Slides have been created by Prof. Michele Focchi

webpage: https://mfocchi.github.io/Teaching/

E0.1-Representation of orientation

Rotation Matrix

RIGID BODY $\triangleq$ 3D object where the distance between any couple of points remains constant

Rigid motion $\triangleq$ motion of a rigid body in space

Manipulators are serial chains of rigid bodies (links) connected by joints

MATH REVIEW

- **vector norm**: $\|\cdot\|$: induces a metric in the vector space introducing the notion of distance
  - is a scalar, always non negative (≥ 0)
  - represents the magnitude of a vector
  - euclidean norm:

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} \qquad v \in \mathbb{R}^n$$

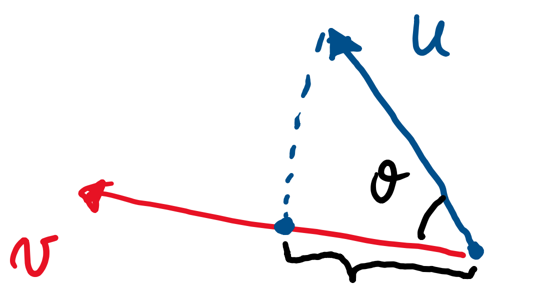
- **unit vector**: $\hat{u} \in V$ (V space with a norm) is a vector of lenghts 1 co-directional with u

$$\hat{u} = \frac{u}{\|u\|} \qquad \text{e.g.} \quad [1, 0, 0], [0, 1, 0]$$

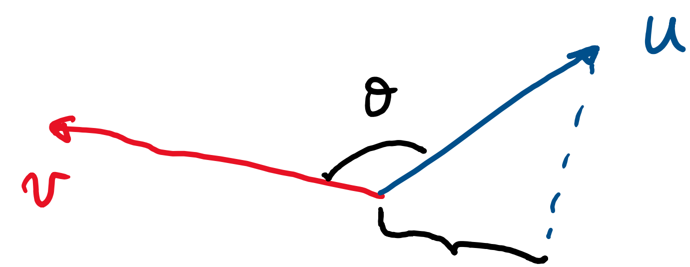
- dot / scalar product of vectors $v, u \in \mathbb{R}^n$ is a scalar defined by:

$$\langle v, u \rangle = v \cdot u = v^T u = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

- intuitively is the projection of u onto v
- is commutative : $u^T v = v^T u$
- $\|v\|_2 = \sqrt{v^T v}$
- in $\mathbb{R}^3$: $v^T u = \|v\| \|u\| \cos\theta$



$\underbrace{\qquad\qquad}_{v^T u > 0}$



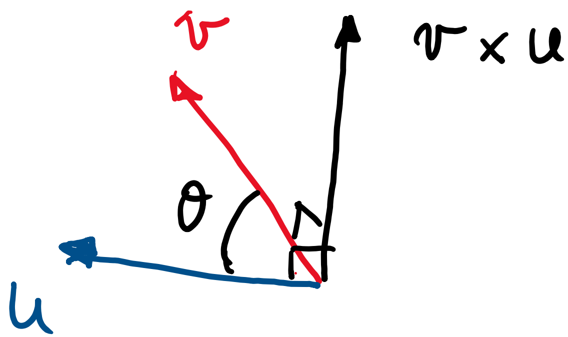
$\underbrace{\qquad\qquad}_{v^T u < 0}$!

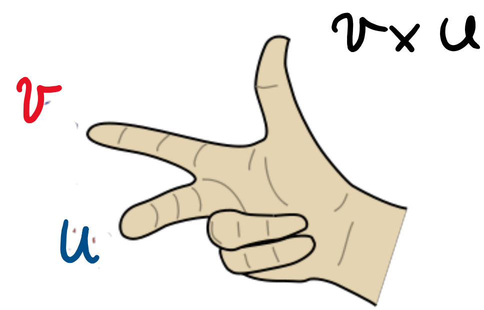
- cross / vector product : $v \times u$ is a **vector** that:

① is only defined in $\mathbb{R}^3$

② is orthogonal to the plane defined by u and v

③ direction defined by right hand rule





④ $\|v \times u\| = \|v\| \, \|u\| \sin\theta$

⑤ $$v \times u = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_2 u_3 - v_3 u_2 \\ -(v_1 u_3 - v_3 u_1 \\ v_1 u_2 - v_2 u_1 \end{bmatrix}$$

6. a cross product can also be written as the product of a matrix with a vector

$$v \times u = \underbrace{\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$[v]_\times \triangleq$ skew symmetric matrix associated to the cross product

- **gradient:** derivative of a multivariate function $f: \mathbb{R}^n \to \mathbb{R}$ is the column vector:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \qquad f(x_1, \dots x_n) \in \mathbb{R}$$

- **Jacobian:** if we have $f: \mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix where the entry $J_{ij} = \frac{\partial f_i}{\partial x_j}$

$f = (f_1(x), \dots, f_m(x))$

$$J = \left[\frac{\partial f}{\partial x_1} \,\vdots\, \cdots \,\vdots\, \frac{\partial f}{\partial x_n} \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla f_1^T \\ \vdots \\ \nabla f_m^T \end{bmatrix}$$

TRIGONOMETRIC FORMULAS

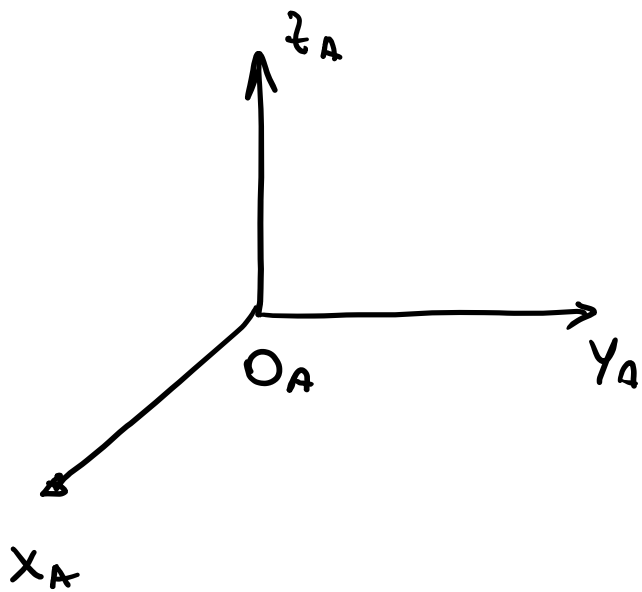
Sum and addition:

- $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$
- $\cos(\alpha \mp \beta) = \cos\alpha \cos\beta \pm \sin\alpha \sin\beta$

REFERENCE FRAME (COORDINATE SYSTEM)

$\triangleq$ Tuple (O_A, x_A, y_A, z_A) where O_A is the origin and x_A, y_A, z_A are the axes

- We consider right handed frames $(x_A \times y_A = z_A)$



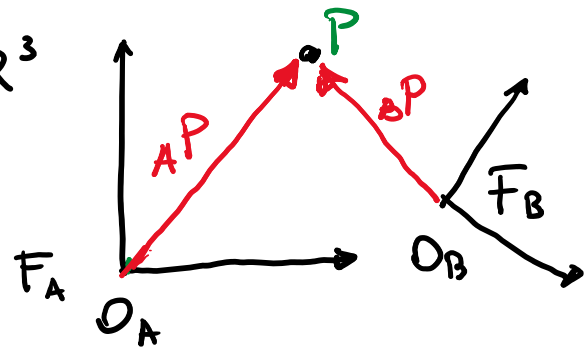
In an Euclidean space the axes are $\perp$ each other

GEOMETRICAL AND PHYSICAL VECTORS

We can use vectors to represent:

A - geometrical points (i.e. position of a point in a rigid body, vertex of a polygon)
They are called geometrical vectors

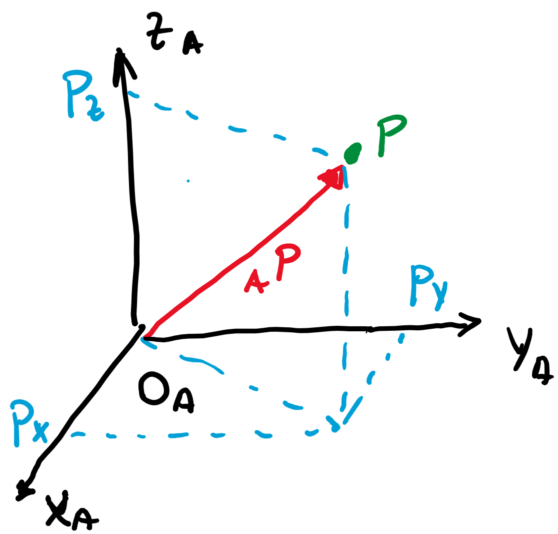
$P \in \mathbb{R}^3$



- a geometrical vector (e.g. $_AP$) is the representation of a geometrical point (e.g. P) and depends on the reference frame (e.g. F_A or F_B)
- $_AP$ represents the directed segment $\overrightarrow{O_A P}$, the left subscript A represents the frame where it is expressed

EXAMPLE of geometrical vector = position of a point in space

$_A P = [P_x, P_y, P_z] \in \mathbb{R}^3$ A : coord. system where it is expressed

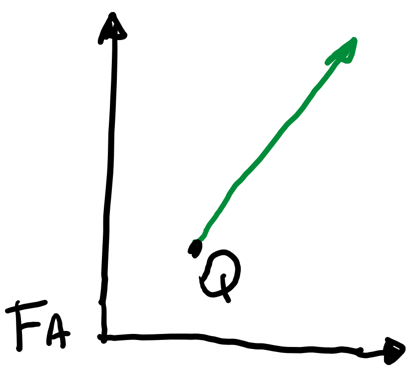


NOTE:

- P is the geometric entity (independent from the choice of frame)
- $_A P$ is its representation in frame A (set of numbers called coordinates)
- $_A P$ represents the distance from O_A
  - magnitude (norm)
  - specifies direction (versor)

- P_x, P_y, P_z are the cartesian coordinates and depend on the choice of the reference frame
- also cylindrical coordinates could be used

B - physical quantities (e.g. linear/angular velocities, accelerations, force, moments) can be represented by physical vectors



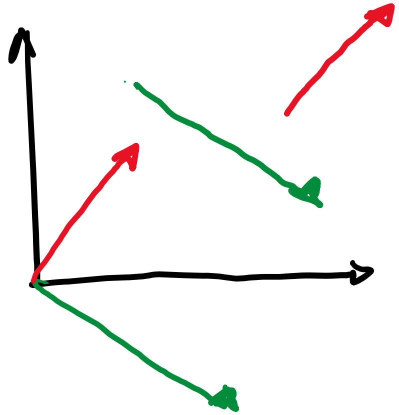
They have:

- magnitude
- direction
- application point Q

if Q:

① does not have physical/geometric meaning (e.g. velocity of a point can be translated every where without changing its meaning)

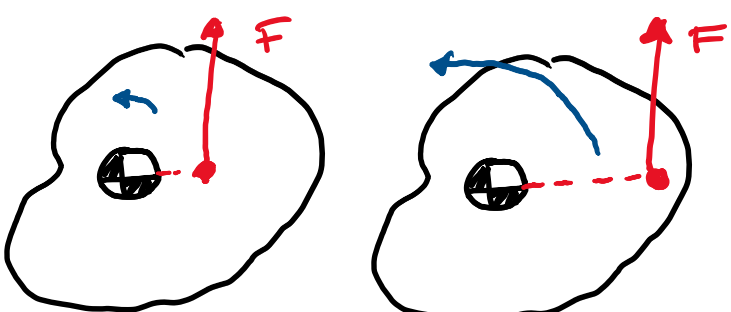
⇒ free vectors



free vectors are the same if they have same magnitude and direction, application point is not fixed

(B) if Q cannot be changed without affecting the physical effect that produces ⇒ applied vectors

EXAMPLE : a force acting on a rigid body, if we change the application point the moment about the center of mass changes

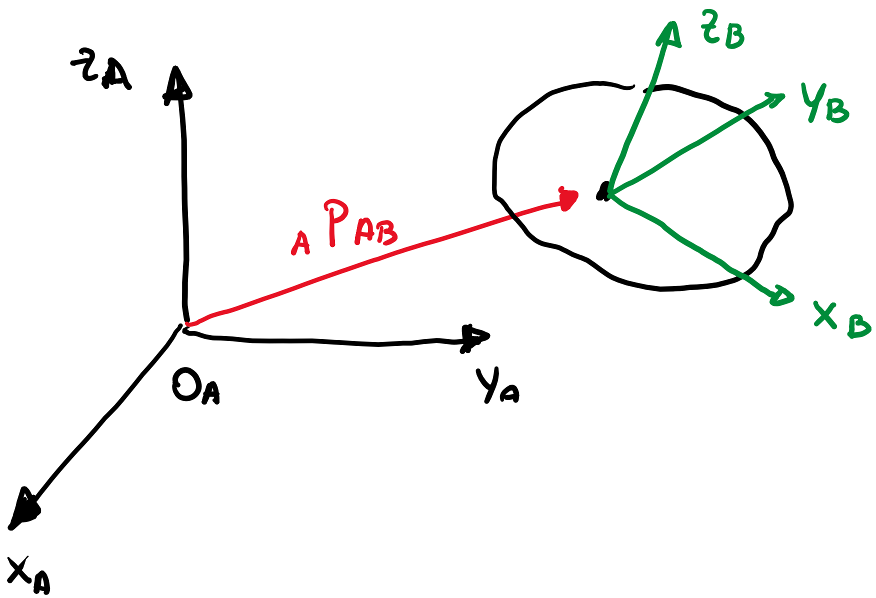


if we are interested only in translational motion then F becomes a free vector

POSE OF A RIGID BODY

≜ representation of position / orientation of a rigid body, determined by:

1. position of one of its points (e.g. CoM)
2. its orientation w.r.t a given frame (e.g. A)



⟹ a rigid body can also have a coordinate system that it is rigidly attached to the body (moves with the boby but is not necessarily inside the body)

to describe orientation of rigid body:

1. attach coord system B to body (O_B, x_B, y_B, z_B)
2. Describe its orientation w.r.t. fixed frame A

REPRESENTATION OF ORIENTATION: ROTATION MATRIX

$$ {}_A R_B = \left[\, {}_A x_B \;\vdots\; {}_A y_B \;\vdots\; {}_A z_B \, \right] $$

→ unit axes of frame B expressed in frem A

↳ Projected onto frame A axes

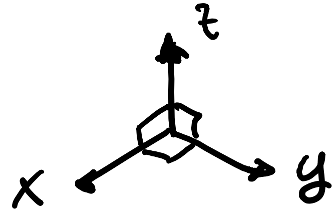
① INTERPRETATION: ORIENTATION OF FRAME

${}_A R_B$ represents the orientation of frame B w.r.t frame A

└ PROPERTIES:

(P1) 3×3 matrix (9 entries) (in 2D is 2×2)

(P2) orthonormal = orthogonal ⊕ normal



orthogonal ⇓ columns orthogonal each other

$${}_A x_B^T \, {}_A y_B = 0$$
$${}_A x_B^T \, {}_A z_B = 0$$
$${}_A y_B^T \, {}_A z_B = 0$$

normal ⇓ columns are unit vectors

$${}_A x_B^T \, {}_A x_B = 1 \approx \|{}_A x_B\|^2 = 1$$
$${}_A y_B^T \, {}_A y_B = 1$$
$${}_A z_B^T \, {}_A z_B = 1$$

6 independent constraints ⇐

9 - 6 = 3 DoFs

↳ non minimal repr.

$$\boxed{R^{-1} = R^T} \Rightarrow R^T R = R R^T = I$$

(P3) $\det R = +1 \Rightarrow$ proper rotations (ccw wrt rot. axis)

EXAMPLE

P3 not ok

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \det = -1 \Rightarrow \cancel{R}$$

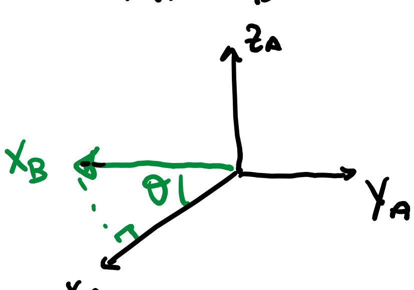
Now let's have a better look at ${}_A R_B$

${}_A x_B$, ${}_A y_B$, ${}_A z_B$: projections of frame B axes onto frame A

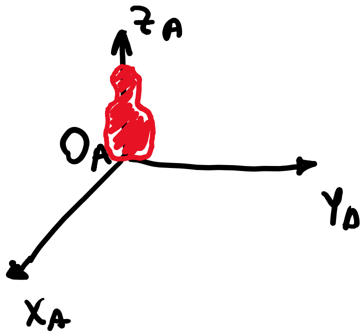
$${}_A R_B = \left[\begin{array}{c|c|c} x_A^T x_B & x_A^T y_B & x_A^T z_B \\ y_A^T x_B & y_A^T y_B & y_A^T z_B \\ z_A^T x_B & z_A^T y_B & z_A^T z_B \end{array}\right]$$

The third column is direction cosines of z_B axis w.r.t frame A axes

$$x_A^T x_B = \underbrace{\|x_A\| \|x_B\|}_{1 \text{ (unit vectors)}} \cos\theta = \cos\theta$$

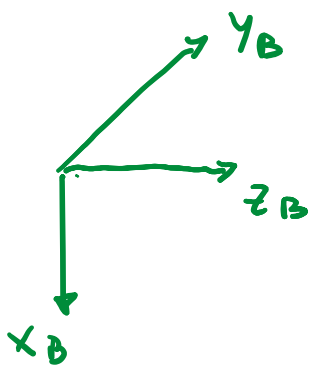


EXAMPLE



$$
{}_A R_B = \begin{array}{ccc}
\left[\begin{array}{c|c|c} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right] \\
{}_A x_B \quad\; {}_A y_B \quad\; {}_A z_B
\end{array}
$$

how frame B looks?



${}_A z_B$ → aligned with Y_A

→ orthogonal with X_A, Z_A

${}_A y_B$ → orthogonal to Y_A, Z_A

opposite to X_A

${}_A x_B$ → orthogonal to X_A, Y_A

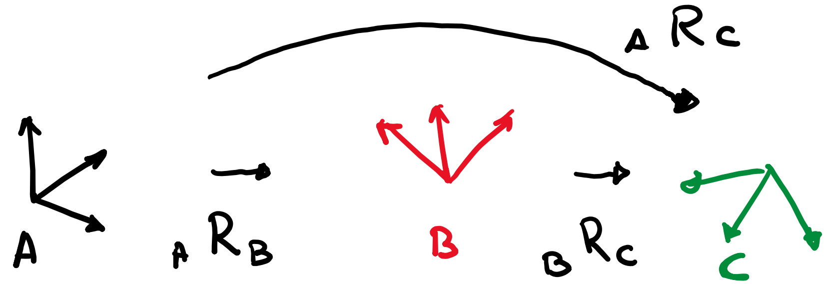
opposite to Z_A

To know frame B you need to know frame A

not just numbers

COMPOSITION OF ROTATIONS

$${}_A R_{\cancel{B}} \cdot {}_{\cancel{B}} R_C = {}_A R_C$$



orient. of B wrt A

orient of C wrt B

NB product of (rotation) matrices is NOT commutative!

$${}_A R_B \, {}_B R_C \neq {}_B R_C \, {}_A R_B$$

ROTATION MATRIX INVERSE

${}_B R_A$ = represents the orientation of frame A as seen from frame B $\Rightarrow$ inverse operation of ${}_A R_B$

$$\Rightarrow \quad {}_A R_B \; {}_B R_A = {}_A R_A = I$$

$$\Rightarrow \quad \boxed{{}_A R_B^{-1} = {}_A R_B^{T} = {}_B R_A}$$

ROTATIONS GROUP

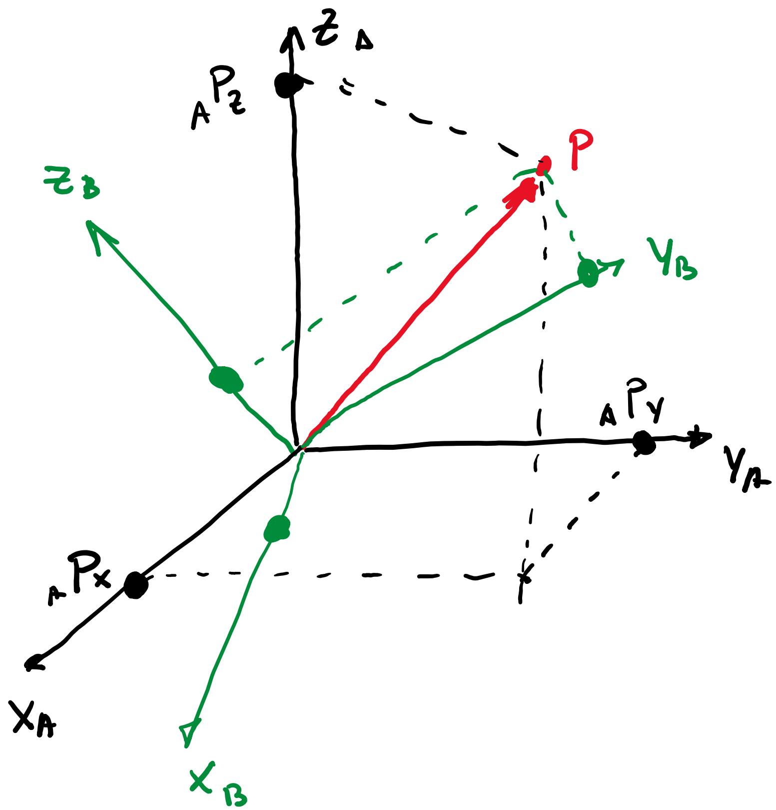
Rot. matrix form a group called special (det=1) orthogonal group of dim 3 : SO(3)

Ⓐ product of elements (composition) it is still an element of the group

Ⓑ $\exists$ neutral element : $I \Rightarrow R I = R$

Ⓒ $\exists$ inverse of element : $R^T \Rightarrow R R^T = I$

② INTERPRETATION: CHANGE OF COORDINATES OF VECTOR



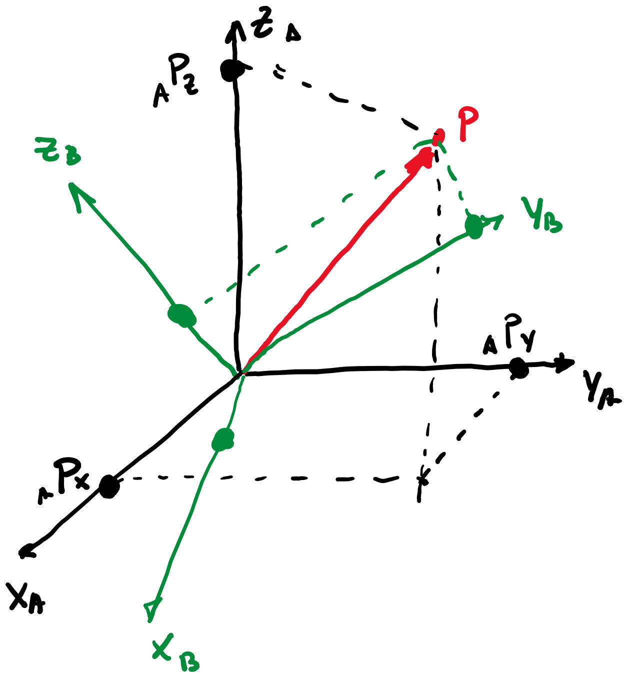
- assume point does not move
- ${}_A R_B$ can be seen also as a linear operator that maps a vector from frame B to frame A

let's first rewrite ${}_A P$ differently:

$$
{}_A P = \begin{bmatrix} {}_A P_x \\ {}_A P_y \\ {}_A P_z \end{bmatrix} = I \, {}_A P = \underset{\begin{bmatrix}1\\0\\0\end{bmatrix}}{{}_A \hat{x}_A} \, {}_A P_x + \underset{\begin{bmatrix}0\\1\\0\end{bmatrix}}{{}_A \hat{y}_A} \, {}_A P_y + \underset{\begin{bmatrix}0\\0\\1\end{bmatrix}}{{}_A \hat{z}_A} \, {}_A P_z
$$

•

- now if we have available the coordinates of P w.r.t frame B and we want To get ${}_A P$:



$$
{}_A P = \left[{}_A X_B \;\vdots\; {}_A Y_B \;\vdots\; {}_A Z_B \right] \underbrace{\begin{bmatrix} {}_B P_x \\ {}_B P_y \\ {}_B P_z \end{bmatrix}}_{}^{{}_B P} = {}_A \hat{X}_B \left({}_B P_x \right) + {}_A \hat{Y}_B \left({}_B P_y \right) + {}_A \hat{Z}_B \left({}_B P_z \right)
$$

components of P in B frame

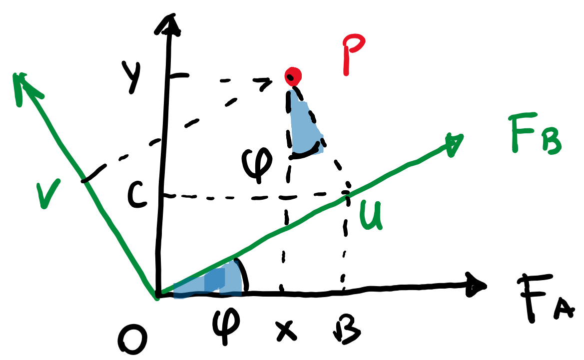
${}_A R_B$ performs a "change" of coordinates of vector P from fame B To frame A

- To get ${}_B P$ from ${}_A P$:

$${}_A P = {}_A R_B \; {}_B P$$

$${}_B P = \left({}_A R_B\right)^{-1} {}_A P = {}_A R_B^{\,T} \; {}_A P = {}_B R_A \; {}_A P$$

ELEMENTARY ROTATION AROUND z AXIS



- F_A, F_B have common origin
- ${}_B P = (u, v, w) \Rightarrow {}_A P = (x, y, z)$

$$x = OB - xB = u\cos\varphi - v\sin\varphi$$

$$y = OC + CY = u\sin\varphi + v\cos\varphi$$

$$z = w \quad \rightarrow \text{does not change}$$

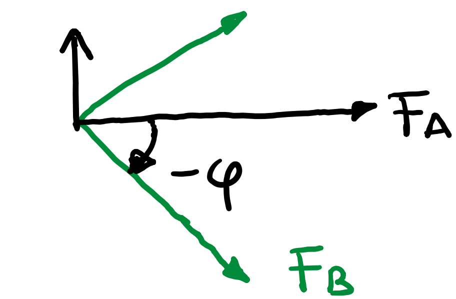
let's reorganize in matrix form:

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{{}_A P} = \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z(\varphi)} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{{}_B P}$$

↳ elementary change of orient. about z axis

- (cyan) z axes of A and B frames are coincident
- (green) directional cosines of x_B about frame A
- (magenta) directional cosines of y_B about frame A

- $R_z(-\varphi) = R_z(\varphi)^T$



$R_z^T(\varphi) =$ orient. of F_A wrt F_B

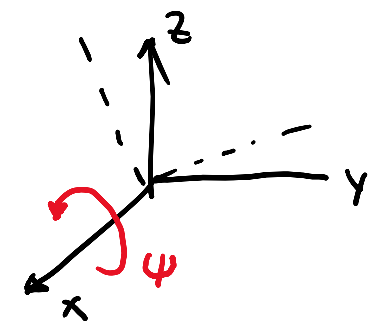
$\Downarrow$

if I rotate F_A by φ to get F_B then I need to rotate F_B of $-\varphi$ to get frame F_A

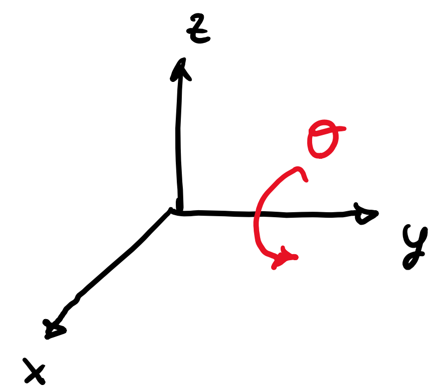
EXERCISE: ELEMENTARY ROTATIONS

get elementary rotations around x axis, y axis

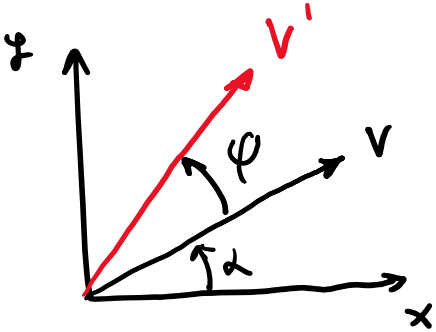
$$R_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$



$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



③ INTERPRETATION : ROTATION OF A VECTOR INSIDE A FRAME



- operator on vectors

$$V' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\to\, v}$$

- assume rotation around z $(z' = z)$

$$x = \|V\| \cos\alpha \qquad x' = \|V\| \cos(\alpha + \varphi) = \|V\| (\cos\alpha \cos\varphi - \sin\alpha \sin\varphi)$$
$$= x \cos\varphi - y \sin\varphi$$

$$y = \|V\| \sin\alpha \qquad y' = \|V\| \sin(\alpha + \varphi) = \|V\| (\sin\alpha \cos\varphi + \cos\alpha \sin\varphi)$$
$$= x \sin\varphi + y \cos\theta$$

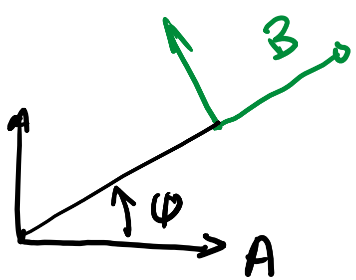
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\varphi) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- holds also for generic orientations : $R(\psi, \theta, \varphi)$

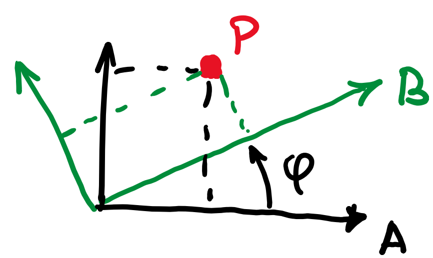
SUMMARY ROTATION MATRIX INTERPRETATIONS

same rotation matrix ${}_A R_B$ can represent:

① orientation of a frame B wrt another frame A



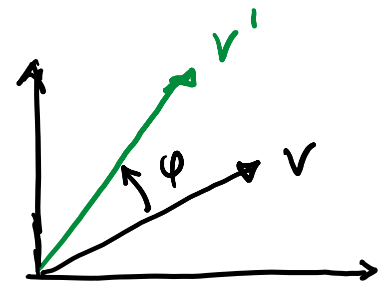
② change of coordinates of vector from rotated frame B to original frame A



$${}_A P = {}_A R_B \; {}_B P$$

$\downarrow$ e.g $R_z(\varphi)$

③ operator which rotates vectors



$$v' = R(\psi, \theta, \psi) \, v$$

DERIVATIVE OF A ROTATION MATRIX

$${}_A R_B \; {}_A R_B^T = I \quad \text{orthogonality}$$

$$\downarrow d/dt$$

$$\underbrace{{}_A \dot{R}_B \; {}_A R_B^T}_{S} + {}_A R_B \; {}_A \dot{R}_B^T = \emptyset \qquad S + S^T = 0$$

Skew-symmetric

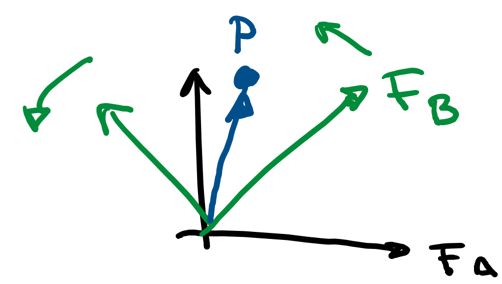
$${}_A \dot{R}_B \; {}_A R_B^T = S \quad \Rightarrow \quad {}_A \dot{R}_B = S \; {}_A R_B$$

- how does S look like?

- consider constant vector P in a rotating frame B, its coords. in A are:

$${}_A P = {}_A R_B(t) \; {}_B P$$

↳ Time varying



$$ {}_A\dot{P} = {}_A\dot{R}_B\,{}_BP + {}_AR_B\,\cancel{{}_B\dot{P}}^{=0} = {}_A\dot{R}_B\,{}_BP $$

$$ {}_A\dot{P} = S\,\underbrace{{}_AR_B\;{}_BP}_{{}_AP} = S\,{}_AP \qquad (1) $$

- from mechanics: velocity of point in a frame rotating at ω is:

$$ {}_A\dot{P} = {}_A\omega \times {}_AP \qquad (2) $$

$$ (1) = (2) \Rightarrow S = [{}_A\omega]_\times $$

$$ \boxed{{}_A\dot{R}_B = [{}_A\omega]_\times\,{}_AR_B} $$

that is equivalent to:

$$ \boxed{{}_A\dot{R}_B = S\,{}_AR_B = {}_A\omega \times {}_AR_B} $$

SKEW-SYMMETRIC MATRIX

- square matrix
- $A = -A^T$ (0 on diagonal, $A_{ij} = -A_{ji}$ off-diagonal)
- $x^T A x = 0 \quad \forall x$
- 3×3 skew sym. matrix can be used to represent cross product $a \times b$ as matrix multiplication $[a]_\times b$

$$a = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad S(a) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad \text{or} \quad [a]_\times$$

$\Rightarrow$ in this special case the following properties hold:

- $[Ra]_\times = R[a]_\times R^T$
- $[a]_\times b = -[b]_\times a$
- $[a]_\times [a]_\times = \underbrace{a a^T}_{\text{outer product}} - \underbrace{a^T a}_{\|a\|^2} I_{3\times3}$

SYMMETRIC MATRIX

- square matrix
- $A = A^T$
- $x^T A x \neq 0 \quad \forall x \neq 0$
- real eigenvalues

from generic A

$$\frac{A + A^T}{2} \qquad \frac{A - A^T}{2}$$

$\downarrow$ symmetric $\qquad \downarrow$ skew-sym

POSITIVE DEFINITE MATRIX

- symmetric matrix
- $x^T A x > 0 \quad \forall x \neq 0$
- positive real eigenvalues

E0.2-Representation of orientation

Euler angles

- rotation matrices have some disadvantages as representation of orientation:

① DISADVANTAGE: NON MINIMAL REPRESENTATION

- Rotation matrix is not a minimal representation

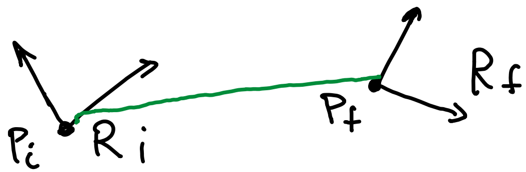
 9 parameters

− 6 constraints

$$\begin{array}{ll} u^T u = 1 & u^T v = 0 \\ v^T v = 1 & v^T w = 0 \\ w^T w = 1 & u^T w = 0 \\ (3) & (3) \end{array}$$

= 3 independent variables

② DISADVANTAGE : INTERPOLATION



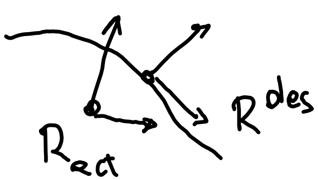
position: $P(s) = P_i + (P_f - P_i)s \quad s \in [0,1]$

orientation: $R(s) = R_i + (R_f - R_i)s$?

NO!

$\forall s$ $R(s)$ might not be a rotation matrix (eg constraints not satisfied

③ DISADVANTAGE : DEFINITION OF ORIENT. ERROR



$R_{error} \neq R^{des}(t) - R_{act}(t)$

how can be able to treat orient/pos. errors the same?

EULER ANGLES

(φ, θ, ψ) is not a vector but just a sequence of angles
cannot be combined as a vector space.

- define a sequence of rotations around independent axes of reference frame
  - around fixed (a_i) / moving (a_i') axes
  - 12 + 12 possible combinations without contiguous repetitions of axes (no xxz, $yz'z'$)

xyx = rotation about x / rotation about y / about x

$\Phi =$

xyx	zxy
xyz	zxz
xzx	zyx
xzy	zyz
yxy	
yxz	
yzx	
yzy	

■ proper angles

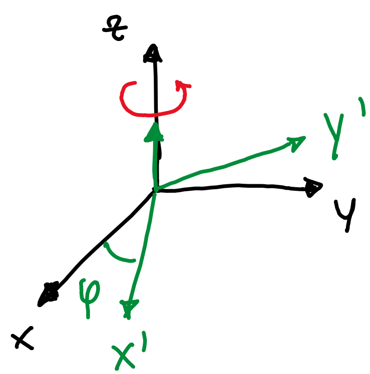
□ Tait bryan angles

→ same for moving axes

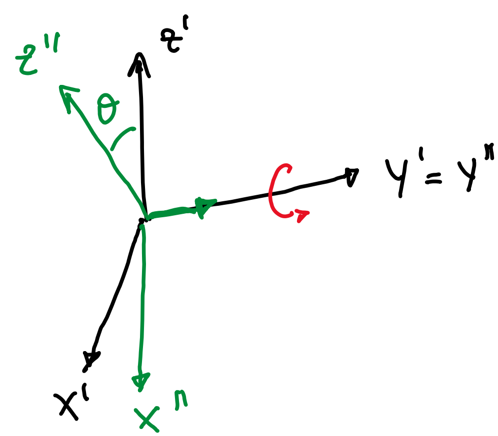
e.g. $x\,y'\,x''$

- only 12 matter $\boxed{(a_1,\alpha_1)(a_2,\alpha_2)(a_3,\alpha_3) = (a_3',\alpha_3)(a_2',\alpha_2)(a_1',\alpha_1)}$

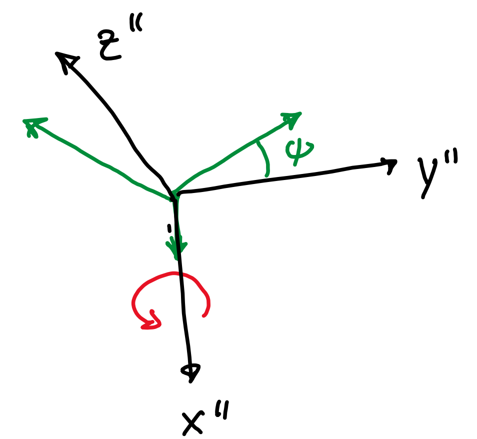
- we will consider only moving axes: frame w.r.t each rotation occurs is about the current frame not the original one (fixed axis)
- we choose a sequence $\Phi = z\,y'\,x''$ that is mostly used in robotics



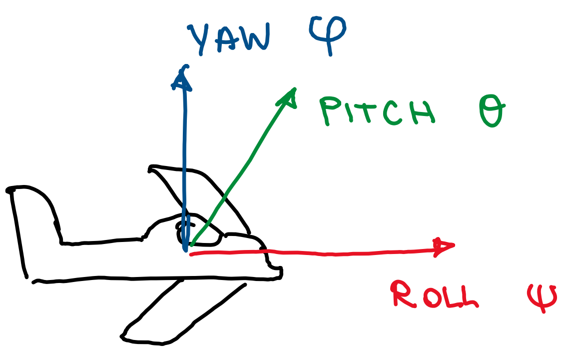
φ around z



θ around y'

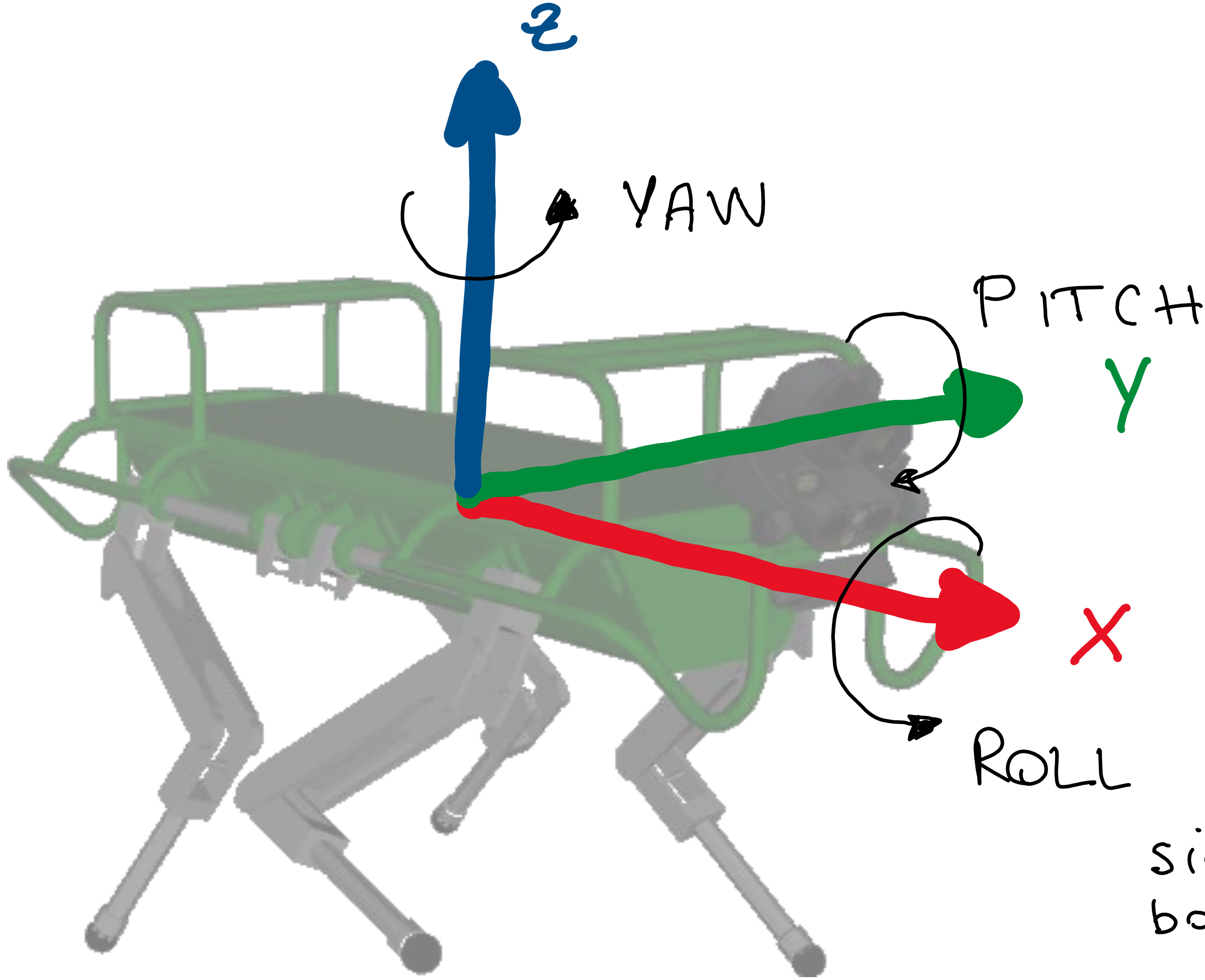


ψ around x''

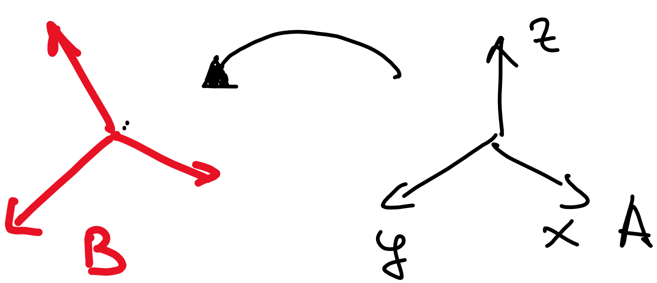


NOTE:

z, y', x'' will not be orthogonal!



siciliano's book



Rotation about x : ROLL ψ

Rotation about y : PITCH θ

Rotation about z : YAW φ

DIRECT PROBLEM: WITH MOVING AXES

To obtain the final orientation associated to $\phi = (\varphi, \theta, \psi)$ is:

$$\boxed{R(\phi) = R_z(\varphi)\, R_{y'}(\theta)\, R_{x''}(\psi)}$$

$z\, y'\, x''$ axes moving

order of rotations $\longrightarrow$ $z \rightarrow y' \rightarrow x''$

- With successive rotations (moving axes) post-multiply the previous rotation with the following one
- The rotations are elementary rotations

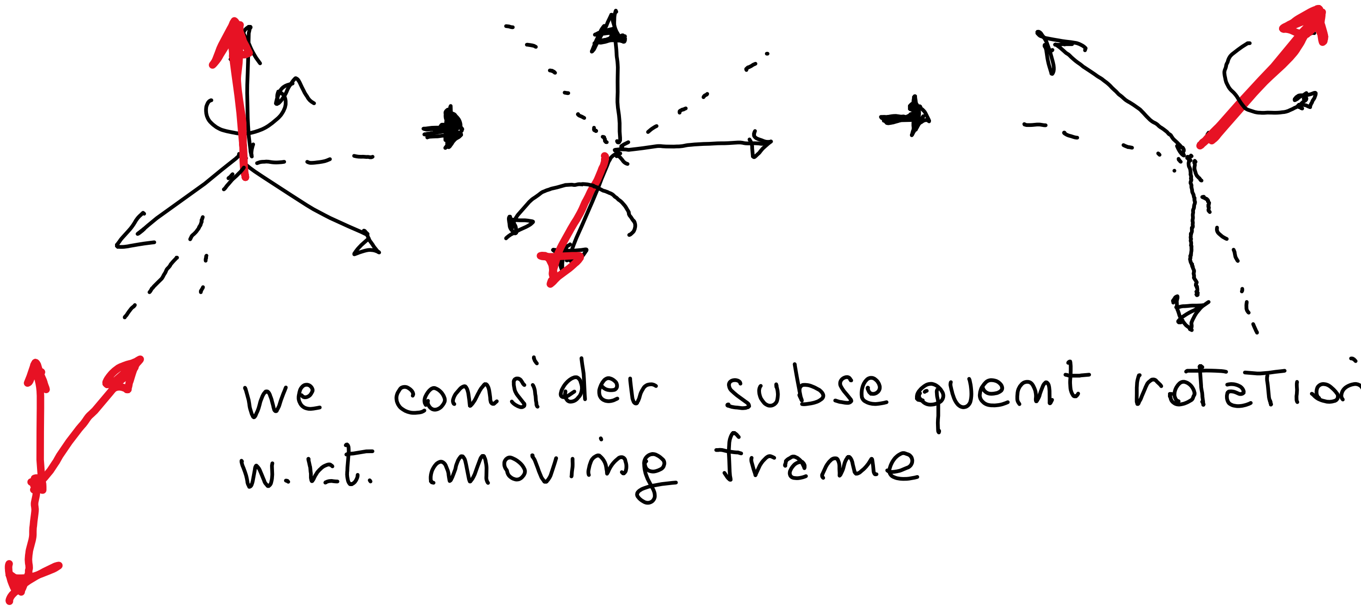
$$R_z = \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_y = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \quad R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}$$

Elementary rotations

$$R(\emptyset) = \begin{bmatrix} c\varphi\, c\theta & c\varphi\, s\theta\, s\psi - s\varphi\, c\psi & c\varphi\, s\theta\, c\psi + s\varphi\, s\psi \\ s\varphi\, c\theta & s\varphi\, s\theta\, s\psi + c\varphi\, c\psi & s\varphi\, s\theta\, c\psi - c\varphi\, s\psi \\ -s\theta & c\theta\, s\psi & c\theta\, c\psi \end{bmatrix} \quad (1)$$

$\boxed{\text{exe}}$ Matlab

ISSUE OF NON ORTHOGONAL AXES



we consider subsequent rotations w.r.t. moving frame

- we cannot compute orientation error as:

$$e_o = (\psi^d - \psi, \theta^d - \vartheta, \varphi^d - \varphi)$$

- we need to map them to other representations (e.g. angle-axis, Rotation matrix)

DIRECT PROBLEM: WITH FIXED AXES

- NB you will get the same rotation matrix if you rotate around fixed axes but with reverse order.

$$R(\phi) = \underbrace{\bar{R}_x(\psi)\ \bar{R}_y(\theta)\ \bar{R}_z}_{\text{fixed frame axes}}(\varphi)$$

XYZ

fixed axes

NOTE: The sequence XYZ about fixed frame axes is equivalent to the sequence ZYX about moving axes

$$\bar{R}_x(\psi)\ \bar{R}_y(\theta)\ \bar{R}_z(\varphi) = R_z(\varphi)\ R_{y'}(\theta)\ R_{x''}(\psi)$$

FIXED XYZ → These are not elementary matrix

MOVING ZYX → These are elementary matrix

ROTATION MATRIX TO EULER ANGLES (INVERSE PROBLEM)

$$R(\Phi) = \begin{bmatrix} c\varphi \, c\theta & c\varphi \, s\theta \, s\psi - s\varphi \, c\psi & c\varphi \, s\theta \, c\psi + s\varphi \, s\psi \\ s\varphi \, c\theta & s\varphi \, s\theta \, s\psi + c\varphi \, c\psi & s\varphi \, s\theta \, c\psi - c\varphi \, s\psi \\ -s\theta & c\theta \, s\psi & c\theta \, c\psi \end{bmatrix} \quad (1)$$

Inspecting (1) the solutions for ψ, θ, φ (Roll/pitch/yaw) are:

$$\frac{r_{21}}{r_{11}} = \frac{s\varphi \, \cancel{c\theta}}{c\varphi \, \cancel{c\theta}} = \tan\varphi \implies \varphi = \operatorname{atan2}(\overset{\sin}{r_{21}}, \overset{\cos}{r_{11}})$$

$$\theta = \operatorname{atan2}\left(-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\right)$$

$\implies$ There is a pair of solutions

$$\psi = \operatorname{atan2}\left(\frac{r_{32}}{c\theta}, \frac{r_{33}}{c\theta}\right)$$

$\implies c\theta = 0$??

SINGULARITIES FOR ZY'X'' SEQUENCE

- solutions degenerate when $c\theta = 0$ (e.g. when roll and yaw axes align (GIMBAL LOCK). $\boxed{\theta = \pm\frac{\pi}{2}}$

$$R_s(\phi)\Big|_{\theta=\frac{\pi}{2}} = \begin{bmatrix} 0 & s\psi\, c\varphi - c\psi\, s\varphi & c\psi\, c\varphi + s\psi\, s\varphi \\ 0 & s\psi\, s\varphi + c\psi\, c\varphi & c\psi\, s\varphi - s\psi\, c\varphi \\ -1 & 0 & 0 \end{bmatrix}$$

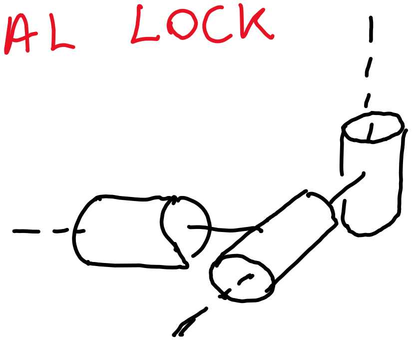
- it is possible only to determine the sum or difference $\varphi + \psi$ or $\varphi - \psi$ from R

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

$$R(\phi) = \begin{bmatrix} 0 & \sin(\psi - \varphi) & \cos(\psi - \varphi) \\ 0 & \cos(\psi - \varphi) & -\sin(\psi - \varphi) \\ -1 & 0 & 0 \end{bmatrix}$$

changing ψ or $(-\varphi)$ has The same effect

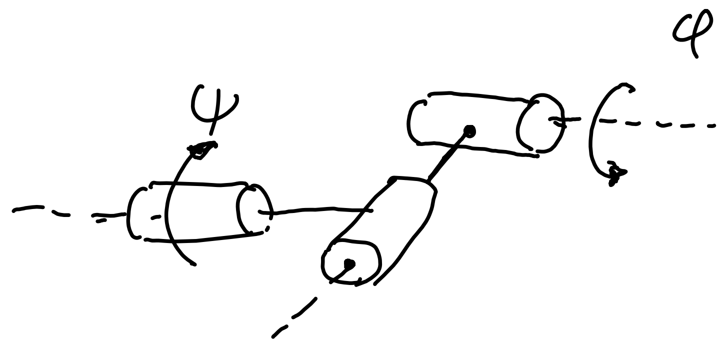
GIMBAL LOCK



$\psi = 0$

$\theta = 0$

$\varphi = 0$

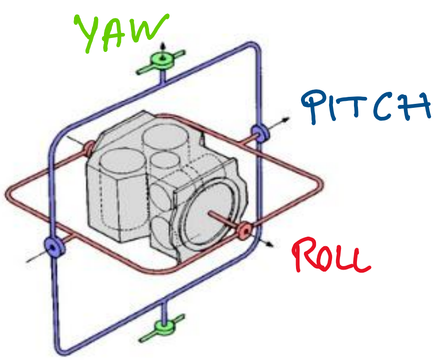


$\psi = ?$

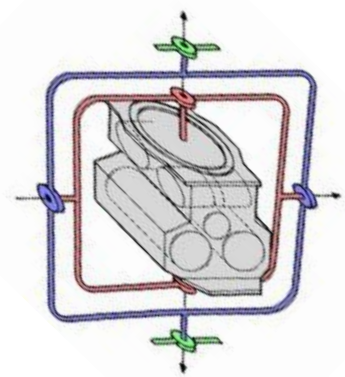
$\theta = -\pi/2$

$\varphi = ?$

cannot distinguish between ψ and $(-\varphi)$



Normal



- singularity: more sets of euler angles correspond to the same rotation matrix

e.g. $R\left(\frac{\pi}{4}, \frac{\pi}{2}, 0\right) = R\left(0, \frac{\pi}{2}, -\frac{\pi}{4}\right) = R\left(\frac{\pi}{8}, 0, -\frac{\pi}{8}\right)$

- The chosen zyx representation is not able to capture the orientation associated at R_s
- you can repeat the analysis for each sequence and will find:
  - 2 solutions
  - singularities for some values of angles

EULER RATES AND OMEGA

$\omega = [\omega_x, \omega_y, \omega_z]$ lives in an euclidean space (with $\perp$ axes)

contributions can be summed (in any order)

$\left.\begin{array}{ll} \phi = [\varphi, \theta, \psi] & \text{euler angles} \\ \dot{\phi} = [\dot{\psi}, \dot{\theta}, \dot{\varphi}] & \text{euler angles rates} \end{array}\right\} \notin$ vector space

$\Rightarrow$ a sequence of 2 rotations is NOT obtained summing the corresponding minimal representations

in general: $\quad \omega \neq \dfrac{d\phi}{dt} = \dot{\phi}$

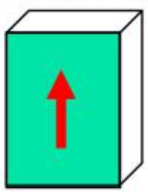
W IS NOT AN EXACT DIFFERENTIAL

$$\int \dot{\phi}\, dt = \phi$$

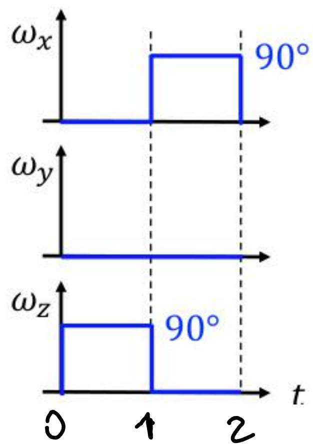
$\int \omega\, dt \neq \phi$ $\quad$ $\int \omega$ has no physical interpretation

Let's consider 2 path of integration $T=2$

initial orientation

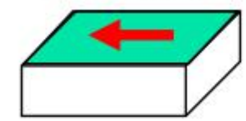


$R_i = I$

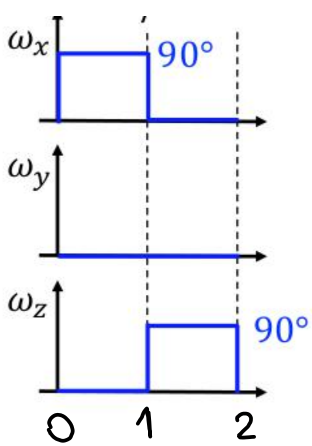


$$\int_0^1 \omega\, dt + \int_1^2 \omega\, dt = \begin{bmatrix} 0 \\ 0 \\ 90^\circ \end{bmatrix} + \begin{bmatrix} 90^\circ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 90^\circ \\ 0 \\ 90^\circ \end{bmatrix}$$

first final orientation



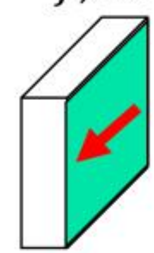
$R_{f,ZX}$



$$\int_0^2 \omega\, dt = \begin{bmatrix} 90^\circ \\ 0 \\ 90^\circ \end{bmatrix}$$

same value but different final orientation

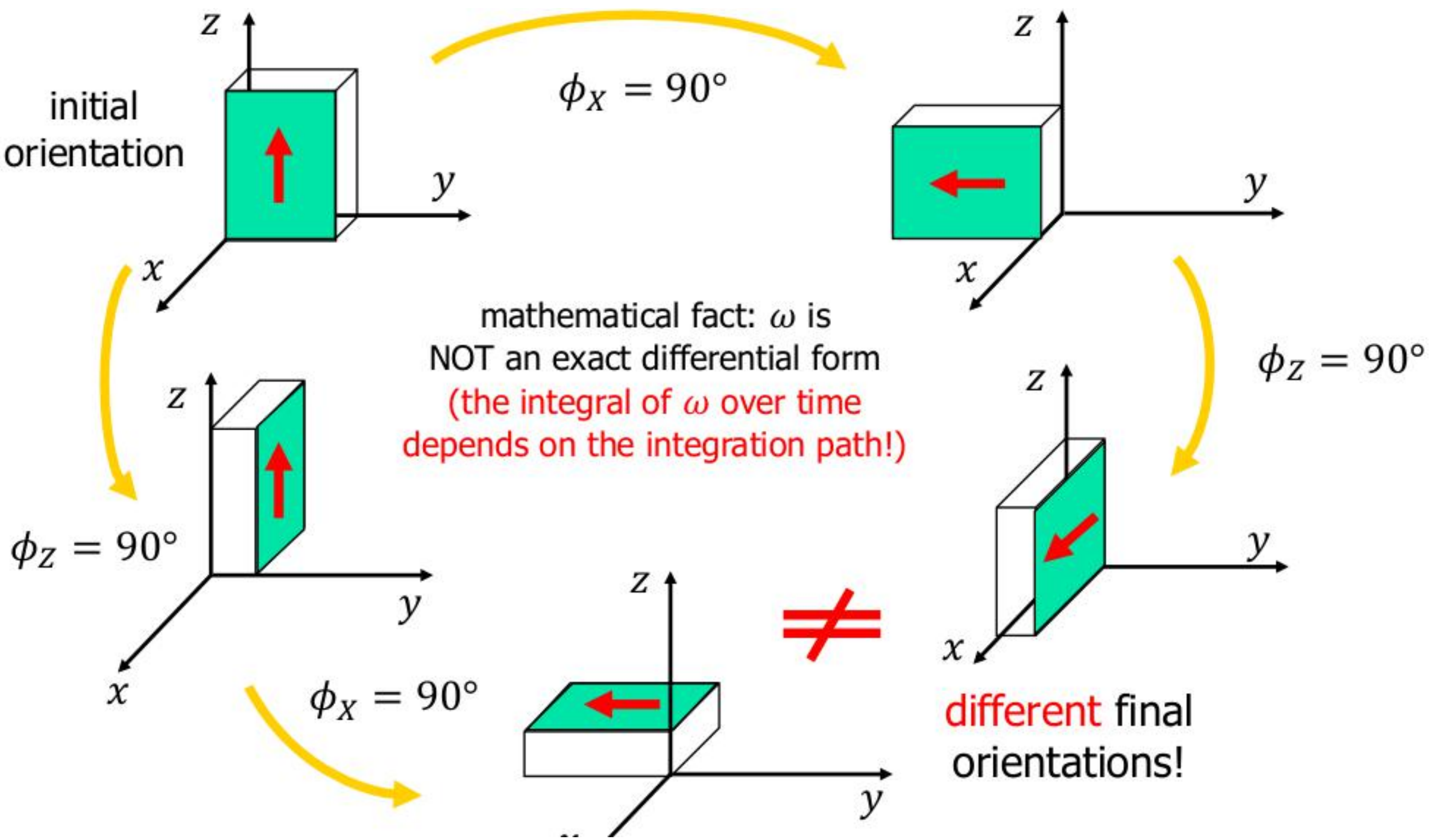
$R_{f,XZ}$



...final orientation

while for linear velocity: $v = \frac{dp}{dt}$

- when you consider translations does not matter in which order of directions you translate while finite rotations $\boxed{\text{do not commute}}$



- it is possible to find the relationship between ω and $\dot{\Phi}$ for a given orientation ϕ (zyx)

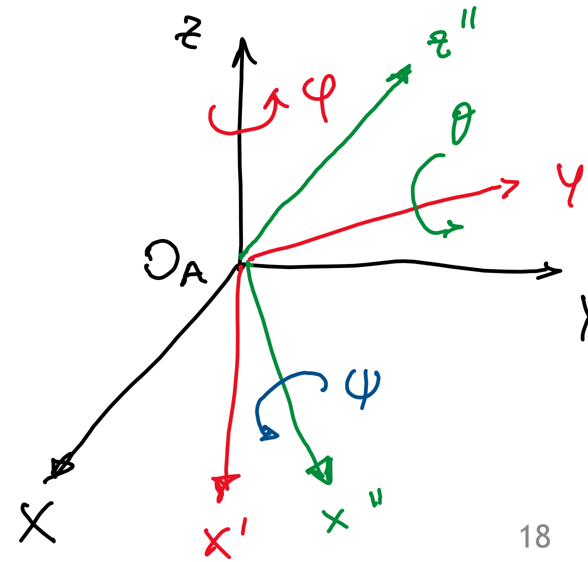
- let's consider the sequence $z\,y'\,x''$ about moving axes

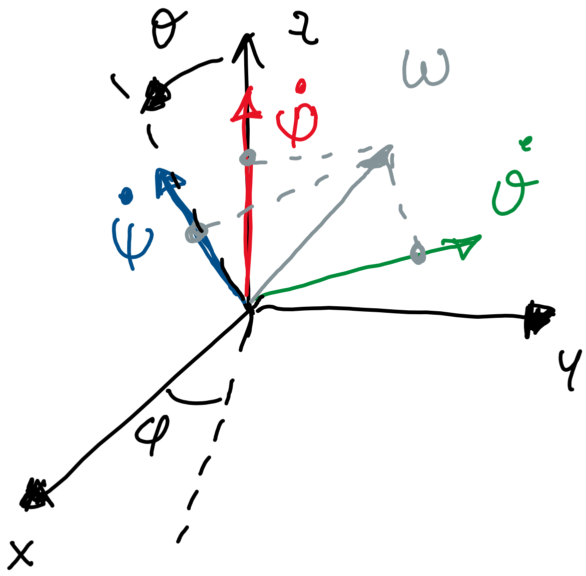
- compute contributions of $\dot{\psi}, \dot{\theta}, \dot{\varphi}$ to $\omega_x, \omega_y, \omega_z$

as a result of $\dot{\varphi}$ about z: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\varphi}$

as a result of $\dot{\theta}$ about y': $\begin{bmatrix} -s\varphi \\ c\varphi \\ 0 \end{bmatrix} \dot{\theta}$

as a result of $\dot{\psi}$ about x'': $\begin{bmatrix} c\varphi\, c\theta \\ s\varphi\, c\theta \\ -s\theta \end{bmatrix} \dot{\psi}$





(zyx)

$$
{}_A\omega = \overbrace{\begin{bmatrix} c\varphi c\theta & -s\varphi & 0 \\ s\varphi c\theta & c\varphi & 0 \\ -s\theta & 0 & 1 \end{bmatrix}}^{T_{RPY}(\varphi,\theta)} \begin{bmatrix} \dot{\psi} \\ \dot{\vartheta} \\ \dot{\varphi} \end{bmatrix}
$$

1° col. in $R_z(\varphi)\,R_y(\theta)$ 2° col. in $R_z(\varphi)$ z

- the 3 contributions are summed vectors
- The mapping T depends on the orientation Φ
- $\det(T_{RPY}(\phi)) = \cos\theta$ if $\theta = \pm\frac{\pi}{2}$ $\det(T) = 0$

Singularity: T cannot be inverted for $\theta = \pm\frac{\pi}{2}$

$\Rightarrow$ $\exists$ angular velocities that cannot be represented with $\dot{\phi}$

- diffrent T and singularity for any euler sequence

SUMMARY EULER ANGLES

⊕ minimal representation (3 params)

⊖ have singularity (both in position and velocity)

⊖ multiple representation

⊖ Not orthonormal axes (subsequent rotations)

⊕ easy to be numerically integrated