CANONICAL AND STANDARD FORMS

Minterms

- A product term containing all n variables of the function in either true or complemented form is called as the Minterm.
- Each minterm is obtained by an AND operation of the variables in their true form or complemented form.
- For a two-variable function, four different combinations are possible, such as, $A'B'$, $A'B$, AB', and AB. Each of these four AND terms is called a *minterm*, or a *standard product*.
- In the minterm, a variable will possess the value 1 if it is in true or uncomplemented form, whereas, it contains the value 0 if it is in complemented form.

- For a three-variable function, Eight different combinations (Minterms) are possible.
- So, if the number of variables are n, then the possible number of Minterms are 2^n

A	B	C	Minterm
0	0	0	$A'B'C'$
0	0	1	$A'B'C$
0	1	0	$A'BC'$
0	1	1	$A'BC$
1	0	0	$AB'C'$
1	0	1	$AB'C$
1	1	0	ABC'
1	1	1	ABC

- The Minterms are denoted by m_j, (m_0, m_1, m_2, m_3........) where the subscript j represents the decimal codes of the combinations.

A	B	C	Minterm	Designation
0	0	0	A'B'C'	m_0
0	0	1	A'B'C	m_1
0	1	0	A'BC'	m_2
0	1	1	A'BC	m_3
1	0	0	AB'C'	m_4
1	0	1	AB'C	m_5
1	1	0	ABC'	m_6
1	1	1	ABC	m_7

Maxterms

- A sum term containing all n variables of the function in either true or complemented form is called the Maxterm.
- Each maxterm is obtained by an OR operation of the variables in their true form or complemented form.
- For a two-variable function, four different combinations are possible, such as, $A' + B'$, $A' + B$, $A + B'$, and $A + B$. These sum terms are called the standard sums or Maxterms.
- In the maxterm, a variable will possess the value 0, if it is in true or uncomplemented form, whereas, it contains the value 1, if it is in complemented form.

- For a three-variable function, Eight different combinations (Maxterms) are possible.
- So, if the number of variables are n, then the possible number of Maxterms are 2^n .

A	B	C	Maxterm
0	0	0	$A + B + C$
0	0	1	$A + B + C'$
0	1	0	$A + B' + C$
0	1	1	$A + B' + C'$
1	0	0	$A' + B + C$
1	0	1	$A' + B + C'$
1	1	0	$A' + B' + C$
1	1	1	$A' + B' + C'$

- Maxterms are represented as M_j ($M_0, M_1, M_2, \ldots\ldots\ldots$) where j represents their decimal code.

A	B	C	Maxterm	Designation
0	0	0	A+B+C	M_0
0	0	1	A+B+C'	M_1
0	1	0	A+B'+C	M_2
0	1	1	A+B'+C'	M_3
1	0	0	A'+B+C	M_4
1	0	1	A'+B+C'	M_5
1	1	0	A'+B'+C	M_6
1	1	1	A'+B'+C'	M_7

Note

$$\overline{m_j} = M_j \quad \text{or} \quad \overline{M_j} = m_j$$

There are two ways of representing Boolean expressions.

1. Canonical form
2. Standard form

Canonical Form

- Boolean functions expressed as a Sum of Minterms or Product of Maxterms are said to be in *canonical form* .
- For n binary variables, one can obtain 2^n distinct Minterms or Maxterms and that any Boolean function can be expressed as a sum of Minterms or product of Maxterms.

Sum of Minterms

- When a Boolean function is expressed as the logical sum of all the Minterms for which the value of the function is 1 in the truth table, it is referred to as the canonical sum of product expression.
- The same can be expressed in a compact form by listing the corresponding decimal-equivalent codes of the Minterms containing a function value of 1
- Eg. $F(A,B,C) = \Sigma(2,4,5,6)$

$$= m_2 + m_4 + m_5 + m_6$$

$$= A'BC' + AB'C' + AB'C + ABC'$$

where Σ (3,5,6) represents the summation of Minterms corresponding to decimal codes 3, 5, and 6.

$$Y = A'BC' + AB'C' + AB'C + ABC'$$

Inputs			*Output*	*Product terms*	*Sum terms*
A	*B*	*C*	*Y*		
0	0	0	0		$A + B + C$
0	0	1	0		$A + B + C'$
0	1	0	1	$A'BC'$	
0	1	1	0		$A + B' + C'$
1	0	0	1	$AB'C'$	
1	0	1	1	$AB'C$	
1	1	0	1	ABC'	
1	1	1	0		$A' + B' + C'$

- If the function is not in this form, it can be obtained in following ways.

1. Check each term in the given logic function. Retain if it is a minterm, continue to examine the next term in the same manner.

2. Examine for the variables that are missing in each product which is not a minterm. If the missing variable in the minterm is X, multiply that minterm with $(X+X')$.

3. Multiply all the products and discard the redundant terms.

E.g.: Obtain the canonical sum of product form of the following function:

$F(A, B) = A + B$

Solution. The given function contains two variables A and B. The variable B is missing from the first term of the expression and the variable A is missing from the second term of the expression. Therefore, the first term is to be multiplied by $(B + B')$ and the second term is to be multiplied by $(A + A')$ as demonstrated below.

$$\begin{aligned} F(A, B) &= A + B \\ &= A.1 + B.1 \\ &= A(B + B') + B(A + A') \\ &= AB + AB' + AB + A'B \\ &= AB + AB' + A'B \text{ (as } AB + AB = AB) \end{aligned}$$

Hence the canonical sum of the product expression of the given function is

$$F(A, B) = AB + AB' + A'B.$$

Product of Maxterms

- When a Boolean function is expressed as the logical product of all the Maxterms from the rows of a truth table, for which the value of the function is 0, it is referred to as the canonical product of sum/maxterms expression.
- The same can be expressed in a compact form by listing the corresponding decimal equivalent codes of the Maxterms containing a function value of 0.

Eg:- $F(A,B,C) = \Pi(0,1,3,7)$

$$= M_0 . M_1 . M_{3.} M_7$$

$$= (A + B + C)(A + B + C')(A + B' + C')(A' + B' + C')$$

where $\Pi(0,1,3,7)$ represents the product of Maxterms corresponding to decimal codes 0, 1,3 and 7.

$$Y = (A + B + C)(A + B + C')(A + B' + C')(A' + B' + C')$$

Inputs			*Output*	*Product terms*	*Sum terms*
A	*B*	*C*	*Y*		
0	0	0	0		$A + B + C$
0	0	1	0		$A + B + C'$
0	1	0	1	$A'BC'$	
0	1	1	0		$A + B' + C'$
1	0	0	1	$AB'C'$	
1	0	1	1	$AB'C$	
1	1	0	1	ABC'	
1	1	1	0		$A' + B' + C'$

The canonical product of sums form of a logic function can be obtained by using the following procedure.

1. Check each term in the given logic function. Retain it if it is a maxterm, continue to examine the next term in the same manner.

2. Examine for the variables that are missing in each sum term that is not a maxterm. If the missing variable in the maxterm is X, add that maxterm with (X.X′).

3. Expand the expression using the properties and postulates as described earlier and discard the redundant terms.

Eg: Obtain the canonical product of the sum form of the following function.

$$F(A, B, C) = (A + B')(B + C)(A + C')$$

Solution. In the above three-variable expression, C is missing from the first term, A is missing from the second term, and B is missing from the third term. Therefore, CC′ is to be added with first term, AA′ is to be added with the second, and BB′ is to be added with the third term. This is shown below.

$$\begin{aligned}
F(A, B, C) &= (A + B')(B + C)(A + C') \\
&= (A + B' + 0)(B + C + 0)(A + C' + 0) \\
&= (A + B' + CC')(B + C + AA')(A + C' + BB') \\
&= (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C')(A + B' + C') \\
&\qquad \text{[using the distributive property, as } X + YZ = (X + Y)(X + Z)] \\
&= (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C') \\
&\qquad [\text{as } (A + B' + C')(A + B' + C') = A + B' + C']
\end{aligned}$$

Hence the canonical product of the sum expression for the given function is

$$F(A, B, C) = (A + B' + C)(A + B' + C')(A + B + C)(A' + B + C)(A + B + C')$$

Example: Express the Boolean function $F = A + B'C$ as a sum of Minterms.

The function has three variables: A, B, and C.

$$A = A(B + B') = AB + AB' = AB(C + C') + AB'(C + C')$$

$$= ABC + ABC' + AB'C + AB'C'$$

$$B'C = (A + A')B'C = AB'C + A'B'C$$

Combining all terms, we have

$$F = A + B'C = ABC + ABC' + AB'C + AB'C' + AB'C + A'B'C$$

$$F = A'B'C + AB'C' + AB'C + ABC' + ABC$$

$$= m_1 + m_4 + m_5 + m_6 + m_7$$

$$F(A, B, C) = \sum(1, 4, 5, 6, 7)$$

Example: Express the Boolean function $F = xy + x'z$ as a product of Maxterms.

First, convert the function into OR terms/maxterms

$F = xy + x'z = (xy + x')(xy + z) = (x + x')(y + x')(x + z)(y + z)$

$\quad = (x' + y)(x + z)(y + z)$

The function has three variables: x, y, and z.

$x' + y = x' + y + zz' = (x' + y + z)(x' + y + z')$

$x + z = x + yy' + z = (x + y + z)(x + y' + z)$

$y + z = xx' + y + z = (x + y + z)(x' + y + z)$

Combining all the terms, we have

$F = (x + y + z)(x + y' + z)(x' + y + z)(x' + y + z')$

$\quad = M_0 \,.\, M_2 \,.\, M_4 \,.\, M_5$

$F(x, y, z) = \prod(0, 2, 4, 5)$

Example: **Express each function in sum-of-minterms and product-of-maxterms form.**

1. $(xy + z)(y + xz)$
2. $(A' + B)(B' + C)$

Solution:

1. $(xy + z)(y + xz)$

$= xyy + xyxz + yz + xzz$

$= xy + xyz + yz + xz$

$= xy(z + z') + (x + x')yz + x(y + y')z + xyz$

$= xyz + xyz' + xyz + x'yz + xyz + xy'z + xyz$

$= x'yz + xy'z + xyz' + xyz$

$= m_3 + m_5 + m_6 + m_7$

$F(x, y, z) = \sum(3, 5, 6, 7) = \prod(0, 1, 2, 4)$

Solution:

2. $(A' + B)(B' + C)$

$= (A' + B + CC')(AA' + B' + C)$

$= (A' + B + C)(A' + B + C')(A + B' + C)(A' + B' + C)$

$= M_4 . M_5 . M_2 . M_6$

$F(A, B, C) = \prod(2, 4, 5, 6) = \sum(0, 1, 3, 7)$

or

$(A' + B)(B' + C) = A'B' + A'C + BB' + BC$

$= A'B'(C + C') + A'(B + B')C + (A + A')BC$

$= A'B'C + A'B'C' + A'BC + A'B'C + ABC + A'BC$

$= m_1 + m_0 + m_3 + m_7$

$F(A, B, C) = \sum(0, 1, 3, 7) = \prod(2, 4, 5, 6)$

Conversion between Canonical Forms

- The complement of a function expressed as the sum of Minterms equals the sum of Minterms missing from the original function.
- This is because the original function is expressed by those Minterms which make the function equal to 1, whereas its complement is a 1 for those Minterms for which the function is a 0.

Example:

- Consider the function

$$F(A, B, C) = \sum(1, 4, 5, 6, 7)$$

- This function has a complement that can be expressed as

$$F'(A, B, C) = \sum(0, 2, 3) = m_0 + m_2 + m_3$$

- Now, if we take the complement of F' by DeMorgan's theorem, we obtain F in a different form:

$$F = (m_0 + m_2 + m_3)' = m'_0 \,.\, m'_2 \,.\, m'_3$$
$$= M_0 M_2 M_3 = \prod(0, 2, 3)$$

Example: Convert each of the following to the other canonical form:

1. $F(x, y, z) = \sum(1, 3, 5)$
2. $F(x, y, z) = \prod(0, 3, 6, 7)$
3. $F(A, B, C, D) = \sum(0, 2, 6, 11, 13, 14)$
4. $F(A, B, C, D) = \prod(0, 1, 3, 4, 6, 8, 11, 12)$

Solution:

1. $F(x, y, z) = \sum(1, 3, 5)$

   $= \prod(0, 2, 4, 6, 7)$

2. $F(x, y, z) = \prod(0, 3, 6, 7)$

   $= \sum(1, 2, 4, 5)$

3. $F(A, B, C, D) = \sum(0, 2, 6, 11, 13, 14)$

   $= \prod(1, 3, 4, 5, 7, 8, 9, 10, 12, 15)$

4. $F(A, B, C, D) = \prod(0, 1, 3, 4, 6, 8, 11, 12)$

   $= \sum(2, 5, 7, 9, 10, 13, 14, 15)$

Example: **Express the complement of the following functions in sum-of-minterms form:**

1. $F(x, y, z) = \sum(0, 3, 6, 7)$
2. $F(A, B, C, D) = \sum(2, 4, 7, 10, 12, 14)$
3. $F(A, B, C) = \prod(3, 5, 7)$
4. $F(w, x, y, z) = \prod(0, 1, 3, 4, 6, 8, 11, 12)$

Solution:

1. $F(x, y, z) = \sum(0, 3, 6, 7) = \prod(1, 2, 4, 5)$

   $F'(x, y, z) = \sum(1, 2, 4, 5)$
2. $F(A, B, C, D) = \sum(2, 4, 7, 10, 12, 14)$

   $F'(A, B, C, D) = \sum(0, 1, 3, 5, 6, 8, 9, 11, 13, 15)$
3. $F(A, B, C) = \prod(3, 5, 7) = \sum(0, 1, 2, 4, 6)$

   $F'(A, B, C) = \sum(3, 5, 7)$
4. $F(w, x, y, z) = \prod(0, 1, 3, 4, 6, 8, 11, 12)$

   $F'(w, x, y, z) = \sum(0, 1, 3, 4, 6, 8, 11, 12)$

Standard Forms

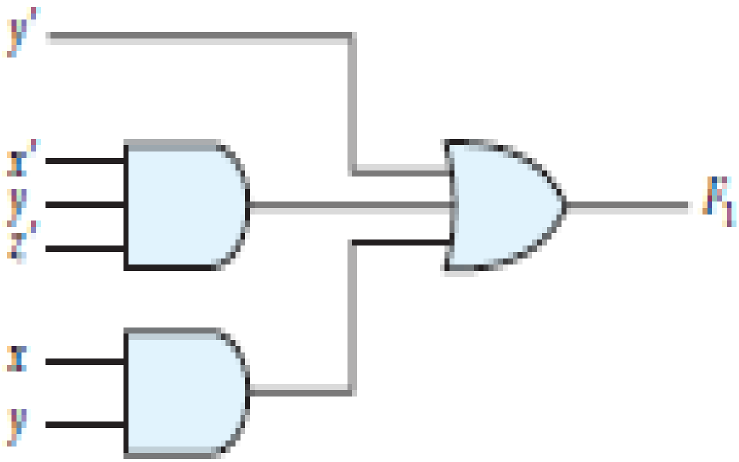
- Another way to express Boolean functions is in *standard* form.
- In this configuration, the terms that form the function may contain one, two, or any number of literals.
- There are two types of standard forms: the sum of products and products of sums.

- The sum of products is a Boolean expression containing AND terms, called product terms, with one or more literals each. The sum denotes the OR operation of these terms.
- An example of a function expressed as a sum of products is $F_1 = y' + xy + x'yz'$
- The expression has three product terms, with one, two, and three literals. The sum is an OR operation.

- A product of sums is a Boolean expression containing OR terms, called sum terms. Each term may have any number of literals. The product denotes the AND operation of these terms.
- An example of a function expressed as a product of sums is $F_2 = x(y' + z)(x' + y + z')$
- This expression has three sum terms, with one, two, and three literals. The product is an AND operation.

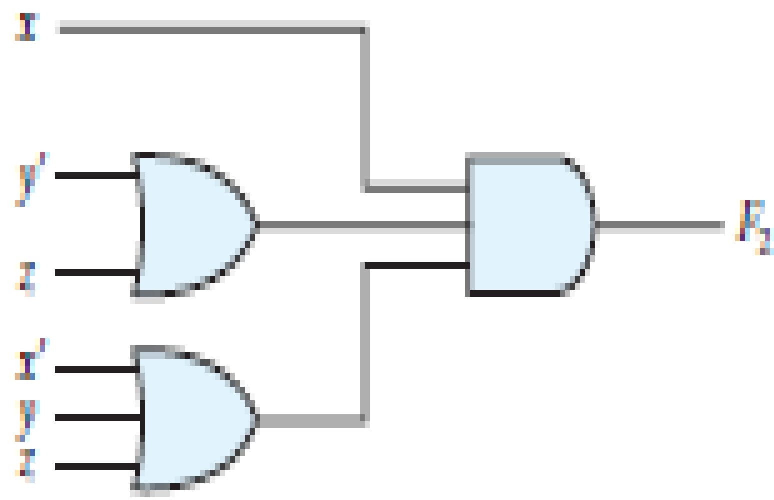
Sum of Products

$$F_1 = y' + xy + x'yz'$$



Product of Sums

$$F_2 = x(y' + z)(x' + y + z')$$



Example: **Convert each of the following expressions into sum of products and product of sums:**

1. $(AB + C)(B + C'D)$
2. $x' + x(x + y')(y + z')$

Solution:

1. <u>Sum of Products</u>

$$(AB + C)(B + C'D)$$

$$= ABB + ABC'D + BC + CC'D$$

$$= AB + ABC'D + BC$$

$$= AB(1 + C'D) + BC = AB + BC$$

<u>Product of Sums</u>

$$(AB + C)(B + C'D)$$

$$= (A + C)(B + C)(B + C')(B + D)$$

Solution:

2. <u>Sum of Products</u>

$x' + x(x + y')(y + z')$

$= x' + x(xy + xz' + y'y + y'z')$

$= x' + xxy + xxz' + xy'z'$

$= x' + xy + xz' + xy'z'$

<u>Product of Sums</u>

$x' + x(x + y')(y + z')$

$= (x' + x)(x' + x + y')(x' + y + z')$

$= 1 \,.\, 1 \,.\, (x' + y + z')$

$= (x' + y + z')$