The Double Pendulum Problem

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Abstract

The double pendulum will be analyzed using Lagrangian and Hamiltonian methods. The system's behavior is then visualized using Matlab's ODE45 routine. Finally, the different behaviors are categorized.

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1. Introduction

In order to analyze the system by using Newton's second law, all the forces acting on the system have to be known and be able to be expressed. Because the double pendulum problem is quite complex, Lagrangian and Hamiltonian equations are used to simplify the analysis. This approach is more suitable because it uses

- energy, a scalar quantity, as a basis for analysis rather than force, a vector quantity, and
- generalized coordinates of any suitable quantity, rather than the limiting rectangular, polar, or spherical coordinates.

The Lagrangian formalism will render two second-order differential equations that are functions of the generalized coordinates. These equations will be rewritten in Hamiltonian formalism which generates four first-order differential equations that are functions of the generalized coordinates and momenta. Furthermore, the use of Lagrangian and Hamiltonian formalism to analyze the system is justified as two important conditions are met:

1. The energy of the system is conserved, as the double pendulum rotates in the conservative force field of the earth
2. Angular momentum is conserved as dissipative friction and air resistance forces are ignored

2. Derivation

A double pendulum consists of two pendulums with the first suspended from a point in space and the second suspended from the end of the first. Consider the double pendulum shown in Figure 1 with masses m_1 and m_2 attached by rigid massless rods of lengths l_1 and l_2. To simplify the analysis of the system, the system's motion will be constricted to a plane. Let the angles the two rods make with the vertical be denoted as θ_1 and θ_2, and let gravity be given by g.

2.1. Position Equations

Then the positions of m_1 and m_2 in Cartesian coordinates are described by (x_1, y_1) and (x_2, y_2), respectively, and are given by

$$x_1 = l_1 \sin\theta_1 \tag{1}$$ eq1

$$y_1 = -l_1 \cos\theta_1 \tag{2}$$ eq1b

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \tag{3}$$ eq1c

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2. \tag{4}$$ eq1d

2.2. Energy Equations

The *potential energy* of the system is found by combining the potential energy of each mass as follows

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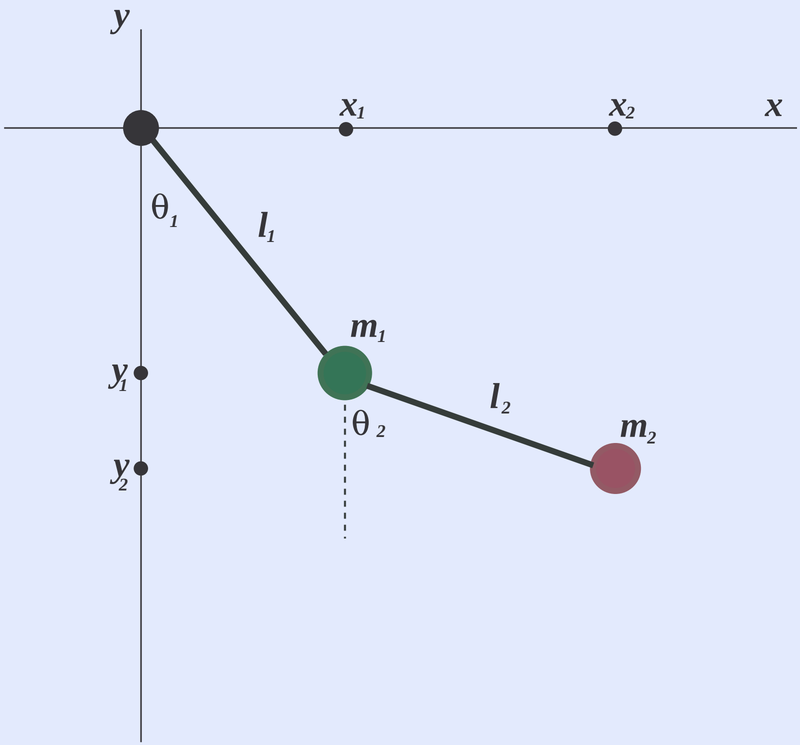

Figure 1: A double pendulum

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$$
\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 && (5)\\
&= m_1 g(-l_1 \cos\theta_1) + m_2 g(-l_1\cos\theta_1 - l_2\cos\theta_2)\\
&= -(m_1+m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2. && (6)
\end{aligned}
$$

The *kinetic energy* of the system is given by

$$
\begin{aligned}
T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 && (7)\\
&= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)\\
&= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]. && (8)
\end{aligned}
$$

2.3. Lagrangian Function

Since the motion of the system takes place in a conservative force field, V is a function of the generalized coordinates and not the generalized velocities $V =$

$V(\theta_1, \theta_2)$. The Lagrangian function is defined as

$$L \equiv T - V. \tag{9}$$

Thus the Lagrangian function of the generalized coordinates (θ_1, θ_2) is

$$\begin{aligned} L = {} & \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ & + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2. \end{aligned} \tag{10}$$

eq10

2.4. Lagrange's Differential Equations

Lagrange's differential equation for θ_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0. \tag{11}$$

eq11

From Equation 10,

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{12}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = {} & (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ & - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \end{aligned} \tag{13}$$

$$\frac{\partial L}{\partial \theta_1} = l_1 (m_1 + m_2) g \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2), \tag{14}$$

and substituting these in Equation 11, we obtain

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + l_1(m_1+m_2)g\sin\theta_1 = 0. \tag{15}$$

eq15

Similarly, Lagrange's differential equation for θ_2 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0. \tag{16}$$

Once again from Equation 10,

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2) \tag{17}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) \tag{18}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - l_2 m_2 g \sin\theta_2, \tag{19}$$

and substituting these in equation (16) gives

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + l_2 m_2 g \sin\theta_2 = 0. \tag{20}$$

eq20

Equation 15 and Equation 20 are coupled second-order ordinary differential equations that describe the motion of the system. Note that these two equations

depend on four unknowns. In order to solve the system numerically using Matlab's ODE45 routine, a system of four first-order ordinary differential equations is needed.

2.5. Generalized Momenta

Since the system is closed, the equations of motion can also be written in the Hamiltonian formalism. This introduces the generalized momenta p_1 and p_2, so the motion of the system depends on four initial conditions $(\theta_1, \theta_2, p_1, p_2)$. Computing the generalized momenta gives

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{21}$$

eq21

$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2). \tag{22}$$

eq22

2.6. Hamiltonian Function

The Hamiltonian function of the generalized coordinates(θ_1, θ_2) and generalized momenta $(p_1, p_2)$is defined as

$$H \equiv T + V \tag{23}$$

The Hamiltonian can also be written in terms of the Lagrangian

$$\begin{aligned} H = \theta_i p_i - L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ - (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned} \tag{24}$$

eq24

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Solving the generalized momenta, Equation 21 and Equation 22, for $\dot{\theta}_1$ and $\dot{\theta}_2$ and substituting back into equation Equation 24 gives

$$H = \frac{l_2^2 m_2 p_2^2 + l_1^2 (m_1 + m_2) p_2^2 - 2 m_2 l_1 l_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]}$$

$$- (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \tag{25}$$

2.7. Hamilton's Equations of Motion

Since the Hamiltonian is now a function of all the initial conditions θ_1, θ_2, p_1, and p_2, the equation can be separated into four first-order differential equations by taking partial derivatives with respect to the appropriate variables

$$\dot{\theta}_1 = \frac{\partial H}{\partial p_1} = \frac{l_2 p_1 - l_1 p_2 \cos\theta_1 - \theta_2}{l_1^2 l_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \tag{26}$$

$$\dot{\theta}_2 = \frac{\partial H}{\partial p_2} = \frac{l_1 (m_1 + m_2) p_2 - l_2 m_2 p_1 \cos(\theta_1 - \theta_2)}{l_1 l_2^2 m_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \tag{27}$$

$$\dot{p_1} = -\frac{\partial H}{\partial \theta_1} = -(m_1 + m_2) g l_1 \sin\theta_1 - C_1 + C_2 \tag{28}$$

$$\dot{p_2} = -\frac{\partial H}{\partial \theta_2} = -m_2 g l_2 \sin\theta_2 + C_1 - C_2 \tag{29}$$

$$C_1 \equiv \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{l_1 l_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \tag{30}$$

$$C_2 \equiv \frac{l_2^2 m_2 p_1^2 + l_1^2 (m_1 + m_2) p_2^2 - l_1 l_2 m_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]^2} \sin[2(\theta_1 - \theta_2)] \tag{31}$$

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3. Finding a Numerical Solution

3.1. Creating a Function M-File

Solutions for θ_1 and θ_2 can be found by solving the system of first-order differential equations numerically using Matlab's ODE45 routine. The first step is to create a function m-file in the Matlab editor.

```
function xprime=doublependulum(t,x,flag,g,l1,l2,m1,m2)

C1=(x(3).*x(4).*sin(x(1)-x(2)))./...
   l1*l2*(m1+m2*(sin(x(1)-x(2))).^2);
C2=((l2^2*m2*(x(3)).^2+l1^2*(m1+m2)*...
   (x(4)).^2-l1*l2*m2*x(3).*x(4).*cos(x(1)-x(2)))...
   ./2*l1^2*l2^2*(m1+m2*(sin(x(1)-x(2))).^2).^2)...
   .*sin(2*(x(1)-x(2)));

xprime=zeros(4,1);
xprime(1)=(l2*x(3)-l1*x(4).*cos(x(1)-x(2)))...
   ./l1^2*l2*(m1+m2*(sin(x(1)-x(2))).^2);
xprime(2)=(l1*(m1+m2)*x(4)-l2*m2*x(3).*cos(x(1)-x(2)))...
   ./l1*l2^2*m2*(m1+m2*(sin(x(1)-x(2))).^2);
xprime(3)=-(m1+m2)*g*l1*sin(x(1))-C1+C2;
xprime(4)=-m2*g*l2*sin(x(2))+C1-C2;
```

This m-file defines the system of four first-order differential equations, where `flag` sets the parameters of the system.

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3.2. Using ODE45

Below is an example of how to execute Matlab's ODE45 routine. First the function is initiated, then a time interval is given followed by the initial conditions of the system, and finally the parameters of the system are given values. Four numerical solutions are produced by this example, where x is a vector that contains the solutions, $x = [\theta_1(t), \theta_2(t), p_1(t), p_2(t)]$.

```
[t,x]=ode45('doublependulum',[0,25],[pi;pi;0;0],[],9.8,1,1,1,1);
```

To call up the numerical equation θ_1, the first column of the vector `x` is initiated `x(:,1)`. Solutions for θ_2, p_1, and p_2 are produced by calling the appropriate column of the vector `x`. These solutions can then be plugged into the position Equation 1, Equation 2, Equation 3, and Equation 4 to model the behavior of the masses in the system.

```
x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=sin(x(:,1))+sin(x(:,2));
y2=-cos(x(:,1))-cos(x(:,2));
```

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4. Behaviors of the System

Depending on the given initial conditions, the double pendulum exhibits three types of behavior:

- Periodic motion
- Quasiperiodic motion
- Chaotic motion

4.1. Periodic Motion

In periodic behavior, the system's motion is cyclic. This occurs when the energy of the system is either small or extremely large. Figure 2 shows the trace of m_2 when the system is given a substantial amount of energy. The mass stays on the same path indefinitely, or the motion is periodic. The following is the code that produced Figure 2.

```
[t,x]=ode45('doublep',[0,0.7],[pi;pi;10;-8.898788999]);
x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=sin(x(:,1))+sin(x(:,2));
y2=-cos(x(:,1))-cos(x(:,2));
plot(x2,y2)
axis([min(-2) max(2)...
      min(-2) max(2)])
```

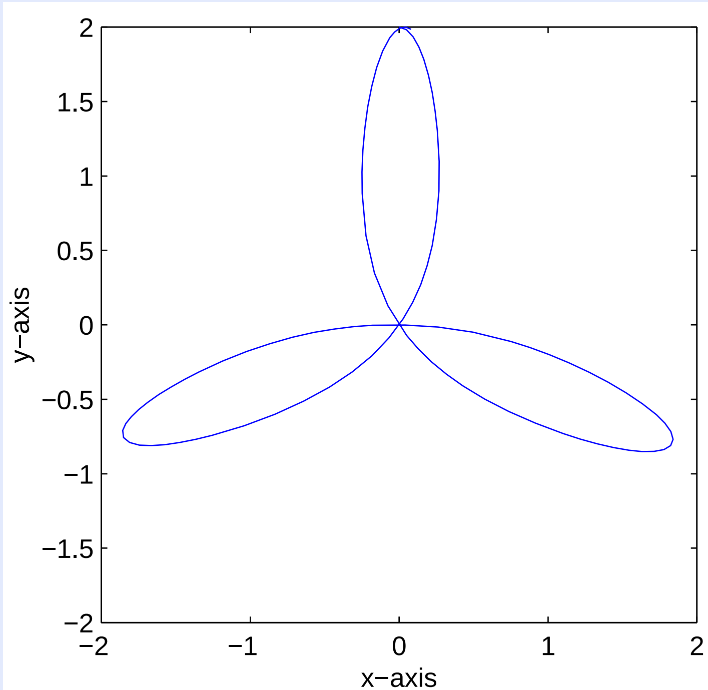

Figure 2: Periodic Behavior

4.2. Quasiperiodic Motion

In quasiperiodic behavior, the system's motion is almost periodic. This behavior occurs at all energy levels, yet is rare at middle range energy values. Figure 3 shows the trace of m_2 when the system is given appropriate values to produce quasiperiodic behavior. Here is the code that produced Figure 3.

```
[t,x]=ode45('doublep',[0,1],[pi;pi;20;-50]);
x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=sin(x(:,1))+sin(x(:,2));
y2=-cos(x(:,1))-cos(x(:,2));
plot(x2,y2)
axis([min(-2) max(2) ...
      min(-2) max(2)])
```

4.3. Chaotic Motion

In chaotic behavior, the system's motion is unpredictable and non-repetitive. Two systems with almost identical initial conditions show very different long term behavior. The system produces chaotic behavior at all energy levels unless the system is given an infinite amount of energy where only periodic and quasiperiodic behavior exist. Figure 4 is an example of chaotic behavior exhibited by the system. The code for Figure 4 follows.

```
[t,x]=ode45('doublep',[0,25],[pi;pi;0;0]);
x1=sin(x(:,1));
y1=-cos(x(:,1));
x2=sin(x(:,1))+sin(x(:,2));
y2=-cos(x(:,1))-cos(x(:,2));
plot(x2,y2)
axis([min(-2) max(2) ...
      min(-2) max(2)])
```

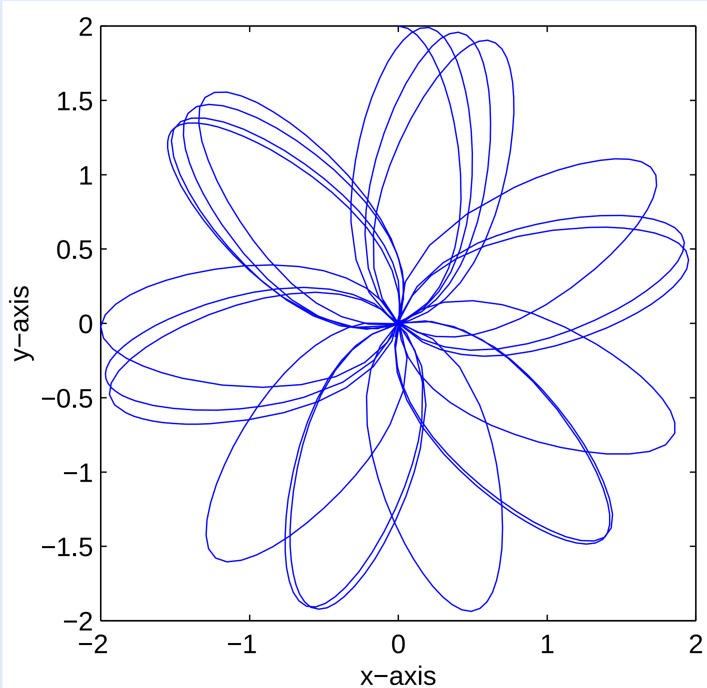

Figure 3: Quasiperiodic Behavior

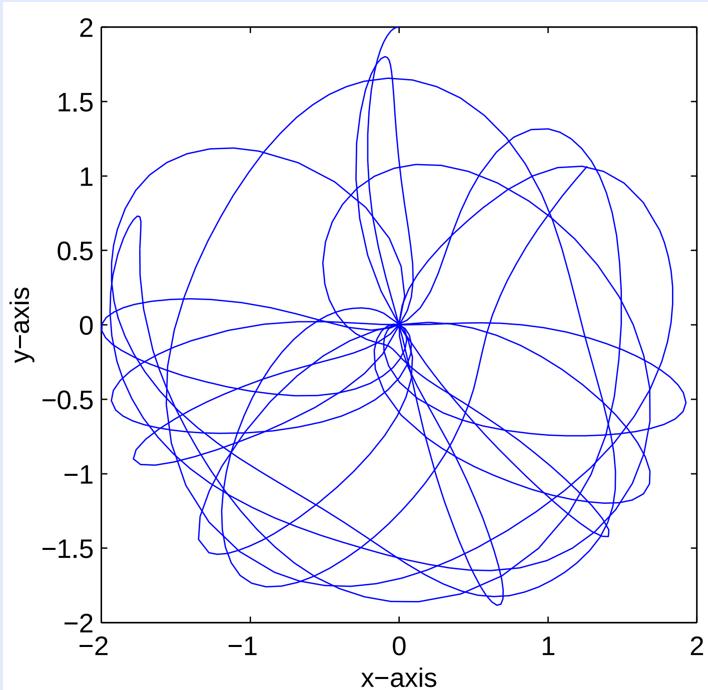

Figure 4: Chaotic Behavior

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