CM1103: Exercises 3 solutions – Counting

1. (a) $\{AB, AC, AD, BC, BD, CD\}$ – 6 ways (order of each pair is irrelevant)
   (b) $\{AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC\}$ – 12 ways (order of each pair denotes 1st & 2nd prize)
   (c) $\{AA, AB, AC, AD, BB, BC, BD, CC, CD, DD\}$ – 10 –ways (order of each pair is irrelevant)
   (d) $\{AA, AB, BA, AC, CA, AD, DA, BB, BC, CB, BD, DB, CC, CD, DC, DD$ – 16 ways (order of each pair denotes 1st & 2nd prize)

2.

(a) 120	(b) 720	(c) 56	(d) 8
(e) 20	(f) 1	(g) 28	(h) 8

3. We have 5 rounds of flipping the coin. Suppose the elements of the subsets represent the indices of the rounds that came up heads (and the remainder come up tails). That is, a subset $\{2, 3, 5\}$ represents the sequence $THHTH$.

4. $\boxed{n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1}$
   (a) $1! = 1$ (b) $2! = 2$ (c) $3! = 6$
   (d) $4! = 3! \times 4 = 6 \times 4 = 24$
   (e) $5! = 4! \times 5 = 24 \times 5 = 120$
   Note that here I've applied the *recursive* rule $n! = n \times (n-1)!$ to save doing the multiplication from scratch each time.

5. (a) An r-permutation of a set of cardinality n is an ordered list of r elements chosen from the set.
   (b) An r-combination of a set of cardinality n is a subset of cardinality r chosen from the set.

6. $$\boxed{P(n,r) = \frac{n!}{(n-r)!} \qquad \text{and} \qquad C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}}$$

7. 2-permutations of $\{w, x, y, z\}$ are wx, xw, wy, yw, wz, zw, xy, yx, xz, zx, yz and zy. There are twelve of these, giving $P(4, 2) = 12$.

   Check using formula:
   $$P(4,2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{24}{2} = 12$$

   2-combinations of $\{w, x, y, z\}$ are $\{w, x\}$, $\{w, y\}$, $\{w, z\}$, $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. There are six of these, giving $C(n, r) = 6$

   Check using formula:
   $$C(n,r) = \frac{4!}{2!\,2!} = \frac{24}{2 \times 2} = \frac{24}{2} = 12$$

8. $C(n, r) \le P(n, r)$ with equality if r or n is 0 or 1.

9. (a) $C(12, 3) = 220$ different teams of three can be chosen.

(b) The total number of teams is

number of teams including both Jim and Alma	+	number of teams excluding both Jim and Alma

To calculate the number of teams including both Jim and Alma, we assume that Jim and Alma have already been chosen and require one more team member chosen from the remaining ten astronauts i.e. $C(10,1)$
To calculate the number of teams excluding both Jim and Alma, we need to choose the team of three from the ten other astronauts i.e. $C(10,3)$
So the number of teams is $C(10,1) + C(10,3) = 130$

10. (a) $\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$

    (b) $A = \{x : x \in \mathbb{Z} \text{ and } 2 \leq x < 5\} = \{2,3,4\}$ so $|A| = 3$; $|B| = 20$ is given; $A \cap B = \{2,4\}$ so $|A \cap B| = 2$. Then $|A \cup B| = |A| + |B| - |A \cap B| = 3 + 20 - 2 = 21$

    (c) $\boxed{\begin{aligned} |A \cup B \cup C| = \; & |A| + |B| + |C| - |A \cap B| \\ & - |A \cap C| - |B \cap C| + |A \cap B \cap C| \end{aligned}}$

11. By the product rule, the number of possible tickets is:

$$C(34,5) \times C(14,1) = 278256 \times 14 = 3895584$$

12. (a)
    i. Repetition is not allowed: $P(5,3) = \frac{5!}{2!} = 60$.
    ii. Repetition is allowed: $5^3 =$ ~~75.~~ 125

    (b)
    i. $C(11,4) = 330$
    ii. $C(17,10) = 19448$

    (c) $n_a = 2, n_g = 1, n_r = 1, n_e = 4, n_b = 1, n_l = 1, n_n = 1, n_s = 2$, hence the answer is

$$\frac{13!}{n_a!n_g!n_r!n_e!n_b!n_l!n_1!n_s!} = 64864800$$