CM1103 PROBLEM SOLVING WITH PYTHON

SETS

Dr Jing Wu

School of Computer Science & Informatics

Cardiff University

Sets

Set theory is a formal way of dealing with **collections of things**, e.g. numbers, names, entries in a database, symbols in a language, etc.

Definition (set)
A **set** is a collection of distinguishable objects (i.e. unrepeated, no duplicates), such that given a set and an object, we can decide whether the object belongs to the set or not.

Example
Set of days of the week: $S = \{$ Monday, Tuesday, …, Sunday $\}$
Set of surnames of students in COMSC
Set of positive even numbers less than ten: $T = \{2,4,6,8\}$

Elements

Definition

We refer to the objects in a set as **elements** or **members**.

Definition (Element notation)

The notation $x \in S$ means "x is an element of S".

The notation $x \notin S$ means "x is **not** an element of S".

Example

Set of positive even numbers less than 10: $T = \{2,4,6,8\}$

$6 \in T$, but $7 \notin T$

Examples

Example	
Is $\{1,1,2,3\}$ a valid set?	No
Elements of $\{1, \{1,2\}, 3\}$?	$1, \{1,2\}, 3$
Elements of $\{1, \{1\}, 3\}$?	$1, \{1\}, 3$

- Sets can themselves be elements of other sets.
- $\{1\} \neq 1$

Cardinality

Definition (cardinality)

The **cardinality** of a set S, denoted $|S|$, is the number of elements it contains

S	$\|S\|$
$\{1,2,3\}$	3
$\{1,\{1,2\},3\}$	3
$\{1,\{1\},3\}$	3
$\{\{1,2\}\}$	1

Some “special” sets

- Some “special” sets
  - $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ is the set of **natural** numbers (“counting” numbers)
  - $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ is the set of **integers**.
  - $\mathbb{Q}$ is the set of **rational** numbers – those that can be defined as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, e.g. $\frac{1}{3}$, $\frac{-17}{4}$
    - $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q}$ such that $p, q \in \mathbb{Z}\}$
  - $\mathbb{R}$ is the set of **real** numbers – all decimal numbers, e.g. $\frac{3}{4}$, 42, 0.451937 (possibly with unending digits after the decimal point, e.g. π, 0.333…, $\sqrt{2}$)
  - $\mathbb{C}$ is the set of complex numbers – e.g. $\sqrt{-1}$, but we don’t cover these in this module

Describing Sets Mathematically

- List the elements: $\{0,1,2,3,4,5,6,7,8,9\}$.

  We can also abbreviate this list as $\{0,1,2,\ldots,9\}$.
- Describe the elements in terms of some properties they satisfy:

  $\{x : x \text{ is an integer and } 0 \le x < 10\}$
- Describe the elements as the set of all elements in some other set that satisfy some properties:

  $\{x \in \mathbb{Z} : 0 \le x < 10\}$

- Example: write these sets using set builder notation:

even natural numbers:	$\{x \in \mathbb{N} : x \,\%\, 2 = 0\}$	$\{x : x = 2k \text{ for } k \in \mathbb{N}\}$
real numbers bigger than 10:	$\{x \in \mathbb{R} : x > 10\}$	
odd natural numbers:	$\{x \in \mathbb{N} : x \,\%\, 2 = 1\}$	$\{x : x = 2k + 1 \text{ for } k \in \mathbb{N}\}$

Subsets

Definition (subset)

A is a **subset** of B (written $A \subseteq B$) if every element of A is also an element of B. I.e. $\forall x$, if $x \in A$ then $x \in B$

Definition (equality)

A is **equal** to B (written $A = B$) if they have *exactly* the same elements. I.e. $A \subseteq B$ and $B \subseteq A$

Definition (proper subset)

A is a **proper subset** of B (written $A \subset B$) if it is a subset of B but not equal to B, i.e. $A \subseteq B$ and $A \neq B$, which means every element of A is in B but there is at least one element of B that is not in A.

Examples

Example			
$\mathbb{N} \subseteq \mathbb{Z}$	$\mathbb{Z} \nsubseteq \mathbb{N}$	$\mathbb{N} \subset \mathbb{Z}$	$-1 \in \mathbb{Z}$, but $-1 \notin \mathbb{N}$
$\mathbb{Z} \subseteq \mathbb{Q}$	$\mathbb{Q} \nsubseteq \mathbb{Z}$	$\mathbb{Z} \subset \mathbb{Q}$	$1/3 \in \mathbb{Q}$, but $1/3 \notin \mathbb{Z}$
		$\mathbb{Q} \subset \mathbb{R}$	$\pi \in \mathbb{R}$, but $\pi \notin \mathbb{Q}$

- Which of the following statements is true:

$\{4,1,2\} \subseteq \{1,2,3,4\}$	True
$\{3,1,2\} \subset \{1,2,3\}$	False
$\mathbb{Q} \subseteq \mathbb{N}$	False
$5 \in \{x: x = 5k \text{ for } k \in \mathbb{Z}\}$	True
$\{7,8\} \subseteq \{x \in \mathbb{N}: 1 \leq x \leq 20 \text{ and } x \text{ is even}\}$	False

Sets in Python

- The set type is built into Python

  Create using the set keyword:

```
T = set([2,4,6,8,10,12,14,16,18,20])
T = set([x for x in range(1,21) if x%2 == 0])
```

- Given element a and set A in Python:

  To test whether $a \in A$, use a **in** A

  To test whether $a \notin A$, use a **not in** A

- Given sets A and B in Python:

  To test whether $A \subseteq B$, use A <= B

  To test whether $A = B$, use A == B

  To test whether $A \subset B$, use A < B

Operations on sets

Definition (universal set)

The **universal set** is the set of all possible elements, denoted by $\mathbb{U}$. The exact definition of $\mathbb{U}$ will depend on the context

Definition (empty set)

The **empty set** is the set with zero elements, and is denoted by $\{\}$ or $\emptyset$

Operations on sets

We can combine sets in various ways to form new sets:

Definition (union)

The **union** of sets A and B is denoted $A \cup B$, and is defined as:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Definition (intersection)

The **intersection** of sets A and B is denoted $A \cap B$, and is defined as:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Example

Let $A = \{1,2,3\}$ and $B = \{2,4\}$

$$A \cup B =$$

$$A \cap B =$$

Operations on sets

Definition (difference)

The **difference** of sets A and B is denoted $A - B$, and is defined as:

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Definition (complement)

The **complement** of a set A is denoted $\overline{A}$ (or A^C), and is defined as:

$$\overline{A} = \mathbb{U} - A = \{x : x \in \mathbb{U} \text{ and } x \notin A\}$$

Example

Let $A = \{1,2,3\},\ B = \{2,4\},\ C = \{0,6,8\},\ D = \{8\}$, and $\mathbb{U} = \{x \in \mathbb{N} : 0 \leq x \leq 10\}$

$A - B =$

$C - D =$

$D - C =$

$\overline{C} = \ \mathbb{U} - C =$

Some set properties

Commutative	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity	$A \cap \mathbb{U} = A$	$A \cup \emptyset = A$
Negation	$A \cup \overline{A} = \mathbb{U}$	$A \cap \overline{A} = \emptyset$
Idempotent	$A \cap A = A$	$A \cup A = A$
De Morgan	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
Complement	$\overline{\mathbb{U}} = \emptyset$	$A - B = A \cap \overline{B}$

Python equivalents

- Given sets A and B in Python:

  Their union $A \cup B$ is given by A | B

  Their intersection $A \cap B$ is given by A & B

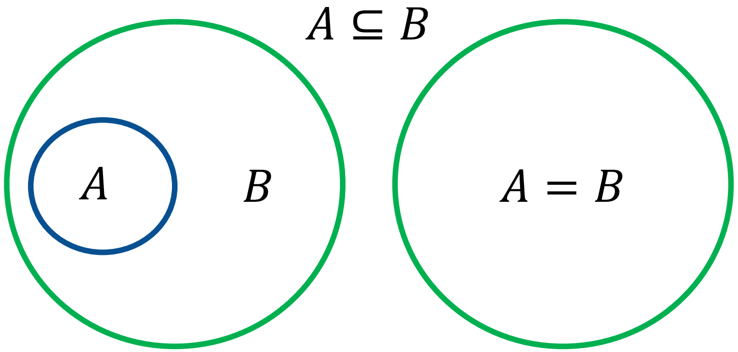
  Their difference $A - B$ is given by A - B

  The complement of A is given by …?

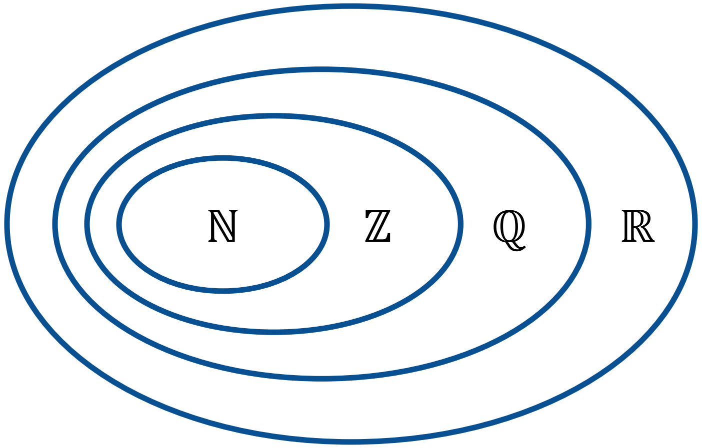
Venn diagrams

Venn diagrams are schematic diagrams that allow us to visualise the relationship between collections of sets.

The Venn diagram representations for subsets



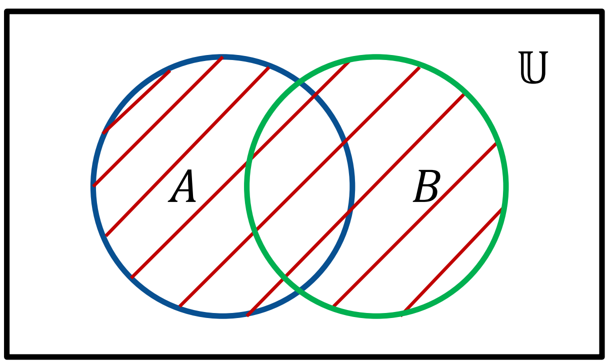
Relations among sets of numbers



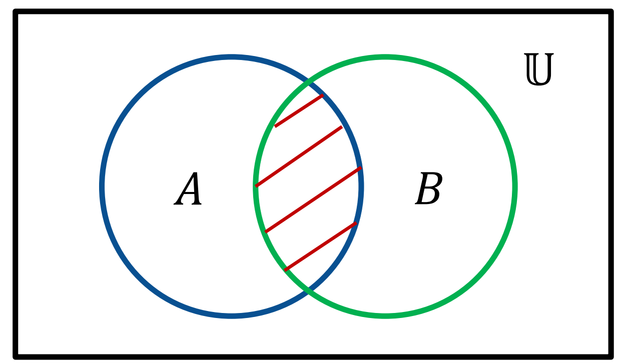
Venn diagrams

The Venn diagram representations for union, intersection, difference, and complement. The shaded region represents:

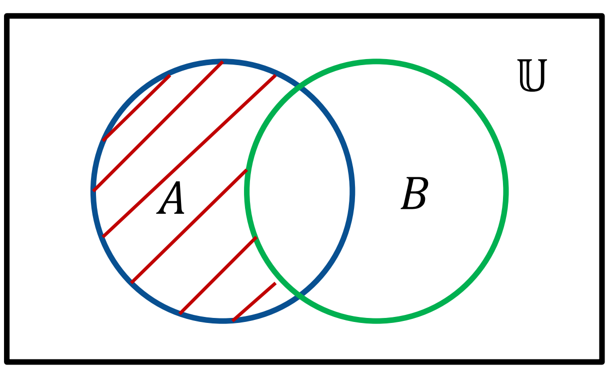
$A \cup B$



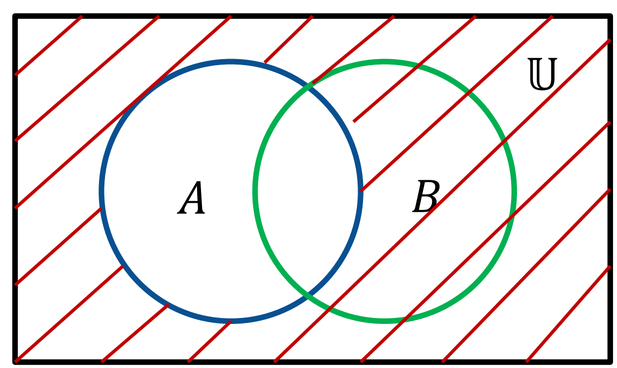
$A \cap B$



$A - B$



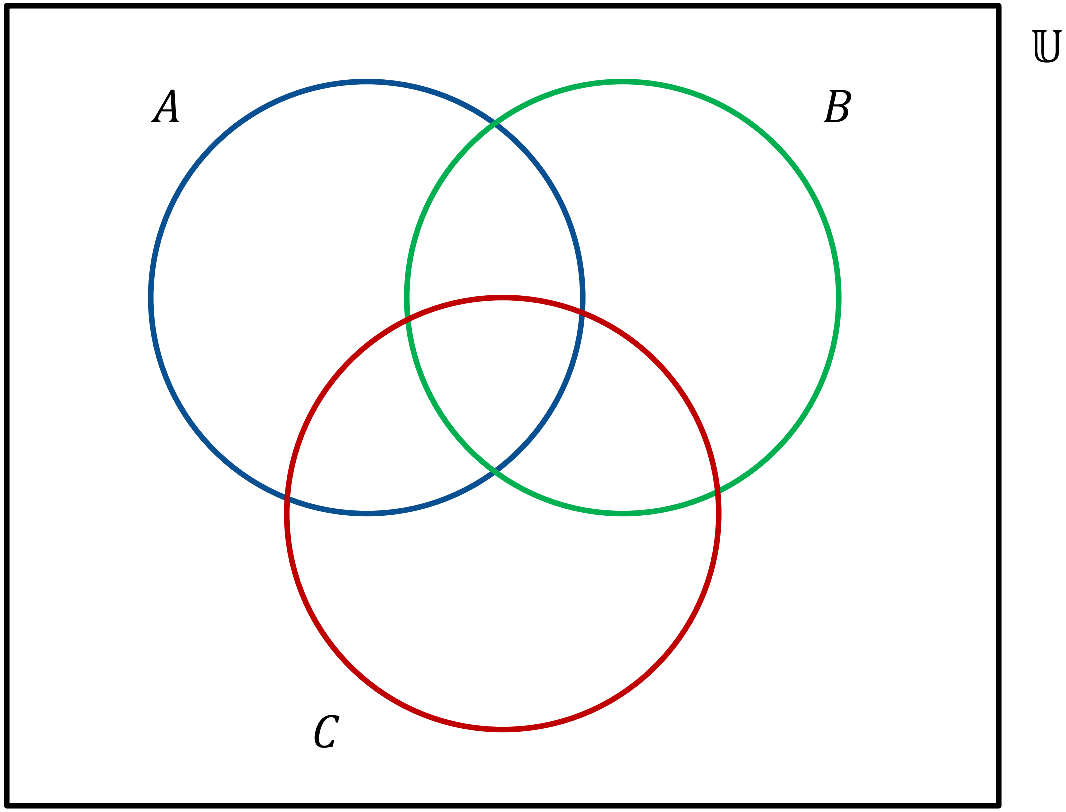
$\overline{A}$



Venn diagrams

Example: 3 sets

Let $A = \{0,1,2,3,4\}$, $B = \{2,4,6\}$, $C = \{0,4,5,6,8\}$



Power set

Definition (power set)

The power set of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S

Example

If $S = \{1,2,3\}$

$\mathcal{P}(S) =$

Cartesian product

Definition

The *Cartesian product* $A \times B$ of sets A and B is the set of all ordered pairs, (a, b), where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A \ \ and \ \ b \in B\}$$

Example

If $A = \{x, y\}$, $B = \{1, 2, 3\}$

$A \times B =$

Cartesian Products

Definition

The *Cartesian product* of sets $A_1, A_2, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \colon a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Example

If $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

$A \times B \times C =$

In Python: itertools.product(…)

Disjoint sets

Definition (disjoint)

Two sets A and B are **disjoint** if they have no common elements, which means $A \cap B = \emptyset$

Are the following sets disjoint?

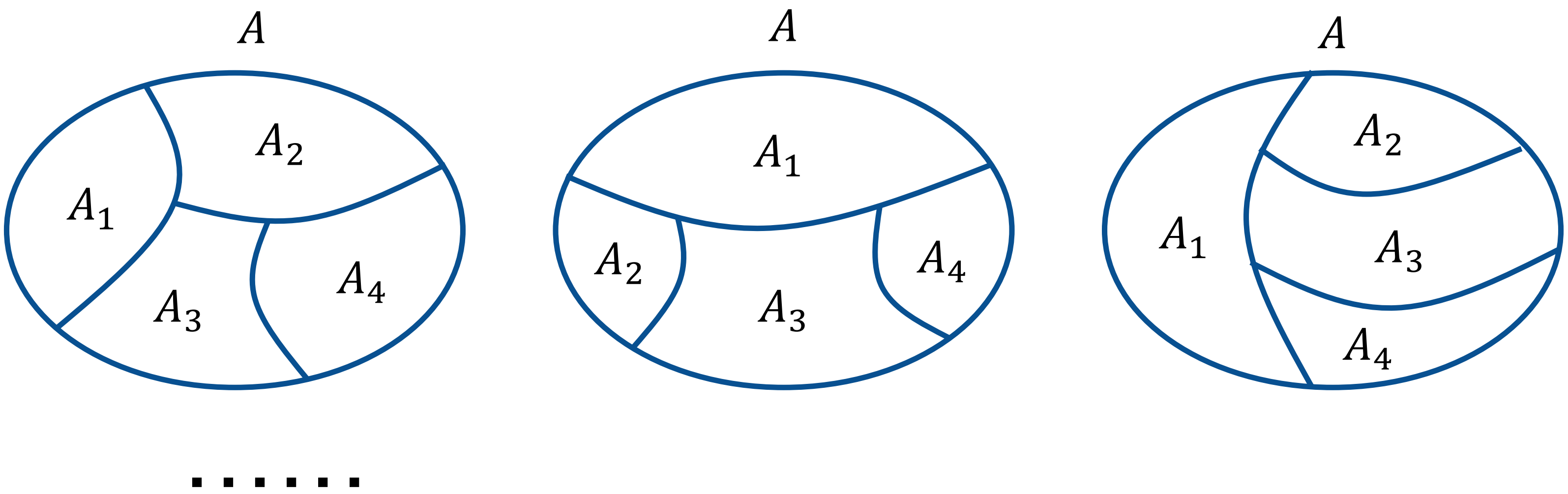
$\{1,2,3\}, \{x, y\}$	
$\{1,2,3\}, \{2, x, y\}$	

Partitions

Definition (partition)

A **partition** of a set A is a set of sets $P = \{A_1, A_2, \cdots, A_n\}$ such that:

1. None of $A_1, A_2, \cdots, A_n$ are empty
2. $A_1 \cup A_2 \cup \cdots \cup A_n = A$
3. **Every** pair of sets from $A_1, A_2, \cdots, A_n$ are disjoint (**mutually disjoint**)



Partitions

Example

Let $A = \{0, 1, 2, 3, 4\}$, which of the following are partitions of A

$\{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}\}$	
$\{\{0, 1\}, \{2, 3\}, \{4\}\}$	
$\{\{0, 2\}, \{1, 3\}, \{4\}\}$	
$\{\{0, 2\}, \{1, 3\}, \{0, 4\}\}$	
$\{\emptyset, \{0,1,2,3,4\}\}$	
$\{\{0, 2, 4\}, \{1, 3, 5\}\}$	

Example

Let $A_1 = \{n : n = 2k \text{ for } k \in \mathbb{N}\}$
$A_2 = \{n : n = 2k + 1 \text{ for } k \in \mathbb{N}\}$.

Is $\{A_1, A_2\}$ a partition of $\mathbb{N}$?

Inclusion-exclusion principle

Definition (inclusion-exclusion principle)

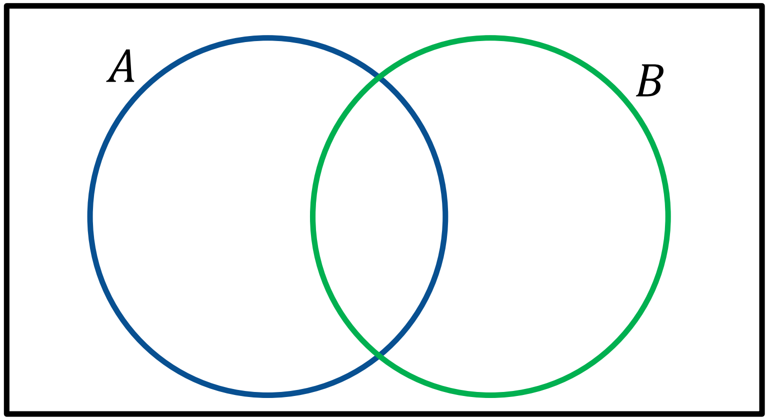
For two (finite) sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets A, B, and C:

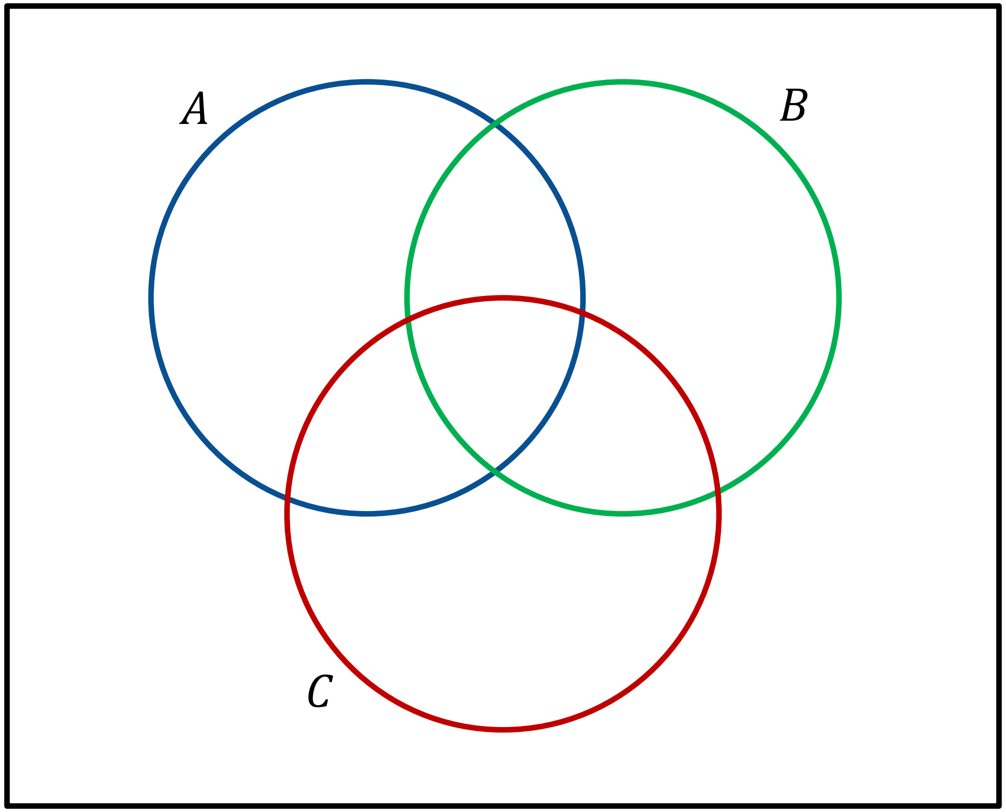
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Inclusion-exclusion principle

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-exclusion principle

Example: 2 sets	
Let $A = \{0,1,2,3,4\}, \quad B = \{2,4,6\}$.	$\lvert A\rvert = 5, \qquad \lvert B\rvert = 3$

Example: 3 sets	
Let $A = \{0,1,2,3,4\}$, $B = \{2,4,6\}$, $C = \{0,4,5,6\}$.	$\lvert A\rvert = 5,\ \lvert B\rvert = 3, \lvert C\rvert = 4$

Examples

Let $\mathbb{U} = \mathbb{N}$, $A = \{1, 3\}$, $B = \emptyset$, and $C = \{1, 2, 3, 4, 5\}$	
What is $A \cap B$?	
What is $A \cup B$?	
What is $A \cup C$?	
Is $A \subset B$?	
Is $B \subset A$?	
Give two disjoint sets whose union is C	

Examples

$\boldsymbol{S}$	$\lvert\boldsymbol{S}\rvert$
$\{1,2,3,4\}$	
$\{\{1,2\},\{3,4\}\}$	
$\{n \in N: 0 < n < 10\}$	
$\emptyset$	
$\{\emptyset\}$	
$\{\{\emptyset\}\}$	
$\mathbb{N}$	

Functions

Definition (Function)

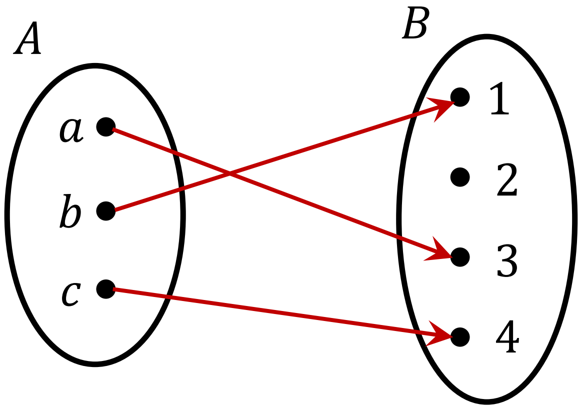
Let A and B be nonempty sets. A *function* $f: A \to B$ is a relationship between elements of A and B such that each element $a \in A$ is related to exactly one element $f(a) \in B$.

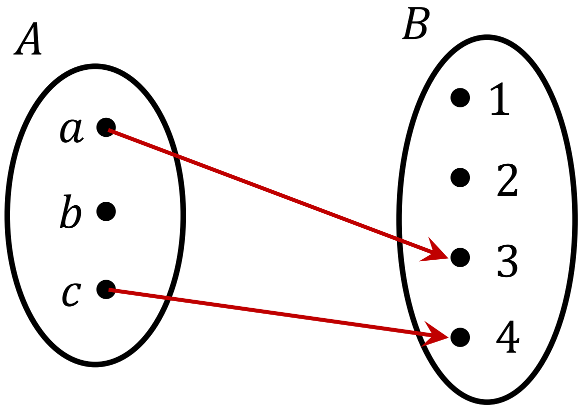
Definition (Domain)

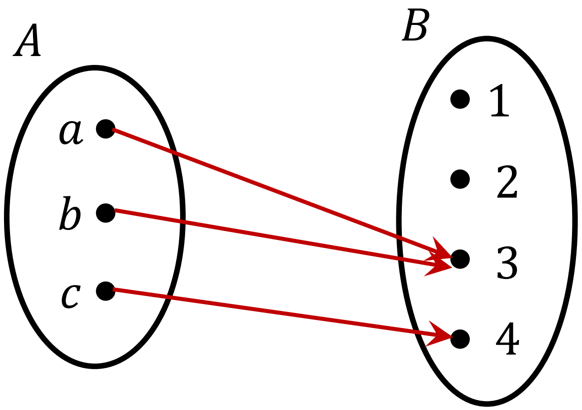
If $f: A \to B$ is a function, then A is the *domain* of f and B is the *co-domain*.

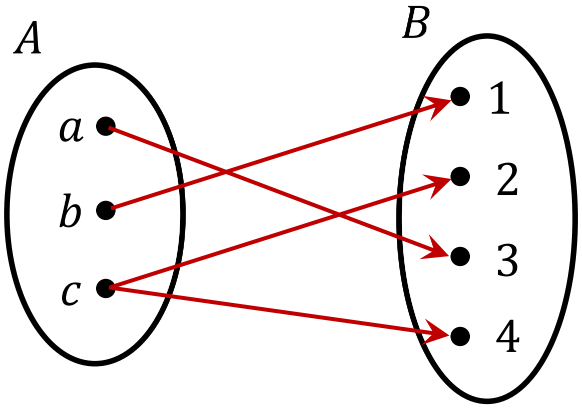
Functions

Example: which of the following arrow diagrams define functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$?









Functions

The **squaring function** $f: \mathbb{R} \to \mathbb{R}$ is defined by the formula

$f(x) = x^2$ for all real numbers x.

The **successor function** $\mathrm{g}: \mathbb{Z} \to \mathbb{Z}$ is defined by the formula

$\mathrm{g}(n) = n + 1.$

An example of a **constant function** is $h: \mathbb{Q} \to \mathbb{Z}$ defined by the formula

$h(r) = 2$ for all rational numbers r.

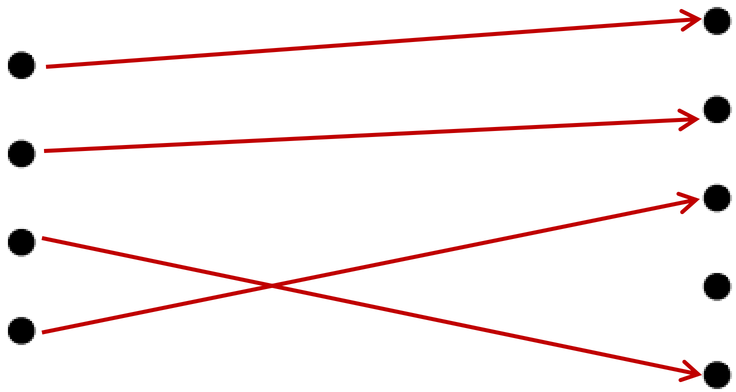
The **identity function** on X, $i_X: X \to X$, is defined by the formula

$i_X(x) = x$ for all x in X.

Injective functions

Definition (Injective function)

$f: A \to B$ is an *injective* (*one-to-one*) *function* if and only if $f(a) \neq f(b)$ whenever $a \neq b$.



$|A| \leq |B|$

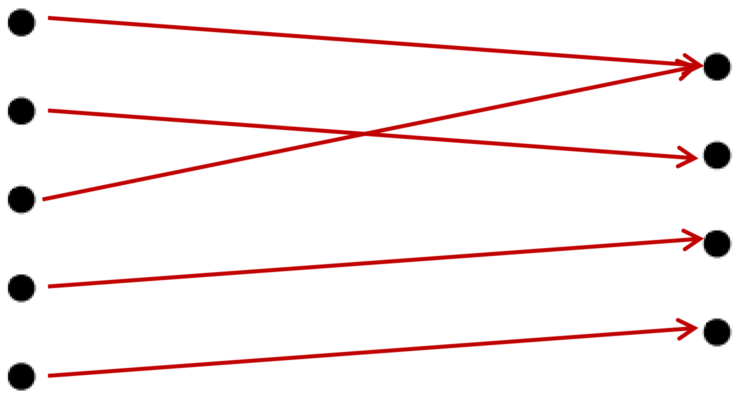
Example: are the following injective functions?

$f: \mathbb{N} \to \mathbb{R}, \quad f(x) = \sqrt{x}$	
$f: \mathbb{Z} \to \mathbb{N}, \quad f(x) = x^2$	
$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 2^x$	

Surjective functions

Definition (Surjective function)

$f: A \rightarrow B$ is a *surjective (onto) function* if and only if for every $b \in B$ there is an element $a \in A$ with $f(a) = b$.



$|A| \geq |B|$

Example: surjective functions

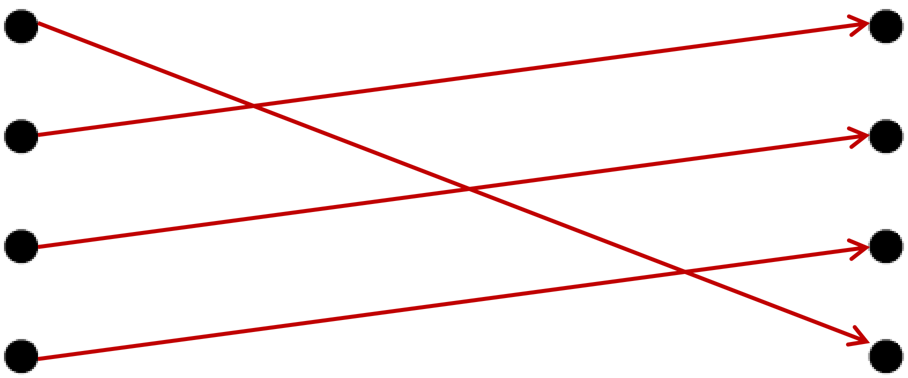
$f: \mathbb{Z} \rightarrow \mathbb{N}, \quad f(x) = abs(x)$

$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x$

Bijective functions

Definition (Bijective function)

$f: A \to B$ is a *bijective function* (*one-to-one correspondence*) if it is both injective and surjective.



$|A| = |B|$

Example: surjective functions

$f: \mathbb{R} \to \mathbb{R}$,	$f(x) = x$
$f: \mathbb{R} \to \mathbb{R}$,	$f(x) = x^3$

Finite and infinite sets

Definition (Finite sets)

A set S is *finite*: either $S = \emptyset$, or there is a bijection $f\colon \{1, \cdots, n\} \to S$.

A set is *infinite* if it is not finite.

Example: { Cardiff, London, Liverpool} is a finite set

$$\begin{array}{ccc} \text{Cardiff,} & \text{London,} & \text{Liverpool} \\ \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 \end{array}$$

Definition

Sets A and B have the same cardinality if and only if there is a bijection $f\colon A \to B$.

Finite and infinite sets

Example: $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.

Countable sets

Definition (Countable)

A set S is *countable* if it is either finite or has the same cardinality as $\mathbb{N}$.

Theorem

$\mathbb{Q}$ is countable.

Uncountable sets

Definition (Uncountable)

A set S is *uncountable* if it is not countable. It has cardinality $|S| > \aleph_0$ (aleph null – cardinality of natural numbers $\mathbb{N}$).

Example:

$|\mathbb{R}| = \mathfrak{c}$ (cardinality of the continuum).
$\mathfrak{c} = 2^{\aleph_0} > \aleph_0$, therefore $\mathbb{R}$ is uncountable.

How to prove $\mathbb{R}$ is uncountable?

Proof by contradiction.

Proof: Susanna S. Epp, "Discrete Mathematics with Applications" Fourth Edition, pp. 434 – 436.

Application: Grand Tour Winners

A Grand Tour refers to one of the three major European professional cycling stage races: Tour de France, Giro d’Italia and uelta a España.

data.py contains names of winners of each Grand Tour.

Use set operations to find out who have won all three of the Grand Tours.

Summary

You should be able to:

- Represent collections of objects as sets
- Use correct set notation
- Use set operations (union, intersection, difference, complement)
- Use Venn diagrams to visualise sets
- Calculate the partitions, power-sets, Cartesian products of given sets.
- Use the inclusion-exclusion principle to count the elements of sets
- Use Python to create, count & manipulate simple sets
- Understand the use of functions to describe the relations between sets

Definitions covered

- Set, element and cardinality
- Special sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{U}$, $\emptyset$
- $\subseteq$, $\subset$
- $\cup$, $\cap$, $-$, $\overline{A}$, $\mathcal{P}$, $\times$ with terms (union, intersection, etc.)
- Partitions
- Inclusion-exclusion principle
- Functions, injective functions, surjective functions, bijective functions
- Finite sets, infinite sets, countable sets, uncountable sets