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Feedback linearization of transverse dynamics for periodic orbits ☆

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Abstract

In this paper we give necessary and sufficient conditions for feedback linearization of the transverse dynamics (TFL) of a nonlinear affine single-input system in a neighborhood of a periodic orbit. The TFL procedure provides a means of finding coordinates that are tuned to the structure of the control system with respect to the periodic orbit. An autonomous feedback control providing exponential stability of the periodic orbit is easily designed in the transverse coordinate system.

Keywords: Nonlinear systems; Fedback linearization; Periodic orbits; Transverse dynamics

0. Introduction

Stable maneuvering of a nonlinear system is an important goal in many fields including the flight of aerospace vehicles, robotic manipulation, and the manufacture of sophisticated materials. This goal can often be accomplished by providing a stable orbit (or, more generally, a maneuver) for the system by stabilizing the dynamics *transverse* to that orbit. In this paper, we explore the structure of the transverse dynamics for the special case when the maneuver is a periodic orbit.

Consider the smooth dynamical system

$$\dot{x}=f(x)+g(x)u \tag{1}$$

on $\mathbb{R}^n$ and suppose that $\eta \subset \mathbb{R}^n$ is a periodic orbit of (1) with minimal period T when $u \equiv 0$.

We are studying when it is possible to find new coordinates $(\theta, \rho_1, \dots, \rho_{n-1})$ and control v so that, after change of coordinates and feedback $u = k(x) + l(x)v$, the dynamics of (1) in a neighborhood of the periodic

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orbit η have the form

$$\begin{aligned}
\dot{\theta} &= 1 + f_1(\theta,\rho) + g_0(\theta,\rho)v, \\
\dot{\rho}_1 &= \rho_2, \\
&\vdots \\
\dot{\rho}_{n-2} &= \rho_{n-1}, \\
\dot{\rho}_{n-1} &= v,
\end{aligned} \tag{2}$$

where $f_1(\cdot,\cdot)$ satisfies $f_1(\theta,0) = 0$. The variable $\theta \in S^1 = [0,T]$ (we identify 0 and T) parametrizes the periodic orbit η and the coordinates $(\rho_1,\dots,\rho_{n-1})$ parametrize the transverse dynamics.

A system (1) which admits such a feedback transformation will be called *(globally) transversely feedback linearizable along* η. In this paper we give necessary and sufficient conditions for transverse feedback linearizability for affine single-input nonlinear systems.

We will also consider systems (1) which, even though not globally transversely feedback linearizable, they are *locally transversely feedback linearizable* in the sense that one can cover a neighborhood of η with a finite number of open neighborhoods such that the dynamics of (1) in every neighborhood has form (2).

Feedback linearization of transverse dynamics can be applied to design controllers stabilizing the transverse dynamics of (1), so that all trajectories of the closed-loop system with initial conditions close to η will asymptotically approach η.

The idea of transverse linearization is not restricted to *periodic* orbits. Indeed, one can attempt to linearize the transverse dynamics for any orbit passing through *any* point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \neq 0$.

Note that the paper [6] deals with a problem similar to the one considered in the present paper. The results in [6] would apply to the present situation if we required $f_1(\theta,\rho) = g_0(\theta,\rho) = 0$ in (2).

The paper is organized as follows. In Section 1 we introduce and study a new notion of controllability, called transverse linear controllability, that assures that the linearized transverse dynamics is controllable along the periodic orbit. We give a coordinate-free description of transverse linear controllability and study its relationships with the usual linear controllability. In Section 2 we give necessary and sufficient conditions for global and local transverse linearization of system (1). In Section 3 we provide an example of construction of a transformation linearizing the transverse dynamics for a system in $\mathbb{R}^3$.

1. On transverse linear controllability

We say that $(\theta, \rho) = (\theta, \rho_1,\dots, \rho_{n-1})$ is a set of *transverse coordinates* around η if the mapping $x \mapsto (\theta, \rho)$ is a diffeomorphism on a neighborhood of η and $\rho = 0$, $\dot{\theta} = 1$ on η. The requirement that $\dot{\theta} = 1$ on η is an arbitrary, but convenient, way to fix the parametrization of η.

Note that, for *any* transverse coordinates (θ,ρ), the system (1) has the form (cf. [2-4])

$$\begin{aligned}
\dot{\theta} &= 1 + f_1(\theta, \rho) + g_0(\theta, \rho)u, \\
\dot{\rho} &= A(\theta)\rho + b(\theta)u + f_2(\theta, \rho) + g_1(\theta, \rho)u,
\end{aligned} \tag{3}$$

where a subscript j indicates that the function (or vector field) is order j in the transverse coordinate ρ, so that, e.g., $f_2(\cdot, \cdot)$ satisfies $f_2(\theta,0) = 0$ and $D_2 f_2(\theta,0) = 0$.

The $(n-1)$-dimensional (periodic) time-varying linear system derived from (3) given by

$$\frac{d\rho}{d\theta} = A(\theta)\rho + b(\theta)u \tag{4}$$

is called the *transverse linearization* of the system (1) along η (with respect to (θ, ρ) coordinates).

The notions of transverse coordinates and transverse linearization are not restricted to *periodic* orbits. Indeed, one can find *local* transverse coordinates about the orbit passing through *any* point $x_0 \in \mathbb{R}^n$ such that $f(x_0) \neq 0$.

We say that (1) is *linearly controllable* at $x \in \mathbb{R}^n$ if

$$\dim \operatorname{span}\{g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)\} = n. \tag{5}$$

It is well known that, if $x_0 \in \mathbb{R}^n$ is an equilibrium point of the the undriven system ($f(x_0) = 0$), then (5) is satisfied iff the linearization of (1) about x_0 is controllable, i.e.,

$$\operatorname{rank}\left[b, Ab, \ldots, A^{n-1}b\right] = n,$$

where $A = Df(x_0)$ and $b = g(x_0)$. We say that the system (1) is *linearly controllable* on a subset of $\mathbb{R}^n$ if it is linearly controllable at every point of this subset.

Note that the transverse dynamics of the system (2) is linearly controllable. This motivates the following (coordinate independent) definition. We call (1) *transversely linearly controllable* at $x \in \mathbb{R}^n$ if

$$\dim \operatorname{span}\{f(x), g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\} = n. \tag{6}$$

We say that (1) is *transversely linearly controllable* on a subset of $\mathbb{R}^n$ if (1) is transversely linearly controllable at every point of this subset. The following result shows that (6) is a test of (instantaneous) linear controllability.

Proposition 1.1. *The system* (1) *is transversely linearly controllable at* $x \in \eta$ *if and only if the transverse linearization* (4) *is instantaneously controllable at* θ *where* x *is mapped to* $(\theta, 0)$ *under the coordinate change.*

Proof. Since the condition (6) is coordinate independent, we may establish the equivalence by calculating the required distribution in (θ, ρ) coordinates. Direct calculation shows that

$$f|_{\rho=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad g|_{\rho=0} = \begin{pmatrix} * \\ b(\theta) \end{pmatrix}, \qquad ad_f g|_{\rho=0} = \begin{pmatrix} * \\ b'(\theta) - A(\theta)b(\theta) \end{pmatrix}, \ldots,$$

where $*$ indicates a do not care value and $b' = db/d\theta$. We see that (6) is satisfied if and only if

$$\operatorname{rank}\left[b, \mathscr{A}b, \ldots, \mathscr{A}^{n-2}b\right](\theta) = n - 1,$$

where $\mathscr{A}$ denotes the operator $h(\theta) \mapsto h'(\theta) - A(\theta)h(\theta)$ for $h: \mathbb{R} \to \mathbb{R}^{n-1}$. This is precisely the condition for instantaneous linear controllability of a time-varying linear system (cf. [1]). □

The notion of transverse linear controllability is coordinate and feedback invariant. One can easily check that the transverse linear controllability along the periodic orbit η is necessary for transverse feedback linearization along η.

Dynamically, if the system is transversely linearly controllable, then we can find controls that easily steer the transverse directions while we flow downstream along with the orbit. Note that it may still be possible to steer the system through higher-order brackets even when the system is not transversely linearly controllable. Indeed, it has been shown in [5] that, if there is a point on the orbit such that either (5) or (6) (or a (6)-like condition with $ad_f^k g$, $k = 0, 1, \ldots$ included) is satisfied, then there is a neighborhood of the orbit such that the system may be steered between any two points in that neighborhood by an appropriate control.

The following result shows that linear controllability and transverse linear controllability are not completely independent.

Proposition 1.2. *Suppose that* (1) *is linearly controllable at* $x_0 \in \mathbb{R}^n$. *Then for every open neighborhood* S *of* x_0, *there is a point* $x_1 \in S$ *such that* (1) *is transversely linearly controllable at* x_1.

Proof. Note that there is an open neighborhood O of x_0 such that (5) is satisfied at every $x \in O$. Suppose that there is an open neighborhood $S \subset O$ of x_0 such that (6) fails at every $x \in S$. Then, there are smooth functions a_i such that

$$a_0 f + a_1 g + a_2 ad_f g + \cdots + a_{n-1} ad_f^{n-2} g = 0 \tag{7}$$

for all $x \in S$ and, for each $x \in S$, at least one a_i is nonzero. Taking the Lie bracket of f with both sides of (7) we see that

$$\begin{aligned}&(L_f a_0) f + (L_f a_1) g + (L_f a_2 + a_1) ad_f g + \cdots + \\ &\quad (L_f a_{n-1} + a_{n-2}) ad_f^{n-2} g + a_{n-1} ad_f^{n-1} g = 0\end{aligned} \tag{8}$$

for all $x \in S$. Now, by linear controllability, $ad_f^{n-1} g$ is independent of $ad_f^j g$ for $j = 0, \ldots, n-2$, and therefore of f. Thus, (8) implies that a_{n-1} and, hence, $L_f a_{n-1}$ are identically zero on S. Similarly, by taking further brackets of f with (8), we may conclude that $a_i \equiv 0$ on S for $i = 0, \ldots, n-2$. This contradicts the hypothesis that (6) fails at *every* $x \in S$. $\square$

Since the points of transverse linear controllability form an open set, the above result implies that the points of transverse linear controllability are *dense* in the set of points of linear controllability.

Note that at a point x_0 a system can be both linearly controllable but not transversely linearly controllable (e.g., a linearly controllable system at an equilibrium point of f) or transversely linearly controllable but not controllable (e.g., $\dot{\theta}_1 = 1, \dot{\rho} = u$).

2. Transverse feedback linearization

The main result of this paper is as follows.

Theorem 2.1. *Let* η *be a periodic orbit of the undriven system* (1) *with* $u \equiv 0$. *Then, the system* (1) *is transversely feedback linearizable along the periodic orbit* η *if and only if it is transversely linearly controllable along* η *and there exists a smooth function* α *in a neighborhood* $\mathcal{N}$ *of* η *such that*

(i) $d\alpha \neq 0$ *on* η.
(ii) $\alpha = 0$ *on* η.
(iii) $L_{ad_f^i g}\alpha = 0$ *in* $\mathcal{N}$ *for* $i = 0, \ldots, n-3$.

Proof. ($\Rightarrow$) Transverse linear controllability along η is obvious. We will show that conditions (i)–(iii) are satisfied for $\alpha := \rho_1$. Conditions (i) and (ii) are obviously satisfied. To verify that condition (iii) holds, it is sufficient to note that in (θ, ρ) coordinates we have

$$
\begin{aligned}
g(\theta,\rho) &= g_0(\theta,\rho)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\rho_{n-1}},\\
ad_f g(\theta,\rho) &= g_1(\theta,\rho)\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\rho_{n-2}},\\
ad_f^2 g(\theta,\rho) &= g_2(\theta,\rho)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\rho_{n-3}},\\
&\vdots\\
ad_f^{n-3} g(\theta,\rho) &= g_{n-3}(\theta,\rho)\frac{\partial}{\partial\theta} + (-1)^{n-3}\frac{\partial}{\partial\rho_2},
\end{aligned}
\tag{9}
$$

where $g_i(\theta,\rho)$, $i = 1,\dots,n-3$ are smooth functions depending on $g_0(\theta,\rho), f_1(\theta,\rho)$ and their partial derivatives.

($\Leftarrow$) Let v be a smooth vector field satisfying $L_v\alpha = 1$ in a neighborhood of η. By the assumption of transverse linear controllability, the vector fields $f, g, ad_f g, \dots, ad_f^{n-3} g$ are linearly independent in a neighborhood of η. Since α is constant along $g, ad_f g, \dots, ad_f^{n-3} g$ and $L_f\alpha = 0$, $L_v\alpha = 1$ on η, we conclude that f, v, g, $ad_f g, \dots$, $ad_f^{n-3} g$ are linearly independent in a neighborhood of η. Fix a point x_0 on η. In a neighborhood of η one can *reach* any point x by *traveling* along vector fields f, v, g, $ad_f g, \dots$, $ad_f^{n-3} g$ with *times* $s_0, s_1, \dots, s_{n-1}$, i.e., the mapping $s \mapsto x$ given by ($\phi_s^h(\cdot)$ is the flow of a vector field h)

$$
x = \phi_{s_{n-1}}^{g} \circ \phi_{s_{n-2}}^{ad_f g} \circ \cdots \circ \phi_{s_2}^{ad_f^{n-3} g} \circ \phi_{s_1}^{v} \circ \phi_{s_0}^{f}(x_0) \tag{10}
$$

is a local diffeomorphism between the cylinder $S^1 \times \mathbb{R}^{n-1}$ and a tubular neighborhood of η (cf. [7]). Now the value of α at x is exactly $s_1(x)$. This is clear for the points x that can be reached from η by flowing along v (i.e., $s_2 = \cdots = s_{n-1} = 0$). Furthermore, condition (iii) implies that the value of α is unchanged for nonzero $s_2, \dots, s_{n-1}$. Essentially, we use s_0, s_1 to reach the appropriate leaf of the foliation determined by the value of α. Put

$$
\begin{aligned}
\theta &:= s_0,\\
\rho_1 &:= \alpha,\\
\rho_2 &:= L_f\alpha,\\
&\vdots\\
\rho_{n-1} &:= L_f^{n-2}\alpha.
\end{aligned}
\tag{11}
$$

To verify that $(\theta, \rho_1, \dots, \rho_{n-1})$ are valid coordinates it suffices to show that they have linearly independent differentials on η. Observe that

$$
(\mathrm{d}\theta \wedge \mathrm{d}\rho_{n-1} \wedge \cdots \wedge \mathrm{d}\rho_1)(f,\, g,\, ad_f g, \dots,\, ad_f^{n-3},\, v) = \det S,
$$

where

$$S = \begin{bmatrix} L_f\theta & L_g\theta & L_{ad_f g}\theta & \cdots & L_v\theta \\ L_f L_f^{n-2}\alpha & L_g L_f^{n-2}\alpha & L_{ad_f g}L_f^{n-2}\alpha & \cdots & L_v L_f^{n-2}\alpha \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ L_f^2\alpha & L_g L_f\alpha & L_{ad_f g}L_f\alpha & \cdots & L_v L_f\alpha \\ L_f\alpha & L_g\alpha & L_{ad_f g}\alpha & \cdots & L_v\alpha \end{bmatrix}.$$

Now, $L_{ad_f^i g}\alpha = 0$ for $i < n-2$ around η implies that $L_{ad_f^j g}L_f^k\alpha = 0$ for $j+k < n-2$ around η. Also, $L_{ad_f^j g}L_f^k\alpha \neq 0$ for $j+k = n-2$ in a neighborhood of η. Otherwise, since (1) is transversely linearly controllable and $d\alpha \neq 0$, $L_f\alpha$ would be nonzero around η which contradicts the fact that $L_f\alpha(x) = 0$ for $x \in \eta$. Collecting these facts, we see that, on η, the matrix S is upper triangular with nonzero diagonal elements so that (θ, ρ) are valid coordinates. In (θ, ρ) coordinates, the system (1) takes the form

$$\begin{aligned} \dot\theta &= 1 + \tilde f_0(\theta,\rho) + \tilde g_0(\theta,\rho)u, \\ \dot\rho_1 &= \rho_2, \\ &\vdots \\ \dot\rho_{n-2} &= \rho_{n-1}, \\ \dot\rho_{n-1} &= p(\theta,\rho) + r(\theta,\rho)u, \end{aligned} \tag{12}$$

where $p(\theta, \rho) := L_f^{n-1}\alpha$, $r(\theta, \rho) := L_g L_f^{n-2}\alpha$. Since $r(\theta, \rho) \neq 0$ in a neighborhood of η, the preliminary feedback $u = r(\theta, \rho)^{-1}(-p(\theta, \rho) + v)$ puts the system into the desired form (2). □

Now we provide a sufficient condition for existence of function α satisfying conditions (i)–(iii) and hence for the transverse feedback linearization.

Theorem 2.2. *Let η be a periodic orbit of the undriven system* (1) *with $u \equiv 0$. Then, the system* (1) *is transversely feedback linearizable along the periodic orbit η if it is transversely linearly controllable along η and the distribution*

$$\mathscr{D} := \text{span}\{g, ad_f g, \ldots, ad_f^{n-3}g\}.$$

is involutive.

Proof. Let v be any smooth vector fields such that f, v, g, $ad_f g, \ldots, ad_f^{n-3}g$ are linearly independent in a neighborhood of η. (It follows from the transverse linear controllability that such v exists, for instance, one can choose $v := ad_f^{n-2}g$.) Now, as in the proof of part ($\Leftarrow$) of Theorem 2.1, we construct $s_0, s_1, \ldots, s_{n-1}$ by *traveling* along the vector fields f, v, g, $ad_f g, \ldots, ad_f^{n-3}g$. It is easy to verify that $\alpha := s_1$ satisfies conditions (i) and (ii). Condition (iii) follows from involutivity of $\mathscr{D}$. □

Remark 2.1. In practice, the construction of the function α (for involutive $\mathscr{D}$) by the flow of a vector field v may be difficult. Instead, one can proceed as follows. If the system (1) is transversely linearly controllable, then $\dim \mathscr{D} = n-2$. When $\mathscr{D}$ is an involutive distribution, there are two independent functions α_1, α_2 that are constant along $\mathscr{D}$. One may try to construct directly a function $\alpha = \alpha(\alpha_1, \alpha_2)$ that has value zero on η. Such a function is guaranteed to exist. Then $\rho_1 := \alpha$ and its Lie derivatives along f can be used as transverse coordinates. Any convenient method for parametrizing a family of transverse sections (e.g., orthogonal plane) can be used to provide the θ coordinate. An example of this technique is given in the next section. □

η along v.) Note that $\mathcal{M}$ is a two-dimensional smooth submanifold of $\mathcal{N}$ containing η. Moreover, $T_x\mathcal{M}$ is transversal to $\mathcal{D}_1(x)$ in $\mathcal{M}$, so that $\dim T_x\mathcal{M} \cap \mathcal{D}_1(x) = 1$ in $\mathcal{M}$. For $x \in \mathcal{M}$ we define $\theta(x) := s_0(x)$. We are going to construct a smooth vector field $\tilde{f}$ on $\mathcal{M}$ such that $\tilde{f}(x) \in T_x\mathcal{M} \cap \mathcal{D}_1(x)$ for $x \in \mathcal{M}$, $\tilde{f} = f$ on η, and $L_{\tilde{f}}\theta = 1$ on $\mathcal{M}$. First of all, since $T\mathcal{M} \cap \mathcal{D}_1$ is a one-dimensional smooth distribution in $\mathcal{M}$, there is a smooth vector field $\hat{f}$ in $\mathcal{M}$ that spans $T_x\mathcal{M} \cap D_1$ (locally it is obvious, global construction on $\mathcal{M}$ can be obtained using partitions of unity). Note that $L_{\hat{f}}\theta \neq 0$ on η so that we can assume that the same holds in $\mathcal{M}$ (making $\mathcal{M}$ smaller, if necessary). Let $\tilde{f} := (1/L_{\hat{f}}\theta\hat{f})$. It is easy to verify that this vector field has the desired property $L_{\tilde{f}}\theta = 1$ on $\mathcal{M}$ and thus, in particular, $\tilde{f} = f$ on η. We can assume (making $\mathcal{M}$ smaller, if necessary) that $\tilde{f}, v, g, ad_f g, \ldots, ad_f^{n-3} g$ are linearly independent for $x \in \mathcal{M}$. In particular, $\tilde{f}, g, ad_f g, \ldots, ad_f^{n-3} g$ span $\mathcal{D}_1$ for $x \in \mathcal{M}$. Let $y \in \eta$ be arbitrary. In a neighborhood of y one can *reach* any point x by *traveling* along vector fields $v, \tilde{f}, g, ad_f g, \ldots, ad_f^{n-3} g$ with *times* $p_0^y, p_1^y, \ldots, p_{n-1}^y$, i.e., the mapping $p^y \mapsto x$ given by ($\phi_p^h(\cdot)$ is the flow of a vector field h)

$$x = \phi^g_{p^y_{n-1}} \circ \phi^{ad_f g}_{p^y_{n-2}} \circ \cdots \circ \phi^{ad_f^{n-3} g}_{p^y_2} \circ \phi^{\tilde{f}}_{p^y_1} \circ \phi^{v}_{p^y_0}(y) \tag{14}$$

is a local diffeomorphism between a cube in $\mathbb{R}^n$ and an open neighborhood $\mathcal{O}_y$ of y. Note that a finite number, say m, of such open neighborhoods $\mathcal{O}_{y_i}$, $i = 1, \ldots m$ covers η. In each $\mathcal{O}_{y_i}$ we define $\alpha^i := p_0^{y_i}$. It can be easily checked that α_i satisfies conditions (i)–(iii) of Theorem 2.1 in $\mathcal{O}_{y_i}$. As in the proof of Theorem 2.1, one can show that

$$\begin{aligned}
\theta^i &:= p_1^{y_i}, \\
\rho_1^i &:= \alpha^i, \\
\rho_2^i &:= L_f \alpha^i, \\
&\vdots \\
\rho_{n-1}^i &:= L_f^{n-2} \alpha^i.
\end{aligned} \tag{15}$$

are valid coordinates in $\mathcal{O}_{y_i}$ and the dynamics of (1) in those coordinates (after a preliminary feedback) has form (2). □

Remark 2.2. Let us assume that the hypothesis of Theorem 2.3 is satisfied. Then it can be shown that one needs at most two open sets $\mathcal{O}_{y_i}$ to cover η, i.e., one can cover η with exactly two coordinate charts in which the transverse dynamics (after a preliminary feedback) is linear. This is due to the fact that sufficiently close to η one can travel in $\mathcal{M}$ along $\tilde{f}$ for a long time, making a full circle without leaving $\mathcal{M}$ (i.e., the Poincare return map is well-defined sufficiently close to the orbit). Actually, for arbitrary $y_1, y_2 \in \eta$, $y_1 \neq y_2$, one can construct two overlapping open neighborhoods $\mathcal{O}_{y_i}, i = 1,2$ covering η, as in the proof of Theorem 2.3. The reason why one cannot, in general, construct one such neighborhood covering the whole η (as it was possible in the proof of Theorem 2.2) is that the orbits of $\tilde{f}$ on $\mathcal{M}$ do not have to close up. Still, it may happen that the orbits $\tilde{f}$ on $\mathcal{M}$ do close up (i.e., the flow of $\tilde{f}$ on $\mathcal{M}$ consists of periodic orbits foliating $\mathcal{M}$). In this case one can construct the linearizing transverse coordinates that work globally around η, as in the case of Theorem 2.2. □

Remark 2.3. If the hypothesis of Theorem 2.3 is satisfied, local construction of α^i by flowing along v may not be practical. Instead, one can try to solve (locally) the set of PDE's $L_{h_i}\alpha = 0$, where $h_i, i = 1, \ldots, n-1$ span $\mathcal{D}_1$. An example is provided in the next section. □

Note that every globally transversely linearizable system is, in particular, locally transversely linearizable. Therefore, we can formulate the following corollary from Theorems 2.1 and 2.3, and Proposition 2.1.

To see that the condition of involutivity of $\mathscr{D}$ is not necessary for transverse feedback linearizability, consider the following example.

Example 2.1. Consider the system

$$\begin{aligned}\dot{\theta} &= 1 + \rho_3^2,\\ \dot{\rho}_1 &= \rho_2,\\ \dot{\rho}_2 &= \rho_3,\\ \dot{\rho}_3 &= u.\end{aligned} \tag{13}$$

This system is already in the desired form (2), so that it is clearly transversely feedback linearizable. We have

$$\mathscr{D} = \operatorname{span}\{g, ad_f g\} = \operatorname{span}\left\{\frac{\partial}{\partial \rho_3}, -2\rho_3\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \rho_2}\right\},$$

and

$$[g, ad_f g] = 2\frac{\partial}{\partial \theta},$$

so that $\mathscr{D}$ is not involutive.

A necessary condition for transverse feedback linearizability of (1) can be formulated in terms of the distribution

$$\mathscr{D}_1 := \text{Involutive closure of } \mathscr{D} \quad (\text{i.e.}, \mathscr{D}_1 = \mathscr{D} + [\mathscr{D}, \mathscr{D}] + \cdots).$$

Namely, we have the following result.

Proposition 2.1. *Let η be a periodic orbit of the undriven system* (1) *with $u \equiv 0$. Then, the system* (1) *is transversely feedback linearizable along the periodic orbit η only if the distribution $\mathscr{D}_1$ is at most $(n-1)$-dimensional.*

Proof. Expressing $ad_f^i g, i = 0, \ldots, n-3$, in (θ, ρ)-coordinates (cf. (10)), we see that $\mathscr{D}$, and hence $\mathscr{D}_1$, is spanned by a combination of vector fields $\partial/\partial\theta$, $\partial/\partial\rho_i$, $i = 2, \ldots, n-1$. But $\dim \operatorname{span}\{\partial/\partial\theta, \partial/\partial\rho_2, \ldots, \partial/\partial\rho_{n-1}\} = n-1$. □

It happens that if (1) is transversely linearly controllable and $\dim \mathscr{D}_1 = n-1$ in a neighborhood of η then (1) is *locally transversely feedback linearizable* in the sense that one can cover a neighborhood of η by a finite number m of (overlapping) coordinate charts (θ^j, ρ^j), $j = 1, \ldots, m$, in which the dynamics have form (2). We have the following result.

Theorem 2.3. *Let η be a periodic orbit of the undriven system* (1) *with $u \equiv 0$. Then, the system* (1) *is locally transversely feedback linearizable along the periodic orbit η if it is transversely linearly controllable along η, $\dim \mathscr{D}_1 = n-1$ in a neighborhood of η, and $f \in \mathscr{D}_1$ on η.*

Proof. Suppose that η is parametrized by $\theta \in S^1 = [0, T]$, with 0 and T being identified, and $L_f\theta = 1$ on η. Let v be any smooth vector field such that f, v, g, $ad_f g, \ldots, ad_f^{n-3} g$ are linearly independent in a neighborhood of η. (It follows from the transverse linear controllability that such v exists, for instance, one can choose $v := ad_f^{n-2} g$.) Note that v is transversal to $\mathscr{D}_1$ in a neighborhood of η. As in the proof of part ($\Leftarrow$) of Theorem 2.1, we construct $s_0, s_1, \ldots, s_{n-1}$-coordinates in a neighborhood $\mathscr{N}$ of η by *traveling* along vector fields $f, v, g, ad_f g, \ldots, ad_f^{n-3} g$ from a distinguished point $x_0 \in \eta$ corresponding to $\theta = 0$. Let $\mathscr{M} := \{x \in \mathscr{N} \mid s_2(x) = s_3(x) = \cdots s_{n-1}(x) = 0\}$ (i.e., $\mathscr{M}$ is the set of points in $\mathscr{N}$ that can be reached from

Theorem 2.4. *Let η be a periodic orbit of the undriven system* (1) *with $u \equiv 0$. Assume that* $\dim \mathscr{D}_1$ *is constant in a neighborhood of η. Then, the system* (1) *is locally transversely feedback linearizable along the periodic orbit η if and only if it is transversely linearly controllable along η, and either dim $\mathscr{D}_1 = n-2$ in a neighborhood of η (i.e., $\mathscr{D}_1 = \mathscr{D}$), or dim $\mathscr{D}_1 = n-1$ in a neighborhood of η, and $f \in \mathscr{D}_1$ on η.*

Observe that for systems in $\mathbb{R}^3$ transverse feedback linearizability condition is generic with respect to *points*. One might tend to think that it is also a generic condition with respect to *orbits*. This is not the case. It is true that the set of transversely linearizable periodic orbits is *open*. But it is not *dense*. To see that, consider f and g such that there is a two-dimensional surface Ω in $\mathbb{R}^3$ with the property that $\det[f, g, ad_f g] = 0$ on Ω and $\det[f, g, ad_f g]$ changes sign on Ω. Note that transverse linear controllability fails on Ω. A small perturbation of f and g *perturbs* Ω a little, but Ω does not disappear. If a periodic orbit of f intersects Ω *transversely*, a periodic orbit of a perturbed system (if it persists, which it does if the Floquet multipliers have absolute values different from 1) will also intersect the (perturbed) set Ω. Thus, transverse feedback linearizability fails even when f and g are slightly perturbed.

3. Examples

Example 3.1.

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_3 + x_1 u, \\ \dot{x}_2 &= -x_1 + x_2 x_3 + x_2 u, \\ \dot{x}_3 &= u. \end{aligned} \tag{16}$$

Note that the undriven system ($u \equiv 0$) has a family of periodic orbits

$$\eta_R = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = R^2, x_3 = 0\}.$$

We have $ad_f g = -x_1 \partial/\partial x_1 - x_2 \partial/\partial x_2$ and $\det[f, g, ad_f g] = x_1^2 + x_2^2$, so that the system is transversely linearly controllable along any periodic orbit η_R if $R > 0$. (Note also that since $ad_f^2 g = 0$ the system is not linearly controllable.) The system is trivially (globally) transversely feedback linearizable as $\mathscr{D} = \text{span}\,\{g\}$ is a one-dimensional (and hence involutive) distribution. It is easy to verify that the functions $\alpha_1(x) := x_1 e^{-x_3}$ and $\alpha_2(x) := x_2 e^{-x_3}$ are constant along g. One can observe that the function

$$\alpha := \log \frac{(\alpha_1(x)^2 + \alpha_2(x)^2)^{1/2}}{R} - x_3 = \tfrac{1}{2} \log(x_1^2 + x_2^2) - \log R - x_3$$

is zero on η_R. We have $L_f \alpha = x_3$, $p(x) := L_f^2 \alpha = 0$ and $r(x) := L_g L_f \alpha = 1$. Defining

$$\begin{aligned} \theta &:= -\tan^{-1}\left(\frac{x_2}{x_1}\right), \\ \rho_1 &:= \tfrac{1}{2} \log(x_1^2 + x_2^2) - \log R - x_3, \\ \rho_2 &:= x_3 \end{aligned} \tag{17}$$

(an appropriate definition of $\tan^{-1}$ is used to define θ for all $(x_1, x_2) \neq (0,0)$) the system (in (θ, ρ) coordinates) is given by

$$\begin{aligned} \dot{\theta} &= 1, \\ \dot{\rho}_1 &= \rho_2, \\ \dot{\rho}_2 &= u. \end{aligned} \tag{18}$$

Note that no preliminary feedback is needed to put the system into the form (2).

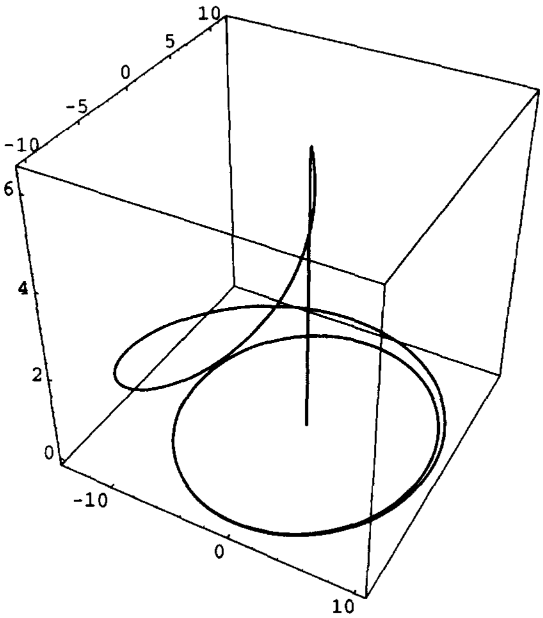

Fig. 1. Closed-loop trajectory.

The system expressed in (θ, ρ) coordinates can, for example, be used in the design of a stabilizing feedback. Provided the coordinate change maps *onto* $S^1 \times \mathbb{R}^{n-1}$, i.e., ρ is *unrestricted*, the domain of attraction of η_R will coincide with the region (in $\mathbb{R}^3$) on which the change of coordinates is one-to-one.

Fig. 1 shows the closed-loop trajectory for an initial condition close to the origin for the feedback $u = -\rho_1 - \sqrt{3}\rho_2$.

Example 3.2. Consider the system defined on $S^1 \times \mathbb{R}^3$ (which can be thought of as being embedded in $\mathbb{R}^4$)

$$\begin{aligned} \dot{\theta} &= 1 + x_3^2, \\ \dot{x}_1 &= x_2 + \gamma(\theta)x_1x_3^2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \end{aligned} \tag{19}$$

where $\theta \in S^1 = [0, T]$, with 0 and T being identified, $\gamma(\theta)$ is a smooth periodic function. The undriven system has a periodic orbit $x_1 = x_2 = x_3 = 0$, $\theta \in S^1 = [0, T]$. $(x_1, x_2, x_3) \in \mathbb{R}^3$ represent the transverse dynamics. One can verify that $\mathscr{D}_1 = \text{span}\,\{\partial/\partial x_2, \partial/\partial x_3, \partial/\partial\theta + \gamma(\theta)x_1\partial/\partial x_1\}$. One can check that the system is transversely linearly controllable, $f \in \mathscr{D}_1$ for $x \in \eta$, and $\dim \mathscr{D}_1 = 3$ on $S^1 \times \mathbb{R}^3$, so that the hypothesis of Theorem 2.3 is satisfied, and thus the system is locally transversely linearizable. To find a (locally) linearizing function α one can solve the set of PDE's $\partial\alpha/\partial x_2 = 0, \partial\alpha/\partial x_3 = 0, \partial\alpha/\partial\theta + \gamma(\theta)x_1\partial\alpha/\partial x_1 = 0$. A general solution of this set of PDE's is of the form $\alpha = F(x_1 e^{-\int\gamma(\theta)\,d\theta})$ (an additional requirement $F(0) = 0$ guarantees that α vanishes on η). In general, this solution is only local, for the function $e^{-\int\gamma(\theta)d\theta}$ does not have to be periodic in θ. For instance, for $\gamma(\theta) = 1$, we have $\alpha = F(x_1 e^{-\theta})$, which is not periodic in θ, so that the system is only locally transversely linearizable. However, if $\int_0^T \gamma(\theta)\,d\theta = 0$, then α is periodic in θ, and construction is global around η. For instance, for $T = \pi$, $\gamma(\theta) = \cos(\theta)$, we have $\alpha = F(x_1 e^{\sin(\theta)})$, which is periodic in θ, and hence the system is globally transversely linearizable.

To illustrate the proof of Theorem 2.3, observe that one can choose $v = \partial/\partial x_1$. As $\mathcal{M}$ one can choose the cylinder $S^1 \times R$. Then $\tilde{f} = \partial/\partial\theta + \gamma(\theta)x_1\partial/\partial x_1$. Note that the orbits of $\tilde{f}$ close up (i.e., the flow of $\tilde{f}$ is periodic) if and only if $\int_0^T \gamma(\theta)\,\mathrm{d}\theta = 0$.

4. Conclusion

We have presented necessary and sufficient conditions for global and local transverse feedback linearizability of an affine single-input nonlinear system about a periodic orbit. These conditions are similar in nature to the well-known conditions for feedback linearization. The application of these results was shown using a system defined on $\mathbb{R}^3$.

Transverse feedback linearization can be used as one step in the design of a controller for stabilizing the periodic orbit. More importantly, the transverse feedback linearization procedure provides a technique for finding coordinates that are tuned to the control structure of the system with respect to the periodic orbit. Indeed, these techniques are applicable to a much larger class of orbits (e.g., maneuvers). For this reason, we expect these results to be valuable in the analysis and design of more general *maneuvering* control systems.

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