Problem Set #1

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Exercise 1.3

Which of the following are algebras? Which are σ-algebras?

- $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open}\}$

    1. $\emptyset \in \mathcal{G}_1$, as the empty set is a subset of the field of real numbers.
    2. Given $A \in \mathcal{G}_1$, then $A^c \notin \mathcal{G}_1$, as A^c is a closed set by definition (a closed set is defined as the complement set of an open set).

    Therefore, $\mathcal{G}_1$ is neither an algebra or a σ-algebra.

- $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], (a, \infty)\}$

    1. If we assume that $a = b$ gives the empty set, then $\emptyset \in \mathcal{G}_2$.
    2. Given $A \in \mathcal{G}_2$, then A^c is a finite union of intervals of the form $(a, b]$, $(-\infty, b]$, (a, ∞) as well. Thus, $A^c \in \mathcal{G}_2$.
    3. $\mathcal{G}_2$ is, by definition, closed under finite unions (but not under countable unions).

    Therefore, $\mathcal{G}_2$ is an algebra but not a σ-algebra.

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], (a, \infty)\}$

    1. As before, if we assume that $a = b$ gives the empty set, then $\emptyset \in \mathcal{G}_3$.
    2. Given $A \in \mathcal{G}_3$, then A^c is a countable union of intervals of the form $(a, b]$, $(-\infty, b]$, (a, ∞) as well. Thus, $A^c \in \mathcal{G}_3$.
    3. $\mathcal{G}_3$ is, by definition, closed under countable unions.

    Therefore, $\mathcal{G}_3$ is a σ-algebra, and consequently an algebra.

Exercise 1.7

If X is a nonempty set and $\mathcal{A}$ is any σ-algebra, then why $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$?
By definition, X is nonempty and therefore it cannot be a σ-algebra. If we add the empty set to obtain $\{\emptyset, X\}$, then it is trivial to see how this is the smallest combination of sets that generates a σ-algebra.
The power set $\mathcal{P}(X)$ is defined as the sets of all the possible subsets of X. Thus, all the possible σ-algebras are already contained in the in $\mathcal{P}(X)$ and there is no subset that can be added.

Exercise 1.10

Prove that the intersection of σ-algebras $\bigcap_\alpha \mathcal{S}_\alpha$ is a σ-algebra as well.

1. Since each $\mathcal{S}_\alpha$ is a σ-algebra by definition, it means that each family of subsets contains the empty set. Therefore, the intersection of all the $\mathcal{S}_\alpha$ contains as well the empty set: $\emptyset \in \bigcap_\alpha \mathcal{S}_\alpha$.

2. $A \in \bigcap_\alpha \mathcal{S}_\alpha \Rightarrow A \in \mathcal{S}_\alpha\ \forall \alpha$. Since each $\mathcal{S}_\alpha$ is a σ-algebra, we have that also $A^c \in \mathcal{S}_\alpha\ \forall \alpha$. Therefore, it is also true that $A^c \in \bigcap_\alpha \mathcal{S}_\alpha$.

3. Let $A_i \in \bigcap_\alpha \mathcal{S}_\alpha$ for $i \neq \alpha$. Then $A_i \in \mathcal{S}_\alpha\ \forall i, \alpha$. By definition of σ-algebra, we also have that $\bigcup_i A_i \in \mathcal{S}_\alpha\ \forall \alpha$. Thus, $\bigcup_i A_i \in \bigcap_\alpha \mathcal{S}_\alpha$.

Exercise 1.22
Prove the following:

- μ is monotone.
  Let $A \subset B$, then we can consider B as the union of two disjoint sets: $B = A \cup (B \setminus A)$. Therefore, since the measure is a positive and addictive function, we have that $\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\geq 0} \geq \mu(A)$.

- μ is countably addictive.
  $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i) - \underbrace{\sum \mu(\text{all possible intersections})}_{\geq 0} \leq \sum_{i=1}^\infty \mu(A_i)$.

Exercise 1.23
Show that $\lambda(A) = \mu(A \cap B)$ is a measure.

1. $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

2. $\lambda(\bigcup_{i=1}^\infty A_i) = \mu(\bigcup_{i=1}^\infty A_i \cap B) = \mu(\bigcup_{i=1}^\infty (A_i \cap B)) = \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i)$, where $A_i \neq A_j$ for $i \neq j$.

Exercise 1.26
Prove that $(A_1 \supset A_2 \supset A_3 \supset \cdots, A_i \in \mathcal{S}, \mu(A_1) < \infty) \Rightarrow (\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{i=1}^\infty A_i))$.
Let $B_i = A_1 \setminus A_i$ and $\mu(B_i) = \mu(A_1) - \mu(A_i)$.
We have that $\mu(\bigcup_{n=1}^\infty B_n) = \mu(A_1 \setminus \bigcap_{n=1}^\infty A_n) = \mu(A_1) - \mu(\bigcap_{n=1}^\infty A_n)$.
We also know (Thm 1.25) that $\mu(\bigcup_{n=1}^\infty B_n) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$.
It follows that $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{n=1}^\infty A_n)$.