Available online at www.sciencedirect.com
ScienceDirect
Computer Aided Geometric Design 25 (2008) 274–297
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

Identification of spatial PH quintic Hermite interpolants with near-optimal shape measures

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Received 1 June 2007; received in revised form 26 September 2007; accepted 26 September 2007

Available online 14 December 2007

Abstract

The problem of specifying the two free parameters that arise in spatial Pythagorean-hodograph (PH) quintic interpolants to given first-order Hermite data is addressed. Conditions on the data that identify when the "ordinary" cubic interpolant becomes a PH curve are formulated, since it is desired that the selection procedure should reproduce such curves whenever possible. Moreover, it is shown that the arc length of the interpolants depends on only one of the parameters, and that four (general) helical PH quintic interpolants always exist, corresponding to extrema of the arc length. Motivated by the desire to improve the fairness of interpolants to general data at reasonable computational cost, three selection criteria are proposed. The first criterion is based on minimizing a bivariate function that measures how "close" the PH quintic interpolants are to a PH cubic. For the second criterion, one of the parameters is fixed by first selecting interpolants of extremal arc length, and the other parameter is then determined by minimizing the distance measure of the first method, considered as a univariate function. The third method employs a heuristic but efficient procedure to select one parameter, suggested by the circumstances in which the "ordinary" cubic interpolant is a PH curve, and the other parameter is then determined as in the second method. After presenting the theory underlying these three methods, a comparison of empirical results from their implementation is described, and recommendations for their use in practical design applications are made.

Keywords: Pythagorean-hodograph curves; Hermite interpolation; Quaternion representation; Helices; Arc length; Energy integral

1. Introduction

A Pythagorean-hodograph (PH) curve $\mathbf{r}(t)$ is characterized by the fact that its parametric speed

$$\sigma(t) = \left|\mathbf{r}'(t)\right| = \frac{\mathrm{d}s}{\mathrm{d}t}, \tag{1}$$

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2007.09.007

where s is the cumulative arc length, is a *polynomial* in the curve parameter t. This distinctive feature of PH curves offers many advantages in applications. For a brief review of the construction and properties of PH curves, see Farouki (2002).

Except in the simplest cases, the non-linear nature of PH curves precludes a direct specification by Bézier/B-spline control polygons. Instead, solution of the first-order Hermite interpolation problem is a standard approach to the construction of planar or spatial PH curves satisfying prescribed geometrical constraints. The use of an appropriate representation—in terms of complex variables for planar PH curves (Farouki, 1994) and quaternions (Choi et al., 2002; Farouki et al., 2002a) for spatial PH curves—greatly facilitates the construction and analysis of such interpolants.

The problem of Hermite interpolation by PH curves inherently admits a *multiplicity* of formal solutions, and the issue of selecting a "good" or "best" interpolant among the complete set of nominal solutions must be addressed. For example, interpolation of planar first-order Hermite data by PH quintics generically incurs four distinct solutions (Farouki and Neff, 1995), and a number of methods are available to identify the "good" interpolant (Choi et al., 2007; Farouki and Neff, 1995; Moon et al., 2001). These are based upon either absolute shape measures, or a comparison with the unique "ordinary" cubic interpolant, and include an *a priori* method for constructing the "good" solution alone, under mild constraints on the Hermite data.

Interpolation of first-order spatial Hermite data by PH quintics is a more challenging problem (Farouki et al., 2002b), involving a *two-parameter family* of solutions rather than a finite multiplicity.[1] In this context, the *quaternion representation* (Choi et al., 2002; Farouki et al., 2002a) is typically used—although an alternative approach in terms of the Clifford algebra *geometric product* is described in Perwass et al. (2007). The two free parameters that characterize spatial PH quintic Hermite interpolants are angular variables, α and β, and the identification of "good" or "optimal" values for them is a challenging open problem that must be addressed, since the shape of the PH quintic interpolants may depend sensitively on these parameters.

One approach to specifying α and β is to require the PH quintic Hermite interpolant to be a *helical* curve (Farouki et al., 2004), i.e., a curve whose tangent $\mathbf{t}$ maintains a constant angle ψ with a fixed direction $\mathbf{a}$ in space, the *axis* of the helix. In (Farouki et al., 2004) it was shown that any helical polynomial space curve must be a PH curve and that, whereas all spatial PH cubics are helical, the helical quintics form a proper subset of the spatial PH quintics. As noted in Farouki et al. (2004), when the tangents at the start and end points are not parallel, the helical quintics can be divided into two (non-disjoint) classes, the *general helices* and *monotone helices*. The monotone helical quintics are not sufficiently flexible to interpolate arbitrary Hermite data (Farouki et al., 2004). Although no formal proof for the existence of general helical PH quintic interpolants to arbitrary data was given in Farouki et al. (2004), strong empirical evidence was cited therein that this is true.

It was shown in Farouki et al. (2004) that, to obtain a general helical interpolant, the angle β must satisfy a certain trigonometric equation and also a sign condition. The solutions of the trigonometric equation may be associated with the real roots of a quartic polynomial—see Eq. (51) in Farouki et al. (2004)—and each of these roots, if admissible, yields two different general helical PH quintic interpolants with the same axis, associated with different values of the other angular parameter α. Thus, the additional cost of obtaining a helical PH quintic interpolant is essentially that of solving a quartic equation by Ferrari's method (Uspensky, 1948).

As an initial result, we determine in this paper conditions on the given data that identify when the "ordinary" cubic Hermite interpolant is actually a PH curve, motivated by the desire that the (α, β) selection criteria should identify a spatial PH cubic interpolant when the data are compatible with its existence. The properties of helical PH quintics are then further analyzed and their use in selecting Hermite interpolants is considered. Expressions for the helix axis and pitch angle are derived for both monotone and general helices, and it is shown that the arc lengths of the PH quintic Hermite interpolants depend on only one of the angular parameters,[2] β. As an ancillary result, it is shown that general helical PH quintic interpolants exist for arbitrary Hermite data, and correspond to solutions of minimum or maximum arc length.

[1] In the related study (Sir and Jüttler, 2005) the asymptotic order of approximation to an analytic curve, from which the data is presumed to have been sampled, is used to select these parameters. See also Sir and Jüttler (2007) which is concerned with interpolation of second-order Hermite data by spatial PH curves of degree 9.

[2] This result was independently discovered in the Clifford algebra formulation (Perwass et al., 2007)—it was previously identified (though not published) in terms of the quaternion representation.

These results are used to formulate practical criteria for selecting the two angular parameters α, β that arise in spatial PH quintic Hermite interpolants. Three different criteria for choosing these parameters are introduced, that all generate the cubic interpolant when possible. In the first criterion (BV), the parameters α and β are computed simultaneously, while in the second (HC) and third (CC) criteria, their computation is sequential.

In BV, the angular parameters are computed by minimizing a bivariate function over the domain $(\alpha, \beta) \in [0, 2\pi] \times [0, 2\pi]$ that essentially measures the distance of the PH quintic interpolants from a PH cubic. Thus, a suitable bivariate numerical optimization scheme is required for this criterion. With HC, β is computed first by specifying interpolants of maximum arc length (since, in fact, these typically have smoother shapes). The value for α is then selected among the two solutions of a homogeneous trigonometric equation, obtained from the same optimality criterion as in BV, but expressed in terms of α alone. Finally, CC is a heuristic but more efficient selection scheme, that also determines β first, then α. Specifically, β is computed using a strategy suggested by the approach appropriate to the case in which the "ordinary" cubic interpolant is a PH curve, then α is computed as in method HC.

Computed examples are used to compare the shape of the "best" general helical PH quintic interpolant with those of the three PH quintic interpolants produced by the new selection criteria. Although the helical interpolant has the special geometrical property of a constant curvature/torsion ratio, these examples show that the BV, HC, CC selection criteria can yield interpolants with lower fairness (energy) measures. In particular, since the computational cost of CC is lowest, this criterion seems quite suitable when fair C^1 quintic PH splines interpolating general Hermite data are to be constructed (Farouki et al., 2003).

The remainder of the paper is organized as follows. Section 2 summarizes basic ideas from the algebra of quaternions, and presents some identities that are required subsequently (because of their technical nature, the proofs are given in Appendix A). After a brief review of the quaternion representation of spatial PH curves in Section 3, an analysis of helical PH quintic curves, and the determination of their axes and pitch angles, is presented in Section 4. Section 5 establishes conditions on the Hermite data for the existence of a PH cubic interpolant, while Section 6 states the PH quintic Hermite interpolation problem in a more "geometric" form than on previous occasions. Section 7 analyzes the arc length of spatial PH quintics, and demonstrates that general helical PH quintics interpolants exist for arbitrary Hermite data, and exhibit extremal arc lengths. The criteria for selecting the two free parameters of PH quintic Hermite interpolants are then described in Section 8, and Section 9 presents some computed examples using these criteria. Section 10 concludes by summarizing and assessing the key results of this paper.

2. Basic quaternion algebra results

Quaternions are "four-dimensional numbers" of the form

$$\mathcal{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad \text{and} \quad \mathcal{B} = b + b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}, \tag{2}$$

where the *basis elements* 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Here 1 is the usual real unit: its product with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ leaves them unchanged. Preserving the order of terms in products, we infer from the above that

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{3}$$

Thus, since the products of the basis elements are non-commutative, we have $\mathcal{AB} \neq \mathcal{BA}$ in general. Quaternion multiplication is associative, however, so that $(\mathcal{AB})\mathcal{C} = \mathcal{A}(\mathcal{BC})$ for any three quaternions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

Regarding $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as unit vectors in Cartesian coordinates, we may regard $\mathcal{A}$ as comprising "scalar" and "vector" parts,[3] a and $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$. We write $\mathcal{A} = (a, \mathbf{a})$—conversely,

$$a = \mathrm{scal}(\mathcal{A}) \quad \text{and} \quad \mathbf{a} = \mathrm{vect}(\mathcal{A}). \tag{4}$$

All real numbers and three-dimensional vectors are subsumed as "pure scalar" and "pure vector" quaternions, of the form $(a, \mathbf{0})$ and $(0, \mathbf{a})$, respectively. For brevity, we often denote such quaternions by simply a and $\mathbf{a}$. The sum and

[3] Also known as the "real" and "imaginary" parts—the square of a "pure imaginary" quaternion is always a negative real number.

product of the two quaternions $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ may be concisely expressed (Roe, 1993) in terms of the familiar vector dot and cross products as

$$\mathcal{A} + \mathcal{B} = (a + b, \mathbf{a} + \mathbf{b}), \tag{5}$$

$$\mathcal{AB} = (ab - \mathbf{a} \cdot \mathbf{b}, a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b}). \tag{6}$$

Every quaternion $\mathcal{A} = (a, \mathbf{a})$ has a *conjugate*, $\mathcal{A}^* = (a, -\mathbf{a})$, and a *magnitude* equal to the non-negative real number $|\mathcal{A}|$ defined by

$$|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = a^2 + |\mathbf{a}|^2. \tag{7}$$

One can readily verify that the conjugates of products satisfy the rule

$$(\mathcal{AB})^* = \mathcal{B}^*\mathcal{A}^*. \tag{8}$$

If $|\mathcal{A}| = 1$, we say that $\mathcal{A}$ is a *unit* quaternion. The unit quaternions form a (non-commutative) *group* under multiplication, since the product of two unit quaternions is always a unit quaternion. Unit quaternions are necessarily of the form $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\mathbf{n})$ for some angle θ and unit vector $\mathbf{n}$. For any pure vector quaternion $\mathbf{q}$ and unit quaternion $\mathcal{U}$, the product $\mathcal{U}\,\mathbf{q}\,\mathcal{U}^*$ always yields a pure vector quaternion, that corresponds to a rotation of $\mathbf{q}$ through angle θ about the axis defined by $\mathbf{n}$ (Roe, 1993). Note also that the unit quaternion $-\mathcal{U} = (-\cos\frac{1}{2}\theta, -\sin\frac{1}{2}\theta\mathbf{n})$ specifies a rotation through $2\pi - \theta$ about $-\mathbf{n}$, and thus has exactly the same effect as $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\mathbf{n})$.

2.1. *Solutions of the equation* $\mathcal{A}\mathbf{u}\mathcal{A}^* = \mathbf{d}$

Consider the quaternion equation

$$\mathcal{A}\mathbf{u}\mathcal{A}^* = \mathbf{d}, \tag{9}$$

where $\mathbf{u}$ is a unit vector, and $\mathbf{d}$ is any non-zero vector not aligned with $-\mathbf{u}$. The quaternion solutions of (9) comprise a one-parameter family (Farouki et al., 2002b), which can be conveniently described in terms of the unit vectors

$$\boldsymbol{\delta} = \frac{\mathbf{d}}{|\mathbf{d}|} \quad \text{and} \quad \mathbf{n} = \frac{\mathbf{u} + \boldsymbol{\delta}}{|\mathbf{u} + \boldsymbol{\delta}|}.$$

Here $\boldsymbol{\delta}$ is a unit vector in the direction of $\mathbf{d}$, while $\mathbf{n}$ is the (unit) *bisector* of $\boldsymbol{\delta}$ and $\mathbf{u}$. The solutions to (9) can then be written as

$$\mathcal{A} = \sqrt{|\mathbf{d}|}\,\mathbf{n}\exp(\phi\mathbf{u}), \tag{10}$$

where ϕ is a free angular variable, and we invoke the short-hand notation

$$\exp(\phi\mathbf{u}) = \cos\phi + \sin\phi\,\mathbf{u}.$$

Note that, in expression (10), $\sqrt{|\mathbf{d}|}$ is a scalar, $\mathbf{n}$ is a pure vector quaternion, and $\exp(\phi\mathbf{u})$ is a unit quaternion with vector part in the direction of $\mathbf{u}$ (note also that, when a combination of scalars, vectors, and quaternions is simply written in juxtaposition, the *quaternion* product is implied).

For completeness we remark that if $\boldsymbol{\delta} = -\mathbf{u}$ the general solution of (9) is

$$\mathcal{A} = \sqrt{|\mathbf{d}|}(\boldsymbol{\delta}_1^{\perp}\cos\phi + \boldsymbol{\delta}_2^{\perp}\sin\phi),$$

where the vectors $\boldsymbol{\delta}$, $\boldsymbol{\delta}_1^{\perp}$, $\boldsymbol{\delta}_2^{\perp}$ comprise an orthonormal basis for $\mathbb{R}^3$.

2.2. *Some special quaternion identities*

Because they are rather long, the proofs of the following two propositions are deferred to Appendix A. These results will be used in subsequent analyses.

Proposition 1. *Let* $\mathbf{u}$ *be a unit vector, and* $\mathcal{A}$ *and* $\mathcal{B}$ *be quaternions such that* $\mathbf{d}_{\mathcal{A}} = \mathcal{A}\mathbf{u}\mathcal{A}^*$ *and* $\mathbf{d}_{\mathcal{B}} = \mathcal{B}\mathbf{u}\mathcal{B}^*$ *satisfy* $\mathbf{d}_{\mathcal{A}} \times \mathbf{d}_{\mathcal{B}} \neq \mathbf{0}$. *Then the quaternions* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}\mathbf{u}$, $\mathcal{B}\mathbf{u}$ *are linearly independent.*

Also, given the unit vector $\mathbf{u}$ and quaternions $\mathcal{A}$, $\mathcal{B}$ we introduce the following four quaternions:

$$\begin{aligned}
&\mathcal{Q}_{\mathcal{A}} = (q_{\mathcal{A}}, \mathbf{q}_{\mathcal{A}}) = (\mathcal{A}\mathcal{A}^*, \mathcal{A}\mathbf{u}\mathcal{A}^*),\\
&\mathcal{Q}_{\mathcal{B}} = (q_{\mathcal{B}}, \mathbf{q}_{\mathcal{B}}) = (\mathcal{B}\mathcal{B}^*, \mathcal{B}\mathbf{u}\mathcal{B}^*),\\
&\mathcal{V} = (v, \mathbf{v}) = (\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*, \mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*),\\
&\mathcal{S} = (s, \mathbf{s}) = (-\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*, \mathcal{A}\mathcal{B}^* - \mathcal{B}\mathcal{A}^*).
\end{aligned} \tag{11}$$

For these quaternions, the following relations can be verified.

Proposition 2. *Let $(q, \mathbf{q})$ be either $\mathcal{Q}_{\mathcal{A}}$ or $\mathcal{Q}_{\mathcal{B}}$. Then we have*

$$\mathbf{v} \cdot \mathbf{s} = vs, \qquad |\mathbf{v}|^2 - v^2 = |\mathbf{s}|^2 - s^2, \qquad \mathbf{v} \cdot \mathbf{q} = vq, \qquad \mathbf{s} \cdot \mathbf{q} = sq. \tag{12}$$

3. Quaternion form of PH space curves

A spatial PH curve $\mathbf{r}(t)$ is characterized by the property that its parametric speed (1) is a polynomial in the curve parameter t. This is equivalent (Choi et al., 2002; Farouki et al., 2002a) to the requirement that the hodograph $\mathbf{r}'(t)$ can be expressed as a quaternion product of the form

$$\mathbf{r}'(t) = \mathcal{A}(t)\mathbf{u}\mathcal{A}^*(t), \tag{13}$$

where $\mathbf{u}$ is any fixed unit vector and $\mathcal{A}(t)$ is a quaternion polynomial of degree m for a degree-n PH curve, $n = 2m + 1$. We specify $\mathcal{A}(t)$ in Bernstein form,

$$\mathcal{A}(t) = \sum_{i=0}^{m} \mathcal{A}_i b_i^m(t), \quad \text{where } b_i^m(t) = \binom{m}{i}(1-t)^{m-i}t^i. \tag{14}$$

In particular for cubic and quintic PH curves we have, respectively,

$$\mathcal{A}(t) = \mathcal{A}_0(1-t) + \mathcal{A}_1 t, \tag{15}$$

$$\mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2. \tag{16}$$

Note that, if $\mathcal{Q}$ is any quaternion satisfying $\mathcal{Q}\mathbf{u}\mathcal{Q}^* = \mathbf{u}$, the hodograph (13) can be also written as $\mathbf{r}'(t) = \mathcal{C}(t)\mathbf{u}\mathcal{C}^*(t)$, where $\mathcal{C}(t)$ is defined by (14) with the coefficients $\mathcal{A}_i$, $i = 0, \ldots, n$, replaced with $\mathcal{C}_i = \mathcal{A}_i\mathcal{Q}$, $i = 0, \ldots, n$. This implies that one component of one of the quaternions $\mathcal{A}_i$ can be freely chosen.

4. Helical PH curves

A helix is a curve whose unit tangent $\mathbf{t}$ maintains a constant angle ψ with a fixed direction in space, the *axis* of the helix. If $\mathbf{a}$ is a unit vector specifying the axis, we have

$$\mathbf{t} \cdot \mathbf{a} = \cos\psi = \text{constant}, \tag{17}$$

and hence the helix is called a "curve of constant slope". Since $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$, this implies that any helical polynomial curve is a PH curve (Farouki et al., 2004). In particular, it has been noted that, whereas all spatial PH cubics are helical (Farouki and Sakkalis, 1994), the helical PH quintics form a proper subset (Farouki et al., 2004) of all spatial PH quintics.

Concerning PH quintics, it was shown in Appendix B of Farouki et al. (2004) that, when the four quaternions

$$\mathcal{A}_0, \quad \mathcal{A}_2, \quad \mathcal{A}_0\mathbf{u}, \quad \mathcal{A}_2\mathbf{u} \tag{18}$$

are linearly independent, the PH quintic is helical if and only if

$$\gamma_0 = \gamma_2 = 0 \quad \text{or} \quad 4(c_0 + \gamma_0\mathbf{u})(c_2 + \gamma_2\mathbf{u}) = 1, \tag{19}$$

where we set

$$\mathcal{A}_1 = c_0\mathcal{A}_0 + \gamma_0\mathcal{A}_0\mathbf{u} + c_2\mathcal{A}_2 + \gamma_2\mathcal{A}_2\mathbf{u}. \tag{20}$$

Hence, we can divide helical quintics into two classes.[4] The first corresponds to helices with $\gamma_0 = \gamma_2 = 0$, so that $\mathcal{A}_1$ is a linear combination of $\mathcal{A}_0$ and $\mathcal{A}_2$. We call this type the *general* helical PH quintics: the hodograph components have no common factor, and the tangent $\mathbf{t}$ is capable of reversing its sense of rotation about $\mathbf{a}$ (Farouki et al., 2004). The second class is called the *monotone* helical PH quintics (Farouki et al., 2004)—the hodograph components have a common quadratic factor, and the tangent $\mathbf{t}$ maintains a fixed sense of rotation about $\mathbf{a}$. Now for a PH quintic defined by (13) and (16) satisfying

$$\mathbf{r}'(0) \times \mathbf{r}'(1) \neq \mathbf{0},$$

the four quaternions (18) are linearly independent by Proposition 1, and we can conclude that the curve is helical if and only if (19) holds—i.e., if and only if it is a general helix or a monotone helix.

We now determine the axis $\mathbf{a}$ and pitch angle ψ for both general and monotone helical PH quintics. Consider the hodograph of a PH quintic,

$$\mathbf{r}'(t) = \sum_{i=0}^{4} \mathbf{d}_i b_i^4(t), \tag{21}$$

where

$$\begin{aligned}
&\mathbf{d}_0 = \mathcal{A}_0\mathbf{u}\mathcal{A}_0^*,\\
&\mathbf{d}_1 = \tfrac{1}{2}(\mathcal{A}_0\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_1\mathbf{u}\mathcal{A}_0^*),\\
&\mathbf{d}_2 = \tfrac{1}{6}(\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + 4\mathcal{A}_1\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*),\\
&\mathbf{d}_3 = \tfrac{1}{2}(\mathcal{A}_1\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_1^*),\\
&\mathbf{d}_4 = \mathcal{A}_2\mathbf{u}\mathcal{A}_2^*.
\end{aligned} \tag{22}$$

The corresponding parametric speed (1) is a quartic polynomial,

$$\sigma(t) = \mathcal{A}(t)\mathcal{A}^*(t) = \sum_{i=0}^{4} \sigma_i b_i^4(t), \tag{23}$$

with Bernstein coefficients defined by

$$\begin{aligned}
&\sigma_0 = \mathcal{A}_0\mathcal{A}_0^*,\\
&\sigma_1 = \tfrac{1}{2}(\mathcal{A}_0\mathcal{A}_1^* + \mathcal{A}_1\mathcal{A}_0^*),\\
&\sigma_2 = \tfrac{1}{6}(\mathcal{A}_0\mathcal{A}_2^* + 4\mathcal{A}_1\mathcal{A}_1^* + \mathcal{A}_2\mathcal{A}_0^*),\\
&\sigma_3 = \tfrac{1}{2}(\mathcal{A}_1\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_1^*),\\
&\sigma_4 = \mathcal{A}_2\mathcal{A}_2^*.
\end{aligned} \tag{24}$$

Setting $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{B} = \mathcal{A}_2$ in (11), we obtain

$$\begin{aligned}
&(q_0, \mathbf{q}_0) = (\mathcal{A}_0\mathcal{A}_0^*, \mathcal{A}_0\mathbf{u}\mathcal{A}_0^*),\\
&(q_2, \mathbf{q}_2) = (\mathcal{A}_2\mathcal{A}_2^*, \mathcal{A}_2\mathbf{u}\mathcal{A}_2^*),\\
&(v, \mathbf{v}) = (\mathcal{A}_0\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_0^*, \mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*),\\
&(s, \mathbf{s}) = (-\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*, \mathcal{A}_0\mathcal{A}_2^* - \mathcal{A}_2\mathcal{A}_0^*).
\end{aligned} \tag{25}$$

Thus, using (20), we can write

$$\begin{aligned}
&\mathbf{d}_0 = \mathbf{q}_0, \qquad \mathbf{d}_1 = c_0\mathbf{q}_0 + \tfrac{1}{2}(c_2\mathbf{v} + \gamma_2\mathbf{s}),\\
&\mathbf{d}_2 = \tfrac{1}{6}\big[4(c_0^2 + \gamma_0^2)\mathbf{q}_0 + 4(c_2^2 + \gamma_2^2)\mathbf{q}_2 + (1 + 4c_0c_2 + 4\gamma_0\gamma_2)\mathbf{v} + 4(c_0\gamma_2 - c_2\gamma_0)\mathbf{s}\big],\\
&\mathbf{d}_3 = c_2\mathbf{q}_2 + \tfrac{1}{2}(c_0\mathbf{v} - \gamma_0\mathbf{s}), \qquad \mathbf{d}_4 = \mathbf{q}_2.
\end{aligned}$$

[4] These two classes are not disjoint—it can be shown, for example, that degree-elevated PH cubics belong to both of them.

Analogously, from (24) we obtain

$$\sigma_0 = q_0, \qquad \sigma_1 = c_0 q_0 + \tfrac{1}{2}(c_2 v + \gamma_2 s),$$
$$\sigma_2 = \tfrac{1}{6}\big[4(c_0^2+\gamma_0^2)q_0 + 4(c_2^2+\gamma_2^2)q_2 + (1+4c_0c_2+4\gamma_0\gamma_2)v + 4(c_0\gamma_2 - c_2\gamma_0)s\big],$$
$$\sigma_3 = c_2 q_2 + \tfrac{1}{2}(c_0 v - \gamma_0 s), \qquad \sigma_4 = q_2.$$

In particular, for a general helical PH quintic, we have

$$\begin{aligned}
&\mathbf{d}_0 = \mathbf{q}_0, \qquad \mathbf{d}_1 = c_0\mathbf{q}_0 + \tfrac{1}{2}c_2\mathbf{v},\\
&\mathbf{d}_2 = \tfrac{1}{6}\big[4c_0^2\mathbf{q}_0 + 4c_2^2\mathbf{q}_2 + (1+4c_0c_2)\mathbf{v}\big],\\
&\mathbf{d}_3 = c_2\mathbf{q}_2 + \tfrac{1}{2}c_0\mathbf{v}, \qquad \mathbf{d}_4 = \mathbf{q}_2,\\
&\sigma_0 = q_0, \qquad \sigma_1 = c_0 q_0 + \tfrac{1}{2}c_2 v,\\
&\sigma_2 = \tfrac{1}{6}\big[4c_0^2 q_0 + 4c_2^2 q_2 + (1+4c_0c_2)v\big],\\
&\sigma_3 = c_2 q_2 + \tfrac{1}{2}c_0 v, \qquad \sigma_4 = q_2.
\end{aligned} \tag{26}$$

On the other hand, for a monotone helical PH quintic, we have

$$\begin{aligned}
&\mathbf{d}_0 = \mathbf{q}_0, \qquad \mathbf{d}_1 = c_0\mathbf{q}_0 + \frac{c_0\mathbf{v} - \gamma_0\mathbf{s}}{8(c_0^2+\gamma_0^2)},\\
&\mathbf{d}_2 = \frac{1}{6}\left[4(c_0^2+\gamma_0^2)\mathbf{q}_0 + \frac{\mathbf{q}_2}{4(c_0^2+\gamma_0^2)} + \frac{2c_0}{c_0^2+\gamma_0^2}(c_0\mathbf{v} - \gamma_0\mathbf{s})\right],\\
&\mathbf{d}_3 = \frac{c_0\mathbf{q}_2}{4(c_0^2+\gamma_0^2)} + \frac{1}{2}(c_0\mathbf{v} - \gamma_0\mathbf{s}), \qquad \mathbf{d}_4 = \mathbf{q}_2,\\
&\sigma_0 = q_0, \qquad \sigma_1 = c_0 q_0 + \frac{c_0 v - \gamma_0 s}{8(c_0^2+\gamma_0^2)},\\
&\sigma_2 = \frac{1}{6}\left[4(c_0^2+\gamma_0^2)q_0 + \frac{q_2}{4(c_0^2+\gamma_0^2)} + \frac{2c_0}{c_0^2+\gamma_0^2}(c_0 v - \gamma_0 s)\right],\\
&\sigma_3 = \frac{c_0 q_2}{4(c_0^2+\gamma_0^2)} + \frac{1}{2}(c_0 v - \gamma_0 s), \qquad \sigma_4 = q_2.
\end{aligned} \tag{27}$$

We are now ready to prove the following results characterizing the axes and pitch angles of helical PH quintics (see also Farouki et al. (2004) for the case of general helices).

Proposition 3. *For a general helical PH quintic*

$$\mathbf{a} = \frac{\mathbf{s}}{|\mathbf{s}|} \quad \text{and} \quad \cos\psi = \frac{s}{|\mathbf{s}|}, \tag{28}$$

while for a monotone helical PH quintic

$$\mathbf{a} = \frac{c_0\mathbf{s} + \gamma_0\mathbf{v}}{|c_0\mathbf{s} + \gamma_0\mathbf{v}|} \quad \text{and} \quad \cos\psi = \frac{c_0 s + \gamma_0 v}{|c_0\mathbf{s} + \gamma_0\mathbf{v}|}, \tag{29}$$

where $(v, \mathbf{v})$ *and* $(s, \mathbf{s})$ *are as defined in* (25).

Proof. From (17) and (21)–(24), the following relations must be satisfied

$$\mathbf{d}_i \cdot \mathbf{a} = \sigma_i \cos\psi, \quad i = 0, \ldots, 4. \tag{30}$$

Now in view of (26), for a general helix this is equivalent to proving that

$$\mathbf{q}_0 \cdot \mathbf{a} = q_0 \cos\psi, \qquad \mathbf{q}_2 \cdot \mathbf{a} = q_2 \cos\psi, \qquad \mathbf{v} \cdot \mathbf{a} = v \cos\psi.$$

Choosing $\mathbf{a}$ and $\cos\psi$ as in (28), the result is a consequence of Proposition 2. Considering (27), proving (30) in the case of a monotone helix is equivalent to showing that

$$\mathbf{q}_0 \cdot \mathbf{a} = q_0 \cos\psi, \qquad \mathbf{q}_2 \cdot \mathbf{a} = q_2 \cos\psi, \qquad (c_0\mathbf{v} - \gamma_0\mathbf{s}) \cdot \mathbf{a} = (c_0 v - \gamma_0 s)\cos\psi.$$

Choosing $\mathbf{a}$ and $\cos\psi$ as in (29), the above relations are a consequence of Proposition 2 and the fact that

$$(c_0\mathbf{v} - \gamma_0\mathbf{s}) \cdot (c_0\mathbf{s} + \gamma_0\mathbf{v}) = (c_0^2 - \gamma_0^2)\mathbf{v} \cdot \mathbf{s} + c_0\gamma_0\big(|\mathbf{v}|^2 - |\mathbf{s}|^2\big),$$
$$(c_0 v - \gamma_0 s)(c_0 s + \gamma_0 v) = (c_0^2 - \gamma_0^2)vs + c_0\gamma_0(v^2 - s^2). \qquad \square$$

Concerning the interpolation of arbitrary Hermite data, an example was presented in Farouki et al. (2004) to show that this is not always possible using monotone helical PH quintics. Although no proof for the existence of general helical PH quintic interpolants to arbitrary Hermite data was given in Farouki et al. (2004), no counter-example was found. We now prove, in Section 7 below, that four general helical PH quintic interpolants always exist for arbitrary Hermite data. Moreover, we show that such interpolants correspond to extrema of the arc length L among all PH quintic interpolants to the given Hermite data.

5. Existence of PH cubic interpolants

We now determine conditions under which initial and final points $\mathbf{p}_i$, $\mathbf{p}_f$ and derivatives $\mathbf{d}_i$, $\mathbf{d}_f$ can be interpolated by a spatial PH cubic, i.e., we identify data sets for which the "ordinary" cubic interpolant is a PH curve. Setting

$$\mathbf{w} = 3(\mathbf{p}_f - \mathbf{p}_i) - (\mathbf{d}_i + \mathbf{d}_f) \tag{31}$$

and introducing the vectors

$$\boldsymbol{\delta}_i = \frac{\mathbf{d}_i}{|\mathbf{d}_i|}, \qquad \boldsymbol{\delta}_f = \frac{\mathbf{d}_f}{|\mathbf{d}_f|}, \qquad \mathbf{z} = \frac{\boldsymbol{\delta}_i \times \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|}, \tag{32}$$

we have:

Proposition 4. *The cubic Hermite interpolant to the data points* $\mathbf{p}_i$, $\mathbf{p}_f$ *and derivatives* $\mathbf{d}_i$, $\mathbf{d}_f$ *is a PH curve if and only if*

$$\mathbf{w} \cdot (\boldsymbol{\delta}_i - \boldsymbol{\delta}_f) = 0 \tag{33}$$

and

$$\left(\mathbf{w} \cdot \frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|}\right)^2 + \frac{(\mathbf{w} \cdot \mathbf{z})^2}{|\mathbf{z}|^4} = |\mathbf{d}_i||\mathbf{d}_f|. \tag{34}$$

Proof. The ordinary cubic Hermite interpolant may be expressed in Bézier form as

$$\mathbf{r}(t) = \mathbf{p}_i b_0^3(t) + \big(\mathbf{p}_i + \tfrac{1}{3}\mathbf{d}_i\big)b_1^3(t) + \big(\mathbf{p}_f - \tfrac{1}{3}\mathbf{d}_f\big)b_2^3(t) + \mathbf{p}_f b_3^3(t), \tag{35}$$

and, from (31), its hodograph is

$$\mathbf{r}'(t) = \mathbf{d}_i b_0^2(t) + \mathbf{w} b_1^2(t) + \mathbf{d}_f b_2^2(t). \tag{36}$$

From (13) and (15), we note that (35) is a PH cubic curve if two quaternions $\mathcal{A}_0$, $\mathcal{A}_1$ exist such that

$$\mathbf{r}'(t) = \mathcal{A}_0\mathbf{u}\mathcal{A}_0^* b_0^2(t) + \tfrac{1}{2}(\mathcal{A}_0\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_1\mathbf{u}\mathcal{A}_0^*)b_1^2(t) + \mathcal{A}_1\mathbf{u}\mathcal{A}_1^* b_2^2(t).$$

Assuming (without loss of generality) that $\mathbf{u} = \boldsymbol{\delta}_i$, the above hodograph agrees with (36) if

$$\mathcal{A}_0\boldsymbol{\delta}_i\mathcal{A}_0^* = \mathbf{d}_i, \qquad \mathcal{A}_1\boldsymbol{\delta}_i\mathcal{A}_1^* = \mathbf{d}_f, \qquad \mathcal{A}_0\boldsymbol{\delta}_i\mathcal{A}_1^* + \mathcal{A}_1\boldsymbol{\delta}_i\mathcal{A}_0^* = 2\mathbf{w}. \tag{37}$$

Again, without loss of generality, from the first of equations (37) we can assume that

$$\mathcal{A}_0 = \sqrt{|\mathbf{d}_i|}\,\boldsymbol{\delta}_i, \tag{38}$$

and we then obtain

$$\mathcal{A}_0\boldsymbol{\delta}_i\mathcal{A}_1^* + \mathcal{A}_1\boldsymbol{\delta}_i\mathcal{A}_0^* = \sqrt{|\mathbf{d}_i|}(\mathcal{A}_1 - \mathcal{A}_1^*) = 2\sqrt{|\mathbf{d}_i|}\,\mathrm{vect}(\mathcal{A}_1). \tag{39}$$

On the other hand, from Section 2.1 we know that a solution $\mathcal{A}_1$ of the second equation in (37) has the general form

$$\mathcal{A}_1 = \sqrt{|\mathbf{d}_f|}\,\frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|}(\cos\phi + \sin\phi\,\boldsymbol{\delta}_i). \tag{40}$$

From the quaternion product rules, we can write

$$\mathrm{vect}(\mathcal{A}_1) = \sqrt{|\mathbf{d}_f|}\left(\cos\phi\,\frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|} - \sin\phi\,\mathbf{z}\right), \tag{41}$$

where $\mathbf{z}$ is defined by (32).

Now since the three vectors $\boldsymbol{\delta}_i + \boldsymbol{\delta}_f$, $\mathbf{z}$, $\boldsymbol{\delta}_i - \boldsymbol{\delta}_f$ satisfy

$$(\boldsymbol{\delta}_i + \boldsymbol{\delta}_f)\cdot\mathbf{z} = 0, \qquad \mathbf{z}\cdot(\boldsymbol{\delta}_i - \boldsymbol{\delta}_f) = 0, \qquad (\boldsymbol{\delta}_i - \boldsymbol{\delta}_f)\cdot(\boldsymbol{\delta}_i + \boldsymbol{\delta}_f) = 0,$$

they are mutually orthogonal, and from (39) and (41) we immediately infer that the third condition in (37) can be satisfied only if (33) holds. This means that $\mathbf{w}$ must be a linear combination of $\boldsymbol{\delta}_i + \boldsymbol{\delta}_f$ and $\mathbf{z}$. Writing

$$\mathbf{w} = \mu(\boldsymbol{\delta}_i + \boldsymbol{\delta}_f) + \nu\mathbf{z} \tag{42}$$

and taking the dot product of both sides with $\boldsymbol{\delta}_i + \boldsymbol{\delta}_f$ and $\mathbf{z}$ gives

$$\mu = \mathbf{w}\cdot\frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|^2} \quad\text{and}\quad \nu = \frac{\mathbf{w}\cdot\mathbf{z}}{|\mathbf{z}|^2}. \tag{43}$$

Then if (33) holds, by comparing (37), (39) and (41) with (42) and (43), we may conclude that if the "ordinary" cubic Hermite interpolant is a PH curve, the value of the angular parameter ϕ in (40) must be such that

$$\cos\phi = \frac{1}{\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}}\mathbf{w}\cdot\frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|}, \qquad \sin\phi = \frac{-1}{\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}}\frac{\mathbf{w}\cdot\mathbf{z}}{|\mathbf{z}|^2}. \tag{44}$$

The identity $\cos^2\phi + \sin^2\phi = 1$ implies that the vector $\mathbf{w}$ must satisfy the constraint (34).

The same arguments show that conditions (33) and (34) are also sufficient to ensure that the cubic Hermite interpolant to the data is a PH curve. □

6. PH quintic Hermite interpolants

The hodograph of a spatial PH quintic is defined by substituting the quadratic quaternion polynomial (16) into expression (13), where $\mathbf{u}$ is an arbitrary unit vector. Integrating this hodograph then gives the Bézier form

$$\mathbf{r}(t) = \sum_{i=0}^{5}\mathbf{p}_i b_i^5(t) \tag{45}$$

of the PH quintic, with control points $\mathbf{p}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$ defined by

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \tfrac{1}{5}\mathcal{A}_0\mathbf{u}\mathcal{A}_0^*,\\
\mathbf{p}_2 &= \mathbf{p}_1 + \tfrac{1}{10}(\mathcal{A}_0\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_1\mathbf{u}\mathcal{A}_0^*),\\
\mathbf{p}_3 &= \mathbf{p}_2 + \tfrac{1}{30}(\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + 4\mathcal{A}_1\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*),\\
\mathbf{p}_4 &= \mathbf{p}_3 + \tfrac{1}{10}(\mathcal{A}_1\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_1^*),\\
\mathbf{p}_5 &= \mathbf{p}_4 + \tfrac{1}{5}\mathcal{A}_2\mathbf{u}\mathcal{A}_2^*.
\end{aligned} \tag{46}$$

Interpolation of the end-derivatives yields the equations

$$\mathcal{A}_0\mathbf{u}\mathcal{A}_0^* = \mathbf{d}_i \quad\text{and}\quad \mathcal{A}_2\mathbf{u}\mathcal{A}_2^* = \mathbf{d}_f \tag{47}$$

for $\mathcal{A}_0$ and $\mathcal{A}_2$. Moreover, with $\mathbf{p}_0 = \mathbf{p}_i$ as integration constant, interpolation of the end points gives the condition

$$\begin{aligned}
\int_0^1 \mathcal{A}(t)\mathbf{u}\mathcal{A}^*(t)\,\mathrm{d}t = \mathbf{p}_f - \mathbf{p}_i &= \tfrac{1}{5}\mathcal{A}_0\mathbf{u}\mathcal{A}_0^* + \tfrac{1}{10}(\mathcal{A}_0\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_1\mathbf{u}\mathcal{A}_0^*)\\
&\quad + \tfrac{1}{30}(\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + 4\mathcal{A}_1\mathbf{u}\mathcal{A}_1^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*)\\
&\quad + \tfrac{1}{10}(\mathcal{A}_1\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_1^*) + \tfrac{1}{5}\mathcal{A}_2\mathbf{u}\mathcal{A}_2^*.
\end{aligned} \tag{48}$$

Since Eqs. (47) are of the form (9), they can be solved directly to obtain

$$\mathcal{A}_0 = \sqrt{|\mathbf{d}_i|}\,\mathbf{n}_i\exp(\phi_0\mathbf{u}), \qquad \mathcal{A}_2 = \sqrt{|\mathbf{d}_f|}\,\mathbf{n}_f\exp(\phi_2\mathbf{u}), \tag{49}$$

where ϕ_0 and ϕ_2 are free angular variables, and using (32) we define

$$\mathbf{n}_i = \frac{\boldsymbol{\delta}_i + \mathbf{u}}{|\boldsymbol{\delta}_i + \mathbf{u}|}, \qquad \mathbf{n}_f = \frac{\boldsymbol{\delta}_f + \mathbf{u}}{|\boldsymbol{\delta}_f + \mathbf{u}|}.$$

Knowing $\mathcal{A}_0$ and $\mathcal{A}_2$, and using (47), Eq. (48) can be reduced to

$$\mathcal{B}\mathbf{u}\mathcal{B}^* = \mathbf{d}$$

where $\mathcal{B} = 3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2$, and we define

$$\mathbf{d} = \mathbf{c} + 5(\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^*), \tag{50}$$

$$\mathbf{c} = 120(\mathbf{p}_f - \mathbf{p}_i) - 15(\mathbf{d}_i + \mathbf{d}_f). \tag{51}$$

This equation is again of the form (9), and its general solution is

$$\mathcal{B} = \sqrt{|\mathbf{d}|}\,\mathbf{n}\exp(\phi_1\mathbf{u}) \tag{52}$$

where ϕ_1 is another free angular variable, and we set

$$\boldsymbol{\delta}_d = \frac{\mathbf{d}}{|\mathbf{d}|} \quad\text{and}\quad \mathbf{n} = \frac{\boldsymbol{\delta}_d + \mathbf{u}}{|\boldsymbol{\delta}_d + \mathbf{u}|}. \tag{53}$$

Note that $\mathcal{B}$, and thus $\mathcal{A}_1 = \frac{1}{4}\mathcal{B} - \frac{3}{4}(\mathcal{A}_0 + \mathcal{A}_2)$, depends on ϕ_0, ϕ_2 as well as ϕ_1, due to the dependence of $\mathbf{d}$ on those variables. However, one can show (Farouki et al., 2002b) that the hodograph (13) depends only on the *differences* of ϕ_0, ϕ_1, ϕ_2. Thus, we can take $\phi_1 = 0$ without loss of generality, and regard the PH quintic interpolant as dependent on just the two angular parameters defined by

$$\alpha = \tfrac{1}{2}(\phi_0 + \phi_2) \quad\text{and}\quad \beta = \phi_2 - \phi_0. \tag{54}$$

The three quaternions $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ defining a PH quintic Hermite interpolant can then be expressed in terms of (54) as

$$\begin{aligned}
\mathcal{A}_0 &= \sqrt{|\mathbf{d}_i|}\,\mathbf{n}_i\exp\big((\alpha - \tfrac{1}{2}\beta)\mathbf{u}\big),\\
\mathcal{A}_2 &= \sqrt{|\mathbf{d}_f|}\,\mathbf{n}_f\exp\big((\alpha + \tfrac{1}{2}\beta)\mathbf{u}\big),\\
\mathcal{A}_1 &= \tfrac{1}{4}\sqrt{|\mathbf{d}|}\,\mathbf{n} - \tfrac{3}{4}(\mathcal{A}_0 + \mathcal{A}_2).
\end{aligned} \tag{55}$$

6.1. Reduction of quaternion expressions

We now analyze certain quaternion expressions that will subsequently prove useful. With some algebra, one can verify that the following relations hold for any spatial PH quintic interpolant:

$$\begin{aligned}
\mathbf{v} &= \mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^* = \mathbf{e}\cos\beta + \mathbf{f}\sin\beta,\\
v &= \mathcal{A}_0\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_0^* = e\cos\beta + f\sin\beta,\\
\mathbf{s} &= \mathcal{A}_0\mathcal{A}_2^* - \mathcal{A}_2\mathcal{A}_0^* = \mathbf{f}\cos\beta - \mathbf{e}\sin\beta,\\
s &= \mathcal{A}_2\mathbf{u}\mathcal{A}_0^* - \mathcal{A}_0\mathbf{u}\mathcal{A}_2^* = f\cos\beta - e\sin\beta,
\end{aligned} \tag{56}$$

where the vectors $\mathbf{e}$, $\mathbf{f}$ and the scalars e, f are given by

$$
\begin{aligned}
\mathbf{e} &= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\big[(\mathbf{u}\cdot\mathbf{n}_f)\mathbf{n}_i + (\mathbf{u}\cdot\mathbf{n}_i)\mathbf{n}_f - (\mathbf{n}_i\cdot\mathbf{n}_f)\mathbf{u}\big] \\
&= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\frac{(1-\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_f)\mathbf{u} + (1+\mathbf{u}\cdot\boldsymbol{\delta}_i)\boldsymbol{\delta}_f + (1+\mathbf{u}\cdot\boldsymbol{\delta}_f)\boldsymbol{\delta}_i}{|\mathbf{u}+\boldsymbol{\delta}_i||\mathbf{u}+\boldsymbol{\delta}_f|}, \\
e &= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\mathbf{n}_i\cdot\mathbf{n}_f, \\
\mathbf{f} &= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\mathbf{n}_f\times\mathbf{n}_i = 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\frac{\mathbf{u}\times\boldsymbol{\delta}_i - \mathbf{u}\times\boldsymbol{\delta}_f - \boldsymbol{\delta}_i\times\boldsymbol{\delta}_f}{|\mathbf{u}+\boldsymbol{\delta}_i||\mathbf{u}+\boldsymbol{\delta}_f|}, \\
f &= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\mathbf{u}\cdot(\mathbf{n}_i\times\mathbf{n}_f).
\end{aligned} \tag{57}
$$

Thus, the vector (50) depends only on β, and has the analytic expression

$$\mathbf{d}(\beta) = \mathbf{c} + 5(\cos\beta\,\mathbf{e} + \sin\beta\,\mathbf{f}). \tag{58}$$

As β varies from 0 to 2π, $\mathbf{d}(\beta)$ traces an ellipse in $\mathbb{R}^3$ with center $\mathbf{c}$, residing in the plane with normal in the direction of

$$\mathbf{e}\times\mathbf{f} = 4|\mathbf{d}_i||\mathbf{d}_f|\big[(\mathbf{u}\cdot\mathbf{n}_f)\mathbf{n}_f - (\mathbf{u}\cdot\mathbf{n}_i)\mathbf{n}_i\big].$$

Correspondingly, the unit vector $\mathbf{n}$ defined by (53) also depends on β only.

Now from Proposition 2, we can deduce that

$$\mathbf{e}\cdot\mathbf{f} = -ef \quad\text{and}\quad |\mathbf{e}|^2 - e^2 = |\mathbf{f}|^2 - f^2. \tag{59}$$

Assuming that $\mathbf{u} = \boldsymbol{\delta}_i$ gives the simplifications

$$\mathbf{n}_i = \boldsymbol{\delta}_i \quad\text{and}\quad \mathbf{n}_f = \frac{\boldsymbol{\delta}_i+\boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i+\boldsymbol{\delta}_f|},$$

and consequently

$$
\begin{aligned}
\mathbf{e} &= 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\frac{\boldsymbol{\delta}_i+\boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i+\boldsymbol{\delta}_f|}, \qquad e = 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\boldsymbol{\delta}_i\cdot\frac{\boldsymbol{\delta}_i+\boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i+\boldsymbol{\delta}_f|}, \\
\mathbf{f} &= -2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\,\frac{\boldsymbol{\delta}_i\times\boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i+\boldsymbol{\delta}_f|}, \qquad f = 0.
\end{aligned} \tag{60}
$$

Hence, in this case, we have

$$\mathbf{e}\cdot\mathbf{f} = 0, \qquad |\mathbf{e}|^2 = |\mathbf{f}|^2 + e^2, \qquad e = \mathbf{e}\cdot\boldsymbol{\delta}_i. \tag{61}$$

7. Arc length of PH quintics

The arc length of a spatial PH quintic is $L = \frac{1}{5}(\sigma_0+\sigma_1+\sigma_2+\sigma_3+\sigma_4)$, where the coefficients of the parametric speed $\sigma(t)$ are specified by (24). Using (55), this can be reduced to

$$L = \frac{1}{120}\big[15\big(|\mathbf{d}_i| + |\mathbf{d}_f|\big) + |\mathbf{d}| - 5(\mathcal{A}_0\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_0^*)\big]. \tag{62}$$

Since the vector $\mathbf{d}$ defined by (50) depends only on the angle β, as expressed in (58), and $\mathcal{A}_0\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_0^* = e\cos\beta + f\sin\beta$, with e and f defined in (57), L is a function of β alone. Taking $\mathbf{u} = \boldsymbol{\delta}_i$ and making use of (61), we obtain

$$|\mathbf{d}(\beta)| = \sqrt{|\mathbf{c}|^2 + 25|\mathbf{f}|^2 + 10\mathbf{c}\cdot(\mathbf{e}\cos\beta + \mathbf{f}\sin\beta) + 25e^2\cos^2\beta}. \tag{63}$$

Then we deduce that (62) is defined by the univariate function

$$L(\beta) = \frac{1}{120}\big[15\big(|\mathbf{d}_i| + |\mathbf{d}_f|\big) + |\mathbf{d}(\beta)| - 5e\cos\beta\big]. \tag{64}$$

Now since $L(\beta)$ is a continuous function, it assumes minimum and maximum values in the compact interval $[-\pi, +\pi]$. Moreover, since $L(\beta)$ is a periodic C^1 function if $\mathbf{d}\cdot\mathbf{d} > 0$, the extremal values are stationary points.

The stationary points of $L(\beta)$ are the roots of the equation

$$\mathbf{c}\cdot(-\mathbf{e}\sin\beta + \mathbf{f}\cos\beta) - 5e^2\sin\beta\cos\beta = -e\sin\beta\,\big|\mathbf{d}(\beta)\big|. \tag{65}$$

To determine these roots, we make the substitutions

$$\cos\beta = \frac{1-\tau^2}{1+\tau^2}, \quad \sin\beta = \frac{2\tau}{1+\tau^2}, \quad \text{where } \tau = \tan\tfrac{1}{2}\beta. \tag{66}$$

Squaring both sides in (65), we obtain

$$
\begin{aligned}
&\big[-5e^2(1-\tau^2)2\tau - \mathbf{c}\cdot\mathbf{e}\,2\tau(1+\tau^2) + \mathbf{c}\cdot\mathbf{f}\,(1-\tau^2)(1+\tau^2)\big]^2 \\
&\quad - e^2 4\tau^2\big[\mathbf{c}\cdot\mathbf{c}\,(1+\tau^2)^2 + 25\,\mathbf{e}\cdot\mathbf{e}\,(1-\tau^2)^2 + 25\,\mathbf{f}\cdot\mathbf{f}\,4\tau^2 \\
&\quad + 10\,\mathbf{c}\cdot\mathbf{e}\,(1-\tau^2)(1+\tau^2) + 10\,\mathbf{c}\cdot\mathbf{f}\,2\tau(1+\tau^2)\big] = 0
\end{aligned} \tag{67}
$$

on substituting from (66) and using $\mathbf{f}\cdot\mathbf{f} - \mathbf{e}\cdot\mathbf{e} = -e^2$ from (59) and (61). Collecting powers of $(1+\tau^2)$ and using (61), we can rewrite this equation as

$$
\begin{aligned}
&100e^2\tau^2\big[e^2(1-\tau^2)^2 - (e^2 + \mathbf{f}\cdot\mathbf{f})(1-\tau^2)^2 - \mathbf{f}\cdot\mathbf{f}\,4\tau^2\big] \\
&\quad - 10e^2\,\mathbf{c}\cdot\mathbf{f}\big[(1-\tau^2)^2 + 4\tau^2\big]2\tau(1+\tau^2) + (1+\tau^2)^2\big[(\mathbf{c}\cdot\mathbf{e})^2 4\tau^2 \\
&\quad + (\mathbf{c}\cdot\mathbf{f})^2(1-\tau^2)^2 - 2(\mathbf{c}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f})2\tau(1-\tau^2) - e^2\,\mathbf{c}\cdot\mathbf{c}\,4\tau^2\big] = 0,
\end{aligned}
$$

and hence we obtain

$$(1+\tau^2)^2(\tilde{d}_0 + \tilde{d}_1\tau + \tilde{d}_2\tau^2 + \tilde{d}_3\tau^3 + \tilde{d}_4\tau^4) =: (1+\tau^2)^2\tilde{P}(\tau) = 0, \tag{68}$$

where

$$
\begin{aligned}
\tilde{d}_0 &:= (\mathbf{c}\cdot\mathbf{f})^2, \\
\tilde{d}_1 &:= -20e^2\mathbf{c}\cdot\mathbf{f} - 4(\mathbf{c}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f}), \\
\tilde{d}_2 &:= -100e^2\mathbf{f}\cdot\mathbf{f} + 4(\mathbf{c}\cdot\mathbf{e})^2 - 2(\mathbf{c}\cdot\mathbf{f})^2 - 4e^2\mathbf{c}\cdot\mathbf{c}, \\
\tilde{d}_3 &:= -20e^2\mathbf{c}\cdot\mathbf{f} + 4(\mathbf{c}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f}), \\
\tilde{d}_4 &:= (\mathbf{c}\cdot\mathbf{f})^2.
\end{aligned} \tag{69}
$$

Summarizing, the stationary points of $L(\beta)$ are identified via (66) by the real roots (at least two) of the quartic equation

$$\tilde{P}(\tau) = 0. \tag{70}$$

Proposition 5. *When $|\mathbf{d}(\beta)| > 0$ and $\boldsymbol{\delta}_f \neq -\mathbf{u}$, the arc length function $L(\beta)$ given by (64) has only one minimum and one maximum.*

Proof. Consider the auxiliary function

$$\ell(\beta) = \frac{1}{120}\big[|\mathbf{d}(\beta)| + 5e\cos\beta\big], \tag{71}$$

whose stationary points are the roots of the equation

$$\mathbf{c}\cdot(-\mathbf{e}\sin\beta + \mathbf{f}\cos\beta) - 5e^2\sin\beta\cos\beta = e\sin\beta|\mathbf{d}(\beta)|. \tag{72}$$

Comparing (65) and (72), it is clear that the stationary points of $\ell(\beta)$ are given via (66) by (at least two) real roots of Eq. (70) but no solutions of (65) can be solutions[5] of (72). On the other hand, by continuity arguments both (65) and (72) have at least two solutions. Thus, we conclude that $\tilde{P}(\tau)$ has exactly four real roots: two of them correspond to solutions of (72), and the remaining two correspond to the only two solutions of (65). □

[5] Assuming $\mathbf{d}\cdot\mathbf{d} > 0$ and $\boldsymbol{\delta}_f \neq -\mathbf{u}$ the only common solution of the two equations is β such that $\sin\beta = 0$, and this implies $\mathbf{c}\cdot\mathbf{f} = 0$, which is the case of planar data that does not concern us here.

The next proposition shows that any solution of (65) identifies an $L(\beta)$ corresponding to the arc length of a general helical PH quintic interpolant. Hence, the general helical PH quintic interpolants always have minimal or maximal arc length.

Proposition 6. *If the PH quintic Hermite interpolant defined by* (45), (46), *and* (55) *is a general helix, the corresponding angle β is a stationary point of the arc length function $L(\beta)$.*

Proof. The unit axis vector $\mathbf{a}$ of the helix and the constant $\cos\psi$ are as defined in (28). Using (56), we can write

$$\mathbf{a} = \frac{\mathbf{f}\cos\beta - \mathbf{e}\sin\beta}{|\mathbf{f}\cos\beta - \mathbf{e}\sin\beta|}. \tag{73}$$

In addition, assuming without loss of generality that $\mathbf{u} = \boldsymbol{\delta}_i$, and using (25), (56), and (60), we obtain

$$\cos\psi = -\frac{e\sin\beta}{|\mathbf{f}\cos\beta - \mathbf{e}\sin\beta|}. \tag{74}$$

Now (17) implies that

$$\int_0^1 \left|\mathbf{r}'(t)\right|\cos\psi\,\mathrm{d}t = \int_0^1 \mathbf{r}'(t)\cdot\mathbf{a}\,\mathrm{d}t = (\mathbf{p}_f - \mathbf{p}_i)\cdot\mathbf{a},$$

so the arc length $L(\beta)$ must be equal to $(\mathbf{p}_f - \mathbf{p}_i)\cdot\mathbf{a}/\cos\psi$. Hence, using (64), (73), and (74), we obtain

$$15\big(|\mathbf{d}_i| + |\mathbf{d}_f|\big) + \big|\mathbf{d}(\beta)\big| - 5e\cos\beta = \frac{120}{e\sin\beta}(\mathbf{p}_f - \mathbf{p}_i)\cdot(\mathbf{e}\sin\beta - \mathbf{f}\cos\beta). \tag{75}$$

Using the vector $\mathbf{c}$ defined by (51), this can be re-written as

$$15\big(|\mathbf{d}_i| + |\mathbf{d}_f|\big)e\sin\beta + \big|\mathbf{d}(\beta)\big|e\sin\beta - 5e^2\sin\beta\cos\beta = \big(\mathbf{c} + 15(\mathbf{d}_i + \mathbf{d}_f)\big)\cdot(\mathbf{e}\sin\beta - \mathbf{f}\cos\beta). \tag{76}$$

Now the simplified expressions (60) for $\mathbf{e}$ and $\mathbf{f}$ imply that $\mathbf{f}\cdot\mathbf{d}_i = \mathbf{f}\cdot\mathbf{d}_f = 0$ and $\mathbf{e}\cdot(\mathbf{d}_i + \mathbf{d}_f) = e(|\mathbf{d}_i| + |\mathbf{d}_f|)$. Thus (76) simplifies to (65). □

We now show that four distinct general helical PH quintic interpolants exist for arbitrary spatial Hermite data. The construction of general helical PH quintic Hermite interpolants can be reduced (Farouki et al., 2004) to determining values of the angle β that satisfy

$$\frac{(\mathbf{c},\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}\,\frac{(\mathbf{d}_i,\mathbf{c},\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} = \left[\frac{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{c})}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} + 5\right]^2 \tag{77}$$

where $(\mathbf{x},\mathbf{y},\mathbf{z})$ denotes the triple product $(\mathbf{x}\times\mathbf{y})\cdot\mathbf{z}$, and $\mathbf{d}_i$, $\mathbf{d}_f$, $\mathbf{c}$, $\mathbf{e}$, $\mathbf{f}$ are as defined above. Substituting from (66), this corresponds to computing the real roots of the quartic equation

$$d_4\tau^4 + d_3\tau^3 + d_2\tau^2 + d_1\tau + d_0 = 0 \tag{78}$$

in τ, with coefficients

$$\begin{aligned}
d_4 &= (\mathbf{c},\mathbf{d}_f,\mathbf{e})(\mathbf{d}_i,\mathbf{c},\mathbf{e}) - (\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}-5\mathbf{e})^2,\\
d_3 &= -2(\mathbf{c},\mathbf{d}_f,\mathbf{e})(\mathbf{d}_i,\mathbf{c},\mathbf{f}) - 2(\mathbf{d}_i,\mathbf{c},\mathbf{e})(\mathbf{c},\mathbf{d}_f,\mathbf{f}) - 20(\mathbf{d}_i,\mathbf{d}_f,\mathbf{f})(\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}-5\mathbf{e}),\\
d_2 &= -2(\mathbf{c},\mathbf{d}_f,\mathbf{e})(\mathbf{d}_i,\mathbf{c},\mathbf{e}) + 4(\mathbf{c},\mathbf{d}_f,\mathbf{f})(\mathbf{d}_i,\mathbf{c},\mathbf{f}) - 100(\mathbf{d}_i,\mathbf{d}_f,\mathbf{f})^2 - 2(\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}-5\mathbf{e})(\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}+5\mathbf{e}),\\
d_1 &= 2(\mathbf{c},\mathbf{d}_f,\mathbf{f})(\mathbf{d}_i,\mathbf{c},\mathbf{e}) + 2(\mathbf{d}_i,\mathbf{c},\mathbf{f})(\mathbf{c},\mathbf{d}_f,\mathbf{e}) - 20(\mathbf{d}_i,\mathbf{d}_f,\mathbf{f})(\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}+5\mathbf{e}),\\
d_0 &= (\mathbf{c},\mathbf{d}_f,\mathbf{e})(\mathbf{d}_i,\mathbf{c},\mathbf{e}) - (\mathbf{d}_i,\mathbf{d}_f,\mathbf{c}+5\mathbf{e})^2.
\end{aligned} \tag{79}$$

On the other hand it was also shown in Farouki et al. (2004) that, to conform to the definition of a general helical PH quintic interpolant, the value of the angular variable β associated with a real root of (78) must be such that

$$\frac{(\mathbf{c},\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} \geqslant 0, \qquad \frac{(\mathbf{d}_i,\mathbf{c},\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} \geqslant 0. \tag{80}$$

Two distinct general helical PH quintic interpolants are associated with each root of (78) for which these inequalities hold—see (Farouki et al., 2004)—and they share the same axis and arc length, since they have the same β value.

Using MAPLE, we find that the coefficients (69) and (79) are related by

$$d_i = K\tilde{d}_i, \quad i = 0,\ldots,4 \tag{81}$$

for some non-zero constant K (assuming that $\mathbf{u} = \boldsymbol{\delta}_i$). Since (70) and (78) have the same roots, we conclude that (78) has exactly four real roots by the proof of Proposition 5. On the other hand, we have already shown that there are at most two admissible roots, because the corresponding β value must be a solution of (65)—see Proposition 6. In the following proposition, we show that there are always exactly *two* admissible roots, and hence four general helical PH quintic interpolants.

Proposition 7. *For arbitrary Hermite data, each of the two values for the angular variable β that identify extrema of the arc length function $L(\beta)$ allows us to define two general helical PH quintic interpolants.*

Proof. We have already verified that each of the two β values that identify extrema of $L(\beta)$ is associated with a (real) root of (78)—see Proposition 5. Thus, assuming (without loss of generality) that $\mathbf{u} = \boldsymbol{\delta}_i$, we must show that satisfaction of (65) implies both the inequalities in (80). Concerning this, we observe that the two quantities in (80) have the same sign when β corresponds to any (real) solution of (78), i.e., to any solution of (77). Thus, it is more convenient to prove that the inequality

$$|\mathbf{d}_i|\frac{(\mathbf{c},\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} + |\mathbf{d}_f|\frac{(\mathbf{d}_i,\mathbf{c},\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)} \geqslant 0 \tag{82}$$

holds. From the simplified expressions for $\mathbf{e}$ and $\mathbf{f}$ in (60) we have $(\mathbf{d}_i,\mathbf{d}_f,\mathbf{e}) = 0$ and $(\mathbf{d}_i,\mathbf{d}_f,\mathbf{f}) \leqslant 0$. Then the previous inequality can be re-written as

$$\frac{(\mathbf{c},\boldsymbol{\delta}_f - \boldsymbol{\delta}_i,\mathbf{e}\cos\beta + \mathbf{f}\sin\beta)}{\sin\beta} \leqslant 0,$$

i.e.,

$$\mathbf{c}\cdot\big[(\boldsymbol{\delta}_f - \boldsymbol{\delta}_i)\times\mathbf{e}\big]\cot\beta + \mathbf{c}\cdot\big[(\boldsymbol{\delta}_f - \boldsymbol{\delta}_i)\times\mathbf{f}\big] \leqslant 0.$$

Now using (60), with some algebra we obtain

$$(\boldsymbol{\delta}_f - \boldsymbol{\delta}_i)\times\mathbf{e} = 2\mathbf{f}, \qquad (\boldsymbol{\delta}_f - \boldsymbol{\delta}_i)\times\mathbf{f} = -(1-\lambda)\mathbf{e},$$

where $\lambda = \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_f$. Then, we may re-write (82) as

$$2\mathbf{c}\cdot\mathbf{f}\cot\beta - (1-\lambda)\mathbf{c}\cdot\mathbf{e} \leqslant 0. \tag{83}$$

Now from (65), we have

$$(\mathbf{c}\cdot\mathbf{f})\cot\beta - \mathbf{c}\cdot\mathbf{e} = 5e^2\cos\beta - e|\mathbf{d}(\beta)|.$$

Then, since $e \geqslant 0$ from (61), we can write (83) as

$$|\mathbf{d}(\beta)| \geqslant 5e\cos\beta + \frac{1+\lambda}{2e}\mathbf{c}\cdot\mathbf{e}. \tag{84}$$

Now, using the simplified expressions in (60), we have

$$e = |\mathbf{e}|\sqrt{\frac{1+\lambda}{2}},$$

and (84) becomes

$$|\mathbf{d}(\beta)| \geqslant \frac{e}{|\mathbf{e}|}\left[5|\mathbf{e}|\cos\beta + \frac{\mathbf{c}\cdot\mathbf{e}}{|\mathbf{e}|}\right].$$

On the other hand, considering that $\mathbf{e}\cdot\mathbf{f} = 0$, from the expression for $\mathbf{d}(\beta)$ in (58), we obtain

$$5|\mathbf{e}|\cos\beta + \frac{\mathbf{c}\cdot\mathbf{e}}{|\mathbf{e}|} = \mathbf{d}(\beta)\cdot\frac{\mathbf{e}}{|\mathbf{e}|}.$$

Thus, since $|\mathbf{v}| \geqslant |\mathbf{v}\cdot\mathbf{u}|$ for any vector $\mathbf{v}$ and unit vector $\mathbf{u}$, we have $e < |\mathbf{e}|$ and the proposition is proved. □

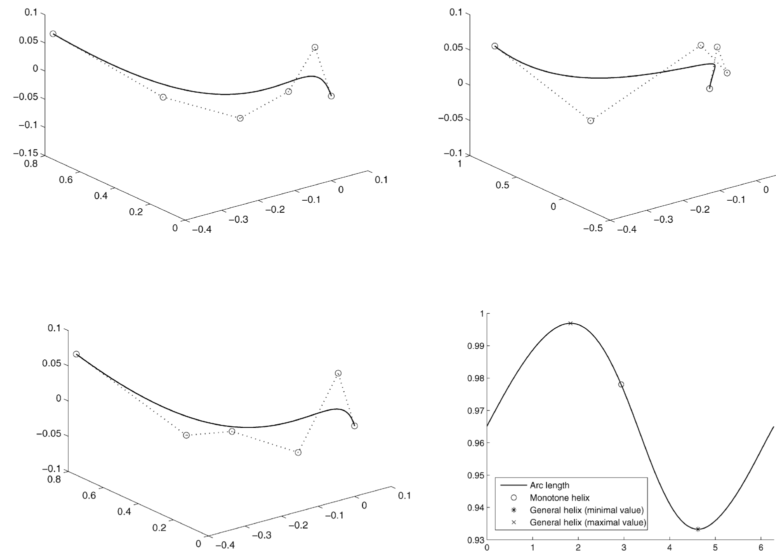

Fig. 1. Helical PH quintic interpolants to the Hermite data $\mathbf{p}_i = (0,0,0)$, $\mathbf{d}_i = (0.48147, 1.47196, 0.13832)$ and $\mathbf{p}_f = (-0.38943, 0.77619, 0.06792)$, $\mathbf{d}_f = (-1.09182, 0.86153, 0.63159)$. Top left: general helix, with maximum arc length; top right: general helix, with minimum arc length; bottom left: a monotone helix; bottom right: the arc length plotted as a function of β.

Remark 1. The two general helical PH quintic interpolants associated with each of the two extrema of the arc length $L(\beta)$ on $[0, 2\pi]$ have not only the same arc length, but also the same axis, since they share the same β value.

Remark 2. It can be verified by a numerical example that, in general, the arc length of a monotone-helical PH quintic interpolant (when it exists) is not a critical value of $L(\beta)$—see Fig. 1.

8. Selection of angular parameters

As observed in Section 6, the solution to the PH quintic Hermite interpolation problem, specified by (55), depends on the free angular parameters α, β. We wish to compute "optimal" choices for these parameters. As one criterion of optimality, we consider minimization of the quantity

$$F(\alpha, \beta) = \left|\mathcal{A}_1 - \tfrac{1}{2}(\mathcal{A}_0 + \mathcal{A}_2)\right|^2, \tag{85}$$

since $F(\alpha, \beta) = 0$ identifies the unique condition under which the quadratic polynomial (16) is actually a *degree-elevated linear polynomial*, so the curve becomes a degree-elevated PH cubic.

In general, it is not possible to achieve $F = 0$ under the given interpolation constraints (see Section 5), but we regard the goal of placing $\mathcal{A}_1$ as close as possible to $\frac{1}{2}(\mathcal{A}_0 + \mathcal{A}_2)$ as being motivated by the desire for a "reasonable" quadratic pre-image curve $\mathcal{A}(t)$ in $\mathbb{H}$ for the hodograph (13). An additional motivation for the chosen F is that its dependence on α, β can be explicitly derived, and its derivatives with respect to these parameters admit closed-form expressions. Substituting from (55) and simplifying, we obtain

$$\begin{aligned}
16F &= |\mathbf{d}(\beta)| \\
&\quad - 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_i|}\left[\cos\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{n}_i - \sin\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_i\right] \cdot \mathbf{n}(\beta) \\
&\quad - 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_f|}\left[\cos\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{n}_f - \sin\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_f\right] \cdot \mathbf{n}(\beta) \\
&\quad + 25\left[|\mathbf{d}_i| + |\mathbf{d}_f| + 2\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\left(\mathbf{n}_i \cdot \mathbf{n}_f \cos\beta + \mathbf{u} \cdot (\mathbf{n}_i \times \mathbf{n}_f)\sin\beta\right)\right].
\end{aligned}$$

To determine the extrema of F, we require its partial derivatives F_α, F_β. The derivative of the vector (58) is given by

$$\mathbf{d}'(\beta) = 5(\cos\beta\,\mathbf{f} - \sin\beta\,\mathbf{e}).$$

Its magnitude and the square root of its magnitude have the derivatives

$$|\mathbf{d}(\beta)|' = \frac{\mathbf{d}(\beta) \cdot \mathbf{d}'(\beta)}{|\mathbf{d}(\beta)|}, \qquad \sqrt{|\mathbf{d}(\beta)|}' = \frac{|\mathbf{d}(\beta)|'}{2\sqrt{|\mathbf{d}(\beta)|}}.$$

For the unit vector $\boldsymbol{\delta}$ and the bisector $\mathbf{n}$, the derivatives may be written as

$$\begin{aligned}
\boldsymbol{\delta}'(\beta) &= \frac{|\mathbf{d}(\beta)|^2\mathbf{d}'(\beta) - [\mathbf{d}(\beta) \cdot \mathbf{d}'(\beta)]\mathbf{d}(\beta)}{|\mathbf{d}(\beta)|^3}, \\
\mathbf{n}'(\beta) &= \frac{|\mathbf{u} + \boldsymbol{\delta}(\beta)|^2\boldsymbol{\delta}'(\beta) - [(\mathbf{u} + \boldsymbol{\delta}(\beta)) \cdot \boldsymbol{\delta}'(\beta)]\boldsymbol{\delta}(\beta)}{|\mathbf{u} + \boldsymbol{\delta}(\beta)|^3}.
\end{aligned}$$

We can then formulate the partial derivatives of F explicitly as

$$\begin{aligned}
16F_\alpha &= 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_i|}\left[\sin\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{n}_i + \cos\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_i\right] \cdot \mathbf{n}(\beta) \\
&\quad + 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_f|}\left[\sin\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{n}_f + \cos\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_f\right] \cdot \mathbf{n}(\beta), \\
16F_\beta &= |\mathbf{d}(\beta)|' - 10\sqrt{|\mathbf{d}(\beta)|}'\sqrt{|\mathbf{d}_i|}\left[\cos\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{n}_i - \sin\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_i\right] \cdot \mathbf{n}(\beta) \\
&\quad - 10\sqrt{|\mathbf{d}(\beta)|}'\sqrt{|\mathbf{d}_f|}\left[\cos\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{n}_f - \sin\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_f\right] \cdot \mathbf{n}(\beta) \\
&\quad - 5\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_i|}\left[\sin\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{n}_i + \cos\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_i\right] \cdot \mathbf{n}(\beta) \\
&\quad + 5\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_f|}\left[\sin\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{n}_f + \cos\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_f\right] \cdot \mathbf{n}(\beta) \\
&\quad - 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_i|}\left[\cos\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{n}_i - \sin\left(\alpha - \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_i\right] \cdot \mathbf{n}'(\beta) \\
&\quad - 10\sqrt{|\mathbf{d}(\beta)||\mathbf{d}_f|}\left[\cos\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{n}_f - \sin\left(\alpha + \tfrac{1}{2}\beta\right)\mathbf{u} \times \mathbf{n}_f\right] \cdot \mathbf{n}'(\beta) \\
&\quad + 50\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\left(\mathbf{u} \cdot (\mathbf{n}_i \times \mathbf{n}_f)\cos\beta - \mathbf{n}_i \cdot \mathbf{n}_f \sin\beta\right).
\end{aligned}$$

An approximation of the values (α, β) that identify the global minimum of F over the domain $[0, 2\pi]^2$ can be determined by means of an appropriate numerical scheme. We refer to this procedure for selecting the free angular parameters as the BV (bivariate) criterion.

A simplification of the above method is embodied in the following two-step scheme. First, the value of β is computed by finding the solution of (65)—using the substitution (66)—that identifies curves of maximal arc length (since such interpolants are usually found to have smoother shapes). An α value is then selected by minimizing (85) considered as a function of α alone. The desired α value can be explicitly obtained as root of F_α, i.e., by solving a homogeneous trigonometric equation. In this case, we have

$$\tan\alpha = \frac{[\sqrt{|\mathbf{d}_i|}(\tan\frac{1}{2}\beta\mathbf{n}_i - \mathbf{u} \times \mathbf{n}_i) - \sqrt{|\mathbf{d}_f|}(\tan\frac{1}{2}\beta\mathbf{n}_f + \mathbf{u} \times \mathbf{n}_f)] \cdot \mathbf{n}(\beta)}{[\sqrt{|\mathbf{d}_i|}(\mathbf{n}_i + \tan\frac{1}{2}\beta\mathbf{u} \times \mathbf{n}_i) + \sqrt{|\mathbf{d}_f|}(\mathbf{n}_f - \tan\frac{1}{2}\beta\mathbf{u} \times \mathbf{n}_f)] \cdot \mathbf{n}(\beta)}. \tag{86}$$

We call this method of selecting α, β the HC (helical-cubic) criterion.

Finally, we suggest a heuristic criterion based on the results of Section 5 for selecting the two free parameters α and β. In this approach, we select the angle β by requiring that

$$\mathcal{A}_0\mathbf{u}\mathcal{A}_2^* + \mathcal{A}_2\mathbf{u}\mathcal{A}_0^* = \mathbf{w}_h, \tag{87}$$

where $\mathbf{w}_h$ is a vector orthogonal to $\boldsymbol{\delta}_f - \boldsymbol{\delta}_i$ of suitable length, to make the imposition of condition (87) possible. More precisely, if $\mathbf{w}$ and $\mathbf{z}$ are defined as in Section 5, we set

$$\mathbf{w}_0 = \mathbf{w} - \left(\mathbf{w} \cdot \frac{\boldsymbol{\delta}_f - \boldsymbol{\delta}_i}{|\boldsymbol{\delta}_f - \boldsymbol{\delta}_i|}\right)\frac{\boldsymbol{\delta}_f - \boldsymbol{\delta}_i}{|\boldsymbol{\delta}_f - \boldsymbol{\delta}_i|},$$

and define $\mathbf{w}_h$ from (34) by

$$\mathbf{w}_h = \mathbf{w}_0\sqrt{|\mathbf{d}_i||\mathbf{d}_f|}\left[\left(\mathbf{w}_0 \cdot \frac{\boldsymbol{\delta}_i + \boldsymbol{\delta}_f}{|\boldsymbol{\delta}_i + \boldsymbol{\delta}_f|}\right)^2 + \left(\frac{\mathbf{w}_0 \cdot \mathbf{z}}{|\mathbf{z}|^2}\right)^2\right]^{-\frac{1}{2}}. \tag{88}$$

The angle α is then determined as in HC, i.e., using (86). We call this method of selecting the free angular parameters the CC (cubic–cubic) criterion.

Remark 3. The BV, HC, and CC criteria all produce a PH cubic interpolant when the given Hermite data are compatible with its existence. Furthermore, in such cases the interpolant will be a helix, since all PH cubics are helical.

9. Numerical results

To illustrate the performance of the above selection criteria, we now present examples of the first-order PH quintic Hermite interpolants they generate, and compare these curves to the general helical PH quintic interpolants—referred to here as HL. For the five data sets listed in Table 1, we evaluate the integral shape measures

$$L = \int_0^1 \sigma \, \mathrm{d}t, \qquad E_{\mathrm{RMF}} = \int_0^1 \kappa^2\sigma \, \mathrm{d}t, \qquad E = \int_0^1 \omega^2\sigma \, \mathrm{d}t, \tag{89}$$

where σ is the parametric speed, κ is the usual curvature, and $\omega = \sqrt{\kappa^2+\tau^2}$ is the *total curvature*, which depends also on the torsion τ (Kreyszig, 1959). L is the arc length, while E_{RMF} and E are energy measures corresponding to different adapted orthonormal frames along the curves—the basis vectors in the curve normal plane indicate the "twist" of the curve.

The energy measure E, employed in (Farouki et al., 2002b), corresponds to using the Frenet frame to specify the twist of the curve, since the total curvature ω represents the rotation rate of the Frenet frame (Kreyszig, 1959). However, as noted in (Farouki and Han, 2003), E_{RMF} is preferable as an intrinsic shape measure for space curves, since it gives the least possible energy value among all adapted frames—namely, the value corresponding to a *rotation-minimizing frame* (with rotation rate κ).

The computed values of the integrals (89) are summarized in Table 2. To impart an idea of how these values relate to the overall range for each shape measure, we also present in Table 3 the "percent values" defined by

$$\%X = 100 \times \frac{X - X_{\min}}{X_{\max} - X_{\min}}, \tag{90}$$

where X is any one of the quantities (89), with extremum values $X_{\min}$, $X_{\max}$. The arc length L can be obtained analytically, but numerical quadrature is required for E and E_{RMF}. Extremal values of these quantities are estimated by evaluating them on a uniform 126×126 grid over $(\alpha,\beta) \in [0,2\pi]^2$ and augmenting these grid values with those obtained using the selection criteria HL, HC, BV, CC. The spatial PH quintics corresponding to the (α,β) values obtained from each of the selection criteria are illustrated in Figs. 2–6.

In general, L is not a very reliable shape quality indicator, and its range of variation is rather small. Therefore, we focus on E and E_{RMF}—in particular, the latter, since it is the least energy among all possible adapted frames on space curves. From Tables 2 and 3, we observe that all four of the methods HL, HC, BV, CC are in reasonable agreement in selecting a "good" spatial PH quintic interpolant among the two-parameter family of formal solutions. The largest discrepancy occurs in case #5 (where $\mathbf{d}_i$ and $\mathbf{d}_f$ are of disparate magnitudes)—in this case, HL gives an appreciably poorer choice than HC, BV, CC. In the four cases other than #4 (for which the curves are identical), CC gives the least E_{RMF} value in two of them, and BV in the other two.

However, the percent differences listed in Table 3 are rather insignificant, typically $\ll 0.1\%$. This indicates that, while all four selection criteria do an excellent job of identifying PH quintic interpolants with near-optimal shape (i.e., least E_{RMF}), there exist curves among the two-parameter family of formal solutions with *much worse* shape quality. A random or *ad hoc* choice for (α,β) might easily result in one of these poorly-shaped interpolants. The selection criteria proposed herein are thus of great practical importance.

Since the CC criterion is computationally much less expensive than BV, it appears to be the best "pragmatic" selection scheme. Note that HL does not produce a least E_{RMF} value in any of the test cases (except #4), so the helicity property is not necessarily *per se* a "good shape" indicator.

We conclude this section by briefly comparing our results with those of a recent study (Sir and Jüttler, 2005) that also deals with the problem of first-order Hermite interpolation by spatial PH quintics. In Sir and Jüttler (2005) the authors prove that setting the two free angular parameters equal to 0 results in a solution that, among other properties, yields fourth-order convergence to an analytic curve, from which the data is presumed to have been sampled.

However, we would like to emphasize here that, whenever we are interested in a *fixed* (finite) length of the sampling interval—rather than the solution behavior as this interval diminishes to zero—only data-dependent selection criteria for the free angular parameters yield "good" C^1 PH curve Hermite interpolants. To illustrate this, Fig. 7 depicts (on the left) the PH quintic Hermite interpolant to the data of test case #3, taking $\alpha = \beta = 0$. The corresponding percent values for the quantities (89) are $\%L = 2.6$, $\%E = 0.38836$, and $\%E_{\mathrm{RMF}} = 34.072$. Thus, using $\alpha = \beta = 0$ gives an L close to the minimum value, but E and E_{RMF} are *much larger* than obtained with any of the selection criteria HL, HC, BV, CC.

Fig. 7 also shows (on the right) the C^1 PH quintic spline interpolating Hermite data obtained by evaluating at $t = 0, \frac{1}{2}, 1$ the "ordinary" cubic interpolant to the Hermite data of case #3, always taking $\alpha = \beta = 0$. On the other hand, Fig. 8 confirms that this choice is appropriate when asymptotic convergence of the result is of primary concern, by illustrating the behavior of approximations to the cubic using 4 and 8 PH quintic interpolants, defined by the choices $\alpha = \beta = 0$ in each case.

Table 1
Derivative data for the five test curves. In each case, the end points are $\mathbf{p}_i = (0,0,0)$ and $\mathbf{p}_f = (1,1,1)$—except for case #4, in which the end point $\mathbf{p}_f = (0.15396, -0.60997, 0.40867)$ is chosen such that the "ordinary" cubic Hermite interpolant is actually a PH curve

	$\mathbf{d}_i$	$\mathbf{d}_f$
case #1	(1.0, 0.0, 1.0)	(0.0, 1.0, 1.0)
case #2	(−0.8, 0.3, 1.2)	(0.5, −1.3, −1.0)
case #3	(0.4, −1.5, −1.2)	(−1.2, −0.6, −1.2)
case #4	(−0.8, 0.3, 1.2)	(0.5, −1.3, −1.0)
case #5	(10.0, 0.0, 10.0)	(0.0, 1.0, 1.0)

Table 2
Values of the integrals (89) for the five sets of Hermite data in Table 1

	HL	HC	BV	CC
		case #1		
L	1.8254	1.8254	1.8164	1.8233
E	4.9737	4.9737	3.4003	4.0583
E_{RMF}	1.2736	1.2736	1.2782	1.2622
		case #2		
L	2.3597	2.3597	2.3551	2.3569
E	8.7789	8.7037	8.5180	8.5315
E_{RMF}	8.4383	8.3502	8.3022	8.2987
		case #3		
L	2.8780	2.8780	2.8754	2.8723
E	16.2503	16.2491	16.1802	16.1989
E_{RMF}	16.1767	16.1753	16.1459	16.1663
		case #4 (PH cubic)		
L	1.1469	1.1469	1.1469	1.1469
E	7.7459	7.7459	7.7459	7.7459
E_{RMF}	7.1044	7.1044	7.1044	7.1044
		case #5		
L	3.3489	3.3489	3.2865	3.3433
E	21.9795	23.0214	20.7990	21.7361
E_{RMF}	19.1460	16.1940	15.6567	15.6787

Table 3
Percent values of the shape integrals (89), as defined by (90)

	HL	HC	BV	CC
		case #1		
$\%L$	100.0	100.0	81.42	95.63
$\%E$	0.017063	0.017063	0.001682	0.008115
$\%E_{\mathrm{RMF}}$	0.000119	0.000119	0.000168	0.000000
		case #2		
$\%L$	100.0	100.0	98.42	99.04
$\%E$	0.001426	0.001015	0.000000	0.000074
$\%E_{\mathrm{RMF}}$	0.010571	0.005230	0.002322	0.002108
		case #3		
$\%L$	100.0	100.0	99.56	99.01
$\%E$	0.000635	0.000626	0.000064	0.000216
$\%E_{\mathrm{RMF}}$	0.053331	0.051964	0.022791	0.043083
		case #4 (PH cubic)		
$\%L$	100.0	100.0	100.0	100.0
	0.000134	0.000134	0.000134	0.000134
$\%E_{\mathrm{RMF}}$	0.002277	0.002277	0.002277	0.002277
		case #5		
$\%L$	100.0	100.0	91.28	99.22
$\%E$	0.000341	0.000429	0.000241	0.000320
$\%E_{\mathrm{RMF}}$	0.000624	0.000361	0.000313	0.000315

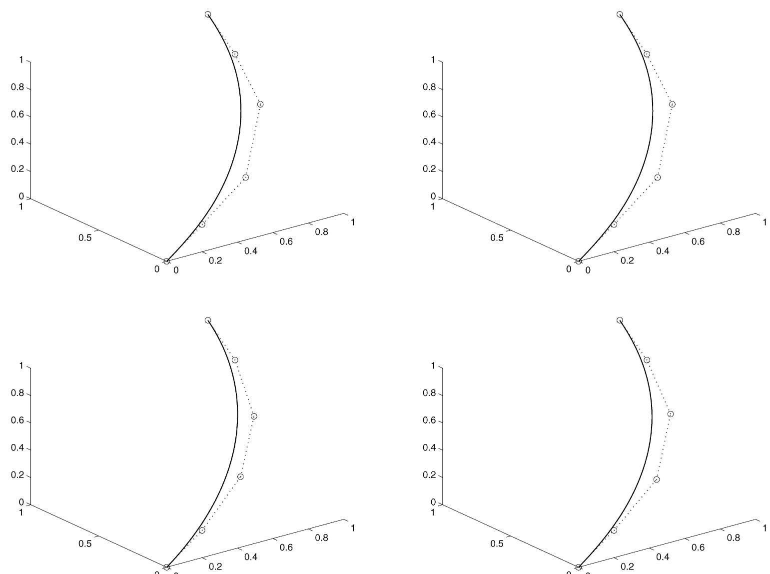

Fig. 2. The four PH quintic interpolants to the Hermite data of case #1—HL (top left), HC (top right), BV (bottom left), and CC (bottom right).

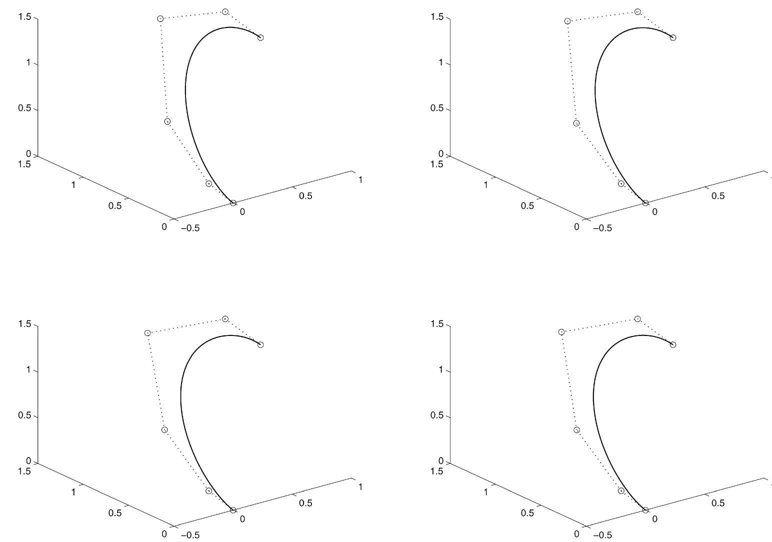

Fig. 3. The four PH quintic interpolants to the Hermite data of case #2—HL (top left), HC (top right), BV (bottom left), and CC (bottom right).

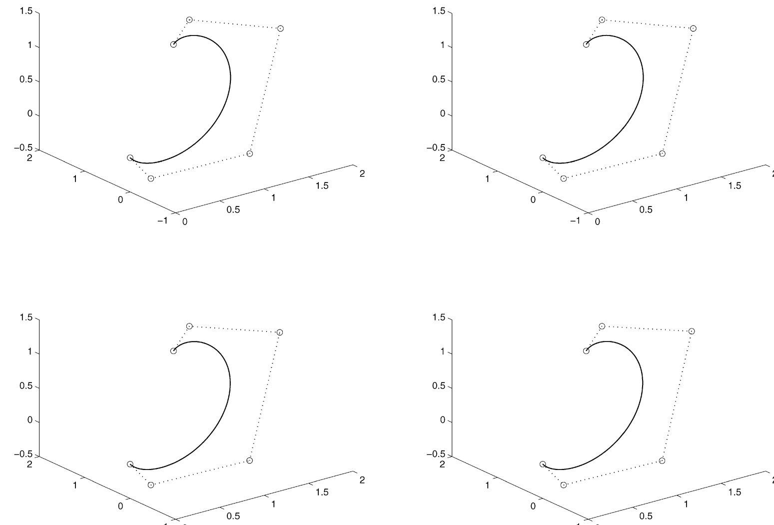

Fig. 4. The four PH quintic interpolants to the Hermite data of case #3—HL (top left), HC (top right), BV (bottom left), and CC (bottom right).

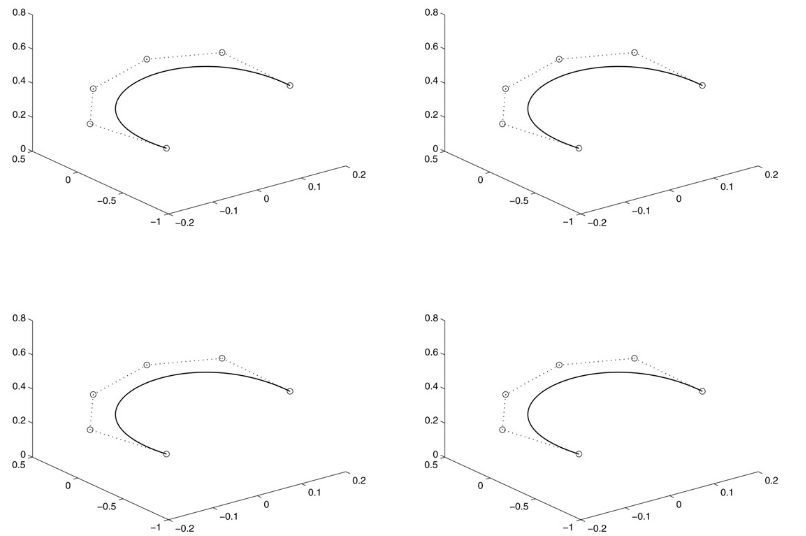

Fig. 5. The four PH quintic interpolants to the Hermite data of case #4—HL (top left), HC (top right), BV (bottom left), and CC (bottom right). In this case, the four interpolants are actually identical, and coincide with the spatial PH cubic that is compatible with the prescribed Hermite data.

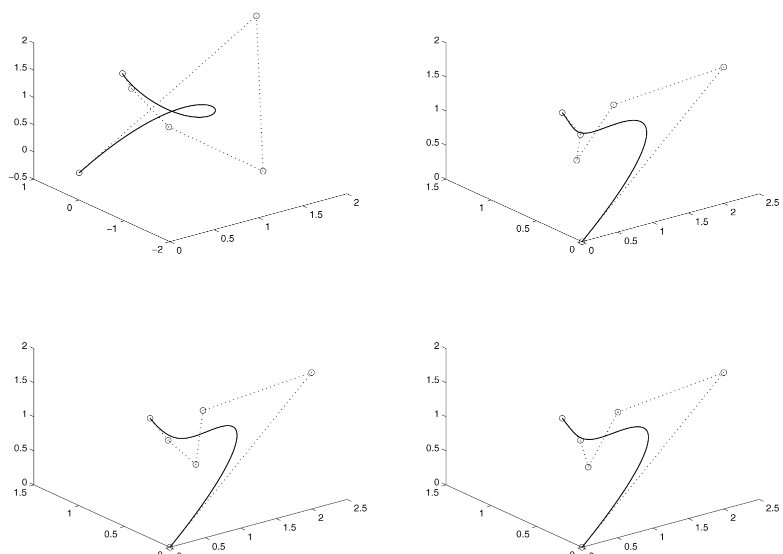

Fig. 6. The four PH quintic interpolants to the Hermite data of case #5—HL (top left), HC (top right), BV (bottom left), and CC (bottom right).

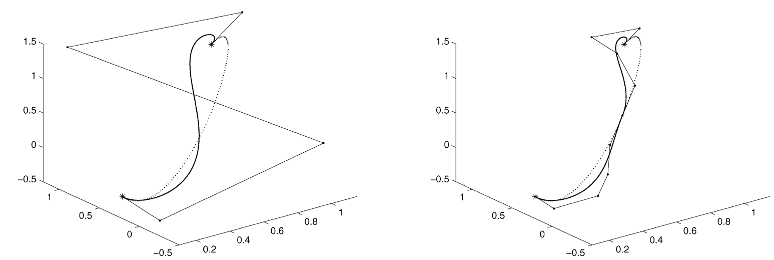

Fig. 7. C^1 PH quintic spline curves (solid lines) interpolating the cubic curve (dotted lines) defined by the Hermite data of test case #3, and their control polygons. Here, the choices $\alpha = \beta = 0$ are always used. The number of approximating PH quintic segments is 1 on the left, and 2 on the right.

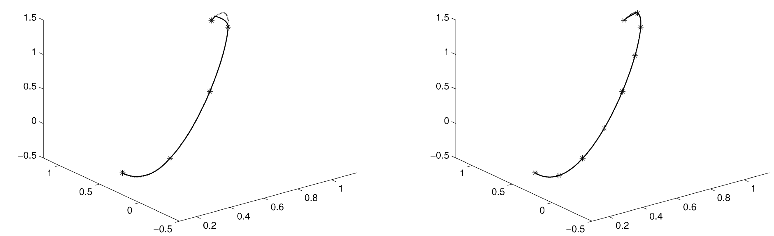

Fig. 8. C^1 PH quintic spline curves (solid lines) interpolating the cubic (dotted lines) defined by the Hermite data of test case #3, with $\alpha = \beta = 0$. The number of PH quintic segments is 4 on the left, and 8 on the right.

10. Closure

Several criteria have been proposed for determining the two free parameters inherent in the spatial PH quintic Hermite interpolation problem, and their performance has been tested through some computed examples. Conditions on the Hermite data for the existence of PH cubic interpolants were identified, and the selection criteria were designed to yield such cubics when possible. Furthermore, it was proved that four helical PH quintic Hermite interpolants always exist, and that they represent extrema of the arc length (dependent on only one of the free parameters). The "CC" criterion appears to be an excellent pragmatic selection scheme, in terms of its computational simplicity and the near-optimal shape quality of the interpolants it yields.

Appendix A. Proofs of Propositions 1 and 2

The proofs for Propositions 1 and 2 are presented here.

Proof of Proposition 1. We argue by contradiction. First, from the quaternion product rules, one can easily verify that $\mathcal{A}$ and $\mathcal{A}\mathbf{u}$ are linearly independent. Now suppose scalars λ, μ with $\lambda^2 + \mu^2 > 0$ exist, such that $\mathcal{B} = \lambda\mathcal{A} + \mu\mathcal{A}\mathbf{u}$. Then we would have

$$\mathcal{B}\mathbf{u}\mathcal{B}^* = (\lambda\mathcal{A} + \mu\mathcal{A}\mathbf{u})\mathbf{u}(\lambda\mathcal{A}^* - \mu\mathbf{u}\mathcal{A}^*) = (\lambda^2 + \mu^2)\mathcal{A}\mathbf{u}\mathcal{A}^*,$$

which contradicts the assumption that $\mathbf{d}_{\mathcal{A}} = \mathcal{A}\mathbf{u}\mathcal{A}^*$ and $\mathbf{d}_{\mathcal{B}} = \mathcal{B}\mathbf{u}\mathcal{B}^*$ satisfy $\mathbf{d}_{\mathcal{A}} \times \mathbf{d}_{\mathcal{B}} \neq \mathbf{0}$. Therefore, $\mathcal{A}$, $\mathcal{A}\mathbf{u}$, $\mathcal{B}$ must be linearly independent. Likewise, if we assume $\mathcal{B}\mathbf{u} = \lambda\mathcal{A} + \mu\mathcal{A}\mathbf{u}$, we would have $\mathcal{B} = -\lambda\mathcal{A}\mathbf{u} + \mu\mathcal{A}$, and by the preceding argument we infer that $\mathcal{A}$, $\mathcal{A}\mathbf{u}$, $\mathcal{B}\mathbf{u}$ are also linearly independent.

Finally, suppose scalars λ, μ, ν exist, such that $\mathcal{B} = \lambda\mathcal{A} + \mu\mathcal{A}\mathbf{u} + \nu\mathcal{B}\mathbf{u}$. Then we would have $\mathcal{B}\mathbf{u} = \lambda\mathcal{A}\mathbf{u} - \mu\mathcal{A} - \nu\mathcal{B}$, and hence

$$\mathcal{B} = (\lambda - \mu\nu)\mathcal{A} + (\mu + \lambda\nu)\mathcal{A}\mathbf{u} - \nu^2\mathcal{B}.$$

But this implies that

$$\mathcal{B} = \frac{\lambda - \mu\nu}{1 + \nu^2}\mathcal{A} + \frac{\mu + \lambda\nu}{1 + \nu^2}\mathcal{A}\mathbf{u},$$

in contradiction to the conclusion that $\mathcal{A}$, $\mathcal{A}\mathbf{u}$, $\mathcal{B}$ are linearly independent. This completes the proof. □

Proof of Proposition 2. From (6) the scalar product $\mathbf{v} \cdot \mathbf{s}$ can be written as the quaternion expression $-\frac{1}{2}(\mathbf{vs} + \mathbf{sv})$. Then the first relation in (12) becomes $\mathbf{vs} + \mathbf{sv} = -(vs + sv)$. Now we have

$$\begin{aligned}\mathbf{vs} + \mathbf{sv} &= (\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathcal{B}^* - \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{B}\mathcal{A}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathcal{A}^*) \\ &\quad + (\mathcal{A}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{A}\mathcal{B}^*\mathcal{B}\mathbf{u}\mathcal{A}^* - \mathcal{B}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*) \\ &= \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}-(vs + sv) &= (\mathcal{A}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{A}\mathcal{B}^*\mathcal{B}\mathbf{u}\mathcal{A}^* + \mathcal{B}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*) \\ &\quad + (\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathcal{B}^* + \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{B}\mathcal{A}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathcal{A}^*) \\ &= \mathcal{A}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^* + \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathcal{A}^*.\end{aligned}$$

Thus, the first relation in (12) holds.

Consider now the second relation in (12). Noting that $|\mathbf{v}|^2$ and $|\mathbf{s}|^2$ amount to the quaternion expressions $-\mathbf{vv}$ and $-\mathbf{ss}$, this relation becomes $-\mathbf{vv} + \mathbf{ss} = v^2 - s^2$. Now we have

$$\begin{aligned}-\mathbf{vv} + \mathbf{ss} &= -(\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{B}\mathbf{u}\mathcal{A}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*) \\ &\quad + (\mathcal{A}\mathcal{B}^*\mathcal{A}\mathcal{B}^* - \mathcal{A}\mathcal{B}^*\mathcal{B}\mathcal{A}^* - \mathcal{B}\mathcal{A}^*\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*\mathcal{B}\mathcal{A}^*) \\ &= -\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*\mathcal{B}\mathcal{A}^*.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}v^2 - s^2 &= (\mathcal{A}\mathcal{B}^*\mathcal{A}\mathcal{B}^* + \mathcal{A}\mathcal{B}^*\mathcal{B}\mathcal{A}^* + \mathcal{B}\mathcal{A}^*\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*\mathcal{B}\mathcal{A}^*) \\ &\quad - (\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{B}\mathbf{u}\mathcal{A}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*) \\ &= \mathcal{A}\mathcal{B}^*\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*\mathcal{B}\mathcal{A}^* - \mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{B}^* - \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*.\end{aligned}$$

This establishes the second relation in (12).

Now consider the remaining two relations in (12) where, for example, we take $\mathcal{Q} = \mathcal{Q}_{\mathcal{A}}$. Again, we have $\mathbf{v} \cdot \mathbf{q} = -\frac{1}{2}(\mathbf{vq} + \mathbf{qv})$. Thus, the third relation becomes $\mathbf{vq} + \mathbf{qv} = -2vq$. Now, we have

$$\begin{aligned}\mathbf{vq} + \mathbf{qv} &= (\mathcal{A}\mathbf{u}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{A}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{A}^*) + (\mathcal{A}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{A}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathbf{u}\mathcal{A}^*) \\ &= \mathcal{A}\mathbf{u}(\mathcal{B}^*\mathcal{A} + \mathcal{A}^*\mathcal{B})\mathbf{u}\mathcal{A}^* - \mathcal{A}\mathcal{A}^*(\mathcal{B}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*) \\ &= -\mathcal{A}\mathcal{A}^*(\mathcal{B}^*\mathcal{A} + \mathcal{A}^*\mathcal{B}) - \mathcal{A}\mathcal{A}^*(\mathcal{B}\mathcal{A}^* + \mathcal{A}\mathcal{B}^*) \\ &= -2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{B}^*\mathcal{A}) - 2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{A}\mathcal{B}^*) = -4\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{A}\mathcal{B}^*).\end{aligned}$$

On the other hand, we also have

$$vq = (\mathcal{A}\mathcal{B}^* + \mathcal{B}\mathcal{A}^*)\mathcal{A}\mathcal{A}^* = 2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{A}\mathcal{B}^*).$$

This establishes the third relation in (12). Finally, the fourth relation in (12) can be written as $\mathbf{sq} + \mathbf{qs} = -2sq$. Now we have

$$\begin{aligned}\mathbf{sq} + \mathbf{qs} &= (\mathcal{A}\mathcal{B}^*\mathcal{A}\mathbf{u}\mathcal{A}^* - \mathcal{B}\mathcal{A}^*\mathcal{A}\mathbf{u}\mathcal{A}^*) + (\mathcal{A}\mathbf{u}\mathcal{A}^*\mathcal{A}\mathcal{B}^* - \mathcal{A}\mathbf{u}\mathcal{A}^*\mathcal{B}\mathcal{A}^*) \\ &= \mathcal{A}(\mathcal{B}^*\mathcal{A}\mathbf{u} - \mathbf{u}\mathcal{A}^*\mathcal{B})\mathcal{A}^* - \mathcal{A}\mathcal{A}^*(\mathcal{B}\mathbf{u}\mathcal{A}^* - \mathcal{A}\mathbf{u}\mathcal{B}^*) \\ &= 2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{B}^*\mathcal{A}\mathbf{u}) - 2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{B}\mathbf{u}\mathcal{A}^*) = 4\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{A}\mathbf{u}\mathcal{B}^*).\end{aligned}$$

On the other hand, we also have

$$sq = (-\mathcal{A}\mathbf{u}\mathcal{B}^* + \mathcal{B}\mathbf{u}\mathcal{A}^*)\mathcal{A}\mathcal{A}^* = -2\mathcal{A}\mathcal{A}^*\mathrm{scal}(\mathcal{A}\mathbf{u}\mathcal{B}^*).$$

Hence, the fourth relation in (12) is established. □

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