CM1103 Week 8: Exercises 3 – Counting: solution

Optional:

13. $C(n,k-1)+C(n,k)=\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}=\frac{kn!}{k(k-1)!(n-k+1)!}+\frac{(n-k+1)n!}{k!(n-k+1)(n-k)!}=$
$\frac{kn!}{k!(n-k+1)!}+\frac{(n+1)n!-kn!}{k!(n-k+1)!}=\frac{(n+1)!}{k!(n+1-k)!}=C(n+1,k)$

14. There are two ways to prove it.

Soln 1: proof by induction

Base step: when $n=1$, $C(1,0)+C(1,1)=1+1=2^1$. The equation is true.

Inductive step: suppose the equation is true for $n=k$, i.e.

$$C(k,0)+C(k,1)+C(k,2)+\cdots+C(k,k)=2^k$$

[We need to prove it is true for $n=k+1$]

When $n=k+1$,

$C(k+1,0)+C(k+1,1)+C(k+1,2)+C(k+1,3)+\cdots+C(k+1,k)+C(k+1,k+1)$

(using Pascal's Identity)

$=C(k+1,0)+C(k,0)+C(k,1)+C(k,1)+C(k,2)+C(k,2)+C(k,3)+\cdots+C(k,k-1)+C(k,k)+C(k+1,k+1)$

(Since $C(k+1,0)=C(k,0), and\ C(k,k)=C(k+1,k+1)$)

$=2\times(C(k,0)+C(k,1)+C(k,2)+\cdots+C(k,k-1)+C(k,k))$

(from supposition the equation is true for $n=k$)

$=2\times 2^k=2^{k+1}$.

Soln 2: direct proof using definition of power set and its cardinality

Let $S=\{1,2,3,\dots,n\}$, its subsets can be divided up into subsets with 0 element, 1 element, 2 elements, …, n elements. Counting combinations, we get the number of subsets of S equals $C(n,0)+C(n,1)+C(n,2)+\cdots+C(n,n)$. All the subsets constitute the power set of S, and we know the power set of S has cardinality of 2^n, so,

$$C(n,0)+C(n,1)+C(n,2)+\cdots+C(n,n)=2^n$$