# NANO106 Handout 9 - Einstein Notation

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## 1 Introduction

This document provides a summary of the Einstein notation.

## 2 General Principle

The Einstein notation, or Einstein summation convention, is a convetion that implies summation over repeated indices. It allows us to write many relationships in a much more succinct manner. For the purposes of crystallography, we will mainly be working with the range of indices over the set {1, 2, 3}. Therefore,

$$y = c_i a_i = \sum_{i=1}^{3} c_i a_i = c_1 a_1 + c_2 a_2 + c_3 a_3$$

To illustrate the power of this notation, we will now write many expressions in linear algebra in Einstein notation.

### 2.1 Scalar or dot product of two column vectors

$$\mathbf{x} \cdot \mathbf{y} = x_i y_i$$

### 2.2 Product of a matrix with column vector

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In Einstein notation, this is written as:

$$y_i = a_{ij}x_j$$

It is important to note that non-repeated indices are "dummy" indices and have no special meaning. Only *repeated* indices on the same side of the equation implies summation.

## 2.3 Product of row vector with matrix

$$\mathbf{y}^T = \mathbf{x}^T\mathbf{A} \quad = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

In Einstein notation, this is written as:

$$y_j = a_{ij}x_i$$

Be very careful about the indices! Note the difference in the order of the indices in this expression compared to that of the product of the matrix with a column vector.

# 3 Using Einstein notation in tensors

Tensors are geometric objects that describe linear relations between vectors, scalars, and other tensors. We will be using Einstein notation extensively to work with tensors.

- Rank–0 tensor: Scalar, e.g., temperature
- Rank–1 tensor: Vector, e.g., pyroelectricity ($p_i$)
- Rank–2 tensor: Matrix, e.g., diffusivity ($D_{ij}$)
- Rank>3 tensor: Tensor, e.g., piezoelectric coefficient ($d_{ijk}$), elastic constants ($c_{ijkl}$)

Some examples of denoting relationships using Einstein notation.

- Relationship between diffusion flux and concentration gradient

$$J_i = D_{ij}\nabla\phi_j$$

- Relationship between stress and strain

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

# 4 The Kronecker Delta and Permutation Symbol

To use the Einstein summation notation effectively, we need to introduce two additional symbols. Note that this is really more for your information on the power of this notation. We will be using the Kronecker Delta, but in this course, we will not actually be using the permutation symbol. It is included for completeness in case you see this in future.

## 4.1 Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

## 4.2 Levi-Civita or Permutation symbol

$$\epsilon_{ijk} = \begin{cases} 0, & \text{for } i = j,\ j = k \text{ or } i = k \\ +1, & \text{for } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1, & \text{for } (i,j,k) \in \{(1,3,2),(2,1,3),(3,2,1)\} \end{cases}$$

## 4.3 Some useful relations

$$\begin{aligned} \delta_{ij}\delta_{jk} &= \delta_{ik} \\ \delta_{ij}\epsilon_{ijk} &= 0 \\ \epsilon_{ipq}\epsilon_{jpq} &= 2\delta_{ij} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \\ \epsilon_{ijk}\epsilon_{pqk} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \end{aligned}$$

## 4.4 Cross-product in Einstein notation

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e_k}$$

## 4.5 Proof of triple product using Einstein notation

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_i \mathbf{e_i} \cdot \epsilon_{jkm} b_j c_k \mathbf{e_m} \\ &= \epsilon_{jkm} a_i b_j c_k \mathbf{e_i} \cdot \mathbf{e_m} \\ &= \epsilon_{jkm} a_i b_j c_k \delta_{im} \\ &= \epsilon_{jki} a_i b_j c_k \\ &= \epsilon_{ijk} a_i b_j c_k \end{aligned}$$