Slides have been created by Prof. Michele Focchi

webpage: https://mfocchi.github.io/Teaching/

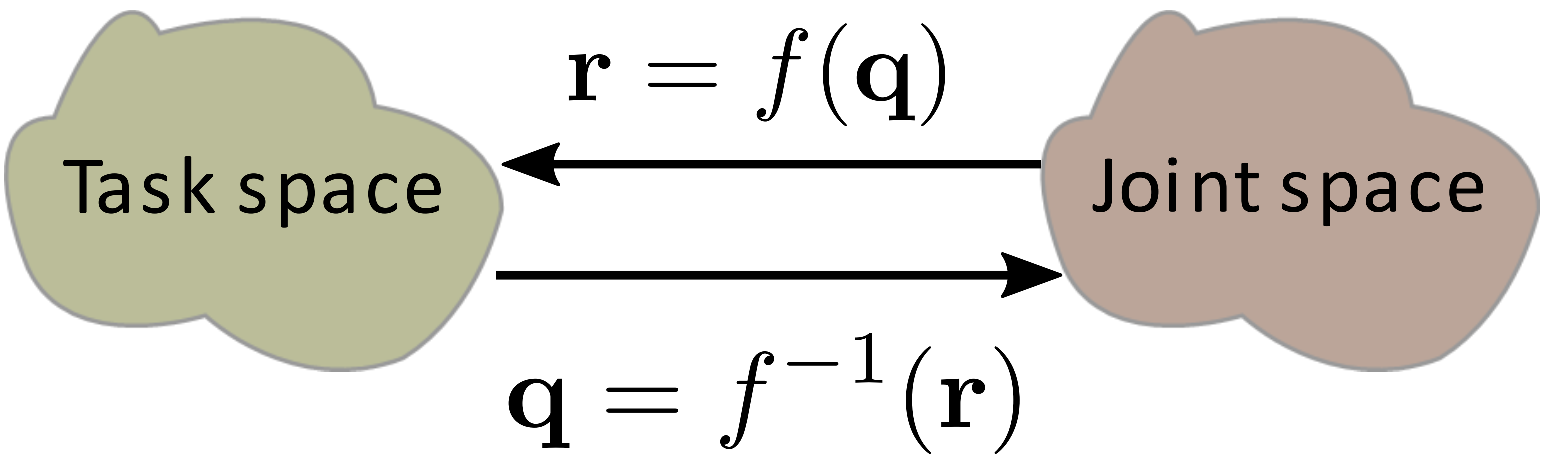
E3 – KINEMATICS

Direct Differential Kinematics

Previous Lectures

Direct Kinematics:
Compute the **pose** of an end-effector (*task space*) as a function of the joint variables (*joint space*).

Inverse Kinematics:
Compute the joint **position** variables (*joint space*) as a function of the desired end-effector pose (*task space*)



This Lecture: Direct Differential Kinematics

(Direct) *Differential* Kinematics

Express the end-effector **motion** (linear/angular **velocity**) as a function of the joint velocities.

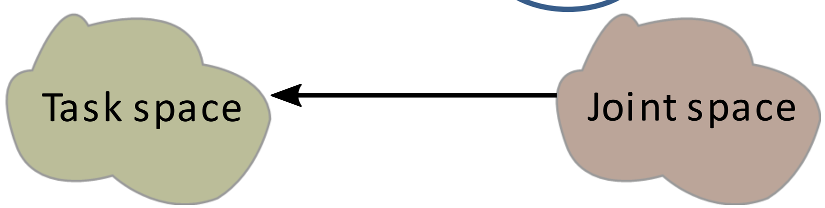
This velocity mapping can be obtained in two ways:

From joint space to chosen **task space**

$$\dot{\mathbf{r}} = \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_A(\mathbf{q}) \dot{\mathbf{q}}$$

Analytical Jacobian

time differentiation of the direct kinematics

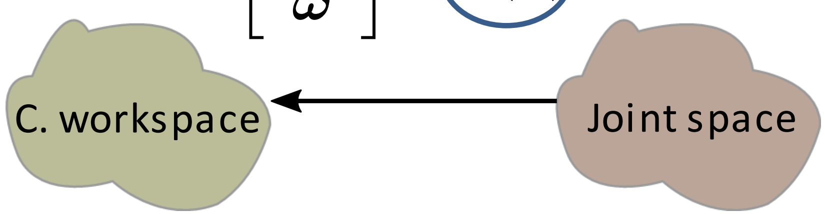


From joint space to **cartesian workspace** (linear and angular velocity)

$$\begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}}$$

Geometric Jacobian

directly at the differential level, relying on geometry



Recap on notation

$$a, b$$

lower-case, non bold: **scalars**

$$\mathbf{a}, \mathbf{p}$$

lower-case, bold: **vectors**

$$\mathbf{A}, \mathbf{B}, \mathbf{R}$$

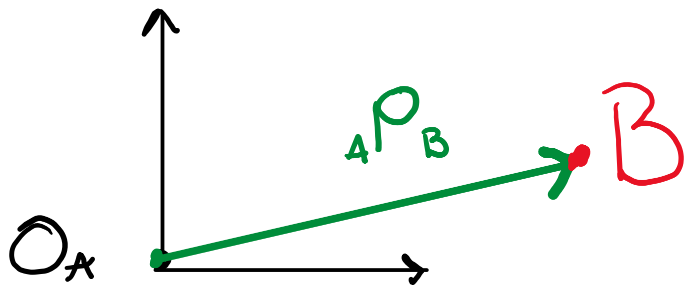
upper-case, bold: **matrices**

*does not apply to subindices

Recap on notation

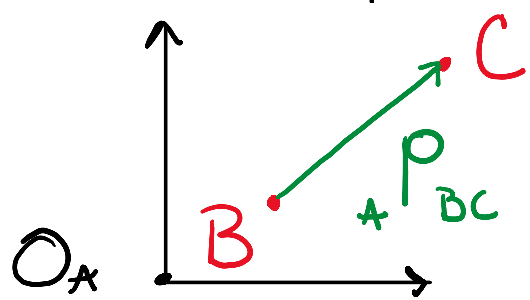
$${}_{\mathrm{A}}\mathbf{p}_{\mathrm{B}}$$

position vector representing point B expressed in frame **A**



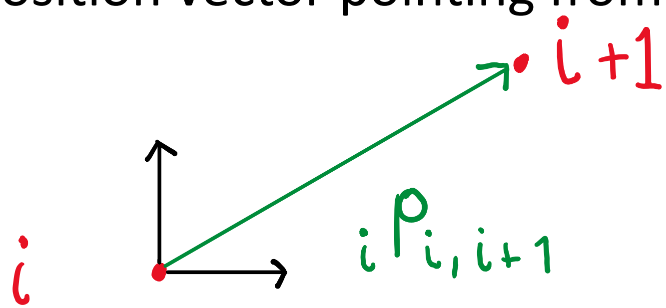
$${}_{\mathrm{A}}\mathbf{p}_{\mathrm{BC}}$$

position vector pointing from **B** to **C** expressed in frame **A**



$${}_{i}\mathbf{p}_{i,i+1}$$

position vector pointing from ***i*** to ***i+1*** expressed in frame ***i***



Recap on notation

$\mathbf{{}^{i}\omega_{i,i+1}}$ angular velocity from link ***i*** to link ***i+1*** expressed in frame ***i***

$\mathbf{{}^{W}\omega_{i,i+1}}$ angular velocity from link ***i*** to link ***i+1*** expressed in the "world frame" **W**

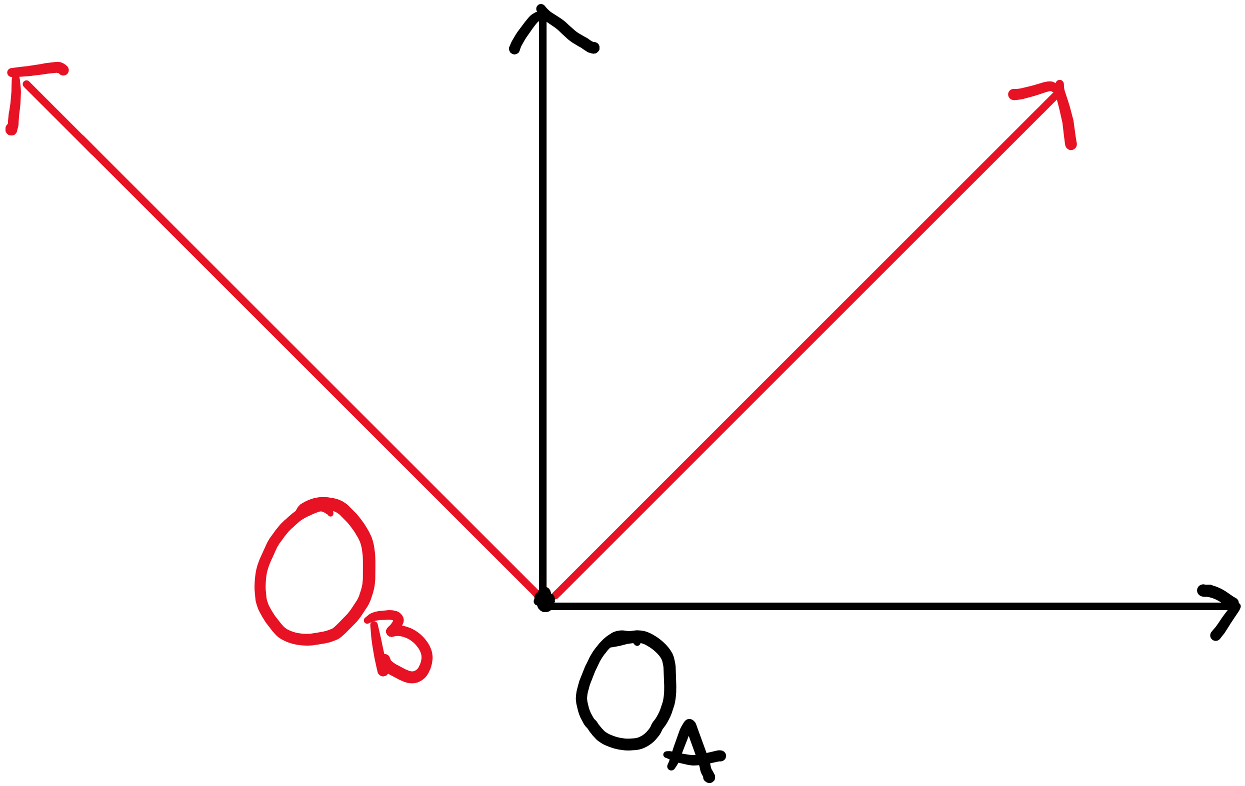
Note: To simplify notation, when expressing angular velocities from the frame in which they are expressed, we omit the corresponding right subscript:

$$\mathbf{{}^{i}\omega_{\cancel{i},i+1} = {}^{i}\omega_{i+1}}$$

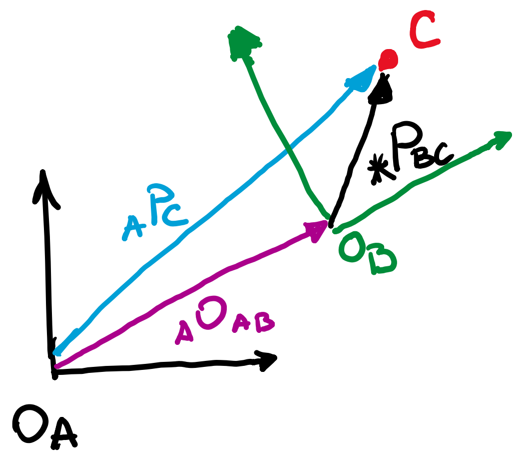
$$\mathbf{{}^{W}\omega_{\cancel{W},i} = {}^{W}\omega_{i}}$$

Recap on notation

${}_{A}\mathrm{R}_{B}$ rotation matrix **R** from frame **A** to frame **B**



Kinematic analysis with moving frame



*Can be expressed w.r.t. F_A or F_B

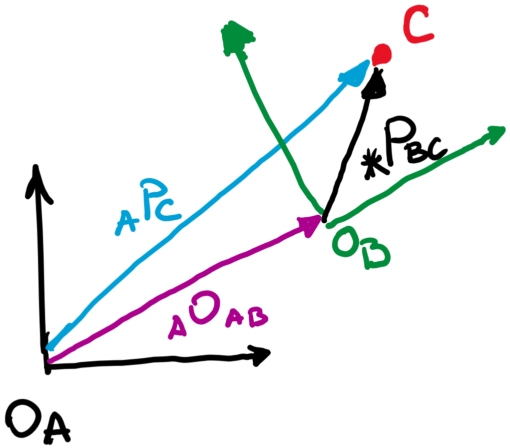
- Frame ***FA*** is **fixed**
- Frame ***FB*** is **moving**
  → translating: $_A\dot{O}_{AB}$
  → rotating: $_A\omega$ (angular velocity expressed in ***FA***)
- The position of point **c** attached to moving frame ***FA*** is:

$$_A\mathrm{p_C} = {}_A\mathrm{O_{AB}} + {}_A\mathrm{p_{BC}} \quad (1)$$

The velocity of point **c** expressed in ***FA*** is obtained by taking the time derivative of (1)

$$_A v_\mathrm{C} = \frac{d}{dt}\,{}_A\mathrm{p_C} = {}_A\dot{\mathrm{O}}_\mathrm{AB} + \frac{d}{dt}\,{}_A\mathrm{p_{BC}}$$

Kinematic analysis with moving frame



The position of point **c** attached to moving frame ***FB***

$$ {}_{\mathrm{A}}\mathrm{p}_{\mathrm{C}} = {}_{\mathrm{A}}\mathrm{O}_{\mathrm{AB}} + {}_{\mathrm{A}}\mathrm{p}_{\mathrm{BC}} \quad (1) $$

Rewriting (1) in a different way:

$$ {}_{\mathrm{A}}\mathrm{p}_{\mathrm{C}} = {}_{\mathrm{A}}\mathrm{O}_{\mathrm{AB}} + {}_{\mathrm{A}}\mathrm{R}_{\mathrm{B}}\,{}_{\mathrm{B}}\mathrm{p}_{\mathrm{BC}} \quad (2) $$

If we take the time derivative of (2)

$$ {}_{\mathrm{A}}v_{\mathrm{C}} = \frac{d}{dt}\,{}_{\mathrm{A}}\mathrm{p}_{\mathrm{C}} = \boxed{{}_{\mathrm{A}}\dot{\mathrm{O}}_{\mathrm{AB}}} + {}_{\mathrm{A}}\mathrm{R}_{\mathrm{B}}\,{}_{\mathrm{B}}\dot{\mathrm{p}}_{\mathrm{BC}} + \boxed{{}_{\mathrm{A}}\dot{\mathrm{R}}_{\mathrm{B}}\,{}_{\mathrm{B}}\mathrm{p}_{\mathrm{BC}}} $$

Velocity of **c** seen from frame ***FA***

Velocity of frame ***FB*** seen from ***FA***

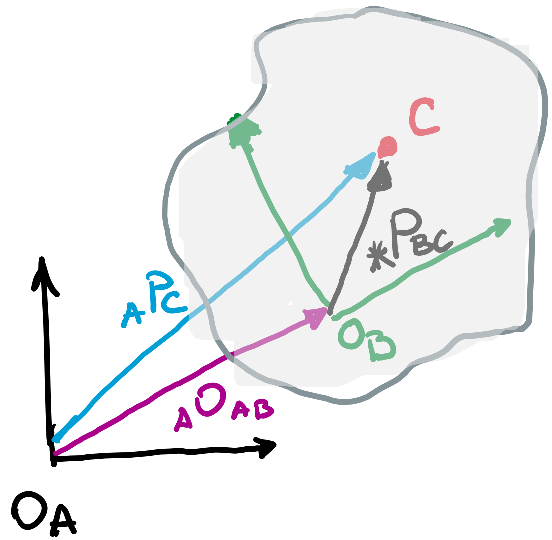
Variation due to motion of **c** w.r.t. ***FB***

Variation due to rotation of frame***FB***

$\boxed{}$ Due to the motion of frame ***FB***

What happens in the case of a **rigid body** (e.g., a link)?

Velocity of a point on a rigid body



Consider now that **c** and ***FB*** are now part of a rigid body

This means there is no relative motion of **c** w.r.t. ***FB*** due to **rigidity constraints** (i.e., **c** and ***FB*** are rigidly attached)

Then from our last expression of the velocity taken from (2):

Since $_{\mathrm{B}}\mathbf{p}_{\mathrm{BC}}$ is constant

$$_{\mathrm{A}}v_{\mathrm{C}} = \frac{d}{dt}\,{}_{\mathrm{A}}\mathbf{p}_{\mathrm{C}} = {}_{\mathrm{A}}\dot{\mathrm{O}}_{\mathrm{AB}} + {}_{\mathrm{A}}\mathrm{R}_{\mathrm{B}}\,{}_{\mathrm{B}}\dot{\mathbf{p}}_{\mathrm{BC}} + {}_{\mathrm{A}}\dot{\mathrm{R}}_{\mathrm{B}}\,{}_{\mathrm{B}}\mathbf{p}_{\mathrm{BC}}$$

Derivative of a rotation matrix*:

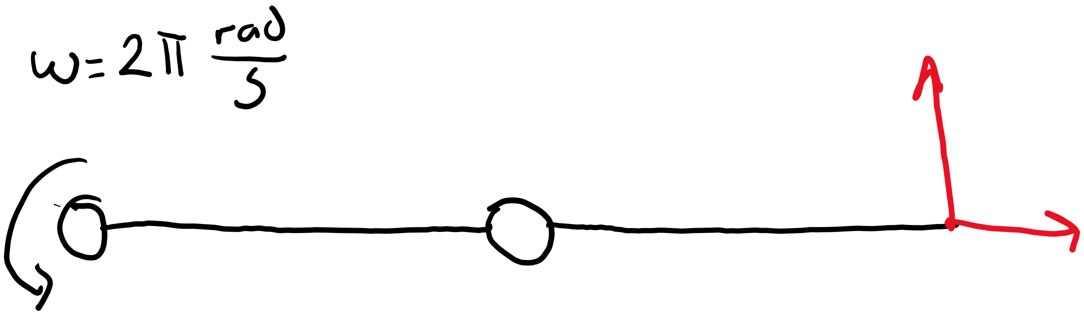
$$_{\mathrm{A}}\dot{\mathrm{R}}_{\mathrm{B}} = {}_{\mathrm{A}}\omega \times {}_{\mathrm{A}}\mathrm{R}_{\mathrm{B}}$$

*Slide 32, E0.1

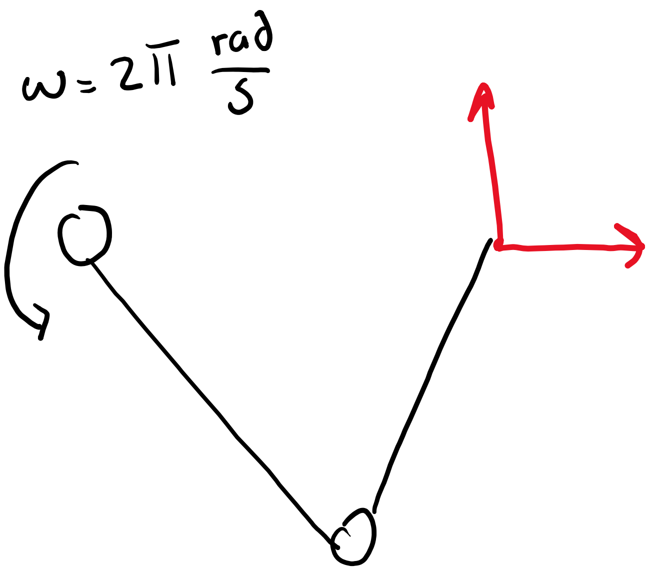
$$_{\mathrm{A}}v_{\mathrm{C}} = \frac{d}{dt}\,{}_{\mathrm{A}}\mathbf{p}_{\mathrm{C}} = {}_{\mathrm{A}}\dot{\mathrm{O}}_{\mathrm{AB}} + {}_{\mathrm{A}}\omega \times \underbrace{{}_{\mathrm{A}}\mathrm{R}_{\mathrm{B}}\,{}_{\mathrm{B}}\mathbf{p}_{\mathrm{BC}}}_{{}_{A}P_{BC}}$$

$$\boxed{{}_{\mathrm{A}}v_{\mathrm{C}} = {}_{\mathrm{A}}\dot{\mathrm{O}}_{\mathrm{AB}} + {}_{\mathrm{A}}\omega \times {}_{\mathrm{A}}\mathbf{p}_{\mathrm{BC}}}$$

Velocity of a point on a rigid body (angular velocity)

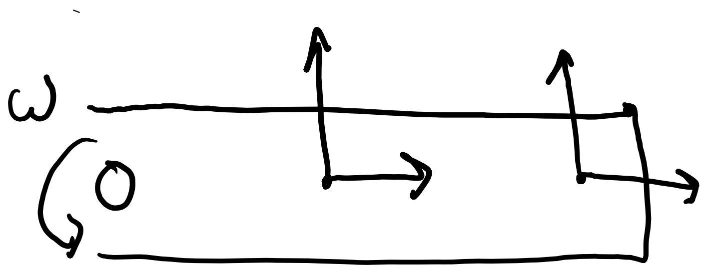


After 1 second, both frames at the end of the arm return to the same position



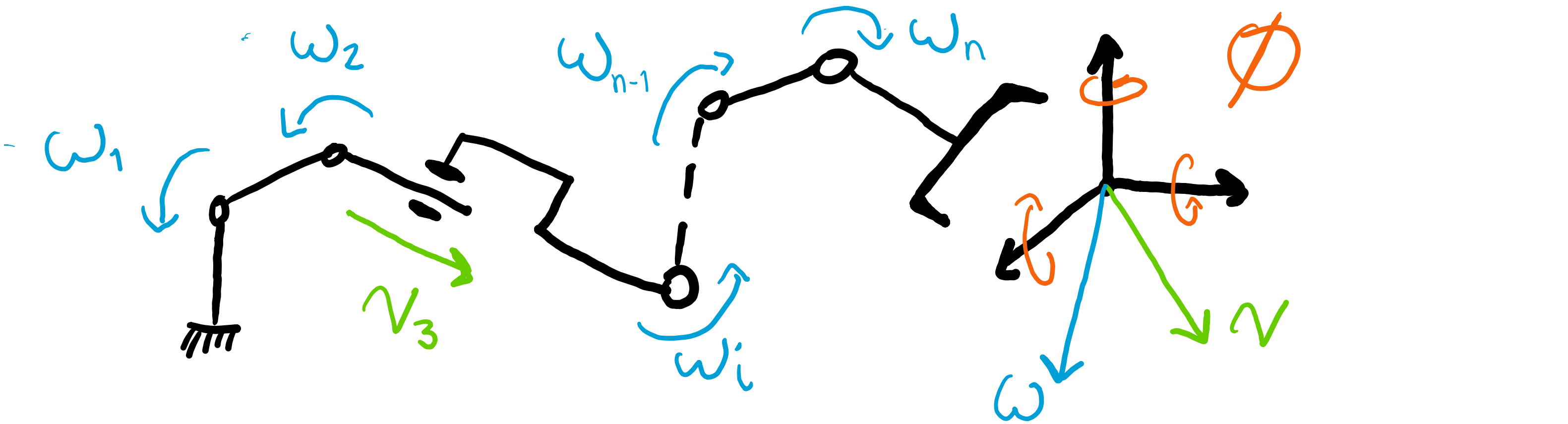
Velocity of a point on a rigid body

Similarly, in the case of a rigid body the **rate of change** in the **orientation** of any point is the same along the entire body



Angular velocity is the same for the entire rigid body

Velocity of a Rigid Body Link of a Manipulator



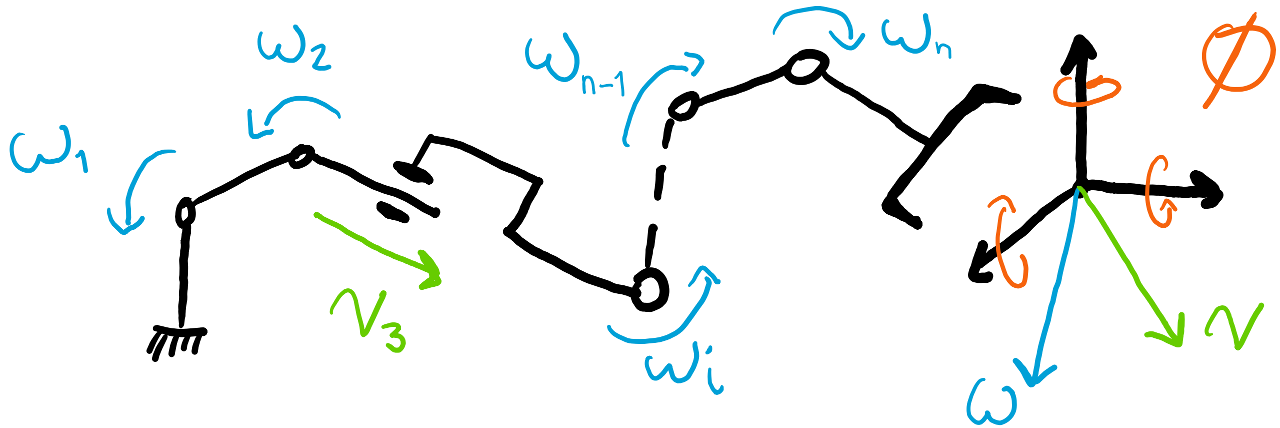
There are two sets of differential quantities (i.e., velocities) that we are interested in:

- The linear $\boldsymbol{v}$ and angular $\boldsymbol{\omega}$ velocities of the end-effector
  - These two quantities are members of a **vector space**
  - They can be obtained by adding the contributions of each single linear ($\boldsymbol{v}_i$) and angular ($\boldsymbol{\omega}_i$) joint velocity

- The rotational rate of the end-effector $\dot{\boldsymbol{\Phi}}$
  - $\boldsymbol{\Phi}$ are the angles representing the sequence of rotations corresponding to a minimal representation (generally arranged in the form of a vector, e.g., roll, pitch, yaw)
  - $\dot{\boldsymbol{\Phi}}$ is not a member of a **vector space**, i.e., it cannot be obtained by adding contributions of each single joint

Important: In general, $\dot{\boldsymbol{\Phi}} \neq \boldsymbol{\omega}$

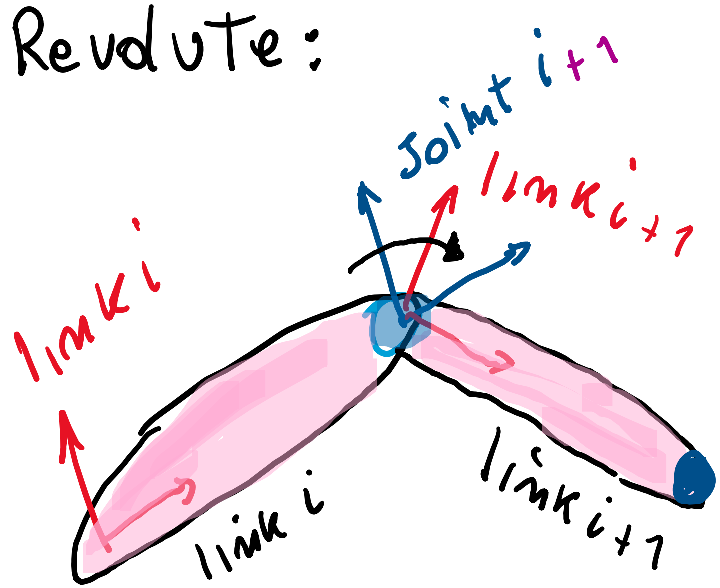
Velocity of a Rigid Body Link of a Manipulator

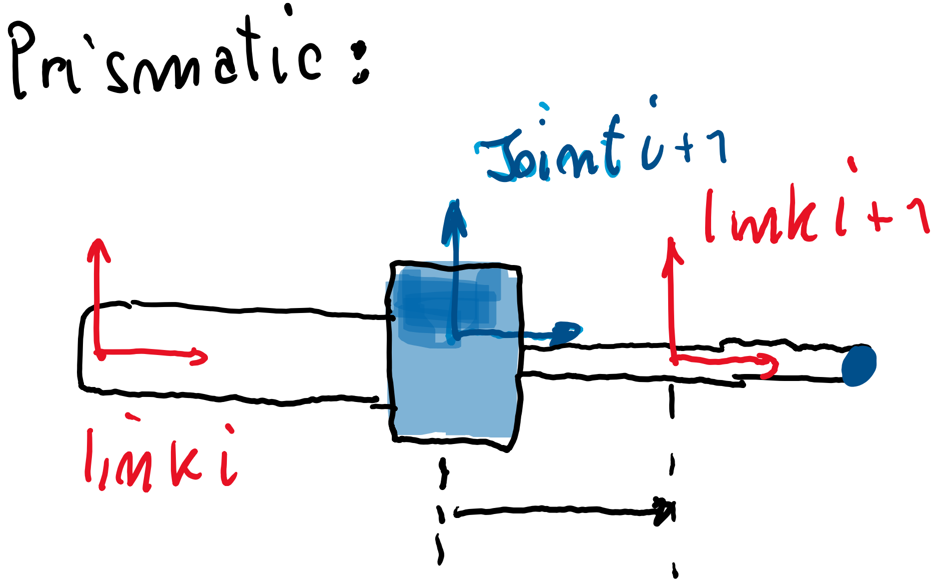


We will proceed to obtain expressions for the linear (v_{i+1}) and angular (ω_{i+1}) velocities for a single rigid body link with respect to its predecessor (namely i)

Once we compute these two quantities in function of their previous link, we will be able to compute the velocity of any link as long as the direct kinematics of the complete chain are known

Recap on frames

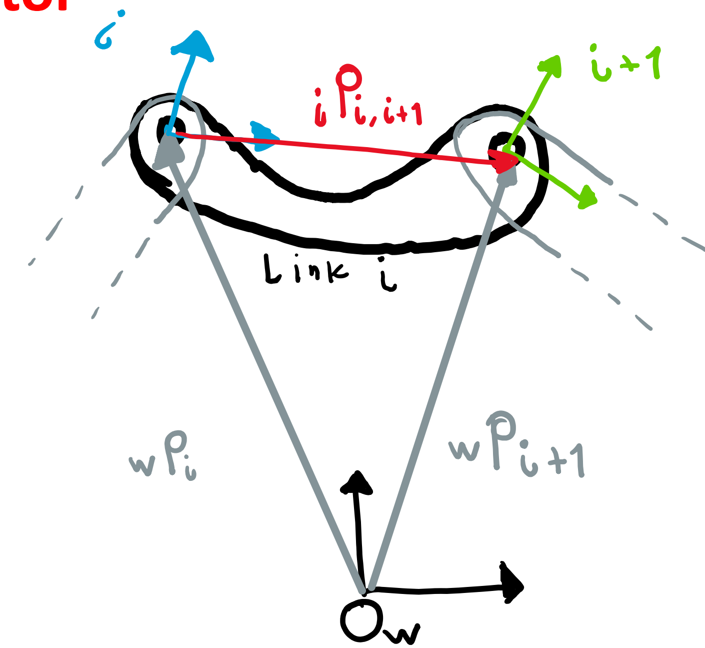




*E1 slide 21

** This is different to Siciliano's book

Velocity of a Rigid Body Link of a Manipulator (Linear velocity v_{i+1})



Consider an arbitrary link i of a kinematic chain attached to joints i and $i+1$

We also define a fixed frame called the **world frame W** (this was previously referred as $\boldsymbol{FA}$).

We can express the position of link $i+1$ in terms of link i (namely, its predecessor) in **W** as:

$$
{}_{\mathbf{W}}\mathbf{p}_{i+1} = {}_{\mathbf{W}}\mathbf{p}_i + {}_{\mathbf{W}}\mathbf{R}_i \, {}_i\mathbf{p}_{i,i+1}
$$

where ${}_i\mathbf{p}_{i,i+1}$ is the directed vector from link frame i to link frame $i+1$ express in frame i

Derivative of a rotation matrix*:

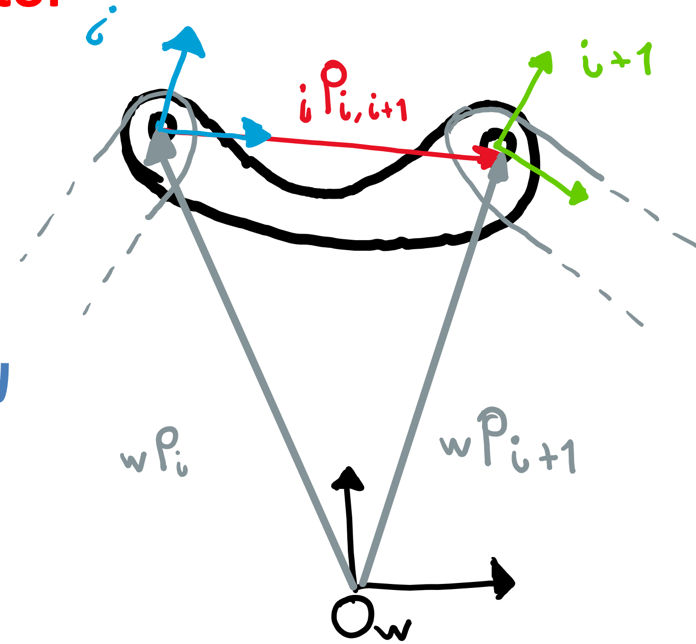
$$
{}_{\mathbf{A}}\dot{\mathbf{R}}_{\mathbf{B}} = {}_{\mathbf{A}}\boldsymbol{\omega} \times {}_{\mathbf{A}}\mathbf{R}_{\mathbf{B}}
$$

*Slide 32, E0.1

If we differentiate with respect to time we obtain:

$$
{}_{\mathbf{W}}\dot{\mathbf{p}}_{i+1} = {}_{\mathbf{W}}\dot{\mathbf{p}}_i + {}_{\mathbf{W}}\mathbf{R}_i \, {}_i\dot{\mathbf{p}}_{i,i+1} + {}_{\mathbf{W}}\boldsymbol{\omega}_i \times {}_{\mathbf{W}}\mathbf{R}_i \, {}_i\mathbf{p}_{i,i+1}
$$

Velocity of a Rigid Body Link of a Manipulator (Linear velocity v_{i+1})



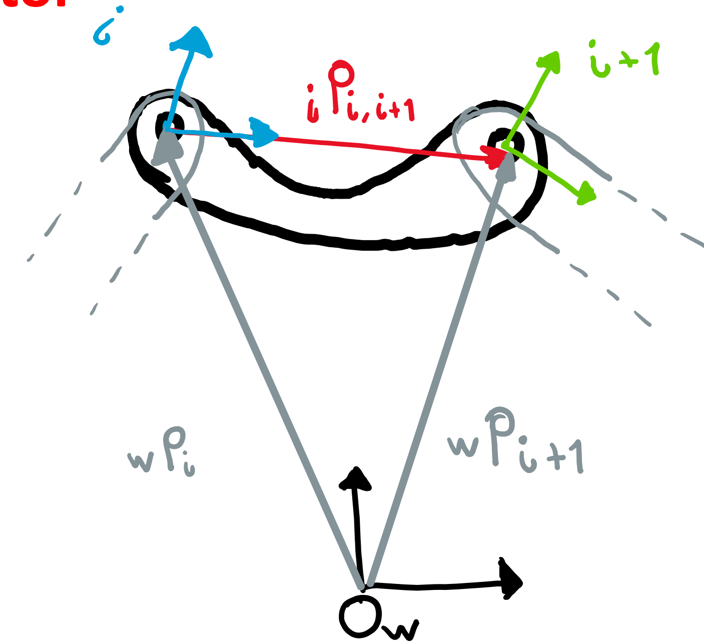
$${}_{w}\dot{\mathbf{p}}_{i+1} = {}_{w}\dot{\mathbf{p}}_i + \underbrace{{}_{w}\mathbf{R}_i\,{}_{i}\dot{\mathbf{p}}_{i,i+1}}_{{}_{w}\boldsymbol{v}_{i,i+1}} + {}_{w}\boldsymbol{\omega}_i \times \underbrace{{}_{w}\mathbf{R}_i\,{}_{i}\mathbf{p}_{i,i+1}}_{{}_{w}\mathbf{p}_{i,i+1}}$$

$${}_{w}\dot{\mathbf{p}}_{i+1} \rightarrow {}_{w}\boldsymbol{v}_{i+1} \qquad {}_{w}\dot{\mathbf{p}}_i \rightarrow {}_{w}\boldsymbol{v}_i$$

Linear velocity of link $i+1$ as a function of the linear and angular velocities of i

$$\boxed{{}_{w}\boldsymbol{v}_{i+1} = {}_{w}\boldsymbol{v}_i + {}_{w}\boldsymbol{v}_{i,i+1} + {}_{w}\boldsymbol{\omega}_i \times {}_{w}\mathbf{p}_{i,i+1}}$$

Velocity of a Rigid Body Link of a Manipulator (Angular velocity $\boldsymbol{\omega}_{i+1}$)



Consider the same arbitrary link $\boldsymbol{i}$. To obtain the expression for the angular velocity $\boldsymbol{\omega}_i$ it is worth starting from the composition of rotation matrices:

$$ {}_{\mathrm{W}}\mathbf{R}_{i+1} = {}_{\mathrm{W}}\mathbf{R}_i \, {}_i\mathbf{R}_{i+1} $$

The time derivative of the previous expression is:

$$ {}_{\mathrm{W}}\dot{\mathbf{R}}_{i+1} = {}_{\mathrm{W}}\dot{\mathbf{R}}_i \, {}_i\mathbf{R}_{i+1} + {}_{\mathrm{W}}\mathbf{R}_i \, {}_i\dot{\mathbf{R}}_{i+1} $$

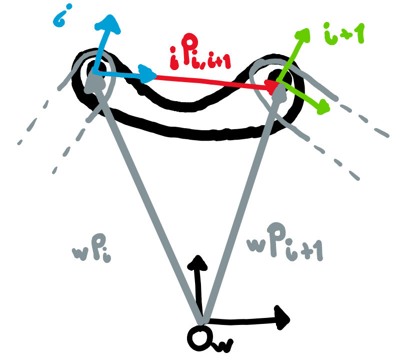
Recall:

$$ {}_{\mathrm{W}}\dot{\mathbf{R}}_i = {}_W\boldsymbol{\omega}_i \times {}_{\mathrm{W}}\mathbf{R}_i $$

$$ {}_{\mathrm{W}}\boldsymbol{\omega}_i \times {}_{\mathrm{W}}\mathbf{R}_i = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_i) \, {}_{\mathrm{W}}\mathbf{R}_i $$

$$ \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1}) \, {}_{\mathrm{W}}\mathbf{R}_{i+1} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_i) \, {}_{\mathrm{W}}\mathbf{R}_i \, {}_i\mathbf{R}_{i+1} + {}_{\mathrm{W}}\mathbf{R}_i \mathbf{S}({}_i\boldsymbol{\omega}_{i,i+1}) \, {}_i\mathbf{R}_{i+1} $$

Velocity of a Rigid Body Link of a Manipulator (Angular velocity ω_{i+1})



$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_i)\underbrace{{}_{\mathrm{W}}\mathbf{R}_i\,{}_i\mathbf{R}_{i+1}}_{{}_{\mathrm{W}}\mathbf{R}_{i+1}} + {}_{\mathrm{W}}\mathbf{R}_i\mathbf{S}({}_i\boldsymbol{\omega}_{i,i+1})\,{}_i\mathbf{R}_{i+1}$$

$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_i)\,{}_{\mathrm{W}}\mathbf{R}_{i+1} + \underbrace{{}_{\mathrm{W}}\mathbf{R}_i\mathbf{S}({}_i\boldsymbol{\omega}_{i,i+1})\,{}_i\mathbf{R}_{i+1}}_{A}$$

$$A = \underbrace{{}_{\mathrm{W}}\mathbf{R}_i\mathbf{S}({}_i\boldsymbol{\omega}_{i,i+1})\,{}_i\mathbf{R}_{\mathrm{W}}}_{\mathbf{S}({}_{\mathrm{W}}\mathbf{R}_i\,{}_i\boldsymbol{\omega}_{i,i+1})}\underbrace{\;}_{\mathbf{I}}\underbrace{{}_{\mathrm{W}}\mathbf{R}_i\,{}_i\mathbf{R}_{i+1}}_{{}_{\mathrm{W}}\mathbf{R}_{i+1}}$$

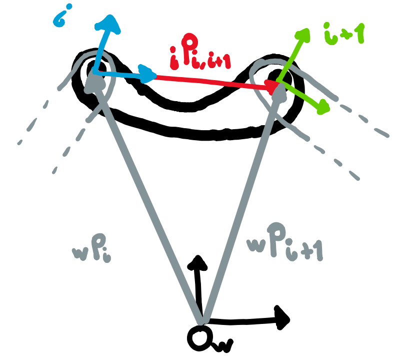
Recall*:

$${}_{\mathrm{A}}\mathbf{R}_{\mathrm{B}}\mathbf{S}({}_B\boldsymbol{\omega})\,{}_{\mathrm{B}}\mathbf{R}_{\mathrm{A}} = \mathbf{S}({}_{\mathrm{A}}\mathbf{R}_{\mathrm{B}}\,{}_B\boldsymbol{\omega})$$

$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_i)\,{}_{\mathrm{W}}\mathbf{R}_{i+1} + \mathbf{S}({}_{\mathrm{W}}\mathbf{R}_i\,{}_i\boldsymbol{\omega}_{i,i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1}$$

*E0.1 slide 34

Velocity of a Rigid Body Link of a Manipulator (Angular velocity ω_{i+1})



$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i})\,{}_{\mathrm{W}}\mathbf{R}_{i+1} + \mathbf{S}({}_{\mathrm{W}}\mathbf{R}_{i}\,{}_{i}\boldsymbol{\omega}_{i,i+1})\,{}_{\mathrm{W}}\mathbf{R}_{i+1}$$

Post-multiplying by ${}_{i+1}\mathbf{R}_{\mathrm{W}}$

$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1})\underbrace{{}_{\mathrm{W}}\mathbf{R}_{i+1}\,{}_{i+1}\mathbf{R}_{\mathrm{W}}}_{\mathbf{I}} = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i})\underbrace{{}_{\mathrm{W}}\mathbf{R}_{i+1}\,{}_{i+1}\mathbf{R}_{\mathrm{W}}}_{\mathbf{I}} + \mathbf{S}(\underbrace{{}_{\mathrm{W}}\mathbf{R}_{i}\,{}_{i}\boldsymbol{\omega}_{i,i+1}}_{{}_{\mathrm{W}}\boldsymbol{\omega}_{i,i+1}})\underbrace{{}_{\mathrm{W}}\mathbf{R}_{i+1}\,{}_{i+1}\mathbf{R}_{\mathrm{W}}}_{\mathbf{I}}$$

$$\mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i+1}) = \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i}) + \mathbf{S}({}_{\mathrm{W}}\boldsymbol{\omega}_{i,i+1})$$

Which leads to

$$\boxed{{}_{\mathrm{W}}\boldsymbol{\omega}_{i+1} = {}_{\mathrm{W}}\boldsymbol{\omega}_{i} + {}_{\mathrm{W}}\boldsymbol{\omega}_{i,i+1}}$$

Angular velocity of link $i+1$ as a function of the angular velocity of i

Velocity of a Rigid Body Link of a Manipulator

Linear velocity:

$$\mathbf{w}\boldsymbol{v}_{i+1} = \mathbf{w}\boldsymbol{v}_i + \mathbf{w}\boldsymbol{v}_{i,i+1} + \mathbf{w}\boldsymbol{\omega}_i \times \mathbf{w}\mathbf{p}_{i,i+1}$$

Angular velocity:

$$\mathbf{w}\boldsymbol{\omega}_{i+1} = \mathbf{w}\boldsymbol{\omega}_i + \mathbf{w}\boldsymbol{\omega}_{i,i+1}$$

Obtained from predecessor link

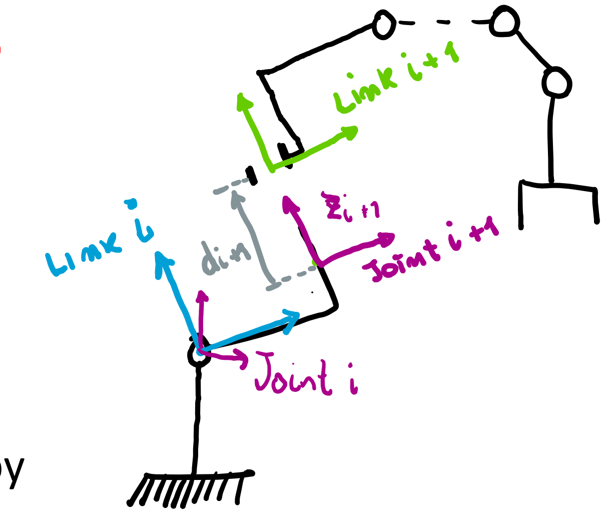
Known from direct kinematics

Depend on the type of joint

We will use these expressions to compute the velocity from the beginning (base link) to the end (end-effector) of the kinematic chain

The linear and angular velocities have specific expressions depending on the type of joint (i.e., prismatic or revolute)

Velocity of a Rigid Body Link of a Manipulator (attached to a prismatic joint)



Consider link $i+1$ supported by joint $i+1$

The angular velocity of link $i+1$ with respect to link i is given by

$$\mathrm{w}\boldsymbol{\omega}_{i,i+1} = \mathbf{0}$$

Since there is no change in orientation from i to $i+1$

The linear velocity of link $i+1$ with respect to link i is given by

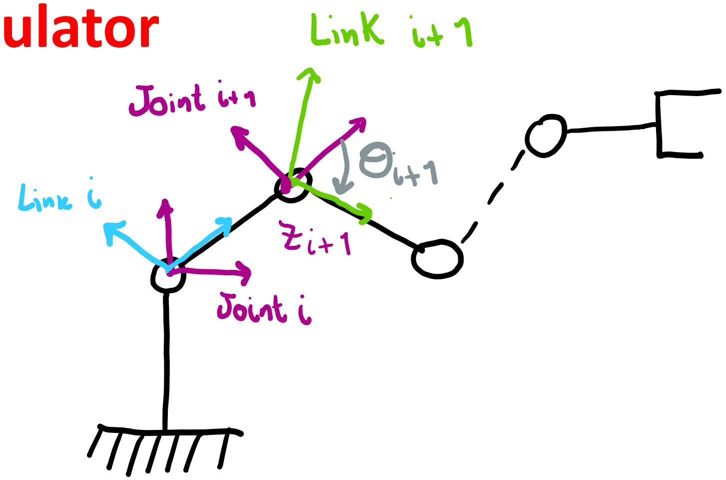
$$\mathrm{w}\boldsymbol{v}_{i,i+1} = \dot{d}_{i+1}\mathbf{z}_{i+1}$$

$\mathbf{z}_{i+1}$ → Line of action on joint $i+1$ (slide 21 E1)

And the expressions for linear and angular velocity become

$$\boxed{\begin{aligned} \mathrm{w}\boldsymbol{v}_{i+1} &= \mathrm{w}\boldsymbol{v}_i + \dot{d}_{i+1}\mathbf{z}_{i+1} + \mathrm{w}\boldsymbol{\omega}_i \times \mathrm{w}\mathbf{p}_{i,i+1} \\ \mathrm{w}\boldsymbol{\omega}_{i+1} &= \mathrm{w}\boldsymbol{\omega}_i \end{aligned}}$$

Velocity of a Rigid Body Link of a Manipulator (attached to a revolute joint)



Consider link $i+1$ supported by joint $i+1$

The angular velocity of link $i+1$ with respect to link i is given by

$$\mathbf{w}\boldsymbol{\omega}_{i,i+1} = \dot{\theta}_{i+1}\mathbf{z}_{i+1}$$

Axis of rotation acting of joint $i+1$ (slide 21 E1)

The linear velocity of link $i+1$ with respect to link i is given by

$$\mathbf{w}\boldsymbol{v}_{i,i+1} = \mathbf{w}\boldsymbol{\omega}_i \times \mathbf{w}\mathbf{p}_{i,i+1}$$

And the expressions for linear and angular velocity become

$$\boxed{\begin{aligned} \mathbf{w}\boldsymbol{v}_{i+1} &= \mathbf{w}\boldsymbol{v}_i + \mathbf{w}\boldsymbol{\omega}_i \times \mathbf{w}\mathbf{p}_{i,i+1} \\ \mathbf{w}\boldsymbol{\omega}_{i+1} &= \mathbf{w}\boldsymbol{\omega}_i + \dot{\theta}_{i+1}\mathbf{z}_{i+1} \end{aligned}}$$

Velocity of the end-effector

The previous expressions help us to compute the linear and angular velocities of any rigid body link with respect to its predecessor for both prismatic and revolute joints.

Let us group the joint variables in the column vector:

$$\mathbf{q} = \begin{bmatrix} \theta_1 & \theta_2 & d_3 & \theta_i & \dots & \theta_n \end{bmatrix}^T$$

Our goal is to describe the end-effector linear velocity ${}_{\mathbf{W}}\boldsymbol{v}_e$ and angular velocity ${}_{\mathbf{W}}\boldsymbol{\omega}_e$ as a function of the joint positions $\mathbf{q}(t)$ and joint velocities $\dot{\mathbf{q}}(t)$, namely

$$\begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} {}_{\mathbf{W}}\boldsymbol{v}_i \\ \sum_{i=1}^{n} {}_{\mathbf{W}}\boldsymbol{\omega}_i \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

Where the matrix $\mathbf{J}(\mathbf{q})$ is commonly known as the **Geometric Jacobian** or **Manipulator Jacobian**

Velocity of the end-effector

The Geometric Jacobian $\mathbf{J}(\mathbf{q})$ can be represented by the Jacobians corresponding to the linear ($\mathbf{J}_P(\mathbf{q})$) and angular ($\mathbf{J}_O(\mathbf{q})$) contribution of the velocities

$$\begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{J}_P(\mathbf{q}) \\ \mathbf{J}_O(\mathbf{q}) \end{bmatrix} \dot{\mathbf{q}}$$

Where $\mathbf{J}_P(\mathbf{q})$ and $\mathbf{J}_O(\mathbf{q})$ are comprised by the sum of the contributions of each joint to the linear and angular velocity, respectively, i.e.,

$$\mathbf{J}_P(\mathbf{q})\dot{\mathbf{q}} = \sum_{i=1}^{n} \mathbf{J}_{Pi}\dot{\mathbf{q}}$$

$$\mathbf{J}_O(\mathbf{q})\dot{\mathbf{q}} = \sum_{i=1}^{n} \mathbf{J}_{Oi}\dot{\mathbf{q}}$$

It is convenient to derive separately the expressions for $\mathbf{J}_P(\mathbf{q})$ and $\mathbf{J}_O(\mathbf{q})$. The expressions corresponding to their contribution change depending on the type of joint (i.e., prismatic or revolute)

Geometric Jacobian:

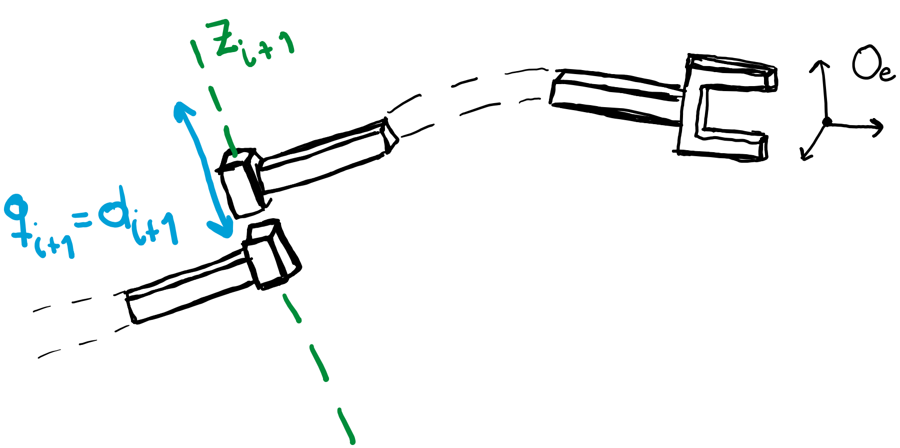
Linear velocity contribution (v, $\mathbf{J_{Pi}}$)

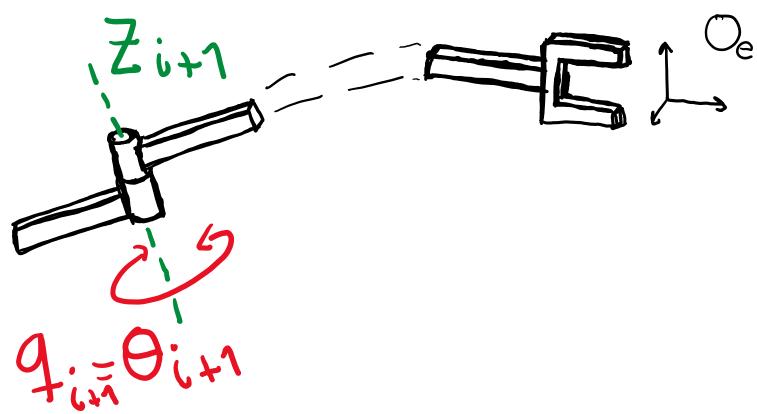
We can write the velocity of the end-effector by obtaining the derivative of its position using the chain rule

$$_{\mathbf{W}}\boldsymbol{v}_e = {}_{\mathbf{W}}\dot{\mathbf{p}}_e = \sum_{i=1}^{n} \frac{\partial\, {}_{\mathbf{W}}\mathbf{p}_e}{\partial q_i}\dot{q}_i = \sum_{i=1}^{n} \mathbf{J}_{Pi}\dot{q}_i$$

We need to compute the contribution to the linear velocity of joint $i+1$ to the origin of the frame attached to the end-effector (e).

There are two possible cases depending on the type of joint (i.e, prismatic or revolute)

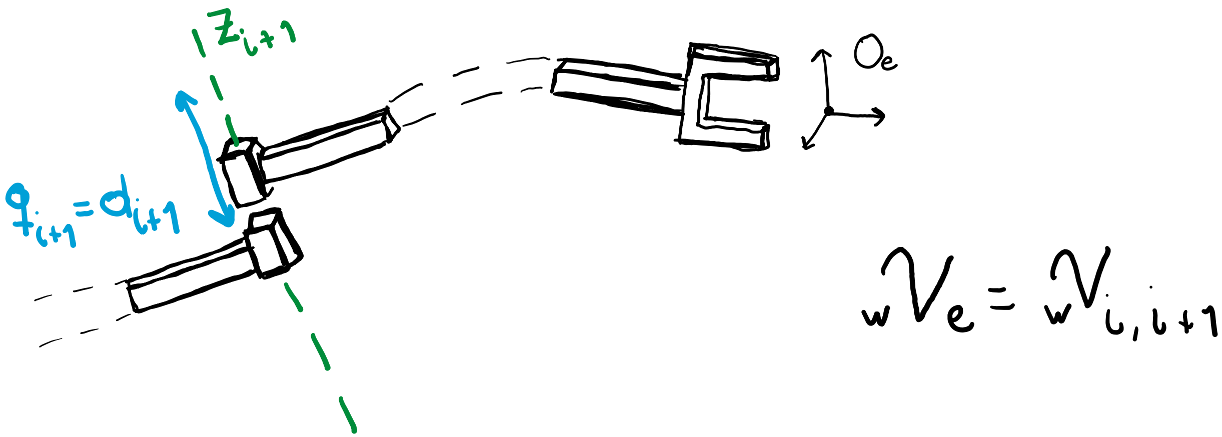




Geometric Jacobian:

Linear velocity contribution (v, $\mathbf{J}_{\mathrm{Pi}}$) prismatic joint

In the case of a prismatic joint, we consider that the links beyond the $i + 1$-th one are “frozen” and can be considered as a singled rigid body



With this in mind, the linear velocity contribution from joint i to joint e is given only by the contribution on $i + 1$

$$_\mathbf{W}\boldsymbol{v}_{i,i+1} = \dot{d}_{i+1}\mathbf{z}_{i+1}$$ slide 22

Where d_{i+1} is a joint space coordinate of the manipulator (i.e., q_{i+1}), which leads to

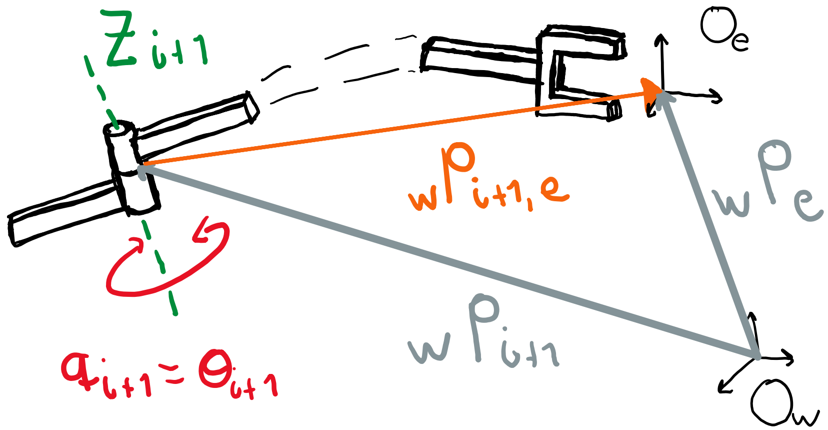
$$\dot{q}_{i+1}\mathbf{J}_{Pi+1} = \dot{d}_{i+1}\mathbf{z}_{i+1}$$

$$\boxed{\mathbf{J}_{Pi+1} = \mathbf{z}_{i+1}}$$

Geometric Jacobian:

Linear velocity contribution (v, J_{Pi}) revolute joint

In the case of a revolute joint, the contribution to the linear velocity needs to be computed directly with reference to the origin of the end-effector



This is given by

$$_{\mathrm{W}}\boldsymbol{v}_{i+1,e} = \boxed{_{\mathrm{W}}\boldsymbol{\omega}_{i,i+1}} \times \boxed{_{\mathrm{W}}\mathbf{p}_{i+1,e}} = \boxed{\dot{\theta}_{i+1}\mathbf{z}_{i+1}} \times (\boxed{_{\mathrm{W}}\mathbf{p}_e - {_{\mathrm{W}}\mathbf{p}_{i+1}}})$$ slide 23

Where θ_{i+1} is a joint space coordinate of the manipulator (i.e., q_{i+1}), which leads to

$$\dot{q}_{i+1}\mathbf{J}_{Pi+1} = \dot{\theta}_{i+1}\mathbf{z}_{i+1} \times ({_{\mathrm{W}}\mathbf{p}_e} - {_{\mathrm{W}}\mathbf{p}_{i+1}})$$

$$\boxed{\mathbf{J}_{Pi+1} = \mathbf{z}_{i+1} \times ({_{\mathrm{W}}\mathbf{p}_e} - {_{\mathrm{W}}\mathbf{p}_{i+1}})}$$

Geometric Jacobian:

Angular velocity contribution (ω, $\mathbf{J}_{Oi}$) prismatic joint

The angular velocity of link i is given by

$$\mathbf{w}\boldsymbol{\omega}_{i+1} = \mathbf{w}\boldsymbol{\omega}_i + \mathbf{w}\boldsymbol{\omega}_{i,i+1}$$

We can use this expression to find angular velocity at the end-effector frame as

$$\mathbf{w}\boldsymbol{\omega}_e = \mathbf{w}\boldsymbol{\omega}_n = \sum_{i=1}^{n} \boxed{\mathbf{w}\boldsymbol{\omega}_{i,i+1}} = \sum_{i=1}^{n} \mathbf{J}_{Oi+1}\dot{q}_{i+1}$$

In the case of a **prismatic** joint, we know that

$$\mathbf{w}\boldsymbol{\omega}_{i,i+1} = \mathbf{0}$$ slide 22

Which leads to

$$\dot{q}_{i+1}\mathbf{J}_{Oi+1} = 0$$

$$\boxed{\mathbf{J}_{Oi+1}} = 0$$

Geometric Jacobian:
Angular velocity contribution (ω, $\mathbf{J}_{Oi}$) revolute joint

The angular velocity of link i is given by

$$\mathrm{w}\boldsymbol{\omega}_{i+1} = \mathrm{w}\boldsymbol{\omega}_i + \mathrm{w}\boldsymbol{\omega}_{i,i+1}$$

We can use this expression to find angular velocity at the end-effector frame as

$$\mathrm{w}\boldsymbol{\omega}_e = \mathrm{w}\boldsymbol{\omega}_n = \sum_{i=1}^{n} \boxed{\mathrm{w}\boldsymbol{\omega}_{i,i+1}} = \sum_{i=1}^{n} \mathbf{J}_{Oi+1}\dot{q}_{i+1}$$

In the case of a **revolute** joint, we know that

$$\mathrm{w}\boldsymbol{\omega}_{i,i+1} = \dot{\theta}_{i+1}\mathbf{z}_{i+1}$$ slide 23

Where $\dot{\theta}_{i+1}$ is the derivative of a joint space coordinate (i.e., $\dot{q}_{i+1}$). This leads to

$$\dot{q}_{i+1}\mathbf{J}_{Oi+1} = \dot{\theta}_{i+1}\mathbf{z}_{i+1}$$

$$\boxed{\mathbf{J}_{Oi+1} = \mathbf{z}_{i+1}}$$

Geometric Jacobian:

Summary

The Geometric Jacobian is given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{P1} & \dots & \mathbf{J}_{Pi} \\ \mathbf{J}_{O1} & \dots & \mathbf{J}_{Oi} \end{bmatrix}$$

*Note: $\mathbf{z}_i$ and $_{\mathbf{W}}\mathbf{p}_i$ are only a function of the direct kinematics

Where

$$\begin{bmatrix} \mathbf{J}_{Pi} \\ \mathbf{J}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}_i \\ \mathbf{0} \end{bmatrix} & \text{if } i \text{ is prismatic} \\ \begin{bmatrix} \mathbf{z}_i \times ({}_{\mathbf{W}}\mathbf{p}_e - {}_{\mathbf{W}}\mathbf{p}_i) \\ \mathbf{z}_i \end{bmatrix} & \text{if } i \text{ is revolute} \end{cases}$$

Geometric Jacobian:

Coordinate frame change

The Jacobian that we just derived gives the relation between the linear and angular velocities of the end-effector and the joint velocities in a specific frame (the inertial frame)

$$\begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

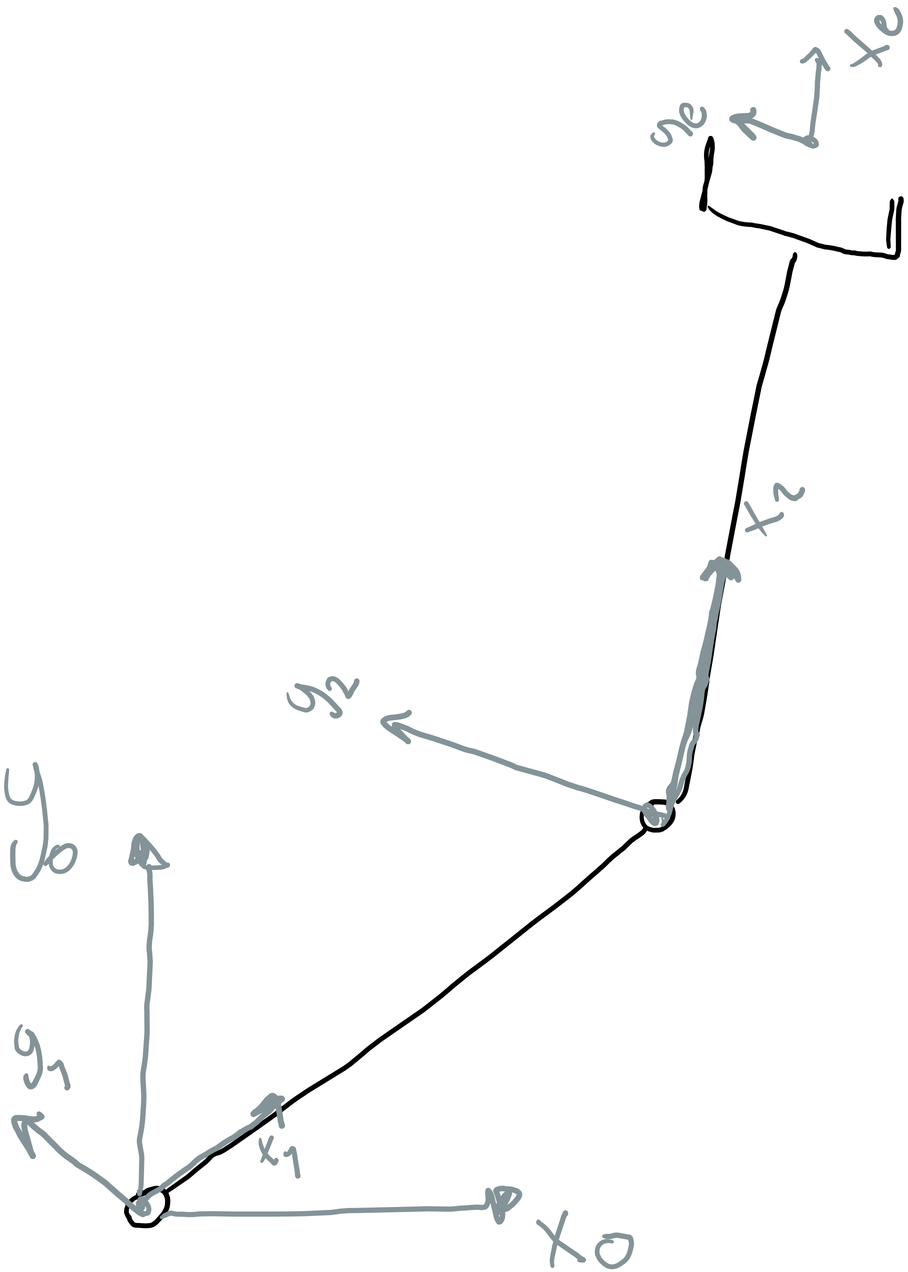
Consider the following coordinate change

$$\begin{bmatrix} {}_{\mathbf{u}}\boldsymbol{v}_e \\ {}_{\mathbf{u}}\boldsymbol{\omega}_e \end{bmatrix} = \begin{bmatrix} {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} \end{bmatrix} \begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix} = \begin{bmatrix} {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} \end{bmatrix} \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

Which leads to the following coordinate change of the Geometric Jacobian

$${}_{\mathbf{u}}\mathbf{J}(\mathbf{q}) = \begin{bmatrix} {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & {}_{\mathbf{u}}\mathbf{R}_{\mathbf{W}} \end{bmatrix} \mathbf{J}(\mathbf{q})$$

Example: 2R planar arm



Frame 0:

Base (fixed)

Frame 1:

Supports link 1

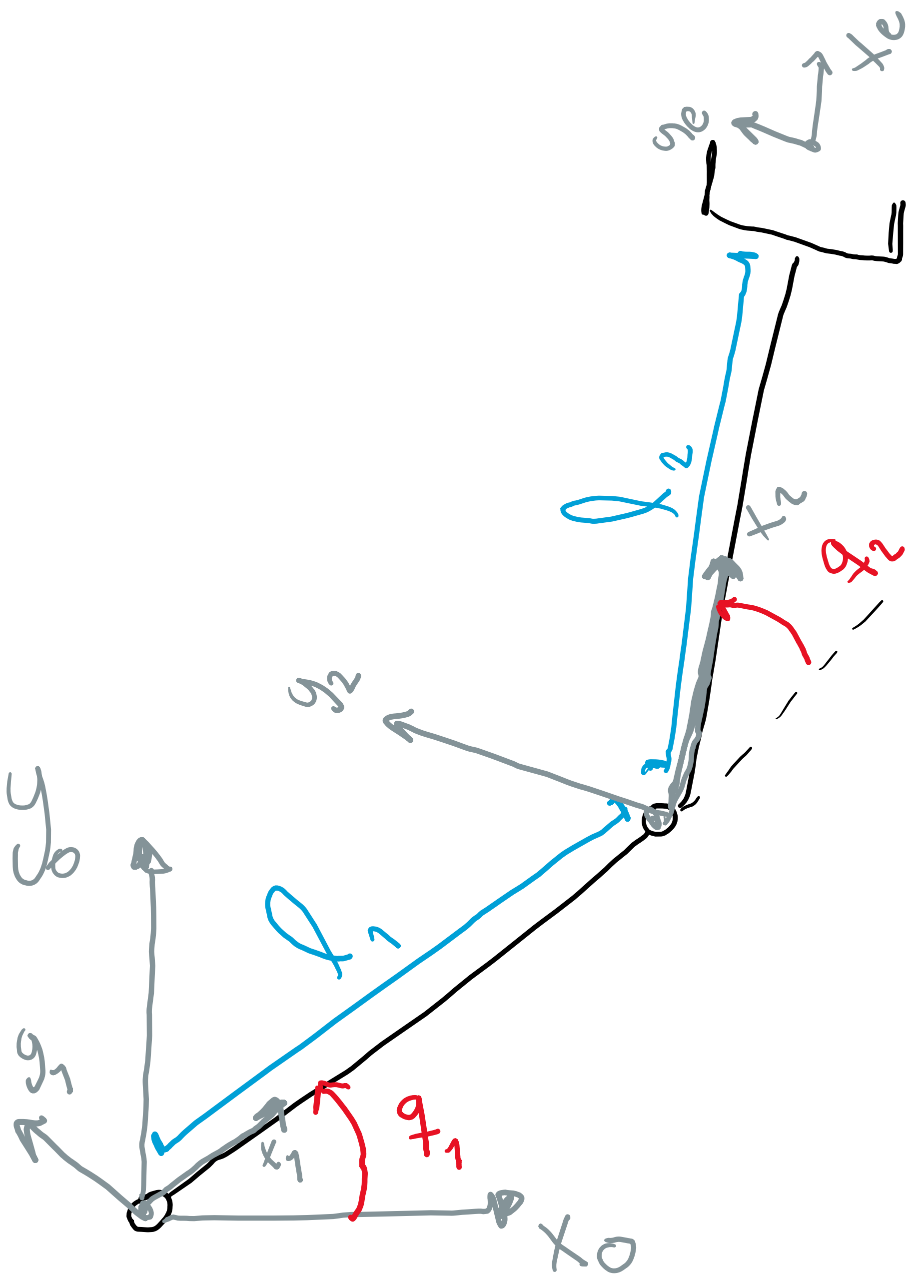
Frame 2:

Supports link 2

Frame e:

End-effector

Example: 2R planar arm



q_1 : Angle between F0 and F1

l_1 : Link 1 length

q_2 : Angle between F1 and F2

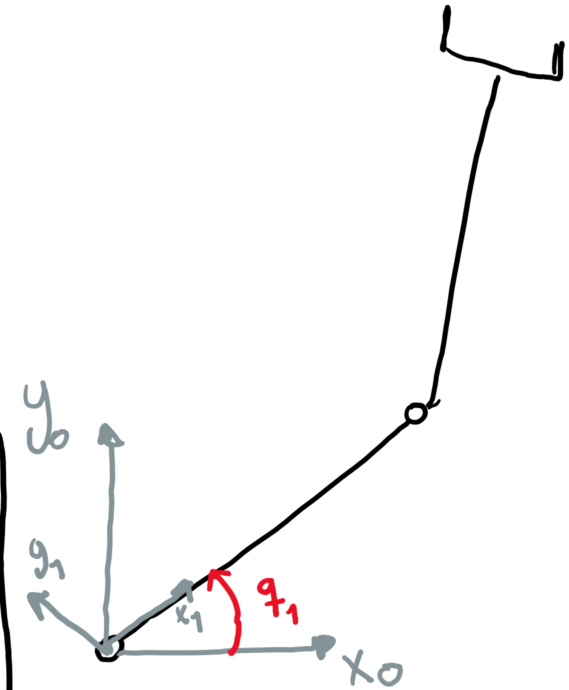
l_2 : Link 2 length

Example: 2R planar arm

Direct Kinematics

From 0 to 1 (pure rotation)

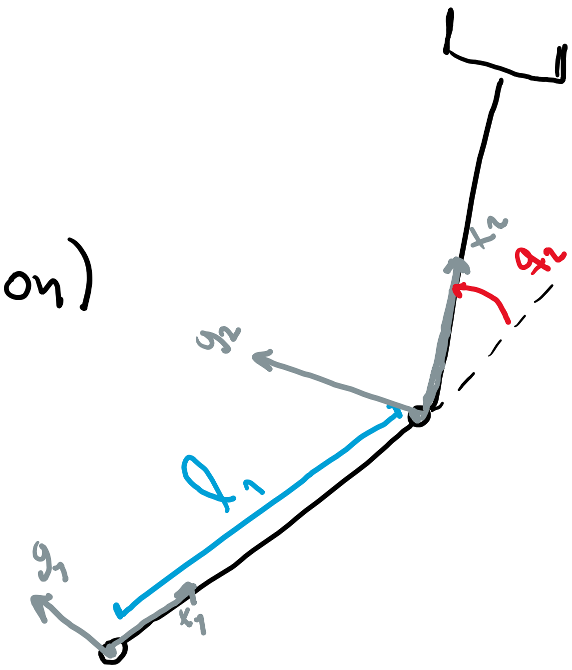
$$
{}_0T_1 = \left[\begin{array}{ccc|c} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$



Example: 2R planar arm

Direct Kinematics

From 1 to 2 (roto-translation)



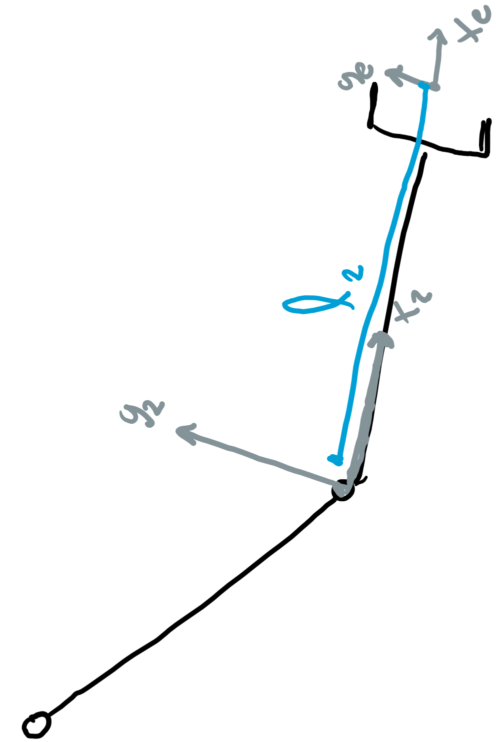
$$
{}_1T_2 = \left[\begin{array}{ccc|c} \cos q_2 & -\sin q_2 & 0 & l_1 \\ \sin q_2 & \cos q_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Example: 2R planar arm

Direct Kinematics

From 2 to e (Rigid transform)

$$_2T_e = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

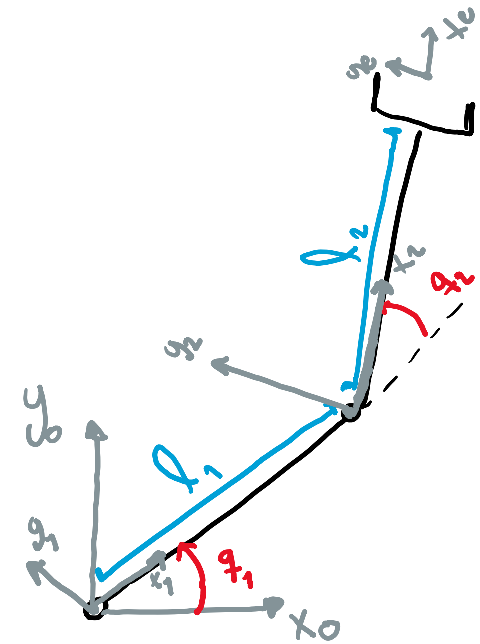


Example: 2R planar arm

Direct Kinematics

From 0 to 2

$${}_0T_2 = {}_0T_1 \, {}_1T_2$$



$${}_0T_2 = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 & l_1 \cos q_1 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 & l_1 \sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

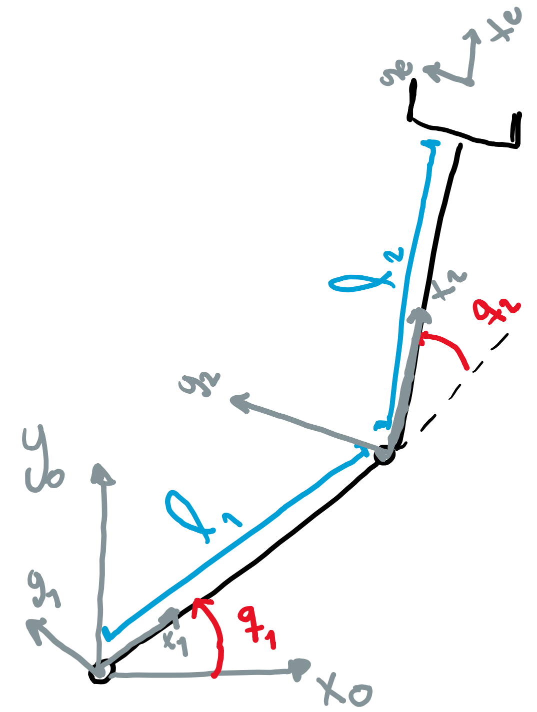
${}_0P_2$

Example: 2R planar arm

Direct Kinematics

From 0 to e

$$ {}_0T_e = {}_0T_2 \; {}_2T_e $$



$$ {}_0T_e = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 & l_2\cos(q_1+q_2) + l_1\cos q_1 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 & l_2\sin(q_1+q_2) + l_1\sin q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}_0P_e $$

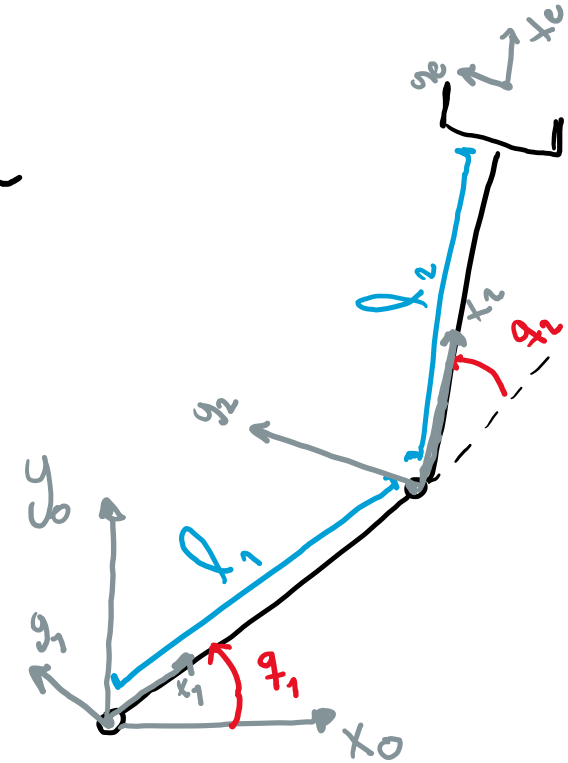
Example: 2R planar arm

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J(q)\dot{q} \qquad \dim(J) = 6 \times 2$$

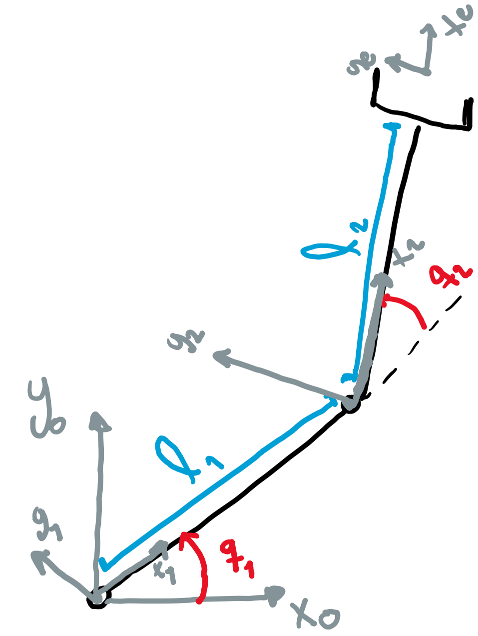
$$\begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \qquad J_i \in \mathbb{R}^6$$

For a revolute joint

$$J_i = \begin{bmatrix} z_i \times ({}_wP_e - {}_wP_i) \\ z_i \end{bmatrix}$$



Example: 2R planar arm



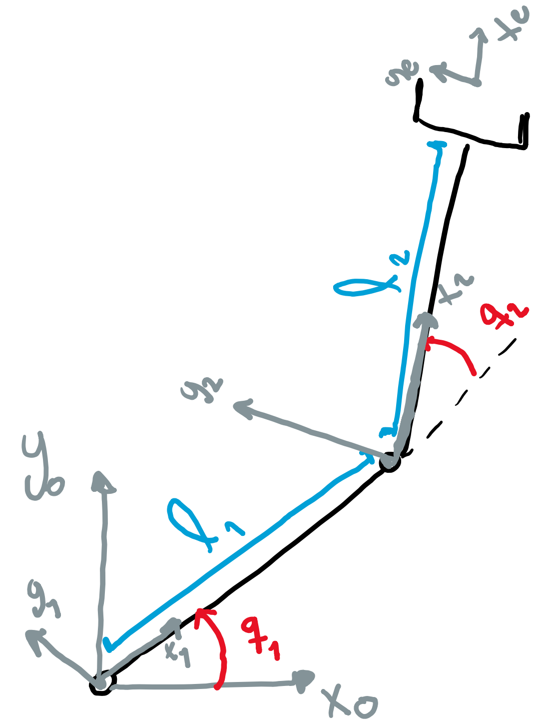
$$J = \begin{bmatrix} z_1 \times \underbrace{({}_0P_e - {}_0P_1)}_{{}_0P_e} & z_2 \times \underbrace{({}_0P_e - {}_0P_2)}_{{}_0P_{2e}} \\ z_1 & z_2 \end{bmatrix}$$

$\uparrow {}_0T_e$

$${}_0P_{2e} = {}_0P_e - \underset{\substack{\uparrow \\ {}_0T_2}}{{}_0P_2}$$

$${}_0P_e = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix} \qquad {}_0P_{2e} = \begin{bmatrix} l_2 c_{12} \\ l_2 s_{12} \\ 0 \end{bmatrix}$$

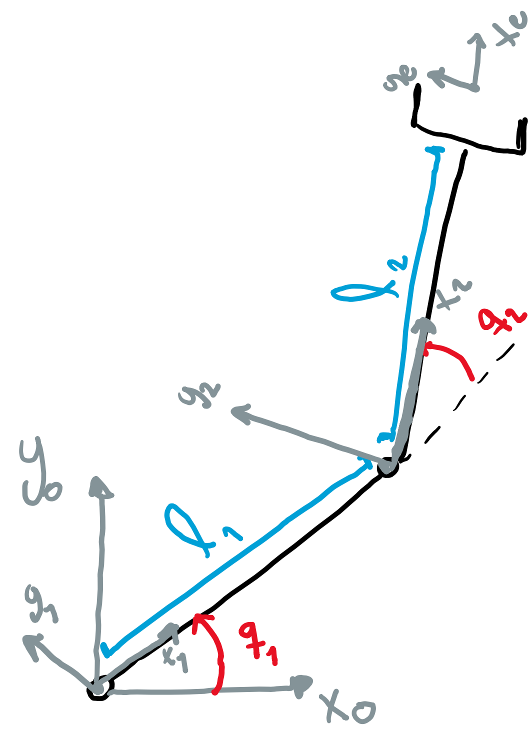
Example: 2R planar arm



Substituting $_0p_e$, $_0p_{2e}$, z_1 and z_2 in $J(q)$

$$J(q) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Geometric Jacobian: some comments



The dimension of $\mathbf{J}(\mathbf{q})$ for the 2R planar robot:

$$\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{6\times 2}$$

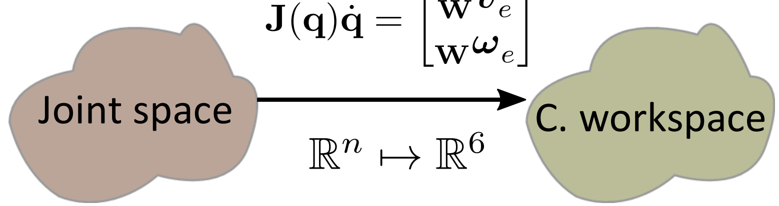
Which allows us to perform the mapping from the joint space to the task space, which in the specific case is

$$\mathbb{R}^2 \mapsto \mathbb{R}^6$$

In general, the dimension of $\mathbf{J}(\mathbf{q})$ is

$$\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{6\times n}$$

Where **6** corresponds to the dimension of the cartesian workspace (3 linear and 3 angular velocities) and **n** is the number of joint space coordinates. $\mathbf{J}(\mathbf{q})$ performs the following mapping

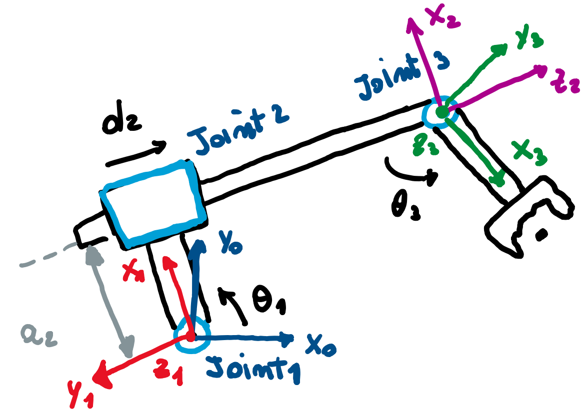


$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix}$$

$$\mathbb{R}^n \mapsto \mathbb{R}^6$$

The maximum number of linear/angular end-effector velocities that can be **independently** assigned is **n**

Exercise: Geometric Jacobian of RPR robot



Local homogeneous transformation matrices (E2, slide 25)

$$ {}_0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

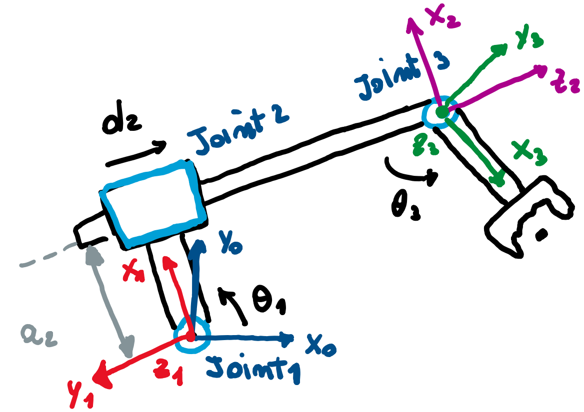
$$ {}_1T_2 = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}_2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}_3T_{ee} = \begin{bmatrix} 0 & 0 & 1 & a_e \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Exercise: Geometric Jacobian of RPR robot

Global homogeneous transformation matrices (E2, slide 25)



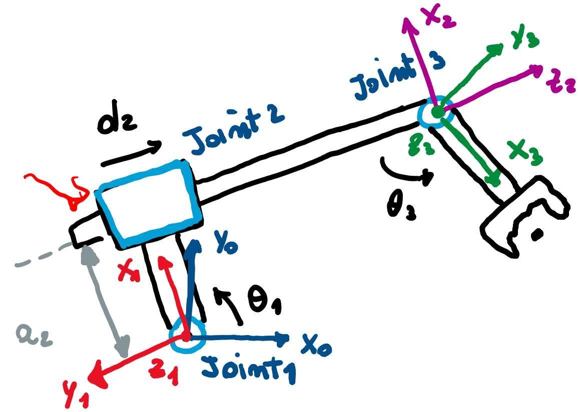
$$
{}_0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}_0p_1
$$

$$
{}_0T_2 = {}_0T_1\,{}_1T_2 = \begin{bmatrix} c_1 & 0 & s_1 & a_2 c_1 + d_2 s_1 \\ s_1 & 0 & -c_1 & a_2 s_1 - d_2 c_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}_0p_2
$$

$$
{}_0T_3 = {}_0T_2\,{}_2T_3 = \begin{bmatrix} c_1 c_3 - s_1 s_3 & -c_1 s_3 - s_1 c_3 & 0 & a_2 c_1 + d_2 s_1 \\ s_1 c_3 + c_1 s_3 & -s_1 s_3 + c_1 c_3 & 0 & a_2 s_1 - d_2 c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}_0p_3
$$

$$
{}_0T_e = {}_0T_3\,{}_3T_e = \begin{bmatrix} -s_{13} & 0 & c_{13} & a_e c_{13} + a_2 c_1 + d_2 s_1 \\ c_{13} & 0 & s_{13} & a_e s_{13} + a_2 s_1 - d_2 c_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}_0p_e
$$

Exercise: Geometric Jacobian of RPR robot



Geometric Jacobian

J_1: Revolute $\qquad J_2$: Prismatic $\qquad J_3$: Revolute

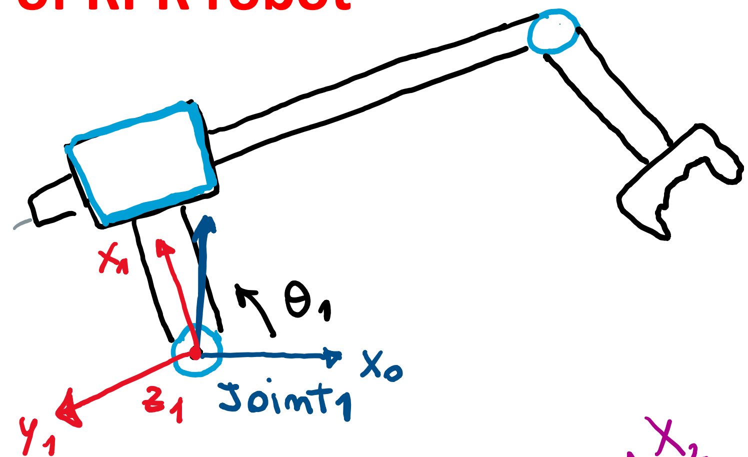
$$J(q) = \begin{bmatrix} z_1 \times ({}_0p_e - {}_0p_1) & z_2 & z_3 \times ({}_0p_e - {}_0p_3) \\ z_1 & 0 & z_3 \end{bmatrix}$$

Remember: z_i w.r.t. W (F0)

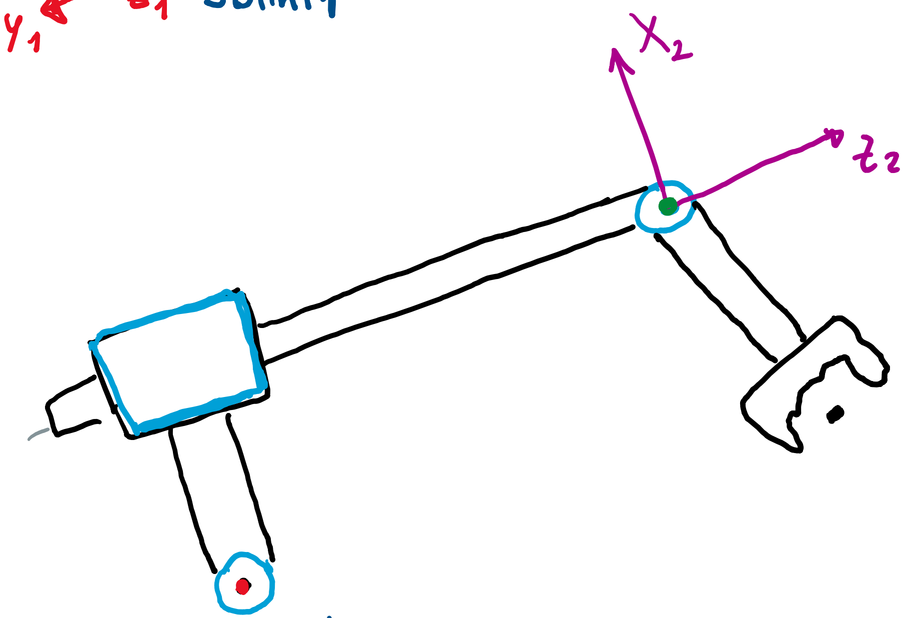
Exercise: Geometric Jacobian of RPR robot

Joint motion axes $\mathbf{z}_i$

$$ {}_1z_1 = {}_wz_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} $$



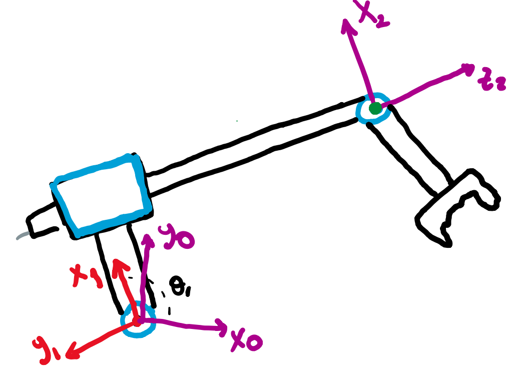
$$ {}_2z_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} $$



$$ {}_wz_2 = \underbrace{{}_wR_2}_{}\,{}_2z_2 $$

Exercise: Geometric Jacobian of RPR robot

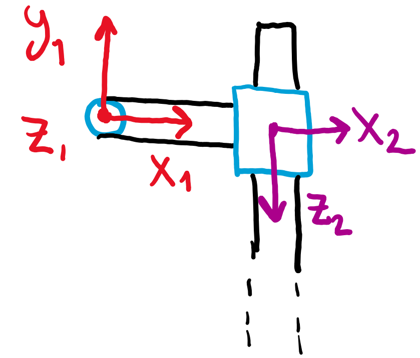
Joint motion axes $\mathbf{z}_i$



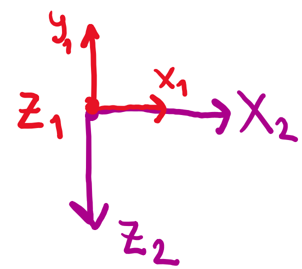
$${}^{w}z_2 = \underbrace{{}^{w}R_2}\;{}^{2}z_2$$

90° about X_1

Rot. due to q_1

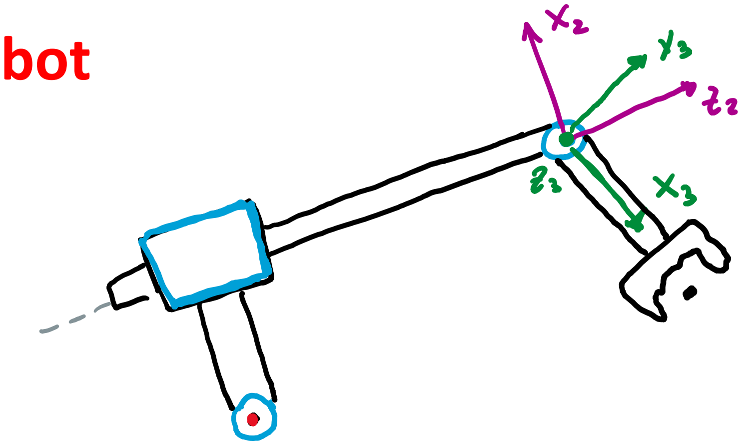


$${}^{w}R_2 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



$${}^{w}z_2 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

Exercise: Geometric Jacobian of RPR robot



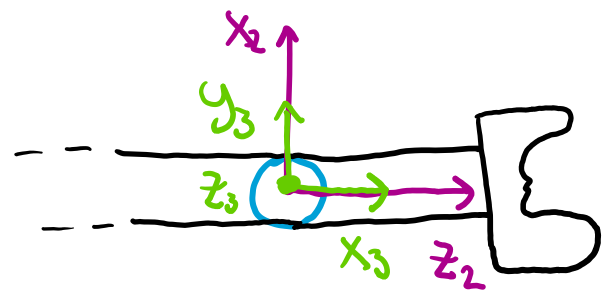
Joint motion axes $\mathbf{z}_i$

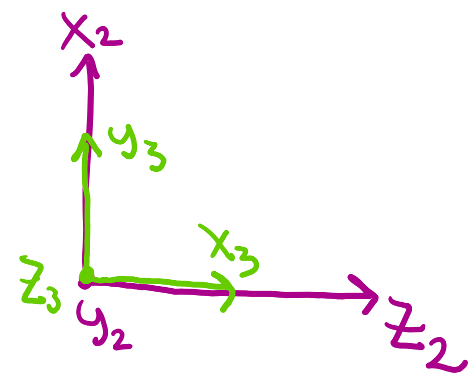
$${}_w z_3 = {}_w R_3 \, {}_3 z_3$$

$${}_w R_3 = {}_w R_2 \, {}_2 R_3$$

$$\begin{array}{cc} {}_w R_2 & -90^\circ \text{ about } x_2 \\ \end{array}$$

$${}_w R_3 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

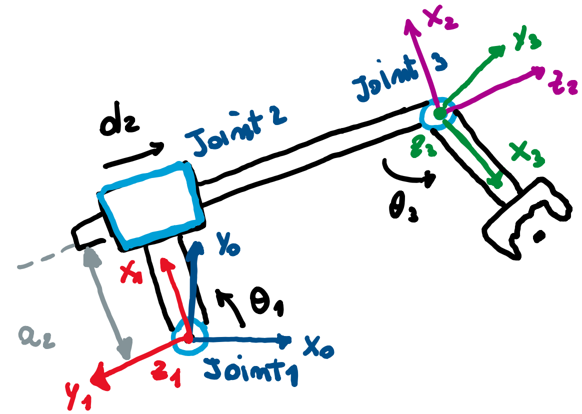




$${}_w z_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise: Geometric Jacobian of RPR robot

Geometric Jacobian



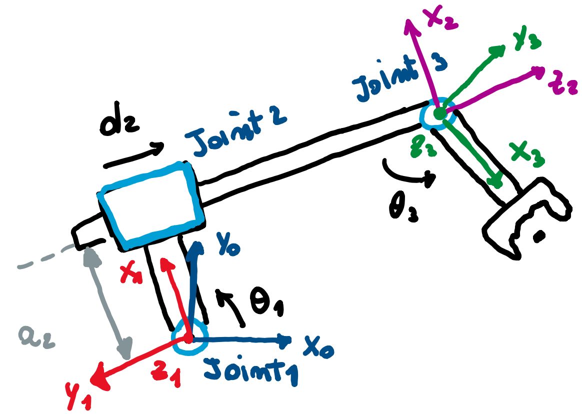
J_2: Prismatic

J_1: Revolute

J_3: Revolute

$$J(q) = \begin{bmatrix} z_1 \times ({}_0P_e - {}_0P_1) & z_2 & z_3 \times ({}_0P_e - {}_0P_3) \\ z_1 & 0 & z_3 \end{bmatrix}$$

Exercise: Geometric Jacobian of RPR robot



Geometric Jacobian

$$ {}_0P_e - {}_0P_1 = \overset{{}_0T_e}{\underline{\begin{bmatrix} a_e C_{13} + a_1 C_1 + d_2 S_1 \\ a_e S_{13} + a_2 S_1 - d_2 C_1 \\ 0 \end{bmatrix}}} + \overset{{}_0T_1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} $$

$$ {}_0P_e - {}_0P_3 = \begin{bmatrix} a_e C_{13} + a_1 C_1 + d_2 S_1 \\ a_e S_{13} + a_1 S_1 - d_2 C_1 \\ 0 \end{bmatrix} - \overset{{}_0T_3}{\begin{bmatrix} a_1 C_1 + d_2 S_1 \\ a_1 S_1 - d_2 C_1 \\ 0 \end{bmatrix}} = \begin{bmatrix} a_e C_{13} \\ a_e S_{13} \\ 0 \end{bmatrix} $$

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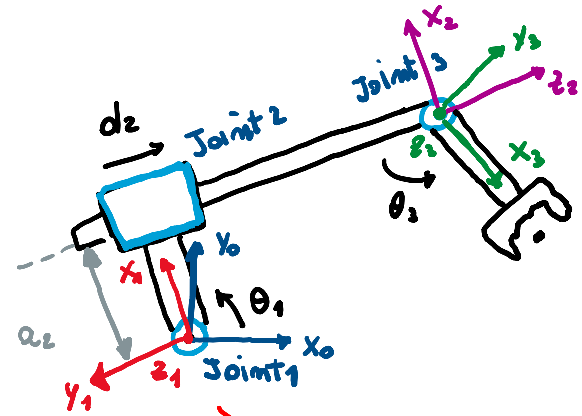
Geometric Jacobian



$$z_1 \times ({}_0P_e - {}_0P_1) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_e C_{13} + a_1 C_1 + d_2 S_1 \\ a_e S_{13} + a_2 S_1 - d_2 S_1 \\ 0 \end{bmatrix}$$

$$[z_1]_x ({}_0P_e)$$

$$= \begin{bmatrix} -a_e S_{13} - a_2 S_1 - d_2 S_1 \\ a_e C_{13} + a_1 C_1 + d_2 S_1 \\ 0 \end{bmatrix}$$

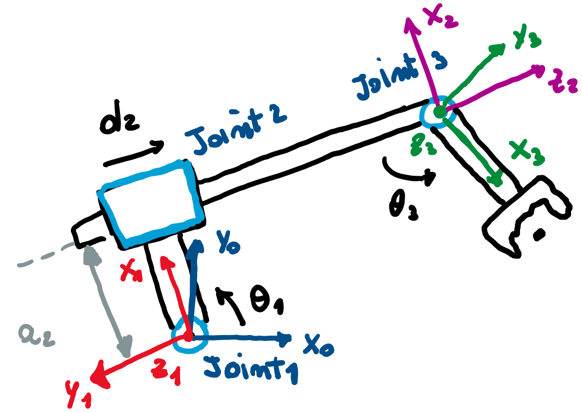
$$S(\omega)$$

$$[\omega]_x = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$z_1 = z_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$z_3 \times ({}_0P_e - {}_0P_3) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_e C_{13} \\ a_e S_{13} \\ 0 \end{bmatrix} = \begin{bmatrix} -a_e S_{13} \\ a_e C_{13} \\ 0 \end{bmatrix}$$

Exercise: Geometric Jacobian of RPR robot



Geometric Jacobian

$$J(q) = \begin{bmatrix} -a_e S_{13} - a_2 S_1 - d_2 S_1 & S_1 & -a_e S_{13} \\ a_e C_{13} + a_1 C_1 + d_2 S_1 & -C_1 & a_e C_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Why not just differentiating the direct kinematics?

The direct kinematics can be expressed in the following way

$$\mathbf{r} = \begin{bmatrix} {}_{W}\mathbf{p}_e(\mathbf{q}) \\ \mathbf{\Phi}(\mathbf{q}) \end{bmatrix} = \mathbf{f}(\mathbf{q})$$

Recall that $\mathbf{\Phi}$ does **not** belong to a vector space and is a set of Euler angles.

We can obtain the derivative of $\mathbf{r}$ with respect to time using the chain rule

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \boxed{\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}} \dot{\mathbf{q}}$$

The expression $\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}}$ is referred to as the **Analytical Jacobian**

$$\dot{\mathbf{r}} = \mathbf{J}_{\mathbf{r}}(\mathbf{q}) \dot{\mathbf{q}}$$

Analytical Jacobian

The function $\mathbf{f}(\mathbf{q})$ is a multivariable vector-valued function, i.e.,

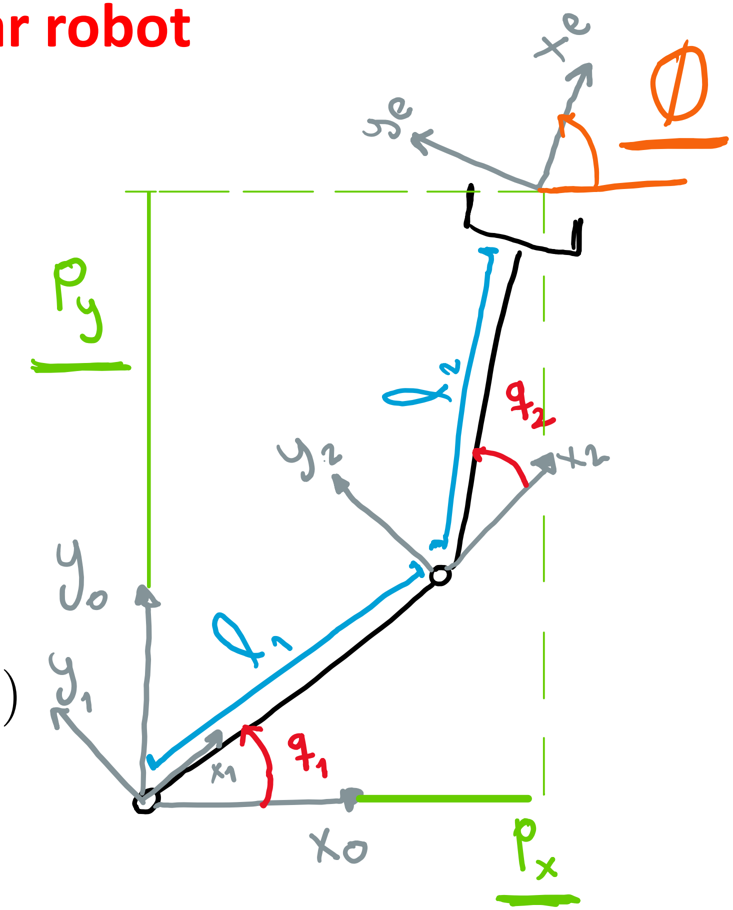
$$\mathbf{f}(\mathbf{q}) = \begin{bmatrix} f_1(\mathbf{q}) \\ f_2(\mathbf{q}) \\ \vdots \\ f_m(\mathbf{q}) \end{bmatrix}$$

The dimension of $\mathbf{f}(\mathbf{q})$ depends on the number of task space coordinates. For example, for the 2R planar robot $\mathbf{f}(\mathbf{q}) \in \mathbb{R}^3$ (p_x, p_y, ϕ, see next slide)

Thus, the derivative of $\mathbf{f}(\mathbf{q})$ with respect to the joint space variables is a **matrix** of the form

$$\frac{\partial \mathbf{f}}{\partial \mathbf{q}}(\mathbf{q}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{q})}{\partial q_1} & \frac{\partial f_1(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_1(\mathbf{q})}{\partial q_n} \\ \frac{\partial f_2(\mathbf{q})}{\partial q_1} & \frac{\partial f_2(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_2(\mathbf{q})}{\partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m(\mathbf{q})}{\partial q_1} & \frac{\partial f_m(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_m(\mathbf{q})}{\partial q_n} \end{bmatrix}$$

Example: analytical Jacobian of 2R planar robot
Method 1



Direct kinematics

$$p_x = \ell_1 \cos(q_1) + \ell_2 \cos(q_1 + q_2)$$
$$p_y = \ell_1 \sin(q_1) + \ell_2 \sin(q_1 + q_2)$$
$$\phi = q_1 + q_2$$

Differentiating with respect to time

$$\dot{p}_x = -\ell_1 \sin(q_1)\dot{q}_1 - \ell_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)$$
$$\dot{p}_y = \ell_1 \cos(q_1)\dot{q}_1 + \ell_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)$$
$$\dot{\phi} = \dot{q}_1 + \dot{q}_2$$

Rearranging

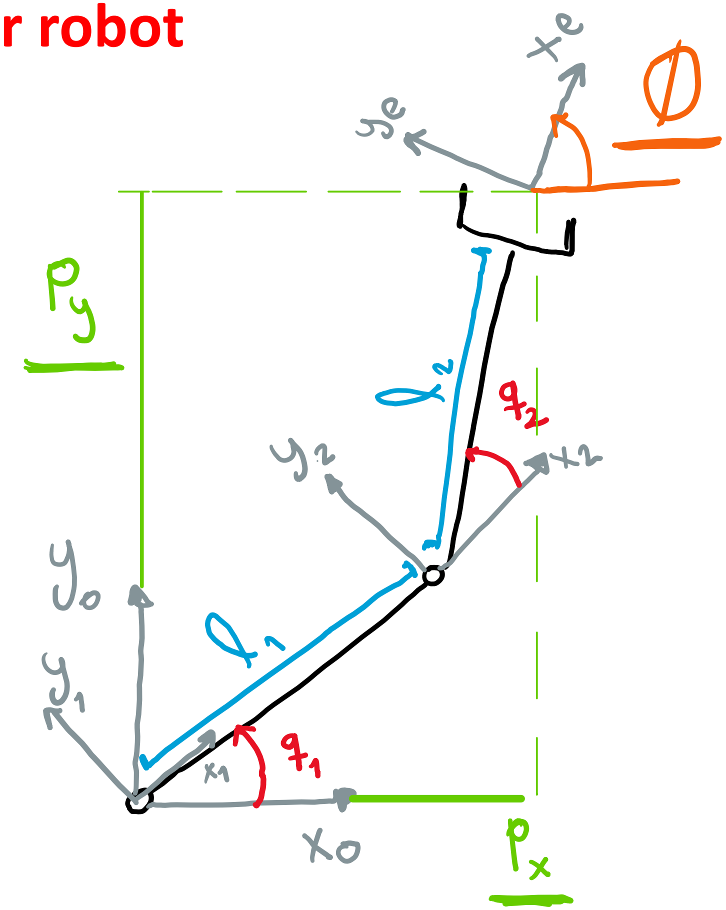
$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} -\ell_1 \sin(q_1) - \ell_2 \sin(q_1 + q_2) & -\ell_2 \sin(q_1 + q_2) \\ \ell_1 \cos(q_1) + \ell_2 \cos(q_1 + q_2) & \ell_2 \cos(q_1 + q_2) \\ 1 & 1 \end{bmatrix}}_{\mathbf{J}_\mathbf{r}(\mathbf{q})} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Example: analytical Jacobian of 2R planar robot

Method 2

Direct kinematics

$$\mathbf{f}(\mathbf{q}) = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} \ell_1 \cos(q_1) + \ell_2 \cos(q_1+q_2) \\ \ell_1 \sin(q_1) + \ell_2 \sin(q_1+q_2) \\ q_1 + q_2 \end{bmatrix}$$



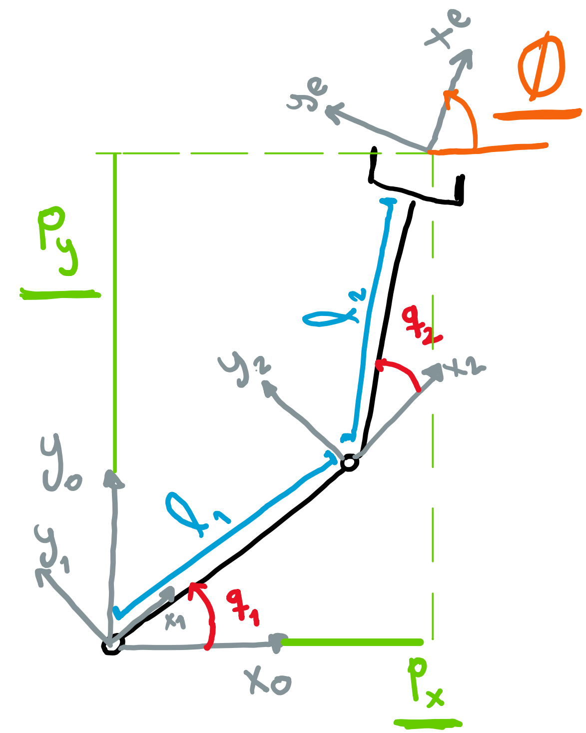
Differentiating with respect to time

$$\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} \\ \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} \\ \frac{\partial \phi}{\partial q_1} & \frac{\partial \phi}{\partial q_2} \end{bmatrix}$$

Which results in

$$\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{J_r}(\mathbf{q}) = \begin{bmatrix} -\ell_1 \sin(q_1) - \ell_2 \sin(q_1+q_2) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos(q_1) + \ell_2 \cos(q_1+q_2) & \ell_2 \cos(q_1+q_2) \\ 1 & 1 \end{bmatrix}$$

Analytical Jacobian: some comments



The dimension of $\mathbf{J}_r(\mathbf{q})$ for the 2R planar robot:

$$\mathbf{J_r}(\mathbf{q}) \in \mathbb{R}^{3\times 2}$$

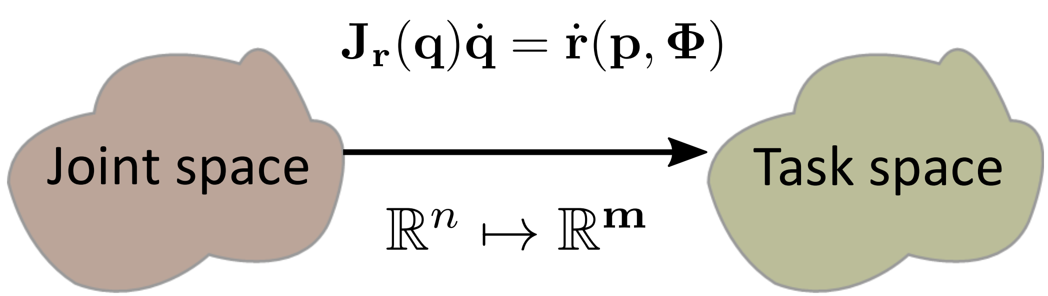
Which allows us to perform the mapping from the joint space to the task space, which in the specific case is

$$\mathbb{R}^2 \mapsto \mathbb{R}^3$$

In general, the dimension of $\mathbf{J}_r(\mathbf{q})$ is

$$\mathbf{J_r}(\mathbf{q}) \in \mathbb{R}^{\mathbf{m}\times n}$$

Where **m** corresponds to number of chosen task space coordinates to describe position and orientation of the end-effector, and **n** is the number of joint space coordinates. $\mathbf{J}_r(\mathbf{q})$ performs the following mapping



Some comments regarding the previous lecture

- Velocities are **free vectors**
  - Their action is not confined to a specific point in space (like position vectors). We are only interested in their magnitude and **direction** (orientation)
- In the definition of the geometric Jacobian, vector $\mathbf{z}_i$ is given with respect to the world frame $\mathbf{W}$

$$\begin{bmatrix} \mathbf{J}_{Pi} \\ \mathbf{J}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} {}_{\mathbf{W}}\mathbf{z}_i \\ \mathbf{0} \end{bmatrix} & \text{if } i \text{ is prismatic} \\ \\ \begin{bmatrix} {}_{\mathbf{W}}\mathbf{z}_i \times ({}_{\mathbf{W}}\mathbf{p}_e - {}_{\mathbf{W}}\mathbf{p}_i) \\ {}_{\mathbf{W}}\mathbf{z}_i \end{bmatrix} & \text{if } i \text{ is revolute} \end{cases}$$

Comparison and Relationship Between J(q) and J_r(q)

Let us compare the geometric and the analytical Jacobian of the 2R planar robot

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & \ell_2 \cos(q_1 + q_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{J_r}(\mathbf{q}) = \begin{bmatrix} -\ell_1 \sin(q_1) - \ell_2 \sin(q_1 + q_2) & -\ell_2 \sin(q_1 + q_2) \\ \ell_1 \cos(q_1) + \ell_2 \cos(q_1 + q_2) & \ell_2 \cos(q_1 + q_2) \\ 1 & 1 \end{bmatrix}$$

The rows corresponding to the task space coordinates that are needed to fully describe position and orientation of the 2R planar robot are the same between $\mathbf{J}$ and $\mathbf{J_r}$, whereas the unnecessary variables are 0 in the case of $\mathbf{J}$.

Comparison and Relationship Between J(q) and J_r(q)

Let us go back to the relationship between angular velocity $\boldsymbol{\omega}$ and $\boldsymbol{\Phi}$*

$$\boldsymbol{\omega} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\phi & 0 \\ \cos\phi\sin\theta & \cos\phi & 0 \\ -\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \begin{array}{l} \text{Yaw rate} \\ \text{Pitch rate} \\ \text{Roll rate} \end{array}$$

$$\boldsymbol{\omega} = \mathbf{T}_{RPY}(\phi, \theta)\dot{\boldsymbol{\Phi}}$$

The linear and angular velocity can be expressed as follows

$$\underbrace{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}}_{\mathbf{J}(\mathbf{q})\dot{\mathbf{q}}} = \begin{bmatrix} \boldsymbol{v} \\ \mathbf{T}_{RPY}(\boldsymbol{\Phi})\dot{\boldsymbol{\Phi}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{RPY}(\boldsymbol{\Phi}) \end{bmatrix}}_{\mathbf{T}_A(\boldsymbol{\Phi})} \underbrace{\begin{bmatrix} \boldsymbol{v} \\ \dot{\boldsymbol{\Phi}} \end{bmatrix}}_{\mathbf{J_r}(\mathbf{q})\dot{\mathbf{q}}}$$

$$\mathbf{J}(\mathbf{q}) = \mathbf{T}_A(\boldsymbol{\Phi})\mathbf{J_r}(\mathbf{q})$$

*E0.3 slide 17

Summary

Velocity of the end effector in terms of linear and angular velocities ($\boldsymbol{v}$ and $\boldsymbol{\omega}$):

$$\begin{bmatrix} {}_{\mathbf{W}}\boldsymbol{v}_e \\ {}_{\mathbf{W}}\boldsymbol{\omega}_e \end{bmatrix} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{J}_P(\mathbf{q}) \\ \mathbf{J}_O(\mathbf{q}) \end{bmatrix} \dot{\mathbf{q}}$$

Where $\mathbf{J_P}$ and $\mathbf{J_O}$ are made up for the linear and angular velocity contributions of each of the joints along the kinematic chain, i.e.,

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{P1} & \cdots & \mathbf{J}_{Pi} \\ \mathbf{J}_{O1} & \cdots & \mathbf{J}_{Oi} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{J}_{Pi} \\ \mathbf{J}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}_i \\ \mathbf{0} \end{bmatrix} & \text{if } i \text{ is prismatic} \\ \begin{bmatrix} \mathbf{z}_i \times ({}_{\mathbf{W}}\mathbf{p}_e - {}_{\mathbf{W}}\mathbf{p}_i) \\ \mathbf{z}_i \end{bmatrix} & \text{if } i \text{ is revolute} \end{cases}$$

Summary

Velocity of the end effector in terms of linear velocity and end-effector orientation ($\boldsymbol{v}$ and $\mathbf{\Phi}$)

$$\dot{\mathbf{r}}(\mathbf{p}, \mathbf{\Phi}) = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J_r}(\mathbf{q}) \dot{\mathbf{q}}$$

Where $\mathbf{f}(\mathbf{q})$ is a vector-valued function and its derivative is given by

$$\frac{\partial \mathbf{f}}{\partial \mathbf{q}}(\mathbf{q}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{q})}{\partial q_1} & \frac{\partial f_1(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_1(\mathbf{q})}{\partial q_n} \\ \frac{\partial f_2(\mathbf{q})}{\partial q_1} & \frac{\partial f_2(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_2(\mathbf{q})}{\partial q_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m(\mathbf{q})}{\partial q_1} & \frac{\partial f_m(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial f_m(\mathbf{q})}{\partial q_n} \end{bmatrix}$$

Summary

There is a relationship between J(q) and J_r(q) that can be obtained from the relation between $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\Phi}}$

$$\underbrace{\begin{bmatrix} v \\ \boldsymbol{\omega} \end{bmatrix}}_{\mathbf{J}(\mathrm{q})\dot{\mathrm{q}}} = \begin{bmatrix} v \\ \mathbf{T}_{RPY}(\boldsymbol{\Phi})\dot{\boldsymbol{\Phi}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{RPY}(\boldsymbol{\Phi}) \end{bmatrix}}_{\mathbf{T}_A(\boldsymbol{\Phi})} \underbrace{\begin{bmatrix} v \\ \dot{\boldsymbol{\Phi}} \end{bmatrix}}_{\mathbf{J}_{\mathbf{r}}(\mathrm{q})\dot{\mathrm{q}}}$$

$$\mathbf{J}(\mathrm{q}) = \mathbf{T}_A(\boldsymbol{\Phi})\mathbf{J}_{\mathbf{r}}(\mathrm{q})$$

Sources

In references folder:

- 11_deluca_rob1_DifferentialKinematics

Other references

- Siciliano, Sciavicco, Oriolo, Villani, *Robotics. Modelling, Planning and Control.* Chapter 3