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Quantentransport in Spindichtesystemen mit dem Memory-Matrix-Formalismus

Masterthesis

von

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Karlsruhe, den 24. Mai 2018

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(**Martin Lietz**)

Acknowledgment

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2 Infinite Conductivity in Translation Invariant Systems

Ever since Drude published his theory about the electrical conductivity in metals [Dru00] at the beginning of the last century, it is well known that non-conversion of electron momentum in a system is required for finite electrical conductivity. In Drude's model, the electrons possess a mean scattering time τ_{el} representing the mean time between two scattering events of an electron and a lattice atom. At each scattering event, the electrons transfer momentum to the lattice atoms, which is the reason electron momentum is not a conserved quantity. In the case of conserved momentum, if an electrical field is exemplary applied, it accelerates electrons up to an infinite velocity, causing infinite conductivity.

In the following section, a momentum conserving system is considered and the electrical conductivity is computed using the memory-matrix-formalism, where an infinite conductivity is expected. Beforehand, a short overview over the memory-matrix-formalism is given, followed by detailed and explicite deviations in chapter **??**.

> overview oder review?

2.1 An Overview over the Memory-Matrix-Formalsim

Let us assume a physical dynamical variable $A(t)$ in an arbitrary system and an arbitrary pertubation too . Our interest is now the time evolution of $A(t)$ depending on the pertubation. Completely general, a physical dynamical variable can be splitted into two parts, a secular one and a non-secular one, which is shown in [Mor65]. The latter represent processes like fluctuations or initial transient processes, which have in common a short lifetime comparing to the secular processes. The dynamic and time evolution of $A(t)$ is therefore dominated by secular processes.

This seperation enables a simple but clever geometrical interpretation, where dynamical variables are considered to be vectors in a vector space. The variables of the unpertubated system represent the basis vectors of this vector space, denoting in the case of the variable $A(t)$ as A-axis. Due to pertubation the direction of the variable $A(t)$ changes with respect to the A-axis. The projection of $A(t)$ onto the A-axis corresponds to the secular part, whereas the perpendicular component represents the non-secular parts.

> besser formulieren

First of all, the mathematical framework for this vector space has to be established. In quantum mechanics, the Liouville space, also called as operator space, is the respective vector space of the memory-matrix-formalism. As indicated by the name *operator space*, the vectors of the Liouville space are operators, all of which being Hermitian. The basis of the Liouville space is signified as $\{|A_i\rangle\}$, where $i = 1, 2, 3, \ldots, n$, and the

corresponding dual space basis is denoted as $\{(A_i|\}$. To complete the definition of any vector space, a scalar product is required. The following one is chosen:

$$\left(A_i(t)\middle|A_j(t')\right) = \frac{1}{\beta}\int\limits_0^\beta d\lambda \left\langle A_i^\dagger(t)A_j(t'+i\lambda)\right\rangle. \tag{2.1}$$

The normal time evolution $A_i(t) = e^{iHt/\hbar}A_i(0)e^{-iHt/\hbar}$ of an operator should be valid so that $A_i(i\lambda) = e^{-\lambda H}A_i(0)e^{\lambda H}$ can be used. If secular processes are neglected, the choice of the scalar product is determined under the aspect that as a consequence the time evolution of $A(t)$ given the most probably path, see [Mor65]. In quantum mechanics, the dynamic of an operator is ususally described by the Heisenberg equation of motion, which is transformed using the dyad product into the Liouville space

$$\left|\dot{A}_i(t)\right) = \frac{i}{\hbar}\left|[H, A_i(t)]\right) = iL\left|A_i(t)\right). \tag{2.2}$$

Here the Hermitian Liouville operator, $L = \hbar^{-1}[H, \bullet]$, is introduced, which is defined by acting onto an arbitrary operator. The formal solution of this equation is given by $|A_i(t)) = \exp(itL)\cdot|A_i(0))$, where the time evolution of an operator is therefore given by the Liouville operator. For our approach a further operator is required in the Liouville space, namely the projection operator. A set of arbitrary operators $\{C_i\}$ is therefore defined. The choice of them is different for each observed problem and unimportant at the moment. The definition of the projection operator in Liouville space follows directly from the projection operator defined in the commonly used Hilbert space in quantum mechanics.

$$P = \sum_{i,j} \frac{\left|C_i(0)\right)\left(C_j(0)\right|}{\left(C_i(0)\middle|C_j(0)\right)} \tag{2.3}$$

The projection operator acting on some vector in Liouville space yields the projection onto the subspace spanned by the operator C_i. Thus, the operator $Q = 1 - P$ yields the corresponding part projected out of the subspace. Furthermore, the projection operator is Hermitian and fullfills the two properties $P^2 = P$ and $PQ = QP = 0$. This completes the required mathematical basis of the memory-matrix formalism.

Correlation functions are the natural approach to describe the reaction of a dynamic variable on a pertubation. In quantum mechanics, the correlation function is defined in Kubo's linear response theory as an integral over an expectation value of two operators, one of them being the investigated operator, and the other one being the coupling operator from the pertubation Hamiltonian. In the Liouville space, the correlation function is defined as

$$\mathcal{C}_{ij}(t) := \left(A_i(t)\middle|A_j(0)\right) = \frac{1}{\beta}\int\limits_0^\beta d\lambda \left\langle A_i^\dagger(t)A_j(i\lambda\hbar)\right\rangle, \tag{2.4}$$

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using the definition of the scalar product (2.1) in the second step. Expressing the time evolution of $A_i(t)$ with the Liouville operator, using the Laplace transformation and a few conversions yields an algebraic matrix equation of the correlation function, which has the form

$$\sum_l \left[\omega\delta_{il} - \Omega_{il} + i\Sigma_{il}(\omega)\right]\mathcal{C}_{lj}(\omega) = \frac{i}{\beta}\chi_{ij}(0), \tag{2.5}$$

where the abbreviations

$$\Omega_{il} := i\beta\sum_k \left(\dot{A}_i\middle|C_k\right)\chi_{kl}^{-1}(0) \quad \text{and} \quad \Sigma_{il}(\omega) := i\beta\sum_k \left(\dot{A}_i\middle|Q\frac{1}{\omega - QLQ}Q\middle|\dot{C}_k\right)\chi_{kl}^{-1}(0) \tag{2.6}$$

are introduced. Both sums over l and k run over the set of operators, defined by the projection operator. Similiarly, the indices i and j have to be chosen from this set of operators so that (2.5) yields n^2 equations, if n is the number of operators in the set. Both abbreviations can be combined to a function $M_{il}(\omega) := \Sigma_{il}(\omega) + i\Omega_{il}$, called the memory function. In this function, $\Sigma_{il}(\omega)$ takes the role of the quantum mechanical self energy, while Ω_{il} represents dissipative effects. Ω_{il} vanishes under to condictions. The model Hamiltonian has to be invariant under time reversal symmetry. Furthermore, both operators, A_i and C_k, have to be the same sign with respect to time reversal symmetry. This assertion is proven in great detail in 5.3.1. In folloes that the memory function is solely determined by $\Sigma_{il}(\omega)$.

The structur of $\Sigma_{il}(\omega)$ resembles the one of the Laplace transformed correlation function, comparing equation (??), with two differences. The expectation value is performed with respect to the operators like $Q|\dot{A}_i)$ instead of $|A_i)$, while only the reduced Liouville operator QLQ is considered. The latter projects at the part of the full Liouville operator, which causes the intrinsic fluctuations of the operator A. In other words, the operator QLQ describes the internal dynamics of all other degrees of freedom of the system, called the *bath*, excluded A. The coupling to the bath is characterizied by the vector $Q|\dot{A}_i)$ and is clearly changing the dynamic behaviour of A.

2.2 Electrical Conductivity in a Momentum Conserving System

In the following section, the electrical conductivity is computed for a system, where momentum is conserved, using the memory-matrix-formalism. An infinite conductivity is expected due to the fact that electrons do not transfer any momentum to other degrees of freedom. Besides conserving momentum, the observed system has to satisfy some additional requirements. Firstly, the current has to be an unconserved quantitiy. Furthermore, the scalar product (J|P) of current and momentum has to be non-zero. In other words, both quantities possess a finite overlap. These two quantities should also generate the subspace into the projection operator projects. The sum in (2.3)

summerizes over all operators C_i, where the choice has now to be done with respect to the electrical conductivity. As electrical conductivity is determined by momentum and current, these are chosen as operators. The last requirement is for the Hamiltonian to be invariant under time reversal symmetry, which is our last requirement.

Nevertheless, one remark has to be made for computing the conductivity. Due to the fact that the Hamiltonian is invariant under time reversal symmetry and the subspace is generated by J and P, which have same signature under time reversal symmetry, no dissipative effects exist. The quantity Ω, defined in (2.6), is therefore zero.

In the strictly general sense, the electrical conductivity is given by the current-current correlation function $\mathcal{C}_{JJ}(\omega)$ (see [Czy17]). The static case is obtained by taking the small frequency limit, $\omega \to 0$, of the conductivity.

$$\sigma_{\mathrm{dc}} = \lim_{\omega \to 0} \sigma(\omega) = \lim_{\omega \to 0} \beta\, \mathcal{C}_{JJ}(\omega) \tag{2.7}$$

The current-current correlation function is given by equation (2.5), which is in the case of the J-P subspace a 2×2 matrix equation. The quantity Σ is simplified by two aspects. Firstly, the momentum is conserved, which means the time derivative of P is zero and therefore all expectation values are trivially zero where $\dot{\mathrm{P}}$ is a taken vector. The remaining expectation values of Σ contains only the current J, but these ones are vanishing as well. The operator QLQ describes the intrinsic fluctuation of $|J)$, which are assumed to be zero. Furthermore, the operator Q acting on $|J)$ yields the coupling to these intrinsic fluctuation and, leading to $Q|J) = 0$. The obtained matrix equation has a very simple form and the current-current correlation function can be directly read out as

$$\mathcal{C}_{JJ}(z) = \frac{i}{\omega}\beta^{-1}\chi_{JJ}(\omega = 0) = \frac{i}{\omega}\mathcal{C}_{JJ}(t = 0), \tag{2.8}$$

where relation (5.51) is being used. The correlation function at time $t = 0$ is given by the scalar product $(J(0)|J(0))$ (see (2.4)). Due to the fact that current and momentum are two closely connected quantities, this is also valid for their correlation functions For this reason, the current-current correlation function is expresed with respect to the momentum-momentum correlation function.

The vector operator $|J(0))$ is seperated into two pieces, one parallel and one perpendicular part, which are labeled as $|J_{||})$ and $|J_{\perp})$, respectivily. As discussed above, every variable generally consists of a secular and non-secular part, which does not mean that both parts need to exist in the observing model. The non-secular part is represented by the perpendicular component of $|J)$. In the experiement, these processes are obseved as a constant slowly fluctuating background, known as noise. The secular part is represented by the parallel component of $|J)$, which is responsible for the electrical condictivity and is visable as a peak in the measurement, depicted in figure 2.1.

Drude's theory of conductivity yields a direct proportionality between J and P (see for example [GM14]). Since momentum is conserved and current is unconserved, in the investigated system this proportionality can't be valid. Nevertheless, both quantities possess a finite overlap, which means that a part of the current is conserved and
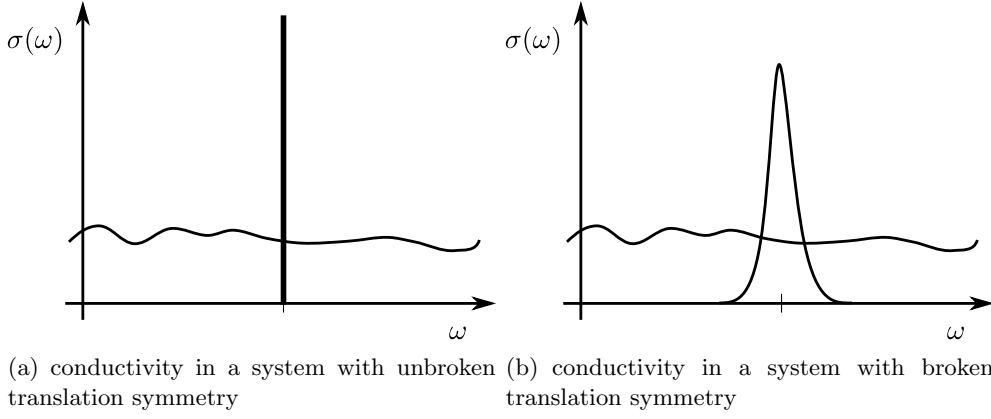
(a) conductivity in a system with unbroken translation symmetry

(b) conductivity in a system with broken translation symmetry

Figure 2.1: caption

therefore parallel to the momentum. This part is the parallel component of J and given by the projection von $|J)$ onto $|P)$.

$$|J_{||}) = \mathcal{P}|J) = \frac{|P)(P|}{(P|P)}|J) = \frac{\chi_{PJ}}{\chi_{PP}}|P) \tag{2.9}$$

Seperating both current operator in $(J(0)|J(0))$ yields four new correlation functions, where the mixed ones are zero, since $|J_{||})$ and $|J_{\perp})$ are perpendicular to each other by construction. For the operators in the parallel correlation function, $(J_{||}(0)|J_{||}(0))$ equation (2.9) is used. In this equation, the correlation function of the parallel components is represented as a momentum-momentum correlation function. Inserting the obtained expression for the current-current correlation function in (2.8) the conductivity is given by

$$\sigma(\omega) = \frac{|\chi_{PJ}|^2}{|\chi_{PP}|}\frac{i}{\omega} + \sigma_{\text{reg}}(\omega), \tag{2.10}$$

where the momentum-momentum correlation function is expressed as static suscep-tibility (see (5.51)), and the regular conductivity $\sigma_{\text{reg}}(\omega) = i(\omega\beta)^{-1}(J_{\perp}|J_{\perp})$ is intro-duced, which representes the noise (see figure 2.1).
In the above equation the quantity ω is a complex number. Since ω has to be a real number for a physical interpetation, it is transformed on the real axis using analytical continuation. The conductivity is then given by

$$\sigma(\omega) = \frac{|\chi_{PJ}|^2}{|\chi_{PP}|}\left(\text{P.V.}\frac{i}{\omega} + \pi\delta(\omega)\right) + \sigma_{\text{reg}}(\omega) \tag{2.11}$$

where $\frac{1}{\omega+i\eta} = \text{P.V.}\frac{1}{\omega} - i\pi\delta(\omega)$ is used and P.V. symbolizies the prinzipel value. Taking now the small frequency limit $\omega \to 0$ the δ-distribution is the dominant term and the static conductivity becomes infinite, which is the exactly expected result. The static

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electrical conductivity is infinite in a momentum conserving system, thus conductivity is getting finite if the momentum is not a conserved quantity any more. The respective underlying symmetry is translation symmetry. For breaking this symmerty, many possibilities are available, but in the following computation only one of them, namely umklapp scattering, is considered.

In the next chapter, the observed spin-fermion model is introduced. Beforehand, quantum phase transitions are reviewed. The conservation of momentum and non-conversation of current is further constituted. Finally, umklapp scattering is introduced for breaking translation symmetry and unconserving momentum.

3 Spin-Fermion-Model

In the following chapter, the spin-fermion-model is introduced for a metal exhibiting a antiferromagnetic quantum phase transition, as presented in [ACS03]. Bosonic spin fluctuations arise in the vicinity of the quantum critical point due to fermionic particle-hole-excitations, enabling an attractive interaction between electrons. This chapter is not displayed a detailed mathematical and microscopic derivation of the spin-fermion-model, but rather it is based on a qualitative description to justified its form. A short overview over quantum phase transitions are established as suggested in [Sac11]. In particular, we focus one's attention on the arising spin fluctuations, agrue them carry large momentum $\mathbf{Q}$ and introduce the damped spin density propagator and their perodicity. Further, we present the basic concepts of hot-spot theory, emerging as points on the Fermi surface in 2D since the magnetic Brillouin zone is cutting it. Besides, the conservation of momentum and non-conservation of current is reviewed for the observed model Hamiltonian. Umklapp scattering is introduced for breaking translation symmetry and the consequently non-conservation of momentum is constituted.

3.1 The Spin-Fermion-Model at the Onset of the Antiferromagnetic Quantum Phase Transition

An antiferromagnetic phase transition is exhibited in many metals at a characteristic temperature T_N, called Néel-temperature. The random ordered spins of lattice atoms are ordered along one axis in consequence of thermal fluctuations, where the nearest neighbors are always aligned in opposite direction. This temperature is changed by tuning a certain parameter as pressure or doping. A schematic and simplified phase diagram is depicted in figure 3.1. A certain value g_c is reached for the tuning parameter g at $T = 0$, decreasing temperature and following the phase line between both magnetic phases. This point is called quantum critical point and quantum fluctuations are the origin of this phase transition in comparsion with phase transitions at finite temperature.

A short and rudimentary describtion of quantum phase transition is given, before starting with a qualitative derivation of the spin-fermion-model. This overview is required for the analytical discussion of our computation later. However, the reason for phase transitions is always level crossing between the ground and an excited state. A band gap Δ is arised, due to the fact that level crossing is forbidden. This band gap is therefore a characteristic energy scale for the quantum phase transition. The
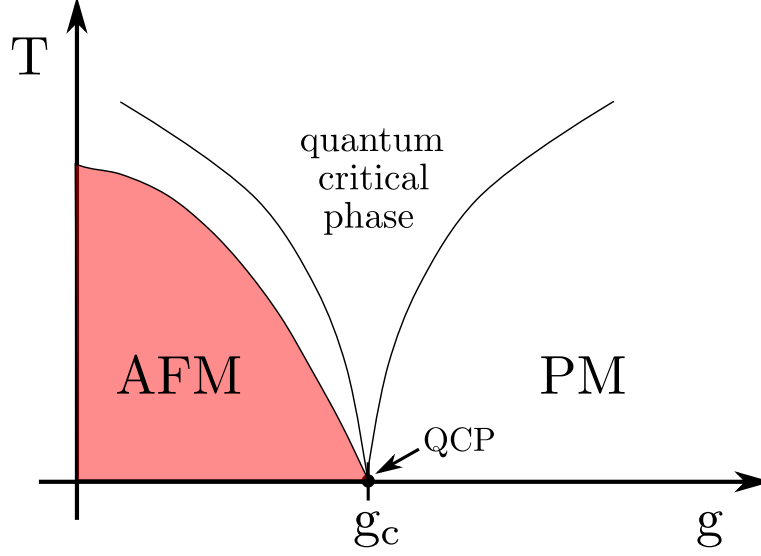
Figure 3.1: A schematic and simplified phase diagram is shown for metals transition from a paramagnetic (PM) into an antiferrmagnetic (AFM) phase depending on a tunung parameter g. The phase line of the AFM phase ends in a quantum critical point (QCP) at $g = g_c$, decreasing temperature down to $T = 0$. At this point, the phase transition is only caused by quantum fluctuations. At $T > 0$, thermal fluctuactions more and more dominates the phase transition. Nevertheless, the physical behaviour is influenced due to quantum fluctuations in a lagre regime, labeled as quantum critical phase.

characteristic energy scale Δ is proportional to the tuning parameter as

$$\Delta \sim J|g - g_c|^{z\nu}, \tag{3.1}$$

considering only phase transitions of second order. Here, $z\nu$ is a critical exponent and J an energy scale of a microscopic coupling [Sac11]. A characteristic length scale ξ is also required for quantum phase transitions. This length scale is called correlation length, which diverges right at the quantum critical point.

$$\xi^{-1} \sim \Lambda|g - g_c|^{\nu}, \tag{3.2}$$

where ν is again a critical exponent and Λ an arbitrary inverse length scale as a momentum cut-off. A directly relation between both characteristic quantities is yielded, inserting (3.2) in (3.1). For finite temperatures, $T > 0$, a second energy scale is given by $k_B T$, where k_B is the Botzmann constant. A proportionality between temperatur and correlation length is yielded, comparing both energy scales

$$T \sim \xi^{-z}. \tag{3.3}$$

The curved conical boundary phase lines of the quantum critical phase is explained with this relation, as depicted in figure 3.1. In the case of small temperatures, the critical exponent z attains the value $z = 2$, so that the phase lines are shaped like a square root. Increasing temperature the phase boundary lines are linear, since the critical exponent is changed to $z = 1$ [PS14]. Inside this regime the physical behaviour of the metals is determined by quantum fluctions.

The spin-fermion-model is now introduced, knowing the physical origin of quantum phase transitions. Instead of an microscopic and detailed mathematical describtion, our focus is based on an introduction with qualitative arguments motivating the model. Metals in the vicinity of a magnetic quantum critical point are described with the spin-fermion-model, considering dermionic quasi-particles and bosonic spin density waves. Thereby, the fermionic propagator is determined by the usual free fermionic Green function. Spin fluctuations are constituted as collective modes and their propagator is characterized by the dynamical magnetic susceptibility.

$$\mathcal{D}_\mu(\mathbf{q}, \omega) = \sum_{\mathbf{Q}} \frac{1}{(\mathbf{q} + \mathbf{Q})^2 + \xi^{-2} - (\omega/v_S)^2}, \tag{3.4}$$

where μ is the spatial direction of the spin density wave, ξ is the magnetic correlation length and v_S is the spin wave velocity. The latter is of the same order as the Fermi velocity since spin fluctuations are originated due to fermions in the vicinity of the Fermi surface. Furthermore, a peak at the momentum vector $\mathbf{Q} = (\pi, \pi)$ is exhibit in the magnetic susceptibility. A strong coupling interaction is therefore implicated between momentum vectors $\mathbf{k}$ and $\mathbf{k} + \mathbf{Q}$.

Phase transitions are always associated by an order parameter, equally the investigated antiferromagnetic phase transition. In this case, the local magnetization is the corresponding order parameter measured by the spin expectation value $\langle \mathbf{S}(\mathbf{r}_i) \rangle$. Naturally, it is summerarized over all spins located at a certain position $\mathbf{r}_i$. The order parameter is finite in the antiferromagnetic ordered phase and vanishes in the paramagnetic disordered phase. Furthermore, the expectation value of the spin operator is spatially modulated according to $\langle S_\mu \rangle \sim \exp(i\mathbf{Q}\mathbf{R})$, where $\mathbf{R}$ is some lattice vector [Wei15]. The order parameter and equally the propagator are therefore periodic quantities. In reciprocal space this is reflected in the periodicity of the magnetic Brillouin zone spanned by the vector $\mathbf{Q}$.

Our describtion of the antiferromagnetic quantum phase transition in the spin-fermion-model is based on a few fundamental assumptions, comparatively to [ACS03]. It is assumed, spin fluctuations are generated over a large range of the tuning parameter g and other low-energy collective degrees of freedom, independent of spin excitations, are neglected. Starting by large values of the tuning parameter g in the paramagnetic phase, the physical behaviour is described by Landau's Fermi liquid theory. Decreasing the tuning parameter and getting closer to the quantum critical point the behaviour is changed to a Non-Fermi liquid behaviour. Only one type of fermions is assumed and the arising collective modes are originated due to permanent interaction between particles and holes. The physics in the vicinity of the quantum critical point is determined by spin excitations, turning into smooth modes. One dominant channel is only

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assumed containing fermion-fermion interaction with energies smaller than a certain energy cut-off Λ.

In the vicinity of the quantum critical point the (magnetic) correlation length ξ and the coupling constant λ is divergent Therefore, the correlation length is much larger than the lattice constant, $\xi \gg a$, and the coupling constant is much larger than the band gap, $\lambda gg \Delta$. The band gap is associated with the energy of spin excitations. In the limit of large distance and small energies or temperatures a mircoscopic consideration of the lattice Hamiltonian is unnecessary. In this low-energy theory, spin fluctuations are described as a three component bosonic field $\Phi_\mu(\mathbf{x}, \tau)$. The field is defined as a sum over all spin oparators analysed in the neighbourhood of the spin's position i and is formally writen as $\Phi_\mu(\mathbf{x}, \tau) \sim \sum_{i \in \Gamma(\mathbf{r}_i)} S_\mu(\mathbf{r}_i)$, where μ is the spatial direction of the field and $\Gamma(\mathbf{r}_i)$ represented the neighbourhood around the spin position $\mathbf{r}_i$ [Sac11]. The magnitude of the bosnic field is chosen arbitrary and the field itself is considered as a real field. Tn the low-energy theory, the obtained effective Hamiltonian for spin fluctuations is then given by

$$H_\Phi = \sum_\mu \int_{\mathbf{k}} \left[-\frac{\mathbf{k}^2}{2} - \frac{r_0}{2} \right] \Phi_\mu(\mathbf{k}, \tau) \Phi_\mu(-\mathbf{k}, \tau) + \frac{v_S^2}{2} \pi_\mu(\mathbf{k}, \tau) \pi_\mu(-\mathbf{k}, \tau), \qquad (3.5)$$

where r is a control parameter and corresponds to the squared inverse correlation length, $r = \xi^{-2}$. The integral is extended over the first Brillouin zone and the sum runs over the spatial direction of the bosonic field.

It has to take into account that the spin densiy waves are damped in consequence of the interaction with fermions. The spin itself does not possess an own damping source. The whole damping is governed decaying of particel-holes-pairs and the full renormalized particel-hole bubble corresponds to the inverse lifetime of spin fluctuations. The interation Hamiltonian between spin density waves and fermions is given by

$$H_{\Psi\Phi} = -\lambda \sum_\mu \int_{\mathbf{k}} \int_{\mathbf{q}} \Phi_\mu(\mathbf{k} - \mathbf{q}, \tau) \left[\Psi_a^\dagger(\mathbf{k}, \tau) \cdot \sigma_\mu \cdot \Psi_b(\mathbf{q}, \tau) + \Psi_b^\dagger(\mathbf{k}, \tau) \cdot \sigma_\mu \cdot \Psi_a(\mathbf{q}, \tau) \right],$$

$$(3.6)$$

where λ is the coupling constant. Both integrals are extended over the first Brillouin zone and the sum runs over all spatial directions of the bosonic field. σ_μ is the Pauli matrix with respect to the spatial direction The two different Fermi surfaces in the obtained model are represented by the indicies a and b at the fermionic fields Ψ. An detailed explanation is presently given below. The imaginary part of the susceptibility has to be computed in the low-energy limit with tools of the quantum field pertubation theory. An explicit calculation is presented in the appendix ??. Furthermore, the dynamic of the spin fluctuations is assumed to be governed by the low-energy fermions. The ω^2-term in the susceptibility (3.4) is neglectabled [ACS03] and at the vicinity of the quantum critical point the correlation length diverges, implying $\xi^{-2} \to 0$. The damped magnetic susceptibility is then given by

$$\mathcal{D}_\mu(\mathbf{q}, i\omega_n) = \sum_{\mathbf{Q}} \frac{1}{(\mathbf{q} + \mathbf{Q})^2 + \gamma |\omega_n|}, \qquad (3.7)$$
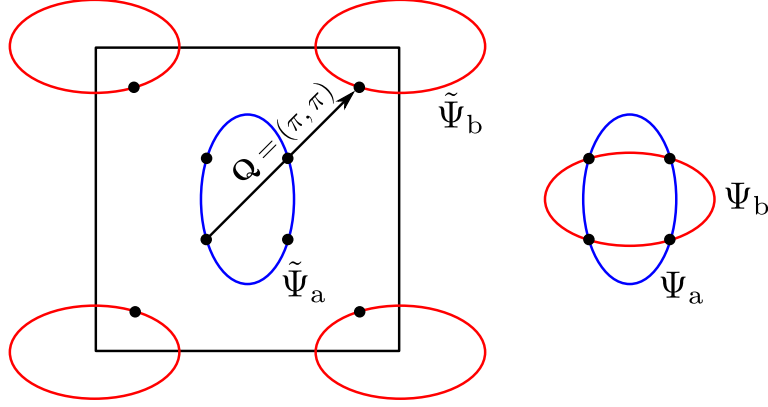
Figure 3.2: One the left hand side the Brillouin zone is depicted containing the Fermi surface of both species of fermions, $\tilde{\Psi}_a$ and $\tilde{\Psi}_b$. While the fermions $\tilde{\Psi}_a$ are located with an anisotropic parapolical dispersion at the origin $(0,0)$, the fermions $\tilde{\Psi}_b$ are located with a $\pi/2$-rotated anisotropic parapolical dispersion at the corners of the Brillouin zone, (π,π) for example. The vector $\mathbf{Q}$ connects certain points on the Fermi surface, called hotspots and represented with dots. As a result of this an attraction between these fermions occurs. On the right hand side the fermions $\tilde{\Psi}_a$ are shifted to the corner at (π,π). The convienence of this represantation is that the coupling between fermions and spin fluctuations is local and independent of the position.

where ω_n represent the bosonic Matsubara frequency. Spin fluctuations are no independent degrees of freedom due to the fact that their damping is strongly conected to fermions. The interaction between fermions on the Fermi surface seperated by the large vector $\mathbf{Q}$ is constituted in the damped susceptibility. These points seperated by $\mathbf{Q}$ on the Fermi surface are called hot-spots in two dimensions and their theory is discussed next. In comparison to a normal metals, the new interaction of spin fluctuations are seperated the Fermi surface in *hot* and *cold* manifolds. A point $\mathbf{k}$ is considered on the Fermi surface of a d-dimensional system and there, the fermion is assumed to has zero energy. This point is connected to another point $\mathbf{k}+\mathbf{Q}$, via the spin density wave, and is required to be also on the Fermi surface. A $d-2$-dimension manifold is yielded for a d-dimensional system due to both constraints. In the case of $d=2$ the *hot* manifold is therefore called a hot-spot. Another visualization of *hot* manifolds is offered by the magnetic Brillouin zone spanned by the vector $\mathbf{Q}$. The intersections of the magnetic Brilouin zone and the Fermi surface are equivalent to the *hot* manifolds.

In our investigated model the fermions are considered as free, up to the interaction with the spin fluctuations. Two species of fermions are assumed, $\tilde{\Psi}_a$ and $\tilde{\Psi}_b$. Their dispersion is anisotropic and parabolic, as suggested in [PS14]. The anisotropic parabolic Fermi surfaced are depicted on the left hand side in figure 3.2. This is not a discrepancy to our previous constraint, assuming one fermion, since both species of fermions

are electrons and differ only in their dispersion. The fermions $\tilde{\Psi}_a$ and $\tilde{\Psi}_b$ are located on an ellipse around the origin $(0,0)$ and (π, π), respectivily. Both ellipses are seperated by the vector $\mathbf{Q} = (\pi, \pi)$ and rotated by $\pi/2$ to each other. For achieving a continuum theory the ellipse centered around (π, π) is shifted by the vector $\mathbf{Q}$ to the origin. Therefore new fermionic fields, $\tilde{\Psi}_a(\mathbf{r}) = \Psi_a(\mathbf{r})$ and $\tilde{\Psi}_b(\mathbf{r}) = \Psi_b(\mathbf{r})\exp(i\mathbf{Q}\mathbf{r})$, are introduced. The new Fermi surfaces of the fermions $\Psi_a(\mathbf{r})$ and $\Psi_b(\mathbf{r})$ are depicted on the right hand side in figure 3.2 The enormous advantage is that the coupling between the fermions, caused by spin density waves, is now local and indipendent of the position. In reciprocal space, the free continuum Hamiltonian of the fermions $\Psi_a(\mathbf{r})$ and $\Psi_b(\mathbf{r})$ is given by

$$H_\Psi = \int_{\mathbf{k}} \left[\epsilon_a(\mathbf{k})\psi_a^\dagger(\mathbf{k}, \tau)\psi_a(\mathbf{k}, \tau) + \epsilon_b(\mathbf{k})\psi_b^\dagger(\mathbf{k}, \tau)\psi_b(\mathbf{k}, \tau) \right], \qquad (3.8)$$

where the intergal is extended over the first Brillouin zone. $\epsilon_a(\mathbf{k})$ and $\epsilon_b(\mathbf{k})$ are the anisotropic parabolical dispersions, corresponding to the fermion species a and b, and are given by

$$\epsilon_a(\mathbf{k}) = \frac{k_x^2}{2m_1} + \frac{k_y^2}{2m_2} - \mu_0 \quad \text{and} \quad \epsilon_a(\mathbf{k}) = \frac{p_x^2}{2m_2} + \frac{p_y^2}{2m_1} - \mu_0, \qquad (3.9)$$

where μ_0 is the chemical potential.

3.2 Momentum Conservation in the Spin-Fermion-Model and the Consequence of Umklapp Scattering

In the low-energy theory, our investigated 2D-model is described by the continuum Hamiltonian H_Ψ for free fermions on the Fermi surface, the effective Hamiltonian H_Φ for spin fluctuations and by the interaction Hamiltonian $H_{\Psi\Phi}$ between both. Directly at the quantum critical point the control parameter r in (3.5) is zero, due to $r = \xi^{-2}$ and the correlation length ξ is divergent. r is thus set to zero in the Hamiltonian H_Φ. The obtained model Hamiltonian is given by $H = H_\Psi + H_\Phi + H_{\Psi\Phi}$. Our further approach is to prove the conservation of momentum and current for this model Hamiltonian. After that, a pertubation Hamiltonian is introduced for considering umklapp scattering since translation symmetry is breaked by this pertubation and momentum is not a conserved quantity any more.

A physical quantity is conserved, if its time derivative vanishes. In quantum mechanics the time derivative is given by Heisenberg equation of motion, $\dot{A}(t) = i[H, A(t)]_-$, where $A(t)$ is an arbitrary operator and its partial time derivative is assumed to be zero. The momentum operator is given by

$$P_j = \int_{\mathbf{k}} k_j \left[\Psi_a^\dagger(\mathbf{k}, \tau)\Psi_a(\mathbf{k}, \tau) + \Psi_b^\dagger(\mathbf{k}, \tau)\Psi_b(\mathbf{k}, \tau) - \pi_\mu(\mathbf{k}, \tau)\Phi_\mu(-\mathbf{k}, \tau) \right], \qquad (3.10)$$

where the spatial direction is indicated by j. The integral is extended over the first Brillouin zone and the sum over μ is implied in the last term. The energy-momentum-tensor, as suggested in [Ili02], is used for the computation, shown in appendix **??**. Further the x-component of current operator is given by

$$
\mathrm{J_x} = -\int_{\mathbf{k}} \left[\frac{k_{\mathrm{x}}}{m_1} \Psi_{\mathrm{a}}^{\dagger}(\mathbf{k}, \tau) \Psi_{\mathrm{a}}(\mathbf{k}, \tau) + \frac{k_{\mathrm{x}}}{m_2} \Psi_{\mathrm{b}}^{\dagger}(\mathbf{k}, \tau) \Psi_{\mathrm{b}}(\mathbf{k}, \tau) \right],
\tag{3.11}
$$

where the integral is also extended over the first Brillouin zone. The y-component of the current operator is yielded changing the index x to y and interchanging the index a and b of the fermionic field operators. Now, the time derivative of both quantities is computed by using the basic bosonic and fermionic commutator relations. An explicite calculation is represented in the appendix **??**. This yields a vanishing time derivative for the the momentum,

$$
\dot{\mathrm{P}}_j = 0,
\tag{3.12}
$$

and the following expression for the time derivative of the x-component of the current

$$
\dot{\mathrm{J}}_x = \lambda \sum_{\mu} \int_{\mathbf{k}} \int_{\mathbf{q}} \Phi_{\mu}(\mathbf{k} - \mathbf{q}, \tau) \left[\left(\frac{q_{\mathrm{x}}}{m_1} - \frac{k_{\mathrm{x}}}{m_2} \right) \Psi_{\mathrm{b}}^{\dagger}(\mathbf{k}, \tau) \cdot \sigma_{\mu} \cdot \Psi_{\mathrm{a}}(\mathbf{q}, \tau) \right.
$$
$$
\left. + \left(\frac{q_{\mathrm{x}}}{m_2} - \frac{k_{\mathrm{x}}}{m_1} \right) \Psi_{\mathrm{a}}^{\dagger}(\mathbf{k}, \tau) \cdot \sigma_{\mu} \cdot \Psi_{\mathrm{b}}(\mathbf{q}, \tau) \right].
\tag{3.13}
$$

The y-component of the current is obtained by changing again the index x to y and interchanging the index a and b of the fermionic field operators. Therefore, momentum is a conserved quantity and the current is an unconserved quantity in the spin-fermion-model, described by the Hamiltonian H. The Hamiltonian is further invariant with respect to time reversal symmetry and a finite overlap is possessed between both quantities. Infinity is obtained for the static electrical conductivity in comparsion to the discussion of the previous chapter. A finite conductivity is only obtained breaking translation symmetry. This is equivalent to an unconserved momentum. Therefore, the model Hamiltonian H is pertubated by umklapp scattering via the Hamiltonian

$$
\mathrm{H_{umklapp}} = \sum_{\mu, \mathbf{G}} \mathrm{J_{\mathbf{G}}} \int_{\mathbf{k}} \Phi_{\mu}(\mathbf{k}, \tau) \Phi_{\mu}(-\mathbf{k} + \mathbf{G}, \tau),
\tag{3.14}
$$

where the integral extends over the first Brillouin zone and the sum over $\mathbf{G}$ included all reciprocal lattice vectors. The quantity $\mathrm{J_{\mathbf{G}}}$ is a coupling constant depending on reciprocal lattice vectors. It is assumed that the coupling is decreasing fast to zero in the limit of large reciprocal lattice vectors ($|\mathbf{G}| \to \infty$). The time derivative of the momentum is given by

$$
\dot{\mathrm{P}}_j(\tau) = i \sum_{\mathbf{G}} \mathrm{J_{\mathbf{G}}} \int_{\mathbf{k}} G_j \Phi_{\mu}(\mathbf{k}, \tau) \Phi_{\mu}(-\mathbf{k} - \mathbf{G}, \tau)
\tag{3.15}
$$

with respect to the new model Hamiltonian $H' = H + H_{umklapp}$. The time derivative of the current is persisted, since the current and pertubation Hamiltonian depends on different field operators and the commutator is therefore trivially zero. The static electrical conductivity is become finite due to the unconserved momentum. In the next chapter, the static electrical conductivity is computed using diagrammatic pertubation theory and the results are analysed.

4 The Static Conductivity in the Spin Fermion Model including Umklapp Scattering

The static electrical conductivity σ_{dc} is computed in this chapter, using the memory-matrix-formalsim. The underlying model is the spin-fermion-model (see chapter 3), including umklapp scattering as a translation symmetry breaking pertubation. In this chapter, our attention is focused on the physical analysis of the obtained expression for the conductivity. An explicit computation is demonstrated in appendix **??** Two physical objects, fermions near the Fermi surface and spin fluctuations, are considered in the spin-fermion-model, introduced in the previous chapter. Spin fluctuations are generated at the vicinity of an antiferromagnetic quantum critical point due to particle-holes-excitations. Fermions, located at the point $\mathbf{k}$ and $\mathbf{k} + \mathbf{Q}$ on the Fermi suface, are connected to each other because of these fluctuations. In the previous chapter, it is shown that the model Hamiltonian of the spin-fermion-model is conserved the momentum. The static electrical conductivity is thus infinite as it is poven in chapter 2. Considering umklapp scattering, the translation symmetry is broken and the momentum is unconserved. As a conseuqence the static conductivity becomes finite, which is computed in the following.

In chapter 5 over the memory-matrix-formalism, the following formula is derivated for the static electrical conductivity, considering the spin-fermion-model and umklapp scattering.

$$\sigma_{\mathrm{dc}} = -\lim_{z \to 0} \frac{z \cdot |\chi_{\mathrm{JP}}(\omega = 0)|^2}{\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z)} \tag{4.1}$$

Here, z is a complex frequency and $\chi_{\mathrm{JP}}(\omega = 0)$ is the static susceptibility between current and momentum. In the denominator the Green function is given by

$$\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z) = \int\limits_{0}^{\infty} \mathrm{d}t \, e^{izt} \left\langle \left[\dot{\mathrm{P}}(t), \dot{\mathrm{P}}(0) \right]_{-} \right\rangle_0 , \tag{4.2}$$

where the usual quantum mechanical commutator is used, indicated with the minus sign at the squared brackets. The expectation value is generated with respect to the unpertubative Hamiltonian H, illustrated with the index 0. Here, the interaction between fermions and spin fluctuations is included. The time derivative of momentum is given by equation (3.15).
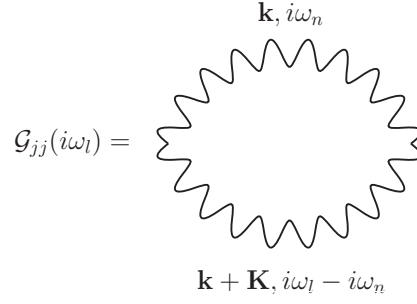
$$\mathcal{G}_{jj}(i\omega_l) =$$

with labels $\mathbf{k}, i\omega_n$ (top) and $\mathbf{k}+\mathbf{K}, i\omega_l - i\omega_n$ (bottom)

Figure 4.1: caption

> Why? -¿ P and J are explicitly time independent

The Green function $\mathcal{G}_{\dot{P}\dot{P}}(\mathbf{k}, z)$ and the static susceptibiliy $\chi_{JP}(\omega = 0)$ are investigated, due to their possible temperature dependence. The latter is expected to be temperature independent. This is computed in detail in the appendix ??, using the diagrammatic pertubation technique. The considered diagrams are the electron-pair bubble for each species of fermions, depicted in figure ??. Therefore, the obtained expression of the susceptibility is given by

$$\chi_{PJ}(\omega = 0) = \frac{\sqrt{m_1 m_2}}{\pi} T \cdot \ln\left(e^{\mu/T} + 1\right), \tag{4.3}$$

where the chemical potential is denoted as μ. The logarithm is approximated in the limit of small temperatures ($\mu \gg T$) to $\ln\left(e^{\mu/T} + 1\right) \to \mu/T$. Inserting this expression, the temperature is canceled and the static susceptibility is given by constant parameters, $\chi_{PJ}(\omega = 0) = \mu\sqrt{m_1 m_2}/\pi$. The chemical potential is assumed as temperature independent in the observed regime.

The only temperature dependence of the static electrical conductivity is hence emerged by the Green function $\mathcal{G}_{\dot{P}\dot{P}}(\mathbf{k}, z)$. Only the free bubble diagram is considered computing the Green function with diagrammatic pertubation theory, which is depicted in figur 4.1. Higher order terms in the diagrammatic treatment are corrections of the bubble diagram and since spin fluctuations are the observing objects only bubble diagrams occur. Therefore the conductivity or resistance is only determined by spin fluctuations. Resistance is a consequence of fermionic coupling to other degrees of freedom, obtained as finite fermionic lifetime. The resistance is thus proportional to the imaginary part of the retarded Green function $\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)$. Appendix ?? is shown the detailed computation of the expression of the retarded Green function.

$$\text{Im}\left\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\right\} = -\frac{4\gamma^2 \beta \omega}{\pi} \sum_{\mathbf{Q}_1, \mathbf{Q}_2} \sum_{\mathbf{G}} |\mathbf{J}_{\mathbf{G}}|^2 \int_0^\infty d\epsilon \, \frac{\epsilon^2 e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}$$
$$\times \int_{\mathbf{k}} G_j^2 \cdot \frac{1}{(\mathbf{k} + \mathbf{G} - \mathbf{Q}_1)^4 + \gamma^2\epsilon^2} \cdot \frac{1}{(\mathbf{k} + \mathbf{Q}_2)^4 + \gamma^2\epsilon^2} \tag{4.4}$$

Here the term under the momentum integral is given by the imaginary part of the damped spin density propagator, denoted in (??). The integral itself is extended over

the first Brillouin zone. The periodicity of the spin susceptibility and pertubation is considered by the sums over $\mathbf{Q}_i$ ($i = 1, 2$) and $\mathbf{G}$, respectivily. In both cases these are reciprocal lattice vectors. Spatial direction of the momentum is represented by the index j and $\beta^{-1} = T$.

The obtained integrals is now investigated with respect to different cases since their are not exactly solvable. A detailed discussion of the single cases and an approximated computation of the integral is demonstrated in the following.

Convergence of the intergal is surely given, if one of the momentum terms, $\mathbf{k} + \mathbf{G} - \mathbf{Q}_1$ or $\mathbf{k} + \mathbf{Q}_2$, is large according to amount. Since both terms are of the power of four, the integral is a fast decreasing function in the large momentum limit. Nevertheless, divergences of the integral are reached, if these terms are zero and this happens in many cases. One of the most divergent cases is assumed in the following approach. The vector $\mathbf{k}$ is limited on the first Brillouin zone and the reciprocal lattice vectors are set to $\mathbf{G} = \mathbf{Q}_1$ and $\mathbf{Q}_2 = 0$. The choice of the reciprocal lattice vectors is arbitrary and it is therefore possible that the j-component of $\mathbf{G}$ is large. This divergence is canceled by the coupling parameter $\mathrm{J}_{\mathbf{G}}$ assumed as fast decreasing for large $|\mathbf{G}|$.

The remaining momentum integral is now exactly integrable. First of all, the momentum and frequency variables are transformed into dimensionles variables, using the transformation rules $\mathbf{k} = \mathbf{y} \cdot \sqrt{\gamma T}$ and $\epsilon = x \cdot T$, respectivily. The new variable $\mathbf{y}$ is further transformed into plane polar coordinates. The upper limit of the radius $|\mathbf{y}|$ is set to infinity, since the integrand is decreasing fast to zero for $|\mathbf{y}| \to \infty$. As a consequence the angular integral is easily evaluated, yielding the factor 2π. The remaining integral is given by the following expression.

$$\mathrm{Im}\left\{\mathcal{G}_{\mathrm{P\dot{P}}}^{\mathrm{ret}}(\mathbf{k}, z)\right\} = -\frac{2 \cdot G_j^2 \cdot |\mathrm{J}_{\mathbf{K}}|^2}{\gamma \cdot \pi^2} \cdot \frac{\omega}{T} \int\limits_0^\infty \mathrm{d}x \, \frac{x^2 e^x}{(e^x - 1)^2} \int\limits_0^\infty \mathrm{d}y \, \frac{y}{\left(y^4 + x^2\right)^2} \qquad (4.5)$$

The y-integral is substituted one last time, using the transformation rule $y^2 = z \cdot x$. The obtained integral is evaluated to $\pi/4$, using the integral formula (**??**). Both limits, $x \to 0$ and $x \to \infty$, of the intergal over x are investigated. In the case of large x, the integral is decreasing fast to zero due to the ratio of the exponential functions. The upper limit is thus set to one and the integrand is expanded for small values of x. In first order the integrand is approximated to $1/x^3$. This is a highly divergent function in the limit $x \to 0$ and as a consequence the static conductivity is divergent as well.

The result of a divergent conductivity is clearly incorrect in a system with broken translation symmetry. In chapter 2, it is shown that translation symmetry and the consequent conserved momentum is caused an infinite conductivity in the static limit. As showing in chapter 3, momentum is conserved in the unpertubated spin-fermion-model (equation (3.12)) and it is thus a system with unbroken translation symmetry. Momentum is unconserved (equation (3.15)), introducing umklapp scattering as an observing pertubation. A finite conductivity is now expected, instead of a divergent one.

Our supposed correct choice of the control parameter r is the origin of this discrepancy. The paramter r is assumed to be zero, since its proportional to ξ^{-2} and the correlation

length ξ is divergent at the quantum critial point. This assumption is incorrect in the sence that the system is not directly at the quantum cirtical point, but rather in the vicinity of him. The system is supposed to be in the quantum critical phase, depicted in the phase diagram 3.1, at a point with tuning parameter $g = g_c$ and temperature $T > 0$. While the temperature is decreased to get closer to the quantum critical point, the tuning parameter is fixed. Considering the finite value of the control parameter r, the imaginary part of the retareded Green function is given by

$$\text{Im}\Big\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\Big\} = -\frac{4\gamma^2\beta\omega}{\pi} \sum_{\mathbf{Q}_1,\mathbf{Q}_2} \sum_{\mathbf{G}} |\mathbf{J_G}|^2 \int\limits_0^\infty d\epsilon\, \frac{\epsilon^2 e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}$$
$$\times \int_{\mathbf{k}} G_j^2 \cdot \frac{1}{((\mathbf{k} + \mathbf{G} - \mathbf{Q}_1)^2 + r)^2 + \gamma^2\epsilon^2} \cdot \frac{1}{((\mathbf{k} + \mathbf{Q}_2)^2 + r)^2 + \gamma^2\epsilon^2} \tag{4.6}$$

The approach is nearly the same for solving the integrals as previously. Firstly, the convergence of large values of momenta is not changed, because of considering r. One of the most divergent cases is stil $\mathbf{G} = \mathbf{Q}_1$ and $\mathbf{Q}_2 = 0$. Furthermore, the divergence, generated by large values of G_j, is still canceled since the coupling parameter $\mathbf{J_G}$ is decreased fast in the limit $|\mathbf{G}| \to \infty$. The remaining momentum integral is of the same structure as above and exactly solvable. Therefore, the momentum integral is firstly transformed into plane polar coordinates, $\mathbf{k} = (k\cos\phi, k\sin\phi)$. The upper limit of k is again set to infinity, since the integrand is fast decreasing to zero for $k \to \infty$ and the contributions are neglectable.

$$\text{Im}\Big\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\Big\} = -\frac{|\mathbf{J_G}|^2 \cdot G_j^2}{\gamma\pi^2} \cdot \frac{\omega}{T} \int\limits_0^\infty d\epsilon\, \frac{\epsilon^2 e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2} \int\limits_0^\infty dk\, \frac{k}{((r + k^2)^2 + \gamma^2\epsilon^2)^2} \tag{4.7}$$

where the angular integral is performed. The obtained momentum integral is substituted using $z = (r + k^2)/\gamma\epsilon$. In comparison to the case above, the lower limit is changed to $r/\gamma\epsilon$. Both cases are equivalent in the limit $r \to 0$. The finite lower boundary guarantees the convergence of the remaining integral over ϵ. The integral over z is performed, utilizing the integral formula (??) again. Dimensionless variables are introduced, using $x = \beta\epsilon$. The obtained expression is given by

$$\text{Im}\Big\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\Big\} = -\frac{|\mathbf{J_G}|^2 \cdot G_j^2}{4\gamma\pi^2} \cdot \frac{\omega}{T}$$
$$\times \int\limits_0^\infty dx\, \frac{1}{x} \frac{e^x}{(e^x - 1)^2} \cdot \left[\pi - \frac{2\,\tilde{r}/x}{(\tilde{r}/x)^2 + 1} - 2\arctan(\tilde{r}/x)\right], \tag{4.8}$$

where the abbreviation $\tilde{r} = r/\gamma T$ is introduced. The integral over x is not analytical exactly solvable. However, the integral is finite in both limits, $x \to 0$ and $x \to \infty$. The latter is surely statisfied. The ratio of exponential functions is fast decreasing to zero and no other term is divergent. In the limit $x \to 0$, the integrand is limited by
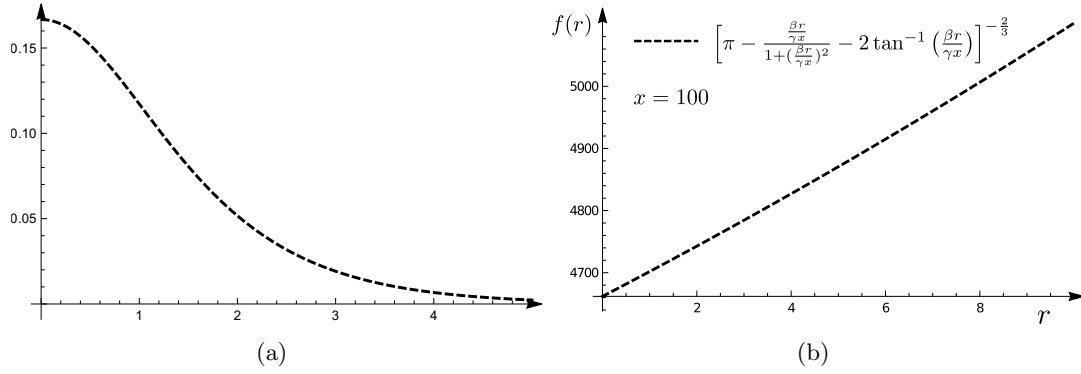
Figure 4.2: caption

$4/3r^3$, which is only divergent for $r \to 0$. The behaviour of the integrand is depicted on left hand side in figure 4.2, setting $\tilde{r} = 2$. A solution of the integral is evaluated determining the behaviour of the expression in squared brackets. For large values of x, a linear r-dependence is given by the power of $-2/3$. This is depicted in the right hand side of figure 4.2, where x is set to 100. An approximated expression is generated for the interand using the method of power counting. The initial point of this approach is the momentum integral in euqtion (4.7). Numerator and differatial yield in total the power of two, while the denominator generates the power of eight. Combining both, the integrand is a function proportional to k^{-6}. The further approach is now to investigate singularities of the integrand. The limit $r \to 0$ or $\epsilon \to 0$ is seperately generated singularities in the function. Both are considered with the power of three, since the power of them is half in comparison to the power of k in equation (4.7). The integrand is therefore approximated by the following expression.

$$\text{Im}\left\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\right\} \approx -\frac{2|\mathrm{J}_{\mathbf{G}}|^2 \cdot G_j^2}{\gamma\pi^2} \cdot \frac{\omega}{T} \int\limits_0^\infty \mathrm{d}x \, \frac{x^2 e^x}{(e^x - 1)^2} \frac{1}{(\tilde{r}^2 + x^2)^{3/2}} \tag{4.9}$$

In figure 4.3, the approximated expression is confronted with the exact one. The behaviour of both are nearly identical and the estimation is inspected as valid. The integrand is still decreasing fast to zero for $x \to \infty$, while the upper limit is set to some arbitrary cut-off Λ. The ratio of the exponential functions are then expanded for small values of x up to the first non-vanishing order. The remained integral is exactly solvable and expanding the solution for small $\tilde{r}/\Lambda$ the solution of the integral is given by

$$\text{Im}\left\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\mathbf{k}, z)\right\} \approx -\frac{2\gamma|\mathrm{J}_{\mathbf{G}}|^2 \cdot G_j^2}{\pi^2\Lambda} \cdot \frac{\omega T}{r^2} \tag{4.10}$$

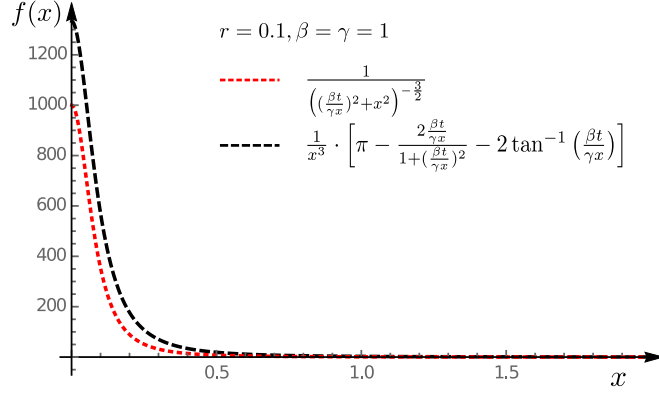Inserting this expression and the approximated static susceptibility of (4.3) into equa-

21

$$f(x)$$

$$r = 0.1, \beta = \gamma = 1$$

$$\frac{1}{\left(\left(\frac{\beta t}{\gamma x}\right)^2 + x^2\right)^{-\frac{3}{2}}}$$

$$\frac{1}{x^3} \cdot \left[\pi - \frac{2\frac{\beta t}{\gamma x}}{1 + \left(\frac{\beta t}{\gamma x}\right)^2} - 2\tan^{-1}\left(\frac{\beta t}{\gamma x}\right)\right]$$

Figure 4.3: caption

tion (4.1), the static electrical conductivity is given by

$$\sigma_{\text{dc}}(T) = \frac{\mu\, m_1\, m_2\, \Lambda}{2\, \gamma\, |\mathbf{J_G}|^2\, G_j^2} \cdot \frac{r^2}{T} \tag{4.11}$$

Beside the temperature itself, the only temperature dependent parameter is the control parameter r, which is proportional to the squard inverse correlation length, $r = \xi^{-2}$. The correlation length is originated due to the fact that the system possesses a quantum phase transition and is temperature dependent is given by $\xi = T^{-1/z}$, as shown in chapter 3. z is a critical exponent and it is chosen in dependence of the distance to the qantum critical point. The corresponding temperature depencence of the control parameter is therefore provided to $r = T^{2/z}$. The resistance is determined by the inverse of the conductivity. Considering only the temperature dependent parameter, the static resistance is given by

$$\rho_{\text{dc}}(T) \sim T^{1 - 4/z} \tag{4.12}$$

Two possible choices are mentioned for the ciritcal exponent z in chapter 3. In the vicinity of the quantum critical point at higher temperatures, z is chosen to be one. The T-dependence of the resistance is then given by T^{-3}. In the limit $T \to 0$, this is highly divergent. z is chosen to be two for lower temperatures. Nevertheless, the T-dependence is evaluated to T^{-1}, which is still divergent in the limit $T \to 0$.

Our physial expectation is not confirmed with this divergent behaviour of the resistance. Experiments of the investigated metals [Löh+07] are shown a linear temperature dependence of the resistance, $\rho(T) = \rho_0 + A\,T$. The behaviour of the resistance is described incorrect with our obtained solution (4.12). A possible origin of this disgrepancy between experiemnt and theory is maybe the diagrammatic pertubation theory. In equation (4.2), the time evolution operator is expanded up to the first non-vanishing order. This yields the free bubble diagram, while correction to the bubble are considered expanding the time evolutaion operator up to the next order.

5 Memory-Matrix-Formalism

The memory-matrix-formalism is a technique to determine the correlation function between two observables. In difference to other methods the Liouville space is considered. The derivation of the memory-matrix-formalism is exhaustively demonstrated in this chapter, as suggested in [For90]. At the beginning Kubo's relaxation function is introduced, since the correlation function is based on it in memory-matrix-formalism. After the derivation the formula for the static electrical conductivity is determined for the spin-fermion-model pertubated by umklapp scattering.

5.1 Kubo relaxation function

A system in equlibrium, represented by the Hamiltonian H_0, is considered. The pertubation $H_1 = -B \cdot F(t)$ is switched on at an arbitrary time t'. The coupling to the system is determined by the operator B and the time evolution of the pertubation is described by $F(t)$, where $F(t)$ is assumed to be zero for times $t < t'$. Our interest is now focused on the reaction of an operator A due to the pertubation. The deviation in comparing to the equilbrum value is given by

$$\delta \langle A(t) \rangle := \langle A \rangle (t) - \langle A(t) \rangle_{H_0} \approx \int_{-\infty}^{\infty} dt' \, \chi_{AB}(t - t') F(t') \tag{5.1}$$

where $\chi_{AB}(t - t')$ is called the retarded susceptibility and is given by

$$\chi_{AB}(t - t') = i\Theta(t - t') \left\langle \left[A_I(t - t'), B_I(0) \right] \right\rangle_{H_0} \tag{5.2}$$

The above equation for $\delta \langle A(t) \rangle$ is denoted as the Kubo formula. In the following, a certain type of pertubations is considered. The time evolution of the pertubation is assumed to be $F(t) = \Theta(-t) \cdot F \cdot e^{-s\tau}$. The pertubation is switched on adiabatically at $t = -\infty$ and is switched off at $t = 0$. This time evolution is inserted into the Kubo formula and $t - t'$ is substituted by τ. The deviation of A(t) is then given by $\delta \langle A(t) \rangle = \Phi_{AB}(t) \cdot Fe^{st}$. The arising function $\Phi_{AB}(t)$ is called Kubo relaxation function and is given by

$$\Phi_{AB}(t) = \frac{i}{\hbar} \lim_{s \to 0} \int_{t}^{\infty} d\tau \, \langle [A_I(\tau), B_I(0)] \rangle_0 \, e^{-s\tau}. \tag{5.3}$$

The lower limit of the integral is determined to t due to the θ-distribution. For a more detailed derivation of the Kubo relaxation function see [Sch08] or [Sch06]. Between

the Kubo relaxation function and the Kubo formula exist three very important relations used during the derivation of the correlation function and the formula of the conductivity.

1. $\chi_{AB}(t) = -\Theta(t)\dfrac{d}{dt}\Phi_{AB}(t)$ (5.4)

2. $\Phi_{AB}(t=0) = \chi_{AB}(\omega=0)$ (5.5)

3. $\Phi_{AB}(\omega) = \dfrac{1}{i\omega}\big[\chi_{AB}(\omega) - \chi_{AB}(\omega=0)\big].$ (5.6)

The evidence of these tree relations are shown in the appendix F. The Kubo relaxation function is now transfered into a commutator independent form, using two identies. The first identity is given by

$$\big\langle\, [A(t), B(t')]\big\rangle = \frac{1}{Z}\,\mathrm{Tr}\big\{[\rho, A(t)]B(t')\big\}, \tag{5.7}$$

where the invariance of the trace with respect to cycling permutation is used. The second identity is denoted as the Kubo-identity.

$$i[\rho, A(t)] = i\Big[\rho A(t) - e^{-\beta H}e^{\beta H}A(t)e^{-\beta H}\Big]$$

$$\Leftrightarrow\ i[\rho, A(t)] = -i\rho\int_0^\beta d\lambda\,\frac{d}{d\lambda}e^{\lambda H}A(t)e^{-\lambda H}$$

$$\Leftrightarrow\ i[\rho, A(t)] = -i\rho\int_0^\beta d\lambda\,[H, A(t-i\lambda)]$$

$$\Leftrightarrow\ i[\rho, A(t)] = -\rho\int_0^\beta d\lambda\,\dot{A}(t-i\lambda) \tag{5.8}$$

Here the analogy between the time evolution of an operator and the exponential function is used in the second step. Furthermore, Heisenberg equation of motion is utilied in the third step. The time derivative of the operator A is symbolizied with the dot above the operator.

$$\Phi_{AB}(t) = -\lim_{s\to0}\int_0^\beta d\lambda\int_t^\infty d\tau\,\Big\langle\dot{A}_I(\tau-i\lambda\hbar)B_I(0)\Big\rangle_0 e^{-s\tau}$$

$$\overset{\mathrm{PI}}{\Leftrightarrow}\ \Phi_{AB}(t) = \int_0^\beta d\lambda\,\langle A_I(t-i\lambda\hbar)B_I(0)\rangle_0 = \int_0^\beta d\lambda\,\langle A_I(t)B_I(i\lambda\hbar)\rangle_0 \tag{5.9}$$

The first line is obtained, inserting both identities. On the righ hand side the integral is evaluated using integration by parts (PI). Afterwards, the limit $s\to0$ is performed, which yields the obatin result for $\Phi_{AB}(t)$. This structure of the Kubo relaxation function is similar to the later chosen scalar product in the memory matrix formalism.

5.1.1 Spectral representation

In the previous section, the dynamical susecptibility χ_{AB} is introduced deviating the Kubo-formula (??). The evolution of a operator is described by this function, while a pertubation is acting to the system. The dynamic susceptibility is seperated into to types, non-dissipative and dissipative processes. In the following, dissipative processes are investigated. The dissipative susceptibility of the form

$$\chi''_{AB}(t - t') = \frac{1}{2\hbar} \left\langle \left[A(t), B(t') \right] \right\rangle \tag{5.10}$$

is considered, where the operators A and B are Hermitian. The property

$$\left(\chi''_{AB}(t - t') \right)^* = -\chi''_{AB}(t - t') \tag{5.11}$$

is valid, since the commutator of two Hermitian operators is anti-Hermitian. In the following, the dynamic susceptibility is expressed using (5.10). The obtained equation is multiplied with $e^{i\omega t}$ and is integrated over time t.

$$\chi_{AB}(t) = \frac{i}{\hbar} \Theta(t) \left\langle [A(t), B(0)] \right\rangle = 2i\Theta(t)\chi''_{AB}(t)$$

$$\Leftrightarrow \chi_{AB}(\omega) = 2i \int\limits_{-\infty}^{\infty} dt\, e^{i\omega t} \Theta(t) \chi''_{AB}(t)$$

$$\Leftrightarrow \chi_{AB}(\omega) = -\frac{1}{\pi} \lim_{\eta \to 0} \int\limits_{-\infty}^{\infty} d\omega' \frac{1}{\omega' + i\eta} \int\limits_{-\infty}^{\infty} dt\, e^{i(\omega - \omega')t} \chi''_{AB}(t)$$

$$\Leftrightarrow \chi_{AB}(\omega) = \frac{1}{\pi} \lim_{\eta \to 0} \int\limits_{-\infty}^{\infty} d\omega' \frac{\chi''_{AB}(\omega')}{\omega' - \omega - i\eta}$$

$$\Leftrightarrow \chi_{AB}(\omega) = \frac{1}{\pi} \mathrm{PV} \int\limits_{-\infty}^{\infty} d\omega' \frac{\chi''_{AB}(\omega')}{\omega' - \omega} + i \int\limits_{-\infty}^{\infty} d\omega'\, \delta(\omega' - \omega)\chi''_{AB}(\omega')$$

$$\Leftrightarrow \chi_{AB}(\omega) = \chi'_{AB}(\omega) + i\chi''_{AB}(\omega) \tag{5.12}$$

On the left hand side the definition of the Fourier transformation is used, while on the right hand side the following definition of the Θ-function is inserted.

$$\Theta_\eta(t) = i \lim_{\eta \to 0} \int\limits_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{e^{-i\omega' t}}{\omega' + i\eta} \tag{5.13}$$

The dynamical susceptibility $\chi_{AB}(\omega)$ is seperated into two parts $\chi'_{AB}(\omega)$ and $\chi''_{AB}(\omega)$, where the latter is the dissipative susceptibility. Assuming the dissipative susceptibility is a real number, than this is also valid for $\chi'_{AB}(\omega)$ and the both functions $\chi'_{AB}(\omega)$ and $\chi''_{AB}(\omega)$ represent real and imaginary part of $\chi_{AB}(\omega)$, respectivily.

5.2 Motivation

A physicist is always interested in the beaviour and time evolution of the observables of the investigates system. In the middle of the last century many physicists worked on the understanding and mathematical description of one physical process, the Brownian motion. One stochasical theory of these certain physical process is based on the Langevin equation

$$\frac{\partial}{\partial t}A(t) - F_{ex}(x,t) + \gamma \cdot A(t) = f(t), \tag{5.14}$$

where $A(t)$ is some arbitrary dynamical observable and $f(t)$ is a random force like white noise for example. The he second term on the left hand side is originated from some external force, which is coupled to the dynamical observalbe $A(t)$. The third term is a damping or friction term. Now let us assume to seperate equation (5.14) into two parts. The first part, called f_1, is a functional of the dynamical observable $A(t')$, where $t_0 \leq t' \leq t$, so that this part is depending on the history of A. The second part f_2 should be depending on all other degrees of freedom of the system. Now f_1 is expanded up to the linear order and all terms of higher order and the part f_2 are summerized to the quantity $F(t)$. The result is a linearized form of the Langevin equation

$$\frac{\partial}{\partial t}A(t) = \int_{t_0}^{t} dt' \, \mathcal{C}(t-t')A(t') + F(t), \tag{5.15}$$

where $\mathcal{C}$ denotes a correlation function and $A(t')$ is the deviation of the invariant part of the Hamiltonian. For large time scales the deviation should be vanish, so the time-integral over $A(t')$ should become zero. For simplification the origin of the time axis is moved to t_0. In general the Laplace transformation of a function is given by

$$\mathcal{L}\{A(t)\} = A(s) = \int_{0}^{\infty} dt \, A(t)e^{-st}. \tag{5.16}$$

Using the Laplace transformation equation (5.15) becomes a algebratic equation of motion. The solution of this equation is given by

$$A(t) = \Xi(t) \cdot A(0) + A'(t) \qquad \text{with} \qquad A'(t) = \int_{0}^{t} dt' \, \Xi(t-t')F(t'), \tag{5.17}$$

where the function $\Xi(t)$ is defined by the Laplace transformation of $\Xi(s) = [s - \mathcal{C}(s)]^{-1}$ and $\mathcal{C}(s)$ is the Laplace transformtion of the correlation function $\mathcal{C}(t)$. The main result of equation (5.17) and the motivation for the following introduced memory-matrix-formalism is that the dynamical observable $A(t)$ can be splitted into two parts.

For the first term on the right hand side the only time-dependence is adverted from the correlation function $\mathcal{C}$, which is clear considing the definition of Ξ. This term included the linear contributions of $A(t)$ by construction. These ones are the mostly important contributions to the time evolutaion of an observable, because they are secular. In contrast the second term on the right hand side is the convolution between the function $\Xi(t - t')$ and the function $F(t')$. The latter summerize all the non-linear effects, fluctuations and intital transient processes, which are all effects with a small lifetimes in contrast with the secular effects. Therefore these effects shouldn't have large influences on the time evolution of an observable, always large time scales in mind.

Beside the physical interpretation a simple geometrical and mathematical one is very usefull. Let us assume a vector space and the observable should be a vector in this vector space. Then the secular term is a projection on the A-axis and the non-secular term is aquivalent to a vector perpendicular to the A-axis. The memory-matrix-formalism take up this simple interpretation of equation (5.17) and put it in a general and exact form, so that it can be used classicaly and quantum mechanicaly.

5.3 Deviation of the Memory-Matrix-Formalism

After this short reminder of the linear response theorie and the investigation of the dynamical susceptibility the groundwork for the deviation of the memory-matrix-formalism is done. This chapter started splitting an arbitrary dynamical observable into two parts, a secular and a non-secular one. The systematical evolution of observables is determined by the secular part. Looking at a system after a pertubation is switched off for a long time the secular part depending the evolution. Furthermore all processes with a short lifetime or small magnitudes compared with the linear term in pertubation series are summerized in the non-secular part. This result is the starting point to a simple geometrical interpretation in a vector space, which we want define in the following.

Therefore the mathematical framework in quantum mechanics has to be clear, why a short review based on [Aud05] is given in the following. A d-dimensional Hilbert-space is mormaly the mathematical working area in quantum mechanics. This vector space is linear, complex and has a defined scalar product. The vectors $|\phi\rangle$, usually denoted in the Dirac-notation, are identified with all possible states for a system. Because the man is always interested in observables, linear operators are defined in the Hilbert-space where the eigenvalues of them conform to the observables. Defining the dyad product $\sum_i |i\rangle\langle i|$ it's not hard to see that any linear operator occupies a dyad decomposition

$$A = \sum_{i,j} |i\rangle\langle i| A |j\rangle\langle j| = \sum_{i,j} A_{ij} |i\rangle \langle j|, \tag{5.18}$$

where $A_{ij} := \langle i| A |j\rangle$ is a matrix element of the linear operator. The dyad product of an operator is now used to introduce a new vector space of all linear operators

acting on the d-dimensional Hilbert-space which is called the Liouville-space $\mathbb{L}$ or also operator space.

The Liouville-space is a linear and complex vector space equally to the Hilbert-space. The difference between both are the vectors or elements living in the space. In the Liouville-space the vectors are linear operators $A, B, \ldots$ which are acting on some Hilbert-space. In other words this means that the dyad decomposition of an vector in the d-dimensional Hilbert-space is the new vector in the Liouville-space. So some vector in the Liouville-space is notated as

$$|A) := \sum_{ij}^{d} A_{ij}||i\rangle\langle j|). \tag{5.19}$$

Similiarly to the quantum mechanic the Dirac notation is used with the difference that round brackets are used instead of angle brackets to distinghush both spaces. Out of the definition (5.36) it's clear, that the basis in the Liouville-space is build by the d^2 dyads of the Hilbert-space . The dimension of the Liouville-space is therefore d^2. Equally to a Hilbert-space there are many other oppertunities to choose the basis in the Liouville space $\mathbb{L}$, but the defintion in (5.36) should be the one we are working with.

In the following the basis of our Liouville space is denoted with $\{|A_i)\}$ where $i = 1, 2, 3, \ldots, n$ and A_i is an operator. The corresponding basis of the dual space is given by $\{(A_i|\}$, similarly to the Hilbert space. The last needed element of our Liouville space is a scalar product which has to fullfill the following three condictions.

1. $(A_i|A_j) = (A_j|A_i)^*$ \hfill (5.20)

2. $(A_i|B) = c_1(A_i|A_j) + c_2(A_i|A_k),$

    where $\quad B = c_1 A_j + c_2 A_k \quad$ and $\quad c_1, c_2 \in \mathbb{C}$ \hfill (5.21)

3. $(A_i|A_i) \geq 0 \quad$, where equallity is fulfilled if $\quad A_i = 0.$ \hfill (5.22)

Beside these the choice of the scalar product is arbitrary. For the moment let us choose

$$\left(A_i(t)\big|A_j(t')\right) = \frac{1}{\beta}\int_0^\beta \mathrm{d}\lambda \left\langle A_i^\dagger(t) A_j(t' + i\lambda\hbar)\right\rangle \tag{5.23}$$

as our scalar product, where the normal time evolution of an operator $A_i(t) = e^{iHt/\hbar} A_i(0) e^{-iHt/\hbar}$ is valid, so that $A_i(i\lambda\hbar) = e^{-\lambda H} A_i(0) e^{\lambda H}$ can be used. Now we have to proof if the condictions are fullfilled by the choice of our scalar product. Let's get started with the second one because it's easily shown transforming the expactation value into the trace representation and then using the properties of the trace.

Motivation why we choose this kind of a scalar product

maybe the computation isn't needed here

$$\left(A_i(t)\big|B(t')\right) = \frac{1}{\beta}\int_0^\beta \mathrm{d}\lambda \frac{1}{Z} \mathrm{Tr}\Big\{\rho A_i^\dagger(t)\Big[c_1 A_j(t' + i\lambda\hbar) + c_2 A_k(t' + i\lambda\hbar)\Big]\Big\}$$

$$\Leftrightarrow \ \big(A_i(t)\big|B(t')\big) = c_1\big(A_i(t)\big|A_j(t')\big) + c_2\big(A_i(t)\big|A_k(t')\big) \tag{5.24}$$

The first and third condition can be shown by transforming the scalar product in the spectral representation. Therefore the trace is writen explicitly as a sum over all states and the unity operator written as a sum over all projection operators are inserted between both operators A_i and A_j.

$$\big(A_i(t)\big|A_j(t')\big) = \frac{1}{\beta \cdot Z} \int\limits_0^\beta d\lambda \sum_{n,m} \langle n|\, e^{-\beta H} A_i^\dagger(t)\, |m\rangle\, \langle m|\, e^{-\lambda H} A_j(t') e^{\lambda H}\, |n\rangle$$

$$\Leftrightarrow \ \big(A_i(t)\big|A_j(t')\big) = \frac{1}{\beta \cdot Z} \sum_{n,m} \langle n|A_i^\dagger(t)|m\rangle\, \langle m|A_j(t')|n\rangle\, e^{-\beta E_n} \int\limits_0^\beta d\lambda\, e^{\lambda(E_n - E_m)}$$

$$\Leftrightarrow \ \big(A_i(t)\big|A_j(t')\big) = \frac{1}{\beta \cdot Z} \sum_{n,m} \langle n|A_i^\dagger(t)|m\rangle\, \langle m|A_j(t')|n\rangle\, \frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \tag{5.25}$$

The complex conjugated of the expectation value is considered in the Liouville space and using $\langle n|A_j^\dagger(t)|m\rangle^* = \langle m|A_j(t)|n\rangle$ let us find instantly the first condition. Notice that on the right hand side of equation (5.42) only the expactation values are complex quantities. For them the complex conjugation yields

$$\Big(\langle n|A_i^\dagger(t)|m\rangle\, \langle m|A_j(t')|n\rangle \Big)^* = \langle n|A_j^\dagger(t')|m\rangle\, \langle m|A_i(t)|n\rangle \tag{5.26}$$

and inserting back $(A_i(t)|A_j(t'))^*$ is exactly the same as (5.42). Proofing the third condition it has to be set $A_j(t') = A_i(t)$ in equation (5.42), which one the right hand side yields

$$\frac{1}{\beta \cdot Z} \sum_{n,m} |\,\langle m|A_i(t)|n\rangle\,|^2 \frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m}. \tag{5.27}$$

It's clear that the squared expactation value is always non-negative. The fraction is positive too, which is easily seen by proofing the two cases $E_n > E_m$ and $E_n < E_m$. Therefore the expactation value $\big(A_i(t)\big|A_i(t)\big) \geq 0$ and equality is only possible if $A_i = 0$. All three conditions are well proven and the chosen scalar product is really one's. At this point all the mathematical ground work is done, we know how the vectors look in the Liouville space and we have a well defined scalar product.

The goal of every physical theory is to describe the measurment results. Typically in statistical or quantum mechanics this is done via correlation functions. So our goal is now to find a useable expression for correlation functions in our new Liouville space. The natural starting point describing the time evolution of an operator A_i is in quantum mechanics the Heisenberg equation of motion

$$\frac{d}{dt} A_i(t) = \dot A_i(t) = \frac{i}{\hbar}[H, A_i(t)] = iL A_i(t) \tag{5.28}$$

where the operators are in the Heisenberg representation and the Hermitian Liouville operator $L = \hbar^{-1}[H, \bullet]$ is introduced, defined by his action on an arbitrary operator. The formal solution of equation (5.45) is given by

$$A_i(t) = e^{itL}A_i(0) = e^{itH/\hbar}A_i(0)e^{-itH/\hbar}. \tag{5.29}$$

In the second step only the definition of the Liouville operator and some algebraic transformations are used. In this notation it is more clearly that the time evolution of an operator is given by the Liouville operator. The same result is obtained in the Liouville space if the Liouville operator is acting on the basis vectors. This isn't really surprisingly because only the dyad product has to be insert in equation (5.45), which results in

$$\left|\dot{A}_i(t)\right) = \frac{i}{\hbar}\left|[H, A_i(t)]\right) = iL\left|A_i(t)\right) \tag{5.30}$$

for the equation of motion in the Liouville space and there formal solution is given by

$$\left|A_i(t)\right) = e^{itL}\left|A_i(0)\right). \tag{5.31}$$

Beside the Liouville operator one more operator has to be introduced for the deviation of the correlation function, called the projection operator. Therefore let us define a set of operators $\{C_i\}$, where the choice of these operators are differently depending on the investigated system and correlation function. In the later computation of a certain problem the choice of the operators is discussed in more detail. For the moment it's sufficient to know that the set of operators exists. Directly follwing out the definition of the projection operator in quantum mechanics the projection operator in the Liouville space looks like

$$P = \sum_{i,j}\left|C_i(0)\right)\left(C_i(0)\middle|C_j(0)\right)^{-1}\left(C_j(0)\right|. \tag{5.32}$$

The action of P on some vector $\left|A(t)\right)$ in the Liouville space yields the parallel components to the chosen operators C_i, which is the projection from $\left|A(t)\right)$ at the vector subspace spanned by C_i. The corresponding vertical component of $\left|A(t)\right)$ with respect to the operators C_i is given by $Q = 1 - P$, which is the projection out of the vector subspace. Naturally the projection operator is fullfilled the two properties $P^2 = P$ and $PQ = QP = 0$ of a projection operator, which follows immediately from the definition of P.

After the deviation of the time evolution of an operator and the projection operator in Liouville space the correlation function can be defined as

$$\mathcal{C}_{ij}(t) = \left(A_i(t)\middle|A_j(0)\right) \overset{(5.40)}{=} \frac{1}{\beta}\int_0^\beta \mathrm{d}\lambda\left\langle A_i^\dagger(t)A_j(i\lambda\hbar)\right\rangle, \tag{5.33}$$

where in the last step the definition of the scalar product is only inserted. Comparing equation (5.50) with (5.25) our choice of the correlation function is more clear. The

defined correlation function is proportional to the Kubo relaxation function, which describes the system's reaction on a switched off pertubation, how we learned in section 5.2.1. For $t = 0$ the correlation function is also proportional to the Fourier transformated susebtibility

$$\mathcal{C}_{ij}(t = 0) = \frac{1}{\beta}\Phi_{ij}(t = 0) = \frac{1}{\beta}\chi_{ij}(\omega = 0), \tag{5.34}$$

which directly results from equation (5.20). Equation (5.48) is used to bring the time evolution of the correlation function in more suitable expression

$$\mathcal{C}_{ij}(t) = \big(A_i(0)\big|A_j(-t)\big) = \big(A_i(0)\big|e^{-itL}\big|A_j(0)\big), \tag{5.35}$$

which opens the possibility for using the Laplace transformation. Instead of the definition in equation (5.16) here a form of the Laplace transformation is used where s is substituted by $-i\omega$ which is nothing else as a rotation of the definition area by $\pi/2$. Multipling the last equation with $e^{i\omega t}$ and intgrate from zero to infinty with resprct to t yields

$$\mathcal{C}_{ij}(\omega) = \big(A_i\big|\int_0^\infty dt\, e^{i(\omega - L)t}\big|A_j\big) = \big(A_i\big|\frac{i}{\omega - L}\big|A_j\big), \tag{5.36}$$

where for reasons of clarity and comprehensibility from now on the argument $t = 0$ isn't anymore written at the basis vectors. Now the relation $L = LQ + LP$ which follows immediatly by using the definition of P and Q and the identity $(X + Y)^{-1} = X^{-1} - X^{-1}Y(X + Y)^{-1}$ is used to simplify the correlation function, where $X = \omega - LQ$ and $Y = -LP$.

$$\mathcal{C}_{ij}(\omega) = \big(A_i\big|\frac{i}{\omega - LQ - LP}\big|A_j\big)$$

$$\Leftrightarrow \mathcal{C}_{ij}(\omega) = \big(A_i\big|\frac{i}{\omega - LQ}\big|A_j\big) + \big(A_i\big|\frac{1}{\omega - LQ}LP\frac{i}{\omega - L}\big|A_j\big) \tag{5.37}$$

The both terms on the right hand side are considered seperatly. Lez us starting with the first one. The fraction can be written as the geometric series assuming $\frac{LQ}{\omega} < 1$, which means that the pertubation is small compared to other quantities in the system.

$$\frac{i}{\omega - LQ} = \frac{i}{\omega}\left[1 + \frac{LQ}{\omega} + \left(\frac{LQ}{\omega}\right)^2 + \dots\right] \tag{5.38}$$

Each term of the series in the squard brackets acting on the operator $\big|A_j\big)$. Remember this is the operator at time $t = 0$, which means that no vertical component exists and therefore $Q\big|A_j\big) = 0$. Every term except the first one conatins an operator Q, so the first term of the correlation function yields

$$\big(A_i\big|\frac{i}{\omega - LQ}\big|A_j\big) = \frac{i}{\omega}\big(A_i\big|A_j\big) = \frac{i}{\omega}\mathcal{C}_{ij}(0). \tag{5.39}$$

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At the second term only the back is considered. Here only the explicit expression of the propagator is inserted, which yields the definition of the Laplace transformed correlation function.

$$P\frac{i}{\omega - L}|A_j) = \sum_{k,l}|C_k)(C_k|C_l)^{-1}(C_l|\frac{i}{\omega - L}|A_j) = \sum_{k,l}|C_k)\mathcal{C}_{kl}^{-1}(0)\mathcal{C}_{lj}(\omega) \quad (5.40)$$

Inserting back both simplifications the correlation function is get the formal expression

$$\mathcal{C}_{ij}(\omega) = \frac{i}{\omega}\mathcal{C}_{ij}(0) + \sum_{k,l}\left(A_i\Big|\frac{1}{\omega - LQ}L\Big|C_k\right)\mathcal{C}_{kl}^{-1}(0)\mathcal{C}_{lj}(\omega). \quad (5.41)$$

The form of the resulting correlation function is now much more useable beside the fraction in the expectation value, but this one can be simplified too. Therefore a null $LQ - LQ$ is added in the nominator.

$$\frac{\omega}{\omega - LQ} = \frac{\omega - LQ + LQ}{\omega - LQ} = 1 + \frac{LQ}{\omega - LQ} \quad (5.42)$$

Multipling with ω and inserting the rearrangement of the fraction yields an algebratic equation for the correlation function.

$$\omega\mathcal{C}_{ij}(\omega) = i\mathcal{C}_{ij}(0) + \sum_{k,l}\left(A_i\Big|\frac{\omega}{\omega - LQ}L\Big|C_k\right)\mathcal{C}_{kl}^{-1}(0)\mathcal{C}_{lj}(\omega)$$

$$\Leftrightarrow \omega\mathcal{C}_{ij}(\omega) = i\mathcal{C}_{ij}(0) + \sum_{k,l}\left(A_i\Big|1 + \frac{LQ}{\omega - LQ}L\Big|C_k\right)\mathcal{C}_{kl}^{-1}(0)\mathcal{C}_{lj}(\omega)$$

$$\Leftrightarrow \omega\sum_{l}\delta_{il}\mathcal{C}_{lj}(\omega) = i\frac{1}{\beta}\chi_{ij}(0) + \sum_{l}\Big[\Omega_{il} - i\Sigma_{il}(\omega)\Big]\mathcal{C}_{lj}(\omega)$$

$$\Leftrightarrow \sum_{l}\Big[\omega\delta_{il} - \Omega_{il} + i\Sigma_{il}(\omega)\Big]\mathcal{C}_{lj}(\omega) = i\frac{1}{\beta}\chi_{ij}(0) \quad (5.43)$$

where equation (5.20) is used and the abbreviations

$$\Omega_{il} := \beta\sum_{k}\left(A_i\big|L\big|C_k\right)\chi_{kl}^{-1}(0) \quad \text{and} \quad \Sigma_{il}(\omega) := i\beta\sum_{k}\left(A_i\Big|\frac{LQ}{\omega - LQ}L\Big|C_k\right)\chi_{kl}^{-1}(0)$$

$$(5.44)$$

are inserted. Equation (5.60) is the wanted algebraic equation for the correlation function. The sums over k and l are originated from the utilization of the projection operator. Therefore each sum is summarized over all operators included in the set $\{C_i\}$ of selected operators. The indicies i and j have to be chosen out of this set too, so that equation (5.60) yields n^2 algebraic equations if n is the number of operators in $\{C_i\}$.

write that the founded expression is an exact one

It's useful for later computations to write the even defined abbreviations in another form. For Ω_{il} not much work is to do, because we use only equation (5.47), so that

$$\Omega_{il} = i\beta \sum_k \left(\dot{A}_i \middle| C_k\right)\chi_{kl}^{-1}(0).$$ (5.45)

For the rearrangement of the second abbreviation equation (5.47) is used at the first step too and the fraction is written as the geometric series at the second step. Then in every term the relation $Q = Q^2$ is inserted, where the proof is an easy finger excercise. After factorizing one Q to each vector operator the geometric series is written back as a fraction.

$$\Sigma_{il}(\omega) = \frac{i\beta}{\omega} \sum_k \left(\dot{A}_i \middle| Q\left[1 + \frac{LQ}{\omega} + \left(\frac{LQ}{\omega}\right)^2 + \dots\right] \middle| \dot{C}_k\right)\chi_{kl}^{-1}(0)$$

$$\Leftrightarrow \Sigma_{il}(\omega) = \frac{i\beta}{\omega} \sum_k \left(\dot{A}_i \middle| Q^2 + \frac{Q^2 LQ^2}{\omega} + \frac{Q^2 LQ^2 LQ^2}{\omega^2} + \dots \middle| \dot{C}_k\right)\chi_{kl}^{-1}(0)$$

$$\Leftrightarrow \Sigma_{il}(\omega) = \frac{i\beta}{\omega} \sum_k \left(\dot{A}_i \middle| Q\left[1 + \frac{QLQ}{\omega} + \left(\frac{QLQ}{\omega}\right)^2 + \dots\right] Q \middle| \dot{C}_k\right)\chi_{kl}^{-1}(0)$$

$$\Leftrightarrow \Sigma_{il}(\omega) = i\beta \sum_k \left(\dot{A}_i \middle| Q\frac{1}{\omega - QLQ}Q \middle| \dot{C}_k\right)\chi_{kl}^{-1}(0)$$ (5.46)

After all this exhausting mathematical and algebraical conversions the correlation function in the memory matrix formalsim is in a useful and workable form. In equation (5.60) the abbreviations can be combined to one function $M_{il}(\omega) := \Sigma_{il}(\omega) + i\Omega_{il}$. The symbol Σ is selected in dependence on the quantum mechanical self energy. The function $M(\omega)$ is called the mass operator in quantum field theory and the memory function in non-equilibrium physics.

Let us discuss the physical meaning of Ω and $\Sigma(\omega)$ in more detail. The quantity Ω always vanishs in the case if the considered Hamiltonian occupies time reversal symmetry and if the operators A_i and A_k transform with the same signature with respect to time reversal symmetry since then $\left(\dot{A}_i \middle| A_k\right) = 0$. This assertion is immediatly proven extensivly in the section below. In this case the memory function is solely given by the function $\Sigma(\omega)$. Comparising (5.63) with the definition (5.50) of correlation functions exhibits $\Sigma(\omega)$ has the structur of a correlation function but $\Sigma(\omega)$ occupies to differences. On the one hand only $Q\middle|\dot{A})$ which is perpendicular to $\middle|A)$ forms the basis vectors of the expectation value. On the other hand only the reduced Liouville operator QLQ contribute to the expectation value. Let us understand these two object better.

The latter one projects at the part of L, the full Liouville operator, which causes the intrinsic fluctuations of the operator A. This means that the function $\Sigma(\omega)$ describes the dynamic of the operators, which is the interesting part for us. In other words the operators QLQ describes the internal dynamics of all other degrees of freedom of the system excluded A, called the "bath". Then $Q\middle|\dot{A})$ characterizes the coupling to the bath and it's clear, that the dynamic of the bath changes the behaviour of A.

5.3.1 Time Reversal Symmetry

Even above the assertion was postulated that the quantity Ω_{il} vanishs if the considered Hamiltonian is symmetric and if the operators $\dot{\mathrm{A}}$ and A have different signature under time reversal symmetry. In the following section the evidence of this statement is proven. Our starting point is the introduction of the time reversal operator T via the transformation rule

$$\mathrm{A}(t) \to \mathrm{A}'(t) = \mathrm{T}\mathrm{A}(t)\mathrm{T}^{-1} = \epsilon_\mathrm{A}\mathrm{A}(-t), \tag{5.47}$$

where ϵ_A supposees two different values, $+1$ or -1. The first one is taken if the physical quantity is something like a position or electrical field and the latter is taken if the physical quantity is a momentum, angular momentum or magnetic field. Now let us firstly investigate the action of T at the time evolution of an operator.

$$\mathrm{T}e^{i\mathrm{H}t/\hbar}\mathrm{T}^{-1} = e^{-i\mathrm{H}t/\hbar} \tag{5.48}$$

Remembering the Hamiltonian is assumed as invariant under time reversal symmetry, so that the only changed quantity is the explicit time argument t. At next the time derivative of the time evolution of an operator is investigated which yields

$$\mathrm{T}\frac{\partial}{\partial t}e^{i\mathrm{H}t/\hbar}\mathrm{T}^{-1} = \frac{i}{\hbar}\mathrm{T}\mathrm{H}e^{i\mathrm{H}t/\hbar}\mathrm{T}^{-1} = \frac{i}{\hbar}\mathrm{T}\mathrm{H}\mathrm{T}^{-1}\mathrm{T}e^{i\mathrm{H}t/\hbar}\mathrm{T}^{-1} = \frac{i}{\hbar}\mathrm{H}e^{-i\mathrm{H}t/\hbar}, \tag{5.49}$$

where in the second step the unit elememt $\mathbb{1} = \mathrm{T}\mathrm{T}^{-1}$ is inserted. Setting $t = 0$ and multipling with T from the right hand side immediatly the commutator relation between the time reversal operator and the Hamiltonian is given by

$$\mathrm{TH} = \mathrm{HT} \qquad \Leftrightarrow \qquad [\mathrm{H}, \mathrm{T}] = 0. \tag{5.50}$$

This is aquivalent to the assumption that the Hamiltonian is invariant with respect to time reversal symmetry. This is all we need to know about the time reversal operator to prove the assertion. The expectation value of a Hermitain operator can be easily manipulated with the aim of the time reversal operator T.

$$\langle \mathrm{B} \rangle = \frac{1}{Z}\operatorname{Tr}\left\{ e^{-\beta\mathrm{H}}\mathrm{T}\mathrm{B}\mathrm{T}^{-1} \right\} = \left\langle \left(\mathrm{T}\mathrm{B}\mathrm{T}^{-1}\right)^\dagger \right\rangle \tag{5.51}$$

where the invariance of the trace with respect to cycling permutation and the commutator relation between T and H is used in the first step. The anti-unitarity of the time reversal operator and the hermiticity of B is utilized in the last step. The same stuff is done with the commutator between two Hermitian operators.

$$\left(\mathrm{T}\big[\mathrm{A}(t), \mathrm{B}(t')\big]\mathrm{T}^{-1} \right)^\dagger = \epsilon_\mathrm{A}\epsilon_\mathrm{B}\left(\big[\mathrm{A}(-t), \mathrm{B}(-t')\big] \right)^\dagger = -\epsilon_\mathrm{A}\epsilon_\mathrm{B}\big[\mathrm{A}(-t), \mathrm{B}(-t')\big] \tag{5.52}$$

Finitely coming to the quantity Ω_{il} which is proportional to the correlation function $\big(\dot{\mathrm{A}}_i \big| \mathrm{A}_k\big)$ between a time derivative quantity $\dot{\mathrm{A}}_i$ and the quantity A_k, as seen in equation

(5.62). Using equation (5.33) yields

$$i\beta\big(\dot{A}_i\big|A_k\big) = i\chi_{\dot{A}_iA_k}(\omega = 0) = i\,\text{P.V.}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{\pi}\frac{\chi''_{\dot{A}_iA_k}(\omega')}{\omega'} - \lim_{\omega\to 0}\chi''_{\dot{A}_iA_k}(\omega), \qquad (5.53)$$

where the dissipative susceptibility $\chi''_{\dot{A}_iA_k}(\omega)$ occurs, which differs a little but significant from the dissipative susceptibility $\chi''_{A_iA_k}(\omega)$ defined in section 5.2.3. To find the relation between both the derivative of (5.31) is formed.

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi''_{A_iA_k}(t) = \frac{1}{2\hbar}\left\langle\left[\dot{A}_i(t), A_k(0)\right]\right\rangle = \chi''_{\dot{A}_iA_k}(t) \qquad (5.54)$$

Express on both sides the susceptibilities by her Fourier transform yields the wanted relation between both.

$$\chi''_{\dot{A}_iA_k}(\omega) = -i\omega\chi''_{A_iA_k}(\omega) \qquad (5.55)$$

Inserting this in (5.73) yields

$$i\beta\big(\dot{A}_i\big|A_k\big) = \text{P.V.}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{\pi}\chi''_{A_iA_k}(\omega') \qquad (5.56)$$

where the limit in the second term is taken. This result entails two very important advantages. One the one hand the physical meaning of the quantity Ω_{il} becomes clearer, because Ω_{il} is directly associated with dissipative processes via $\chi''_{A_iA_k}(\omega')$. On the other hand the founded expression establishs the possibility to analyse the behaviour of Ω_{il} under time reversal symmetry.

$$\chi''_{A_iA_k}(t - t') = \frac{1}{2\hbar}\left\langle\left[A_i(t), A_k(t')\right]\right\rangle \qquad (5.57)$$

The expectation value is rewriten using equation (5.68) which establishes the oppertunity to use (5.69).

$$\chi''_{A_iA_k}(t - t') = -\epsilon_{A_i}\epsilon_{A_k}\chi''_{A_iA_k}(t' - t), \qquad (5.58)$$

where the relation (5.32) is used. The Laplace transformation of this equation yields

$$\chi''_{A_iA_k}(\omega) = -\epsilon_{A_i}\epsilon_{A_k}\chi''_{A_iA_k}(-\omega) = \epsilon_{A_i}\epsilon_{A_k}\chi''_{A_iA_k}(\omega), \qquad (5.59)$$

where the antisymmetry of the commutator with respect of interchanging both operators is utilized. Two cases has to be investigated by analyzing the analytical properties of $\chi''_{A_iA_k}(\omega)$, which are $\epsilon_{A_i} = \epsilon_{A_k}$ and $\epsilon_{A_i} \neq \epsilon_{A_k}$. The analysis gives us the required properties to compute the integral over the dissipative susceptibility.

1. case: $\epsilon_{A_i} = \epsilon_{A_k}$

This yields $\chi''_{A_i A_k}(\omega) = \chi''_{A_k A_i}(\omega)$ which means that the dissipative susceptibility is symmetrical under interchanging A_i and A_k. Further the dissipative sysceptibility is an antisymmetrical function, because $\chi''_{A_i A_k}(\omega) = -\chi''_{A_i A_k}(-\omega)$. Consider the complex conjugated of $\chi''_{A_i A_k}(\omega)$ yields that the dynamical susceptibility is a real number.

$$\left(\chi''_{A_i A_k}(\omega)\right)^* = -\int\limits_{-\infty}^{\infty} dt\, e^{-i\omega(t-t')} \chi''_{A_i A_k}(t-t') = -\chi_{A_i A_k}(-\omega) = \chi_{A_i A_k}(\omega) \quad (5.60)$$

where equation (5.32) is used.

2. case: $\epsilon_{A_i} \neq \epsilon_{A_k}$

If the signature of A_i and A_k is different under time reversal symmetry than the dissipative susceptibility is antisymmetric under the interchange of the both operators so that this yields $\chi''_{A_i A_k}(\omega) = -\chi''_{A_k A_i}(\omega)$. For the same reason $\chi''_{A_i A_k}(\omega)$ is a symmetrical function because $\chi''_{A_i A_k}(\omega) = \chi''_{A_i A_k}(-\omega)$. Towards the first case the dissipative susceptibility is an imaginary number which ensures by the complex conjugation likewise.

$$\left(\chi''_{A_i A_k}(\omega)\right)^* = -\int\limits_{-\infty}^{\infty} dt\, e^{-i\omega(t-t')} \chi''_{A_i A_k}(t-t') = -\chi_{A_i A_k}(-\omega) = -\chi_{A_i A_k}(\omega) \quad (5.61)$$

Back to equation (5.73). We see that the integral vanishs in the first case because the suscebtibility is an odd function. This means that $i\beta(\dot{A}_i|A_k)$ is always zero when the operators A_i and A_k have the same signature with respect to time reversal symmetry, assumed the Hamiltonian is invariant under time reversal symmetry. This assertion is aquivalent to the one that $\dot{A}_i$ and A_k have opposite signature with respect to time reversal symmetry. This is exactly the assertion which we want to prove.

6 Conclusion

A Computation of the Damped Spin Density Propagator

In the present appendix, the propagator of damped spin density fluctuations is computed, considering the first order in pertubation theory. Our approach is seperated in two parts. Firstly, the free spin density propagator is determined, using the equation of motion for Green functions, as suggested in [EG79] or many other textbooks. The damped propagator is then calculated, using the diagrammatic pertubation theory. The obtained propagator is transformed into Matsubara frquency space, since the computation of the conductivity is treated into there.

A.1 Computation of the Free Propagator

Free propagators are easily computed, using the equation of motion of Green functions. The equation of motion is obtained, derivating the Green function with respect to the time t. In Fourier space the equation of motion is given by

$$\omega \langle\langle A; B \rangle\rangle_{\omega}^{j} = \langle [A, B]_{\eta} \rangle + \langle\langle [A, H]_{-}; B \rangle\rangle_{\omega}^{j}, \tag{A.1}$$

where the typ of the Green function (retarded, advanced and causal) is denoted with j and the frequency dependence of the Green function is represented with an index ω. The Green function itself is symbolized by double angle brackets. This equation is an algebraic equation or more precisely an infinite algebraic equation chain for Green functions. A new more complicated Green function appears on the right hand side. For this one exists a new equation chain with a more complicted Green function, and so on. Nevertheless, the equation chain is not infinity in the case of free propagators. The dynamic of free spin density waves is described by the Hamiltonian H_{Φ} (see equation (3.5). The operators A and B are replaced by bosonic field operators $\Phi_{\mu}(\mathbf{k} + \mathbf{G}, t)$ and $\Phi_{\mu}(-\mathbf{k} - \mathbf{G}, t)$, respectively. During the computation, the abbreviation $\langle\langle \Phi_{\mu}; \Phi_{\mu} \rangle\rangle_{\omega}$ is used, where the argument of the field operators is dropped. The first field operator is always considered with the momentum argument $\mathbf{k} + \mathbf{G}$, while the second one has opposite sign. The time argument is equal for both operators. The equation of motion is given by

$$\omega \langle\langle \Phi_{\mu}; \Phi_{\mu} \rangle\rangle_{\omega} = \langle [\Phi_{\mu}(\mathbf{k} + \mathbf{G}), \Phi_{\mu}(-\mathbf{k} - \mathbf{G})] \rangle + \langle\langle [\Phi_{\mu}(\mathbf{k} + \mathbf{G}), H_{\Phi}]; \Phi_{\mu} \rangle\rangle_{\omega} \tag{A.2}$$

Both commuators are evaluated on the right hand side, using the bosonic commutator relations. The inhomogeneity is trivially zero, since the commutator relation is zero

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per definition. The commutator, contained in the Green function, is given by

$$[\Phi_\mu(\mathbf{k}+\mathbf{G},t),\mathrm{H}_\Phi] = -\frac{v_\mathrm{S}^2}{2}\sum_\lambda\int_\mathbf{p}[\Phi_\mu(\mathbf{k}+\mathbf{G},t),\pi_\lambda(\mathbf{p},t)\pi_\lambda(-\mathbf{p},t)]$$

$$\Leftrightarrow [\Phi_\mu(\mathbf{k}+\mathbf{G},t),\mathrm{H}_\Phi] = -\frac{v_\mathrm{S}^2}{2}\sum_\lambda\int_\mathbf{p}\Big[\pi_\lambda(\mathbf{p},t)[\Phi_\mu(\mathbf{k}+\mathbf{G},t),\pi_\lambda(-\mathbf{p},t)]$$

$$+[\Phi_\mu(\mathbf{k}+\mathbf{G},t),\pi_\lambda(\mathbf{p},t)]\pi_\lambda(-\mathbf{p},t)\Big]$$

$$\Leftrightarrow [\Phi_\mu(\mathbf{k}+\mathbf{G},t),\mathrm{H}_\Phi] = -iv_\mathrm{S}^2\pi_\mu(\mathbf{k}+\mathbf{G},t). \tag{A.3}$$

The obtained result of the commutator is inserted in equation (A.2). This relation is the first term of the equation chain.

$$\omega\big\langle\!\big\langle\Phi_\mu;\Phi_\mu\big\rangle\!\big\rangle_\omega = -iv_\mathrm{S}^2\big\langle\!\big\langle\pi_\mu;\Phi_\mu\big\rangle\!\big\rangle_\omega \tag{A.4}$$

Equally to the initial Green function an algebraic equation chain is established for the Green function of the right hand side.

$$\omega\big\langle\!\big\langle\pi_\mu;\Phi_\mu\big\rangle\!\big\rangle_\omega = \big\langle[\pi_\mu(\mathbf{k}+\mathbf{G},t),\Phi_\lambda(-\mathbf{k}-\mathbf{G},t)]\big\rangle + \big\langle\!\big\langle[\pi_\mu(\mathbf{k}+\mathbf{G},t),\mathrm{H}_\Phi];\Phi_\mu\big\rangle\!\big\rangle_\omega \tag{A.5}$$

The inhomogeneity is directly given by the commutator relations and yields $-i$. The commutator of the Green function is evaluated to $[\pi_\mu(\mathbf{k}+\mathbf{G},t),\mathrm{H}_\Phi] = i\big((\mathbf{k}+\mathbf{G})^2 + r_0\big)\Phi_\mu(\mathbf{k}+\mathbf{G},t)$. Inserting our results in the (A.5), the inital Green function is obtained on the right hand side. This and (A.4) are an equation system, which is solved with respect to $\big\langle\!\big\langle\Phi_\mu;\Phi_\mu\big\rangle\!\big\rangle_\omega$.

$$\mathcal{D}_\mu^{(0)}(\mathbf{k},\omega) := \big\langle\!\big\langle\Phi_\mu;\Phi_\mu\big\rangle\!\big\rangle_\omega = \sum_\mathbf{G}\frac{1}{(\mathbf{k}+\mathbf{G})^2 + r_0 - (flatfrac\omega v_\mathrm{S})^{-2}} \tag{A.6}$$

A.2 Consideration of Damping in the Spin Density Propagator

An interaction between electrons on different Fermi surfaces is considered in the spin-fermion-model. This interaction is originated by spin fluctuations and their damping is governed by the decay of partile-hole-excitations. The usual diagrammatic pertubation theory is used for considering the damping of the spin fluctuations. The full spin density wave propagator is given by

$$\mathcal{D}_\mu(\mathbf{k},t-t') = -i\big\langle\mathcal{T}_t\mathrm{U}(\infty,-\infty)\Phi_\mu(\mathbf{k}+\mathbf{G},t)\Phi_\mu(-\mathbf{k}-\mathbf{G},t')\big\rangle_0 \tag{A.7}$$

where $\mathcal{T}_t$ is the time ordering operator. The index 0 denotes that the expectation value is performed with respect to the unpertubated Hamiltonian. The interaction is only incorporated through the time evolution operator U, which is given by

$$\mathrm{U}(t,t') = \exp\Big(-i\int_{t'}^t\mathrm{d}t_1\,\mathrm{H}_\mathrm{int}(t_1)\Big). \tag{A.8}$$

The full Green function $\mathcal{D}_\mu$ is developed into a Dyson series, similar to the derivation of Landau's Fermi liquid theory illustrated in many textbooks about quantum field theory in condensed matter [Nol09]. The free propagator is denoted with an upper bracketed index 0. A new object Π is introduced, called the polaraization operator. In general, this object is an infinite series, containing the interaction between spin fluctuation and fermions. The Dyson equation is solved using the geometric sereis.

$$\mathcal{D}_\mu = \mathcal{D}_\mu^{(0)} + \mathcal{D}_\mu^{(0)} \Pi_\mu \mathcal{D}_\mu \qquad \Rightarrow \qquad \mathcal{D}_\mu = \frac{1}{\left(\mathcal{D}_\mu^{(0)}\right)^{-1} - \Pi_\mu} \tag{A.9}$$

The polarization operator is sperated into a real and imaginary part. The latter is interpreted as a lifetime, while the real part is an energy correction. In consequence of an finite lifetime the spin fluctuations are damped. Our next treatment is to identify the leading diagrams contained in the polarization operator.

Therefore, the time evolution operator is expanded up to the second order. The zeroth order yields the free propagator, calculated above, while the first order vanishes. One bosonic field operator is contained in the interaction Hamiltonian $H_{\Psi\Phi}$ (3.6). Furthermore, two bosonic field operators are given by the definition of the Green function. An expectation value of three bosonic operators is generated in the first order. Wick's theorem yields that each expectation value of an odd number of operators is always zero.

The first correction to the free Green function is considered in the second order of series expansion. For the fermionic expectation values, Wick's theorem can not be used as simple as usual, since the fermionic field operators are connected via Pauli matrizies. The fermionic expectation value is given in the following form:

$$\text{EV} := \left\langle \mathcal{T}_t\, \Psi_\alpha^\dagger(\mathbf{p}_4, t_2) \cdot \sigma_{\lambda'} \cdot \Psi_\beta(\mathbf{p}_3, t_2) \cdot \Psi_\gamma^\dagger(\mathbf{p}_2, t_1) \cdot \sigma_\lambda \cdot \Psi_\beta(\mathbf{p}_1, t_1) \right\rangle. \tag{A.10}$$

Here $\alpha, \beta, \gamma, \delta \in \{a, b\}$, where two greek letters have to be an "a", while the other ones a "b". The quantum numbers of the field operators are denoted with ν_i, where $i = 1, 2, 3, 4$. These kins of expectation values are evaluated using Fierz identity. A product of two Pauli matricies are rewriten as the following relation, where δ is the Kroncker symbol.

$$\sum_{\mu=1}^{3} \sigma_{ij}^\mu \sigma_{kl}^\mu = 2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} \tag{A.11}$$

For using Fierz identity the product of field operators and Pauli matricies is written in component representation. For reasons of clarity, the argument of the field operators are abbreviated by ν_i, $i = 1, 2, 3, 4$.

$$\text{EV}_\text{F} = \sum_\mu \left\langle \left(\Psi_\alpha^\dagger(\nu_1)\right)_i \cdot \sigma_{ij}^\mu \cdot \left(\Psi_\beta(\nu_2)\right)_j \cdot \left(\Psi_\gamma^\dagger(\nu_3)\right)_k \cdot \sigma_{kl}^\mu \cdot \left(\Psi_\delta(\nu_4)\right)_l \right\rangle$$

$$\Leftrightarrow \text{EV}_\text{F} = \left(2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}\right) \left\langle \left(\Psi_\alpha^\dagger(\nu_1)\right)_i \left(\Psi_\beta(\nu_2)\right)_j \left(\Psi_\gamma^\dagger(\nu_3)\right)_k \left(\Psi_\delta(\nu_4)\right)_l \right\rangle$$

$$\Leftrightarrow \ \mathrm{EV_F} = 2 \left\langle \Psi_\alpha^\dagger(\nu_1)\Psi_\delta(\nu_4)\Psi_\beta(\nu_2)\Psi_\gamma^\dagger(\nu_3) \right\rangle \delta_{\nu_3,\nu_4} - \left\langle \Psi_\alpha^\dagger(\nu_1)\Psi_\beta(\nu_2)\Psi_\gamma^\dagger(\nu_3)\Psi_\delta(\nu_4) \right\rangle \tag{A.12}$$

The arising δ-distribution is generated interchanging the field operators. In the next step, the remained expression is investigated under an explicitly choice of the greek letters. Two of the greek letters have to be always equal. Therefore only expectation vaules contribute, where one creation and one annhiliation operator are dedicated by the same fermion species, a or b. Otherwise, the expectation value is zero. Equation (A.12) is used to express the obtained fermionic expectation values as (A.10). For using the definition of the Green function, the operaters are sorted, that all annihilaition operators are on the left side of the creation operatrs. A δ-distribution is produced each time commutating a creation and an annihilation operator. The obtained expression for $\mathcal{D}$ in second order pertubation theory is given by

$$\mathcal{D}_\mu^{(2)}(\mathbf{k},\omega) = (-i)^3 \lambda^2 \int\limits_{-\infty}^\infty \mathrm{d}t_1\,\mathrm{d}t_2 \int_{\mathbf{p}_1}\int_{\mathbf{p}_2}$$

$$\times \left\langle \mathcal{T}_t \Phi_{\lambda'}(\mathbf{p}_2 - \mathbf{p}_1, t_2)\Phi_\lambda(\mathbf{p}_1 - \mathbf{p}_2, t_1)\Phi_\mu(\tilde{\mathbf{k}}, t)\Phi_\mu(-\tilde{\mathbf{k}}, t') \right\rangle_0$$

$$\times \left(\left\langle \mathcal{T}_t \Psi_\mathrm{a}(\mathbf{p}_2, t_1)\Psi_\mathrm{a}^\dagger(\mathbf{p}_2, t_2) \right\rangle_0 \left\langle \mathcal{T}_t \Psi_\mathrm{b}(\mathbf{p}_1, t_2)\Psi_\mathrm{b}^\dagger(\mathbf{p}_1, t_1) \right\rangle_0 \right.$$

$$\left. + \left\langle \mathcal{T}_t \Psi_\mathrm{b}(\mathbf{p}_2, t_1)\Psi_\mathrm{b}^\dagger(\mathbf{p}_2, t_2) \right\rangle_0 \left\langle \mathcal{T}_t \Psi_\mathrm{a}(\mathbf{p}_1, t_2)\Psi_\mathrm{a}^\dagger(\mathbf{p}_1, t_1) \right\rangle_0 \right). \tag{A.13}$$

For reasons of clarity, the momentum vectors are abbreviated as $\tilde{\mathbf{k}} = \mathbf{k} + \mathbf{G}$. Wick's theorem is now Utilizied for the bosonic expectation value, which yields two possible contractions. All these obtained diargams are of the same structure. These diagrams can be transfer to each other by interchange the acting point of the bosonic propagators or by interchange the fermionic propagators Thereforem all diagrams yield the same contribution and it is enough to compute one of them. This diagram is depicted in figure A.1 and is called bubble diagram. The bubble diagram is the lowest order of the full polarization operator. In the following this diagram is computed in detail. The first order of the polarization bubble is given by

$$\Pi_\mu^{(0)}(\mathbf{k},\omega) = i \sum_{\mathbf{P}} \int\limits_{|p|\leq p_\mathrm{F}} \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \int\limits_{-\infty}^\infty \frac{\mathrm{d}\epsilon}{2\pi} \, \mathcal{G}_\mathrm{a}^{(0)}\left(\mathbf{p} + \frac{\mathbf{k}}{2}, \epsilon + \frac{\omega}{2}\right)\mathcal{G}_\mathrm{b}^{(0)}\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \epsilon - \frac{\omega}{2}\right). \tag{A.14}$$

Here the outer momentum and frequency is shfited to symmetrizie the argument of both fermionic Green functions. The Green function is given by

$$\mathcal{G}_\alpha^{(0)}(\mathbf{k},\omega) := \left\langle\left\langle \Psi_\alpha; \Psi_\alpha^\dagger \right\rangle\right\rangle_\omega = \sum_{\mathbf{G}} \frac{1}{\omega - \epsilon_\alpha(\mathbf{k} + \mathbf{G})}, \tag{A.15}$$

where the species of the corresponding fermions is indicated with the index $\alpha = \mathrm{a, b}$. First of all, the dispersion relation of the fermions is investigated. Spin fluctuations
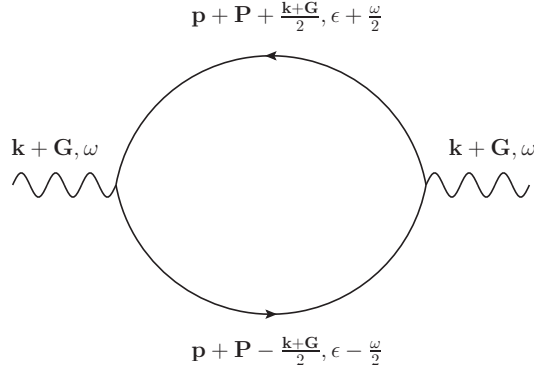
Figure A.1: caption

interact only with fermions near the Fermi surface. Under this condition, the dispersion relation of the fermions of species a is expanded.

$$
\epsilon_\mathrm{a}(\mathbf{p} + \frac{\mathbf{k} + \mathbf{G}}{2}) = \frac{\left(p_x + \frac{k_x + G_x}{2}\right)^2}{2m_1} + \frac{\left(p_y + \frac{k_y + G_y}{2}\right)^2}{2m_2}
$$

$$
\Leftrightarrow \epsilon_\mathrm{a}(\mathbf{p} + \frac{\mathbf{k} + \mathbf{G}}{2}) = \frac{p_x^2}{2m_1} + \frac{1}{2}\frac{p_x}{m_1}(k_x + G_x) + \frac{(k_x + G_x)^2}{8m_1}
$$

$$
+ \frac{p_y^2}{2m_2} + \frac{1}{2}\frac{p_y}{m_2}(k_y + G_y) + \frac{(k_y + G_y)^2}{8m_2}
$$

$$
\Leftrightarrow \epsilon_\mathrm{a}(\mathbf{p} + \frac{\mathbf{k} + \mathbf{G}}{2}) \approx \xi_\mathrm{a} + \frac{1}{2}\mathbf{v}_{\mathrm{a,F}}(\mathbf{k} + \mathbf{G}) + \mu \tag{A.16}
$$

where the quadratic term with respect to $\mathbf{k}$ is neglected and ξ_a is the dispersion denoted in equation (3.9). Moreover, the velocity $\mathbf{v}_a$ of the fermion of species a is introduced. Since only fermions near the Fermi surface are considered the velocity is assumed to be the Fermi velocity. The same procudure is valid for the fermions of species b. Finally, the normalized momentum vector $\mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}$ is introduced. The Fermi velocity of fermions of species a is then given by $\mathbf{v}_{\mathrm{a,F}} = v_{\mathrm{a,F}}\mathbf{n}$ and equally for b. The scalar product between the normalized momentum vector $\mathbf{n}$ and the bosonic spin density wave vector $\mathbf{k} + \mathbf{G}$ is rewriten as the magnitude multiplied with $\cos(\vartheta)$, where ϑ is the angle between both.

In the investigated model, the points of interaction between the fermions caused by spin fluctuations are hot-spots (see chapter 3). This are certain points on the Fermi surface seperated by the momentum vector $\mathbf{Q}$. The energy and the magnitudes of the Fermi velocities are equal on these points connected wby $\mathbf{Q}$.

$$
\xi := \xi_\mathrm{a} = \xi_\mathrm{b} \qquad v_\mathrm{F} := v_{\mathrm{a,F}} = v_{\mathrm{b,F}} \tag{A.17}
$$

The direction of the velocities have to be unequal Otherwise, the angle ϑ is 0 or π and the imaginary part of Π_μ is zero. Using these conversions and assumptions, the

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polarization operator is given by

$$\Pi_\mu^{(0)}(\mathbf{k}, \omega) = i\nu_{\mathrm{F}} \int\limits_0^\pi \mathrm{d}\vartheta \int\limits_{\xi \leq \xi_{\mathrm{F}}} \mathrm{d}\xi \int\limits_{-\infty}^\infty \frac{\mathrm{d}\epsilon}{2\pi}$$

$$\times \frac{1}{\epsilon + \frac{\omega}{2} - \xi - \frac{1}{2}v_{\mathrm{F}}|\mathbf{k} + \mathbf{G}|\cos(\vartheta) + i\eta\,\mathrm{sign}(\epsilon + \frac{\omega}{2})}$$

$$\times \frac{1}{\epsilon - \frac{\omega}{2} - \xi + \frac{1}{2}v_{\mathrm{F}}|\mathbf{k} + \mathbf{G}|\cos(\vartheta) + i\eta\,\mathrm{sign}(\epsilon - \frac{\omega}{2})}. \tag{A.18}$$

Here, the two dimensional momentum integral is firstly transformed in plane polar coordinates and then the k-integral is transformed into an energy integral over the density of states. The density of states is approximated with the value of the density of states at the Fermi surface, $\nu_{\mathrm{F}} := \nu(\xi_{\mathrm{F}})$, since only fermions near the Fermi surface are considered. The energy integral is limited by the Fermi energy ξ_{F}.

The further treatment is started by computing the frequency integral. The integral over ϵ is transformed into a complex contour integral, where the contour Γ is chosen in two different ways. According to the singularities of the integrand the countour is closed in the upper or lower half plane using a semicircle with radius infinity. In both cases the contribution of the semicircle is zero, since the integrand is proportional to $1/\epsilon$. The equality between the integral along the real axis and the complex contour integral is ensured due to the non-contributing of the semicircle. The investigated integrand has two singularities at

$$\epsilon_1 := \xi - \frac{1}{2}\big(\omega - v_{\mathrm{F}}|\mathbf{k} + \mathbf{G}|\cos(\vartheta)\big) \quad \text{and} \quad \epsilon_2 := \xi + \frac{1}{2}\big(\omega - v_{\mathrm{F}}|\mathbf{k} + \mathbf{G}|\cos(\vartheta)\big), \tag{A.19}$$

where both singularities are of first order. The poles are located in the upper or lower plane, according to the signum function in the denominator. In total four different constitutions are possible. On the one hand, both singularities can be located in the lower or upper complex plane, which yield in both cases zero. On the other hand, one pole is sited in the upper plane, while the other one is sited in the lower plane, or vice versa, where both yield the same contribution

1. case: $\mathrm{sign}(\epsilon + \frac{\omega}{2}) = \mathrm{sign}(\epsilon - \frac{\omega}{2})$

Both singularities are located in the upper or in the lower complex half plane. The contour is therefore closed in the upper or lower plane, respectivily, enclosing both poles . In the first case the winding number is 1, since the pole is enclosed counterclockwise. Accordingly, the winding number is -1 in the second case.

$$\mathrm{I}_\omega^{\mathrm{e}} := i \oint_\Gamma \frac{\mathrm{d}\omega}{2\pi i} \frac{1}{\omega - \omega_1 \mp i\eta} \cdot \frac{1}{\omega - \omega_2 \mp i\eta}$$

$$\Leftrightarrow \mathrm{I}_\omega^{\mathrm{e}} = \pm i\left[\frac{1}{\omega_2 - \omega_1} + \frac{1}{\omega_1 - \omega_2}\right] = 0, \tag{A.20}$$

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where the index "e" stands for even and means, that both poles are located in the same half plane.

2. case: $\mathrm{sign}(\epsilon + \frac{\omega}{2}) \neq \mathrm{sign}(\epsilon - \frac{\omega}{2})$

The singularities are sited in different complex half planes, one in the upper and accordingly the other in the lower half plane. It is arbitrary in which half plane the contour is closed as long as one pole is enclosed. In the following computation the contour is closed in the upper half plane.

$$\mathrm{I}_\omega^{\mathrm{o}} := i \oint_\Gamma \frac{\mathrm{d}\omega}{2\pi i} \frac{1}{\omega - \omega_1 \mp i\eta} \cdot \frac{1}{\omega - \omega_2 \pm i\eta}$$
$$\Leftrightarrow \mathrm{I}_\omega^{\mathrm{o}} = \frac{\pm i}{\omega_2 - \omega_1 \pm i\eta} \tag{A.21}$$

where the index "o" stands for odd and mean, that both poles are located in opposite half planes.

After inserting both expressions for the singularities ω_1 and ω_2, the intergand is independent with respect to ξ The signum functions in (A.18) is equivalent to the signum functions $\mathrm{sign}(\xi \pm \frac{1}{2} v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta))$ in energy representation. The energy ξ is dropped, since the integrand is independent of ξ, and the constant positive factors are neglectable. Then, the signum function is expressed as $\mathrm{sign}[|\mathbf{k} + \mathbf{G}| \cos(\vartheta))$. Finally, the integrals limits are set to $\pm \frac{1}{2} v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta)$, which follows directly from the localization of both poles. ω_1 is assumed to be in the lower half plane, while ω_2 is then assumed in the opposite one. This implies that $\epsilon + \frac{\omega}{2} > 0$ and $\epsilon - \frac{\omega}{2} < 0$, respectivily. Transforming into energy representation the obtained expression yield the definition interval of ξ. The polarization operator is therefore given by

$$\Pi_\mu^{(0)}(\mathbf{k}, \omega) = i\nu_{\mathrm{F}} \int_0^\pi \mathrm{d}\vartheta \int_{-\xi_0}^{\xi_0} \mathrm{d}\xi \frac{i \, \mathrm{sign}(|\mathbf{k} + \mathbf{G}| \cos(\vartheta))}{\omega - v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta) + i\eta \, \mathrm{sign}(|\mathbf{k} + \mathbf{G}| \cos(\vartheta))}, \tag{A.22}$$

where the abbreviation $\xi_0 = v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta)/2$ is introduced. The integration over ξ yields the factor $v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta)$. Subsequently the signum function is reexpressed in frequency representation corresponding to $\mathrm{sign}(\omega)$. Since our interest is attracted to damping of the spin fluctuations, the imaginary part of the polarization operator is taken:

$$\mathrm{Im}\left\{\Pi_\mu^{(0)}(\mathbf{k}, \omega)\right\} = \nu_{\mathrm{F}} \pi \int_0^\pi \mathrm{d}\vartheta \, v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta) \delta(\omega - v_{\mathrm{F}} |\mathbf{k} + \mathbf{G}| \cos(\vartheta)). \tag{A.23}$$

Here the formula $\frac{1}{x \pm i\eta} = \mathrm{P.V.} \frac{1}{x} \mp i\pi\delta(x)$ is used. The obtained δ-distribution has a function $g(\vartheta)$ as argument. Using formula

$$\delta(g(x)) = \sum_{i=1} \frac{\delta(x - x_{0,i})}{|g'(x_{0,i})|}, \tag{A.24}$$

the δ-distribution is rewriten as a sum over δ-distributions with the zeros of $g(\theta)$ as argument. In the above formula, the sum runs over all zeros of $g(x)$, which are denoted as $x_{0,i}$. The derivative with respect to x is labeled with a prime at g and is evaluated at the corresponding zero $x_{0,i}$.

In our observed case, the argument of $\delta(g(\vartheta))$ contains only one zero at $\vartheta_0 = \cos^{-1}(\omega/(v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|))$ inside the integral boundaries. The δ-distribution is rewriten as:

$$\delta(\omega - v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|\cos(\vartheta)) = \Big(v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|\cdot|\sin(\vartheta_0)|\Big)^{-1}\delta(\vartheta - \vartheta_0)$$

$$\Rightarrow \delta(\omega - v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|\cos(\vartheta)) \approx \Big(v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|\Big)^{-1}\delta(\vartheta - \vartheta_0). \tag{A.25}$$

In the second line the limit of small frequencies ω is taken, which is valid in our observed low-energy theory in the spin fermion model. The integration over ϑ is now performed and the imaginary part of the polarization operator is given by

$$\mathrm{Im}\Big\{\Pi_\mu^{(0)}(\mathbf{k},\omega)\Big\} = \frac{\nu_{\mathrm{F}}\pi}{v_{\mathrm{F}}|\mathbf{k}+\mathbf{G}|}\cdot\omega = \gamma\omega, \tag{A.26}$$

where the damping parameter γ is introduced. As above demonstrated, the damped spin density propagator is determined by the Dyson equation (A.9). The polarization operator is reperated into real and imaginary part. Inserting the free spin density propagator and the obtained result for $\mathrm{Im}\{\Pi_\mu\}$, the damped spin density popagator is obtained to

$$\mathcal{D}_\mu(\mathbf{k},\omega) = \sum_{\mathbf{G}} \frac{1}{(\mathbf{k}+\mathbf{G})^2 + r - (\omega/v_{\mathrm{S}})^{-2} - i\gamma\omega} \tag{A.27}$$

where the abbreviation $r = r_0 - \mathrm{Re}\{\Pi_\mu(\mathbf{k},\omega)\}$ is used. In the vicinity of the quantum critical point the real part of Π_μ is canceled by r_0, since the correlations length is divergent and $r = \xi^{-2}$ In low-energy theory, the ω^2-term is further neglectable.

A.3 Transformation into Matsubara Frequency Space

The spin density wave propagator, obtained in the previous section, is a complex function of the real quantity ω. In the following, the quantity ω is transformed into an imaginary quantity $i\omega_n$, known as Matsubara frequancies. The real and imaginary part of the propagator $\mathcal{D}_\mu(\mathbf{k},i\omega_n)$ with respect to Matsubara frequencies is evaluated using the Kramers-Kronig relations [Sch08]. The propagator in (A.27) is seperated into real and imaginary part, extending with the complex conjugated denominater.

$$\mathcal{D}_\mu(\mathbf{k},\omega) = \sum_{\mathbf{G}} \left[\frac{(\mathbf{k}+\mathbf{G})^2}{(\mathbf{k}+\mathbf{G})^4 + (\gamma\omega)^2} + i\frac{\gamma\omega}{(\mathbf{k}+\mathbf{G})^4 + (\gamma\omega)^2}\right], \tag{A.28}$$

where r and ξ is set to zero. For the following treatment it is usefull to investigate the symmetries of the real and imaginary part. The real part is a symmetric function and

the imaginary part is an antiysmmetric funtion, with respect to ω. The imarginary part of $\mathcal{D}_\mu(\mathbf{k}, i\omega_n)$ is evaluated to zero, since its defined as the integral over the real part of $\mathcal{D}_\mu(\mathbf{k}, \omega)$ divided by ω. The integrand is therefore an odd function and the integral over all frequencies ω is zero. In Matsubara frequency space, the damped propagator is given by its real part.

$$\mathcal{D}_\mu(\mathbf{k}, i\omega_n) = \text{Re}\{\mathcal{D}_\mu(\mathbf{k}, i\omega_n)\} = \frac{1}{\pi} \, \text{P.V.} \int\limits_{-\infty}^{\infty} d\omega \, \frac{\text{Im}\{\mathcal{D}_\mu(\mathbf{k}, \omega)\}}{\omega - i\omega_n} \tag{A.29}$$

where P.V. represents that the integrals is evaluated using the principal value. The integral is speperated into two parts. In the first one ω is substituted by $-\omega$ and the antismmetry of $\text{Im}\{\mathcal{D}_\mu(\mathbf{k}, \omega)\}$ is utilizied.

$$\mathcal{D}_\mu(\mathbf{k}, i\omega_n) = \frac{1}{\pi} \int\limits_{0}^{\infty} d\omega \, \text{Im}\{\mathcal{D}_\mu(\mathbf{k}, \omega)\} \left[\frac{1}{\omega + i\omega_n} + \frac{1}{\omega - i\omega_n} \right]$$

$$\Leftrightarrow \mathcal{D}_\mu(\mathbf{k}, i\omega_n) = \frac{2}{\pi} \sum_{\mathbf{G}} \int\limits_{0}^{\infty} d\omega \, \frac{\gamma\omega}{(\mathbf{k} + \mathbf{G})^4 + (\gamma\omega)^2} \cdot \frac{\omega}{\omega^2 + \omega_n^2}$$

$$\Leftrightarrow \mathcal{D}_\mu(\mathbf{k}, i\omega_n) = \sum_{\mathbf{G}} \frac{1}{(\mathbf{k} + \mathbf{G})^2 + \gamma|\omega_n|} \tag{A.30}$$

In the last step the integral formula

$$\int\limits_{0}^{\infty} dx \, \frac{x}{a^2 + x^2} \cdot \frac{x}{y^2 + x^2} = \frac{\pi}{2} \frac{1}{a + |y|} \tag{A.31}$$

is used, which can be shown by transforming into a complex contour integral.

B Commutator relation

The time derivative of the momentum and current operator are designated for the case of non-pertubation and pertubation in chapter 3. In this appendix, the explicit computation is demonstrated of them. Our starting point is the Lagrange density $\mathcal{L} = \mathcal{L}_\Psi + \mathcal{L}_\Phi + \mathcal{L}_{\Psi\Phi}$, as suggested in [PS14]. In Matsubara time $\tau = it$, the single Lagrange densities are given by

$$\mathcal{L}_\Psi = \boldsymbol{\Psi}^\dagger(\mathbf{x}, \tau) \begin{pmatrix} \sigma_0(\partial_\tau - \mu_0 + \xi_a) & 0 \\ 0 & \sigma_0(\partial_\tau - \mu_0 + \xi_b) \end{pmatrix} \boldsymbol{\Psi}(\mathbf{x}, \tau) \tag{B.1}$$

$$\mathcal{L}_\Phi = \frac{1}{2}\big(\partial_i \Phi_\mu(\mathbf{x}, \tau)\big)\big(\partial_i \Phi_\mu(\mathbf{x}, \tau)\big) + \frac{\epsilon}{2}\big(\partial_\tau \Phi_\mu(\mathbf{x}, \tau)\big)\big(\partial_\tau \Phi_\mu(\mathbf{x}, \tau)\big)$$
$$+ \frac{u}{6}\big(\Phi_\mu(\mathbf{x}, \tau)\Phi_\mu(\mathbf{x}, \tau) + \frac{3}{g}\big)^2 \tag{B.2}$$

$$\mathcal{L}_{\Psi\Phi} = \lambda \Phi_\mu(\mathbf{x}, \tau)\Big[\Psi_\mathrm{a}^\dagger(\mathbf{x}, \tau)\sigma_\mu \Psi_\mathrm{b}(\mathbf{x}, \tau) + \Psi_\mathrm{b}^\dagger(\mathbf{x}, \tau)\sigma_\mu \Psi_\mathrm{a}(\mathbf{x}, \tau)\Big] \tag{B.3}$$

Here $\boldsymbol{\Psi} = (\Psi_\mathrm{a}, \Psi_\mathrm{b})$ and Ψ_a, Ψ_b are two-component spinors and Φ_μ is a SO(3) vector boson order parameter. The chemical potential is denoted with μ_0 and the Pauli matrix are labeled with σ_0 and σ_μ. ϵ is the squared inverse spin velocity v_S^{-2}. In the Lagrangian $\mathcal{L}_\Phi$, the last term is neglectable in the low-energy theory of the spin fermion model and is therefore dropped in the following. The anisotropic parabolical dispersion relations ξ_a and ξ_b are given by

$$\xi_\mathrm{a} = -\frac{\partial_x^2}{2m_1} - \frac{\partial_y^2}{2m_2} \qquad \xi_\mathrm{b} = -\frac{\partial_x^2}{2m_2} - \frac{\partial_y^2}{2m_1} \tag{B.4}$$

The Lagrangian for umklapp scattering is assumed to be

$$\mathcal{L}_\mathrm{umklapp} = -\mathrm{J}(\mathbf{R})\Phi_\mu(\mathbf{x}, \tau)\Phi_\mu(\mathbf{x}, \tau) \tag{B.5}$$

where $\mathrm{J}(\mathbf{R})$ is a coupling parameter and $\mathbf{R}$ is a lattice vector. The following treatment is now to compute the Hamiltonian, momentum and current, using the energy-momentum-tensor

$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\sum_{\{\zeta_n\}}\left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \zeta_n)}\right)(\partial_\nu \zeta_n) + (\partial_\nu \zeta_n)^\dagger\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \zeta_n)}\right)^\dagger\right] \tag{B.6}$$

and the formula for the current density

$$\mathcal{J}_\mu = -i \sum_{\{\zeta_n\}} \epsilon(\zeta_n) \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \zeta_n)} \right) \zeta_n - \zeta_n^\dagger \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \zeta_n)} \right)^\dagger \right], \tag{B.7}$$

where $\{\zeta_n\} = \{\Psi_\text{a}, \Psi_\text{b}, \Phi_\mu\}$ and $\epsilon(\zeta_n)$ is 0 in the case of real fields and 1 in the case of complex fields. The energy and the j-component of the momentum is represented by $\mathcal{T}_{00}$ and $\mathcal{T}_{0j}$ of the energy-momentum-tensor, respectivily. The j-component of the current is represented by $\mathcal{J}_j$. In the spin-fermion-model, the bosonic field operator is considered as real, so that $\Phi_\mu^\dagger = \Phi_\mu$.

C Conversion of $\left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right)$

In this appendix the short conversion of the expectation value $\left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right)$ is done. Therefore only realtions are used which we find in chapter **??**. Firstly the expectation value is writen as the correlation function in the frequency space using equation (5.53).

$$\left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -i\mathcal{C}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\omega)$$

$$\overset{\text{(F.6)}}{\Leftrightarrow} \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -i\int\limits_0^\infty \mathrm{d}t\, e^{i\omega t}\mathcal{C}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(t)$$

$$\overset{\text{(5.50)}}{\Leftrightarrow} \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -\frac{i}{\beta}\int\limits_0^\infty \mathrm{d}t\, e^{i\omega t}\int\limits_0^\beta \mathrm{d}\lambda \left\langle \dot{\mathrm{P}}^\dagger(t)\dot{\mathrm{P}}(0)\right\rangle$$

$$\overset{\text{(5.25)}}{\Leftrightarrow} \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -\frac{i}{\beta}\Phi_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\omega)$$

$$\overset{\text{(5.20)}}{\Leftrightarrow} \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -\frac{\omega^{-1}}{\beta}\Big[\chi_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\omega) - \underbrace{\chi_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\omega = 0)}_{=0}\Big]$$

$$\Leftrightarrow \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -\frac{\omega^{-1}}{\beta}\int\limits_{-\infty}^\infty \mathrm{d}t\, e^{i\omega t}\chi_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(t)$$

$$\overset{\text{(??)}}{\Leftrightarrow} \left(\dot{\mathrm{P}}\middle|(\omega - \mathrm{L}_0)^{-1}\middle|\dot{\mathrm{P}}\right) = -\frac{i\omega^{-1}}{\hbar\beta}\int\limits_0^\infty \mathrm{d}t\, e^{i\omega t}\left\langle\Big[\dot{\mathrm{P}}(t),\dot{\mathrm{P}}(0)\Big]\right\rangle_0 \qquad \text{(C.1)}$$

In line 5 the susceptibility at frequency $\omega = 0$ is set to zero, because at $\omega = 0$ the susceptibility corresponds to the Kubo relaxation function at $t = 0$. This function is zero because $\dot{\mathrm{P}} = 0$ at $t = 0$.

D Computation of the Static Susceptibility

In this appendix, the static susceptibility $\chi_{\mathrm{JP}}(\omega = 0)$ is explicitely calculated. The temperature dependence of χ_{JP} is expected to be non-existent, since momentum and current are not explicitly time dependent. Using equation (**??**), the static susceptibility is directly proportional to the Kubo relaxation function (5.3) at $t = 0$.

$$\chi_{\mathrm{PJ}}(\omega = 0) = \Phi_{\mathrm{PJ}}(t = 0) = i \int\limits_0^\infty \mathrm{d}t' \left\langle \left[\mathrm{P}_j(t'), \mathrm{J}_j(0) \right] \right\rangle \tag{D.1}$$

In the formula above, the limit $s \to 0$ is dropped, since the integral is assumed to be convegent. The spatial direction of P and J is signified by the index j. The integral is transformed into Matsubara time $\tau = it$, firstly. The Jacobi determinate is $-i$ and the integral's limits is set to 0 and β. Observing only the zeroth order in pertubation theory the static susceptibility is given by

$$\chi_{\mathrm{PJ}}(\omega = 0) = \int\limits_0^\beta \mathrm{d}\tau \left\langle \mathcal{T}_\tau \mathrm{P}_j(\tau) \mathrm{J}_j(0) \right\rangle_0. \tag{D.2}$$

The momentum and current operators are named in equation (3.10) and (3.11), respetivily. Inserting them, expectation values are generated including four operators. Only if these operators are of the same species, Φ_μ, Ψ_{a} or Ψ_{b}, the expectation values yield connected diagrams. This happens two times in the case of fermionic operators. Wick's theorem is used to seperate the remaining two expectation values in terms of free Green functions. The contraction of operators with different time argument is generated the two bubble diagrams, depicted in figure D.1 The static susceptibility is given by

$$\chi_{\mathrm{PJ}}(\omega = 0) = \int\limits_0^\beta \mathrm{d}\tau \int_{\mathbf{k}} k_j^2 \left[\frac{1}{m_1} \mathcal{G}_{\mathrm{a}}^{(0)}(\mathbf{k}, -\tau) \mathcal{G}_{\mathrm{a}}^{(0)}(\mathbf{k}, \tau) + \frac{1}{m_2} \mathcal{G}_{\mathrm{b}}^{(0)}(\mathbf{k}, -\tau) \mathcal{G}_{\mathrm{b}}^{(0)}(\mathbf{k}, \tau) \right]. \tag{D.3}$$

Here the free fermionic propagator $\mathcal{G}_\alpha^{(0)}(\mathbf{k}, \tau) = - \left\langle \mathcal{T}_\tau \Psi_\alpha(\mathbf{k}, \tau) \Psi_\alpha^\dagger(\mathbf{k}, 0) \right\rangle$ with $\alpha \in \{\mathrm{a, b}\}$ is inserted. Both fermionic Green functions are transformed into the Matsubara frequency space. The only τ-dependence is thus given by the exponential functions. The τ-integral is evaluated, using the definition of the δ-distribution in Matsubara space,

$$\chi_{\text{PJ}}(\omega = 0) =$$

Figure D.1: caption

$\int_0^\beta d\tau \exp(i(\omega_m - \omega_n)\tau) = \beta\delta(\omega_m - \omega_n)$. One of the two sums over Matsubara frequencies is easily performed due to this δ-distribution. The obtained expression is given by

$$\chi_{\text{PJ}}(\omega = 0) = \frac{1}{\hbar} \int_{\mathbf{k}} k_j^2 \left[\frac{1}{m_1} S_{\text{a}}(\omega_n) + \frac{1}{m_2} S_{\text{b}}(\omega_n) \right]. \tag{D.4}$$

The Matsubara sum $S_\alpha(\omega_n) = \beta^{-1} \sum_{\omega_n} \mathcal{G}_\alpha^{(0)}(\mathbf{k}, \omega_n) \mathcal{G}_\alpha^{(0)}(\mathbf{k}, \omega_n)$, where $\alpha = \text{a, b}$, is defined. This sum is transformed into a complex contour integral, offering the use of the usual complex analysis. The product of Green functions multiplied with the Fermi distribution are generated the integrand of this contour integral. The full transformation rule is given by

$$S = \frac{1}{\beta} \sum_{\omega_n} \mathcal{G}(\omega_n) = -\frac{1}{2\pi i} \oint_\Gamma dz\, n_{\text{F}}(z) \mathcal{G}(z). \tag{D.5}$$

The choice of the contour is arbitrary beside one condiction. All singularies of the Green functions are excluded in the contour, while the infinite poles $\omega_n = (2n + 1)\pi/\beta$, where $n \in \mathbb{Z}$, of the Fermi distribution are included in the contour. Our starting contour Γ is depicted on the left hand side in figure D.2.
The contour is generated by to path along the imaginary axis, from $i\infty$ to $-i\infty$ and vice versa. Both paths are connected in plus and minus infinity and shifted infinitesimal aside the imaginary axis. This contour is now expaned to a circle with radius infinity, where always the singularities of the Green function is excluded. The obtained contour is depicted on the right hand side in figure D.2. The same magnitude with opposite sign is generated by the paths from infinity to the poles and vice versa. Therefore, the contour is seperated into small circles around the poles of the Green functions and into one circle with infinite radius. The contribution of the latter is zero, since the Green function is proportional to $1/z$. In the remaining contour integrals the product of Green functions is now inserted, where the single free fermionic Green function is given by

$$\mathcal{G}_\alpha(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_\alpha(\mathbf{k})}. \tag{D.6}$$

$\epsilon_\alpha(\mathbf{k})$, where $\alpha = \text{a, b}$, is the fermion's dispersion and also the singularity of the Green function. The explicite expression is denoted in equation (3.9). Besides this singularity,
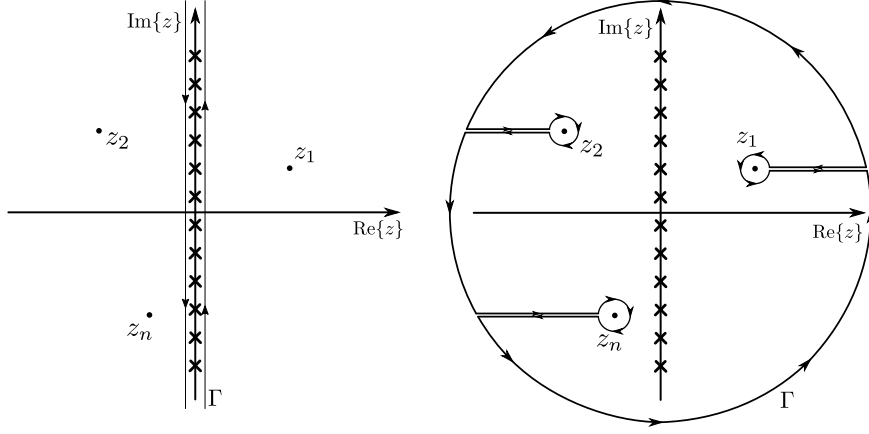
Figure D.2: caption

the Green function is holomoph in the whole complex plane. The contour integral is evaluated using the resiuum theorem, which yields

$$\chi_{\mathrm{PJ}}(\omega = 0) = -\int_{\mathbf{k}} k_j^2 \left[\frac{1}{m_1} \frac{\mathrm{d}n_{\mathrm{F}}(\epsilon_{\mathrm{a}}(\mathbf{k}))}{\mathrm{d}\epsilon_{\mathrm{a}}(\mathbf{k})} + \frac{1}{m_2} \frac{\mathrm{d}n_{\mathrm{F}}(\epsilon_{\mathrm{b}}(\mathbf{k}))}{\mathrm{d}\epsilon_{\mathrm{b}}(\mathbf{k})} \right] \tag{D.7}$$

The derivatives of the distribution function with respect to the dispersion relation are generated, since the integrand has a singularity of second order at $z_0 = \epsilon_\alpha(\mathbf{k})$. These two obtained integrals are exactly solvable. Therefore the inetegrals are transformed into plane polar coordinates, where the transformation rule is given by $(k_x, k_y) = (q\sqrt{2m_{1,2}}\cos(\phi), q\sqrt{2m_{2,1}}\sin(\phi))$. Two typs of transformation are used, since the dispersion relation of the two species of fermions differs in the contained mass terms. The only angular dependece is originated by the k_j^2-term. This term yields $\cos^2(\phi)$ or $\sin^2(\phi)$ for the x- or y-direction, respectivily. Since the limits of the ϕ-integral are 0 and 2π, the contribution is π in both cases. The upper limit of the q-integral is set to infity, since the integrand is decreasing fast to zero for large values of q.

$$\chi_{\mathrm{PJ}}(\omega = 0) = \frac{8\beta\pi}{(2\pi)^2} \sqrt{m_1 m_2} \int_0^\infty \mathrm{d}q \, q^3 \frac{e^{\beta(q^2 - \mu)}}{(e^{\beta(q^2 - \mu)} + 1)^2} \tag{D.8}$$

The obtained integral is solved, substituting $x = \beta(q^2 - \mu)$. The first of the two obtained integrals is evaluated using integration by parts, while the derivative of the Fermi distribution with respect to x is originated in the second integral. Performing these integrals the static susceptibility of J and P is given by

$$\chi_{\mathrm{PJ}}(\omega = 0) = \frac{\sqrt{m_1 m_2}}{\pi\beta} \ln\left(e^{\beta\mu} + 1\right) \tag{D.9}$$

In the case of $\mu \gg k_{\mathrm{B}}T$, the exponential function is large compared to 1 and neglectable. The static susceptibility is then temperature independent in the leading

order and given by

$$\chi_{\text{PJ}}(\omega = 0) \to \frac{\mu\sqrt{m_1 m_2}}{\pi} \qquad (T \to 0) \qquad\qquad\qquad \text{(D.10)}$$

E Finite Conductivity because of Breaking the Translation Symmetry via Umklapp Scattering

The computation of the Green function $\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z)$ in (4.1) is displayed in detail in the following appendix. The Green function is given by equation (4.2). For using the diagrammatic technique the integral is transformed into Matsubara time $\tau = it$. The magnitude of the Jacobi determinate is $-i$ and the upper limit is changed to $\beta = T^{-1}$. Each time derivative is further generated an imaginary unit i. The Green function is represented in Matsubara time as

$$\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z) = i \int\limits_0^\beta \mathrm{d}\tau\, e^{z\tau} \left\langle \mathcal{T}_\tau \dot{\mathrm{P}}_j(\tau) \dot{\mathrm{P}}_j(0) \right\rangle_{\mathrm{H}}. \tag{E.1}$$

z is a complex frequency and the direction of the momentum is marked by the index $j = x, y$. The index H denotes, that the expectation value is evaluated with respect to Hamiltonian $\mathrm{H} = \mathrm{H}_\Psi + \mathrm{H}_\Phi + mt H_{\Psi\Phi}$, where the interaction between fermions and spin fluctuations is considered by the last term. In Matsubara interaction representation, the appearing time evoluation operator is expanded up the the zeroth order. As a consequance, corrections of the Green function, caused by the fermion-spin fluctuation interaction, are neglected. Only the two time derivatives of momentum are therefore contained in the expectation value. This is the only origin, where the pertubation of umklapp scattering is considered. The Green function is given by

$$\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z) = i \int\limits_0^\beta \mathrm{d}\tau\, e^{z\tau} \left\langle \mathcal{T}_\tau \dot{\mathrm{P}}_j(\tau) \dot{\mathrm{P}}_j(0) \right\rangle_{\mathrm{H}_0} \tag{E.2}$$

where the expectation value is now evaluated with respect to the free Hamiltonian $\mathrm{H}_0 = \mathrm{H}_\Psi + \mathrm{H}_\Phi$. The time derivative of momentum is designated in equation (3.15). Inserting this expression, $\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z)$ is given by

$$\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z) = -i \sum_{\mu,\lambda} \sum_{\mathbf{G}_1,\mathbf{G}_2} \mathrm{J}_{\mathbf{G}_1} \mathrm{J}_{\mathbf{G}_2} \int\limits_0^\beta \mathrm{d}\tau\, e^{z\tau} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} G_{1,j} G_{2,j}$$
$$\times \left\langle \mathcal{T}_\tau \Phi_\mu(\mathbf{k}_1, \tau) \Phi_\mu(-\mathbf{k}_1 - \mathbf{G}_1, \tau) \Phi_\lambda(\mathbf{k}_2, 0) \Phi_\lambda(-\mathbf{k}_2 - \mathbf{G}_2, 0) \right\rangle_{\mathrm{H}_0}. \tag{E.3}$$

$$\mathcal{G}_{jj}(i\omega_l) =$$

$$\mathbf{k}, i\omega_n$$

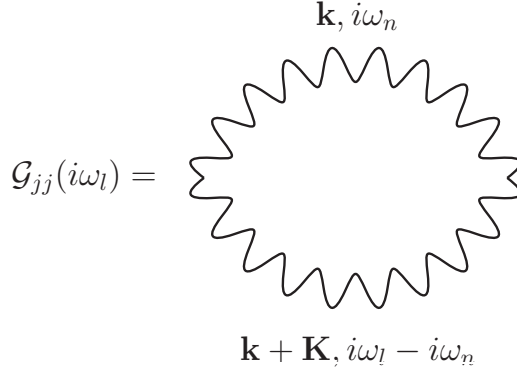$$\mathbf{k} + \mathbf{K}, i\omega_l - i\omega_n$$

Figure E.1: caption

Here, the moment integrals are extended over the first Brillouin zone, while the sum over $\mathbf{G}_1$ and $\mathbf{G}_2$ runs over all reciprocal lattice vectors. $\mathbf{G}_i$, where $i = 1, 2$, is the coupling parameter with respect to the reciprocal lattice vector. The sum over μ and λ runs over the spatial direction of the three component bosonic field, while the direction of P is indicated by j.

Wick's theorem is used to seperate the expectation value in terms of free propagators. Three contraction are possible for a an expectation value, including four bosonic, real field operators. One of them is not contributed, since the operators to equal times are contracted. The remaining two diagrams are both bubble diagrams and topological equivalent. The diagram is depicted in figure E.1.

Acting Wick's theorem in momentum representation, each interchange of the bosonic operators yields a δ-distribution, or more presicely, the commutator relation. This signifies the momentum conservation at each vertex. One momentum integral one sum over reciprocal lattice vector and the sum over λ are therefore evaluated directly. The remaining vectors are renamed into $\mathbf{k}$ and $\mathbf{G}$ for the momentum and reciprocal lattice vector, respectivily.

$$\mathcal{G}_{\dot{\mathrm{P}}\dot{\mathrm{P}}}(\mathbf{k}, z) = -2 \sum_{\mathbf{G}} |\mathrm{J}_{\mathbf{G}}|^2 \int\limits_0^\beta \mathrm{d}\tau\, e^{z\tau} \int_{\mathbf{k}} G_j^2$$

$$\times \langle \mathcal{T}_\tau \Phi_\mu(\mathbf{k}, \tau) \Phi_\mu(-\mathbf{k}, 0) \rangle_{\mathrm{H}_0} \langle \mathcal{T}_\tau \Phi_\mu(-\mathbf{k} - \mathbf{G}, \tau) \Phi_\mu(\mathbf{k} + \mathbf{G}, 0) \rangle_{\mathrm{H}_0}, \qquad (\mathrm{E.4})$$

The factor of 2 is originated by the sum over both identical diagrams. Furthermore, the relation $\mathrm{J}_{-\mathbf{G}} = \mathrm{J}_{\mathbf{G}}^*$ is used, where the complex conjugated is denoted by the asterisk. The obtained expectation values are the free spin density propagtors $\mathcal{D}_\mu^{(0)}(\mathbf{k}, \tau)$. Both propagators are transformed into Matsubara frequency space, using the transformation rule

$$\mathcal{D}_\mu^{(0)}(\mathbf{k}, \tau) = \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \mathcal{D}_\mu^{(0)}(\mathbf{k}, i\omega_n) \qquad (\mathrm{E.5})$$
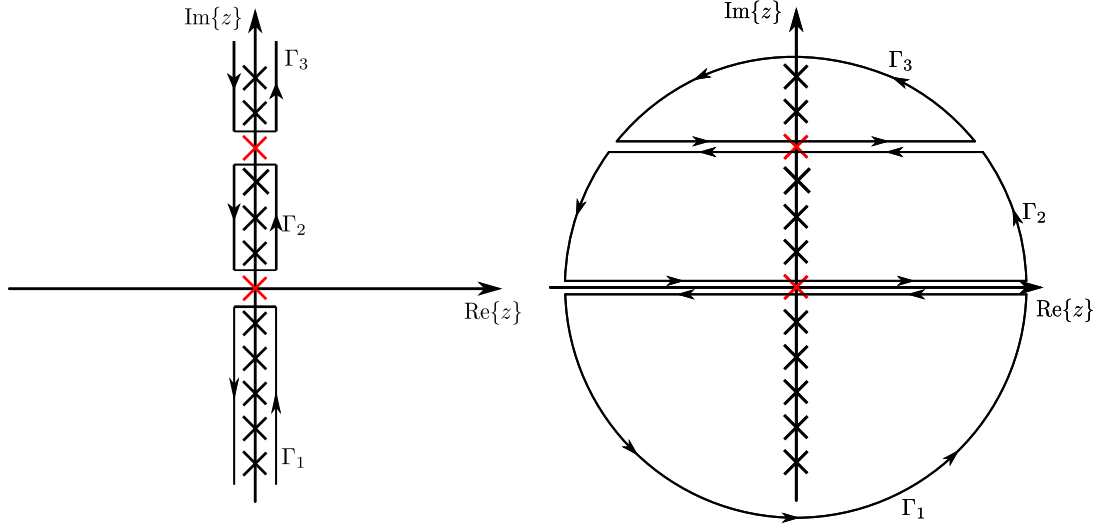
Figure E.2: caption

where the summation runs over all bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, where $n \in \mathbb{Z}$. The frequency argument of the Green function is set to $z = i\omega_l$, since the Green function is now investigated onto the imaginary frequency axis. The exponential functions are generated the only τ-dependence and the τ-integral is evaluated to $\beta\delta_{\omega_m+\omega_l-\omega_n}$. One of the two sums over Matsubara frequencies is evaluated as well. The following expression is obtained:

$$\mathcal{G}_{\dot{P}\dot{P}}(\mathbf{k}, i\omega_l) = -2 \sum_{\mathbf{G}} |\mathbf{J}_{\mathbf{G}}|^2 \int_{\mathbf{k}} G_j^2 \frac{1}{\beta} \sum_{\omega_n} \mathcal{D}_\mu(\mathbf{k}, i\omega_n) \, \mathcal{D}_\mu(-\mathbf{k} - \mathbf{G}, i\omega_l - i\omega_n). \quad \text{(E.6)}$$

In the following the Matsubara sum is evaluated. Therefore, the singularities of both propgators are observed. A discontinuity of the propagator, denoted in (3.7), is generated if the term of Matsubara frequancies in the denominator becomes to be zero. This type of singularity is called branch cut, since the whole horizontal line along the singularity is forbidden. In the investigated case, two singularity of them arise, where one is sited at $\omega_n = 0$ and the other one is sited at $\omega_n = \omega_l$. To transform the Matsubara sum into a complex contour integral, both singularities are seperated from the sum. For reasons of clarity the momentum argument of the second propagator is abbreviatied by $\mathbf{k}' = -\mathbf{k} - \mathbf{K}$ and the momentum arguments are written as a lower index.

$$S(i\omega_l) := \frac{1}{\beta} \sum_{\omega_n} \mathcal{D}_{\mu,\mathbf{k}}(i\omega_n) \, \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - i\omega_n)$$

$$\Leftrightarrow \ S(i\omega_l) := \frac{1}{\beta} \mathcal{D}_{\mu,\mathbf{k}}(0) \, \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l) + \frac{1}{\beta} \mathcal{D}_{\mu,\mathbf{k}}(i\omega_l) \, \mathcal{D}_{\mu,\mathbf{k}'}(0)$$

$$+ \frac{1}{\beta} \sum_{\substack{n \neq 0 \\ n \neq l}} \mathcal{D}_{\mu,\mathbf{k}}(i\omega_n)\, \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - i\omega_n) \tag{E.7}$$

The remaining sum, labeled with $\mathrm{I}_{MS}(\omega_l)$ in the following, is now transformed into a complex contour integral. All Matsubara frequencies are included into the contour Γ, beside the two seperated frequencies. Similar to the case of simple poles, see D, the contour runs along the imaginary axis. The contour is closed infinitesimal above and below the banches, since crossing of the branches is forbidden. The contour Γ is depicted on the left hand side in figure E.2. Since the spin propagators are bosonic, the Bose distribution $n_{\mathrm{B}}(z)$ is considered, transforming the sum into a contour integral.

$$\mathrm{I}_{MS}(\omega_l) := \frac{1}{\beta} \sum_{\substack{n \neq 0 \\ n \neq l}} \mathcal{D}_{\mu,\mathbf{k}}(i\omega_n)\, \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - i\omega_n) = \oint_\Gamma \frac{\mathrm{d}z}{2\pi i} n_{\mathrm{B}}(z) \mathcal{D}_{\mu,\mathbf{k}}(z)\, \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - z)$$

$$\tag{E.8}$$

As shown in figure E.2 Γ is seperated in three single contours, denoted as Γ_1, Γ_2 and Γ_3. These three contours are expanded to infity where the contour path is never allowed to cross the horizontal line at ω_l or 0. The obtained contour is depicted on the right hand side in figure E.2.

The contour Γ_1 is taken along the real axis in negative direction and is closed by a semi-circle with infinite radius in the lower complex plane. Γ_2 is the contour between both branch cuts. The contour is started along the real axis in positive direction. At infinity the contour is perfomed up to the branch at ω_n. The path is taken in oppsite direction to minus infinity and from there dwon to the starting point. The last contour Γ_3 is taken along the ω_n-axis in positive direction and is closed by a semi-circle with infinite radius in the upper complex plane. The contours parallel the real and ω_n-axis is set infinitesimal beside these axis, which is indicated with the term $\pm i\eta$. The sign is chosen according to the corresponding contour.

Each one of these three contour integrals can be splitted in parts, where only the parts parallel to the real axis do not vanish. In all other segments the path is lying in infinity and, since the integrand is decreasing faster than $1/z$ for $z \to \infty$, these part are zero. Only, the paths survive along the two branch cuts.

$$\mathrm{I}_{MS}(\omega_l) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi i} \mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - \epsilon) \left[n_{\mathrm{B}}(\epsilon - i\eta)\mathcal{D}_{\mu,\mathbf{k}}(\epsilon - i\eta) - n_{\mathrm{B}}(\epsilon + i\eta)\mathcal{D}_{\mu,\mathbf{k}}(\epsilon + i\eta) \right]$$

$$+ \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi i} \mathcal{D}_{\mu,\mathbf{k}}(i\omega_l + \epsilon) \left[n_{\mathrm{B}}(i\omega_l + \epsilon - i\eta)\mathcal{D}_{\mu,\mathbf{k}'}(\epsilon - i\eta) \right.$$

$$\left. - n_{\mathrm{B}}(i\omega_l + \epsilon + i\eta)\mathcal{D}_{\mu,\mathbf{k}'}(\epsilon + i\eta) \right] \tag{E.9}$$

The limit $\eta \to 0^+$ is implied and the symmetry property with respect to the frequency is used. Now, the Bose distribution is simplified. Firstly the relation $n_{\mathrm{B}}(z + i\omega_l) =$

$n_\mathrm{B}(z)$ is used, which follows directly from the definition of the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, where $n \in \mathbb{Z}$. Inserting them into the Bose distribution, the exponential function is always evaluated to 1, since $\exp(i2\pi) = 1$. The Bose distribution is transformed onto the real axis using analytical continuation. The complex frequency z is expressed as $\epsilon \pm i\eta$, where the limit $\eta \to 0^+$ is implied. The exponential function in $n_\mathrm{B}(\epsilon \pm i\eta)$ is expanded up the second order, since the first order is chanceled.

$$n_\mathrm{B}(\epsilon \pm i\eta) \approx \frac{\beta^{-1}}{\epsilon \pm i\eta} = \beta^{-1}\Big(\mathrm{P.V.}\frac{1}{\epsilon} \mp i\pi\delta(\epsilon)\Big) \approx n_\mathrm{B}(\epsilon) \mp i\pi\beta^{-1}\delta(\epsilon), \qquad \text{(E.10)}$$

The last equality is only valid for small frequencies. In equation (E.9), the Matsubara frequencies are eliminated in the argument of both Bose distributions. Both squared brackets under the integral are quite identical, beside the different momentum argument of the propagator. Our next approach is writing the Bose distribution and the propagators into real and imaginary parts. The real part of them are symmetric and the imaginary part of them is antisymmetric with respect to change η to $-\eta$. The following expression is obtained for the squared brackets in the first line of (E.9):

$$2i\Big[\mathrm{Re}\{n_\mathrm{B}(\epsilon + i\eta)\}\,\mathrm{Im}\{\mathcal{D}_{\mu,\mathbf{k}}(\epsilon + i\eta)\} + \mathrm{Im}\{n_\mathrm{B}(\epsilon + i\eta)\}\,\mathrm{Re}\{\mathcal{D}_{\mu,\mathbf{k}}(\epsilon + i\eta)\}\Big]. \qquad \text{(E.11)}$$

An analogical expression is obtained for the squared brackets in the second line changing $\mathbf{k}$ to $\mathbf{k}'$ and vice versa. The second term of (E.11) is evaluated first. The imaginary part of the Bose distribution is given by equation (E.10). The equivalent procedure is performed for the second bracket in (E.9). Both relations are integrated with respect to ϵ. Using relation $\mathrm{Re}\{\mathcal{D}_{\mu,\mathbf{k}}(0)\} = \mathcal{D}_{\mu,\mathbf{k}}(0)$, the obtained expression is equivalent to the seperated terms in equation (E.7), beside a minus sign.

$$-\frac{1}{\beta}\mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l)\mathcal{D}_{\mu,\mathbf{k}}(0) - \frac{1}{\beta}\mathcal{D}_{\mu,,\mathbf{k}}(i\omega_l)\mathcal{D}_{\mu,\mathbf{k}'}(0) \qquad \text{(E.12)}$$

These both therms are chanceled the seperated terms in (E.7) The remaining contributions of $S(i\omega_l)$ are given by the first term of equation (E.11) and the corresponding term for the second squared brackets in (E.9). The real part of the Bose distribution $n_\mathrm{B}(\epsilon + i\eta)$ is the Bose distribution itself (see (E.10)) Since the imaginary part of $\mathcal{D}_\mu$ is independent of the additional part $i\eta$ the Matsubara sum is given by

$$S(i\omega_l) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{\pi} n_\mathrm{B}(\epsilon)\Big[\mathcal{D}_{\mu,\mathbf{k}'}(i\omega_l - \epsilon)\,\mathrm{Im}\{\mathcal{D}_{\mu,\mathbf{k}}(\epsilon)\} + \mathcal{D}_{\mu,\mathbf{k}}(i\omega_l + \epsilon)\,\mathrm{Im}\{\mathcal{D}_{\mu,\mathbf{k}'}(\epsilon)\}\Big]$$
$$\text{(E.13)}$$

$S(i\omega_l)$ is now transformed on the real axis, using analytical continuation, $i\omega_l = \omega + i\eta =: \tilde{\omega}$. The Matsubara sum is given by

$$S(\omega) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{\pi} n_\mathrm{B}(\epsilon)\Big[\mathcal{D}^*_{\mu,\mathbf{k}'}(\epsilon - \omega)\,\mathrm{Im}\{\mathcal{D}_{\mu,\mathbf{k}}(\epsilon)\} + \mathcal{D}_{\mu,\mathbf{k}}(\epsilon + \omega)\,\mathrm{Im}\{\mathcal{D}_{\mu,\mathbf{k}'}(\epsilon)\}\Big] \qquad \text{(E.14)}$$

Here, relation $\mathcal{D}_{\mu,\mathbf{k}'}(\omega - \epsilon) = \mathcal{D}_{\mu,\mathbf{k}'}^*(\epsilon - \omega)$ is used, following from equation (A.27). The frequency argument is shifted from $\epsilon - \omega$ to ϵ in the first term. The imaginary part of the retarded Green function is used in the calculation of the conductivity or resistance. Considering equation (E.6), the Matsubara sum is the only observed complex quantity. The imaginary part of the Matsubara sum is given by

$$\text{Im}\{S(\omega)\} = \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} \left[n_{\text{B}}(\epsilon) - n_{\text{B}}(\epsilon + \omega) \right] \text{Im}\{\mathcal{D}_{\mu}(-\mathbf{k} - \mathbf{G}, \epsilon)\} \, \text{Im}\{\mathcal{D}_{\mu}(\mathbf{k}, \epsilon + \omega)\}$$

(E.15)

where $\text{Im}\{\mathcal{D}^*\} = -\text{Im}\{\mathcal{D}\}$ is used. The imaginary part of the retarded Green function is obtained by inserting the expression for $\text{Im}\{S(\omega)\}$.

$$\text{Im}\left\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\omega)\right\} = -2 \sum_{\mathbf{G}} |\mathbf{J}_{\mathbf{G}}|^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{\pi} \left[n_{\text{B}}(\epsilon) - n_{\text{B}}(\epsilon + \omega) \right]$$
$$\times \int_{\mathbf{k}} G_j^2 \, \text{Im}\{\mathcal{D}_{\mu}(-\mathbf{k} - \mathbf{G}, \epsilon)\} \, \text{Im}\{\mathcal{D}_{\mu}(\mathbf{k}, \epsilon + \omega)\}, \qquad (E.16)$$

The imaginary part of the spin propagator is denoted in equation (A.28). Small frequencies ω are considered, since our interest is based on the static electrical conductivity. Spin propagators and Bose distributions are investigated seperately in this limit. The latter is expanded up to the first order, since the zeroth order vanishes. In the case of spin propagators, ω is simply set to zero. The first derivative of the Bose distribution and both spin propagators are even function with respect to ϵ. Therefore, the lower boundary of the integral is set to zero and the integral is multiplied by 2. Inserting the explicit expressions for the imaginary parts of the propagators, the imaginary part of the retarded Green function is obtained.

$$\text{Im}\left\{\mathcal{G}_{\dot{P}\dot{P}}^{\text{ret}}(\omega)\right\} = -\frac{4\gamma^2 \beta \omega}{\pi} \sum_{\mathbf{Q}_1, \mathbf{Q}_2} \sum_{\mathbf{G}} |\mathbf{J}_{\mathbf{G}}|^2 \int_0^{\infty} d\epsilon \, \frac{\epsilon^2 e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}$$
$$\times \int_{\mathbf{k}} G_j^2 \cdot \frac{1}{(\mathbf{k} + \mathbf{G} - \mathbf{Q}_1)^4 + \gamma^2 \epsilon^2} \cdot \frac{1}{(\mathbf{k} + \mathbf{Q}_2)^4 + \gamma^2 \epsilon^2} \qquad (E.17)$$

The conservation of momentum connected with an unbroken translation symmetry yields a infinite electrical conductivity, which is computated in the previous section. In the next calculation a system with broken translation symmetry is considered. The assumed symmetry breaking pertubation is umklapp scattering, where the Hamiltonian is given by equation (). In ... it is shown that this pertubation is the reason for an unconserved momentum. Thus the above disscusion about the Drude weight and conductivity let us expect that the conductivity is lessened to a finite value. The static electrical conductivity is given by equation (**??**) in general. Again the memory matrix formalsim is now used to compute the current-current correlation function given by

link zu umklapp hamiltonian

link zum abschnitt in dem gezeigt wird das P nicht mehr erhalten ist

the formal equation

$$\sum_l \left[\omega\delta_{il} - \Omega_{il} + i\Sigma_{il}(\omega)\right]\mathcal{C}_{lj}(\omega) = \frac{i}{\beta}\chi_{ij}(0) \tag{E.18}$$

where Ω_{il} and $\Sigma_{il}(\omega)$ are given by

$$\Omega_{il} = i\beta\sum_k \left(\dot{A}_i\big|C_k\right)\chi_{kl}^{-1}(0) \quad \text{and} \tag{E.19}$$

$$\Sigma_{il}(\omega) = i\beta\sum_k \left(\dot{A}_i\big|Q\frac{1}{\omega - QLQ}Q\big|\dot{C}_k\right)\chi_{kl}^{-1}(0). \tag{E.20}$$

Always the first step is to think about the vector subspace, generated by the vectors of the projection operator. Computing the electrical conductivity the current and the momentum operator are usually the operators of interest. Therefore our decision is made and our subspace is generated by these two operators. What does this choice of operators mean for the quantities Ω_{il} and $\Sigma_{il}(\omega)$? Starting with the first one. Ω_{il} vanishs if two properties are valid. The first one is, that the considered Hamiltonian has to be invariant with respect to time reversal symmetry. The unpertubated Hamiltonian (...) and the pertubation Hamiltonian (...) occupy this condiction which is trivially to prove. The second property is that both operators labeled with A_i and C_k must have the same signature under time reversal symmetry. Both operators can be either J or P, where both have the same signature under time reversal symmetry. Therefore in all cases the quantity Ω_{il} is zero. In $\Sigma_{il}(\omega)$ the expecation value is formed with respect of the derivative of vector operators, which are $|\dot{J})$ and $|\dot{P})$. In the discussion above a translation invariant system is assumed why the derivative of the momentum vanishes. Now the momentum isn't anymore conserved and therefore the derivative yields a finite value.

For further assertions the action of the operator Q has to be investigated on both vector operator. $Q|\dot{C}_k)$ describes the coupling to all other degrees of freedom which aren't included in the subspace. Firstly remember that umklapp scattering is the considered pertubation. What does this pertubation change in our system? It breaks translation symmetry which yields some finite value for $\dot{P}$ instead of zero in the unpertubated system. This means the complete unconserved part of the momentum is coupled to the crystal lattice which is clearly a degree of freedom out off the J-P subspace. This is the reason why $Q|\dot{P}) = |\dot{P})$. Further the pertubation doesn't change the quantity $|\dot{J})$. The unconserved current yields from the interaction between the electrons lives on different Fermi surfaces coupeld via spin density waves. This process is included in the J-P subspace and therefore $Q|\dot{J}) = 0$. This signifies for the memory function that Σ_{il} doesn't vanish if $i = P$ and vanish if $i = J$.

In summary umklapp scattering yields a non-zero contribution to the memory function $\Sigma_{il}(\omega)$ and is therefore a correction of the correlation function instead of the unpertubated case where the memory function is zero. Equation (5.60) yields 4 equations in the J-P subspace, which can be writen as a matrix equation.

$$\begin{pmatrix} \omega & 0 \\ -i\Sigma_{PJ}(\omega) & \omega - i\Sigma_{PP}(\omega) \end{pmatrix} \cdot \begin{pmatrix} \mathcal{C}_{JJ}(\omega) & \mathcal{C}_{JP}(\omega) \\ \mathcal{C}_{PJ}(\omega) & \mathcal{C}_{PP}(\omega) \end{pmatrix} = \frac{i}{\beta}\begin{pmatrix} \chi_{JJ}(0) & \chi_{JP}(0) \\ \chi_{PJ}(0) & \chi_{PP}(0) \end{pmatrix} \tag{E.21}$$

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vllt noch etwas ausführlicher schreiben

link zum ungestörten Hamiltonian

link zum umklapp Hamiltonian

Before the computation is going on we want to make a short remark. Equation (5.60) is an exact algebraic matrix equation. At the derivation no assumtions are made and up to this point we have also made no assumptions. All the conversion we have done are exact and only depending on the considered model.

The electrical conductivity is given by the J-J correlation function, which has the formal expression

$$\mathcal{C}_{\mathrm{JJ}}(\omega) = \big(\mathrm{J}\big| \frac{i}{\omega - \mathrm{L}} \big| \mathrm{J}\big) \tag{E.22}$$

in frequenzy space. Equally to the case of conserved momentum nothing bars us to split the current into one parallel and one vertical part, where the parallel part is pointed in the direction of the secular component of J. The appearing mixed correlation functions vanishes because $|\mathrm{J}_{||})$ and $|\mathrm{J}_{\perp})$ are orthogonal. How we have seen in the previous section the background or noise originated by fluctuation and other random processes, which is represented by the correlation function of the vertical component. This term isn't necessary to write it down every time why we drop it. For a discussion in more detail the work of Jung [Jun07] is suggested. However the only important part for us is the parallel component of the correlation function. On the other hand the parallel componend of the correlation function is given by the projection of J onto P, see equation (??). Thus the J-J correlation function is rewriten in a P-P correlation function mutiplied with a fraction of susceptibilities.

$$\mathcal{C}_{\mathrm{JJ}}(\omega) = \big(\mathrm{J}_{||}\big| \frac{i}{\omega - \mathrm{L}} \big| \mathrm{J}_{||}\big) = \frac{|\chi_{\mathrm{PJ}}|^2}{|\chi_{\mathrm{PP}}|^2} \mathcal{C}_{\mathrm{PP}}(\omega) \tag{E.23}$$

The P-P correlation function can be readed out of equation (E.21). Therefore the inverse of the memory matrix has to be multiplied from the left hand side. The P-P correlation function is then given by

$$\mathcal{C}_{\mathrm{PP}}(\omega) = \frac{i}{\beta} \cdot \frac{i\Sigma_{\mathrm{PJ}}(\omega)\chi_{\mathrm{JP}}(0)}{\omega\big(\omega - i\Sigma_{\mathrm{PP}}(\omega)\big)} + \frac{i}{\beta} \cdot \frac{\chi_{\mathrm{PP}}(0)}{\omega - i\Sigma_{\mathrm{PP}}(\omega)} \approx \frac{i}{\beta} \cdot \frac{i\chi_{\mathrm{PP}}(0)}{\Sigma_{\mathrm{PP}}(\omega)} \tag{E.24}$$

where in the last step the limit of small frequencies is taken. Then on the one hand the first term is neglectable compared to the second one. On the other hand is $\omega \ll \Sigma_{\mathrm{PP}}(\omega)$. Thus in the second term ω is neglectable against $\Sigma_{\mathrm{PP}}(\omega)$. In summary the static conductivity is given by

$$\sigma_{\mathrm{dc}} = \lim_{\omega \to 0} \beta \mathcal{C}_{\mathrm{JJ}}(\omega) = \frac{i}{\beta} \lim_{\omega \to 0} \frac{|\chi_{\mathrm{PJ}}|^2}{\chi_{\mathrm{PP}}} \frac{i\beta}{\Sigma_{\mathrm{PP}}(\omega)} \tag{E.25}$$

The memory function $\Sigma_{\mathrm{PP}}(\omega)$ is definied in equation (5.63). Because of the considered Hamiltonian only the term included $\dot{\mathrm{P}}$ yields a non-zero contribution. Further the operator QLQ can be approximated by L_0 the Liouville operator of the unpertubated system. The final expression for the dc-conductivity is given by

$$\sigma_{\mathrm{dc}} \approx \frac{i}{\beta} \lim_{\omega \to 0} |\chi_{\mathrm{PJ}}|^2 \big(\dot{\mathrm{P}}\big| \frac{1}{\omega - \mathrm{L}_0} \big| \dot{\mathrm{P}}\big)^{-1} \tag{E.26}$$

In a short conversion the expectation value can be expressed as a time integral over the $\dot{\text{P}}$-$\dot{\text{P}}$ susceptibility. This expression is more usefull for explicite computations, because it allows us using pertubation theory. For the detailed conversion see appendix C. The static electrical conductivity is finaly given by

$$\sigma_{\text{dc}} \approx -\hbar \lim_{\omega \to 0} \frac{\omega |\chi_{\text{JP}}(\omega = 0)|^2}{\int\limits_0^\infty \mathrm{d}t\, e^{i\omega t} \left\langle \left[\dot{\text{P}}(t), \dot{\text{P}}(0)\right]\right\rangle_0}, \tag{E.27}$$

which is used in the compuation below. The calculation is splitted into two parts. At first the computation of the denominator is perfermed, which gives us the temperature dependence of the conductivity. Further the J-P susceptibility has to be calculated. In first order for this quantity is expected no temperature dependence, but we have to convience us from this.

$-\hbar$???

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F Properties of the Kubo Relaxation Function

In section 5.2.1 the Kubo relaxation function

$$\Phi_{AB}(t) = \frac{i}{\hbar} \lim_{s \to 0} \int_t^\infty d\tau \, \langle [A_I(\tau), B_I(0)] \rangle_0 \, e^{-s\tau}. \tag{F.1}$$

and the three relations

1. $\chi_{AB}(t) = -\Theta(t) \dfrac{d}{dt} \Phi_{AB}(t)$ (F.2)

2. $\Phi_{AB}(t=0) = \chi_{AB}(\omega = 0)$ (F.3)

3. $\Phi_{AB}(\omega) = \dfrac{1}{i\omega} \big[\chi_{AB}(\omega) - \chi_{AB}(\omega = 0) \big].$ (F.4)

connecting the dynamical susceptibility χ_{AB} with Φ_{AB} are introduced. In the following we want to proof these three relations.

The first one is easily gotten by derivating the Kubo relaxation function with respect to t and comparing the result with the definition of the dynamical susceptibility (**??**).

$$-\Theta(t) \frac{d}{dt} \Phi_{AB}(t) = \frac{i}{\hbar} \Theta(t) \, \langle [A_I(t), B_I(0)] \rangle_0 = \chi_{AB}(t) \tag{F.5}$$

The second relation is found with the aim of the Laplace transformation of the Kubo relaxation function.

$$\Phi_{AB}(\omega) = \int_0^\infty dt \, \Phi_{AB}(t) e^{i\omega t} \tag{F.6}$$

In this definition of the Laplace transformation compared to (5.16) we set $s = -i\omega$ which correspond to a rotation of $\frac{\pi}{2}$ of the definition space . Using (F.6) after setting $t = 0$ in (F.1) yield

reference to a book of laplace transformation

$$\Phi_{AB}(t=0) = \frac{i}{\hbar} \lim_{s \to 0} \int_0^\infty d\tau \, \langle [A_I(\tau), B_I(0)] \rangle_0 \, e^{-s\tau}$$

$$\Leftrightarrow \; \Phi_{AB}(t=0) = \frac{i}{\hbar} \lim_{\substack{s \to 0 \\ \omega \to 0}} \int_{-\infty}^\infty d\tau \, \Theta(\tau) \, \langle [A_I(\tau), B_I(0)] \rangle_0 \, e^{i\omega\tau} e^{-s\tau}$$

$$\Leftrightarrow \quad \Phi_{\mathrm{AB}}(t = 0) = \lim_{\omega \to 0} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \chi_{\mathrm{AB}}(\tau) e^{i\omega\tau}$$

$$\Leftrightarrow \quad \Phi_{\mathrm{AB}}(t = 0) = \chi_{\mathrm{AB}}(\omega = 0), \tag{F.7}$$

where it is assumed the susceptibility is a good function in the sense they decay fast enough and the convergence generating faktor is negligible. The third relation is computated with the aim of the first and second relation. Therefore relation one is multiplied with $e^{i\omega t}$ and is integrated with respect to t.

$$\int_{0}^{\infty} \mathrm{d}t \, e^{i\omega t} \chi_{\mathrm{AB}}(t) = -\int_{0}^{\infty} \mathrm{d}t \, e^{i\omega t} \frac{\mathrm{d}}{\mathrm{d}t} \Phi_{\mathrm{AB}}(t)$$

$$\overset{\mathrm{PI}}{\Leftrightarrow} \quad \chi_{\mathrm{AB}}(\omega) = -e^{i\omega t} \Phi_{\mathrm{AB}}(t)\Big|_{0}^{\infty} + i\omega \int_{0}^{\infty} \mathrm{d}t \, e^{i\omega t} \Phi_{\mathrm{AB}}(t)$$

$$\Leftrightarrow \quad \chi_{\mathrm{AB}}(\omega) = \Phi_{\mathrm{AB}}(t = 0) + i\omega \Phi_{\mathrm{AB}}(\omega)$$

$$\Leftrightarrow \quad \Phi_{\mathrm{AB}}(\omega) = \frac{1}{i\omega} \big[\chi_{\mathrm{AB}}(\omega) - \chi_{\mathrm{AB}}(\omega = 0) \big] \tag{F.8}$$

In the first step the right hand side is integrated by parts and in last step the first relation and (F.6) is used. So the third relation gives us the dependence between the Kubo relaxation function and the dynamical susceptibility in frequency space.

Todo list

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