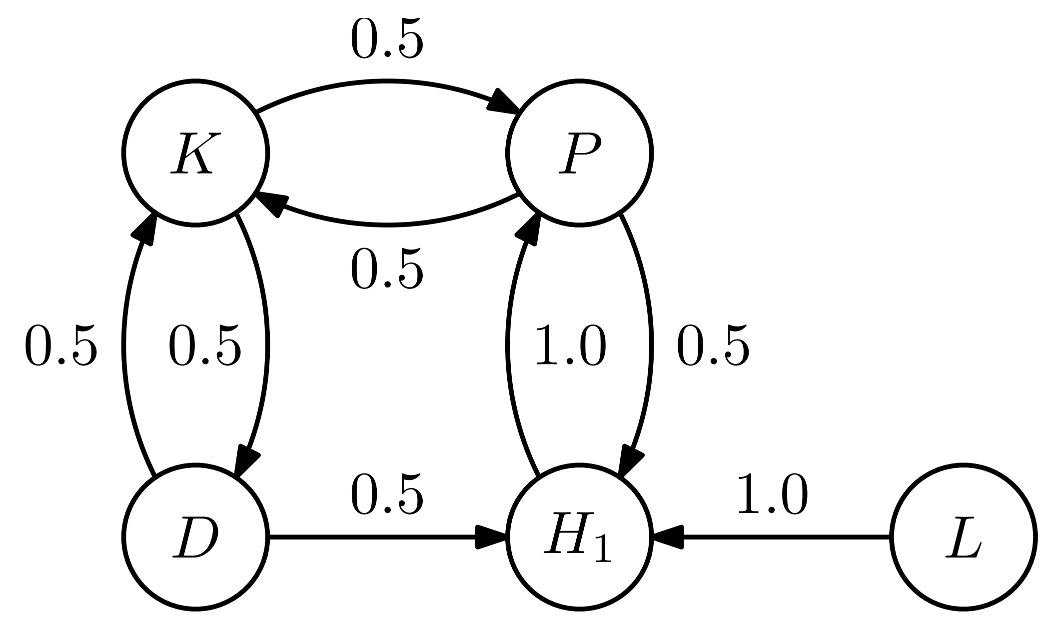
$$
\begin{array}{c|ccccc}
 & L & K & P & D & H \\
L & 0 & 0 & 0 & 0 & 1 \\
K & & & & & \\
P & & & & & \\
D & & & & & \\
H & & & & &
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

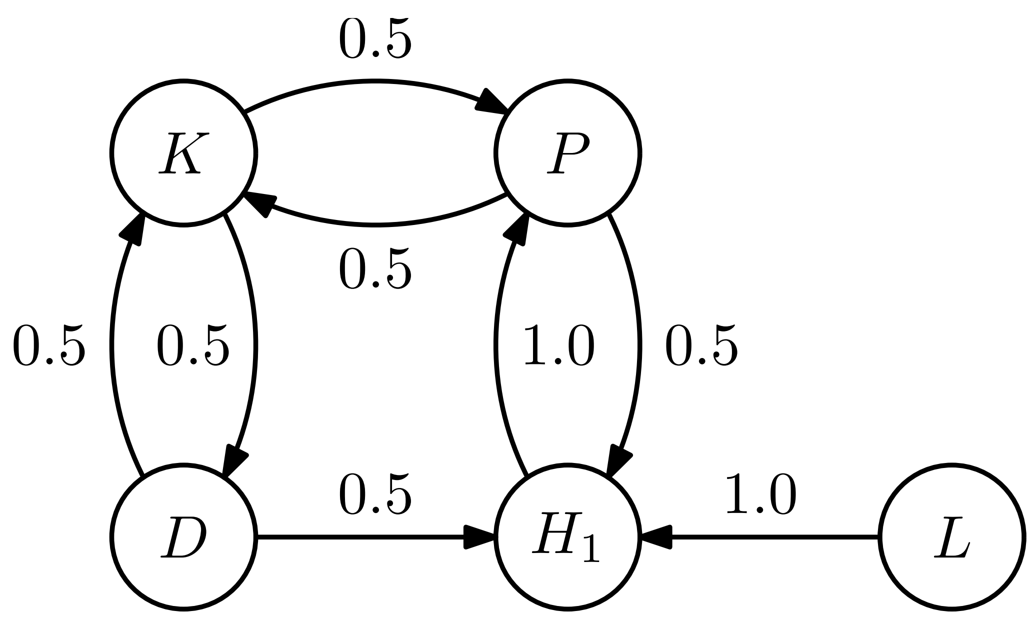
$$
\begin{array}{c|ccccc}
 & L & K & P & D & H \\
\hline
L & 0 & 0 & 0 & 0 & 1 \\
K & 0 & 0 & 0.5 & 0.5 & 0 \\
P & & & & & \\
D & & & & & \\
H & & & & & \\
\end{array}
$$



Transition probability matrix

- **Example:** the household robot

$$
\begin{array}{c}
 \\ L \\ K \\ P \\ D \\ H
\end{array}
\begin{array}{c}
\begin{array}{ccccc} L & K & P & D & H \end{array} \\
\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$



Markov chain

- Compact representation/specification of a MC
  - A Markov chain can be represented as a pair $(\mathscr{X}, \mathbf{P})$
    - $\mathscr{X}$ is the set of possible states
    - $\mathbf{P}$ is the transition probability matrix
- We write $\mathbf{P}(y \mid x)$ to denote the element xy of matrix $\mathbf{P}$

Predictions with MCs

Predicting the weather…

- Today is **Sunny**
- What is the weather **tomorrow**?

- **Rainy**, with 25% probability
- **Sunny**, with 75% probability

$$\begin{array}{c} \\ S \\ R \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{bmatrix} \boxed{0.75 \quad 0.25} \\ 0.45 \quad 0.55 \end{bmatrix} \end{array}$$

Predicting the weather…

- Today is **Rainy**
- What is the weather **tomorrow**?

- **Rainy**, with 55% probability
- **Sunny**, with 45% probability

$$
\begin{array}{c c}
 & \begin{array}{cc} S & R \end{array} \\
\begin{array}{c} S \\ R \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ \boxed{0.45 \quad 0.55} \end{bmatrix}
\end{array}
$$

Predicting the weather…

- Today is **Sunny**
- How likely is the weather **Sunny** the day after tomorrow?

$$
\begin{aligned}
&\mathbb{P}\left[\mathrm{x}_2 = S \mid \mathrm{x}_0 = S\right] \\
&= \mathbb{P}\left[\mathrm{x}_2 = S \mid \mathrm{x}_1 = S\right] \mathbb{P}\left[\mathrm{x}_1 = S \mid \mathrm{x}_0 = S\right] \\
&\quad + \mathbb{P}\left[\mathrm{x}_2 = S \mid \mathrm{x}_1 = R\right] \mathbb{P}\left[\mathrm{x}_1 = R \mid \mathrm{x}_0 = S\right] \\
&= 0.75 \times 0.75 + 0.45 \times 0.25 \\
&= 67.5\%
\end{aligned}
$$

$$
\begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}
\begin{bmatrix} 0.75 & 0.25 \\ 0.45 & 0.55 \end{bmatrix}
$$

Transition probabilities

- We observe that

$$[\mathbf{P} \times \mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+2} = y \mid \mathrm{x}_t = x\right]$$

- In general,

$$[\mathbf{P}^k]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+k} = y \mid \mathrm{x}_t = x\right]$$

k-step transition probabilities

Predicting the position…

- Robot is in the **Living room**
- Where will it be 5 time steps from now?
- We compute $\mathbf{P}^5$

$$
\begin{array}{c}
\begin{array}{ccccc} \mathrm{L} & \mathrm{K} & \mathrm{P} & \mathrm{D} & \mathrm{H} \end{array} \\
\mathbf{P}^5 = \begin{bmatrix}
0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0 & 0.75 & 0.25 & 0 \\
0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0.5 & 0 & 0 & 0.5 \\
0 & 0 & 0.75 & 0.25 & 0
\end{bmatrix}
\end{array}
$$

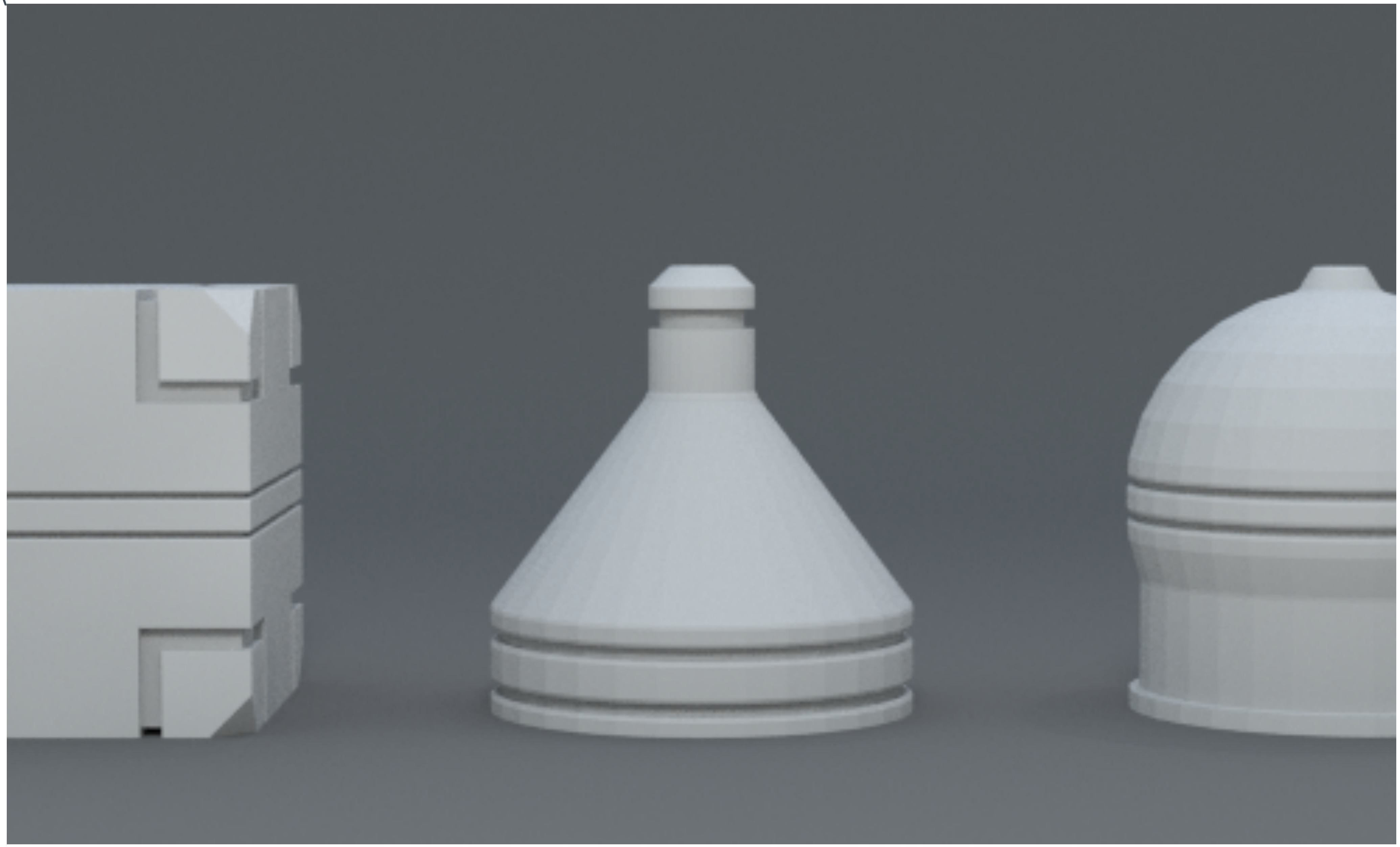
- Kitchen: 50% prob.; Hallway 1: 50% probability.

Predicting the position…

- In Python…

fmelo — python — 80×24

```
[>>> import numpy as np                                                        ]
[>>> P = np.array([[0, 0, 0, 0, 1], [0, 0, .5, .5, 0], [0, .5, 0, 0, .5], [0, .5,]
 0, 0, .5], [0, 0, 1, 0, 0]])
[>>> np.linalg.matrix_power(P, 5)                                              ]
array([[ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.  ,  0.75,  0.25,  0.  ],
       [ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.5 ,  0.  ,  0.  ,  0.5 ],
       [ 0.  ,  0.  ,  0.75,  0.25,  0.  ]])
>>>
```



MC examples (modeling)

Example 1. The Gambler

- A gambler, Adam, enters a casino with M euros
- Adam decides to play a chance game:
  - At each round of the game, he bets 1 euro;
  - If Adam wins the bet, he receives his euro back plus one additional euro
  - If Adam loses the bet, he loses his euro
  - Adam wins with probability p

Example 1. The Gambler

- The game stops when:
    - Adam is out of money
    - Adam doubles his money

1. Is this a Markov chain?

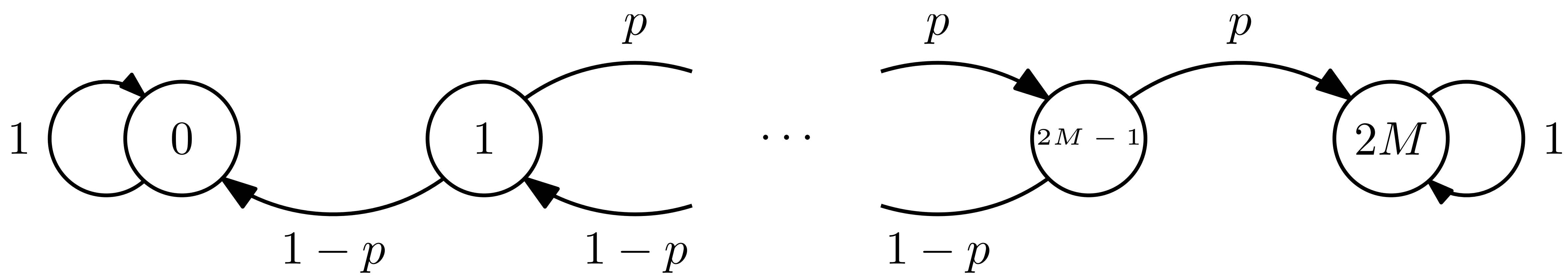
- **Yes**
  - The money at step $t + 1$ depends only on the money at step t

2. What are the states?

- The possible amounts of money:
  - $\mathscr{X} = \{0, 1, 2, \ldots, M, \ldots, 2M\}$

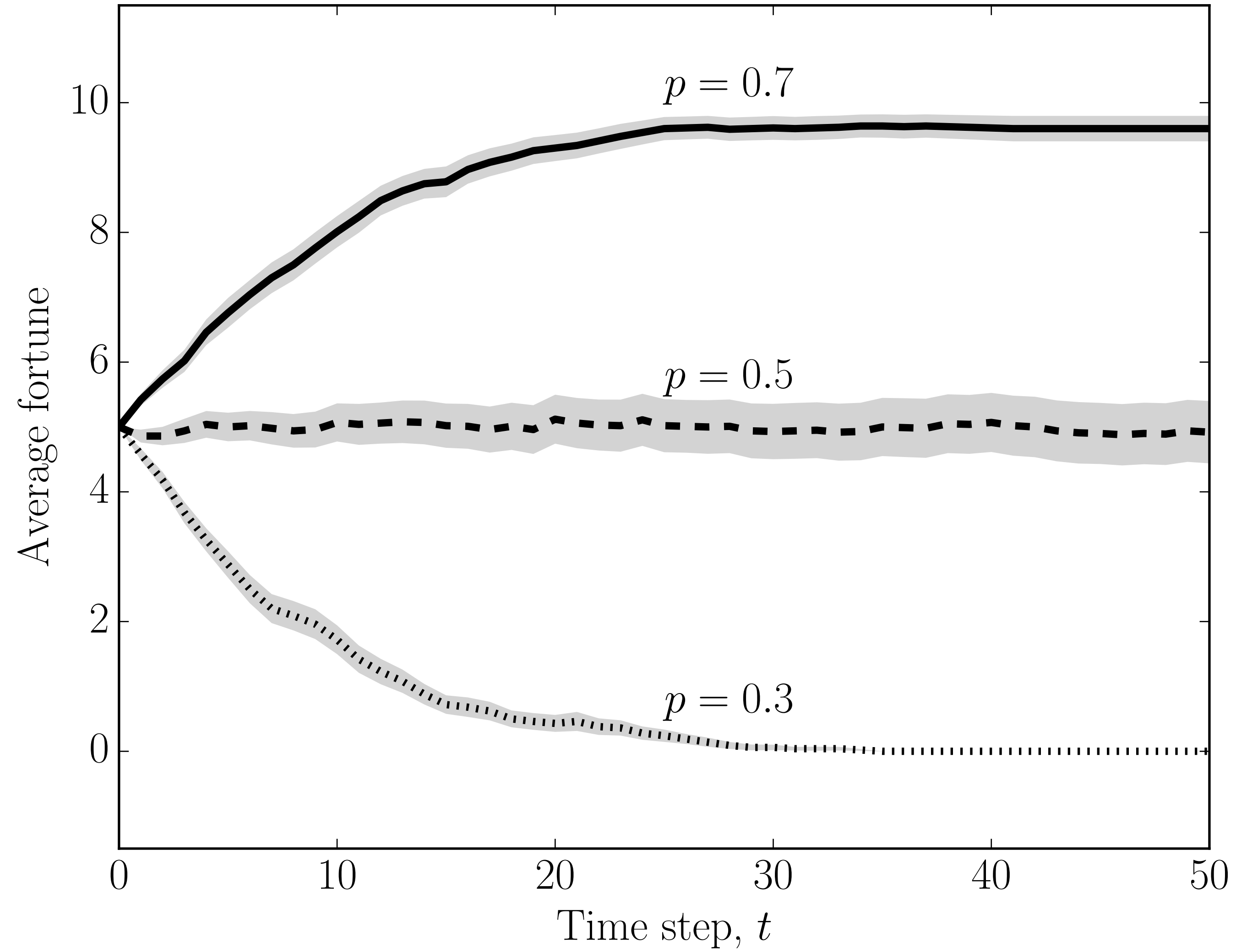
3. Transition probabilities?

- We can represent the game with a transition diagram:



What does it look like?

- We can check what happens to Adam's money for different values of *p*:



Example 2. PageRank

- PageRank is the algorithm used by Google to rank a set of connected documents
- It simulates a "random bot" navigating the web of retrieved documents

Example 2. PageRank

- Upon visiting a page, the bot randomly moves to one of the pages linked by the current one

- The rank corresponds to the “amount of time” that the bot spends on each page

1. Is this a Markov chain?

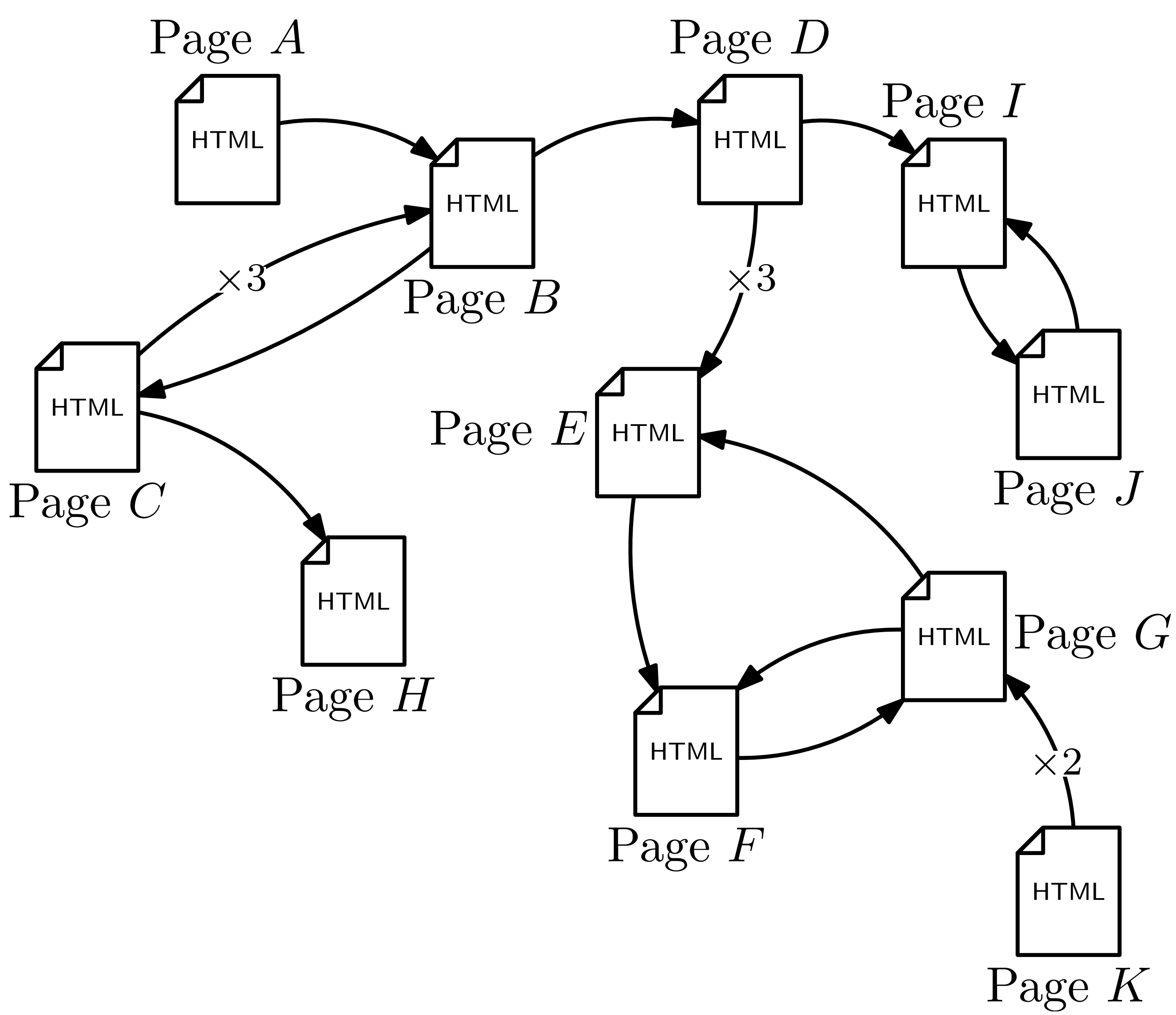
- **Yes**
  - The next page visited by the bot depends only on the current page (namely, its links)

2. What are the states?

- The web of pages

2. What are the states?

- For example, for the documents:

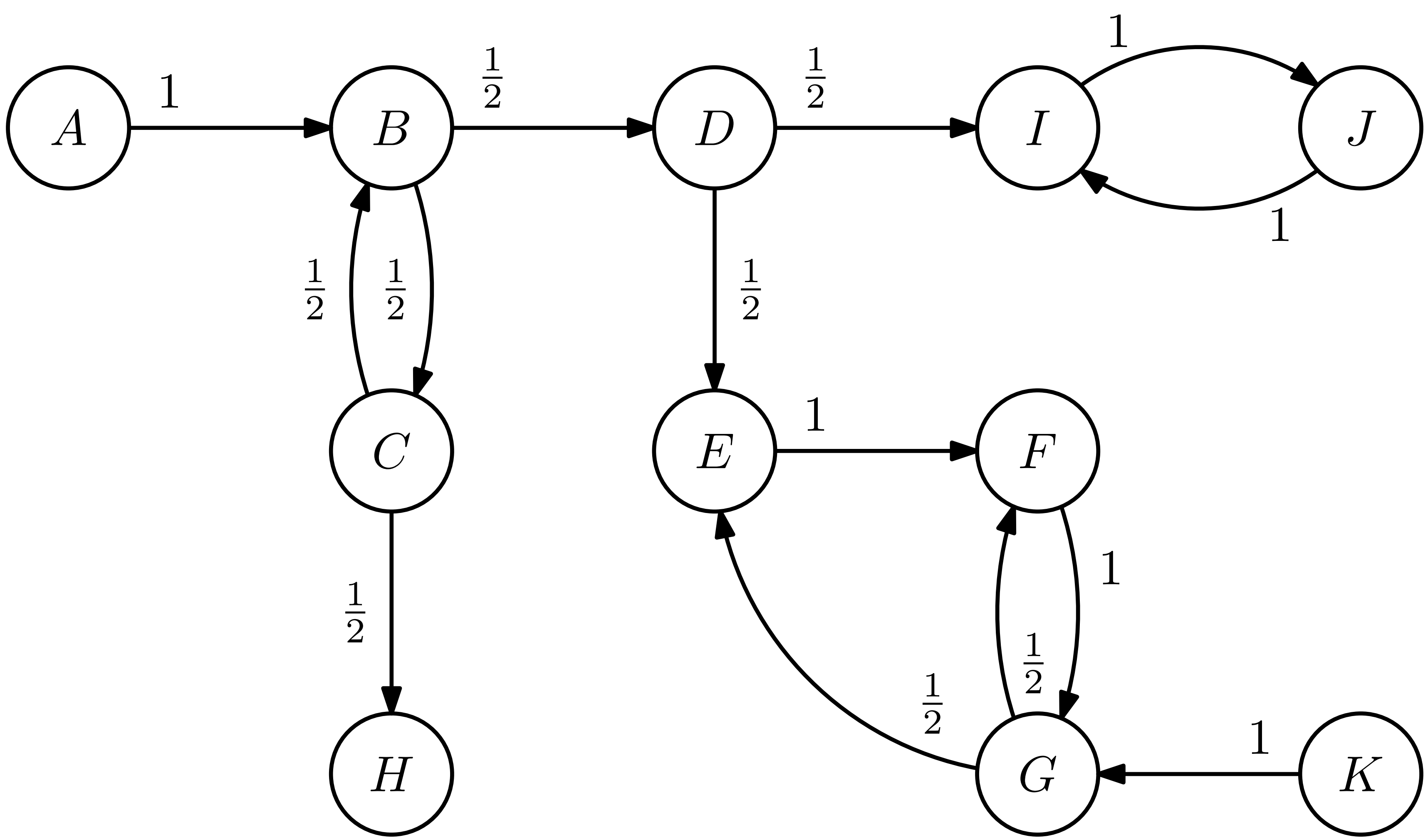


2. What are the states?

- We have that
  - $\mathscr{X} = \{A, B, C, D, E, F, G, H, I, J, K\}$

3. Transition probabilities?

- We can represent the motion of the bot as a transition diagram



... we'll revisit this diagram later on.

Stability of MCs

What is stability?

- It is often important to understand how a MC behaves in the long run

  - **Stability** concerns the **long-term behavior** of the chain

What is stability?

- We may want to know:
  - Depending on where the chain starts, can it **reach** any other state?
  - Is the behavior of the chain **cyclic**?
  - How **frequently** does the chain visit each state?

Irreducibility

- A state y can be reached from a state x if

$$\mathbf{P}^t(y \mid x) > 0$$

Positive probability of visiting *y* after visiting *x*

for some *t*

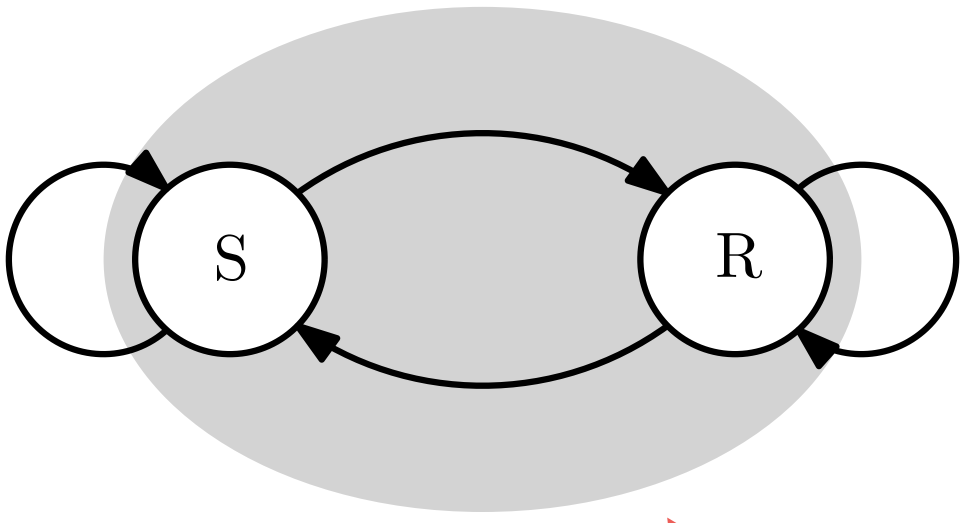
- A chain is **irreducible** if any state *y* can be reached from any other state *x*

Irreducibility

- We can split the state space of a chain in **communicating classes** (sets of states that are mutually reachable)

Irreducibility

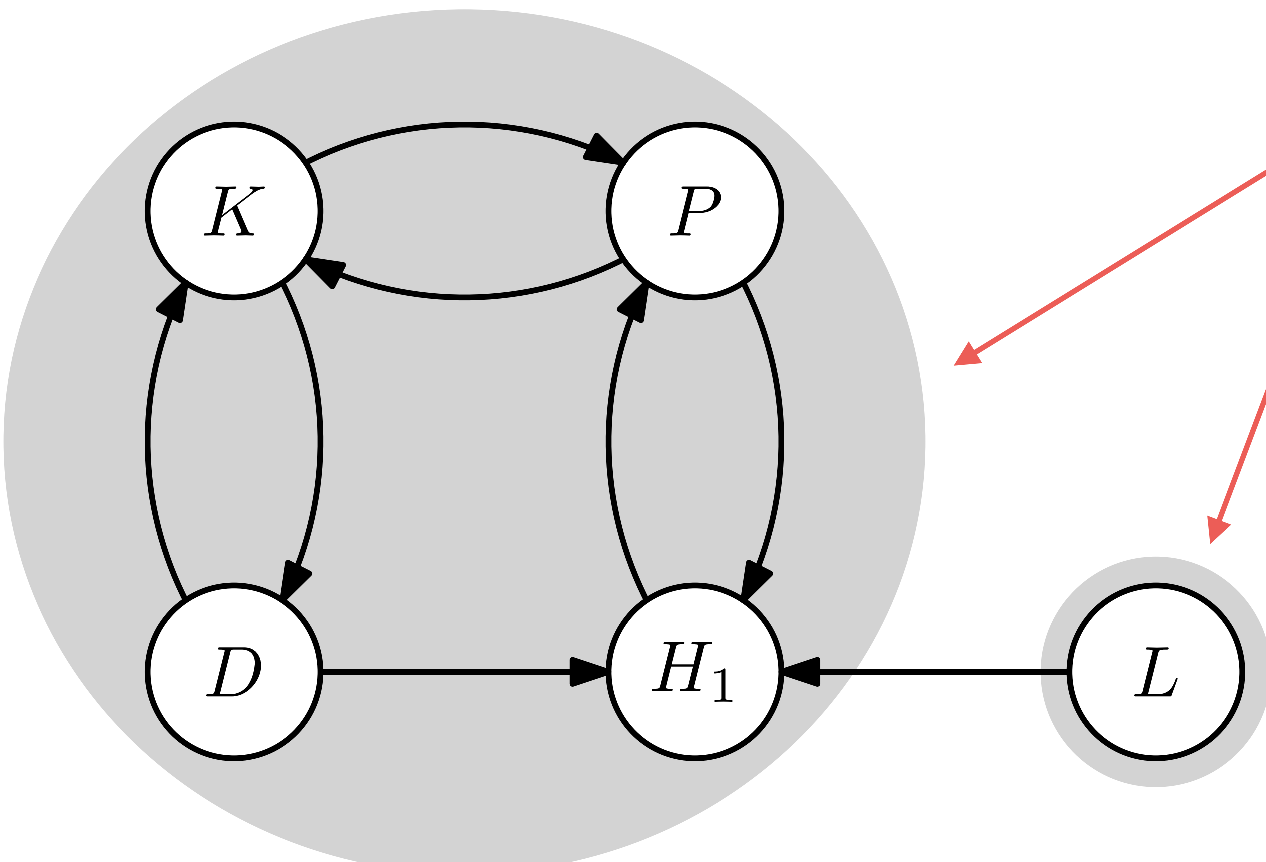
- In the weather example:



Single communicating class
$\Rightarrow$ irreducible chain

Irreducibility

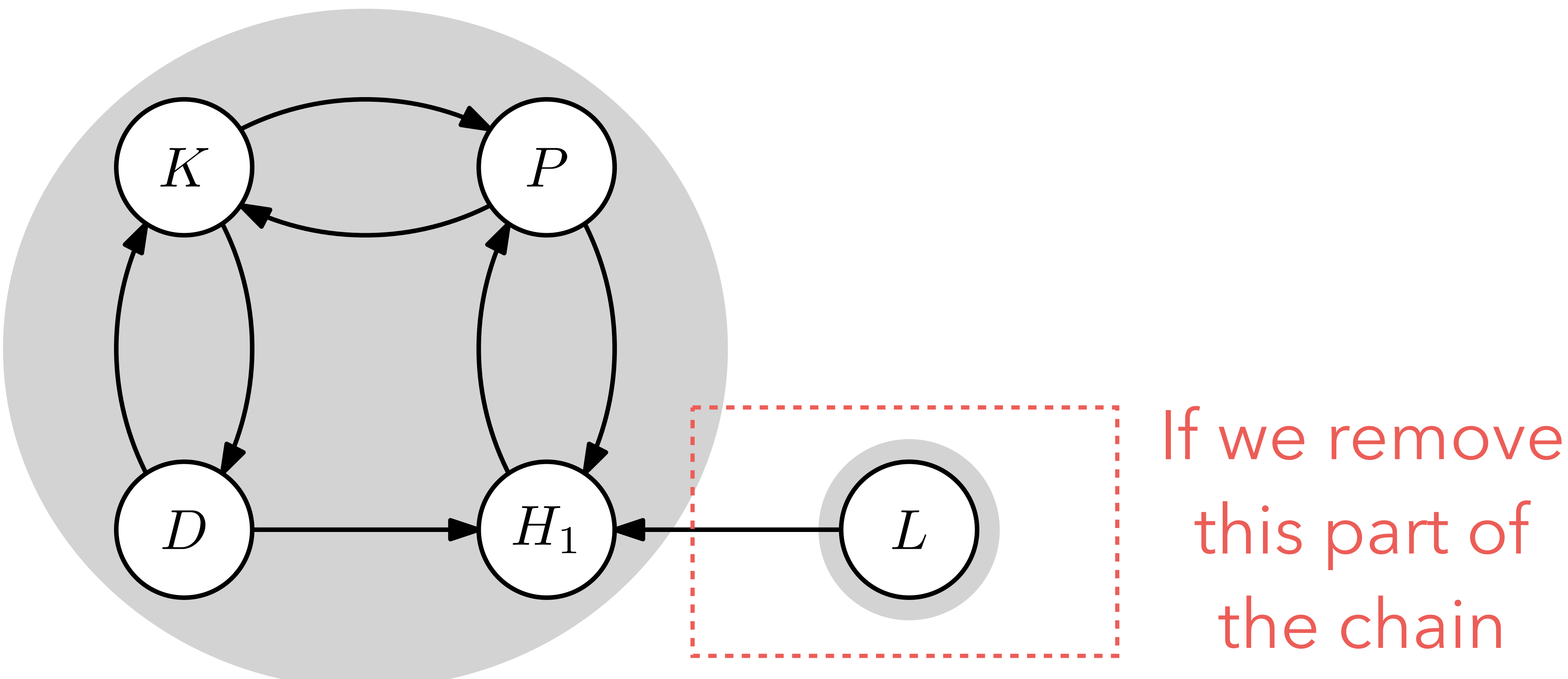
- In the robot example:



Two communicating classes
$\Rightarrow$ not irreducible

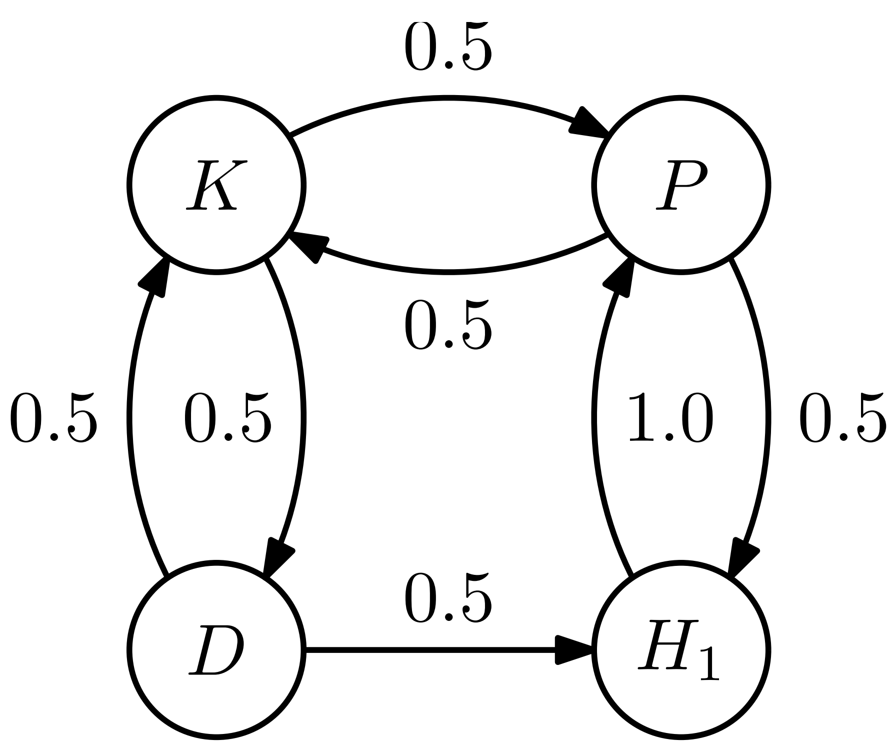
Irreducibility

- In the robot example:



Irreducibility

- In the robot example:



How many communicating classes?

Aperiodicity

- The period of a state x is…
  - the **greatest common divider**…
  - of all time steps in which x can be visited…
  - if the chain departs from x

Aperiodicity

- Formally, the **period of *x*** is the number

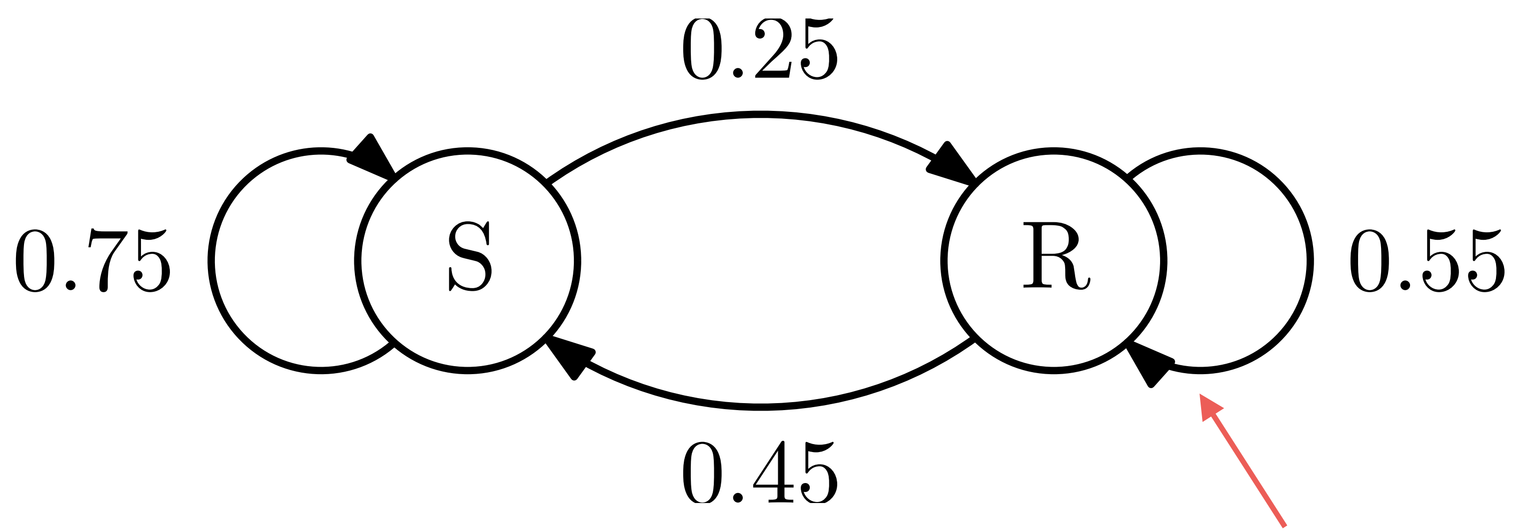
$$d_x = \gcd \left\{t \in \mathbb{N} \mid \mathbf{P}^t(x \mid x) > 0, t > 0\right\}$$

- A state *x* is **aperiodic** if $d_x = 1$

- A chain is **aperiodic** if all states are aperiodic, and **periodic** otherwise

Aperiodicity

- In the weather example:

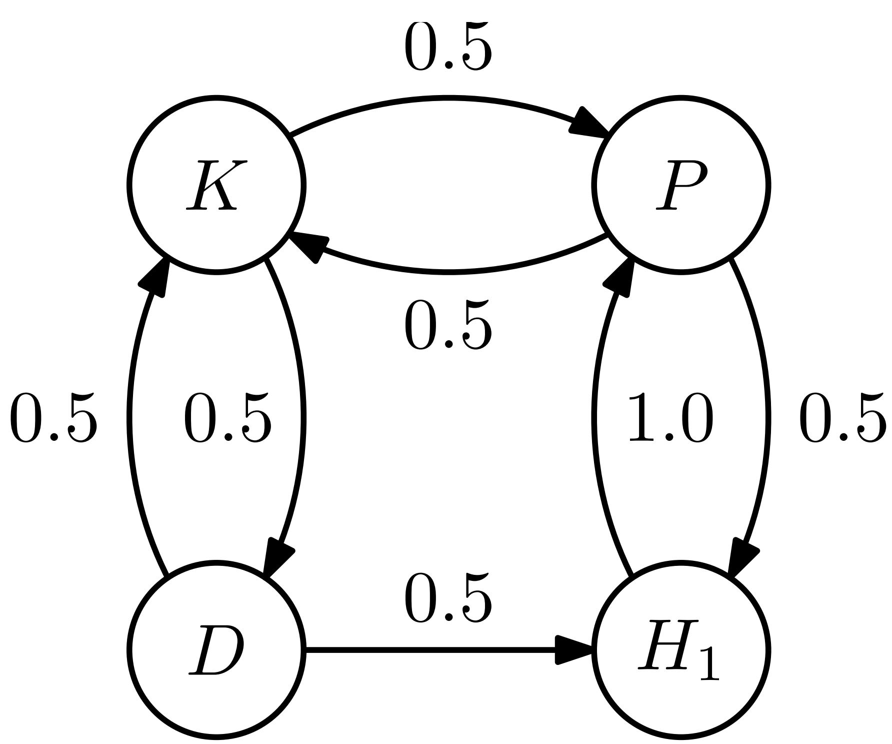


$\mathbf{P}(R \mid R) > 0$

$\Rightarrow$ aperiodic state

Aperiodicity

- In the robot example:



If the chain departs from *P*, when does it return?

↓

Several possibilities:

t = 2 (goes to H_1 and returns)

t = 2 (goes to *K* and returns)

t = 4 (goes to K, D and returns)

…

The period is *d* = 2

Stationary distribution

- Let μ be a **distribution** over $\mathcal{X}$

- In practical terms μ is a row vector

$$\boldsymbol{\mu} = \begin{bmatrix} \mu(x_1) & \mu(x_2) & \dots & \mu(x_{|\mathcal{X}|}) \end{bmatrix}$$

Component μ(*x*) is the probability of *x* according to μ

where

$$\sum_{x \in \mathcal{X}} \mu(x) = 1$$

Stationary distribution

- μ can represent, for example,
  - … the initial distribution for the chain;
  - … the predicted distribution after *t* steps;
  - … etc.

Stationary distribution

- The distribution μ is **stationary** if

$$\mu(x) = \sum_{y \in \mathcal{X}} \mu(y)\mathbf{P}(x \mid y)$$

- In other words, if the state at time t follows the distribution μ and μ is stationary, then the state at time $t + 1$ also follows the distribution μ

- The stationary distribution corresponds to **stable behavior of the chain**

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Key stability results

An irreducible and aperiodic Markov chain **possesses a stationary distribution**.

Positive chain

For an irreducible and aperiodic Markov chain with statio-nary distribution μ^*,

$$\lim_{t \to \infty} \mu_0 \mathbf{P}^t = \mu^*$$

for any initial distribution μ_0.

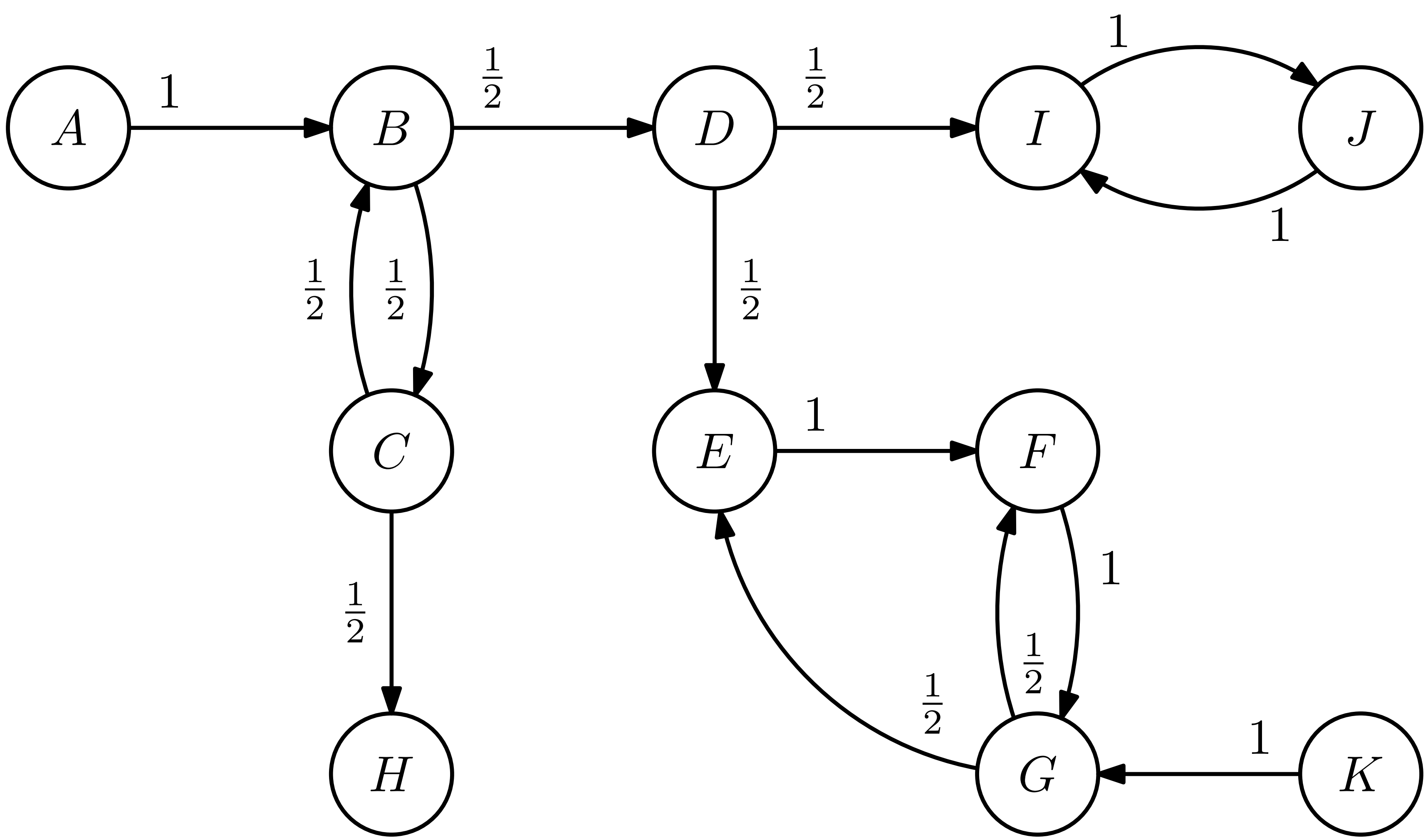
Ergodic chain

Summarizing...

- A positive chain possesses a stationary distribution μ:
  - If x_t is distributed according to μ, then so is x_{t+1}
- An ergodic chain eventually reaches the stationary distribution

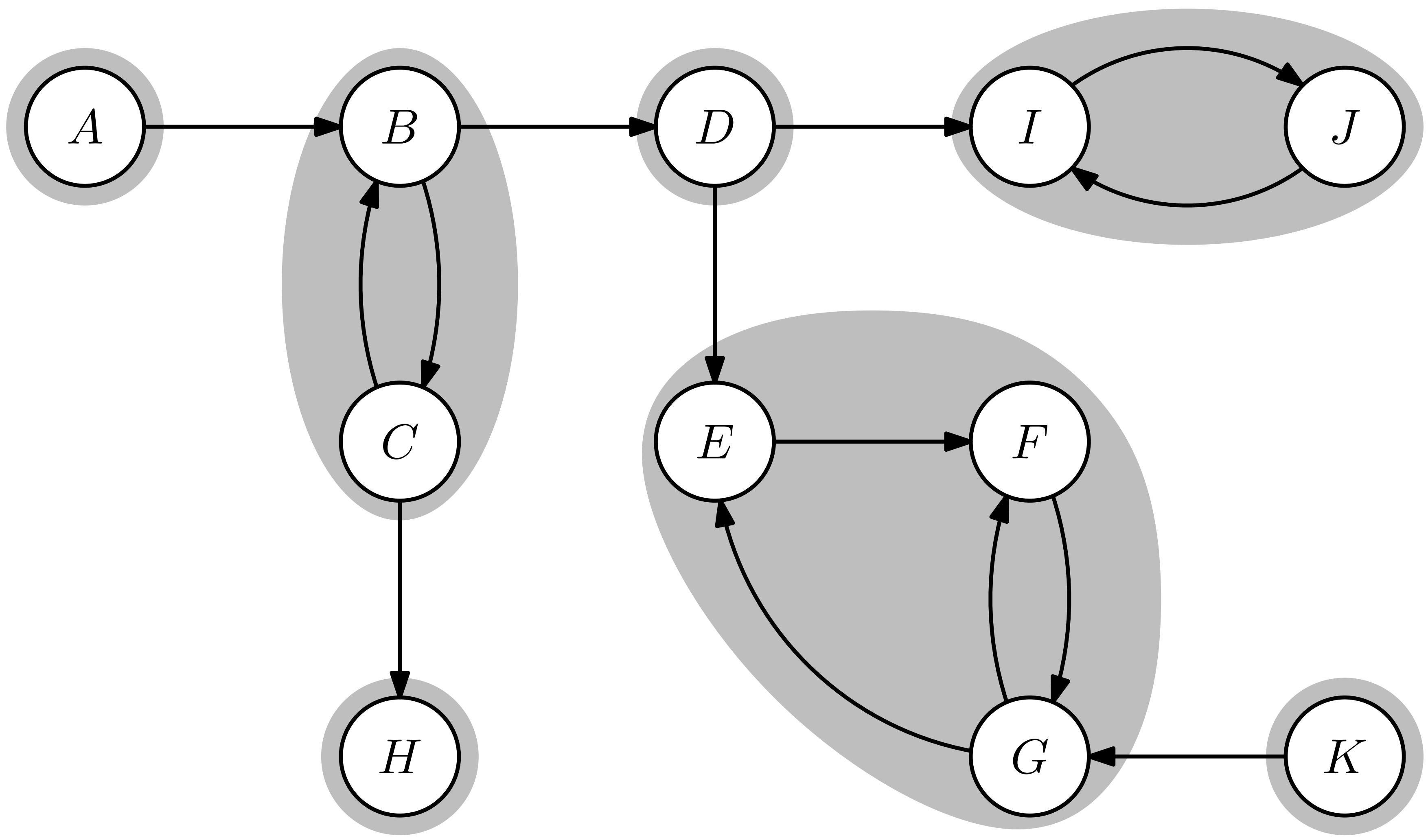
Returning to PageRank

- Is the chain irreducible?



Returning to PageRank

- Is the chain irreducible? No.



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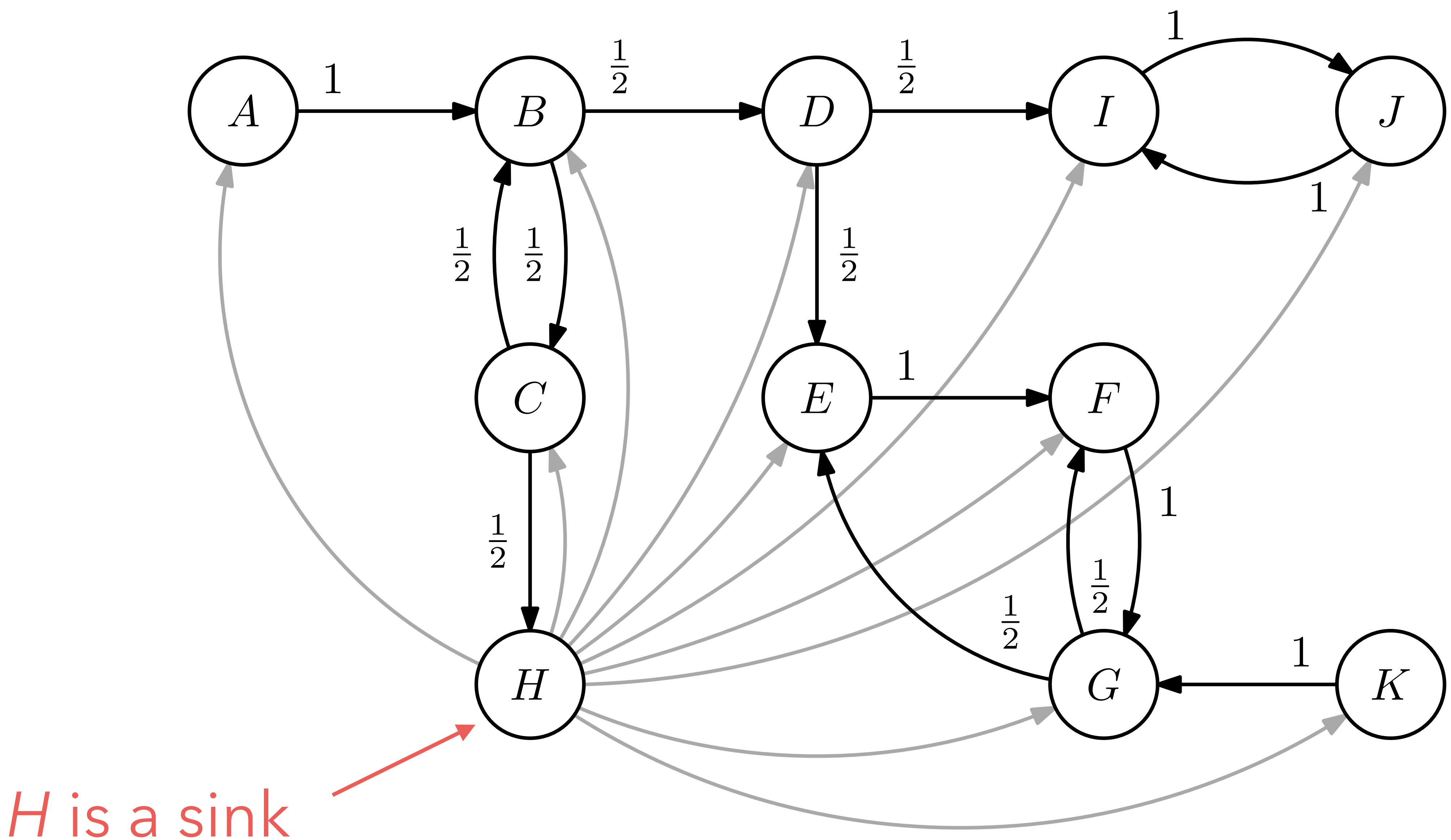
Returning to PageRank

- PageRank introduces two modifications to the chain:
  - Sinks link to all other nodes
  - There is a probability $1 - \gamma$ of "teleporting" to an arbitrary node

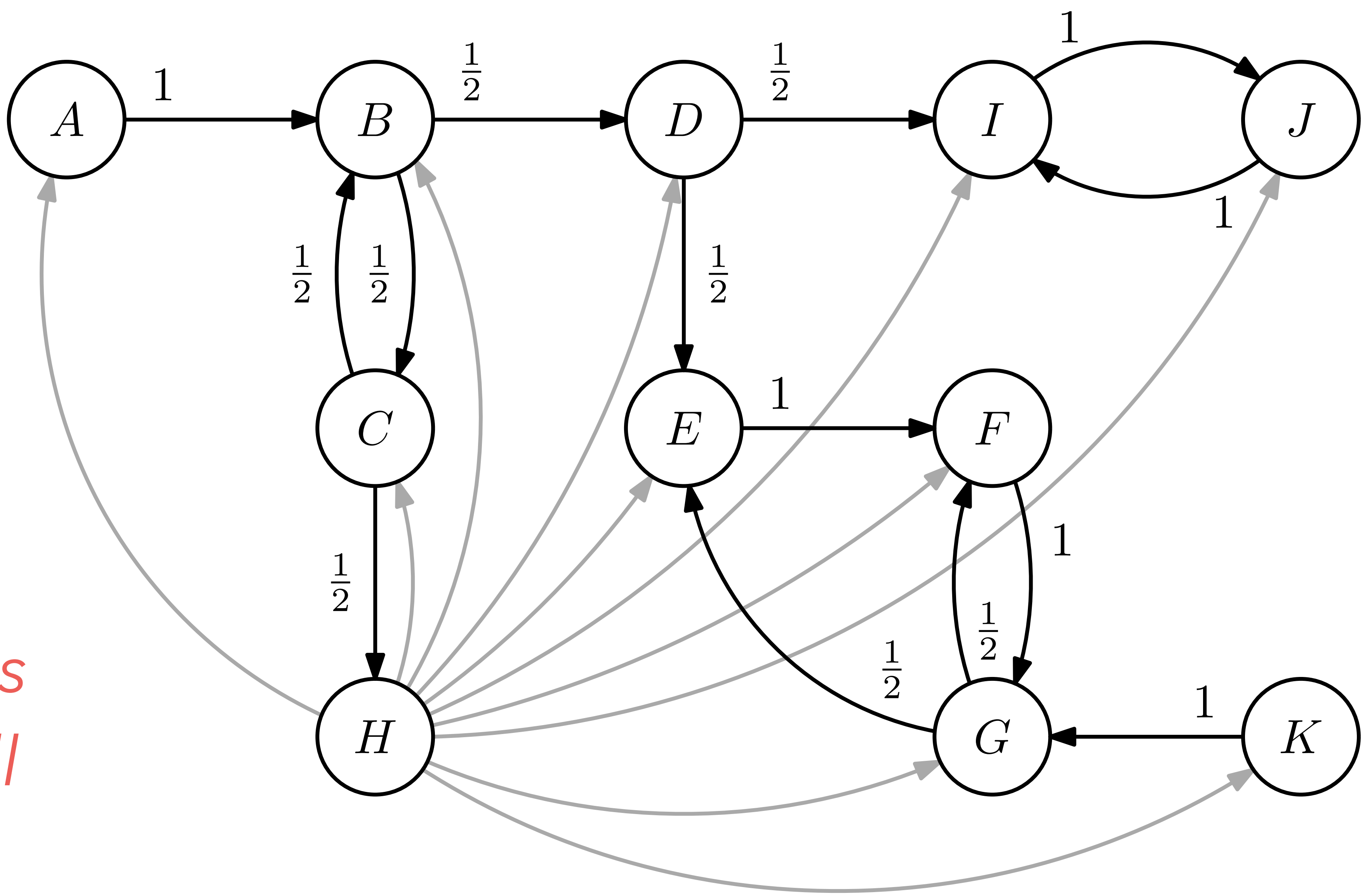
Returning to PageRank

- In our case:



Returning to PageRank

- In our case:



All nodes link to all nodes

Returning to PageRank

- Is the chain irreducible?
  - Yes
- Is the chain aperiodic?
  - Yes
- … then there is a stationary distribution

Returning to PageRank