FAI LAB 7
First-Order logic

Paolo Morettin

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First-Order Logic (FOL)

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- **Objects**: $Paolo$, $Roberto$, 3
- **Relations**: $SurvivedPhD(\cdot)$, $Paper(\cdot)$, $Wrote(\cdot, \cdot)$, $Coauthors(\cdot, \cdot)$
- **Functions**: $CV(\cdot)$, $PublicationDate(\cdot)$, $ErdosNumber(\cdot)$, $(\cdot + \cdot)$, ...

$$SurvivedPhD(Paolo) \wedge Coauthors(Paolo, Roberto)$$

$$|CV(Roberto)| > |CV(Paolo)|$$

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$$|CV(Roberto)| > |CV(Paolo)|$$

Variables and quantifiers we can compactly describe knowledge over many/infinite objects:

$$\forall x \,.\, \forall y \,.\, [Coauthor(x,y) \leftrightarrow Coauthor(y,x)]$$
$$\forall x \,.\, [HasPhD(x) \rightarrow \exists y \,.\, (Paper(y) \wedge AuthorOf(x,y))]$$
$$\forall x \,.\, \forall y \,.\, [Coauthors(x,y) \rightarrow \exists z \,.\, (AuthorOf(x,z) \wedge AuthorOf(y,z))]$$

Example: Erdös numbers

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Exercises: NL to FOL

Write the following natural language sentence in FOL using the following predicates/functions: $\mathit{IsFriend}(\cdot,\cdot)$, $\mathit{IsEnemy}(\cdot,\cdot)$, $\mathit{Hates}(\cdot,\cdot)$

- *Karen has a friend who has a friend who hates Karen.*
- *Hate might not be mutual.*
- *The enemy of an enemy is a friend.*
- *People can't have friends if they hate themselves.*

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$$\exists x \,.\, \exists y \,.\, [\mathit{IsFriend}(x, \mathit{Karen}) \wedge \mathit{IsFriend}(y, x) \wedge \mathit{Hates}(y, \mathit{Karen})]$$

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$$\exists x \,.\, \exists y \,.\, [\mathit{Hates}(x, y) \wedge \neg \mathit{Hates}(y, x)]$$

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- *The enemy of an enemy is a friend.*

$$\forall x \,.\, \forall y \,.\, \forall z \,.\, [(\mathit{IsEnemy}(x, z) \wedge \mathit{IsEnemy}(y, x)) \rightarrow \mathit{IsFriend}(y, z)]$$

- *People can't have friends if they hate themselves.*

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$$\exists x \, . \, \exists y \, . \, [IsFriend(x, Karen) \wedge IsFriend(y, x) \wedge Hates(y, Karen)]$$

- *Hate might not be mutual.*

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- *The enemy of an enemy is a friend.*

$$\forall x \, . \, \forall y \, . \, \forall z \, . \, [(IsEnemy(x, z) \wedge IsEnemy(y, x)) \rightarrow IsFriend(y, z)]$$

- *People can't have friends if they hate themselves.*

$$\forall x \, . \, [Hates(x, x) \rightarrow \forall y \, . \, \neg IsFriend(y, x)]$$

Exercises: NL to FOL

Write the following natural language sentence in FOL using the following predicates/functions: $\mathit{Movie}(\cdot)$, $\mathit{FeaturedIn}(\cdot,\cdot)$, $\mathit{Loves}(\cdot,\cdot)$, $\mathit{Year}(\cdot)$

- *Alicia loves every movie featuring Julia.*
- *Alicia and Julia are featured in a movie together.*
- *Alicia doesn't love every movie filmed in the 80s.*
- *Julia never featured in a movie before 1987.*

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$$\exists x \;.\; [Movie(x) \land (Year(x) \in [1980, 1989]) \land \neg Loves(Alicia, x)]$$

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- *Julia never featured in a movie before 1987.*

$$\neg \exists x \,.\, [Movie(x) \wedge (Year(x) < 1987) \wedge FeaturedIn(Julia, x)]$$

Exercises: FOL semantics

- The formula $\exists x \,.\, (x + x = x)$ is satisfiable.
- The formula $\exists x \,.\, (x + x = x)$ is valid.
- The formula $\neg\exists x \,.\, (x + x = x)$ is satisfiable.
- The formula $\neg\exists x \,.\, (x + x = x)$ is valid.
- The formula $\exists x \,.\, (x < x)$ is satisfiable.
- The formula $\exists x \,.\, [P(x) \wedge \neg P(x)]$ is satisfiable.
- The formula $\forall x \,.\, [P(x) \vee \neg P(x)]$ is satisfiable.
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- The formula $\exists x\,.\,(x + x = x)$ is satisfiable. **True**
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FOL semantics: a warning

$$\exists x \,.\, (x < x)$$

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$$\exists x \,.\, (x < x)$$

$$\exists x \,.\, (x \because x)$$

$$\exists x \,.\, (x \sphericalangle x)$$

FOL semantics: a warning

$$\exists x \,.\, (x < x)$$

$$\exists x \,.\, (x \therefore x)$$

$$\exists x \,.\, (x \sphericalangle x)$$

$$\exists x \,.\, P(x,x)$$