Problem Set #1

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Exercise 1

In the Brock and Mirman's model the households solves the following dynamic program:

$$V\left(K_t, z_t\right) = \max_{K_{t+1}} \ln\left(e^{z_t}K_t^\alpha - K_{t+1}\right) + \beta E_t\left\{V\left(K_{t+1}, z_{t+1}\right)\right\},$$

where the law of motion is:

$$z_{t+1} = \rho z_t + \varepsilon_t; \quad \varepsilon_t \sim i.i.d\left(0, \sigma^2\right).$$

The associated Euler equation is:

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \beta E_t\left\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right\}.$$

In order to find an algebraic solution for the policy function we use the "guess and verify" method. We guess that the policy function is in the form of $K_{t+1} = Ae^{z_t}K_t^\alpha$ and we substitute it in the Euler equation. We obtain:

$$\frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} = \beta E_t\left\{\frac{\alpha e^{z_{t+1}}A^{\alpha-1}e^{(\alpha-1)z_t}K_t^{\alpha^2-\alpha}}{e^{z_{t+1}}A^\alpha e^{\alpha z_t}K_t^{\alpha^2} - A^{1+\alpha}e^{z_{t+1}}e^{\alpha z_t}K_t^{\alpha^2}}\right\},$$

$$\iff \frac{1}{(1-A)e^{z_t}K_t^\alpha} = \beta E_t\left\{\frac{\alpha A^{-1}e^{-z_t}K_t^{-\alpha}A^\alpha e^{\alpha z_t}K_t^\alpha e^{z_{t+1}}}{(1-A)A^\alpha e^{\alpha z_t}K_t^{\alpha^2}e^{z_{t+1}}}\right\},$$

$$\iff \frac{1}{(1-A)e^{z_t}K_t^\alpha} = \beta E_t\left\{\frac{\alpha}{(1-A)Ae^{z_t}K_t^\alpha}\right\},$$

$$\iff \alpha\beta = \frac{(1-A)Ae^{z_t}K_t^\alpha}{(1-A)e^{z_t}K_t^\alpha},$$

$$\iff A = \alpha\beta.$$

Therefore the policy function is $K_{t+1} = \alpha\beta\ e^{z_t}K_t^\alpha$.

Exercise 2

Our baseline model with:

$$u\left(c_t, \ell_t\right) = \ln c_t + a\ln\left(1-\ell_t\right),$$

and

$$f\left(K_t, L_t, z_t\right) = e^{z_t}K_t^\alpha L_t^{1-\alpha},$$

has the following characterizing equations:

$$c_t = (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t - k_{t+1},$$

$$
\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left\{ \frac{1}{c_{t+1}} \left[\left(r_{t+1} - \delta \right) \left(1 - \tau \right) + 1 \right] \right\}, \\
\frac{a}{1-\ell_t} &= \frac{1}{c_t} w_t (1-\tau), \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}, \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} \ell_t^{-\alpha}, \\
\tau \left[w_t \ell_t + \left(r_t - \delta \right) k_t \right] &= T_t, \\
z_t = (1-\rho_z) \overline{z} + \rho_z z_{t-1} + \epsilon_t^z; &\quad \epsilon_t^z \sim \text{ i.i.d.} \left(0, \sigma_z^2 \right).
\end{aligned}
$$

In this case we can't use the same tricks as in Exercise 1 to solve for the policy function because the model is too complex and it's impossible to guess a good algebraic solution.

Exercise 3

Our baseline model with:

$$
u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln\left(1 - \ell_t\right),
$$

and

$$
f\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha},
$$

has the following characterizing equations:

$$
\begin{aligned}
c_t &= (1-\tau) \left[w_t \ell_t + \left(r_t - \delta \right) k_t \right] + k_t + T_t - k_{t+1}, \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} \left[\left(r_{t+1} - \delta \right) \left(1 - \tau \right) + 1 \right] \right\}, \\
\frac{a}{1-\ell_t} &= c_t^{-\gamma} w_t (1-\tau), \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}, \\
w_t &= (1-\alpha) e^{z_t} k_t^{\alpha} \ell_t^{-\alpha}, \\
\tau \left[w_t \ell_t + \left(r_t - \delta \right) k_t \right] &= T_t, \\
z_t = (1-\rho_z) \overline{z} + \rho_z z_{t-1} + \epsilon_t^z; &\quad \epsilon_t^z \sim \text{ i.i.d.} \left(0, \sigma_z^2 \right).
\end{aligned}
$$

Exercise 4

Our baseline model with:

$$
u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \frac{\left(1 - \ell_t\right)^{1-\xi} - 1}{1-\xi},
$$

and

$$
f\left(K_t, L_t, z_t\right) = e^{z_t} \left[\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta} \right]^{\frac{1}{\eta}},
$$

has the following characterizing equations:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1},$$

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\},$$

$$a(1-\ell_t)^{-\xi} = c_t^{-\gamma} w_t(1-\tau),$$

$$r_t = \alpha e^{z_t} k_t^{\eta-1}\left[\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}\right]^{\frac{1-\eta}{\eta}},$$

$$w_t = (1-\alpha)e^{z_t}\ell_t^{\eta-1}\left[\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}\right]^{\frac{1-\eta}{\eta}},$$

$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t,$$

$$z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d.}\left(0,\sigma_z^2\right).$$

Exercise 5
Assume $\ell_t = 1$. For the market clearing conditions we also have that $L_t = \ell_t = 1$. Our baseline model with:

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma},$$

and

$$f\left(K_t,L_t,z_t\right) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha},$$

has the following characterizing equations:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1},$$

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\},$$

$$c_t^{-\gamma} w_t(1-\tau) = 0,$$

$$r_t = \alpha k_t^{\alpha-1}\left(\ell_t e^{z_t}\right)^{1-\alpha},$$

$$w_t = (1-\alpha)k_t^{\alpha}\ell_t^{-\alpha}e^{(1-\alpha)z_t},$$

$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t,$$

$$z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d.}\left(0,\sigma_z^2\right).$$

The steady state equations, considering $\ell_t = 1$, are:

$$\bar{c} = (1-\tau)\left[\bar{w} + (\bar{r}-\delta)\,\bar{k}\right] + \bar{T},$$

$$\bar{c}^{-\gamma} = \beta E_t\left\{\bar{c}^{-\gamma}\left[(\bar{r}-\delta)(1-\tau)+1\right]\right\},$$

$$\bar{c}^{-\gamma}\bar{w}(1-\tau) = 0,$$

$$\bar{r} = \alpha\bar{k}^{\alpha-1}e^{(1-\alpha)\bar{z}},$$

$$\bar{w} = (1-\alpha)\bar{k}^{\alpha}e^{(1-\alpha)\bar{z}},$$

$$\tau\left[\bar{w} + (\bar{r}-\delta)\,\bar{k}\right] = \bar{T}.$$

Therefore, the steady state for the value of k is given by:

$$\bar{k} = \left[\frac{1}{\alpha}\left(\frac{1-\beta}{\beta(1-\tau)} + \delta\right)\right]^{\frac{1}{\alpha-1}} e^{\bar{z}}.$$

Please, look at the computational part on the Jupyter Notebook "PS1.ipynb".

Exercise 6
Our baseline model with:

$$u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi} - 1}{1-\xi},$$

and

$$f\left(K_t, L_t, z_t\right) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha},$$

has the following characterizing equations:

$$\begin{aligned}
c_t &= (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t - k_{t+1}, \\
c_t^{-\gamma} &= \beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1} - \delta\right)(1-\tau) + 1\right]\right\}, \\
a(1-\ell_t)^{-\xi} &= c_t^{-\gamma} w_t (1-\tau), \\
r_t &= \alpha k_t^{\alpha-1}\left(\ell_t e^{z_t}\right)^{1-\alpha}, \\
w_t &= (1-\alpha) k_t^{\alpha}\ell_t^{-\alpha} e^{(1-\alpha)z_t}, \\
\tau\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] &= T_t, \\
z_t = (1-\rho_z)\,\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; &\quad \epsilon_t^z \sim \text{ i.i.d. }\left(0, \sigma_z^2\right).
\end{aligned}$$

The steady state equations, considering $\ell_t = 1$, are:

$$\begin{aligned}
\bar{c} &= (1-\tau)\left[\bar{w}\bar{\ell} + \left(\bar{r} - \delta\right)\bar{k}\right] + \bar{T}, \\
\bar{c}^{-\gamma} &= \beta E_t\left\{\bar{c}^{-\gamma}\left[\left(\bar{r} - \delta\right)(1-\tau) + 1\right]\right\}, \\
a(1-\bar{\ell})^{-\xi} &= \bar{c}^{-\gamma}\bar{w}(1-\tau), \\
\bar{r} &= \alpha\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha}e^{(1-\alpha)\bar{z}}, \\
\bar{w} &= (1-\alpha)\bar{k}^{\alpha}\bar{\ell}^{-\alpha}e^{(1-\alpha)\bar{z}}, \\
\tau\left[\bar{w}\bar{\ell} + \left(\bar{r} - \delta\right)\bar{k}\right] &= \bar{T}.
\end{aligned}$$

Please, look at the computational part on the Jupyter Notebook "PS1.ipynb".

Exercise 7
Please, look at the computational part on the Jupyter Notebook "PS1.ipynb".