Third Year B.S. (Honors) 2022-2023

Department of Applied Mathematics, University of Dhaka

Course Title: Math Lab III (Matlab), Course No.: AMTH 350

Assignment 03

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Use Matlab to solve each of the following problems.

1. The one dimensional homogeneous heat equation is $u_t = a^2 u_{xx}$, where u defines the temperature and a^2 is a positive constant known as the diffusivity. Use symbolic variables and `symunit` package to assign units to each of the variables and parameters present in the heat equation. Also, check units consistency of this equation (*hint: compute the derivatives*).

2. Consider a thin wire made of homogeneous material the diameter of which coincides with the x-axis from $x = 0$ to $x = L$. It is assumed that the initial temperature, f of the rod is specified as a function of the distance x from one end of the rod ($x = 0$). The temperature distribution $u(x, t)$ at some later time in the absence of any heat source is then a solution of the one-dimensional, homogeneous heat equation. Suppose the two ends are suddenly placed in contact with ice packs at $0°C$ at time $t = 0$ and this temperature is maintained at all times. Then the temperature distribution of this rod is given by

$$u(x,t) = \Sigma_{n=1}^{\infty} c_n \frac{\sin n\pi x}{L} e^{\frac{-a^2 n^2 \pi^2 t}{L^2}}$$

   (a) Verify that the solution given satisfies the one-dimensional, homogeneous equation.

   (b) Suppose, $f(x) = 10$. Compute the coefficients, c_n.

   (c) Assume, $a^2 = 1.71$, and $L = \pi$. Plot the temperature, $u(x, t)$ for first five non-zero values of n.

   (d) Express the temperature, $u(x, t)$ as a sum of the terms you obtained in part(b) and hence plot it.

   (e) Plot the temperature at the center of the rod as a function of time.

   (f) What would be the temperature of the rod if it is kept for sufficiently long time?

3. Consider the following heat conduction problem through a thin rod:

$$u_t = a^2 u_{xx};\ 0 < x < 10;\ t > 0$$

$$B.C.:\ u(0,t) = 10;\ u(10,t) = 30;\ t > 0$$

$$I.C.:\ u(x,0) = 0;\ 0 < x < 10$$

The solution of this problem is

$$u(x,t) = 10 + 2x + \frac{20}{\pi}\Sigma_{n=1}^{\infty} \frac{(-1)^n 3 - 1}{n} \sin(\frac{n\pi x}{10}) e^{\frac{-a^2 n^2 \pi^2 t}{100}}$$

Consider four rods, each of which is made entirely of each of these four materials, Silver (1.71), Copper (1.14), Aluminum (0.86) and Cast iron (0.12) (bracketed values indicate corresponding diffusivity).

5/5

(a) Create a table demonstrating the temperature, $u(x,t)$ at $t = 1$ for each rod (take 0.5 as spatial increment).

(b) Repeat part (a) for $t = 10$ and $t = 20$ and hence comment how the diffusivity of material affects the heat conduction.

(c) Use first three non-zero terms to visualize the temperature distribution in each rod.

(d) How long it will take for the center of each rod to reach a temperature of 15 units?

4. Consider a one-dimensional road with traffic flow described by the following quasi-linear PDE:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

where $\rho(x,t)$ is the traffic density (vehicles per unit length), and $v(\rho)$ is the velocity of the traffic as a function of density. Given:
- Length of the road, $L = 10$ km
- Maximum velocity, $v_{\max} = 120$ km/h
- Density-dependent slowing down parameter, $a = 0.02$ vehicles/(km^2)
- Initial density distribution, $\rho_0(x) = 20$ vehicles/km for $0 < x < L$

Considering the scenario of red light turning green, set up the initial conditions, and implement the method of characteristic using MATLAB to simulate the traffic density evolution over time and analyze the system's behavior.

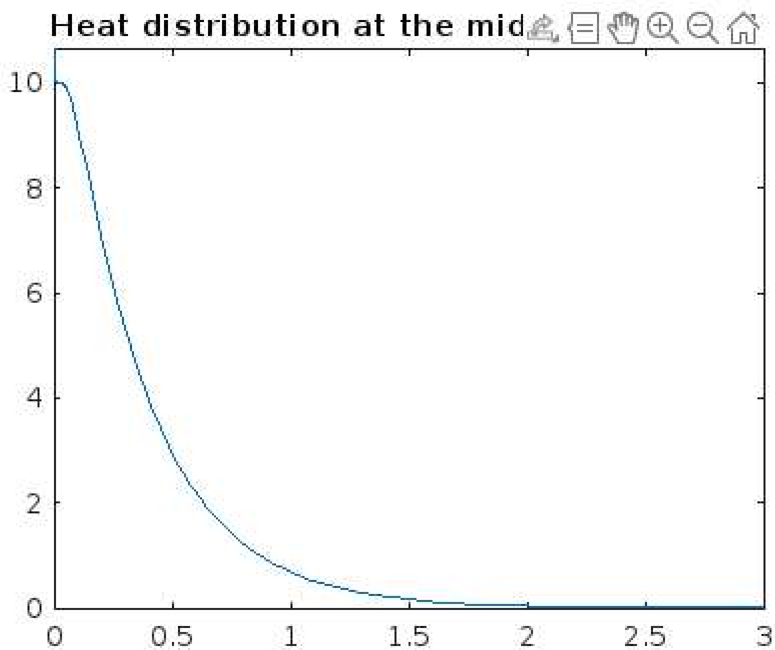
5. Check the transversality condition for the PDE $u_x + 3y^{\frac{2}{3}}u_y = 2$, subject to initial condition, $u(x,1) = 1 + x$ and hence solve using method of characteristic. Plot the solution surface along with characteristic curves.

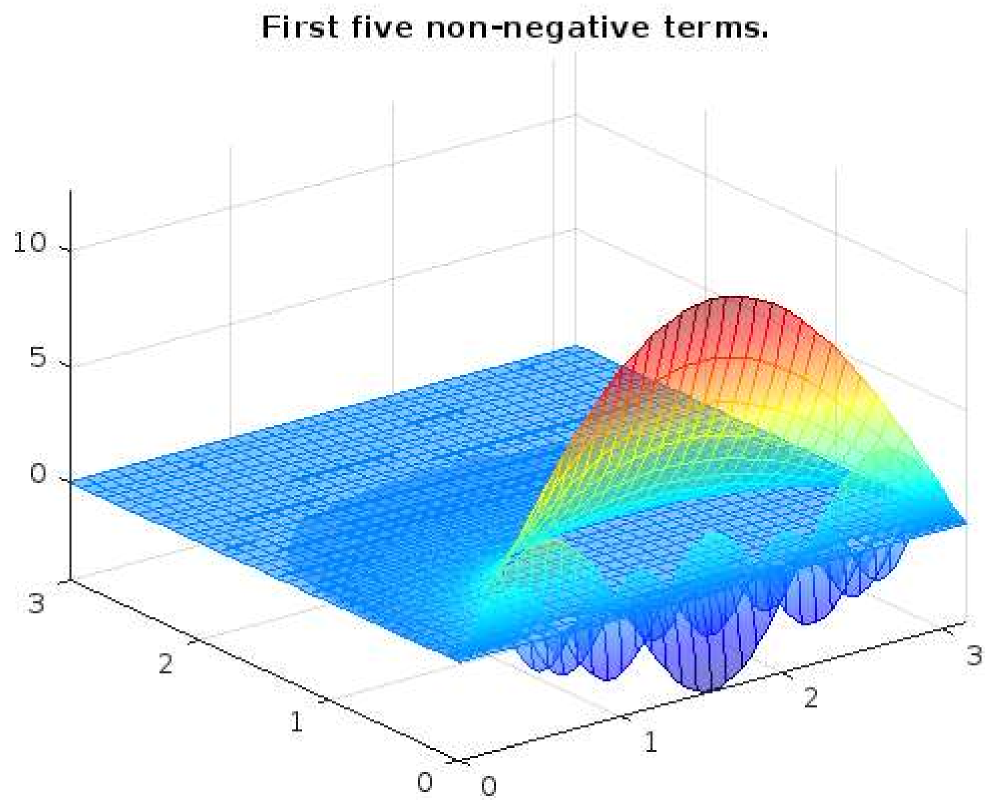
```
%a3q1
clc;
syms U(x,t) a
unit=symunit;
U=U*unit.K;
x=x*unit.m;
t=t*unit.t;
a=a*unit.m^2/unit.t;
eqn=diff(U,t)==a*diff(U,x,x);
disp('Differential term with respect to time:');
diff(U,t)
disp('Differential term with respect to space:');
a*diff(U,x,x)
```

```
Differential term with respect to time:

ans(x, t) =

diff(U(x, t), t)*([K]/[t])

Differential term with respect to space:

ans(x, t) =

a*diff(U(x, t), x, x)*([K]/[t])
```

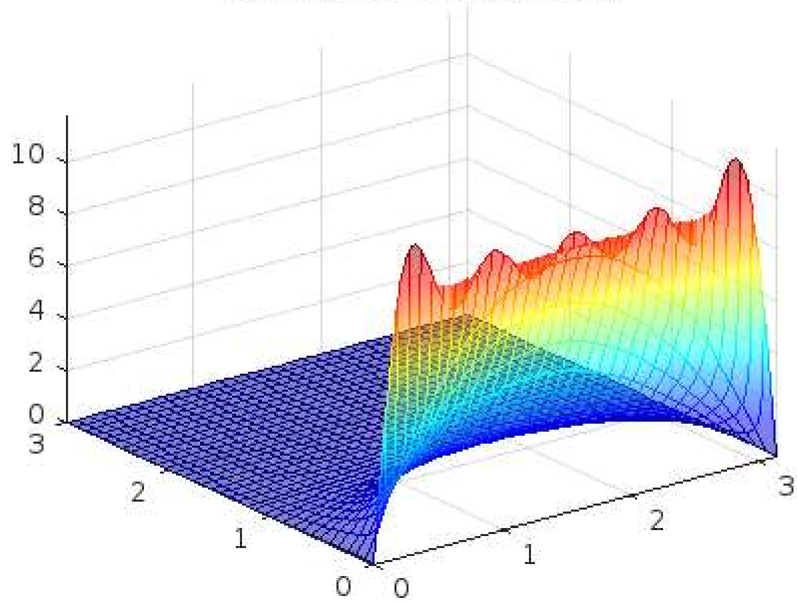
```
%a3q2
clc;
syms t x n c_n(n) L a f(x) u_n(x, t, n) lambda(n) heat(x, t)

assume(n, "integer")
assume([L a], "positive")

heat(x, t) = 0;

lambda(n) = n*pi/L;

u_n(x, t, n) = c_n .* sin(x*lambda(n)) .* exp(-t*(a^2)*lambda(n).^2);

eqn = simplify(diff(u_n, t) - (a^2)*diff(u_n, x, x));
if eqn == 0
    fprintf('The equation is satisfied.\n')
else
    fprintf('The equation is not satisfied.\n')
end

f(x) = 10;

c_n(n) = (2/L) * int(f(x) .*sin(lambda(n).*x), x, 0, L);

u_n_plot(x, t, n) = subs(u_n(x, t, n), sym(["L" "a" "c_n"]), [pi 1.71 c_n]);

for i = 1:9
    tfinal = 3;
    u_plot(x, t) = u_n_plot(x, t, i);
    heat(x, t) = heat(x, t) + u_plot(x, t);
    fsurf(u_plot, [0 pi 0 tfinal], "EdgeColor","interp"), colormap jet
    hold on
    alpha(0.5)
    title(['First five non-negative terms.']);
end

figure
fsurf(heat, [0 pi 0 tfinal], "EdgeColor","interp"), colormap jet
alpha(0.5)
title(['Total heat distribution.']);

figure
heat_mid = heat(pi/2, t);
fplot(heat_mid, [0 3])
title(['Heat distribution at the middle over time.']);
```

The equation is satisfied.

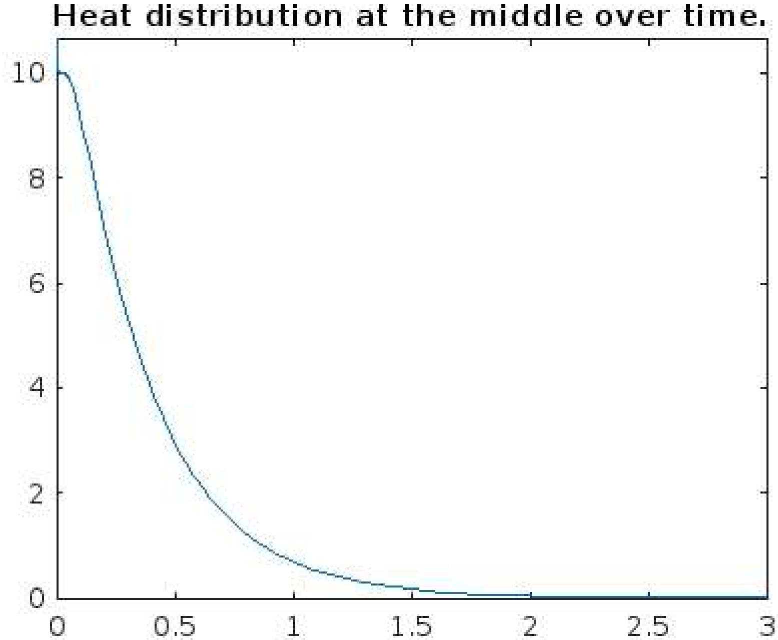

Heat distribution at the mid
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3


First five non-negative terms.
10
5
0
3
2
1
0
0
1
2
3

Total heat distribution.



Heat distribution at the middle over time.



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```
%a3q3
clc;

syms x_val t_val u_val

silverDist = heatDist(x_val, t_val, 1.71);
copperDist = heatDist(x_val, t_val, 1.14);
aluminumDist = heatDist(x_val, t_val, 0.86);
ironDist = heatDist(x_val, t_val, 0.12);

x = linspace(0, 10, 21);

createHeatDistributionTable(1);
createHeatDistributionTable(10);
createHeatDistributionTable(20);

silverMid(t_val) = silverDist(5, t_val);
for i = 0:.1:30
    if(round(silverMid(i)) == 15)
        tSilverMid = i;
        break
    end
end
tSilverMid

copperMid(t_val) = copperDist(5, t_val);
for i = 0:.1:30
    if(round(copperMid(i)) == 15)
        tCopperMid = i;
        break
    end
end
tCopperMid

aluminumMid(t_val) = aluminumDist(5, t_val);
for i = 0:.1:30
    if(round(aluminumMid(i)) == 15)
        tAluminumMid = i;
        break
    end
end
tAluminumMid

tIronMid = 0;
ironMid(t_val) = ironDist(5, t_val);
for i = 1000:1:1300
    if(round(ironMid(i)) == 15)
        tIronMid = i;
        break
    end
end
tIronMid
```

```
figure
fsurf(silverDist, [0 10 0 30], "EdgeColor","interp"), colormap jet
title('Heat distribution in a thin silver rod')
xlabel('x')
ylabel('t')
zlabel('u(x, t)')
box on
alpha(0.5)

figure
fsurf(copperDist, [0 10 0 30], "EdgeColor","interp"), colormap jet
title('Heat distribution in a thin copper rod')
xlabel('x')
ylabel('t')
zlabel('u(x, t)')
box on
alpha(0.5)

figure
fsurf(aluminumDist, [0 10 0 30], "EdgeColor","interp"), colormap jet
title('Heat distribution in a thin aluminum rod')
xlabel('x')
ylabel('t')
zlabel('u(x, t)')
box on
alpha(0.5)

figure
fsurf(ironDist, [0 10 0 30], "EdgeColor","interp"), colormap jet
title('Heat distribution in a thin iron rod')
xlabel('x')
ylabel('t')
zlabel('u(x, t)')
box on
alpha(0.5)

function createHeatDistributionTable(t)
    syms x_val t_val u_val
    x = linspace(0, 10, 21);

    silverDist = heatDist(x_val, t_val, 1.71);
    copperDist = heatDist(x_val, t_val, 1.14);
    aluminumDist = heatDist(x_val, t_val, 0.86);
    ironDist = heatDist(x_val, t_val, 0.12);

    sil = round(vpa(silverDist(x, t), 7), 6);
    cop = round(vpa(copperDist(x, t), 7), 6);
    alu = round(vpa(aluminumDist(x, t), 7), 6);
    irn = round(vpa(ironDist(x, t), 7), 6);

    disp(table(x', sil', cop', alu', irn', 'VariableNames', {'x', 'Silver',
'Copper', 'Aluminum', 'Iron'}));
end
```

```
function heat = heatDist(x, t, a)
    syms n c_n(n) L f(x) u_n(x, t, n) lambda(n) heat(x, t)

    assume(n, "integer")
    assume([L a], "positive")

    heat(x, t) = 10 + 2*x;

    lambda(n) = n*pi/L;

    u_n(x, t, n) = (20/pi) .*(((3*((-1)^n))-1)/n) .* sin(x*lambda(n)) .* exp(-
t*(a^2)*(lambda(n).^2));

    u_n_plot(x, t, n) = subs(u_n(x, t, n), sym(["L" "a"]), [10 a]);
    for i = 1:50
        u_plot(x, t) = u_n_plot(x, t, i);
        heat(x, t) = heat(x, t) + u_plot(x, t);
    end
end
```

x	*Silver*	*Copper*	*Aluminum*	*Iron*
0	*10.0*	*10.0*	*10.0*	*10.0*
0.5	*8.364143*	*7.564586*	*6.809929*	*0.047454*
1	*6.798087*	*5.350807*	*4.109533*	*-0.009952*
1.5	*5.363947*	*3.521625*	*2.174542*	*-0.005671*
2	*4.110384*	*2.147785*	*1.000865*	*0.007038*
2.5	*3.070168*	*1.2099*	*0.398266*	*0.004108*
3	*2.261658*	*0.628138*	*0.136383*	*-0.006242*
3.5	*1.693896*	*0.301017*	*0.040054*	*-0.003572*
4	*1.37413*	*0.136925*	*0.010083*	*0.006404*
4.5	*1.316116*	*0.071895*	*0.00234*	*0.003231*
5	*1.547243*	*0.077057*	*0.001575*	*-0.007206*
5.5	*2.112609*	*0.163995*	*0.00653*	*-0.002858*
6	*3.074501*	*0.394938*	*0.030185*	*0.008712*
6.5	*4.50635*	*0.898621*	*0.120154*	*0.002248*
7	*6.481256*	*1.883283*	*0.409149*	*-0.01135*
7.5	*9.056407*	*3.629437*	*1.194798*	*-0.000879*
8	*12.256065*	*6.443299*	*3.002595*	*0.016408*
8.5	*16.056799*	*10.564865*	*6.523627*	*-0.003675*
9	*20.378855*	*16.052419*	*12.328599*	*-0.027561*
9.5	*25.086718*	*22.693759*	*20.429787*	*0.128964*
10	*30.0*	*30.0*	*30.0*	*30.0*

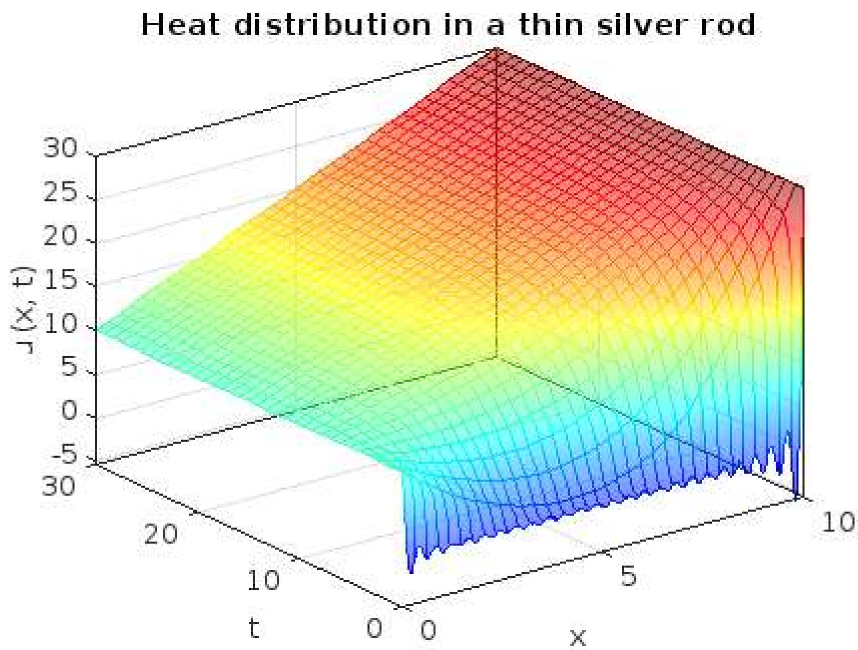
x	*Silver*	*Copper*	*Aluminum*	*Iron*
0	*10.0*	*10.0*	*10.0*	*10.0*
0.5	*10.777734*	*9.906949*	*9.180929*	*3.514942*
1	*11.560939*	*9.839968*	*8.399914*	*0.624074*
1.5	*12.354952*	*9.824565*	*7.69461*	*0.051886*

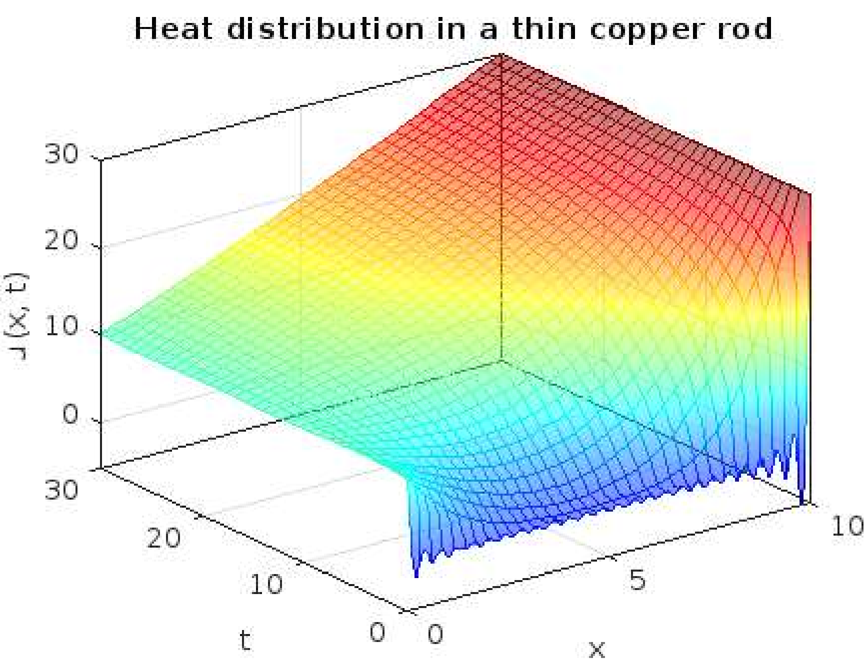
2	13.164845	9.885137	7.101851	0.001939
2.5	13.995298	10.044423	6.657203	0.000032
3	14.850486	10.322985	6.39445	0
3.5	15.733974	10.738715	6.345007	0
4	16.648632	11.306386	6.537219	0
4.5	17.596563	12.037246	6.99555	0
5	18.57905	12.938685	7.739651	0
5.5	19.596525	14.013981	8.783341	0
6	20.64856	15.262134	10.133564	0
6.5	21.733874	16.677808	11.789404	0
7	22.850369	18.251384	13.741272	0.000001
7.5	23.995175	19.969137	15.970377	0.000096
8	25.164728	21.813536	18.448609	0.005818
8.5	26.354852	23.763658	21.138904	0.155658
9	27.560866	25.795716	23.996156	1.872223
9.5	28.777695	27.883684	26.968656	10.544825
10	30.0	30.0	30.0	30.0

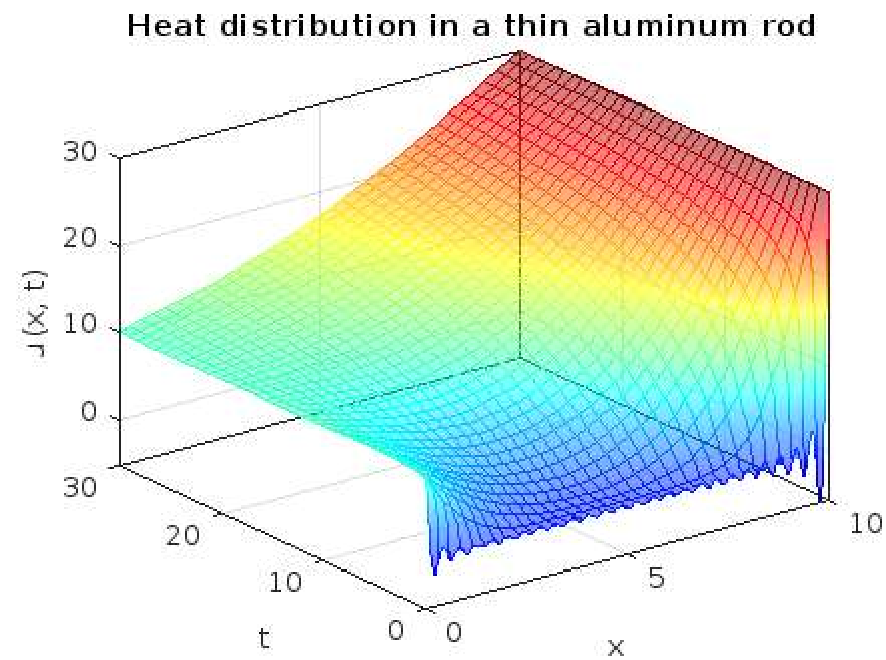
x	Silver	Copper	Aluminum	Iron
0	10.0	10.0	10.0	10.0
0.5	10.987596	10.69375	10.080506	5.100192
1	11.975498	11.395036	10.183234	1.876323
1.5	12.964003	12.111208	10.329903	0.481068
2	13.953395	12.849253	10.541223	0.08408
2.5	14.943934	13.615617	10.836422	0.009876
3	15.935853	14.416052	11.232788	0.000772
3.5	16.929352	15.255475	11.745248	0.00004
4	17.924591	16.13784	12.385999	0.000001
4.5	18.921686	17.066048	13.164186	0
5	19.92071	18.041871	14.085647	0
5.5	20.921686	19.065911	15.152737	0
6	21.924591	20.137578	16.364222	0.000004
6.5	22.929352	21.255115	17.715274	0.00012
7	23.935853	22.415629	19.197551	0.002317
7.5	24.943934	23.615171	20.799373	0.029628
8	25.953395	24.848829	22.505987	0.25224
8.5	26.964003	26.110848	24.299929	1.443205
9	27.975498	27.394774	26.161457	5.62897
9.5	28.987596	28.693612	28.069057	15.300575
10	30.0	30.0	30.0	30.0

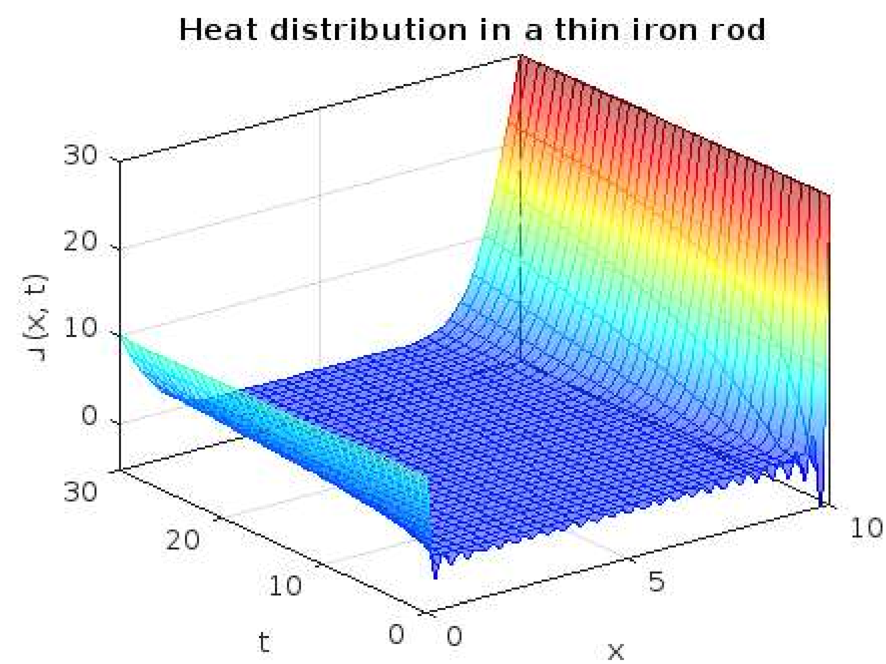
```
tSilverMid =

    5.4000


tCopperMid =

    12
```

```
tAluminumMid =

    21


tIronMid =

          1079
```









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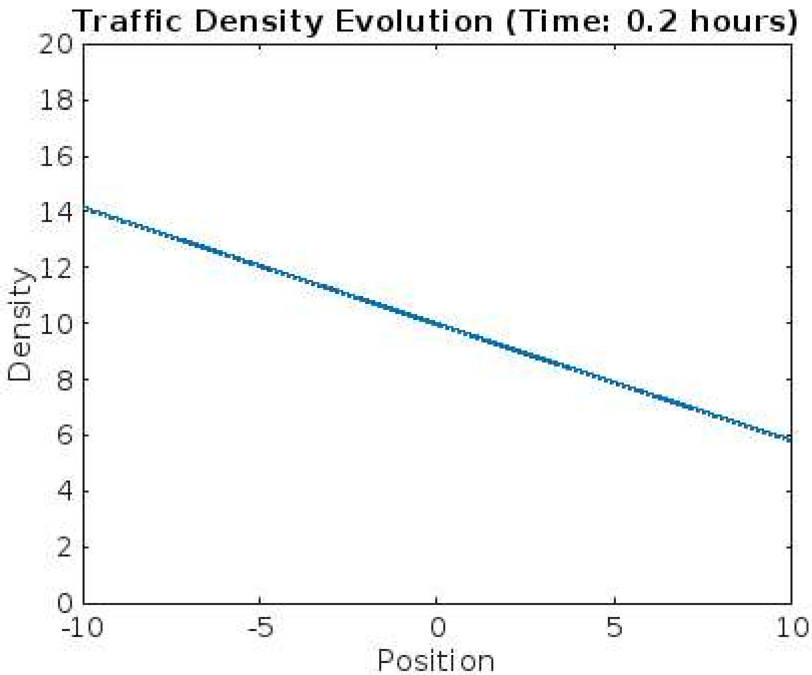
```
%a3q4
road_length = 10;
max_speed = 120;
initial_density = 20;
max_density = initial_density;

time_step = 0.001;
total_time = 0.2;
num_steps = total_time / time_step;

position = linspace(-40, 40, 1000);
density = zeros(size(position));
density(1:end) = initial_density;

colormap('jet');

figure;

for step = 1:num_steps
    velocity = max_speed * (1 - 2 * density / max_density);
    displacement = max_speed * (1 - 2 * density / max_density) * time_step;
    position = position - displacement;
    density = (max_density / 2) * (1 - position ./ (max_speed * step *
time_step));

    % Boundary conditions
    density(position < -max_speed * step * time_step) = max_density;
    density(position > max_speed * step * time_step) = 0;

    plot(position, density, 'LineWidth', 2);
    xlabel('Position');
    ylabel('Density');
    title(['Traffic Density Evolution (Time: ' num2str(step*time_step) '
hours)']);

    ylim([0, max_density]);
    xlim([-10, 10]);

    drawnow;
end
```



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```
%a3q5
clc;
c=1;a=1;
f=@(x,y,u,DuDx) 2 - (1/3)*y^(2/3);
x=linspace(0, 2, 50);
y=linspace(1, 2, 50);
[X,Y]=meshgrid(x,y);
t=linspace(0,1,50);
u0=@(x) 1+x;
sol=pdepe(0, @pdefun,u0, @bcfun,x,t);
u=sol(:,:,1)';
figure;
surf(X,Y,u);
xlabel('x');
ylabel('y');
zlabel('u');
title('Solution Surface');
hold on;
for i=1:length(y)
    y_char=y(i)*ones(size(x));
    x_char=x;
    z_char=u0(x)+c*t(i);
    plot3(x_char, y_char, z_char,'r');
end
hold off
```