Robotics 1
Exercise Solver

November 9, 2023

Contents

1 Rotation Matrices

1.1 Check if R is a rotation matrix

To check if R is a rotation matrix we have to check:

- det(R) = 1
- Orthogonality: $R^T R = I$
- Normality: for each column R_i of R, $||R_i|| = 1$

1.2 Rotation direct problem

To find R from θ and **r** we use the Rodrigues' rotation formula:

$$R(\theta, r) = rr^T + (I - rr^T)\cos(\theta) + (S(r))\sin(\theta) \tag{1}$$

where $S(r)$ is the skew-symmetric matrix of **r**:

$$S(r) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{2}$$

1.3 Rotation inverse problem

To find θ and **r** from R we fisrt check if there is a singularity:

$$\sin(\theta) = \frac{1}{2}\left(\sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2}\right) \tag{3}$$

1.3.1 Singularity (hence $\sin(\theta) = 0$)

If it is a singularity we can find **r** and θ: if θ is 0: there is no solution for r.
if θ is $\pm\pi$:
we set $\sin(\theta) = 0$, $\cos(\theta) = -1$ and we find **r**:

$$\mathbf{r} = \frac{1}{2}\begin{bmatrix} \pm\sqrt{R_{11} + 1} \\ \pm\sqrt{R_{22} + 1} \\ \pm\sqrt{R_{33} + 1} \end{bmatrix} \tag{4}$$

To decide the signs of the elements of **r** we can use the following criteria:

- $r_x r_y = R_{12}/2$
- $r_x r_z = R_{13}/2$
- $r_y r_z = R_{23}/2$

1.3.2 Not singularity

If the singularity is not present we can find θ and $\mathbf{r}$:
Note: we obtain two solutions for θ and cosequently r

$$\cos(\theta) = (R_{11} + R_{22} + R_{33} - 1) \tag{5}$$

$$\sin(\theta) = \pm\sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2} \tag{6}$$

$$\theta = \text{atan2}\,(\sin\theta, \cos\theta) \in (-\pi, \pi] \tag{7}$$

$$\mathbf{r} = \frac{1}{2\sin(\theta)}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \tag{8}$$

2 Roll Pitch Yawn

2.1 RPY direct problem

To find R from ψ, θ and ϕ we use the following formula:

$$R(\psi, \theta, \phi) = R_z(\phi)R_y(\theta)R_x(\psi) \tag{9}$$

Note: the order of the angles is reversed!

2.2 Inverse Problem

Given a rotation matrix R we can find angles of rotation ψ, θ and ϕ: Fisrt we check if there is a singularity (if $R_{32}^2 + R_{33}^2 = 0$), we then have two cases:

- **No Singularity** — We can find all three parameters of the rotational matrix R

$$\theta = \text{atan2}\left(-R_{31}, \pm\sqrt{R_{32}^2 + R_{33}^2}\right) \tag{10}$$

$$\phi = \text{atan2}\,(R_{21}/\cos(\theta), R_{11}/\cos(\theta)) \tag{11}$$

$$\psi = \text{atan2}\,(R_{32}/\cos(\theta), R_{33}/\cos(\theta)) \tag{12}$$

- **Singularity** — We cannot find all three angles, only θ and a combination of ϕ and ψ, the formula for these combinations is:

$$\begin{cases} \phi - \psi = \text{atan2}\,\{R_{2,3}, R_{1,3}\} & \text{if } \theta = \frac{\pi}{2} \\ \phi + \psi = \text{atan2}\,\{-R_{2,3}, R_{2,2}\} & \text{if } \theta = -\frac{\pi}{2} \end{cases} \tag{13}$$

3 DH frames

3.1 Axis assignment

- z_i along the direction of joint i+1.
- x_i along the common normal between z_i and z_{i-1}.
- y_i completes the right-handed coordinate system.

3.2 DH table

- θ_i angle between x_{i-1} and x_i measured about z_{i-1}.
- d_i distance between x_{i-1} and x_i measured along z_{i-1}.
- a_i distance between z_{i-1} and z_i measured along x_i.
- α_i angle between z_{i-1} and z_i measured about x_i.

3.3 Transformation matrix from DH parameters

$$
{}^{i-1}A_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}
$$

3.4 DH parameters from transformation matrix

First we have to check that the first three by three submatrix is a rotation matrix (see section 1.1).
Then we can find the parameters:

$$
\begin{aligned}
\theta_i &= \operatorname{atan2}\left(R_{12}, R_{11}\right) && (15)\\
\alpha_i &= \operatorname{atan2}\left(R_{32}, R_{33}\right) && (16)\\
d_i &= R_{34} && (17)\\
a_i &= R_{14}\cos(\theta_i) + R_{24}\sin(\theta_i) && (18)\\
& && (19)
\end{aligned}
$$

4 Workspace

4.1 2-DOF robot

The primary workspace is defined by two concentric circles of radius r_1 and r_2 where:

$$r_1 = |l_1 - l_2| \tag{20}$$

$$r_2 = l_1 + l_2 \tag{21}$$

4.2 3-DOF robot

The primary workspace is defined by two concentric spheres of radius r_{in} and r_{out} where:

$$r_{in} = l_{min} + l_{med} + l_{max} \tag{22}$$

$$r_{out} = \max(0, l_{max} - l_{med} - l_{min}) \tag{23}$$

where:

- l_{min} is the length of the shortest link
- l_{med} is the length of the medium link
- l_{max} is the length of the longest link