2. Time-budget-constrained multi-state flow network (TBCMFN) model with correlated faults and non-integer capacities

This section describes a time-budget-constrained two-terminal multi-state flow network model with correlated faults and non-integer capacities. The model is represented by:

- a graph (digraph) $G(N, A)$, where $N = [v_1, \ldots, v_n]$ is a list of nodes and $A = [a_1, \ldots, a_m]$ is a list of links (edges, arcs);
- the source node $s \in N$ from which flow units are sent and the sink $t \in N$ where flow units terminate ultimately;
- the components number vector $W = (w_1, \ldots, w_m)$, where $w_i \in \mathbb{N}$ denotes the number of components in the i-th link a_i;
- the component capacity vector $C = (c_1, \ldots, c_m)$, where $c_i \in \mathbb{R}_+$ denotes the capacity of each component in a_i;
- the lead time vector $L = (l_1, \ldots, l_m)$, where l_i denotes the lead time integer value of link a_i which represents the initialization time before the routing, on a_i, of each unit of a flow;
- the transmission cost vector $B = (b_1, \ldots, b_m)$, where b_i denotes the transmission cost of a_i for sending each unit of data;
- the component reliability vector $R = (r_1, \ldots, r_m)$, where r_i represents the probability that the component provides its capacity in a_i;
- the vector of the correlation between the faults of the components $\rho = (\rho_1, \ldots, \rho_m)$, where ρ_i represents the correlation between the faults of the components in a_i.

Let $[a; b]$ denote the set of integers between a and b. A vector $X = (x_1, \ldots, x_m)$ is a *system state vector* (SSV) if there exists a vector $y = (y_1, \ldots, y_m) \in [0; w_1] \times [0; w_2] \times \ldots \times [0; w_m]$ and $x_i = y_i \cdot c_i$ for $i \in [1; m]$. The set of all

SSVs will be denoted by Ω. The *greatest capacity vector* in Ω is denoted by $M = (M_1, \ldots, M_m)$ and is defined as follows: $M_j = w_j \cdot c_j$ for all $j \in [1; m]$.

Let the expected reliability of each component be r, ρ be the correlation between each pair of components i and j, $i \neq j$, X is a random variable representing the number of working components. According to [47, eq. (11)], for $k \in \{1, \ldots, n\}$, the reliability of an n-component k-out-of-n: good system built from correlated identical components can be written as follows:

$$Pr(X \geq k) = \frac{1}{\beta} \sum_{i=0}^{n-k} \binom{n}{i} (r\beta)^{n-i} (1 - r\beta)^i,$$

where $\beta = 1 + \frac{\rho(1-r)}{r}$. It follows that

$$\begin{aligned} Pr(X = k) &= Pr(X \geq k) - Pr(X \geq k+1) \\ &= \frac{1}{\beta} \binom{n}{k} (r\beta)^k (1 - r\beta)^{n-k} \text{ for } k \in \{1, \ldots, n\} \end{aligned}$$

and

$$\begin{aligned} Pr(X = 0) &= 1 - Pr(X \geq 1) \\ &= 1 - \frac{1}{\beta} \sum_{i=0}^{n-1} \binom{n}{i} (r\beta)^{n-i} (1 - r\beta)^i \\ &= 1 - \frac{1}{\beta} (1 - (1 - r\beta)^n) \end{aligned}$$

since $\sum\limits_{i=0}^{n} \binom{n}{i} (r\beta)^{n-i} (1 - r\beta)^i = 1$. So, probability that link a_i exhibits capacity state $k \cdot c_i$, where $k \in [0; w_i]$, is equal to

$$Pr(x_i = k \cdot c_i) = \begin{cases} \frac{1}{\beta_i} \binom{w_i}{k} (r_i\beta_i)^k (1 - r_i\beta_i)^{w_i - k}, & \text{if } k \geq 1 \\ 1 - \frac{1}{\beta_i} (1 - (1 - r_i\beta_i)^{w_i}), & \text{if } k = 0 \end{cases} \tag{1}$$

where

$$\beta_i = 1 + \frac{\rho_i(1 - r_i)}{r_i} \tag{2}$$

In this article, we are interested in the evaluation of the network reliability at level $(d, T, b) - R_{(d,T,b)}$, defined as the probability of transmitting d units of data from a source node to a destination node through a single minimal path within the time of T and budget of b, under the following assumptions:

- The capacity of link $a_i \in A$ takes random values from $\{0, c_i, 2c_i \ldots, w_i c_i\}$ according to Eq. (1) for $i = 1, 2, \ldots, m$.
- The capacities of different links are statistically independent.
- Every node is perfectly reliable, i.e., deterministic.
- Flow in the network satisfies the flow conservation law.
- The data is transmitted from a source node to a destination node through a single path.
- Component reliabilities (faults) in one link can be correlated, and the correlation between them equals $\rho_i \in [0; 1]$.

The problem of the assessment of $R_{(d,T,b)}$ under the above settings and assumptions will be called the *Generalized Quickest Path Reliability Problem*.

2.1. Network reliability at level (d, T, b)

To define $R_{(d,T,b)}$ formally, we need to introduce additional definitions. A *path* is a list of adjacent links through which the data can be transmitted from source node s to sink node t, and a *minimal path* (MP) is a path whose proper sub-lists are not paths from s to t. Let MPs denote the list of all minimal paths.

For $P \in MPs$ and the SSV $X \in \Omega$, *the transmission time* of d units of flow from s to t through P under X, follows from Eq. (3), where the *lead time* of P, which corresponds to the setup time of the path P , is given by Eq. (4); the capacity of P under X is given by Eq. (5) and $\lceil x \rceil$ is the suffix of x, i.e., it is the smallest integer greater or equal to x

$$\tau(P, d, X) = \begin{cases} L(P) + \left\lceil \frac{d}{C(P,X)} \right\rceil & \text{if } C(P, X) > 0 \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

$$L(P) = \sum_{a_i \in P} l_i \tag{4}$$

$$C(P, X) = \min_{a_i \in P} x_i \tag{5}$$

The aforementioned model emerges from diverse physical systems, such as convoy-like traffic flows [48], evacuation problems [49], and data packets routing in Internet networks [50].

The transmission time T is *attainable* for d in a flow network if there exists a minimal path P and SSV $X \in \Omega$ in this network such that $\tau(P, d, X) = T$.

The *transmission cost of path* P per unit of data is defined by Eq. (6) and the *transmission cost of sending* d *units of data through path* P is given by Eq. (7)

$$B(P) = \sum_{a_i \in P} b_i \tag{6}$$

$$B(P, d) = d \cdot B(P) \tag{7}$$

The *transmission time to send* d *units of data from node* s *to node* t *under* X can be defined as the transmission time of the quickest MP P, which satisfies $B(P, d) \leq b$, i.e.,

$$T(d, X) = \min_{P \in MPs} \{\tau(P, d, X) \colon B(P, d) \leq b\} \tag{8}$$

Now we can express the formula for the network reliability at level (d, T, b):

$$R_{(d,T,b)} = Pr(\{X \in \Omega \colon \quad T(d, X) \leq T\}) \tag{9}$$

2.2. (d, T, b)*-MPs*

However, computing $R_{(d,T,b)}$ using formula Eq. (9) can be impractical and inefficient since the space Ω of all SSVs can consist of many vectors. Therefore, we propose the use of the concept (d, T, b)-MPs to enhance the process of network reliability evaluation.

Definition 1. *A system state vector* X *is called a* (d, T, b)-MP candidate *if* $T(d, X) \leq T$*. Moreover, if* $T(d, Y) > T$ *for any SSV* $Y < X$*, then* X *is called a* (d, T, b)-MP.

Now, we can formulate lemma allowing us to compute $R_{(d,T,b)}$ more efficiently