Problem Set #1

Introduction to Measure Theory, Jan Ertl
Martina Fraschini

Exercise 1.3

Which of the following are algebras? Which are σ-algebras?

- $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open}\}$

    1. $\emptyset \in \mathcal{G}_1$, as the empty set is a subset of the field of real numbers.
    2. Given $A \in \mathcal{G}_1$, then $A^c \notin \mathcal{G}_1$, as A^c is a closed set by definition (a closed set is defined as the complement set of an open set).

    Therefore, $\mathcal{G}_1$ is neither an algebra or a σ-algebra.

- $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], (a, \infty)\}$

    1. If we assume that $a = b$ gives the empty set, then $\emptyset \in \mathcal{G}_2$.
    2. Given $A \in \mathcal{G}_2$, then A^c is a finite union of intervals of the form $(a, b]$, $(-\infty, b]$, (a, ∞) as well. Thus, $A^c \in \mathcal{G}_2$.
    3. $\mathcal{G}_2$ is, by definition, closed under finite unions (but not under countable unions).

    Therefore, $\mathcal{G}_2$ is an algebra but not a σ-algebra.

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], (a, \infty)\}$

    1. As before, if we assume that $a = b$ gives the empty set, then $\emptyset \in \mathcal{G}_3$.
    2. Given $A \in \mathcal{G}_3$, then A^c is a countable union of intervals of the form $(a, b]$, $(-\infty, b]$, (a, ∞) as well. Thus, $A^c \in \mathcal{G}_3$.
    3. $\mathcal{G}_3$ is, by definition, closed under countable unions.

    Therefore, $\mathcal{G}_3$ is a σ-algebra, and consequently an algebra.

Exercise 1.7

If X is a nonempty set and $\mathcal{A}$ is any σ-algebra, then why $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$?
By definition, X is nonempty and therefore it cannot be a σ-algebra. If we add the empty set to obtain $\{\emptyset, X\}$, then it is trivial to see how this is the smallest combination of sets that generates a σ-algebra.
The power set $\mathcal{P}(X)$ is defined as the sets of all the possible subsets of X. Thus, all the possible σ-algebras are already contained in the in $\mathcal{P}(X)$ and there is no subset that can be added.

Exercise 1.10

Prove that the intersection of σ-algebras $\bigcap_\alpha \mathcal{S}_\alpha$ is a σ-algebra as well.

1. Since each $\mathcal{S}_\alpha$ is a σ-algebra by definition, it means that each family of subsets contains the empty set. Therefore, the intersection of all the $\mathcal{S}_\alpha$ contains as well the empty set: $\emptyset \in \bigcap_\alpha \mathcal{S}_\alpha$.
2. $A \in \bigcap_\alpha \mathcal{S}_\alpha \Rightarrow A \in \mathcal{S}_\alpha\ \forall \alpha$. Since each $\mathcal{S}_\alpha$ is a σ-algebra, we have that also $A^c \in \mathcal{S}_\alpha\ \forall\alpha$. Therefore, it is also true that $A^c \in \bigcap_\alpha \mathcal{S}_\alpha$.
3. Let $A_i \in \bigcap_\alpha \mathcal{S}_\alpha$ for $i \neq \alpha$. Then $A_i \in \mathcal{S}_\alpha\ \forall i, \alpha$. By definition of σ-algebra, we also have that $\bigcup_i A_i \in \mathcal{S}_\alpha\ \forall\alpha$. Thus, $\bigcup_i A_i \in \bigcap_\alpha \mathcal{S}_\alpha$.

Exercise 1.22
Prove the following:

- μ is monotone.
  Let $A \subset B$, then we can consider B as the union of two disjoint sets: $B = A \cup (B \setminus A)$. Therefore, since the measure is a positive and addictive function, we have that $\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\geq 0} \geq \mu(A)$.

- μ is countably addictive.
  $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i) - \underbrace{\sum \mu(\text{all possible intersections})}_{\geq 0} \leq \sum_{i=1}^\infty \mu(A_i)$.

Exercise 1.23
Show that $\lambda(A) = \mu(A \cap B)$ is a measure.

1. $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
2. $\lambda(\bigcup_{i=1}^\infty A_i) = \mu(\bigcup_{i=1}^\infty A_i \cap B) = \mu(\bigcup_{i=1}^\infty (A_i \cap B)) = \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i)$, where $A_i \neq A_j$ for $i \neq j$.

Exercise 1.26
Prove that $(A_1 \supset A_2 \supset A_3 \supset \cdots, A_i \in \mathcal{S}, \mu(A_1) < \infty) \Rightarrow (\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{i=1}^\infty A_i))$.
Let $B_i = A_1 \setminus A_i$ and $\mu(B_i) = \mu(A_1) - \mu(A_i)$.
We have that $\mu(\bigcup_{n=1}^\infty B_n) = \mu(A_1 \setminus \bigcap_{n=1}^\infty A_n) = \mu(A_1) - \mu(\bigcap_{n=1}^\infty A_n)$.
We also know (Thm 1.25) that $\mu(\bigcup_{n=1}^\infty B_n) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
It follows that $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n=1}^\infty A_n)$.

Exercise 2.10
Since μ^* is an outer measure, and therefore countable subadditive, it follows that

$$\mu^*(B \cap E) + \mu^*(B \cap E^c) \geq \mu^*\big((B \cap E) \cup (B \cap E^c)\big) = \mu^*(B).$$

From the Carathodory construction we also have that $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$.
Since both the propositions are true, we have that $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

Exercise 2.14
$\mathcal{A} = \{A : A$ is a finite disjoint union of intervals of the form $(a, b], (-\infty, b], (a, \infty)\}$. The σ-algebra generated by it, $\sigma(\mathcal{A})$, contains all open sets in $\mathbb{R}$. Since the Borel σ-algebra, $\sigma(\mathcal{O})$, is the smallest σ-algebra that contains all open sets, we can deduce that $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$. Finally, by the Carathodory extension theorem we have that $\sigma(\mathcal{O}) = \sigma(\mathcal{A}) \subset \mathcal{M}$.

Exercise 3.1
Let B be a countable subset of $\mathbb{R}$. Then the Lebesgue outer measure given by $\mu^*(B) := \inf \sum_{n=1}^{\infty} (b_n - a_n) : B \subset \cup_{i=1}^{\infty} (a_i, b_i]$ achieves its infimum for the partition given by the set B, with $a_i = b_i = B_i$ and B_i the i-th element of the set B. Hence, the result.

Exercise 3.7
$\{x \in X : f(x) < a\} \in \mathcal{M}$, therefore also it complement $\{x \in X : f(x) \geq a\} \in \mathcal{M}$. We also have that $\{x \in X : f(x) > a\} \subset \{x \in X : f(x) \geq a\}$, and consequently also $\{x \in X : f(x) > a\} \in \mathcal{M}$. As before, also the complement of this last set $\{x \in X : f(x) \leq a\} \in \mathcal{M}$. We are explicitly using the Borel σ-algebra.

Exercise 3.10
Let's suppose that $\sup_{n\in\mathbb{N}} f_n(x)$ and $\inf_{n\in\mathbb{N}} f_n(x)$ are measurable. This imply that also $\max(f, g)$, $\min(f, g)$ and $|f|$ are measurable. If F(f(x), g(x)) is measurable, then also $f + g$ and $f - g$ are measurable.

Exercise 3.17
Define $N \geq M$ and $\frac{1}{2^N} < \epsilon$. Since f is bounded, M does not depend on x. Therefore, we have uniform convergence.

Exercise 4.13
Let f be measurable and $|f| < M$. By definition, $|f| = f^+ + f^-$ and $f^+, f^- \geq 0$. We have that

$$\int_E f^+ d\mu \leq \int_E (f^+ + f^-) d\mu = \int_E |f| d\mu = M\mu(E) < \infty$$
$$\int_E f^- d\mu \leq \int_E (f^+ + f^-) d\mu = \int_E |f| d\mu = M\mu(E) < \infty$$

Since both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, we say that $f \in \mathscr{L}^1(\mu, E)$.

Exercise 4.14
Let $\mathcal{A} \subset E$ be the subset where f is not finite. By contrast, let's assume that $\forall B : \mathcal{A} \subset B \subset E$ we have that $\mu(B) > 0$. Since μ is monotone, by the definiton of Lebesgue integral we have that $\int_E |f| d\mu > \int_{\mathcal{B}} |f| d\mu = \infty$. This would mean that $f \notin \mathcal{L}^1(\mu, E)$ and this is a contradiction.

Exercise 4.15
Let $f, g \in \mathscr{L}^1(\mu, E)$ and $f \leq g$. We have that f, g are also measurable. Consequently it's already proved in the Lecture Notes that $\int_E f d\mu \leq \int_E g d\mu$.

Exercise 4.16
$A \subset E$. By definition of Lebesgue integral, we have that $\int_A |f| d\mu \leq \int_E |f| d\mu < \infty$. This means that $f \in \mathcal{L}^1(\mu, A)$.

Exercise 4.21
If $B \subset A$, we can define $A = B \cup (A \setminus B)$. We also know that

$$\int_{B \cup (A \setminus B)} f dx = \int_B f dx + \int_{(A \setminus B)} f dx.$$

If $\mu(A \setminus B) = 0$, then we have that $\int_{(A \setminus B)} f dx = 0$ and $\int_A f dx = \int_B f dx$.