1. NALOGA

a)

Povprečni dohodek

Populacija je velikosti 43886. Vzamemo enostavni slučajni vzorec 400 enot.
Sledi: $N = 43886, n = 400$.
Naj bo X_k skupni dohodek v k-ti družini.
Torej je povprečni dohodek v Kindergradu:

$$\overline{X} = \frac{X_1 + ... + X_n}{n}.$$

Standardna napaka

Vemo: $se(\overline{X}) = \sqrt{var(\overline{X})}$. Ker imamo enostavni slučajni vzorec, vemo tudi, da je

$$var(\overline{X}) = \frac{1}{n}\frac{N-n}{N-1}\sigma^2$$

kjer je σ^2 populacijska varjanca. Nepristranska cenilka za σ^2 je

$$\hat{\sigma}^2 = \frac{N-1}{N(n-1)}\sum_{k=1}^{n}(X_k - \overline{X})^2.$$

Sledi:

$$\widehat{se(\overline{X})} = \sqrt{\frac{1}{n}\frac{N-n}{N(n-1)}\sum_{k=1}^{n}(X_k - \overline{X})^2}.$$

Interval zaupanja

Iz navodil sledi, da je interval zaupanja enak $\overline{X} \pm 1,96 \cdot se(\overline{X})$.

Končne vrednosti

b)

Če stratificiramo, mora veljati

$$\frac{n_k}{n} = \frac{N_k}{N}, \sum_{k=1}^{k} n_k = n.$$

V našem primeru startificiramo po četrtih, torej $k = 4$. Vemo:
$N_1 = 10149$ (*severna četrt*),
$N_2 = 10390$ (*vzgodna četrt*),

$N_3 = 13457$ (*južna četrt*),
$N_4 = 9890$ (*zahodna četrt*). Če malo obrnemo zgornjo enakost, dobimo

$$n_k = \frac{N_k}{N} n.$$

Izračunamo za $k = 1, 2, 3, 4$ in upoštevamo vrednosti N_1, N_2, N_3, N_4 ter $N = 43886$.
Dobimo:

$$n_1 = \frac{10149}{43886} \cdot 400 = 92,5033 \rightarrow n_1 = 92,$$

$$n_2 = \frac{10390}{43886} \cdot 400 = 94,699 \rightarrow n_2 = 95,$$

$$n_3 = \frac{13457}{43886} \cdot 400 = 122,654 \rightarrow n_3 = 123,$$

$$n_4 = \frac{9890}{43886} \cdot 400 = 90,142 \rightarrow n_4 = 90.$$

Preverimo:

$$\sum_{k=1}^{4} n_k = 92 + 96 + 123 + 90 = 400.$$

Naj bo sedaj X_{kj} povprečni dohodek j-te družine v k-tem stratumu.
Povprečni dohodek družine se sedaj izraža kot:

$$\overline{X} = \frac{1}{n} \sum_{k=1}^{\#stratumov} \sum_{j=1}^{n_k} X_{kj}.$$

Standarna napaka $se(\overline{X}) = \sqrt{var(\overline{X})}$:

$$var(\overline{X}) = \sum_k w_k^2 var(\overline{X_k})) = \sum_k w_k^2 \cdot \frac{\hat{\sigma}_k^2}{n_k} \cdot \frac{N_k - n_k}{N_k - 1},$$

kjer je $w_k = \frac{N_k}{N}$ delež, σ_k^2 pa populacijska varjanca v k-tem stratumu. Torej:

$$\hat{\sigma}_k^2 = \frac{N_k - 1}{N_k(n_k - 1)} \sum_{j=1}^{n_k} (X_{kj} - \overline{X_k})^2,$$

kjer je X_k povprečje k-tega stratuma.
Interval zaupanja: $\overline{X} \pm 1,96 \cdot se(\overline{X})$.
Vstavimo podatke in dobimo:

c)

2. NALOGA

3.NALOGA

a)

$$X \sim f(x,\alpha) = \begin{cases} \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}, & 0 < x < 1. \\ 0, & sicer. \end{cases}$$

Vemo:

$$E(X) = \frac{1}{3}, var(X) = \frac{2}{9(3\alpha+1)}.$$

Ker je:

$$var(X) = E(X^2) - E(X)^2,$$

sledi:

$$E(X^2) = var(X) + E(X)^2 = \frac{2}{9(3\alpha+1)} + \frac{1}{9} = \frac{\alpha+1}{3(3\alpha+1)}$$

$E(X^2)$ je drugi moment slučajne spremenljivke X, torej:

$$E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Zgornji enačbi izenačimo in poračunamo α:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\alpha+1}{3(3\alpha+1)}$$

$$\frac{9\alpha+3}{n}\sum_{i=1}^{n} X_i^2 = \alpha+1$$

$$\frac{9\alpha}{n}\sum_{i=1}^{n} X_i^2 - \alpha = 1 - \frac{3}{n}\sum_{i=1}^{n} X_i^2$$

$$\alpha = \frac{1 - \frac{3}{n}\sum_{i=1}^{n} X_i^2}{\frac{9}{n}\sum_{i=1}^{n} X_i^2 - 1}$$

b)

$$f_X(x,\alpha) = \begin{cases} \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}, & 0 < x < 1. \\ 0, & sicer. \end{cases}$$

$$L_1(\alpha|x) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}$$

$$L(\alpha|x_1, ..., x_n) = L_1(\alpha|x_1) \cdot ... \cdot L_1(\alpha|x_n) =$$
$$\left(\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}\right)^n x_1^{\alpha-1} \cdot ... \cdot x_n^{\alpha-1}(1-x_1)^{2\alpha-1} \cdot ... \cdot (1-x_n)^{2\alpha-1}$$

$$l(\alpha|x_1, ..., x_n) = ln(L(\alpha|x_1, ..., x_n)) = l_1(\alpha|x_1) + ... + l_1(\alpha|x_n)$$

$$l_1(\alpha|x) = ln(L_1(\alpha|x)) =$$
$$ln(\Gamma(3\alpha)) - ln(\Gamma(\alpha)) - ln(\Gamma(2\alpha)) + (\alpha-1)ln(x) + (2\alpha-1)ln(1-x)$$

$$l(\alpha|x) =$$
$$\sum_{i=1}^{n} (ln(\Gamma(3\alpha)) - ln(\Gamma(\alpha)) - ln(\Gamma(2\alpha)) + (\alpha-1)ln(x_i) + (2\alpha-1)ln(1-x_i))$$

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\Gamma(3\alpha)}\Gamma^{'}(3\alpha)3 - \frac{1}{\Gamma(\alpha)}\Gamma^{'}(\alpha) - \frac{1}{\Gamma(2\alpha)}\Gamma^{'}(2\alpha)2 + ln(x_i) + 2ln(1-x_i)$$

$$\sum_{i=1}^{n} \frac{1}{\Gamma(3\alpha)}\Gamma^{'}(3\alpha)3 - \frac{1}{\Gamma(\alpha)}\Gamma^{'}(\alpha) - \frac{1}{\Gamma(2\alpha)}\Gamma^{'}(2\alpha)2 = ln(\frac{1}{x_i(1-x_i)})$$

Cenilka obstaja, ko ima zgornja enačba rešitev.
Uporabimo funkcijo digamma in rešimo do konca? no clue.

c)

$var(\hat{\alpha}) = \frac{1}{nI_1(\hat{\alpha})}$.

$$I_1(\hat{\alpha}) = -E\left[\frac{\partial^2 l_1(\alpha|x)}{\partial \alpha^2}\right]$$

$$\frac{\partial^2 l_1(\alpha|x)}{\partial \alpha^2} =$$
$$3\frac{\Gamma^{''}(3\alpha)\Gamma(3\alpha) - \Gamma^{'}(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma^{''}(\alpha)\Gamma(\alpha) - \Gamma^{'}(\alpha)^2}{\Gamma(\alpha)^2} - 2\frac{\Gamma^{''}(2\alpha)\Gamma(2\alpha) - \Gamma^{'}(2\alpha)^2}{\Gamma(2\alpha)^2}$$

$$var(\hat{\alpha}) = \frac{1}{n}\frac{1}{3\frac{\Gamma^{''}(3\alpha)\Gamma(3\alpha)-\Gamma^{'}(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma^{''}(\alpha)\Gamma(\alpha)-\Gamma^{'}(\alpha)^2}{\Gamma(\alpha)^2} - 2\frac{\Gamma^{''}(2\alpha)\Gamma(2\alpha)-\Gamma^{'}(2\alpha)^2}{\Gamma(2\alpha)^2}}$$