

Robotics 1

Exercise Solver

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1 DC motors

1.1 Electrical and mechanical balance

$$V_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_{emf}(t) \quad (1)$$

$$v_{emf}(t) = k_v \omega(t) \quad (2)$$

in control domain:

$$V_a = (R_a + sL_a)I_a + V_{emf} \quad (3)$$

$$V_{emf} = k_v \omega \quad (4)$$

where V_a is the voltage applied to the motor, R_a is the armature resistance, L_a is the armature inductance, i_a is the armature current, v_{emf} is the back emf, k_v is the back emf constant and ω is the angular velocity of the motor.

$$\tau_m(t) = I_m(t) \frac{d\omega(t)}{dt} + F_m \omega(t) + \tau_{load}(t) \quad (5)$$

$$\tau_m(t) = k_t i_a(t) \quad (6)$$

in control domain:

$$T_m = (sI_m + F_m)\omega + T_{load} \quad (7)$$

$$T_m = k_t I_a \quad (8)$$

where τ_m is the motor torque, I_m is the motor inertia, F_m is the motor friction, τ_{load} is the load torque and k_t is the torque constant. **Note:** $k_v = k_t$ **numerically!**

1.2 Reduction ratio

The reduction ratio of a the ransmission chain is the product of the reduction ratios of the single elements of the chain:

$$\eta = \sum_{i=1}^n \eta_i \quad (9)$$

1.2.1 Harmonic drives

$$\eta = \frac{\#theet_{FS}}{\#theet_{CS} - \#theet_{FS}} = \frac{\#theet_{FS}}{2} \quad (10)$$

$$\#theet_{FS} = \#theet_{CS} - 2 \quad (11)$$

1.2.2 Standard gears

Given two gears of radius r_1 and r_2 the reduction ratio is:

$$\eta = \frac{r_2}{r_1} \quad (12)$$

1.3 Optimal reduction ratio

$$\eta_{opt} = \sqrt{\frac{J_{load}}{J_{motor}}} \quad (13)$$

1.4 Optimal torque

We impose the relation between the angular acceleration of the load and the motor:

$$\dot{\theta}_m = \eta \dot{\theta}_l \quad (14)$$

$$\tau_m = J_m * \dot{\theta}_m + \frac{1}{\eta} (J_l * \dot{\theta}_l) \quad (15)$$

2 Encoders

2.1 Absolute encoders

The resolution of an absolute encoder is given by:

$$res = \frac{2\pi}{2^{N_t}} \quad (16)$$

where N_t is the number of bits of the encoder. **Note: the resolution changes from base to link end!**

$$res_{base} = res_{link}/L \quad (17)$$

where L is the length of the link.

2.2 Incremental encoders

The resolution of an incremental encoder is given by:

$$rse = \frac{2\pi}{2^{N_t}} \quad (18)$$

The number of bit of the encoder is given by:

$$N_t = \log_2(N_p) \quad (19)$$

where N_p is the number of pulses per turn of the encoder.

2.3 Multi-turn encoders

The number of bits to count the turns in a multi-turn encoder is given by:

$$N_t = \log_2(N_{turns}) \quad (20)$$

where N_{turns} is the number of turns of the encoder. The number of turns of the encoder is given by:

$$N_{turns} = \frac{\Delta\theta_{max} * n_r}{2\pi} \quad (21)$$

where $\delta\theta_{max}$ is the maximum angle of the encoder and n_r is the reduction ratio.

3 Rotation Matrices

3.1 Check if R is a rotation matrix

To check if R is a rotation matrix we have to check:

- $\det(R) = 1$
- Orthogonality: $R^T R = I$
- Normality: for each column R_i of R, $\|R_i\| = 1$

3.2 General Rotation

$${}^A R_B = \begin{bmatrix} x_A x_B & y_A x_B & z_A x_B \\ x_A y_B & y_A y_B & z_A y_B \\ x_A z_B & y_A z_B & z_A z_B \end{bmatrix} \quad (22)$$

3.3 Rotation direct problem

To find R from θ and \mathbf{r} we use the Rodrigues' rotation formula:

$$R(\theta, r) = rr^T + (I - rr^T) \cos(\theta) + (S(r)) \sin(\theta) \quad (23)$$

where $S(r)$ is the skew-symmetric matrix of \mathbf{r} :

$$S(r) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \quad (24)$$

3.4 Rotation inverse problem

To find θ and \mathbf{r} from R we first check if there is a singularity:

$$\sin(\theta) = \frac{1}{2} \left(\sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2} \right) \quad (25)$$

3.4.1 singularity (hence $\sin(\theta) = 0$)

If it is a singularity we can find \mathbf{r} and θ : if θ is 0: there is no solution for \mathbf{r} .
if θ is $\pm\pi$:

we set $\sin(\theta) = 0$, $\cos(\theta) = -1$ and we find \mathbf{r} :

$$\mathbf{r} = \begin{bmatrix} \pm \sqrt{\frac{R_{11}+1}{2}} \\ \pm \sqrt{\frac{R_{22}+1}{2}} \\ \pm \sqrt{\frac{R_{33}+1}{2}} \end{bmatrix} \quad (26)$$

To decide the signs of the elements of \mathbf{r} we can use the following criteria:

- $r_x r_y = R_{12}/2$
- $r_x r_z = R_{13}/2$
- $r_y r_z = R_{23}/2$

3.4.2 not singularity

If the singularity is not present we can find theta and \mathbf{r} :

Note: we obtain two solutions for θ and cosequently \mathbf{r}

$$\cos(\theta) = (R_{11} + R_{22} + R_{33} - 1) \quad (27)$$

$$\sin(\theta) = \pm \sqrt{(R_{32} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2} \quad (28)$$

$$\theta = \text{atan2}(\sin \theta, \cos \theta) \in (-\pi, \pi] \quad (29)$$

$$\mathbf{r} = \frac{1}{2 \sin(\theta)} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \quad (30)$$

4 Euler

4.1 Euler direct problem

To find R from ϕ , θ and ψ around axis X,Y,Z we use the following formula:

$$R(\psi, \theta, \phi) = R_x(\phi) R_y(\theta) R_z(\psi) \quad (31)$$

4.2 Inverse Problem

Given a rotation matrix R we can find ϕ , θ and ψ : Fisrt check if there is a singularity (if $\theta = 0$ or $\pm\pi$).

4.2.1 singularity (hence $R_{13}^2 + R_{23}^2 = 0$)

If it is a singularity we can find $\phi + \psi$ and $\phi - \psi$

4.3 not singularity

If it is not a singularity we can find ϕ , θ and ψ :

$$\theta = \text{atan2} \left(\pm \sqrt{R_{13}^2 + R_{23}^2}, R_{33} \right) \quad (32)$$

$$\phi = \text{atan2} (R_{13}/\sin(\theta), -R_{23}/\sin(\theta)) \quad (33)$$

$$\psi = \text{atan2} (R_{31}/\sin(\theta), R_{32}/\sin(\theta)) \quad (34)$$

5 Roll Pitch Yawn

5.1 RPY direct problem

To find R from ψ , θ and ϕ we use the following formula:

$$R(\phi, \theta, \psi) = R_z(\phi)R_y(\theta)R_x(\psi) \quad (35)$$

Note: the order of the angle is reversed!

5.2 Inverse Problem

Given a rotation matrix R we can find ψ , θ and ϕ : First check if there is a singularity (if $R_{32}^2 + R_{33}^2 = 0$).

5.2.1 singularity (hence $R_{32}^2 + R_{33}^2 = 0$)

If it is a singularity we can find $\phi + \psi$ and $\phi - \psi$:

$$\phi + \psi = \text{atan2} (-R_{23}, R_{13}) \text{ or} \quad (36)$$

$$\phi + \psi = \text{atan2} (-R_{12}, R_{22}) \quad (37)$$

$$\phi - \psi = \text{atan2} \left(R_{31}, \pm \sqrt{R_{32}^2 + R_{33}^2} \right) \quad (38)$$

5.2.2 not singularity

If it is not a singularity we can find ψ , θ and ϕ :

$$\theta = \text{atan2} \left(-R_{31}, \pm \sqrt{R_{32}^2 + R_{33}^2} \right) \quad (39)$$

$$\phi = \text{atan2} (R_{21}/\cos(\theta), R_{11}/\cos(\theta)) \quad (40)$$

$$\psi = \text{atan2} (R_{32}/\cos(\theta), R_{33}/\cos(\theta)) \quad (41)$$

6 DH frames

6.1 Assign axis

- z_i along the direction of joint $i+1$.
- x_i along the common normal between z_i and z_{i-1} .
- y_i completes the right-handed coordinate system.

6.2 DH table

- θ_i angle between x_{i-1} and x_i measured about z_{i-1} .
- d_i distance between x_{i-1} and x_i measured along z_{i-1} .
- a_i distance between z_{i-1} and z_i measured along x_i .
- α_i angle between z_{i-1} and z_i measured about x_i .

6.3 Transformation matrix from DH parameters

$${}^{i-1}A_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \cos(\alpha_i) & \sin(\theta_i) \sin(\alpha_i) & a_i \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \cos(\alpha_i) & -\cos(\theta_i) \sin(\alpha_i) & a_i \sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (42)$$

6.4 DH parameters from transformation matrix

First we have to check that the first three by three submatrix is a rotation matrix (see section 3.1).

Then we can find the parameters:

$$\theta_i = \text{atan2}(R_{12}, R_{11}) \quad (43)$$

$$\alpha_i = \text{atan2}(R_{32}, R_{33}) \quad (44)$$

$$d_i = R_{34} \quad (45)$$

$$a_i = R_{14} \cos(\theta_i) + R_{24} \sin(\theta_i) \quad (46)$$

$$(47)$$

7 Workspace

7.1 2-DOF robot

The primary workspace is defined by two concentric circles of radius r_1 and r_2 where:

$$r_1 = |l_1 - l_2| \quad (48)$$

$$r_2 = l_1 + l_2 \quad (49)$$

7.2 3-DOF robot

The primary workspace is defined by two concentric spheres of radius r_{in} and r_{out} where:

$$r_{out} = l_{min} + l_{med} + l_{max} \quad (50)$$

$$r_{in} = \max(0, l_{max} - l_{med} - l_{min}) \quad (51)$$

where:

- l_{min} is the length of the shortest link
- l_{med} is the length of the medium link
- l_{max} is the length of the longest link

8 Inverse Kinematic

8.1 Trigonometry

$$\cos(\theta + \phi) = \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi) \quad (52)$$

$$\sin(\theta + \phi) = \sin(\theta) \cos(\phi) + \cos(\theta) \sin(\phi) \quad (53)$$