WEAK AND STRONG TIES IN SOCIAL NETWORK

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Abstract. We study how different types of social connections between people shape their social network. There are two possible types of ties between individuals, strong and weak, that differ in maintenance costs and reliability. A network formation game is played in which agents choose the number of ties of each type to maximize their chances of hearing about a job opportunity. We find that in equilibrium, people maintain both types of connections. Furthermore, in the socially optimal symmetric network there are more strong ties than in the equilibrium one.

1. Introduction

Many papers say [8], [9] that networks are important vehicles for passing information in various economic situations. As, nowadays, information is one of the most valuable resources, a deeper understanding of how it travels through the network can significantly benefit different aspects of our life.

As we know from our daily life experience, in social networks there are more than just two types of connections between people (whether they know or do not know each other). Different types of these connections between individuals have different effects. In this paper, we are going to assume that there are two main types of connections[1]: close *friends* and distant *acquaintances*. The question that we ask is: how do these various connections shape the network between individuals?

Social networks are known for delivering various types of information. One of the prime examples of this is learning about a job opening from someone in your social circle. According to a recent survey [11] published by LinkedIn, around 85% of all jobs are filled via networking. But in doing so should we rely more on our Vadim: group of friends

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[1] Given that two people know each other

or acquaintances? What roles do these groups play in providing agents with informational opportunities?

In this paper we study a two-stage model: first, agents strategically form a network; second, information is planted in the network and is spread there for the next 2 time periods. Each player gets utility 1 if the information reaches her and 0 otherwise. Society is divided into villages: people within the same village have an opportunity to be friends with each other and players in different villages have an opportunity to be acquaintances. It is a real-life observation of the fact that it is easier to maintain a friendship with someone who is geographically, or in some other sense, close to you. Agents play a network formation game, where they choose how much effort they want to put into socializing with other people to connect to them. Both agents need to exert a non zero effort in order for them to have a chance to become connected. Once the network has realized, people have to pay different costs to sustain two types of relationships. Friendship is more difficult to maintain but it is a more reliable source of information compared to acquaintanceship. After the agents have made their decisions, the network emerges and defines links between people. In the second stage, a random person in the network receives a piece of valuable information, and we allow this information to travel through the network for two periods.

In the case when information travels only for 1 period, the game becomes trivial. For the two types of connections, one will be more beneficial than the other in terms of the difference between the associated utility and the cost. Thus, agents will choose the more valuable one and completely disregard the other type of link. In other words, you should have only friends or only acquaintances. Also, in the seminal paper Gronovetter [12] states that most job offers come from acquaintances (also called "weak ties") rather than from friends ("strong ties"). This phenomenon the is called "strength of weak ties" and many papers argue that they are significantly more helpfull [2], [15] than the strong connections. We know from the sociological literature that social ties effect our well-being [5], [6], [13], [7]. But why do we observe both types of connections in real life from economic stand point? Should we keep both strong and weak links or should we disregard one of them?

We believe, the answer lies within the network's topology. In equilibrium, if B and C are both friends with A, then they are more likely to have a link between each other rather than if they were both A's acquaintances. It means, that the friends' graph is denser compared to the acquaintances' one. This leads to the following intuition for observing both types of connections in real-life networks. Agent does not want to spend her whole budget on the strong ties as, at some point, a new friend is not going to bring many players into her network because of the high clustering[2] of the friends' graph. On the other hand, a new acquaintance will indirectly connect her to many people who she does not know yet because their graph is sparser. At the same time, not having any friends is not optimal either because in that case, players do not have any strong-weak or weak-strong connections that constitute a bulk of the agent's network. Furthermore, as there are no friends yet, their graph is sparse and does not suffer from a high clustering described above.

In this paper, we establish the uniqueness of a non-trivial symmetric Nash equilibrium. In this equilibrium, it is the best response for people to have both friends and acquaintances at the same time. This explains why we observe both types of connections between individuals. We find that a bulk of agent's utility comes from weak-strong, strong-weak and weak-weak connections and as a result, agents always want to have these ties present in their network. We also perform multiple comparative statics of this equilibrium and check whether the friends' graph has a higher clustering coefficient than the acquaintances' one. Finally, we compare it to the optimal symmetric network and find that there are more strong ties in the latter one. People underinvest in their strong connections in the equilibrium. This suggests that if a social planner would "subsidize" strong ties or "tax" weak ones it would increase social welfare. The intuition for this is that it is socially beneficial not only to get the signal but also to share it with others.

The paper is organized in the following way. In the next chapter, we present our model and explain its different mechanics and properties. In the second one, we present the analytical findings that we described above. Next, we provide some computational results and graphs to get a better feeling of whether our solution/equilibrium is a plausible approximation of the social networks that we observe. We conclude in the final chapter.

1.1. **Literature review.** In the seminal paper by Granovetter [12] the author described a surprising finding of his field experiment: people find a new job through weak ties more often than from the strong ones (27% vs 16%).

Many papers agree with Granovetter's point of view. Contandriopoulos et. al [2] claims that it is significantly more valuable to play a "bridge" role in the network. This corresponds to having weak ties

[2]Informally, clustering coefficient, or just clustering, says how likely two people who have a common friend/acquaintance know each other.

rather than "redundant" strong links: a researcher could gain up to one h-index point by making 3 weak connections. Furthermore, Weng et. al [15] find evidence that weak ties are used to collect "important" information.

There are empirical papers that address the ties' strength and their ability to help find a new job. Gee and et. al [3] construct a very large data set of deidentified individuals from 55 countries. They find that even though it is more likely to hear about a job opening from a week tie, the likelihood of working at a place where your friend/acquaintance works is increasing with the strength of your connection.

Furthermore, Kuzubas and Szabo [10] find that in Indonesia, people tend to use their strong ties more than the weak ones when the number of the latter ones is either small or large side of the range. At the same time, even with a medium-sized weak network, the probability of getting a job through a strong tie decreases by only 16%. However, positions found through weak links correspond to 10% higher wages.

In terms of theoretical work, there are fewer papers in this area. Golub and Livne [4] study how different levels of agent's neighborhoods affect the equilibrium network. They have a similar insight to ours, that player's utility does not only depend on her friends but also her friends of friends. Authors find there are two equilibrium regimes in which the realized network can be either connected or fragmented when the costs are not too convex. However, they do not consider different types of connections that we observe in real life.

Boorman [1] tries to capture the trade-off between strong and weak ties. One of his main results is that, depending on the probability of being unemployed, all-weak and all-strong networks are going to be equilibrium. The author assumes that there is no clustering not only in the acquaintances' network but also in the friends one which is a very unrealistic assumption. Furthermore, it seems that he does not take into account the agent's neighborhoods at a distance of more than 1, which we believe plays a crucial role in a deeper understanding of network effects.

Tümen [14] claims that different types of links are used in different periods of our life. He connects weak links with an early career stable settlement while strong ties are associated with the amplified mobility that generates mismatch.

2. Model

In this paper we study the following two-stage model: in the first stage agents strategically form a network, and in the second stage, information spreads and defines the agents' utilities. We will start with the second stage.

There is a network of people with two different types of connections between each other. We represent it by an undirected graph (V, E), where V is a set of vertices and E is a set of edges. We write E as the disjoint union of E_s, the set of *strong ties*, and E_w, the set of *weak ties*. For $i, j \in V$ let $d(i, j)$ denote the length of the shorterst path between i and j.[3] Note that $d(i, j) = d(j, i)$, as the graph is undirected. We denote by $N_k(i)$ the neighborhood of agent i at distance k, which is a set of agents that are exactly at distance k from i:

$$N_k(i) = \{j \in V \,|\, d(i, j) = k\}.$$

Recall that a clustering coefficient (CC) of a graph is defined as

$$CC = \frac{3 \times \text{number of triangles}}{\text{number of all triplets}}.$$

In period 1 nature randomly selects a person i who receives a signal/piece of valuable information. In the first period, this player sends information to all of her friends and acquaintances. It travels through a strong tie with probability 1 and through a weak one with probability π_w[4]. In the second period, everyone who has received the signal from agent i sends it to Vadim: his friends and acquaintances in the same way. If this information reaches a player in at least one of these two periods, then her utility is 1, and 0 otherwise.

We now define the first stage when agents form the underlying network, (V, E). Society is divided into $N + 1$ "villages", with K agents in each one. People in the same village can potentially be friends with each other and players in different ones can potentially be acquaintances.

For each player $j \neq i$, agent i chooses an effort level, $1 \geq p_{ij} \geq 0$, that she is willing to spend to become his friend or acquaintance, depending on whether they are in the same village or not. Thus, a strategy for player i are effort levels for all other players, $\{p_{ij}\}$ $\forall j \in V \setminus \{i\}$. We are going to focus on symmetric strategies: agent chooses one effort level for people in her own village and another level for people outside of it. Thus, a strategy for player i simplifies to $\{p_i, q_i\}$, where p_i is a socialization level within the village and q_i is a socialization level outside of it. After the strategies have been chosen, two agents i and

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[3] A path between two nodes, $i,\, j \in V$, is a sequence of nodes $n_1, n_2, \ldots n_t \in V$, such that $n_1 = i$, $n_t = j$ and $\forall\, i \in \{1, \ldots t-1\}$ $e_{n_i n_{i+1}} \in E$.

[4] So with probability $1 - \pi_w$ information does not reach the other end of the week link.

j, who are in the same village, become friends with probability $p_i p_j$ and, if they are in different villages, they become acquaintances with probability $q_i q_j$.

Maintainting each strong and each weak link requires c_{strong} and c_{weak} units of time respectively. People have a time budget, B, that they can spend on their social circles. Their utility vanishes if they exceed the budget and does not depend on c_{strong} and c_{weak} otherwise. To simplify our presentation in the following sections we normalize c_{strong} to be 1 and allow c_{weak} and π_w to vary.

Here is a description of the game. Agent i, who is in a village $\mathcal{V}_i$, chooses her effort levels $1 \geq p_i, q_i \geq 0$ and becomes friends with player $j \in \mathcal{V}_i$ with probability $p_i p_j$ and she becomes acquainted with player $\ell \notin \mathcal{V}_i$ with probability $q_i q_\ell$. Agent i maximizes her expected utility U_N, where N stands for the number of villages other than hers, given by this expression:

$$U_N(p_i, q_i, p_{-i}, q_{-i}) = \begin{cases} \mathbb{E}_{(V,E)}\left(\mathbb{P}(i \text{ gets the signal})\right), & \text{if } c_{strong} p_i \sum\limits_{\substack{j \in \mathcal{V}_i \\ j \neq i}} p_j + c_{weak} q_i \sum\limits_{\ell \notin \mathcal{V}_i} q_\ell \leq B \\ 0, & \text{otherwise,} \end{cases}$$

where $\{p_{-i}, q_{-i}\}$ stand for a strategy profile of every player except i and we are taking expectation over realizations of the random graph.

We are going to focus on symmetric equilibria, $p_i = p$ and $q_i = q$ $\forall\, i$. There are two trivial equilibria: 1) $\forall\, i\ p_i = 0$; 2) $\forall\, i\ q_i = 0$. If everyone else is not exerting any effort for one type of connections then it is strictly dominated for player i to choose a non zero effort level for it either Vadim: either?. Whichever strategy player i chooses, this type of connection is not going to realize as the probability of it is the product of effort levels.

But as soon as p and q for other players are positive agent i is going to choose p_i and q_i such that she uses all of the budget that is available to her, B. Otherwise, she can increase one of her effort levels and benefit from increasing the expected number of her connections. Therefore, we can treat this objective as if we have equality in the budget constraint.

From now on, we refer to nontrivial symmetric equilibria as equilibria unless stated otherwise. In the next section, we prove there is only one equilibrium of this game and investigate its properties, including comparative statics results.

3. Analytical results

We start our analysis by noting that the probability the signal reaches agent i can depend on the realized graph in a very complex way. As a simple example, consider the case when there are only two weak-strong paths between i and j and the latter one is given the signal by nature. Then the probability that it reaches i is $1-(1-\pi_w)^2 = 2\pi_w - \pi_w^2$, instead of π_w if there was only one path. Fortunately, as we increase the number of villages, $N+1$, the chance that there are multiple connections between i and some other person j, that include at least one weak link, vanishes. This fact allows us to find a limit utility function U_∞ such that U_N uniformly converges to it as we increase $N+1$.

We are going to prove that when agents are maximizing U_∞ instead of U_N, there is a unique equilibrium, and as a result of this, there are equilibria of the initial function within the ε-neighborhood of it. Furthermore, there are no equilibria of the initial function that are not in a neighborhood of some equilibrium of U_∞. Moreover, it is clear to see that this equilibrium of the limit utility function is also an ε-equilibrium of the initial game. Let us first explain how we find this limit utility function and then write it down explicitly.

Notice that when there are no overlaps in paths, that we described above (of length at most 2, which have at least one weak link in them), there are only 5 groups of people in our[5] social circle that affect our utility. We describe each group by the type of the shortest path between us: strong, weak, strong-strong, weak-strong and strong-weak (together in one group), weak-weak. Let us multiply the number of people in the weak, strong-weak and weak-strong groups by π_w, and the number of people in the weak-weak group by π_w^2. If we now add the number of people in every group (after three of them have been normalized by π_w and π_w^2 in the previous sentence) and add 1 (to include ourselves), then this sum, divided by the total number of people in society, is equal to the probability that the signal reaches us. Thus, by choosing p_i, q_i that maximize this number player i maximizes her utility. This happens because: 1) every individual is equally likely to be chosen by nature; 2) the signal travels for 2 time periods and 3) there are no overlaps in paths of length 1 or 2, containing at least one weak tie, between us and some other player.

In Proposition 7 we evaluate how many people player i expects to have in each of these 5 groups. Denote the number of people in all those groups, some of which are normalized by π_w and π_w^2 correspondingly,

[5]We use "our" and i's interchangeably.

by $\widetilde{U}_N$

$$\widetilde{U}_N(p_i, q_i, p, q) = \Big(1 + (K-1)p_i p + \pi_w K N q_i q + (K-1)(1-p_i p)(1-(1-p_i p^3)^{K-2}) + \\ + \pi_w K(K-1)N\left(p_i p q^2 + p^2 q_i q\right) + \pi_w^2 q_i q^3 K^2 N^2\Big)$$

The second and the third terms of $\widetilde{U}_N$ calculate how many strong and weak ties we have at distance 1. The next two terms are a bit more complicated. The fourth term stands for the number of friends of friends that are not our direct friends. The fifth one calculates the number of people that are at distance 2 such that there is either a *weak-strong* or *strong-weak* path between us. The last term calculates the number of people that are at distance 2 from us, connected through a *weak-weak* path[6]. The 1 stands for the agent i herself. We need to multiply the second and the last two terms by π_w and π_w^2 respectively, because information travels through a weak tie with probability π_w. So, for example, the expected utility from a group of weak links is proportional to the number of them, KNq_iq, multiplied by π_w.

Our $\widetilde{U}_N$ is a polynomial of two variables, p_i, q_i, (keeping p and q fixed) that we need to maximize with respect to p_i and q_i subject to the budget constraint and symmetric equilibrium assumption ($p_i = p$, $q_i = q$). Notice, that even though its order is fairly high it does not depend on N. It is optimal for agent i to choose p_i and q_i such that her budget constraint is satisfied with equality. Thus, we can use this equation to write q_i (and q after applying a symmetric strategy assumption) in terms of p_i (p) and then substitute it back into $\widetilde{U}_N$ so it becomes a polynomial of one variable $\widetilde{U}_N(p_i, q_i(p_i), p, q(p))$. Keep in mind, that even after we do this it is still tricky in general to get analytical results about a maximum of a high order polynomial, especially on $[0, 1]$. But before we proceed we need to introduce some notations to make formulas more readable and intuitive.

Denote by M_{strong} and M_{weak} the following quantities

$$M_{strong} = \frac{B}{(K-1)} \\ M_{weak} = \frac{B}{KNc_{weak}}. \tag{1}$$

[6]We multiply the number of weak-weak connections by $N/(N-1)$ which converges to 1 as we increase N.

These quantities come from $(K-1)p_ip + KNq_iqc_{weak} = B$, where we replaced c_{strong} with 1. Because all effort levels are non negative, M_{strong} and M_{weak} are the upper bounds for p_ip and q_iq respectively. These bounds are achieved when the probability of having another type of connection is equal to 0. So, if player i decides not to have any acquaintances, $q_i = 0$, then $p_ip = M_{strong}$ and she has $M_{strong}(K-1)$ friends in expectation. Similarly, if she decides not to have any friends, $p_i = 0$, then $q_iq = M_{weak}$ and she is going to have $M_{weak}KN$ weak links in expectation. But the most helpful part of this notation is that we can express q_iq in terms of p_ip in a short form as $q_iq = M_{weak} - \frac{(K-1)}{KNc_{weak}}p_ip$.

Now, if we substitute q_iq in $\widetilde{U}_N$ using the equation above we get the limit uitlity function U_∞

$$\begin{aligned}U_\infty =& 1 + (K-1)p_ip + \frac{H}{c_{weak}}(K-1)(M_{strong} - p_ip) + (K-1)(1-p_ip)(1-(1-p_ip^3)^{K-2}) + \\ &+ \frac{H}{c_{weak}}(K-1)^2\left(p_ip(M_{strong} - p^2) + p^2(M_{strong} - p_ip)\right) + \\ &+ \frac{H^2}{c_{weak}^2}(M_{strong} - p_ip)(M_{strong} - p^2)(K-1)^2.\end{aligned}$$

As we can see, U_∞ does not depend on N, which conforms with its name. Denote by p_∞ an equilibrium (trivial or non trivial) of U_∞ and by p_N an equilibrium of U_N. We show that U_N uniformly converges to U_∞. This, in its turn, implies that for any p_∞ there is a p_N within the ε-neighborhood of it and vice versa.

Proposition 1. *For any $\varepsilon > 0$ $\exists\, \bar{N}$ such that $\forall\, N > \bar{N}$ and any fixed K*

$$|U_\infty(p_i, q_i, p, q) - U_N(p_i, q_i, p, q)| < \varepsilon$$

Furthermore, $\forall\, \varepsilon > 0$ $\exists\, \bar{N}$ such that $\forall N > \bar{N}$ and for any equilibrium of U_∞, p_∞, there exists an equilibrium of U_N, p_N, within the ε-neighborhood of p_∞. Moreover, there are not any equilibria of U_N outside of those neighborhoods.

From now on, we are going to assume that the agents' utilities are represented by the latter one, study the corresponding equilibria and then connect them back to the ones of the initial function. Replacing U_N with U_∞ leads to the following objective function of player i:

$$\max_{p_i} \Bigg(1 + (K-1)p_i p + \frac{H}{c_{weak}}(K-1)(M - p_i p) + (K-1)(1-p_i p)(1-(1-p_i p^3)^{K-2}) + \\ + \frac{H}{c_{weak}}(K-1)^2 \left(p_i p(M-p^2) + p^2(M-p_i p)\right) + \\ + \frac{H^2}{c_{weak}^2}(M-p_i p)(M-p^2)(K-1)^2 \Bigg)$$

$$\text{subject to:} \quad (K-1)p_i p + KNq_i q\, c_{weak} = B, \\ 1 \geq p_i \geq 0, \\ 1 \geq q_i \geq 0.$$

Remember, that in order to find symmetric equilibria we need to find solutions to $(\partial U_\infty / \partial p_i)\,|_{p_i = p} = 0$. We already mentioned that there are two trivial equilibria. The first main result that we have is that there is only one non-trivial symmetric Nash equilibrium of U_∞, which we call p^*. It is also an ε-equilibrium of the initial game[7]. As a consequence of this and the previous proposition, there is an equilibrium of the initial utility function, U_N, in ε-neighborhood of p^*.

Theorem 2. *There is a unique equilibrium of U_∞, p^*, if:*

- $\frac{H}{c_{weak}} < 1$
- $\frac{H}{c_{weak}} > e^{-\frac{B^2(K-3)}{(K-1)^2}} \left(1 - \frac{B}{K-1}\right)$

Otherwise, there are only trivial equilibria.

Because there are more potential acquaintances available (NK) than friends $(K-1)$ we expect that the latter graph will be more clustered than the other. This leads to the following trade-off. When choosing between having another strong or weak tie, agents need to take two things into account: 1) how much utility does a new connection bring at distance 1 and 2) how much does it bring at distance 2. The first one asks about the relationship between c_{strong}, c_{weak} and π_w to determine which type of link is more appealing if we were only getting utility from our *direct* ties (the ones that are at distance 1). The second one, more subtle, asks how many new connections does the new tie bring to our current neighborhood.

For example, if it is more beneficial at distance 1 to have a friend we still might not want to spend all our time on the strong ties. Because, at some point, our friends' graph will be so clustered that a new strong connection will not significantly change our neighborhood[8], i.e.

[7]We leave this as an exercise.

[8]And, as a consequence, our expected utility.

friends of my new friend are likely to be either my friends or friends of my friends already. Therefore, it is more beneficial to trade this new strong connection for a few weak ones, which will increase our utility by indirectly connecting us to people we do not know yet[9]. Alternatively, consider the case when we only have acquaintances and assume that we trade some of them for a few friends. The friends' graph is sparse at this point[10] so there is no clustering disadvantage that was described above. Furthermore, not only strong links are more beneficial at distance 1, but they also give us access to their own friends and acquaintances through a very *reliable* connection. So it is not optimal to have only one type of link in the equilibrium. Vadim: maybe make a visualization, picture here

The uniqueness of a non-trivial equilibrium is a nice result as there is no ambiguity about which outcome is going to be implemented by society. This theorem tells us that unless the price for maintaining weak connections is extreme, we should observe both, weak and strong, connections between people.

Two constraints above are necessary for the existence of a nontrivial equilibrium. The first condition ensures that the equilibrium function, $\partial U_\infty / \partial p_i \Big|_{p_i=p}$ is nonnegative around $p = 0$. The second one, on the other hand, makes sure that it becomes negative at some point on $[0, \sqrt{M_{strong}}]$. Therefore, $\partial U_\infty / \partial p_i \Big|_{p_i=p}$ crosses 0, as it is continuous.

But there is more to these constraints than the technical explanations. If c_{weak} is smaller than π_w then there are only trivial equilibrium solutions. The reason why the nontrivial one disappears is that weak connections are not only more appealing at distance 2 as their graph is more disperse, hence they can potentially bring in more people in our network, but they are also very beneficial at distance 1. A direct benefit from a friend is $1 - c_{strong} = 0$ and from an acquaintance is $\pi_w - c_{weak} > 0$. So in a case when $\pi_w / c_{weak} > 1$, it is rational to disregard strong connections and focus only on the weak ones.

Now let us look at the second constraint. Let us fix the left-hand side and change K first. When K is significantly big, the friends' graph becomes very sparse as the village gets larger, and the right-hand side of the constraint approaches 1. This requires c_{weak} to be close to π_w to make weak connections more attractive. Otherwise, we prefer to spend our budget only on friends. Which leads to the other trivial

[9]Because the acquaintances' graph is sparser than the friends' one.

[10]We do not have any friends yet, so from our point of view it is sparse/there is no clustering.

equilibrium. The same thing occurs if B is small. However, if we do have a lot of time, B is large, then the right-hand side decreases, which relaxes the bound on c_{weak} and allows it to be bigger. When we have a lot of time, keeping c_{weak} fixed, our friends' graph will become very clustered at some point and we would like to have a few acquaintances to maximize the effective size of our neighborhood.

The second part of Theorem 2 validates our choice of the approximation function. It shows that there is a nontrivial symmetric initial equilibrium (possibly multiple ones) within ε-neighborhood of the approximate equilibrium. Thus, we can learn about its behavior by analysing p^*.

While proving this proposition, we find that *weak-strong*, *strong-weak* and *weak-weak* ties constitute the bulk of agent's utility. It does make sense if we think about it. Imagine that you are looking for a job at Google and you happen to know someone who works there. Why is this connection valuable to you? Is it because exactly this person is going to offer you a job? This is very unlikely. But she has many colleagues and one of them might give you an offer after she recommends you to him. So it makes a lot of sense to look at the number and the types of connections we have not only at distance 1, but also at distance 2.

Let us now analyze this approximate equilibrium p^* and see what happens to it as we change the parameters of the game. Does p^* increase when we increase the cost of maintaining weak ties, c_{weak}, or increase the number of people in the village, K? For this, we have the following comparative statics result.

Proposition 3. *As we do one of the following:*

- ***increase*** *the cost of weak ties,* c_{weak}*;*
- ***decrease*** *the probability of information traveling through a weak tie,* π_w*;*
- ***decrease*** *the number of people in the village,* K

then the equilibrium effort level for making friends, p^**,* ***increases.***

The first two cases are very intuitive: as we make a weak tie option more attractive we trade some of our friends for acquaintances, as we decrease p^* and, hence, decrease the expected number of strong ties. But the last case, when we change K, seems more interesting in some sense. We have some fixed time budget that we can spend on our connections, so increase in K implies that the upper bound on p, M_{strong}, has to decrease. So it is not surprising that the equilibrium value p^* also decreases. However, the expected number of friends increases when we increase K as we will show in the next section.

Let us give an intuition for this result. Remember that Erdős - Rényi random graphs look like trees for small values of p and are significantly clustered for large values. If the latter one occurs, we have this trade-off between strong and weak ties: strong ties might give you higher utility at distance 1 but are more redundant at distance 2 because of the clustering. When we increase K, M_{strong} decreases which forces p to stay small enough so that the friends' graph does not get very clustered. Hence the strong ties are more attractive than they were before because now they do not have (or have less) disadvantage at distance 2: they are also fairly sparse and bring more diverse information from distance 2. This is why the expected number of friends increases: p^{*2} does decrease but not as fast as K increases, so their product, $(K-1)p^{*2}$, increases.

There are two more questions that we want to answer in this section. First, in equilibrium, is friends' graph more clustered than the acquaintances' graph? Second, how does the socially optimal symmetric network differ from the equilibrium network? Should we have more friends or acquaintances to maximize the social welfare?

Let us start with the former one. We talked about the trade-off that we observe in this model: how much utility each type of link gives us at distance 1 vs how much it brings at distance 2 (how clustered is the graph for this connection type). We assumed that the friends' graph is more clustered than the acquaintances' one. But we did not, technically, force the CC for the former graph to be higher than the CC for the latter one. The following lemma makes sure that in the equilibrium the friends' network is indeed more clustered.

Lemma 4. *If $c_{weak} > \pi_w \frac{B}{B-1} + \frac{(K-1)(B+1)}{K(B-1)N}$ then the clustering coefficient of the acquaintances' graph is smaller than the CC of the friends' graph.*

Thus, when c_{weak} is a bit bigger than its general bound, π_w (from Theorem 2), friends' graph is indeed more clustered. This tells us that our equilibrium graph mimics a very essential property of the real life networks.

Now that we have a good idea of how our equilibrium looks like and behaves we can compare it with another network, the socially optimal one. To elaborate: what would be a socially efficient symmetric random network if we could tell people what strategy to play? In mathematical terms, what if we apply symmetry to $U_\infty(p_i, q_i(p_i), p, q)\big|_{p_i=p}$ and then maximize it with respect to p? Would we have more acquaintances or friends compared to the equilibrium graph? Would the new graph look very different from the equilibrium network? We have a very interesting result that the former one is not going to be dramatically different (solution probabilities and the equilibrium functions are going to be similar) but we are going to have more friends than in the latter one. So in a utopian world, where everyone can coordinate what to do, people should have more friends than in the equilibrium.

Theorem 5. *Let $p_{optimal}$ be a solution to $U'_\infty(p, q(p), p, q(p)) = 0$. If there exists an equilibrium p^* and*

- $\pi_w/c_{weak} \geq (1 - p^{*4})^{K-2}$

then $p_{optimal} > p^*$. In other words, there are more friends in the optimal network than in the equilibrium one.

The intuition for this result is that if we want to maximize social welfare we not only care about getting the signal but also about sharing it with others, and the strong ties are more reliable sources of information.

This completes our analytical analysis and in the next section, we are going to present some graphs and quantitative results for the equilibrium.

Quantitative results

In this sections we are going to show a few more interesting properties of the equilibrium and provide some graphs for visualisation.

We will start with a graph of how many connections of different types people have in equilibrium, Figure 1. Recall that we fixed c_{strong} to be equal to 1 and vary π_w, c_{weak}. As we saw in the last section we care about their relative value towards each other. Let us fix $\pi_w = 0.4$ and vary c_{weak}.

Theorem 2 tells us that c_{weak} has to be bigger than π_w so we choose c_{weak}-axis to be from 0.5 to 1.5.

As we can see weak-strong and strong-weak ties (multiplied by π_w) correspond to the bulk of all agent's connections. Hence, they constitute the most utility out of all other types of connecitons. To have weak-strong and strong-weak connections present in the network agents have to have a positive number of both weak and strong individual ties. This gives an explanation why we observe both types of connections in a real life, even though it might look unintiuitive at first.

The second biggest contriubtors to agent's utility are either weak-weak or strong-strong connections depending on the parameters of the model. As we see, most of agent's utility comes not from her direct links but from the indirect ones, which makes sense if we recall an example with Google's job offer after Theorem 2.

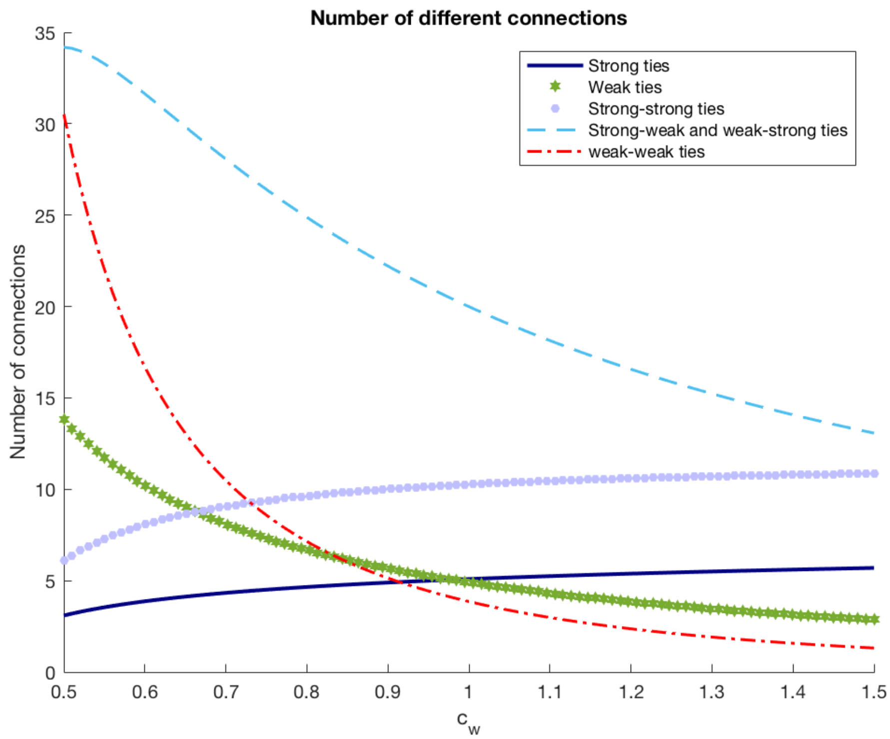

FIGURE 1. Number of different ties (normalized by π_w appropriately) people have in equilibrium when $K = 20$, $N = 100$, $c_{strong} = 1$, $\pi_w = 0.4$, $c_{weak} \in (0.45, 1.55)$.

Notice, how the number of friends vanishes as c_{weak} becomes closer to π_w. It appears almost impossible to not satisfy the other constraint (when the weak connections disappear in equilibrium) of Theorem 2 as c_{weak} would have to be bigger than 209 (keeping other parameters the same).

The second property that we would like to show is mentioned after the Propositions 3. We proved that as we increase the number of people in each village, K, the equilibrium probability of making a new friend, p^{*2}, decreases. However, the number of friends that an agent ends up having in equilibrium increases. Figure 2 illustrates these results.

As we said before, p^2 intuitively should decrease because its upped bound, M_{strong}, decreases when K increases. But the former one does not do it quick enough relatively to the increase in K and, as a result of this, their product, $(K-1)p^{*2}$, increases.

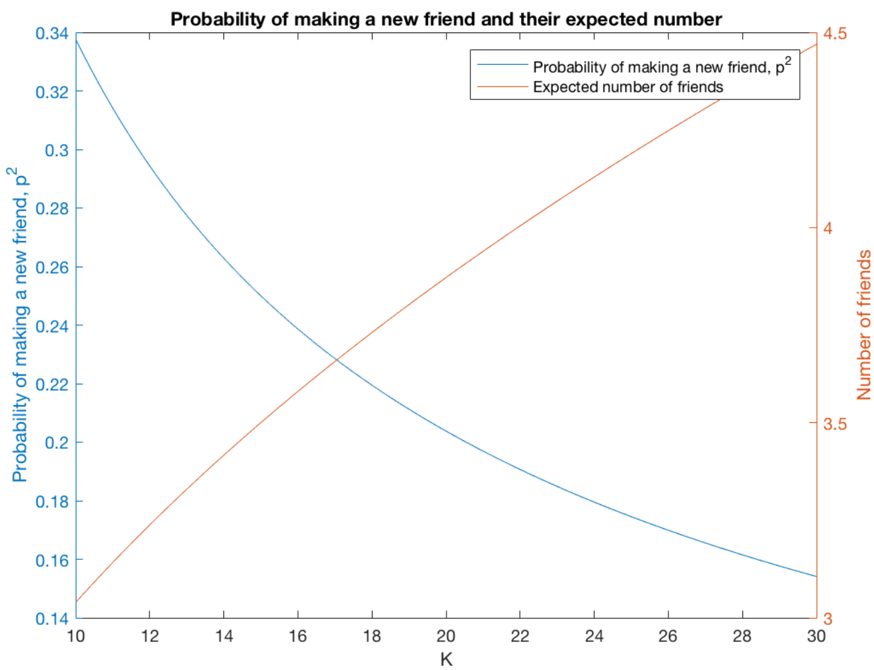

FIGURE 2. Value of p^{*2} as we change the number of people in the village, K.

When we increase the number of people in the village, keeping the budget fixed, the friends graph becomes sparser. Therefore, strong links have less disadvantage at distance 2. This motivates people to trade some of their acquaintances for new friends.

The last graph we would like to present compares the equilibrium and the optimal networks. In Theorem 5 we proved that it is socially optimal for all agents to increase the number of friends. In Figure 3 we can see that $p_{optimal}$ is indeed bigger than p^*. Interestingly, these values are very close to each other, so our equilibrium network does not differ a lot from the optimal one. This implies that society on its own can achieve a fairly efficient outcome without any interference from the social planner. This is a very positive result.

4. Conclusion

In this paper, we find necessary and sufficient conditions for the existence of a non-trivial equilibrium in which players choose both weak and strong ties. The reason why it is optimal for agents to choose both types of connections is because the bulk of their utility comes

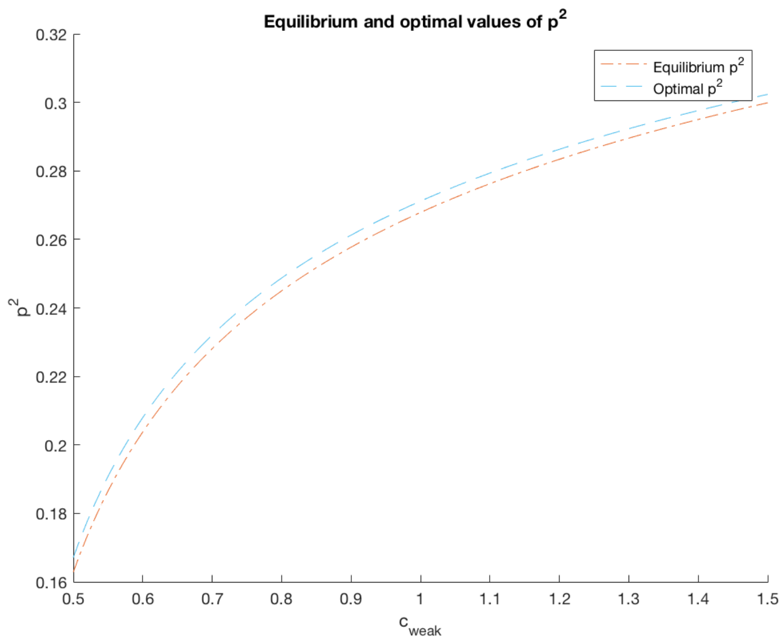

FIGURE 3. Value of p^{*2} and the number of strong connections in the equilibrium and optimal networks as we change the cost of weak ties, c_{weak}.

from weak-strong and strong-weak ties, which require positive amounts of both types of links to be present in the network. This provides an explanation for why we observe strong and weak ties in real life at the same time even though it might seem unintuitive at first. In the equilibrium (under a mild condition), the friends' graph is more clustered than the acquaintances' one, which complies with empirical evidence. We also provide comparative statics of the equilibrium.

Furthermore, we compare the equilibrium network with the socially optimal symmetric one. These networks are surprisingly similar, but in the latter, agents have more friends. Intuitively, when maximizing social welfare, agents care not only about receiving the information but also sharing it with others. And the strong ties are more reliable in this case.

We would also like to note a few other aspects of this paper. There are two main types of network models. The first one works with random graphs to represent society and ties between people. Whereas the second one uses game theory to make sure every link is consensual by both sides. Models of the latter type often require either a lot of symmetry from the network or simplifying assumptions due to very complex combinatorics issues. They also produce multiple equilibria, some of which do not make sense form a network perspective. At the same time, the network does not appear completely randomly but depends on agents' decisions and choices. In this paper, similar to Golub and Livne in [4], we are bringing these two approaches together as well as forming a bridge between sociological and economic literature.

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Appendix A. First

Before we prove Proposition 1 we need a few auxiliary results. As we said above, when the number of villages increases, the probability that there are multiple connections between i and some other person j of length at most 2, that include at least one weak link, vanishes. Furthermore, conditioning on the complement of this event does not change our expected utility at the limit as we show in the following proposition.

Proposition 6. *For a fixed K denote by A^c an event that there are multiple pathes, that include weak tie(s), between agent i and some other player of length 1 or 2 or there is a weak-weak connection that ends in village $\mathcal{V}_i$ in the realized graph (V, E). Then*

$$\lim_{N\to\infty} \mathbb{P}(A^c) = 0$$
$$\lim_{N\to\infty} \Big| U_N(p_i, q_i, p, q)|A - U_N(p_i, q_i, p, q) \Big| = 0$$

Proof of Proposition 6. Before we get to the event A, let us consider an auxiliary event A_0 that player i has no more than S_{weak} acquaintances. We know that the expected number of strong and weak connections are bounded by B and B/c_{weak} respectively because of the budget constraint. Let us prove that the probability that there are more than $S_{weak} = N^{\frac{1}{6}}$ weak connections goes to 0 as N increases. For agent i there are NK potential acquaintances and each link realizes independently with probability $q_i q$. It means that a random variable Z that equals the number of realized weak connections is distributed binomially with parameters $(NK, q_i q)$. There is the following well-known bound for the upper tail of the binomial distribution:

$$\mathbb{P}(Z \geq r) \leq exp\left(-NKD\left(\frac{r}{NK}\,\Big|\Big|\, q_i q\right)\right),$$

where $D(a||p) = a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p}$ is a relative entropy between $Binomial(a)$ and $Binomial(p)$ and $NKq_i q \leq r \leq NK$. Notice, that $q_i q \leq B/(NKc_{weak})$. Hence,

$$\begin{aligned}
\mathbb{P}(Z \geq N^{\frac{1}{6}}) &\leq exp\left(-NK\left(\frac{N^{\frac{1}{6}}}{NK}\log\left(\frac{\frac{N^{\frac{1}{6}}}{NK}}{q_i q}\right)+\left(1-\frac{N^{\frac{1}{6}}}{NK}\right)\log\left(1-\frac{N^{\frac{1}{6}}}{NK}\right)\right)\right)\\
&\leq exp\left(-\left(N^{\frac{1}{6}}\log\left(\frac{\frac{N^{\frac{1}{6}}}{NK}}{q_i q}\right)+\left(NK-N^{\frac{1}{6}}\right)\left(-\frac{N^{\frac{1}{6}}}{NK}+O\left(\frac{1}{N^{\frac{5}{3}}}\right)\right)\right)\right)\\
&\leq exp\left(-\left(N^{\frac{1}{6}}\log\left(\frac{N^{\frac{1}{6}}}{\frac{B}{c_{weak}}}\right)+NK\left(-\frac{N^{\frac{1}{6}}}{NK}+O\left(\frac{1}{N^{\frac{5}{3}}}\right)\right)\right)\right)\\
&\leq exp\left(-N^{\frac{1}{6}}\right).
\end{aligned}$$

In the first inequality we used the fact that $\log(1-a)/(1-p)$ is negative, so decreasing p to 0 only increases the right-hand side. In the second line we used Taylor series for $\log(1-x)$ for small x. To get the third inequality recall that $q_i q \leq B/(NKc_{weak})$. Thus,

$$\mathbb{P}(Z < N^{\frac{1}{6}}) \geq 1 - exp^{-N^{\frac{1}{6}}},$$

which converges to 1 as N increases. Let us condition on the event, A_0, that there are at most $S_{weak} = N^{\frac{1}{6}}$ ties. Then the number of weak, weak-strong, strong-weak and weak-weak ties is bounded by $S_{weak} + 2S_{weak}B + S_{weak}^2 < 3N^{\frac{1}{3}}$. Denote this bound by $\bar{S}$. We are now going to show that the probability that these pathes do not overlap with each other goes to 1 as we increase the number of villages N.

For each path we are going to pick 1 out of N villages where it ends[11]. Note that if all these $\bar{S}$ villages that we chose are distinct then none of the pathes can overlap with each other. Denote by X a number of times that we pick some village that was already chosen before. Let us calculate the expected value of this random variable. We are going to choose villages sequentially with replacement. For the n-th pick denote by $\mathbb{1}_n$ an indicator function which equals to 1 if the n-th village we choose has already been chosen before. Then the expectation of X equals to the expected sum of these indicator functions from 1 to $\bar{S}$.

The first path can not be assigned to the village that was already chosen, as it is the first one. The second one will have the same village as the first path with probability $1/N$. For the third one, the probability is less than $2/N$. Let us eleborate this part. The first two pathes can be either in the same or in different villages with some probabilities, p_1 and $1-p_1$. Then, the probability the third one is in the same village

[11]For each weak-weak path we choose two villages: one for the first weak link and one for the second one.

as one of the first two is $p_1 1/N + (1-p_1)2/N < 2/N$. The analogous argument is applied to the subsequent ones as well. Therefore,

$$\mathbb{E}(X) \leq 0 + \frac{1}{N} + \frac{2}{N} + \cdots + \frac{\bar{S}-1}{N} = \frac{\bar{S}(\bar{S}-1)}{2N}$$

Hence,

$$\frac{S(S-1)}{2N} \geq \mathbb{E}(X) = \sum_{i=1}^{k} \mathbb{P}(X=i)i \geq \sum_{i=1}^{k} \mathbb{P}(X=i) = 1 - \mathbb{P}(X=0)$$

Thus,

$$\mathbb{P}(X=0) \geq 1 - \frac{\bar{S}(\bar{S}-1)}{2N} > 1 - \frac{4N^{\frac{2}{3}}}{2N} = 1 - \frac{2}{N^{\frac{1}{3}}}$$

which goes to 1 as N increases. Let us call the event when $X = 0$, A_1 and its complement A_1^c.

Now we will calculate the probability of the event that there is at least one weak-weak path that comes back to village $\mathcal{V}_i$. Denote by $n_w^{\mathcal{V}_i}$ the number of weak-weak pathes that end up in $\mathcal{V}_i$ and by A_2 an event that $n_w^{\mathcal{V}_i} = 0$. This means that none of the acquaintance of i has an acquaintance in i's village. If everyone has at most $\bar{S}$ acquaintances and each of them has at most $\bar{S}$ weak links then the probability of A_2^c, that at least one weak-weak path comes back to $\mathcal{V}_i$, is equal to

$$\mathbb{P}(n_w^{\mathcal{V}_i} > 0) = 1 - \mathbb{P}(n_w^{\mathcal{V}_i} = 0) = 1 - \left(1 - \frac{1}{N}\right)^{\bar{S}^2},$$

which converges to 0 as $N \to \infty$. Then the probability of either events A_1^c or A_2^c happening is less than or equal to the sum of their corresponding probabilities.

$$\mathbb{P}(A_1^c \cup A_2^c) \leq \mathbb{P}(A_1^c) + (A_2^c) \leq 1 - \left(1 - \frac{1}{N}\right)^{\bar{S}^2} + \frac{\bar{S}(\bar{S}-1)}{2N},$$

which goes to 0 as N goes to ∞. Therefore, $\mathbb{P}(A_1 \cap A_2)$ converges to 1.

Notice, that we calculated probabilities of $A_1 \cap A_2$ conditioned on A_0. But because all their corresponding probabilities converge to 1 the limit of the probability of the unconditional event $A_1 \cap A_2$ is also 1.

$$\lim_{N\to\infty} \mathbb{P}(A_1 \cap A_2) = \lim_{N\to\infty} \Big(\mathbb{P}(A_1 \cap A_2 | A_0)\mathbb{P}(A_0) + \mathbb{P}(A_1 \cap A_2 | A_0^c)\mathbb{P}(A_0^c)\Big) = 1.$$

Now we will calculate the absolute difference in expected utility, U_N, when we condition on $A = A_1 \cap A_2$ and when we do not.

$$
\begin{aligned}
|\mathbb{E}(U|A)-\mathbb{E}(U)| &= \left|\frac{\mathbb{E}(U\mathbb{1}_A)}{1-\mathbb{P}(A^c)}-\mathbb{E}(U)\right| \\
&= \left|\frac{\mathbb{E}(U)\mathbb{P}(A^c)-\mathbb{E}(U\mathbb{1}_{A^c})}{1-\mathbb{P}(A^c)}\right| \\
&\leq 4||U||_\infty \mathbb{P}(A^c),
\end{aligned}
$$

as the denominator is bigger than $1/2$ and the numerator is less than $2||U||_\infty \mathbb{P}(A^c)$. The right-hand side converges to 0, because U is bounded and $\mathbb{P}(A^c)$ goes to 0 as N increases. Therefore, we can use our approximation of the utility function.

□

Now, if we calculate $U_N|A$ we will almost have U_∞. Recall, that when we condition on the event A, there are only 5 groups of people that affect i's utility: $N_1^s(i)$, $N_1^w(i)$, $N_1^{ss}(i)$, $N_2^{ws}(i)$, N_2^{ww}. The following proposition calculates the expected number of people in these different types of neighborhoods and $U_N|A$ itself.

Proposition 7.

$$
\begin{aligned}
U(p_i,q_i,p,q)|A = \Big(&1+(K-1)p_ip+HKNq_iq+(K-1)(1-p_ip)(1-(1-p_ip^3)^{K-2})+ \\
&+H\left(p_ipq^2K(K-1)N+p^2q_iqK(K-1)N\right)+H^2q_iq^3K^2N(N-1)\Big)
\end{aligned}
$$

Proof of Proposition 7. Before we calculate the expected number of people in different groups we need some notation. Denote by

- $N_1^s(i)$ – a set of strong connections of i;
- $N_1^w(i)$ – a set of weak connections of i;
- $N_2^{ss}(i)$ – a set of strong-strong connections of i, that are not in $N_1^s(i)$;
- $N_2^{ws}(i)$ – a set of weak-strong or strong-weak connections of i;
- $N_2^{ww}(i)$ – a set of weak-weak connections of i.Let us proceed.

Expectation is a linear operator, hence we can calculate the expected number of people for each of those five groups seperately. Let us start with $N_i^s(i)$. There are $K-1$ people in the village besides player i and she is going to connect to each of them with probability p_ip. Therefore, $\mathbb{E}_{(V,E)}|N_i^s(i)|=(K-1)p_ip$. Analogously, $\mathbb{E}_{(V,E)}|N_i^w(i)|=KNq_iq$.

In order for an agent to be in $N_2^{ss}(i)$ she can not be connected directly to i but there has to be exactly one person between them. The probability that player j is not connected to i through person k (i,j,k are in the same village $\mathcal{V}_i$) is $(1-p_ip_kp_kp_j)$. As all edges appear (or not) independently of each other, the probability that i is connected to j at distance 2 is

$$1 - \prod_{\substack{k \in \mathcal{V}_i \\ k \neq i,j}} (1 - p_i p_k p_k p_j).$$

The subtrahend is the probability that i is not connected to j through any player k in the same village. After we apply symmetry (every player, except i, plays strategy (p, q)) and remember that j can not be at distance 1 from i, i.e. there can not be a strong edge between i and j, we get the desired formula:

$$\mathbb{E}_{\{p_t,q_t\}}|N_2^{ss}(i)|= (K-1)(1-p_i p)(1-(1-p_i p^3)^{K-2})$$

Now let us calculate the expected number of strong-weak connections. For each individual j who is a firend of player i, $i, j \in \mathcal{V}_i$, we need to calculate how many acquaintances j has, n_j^w, and add them up.

$$\mathbb{E}\left(\sum_{j \in \mathcal{V}_i} \mathbb{1}_{e_{ij} \in E_s} n_j^w\right) = \sum_{j \in \mathcal{V}_i} \mathbb{E}\left(\mathbb{1}_{e_{ij} \in E_s} n_j^w\right) = \sum_{j \in V_i} \mathbb{E}\left(\mathbb{1}_{e_{ij} \in E_s}\right) \mathbb{E}\, n_j^w = p_i p(K-1) q^2 K N.$$

In the equations above we used linearity of expectation and the fact that $\mathbb{1}_{e_{ij} \in E_s}$ and n_j^w are independent, hence, expectation of their product is the product of their expectations.

For the weak-strong and weak-weak connections we can do the analogous calculations. Denote by n_k^s the number of strong friends that player k,$k \notin \mathcal{V}_i$, has.

$$N_2^{ws}(i) = \mathbb{E}\left(\sum_{k \notin \mathcal{V}_i} \mathbb{1}_{e_{ik} \in E_w} n_k^s\right) = \sum_{k \notin \mathcal{V}_i} \mathbb{E}\left(\mathbb{1}_{e_{ik} \in E_w} n_k^s\right) = \sum_{k \notin \mathcal{V}_i} \mathbb{E}\left(\mathbb{1}_{e_{ik} \in E_w}\right) \mathbb{E}\, n_k^s = K N q_i q (K-1) p^2,$$

where n_k^s and $\mathbb{1}_{e_{ik} \in E_w}$ are independent.

For the weak-weak links denote by n_k^w the number of acquaintances player $k \notin \mathcal{V}_i$ has.

$$N_2^{ww}(i) = \mathbb{E}\left(\sum_{k \notin \mathcal{V}_i} \mathbb{1}_{e_i k \in E_w} n_k^w\right) = \sum_{k \notin \mathcal{V}_i} \mathbb{E}\left(\mathbb{1}_{e_i k \in E_w}\right) \mathbb{E}\, n_k^w = N K q_i q (N-1) K q^2.$$

In the last equation we have $N(N-1)$ instead of N^2 because in the event A weak-weak links do not come back to the same village that player i is. This means that i's acquaintances can only pick from $N-1$ other villages to create weak-weak connections.

Recall that because we are conditioning on event A then none of these pathes, that we calculated above, overlap with each other. Now we just subsitute these calculations into the objective function to complete the proof.

□

Now we are ready to prove Proposition 1.

Proof of Proposition 1. From Proposition 6 we know that U_N uniformly converges to $U|A$. Define L to be $M_{weak}(K-1)/B$. Then we have $q_i q = L(M_{strong} - p_i p)$. Notice that

$$\begin{aligned} U|A = \Big(& 1 + (K-1)p_i p + \pi_w KNL(M_{strong} - p_i p) + (K-1)(1-p_i p)(1-(1-p_i p^3)^{K-2}) + \\ &+ \pi_w \left(p_i p L(M_{strong} - p^2)K(K-1)N + p^2 L(M_{strong} - p_i p)K(K-1)N\right) + \\ &+ \pi_w^2 L(M_{strong} - p_i p)L(M_{strong} - p^2)K^2 N(N-1)\Big) \\ =& 1 + (K-1)p_i p + \frac{\pi_w}{c_{weak}}(K-1)(M_{strong} - p_i p) + (K-1)(1-p_i p)(1-(1-p_i p^3)^{K-2}) + \\ &+ \frac{\pi_w}{c_{weak}}(K-1)^2 \left(p_i p(M_{strong} - p^2) + p^2(M_{strong} - p_i p)\right) + \frac{\pi_w^2 (N-1)}{c_{weak}^2 N}(M_{strong} - p_i p)\times \\ &\times (M_{strong} - p^2)(K-1)^2 \\ =& U_\infty - \frac{\pi_w^2}{c_{weak}^2}\frac{1}{N}(M_{strong} - p_i p)(M_{strong} - p^2)(K-1)^2 \\ =& U_\infty + O\left(\frac{1}{N}\right), \end{aligned}$$

where in the first equation we used notation from (1).

This means that $U|A$ uniformly converges to U_∞. Therefore, U_N also uniformly converges to U_∞.

Let us show that equilibrium of U_N, p_N, can only be within ε-neighborhood of the equilibrium of U_∞. First, notice that U_N and U_∞ both have two trivial equilibria: 1) all players invest only in friends and 2) all players invest only in acquaintances.

Now let us deal with the non trivial one. For each p, $U_\infty(p_i, q_i(p_i), p, q(p))$ has a unique maximum with respect to p_i, as it is a concave function. Denote by $f(p)$ the value of p_i at which the corresponding maximum is attained. This is a continuous function as U_∞ is a polynomial of a fixed degree of p_i and p, hence, a small change in p will require a small change in p_i to maintain $\partial U_\infty/\partial p_i = 0$. Then $h(p) = U_\infty(p, q(p), p, q(p)) - U_\infty(f(p), q(f(p)), p, q(p))$ is also a continuous function due to a triangle inequality. Therefore, h attains its minimum, δ, on $[\varepsilon, p^* - \varepsilon] \cup [p^* + \varepsilon, \sqrt{M_{strong}} - \varepsilon]$ which is positive. Hence, for $\bar{N}$

big enough such that $U_\infty - U_N < \delta/3$ for $N > \bar{N}$ there are no equilibria of the initial utility function ouside of those neighborhoods of the equilibria of U_∞. This concludes one direction.

Now let us show the other direction. There are three equilibria of U_∞: two trivial and one non trivial. Also, U_N has the same two trivial equilibria. Let us prove that there is a non trivial equilibrium of U_N in some negihborhood of p^*. Let $p^1 = p^* - \varepsilon$ and $p^2 = p^* + \varepsilon$. For both p^1 and p^2 let us find N_1 and N_2 such that the maximums of $U_\infty(p_i, q_i(p_i), p^1, q(p^1))$ and $U_\infty(p_i, q_i(p_i), p^2, q(p^2))$ with respect to p_i are within the δ-neighborhood of the maximums of $U_N(p_i, q_i(p_i), p^1, q(p^1))$ and $U_N(p_i, q_i(p_i), p^2, q(p^2))$ correspondingly. Notice, that $U_N(p_i, q_i(p), p, q(p))$ can not have maximums outside of these neighborhoods as $U_\infty(p_i, q_i(p), p, q(p))$ is single-peaked for a fixed p with respect to p_i. Pick δ to be equal to $\min(h(p^1), h(p^2))$. We can do this because U_N uniformly converges U_∞. Let us pick $N_0 = \max(N_1, N_2)$. Then we know that at $p = p^1$ the maximum of $U_N(p_i, q_i(p_i), p^1, q(p^1))$ is to the right of $p_i = p^1$[12] and at $p = p^2$ the maximum of $U_N(p_i, q_i(p_i), p^2, q(p^2))$ is to the left of $p_i = p^2$ for all $N > N_0$. Hence, at some point $p^1 \le p_N \le p^2$ the maximum of $U_N(p_i, q_i(p_i), p, q(p))$ crosses the line of $p_i = p$. This is a symmetric non trivial equilibrium of U_N in p^*'s the neighborhood.

Therefore, there are equilibria of U_N within the ε-neighborhood of (trivial and non trivial) equilibria of U_∞. Moreover, there are no equilibria of the initial utility function outside of those neighborhoods.

□

Proof of Theorem 2. In order to prove this theorem we need to show that $(\partial U_\infty/\partial p_i)\Big|_{p_i=p} = 0$ has a unique non trivial solution.

$$\begin{aligned}
\left.\frac{\partial U_\infty}{\partial p_i}\right|_{p_i=p} =& p{\pi_w}^2K^2N^2L^2\left(p^2-(M_{strong}-p^2)\right)+p\pi_w(K-1)KNL\left((M_{strong}-p^2)-p^2\right)-\\
&-\pi_w(K-1)KNp^3L-p\pi_wKNL+p(K-1)\left(1-p^4\right)^{K-2}+\\
&+(K-2)(K-1)p^3\left(1-p^2\right)\left(1-p^4\right)^{K-3}\\
=&(K-2)(K-1)p^3\left(1-p^2\right)\left(1-p^4\right)^{K-3}+p(K-1)\left(1-p^4\right)^{K-2}+\\
&-\pi_wKNLp\Big(1-(K-1)(M_{strong}-2p^2)+\pi_wKNL(M_{strong}-p^2)\Big)\\
=&(K-2)(K-1)p^3\left(1-p^2\right)\left(1-p^4\right)^{K-3}+p(K-1)\left(1-p^4\right)^{K-2}+\\
&-\frac{\pi_w}{c_{weak}}(K-1)p\left(1-(K-1)(M_{strong}-2p^2)+\frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^2)\right).
\end{aligned}$$

[12] $\partial U_\infty/\partial p_i|_{p_i=p}$ is positive at $p = p^1$ and negative at $p = p^2$.

Notice, that we can do this if $q_i q > 0$, otherwise, all terms that include a weak tie become 0 and we do not differentiate them. When $q_i q = 0$, U_∞ it is a trivial equilibrium of U_∞, as well as of U_N, to choose $p_i = \sqrt{M_{strong}}\ \forall\, i$. Denote $(\partial U_\infty / \partial p_i)\Big|_{p_i=p}$ by F.

$$
\begin{aligned}
F =&(K-2)(K-1)p^3\left(1-p^2\right)\left(1-p^4\right)^{K-3} + p(K-1)\left(1-p^4\right)^{K-2} + \\
&- \frac{\pi_w}{c_{weak}}(K-1)p\left(1-(K-1)(M_{strong}-2p^2) + \frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^2)\right) \\
=&T_1(p) + T_2(p).
\end{aligned}
$$

where T_1 is the term on the 1st line and T_2 – on the 2nd one. We will prove that F has a unique non trivial root.

Under the assumptions of the theorem, F has the following form: it is 0 at $p = 0$, has a positive derivative there (when $c_{weak} > \pi_w$), it is negative at $\sqrt{M_{strong}}$ (when $e^{\frac{-B^2}{K-1}}(1 - M_{strong}) < \pi_w / c_{weak}$) and crosses 0 only once on $(0, \sqrt{M_{strong}})$. Let us show this. It is clear that $F(0) = 0$, so we will skip it.

$$
\begin{aligned}
F'(0) =& (K-1)\Bigg(\frac{\pi_w\left(\pi_w(K-1)\left(3p^2 - M_{strong}\right) - c_{weak}\left(6(K-1)p^2 - KM_{strong} + M_{strong}\right)\right)}{c_{weak}^2} + \\
&+ \frac{1}{c_{weak}^2} - \left(1-p^2\right)^{K-3}\left(p^2+1\right)^{K-4}\left((K-1)(4K-7)p^6 + (6K-11)p^4 + (5-3K)p^2 - \right. \\
&\left. - 1\right)\Bigg)\Bigg|_{p=0} \\
=& (K-1)\left(\frac{\pi_w\left((K-1)M_{strong}(c_{weak}-\pi_w) - c_{weak}\right)}{c_{weak}^2} + 1\right) \\
=& (K-1)\left(\frac{\pi_w\left((K-1)M(c_{weak}-\pi_w) + c_{weak}\left(\frac{c_{weak}}{\pi_w} - 1\right)\right)}{c_{weak}^2}\right) \\
=& (K-1)\left(\frac{\pi_w\left((c_{weak}-\pi_w)\left((K-1)M_{strong} + \frac{c_{weak}}{\pi_w}\right)\right)}{c_{weak}^2}\right) \\
>& 0,
\end{aligned}
$$

as $c_{weak} > \pi_w$.

This implies that within some ε-neighborhood of $p = 0$, F is positive. Now we are going to show that at the other end, at $p = \sqrt{M_{strong}}$, it is negative.

$$
\begin{aligned}
F(\sqrt{M_{strong}}) =& (K-1)\Bigg(M_{strong}^{\frac{1}{2}}(1-M_{strong})^{K-2}(M_{strong}+1)^{K-3}((K-1)M_{strong}+1)- \\
&- \frac{\pi_w M_{strong}^{\frac{1}{2}}}{c_{weak}}(1+(K-1)M_{strong})\Bigg) \\
=& -(K-1)(1+(K-1)M_{strong})M_{strong}^{\frac{1}{2}}\Bigg(\frac{\pi_w}{c_{weak}} - (1-M_{strong}^2)^{K-3}\times \\
&\times (1-M_{strong})\Bigg) \\
=& -(K-1)(1+(K-1)M_{strong})M_{strong}^{\frac{1}{2}}\Bigg(\frac{\pi_w}{c_{weak}} - \left(1-\frac{B^2}{(K-1)^2}\right)^{K-3}\times \\
&\times\left(1-\frac{B}{K-1}\right)\Bigg) \\
\leq& -(K-1)(1+(K-1)M_{strong})M_{strong}^{\frac{1}{2}}\left(\frac{\pi_w}{c_{weak}} - e^{-\frac{B^2(K-3)}{(K-1)^2}}\left(1-\frac{B}{K-1}\right)\right) \\
<& \,0,
\end{aligned}
$$

as $\frac{\pi_w}{c_{weak}} > e^{-\frac{B^2(K-3)}{(K-1)^2}}\left(1-\frac{B}{K-1}\right)$.

The fact that $F(\sqrt{M_{strong}}) < 0$, $F(\varepsilon) > 0$ for some small $\varepsilon > 0$ and it is a continuous function implies that $\exists\, p^* \in (0, \sqrt{M_{strong}})$ such that $F(p^*) = 0$. Now we just need to make sure that such non trivial equilibrium p^* is unique.

Because our function is positive and increasing near the 0 and is negative near $\sqrt{M_{strong}}$ then, at some point, $p^\star$, its derivative has to become negative, $F'(p^\star) < 0$, before $F(p)$ becomes negative.

$$
\begin{aligned}
F'(p) =&(K-1)(1-p^4)^{K-4}(1-p^2)\Bigg(1+(3K-5)p^2-(6K-11)p^4-(K-1)(4K-7)p^6- \\
&-\frac{3(K-1)p^2\left(2-\frac{\pi_w}{c_{weak}}\right)\frac{\pi_w}{c_{weak}}}{(1-p^4)^{K-4}(1-p^2)}\Bigg) + (K-1)\frac{\pi_w}{c_{weak}}\left(M_{strong}(K-1)\left(1-\frac{\pi_w}{c_{weak}}\right)-1\right)
\end{aligned}
$$

We would like $F'(p)$ to stay negative after the first time it becomes less than 0. Then it can not cross 0 more than once. Assume that the constant term above is positive[13]. It is a part of the derivative of the second summand of $F(p)$, $T_2(p)$. Notice, that $T_2(p)$ has to be negative at the non trivial equilibrium, because $T_1(p)$ is always positive and their sum is 0. Also, $T_2(p)$ is concave, $T_2(0) = 0$ and $T_2'(0) \geq 0$, therefore, before T_2 becomes negative, its derivative has to become less than 0. Denote by p_1 a point at which $T_2'(p_1) = 0$. Then we can rewrite $F'(p)$ as

$$F'(p) = (K-1)(1-p^4)^{K-4}(1-p^2)A(p),$$

where

$$A(p) = ap^2 - 2p^2 - bp^4 - cp^6 - \frac{ad(p^2 - p_1^2)}{(1-p^2)\,(1-p^4)^{K-4}} + 1,$$

and $a = 3(K-1)$, $b = (6K-11)$, $c = (K-1)(4K-7)$, $d = (2 - \pi_w/c_{weak})\pi_w/c_{weak}$.

It will be sufficient to show that once A becomes negative it stays negative as $(1-p^4)^{K-4}(1-p^2)$ is always positive for $p \in [0, \sqrt{M_{strong}}]$. Notice that $A(0) > 0$, so before it gets negative its derivative has to become negative.

$$\begin{aligned} A'(p) =& 2p\Bigg(- ad\left(p^2+1\right)^{3-K}\left(1-p^2\right)^{2-K}\left(2(K-4)p^4 + p^2\left((-2K+7){p_1}^2+1\right) - {p_1}^2 + 1\right) + \\ & + a - 2bp^2 - 3cp^4 - 2\Bigg) \end{aligned}$$

Now, call the term in parentheses $B(p)$, then B' is negative for $p > 0$.

[13]If it is negative then instead of having $p^2 - p_1^2$ below we are going to have $p^2 + p_1^2$ which will only help with the analogous proof.

$$
\begin{aligned}
B'(p) =& -4p\left(1-p^4\right)^{-K}\Bigg(\left(1-p^4\right)^K\left(b+3cp^2\right)+ad\left(p^2-1\right)\left(p^2+1\right)^2\times\\
&\times\Bigg(-(K-4)(2K-7)p^6+(2K-7)p^4\left((K-3){p_1}^2-1\right)+p^2\Big(2(K-3){p_1}^2-3K+\\
&+10\Big)+(K-3){p_1}^2-1\Bigg)\Bigg)\\
=& -4p\left(1-p^4\right)^{-K}\Bigg(\left(1-p^4\right)^K\left(b+3cp^2\right)+ad\left(p^2-1\right)\left(p^2+1\right)^2\times\\
&\times\Bigg(-1-(2K-7)p^4\Big((K-4)p^2+1-(K-3){p_1}^2\Big)-p^2\Big(-2(K-3){p_1}^2+2K+\\
&-7\Big)-(K-3)(p^2-{p_1}^2)\Bigg)\Bigg)\\
<&0.
\end{aligned}
$$

Thus, $B'(p)$ is negative. This implies the following. When A' becomes negative it stays negative. As $A(0) > 0$, before A becomes negative its derivative, A', has to become negative. Therefore, once A starts decreasing it continues decreasing from that point on. So when A becomes negative it stays negative and this is what we wanted. Hence, $F(p)$ crosses 0 only once on $(0, \sqrt{M_{strong}})$. Let us call this point p^*. This is a unique non trivial equilibrium of U_∞.

Now, assume that $\pi_w \geq c_{weak}$ then $\pi_w/c_{weak} \geq 1 > (1-M_{strong}^2)^{K-3}(1-M_{strong})$. Furthermore, it means that $F(0) = 0$ and F' is negative on $(0, \sqrt{M_{strong}})$. Thus, there is only one equilibrium of U_∞: $\forall\, i\ p_i = 0$ and $q_i = M_{weak}$. When $\pi_w > c_{weak}$ weak links are more beneficial at both distance 1 and 2 and so no one wants to invest in friends.

On the other hand, if $\pi_w/c_{weak} < (1-M_{strong}^2)^{K-3}(1-M_{strong})$ then $\pi_w < c_{weak}$. It means that $F(\sqrt{M_{strong}}) > 0$. Hence, $F(p) = 0$ does not have a solution and is positive on $(0, \sqrt{M_{strong}}]$. So without the two conditions of the theorem we have only trivial symmetric equilibria.

To finish the proof we will show that we found the maximum and not the minimum.

$$
\begin{aligned}
\left.\frac{\partial^2 U_\infty}{\partial p_i^2}\right|_{p_i=p} =& -(K-2)(K-1)p^4\left(1-p^2\right)^{K-3}\left(p^2+1\right)^{K-4}\left((K-1)p^2+2\right)\\
<&0.
\end{aligned}
$$

Thus, the extremum we found is indeed the maximum. This concludes the proof.

□

Proof of Proposition 3. To prove this proposition, let us calculate derivatives of $F(p)$ with respect to c_{weak}, π_w and K. We know from Theorem 2 the derivative of F at p^* is negative. Hence, if we know the derivatives of F and their signs with respect to those parameters we will be able to calculate the corresponding comparative statics. Notice, that only the second summand of F depends on these 3 parameters, which simplifies calculations.

$$1)\ \ \frac{\partial F(p^*)}{\partial c_{weak}} = \frac{\partial\left(-\frac{\pi_w(K-1)}{c_{weak}}p^*\left(1-(K-1)(M_{strong}-2p^{*2}) + \frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2}\right)\right)}{\partial c_{weak}}$$
$$= \frac{\pi_w p^*\left(c_{weak}\left(1-(K-1)(M_{strong}-2p^{*2})\right) + 2\pi_w(K-1)\left(M_{strong}-p^{*2}\right)\right)}{{c_{weak}}^3}.$$

Remember, that the second summand of F is negative at the equilibrium, $T_2(p^*) < 0$ from Theorem 2, therefore

$$\frac{\partial F(p^*)}{\partial c_{weak}} = \frac{-T_2(p^*) + \frac{\pi_w^2 p^*(K-1)(M_{strong}-p^{*2})}{c_{weak}^2}}{c_{weak}}$$
$$> 0.$$

Therefore, as we increase c_{weak}, p^* also increases.

$$2)\ \ \frac{\partial F(p^*)}{\partial H} = -\frac{p^*\left(1-(K-1)(M_{strong}-2p^{*2}) + 2\frac{\pi_w}{c_{weak}}(K-1)\left(M_{strong}-p^{*2}\right)\right)}{c_{weak}}$$
$$< 0.$$

We get the last inequality in the same way we argued in 1) that $T_2(p^*)/\pi_w$ minus something negative is negative at p^*.

$$3)\ \ \frac{\partial F'(p)}{\partial K} = p\Bigg(\left(1-p^2\right)\left(1-p^4\right)^{K-3}\left(\left((K-1)p^2+1\right)\log\left(1-p^4\right)+p^2\right) -$$
$$- p^2\left(2-\frac{\pi_w}{c_{weak}}\right)\frac{\pi_w}{c_{weak}}\Bigg)$$

Notice, that $\log(1-p^4) < 0$. Furthermore, at the approximate equilibrium p^* we have

$$F(p^*) = 0$$

$$
\begin{aligned}
0 =&(K-1)p^* \left(1-p^{*2}\right)^{K-2} \left(p^{*2}+1\right)^{K-3} \left((K-1)p^{*2}+1\right) - \\
&- \frac{\pi_w(K-1)}{c_{weak}} p^* \left(1-(K-1)(M_{strong}-2p^{*2}) + \frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right).
\end{aligned}
$$

Hence,

$$
\left(1-p^{*2}\right)\left(1-p^{*4}\right)^{K-3} = \frac{\pi_w \left(1-(K-1)(M_{strong}-2p^{*2}) + \frac{\pi_w}{c_{weak}}(K-1)(M-p^{*2})\right)}{c_{weak}\left((K-1)p^{*2}+1\right)}
$$

Let us substitute this into $\partial F'(p)/\partial K$, but skip the term with the logarithm and the p term outside the parentheses.

$$
\begin{aligned}
&\frac{\pi_w \left(1-(K-1)(M_{strong}-2p^{*2}) + \frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right)}{c_{weak}\left((K-1)p^{*2}+1\right)} p^{*2} - p^{*2}\left(2-\frac{\pi_w}{c_{weak}}\right)\frac{\pi_w}{c_{weak}} = \\
&\frac{\pi_w p^{*2}}{c_{weak}((K-1)p^{*2}+1)} \Bigg(1-(K-1)M_{strong}\left(1-\frac{\pi_w}{c_{weak}}\right) + p^{*2}(K-1)\left(2-\frac{\pi_w}{c_{weak}}\right) - \\
&- \left(2-\frac{\pi_w}{c_{weak}}\right)\left((K-1)p^{*2}+1\right)\Bigg) = \\
&\frac{\pi_w p^2}{c_{weak}((K-1)p^2+1)} \left(-1+\frac{\pi_w}{c_{weak}} - (K-1)M_{strong}\left(1-\frac{\pi_w}{c_{weak}}\right)\right) < 0.
\end{aligned}
$$

Thus, the whole $\partial F'(p)/\partial K$ is also negative at $p = p^*$. So the probability of having a strong friend is decreasing as we increase K. This concludes our proof.

□

Proof of Lemma 4. First we are going to show that $p^{*2} > q^{*2}$ in equilibrium. Remember that from the proof of Theorem 2

$$
p^{*2} > M\frac{c_{weak}-\pi_w}{2c_{weak}-\pi_w} - \frac{c_{weak}}{(K-1)(2c_{weak}-\pi_w)} = \frac{(B-1)c_{weak}-B\pi_w}{(K-1)(2c_{weak}-\pi_w)}
$$

as the $T_2(p)$ term has to be negative. Hence,

$$
\begin{aligned}
q^{*2} = L(M-p^{*2}) < \frac{(K-1)}{KNc_{weak}}&\left(\frac{B(2c_{weak}-\pi_w)-(B-1)c_{weak}+B\pi_w}{(K-1)(2c_{weak}-\pi_w)} - \right) \\
&= \frac{(B+1)}{KN(2c_{weak}-\pi_w)}.
\end{aligned}
$$

Let us look at their difference

$$p^{*2} - q^{*2} > \frac{(B-1)c_{weak} - B\pi_w}{(K-1)(2c_{weak} - \pi_w)} - \frac{(B+1)}{KN(2c_{weak} - \pi_w)}$$
$$= \frac{(B-1)c_{weak}KN - B\pi_w KN - (B+1)(K-1)}{(K-1)KN(2c_{weak} - \pi_w)}.$$

For this difference to be greater than 0 we need to have

$$c_{weak} > \frac{B}{(B-1)}\pi_w + \frac{(B+1)(K-1)}{(B-1)KN}.$$

Now let us remember a well-known fact about Erdős - Rényi random graphs: the golbal clustering coefficient for friends' network is equal to $p^{*2} + O\left((KN)^{-0.5}\right)$ and the global CC for acquanintances' network is $q^{*2} + O\left((KN)^{-0.5}\right)$. Using this fact and the inequality that we got above we can conclude that when c_{weak} is not too close to π_w or when N is big enough (so $p^* > 0$ and $q^* < p^*$ for big enough N) the CC of the friends' network is bigger than the CC of the acquaintances'. This concludes the proof. □

Proof of Theorem 5. To prove this theorem we are going to differentiate $U_\infty(p, q(p), p, q(p)) \equiv U_{optimal}(p)$ and find its non-trivial root $p_{optimal}$, which is also unique. Then we will show that $p_{optimal}$ has to be bigger than p^*.

$$U'_{optimal}(p) = 2p(K-1)\Bigg(2(K-2)p^2\left(1-p^2\right)^{K-2}\left(p^2+1\right)^{K-3} + \left(1-p^4\right)^{K-2} -$$
$$- \frac{\pi_w}{c_{weak}}\left(1 - 2(K-1)(M_{strong} - 2p^2)\right) + \frac{2\pi_w}{c_{weak}}(K-1)\left(M_{strong} - p^2\right)\Bigg).$$

As we can see it greatly resembles $\left(\partial U_\infty/\partial p_i\right)\Big|_{p_i=p}$. Before we proceed, let us show that $U'_{optimal}$ has the same shape as $\left(\partial U_\infty/\partial p_i\right)\Big|_{p_i=p}$: at 0 it equals 0 and has a positive derivative. Moreover, once its derivative becomes negative it stays negative.

$$
\begin{aligned}
U'_{optimal}(0) =& 0\\
U''_{optimal}(0) =& \left(\frac{\pi_w\left(2(K-1)M_{strong}\left(1-\frac{\pi_w}{c_{weak}}\right)-6(K-1)p^2\left(2-\frac{\pi_w}{c_{weak}}\right)-1\right)}{c_{weak}}-\right.\\
&-\left(1-p^2\right)^{K-3}\left(p^2+1\right)^{K-4}\left((2K-3)(4K-7)p^6+(8K-15)p^4+(11-6K)p^2-\right.\\
&\left.\left.-1\right)\right)\Bigg|_{p=0}\\
=&\frac{\pi_w\left(2(K-1)M_{strong}\left(1-\frac{\pi_w}{c_{weak}}\right)-1\right)}{c_{weak}}+1\\
>&0,
\end{aligned}
$$

as $c_{weak} > \pi_w$.

$$
\begin{aligned}
U'_{optimal}(\sqrt{M_{strong}}) =& 2\sqrt{M_{strong}}(K-1)\Bigg(\left(1-p^2\right)^{K-2}\left(p^2+1\right)^{K-3}\left((2K-3)p^2+1\right)-\\
&-\frac{\pi_w}{c_{weak}}\left(1-2(K-1)(M_{strong}-2M_{strong})\right)+\frac{2\pi_w}{c_{weak}}(K-1)(M_{strong}-\\
&-M_{strong})\Bigg)\\
=&-\left(\frac{\pi_w}{c_{weak}}-(1-M_{strong})^{-2+K}(1+M_{strong})^{-3+K}\right)(1+(-2+2K)\times\\
&\times M_{strong})-(1-M_{strong})^{-2+K}(1+M_{strong})^{-3+K}M_{strong}\\
<&0.
\end{aligned}
$$

We got the last inequality from the same condition we had in Theorem 2, which is necessary for an existance of the equilibrum, p^*.

$$
\begin{aligned}
U''_{optimal}(p) =& 2(K-1)\Bigg(\left(1-p^2\right)^{K-3}\left(p^2+1\right)^{K-4}\Big(-(2K-3)(4K-7)p^6-(8K-15)p^4+\\
&+(6K-11)p^2+1\Big)+\\
&+\frac{\pi_w}{c_{weak}}\left(2\left((K-1)M_{strong}\left(1-\frac{\pi_w}{c_{weak}}\right)-3(K-1)p^2\left(2-\frac{\pi_w}{c_{weak}}\right)\right)-1\right)\Bigg)
\end{aligned}
$$

As we can see $U''_{optimal}$ is positive at $p=\varepsilon$ for ε small enough, so its derivative has to become negative before $U'_{optimal}$ becomes negative. To prove that once $U'_{optimal}$ becomes negative it stays negative we can do an analogous exercise to the one we did in Theorem 2.

Now let see if the solution to $\partial U_{optimal}/\partial p = 0$, $p_{optimal}$, is bigger than p^*, the non trivial solution to $\left(\partial U_\infty/\partial p_i\right)\Big|_{p_i=p} = 0$. To do this we are going to look at the sign of $U'_{optimal}(p^*)$.

$$\begin{aligned}
U'_{optimal}(p^*) =& 2p^*(K-1)\Bigg(2(K-2)p^{*2}\left(1-p^{*2}\right)^{K-2}\left(p^{*2}+1\right)^{K-3}+\left(1-p^{*4}\right)^{K-2}- \\
&-\frac{\pi_w}{c_{weak}}\left(1-2(K-1)(M_{strong}-2p^{*2})+2\frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right)\Bigg) \\
=& p^*(K-1)\Bigg(4(K-2)p^{*2}\left(1-p^{*2}\right)^{K-2}\left(p^{*2}+1\right)^{K-3}+2\left(1-p^{*4}\right)^{K-2}- \\
&-\frac{\pi_w}{c_{weak}}\left(2-4(K-1)(M_{strong}-2p^{*2})+4\frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right)\Bigg) \\
=& p^*(K-1)\Bigg(2(K-2)p^{*2}\left(1-p^{*2}\right)^{K-2}\left(p^{*2}+1\right)^{K-3}- \\
&-\frac{\pi_w}{c_{weak}}\left(-2(K-1)(M_{strong}-2p^{*2})+2\frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right)\Bigg) \\
=& 2p^*(K-1)\Bigg((K-2)p^{*2}\left(1-p^{*2}\right)^{K-2}\left(p^{*2}+1\right)^{K-3}+ \\
&-\frac{\pi_w}{c_{weak}}\left(-(K-1)(M_{strong}-2p^{*2})+\frac{\pi_w}{c_{weak}}(K-1)(M_{strong}-p^{*2})\right)\Bigg),
\end{aligned}$$

where in the third equation we used the fact that $\left(\partial U_\infty/\partial p_i\right)\Big|_{p_i=p}(p^*) = 0$. We are going to use it one more time now to get

$$U_{optimal}(p^*) = 2p^*(K-1)\left(-\left(1-p^{*4}\right)^{K-2}+\frac{\pi_w}{c_{weak}}\right)$$

Therefore, if

$$\frac{\pi_w}{c_{weak}} > \left(1-p^{*4}\right)^{K-2}$$

then $U'_{optimal}(p^*) > 0$, hence, $p_{optimal} > p^*$.

To conclude the proof, let us show that $p_{optimal}$ is indeed a maximum not a minimum. Notice, that $U_{optimal}(p)$ is a polynomial of one variable that has one extreme point on $(0, \sqrt{M_{strong}}]$. As we showed before, $U'_{optimal}$ is non negative in $\varepsilon-$neighborhood of 0 and positive outside of $p = 0$. Furthermore, $U'_{optimal}(\sqrt{M_{strong}}) < 0$. This means that the maximum of our function is not attained at the endpoints. Thus, the extreme point that we found is the maximum. This concludes our proof.

$\square$

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