Using the Goertzel algorithm as a filter

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***Abstract*—The Goertzel algorithm is fairly known algorithm in tone detection systems. This effective realization of a partial Fourier transform is able to evaluate selected spectral bin by a very simple algorithm. Additionally, the Goertzel algorithm can be used as a digital filter. The paper introduces derivation of the frequency characteristics of the Goertzel filter. Finally, the novel filter structure has been proved to eliminate a decimation property of the filter.**

***Keywords*—Goertzel algorithm; Goertzel filter; Fourier transform**

I. Introduction

The Goertzel algorithm is an effective convolutional form of the discrete Fourier transform for direct computation of the Fourier value at selected frequency position, i.e. evaluates the only selected bin of the Fourier spectrum. The algorithm is mainly used as a tone detector in dual tone multi-frequency systems (DTMF). However, its character allows to use the algorithm as a digital filter [1].

The derivation of the Goertzel algorithm can be found in many books, e.g. [2], on the other hand to be complete, let us briefly show the derivation from the beginning. The basic relation of the discrete Fourier transform (1) can be put into a convolutional form (2), where $x(r)$ is an r^{th} signal sample in time domain and $X(k)$ is a k^{th} bin of the Fourier spectrum. Function $u(n)$ represents a rectangular weighting function.

$$X(k) = \sum_{r=0}^{N-1} x(r) e^{-j2\pi r \frac{k}{N}} \tag{1}$$

$$\begin{aligned} y_k(n) &= \sum_{r=-\infty}^{\infty} x(r) \cdot e^{j2\pi(N-r)\frac{k}{N}} \cdot u(n-r) \\ &= x(n) * e^{j2\pi n \frac{k}{N}}|_{n=N} \end{aligned} \tag{2}$$

An impulse response of the derived filter is then a complex harmonic signal (3) of which length is constrained by a rectangular window.

$$h(n) = e^{j2\pi n \frac{k}{N}} \tag{3}$$

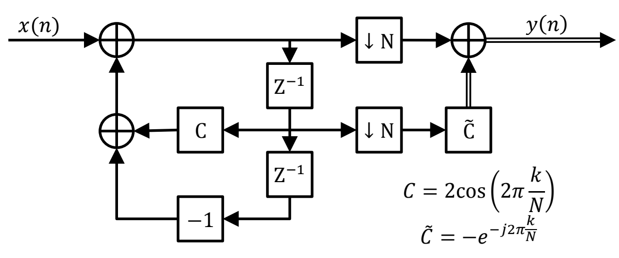
By applying the Z-transform to the impulse response (3), it is possible to find the transfer function of the Goertzel filter (4). More convenient for an implementation of the Goertzel algorithm is the modified form (5), which can be split into the real recursive and the complex direct computational parts.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-1} = \frac{1}{1 - z^{-1} e^{j2\pi \frac{k}{N}}} \tag{4}$$

$$\begin{aligned} H(z) &= \frac{1}{1 - z^{-1} e^{j2\pi \frac{k}{N}}} \cdot \frac{1 - z^{-1} e^{-j2\pi \frac{k}{N}}}{1 - z^{-1} e^{-j2\pi \frac{k}{N}}} \\ &= \frac{1 - z^{-1} e^{-j2\pi \frac{k}{N}}}{1 - 2z^{-1} \cos\left(2\pi \frac{k}{N}\right) + z^{-2}} \end{aligned} \tag{5}$$

The realization of the transfer function (5) is shown in Fig. 1. Notice that the filter has two complex poles located on the unit circle that is a condition of stability. Indeed, the loop is then able to generate an impulse response with stable amplitude. Some uncertainty of the coefficient *C* caused e.g. by a quantization error may result in fading of the impulse response, which can be noticeable especially in the case of large coefficients both *k* and *N*.

Fig. 1. Goertzel filter signal diagram



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The down-sampling blocks stand for the condition in (2) and actually provide the rectangular windowing of the impulse response. The complex output is evaluated by the direct part of the algorithm.

II. TRANSFER FUNCTION OF THE GOERTZEL FILTER

The derivation of filtration characteristics is not so common in literature. By substituting $e^{j\omega}$ into the derived transfer function (4) we will not get the power spectral density due to the constraint of N samples after which the filter is reset. To get the correct frequency characteristics the inverse Fourier transform is applied to the impulse response (3) that is easily reproduced in equation (6).

$$\begin{aligned} H_G(f) &= \int_{-\infty}^{\infty} u(n-r)\cdot e^{j2\pi n\frac{k}{N}}\cdot e^{j2\pi f n}dn \\ &= \frac{1}{N}\int_0^N e^{j2\pi n\frac{k}{N}}\cdot e^{j2\pi f n}dn \end{aligned} \tag{6}$$

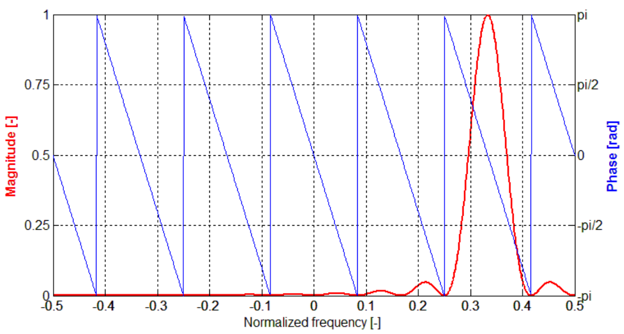
This, after some calculations, yields the complex spectrum (7). Equations (8) and (9) show a power spectral density of the filter and a phase characteristic. An example of the power spectral density and corresponding phase characteristic are shown in Fig. 2.

$$\begin{aligned} H_G(f) &= \frac{e^{j2\pi\left(\frac{k}{N}-f\right)N}-1}{j2\pi\left(\frac{k}{N}-f\right)N} \\ &= \frac{\sin(\pi k-\pi f N)}{\pi k-\pi f N}e^{j\pi\left(\frac{k}{N}-f\right)N} \end{aligned} \tag{7}$$

$$|H_G(f)|^2 = \frac{\sin(\pi k-\pi f N)^2}{\pi^2(k-fN)^2} \tag{8}$$

$$\arg\big(H_G(f)\big) = \pi\left(\frac{k}{N}-f\right)N \tag{9}$$

Fig. 2. Example of frequency characteristics of the Goertzel filter (N = 12, k = 4)

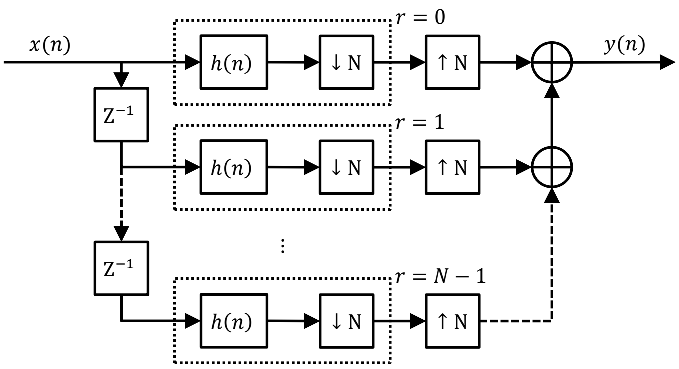


Notice that the PSD has only one-side lobe due to the complex character of the transfer function and filtered signal is then in general also complex. The filter can be tuned to the non-integer frequency, $k \in \mathbb{R}$, that is generally not common for the discrete Fourier transform, and especially its fast realization FFT. If the DFT is used, the frequency will be fixed for each bin and will depend on the length of the DFT N. Fortunately, for the Goertzel algorithm using whatever real frequency from sampled range does not make any problems as has been showed in [1].

III. A NOVEL STRUCTURE OF THE GOERTZEL FILTER

Consider the digital system in Fig. 3. The system consists of N mutually delayed branches of the Goertzel filter. Signals are then successively multiplexed to the output, this operation is depicted by an up-sampling and summing the signals.

Fig. 3. Signal diagram of the Goertzel filter in parallel



The dashed rectangle wraps the Goertzel filter which is represented by its impulse response *h(z)* and the decimation ability with factor N. Each branch is indexed by an index r. Let us express each operation by equations, which are generally known [2]. It is more convenient to derive the expressions in frequency domain. Applying the impulse response to the input signal *x(n)* yields a multiplication in frequency domain as denoted in equation (10).

$$X_h(\omega) = X(\omega)\cdot H(\omega) \tag{10}$$

The next stage, decimation or a down-sampling, adds N copies of the original spectrum shifted by the corresponding fraction of the original sampling frequency (11). Remember that since the input signal is discretized in time the spectrum $X(\omega)$ is already periodical with a period T_S.

$$X_d(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X_h\left(\frac{\omega}{N} - 2\pi\frac{k}{NT_S}\right) \tag{11}$$

Up-sampling does not add any spectra, the only change is expanding of the signal bandwidth, so that the N-times up-sampled bandwidth is wide as the bandwidth of the original signal, but still contains the spectra added by the decimation. The bandwidth expansion can be expressed by equation (12).

$$X_u(\omega) = X_d(\omega N) \tag{12}$$

The delay in time is transformed to the frequency domain as a phase shift of the signal spectrum (13).

$$x_z(n) = x(n - rT_S) \overset{\mathcal{F}}{\rightarrow} X_z(\omega) = X(\omega) \cdot e^{-j\omega rT_S} \tag{13}$$

Putting all together according to Fig. 3 yields a relation for the output spectrum of the proposed system (14).

$$Y(\omega) = \sum_{r=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} H\left(\omega - 2\pi \frac{k}{MT_S}\right) \cdot X\left(\omega - 2\pi \frac{k}{MT_S}\right) \right) \cdot e^{-j\omega rT_S} \tag{14}$$

The order of summation can be exchanged and the copies of filtered spectra may be expressed as a convolution of the original spectrum with the periodic impulse train (15).

$$Y(\omega) = \left(H(\omega) \cdot X(\omega)\right) * \frac{1}{N} \sum_{k=0}^{N-1} \left(\delta\left(\omega - 2\pi \frac{k}{NT_S}\right) \cdot \sum_{r=0}^{N-1} e^{-j\omega rT_S} \right) \tag{15}$$

The impulse train has non-zero value only when its input parameter is zero, so that condition may be applied also to the inner sum item as a substitution of the angle frequency ω. To pass the condition for summation of geometric series the item for $k = 0$ has to be excluded from the sum. Finally, it results in equation (16).

$$Y(\omega) = H(\omega) \cdot X(\omega) \xrightarrow{\mathcal{F}^{-1}} y(n) = x(n) * h(n) \tag{16}$$

It has been found that the proposed system eliminates the decimation property of the Goertzel filter, thus, it may be used as a filter with a specified decimation factor independently on its filtration characteristics.

CONCLUSION

The frequency characteristics introduced clearly describes the filtration abilities of the Goertzel filter. Despite of the filter is recursive, the phase characteristic stays linear. Also the rapid influence of rectangular window is obvious.

The decimation property of the Goertzel filter may sometimes be undesirable, especially when the aliasing due to strong spectral leakage may cause aliasing. The introduced structure has been proved to be able to eliminate such property. The structure is not primarily intended to be used to reach the filter characteristics without decimation for all the working time. Such usage may put the filter at disadvantage compared to other filter structures, but temporarily a filtering system may need a better resolution of the signal – e.g. for synchronization – and then the introduced structure will be useful.

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