Discrete-Time Distributed Population Dynamics for Optimization and Control

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Abstract

Distributed population dynamics offer a game-theoretical framework for distributed decision-making. As such, the application of these methods to network systems has been widely studied in the literature. The currently available theory has focused exclusively on continuous-time formulations of distributed population dynamics. However, given that modern computers can only process information in a discretized fashion, the practical implementation of these methods is inevitably discrete. In consequence, it is paramount to extend the available theory to a more implementable discrete-time approach. For that reason, in this paper we propose a general class of discrete-time distributed population dynamics, and we derive sufficient conditions to guarantee their convergence to an optimal solution of the underlying game. To illustrate the relevance and performance of the proposed methods, we apply the developed theory to distributed optimization and control problems, including a real multi-robot platform, which consider non-complete communication networks and coupled constraints.

Keywords— Distributed optimization; Distributed control; Networked evolutionary game theory; Discrete-time systems.

1 Introduction

Population dynamics provide an evolutionary game theoretical framework to model and analyze the strategic interaction of a large but finite population of agents [28, 10, 25]. Recently, the population dynamics framework has been extended to consider strategic interactions over non-complete networks [4], and the resulting distributed dynamics have been implemented as optimization and control algorithms in several engineering applications [24, 2, 17]. Although many other distributed game theoretical methods exist in the literature [15, 11, 12, 6, 8], an attractive trait of distributed population dynamics is that, when applied as a distributed optimization or control method, they allow the straightforward consideration of some coupled constraints over the variables of the problem. In contrast with other distributed algorithms where coupled constraints are handled through decoupling mechanisms, such as utility design [13] or primal-dual decomposition [30], in the context of population dynamics the coupled constraints are handled by means of some invariance properties of the dynamics [25, 4]. Therefore, if designed properly, population dynamics can satisfy coupled constraints dynamically in the hard sense. This fact makes distributed population dynamics specially attractive for engineering applications in the context of distributed dynamic resource allocation [24].

Even though distributed population dynamics have been widely studied in the literature, to the best of our knowledge all of the available studies have focused only on the continuous-time formulation of such

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dynamics. Whilst such formulation allows the analysis of the dynamics through standard continuous-time nonlinear control theory, it does not admit the direct implementation of the dynamics on modern processors where computations are necessarily discrete. Furthermore, the theoretical analyses developed in continuous-time do not necessarily hold under arbitrary discretizations. To extend the current theory to a more implementable framework, in this paper we propose a general class of discrete-time distributed population dynamics. More precisely, we extend the continuous-time framework of [25, 4] to a novel discrete-time distributed formulation. It is worth mentioning that even though there exist other forms of discrete-time population dynamics, e.g., the biologically inspired discrete-time replicator dynamics [28], such dynamics have not been formulated under a distributed game theoretical framework as in here. Moreover, in this work we study a general family of dynamics that comprises a form of the discrete-time replicator dynamics as a particular case.

To summarize, this paper has four main contributions. First, we propose a novel discrete-time model for a general class of distributed population dynamics. Second, we formally characterize some well-known game theoretical concepts of the proposed dynamics, i.e., Nash stationarity and positive correlation [25]. Third, we obtain sufficient conditions to guarantee the invariance and convergence of the proposed discrete-time distributed dynamics. And fourth, we illustrate the advantages of the dynamics on relevant engineering applications, including an implementation on an actual robotic platform, and we compare our dynamics against other popular and recent distributed algorithms.

The rest of this paper is organized as follows. In Section 2 we provide some preliminary concepts on population games and population dynamics, and we derive our discrete-time distributed population dynamics from their continuous-time counterpart. Then, in Section 3 we characterize the equilibrium points as well as the invariance and convergence properties of the proposed dynamics. Afterwards, in Section 4 we present some numerical examples that illustrate the application of our theoretical results on relevant engineering problems, including the implementation of the dynamics on a real multi-robot platform. Finally, in Section 5 we conclude the paper and mention some future directions.

2 Distributed Population Dynamics: From Continuous to Discrete Time

Population games provide a game-theoretical framework to describe the strategic interaction of a large but finite population of agents [28, 10, 25]. Under this framework, the agents of the population are players engaged in an anonymous game with a finite set of strategies denoted as $\mathcal{V} = \{1, 2, \ldots, n\}$. At any time, the state of the population is described by the vector $\mathbf{x} = [x_i] \in \mathbb{R}^n_{\geq 0}$, where $x_i \in \mathbb{R}_{\geq 0}$ denotes the fraction of players playing the strategy $i \in \mathcal{V}$. Hence, the set of all possible population states is given by

$$\Delta = \left\{ \mathbf{x} \in \mathbb{R}^n_{\geq 0} \,:\, \sum_{i \in \mathcal{V}} x_i = m \right\}, \tag{1}$$

where $m \in \mathbb{R}_{>0}$ corresponds to the total population mass. For the forthcoming analyses and discussions, it is also useful to define the following Δ-related sets:

$$\begin{aligned} \operatorname{aff}(\Delta) &= \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \sum_{i \in \mathcal{V}} x_i = m \right\}, \\ \operatorname{int}(\Delta) &= \Delta \cap \mathbb{R}^n_{>0}. \end{aligned}$$

Here, $\operatorname{aff}(\Delta)$ is the affine hull of Δ, and $\operatorname{int}(\Delta)$ is the interior of Δ. Under the considered framework, each strategy $i \in \mathcal{V}$ has an associated fitness function, $f_i : \Delta \to \mathbb{R}$, that provides the payoff received by a player playing the strategy $i \in \mathcal{V}$ at a given population state $\mathbf{x} \in \Delta$. Throughout, we let $f_i = f_i(\mathbf{x})$, for all $i \in \mathcal{V}$, and we assemble all the fitness functions into the fitness vector $\mathbf{f} = [f_i] \in \mathbb{R}^n$. To decide which strategy to play, the population agents are equipped with a revision protocol, $\rho_{ij}(\mathbf{x}, \mathbf{f})$, that provides the incentive to switch from the strategy $i \in \mathcal{V}$ to the strategy $j \in \mathcal{V}$ [25]. Clearly, such switching incentive depends on the state $\mathbf{x} \in \Delta$ and the corresponding fitness vector $\mathbf{f}$, i.e., $\rho_{ij} : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for all $i, j \in \mathcal{V}$. However, we let $\rho_{ij} = \rho_{ij}(\mathbf{x}, \mathbf{f})$, for all $i, j \in \mathcal{V}$. By interacting with their population peers, the population agents can evaluate their corresponding revision protocols and update their own strategy according to the provided incentives. Due to such revision mechanisms, the (continuous-time) dynamic evolution of the population state is given by

$$\dot{x}_i = \sum_{j \in \mathcal{V}} x_j \rho_{ji} - \sum_{j \in \mathcal{V}} x_i \rho_{ij}, \quad \forall i \in \mathcal{V}, \tag{2}$$

where $x_j \rho_{ji}$ is the mass of agents switching from j to i; and $x_i \rho_{ij}$ is the mass of agents switching from i to j (a complete deduction of such dynamics can be found in [25]). Therefore, if a revision protocol is provided, then the population game can be studied as a dynamical system for which the corresponding equilibrium states can be analyzed using non-linear control theory. In this work, we consider a generalized class of revision protocols that we term as *generalized pairwise comparison protocols* (GPCPs). Namely, a GPCP has the form

$$\rho_{ij} = \phi(i,j) \left| f_j - f_i \right|_+, \quad \forall i, j \in \mathcal{V}, \tag{3}$$

where $|\cdot|_+ = \max(\cdot, 0)$; and $\phi(i,j)$ is a non-negative scalar function that relies on information of the strategies $i, j \in \mathcal{V}$. While not required in general, for the purposes of this paper we let $\phi(i,j) = \phi(x_i, x_j, f_i, f_j)$ with $\phi : \Delta \times \Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$.

Remark 1: Notice that if $\phi(i,j)$ is designed properly, then by replacing (3) into (2) one can obtain various of the classic population dynamics studied in [25]. For instance, $\phi(i,j) = 1$, for all $i, j \in \mathcal{V}$, leads to $\rho_{ij} = |f_j - f_i|_+$ (Smith dynamics). In contrast, $\phi(i,j) = x_j$, for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \Delta$, results in $\rho_{ij} = x_j |f_j - f_i|_+$ (replicator dynamics). And similarly, $\phi(i,j) = 1/x_i$, for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \operatorname{int}(\Delta)$, leads to $\rho_{ij} = (1/x_i) |f_j - f_i|_+$ (projection dynamics). Hence, the proposed GPCP allows us to study several population dynamics in a unified fashion.

Following the ideas of [4], in this paper we consider non well-mixed populations of agents. More precisely, we refer to the tuple $F = (\mathcal{G}, \mathbf{f}, m)$ as a population game, where the interaction framework is governed by a graph $\mathcal{G}$ that determines the interaction and information related constraints that exist between the different strategies of the game. Such interaction graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{W})$, where $\mathcal{V}$ is the set of nodes corresponding to the strategies of the game; $\mathcal{E} \subseteq \{(i,j) : i, j \in \mathcal{V}\}$ is the set of links of possible interaction among the strategies; and $\mathbf{W} = [w_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix that describes the structure of the graph, i.e., $w_{ij} > 0$ if $(i,j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. Moreover, we let $\mathcal{N}_i = \{j \in \mathcal{V} : w_{ij} > 0\} \cup \{i\}$ for all $i \in \mathcal{V}$ and we impose the following Standing Assumption.

Standing Assumption 1: The fitness function f_i depends only on local information available over $\mathcal{N}_i$, for all $i \in \mathcal{V}$.

Under this framework, if $w_{ij} > 0$, then the agents playing the strategy $i \in \mathcal{V}$ are granted information about the strategy $j \in \mathcal{V}$, and thus are allowed to evaluate their corresponding revision protocols to decide whether to switch or not to the strategy $j \in \mathcal{V}$. If $w_{ij} = 0$, on the other hand, then the agents playing strategy $i \in \mathcal{V}$ do not have information regarding the strategy $j \in \mathcal{V}$, and hence cannot directly switch to such strategy. Therefore, if $w_{ij} > 0$, it is said that j is directly reachable from i. Thus, $\mathcal{N}_i$ denotes the set of strategies that are directly reachable from strategy $i \in \mathcal{V}$ (note that every strategy is always directly reachable from itself). Throughout, we assume that $w_{ij} = w_{ji}$, for all $i, j \in \mathcal{V}$, and so it follows that $i \in \mathcal{N}_j \iff j \in \mathcal{N}_i$, for all $i, j \in \mathcal{V}$.

Standing Assumption 2: The interaction graph $\mathcal{G}$ is undirected, i.e., $\mathbf{W} = \mathbf{W}^\top$.

By virtue of the interaction graph $\mathcal{G}$ and the Standing Assumption 2, the (continuous-time) dynamic evolution provided by (2) can be stated more generally as

$$\begin{aligned} \dot{x}_i &= \sum_{j \in \mathcal{V}} w_{ji} x_j \rho_{ji} - \sum_{j \in \mathcal{V}} w_{ij} x_i \rho_{ij}, \quad \forall i \in \mathcal{V} \\ &= \sum_{j \in \mathcal{V}} w_{ij} \left(x_j \rho_{ji} - x_i \rho_{ij} \right), \quad \forall i \in \mathcal{V}. \end{aligned} \tag{4}$$

The authors in [4] show that, for certain types of revision protocols and population games, the distributed population dynamics (4) have some desirable properties. For instance, for certain revision protocols it can be shown that the set Δ is invariant under (4), i.e., if the initial population state is in Δ, then the population state stays in Δ for all time. Moreover, for certain revision protocols and population games it can be shown that the equilibrium points of (4) are asymptotically stable. Hence, if one interprets the fraction of agents selecting each strategy as an optimization or control variable that is to be computed distributedly over a graph $\mathcal{G}$, and one designs the revision protocols and the population game properly, then

(4) can be used as a distributed update mechanism for the variables, and the convergence to a fixed point solution of the population game can be guaranteed subject to the constraints imposed by Δ and $\mathcal{G}$. This is a remarkable property of distributed population dynamics that allows their application to distributed (dynamic) constrained optimization and control problems. As illustrations, authors in [17, 16] apply a version of (4), i.e., the distributed replicator dynamics, to design hierarchical strategies for the distributed control of microgrids. Similarly, authors in [24] illustrate the advantages of (4) in multiple applications of distributed resource allocation. Some examples include distributed lightning control, distributed control of microgrids, and distributed control of large-scale water distribution systems. Additionally, authors in [2] propose a novel revision protocol that, when applied into (4), allows the consideration of additional constraints of the optimization or control problem. As illustration, such novel revision protocol is applied to the optimal frequency control of a power system.

Although the advantages of distributed population dynamics have been illustrated on several relevant industrial applications, all of the available studies have focused exclusively on the continuous-time dynamics, e.g., (4). In consequence, all of the available theoretical analyses are valid only for continuous-time versions of the distributed population dynamics. However, as modern computers cannot process information in continuous-time, an extension of the current theory is required in order to guarantee the applicability of distributed population dynamics to practical implementations where computations are necessarily discrete. To extend the current developments, in this work we propose and analyze a discrete-time version of (4). The advantage of our discrete-time formulation is that it transforms the differential equations of (4) into difference equations with a fixed step size. Therefore, our framework provides a mechanism to design iterative distributed optimization and control algorithms, with practical implementation guarantees, that are suitable for many relevant industrial applications.

To formulate the continuous-time dynamics (4) in discrete-time, we perform a forward Euler discretization. More precisely, by discretizing (4) one obtains

$$x_i[k+1] = x_i[k] + \epsilon \sum_{j\in\mathcal{V}} w_{ij} \left(x_j[k]\rho_{ji} - x_i[k]\rho_{ij} \right), \quad \forall i \in \mathcal{V}, \tag{5}$$

where $k \in \mathbb{Z}$ is the discrete-time index; and $\epsilon \in \mathbb{R}_{>0}$ is the (fixed) step size of the discretization. Under the framework of population games, the discrete-time dynamics (5) can be interpreted in the following way: at time k, the population agents are allowed to make decisions and the rates of change corresponding to such decisions, i.e., $\dot{x}_i$ for all $i \in \mathcal{V}$, are kept fixed during a period of time ϵ. Once the period ϵ is completed, the system transitions to time $k+1$ and the agents are allowed to make new decisions and the corresponding rates of change are again kept fixed during a period ϵ. The whole process repeats indefinitely over time. Hence, in the discrete-time framework we have to design not only the revision protocols and the population game, but also the length of the period ϵ so that the desirable properties of the continuous-time dynamics are preserved. To ease our theoretical analyses, in this work we stick to the synchronous case where the rates of change of all strategies are updated at every k.

Standing Assumption 3: For all $i \in \mathcal{V}$, $x_i[k]$ is updated at every k.

In the following section we provide the theoretical analyses to deduce the conditions required on the revision protocols, the population game, and the discretization period ϵ, so that the desirable properties of the continuous-time dynamics are preserved on the proposed discrete-time distributed framework.

3 Analysis of the Discrete-Time Dynamics

In the forthcoming analyses we will consider the general class of discrete-time dynamics that results from replacing the GPCP of (3) into (5). Namely, such discrete-time dynamics are given by

$$x_i[k+1] = x_i[k] + \epsilon \sum_{j\in\mathcal{V}} w_{ij}\theta(i,j)(f_i - f_j), \quad \forall i \in \mathcal{V}, \tag{6}$$

where[1]

$$\theta(i,j) = \begin{cases} x_j[k]\phi(j,i) & \text{if } f_i > f_j \\ x_i[k]\phi(i,j) & \text{if } f_i < f_j \\ \frac{1}{2}x_j[k]\phi(j,i) + \frac{1}{2}x_i[k]\phi(i,j) & \text{if } f_i = f_j \end{cases}.$$

[1] Note that in (6) the value of $\theta(i,j)$ when $f_i = f_j$ is irrelevant. Yet, we set such value to $(1/2)x_j[k]\phi(j,i) + (1/2)x_i[k]\phi(i,j)$ so that $\theta(i,j) = \theta(j,i)$.

Furthermore, the discrete-time dynamics in (6) can be written in matrix form as

$$\mathbf{x}[k+1] = \mathbf{x}[k] + \epsilon \mathbf{L}\mathbf{f}, \tag{7}$$

with $\mathbf{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ and

$$\ell_{ii} = \sum_{j\in\mathcal{V}\setminus\{i\}} w_{ij}\theta(i,j) \tag{8a}$$

$$\ell_{ij} = -w_{ij}\theta(i,j), \quad i \neq j, \quad \forall i,j \in \mathcal{V}. \tag{8b}$$

Remark 2: It is important to highlight that, whenever time is relevant, the elements of $\mathbf{f}$ and $\mathbf{L}$ are evaluated at time k. Namely, the elements of $\mathbf{f}$ depend on $\mathbf{x}[k]$, and the elements of $\mathbf{L}$ depend both on $\mathbf{f}$ and $\mathbf{x}[k]$. Whenever time is irrelevant, simply drop the time index on $\mathbf{x}[k]$. These dependencies are not shown explicitly in order to ease the notation. However, it is key to keep them in mind for the rest of the paper.

Remark 3: Note that $\mathbf{L}$ is the Laplacian matrix of a (dynamic) graph $\mathcal{D}^{(\mathbf{x})} = \left(\mathcal{V}, \mathcal{E}, \mathbf{D}^{(\mathbf{x})}\right)$ whose adjacency matrix is $\mathbf{D}^{(\mathbf{x})} = \left[d_{ij}^{(\mathbf{x})}\right] \in \mathbb{R}^{n\times n}$, with $d_{ij}^{(\mathbf{x})} = w_{ij}\theta(i,j)$ for all $i,j \in \mathcal{V}$. Moreover, from fact that $w_{ij}\theta(i,j) = w_{ji}\theta(j,i)$, for all $i,j\in\mathcal{V}$, it follows that $\mathbf{L} = \mathbf{L}^\top$.

3.1 Equilibrium Points Analysis

In this section we characterize the equilibrium points of the dynamics (6)-(7) and provide connections between such equilibrium points and the Nash equilibria of the underlying population game. For the forthcoming discussions we will use the terms equilibrium point and rest point interchangeably.

A population state $\mathbf{x}[k] \in \Delta$ is said to be a rest point of the dynamics (6)-(7) if and only if $\mathbf{x}[k] = \mathbf{x}[k+1]$. Thus, given that $\epsilon > 0$, $\mathbf{x} \in \Delta$ is a rest point of (6)-(7) if and only if the corresponding fitness vector $\mathbf{f}$ belongs to the null space of the corresponding matrix $\mathbf{L}$ (by corresponding we mean that $\mathbf{L}$ and $\mathbf{f}$ are evaluated on $\mathbf{x}$ [c.f., Remark 2]). Hence, the set of rest points of the dynamics (6)-(7) can be defined as

$$\mathrm{RP}(\mathbf{L}\mathbf{f}) = \{\mathbf{x}\in\Delta \,:\, \mathbf{L}\mathbf{f} = \mathbf{0}\}.$$

In contrast, given a population game $F = (\mathcal{G}, \mathbf{f}, m)$, the set of global Nash equilibria (GNE) of F is defined as

$$\mathrm{GNE}(F) = \left\{\mathbf{x}\in\Delta \,:\, x_i > 0 \implies f_i = \max_{j\in\mathcal{V}} f_j,\, \forall i \in \mathcal{V}\right\}.$$

Similarly, the set of local Nash equilibria (LNE) of F can be defined as

$$\mathrm{LNE}(F) = \left\{\mathbf{x}\in\Delta \,:\, x_i > 0 \implies f_i = \max_{j\in\mathcal{N}_i} f_j,\, \forall i \in \mathcal{V}\right\}.$$

Therefore, at a global Nash equilibrium no agent has incentives to deviate from its current strategy to any other strategy in $\mathcal{V}$, and, at a local Nash equilibrium no agent has incentives to deviate from its current strategy to any other directly reachable strategy. Clearly, $\mathrm{GNE}(F) \subseteq \mathrm{LNE}(F)$. Moreover, it also holds that $\mathrm{LNE}(F) \subseteq \mathrm{RP}(\mathbf{L}\mathbf{f})$. To show the later, we provide the following Lemmas.

Lemma 1: For any dynamics with the form (6)-(7) it holds that $\mathbf{L}\mathbf{f} = \mathbf{0} \iff w_{ij}\theta(i,j)(f_i - f_j) = 0, \forall i,j\in\mathcal{V}$.

Proof: See Appendix A.1. ■

Lemma 2: For any dynamics with the form (6)-(7) it holds that $\mathbf{x}\in\mathrm{LNE}(F) \implies \mathbf{x}\in\mathrm{RP}(\mathbf{L}\mathbf{f})$. Moreover, if $\phi(i,j) > 0$ for all $i,j\in\mathcal{V}$ and all $\mathbf{x}\in\Delta$, then it follows that

$$\mathbf{x}\in\mathrm{LNE}(F) \iff \mathbf{x}\in\mathrm{RP}(\mathbf{L}\mathbf{f}).$$

Similarly, if $\phi(i,j) > 0$ for all $i,j\in\mathcal{V}$ and all $\mathbf{x}\in\mathrm{int}\,(\Delta)$, then it holds that

$$\mathbf{x}\in\mathrm{int}\,(\Delta)\cap\mathrm{LNE}(F) \iff \mathbf{x}\in\mathrm{int}\,(\Delta)\cap\mathrm{RP}(\mathbf{L}\mathbf{f}).$$

Proof: See Appendix A.2. ■

Lemma 2 provides sufficient conditions to guarantee that the set of local Nash equilibria of the population game F and the set of rest points of (6)-(7) coincide. To guarantee that the sets of global and local Nash equilibria of F coincide, we introduce the concept of support graph. Given a population game $F = (\mathcal{G}, \mathbf{f}, m)$ and a state $\mathbf{x} \in \Delta$, the support graph of the interaction graph $\mathcal{G}$ is defined as

$$\mathcal{S}^{(\mathbf{x})} = \left(\text{supp}(\mathbf{x}),\, \mathcal{E}^{(\mathbf{x})},\, \mathbf{S}^{(\mathbf{x})}\right),$$

where $\text{supp}(\mathbf{x}) = \{i \in \mathcal{V} : x_i > 0\}$ is the support of $\mathbf{x}$; $\mathcal{E}^{(\mathbf{x})} \subseteq \{(i,j) \in \mathcal{E} : i, j \in \text{supp}(\mathbf{x})\}$ is the set of edges of the support graph; and $\mathbf{S}^{(\mathbf{x})} = \left[s_{ij}^{(\mathbf{x})}\right] \in \mathbb{R}^{|\text{supp}(\mathbf{x})| \times |\text{supp}(\mathbf{x})|}$ is the adjacency matrix of $\mathcal{S}^{(\mathbf{x})}$, and its elements are given by $s_{ij}^{(\mathbf{x})} = w_{ij}$, for all $i, j \in \text{supp}(\mathbf{x})$. Namely, for any $\mathbf{x} \in \Delta$ the graph $\mathcal{S}^{(\mathbf{x})}$ is the sub-graph of $\mathcal{G}$ that considers only the strategies in $\text{supp}(\mathbf{x})$. The concept of the support graph allows us to deduce sufficient conditions for the coincidence of LNE(F) and GNE(F).

Assumption 1: The interaction graph $\mathcal{G}$ is well-connected in the following sense: for every $\mathbf{x}^* \in \text{GNE}(F)$, the support graph $\mathcal{S}^{(\mathbf{x}^*)}$ is connected and $\text{supp}(\mathbf{x}^*) \cap \mathcal{N}_j \neq \emptyset$, for all $j \notin \text{supp}(\mathbf{x}^*)$, i.e., for every $i \in \mathcal{V}$ there is some $j \in \mathcal{N}_i$ such that $j \in \text{supp}(\mathbf{x}^*)$.

Lemma 3: If Assumption 1 is satisfied, then it holds that $\mathbf{x} \in \text{LNE}(F) \iff \mathbf{x} \in \text{GNE}(F)$.

Proof: See Appendix A.3. ■

Remark 4: Notice that if $\text{GNE}(F) \subseteq \text{int}(\Delta)$, then Assumption 1 is equivalent to the requirement that the interaction graph $\mathcal{G}$ is connected in the standard sense. Moreover, note that Assumption 1 automatically holds under a complete graph.

Remark 5: By putting together the results of Lemmas 2 and 3, one obtains sufficient conditions to guarantee that the set of rest points of the dynamics (6)-(7) and the set of global Nash equilibria of the underlying population game coincide. Such coincidence characterizes a property termed as Nash stationarity [25], which is of great importance for the convergence analysis of Section 3.3.

3.2 Invariant Set Analysis

In this section we provide sufficient conditions to guarantee the invariance of Δ under the dynamics (6)-(7). Moreover, we also characterize the invariance of $\text{aff}(\Delta)$ and $\text{int}(\Delta)$. Our result is illustrated in the following Theorem.

Theorem 1: Consider the dynamics (6)-(7) and let

$$\bar{\delta} = \sup\left\{\sum_{j \in \mathcal{V}} w_{ij}\phi(i,j)|f_i - f_j| \,:\, i, j \in \mathcal{V},\, \mathbf{x} \in \Delta\right\}, \tag{9}$$

where $\sup\{\mathcal{Z}\}$ denotes the supremum of the set $\mathcal{Z}$. Then, the following statements are all true:

- $\text{aff}(\Delta)$ is invariant under the considered dynamics.
- If $\bar{\delta} \in \mathbb{R}_{>0}$ and $0 < \epsilon \leq 1/\bar{\delta}$, then Δ is invariant under the considered dynamics.
- If $\bar{\delta} \in \mathbb{R}_{>0}$ and $0 < \epsilon < 1/\bar{\delta}$, then $\text{int}(\Delta)$ is invariant under the considered dynamics.

Proof: See Appendix A.5. ■

Remark 6: Theorem 1 provides sufficient conditions to guarantee the applicability of (6)-(7) to constrained optimization or control problems where the variables have to stay within Δ for all times. Note that if $\bar{\delta}$ in (9) is zero, then the dynamics (6)-(7) are at rest for all $\mathbf{x} \in \Delta$ and the initial condition $\mathbf{x}[0] \in \Delta$ characterizes a trivial invariant set. In contrast, note that if $\bar{\delta} = \infty$, then only the first statement of Theorem 1 holds. One example of such dynamics are the projection dynamics for which $\phi(i,j) = 1/x_i$, for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \text{int}(\Delta)$. To preserve the invariance of Δ under such dynamics, an additional projection onto the convex set Δ is required. However, as discussed in [25], such projection typically requires information about all the strategies in $\mathcal{V}$, and, in consequence, is not so trivial to perform under the informational constraints of $\mathcal{G}$.

An interesting line of future research would be to investigate distributed projection mechanisms to preserve the invariance of Δ under more general dynamics. One possibility could be using the proximal operator as a generalized projection mechanism [8, 21].

3.3 Convergence Analysis

In this section we present our results on the convergence of the dynamics (6)-(7). To study the convergence of the proposed discrete-time dynamics, we first characterize an additional property of population dynamics known as positive correlation [25]. Such property is illustrated in the following Lemma.

Lemma 4: Let $\mathbf{x} \in \Delta$ and consider any dynamics with the form of (6)-(7). Then both of the following statements hold:

- $\mathbf{f}^\top \mathbf{L}\mathbf{f} = 0 \iff \mathbf{x} \in \mathrm{RP}(\mathbf{L}\mathbf{f})$
- $\mathbf{x} \notin \mathrm{RP}(\mathbf{L}\mathbf{f}) \implies \mathbf{f}^\top \mathbf{L}\mathbf{f} > 0$

Proof: See Appendix A.4. ■

The property of positive correlation, illustrated in the second statement of Lemma 4, states that whenever the dynamics are not at rest, the angle between the fitness vector $\mathbf{f}$ and the update vector $\mathbf{L}\mathbf{f}$ [c.f., (7)] is acute. This is a remarkable property as it provides a correspondence between the direction given by the fitness vector of the population game, and the direction of movement of the trajectories of the dynamics. When combined with a special class of population games, the positive correlation property allows us to establish convergence results on the proposed discrete-time distributed dynamics.

Definition 1 ([25])*:* Let F be a population game with payoffs defined over the non-negative orthant $\mathbb{R}^n_{\geq 0}$. If there exists a continuously differentiable potential function $p : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ that satisfies $\nabla p(\mathbf{x}) = \mathbf{f}$ for all $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, then F is a full potential game.

Remark 7: An interesting property of full-potential games is that every local maximizer of the potential function $p(\mathbf{x})$ is a global Nash equilibrium of the corresponding population game. In addition, if the potential function is concave, then the set of global Nash equilibria of the corresponding population game is precisely the convex set of maximizers of $p(\mathbf{x})$ over Δ [25, Corollary 3.1.4]. This property makes full-potential games attractive for constrained optimization applications.

With the aid of Definition 1, the positive correlation property, and all of the previously stated results, we now provide our main convergence results on the proposed discrete-time distributed population dynamics.

Theorem 2: Let F be a population game, consider the dynamics (6)-(7), let $\bar{\delta}$ be given by (9), and let

$$\bar{\ell} = \sup \left\{ l_{ii} \in \mathbb{R}_{\geq 0} \,:\, l_{ii} \text{ is as in (8a)},\, i \in \mathcal{V},\, \mathbf{x} \in \mathrm{int}\,(\Delta) \right\}. \tag{10}$$

Moreover, let Assumption 1 be satisfied and suppose that the following conditions hold:

- The population game F is a full-potential game with potential function $p(\mathbf{x})$. Moreover, $p(\mathbf{x})$ is concave, twice continuously differentiable, and L-smooth for the Euclidean norm, i.e., $\|\nabla p(\mathbf{x}) - \nabla p(\mathbf{y})\|_2 \leq L\,\|\mathbf{x} - \mathbf{y}\|_2$ for some $L \in \mathbb{R}_{>0}$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\geq 0}$.
- $\mathbf{x}[0] \in \mathrm{int}\,(\Delta)$, and $\phi(i,j) > 0$ for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \mathrm{int}\,(\Delta)$.

If in addition $\bar{\delta}, \bar{\ell} \in \mathbb{R}_{>0}$ and

$$0 < \epsilon < \min \left\{ \frac{1}{\bar{\delta}},\, \frac{1}{\bar{\ell} L} \right\}, \tag{11}$$

then, the considered dynamics converge asymptotically to a global Nash equilibrium of F.

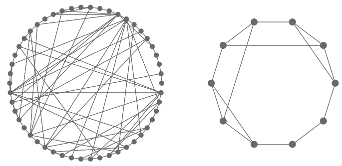
Proof: See Appendix A.6. ■

Corollary 1: Consider the dynamics (6)-(7) and let

$$\bar{\alpha} = \sup \left\{ l_{ii} \in \mathbb{R}_{\geq 0} \,:\, l_{ii} \text{ is as in (8a)},\, i \in \mathcal{V},\, \mathbf{x} \in \mathrm{aff}\,(\Delta) \right\}. \tag{12}$$

Moreover, suppose that the following conditions hold:

- The interaction graph $\mathcal{G}$ is connected.



(a) (b)

Figure 1: Considered random (connected) networks. Models: (a) Barabasi-Albert [1] + Path, (b) Watts-Strogatz [27].

- There exists a twice continuously differentiable function $g : \text{aff}(\Delta) \to \mathbb{R}$ such that $\nabla g(\mathbf{x}) = \mathbf{f}$ for all $\mathbf{x} \in \text{aff}(\Delta)$. Moreover, $g(\mathbf{x})$ is concave and L-smooth for the Euclidean norm, i.e., $\|\nabla g(\mathbf{x}) - \nabla g(\mathbf{y})\|_2 \leq L\|\mathbf{x} - \mathbf{y}\|_2$ for some $L \in \mathbb{R}_{>0}$ and for all $\mathbf{x}, \mathbf{y} \in \text{aff}(\Delta)$.
- $\mathbf{x}[0] \in \text{aff}(\Delta)$, and $\theta(i,j) > 0$ for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \text{aff}(\Delta)$.

If in addition $\bar{\alpha} \in \mathbb{R}_{>0}$ and $0 < \epsilon < 1/(\bar{\alpha}L)$, then, the considered dynamics converge asymptotically to a maximizer of $g(\mathbf{x})$ within $\text{aff}(\Delta)$.

Proof: See Appendix A.7. ■

Remark 8: By virtue of the Standing Assumption 1, Theorem 2 provides sufficient conditions to guarantee the applicability of (6)-(7) to distributed constrained optimization and control problems. Moreover, note that by Theorem 1 the constraints can be satisfied dynamically over the trajectories of the population state, i.e., as hard constraints. This resembles an attractive property of the proposed discrete-time dynamics when considered as a distributed iterative algorithm. Furthermore, such iterative algorithm has a fixed step size which is also a desirable property for optimization methods [29].

Remark 9: Notice that Corollary 1 is an extension of Theorem 2 for dynamics that do not preserve the invariance of Δ or $\text{int}(\Delta)$. Although such dynamics might not be interpretable under the population games framework (as the elements of $\mathbf{x}$ might be negative), they are still attractive from a distributed optimization perspective. An example of such dynamics are the ones that result from setting $\theta(i,j) = 1$, for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \text{aff}(\Delta)$. For such dynamics one gets that $x_i[k+1] = x_i[k] + \epsilon \sum_{j \in \mathcal{V}} w_{ij}(f_i - f_j)$, for all $i \in \mathcal{V}$, which can be thought as a nonlinear consensus algorithm.

Remark 10: Notice that in the proofs of Theorem 2 and Corollary 1, the Laplacian matrix $\mathbf{L}$ appears expressed only at time k (c.f., Remark 2). Hence, our convergence results are also valid for time-varying graphs that remain (sufficiently) well-connected over time. Although we leave the formal study of this claim for a future work, in Section 4.3 we provide some experimental illustrations.

4 Illustrative Examples

In this section, we illustrate the theoretical developments of this work. Namely, we provide numerical comparisons against other distributed methods and we show the application of our discrete-time dynamics to relevant engineering problems.

4.1 Large-Scale Distributed Constrained Optimization

Consider an optimization problem of the form

$$\min_{\mathbf{x}} \quad \sum_{i \in \mathcal{V}} \frac{1}{2}(x_i - a_i)^2 \tag{13a}$$

$$\text{s.t.} \sum_{i \in \mathcal{V}} x_i = m, \text{ and } x_i \geq 0, \ \forall i \in \mathcal{V}. \tag{13b}$$

As illustration, let $\mathcal{V} = \{1, 2, \ldots, 50\}$; set $m = 15$; and let $a_i \sim \text{U}(0, 1)$, for all $i \in \mathcal{V}$, where $\text{U}(0, 1)$ is the uniform distribution over [0, 1]. Optimization problems of the form (13) typically appear in the context

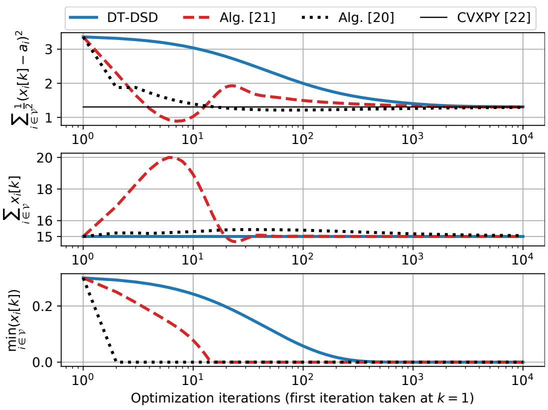


Figure 2: Iterative solution of (13) using the discrete-time distributed Smith dynamics (DT-DSD) and the distributed optimization algorithms developed in [14] and [18]. In all cases the initial condition is $x_i[1] = m/n$, for all $i \in \mathcal{V}$. In the top plot we also present the optimal solution obtained with the (centralized) CVXPY toolbox of [7].

of resource allocation. To treat (13) as a distributed problem, we assume that each x_i is managed by a node in a non-complete network. Without loss of generality, we consider the Barabasi-Albert network shown in Fig. 1a, and we let the corresponding graph $\mathcal{G}$ be a doubly stochastic matrix satisfying the Weights Rule of [18, Assumption 2]. To illustrate the solution of (13) through the developed theory, we let $\mathcal{G}$ be the interaction graph of the population game and we consider a full potential game with potential function $p(\mathbf{x}) = -\sum_{i\in\mathcal{V}}(1/2)(x_i - a_i)^2$. Thus, $f_i = a_i - x_i[k]$, for all $i \in \mathcal{V}$. Moreover, as illustration we set $\phi(i,j) = 1$, for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \Delta$, and we name the resulting dynamics as the discrete-time distributed Smith dynamics (DT-DSD) [c.f., Remark 1]. In Fig. 2 we show the results of applying the DT-DSD and other popular and recent distributed algorithms to solve (13). More precisely, we compare our DT-DSD against a recent distributed optimization method developed in [14], and against the popular distributed projected subgradient algorithm of [18]. The algorithm of [14] is applied directly on (13), whereas the algorithm of [18] is applied on the corresponding dual Lagrangian problem. This is a common practice [29] that is required to decouple the coupled constraints in (13b). Yet, given that in this case there is zero duality gap, solving the dual problem is equivalent to solving the original problem (13). For the DT-DSD we set $\epsilon = (0.9/16)$, which satisfies (11), and for the other algorithms we set the parameters as reported in [14] and [18], respectively. Notice that although the DT-DSD require more iterations to converge to the optimum, our proposed dynamics are the only algorithm that dynamically satisfies the coupled constraint in (13b). In consequence, the solutions provided by the DT-DSD are feasible over all the iterations, and thus can be safely applied even from the first iteration.

4.2 Dynamic Distributed Resource Allocation in a Water Distribution System

A water distribution system (WDS) is a relevant problem for industrial control. The problem is to distribute a limited inflow of water, $q_{in} \in \mathbb{R}_{>0}$, among a set of n tanks to reach a reference water level in each tank (see Fig. 3). The application of continuous-time population dynamics to this problem has been investigated in [4, 23]. In this work, we consider a system similar to the one in [23], but we add more tanks and we consider bidirectional water flows between the tanks. More precisely, we let $\mathcal{T} = \{1, 2, \ldots, 10\}$ be the set of tanks,

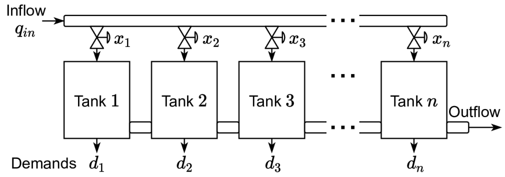


Figure 3: Considered water distribution system.

we let $q_{in} = 4\text{m}^3/\text{s}$, and we consider a WDS with dynamics

$$\begin{aligned}
h_i[k+1] &= h_i[k] + \frac{T_s}{S_i}\left(x_i[k] - d_i[k] + \hat{h}_i[k]\right), \quad \forall i \in \mathcal{T}, \\
\hat{h}_1[k] &= -A_1\sigma(1,2)\sqrt{2g\gamma(1,2)}, \\
\hat{h}_i[k] &= -A_i\sigma(i,i+1)\sqrt{2g\gamma(i,i+1)} \\
&\quad - A_{i-1}\sigma(i,i-1)\sqrt{2g\gamma(i,i-1)}, \ \ \forall i \in \mathcal{T}\setminus\{1,10\}, \\
\hat{h}_{10}[k] &= -A_{10}\sqrt{2gh_{10}[k]} - A_9\sigma(10,9)\sqrt{2g\gamma(10,9)},
\end{aligned}$$

where $\sigma(i,j) = \text{sgn}\left(h_i[k] - h_j[k]\right)$; $\gamma(i,j) = |h_i[k] - h_j[k]|$; and $h_i[k]$, $d_i[k]$, $x_i[k]$ are respectively the water-level height, demand, and water inflow of the i-th tank at time k. The scalar T_s is the sampling time of the discretized dynamics and in our case is taken as 30s; and the scalar g is the acceleration of gravity. Moreover, A_i and S_i are the output-pipe's area and transversal area of the i-th tank, respectively, and we set them according to the values presented in [23]. Furthermore, we assume that each tank $i \in \mathcal{T}$ has a local controller that computes $u_i[k] = k_p\left(r - h_i[k]\right)$, for all k, where $k_p \in \mathbb{R}_{>0}$ is a proportional gain; and $r \in \mathbb{R}_{\geq 0}$ is the reference water level to be achieved at the tanks. The goal is then to project the locally computed values $u_i[k]$, for all $i \in \mathcal{T}$, to the water inflows $x_i[k]$ that satisfy the WDS constraints. Namely, that $\sum_{i\in\mathcal{T}} x_i[k] \leq q_{in}$ and that $x_i[k] \geq 0$ for all $i \in \mathcal{T}$. To solve this problem with population dynamics, we define the set of strategies as $\mathcal{V} = \mathcal{T} \cup \{s\}$, where s is a fictitious strategy to allocate the excess of water inflow. More precisely, with this definition of $\mathcal{V}$ the problem to be solved can be stated as

$$\min_{\mathbf{x}} \quad \sum_{i\in\mathcal{T}} \frac{1}{2}\left(u_i[k] - x_i[k]\right)^2 \tag{14a}$$

$$\text{s.t.} \sum_{i\in\mathcal{T}} x_i[k] + x_s[k] = q_{in}, \text{ and } x_i[k] \geq 0, \ \forall i \in \mathcal{V}. \tag{14b}$$

Thus, by setting $p(\mathbf{x}) = -\sum_{i\in\mathcal{T}}(1/2)\left(u_i[k] - x_i[k]\right)^2$ it follows that $f_i = k_p\left(r - h_i[k]\right) - x_i[k]$, for all $i \in \mathcal{T}$, and $f_s = 0$. Without loss of generality, as before, we assume a doubly stochastic interaction graph $\mathcal{G}$ and we consider the DT-DSD (see Section 4.1). Under such assumption one can set $\epsilon = 0.9/(k_p h_{max} + q_{in})$ to satisfy (11) regardless of the structure of $\mathcal{G}$ (here h_{max} denotes the maximum height of the tanks). For our numerical experiments we set $k_p = 100\text{m}^2/\text{s}$ and $h_{max} = 2\text{m}$. As illustration, Fig. 4 depicts the application of the DT-DSD to the WDS of Fig. 3 both under complete and non-complete graphs. In both cases the fictitious strategy is computed at the first tank, i.e., at the rightmost node in Fig. 1b. Notice that at the beginning of the experiment the tanks start at a random unstable state, which causes a larger setting time when a non-complete graph is considered. However, once the tanks reach an equilibrium state, the performance under both graphs becomes very similar. This is remarkable considering that the complete graph has 45 communication links whereas the non-complete one has only 14.

4.3 Distributed Formation Control of Mobile Robots

Finally, we consider a real multi-robot platform to evaluate the performance of the proposed dynamics on a real-time control implementation (several videos of the actual implementation are available at `youtu.be/1jXpcBvveyQ`). Similar to [3], in this paper we consider a team of robots that seek to achieve a geometric formation within a rectangular region of $\mathbb{R}^2_{>0}$. One of the robots is set as the leader and is assumed to follow predefined trajectory over the rectangular region. The other robots are set as followers that have

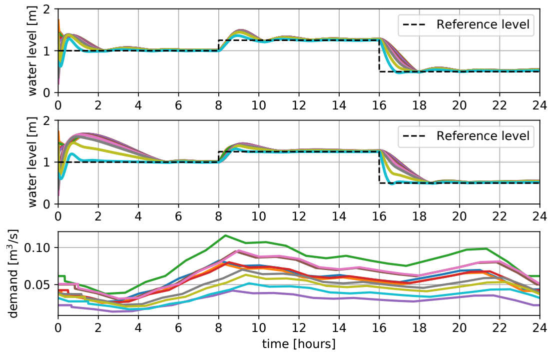


Figure 4: Application of the DT-DSD to the WDS of Fig. 3. The top plot shows the performance under a complete graph; the middle plot shows the performance under the graph of Fig. 1b; and the bottom plot depicts the demands for all tanks [4].

to follow the leader while achieving a desired (arbitrary) geometric formation. The formation is specified in terms of x and y displacements, with respect to the leader robot, that each follower has to achieve (here, x and y denote the coordinate axes of $\mathbb{R}^2$). Namely, each follower is granted a vector $[\psi_i^x,\, \psi_i^y]^\top$, that provides its required x (ψ_i^x) and y (ψ_i^y) displacements with respect to the leader. If each robot is equipped with a local navigation controller that can drive it to a given reference coordinate $(r_i^x,\, r_i^y)$, then the formation control problem reduces to the distributed computation of r_i^x and r_i^y for all $i \in \mathcal{F}$, where $\mathcal{F}$ is the set of follower robots. Therefore, the follower robots have to collectively solve the following optimization problems:

$$\min_{\mathbf{r}} \sum_{i\in\mathcal{F}} \frac{1}{2}\left(c^d[k] + \psi_i^d - r_i^d[k]\right)^2, \quad \forall d \in \{x,\, y\},\, \forall i \in \mathcal{F}, \tag{15}$$

where $(c^x[k],\, c^y[k])$ is the location coordinate of the leader robot at time k. Note that this is a distributed problem as not all robots might have communication with the leader. Hence, following the ideas of [3], we consider two populations games, one for x and one for y, and we set $f_i^d = \psi_i^d - r_i^d[k]$, for all $d \in \{x,\, y\}$ and all $i \in \mathcal{F}$, and $f_\ell^d = -c^d[k]$, for all $d \in \{x,\, y\}$ (here, ℓ denotes the leader robot). Note that with these fitness functions, the unique global Nash equilibrium: $f_i^d = f_\ell^d$, for all $d \in \{x,\, y\}$ and all $i \in \mathcal{F}$, provides the solution of (15). Moreover, to deal with the unconstrained nature of (15) it suffices to set the population masses, m^x and m^y, at sufficiently large values so that any formation within the considered rectangular region of $\mathbb{R}^2_{>0}$ can be achieved. In Fig. 5 we present the application of three instances of (6) to the aforementioned formation problem: i) the discrete-time distributed replicator dynamics (DT-DRD), that result from setting $\phi(i,j)^d = r_j^d$, for all $d \in \{x,\, y\}$, and all $i, j \in \mathcal{F} \cup \{\ell\}$; ii) the DT-DSD presented in Sections 4.1 and 4.2; and iii) the discrete-time linear consensus algorithm of [20], which in this case can be obtained by setting $\theta(i,j)^d = 1$ for all $d \in \{x,\, y\}$, and all $i, j \in \mathcal{F} \cup \{\ell\}$. Clearly, in all cases the robots achieve the desired formation. Notice that even though the consensus algorithm converges faster, this algorithm does not preserve the invariance of Δ (c.f., Theorem 1). Hence, in contrast with the DT-DRD and the DT-DSD, such consensus algorithm cannot be directly applied to distributed resource allocation problems with hard coupled constraints.

5 Concluding Remarks and Future Work

In this paper we have proposed and analyzed a general class of discrete-time distributed population dynamics. In particular we have characterized the equilibrium points of the dynamics under the light of game theory, and we have deduced sufficient conditions to guarantee the invariance and convergence properties of the proposed discrete-time distributed dynamics. Moreover, our theoretical developments have been illustrated in various relevant engineering applications including an implementation on a real robotic platform.

To improve the current developments, future work should extend the presented theory to more general classes of populations games, e.g., stable games [9], and to more general classes of dynamics (c.f., Remark 6). One immediate research line could be the extension of the presented discrete-time framework to the

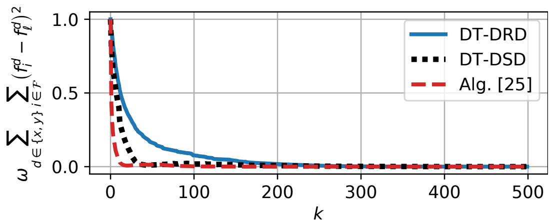


Figure 5: Experimental results for the robotic formation control (ω is a normalization parameter to scale the values to 1). The communication graph for this example is set as a time-varying Watts-Strogatz graph [27] with binary weights, and without loss of generality the desired formation is set as a triangle.

consideration of the novel revision protocols and mixtures of dynamics proposed in [2, 5]. Such novel revision protocols and dynamics extend the applicability of population games to optimization and control problems with more constraints. Therefore, a discrete-time extension of such ideas could further improve the presented results. On a similar vein, an attractive research line would be the consideration of payoff dynamics models [22] in our discrete-time framework. Such models admit the extension of population games to more realistic scenarios with communication delays, anticipatory behaviors and inertia. Furthermore, although the studied dynamics comprise a general family of pairwise comparison protocols, the recent works in [31, 32] have considered some general forms of imitation dynamics that should be further studied under our discrete-time framework. Finally, future work should also focus on the extension of the presented theory to other kinds of distributed applications, e.g., discrete-resource allocation problems [26], as well as the formal characterization of the convergence rate of the proposed dynamics.

A Appendices

A.1 Proof of Lemma 1

(Sufficiency) Note that $(\mathbf{Lf})_i = \sum_{j\in\mathcal{V}} w_{ij}\theta(i,j)(f_i - f_j)$, for all $i \in \mathcal{V}$. Thus, if $w_{ij}\theta(i,j)(f_i - f_j) = 0$, for all $i, j \in \mathcal{V}$, then $\mathbf{Lf} = \mathbf{0}$. (Necessity) Recall Remark 3. Without loss of generality, assume that $\mathcal{D}^{(\mathbf{x})}$ has r connected components indexed by $\mathcal{R} = \{1, 2, \ldots, r\}$, with $1 \leq r \leq n$. From algebraic graph theory we have that the dimension of the null space of $\mathbf{L}$ is equal to r. Therefore, given that $(\mathbf{Lf})_i = \sum_{j\in\mathcal{V}} d_{ij}^{(\mathbf{x})}(f_i - f_j)$, for all $i \in \mathcal{V}$, we have that $\text{null}(\mathbf{L}) = \text{span}\{\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \ldots, \mathbf{z}^{(r)}\}$, where, for all $u \in \mathcal{R}$, the term $\mathbf{z}^{(u)}$ is an $n \times 1$ vector with $z_i^{(u)} = 1$ if $i \in \mathcal{V}$ belongs to the u-th connected component of $\mathcal{D}^{(\mathbf{x})}$, and $z_i^{(u)} = 0$ otherwise. Hence, if $\mathbf{f} \in \text{null}(\mathbf{L})$ and $i, j \in \mathcal{V}$ belong to the same connected component of $\mathcal{D}^{(\mathbf{x})}$, it follows that $f_i = f_j$. In contrast, if $i, j \in \mathcal{V}$ do not belong to the same connected component of $\mathcal{D}^{(\mathbf{x})}$, then $d_{ij}^{(\mathbf{x})} = 0$. In consequence, $\mathbf{Lf} = \mathbf{0}$ implies that $w_{ij}\theta(i,j)(f_i - f_j) = 0$, for all $i, j \in \mathcal{V}$. ■

A.2 Proof of Lemma 2

First we prove that $\mathbf{x} \in \text{LNE}(F) \implies \mathbf{x} \in \text{RP}(\mathbf{Lf})$. If $\mathbf{x} \in \text{LNE}(F)$ and $x_i > 0$, then $f_i \geq f_j$ for all $j \in \mathcal{N}_i$. Moreover, if $f_i > f_j$, then $x_j = 0$ and $\theta(i,j) = \theta(j,i) = 0$. Thus, at any LNE: $(\mathbf{Lf})_i = \sum_{j\in\mathcal{V}} w_{ij}\theta(i,j)(f_i - f_j) = 0$, for all $i \in \mathcal{V}$. In consequence, $\mathbf{Lf} = \mathbf{0}$ and $\mathbf{x} \in \text{RP}(\mathbf{Lf})$. To prove that $\mathbf{x} \in \text{RP}(\mathbf{Lf}) \implies \mathbf{x} \in \text{LNE}(F)$, we use the positivity of $\phi(i,j)$ over Δ. Suppose $\mathbf{x} \in \text{RP}(\mathbf{Lf})$ and $\mathbf{x} \notin \text{LNE}(F)$. Then there is at least one $x_i > 0$ such that $f_i < f_j$ for some $j \in \mathcal{N}_i$. Thus, if $\phi(i,j) > 0$, then $w_{ij}\theta(i,j)(f_i - f_j) < 0$, and, from Lemma 1, $\mathbf{f} \notin \text{null}(\mathbf{L})$. Clearly, this is in contradiction with $\mathbf{x} \in \text{RP}(\mathbf{Lf})$. Hence, $\mathbf{x} \in \text{LNE}(F)$. Finally, using the same contradiction argument as before, if $\mathbf{x} \in \text{int}(\Delta)$ and $\phi(i,j) > 0$ over $\text{int}(\Delta)$, then $\mathbf{x} \in \text{int}(\Delta) \cap \text{RP}(\mathbf{Lf}) \implies \mathbf{x} \in \text{int}(\Delta) \cap \text{LNE}(F)$. ■

A.3 Proof of Lemma 3

It is straightforward to verify that $\mathrm{GNE}(F) \subseteq \mathrm{LNE}(F)$. Hence, it always holds that $\mathbf{x} \in \mathrm{GNE}(F) \implies \mathbf{x} \in \mathrm{LNE}(F)$. Now, to prove that $\mathbf{x} \in \mathrm{LNE}(F) \implies \mathbf{x} \in \mathrm{GNE}(F)$, let $\mathbf{x} \in \mathrm{LNE}(F)$ and consider Assumption 1. Moreover, since every population game admits at least one global Nash equilibrium [25, Theorem 2.1.1], without loss of generality let $\mathbf{x}^* \in \mathrm{GNE}(F)$. From Assumption 1 we have that for every $i \in \mathcal{V}$ there is some $j \in \mathcal{N}_i$ such that $j \in \mathrm{supp}\,(\mathbf{x}^*)$. Additionally, if $\mathbf{x} \in \mathrm{LNE}(F)$, then for every $i \in \mathcal{V}$ with $x_i > 0$ it holds that $f_i \geq f_j$ for all $j \in \mathcal{N}_i$. In consequence, if $\mathbf{x} \in \mathrm{LNE}(F)$ and $\mathbf{x}^* \in \mathrm{GNE}(F)$, then for every $i \in \mathcal{V}$ with $x_i > 0$ it follows that $f_i = f_j = \max_{z \in \mathcal{V}} f_z$ for all $j \in \mathrm{supp}\,(\mathbf{x}^*)$ (otherwise, either $\mathbf{x} \notin \mathrm{LNE}(F)$ or $\mathbf{x}^* \notin \mathrm{GNE}(F)$). Hence, by definition, it must hold that $\mathbf{x} \in \mathrm{GNE}(F)$. ■

A.4 Proof of Lemma 4

Note that $\mathbf{f}^\top \mathbf{L} \mathbf{f}$ is the quadratic form of a Laplacian matrix (c.f., Remark 3). Thus, $\mathbf{f}^\top \mathbf{L} \mathbf{f} = \sum_{(i,j)\in\mathcal{E}} w_{ij}\theta(i,j)(f_i - f_j)^2$. Given that $w_{ij}\theta(i,j) \geq 0$ for all $i,j \in \mathcal{V}$ and all $\mathbf{x} \in \Delta$, it follows that $\sum_{(i,j)\in\mathcal{E}} w_{ij}\theta(i,j)(f_i - f_j)^2 = 0 \iff w_{ij}\theta(i,j)(f_i - f_j) = 0, \forall i,j \in \mathcal{V}$. Therefore, it follows directly from Lemma 1 that $\mathbf{f}^\top \mathbf{L} \mathbf{f} = 0 \iff \mathbf{x} \in \mathrm{RP}(\mathbf{L}\mathbf{f})$. This proves the first statement. To prove the second statement recall that $\mathbf{L} = \mathbf{L}^\top$ (c.f., Remark 3). If $\mathbf{x} \in \Delta$, then $\mathbf{L}$ is also diagonally dominant with non-negative diagonal elements. Hence, $\mathbf{L} \succeq 0$ and $\mathbf{f}^\top \mathbf{L} \mathbf{f} \geq 0$ for all $\mathbf{x} \in \Delta$. Thus, if $\mathbf{x} \in \Delta \setminus \mathrm{RP}(\mathbf{L}\mathbf{f})$, then it holds that $\mathbf{f}^\top \mathbf{L} \mathbf{f} > 0$. ■

A.5 Proof of Theorem 1

The set $\mathrm{aff}\,(\Delta)$ is invariant if for all $k \geq 0$ it holds that $\sum_{i\in\mathcal{V}} x_i[k] = m \implies \sum_{i\in\mathcal{V}} x_i[k+1] = m$. From the Standing Assumptions 2 and 3 and the fact that $w_{ij}\theta(i,j) = w_{ji}\theta(j,i)$, for all $i,j \in \mathcal{V}$, we have that

$$\sum_{i\in\mathcal{V}} \epsilon \sum_{j\in\mathcal{V}} w_{ij}\theta(i,j)(f_i - f_j) = 0, \quad \forall k.$$

Hence, $\sum_{i\in\mathcal{V}} x_i[k+1] = \sum_{i\in\mathcal{V}} x_i[k] = \sum_{i\in\mathcal{V}} x_i[0] = m$, for all $k \geq 0$. Thus $\mathrm{aff}\,(\Delta)$ is invariant under (6)-(7). For Δ to be invariant we require two conditions: i) the invariance of $\mathrm{aff}\,(\Delta)$; and ii) $\mathbf{x}[k] \in \mathbb{R}^n_{\geq 0} \implies \mathbf{x}[k+1] \in \mathbb{R}^n_{\geq 0}$, for all $k \geq 0$. Condition i) holds from our previous discussion. To satisfy condition ii) note that $x_i[k]$ only decreases if there is at least one $j \in \mathcal{N}_i$ such that $f_i < f_j$. Hence, to satisfy condition ii) we can consider the critical (impossible) case where $f_i < f_j$ for all $i,j \in \mathcal{V}$. For such case the dynamics (6) can be written as

$$\begin{aligned} x_i[k+1] &= x_i[k] - \epsilon \sum_{j\in\mathcal{V}} w_{ij} x_i[k] \phi(i,j)|f_i - f_j|, \quad \forall i \in \mathcal{V} \\ &= x_i[k] \left(1 - \epsilon \sum_{j\in\mathcal{V}} w_{ij}\phi(i,j)|f_i - f_j|\right), \quad \forall i \in \mathcal{V}. \end{aligned}$$

Therefore, if $\epsilon > 0$ and $\mathbf{x}[k] \in \mathbb{R}^n_{\geq 0}$, we require that

$$\epsilon \sum_{j\in\mathcal{V}} w_{ij}\phi(i,j)|f_i - f_j| \leq 1, \quad \forall i \in \mathcal{V}. \tag{16}$$

From (9) it holds that $\epsilon \sum_{j\in\mathcal{V}} w_{ij}\phi(i,j)|f_i - f_j| \leq \epsilon\bar{\delta}$, for all $i \in \mathcal{V}$. Thus, if $0 < \epsilon \leq 1/\bar{\delta}$, then (16) is guaranteed for all $i \in \mathcal{V}$ and all $k \geq 0$. Therefore, condition ii) is satisfied for all $k \geq 0$ and Δ is invariant under (6)-(7). Finally, the set $\mathrm{int}\,(\Delta)$ is invariant if both Δ and $\mathbb{R}^n_{>0}$ are invariant, i.e., $\mathbf{x}[k] \in \mathbb{R}^n_{>0} \implies \mathbf{x}[k+1] \in \mathbb{R}^n_{>0}$, for all $k \geq 0$. The first condition holds from the previous discussion. For the second one to hold, note that if $0 < \epsilon < 1/\bar{\delta}$, then (16) is satisfied without binding for all $i \in \mathcal{V}$ and all $k \geq 0$. In consequence, $\mathbf{x}[k] \in \mathbb{R}^n_{>0} \implies \mathbf{x}[k+1] \in \mathbb{R}^n_{>0}$, for all $k \geq 0$, and $\mathrm{int}(\Delta)$ is invariant under (6)-(7). This completes the proof. ■

A.6 Proof of Theorem 2

Given that F is a full-potential game and $p(\mathbf{x})$ is concave, it follows that $\mathrm{GNE}(F)$ is the convex set of maximizers of $p(\mathbf{x})$ (c.f., Remark 7). Therefore, to prove the asymptotic convergence to $\mathrm{GNE}(F)$ from every $\mathbf{x}[0] \in \mathrm{int}\,(\Delta)$, we must prove the following: that $\mathbf{x}[k] \in \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x}[k] \in \mathrm{GNE}(F)$, for all $k \geq 0$; and that $p\,(\mathbf{x}[k+1]) - p\,(\mathbf{x}[k]) > 0$, for all $\mathbf{x}[k] \notin \mathrm{GNE}(F)$ and all $k \geq 0$.

First, we prove that $\mathbf{x}[k] \in \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x}[k] \in \mathrm{GNE}(F)$, for all $k \geq 0$. From Theorem 1 it follows that $\mathrm{int}\,(\Delta)$ is invariant under (6)-(7). In consequence, $\mathbf{x}[0] \in \mathrm{int}\,(\Delta) \implies \mathbf{x}[k] \in \mathrm{int}\,(\Delta)$ for all $k \geq 0$. Moreover, from Lemmas 2 and 3, $\mathbf{x}[k] \in \mathrm{int}\,(\Delta) \cap \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x}[k] \in \mathrm{int}\,(\Delta) \cap \mathrm{LNE}(F) \iff \mathbf{x}[k] \in \mathrm{int}\,(\Delta) \cap \mathrm{GNE}(F)$. Since $\mathbf{x}[0] \in \mathrm{int}\,(\Delta)$ and $\mathrm{int}\,(\Delta)$ is invariant, for all $k \geq 0$ it holds that $\mathbf{x}[k] \in \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x}[k] \in \mathrm{GNE}(F)$. Which is the intended result.

Second, we prove that the potential function $p\,(\mathbf{x}[k])$ is non-decreasing over the trajectories of the considered dynamics and that it stops increasing if and only if $\mathbf{x}[k] \in \mathrm{GNE}(F)$, i.e., if and only if a maximizer of $p(\mathbf{x})$ has been reached. More precisely, we have to show that $p\,(\mathbf{x}[k+1]) - p\,(\mathbf{x}[k]) > 0$ for all $k \geq 0$ and all $\mathbf{x}[k] \notin \mathrm{GNE}(F)$. To do so, denote $\hat{p} = p\,(\mathbf{x}[k+1]) - p\,(\mathbf{x}[k])$ for all k. Given that $\mathbf{x}[k+1] = \mathbf{x}[k] + \epsilon\mathbf{Lf}$, it follows that $\hat{p} = p\,(\mathbf{x} + \epsilon\mathbf{Lf}) - p\,(\mathbf{x})$ (here we have dropped the discrete-time index as all terms are now expressed at time k). Since $p(\mathbf{x})$ is twice continuously differentiable, we can apply $\nabla p(\mathbf{x}) = \mathbf{f}$ in conjunction with Taylor's theorem [19, Theorem 2.1] and the fact that $\mathbf{L} = \mathbf{L}^\top$ (c.f., Remark 3) to obtain $p(\mathbf{x}+\epsilon\mathbf{Lf}) = p(\mathbf{x}) + \epsilon\mathbf{f}^\top\mathbf{Lf} + (\epsilon^2/2)\mathbf{f}^\top\mathbf{L}\mathbf{H}^{(\alpha)}\mathbf{Lf}$, where $\mathbf{H}^{(\alpha)} = \nabla^2 p\,(\mathbf{x} + \alpha\epsilon\mathbf{Lf})$ for some $\alpha \in (0,1)$. Therefore,

$$\hat{p} = \epsilon\left(\mathbf{f}^\top\mathbf{Lf} - \frac{\epsilon}{2}\mathbf{f}^\top\mathbf{L}\bar{\mathbf{H}}^{(\alpha)}\mathbf{Lf}\right),$$

where $\bar{\mathbf{H}}^{(\alpha)} = -\mathbf{H}^{(\alpha)}$. Since $\epsilon > 0$, to guarantee $\hat{p} > 0$ it should hold that $\mathbf{f}^\top\mathbf{Lf} > (\epsilon/2)\mathbf{f}^\top\mathbf{L}\bar{\mathbf{H}}^{(\alpha)}\mathbf{Lf}$. Applying the eigen-decomposition on $\mathbf{L}$, i.e., $\mathbf{L} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top$, it follows that

$$\begin{aligned}
\mathbf{f}^\top\mathbf{L}\bar{\mathbf{H}}^{(\alpha)}\mathbf{Lf} &= \mathbf{f}^\top\mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\mathbf{\Lambda}\mathbf{P}^\top\mathbf{f} \\
&= \mathbf{f}^\top\mathbf{P}\mathbf{\Lambda}^{1/2}\left(\mathbf{\Lambda}^{1/2}\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\mathbf{\Lambda}^{1/2}\right)\mathbf{\Lambda}^{1/2}\mathbf{P}^\top\mathbf{f} \\
&= \mathbf{v}^\top\mathbf{Z}\mathbf{v},
\end{aligned}$$

with $\mathbf{v} = \mathbf{\Lambda}^{1/2}\mathbf{P}^\top\mathbf{f}$ and $\mathbf{Z} = \mathbf{\Lambda}^{1/2}\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\mathbf{\Lambda}^{1/2}$. Since $\mathbf{Z}$ is real and symmetric, it follows that $\mathbf{v}^\top\mathbf{Z}\mathbf{v} \leq \rho\,(\mathbf{Z})\,\mathbf{v}^\top\mathbf{v}$, where $\rho\,(\mathbf{Z})$ is the spectral radius of $\mathbf{Z}$. Moreover, since $\mathbf{Z}$ is real and symmetric it holds that $\rho\,(\mathbf{Z}) = \|\mathbf{Z}\|_2$, where $\|\cdot\|_2$ denotes the induced ℓ_2-norm. In consequence,

$$\begin{aligned}
\mathbf{f}^\top\mathbf{L}\bar{\mathbf{H}}^{(\alpha)}\mathbf{Lf} &\leq \|\mathbf{Z}\|_2\mathbf{v}^\top\mathbf{v} \\
&= \left\|\mathbf{\Lambda}^{1/2}\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\mathbf{\Lambda}^{1/2}\right\|_2\mathbf{f}^\top\mathbf{Lf} \\
&\leq \left\|\mathbf{\Lambda}^{1/2}\right\|_2\left\|\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\right\|_2\left\|\mathbf{\Lambda}^{1/2}\right\|_2\mathbf{f}^\top\mathbf{Lf} \\
&= \rho\left(\mathbf{\Lambda}^{1/2}\right)\rho\left(\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\right)\rho\left(\mathbf{\Lambda}^{1/2}\right)\mathbf{f}^\top\mathbf{Lf} \\
&= \rho\left(\bar{\mathbf{H}}^{(\alpha)}\right)\rho\,(\mathbf{L})\,\mathbf{f}^\top\mathbf{Lf},
\end{aligned}$$

where $\rho\left(\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}\right) = \rho\left(\bar{\mathbf{H}}^{(\alpha)}\right)$ follows from the fact that $\mathbf{P}^\top\bar{\mathbf{H}}^{(\alpha)}\mathbf{P}$ is similar to $\bar{\mathbf{H}}^{(\alpha)}$. Now, recall that $\mathbf{H}^{(\alpha)} = \nabla^2 p\,(\mathbf{x} + \alpha\epsilon\mathbf{Lf})$ for some $\alpha \in (0,1)$. Since $\mathrm{int}\,(\Delta)$ is convex and invariant under the considered dynamics, it follows that $(\mathbf{x} + \alpha\epsilon\mathbf{Lf}) \in \mathrm{int}\,(\Delta)$ for all $\alpha \in (0,1)$. Moreover, given that $-p(\mathbf{x})$ is convex and L-smooth over all $\mathbb{R}^n_{\geq 0}$, it follows that $0 \preceq \bar{\mathbf{H}}^{(\alpha)} \preceq L\mathbf{I}_n$ for all $\mathbf{x} \in \mathrm{int}\,(\Delta) \subset \mathbb{R}^n_{\geq 0}$ and all $\alpha \in (0,1)$ (here $\mathbf{I}_n$ denotes the $n \times n$ identity matrix). Namely, $\rho\left(\bar{\mathbf{H}}^{(\alpha)}\right) \leq L$ for all $\mathbf{x} \in \mathrm{int}\,(\Delta)$ and all $\alpha \in (0,1)$. On the other hand, from the Gershgorin Circle Theorem in conjunction with the form of $\mathbf{L}$ and the invariance of $\mathrm{int}\,(\Delta)$, it follows that $\rho\,(\mathbf{L}) \leq 2\bar{\ell}$, where $\bar{\ell} \in \mathbb{R}_{>0}$ is given by (10). Putting all together leads to $\mathbf{f}^\top\mathbf{L}\bar{\mathbf{H}}^{(\alpha)}\mathbf{Lf} \leq 2L\bar{\ell}\mathbf{f}^\top\mathbf{Lf}$. In consequence, $\hat{p} \leq \epsilon\left(1 - \epsilon L\bar{\ell}\right)\mathbf{f}^\top\mathbf{Lf}$. Furthermore, using Lemma 4 and the fact that $\mathbf{x}[k] \in \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x}[k] \in \mathrm{GNE}(F)$, for all $k \geq 0$, we conclude that if ϵ satisfies (11), then $\hat{p} \geq 0$, for all $k \geq 0$, and $\hat{p} = 0 \iff \mathbf{f}^\top\mathbf{Lf} = 0 \iff \mathbf{x} \in \mathrm{RP}(\mathbf{Lf}) \iff \mathbf{x} \in \mathrm{GNE}(F)$. In consequence, the potential function is non-decreasing under the trajectories of the considered dynamics, and it stops increasing if and only if a maximizer is reached, i.e., a global Nash equilibrium. Therefore, the considered dynamics converge, in the asymptotic sense, to a global Nash equilibrium of F. ■

A.7 Proof of Corollary 1

From Theorem 1 it holds that $\mathrm{aff}\,(\Delta)$ is invariant under (6)-(7). Therefore, $\mathbf{x}[0] \in \mathrm{aff}\,(\Delta) \implies \mathbf{x}[k] \in \mathrm{aff}\,(\Delta)$ for all $k \geq 0$. Moreover, given that $\theta(i,j) > 0$ for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \mathrm{aff}\,(\Delta)$, it holds that $\mathcal{D}^{(\mathbf{x})}$ has the exact same topology as $\mathcal{G}$ for all $\mathbf{x} \in \mathrm{aff}\,(\Delta)$ (c.f., Remark 3). Hence, since $\mathcal{G}$ is connected, it follows that $\mathcal{D}^{(\mathbf{x})}$ is connected for all $\mathbf{x} \in \mathrm{aff}\,(\Delta)$. Furthermore, since $\theta(i,j) > 0$ for all $i, j \in \mathcal{V}$ and all $\mathbf{x} \in \mathrm{aff}\,(\Delta)$, it can be shown, using similar arguments as in the proof of Lemma 4, that $\mathbf{L} \succeq 0$ for all $\mathbf{x}[k] \in \mathrm{aff}\,(\Delta)$, and that $\mathbf{f}^\top \mathbf{L} \mathbf{f} = 0 \iff f_i = f_j$ for all $i, j \in \mathcal{V}$. In addition, from the Karush-Kuhn-Tucker first order conditions and the concavity of $g(\mathbf{x})$, it follows that $f_i = f_j$, for all $i, j \in \mathcal{V}$, if and only if $\mathbf{x}[k] \in \mathrm{argmax}_{\mathbf{x} \in \mathrm{aff}(\Delta)} g(\mathbf{x})$. Using these facts, and similar arguments as the proof of Theorem 2, it can be shown that $g(\mathbf{x})$ is non-decreasing over the trajectories of the considered dynamics, and it stops increasing if and only if a maximizer is reached. Therefore, the considered dynamics converge, in the asymptotic sense, to a maximizer of $g(\mathbf{x})$ within $\mathrm{aff}\,(\Delta)$. ■

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