

## Problem Set #1

DSGE Models, Prof. Kerk Phillips  
Martina Fraschini

### Exercise 1

In the Brock and Mirman's model the households solves the following dynamic program:

$$V(K_t, z_t) = \max_{K_{t+1}} \ln(e^{z_t} K_t^\alpha - K_{t+1}) + \beta E_t \{V(K_{t+1}, z_{t+1})\},$$

where the law of motion is:

$$z_{t+1} = \rho z_t + \varepsilon_t; \quad \varepsilon_t \sim i.i.d(0, \sigma^2).$$

The associated Euler equation is:

$$\frac{1}{e^{z_t} K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2}} \right\}.$$

In order to find an algebraic solution for the policy function we use the “guess and verify” method. We guess that the policy function is in the form of  $K_{t+1} = A e^{z_t} K_t^\alpha$  and we substitute it in the Euler equation. We obtain:

$$\frac{1}{e^{z_t} K_t^\alpha - A e^{z_t} K_t^\alpha} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} A^{\alpha-1} e^{(\alpha-1)z_t} K_t^{\alpha^2-\alpha}}{e^{z_{t+1}} A^\alpha e^{\alpha z_t} K_t^{\alpha^2} - A^{1+\alpha} e^{z_{t+1}} e^{\alpha z_t} K_t^{\alpha^2}} \right\},$$

$$\frac{1}{(1-A)e^{z_t} K_t^\alpha} = \beta E_t \left\{ \frac{\alpha A^{-1} e^{-z_t} K_t^{-\alpha} A^\alpha e^{\alpha z_t} K_t^\alpha e^{z_{t+1}}}{(1-A) A^\alpha e^{\alpha z_t} K_t^{\alpha^2} e^{z_{t+1}}} \right\},$$

$$\frac{1}{(1-A)e^{z_t} K_t^\alpha} = \beta E_t \left\{ \frac{\alpha}{(1-A) A e^{z_t} K_t^\alpha} \right\},$$

$$\alpha \beta = \frac{(1-A) A e^{z_t} K_t^\alpha}{(1-A) e^{z_t} K_t^\alpha},$$

$$A = \alpha \beta.$$

Therefore the policy function is  $K_{t+1} = \alpha \beta e^{z_t} K_t^\alpha$ .

### Exercise 2

Our baseline model with:

$$u(c_t, \ell_t) = \ln c_t + a \ln(1 - \ell_t),$$

and

$$f(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha},$$

has the following characterizing equations:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1},$$

$$\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left\{ \frac{1}{c_{t+1}} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\}, \\
-\frac{a}{1 - \ell_t} &= \frac{1}{c_t} w_t (1 - \tau), \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}, \\
w_t &= (1 - \alpha) e^{z_t} k_t^\alpha \ell_t^{-\alpha}, \\
\tau [w_t \ell_t + (r_t - \delta) k_t] &= T_t, \\
z_t &= (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.} (0, \sigma_z^2).
\end{aligned}$$

In this case we can't use the same tricks as in Exercise 1 to solve for the policy function because the model is too complex and it's impossible to guess a good algebraic solution.

### Exercise 3

Our baseline model with:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \ln(1 - \ell_t),$$

and

$$f(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha},$$

has the following characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1}, \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\}, \\
-\frac{a}{1 - \ell_t} &= c_t^{-\gamma} w_t (1 - \tau), \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1} \ell_t^{1-\alpha}, \\
w_t &= (1 - \alpha) e^{z_t} k_t^\alpha \ell_t^{-\alpha}, \\
\tau [w_t \ell_t + (r_t - \delta) k_t] &= T_t, \\
z_t &= (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.} (0, \sigma_z^2).
\end{aligned}$$

### Exercise 4

Our baseline model with:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi},$$

and

$$f(K_t, L_t, z_t) = e^{z_t} [\alpha K_t^\eta + (1 - \alpha) L_t^\eta]^{\frac{1}{\eta}},$$

has the following characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1}, \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\}, \\
-a(1 - \ell_t)^{-\xi} &= c_t^{-\gamma} w_t (1 - \tau), \\
r_t &= \alpha e^{z_t}, \\
w_t &= (1 - \alpha) e^{z_t}, \\
\tau [w_t \ell_t + (r_t - \delta) k_t] &= T_t, \\
z_t &= (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.} (0, \sigma_z^2).
\end{aligned}$$

### Exercise 5

Assuming  $\ell_t = 1$  and considering the market clearing condition  $L_t = \ell_t = 1$ , we have that our baseline model with:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma},$$

and

$$f(K_t, L_t, z_t) = K_t^\alpha (e^{z_t})^{1-\alpha},$$

has the following characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau) [w_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1}, \\
c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\}, \\
c_t^{-\gamma} w_t (1 - \tau) &= 0, \\
r_t &= \alpha k_t^{\alpha-1} (e^{z_t})^{1-\alpha}, \\
w_t &= 0, \\
\tau [w_t + (r_t - \delta) k_t] &= T_t, \\
z_t &= (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.} (0, \sigma_z^2).
\end{aligned}$$

The steady state version of this equations is:

$$\begin{aligned}
\bar{c} &= (1 - \tau) [\bar{w} + (\bar{r} - \delta) \bar{k}] + \bar{T}, \\
\bar{c}^{-\gamma} &= \beta E_t \left\{ \bar{c}^{-\gamma} [(\bar{r} - \delta) (1 - \tau) + 1] \right\}, \\
\bar{c}^{-\gamma} \bar{w} (1 - \tau) &= 0, \\
\bar{r} &= \alpha \bar{k}^{\alpha-1} (e^{\bar{z}})^{1-\alpha}, \\
\bar{w} &= 0, \\
\tau [\bar{w} + (\bar{r} - \delta) \bar{k}] &= \bar{T}.
\end{aligned}$$