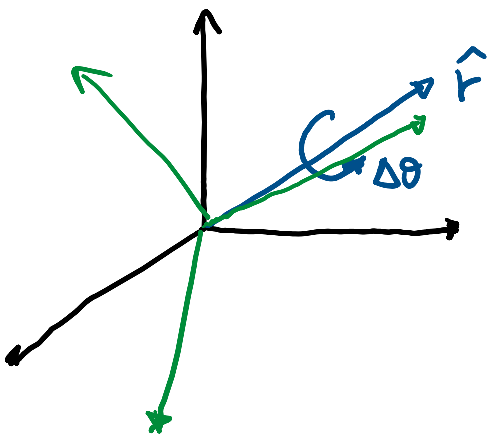
Slides have been created by Prof. Michele Focchi

webpage: https://mfocchi.github.io/Teaching/

E0.3-Representation of orientation

Angle-axis & Unit quaternion

ANGLE-AXIS REPRESENTATION OF ORIENTATION



$\hat{r}$: axis about which rotation is made (unit vector in $\mathbb{R}^3$)

$\Delta\theta$: angle: magnitude of rotation $\in \mathbb{R}^3$ (positive CCW w.r.t. axis, right-hand rule)

$\Downarrow$

$-$ 4 parameters: non minimal representation of orientation

In order to map to rotation matrix we use Rodriguez formula:

$$\boxed{R(\Delta\theta, \hat{r}) = \underbrace{\hat{r}\hat{r}^T}_{\text{outer product}} + (I - \hat{r}\hat{r}^T)\cos\Delta\theta + \underbrace{[\hat{r}]_\times}_{\text{skew-simmetric matrix}} \sin\theta}$$

direct problem

Note:
$(\Delta\theta, \hat{r})$ and $(-\Delta\theta, -\hat{r})$ give same result $\Rightarrow$ mapping is not injective $\Rightarrow$ solving inverse problem gives 2 solutions

PROPERTIES OF $R(\Delta\theta, \hat{r})$

- axis is invariant to rotation : $R(\theta, \hat{r})\hat{r} = \hat{r}$

RECALL :

$A x = \lambda x$ x is an eigenvector of A

$R(\theta, \hat{r}) \rightarrow \hat{r}$ is an eigenvector associated to an eigenvalue $\lambda = 1$

- $\det(R) = \prod \lambda_i = 1$

- $\mathrm{Tr}(R) = \sum \lambda_i = 1 + 2\cos\theta$

- R is not injective map $R(\theta, r) = R(-\theta, -r)$

ROTATION MATRIX TO ANGLE - AXIS - inverse problem

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \Longrightarrow \hat{r}, \Delta\theta \, ?$$

$$R = \hat{r}\hat{r}^T + (I - \hat{r}\hat{r}^T)\cos\theta + [\hat{r}]_\times \sin\theta \rightarrow$$ 9 equations in 4 unknowns

$\Rightarrow \exists$ analytic solution!

- $Tr(R) = 1 + 2\cos\theta = \sum r_{ii}$

$$\Delta\theta = \arccos\left(\frac{r_{11} + r_{12} + r_{13} - 1}{2}\right) \rightarrow$$ provides only values in $[0, \pi]$

- remember that $R - R^T$ is skew symmetric

$$R - R^T = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ R_{21} - R_{12} & 0 & R_{23} - R_{32} \\ R_{31} - R_{13} & R_{32} - R_{23} & 0 \end{bmatrix} = 2\sin\Delta\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

- Therefore we can get the axis:

$$r = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\Delta\theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

$\hookrightarrow$ $\Delta\theta \approx 0$, $\sin\Delta\theta \approx 0$ $\Rightarrow$ axis is not defined
$\Delta\theta \approx \pi$

$\Rightarrow$ singular case

SUMMARY ANGLE - AXIS

⊕ Advantageous in the problem of trajectory planning for orientation (done on $\Delta\theta$)

$$\Delta\theta(s) = \Delta\theta_i + [\Delta\theta_f - \Delta\theta_i] s$$

⊖ Non minimal representation: 4 params

→ components of $\hat{r}$ are not independent but constrained by the condition

$$r_x^2 + r_y^2 + r_z^2 = 1$$

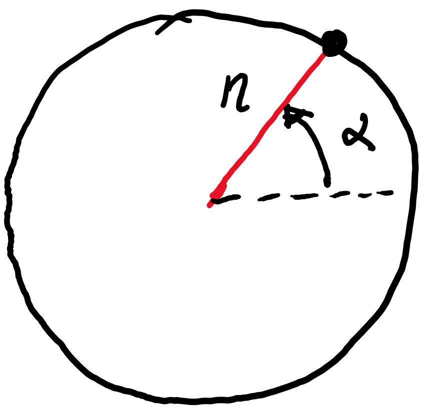
⊖ mapping to rotation matrix not unique

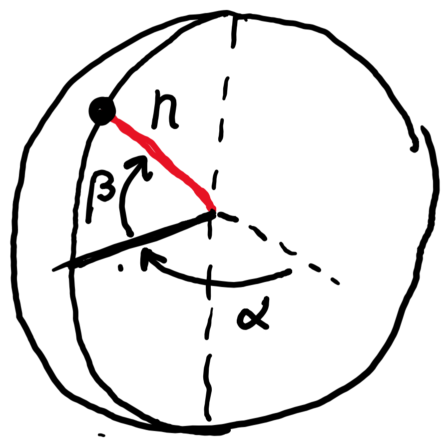
$$R(-\theta, -\hat{r}) = R(\theta, r)$$

UNIT QUATERNION

⊕ 4 params – non minimal
no singularity

⊖ unit norm constraint





?

point on a circle
η, α

point on a sphere
η, α, β

- quaternion is like a point on a 4D sphere
  $\eta, \alpha, \beta, \gamma$

UNIT QUATERNION

To eliminate singular cases of axis/angles and euler angles we can use unit quaternion representation (e.g. to describe satellite orientation)

$$Q = \begin{bmatrix} \eta \\ \varepsilon \end{bmatrix} \begin{matrix} \in \mathbb{R} & \text{scalar part} \\ \in \mathbb{R}^3 & \text{vector part} \end{matrix}$$

- unit norm : $\eta^2 + \varepsilon^T \varepsilon = 1$ $\boxed{\|Q\| = 1}$
- $(\vartheta, \hat{r})$ and $(-\vartheta, -\hat{r})$ give the same quaternion Q
  $\Rightarrow$ The duplication disappears! $\oplus$

To do computations we need a special algebra:

- No rotation $Q = (1, 0^T)$
- inverse of a quaternion $Q^{-1} = \bar{Q} = \begin{bmatrix} \eta \\ -\varepsilon \end{bmatrix}$

- quaternion multiplication

$$Q_3 = Q_2 \otimes Q_1 = \begin{bmatrix} \eta_1 \eta_2 - \varepsilon_1^T \varepsilon_2 \\ \eta_1 \varepsilon_2 + \eta_2 \varepsilon_1 + \varepsilon_1 \times \varepsilon_2 \end{bmatrix}$$

ANGLE-AXIS TO QUATERNION

$$Q = \begin{bmatrix} \eta = \cos\left(\frac{\Delta\theta}{2}\right) \\ \varepsilon = \hat{r} \sin\left(\frac{\Delta\theta}{2}\right) \end{bmatrix}$$

for small $\Delta\theta \approx 0$ $\begin{bmatrix} 1 \\ \frac{\hat{r}}{2} \end{bmatrix}$

QUATERNION TO ANGLE-AXIS

$$\begin{bmatrix} \Delta\theta = 2 \arccos(\eta) \\ \hat{r} = \frac{\varepsilon}{\|\varepsilon\|} = \frac{\varepsilon}{\sqrt{1-\eta^2}} \end{bmatrix}$$

RODRIGUEZ FORMULA FOR QUATERNIONS

$$R = (\eta^2 - \varepsilon^T \varepsilon) I + 2 \varepsilon \varepsilon^T - 2\eta [\varepsilon]_\times$$

References:

- Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors, James Diebel, 2006.