CM1103 PROBLEM SOLVING WITH PYTHON

INTRODUCTION TO PROBABILITY AND RANDOM NUMBERS

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Overview

- Random process, sample space, event, and probability
- Examples
- Probability of more complicated events
- **Random** module in python
- Simulation of random processes
- Pigeonhole principle

Sample space

Definition (sample space)

The **sample space** of an experiment or random process is the set of all possible outcomes

Example:

Experiment	Sample space
Flipping a coin	
Rolling a (six-sided) die	
Both together	

Probability

Definition (event)

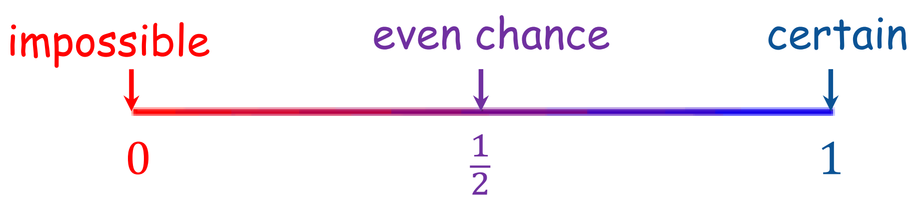
An **event** is a subset of a sample space

Definition (probability)

Given an experiment with a sample space S of equally likely outcomes and an event E, the **probability** of E occurring is:

$$P(E) = \frac{|E|}{|S|}$$

Probability is always between 0 and 1



Example

If we roll a six-sided die once, what is the probability of rolling an even number?

Example

If we flip two coins, what is the probability of getting two heads?

Example

If we flip two coins, what is the probability of getting two heads?

Example

If we roll a pair of six-sided dice once, what is the probability that the total is 6?

All possible outcomes S:

2nd die

1st die						
	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
	$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

Example

One card is drawn at random from a pack of playing cards. What is the probability that the card is a black picture card?

Sample space:

Sum rule

Definition (sum rule)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

It follows from the inclusion-exclusion principle in sets

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|} = P(A) + P(B) - P(A \cap B)$$

Note that if A and B are disjoint,

mutually exclusive, i.e. $A \cap B = \emptyset$

then we get:

$$P(A \cup B) = P(A) + P(B)$$

Examples

If we roll two (6-sided) dice, what is the probability that either both show the same number or their total is odd?

2nd die

1st die

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Examples

A single card is drawn from a pack of 52. Use the sum rule to find the probability that it is:

- An ace or a jack
- A red picture card (J, Q, or K) or a spade
- An ace or a heart

Probability of the Complement

Proposition

Given an event E,

$$P\left(\overline{E}\right) = 1 - P(E)$$

Example

If we roll a pair of six-sided dice, what is the probability that they show different numbers?

Conditional probability

Definition (conditional probability)

Given events A and B, define the **conditional probability** $P(A|B)$ (or the probability of A *given* B) by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

If we roll a pair of six-sided dice once, what is the probability that the total is greater than 6 given at least one of the dice is greater than 3?

2nd die

1st die

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Product rule

Definition (product rule)

$$P(A \cap B) = P(A|B)P(B)$$

Note, if A and B are **independent** (i.e. the occurrence of A is not affected by the (non)occurrence of B, or $P(A|B) = P(A)$), then

$$P(A \cap B) = P(A)P(B)$$

Example

Suppose the chances of having a boy or girl are equal (i.e. ½),

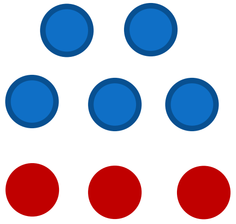
1. If a couple already have a girl, what's the probability of having another girl?
2. What's the probability of having two girls?
3. What's the probability of having one boy one girl?

Examples

Suppose we roll three (6-sided) dice. Use the product rule to work out the probability that all 3 rolls are even.

Suppose we have a bag containing 3 red marbles and 5 green marbles. Use the product rule to calculate the probability of:

1. Drawing 2 red marbles *when the marble is returned to the bag after each draw*
2. Drawing 2 red marbles *when the marble is **not** returned to the bag after each draw*



Examples

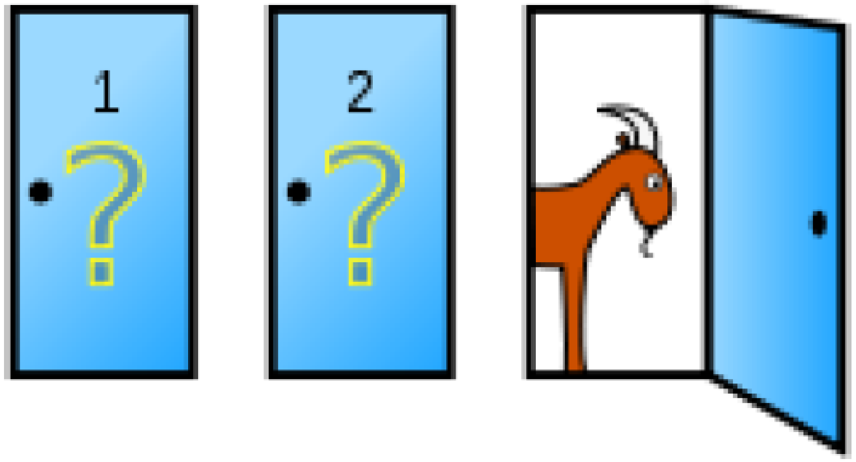
Two cards are drawn at random from a pack of playing cards.
What is the probability of drawing two aces:

1) with replacement? (The card is returned to the pack after each draw.)
2) without replacement? (The card is not returned to the pack.)

Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

Is it to your advantage to switch your choice?



https://www.youtube.com/watch?v=mhlc7peGlGg

Random numbers in Python

Python includes a module **random** that has useful functions to generate pseudorandom numbers

https://docs.python.org/3/library/random.html

- random.random()
- random.uniform(a, b)
- random.randint(a, b)
- random.randrange(a, b)
- random.normalvariate(mu, sigma)
- random.expovariate(lamda)

Example: rolling dice

We can simulate random processes to get empirical values of probabilities, e.g.

- Simulate rolling a six-sided die

  random.randint(1, 6)

- Simulate rolling an n-sided die

  random.randint(1, n)

- Simulate rolling a collection of dice, e.g. m n-sided dice

  for i in range(0, m):

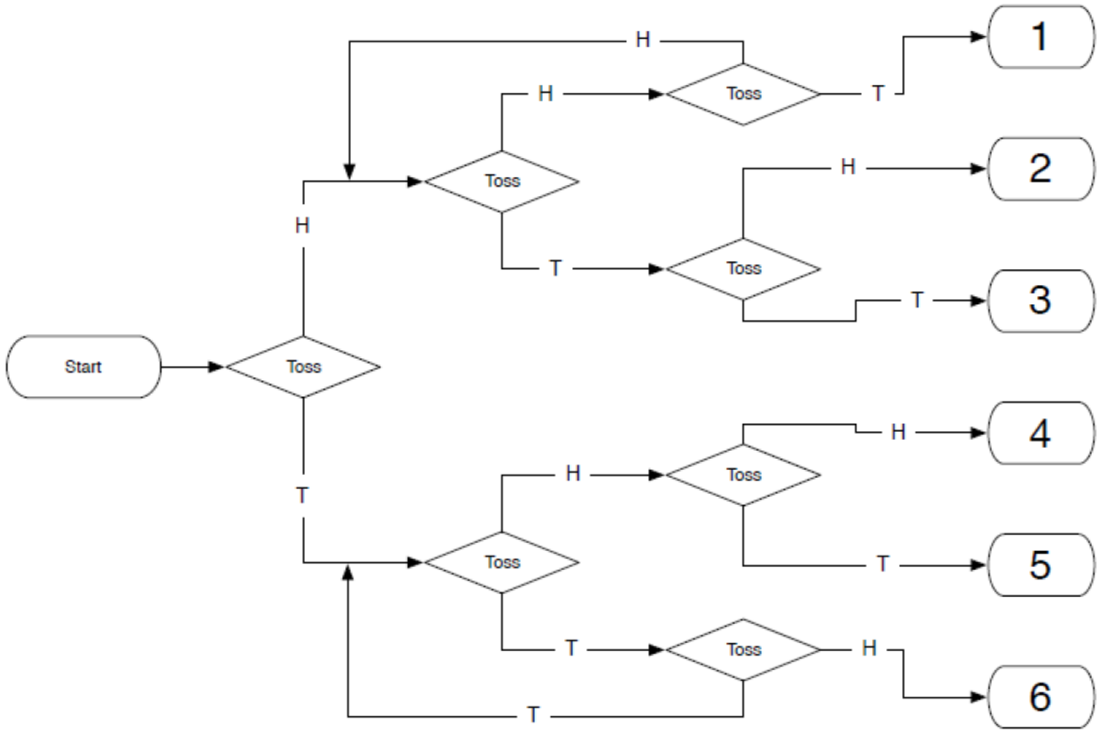
      print(random.randint(1, n))

Algorithm: simulate 6-sided die with a fair coin

Suppose we wanted to play a game that required us to roll a (six-sided) die but we only have a coin. How can we use the coin to get the same result?

The same result:

1) The same sample space $\{1,2,3,4,5,6\}$
2) Equally likely outcomes



Other useful Python functions

- random.choice(x) will select an element randomly from x
- random.shuffle(x) will randomly rearrange x in place

Here x is a list.

Algorithm: Fisher Yates shuffle

How could we implement an unbiased shuffle function?

1) Generate a random permutation of a finite set
2) Unbiased: every permutation is equally likely

Fisher Yates shuffle: Require: list x of length n

For i from $n-1$ down to 1

  Let j be a random integer such that $0 \leq j \leq i$

  Swap elements $x[i]$ and $x[j]$

Algorithm: Fisher Yates shuffle

^{0}a ^{1}b ^{2}c

Step 1: swap
A random index in
[0,1,2]:
3 choices
0

^{0}c ^{1}b ^{2}a

Step 2: swap
A random index in
[0,1]:
2 choices
0

^{0}b ^{1}c ^{2}a

Step 3: swap
A random index in
[0]:
1 choices
0

000 ←→ bca

$3 \times 2 \times 1$ index sequences ←→ 3! permutations

An alternative shuffle

Require: list x of length n

For i from 0 to $n-1$

Let j be a random integer such that $0 \leq j \leq n-1$

Swap elements $x[i]$ and $x[j]$

Step 1: n choices
Step 2: n choices
…
Step n: n choices

n^n index sequences → n! permutations

Occurrences must be unequal: biased!

Pigeonhole principle

Definition (pigeonhole principle)

If $n + 1$ objects are distributed among n boxes, then at least one box must contain more than one objects.

Pigeonhole principle

Summary

You should:

- Understand the definition of probability
- Be able to calculate probabilities by listing all outcomes
- Use the sum and product rules to calculate probability of more complicated events
- Use Python to simulate random events
- Understand the potential pitfalls in calculating probabilities
- Understand the pigeonhole principle

Definitions covered

- Sample space, event, probability
- The sum rule, product rule
- Conditional probability $P(A|B)$
- Pigeonhole principle