CM1103 Week 9 solutions: Exercises 4 – Probability

1. The sample space is the set $\{1, 2, 3, 4, 5, 6\}$. $E = \{3, 4, 5, 6\}$, hence:

$$P(E) = \frac{|E|}{|S|} = \frac{4}{6} = \frac{2}{3}$$

2. E.g. *Disjoint:* Rolling an odd number on a dice, rolling an even number on a dice. *Independent:* rolling a given number on two separate rolls of a dice.

3. (a) $P(\overline{E}) = 1 - P(E)$
   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   (c) Let S be the sample space, $A = E$ and $B = \overline{E}$, then $A \cap B = \emptyset$, so $P(A \cap B) = 0$. Hence, $P(A) + P(B) = P(A \cup B) = P(E \cup \overline{E}) = P(S) = 1$.

4. (a) $P(E) = \frac{1}{6}\frac{1}{6} = \frac{1}{36} = 0.03$
   (b) Let E be the event that at least one six occurs. Then $\overline{E}$ is the event that no sixes occur. $P(\overline{E}) = \frac{5}{6}\frac{5}{6} = \frac{25}{36}$, and hence $P(E) = 1 - P(\overline{E}) = 1 - \frac{25}{36} = \frac{11}{36} = 0.31$.
   (c) $P(E) = \frac{5}{6} = 0.83$

5. (a) $P(E) = \left(\frac{1}{6}\right)^4 = \frac{1}{1296} = 0.0008$
   (b) Let E be the event that at least one six occurs. Then $\overline{E}$ is the event that no sixes occur. $P(\overline{E}) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$, and hence $P(E) = 1 - P(\overline{E}) = 1 - \frac{625}{1296} = \frac{671}{1296} = 0.52$.
   (c) $P(E) = \frac{6.5.4.3}{6^4} = \frac{360}{1296} = 0.28$

6. (a) $\frac{1}{2}$
   (b) $\frac{1}{2}$
   (c) $\frac{8}{52} = \frac{2}{13}$
   (d) $1 - \frac{2}{52} = \frac{50}{52} = \frac{25}{26}$
   (e) $1 - \frac{2}{52} = \frac{50}{52} = \frac{25}{26}$

7. (a) Use the sum rule to add the probability of the 2 disjoint events: R followed by G, and G followed by R.

$$P(E) = \frac{4}{12}\frac{6}{11} + \frac{6}{12}\frac{4}{11} = \frac{4}{11}$$

   (b) The drawings are *independent*, hence:

$$P(E) = \frac{4}{12}\frac{6}{12} + \frac{6}{12}\frac{4}{12} = \frac{1}{3}$$

   (c) $P(E) = \frac{4}{12}\frac{3}{11}\frac{2}{10} = \frac{1}{55}$
   (d) $P(E) = 0$ since there are only 2 blue balls ($P(E) = \frac{2}{12}\frac{1}{11}\frac{0}{10} = 0$)
   (e) There are $P(3, 3)$ orderings in which the three different colours could be picked, and each has probability $\frac{6.4.2}{12.11.10}$ of occurring. Hence:

$$P(E) = P(3, 3).\frac{6.4.2}{12.11.10} = 3!.\frac{2}{55} = \frac{12}{55}$$

8. (a) twice (since $1 - \left(\frac{1}{2}\right)^2 = 0.75$)
   (b) 5 times (since $1 - \left(\frac{1}{2}\right)^5 = 0.96875$)

9. Consider the 5 mutually exclusive events that they score with their 1st, 2nd, 3rd, 4th or 5th penalty and miss the rest. Each of these events has probability $P = \left(\frac{1}{3}\right)^4 \frac{2}{3}$. Hence by the sum rule, the answer is $5.\left(\frac{1}{3}\right)^4 \frac{2}{3} = \frac{10}{243}$.

10. (a) If 1 in 100 people have the disease, the probability of a false positive is:

$$P(Positive\ result|no\ disease).P(No\ disease) = 0.001 \times 0.99 = 0.00099$$

The probability of a true positive is:

$$P(Positive\ result|have\ disease).P(have\ disease) = 0.99 \times 0.01 = 0.0099$$

The probability of a false negative is:

$$P(Negative\ result|have\ disease).P(have\ disease) = (1 - 0.99) \times 0.01 = 0.001$$

(b) If 1 in 10,000 people have the disease, the probability of a false positive is:

$$P(Positive\ result|no\ disease).P(No\ disease) = 0.001 \times 0.9999 = 0.001$$

The probability of a true positive is:

$$P(Positive\ result|have\ disease).P(have\ disease) = 0.99 \times 0.0001 = 0.000099$$

The probability of a false negative is:

$$P(Negative\ result|have\ disease).P(have\ disease) = (1 - 0.99) \times 0.0001 = 0.000001$$

Note that the rarer the disease for the same test accuracy, when we test the population at random, it is more likely that a positive test results from a false positive than for a true positive.

11. Sample space is

$$\begin{array}{llll} \{(1, 1), & (1, 2), & (1, 3), & (1, 4), \\ (2, 1), & (2, 2), & (2, 3), & (2, 4), \\ (3, 1), & (3, 2), & (3, 3), & (3, 4), \\ (4, 1), & (4, 2), & (4, 3), & (4, 4)\} \end{array}$$

The outcomes with a total of at most 4 are $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$. Hence $P(total \leq 4) = \frac{6}{16}$. By the sum rule, $P(total \geq 5) = 1 - \frac{6}{16} = \frac{10}{16}$

12. There are 12 picture cards in a standard deck. Hence, *with* replacement gives:

$$P(E) = \frac{12.12}{52.52} = \frac{9}{169} = 0.0533$$

and *without* replacement:

$$P(E) = \frac{12.11}{52.51} = \frac{11}{221} = 0.0498$$