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Divergence in the Kalman Filter

F. H. SCHLEE,* C. J. STANDISH,† AND N. F. TODA‡
International Business Machines Corporation, Endicott, N. Y.

Under certain conditions, the orbit estimated by a Kalman filter has errors that are much greater than predicted by theory. This phenomenon is called divergence, and renders the operation of the Kalman filter unsatisfactory. This paper investigates the control of divergence in a Kalman filter used for autonomous navigation in a low earth orbit. The system studied utilizes stellar-referenced angle sightings to a sequence of known terrestrial landmarks. A Kalman filter is used to compute differential corrections to spacecraft position, spacecraft velocity, and landmark location. A variety of filter modifications for the control of divergence was investigated. These included the Schmidt-Pines analytical modification and an "empirical" modification based upon Pines' machine noise treatment. Several simplified approximations to the theoretically optimum analytical modifications were also investigated. The principal numerical results are presented in graphs of the magnitude of the error in estimated position and velocity vs time for sixteen orbits. These graphs compare actual position and velocity errors with the theoretical estimates furnished by the trace of the position and velocity covariance matrices. Numerical results indicate that a properly modified filter achieves a steady-state operating level.

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Introduction

THE performance of the Kalman filter under actual operating conditions can be seriously degraded from the theoretical performance indicated by the state covariance matrix. The Kalman filter theoretically produces an increasingly accurate estimate as additional observation data are processed. The magnitude of estimation errors as measured by the determinant of the estimation error covariance matrix is a monotonically decreasing function of the number of observations. However, it has been noted that under actual operating conditions, error levels in the Kalman filter are significantly higher than predicted by theory. Errors can, in fact, increase continuously although additional data are being processed. The possibility of such unstable or divergent behavior was first suggested by Kalman.[1] It was later noted by Pines[2] and Knoll[3] and others in application of the Kalman filter to space navigation and orbit determination.

Autonomous Navigation Filter

A precise determination of the orbit of a space vehicle can be made by employing observational data from on board a spacecraft and a differential correction technique. The data-processing technique, called a filter, assumes a linear relationship between the deviations in the measurements and the corresponding deviations in the orbital elements. These deviations, called observations and states, respectively, are the differences between the measured or estimated values and the reference or nominal values. A large number of observations provides an overdetermined set of linear equations in the unknown states. The filter "fits" an orbit to the observational data by employing some criterion such as least squares, maximum likelihood, or minimum variance.

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* Advisory Engineer, Federal Systems Division, Space Systems Center.

† Senior Mathematician, Federal Systems Division, Space Systems Center; now at Electronics Systems Center, Owego, N. Y.

‡ Senior Engineer, Federal Systems Division, Space Systems Center. Member AIAA.

The traditional filtering methods store the observational data obtained over a long arc, and then process the whole "batch of data" at once to determine a state vector or, equivalently, the orbital elements. Batch processing requires the numerical computation of the inverse of a 6 by 6 or higher-order matrix.

The Kalman filter employs the same linearity assumptions and minimum variance criterion as the batch filter, but processes each piece of data as it is obtained. Because the estimation process is expressed as a set of recursion relations, the Kalman filter is referred to as a recursive process. This recursive estimation process computes a new set of orbital elements after each measurement, thus affording the opportunity to change or rectify the reference trajectory after each measurement. Frequent rectification of the reference orbit reduces the effect of nonlinearities. Except for the residual effects of nonlinearities, a recursive filter is equivalent to an iterative batch process, since both employ the same linearity assumptions and are optimal under the same criteria. In fact, the recursive formulation of the Kalman filter can be derived from the nonrecursive minimum variance filter by employing a matrix inversion lemma.[8]

Causes of Divergence

One cause of filter divergence is discrepancies between the mathematical model used to derive the filter equations and the actual conditions under which the filter must operate. Examples of such discrepancies are the neglect of terms in the gravitational potential or an inaccurate knowledge of the constants in the potential. Another source of divergence is errors in the ballistic coefficient or in the air density model used in computing drag accelerations. Such errors are assumed to be bias errors. Since they affect the dynamical equations of motion, and to differentiate them from observation biases,[4] these errors are termed dynamical biases.

Figure 1 shows the divergence caused by a 25% error in the drag acceleration. A different source of divergence is the round-off errors inherent in the implementation of the filter equations on a finite word length digital computer. Figure 2 shows the effect of computational errors induced by single precision arithmetic (IBM 7090) on the Kalman filter.

One manifestation of machine-caused errors occurs in the computation of the state covariance matrix. After the Kalman filter has been operating for some time, this matrix ceases to be positive definite and symmetric. Filter weighting coefficients computed using this matrix are then wrong

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and, consequently, orbit estimates are incorrect. J. E. Potter and R. Battin[9] derived a variation of the Kalman filter in which the covariance matrix remains at least symmetric and non-negative. This technique eliminates some but not all of the effects of computational errors and at the same time requires a somewhat more complex filter algorithm. In batch processing filters, round-off errors in the filter become evident as serious computational errors in the matrix inversion operation.

Control of Divergence

Several approaches have been suggested for preventing filter divergence. One approach argues that divergence occurs when the filter assigns too small a weight to the last measured data. Thus the current data make only a small correction in the estimate, so small in fact, that errors actually grow because of the natural interaction of position and velocity errors. An obvious "fix" is to more or less arbitrarily increase the weighting of current data. One such fix involves increasing the state covariance matrix while holding the orbital period uncertainty constant. The frequency and amount of the increase must be determined empirically, and may be employed in either recursive or nonrecursive filters.

Schmidt[5] and Pines[2] have suggested analytical modifications to the filter equations to account for dynamical biases without increasing the number of states to be estimated by the filter. Additionally, Pines[2] has developed a simple machine noise modification based on an assumed model of the errors caused by round-off in the digital computer. Other techniques, such as filter reset to keep the diagonal elements of the state covariance matrix above a specified threshold, were investigated by Holick et al.[6]

Ditto[7] experienced divergence in the nonrecursive Gemini-Bayes differential corrector, where numerical problems appear in the inversion of ill-conditioned matrices as well as in the neglect of dynamical biases. The occurrence of filter divergence in a nonrecursive filter is not surprising in view of the equivalence of the Kalman filter and the nonrecursive minimum variance filter.[8] The filter modifications employed by Ditto bear a similarity to those of Schmidt and Pines.

Illustrative Example

To provide the reader with some intuitive understanding of the divergence problem, we give a simple analytical example, where the Kalman filter estimate of a trajectory diverges from the true trajectory as the number of observations increases. This phenomenon is brought about in the present case by the omission of a component of the state vector describing the system. This situation often occurs in practice, either deliberately in order to decrease the complexity of representation of the system or inadvertently because the existence of certain states is unknown. The example considered here is estimation of the altitude X of a vehicle from altimeter data. A Kalman filter is first designed under the assumption that X is a constant, i.e., the vehicle is at some constant altitude. The filter is then shown to diverge when the vehicle is actually climbing (or falling) at a constant rate. A minor modification to the Kalman filter is then shown to eliminate this divergence.

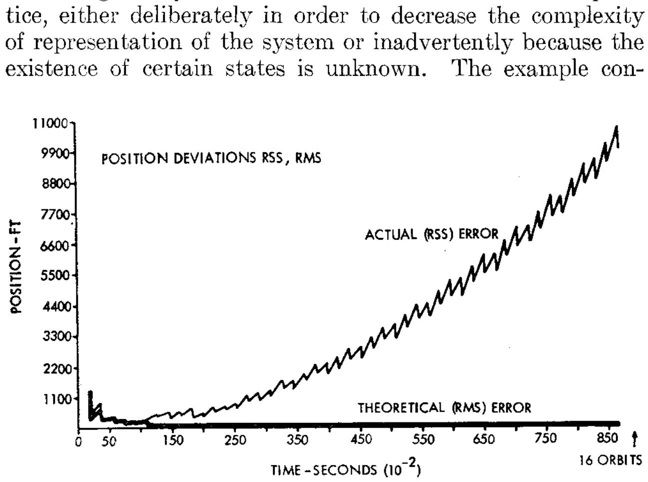

Fig. 1. Kalman filter divergence caused by 25% error in drag acceleration.

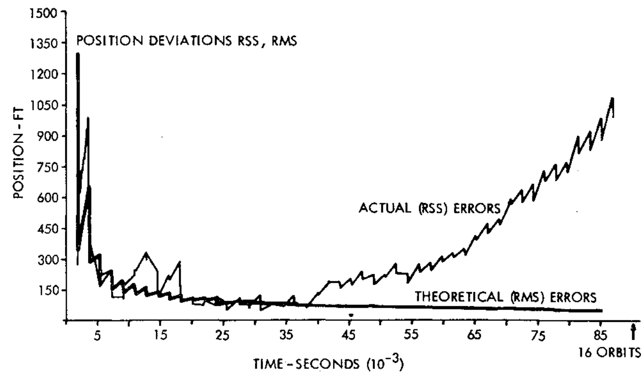

Fig. 2 Divergence of the Kalman filter caused by computational errors resulting from single precision arithmetic (IBM 7090).

To design the Kalman filter, we assume that altitude measurements are corrupted by zero mean independent random noise errors $n(k)$ possessing common standard deviation σ. In addition, the noise errors are assumed bounded, $|n(k)| \leq B$.

The Kalman filter estimate of altitude is given by

$$\left.\begin{aligned} X(k+) &= X(k-) + W(k)[\tilde{y}(k) - y_c(k)] \\ W(k) &= P(k-)/[P(k-) + \sigma^2] \\ P(k-) &= P(k-1+) \qquad X(k-) = X(k-1+) \\ P(k+) &= P(k-) - P(k-)[P(k-) + \sigma^2]^{-1}P(k-) \\ \tilde{y}(k) &= X(k) + n(k) \qquad y_c(k) = X(k-) \end{aligned}\right\} \tag{1}$$

where σ = standard deviation of altimeter noise error $n(k)$ $P(0-)$ = variance of the a priori altitude estimate; and the the symbols $-$ and $+$ appearing in the preceding equations denote estimates immediately prior to and immediately subsequent to an observation.

Suppose now that the vehicle, instead of remaining at a constant altitude, is actually climbing at a constant rate and its trajectory is described by

$$X(t) = X(0) + Vt \tag{2}$$

Suppose further that altitude measurements are taken once per second. An easy calculation yields for the Kalman filter altitude estimate after N observations:

$$X(N+) = X(0) + \frac{VN}{2} + \frac{1}{N+1}\sum_{k=1}^{N} n(k)$$

Subtracting the true trajectory

$$X(N) = X(0) + VN$$

one finds for the error $\epsilon(N)$

$$\epsilon(N) = X(N) - X(N+) = \frac{VN}{2} + \frac{1}{N+1}\sum_{k=1}^{N} n(k) \tag{3}$$

and since noise errors are bounded,

$$|\epsilon(N)| \geq (VN/2) - B$$

Hence the deviation of the Kalman filter estimate from the true trajectory increases with increasing N.

As we have seen, the divergence of the filter was brought about by a constant altitude rate, a phenomenon not provided for in the filter construction. One solution to divergence in this case is to design a filter to estimate both vehicle altitude and altitude rate. Although such a two-state filter would prevent divergence, it would do so at the cost of a considerably increased number of computations. An alternative is to modify the weighting sequence $W(k)$ of the one-state filter to include the effects of a climbing vehicle.

To implement this alternative, we assume that the true trajectory is given by

$$X^*(k+) = X^*(k-) + W(k)^*[\tilde{y}(k) - y_c^*(k)] \tag{4}$$

$$W(k)^* = P^*(k-1)/[P^*(k-1) + \sigma^2] \tag{5}$$

$$P^*(k+) = [1 - W^*(k)]\, P^*(k-) \tag{6}$$

$$P^*(k-) = P^*(k-1+) + \alpha \tag{7}$$

where α is the a priori variance of the rate of change of vehicle altitude.

Denoting the filter error $X(N) - X^*(N+)$ by $\epsilon^*(N)$, one finds from (4)

$$\epsilon^*(N) = \epsilon^*(N-1)[1 - W(N)^*] - V[1 - W(N)^*] + W(N)^* n(N) \tag{8}$$

Combine (5–7) to write

$$P(k+1-) = \frac{\sigma^2 P(k-)}{P(k-) + \sigma^2} + \alpha$$

It is easily seen that $P(k-)$ tends to a limit L as $k \to \infty$ given by

$$L = [\alpha + (\sigma^2 + 4\alpha\sigma^2)^{1/2}]/2$$

and hence $[1 - W(k)^*]$ tends to $L^* = \sigma^2(L + \sigma^2)^{-1}$. Since $|L^*| < 1$,

$$|1 - W^*(k)| < L_0 < 1$$

and for all k sufficiently large, the error recursion relation (8) becomes

$$|\epsilon^*(k)| \leqq L_0|\epsilon^*(k-1)| + L_0(|V| + B)$$

Thus, the $\epsilon^*(k)$ are majorized by a suitable solution of the difference equation

$$z(k) = L_0 z(k-1) + L_0(|V| + B)$$

Since with $|L_0| < 1$ all the solutions $Z(k)$ are bounded, then the $|\epsilon^*(k)|$ are bounded and the modified filter does not diverge.

Derivation of the Modified Equations

In this section, we present the equation for the modified Kalman filter employed in the simulation studies. The development follows the work of Pines[2] and Schmidt.[5]

Let $\mathbf{x}(k)$, $\mathbf{y}(k)$, $\mathbf{u}(k)$ denote true values of the orbital elements, measurements, and unestimated parameters, respectively, and let $\delta\mathbf{x}(k)$, $\delta\mathbf{y}(k)$, $\delta\mathbf{u}(k)$ denote deviations in the preceding quantities at time t_k. These deviations will be called states, observations, and biases, respectively. Let $\mathbf{x}$ ref (k), $\mathbf{y}$ ref (k), $\mathbf{u}$ ref (k) denote reference or nominal values. Then

$$\delta\mathbf{x}(k) = \mathbf{x}\text{ ref }(k) - \mathbf{x}(k)$$

$$\delta\mathbf{y}(k) = \mathbf{y}\text{ ref }(k) - \mathbf{y}(k) \qquad \delta\mathbf{u}(k) = \mathbf{u}\text{ ref }(k) - \mathbf{u}(k)$$

The true dynamic situation represented by a linearization about the reference trajectory is governed by

$$\begin{aligned} \delta\mathbf{x}(k+1) &= \Phi(k+1,k)\delta\mathbf{x}(k) + U(k+1,k)\delta\mathbf{u}(k) \\ \delta\mathbf{u}(k+1) &= \psi(k+1,R)\delta\mathbf{u}(k) \end{aligned} \tag{9}$$

The observables are

$$\delta\mathbf{y}(k) = H(k)\,\delta\mathbf{x}(k) + F(k)\,\delta\mathbf{u}(k) \tag{10}$$

The estimated states are denoted by

$$\begin{aligned} \delta\hat{\mathbf{x}}(k+) &= \mathbf{x}\text{ ref }(k) - \hat{\mathbf{x}}(k+) \\ \delta\hat{\mathbf{x}}(k-) &= \mathbf{x}\text{ ref }(k) - \hat{\mathbf{x}}(k-) \end{aligned} \tag{11}$$

where $k-$ and $k+$ denote estimates immediately before and immediately after the processing of the kth observation.

The measurement deviation $\delta\tilde{\mathbf{y}}(k)$ is given by

$$\delta\tilde{\mathbf{y}}(k) = H(k)\delta\mathbf{x}(k) + F(k)\delta\mathbf{u}(k) + \mathbf{n}(k) \tag{12}$$

where $\mathbf{n}(k)$ is the noise in the kth observation.

We now seek to find an estimate $\delta\hat{\mathbf{x}}(k+)$ of $\delta\mathbf{x}(k)$ of the form

$$\delta\hat{\mathbf{x}}(k+) = \delta\hat{\mathbf{x}}(k-) + W(k)[\delta\tilde{\mathbf{y}}(k) - \delta\mathbf{y}_c(k)] \tag{13}$$

where $\delta\mathbf{y}_c(k)$ is the computed estimate of the kth measurement based on the previous observations and $W(k)$ is chosen to minimize the covariance matrix

$$P(k+) = \overline{[\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+)][\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+)]^T} \tag{14}$$

where $(\bar{\ })$ denotes ensemble average and $(\ )^T$ denotes the matrix transpose.

$$\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+) = \delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-) - W(k)\delta q(k) \tag{15}$$

where

$$\delta q(k) = \delta\tilde{\mathbf{y}}(k) - \delta\mathbf{y}_c(k) \tag{16}$$

Now it is clear that $P(k+)$ given by (14) is a function of the filter weighting coefficients $W(k)$. Minimization of $P(k)$ with respect to $W(k)$ yields

$$W(k) = A(k)[Y(k)]^{-1} \tag{17}$$

where

$$\begin{aligned} A(k) &= \overline{\{\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-)\}\delta q^T(k)} \\ Y(k) &= \overline{\delta q(k)\ \delta q^T(k)} \end{aligned} \tag{18}$$

Now $\delta\mathbf{y}_c(k)$ is the predicted value of the kth observation, computed on the basis of observations taken prior to time t_k. If the initial estimate $\delta\hat{\mathbf{u}}(0)$ of the bias parameter is taken to be zero, and accordingly, zero henceforth, since the bias estimates are not updated, then

$$\delta\mathbf{y}_c(k) = H(k)\,\delta\hat{\mathbf{x}}(k-) \tag{19}$$

Substituting (12) and (19) into (16), we obtain the observed measurement deviation $\delta\mathbf{q}(k)$ as

$$\delta\mathbf{q}(k) = H(k)\{\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-)\} + F(k)\,\delta\mathbf{u}(k) + \mathbf{n}(k) \tag{20}$$

We now introduce the covariance matrices

$$\overline{[\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-1)][\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-)]^T} = P(k-) \tag{21}$$

$$\overline{\mathbf{n}(k)\mathbf{n}^T(k)} = Q(k) \tag{22}$$

$$\overline{\delta\mathbf{u}(k)\ \delta\mathbf{u}^T(k)} = D(k) \tag{23}$$

and the cross-correlation matrix

$$\overline{[\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-)][\delta\mathbf{u}(k)]^T} = C(k-) \tag{24}$$

Following Pines, we make the assumptions that

$$\overline{[\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-)]\mathbf{n}^T(k)} = 0 \tag{25}$$

$$\overline{\delta\mathbf{u}(k)\mathbf{n}^T(k)} = 0 \tag{26}$$

i.e., errors in the orbit estimate based on $k-1$ observations and errors in the bias states are uncorrelated with the noise

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in the kth observation. Subject to these assumptions, one easily expresses $A(k)$ and $Y(k)$ as

$$A(k) = P(k-)H^T(k) + C(k-)F^T(k) \tag{27}$$

$$Y(k) = H(k)\,P(k-)H^T(k) + Q(k) + F(k)D(k)F^T(k) + H(k)\,C(k-)F^T(k) + F(k)C^T(k-)H^T(k) \tag{28}$$

One observes that if bias errors $\delta u(k)$ were neglected then

$$W(k) = P(k-)H(k)^T[H(k)\,P(k-)H^T(k) + Q(k)]^{-1} \tag{29}$$

which is the conventional Kalman filter gain matrix.

Updating and Propagation of $P(k)$, $D(k)$, and $C(k)$

To complete the filter construction, one needs a procedure for updating the matrices $P(k-)$, $C(k-)$ after an observation has been taken, and a procedure for propagating the updated matrices forward to the time of the next observation.

We address the updating problem first. Let

$$P(k+) = \overline{[\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+)][\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+)]^T}$$

denote the covariance matrix of the errors in the state estimates immediately after processing the kth observation. Combining (15–17), one obtains

$$\delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k+) = \delta\mathbf{x}(k) - \delta\hat{\mathbf{x}}(k-) - A(k)Y^{-1}(k)\delta q(k)$$

From this one easily obtains

$$\begin{aligned} P(k+) &= P(k-) - W(k)A^T(k) \\ &= P(k-) - W(k)[H(k)P(k-) + F(k)C(k-)^T] \end{aligned} \tag{30}$$

by employing (17) and (27).

The updated C matrix is obtained by employing (15, 20, 21, and 24) to yield

$$C(k+) = C(k-) - W(k)[H(k)C(k-) + F(k)\,D(k)] \tag{31}$$

The updated matrices $P(k+)$ and $C(k+)$ must now be propagated forward by the transition matrix to obtain $P(k+1-)$, $C(k+1-)$. The propagation of P is accomplished as follows. The true values of the states at t_{k+1} are related to the states and biases at t_k by Eq. (9). The state vector estimates $\delta x(k+1-)$ based on k observations is

$$\delta\hat{\mathbf{x}}(k+1-) = \Phi(k+1, k)\,\delta\hat{\mathbf{x}}(k+)$$

The computer actually produces a "rounded off" version of the foregoing, given by

$$\delta\hat{\mathbf{x}}(k+1-) = \Phi(k+1, k)\,\delta\hat{\mathbf{x}}(k+) + \mathbf{n}_x(k+1) \tag{32}$$

Assuming the "round-off noise" $\mathbf{n}_x(k+1)$ is uncorrelated with the state estimate $\delta\mathbf{x}(k+)$ and biases, one finds

$$\begin{aligned} P(k+1-) = {} & \Phi(k+1, k)P(k+)\Phi^T(k+1, k) + \\ & \Phi(k+1, k)C(k+)U^T(k+1, k) + \\ & U(k+1, k)C^T(k+)\Phi^T(k+1, k) + \\ & U(k+1)D(k)U^T(k+1, k) + Q_x(k+1) \end{aligned} \tag{33}$$

where

$$Q_x(k) = \overline{\mathbf{n}_x(k)\mathbf{n}_x{}^T(k)} \tag{34}$$

also

$$C(k+1) = \Phi(k+1, k)C(k)\psi^T(k+1, k) + U(k+1, k)D(k)\psi^T(k+1, k) \tag{35}$$

$$D(k+1) = \psi(k+1, k)D(k)\psi^T(k+1, k) \tag{36}$$

The modified filter equations previously discussed are applied to three types of error sources: round-off errors due to the finite word length of a digital computer, errors in the gravity constant, and errors in the drag acceleration.

Round-Off Error

Each computation performed in a digital computer will be in error because of the finite word length limitations. For instance, a computer with an eight-digit word length will store a number as a sequence of eight digits: $z_1, z_2, \ldots z_8$ and an exponent p. Thus, the computer stores z as

$$\tilde{z} = \cdot z_1\, z_2 \ldots z_8 10^p$$

The difference between z and $\tilde{z}$ is the round-off error $n_z = z - \tilde{z}$. This error has the same sign as z. Since the computer can include only the first eight digits, the round-off error is of the order of magnitude of 10^{p-8}. Thus, an estimate of the round-off error is

$$n_z = z10^{p-8}/|z| \tag{37}$$

Since, to order-of-magnitude accuracy, $|z| = 10^p$, Eq. (37) reduces to $n_z = z10^{-8}$. This is the round-off error model suggested by Pines[2] and utilized in the present study.

Level I Modification

A so-called Level I modification to the Kalman filter was used to compensate for round-off errors. This modification is based on assuming that all effects of round-off errors can be expressed as an error in (32) and that

$$\mathbf{n}_x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \end{bmatrix} (10^{-\alpha})$$

where $x_1(k+1) \ldots x_6(k+1)$ are the components of the six-dimensional computed state vector at time $k+1$, and α is a parameter that was varied during the study. (Investigations were made between 5.0 and 8.0.)

The covariance matrix $Q_x(k+1)$ given by (22) is then

$$Q_x(k+1) = 10^{-2\alpha} \begin{bmatrix} x_1(k+1)^2 & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & x_6(k+1)^2 \end{bmatrix} \tag{38}$$

Assumptions

This matrix is diagonal because round-off errors are assumed to be independent. Round-off errors are assumed to have no effect upon the observation $\delta\bar{\mathbf{y}}(k+1)$. Thus, the matrix $F(k+1)$ is zero (null). Given $F = 0$, the filter equations (27) and (28) reduce to (29), the unmodified Kalman filter. Round-off errors at time $k+1$ are assumed to be independent of the estimated state vector $\hat{\mathbf{x}}(k)$. Thus, the matrix $C(k-)$ given by (24) is zero (null). Given $F = 0$ and $C = 0$, Eqs. (30) and (33) reduce to

$$P(k+) = P(k-) - W(k)[H(k)P(k-)] \tag{39}$$

$$P(k+1-) = \Phi(k+1, k)P(k+)\Phi^T(k+1, k) + Q_x(k+1) \tag{40}$$

The only difference between the ordinary unmodified Kalman filter and the machine noise modified filter is the added term $Q_x(k+1)$ in Eq. (40), the covariance matrix propagation. Note the similarity between (40) and (7), the analogous relation for the illustrative example. Modifications in both cases are in the form of additions to the filter covariance matrix P.

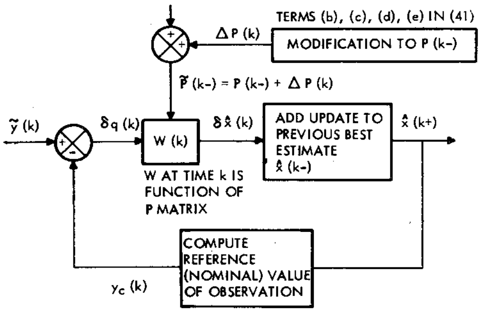

Fig. 3 **Modification to the Kalman filter.**

Dynamical Bias Errors

Bias errors may be divided into two categories: dynamical biases—errors in parameters which enter into the equations of motion of the vehicle; and instrument biases—systematic errors in the observational data (for example, errors caused by misalignment of instruments or miscalibration). We shall limit discussion to the dynamical biases and in particular, errors in the gravity and drag computation.

Form of Modified Filter

The sensitivity matrix F in the modified filter equation is a null matrix because dynamical biases do not affect observational data. With this simplification, the modified filter equations again reduce to (29).

The update equations (30) and (31) become:

$$P(k+) = P(k-) - W(k)\, H(k)\, P(k-)$$

$$C(K+) = C(k-) - W(k)\, H(k)\, P(k-)$$

Propagation equations (33, 35, and 36) become:

$$\begin{aligned} P(k+1-) &= \Phi(k+1,k)P(k+)\Phi^T(k+1,k) && \text{a)} \\ &+ \Phi(k+1,k)C(k+)U^T(k+1,k) && \text{b)} \\ &+ U(k+1,k)C(k+)\Phi^T(k+1,k) && \text{c)} \\ &+ U(k+1,k)DU^T(k+1,k) && \text{d)} \\ &+ Q_x(k+1) && \text{e)} \qquad (41) \end{aligned}$$

$$C(k+1-) = \Phi(k+1,k)\, C(k+) + U(k+1,k)D \qquad (42)$$

$$D(k+1) = D(\text{a constant for all } k) \qquad (43)$$

In this study, the gravity bias was assumed to be a (constant) error in the gravitational constant; the drag bias was assumed to be a (constant) error in the ballistic drag coefficient. Since the biases do not change with time, $\psi(k+1, k)$ in (35) and (36) is the identity matrix and (42) and (43) follow.

Note that the filter equation (29) and the $P(k+)$ update equation are identical to the unmodified Kalman filter. Only the propagation equation (41) differs, and the differences are in the form of additive modification terms. Term a in (41) is the term that appears in the unmodified Kalman filter, terms b and c are modifications due to correlation between the bias state and the estimation error, term d is the modification due to bias uncertainty (as represented by the bias error covariance matrix D, term e is the effect of computational noise. Figure 3 shows the modified Kalman filter.

Simplified Versions of the Modified Filter

Simulation studies were performed using three levels of approximation to the modified filter equations:

Level I: the simplest approximation to the modified filter assumed that $C(k)$ and D in (41) are null matrices with the result:

$$P(k+1-) = \Phi(k+1,k)\, P(k+)\Phi^T(k+1,k) + Q_x(k+1)$$

The matrix $Q_x(k+1)$ is of the form given by (38)

Level II: Only the cross-correlation matrix $C(k)$ in (41) is assumed null. Thus,

$$P(k+1-) = \Phi(k+1,k)\, P(k+)\Phi^T(k+1,k) + U(k+1,k)DU^T(k+1,k) + Q_x(k+1)$$

Level III: All terms in (41) are included in the filter propagation.

There are considerable differences in the computational complexity of the three levels of filter modification. Level I is clearly only a trivial increase in the complexity of the unmodified filter. Level II, however, requires a significantly greater number of computations, primarily in the computation of the bias state transition matrix $U(k+1, 1)$.

The Level III modified filter is more complex than Level II because of the additional terms b and c in (41) and the addition of the update and propagation relations for matrix $C(k)$. Some idea of the relative complexity of the three filter schemes may be derived from the execution times of the computer simulations: a Level I filter simulation executed in 3.00 mins, a comparable Level II filter simulation executed in 6.60 min, and a Level III simulation, 9.18 min.

Simulation Studies

Simulation studies investigated the divergence of the Kalman filter used for autonomous navigation in a low earth orbit. Observations consist of pitch and roll line-of-sight angles derived from optical tracking of known terrestrial landmarks. Figure 4 defines telescope pitch and roll angles. Inertial attitude reference is maintained by an automatic star-tracker system. A Kalman filter is used to compute differential corrections to nine states: three components of spacecraft position, three components of spacecraft velocity, and three components of landmark location.

Divergence was investigated by simulating the entire navigation process including computation of the true orbit and the estimated orbit on an IBM 7090 computer. Investigations considered both a six-state filter, which estimated

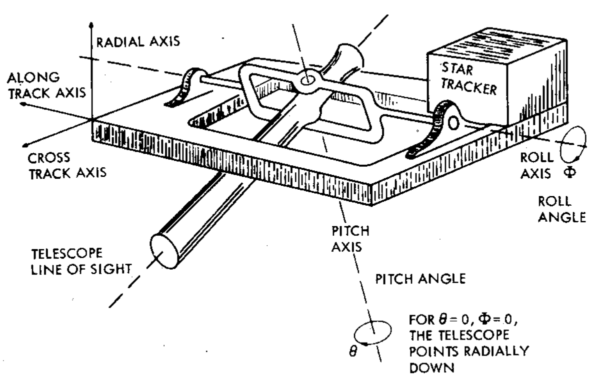

Fig. 4 **Telescope gimbal layout.**

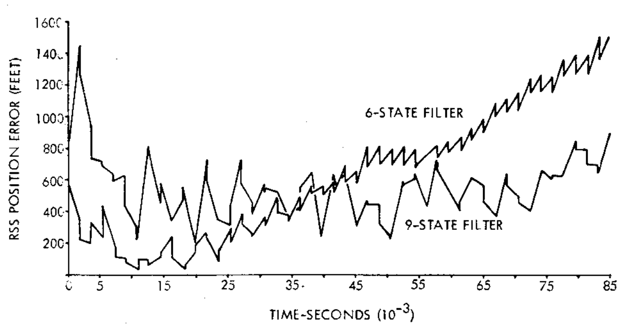

Fig. 5 **Effect of filter state dimension upon divergence.**

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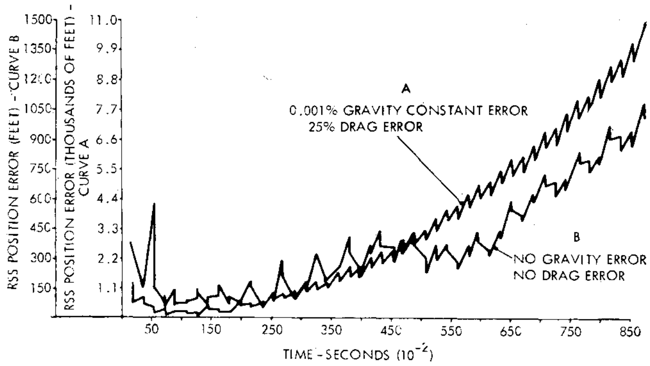

Fig. 6 Effect of dynamic bias errors upon divergence.

only spacecraft states, and a nine-state filter. The effect of adding altimeter measurements was also investigated. Simulation studies utilized a nominal 160-naut-mile circular orbit. Three landmark tracking exercises were performed each orbit. Landmarks were assumed to be first acquired at 30° (forward) pitch. Out-of-plane orientation of the landmark at acquisition was random but limited to ±30° roll angles. Level I, II, and III modified filters, in addition to the unmodified Kalman filter, were studied under a variety of conditions reflecting effects of gravity error, drag error, data rate, pointing accuracy, and landmark location errors.

Simulations showed that an unmodified Kalman filter always diverged after a sufficiently long period of operation. A nine-state variable filter simulation was less divergent than an equivalent six-state filter. Figure 5 shows that the nine-state variable filter diverges later and more slowly than the six-state variable filter. The divergence is caused by round-off in the computations.

The rss position error plotted in Fig. 5 is the length of the vector difference between true spacecraft position and the position estimated by the Kalman filter. Rss position error is thus a measure of the actual navigation error. Rms position error is the square root of the trace of the position covariance matrix. As such, rms is an ensemble statistic that represents the average or expected position error. If there were no machine computation errors and no uncompensated bias errors, the rms position error would be a measure of the actual position errors. Rms errors are also referred to as theoretical errors whereas rss are referred to as actual errors.

Presence of errors in the gravity model and drag parameters used to compute spacecraft ephemeris increased filter divergence. Figure 6 shows the divergence caused by an error of 0.001% in the gravity constant and a 25% error in ballistic drag coefficient. Addition of another measurement such as altimeter data increases the divergence. Simulations show that the rss error increased from 11,000 to 16,000 ft after 16 orbits for the gravity and drag error case

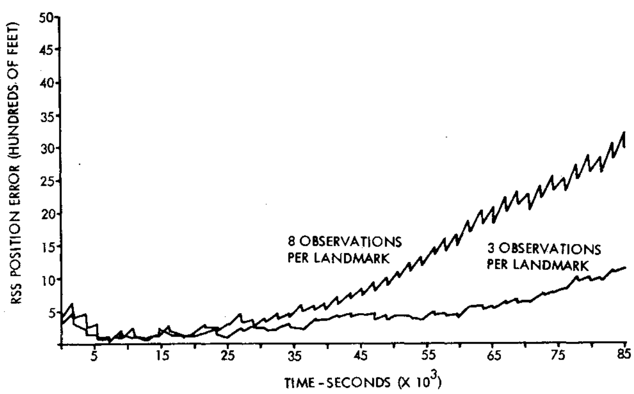

Fig. 7 Effect of data rate upon divergence.

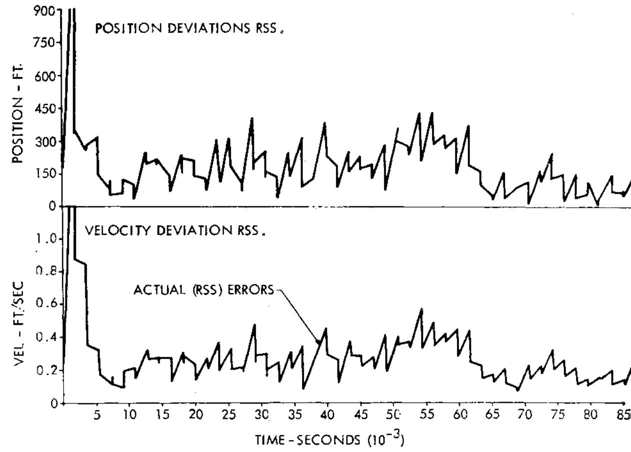

Fig. 8 Level I modified filter.

shown in Fig. 6. An increase in the assumed (a priori) data accuracy likewise aggravates divergence. Simulations that assumed no gravity and no drag error showed a position error of 1085 ft at the end of 16 orbits when the one-sigma telescope pointing error was assumed to be 40 sec, and 1500 ft when the telescope pointing error was assumed to be 10 sec. Increased data rate increases filter divergence. Figure 7 compares a run with eight sightings (eight pairs of pitch and roll angle measurements are made) on each landmark with a run in which only three sightings are made on each landmark.

These results all support the intuitive notion that filter divergence occurs when filter weights become too small. Addition of an altimeter, increased data rate, and increased data accuracy all reduce the filters estimation error covariance matiix and thereby decrease the filter weights. On the other hand, including an additional error source, landmark uncertainty, as in the nine-state filter, increases the filter covariance matrix for sufficiently large landmark uncertainty and would be expected to reduce divergence. The landmark uncertainty used in Fig. 5 is 600 ft (in each axis).

Simulations of the modified filter have shown that divergence can be avoided and that a form of steady-state behavior is achieved. Figure 8 shows that rss error for a Level I modified filter. Figure 9 shows a Level III modified filter. A series of simulations was performed to determine the effects of data rate and magnitude of the α parameter in (38) upon performance of a Level I modified filter. A histogram-like plot was used to quantitatively evaluate these results. These

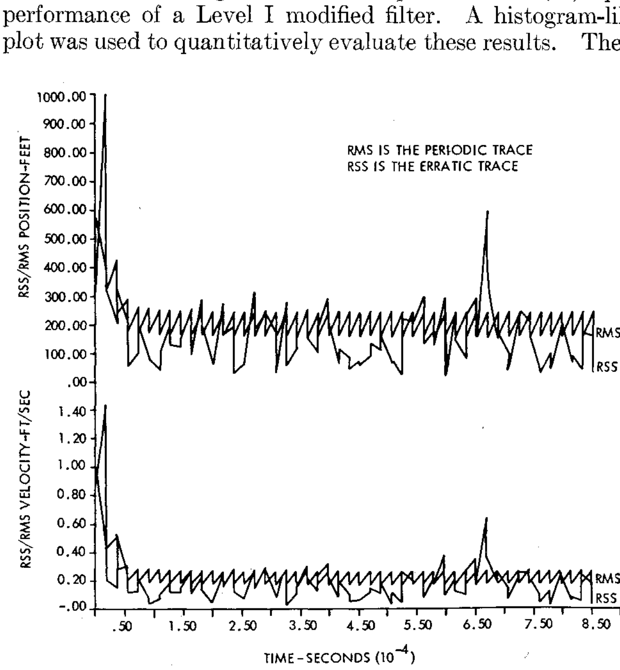

Fig. 9 Level III modified filter.

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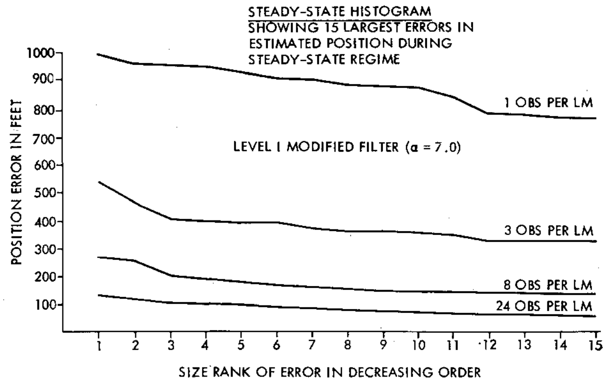

Fig. 10 Effect of varying number of observations per landmark.

plots contain the 15 largest rss position error excursions that occur after steady state is achieved. Figure 10 shows the histograms for a Level I modified filter for various values of the data rate. Figure 11 shows a similar plot for various values of the α modification parameter. A comparison between Level I, Level II, and Level III filters is made in Fig. 12.

Conclusions

Simulation studies have shown that divergence depends upon the a priori instrument accuracy, data rate, and number of observations processed; divergence is more severe for sensors with increased a priori accuracy, for increased data rates, and for landmark tracking systems augmented by a radar altimeter. The addition of three landmark bias states, with moderate standard deviations, retards but does not prevent divergence. Although dynamical biases could be included in the state to be estimated, long-duration Kalman filter operation will still require modifications to control divergence caused by round-off errors.

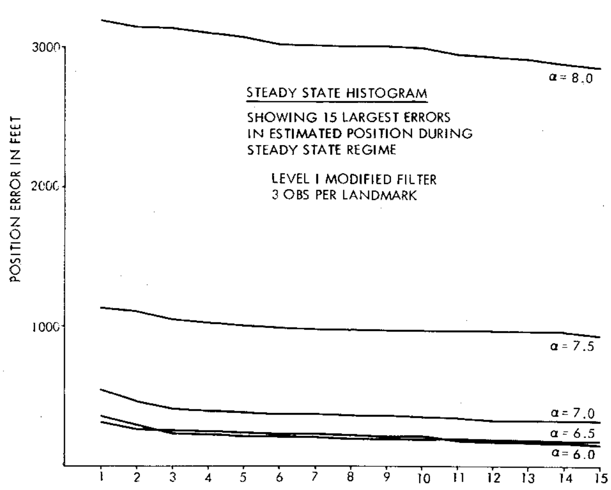

Fig. 11 Effect of varying a parameter.

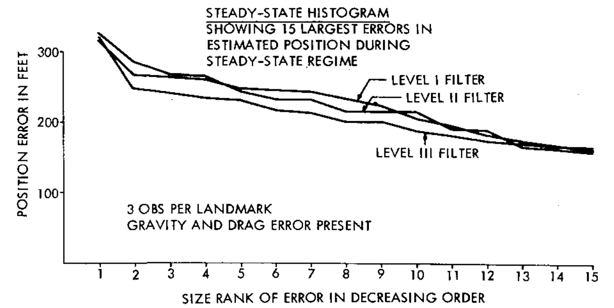

Fig. 12 Comparison of three level filter modifications.

The suppression of divergence by modifications to account for both gravity and drag dynamic biases is demonstrated. The Level I modification is shown to control filter divergence caused by finite word length (single precision IBM 7090) computations, and with properly chosen parameters, is as effective as the Level II and Level III modifications in controlling divergence due to dynamic biases in the drag and gravity models.

Level III modifications would be optimal (since they are derived as a constrained optimal filter) if computer round-off errors were not present. Since round-off errors are always present in a finite word length machine, the Level III modification is not optimal. Thus it is not surprising to find in Fig. 12 that the simple Level I modification performs as well as the complex Level III modifications. It is entirely conceivable that, for certain combinations of dynamic bias error and data rate, data accuracy and computer word length, the Level I filter could perform significantly better than the Level III modified filter. In any given application, simulation studies are necessary to determine the best form of filter modification for divergence control.

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