F0-Dynamics of rigid body-

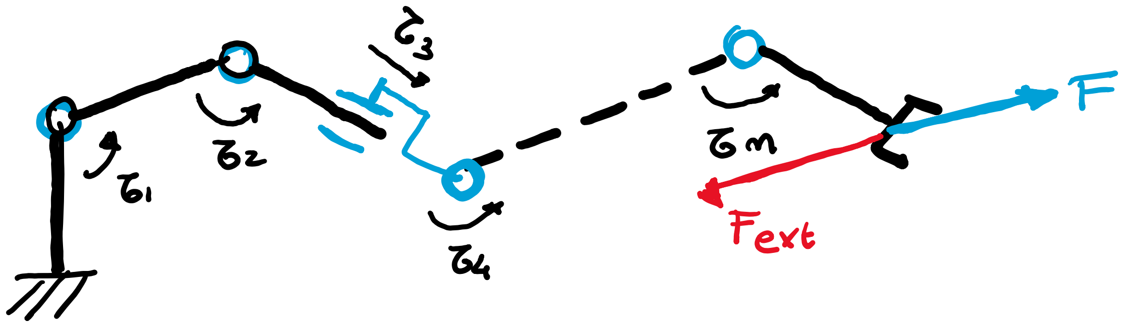
statics

STATICS VS DYNAMICS

DYNAMICS: studies the cause of the motion (fores are involved)

KINEMATICS: Just describes the motion

STATICS: case where we have forces but no motion (equilibrium)

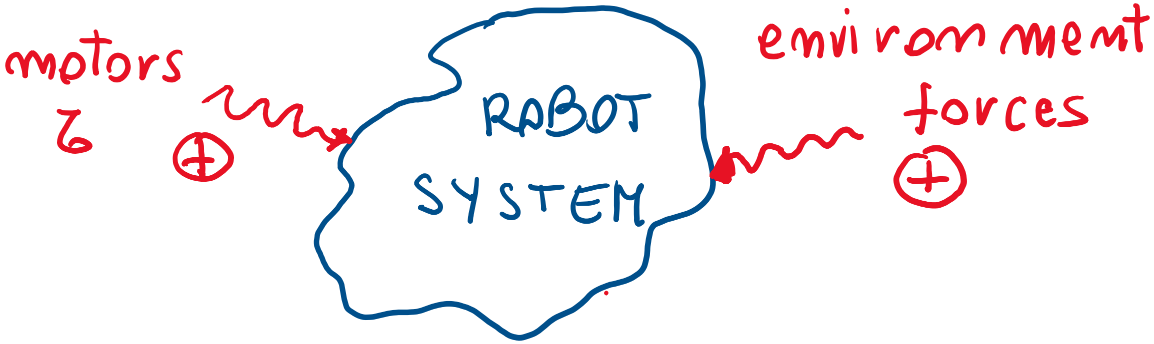


- τ forces / torques exerted by motors at joints (joint efforts)
- F equivalent force (result of joint efforts applied along the structure) exerted by the robot at the end-effector (or any other point...)
- F_{ext} forces by environment at end-effector $\boxed{F_{ext} = -F}$

  (3° principle of dinamics of action/reaction)

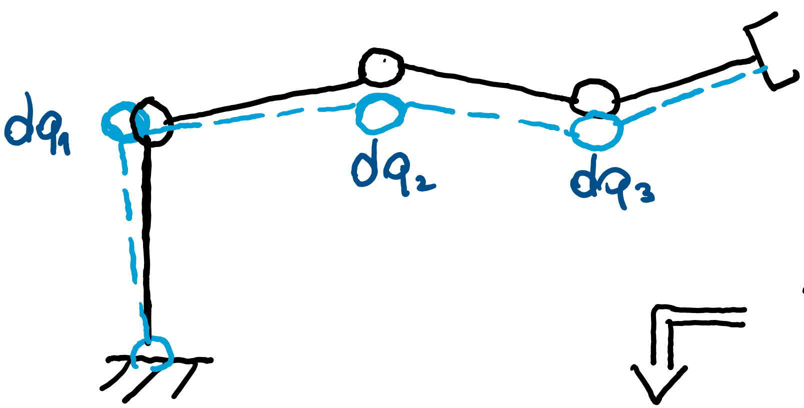
  environment reaction is equal and opposite to robot action

WORK



CONVENTION: positive work is done when forces are applied ON the robot

Question of statics: what are the joint torques that will balance an F_{ext} exerted by the environment?



$$\begin{pmatrix} dp \\ d\phi \end{pmatrix} = J\,dq$$

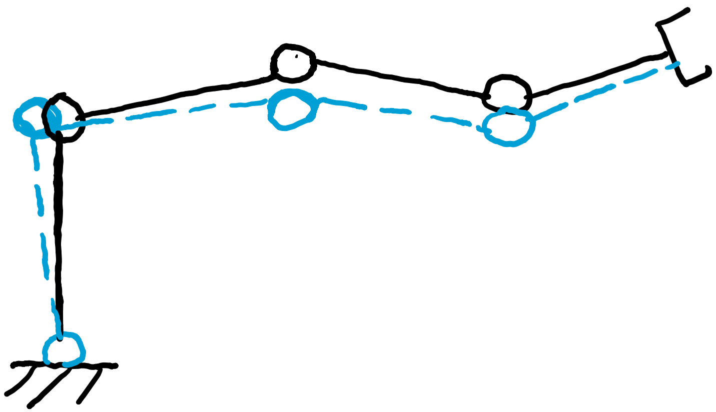
at equilibrium we can define:

dq_i : infinitesimal displ.

δq_i : virtual displacement (satisfy constraints)

- no kin. energy variation
- zero velocity (no dissipation)

PRINCIPLE OF VIRTUAL WORKS



$\triangleq$ sum of virtual works done by all torques/forces acting on the system is zero at equilibrium

$$\underbrace{\tau^T dq}_{\text{work of joints}} - \underbrace{F^T \begin{pmatrix} dp \\ d\phi \end{pmatrix}}_{\text{work of end-effector}} = \tau^T \delta q - F^T J dq = 0$$

$$(\tau^T - F^T J)\, dq = 0 \quad \forall\, dq \quad \Rightarrow \quad \tau^T - F^T J = 0$$

$$\boxed{\tau = J^T F}$$

KINETO - STATIC DUALITY

Joint space $\xrightarrow{J(q)}$ **Task space**

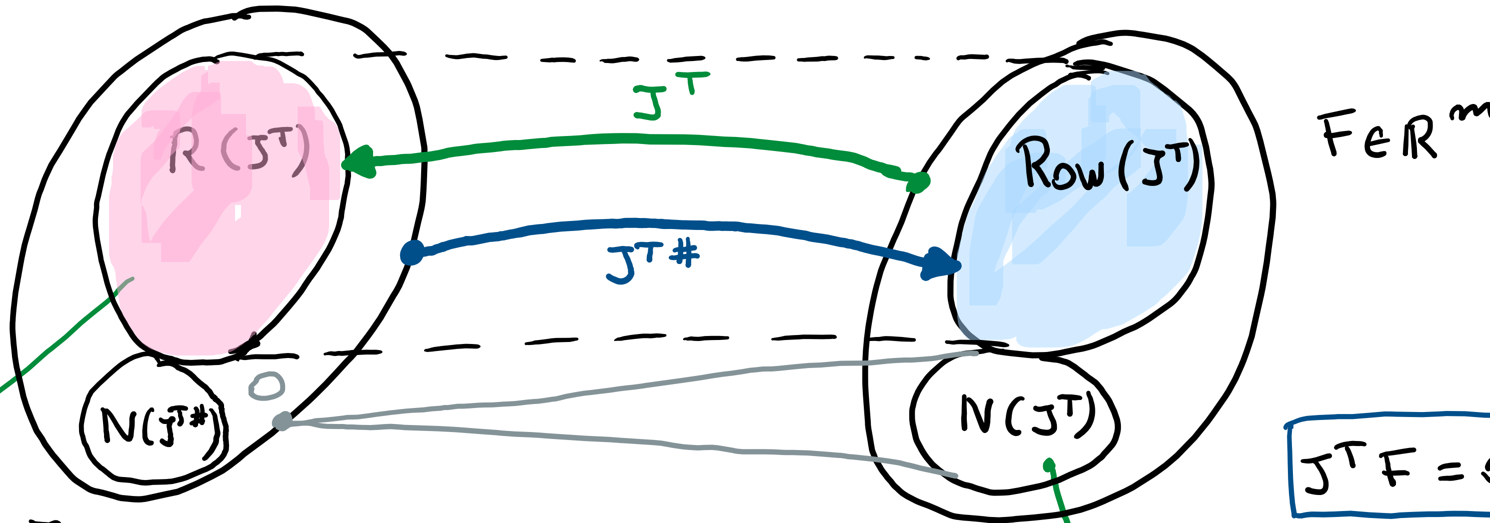
velocity $\dot{q}$
(or displacement dq)

velocity in Cartesian space v
(or E-E displacement $\begin{bmatrix} dp \\ d\phi \end{bmatrix}$)

forces/torques at joints τ $\xleftarrow{J^T(q)}$ forces/moments at the E-E

$\text{rank}(J) = \text{rank}(J^T) \Rightarrow$ singular configurations are the same for force/velocity mappings, and the same analysis holds

similarly To what we did in differential kinematics...
for a given configuration q



$\tau \in R^n$

$\boxed{J^T F = 0}$

Joint Torques That can balance end-effector forces

forces that do not need any Torque To be balanced (They are balanced by reaction forces of The structure)

$$R(J^T) = \{ \tau \in \mathbb{R}^n : \exists F \in \mathbb{R}^m, J^T F = \tau \}$$

$$N(J^T) = \{ F \in R^m : J^T F = 0 \}$$

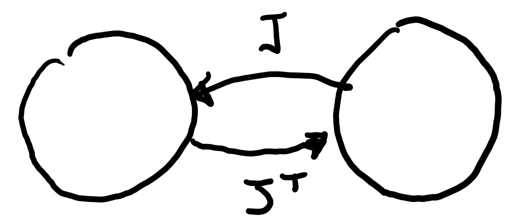
MAPPING END EFFECTOR FORCES

① Non-redundant manipulator $(m = n)$

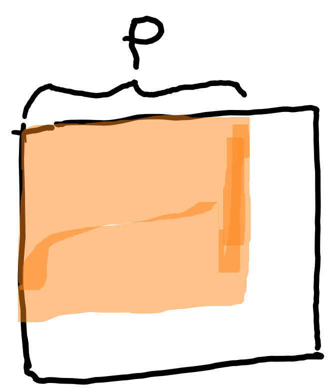
$$\tau = J^T f$$

1.1 $\text{rank}(J/J^T) = m$ $\quad (\det(J) \neq 0)$

$$\exists \cancel{N}(J), \quad \nexists N(J^T)$$

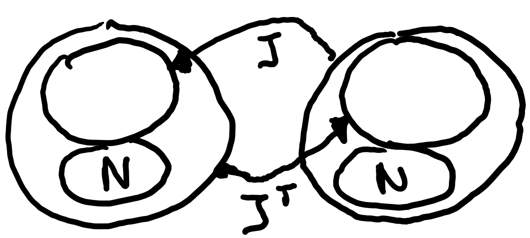


1.2 $\text{rank}(J/J^T) < m$



$$\exists N(J) : \exists \dot{q} \neq 0 : J\dot{q} = 0$$

$$\exists N(J^T) : \exists F \neq 0 : J^T F = 0$$

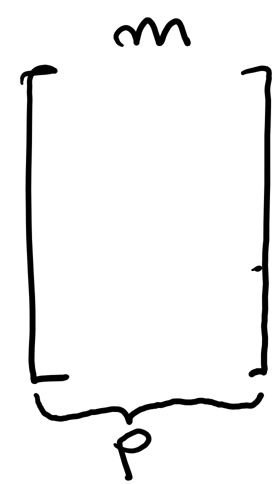


② Redundant manipulator $(m < n)$

J^T is rectangular and skinny

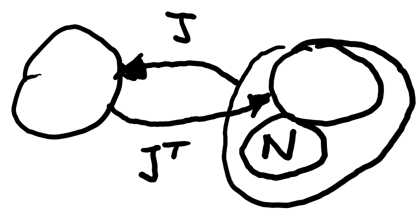
$$J^T = \begin{array}{c c} & m \\ n & \begin{bmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix} \end{array}$$

2.1 if $\operatorname{rank}(J / J^T) = m$ (Full)



$\exists N(J) \Rightarrow \exists \dot{q} \neq 0 : J\dot{q} = 0$

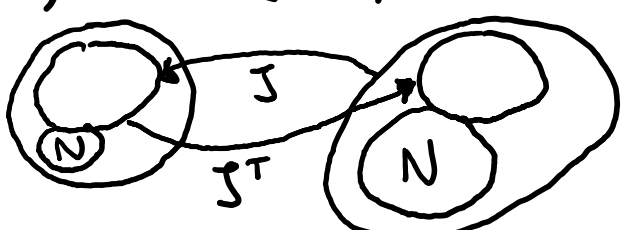
$\nexists N(J^T)$



2.2 if $\operatorname{rank}(J / J^T) < m$

$\exists N(J) \Rightarrow \exists \dot{q} \neq 0 : J\dot{q} = 0$

$\exists N(J^T) \Rightarrow \exists F \neq 0 : J^T F = 0$



EXAMPLE CASE 1.2 $m = n = 2$ $\rho(J) < m = n$

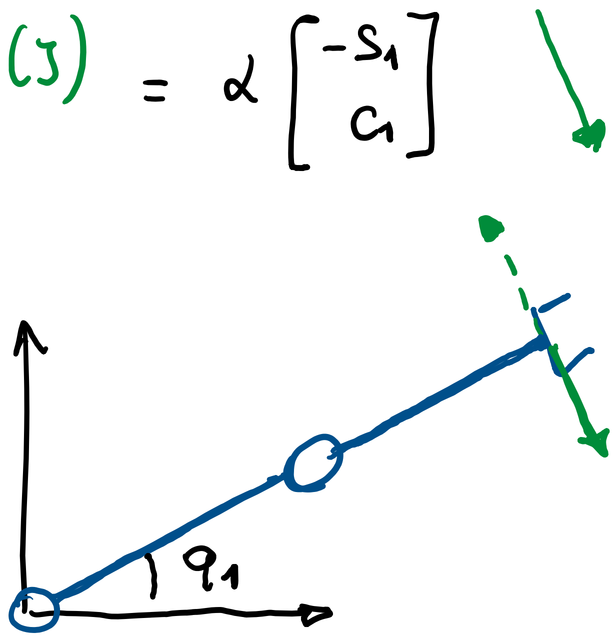
singularity $\Rightarrow q_2 = 0 \Rightarrow \rho(J) = 1$

$$\begin{cases} R(J) & (1) \\ N(J) & (1 = 2 - 1) \\ R(J^T) & (1) \\ N(J^T) & (1 = 2 - 1) \end{cases}$$

$$J/_{q_2=0} = \left[\begin{array}{c|c} -(\ell_1+\ell_2)s_1 & -\ell_2 s_1 \\ (\ell_1+\ell_2)c_1 & \ell_2 c_1 \end{array}\right]$$

$\rightarrow$ second column is $\ell_2/(\ell_1+\ell_2)$ first column not lin. indipendent

$$R(J) = \alpha \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix}$$

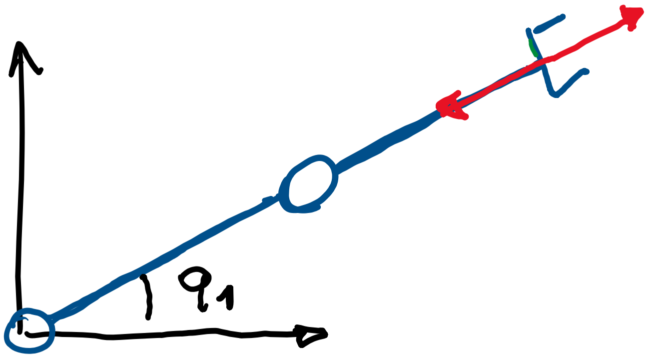


$R(J)$ allowed motions

$$J^T\big/_{q_2=0} = \begin{bmatrix} -(\ell_1+\ell_2)\,s_1 & (\ell_1+\ell_2)\,c_1 \\ -\ell_2\,s_1 & \ell_2\,c_1 \end{bmatrix}$$

$N(J^T)$: find a vector v such that $J^T v = 0$

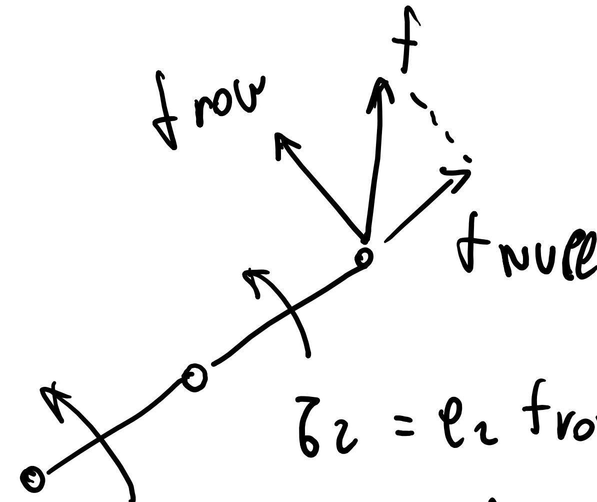
$$J^T \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad N(J^T) = \alpha \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$$



$N(J^T)$ forces absorbed by the structure

$$J^T\big|_{q_2=0} = \begin{bmatrix} -(l_1+l_2)\,s_1 & (l_1+l_2)\,c_1 \\ -l_2 s_1 & l_2 c_1 \end{bmatrix}$$

$$R(J^T) = \alpha \begin{bmatrix} l_1+l_2 \\ l_2 \end{bmatrix}$$



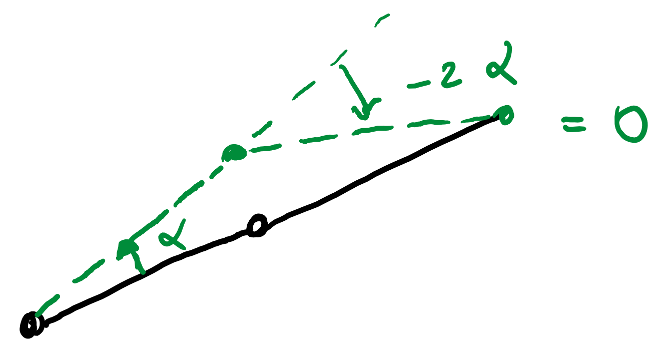
$\tau_2 = l_2\, f_{row}$

$\tau_1 = l_1 + l_2\, f_{row}$

$\tau = \alpha \begin{bmatrix} l_1+l_2 \\ l_1 \end{bmatrix}$

$$J\big|_{q_2=0} = \left[\begin{array}{c|c} -(l_1+l_2)\,s_1 & -l_2 s_1 \\ (l_1+l_2)\,c_1 & l_2 c_1 \end{array}\right]$$

$$N(J) = \alpha \begin{bmatrix} l_2 \\ -(l_1+l_2) \end{bmatrix}$$



RECAP ON SPACES

feasible motions

$R(J^{T\#}) \cong R(J)$

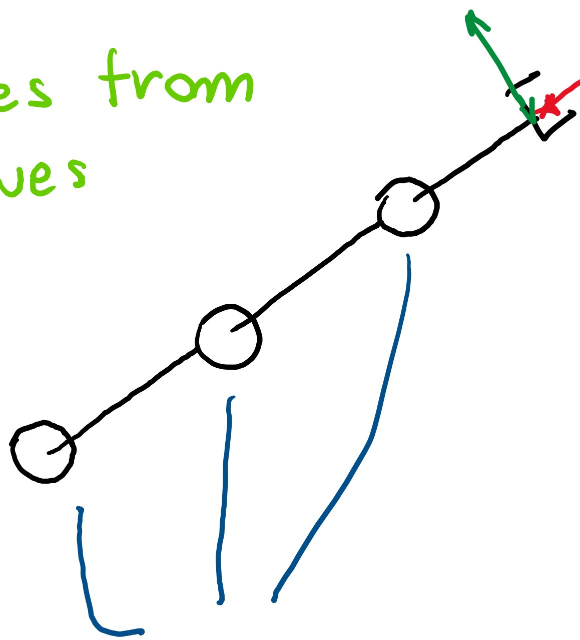
forces from torques

same dimension

$N(J^{\#}) \cong N(J^{T})$

forbidden motion

force from structure



live in joint space

$N(J) \rightarrow$ self motion

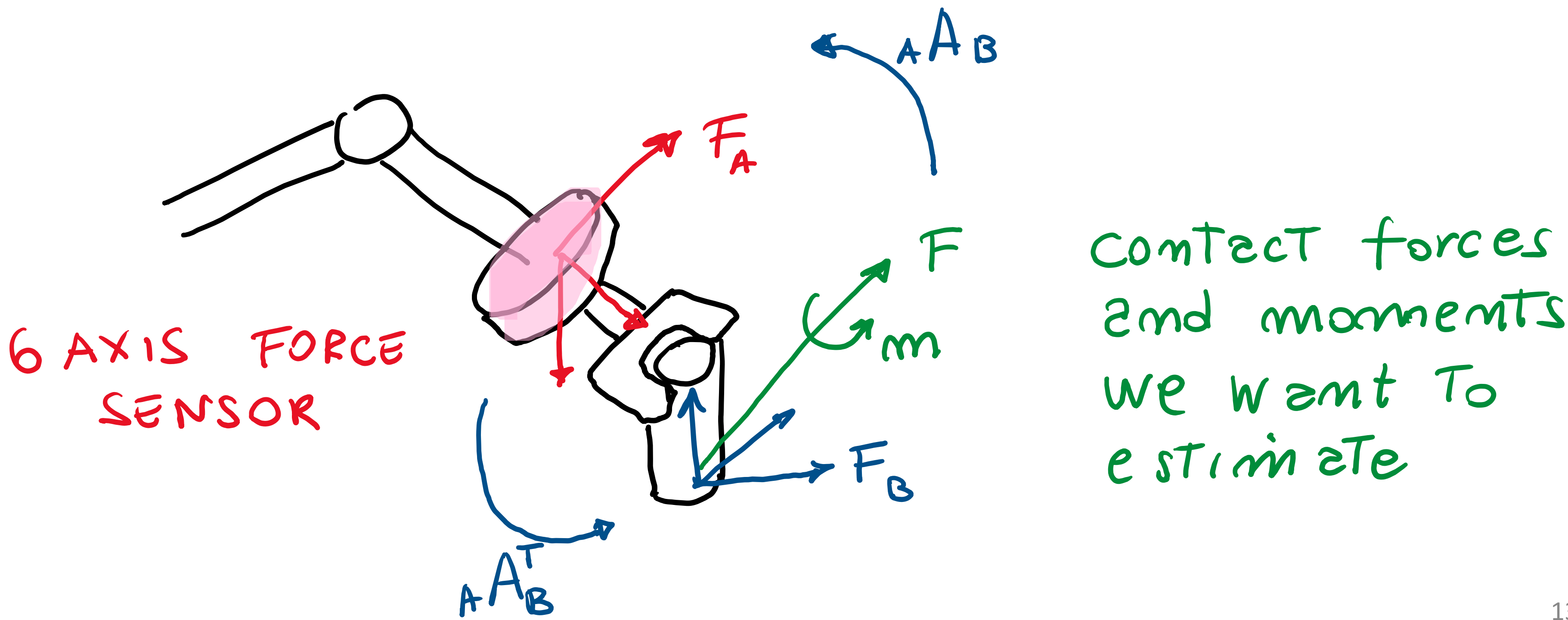
$\cong$

$N(J^{T\#}) \rightarrow$ torques that produce self motion

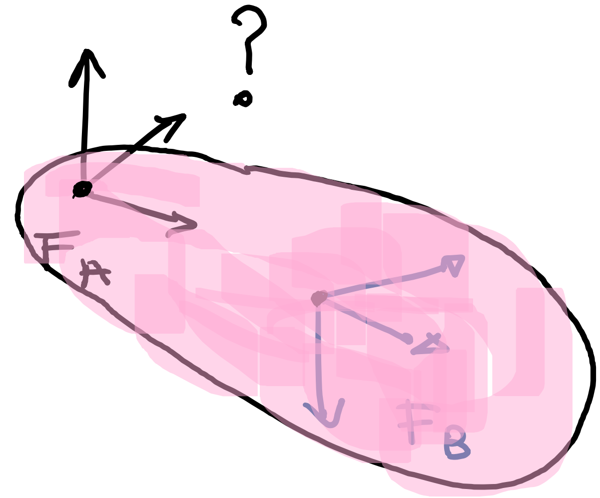
VELOCITY AND FORCE TRANSFORMATIONS

- we can apply virtual work principle to map forces / moments at different places of a rigid body

- example: compute tool forces from force sensor readings



VELOCITY TRANSFORMATION



$$ {}_A v_A = {}_A v_B + {}_A\omega_B \times {}_A p_{BA} $$

$$ = {}_A R_B \, {}_B v_B + {}_A R_B \left({}_B\omega_B \times {}_B p_{BA} \right) $$

$$ = {}_A R_B \, {}_B v_B - {}_A R_B \left(\left[{}_B p_{BA} \right]_\times {}_B\omega_B \right) $$

*rigid body mapping:
lecture E3 slide 22 with $\dot{d}$=0

$$ = \begin{bmatrix} {}_A R_B & -{}_A R_B \left[{}_B p_{BA} \right]_\times \end{bmatrix} \begin{bmatrix} {}_B v_B \\ {}_B\omega_B \end{bmatrix} $$

$$ {}_A\omega = {}_A R_B \, {}_B\omega \quad \Longleftarrow \quad \omega_B = \omega_A = \omega $$

rigid body

$$\boxed{\begin{bmatrix} {}_A v_A \\ {}_A \omega \end{bmatrix} = \underbrace{\begin{bmatrix} {}_A R_B & -{}_A R_B \left[{}_B P_{BA} \right]_\times \\ 0 & {}_A R_B \end{bmatrix}}_{{}_A A_B} \begin{bmatrix} {}_B v_B \\ {}_B \omega \end{bmatrix}}$$

FORCE TRANSFORMATION

We can compute the dual mapping for forces

$$\begin{bmatrix} {}_B f \\ {}_B m \end{bmatrix} = {}_A A_B^T \begin{bmatrix} {}_A f \\ {}_A m \end{bmatrix}$$

$$\begin{bmatrix} {}_A R_B^T & 0 \\ \left({}_A R_B \left[{}_B p_{BA} \right]_\times \right)^T & {}_A R_B^T \end{bmatrix}$$

$$S^T(v) = - S(v)$$

$$= \begin{bmatrix} {}_B R_A & 0 \\ \left[{}_B p_{BA} \right]_\times {}_B R_A & {}_B R_A \end{bmatrix}$$