TÉCNICO LISBOA

Planning, Learning and Decision Making

Lecture 3. Hidden Markov models

Basic notions

The urn problem

The urn problem

- An oracle has available two urns
  - Each urn is filled with black and white balls
  - **Urn 1:** 25% white balls; 75% black balls
  - **Urn 2:** 25% black balls; 75% white balls

The urn problem

- At each step, the oracle draws a ball from one of the urns
- The ball is put back afterwards

The urn problem

- The current urn determines which urn the next ball will come from
  - The next ball will come from the same urn with 80% probability
  - The next ball will come from a different urn with 20% probability

1. Is this a Markov chain?

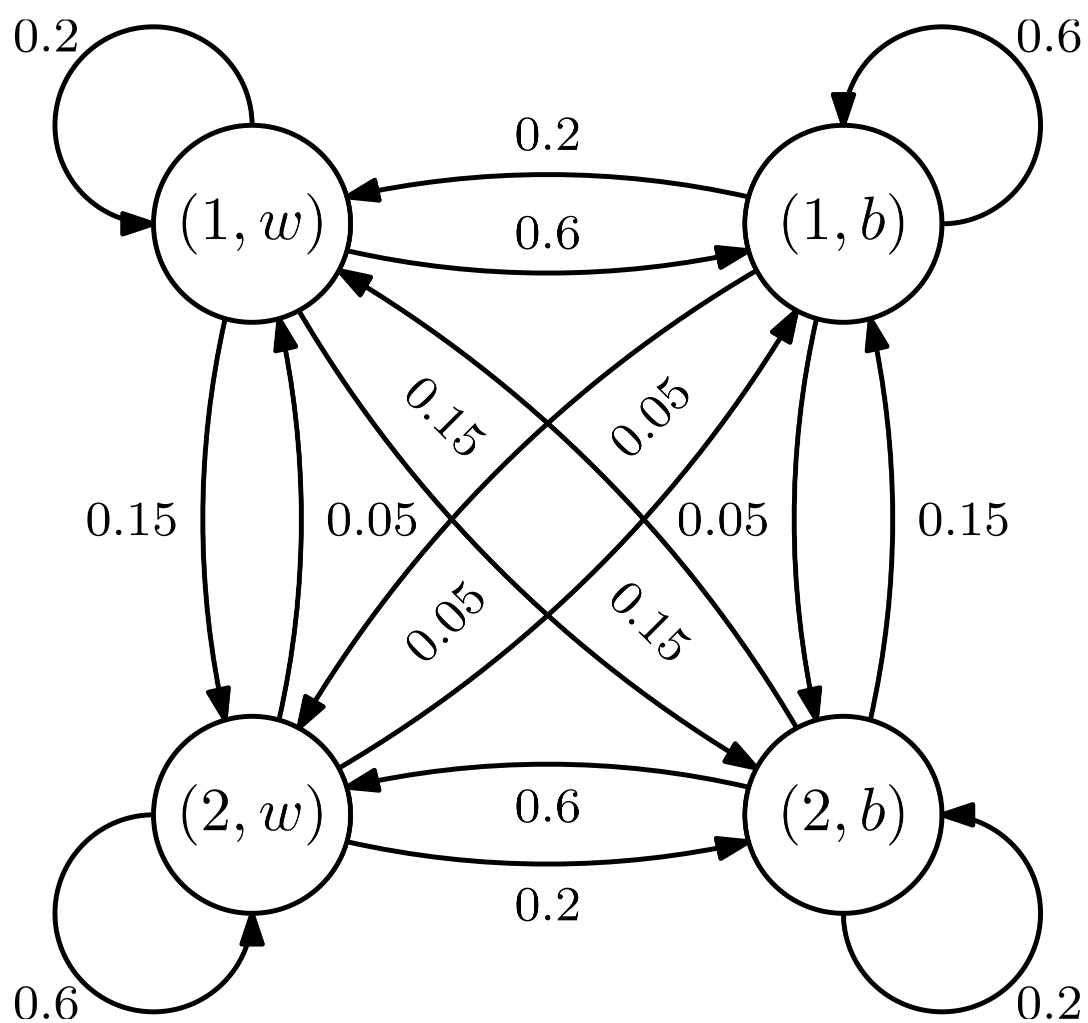
- **Yes**
  - The urn-ball in the next step depends on the urn-ball in the current step

2. What are the states?

- All possible urn-ball pairs:
  - $\mathcal{X} = \{(1, w), (1, b), (2, w), (2, b)\}$

3. Transition probabilities?

- We can represent the game with a transition diagram:



3. Transition probabilities?

- We can also represent the transition probabilities as a matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} (1, w) & (1, b) & (2, w) & (2, b) \end{array} \\
\begin{array}{c} (1, w) \\ (1, b) \\ (2, w) \\ (2, b) \end{array} & \begin{bmatrix} 0.2 & 0.6 & 0.15 & 0.05 \\ 0.2 & 0.6 & 0.15 & 0.05 \\ 0.05 & 0.15 & 0.6 & 0.2 \\ 0.05 & 0.15 & 0.6 & 0.2 \end{bmatrix}
\end{array}
$$

Can we make predictions?

- E.g., suppose that the initial urn-ball was (1, b)

- What is the next state?

$$\begin{bmatrix} 0.2 & 0.6 & 0.15 & 0.05 \\ \boxed{0.2 \quad 0.6 \quad 0.15 \quad 0.05} \\ 0.05 & 0.15 & 0.6 & 0.2 \\ 0.05 & 0.15 & 0.6 & 0.2 \end{bmatrix}$$

Can we make predictions?

- E.g., suppose that the initial urn-ball was (1, b)
- What is the next state?
  - (1, w) with probability 0.2
  - (1, b) with probability 0.6
  - (2, w) with probability 0.15
  - (2, b) with probability 0.05

80% probability

20% probability

Can we make predictions?

- E.g., suppose that the initial urn-ball was (1, b)

- What is the state at time t =3?

$$\begin{bmatrix} 0.15 & 0.46 & 0.29 & 0.10 \\ \boxed{0.15 \quad 0.46 \quad 0.29 \quad 0.10} \\ 0.10 & 0.29 & 0.46 & 0.15 \\ 0.10 & 0.29 & 0.46 & 0.15 \end{bmatrix}$$

Can we make predictions?

- E.g., suppose that the initial urn-ball was (1, b)
- What is the state at time t =3?
  - (1, w) with probability 0.15
  - (1, b) with probability 0.46
  - (2, w) with probability 0.29
  - (2, b) with probability 0.10

The urn problem (revisited)

- At each step, the oracle draws a ball from one of the urns
- The ball is put back afterwards
- **The oracle reveals only the color of the ball**

Partial observability

⬇

The state cannot be fully observed

1. Is this a Markov chain?

- **No!**
  - The information available at each moment (ball color) is not enough to predict next ball color
  - Why?
  - The next distribution depends on the urn!

Can we make predictions?

- Suppose that we know that:
    - At time $t = 0$, the ball was white and drawn from urn 1
    - At time $t = 1$, the ball was black
    - What is the state at time $t = 2$?

Can we make predictions?

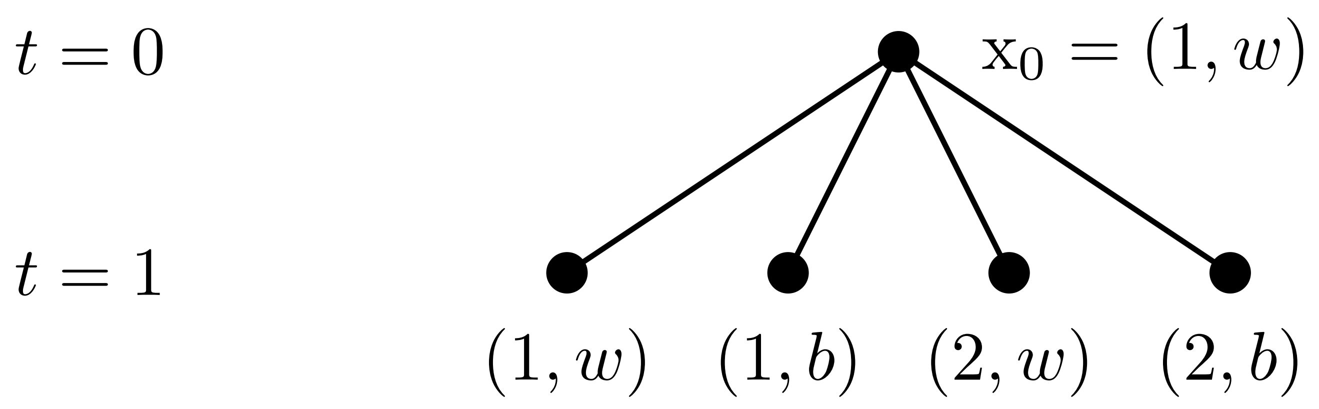
- We have:

$t = 0$

$\bullet \quad \mathrm{x}_0 = (1, w)$

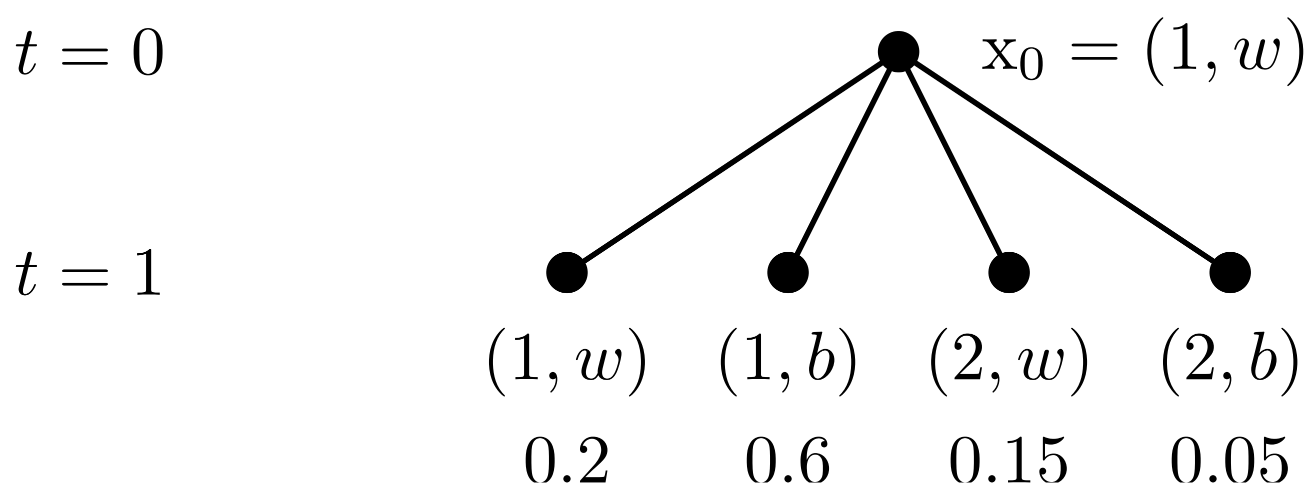
Can we make predictions?

- We have:



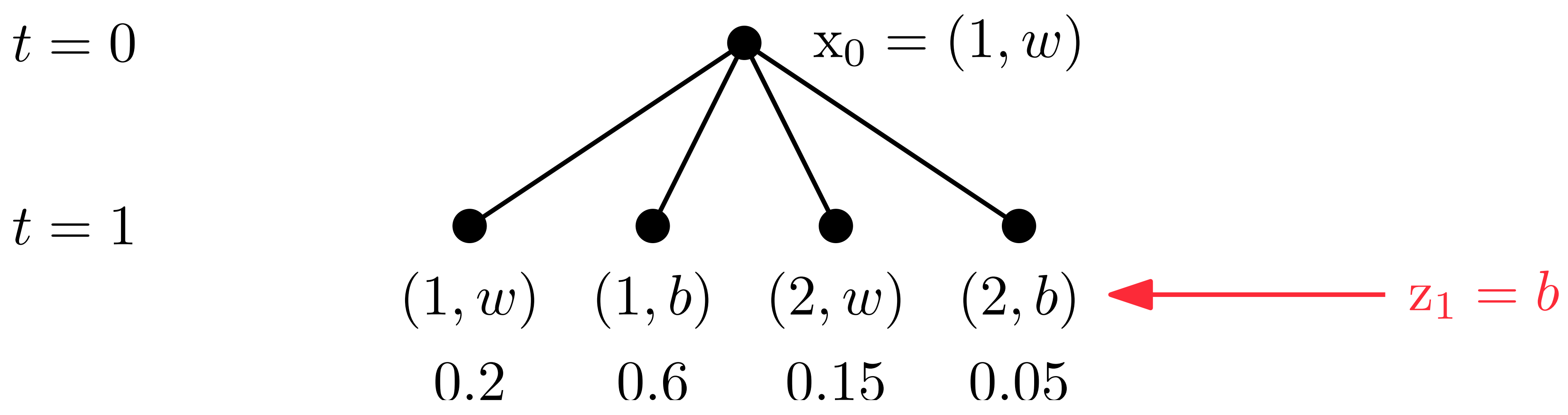
Can we make predictions?

- We have:



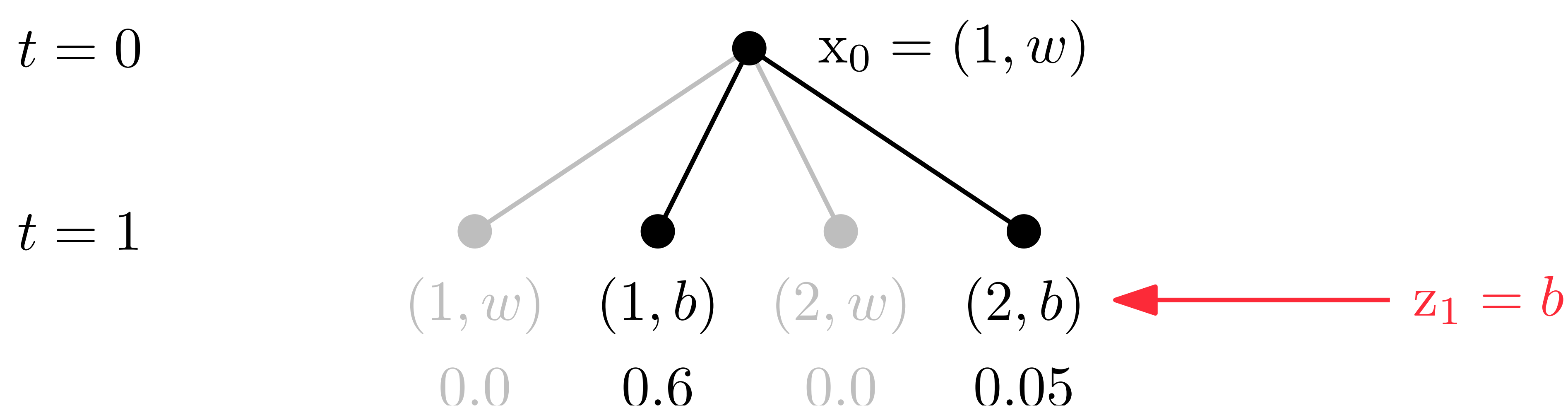
Can we make predictions?

- We have:



Can we make predictions?

- We have:



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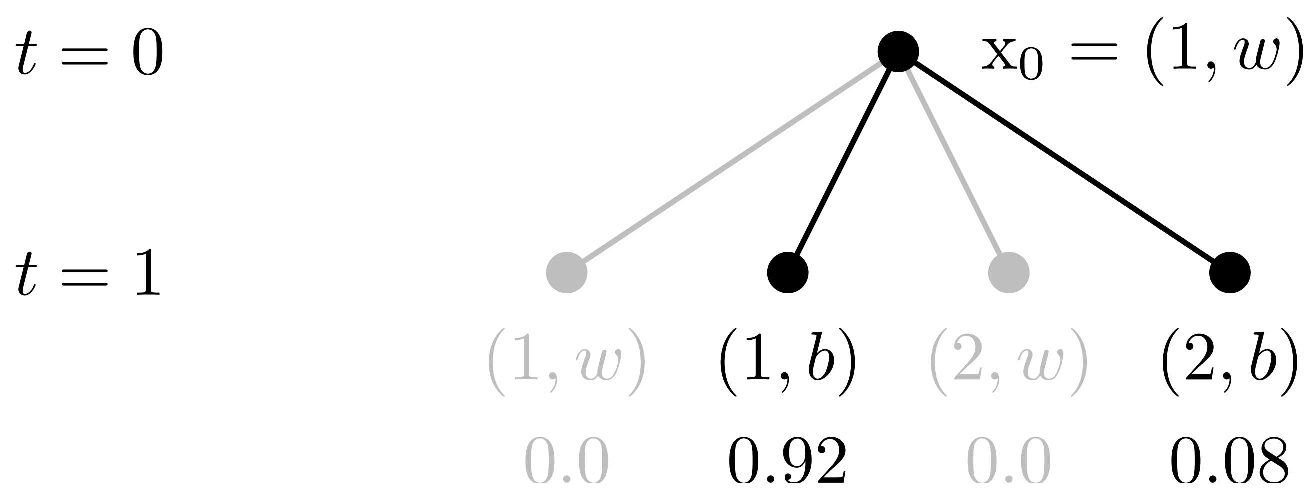

Can we make predictions?

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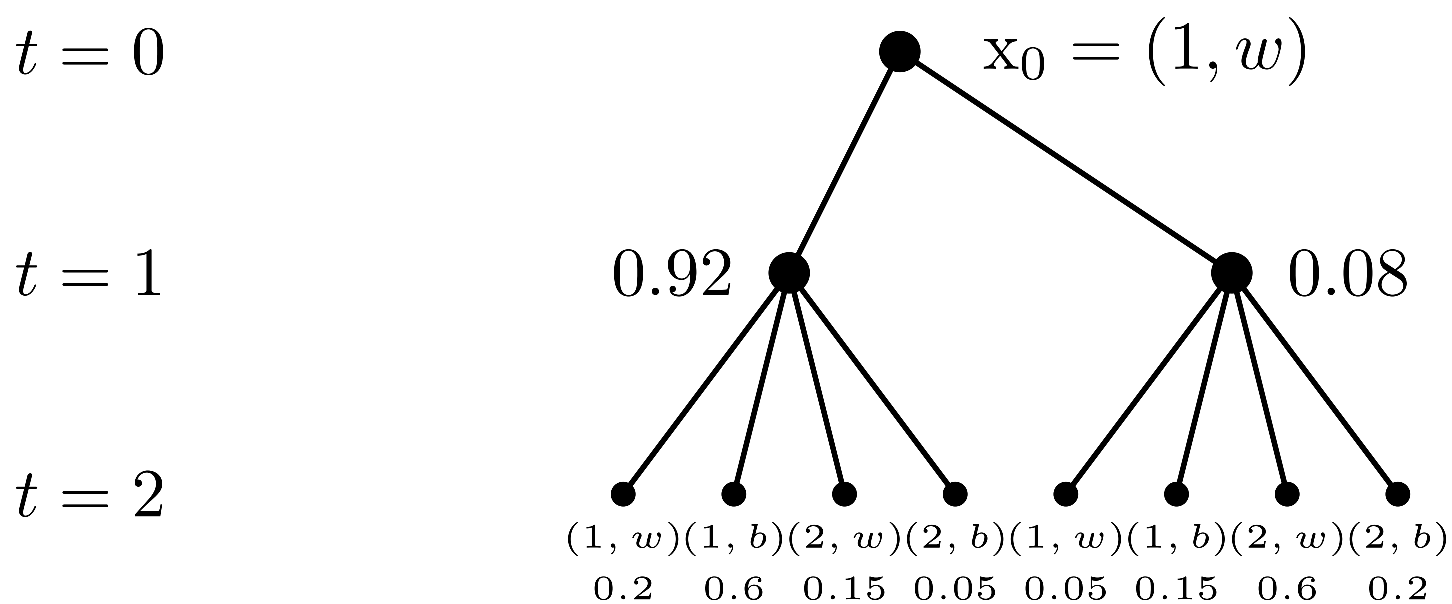

Can we make predictions?

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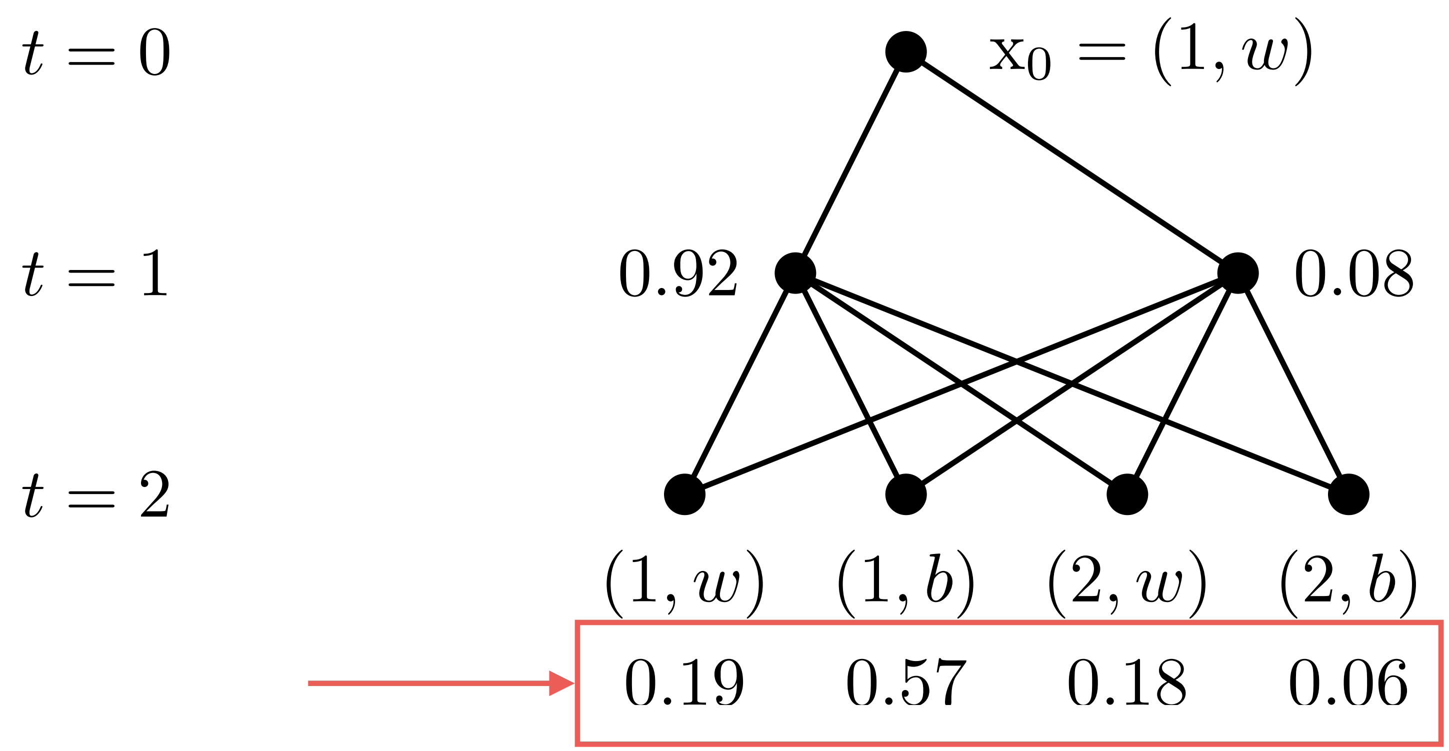


Can we make predictions?

Observation

- In formal terms:

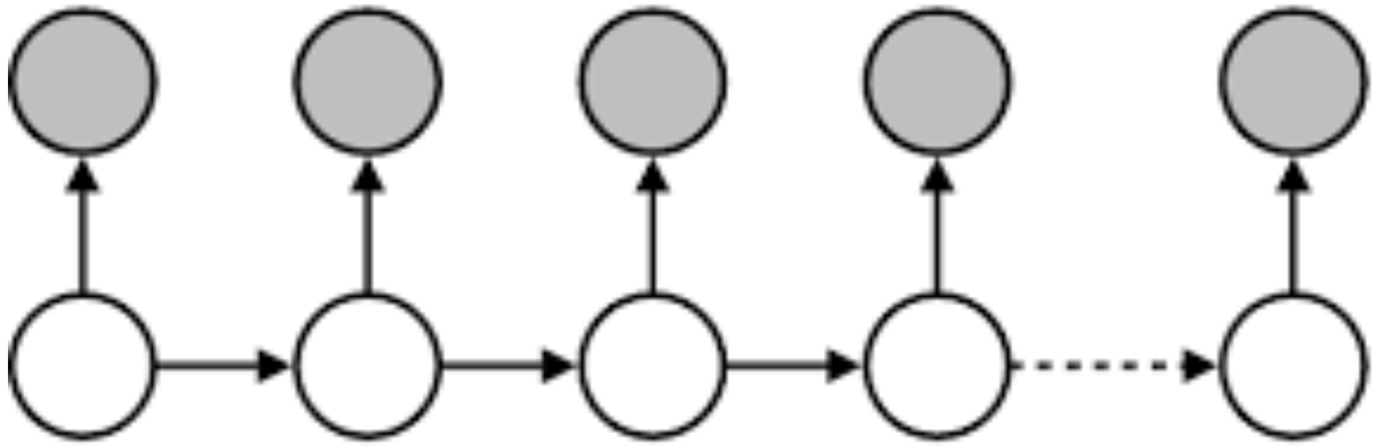
$$\mathbb{P}\left[\mathrm{x}_2 = x \mid \mathrm{x}_0 = (1, w), \mathrm{z}_1 = b\right]$$

$$= \frac{1}{\rho}\Big(\mathbf{P}\big(x \mid (1, b)\big)\mathbf{P}\big((1, b) \mid (1, w)\big) + \mathbf{P}\big(x \mid (2, b)\big)\mathbf{P}\big((2, b) \mid (1, w)\big)\Big)$$

Normalization

First level terms

Combination of similar terms



Hidden Markov models

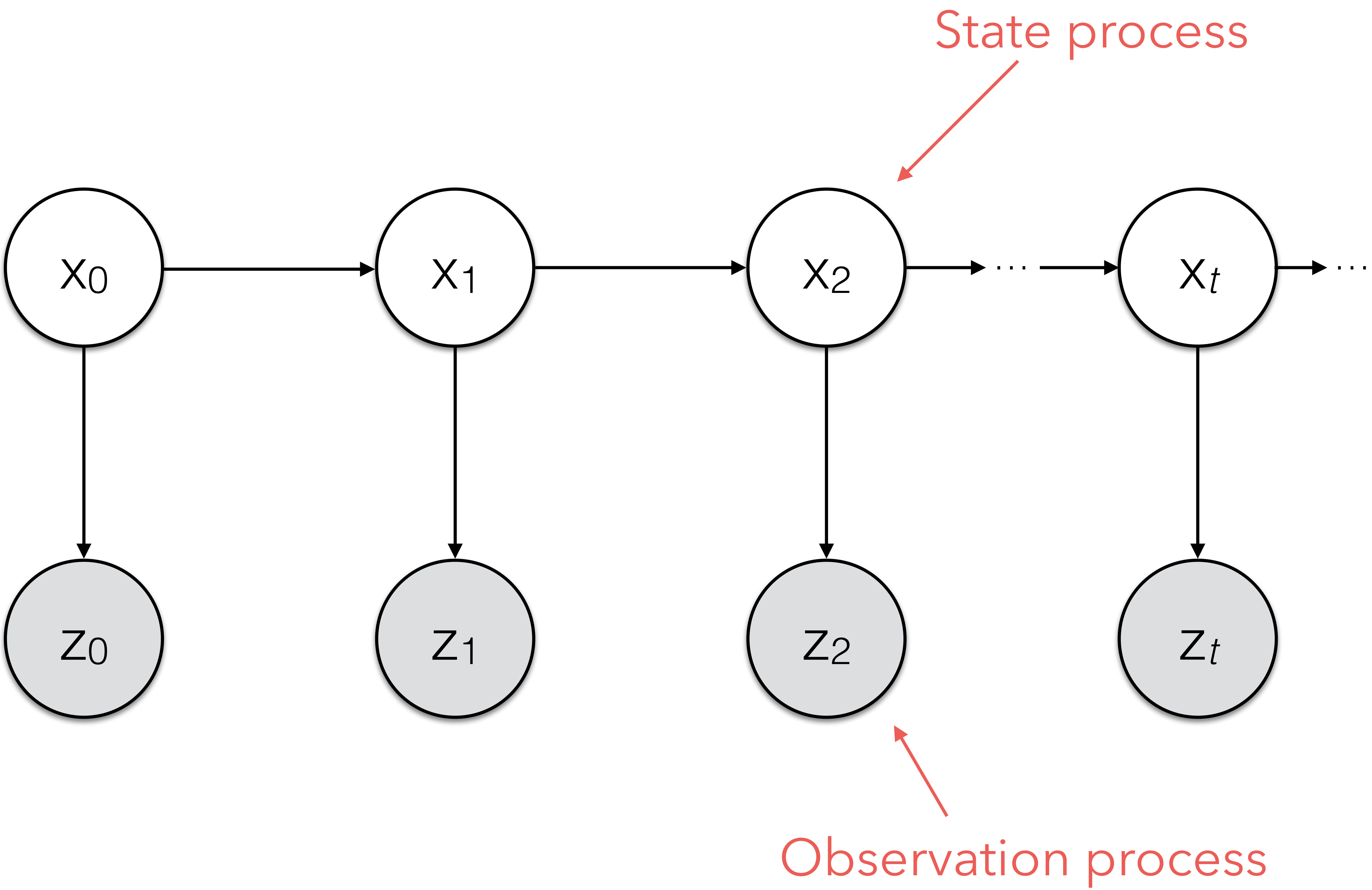
Hidden Markov model

- Model for **sequential process with partial observability**
- Process evolves in **discrete time steps** (as in Markov chains)
- **State of the process** at time step t: x_t
- **Observation** at time step t: z_t

Quantity of interest

Observable

Hidden Markov model



Hidden Markov model

Markov state

The state at instant t is enough to predict the state at instant $t + 1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

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Hidden Markov model

State-dependent observations

The state at instant t is enough to predict the observation at instant t:

$$\mathbb{P}\left[\mathrm{z}_t = z \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t-1} = \boldsymbol{z}_{0:t-1}\right] = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x_t\right]$$

Depends only on the last state

Hidden Markov model

Markov state

The state at instant t is enough to predict the state at instant $t+1$:

$$\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right] = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x_t\right]$$

State-dependent observations

The state at instant t is enough to predict the observation at instant t:

$$\mathbb{P}\left[\mathrm{z}_t = z \mid \mathbf{x}_{0:t} = \boldsymbol{x}_{0:t}, \mathbf{z}_{0:t-1} = \boldsymbol{z}_{0:t-1}\right] = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x_t\right]$$

Hidden Markov model

- Other assumptions (for most of this course):
  - There is only a **finite number** of possible states
  - $\mathcal{X}$ is the set of possible states (**state space**)
  - There is only a **finite number** of observations
  - $\mathcal{Z}$ is the set of possible observations (**observation space**)

Hidden Markov model

- Other assumptions:
  - The transition probabilities $\mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$ do not depend on *t*
  - The observation probabilities $\mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x\right]$ do not depend on *t*

Transition probability matrix

- As with Markov chains, we store the transition probabilities in a **matrix P**

$$[\mathbf{P}]_{xy} = \mathbb{P}\left[\mathrm{x}_{t+1} = y \mid \mathrm{x}_t = x\right]$$

Observation probability matrix

- Similarly, we store the observation probabilities in a **matrix** **O**

$$[\mathbf{O}]_{xz} = \mathbb{P}\left[\mathrm{z}_t = z \mid \mathrm{x}_t = x\right]$$

Number in row *x* column *z* is the probability of observing *z* in *x*

Example

- **The urn problem**

$$
\begin{array}{c|cccc|}
 & (1, w) & (1, b) & (2, w) & (2, b) \\
(1, w) & 0.2 & 0.6 & 0.15 & 0.05 \\
(1, b) & 0.2 & 0.6 & 0.15 & 0.05 \\
(2, w) & 0.05 & 0.15 & 0.6 & 0.2 \\
(2, b) & 0.05 & 0.15 & 0.6 & 0.2
\end{array}
\qquad
\begin{array}{c|cc|}
 & w & b \\
(1, w) & 1 & 0 \\
(1, b) & 0 & 1 \\
(2, w) & 1 & 0 \\
(2, b) & 0 & 1
\end{array}
$$

Summarizing…

- A HMM can be represented compactly as a tuple

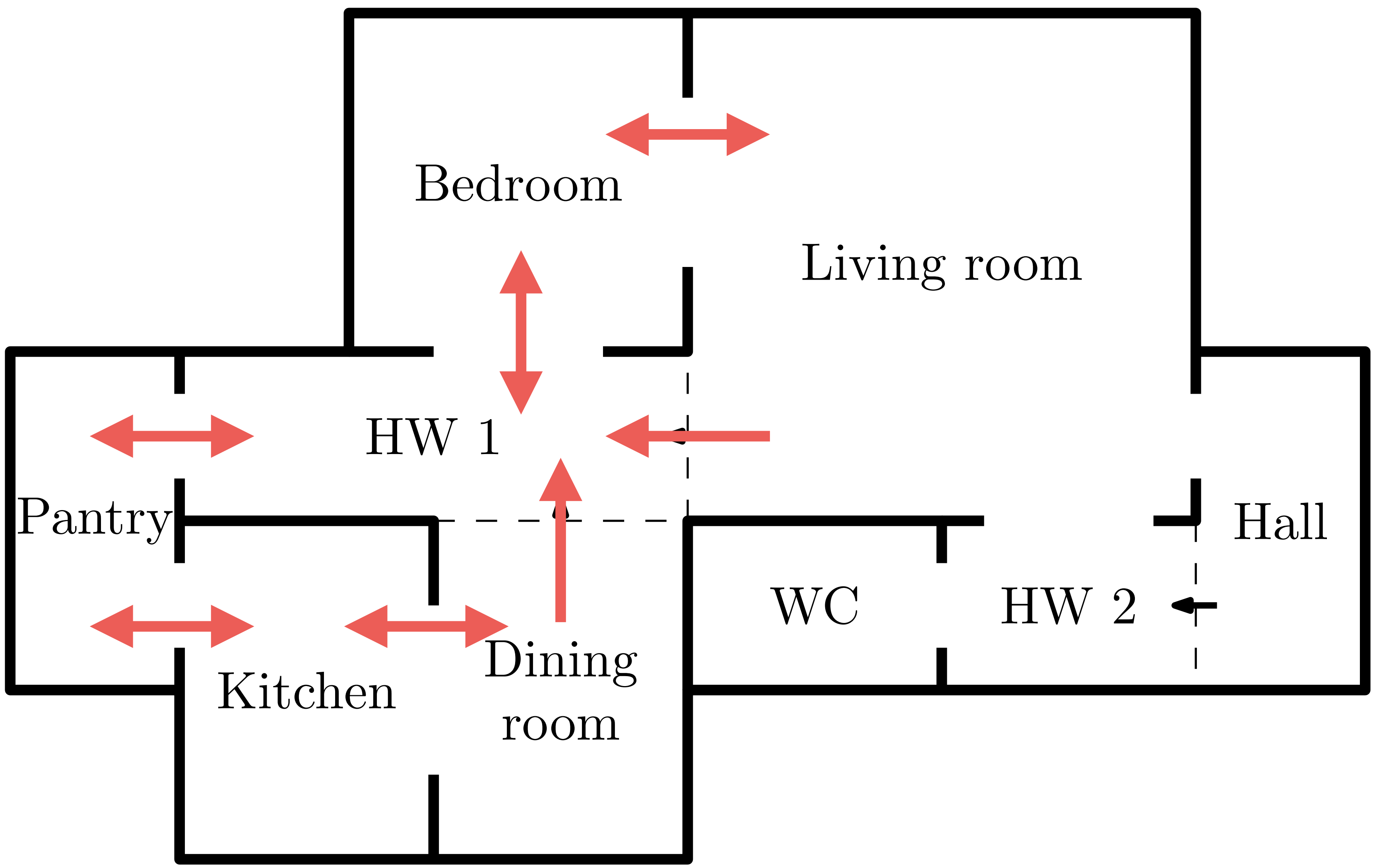
$$(\mathscr{X}, \mathscr{Z}, \mathbf{P}, \mathbf{O})$$

  - $\mathscr{X}$ is the set of possible states
  - $\mathscr{Z}$ is the set of possible observations
  - $\mathbf{P}$ is the transition probability matrix
  - $\mathbf{O}$ is the observation probability matrix

Examples

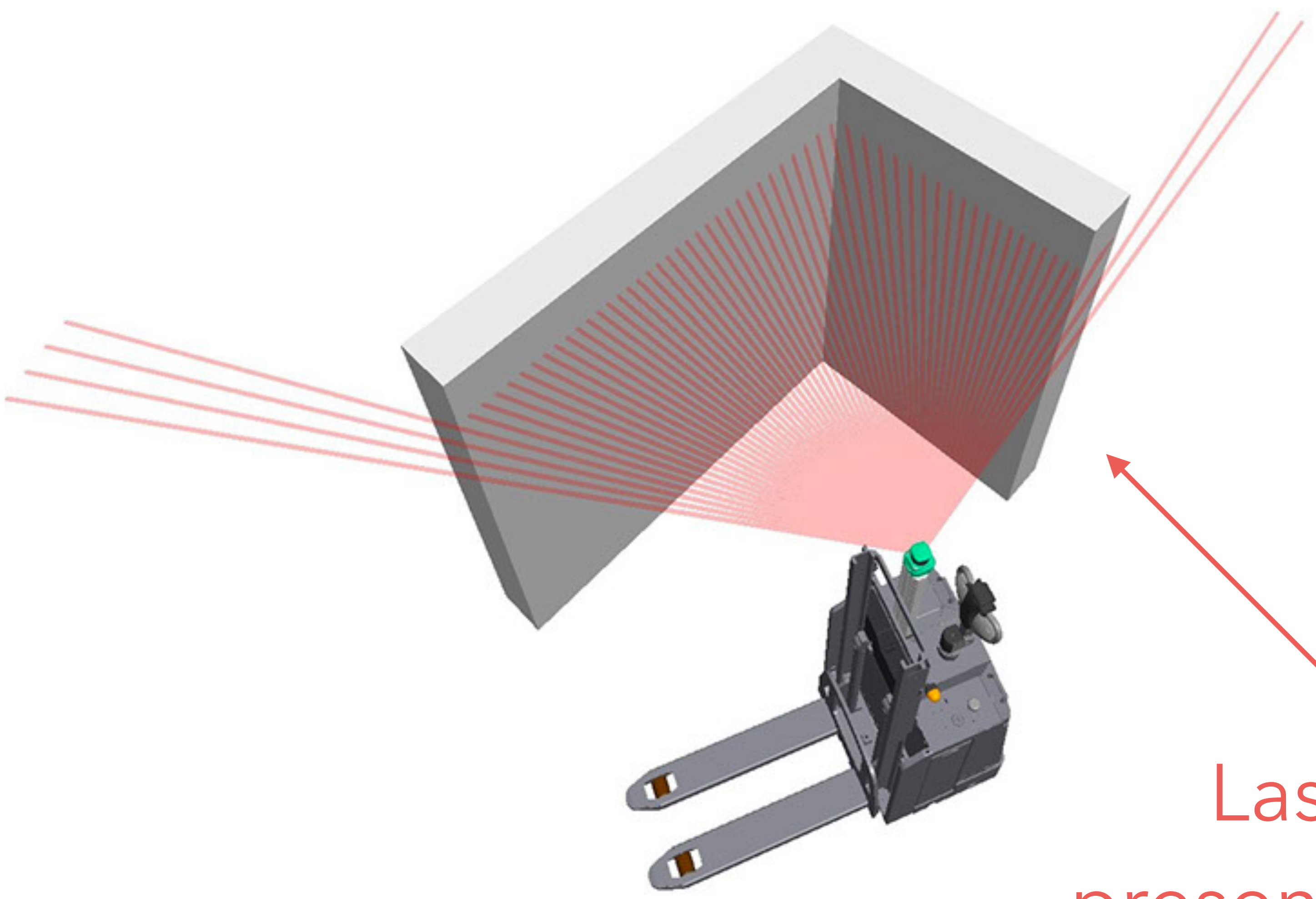
1. Household robot

- Remember?



1. Household robot

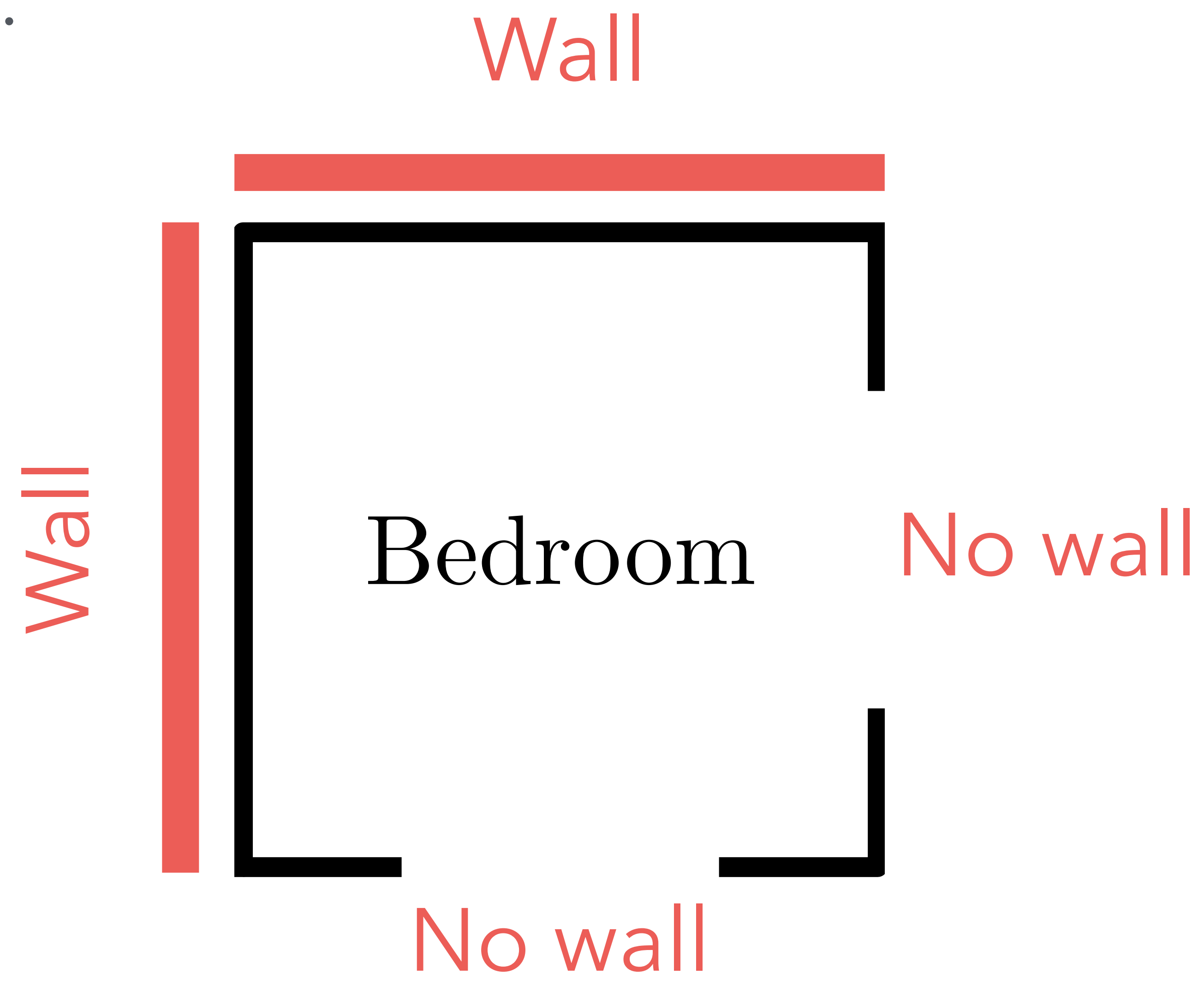
- Robot navigates using a laser



Laser detects presence or absence of walls

1. Household robot

- For example:



1. Household robot

- However, laser is not perfect
  - It fails to detect existing walls with 5% probability
  - It detects non-existing walls with 10% probability (in some situations with 20% probability)
  - Detection of a wall independent of adjacent walls

Is this an HMM?

- State verifies the Markov property?
  - **Yes** - the position of the robot at time $t + 1$ depends only of position of the robot at time t
- Observation depends only on state?
  - **Yes** - the wall detections depends only on robot position

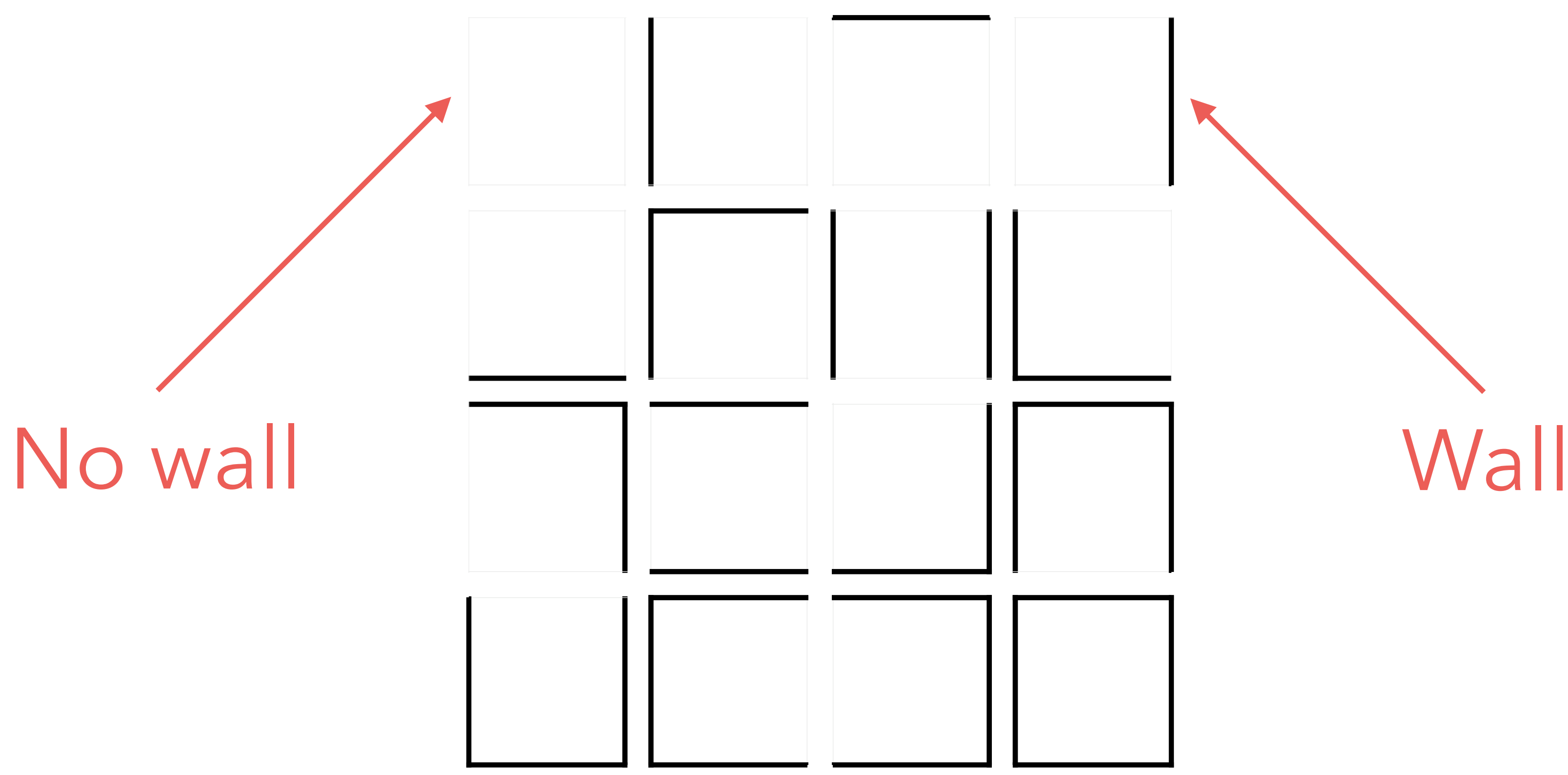
It is an HMM!

What are the states?

- Possible positions of the robot:
  - $\mathscr{X} = \{K, P, D, H_1, B, L\}$

What are the observations?

- Possible wall configurations:



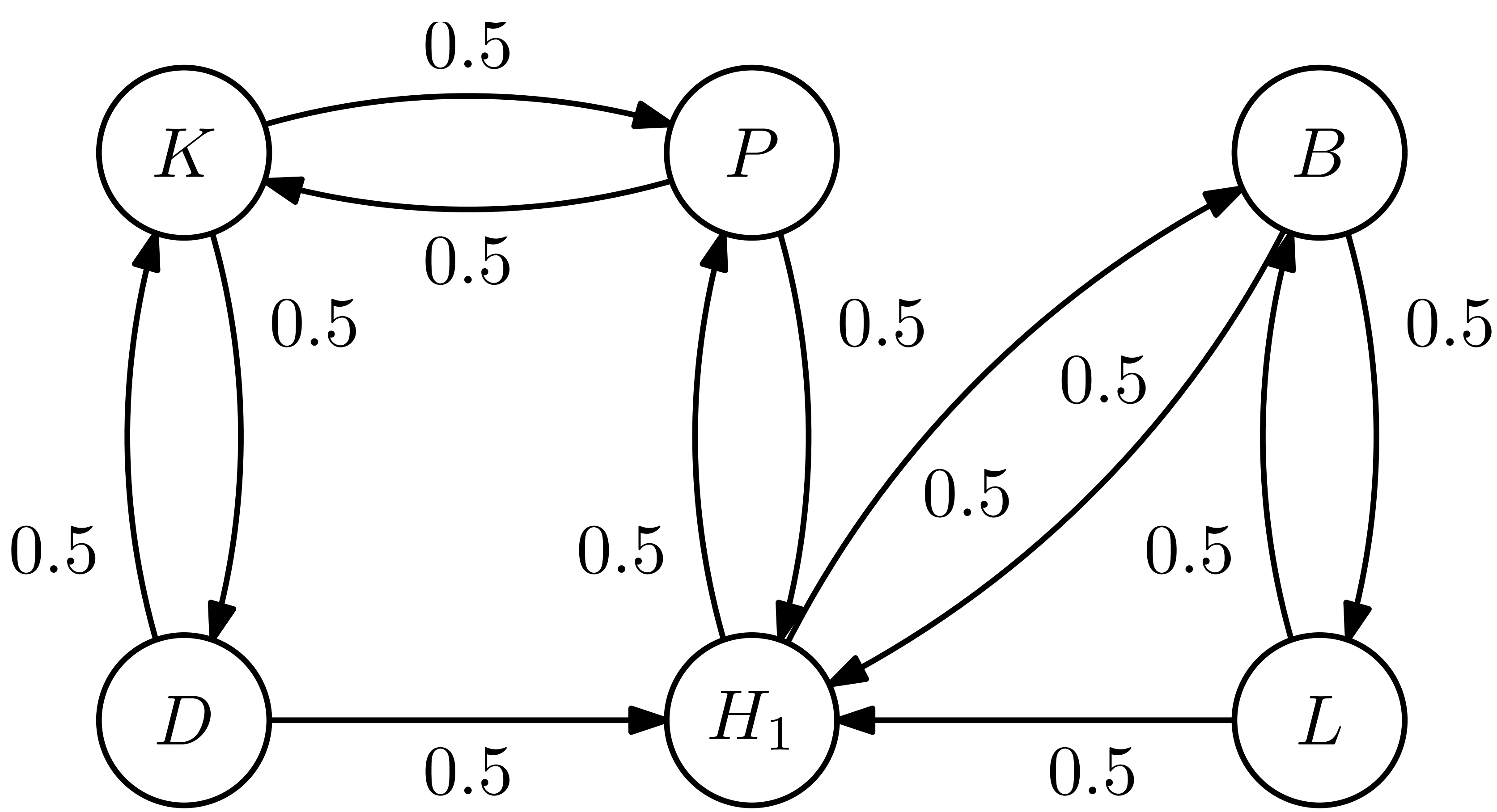
What are the observations?

- Possible wall configurations:
  - $\mathscr{Z} = \{0000, 0001, 0010, 0011,$
    $0100, 0101, 0110, 0111,$
    $1000, 1001, 1010, 1011,$
    $1100, 1101, 1110, 1111\}$

U-D-L-R

Transition probabilities

- We can represent the underlying state transitions using a transition diagram:



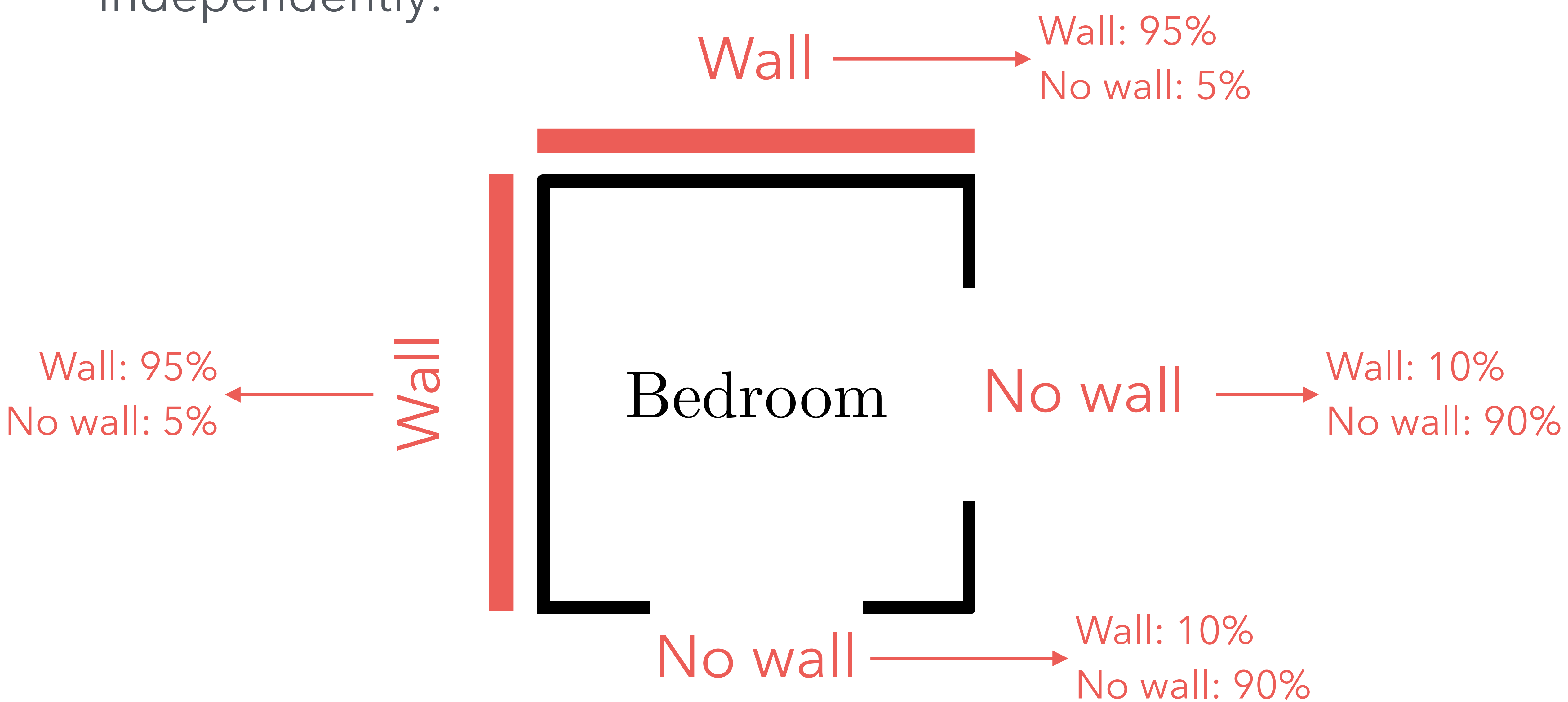
Transition probabilities

- … or as a matrix:

$$\mathbf{P} = \begin{bmatrix} 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 \end{bmatrix}$$

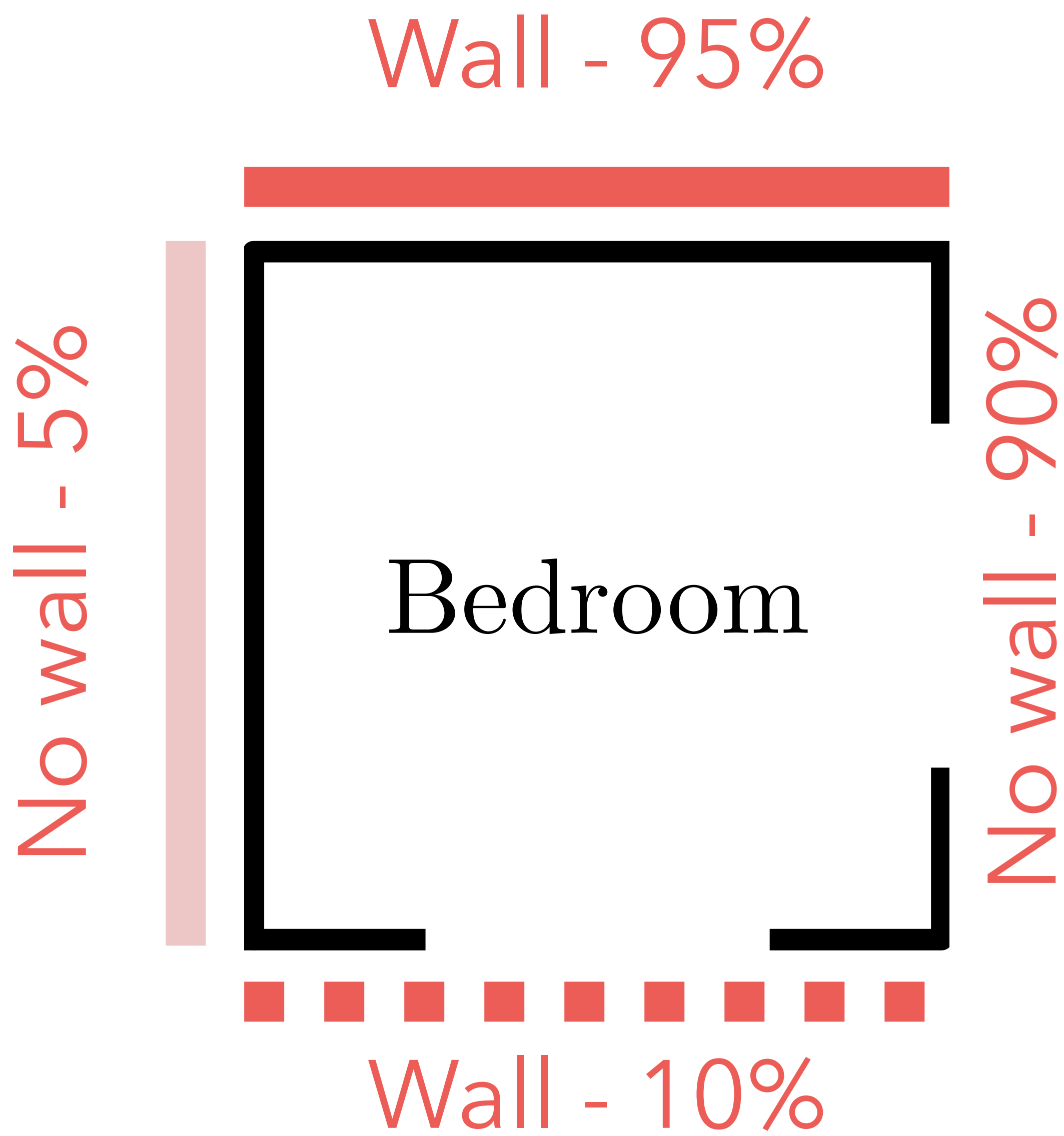
Observation probabilities

- The observation probabilities can be computed for each wall independently:



Observation probabilities

- For example: $\mathbb{P}\left[\mathrm{z}_t = 1100 \mid \mathrm{x}_t = B\right]$



Result: 0.0042

Observation probabilities

- This yields a large matrix:

$$\mathbf{O} = \begin{bmatrix} 0.0018 & 0.0004 & 0.0342 & \dots & 0.0009 & 0.0722 & 0.0180 \\ 0.0001 & 0.0021 & 0.0021 & \dots & 0.0045 & 0.0045 & 0.0857 \\ 0.0020 & 0.0002 & 0.0384 & \dots & 0.0005 & 0.0812 & 0.0090 \\ 0.5184 & 0.0576 & 0.1296 & \dots & 0.0016 & 0.0036 & 0.0004 \\ 0.0020 & 0.0385 & 0.0002 & \dots & 0.0812 & 0.0004 & 0.0090 \\ 0.0324 & 0.0036 & 0.0081 & \dots & 0.0076 & 0.0171 & 0.0019 \end{bmatrix}.$$

HMM problems

HMM problems

- **Estimation:**
  - Estimate HMM trajectories given an HMM model

- **Inference:**
  - Determine an HMM model from data

Estimation

- **Filtering:**

Forward algorithm

  - Given a sequence of observations, estimate the final state

- **Smoothing:**

Viterbi algorithm

  - Given a sequence of observations, estimate the sequence of states

- **Prediction:**

  - Given a sequence of observations, predict future states

Filtering

Filering

- We are given a sequence of observations $\boldsymbol{z}_{0:T}$
- We want to estimate

$$\mathbb{P}_{\mu_0}\left[\mathrm{x}_T = x \mid \mathbf{z}_{0:t} = \boldsymbol{z}_{0:T}\right]$$

where μ_0 is the initial distribution, i.e.,

$$\mu_0(x) = \mathbb{P}\left[\mathrm{x}_0 = x\right]$$

Filtering

- For simplicity, write

$$\mu_{T|0:T}(x) = \mathbb{P}_{\mu_0}\left[\mathrm{x}_T = x \mid \mathbf{z}_{0:t} = \boldsymbol{z}_{0:T}\right]$$

Forward mapping

Forward mapping

Given a sequence of observations $\mathbf{z}_{0:t}$, the forward mapping $\alpha_t : \mathscr{X} \longmapsto \mathbb{R}$ is defined for each t as

$$\alpha_t(x) = \mathbb{P}_{\mu_0}\left[\mathrm{x}_t = x, \mathbf{z}_{0:t} = \boldsymbol{z}_{0:t}\right]$$

So what?

- Forward mapping has several useful properties

1. We can compute $\mu_{T\,|\,0:T}$ from α_T:

$$\mu_{T|0:T}(x) = \frac{\alpha_T(x)}{\sum_{y\in\mathcal{X}} \alpha_T(y)}$$

So what?

- Forward mapping has several useful properties

1. We can compute $\mu_{T\,|\,0:T}$ from α_T

2. The forward mapping can be computed recursively:

$$\alpha_T(x) = \mathbf{O}(z_T \mid x) \sum_{y \in \mathcal{X}} \mathbf{P}(x \mid y)\alpha_{T-1}(y)$$

Forward algorithm

Require: Observation sequence z0:T

1. Initialize $\boldsymbol{\alpha}_0 \leftarrow \operatorname{diag}(\mathbf{O}_{:,z_0})\boldsymbol{\mu}_0^\top$

If we have an initial observation

2. **for** t = 1, …, T **do**

$$\boldsymbol{\alpha}_t \leftarrow \operatorname{diag}(\mathbf{O}_{:,z_t})\mathbf{P}^\top\boldsymbol{\alpha}_{t-1}$$

4. end for

5. **return** $\boldsymbol{\mu}_{T|0:T} = \alpha_T/(\mathbf{1}^\top\boldsymbol{\alpha}_T)$

Example: The urn problem

- Suppose that

$$\boldsymbol{\mu}_0 = \begin{bmatrix} 0.125 & 0.375 & 0.375 & 0.125 \end{bmatrix}$$

- We observe the sequence of observations

$$\boldsymbol{z}_{0:2} = \{w, w, b\}$$

- What is the state at time $t = 2$?

Step 1: Initialize α_0

- $\boldsymbol{\alpha}_0 \leftarrow \operatorname{diag}(\mathbf{O}_{:,z_0})\boldsymbol{\mu}_0^\top$

$$
\begin{array}{c}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
z_0 \quad\quad\;\;
\end{array}
\rightarrow
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 0.125 \\ 0.375 \\ 0.375 \\ 0.125 \end{bmatrix}
\rightarrow
\begin{bmatrix} 0.125 \\ 0 \\ 0.375 \\ 0 \end{bmatrix}
$$

Step 2: Compute α_1

- $\boldsymbol{\alpha}_1 \leftarrow \text{diag}(\mathbf{O}_{:,z_1})\mathbf{P}^\top\boldsymbol{\alpha}_0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.05 & 0.05 \\ 0.6 & 0.6 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.6 & 0.6 \\ 0.05 & 0.05 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0 \\ 0.375 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0.04 \\ 0 \\ 0.24 \\ 0 \end{bmatrix}$$

Step 3: Compute α_2

- $\boldsymbol{\alpha}_2 \leftarrow \mathrm{diag}(\mathbf{O}_{:,z_2})\mathbf{P}^\top \boldsymbol{\alpha}_1$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.05 & 0.05 \\ 0.6 & 0.6 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.6 & 0.6 \\ 0.05 & 0.05 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0 \\ 0.24 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0.06 \\ 0 \\ 0.05 \end{bmatrix}$$

Final step: Compute $\mu_{2 \mid 0:2}$

- We finally get:

$$\begin{aligned}\boldsymbol{\mu}_{2|0:2} &= \alpha_2/(\mathbf{1}^\top \boldsymbol{\alpha}_2) \\ &= \begin{bmatrix} 0 & 0.552 & 0 & 0.448 \end{bmatrix}\end{aligned}$$